SOLUTIONS FOR THE 10^{TH} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

Team Bulgaria

Maximal Cardinalities of Bounded-angled Sets

Author:

Abstract

abstract-text

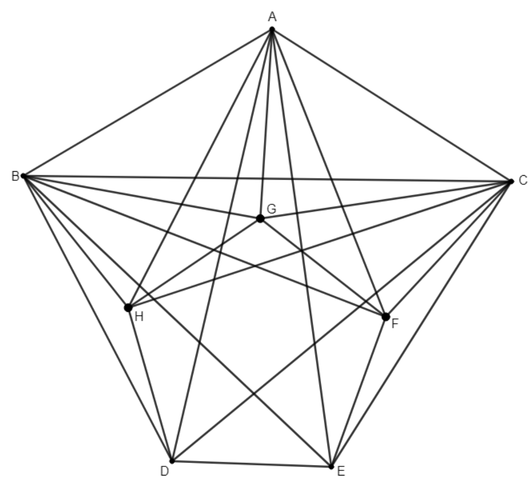
1 Coloring Graphs

1. *Solution.* Firstly, we will denote by:

   $cl(G)$ the clique of maximum length

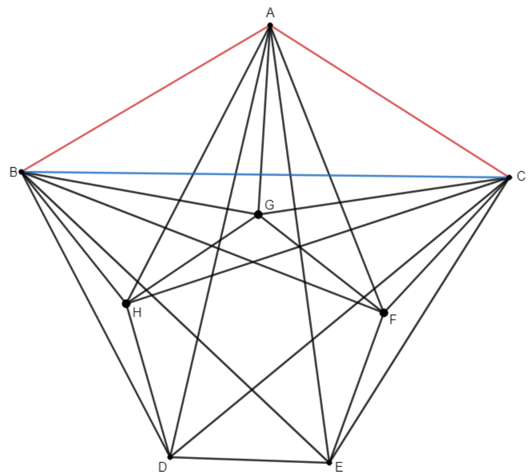
   K_n full graph with n vertices

   We are going to prove that the following graph G does not contain K_6 and for every red-blue edge colouring there exists one-coloured triangle. This means that G is part of the set $X_e(3, 3, 6)$.

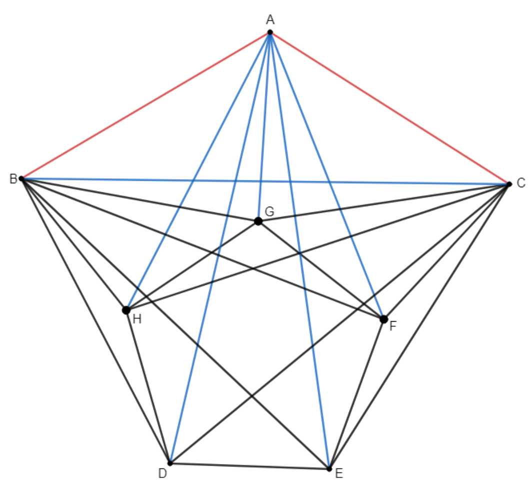
   

   This graph consists of the triangle ABC and the pentagon $DEFGH$. Every vertex from the triangle is connected with every vertex from the pentagon. It is obvious, that there doesn't exist a clique with length 6 because whichever 6 vertices we take, at least three of them will be from the cycle $\{D, E, F, G, H\}$. For every three points from this cycle, at least one pair is not connected by an edge.

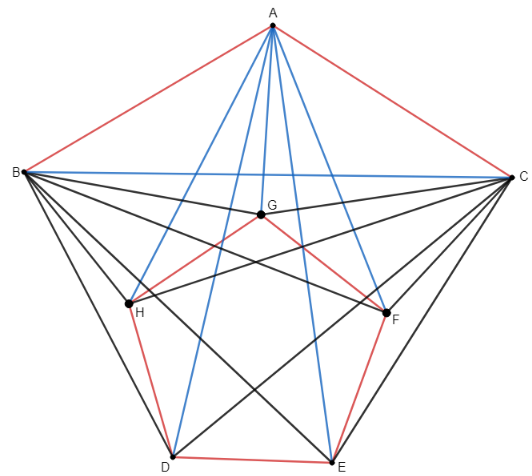
   Let us assume that there is no one-coloured triangle for at least one colouring. Thus, two of the edges of the triangle ABC are in one colour and the other one is different. Without loss of generality, let the AB and BC are in red and BC is blue.



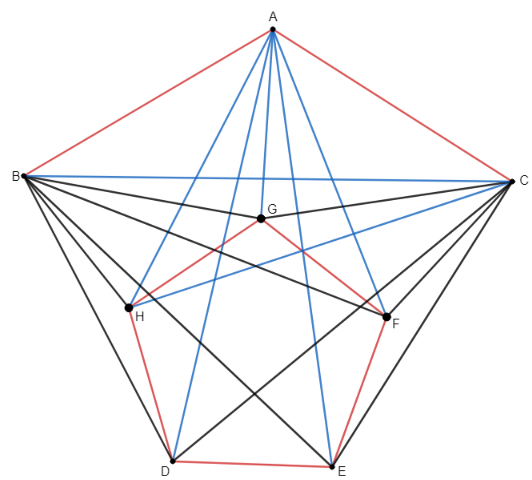
If we assume, that the edge AD is in red, no matter how we colour DB and DC at least one of the triangles ADB, BDC and CDA is one-coloured. Hence, all the edges to the pentagon from A are blue.



