Ejercicio 1.20)

a) Demostrar que $\nabla(fg) = f\nabla g + g\nabla f$:

$$\nabla(fg) = \left(\frac{\partial(fg)}{\partial x}, \ \frac{\partial(fg)}{\partial y}, \ \frac{\partial(fg)}{\partial z} \right)$$

Aplicando la regla del producto en cada componente:

$$= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, \ f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}, \ f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z} \right)$$

Factorizando f y g:

$$= f\left(\frac{\partial g}{\partial x}, \ \frac{\partial g}{\partial y}, \ \frac{\partial g}{\partial z} \right) + g\left(\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial f}{\partial z} \right) = f\nabla g + g\nabla f$$

i) Demostrar que $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$:
Sea $\mathbf{A} = (A_x, A_y, A_z)$ y $\mathbf{B} = (B_x, B_y, B_z)$. El producto cruz es:

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y, \ A_zB_x - A_xB_z, \ A_xB_y - A_yB_x)$$

La divergencia se calcula como:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)$$

Expandiendo las derivadas y reagrupando términos:

$$= B_z\frac{\partial A_y}{\partial x} - B_y\frac{\partial A_z}{\partial x} + B_x\frac{\partial A_z}{\partial y} - B_z\frac{\partial A_x}{\partial y} + B_y\frac{\partial A_x}{\partial z} - B_x\frac{\partial A_y}{\partial z}$$

Reconociendo los rotacionales:

$$= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

c) Demostrar que $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$:
El rotacional en componentes es:

$$\nabla \times (f\mathbf{A}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix}$$

Calculando las componentes:

$$\hat{\mathbf{i}}\left(\frac{\partial(fA_z)}{\partial y} - \frac{\partial(fA_y)}{\partial z} \right) - \hat{\mathbf{j}}\left(\frac{\partial(fA_z)}{\partial x} - \frac{\partial(fA_x)}{\partial z} \right) + \hat{\mathbf{k}}\left(\frac{\partial(fA_y)}{\partial x} - \frac{\partial(fA_x)}{\partial y} \right)$$

Aplicando la regla del producto en cada término:

$$= f\,(\nabla \times \mathbf{A}) + (\nabla f \times \mathbf{A})$$

Reordenando:

$$= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$