Practice problem 2

November 29, 2025

Problem 1. Let $A, B, C, D \in M_{n\times n}(F)$ (i.e. every pair among A, B, C, D commutes). Show that the determinant of the $2n \times 2n$ block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

equals $\det(AD - BC)$.

Problem 2. Let $A \in M_{n\times n}(F)$, $A \neq 0$. If r is any integer with $1 \leq r \leq n$, an $r \times r$ *submatrix* of A means any matrix obtained from A by deleting $(n-r)$ rows and $(n-r)$ columns. The *determinant rank* of A is defined to be the largest positive integer r for which some $r \times r$ submatrix of A has nonzero determinant.

Prove that the determinant rank of A equals the rank of A.

Problem 3. Let $B \in M_{n\times n}(F)$ be fixed. Define linear operators $T_B : M_{n\times n}(F) \to M_{n\times n}(F)$ by $T_B(A) = AB$.

(i) Show that $\det(T_B) = (\det B)^n$.

(ii) Show that the minimal polynomial of the linear operator T_B equals the minimal polynomial of the matrix B.

Problem 4. Let $A, B \in M_{n\times n}(F)$. Show that if A is invertible then there are at most n scalars $c \in F$ for which the matrix $cA + B$ is not invertible.

Problem 5. Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 = I$.

(i) Show that A is invertible and $\det(A) = \pm 1$.

(ii) Show that the eigenvalues of A are 1 and -1.

(iii) Show that A is diagonalizable over $\mathbb{R}$.

(iv) Let m_+ and m_- be the multiplicities of 1 and -1 as eigenvalues. Show that $m_+ + m_- = n$ and $\det(A) = (-1)^{m_-}$.

(v) Compute $\mathrm{tr}(A)$ in terms of m_+ and m_-.

(vi) Find the characteristic polynomial $p_A(x)$ of A.

Problem 6. Let $A \in M_{n\times n}(F)$ be nilpotent: i.e. $A^m = 0$ and $A^{m-1} \neq 0$ for some $m \geq 1$.

(i) Show that $m \leq n$.

(ii) Find $p_A(x)$.

(iii) Show that $\det(I - A) = 1$.
(iv) Show that $\operatorname{tr}(A) = 0$.
(v) $\dim\ker N^k > \dim\ker N^{k-1}$ for any $1 \leq k \leq m$.
(vi) If $m = n \geq 2$, show that there is no $A \in M_{n\times n}(F)$ such that $A^2 = N$.
Hint for (v): Consider v such that $N^{m-1}v \neq 0$. For which l is $N^{m-k}v \in \ker N^l$?
Hint for (vi): What is the minimal polynomial of A?

Problem 7. If $A \in M_{n\times n}(\mathbb{R})$ satisfies $\det(A) < 0$, then A has at least one real eigenvalue.

Problem 8. Let $A \in M_{n\times n}(\mathbb{R})$. If $n = 2m + 1$ is odd, then A has a real eigenvalue.

Problem 9. Let $A \in M_{2\times 2}(F)$. Show that $A = D + N$ where D is diagonalizable, N is nilpotent and $DN = ND$.

Problem 10. Let $A \in M_{n\times n}(\mathbb{C})$ be invertible. Show that if A^m is diagonalizable for some positive integer m, then A is also diagonalizable.
Hint: If $\lambda \neq 0$ then there are m distinct solutions to $x^m = \lambda$ in $\mathbb{C}$.

Problem 11. Let

$$A := \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$$

and define the map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(B) = AB - BA.$$

(i) Show that the identity matrix $I \in M_{2\times 2}(\mathbb{R})$ and A are in $\ker T$.
(ii) Choose a basis β of $M_{2\times 2}(\mathbb{R})$ and compute $[T]_\beta^\beta$.
(iii) Give all eigenvalues of T.
(iv) Give a basis of the eigenspace corresponding to the eigenvalue 0.
(v) Give the characteristic and the minimal polynomials, and the Jordan form of T. Is T diagonalizable?

Problem 12. (i) Let $A, B \in M_{n\times n}(\mathbb{C})$. Show that $p_A(B)$ is invertible if and only if A, B have no common eigenvalue.
(ii) Let $A, B, C \in M_{n\times n}(\mathbb{C})$ be such that $AC = CB$. Show that for any $f(x) \in F[x]$, we have $f(A)X = Xf(B)$.
(iii) Let $A, B, C \in M_{n\times n}(\mathbb{C})$ be such that $AC = CB$. Suppose A, B have no common eigenvalue, show that $C = 0$.

Problem 13. Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Problem 14. (i) Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. Prove that all non-zero eigenvalues of AB are also eigenvalues of BA, and show how the corresponding eigenvectors are related.
(ii) Prove that the prior statement extends to zero eigenvalues when $A, B \in M_{n\times n}(F)$: if AB has zero as an eigenvalue, so does BA.
(iii) Does the extension of part (ii) to zero eigenvalues hold when A and B are more general rectangular matrices, as in part (i)? Prove it or provide a counterexample.