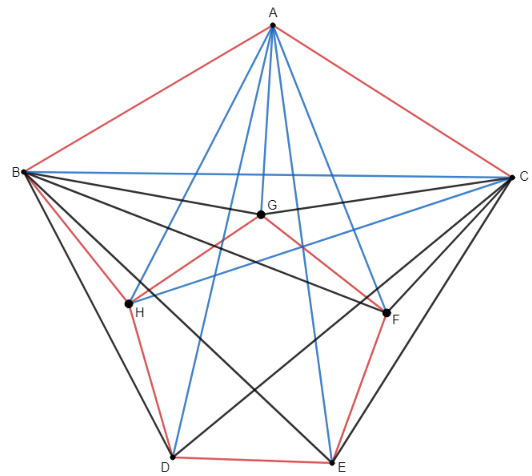
This leads us to the conclusion that the pentagon is red.



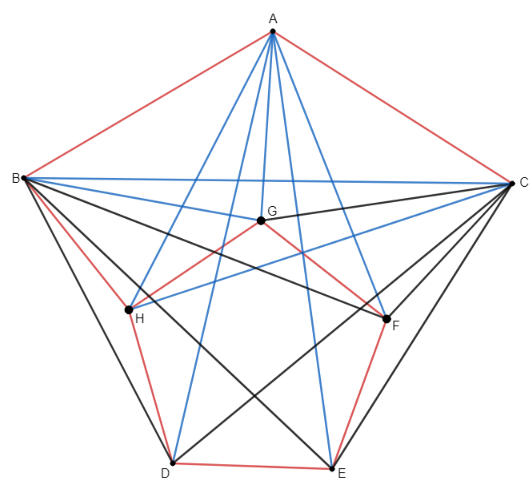
At least one of the edges CH and CG is blue. Let WLG be CH without loss of generality.



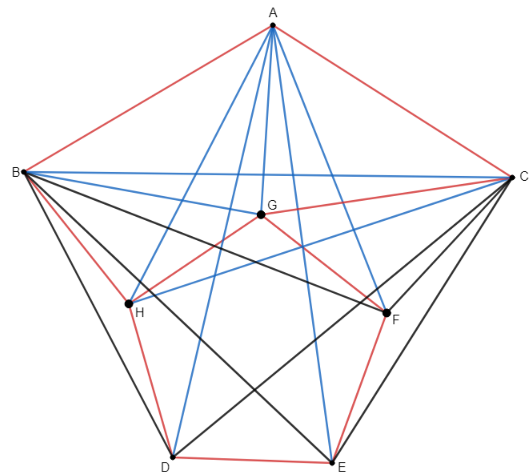
BH should be red, so that the triangle BHC is not blue.



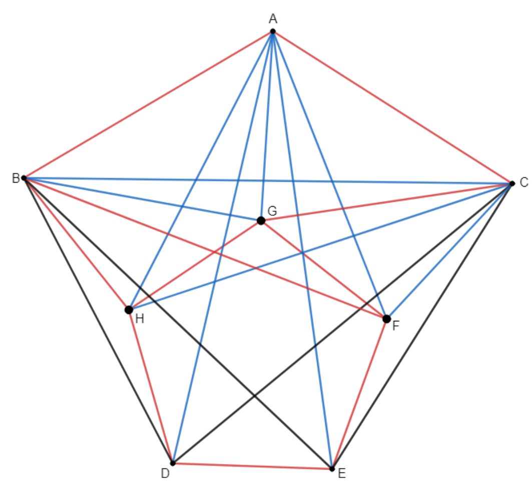
BG should be blue, so that BGH is not red.



