SOLUTIONS FOR THE 11^{TH} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

Team Bulgaria

Graphs of Finite Groups

Author: Ivan Durev

Abstract

abstract-text

Graphs of Finite Groups

1. Firstly, we are going to prove that $1' = 0$.

   Solution. $(m.1)' = m'.1 + m.1' = m'$
   $\Longrightarrow m.1' = 0$ for every $m \in \mathbb{Q}$
   Therefore, $1' = 0$. ■

2. Now we are going to find the value of $(-1)'$.

   Solution. $1' = [(-1)(-1)]' = (-1)'(-1) + (-1)(-1)' = 0$
   Therefore, $(-1)' = 0$ ■

3. We are going to prove that for every $n \in \mathbb{Q} : -n' = (-n)'$

   Solution. $((-1)n)' = (-1)'n + (-1)n' = -n'$ ■

4. **Lemma 1.** Next, we are going to show that:

$$\left(\frac{m}{n}\right)' = \frac{m'n - n'm}{n^2}$$

   Solution.

$$m' = \left(\frac{m}{n}n\right)' = \left(\frac{m}{n}\right)'n + \left(\frac{m}{n}\right)n'$$
$$\Longrightarrow \left(\frac{m}{n}\right)' = \frac{m'n - n'm}{n^2}$$ ■

5. **Lemma 2.** For every $\alpha \in \mathbb{Z}/\{0\}$ and p, which is prime the equation is held:

$$(p^{\alpha})' = \alpha.p^{\alpha-1}$$

(1) $\alpha > 0$ *Solution.* We are going to use induction.
Base. $\alpha = 1$

$$(p^1)' = 1.p^0 = 1$$

Induction hypothesis. $(p^{\alpha})' = \alpha.p^{\alpha-1}$
Inductive step. We need to prove that $(p^{\alpha+1})' = (\alpha+1).p^{\alpha}$

$$(p^{\alpha}.p)' = (p^{\alpha})'p + (p^{\alpha})p' = \alpha p^{\alpha-1}p + p^{\alpha}.1 = (\alpha+1).p^{\alpha}$$ ■

(2) $\alpha < 0$ *Solution.*

$$p^{\alpha} = \frac{1}{p^{|\alpha|}}$$

Using **Lemma 1.:**

$$(p^{\alpha})' = \left(\frac{1}{p^{|\alpha|}}\right)' = \frac{1'p^{|\alpha|} - 1.(p^{|\alpha|})'}{p^{2|\alpha|}} = \frac{\alpha.p^{|\alpha|-1}}{p^{-2\alpha}} = \alpha.p^{\alpha-1}$$ ■

Problem 1a. *Solution.* Every rational number $n > 0, n \neq 1$ can be uniquely described as:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_k^{\alpha_k}$$

where p_i are different primes and $\alpha_i \in \mathbb{Z}/\{0\}$

Lemma 3. We are going to prove by induction that

$$n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$$

Base. $k = 1 : n' = n\frac{\alpha_1}{p_1} = \alpha_1 p^{\alpha_1 - 1}$ (from **Lemma 2.**)
Induction hypothesis. We suppose that $n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$
Inductive step. We need to prove that $n' = n \sum_{i=1}^{k+1} \frac{\alpha_i}{p_i}$
Let n be a natural number with k prime divisors and $m = np_{k+1}^{\alpha_{k+1}}$. Then:

$$m' = (np_{k+1}^{\alpha_{k+1}})' = n' p_{k+1}^{\alpha_{k+1}} + \alpha_{k+1} p_{k+1}^{\alpha_{k+1}-1} n =$$
$$= \left(n \sum_{i=1}^{k} \frac{\alpha_i}{p_i} \right) p_{k+1}^{\alpha_{k+1}} + \alpha_{k+1} p_{k+1}^{\alpha_{k+1}-1} n = \left(np_{k+1}^{\alpha_{k+1}} \right) \sum_{i=1}^{k+1} \frac{\alpha_i}{p_i} = m \sum_{i=1}^{k+1} \frac{\alpha_i}{p_i}$$

Lemma 3. is proven by induction.

