Linear Algebra I: Practice Final

December 12, 2025

Problem 1. Compute the determinant of the following matrix (show your work):

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 4 & 4 \\ 1 & 0 & 3 & 0 & 5 \end{pmatrix}.$$

Problem 2. Let

$$A = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

Is A diagonalizable over $\mathbb{R}$? over $\mathbb{C}$? In any case, find the real canonical form of A. That is find $P \in M_{2\times 2}(\mathbb{R})$ invertible such that $P^{-1}AP$ is of one of the following form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Problem 3. Let

$$A = \frac{1}{2}\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix},$$

and

$$f(x) = \log(1-x) = -x - \frac{x^2}{2} - \cdots = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

(i) Find an invertible matrix P such that $P^{-1}AP = J$ gives the Jordan canonical form of A.

(ii) Find $f(A)$.

Problem 4. Let $A, B \in M_{n\times n}(F)$. If v is an eigenvector of AB with eigenvalue $\lambda \neq 0$, show that Bv is an eigenvector of BA with eigenvalue λ.

Problem 5. Suppose that $A \in M_{2\times 2}(\mathbb{R})$ is symmetric. Prove that A is diagonalizable over $\mathbb{R}$.

Problem 6. Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 = -I$.

(i) Show that $n = 2m$ is even and $\det(A) = \pm 1$.

(ii) Show that i and $-i$ are both eigenvalues of A in $\mathbb{C}$.

(iii) Show that A is diagonalizable over $\mathbb{C}$.

(iv) Show that $\mathrm{tr}(A) = 0$ and that $\dim E(i, A) = \dim E(-i, A) = m$.

(v) Find the characteristic polynomial $p_A(x)$.

(vi) Show that $\det(A) = 1$.

(vii) Find $\det(I - A)$.