CG should be red, so that BGC is not blue

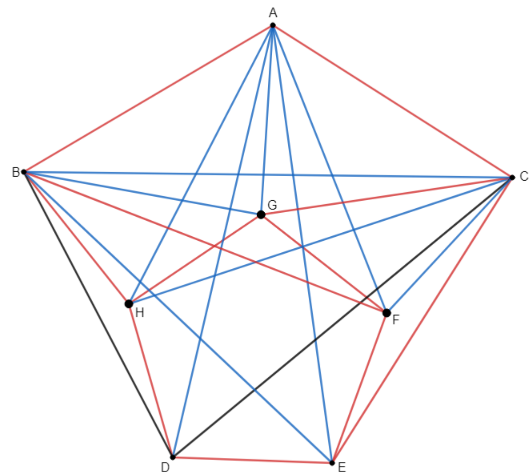


CF is blue



BF is red

BE is blue



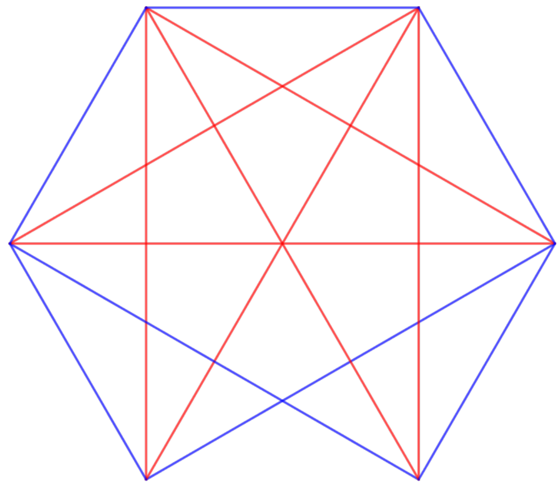
BD and DC are blue, so that BDH and CED are not red. We have a blue triangle BCD. ■

2. (a)

   (b)

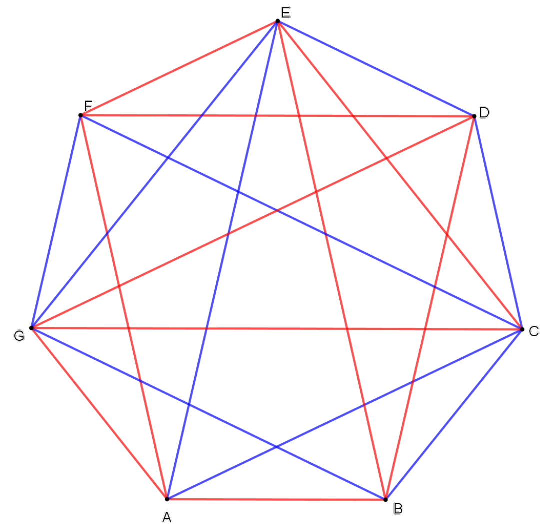
   (c) *Solution.* If graph G has less or equal to $r + b - 2$ vertices, then there exists a colouring with maximum of $r - 1$ red and $b - 1$ blue vertices. Obviously, there is no K_r or K_b monochromatic clique. Let us examine the complete graph K_{r+b-1}. According to the Pigeonhole principle, every colouring leads to at least r red vertices or b blue. The graph is complete and therefore a K_r or K_b clique exists. Thus, $F_v(r, b, p) = r + b - 1$ when $p > r + b - 1$. ■

3. (a) *Solution.* We are going to prove that $F_e(3, 3, 6)$ is equal to 8. We have already given an example for a graph with 8 vertices. Ramsey's number $R(3, 3) = 6$ and therefore our graph must have at least 6 vertices. In order to have 6 vertices and $cl(G) < 6$ to be fulfilled, G must be a subgraph of H, which is K_6 without one edge. An example of a colouring of the graph H that has no one-coloured triangle:



Therefore, $F_e(3,3,6) > 6$.

Let's suppose the graph G has 7 vertices (A, B, C, D, E, F, G). If G is complete, there must be K_6 in it. Let's suppose in G only one edge is missing for it to be complete. We will denote it by AB without loss of generality. Hence, $BCDEFG$ is complete. Therefore, at least 2 edges are missing for it to be complete. If all missing edges for the graph to be complete have a common vertex, the graph made of the rest of the vertices will be complete. Thus, in order for K_6 to not be a part of the graph, at least 2 edges must be missing without a common vertex. Let H is K_7 with 2 edges missing without a common vertex. Every graph G, which does not contain K_6, must be a subgraph of H or a isomorphic to H. An example of red-blue colouring of H without a one-coloured triangle:



Therefore, $F_e(3,3,6) > 7$. As we have an example for a graph with 8 vertices, we can conclude $F_e(3,3,6) = 8$. ■

2 Mystic Powers of Two

The order of the solutions is not the same as the order of Problems. By a row we denote some P_i. We shall begin with proving some necessary facts from Problem No.5 We will look into only such initial data that there exists at least one coloring. It should be noted that permutations of the order of initial data $\{a_i, b_i\}$ give the same number of colourings so only such arranged in some order will be examined.

1. **Problem 5c)** Prove that if s is mystic then $n_i = n_j$ for some $i < j$

   Solution. Let's suppose there are no two equal powers of two in s. If we have $k = m$ and initial data $\{(n_i, n_i)\}_{i=1}^{k}$, there will be only one coloring to match the sequence, which is said to be mystic. That is a contradiction and therefore there is a pair of powers of two in s that are equal. ■

2. **Problem 5b)** Prove that s is mystic if and only if for any $k \geq 1$ and for any initial data $\{(a_i, b_i)\}_{i=1}^{k}$ satisfying $(a_i, b_i) \neq (a_j, b_j)$ for $i \neq j$ there exists only even number of monotonic m-colourings of type s

   Solution. The condition required for a sequence to be mystic is partially covered - for initial data with no two equal sets of numbers. We will consider initial data with two or more coinciding corresponding sets. Let's call the data (a_1, b_1). If the corresponding colours in the equal sets differ by at least one, they can be swapped. This means that for every colouring of the rest of the numbers, there will be two options and therefore the final number of colourings will be even. If the colours in the equal sets match, then recursively we can consider them as one set consisting of the corresponding sums of the two. We can repeat that operation until we get to either the first "If" or a data consisting entirely of different sets, both of which prove the sequence to be mystic. ■

3. **Problem 5d)** Assume s is mystic. Prove that if a maximal power of two in s occurs only once then one can delete it and get a mystic sequence of length $m-1$.

