Problem set 11

Problem 1. Find the minimal polynomial of the following matrices: $\begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. You can use your result from Homework 9 Problem 1.

Problem 2. Let $A \in M_{n\times n}(\mathbb{R})$ be a matrix with distinct positive eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) How many real matrices B satisfy $A = B^2$?

(ii) How many real matrices B satisfy $A = B^3$?

(iii) How many complex matrices B satisfy $A = B^3$? (Include the real matrices B in your count)

Hint: Find the eigenvalues of B. Is B diagonalizable? What are the eigenvectors of B?

Hint: We have seen in Lecture 9 that $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$ has 3 distinct roots in $\mathbb{C}$ where $\omega = \frac{-1+\sqrt{3}i}{2}$.

Problem 3. Suppose $T \in \mathcal{L}(V)$ is invertible and $v_1, \ldots, v_n$ is a basis of V with respect to which the matrix of T is upper triangular, with diagonal entries $\lambda_1, \ldots, \lambda_n$. Show that the matrix of T^{-1} is also upper triangular with respect to the same basis $v_1, \ldots, v_n$, and its diagonal entries are $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$.

Problem 4. Let $A \in M_{n\times n}(F)$.

(i) Suppose A is nilpotent, i.e. there is an integer $k > 0$ such that $A^k = 0$, show that $A^n = 0$ and that A is diagonalizable if and only if $A = 0$. (Note that the integer k is arbitrary and need not satisfy $k \leq n$.)

(ii) Suppose $A \in M_{n\times n}(\mathbb{C})$ and $A^4 = A$. Show that A is diagonalizable over $\mathbb{C}$. What are the possible eigenvalues of A?

(iii) Suppose $m_A(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$. Show that A is invertible if and only if $a_0 \neq 0$. If A is invertible, show that

$$A^{-1} = -\frac{1}{a_0}(A^{m-1} + a_{m-1}A^{m-2} + \cdots + a_2 A + a_1 I).$$

Optional exercises related to Lecture 11

Do NOT submit this with your homework.

The following problems are from Linear algebra done right.

1. Exercise 5B: 1,3,6,7,8
2. Exercise 5C: 1,4,5,7,8
3. Exercise 5D: 1,14,15,17