Problem 15. (i) Let $A, B \in M_{n\times n}(F)$. Prove that if $I - AB$ is invertible then $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

(ii) Use the (i) to prove that, if $A, B \in M_{n\times n}(F)$, then the AB and BA have precisely the same eigenvalues in F (counted with algebraic multiplicity).

(iii) Do AB and BA necessarily have the same minimal polynomial?

Problem 16. Let $A \in M_{n\times n}(F)$ and let $V_1, V_2 \subset F^n$ be subspaces satisfying

$$A(V_1) \subset V_1,\ A(V_2) \subset V_2 \text{ and } F^n = V_1 \oplus V_2$$

Prove that A is similar to a block diagonal matrix

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},$$

where B is an $r \times r$ matrix and C is an $(n-r)\times(n-r)$ matrix for $r = \dim V_1$.

Problem 17. Let A be an $n\times n$ real matrix with distinct (possibly complex) eigenvalues $\lambda_1, \ldots, \lambda_n$, and corresponding eigenvectors $v_1, \ldots, v_n$. Assume that $\lambda_1 = 1$ and that $|\lambda_j| < 1$ for $2 \le j \le n$.

(i) Prove that $\lim_{p\to\infty} A^p v$ exists for any $v \in \mathbb{C}^n$.

(ii) Define $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T(v) = \lim_{p\to\infty} A^p v.$$

Find the dimensions of the kernel and image of T and give bases for both.

Problem 18. Let A be a 3×3 real matrix satisfying the equation

$$16A^3 = 24A^2 - 7A - I.$$

(i) Show that the sequence A^k converges as $k \to \infty$.

(ii) Let B be the matrix limit of the sequence A^k. Show that $B^2 = B$.

Problem 19. Let

$$A = \frac{1}{2}\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix},$$

and

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k.$$

(i) Find an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix.

(ii) Find $f(A)$.

(iii) Verify that $f(A)(I - A) = I$.

Problem 20. Suppose (z_n) is a sequence of complex numbers (not all zero) that satisfy

$$z_{n+1} = z_n + \tfrac{1}{4} z_{n-1}.$$

(i) Show that if $z_0 = 0$ and $z_1 = 1$, then $\dfrac{z_{n+1}}{z_n} \to r$ as $n \to \infty$. Identify r.

(ii) Find numbers a and b so that $|z_n| \to 0$ as $n \to \infty$ if and only if

$$az_1 + bz_0 = 0.$$

Problem 21. In 1970 there were 2 flowers in Wonderland. The next year there were 14 flowers. Then each subsequent year there were exactly twice as many flowers as in the previous year plus three times as many flowers as in the year before the previous one. How many flowers were there in Wonderland in 2021?

Problem 22. The populations of deer and wolves in Yellowstone are well approximated by the recurrence relations:

$$\begin{aligned} d_{n+1} &= \frac{5}{4} d_n - \frac{1}{4} w_n, \\ w_{n+1} &= \frac{1}{4} d_n + \frac{3}{4} w_n, \end{aligned}$$

where d_n and w_n denote the number of deer and wolves, respectively, in year n, for $n = 0, 1, 2, \ldots$.

Assuming that initially there are more deer than wolves (i.e., $w_0 < d_0$), what is the proportion between the number of deer and wolves as $n \to \infty$? That is, compute:

$$\lim_{n \to \infty} \frac{d_n}{w_n}.$$

Problem 23. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 2.$$

Define $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(i) Show that $A \begin{pmatrix} F_{n-2} \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix}$. Conclude that $A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for each nonnegative integer n.

(ii) Find the eigenvalues and eigenvectors of A.

(iii) Use (ii) and (i) to compute $A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Conclude that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for each nonnegative integer n.

Problem 24. Let the cyclic shift mapping S acting on vectors $x \in \mathbb{C}^n$ be defined as:

$$S(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n, x_1).$$

(i) Determine the eigenvalues and eigenvectors of S.
Hint: $x^n = 1$ has solutions $1, \zeta, \cdots, \zeta^{n-1}$ where $\zeta = e^{2\pi i/n}$.
(ii) Consider the $n \times n$ matrices

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_1 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_3 & a_4 & \dots & a_1 & a_2 \\ a_2 & a_3 & \dots & a_n & a_1 \end{pmatrix}$$

Show that the eigenvalues of A are of the form

$$\lambda_{k+1} = a_1 + a_2\omega_k + a_3\omega_k^2 + \cdots + a_n\omega_k^{n-1}, \qquad 0 \le k \le n-1,$$

where $\omega_k = \zeta^k$.
Hint: $A = f(S)$ where $f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$.

(iii) Compute the determinant of $C = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$.

(iv) Think of $\mathbb{C}^n$ as the space of discrete periodic functions, i.e., identify $x_{n+1} \equiv x_1$ and $x_0 \equiv x_n$. Consider the linear transformations corresponding to the discrete first and second derivatives:

$$(\Delta x)_k = x_{k+1} - x_k, \quad (\Delta^2 x)_k = x_{k+1} - 2x_k + x_{k-1}.$$

Show that both Δ and Δ^2 commute with the cyclic shift operator S, and use this fact to find their eigenvalues and eigenvectors.