   Solution. We are going to call the maximal power of two 2^x and the sequence without it s'. Let's suppose s' is not mystic. This means there is at least one data of type s' that leads to an odd number of colourings. If we add to that one more "row" (x, x) and aim to achieve type s, we must colour the new addition in some new colour so as to achieve the maximal power of two or otherwise this (x, x) will not be coloured as it provides greater sums than any of the numbers. We know there is an even number of colourings in this situation because s is mystic. But as the data we are looking into can be $m-1$-coloured in an odd number of ways, and there is no choice about which colour to put on the new "row", we reach a contradiction. Therefore the sequence s' is mystic. ■

4. **Problem 5a)** Prove that if s is mystic then for any $n \geq 0$ sequences $(2^n, 2^{n_1}, ..., 2^{n_m})$ and $(2^{n_1}, ..., 2^{n_m}, 2^n)$ are mystic too.

   Solution. The sequence s is mystic which means that every initial data has an even number of m-colourings. We will denote by s' the new sequence with added 2^n. Let's take a data which has a positive number of $m+1$-colourings of type s'. As 2^n is either last or first, its corresponding colour will also be put respectively on a last or a first number in a row, because in every row the colours are strictly increasing. When in last place, the new numbers, which make up 2^n (it can be only one - 2^n) don't affect the other colors which, we know, have even possible situations. Since the colourings for the two groups are not related, we will multiply the number of colourings of the two. The group not containing the new colours has an even number of colourings because the initial sequence is mystic. This means the total is even too and thus, sequence $(2^{n_1}, ..., 2^{n_m}, 2^n)$ is mystic. Absolutely identical is the proof when $2^n isaddedbefores$, apart from the fact we should increase every colour in the initial data by one, and use only the first colour for the new numbers which should always be first. ■

5. **Problem 1)** Prove that there exists only one mystic sequence (of powers of two) of length two.

   Solution. Let's suppose that a mystic sequence(s) of length two can consist of two different powers of two - 2^x and 2^y. Then it should be true that for

$k = 2$ and initial data $\{(x, x), (y, y)\}$, where $x, y \in \mathbb{N}_0$, an even number of monotonic m-colorings of type s exist. This generates the corresponding sets $P_1 = \{2^x\}$ and $P_2 = \{2^y\}$ However, only one coloring is possible in order to achieve the desired *type*. Thus, a mystic sequence of length two cannot contain two different numbers.

If a mystic sequence of length two exists it can be expressed as $(2^x, 2^x)$ where $x \in \mathbb{N}$. Therefore the same condition must be true again. If we try with $k = 2$ and initial data $\{(x-1, x), (x-1, x-1)\}$ we get the sets $P_1 = \{2^{x-1}, 2^x\}$ and $P_2 = \{2^{x-1}\}$. We must have two colors so as to achieve the *type*. This means that φ_1 should have two outputs - 1 and 2. Combined with the fact that the function is strictly increasing, we get only one coloring - where the two $x-1$ powers are of color 1 and the x power of color 2. Therefore, we shall conclude no sequence of the type $(2^x, 2^x)$ is mystic for $x \in \mathbb{N}$.

According to the above, the only sequence that could be mystic of length two is $(1, 1)$. Having data with $b_i \geq 1$, for some i, is irrelevant because no coloring will reduce the number 2^{b_i} to 1. The smallest number that could be in some set P_j is 1. Given that the sets are increasing, we can conclude sets with more than one power of two in them will lead to zero colorings. This means the only initial data that will have a positive amount of $2-colorings$ is $\{(0, 0), (0, 0)\}$ for $k = 2$. It can be easily confirmed two colorings exist. Therefore the only mystic sequence of length two is $(1, 1)$. ■

6. **Problem 2)** A mystic sequence of powers of two is called minimal if $(2^{n_1}, 2^{n_2}, ..., 2^{n_{m-1}})$ and $(2^{n_2}, 2^{n_3}, ..., 2^{n_m})$ are not mystic. Prove that the following sequences are mystic: $(1, 1)$, $(1, 2, 2)$, $(2, 2, 1)$, $(2, 2, 2, 2)$, $(1, 2, 1, 2, 1)$, $(2, 1, 2, 1, 2)$. Which of them are minimal? Prove that there exists no minimal mystic sequences with components 1's or 2's other than those above.

   Solution. We will denote by C_i the corresponding colours of the numbers in P_i. As proven in Problem 5b) any initial data where $(a_i, b_i) = (a_j, b_j)$ for some $i \neq j$ has automatically an even number of m-colorings. So we shall not look into such.