If $n \in \mathbb{Q}^-$, then $n' = -(-n)'$ (from 3.)
So as a summary of what we have proven so far:

If $n \in \mathbb{Q}/\{-1, 0, 1\} \implies n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$, where $n = \frac{|n|}{n} \prod_{i=1}^{k} p_i^{\alpha_i}$

If $n \in \{-1, 0, 1\} \implies n' = 0$

For every $n \in \mathbb{Q}$ there is a unique value of the function n'. Therefore, there is an unique function. ■

Problem 1b. *Solution.*

Lemma 4. If $n \in \mathbb{N}/\{1\} \implies n' > 0$

Proof. $n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$ (from **Lemma 3.**)

$$n \in \mathbb{N} \implies \alpha_i > 0 \text{ for every } i \in \mathbb{N}$$

$$\implies \sum_{i=1}^{k} \frac{\alpha_i}{p_i} > 0 \implies n' > 0$$

Corollary. If $n \in \mathbb{Z}$ and $n \leq -2 \implies n' < 0$

Now, if $L > 0$:

$$
\begin{aligned}
& (4\lceil L \rceil)' = 4'\lceil L \rceil + 4\lceil L \rceil' = 4(\lceil L \rceil' + \lceil L \rceil) \\
\implies & \lceil L \rceil \geq 1 \implies \lceil L \rceil' \geq 0 \\
\implies & (4\lceil L \rceil)' = 4(\lceil L \rceil' + \lceil L \rceil) \geq 4\lceil L \rceil > \lceil L \rceil \geq L \\
& (-p)' = -1 < L < (4\lceil L \rceil)', \text{ where } p \text{ is a prime.}
\end{aligned}
$$

If $L = 0$

$$(-p)' = -1 < L < p' = 1$$

If $L < 0$

$$
\begin{aligned}
& (4\lfloor L \rfloor)' = 4(\lfloor L \rfloor' + \lfloor L \rfloor) < \lfloor L \rfloor \leq L \\
& (4\lfloor L \rfloor)' < L < p' = 1, \text{ where } p \text{ is a prime.}
\end{aligned}
$$

The same is also held for x'':
If $L > 0$:

$$
\begin{aligned}
& (4\lceil L \rceil)'' = [4(\lceil L \rceil + \lceil L \rceil')]' \\
& \text{Let } \lceil L \rceil + \lceil L \rceil' = m \\
\implies & (4\lceil L \rceil)'' = (4m)' = 4(m' + m) \geq 4m \geq (4\lceil L \rceil)' \\
\implies & p'' = 0 < L < (4\lceil L \rceil)'', \text{ where } p \text{ is a prime.}
\end{aligned}
$$

If $L = 0$:

$$
\begin{aligned}
& (pq)'' = (p+q)' > 0, \text{ where } p \text{ and } q \text{ are primes. (from } \textbf{Lemma 4.}) \\
\implies & (-pq)'' < L < (pq)''
\end{aligned}
$$

If $L < 0$ it can be analogically proved that:

$$(4\lfloor L \rfloor)'' < L < p'' = 0, \text{ where } p \text{ is a prime.} \qquad \blacksquare$$

Problem 2a. *Solution.*
If $a = 1$:

$$
\begin{aligned}
& p' = 1, \text{ for every prime} \\
\implies & \text{The equation } x' = 1 \text{ has infinitely many solutions.} \\
& \text{Also, solutions are the numbers } -\frac{5}{4} \text{ and } -\frac{58}{27}
\end{aligned}
$$

If $a = -1$:

$$
\begin{aligned}
& p' = -1, \text{ for every prime} \\
\implies & \text{The equation } x' = -1 \text{ has infinitely many solutions.} \\
& \text{Also, solutions are the numbers } \frac{5}{4} \text{ and } \frac{58}{27}
\end{aligned}
$$

If $a = 0$:

Let n be such a number so that $n' = 0$

$(-n)' = 0 \implies$ Without loss of generality $n > 0$, $n \neq 1$

$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_k^{\alpha_k}$ and $n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$ (from **Lemma 3.**)

As $n > 0 \implies \sum_{i=1}^{k} \frac{\alpha_i}{p_i} = 0$

If we multiply both sides by $P = \prod_{i=1}^{k} p_i$ we get $\prod_{i=1}^{k} p_i \sum_{i=1}^{k} \frac{\alpha_i}{p_i} = 0$

$p_i \mid 0$ and $p_i \mid \alpha_j \frac{P}{p_j}$ for every i and j so that $i \neq j$

$\implies p_i \mid \alpha_i \frac{P}{p_i} \implies p_i \mid \alpha_i$ for every i

Let $\alpha_i = a_i p_i$

$\implies \sum_{i=1}^{k} a_i = 0$

$\implies$ All solutions of $x' = 0$ are of the form:

$n = \pm \prod_{i=1}^{k} p_i^{a_i p_i}$, where $\sum_{i=1}^{k} a_i = 0$ and $n \in \{-1, 0, 1\}$

Therefore, there are infinitely many solutions.