   Sequence $(1, 1)$ is proven to be mystic in the previous task.

   For sequence $(1, 2, 2)$ three colours are used ($m = 3$). The sum of all numbers is 5. Therefore we can have a maximum $k = 5$ because more than that will lead to a greater sum. Obviously $k = 1$ doesn't work due to the fact each set consists of strictly increasing powers of two.

   - For $k = 2$ only the corresponding sets are suitable

     $$P_1 = \{1, 2\}, P_2 = \{2\}$$

     and can be coloured

$$C_1 = \{1,2\}, C_2 = \{3\}$$
$$C_1 = \{1,3\}, C_2 = \{2\}$$

They can be $3-colored$ in two ways.

- For $k=3$ only the corresponding sets work
$$P_1 = \{1\}, P_2 = \{2\}, P_3 = \{2\}$$
$$P_1 = \{1\}, P_2 = \{1\}, P_3 = \{1,2\}$$
They both contain matching "rows" so they can be coloured in even number of ways.
- For $k=4$ and $k=5$ in every data that can be coloured in a positive number of ways there will be matching "rows" of data and therefore an even number of colourings.

For sequence $(2,2,1)$ it can be proven similarly to the previous.

For sequence $(2,2,2,2)$ we have the following situation: the sum of all sets is 16, which is way greater than 6 (This is the sum of $\{1,2\},\{1\},\{2\}$). This means that we have two matching sets. From Problem 5b) it can be inferred that the sequence is mystic.

For sequence $(1,2,1,2,1)$ the sum of all sets is 7, which by the same logic as in the previous case leads to a mystic sequence.

For sequence $(2,1,2,1,2)$ the sum of all sets is 8, which by the same logic as in the previous case leads to a mystic sequence.

Let us see which sequences are minimal. We have the following statements:

- $\{1,2\}$ and $\{2,2\}$ are not mystic since in Problem 1 we have proven it. Thus, sequences $(1,2,2)$ and $(2,2,1)$ are minimal.
- Sequence $(2,2,2)$ is proven to not be mystic in Problem 6. This means that $(2,2,2,2)$ is minimal.
- Sequence $(1,2,1,2)$ is not mystic if we have the sets $P_1 = \{1,2\}, P_2 = \{1\}, P_3 = \{2\}$. There are three ways of colouring them.
- Sequence $(2,1,2,1)$ is not mystic if we have the sets $P_1 = \{1,2\}, P_2 = \{1\}, P_3 = \{2\}$. There is one way of colouring them.
- As a result of the previous two statements both sequences $(1,2,1,2,1)$, $(2,1,2,1,2)$ are minimal.

We shall prove that there are no other minimal mystic sequences only with components $1,2$. Let us assume, that there is a sequence (minimal and mystic) with 6 or more components. Then the sequences without the first and the last element shouldn't be mystic. They will have 5 or more components, which means that their sum will be at least 5. The sum 5 is reached only when it is $(1,1,1,1,1)$, which is obviously mystic (by Problem 3). The sum 6 is reached only when it is $(1,2,1,1,1)$(in some order), which is obviously mystic (by Problem 3). Now the sum is greater than 6. This is the sum of $\{1,2\},\{1\},\{2\}$. This means that we have two matching sets. From Problem 5b) it can be inferred that the sequence is mystic. ■

7. **Problem 3)** Find all mystic sequences of lengths $3,4,5$.

   Solution. Firstly, we are going to find all the mystic sequences with length 3. They are in the form $\{2^x,2^y,2^z\}$. We already know from Problem 5c) that two of the powers should be equal. Without loss generality we have x=z. We have the following cases:

   - $x > y+1$ Then we use the following data: $\{(x-1,x),(x,x),(y,y)\}$ which has an even number of colourings only when the sequence $(2^x,2^x)$ is mystic. So, from here we are left with the sequences $(1,1,2^y)$ in some order. They are obviously mystic, because the 1-s can be switched.
   - $x = y+1$ Then we use the following data: $\{(x-2,x-1),(x-1,x),(x-2,x-2)\}$. When x is greater than 1, we have only one colouring. Otherwise, we have the sequences (2,2,1) and (1,2,2). We shall notice that the sequence $(2^y,2^x,2^x)$ works with this example and the sequence $(2^x,2^y,2^x)$ can be proven not to be mystic with the initial data $\{(x-1,x),(x,x)\}$.
   - $x < y+1,\ x \neq y$ The order here is without importance. we take the initial data: $\{(x-1,x),(x,x),(y)\}$. We have only one colouring except from the case when the sequence (x,x) is mystic. Again, this leads to $(1,1,2^y)$ in some order.
   - $x = y$ We take initial data: $\{(x-2,x-2),(x-2,x-2),(x-2,x-1),(x-2,x)\}$ and have a contradiction.

   We are going to look through the case for length 4. Mystic sequences are in the form $(2^x,2^y,2^z,2^u)$ in some order. We already know from Problem 5c) that two of the powers should be equal. Without loss generality we have x=z.

   - When only two of the powers are equal analogically to the sequences with length 3, we get to $(1,1,2^y,2^u)$ in some order, which is mystic.
   - Three of the powers are equal. If the fourth one is greater than them,it can be deleted and the sequence $(2^x,2^x,2^x)$ should be mystic. This is impossible as proven above. If the fourth one is smaller than the other three, we have again two cases:
     (a) when $y < x-2$ We are going to use the data: $\{(x-2,x-2),(x-2,x-2),(x-2,x-1),(x-2,x),(y,y)\}$ and have a contradiction.
     (b) when $y = x-1$ We are going to use the data $\{(x-2,x-2),(x-1,x),(x,x),(x-2,x-1)\}$. Here we have 3 colourings, which can be checked if necessary.

(c) when $y = x - 2$
Data: $\{(x-1,x),(x,x),(x-1,x-1),(x-2,x-2)\}$

- If all of the powers are equal, see the matrice in Problem 6.

Now, we are going to take into account the sequences with length 5: by analogical conclusions, the only working sequences are $(1,1,2^x,2^y,2^z)$ in some order. ■

8. **Problem 4)** Prove that sequences $(4,4,4,4,4,4)$ and $(8,8,8,8,8,8,8,8)$ are mystic and minimal.

Solution. We have proven in Problem 5b) that if there are two equal sets in the data there is an even number of ways to colour the data. So to prove a sequence is mystic we have to prove that every data that has at least 1 colouring contains matching "rows". The data with maximal sum that has no equal sets is defined by:

$$\begin{gathered} k=6 \\ (a_1,b_1)=(0,2) \\ (a_1,b_1)=(1,2) \\ (a_1,b_1)=(0,1) \\ (a_1,b_1)=(2,2) \\ (a_1,b_1)=(1,1) \\ (a_1,b_1)=(0,0) \\ P_1=\{1,2,4\} \\ P_2=\{2,4\} \\ P_3=\{1,2\} \\ P_4=\{4\} \\ P_5=\{2\} \\ P_6=\{1\} \end{gathered}$$

The sum is 23. However, the sum of the six 4's is 24. Therefore there must be a repeating row. Following Problem 5b) the sequence $(4,4,4,4,4,4)$ is mystic.

It is minimal and for the proof that $(4,4,4,4,4)$ is not mystic see Problem 6. ■

9. **Problem 6)** Deduce for which n the sequence $(2^n,2^n,...,2^n)$ of length $2n+1$ is mystic.

Solution. Let's consider the following sets:
$P_1=\{1,2,4,8,...,2^n\}$
$P_2=\{2,4,8,...,2^n\}$
.
.

.
$P_n = \{2^{n-1}, 2^n\}$
$P_{n+1} = \{2^n\}$
$P_{n+2} = \{1, 2, 4, 8, ..., 2^{n-1}\}$

They form the following table:

$$\begin{array}{ccccccc}
1 & 2 & 4 & 8 & ... & 2^{n-1} & 2^n \\
2 & 4 & 8 & ... & 2^{n-1} & 2^n & \\
. & & & & & & \\
. & & & & & & \\
. & & & & & & \\
2^{n-1} & 2^n & & & & & \\
2^n & & & & & & \\
1 & 2 & 4 & 8 & ... & 2^{n-1} &
\end{array}$$

There is no choice but to colour every element 2^n in a different colour, which is used for only this number. Otherwise sums bigger than 2^n will occur. There are $n+1$ rows in which 2^n appears. The sum in every column is exactly 2^n, excluding 2^n as it is coloured already in its unique colour. There are n columns, excluding the last one, because it only contains 2^n. This means that if the numbers in each column, except 2^n have the same colour, we get n sums which are all equal to 2^n So in total this produces $n + 1 + n = 2n + 1$ powers, which is exactly the sequence we aim for.

We are going to prove this is the only way of colouring this data. In order to achieve the desired type, one must have all the sums we mentioned no matter how. We are going to call a "normal colour" the one that is equal to the number of the column. This means that in our initial colouring every number has a "normal colour" except 2^n. Suppose there is a column which has more than two colours (excluding 2^n). This means there is a number x which is not of a normal colour. Thus, another, $y = x$, has taken its place, because in order to achieve 2^n as a sum every previous power has to be in the sum. Let's take x to be with the smaller new colour without loss of generality. This means its colour is less than its normal one. Because each column has an increasing by 1 normal colour, if the number before x is of normal then x's colour is less or equal to it. This is a contradiction, because the colours must be strictly increasing in each row. If the previous is not normal then, another, z has its colour. We repeat that until we reach a number, which has a previous one normally coloured and use the above proof. We will reach such because we can't have the numbers before x all reduced by one and still be positive integers. This means a contradiction. Thus, no number can have a non-normal colour and therefore only one colouring exists which means the sequence $(2^n, 2^n, ..., 2^n)$ is not mystic. $\blacksquare$

10. **Problem 7)** Prove that the sequence $(2^n, 2^n, ..., 2^n)$ of length $2n + 2$ is mystic and minimal.

Solution.