■

Problem 2b. *Solution.*

Lemma 5. $n \in \mathbb{Q}$ and $\alpha \in \mathbb{Z} \implies (n^\alpha)' = \alpha n^{\alpha-1} n'$

Proof.

If $\alpha = 0 \implies (n^0)' = 1' = 0$

If $\alpha > 0$ we are going to use induction.

Base. $\alpha = 1 \implies (n^1)' = 1.n^0.n'$

Induction hypothesis. $(n^\alpha)' = \alpha n^{\alpha-1} n'$

Inductive step. $(n^{\alpha+1})' = n'n^\alpha + (n^\alpha)'n = n'n^\alpha + \alpha n^{\alpha-1} n' n =$
$= (\alpha + 1) n^\alpha n'$

If $\alpha < 0 \implies (n^\alpha)' = (\frac{1}{n^{|\alpha|}})' = \frac{1'n^{|\alpha|} - (n^{|\alpha|})'.1}{n^{2|\alpha|}} = \frac{-|\alpha| n^{\alpha-1}}{n^{2|\alpha|}} =$
$= -|\alpha| \frac{1}{n^{|\alpha|+1}} n' = \alpha n^{\alpha-1} n'$

Lemma 5. is now proven

Let $w(x) = \frac{x'}{x}$, defined for $x \in \mathbb{Q}/\{0\}$, be the logarithmic derivative
$\implies w(1) = w(-1) = 0$
The function has the following properties:

Property (1). $w(x) = w(-x)$, for every $n \in \mathbb{Q}/\{0\}$

Property (2). $w(mn) = \dfrac{(mn)'}{mn} = \dfrac{m'n + mn'}{mn} = \dfrac{m'}{m} + \dfrac{n'}{n} = w(m) + w(n)$

Therefore, $w(x)$ is an additive function:

$$w(mn) = w(m) + w(n), \text{ for every } m, n \in \mathbb{Q}/\{0\}$$

For every $n \in \mathbb{Q}/\{0\}$:
$n = \pm p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ and $w(n) = \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$ (from **Lemma 3.**)

Let's start with $b = 0$ and solve $x' = ax$:

$$a = 0 \implies \forall x : x' = 0$$
$$a \neq 0 \text{ If } x = 0 \implies 0' = a.0$$
$$\text{If } x \neq 0 \implies \frac{x'}{x} = a \iff w(x) = a \iff \sum_{i=1}^{k} \frac{\alpha_i}{p_i} = a$$

Let's prove the following **statement**:

$$w(x) = a \text{ has a solution } \iff \text{The denominator of } a, \text{ in lowest terms, is square-free.}$$

Proof of the "$\implies$" direction.

$$\sum_{i=1}^{k} \frac{\alpha_i}{p_i} = a \implies \text{In the denominator of } a$$

there is every prime number p_i
to the power of 1 at most.

Proof of the "$\impliedby$" direction.

$$a = \frac{M}{p_1 p_2 ... p_k}, \; M \in \mathbb{Z} \text{ and } (M; p_1 p_2 ... p_k) = 1$$

Induction base. $k = 1 : a = \dfrac{M}{p_1} = p_1^M$

Induction hypothesis. $w(x) = a$ has a solution when the denominator of a has k different prime divisors.

Inductive step. $a = \dfrac{M}{p_1 p_2 ... p_k p_{k+1}} \implies$ Using Bézout's identity, there exist infinitely many integers u and v, such that: $u.p_1 p_2 ... p_k + v.p_{k+1} = 1$

$$\implies \frac{uM}{p_{k+1}} + \frac{vM}{p_1 p_2 ... p_k} = \frac{M}{p_1 p_2 ... p_k}$$

Now, the ***statement*** *is proven*

As $(p^\alpha y)' = \alpha p^{\alpha-1} y$, for $\forall y : y' = 0$, and using **Problem 2a**:
$\implies$ There exist infinitely many such numbers y.
$\implies$ The equation $w(x) = a$ has infinitely many solutions, for a with square-free denominator, in its lowest terms, and has no solutions in other cases.
We have examined the case for $b = 0$ ■

Now, if $b \neq 0$