3 On Some Sequences Generated by a Function

1. (a) *Solution.* Let $x_1, x_2, ...$ is a given sequence generated by some real number x_0. We define $f(x)$ as

$$f(x) = \left\{ \begin{array}{c} \dfrac{x_1}{x_0}x, \text{ when } x \in (0, x_0] \\ \dfrac{x_{i+1} - x_i}{x_0}x + (i+1)x_i - ix_{i+1}, \text{ when } x \in (ix_0, (i+1)x_0], \text{ for every } i \in \mathbb{N} \end{array} \right\}$$

Obviously $f(ix_0) = x_i$ is true for every $i \in \mathbb{N}$. For $x \in (0, x_0) \cup (ix_0, (i+1)x_0)$ the function is linear and therefore continuous, for every $i \in \mathbb{N}$. We now have to prove it is continuous in ix_0, for every $i \in \mathbb{N}$.

$$\lim_{x \to\ ix_0 - \varepsilon} f(x) = \lim_{\varepsilon \to\ 0} \left(\frac{x_i - x_{i-1}}{x_0}(ix_0 - \varepsilon) + ix_{i-1} - (i-1)x_i \right) =$$

$$= \lim_{\varepsilon \to\ 0} \left(x_i - \frac{x_i - x_{i-1}}{x_0}\varepsilon \right) = x_i$$

This is the left boundary of the function.

$$\lim_{x \to\ ix_0 + \varepsilon} f(x) = \lim_{\varepsilon \to\ 0} \left(\frac{x_{i+1} - x_i}{x_0}(ix_0 + \varepsilon) + (i+1)x_i - ix_{i+1} \right) =$$

$$= \lim_{\varepsilon \to\ 0} \left(x_i + \frac{x_{i+1} - x_i}{x_0}\varepsilon \right) = x_i$$

This is the right boundary of the function.
They are equal. Therefore, the function is continuous for every $i \geq 2, i \in \mathbb{N}$

For $i = 1$:

$$\lim_{x \to\ x_0 - \varepsilon} f(x) = \lim_{\varepsilon \to\ 0} \frac{x_1}{x_0}(x_0 - \varepsilon) = \lim_{\varepsilon \to\ 0}(x_1 - \frac{x_1}{x_0}\varepsilon) = x_1$$

$$\lim_{x\to\ ix_0+\varepsilon} f(x) = \lim_{\varepsilon\to\ 0}\left(\frac{x_2-x_1}{x_0}(x_0+\varepsilon)+2x_1-x_2\right) =$$

$$= \lim_{\varepsilon\to\ 0}\left(x_1+\frac{x_2-x_1}{x_0}\varepsilon\right) = x_1$$

Both are equal and therefore the function is continuous for $i = 1$ too. Thus, the function $f(x)$ is continuous for every $x \in (0, \infty)$ ■

(b) *Solution.* Let $x_0 \in (0, \infty)$ define the sequence $x_i = 2^i$, for every $i \in \mathbb{N}$.

$$|f(x) - f(y)| \leq |x - y|$$

Let $x = ix_0$ and $y = (i + 1)x_0$. Then the above will look like this:

$$|f(ix_0) - f\big((i + 1)x_0\big)| \leq x_0$$

which is equivalent to:

$$|2^i - 2^{i+1}| \leq x_0$$

$$2^i \leq x_0 \text{ for every } i \in \mathbb{N}$$

However, there is no real x_0 to solve this inequality. This means there is no such non-expanding function $f(x)$ for which the inequality is held. ■

2.

3. (a) *Solution.* If $f(x)$ is increasing (strictly increasing) it is obvious that every sequence $(x_n) \in L(f)$ is increasing (strictly increasing).
Let every sequence $(x_n) \in L(f)$ is increasing (strictly increasing). We are going to prove that $f(x)$ is increasing (strictly increasing). Let's suppose there are some values of $x \in (a, b)$ so that $f(x)$ is strictly decreasing for $x \in (a, b)$. There are infinitely many rational numbers in (a, b). Let's take two such rationals from the interval $\frac{p}{q}$ and $\frac{m}{n}$ such that

$$p, q, m, n \in \mathbb{N};\ gcd(p; q) = 1;\ \frac{p}{q} < \frac{m}{n}$$

We are going to look into the sequence with $x_0 = \frac{1}{qn}$. Thus:

$$x_{pn} = f(pnx_0) = f(\frac{p}{q})$$

$$x_{mq} = f(mqx_0) = f(\frac{m}{n})$$

$pn < mq$ but $f(\frac{p}{q}) > f(\frac{m}{n})$ which is a contradiction with the above.

Therefore, not every sequence $(x_n) \in L(f)$ is increasing (strictly increasing). Thus, $f(x)$ is increasing (strictly increasing). ■

4 Pressing Coloured Buttons

1. **Problem 1a)** Suppose that we can only press at least $t = 2$ buttons at a time, no two of which have the same colour. Find, with proof, a necessary and sufficient condition on $a_1, a_2, ..., a_n$, for which it is possible to get all buttons pressed.

   Solution. Let the numbers be $a_1 \leq a_2 \leq .... \leq a_n$ without loss of generality.

   We are going to prove that the necessary and sufficient condition for $t = 2$ is:

   $a_n \leq a_1 + a_2 ...... + a_{n-1}$ The maximum times the biggest number can be reduced or "pressed" is $a_1 + a_2 + ... + a_{n-1}$. So as to reduce it to 0 the other numbers must sum up to at least a_n. This means that the mentioned condition is necessary. We are going to use induction to prove it is sufficient.

   - Base: $n = 3$ and $a_3 \leq a_1 + a_2$
     Let $a_3 - a_2 = x$
     We reduce the buttons until we reach $(a_1 - x, a_2, a_2)$
     We reduce $(a_1 - x)$ times until we reach $(0, a_2 - a_1 + x, a_2 - a_1 + x)$
     We reduce $(a_2 - a_1 + x)$ times until we reach $(0, 0, 0)$
     The numbers of steps equals $a_3 - a_2 + a_1 - a_3 + a_2 + a_2 - a_1 + a_3 - a_2 = a_3$
   - We assume that it is sufficient for n numbers
   - $n = n + 1$ and $a_1 \leq a_2 ..... \leq a_n \leq a_{n+1}$
     $a_{n+1} \leq a_1 + a_2 + ... + a_n$
     We press a_1 times the first and the last buttons and reach $(0, a_2, ..., a_n, a_{n+1} - a_1)$. We should take into account whether the biggest number (either $a_{n+1} - a_1$ or a_n) corresponds to the condition. It is true that:
     - $a_{n+1} - a_1 \leq a_2 + a_3 ..... + a_n$ (already proven necessary)
     - $a_n \leq a_{n+1} - a_1 + a_2 + a_3 + ... + a_{n-1}$ This is true because $a_{n+1} \geq a_n$ and $a_2 \geq a_1$

     Now we have n numbers for which the condition holds and step 2 of the induction assumed it is sufficient for n buttons. Thus, it is sufficient for $n + 1$.

Therefore, we have proven by induction this is the necessary and sufficient condition. ■

2. **Problem 1b)** The above question for $3 \leq t \leq n$.

   Solution. Now we are going to develop the situation for $t \geq 3$.

   The necessary and sufficient condition for the buttons we aim to prove is:

   $a_1 + a_2 + ... + a_n \geq (t-1) * a_{n+1}$

   Again, we are going to make two inductions - firstly for $t = 3$ by n an then summarized for every t.

   - Base $n = 3$: This works only when the number of buttons in each group is equal.
     Base $n = 4$: We define the following operations:
     (*) press a_1, a_2, a_3 - it is applied x times.
     (**) press a_1, a_2, a_4 - it is applied y times
     (***) press a_2, a_3, a_4 - it is applied z times
     (****)press a_1, a_3, a_4 - it is applied w times
     (*****)press a_1, a_2, a_3, a_4 - is is applied v times
     From this, we have that $a_4 = y + z + w + v$, $a_3 = x + z + w + v$, $a_2 = x + y + z + v$, $a_1 = x + y + w + v$
     We notice that $2 * a_4 = 2 * (y + z + w + v) \leq (a_1 + a_2 + a_3)$
   - We assume that this condition is sufficient for every n numbers.
   - $n = n + 1$ We remove a_1 from a_1, a_2 and a_{n+1} The sequence of buttons looks like $(0, a_2 - a_1, a_3......, a_{n+1} - a_1)$
     For the n numbers left it is true that :
     $a_2 - a_1 + a_3 + a_4 + a_5 + ... + a_n + a_{n+1} - a_1 = (a_2 + ... + a_n + a_{n+1}) - 2 * a_1 > 2 * (a_{n+1} - a_1)$ This means that the condition holds for the rest of the n numbers, as we have assumed, and we can reach $(0, 0, 0, ..., 0)$.

   For every t, we make the same induction with the following remarks

   - in step 3 we press a_1 times the buttons in colours $a_1, a_2...., a_{t-1}, a_{n+1}$
   - The base is with t+1 numbers and we define all the operations. For example: (*) press $a_1, a_2...a_t$ (*) press $a_1, a_2, ...at - 1, at + 1$ and so on.

   When we sum all the operations we get the required inequality. ■

3. **Problem 1c)** When the condition holds, find (or give bounds) in terms of t and the a_i-s the minimal number of steps required to do this process.

   Solution. The minimum number of operations is a_n, where a_n is the biggest number of buttons. The buttons of this colour cannot all be pressed with less operations. We are going to provide a strategy:

- base $n = t + 1$
  Let $a_{t+1} - a_t = x$
  $x \leq a_1, a_2..., a_n$ because otherwise $a_1 + a_2 + ... + a_n \geq (t-1) * a_{n+1}$ won't be true

  We remove press x buttons from $a_1, a_2, ..., a_{t-1}, a_t$. Now two of the numbers are the same so we can reduce t by one. We continue doing the same operation until t=2 and all the numbers are equal. The number of operations is $a_{t+1} - a_t + a_t - ... + a_1 - a_1 = a_{t+1}$ ■

References

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