Problem set 12

Problem 1. Find the real canonical form and compute e^A of the following matrices $\begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$.

Problem 2. (i) Prove that the Jordan block

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

is similar to its transpose $J_n(\lambda)^t$.

Hint: Let $e_1, \dots, e_n$ be the standard basis of F^n. Let $P = (e_n, e_{n-1}, \dots, e_1)$. Show that $P^{-1} J_n(\lambda) P = J_n(\lambda)^t$.

(ii) Prove that every $A \in M_{n\times n}(\mathbb{C})$ is similar to its transpose A^t.

(iii) Let $J_n(\lambda) \in M_{n\times n}(\mathbb{C})$ be a Jordan block with $|\lambda| < 1$. Prove that

$$\lim_{N\to\infty} J_n(\lambda)^N = 0.$$

(iv) Let $A \in M_{n\times n}(\mathbb{C})$ be such that all eigenvalues satisfy $|\lambda| < 1$. Prove that

$$\lim_{N\to\infty} A^N = 0.$$

Problem 3. Let $A, N \in M_{n\times n}(F)$ with $N = J_n(0)$ and $AN = NA$. Let $e_1, \dots, e_n$ be the standard basis of F^n. We have seen in Example 11.12 that $N^k e_1 = e_{1+k}$ for $0 \le k \le n-1$.

(i) Suppose $Ae_1 = a_{11}e_1 + a_{21}e_2 + \cdots + a_{n1}e_n$. Let $p(x) = a_{11} + a_{21}x + \cdots + a_{n1}x^{n-1} \in F[x]$. Show that $Ae_1 = p(N)e_1$.

(ii) Show that $Ae_k = p(N)e_k$ for $1 \le k \le n$.

(iii) Conclude that $A = p(N)$.

Problem 4. (i) Find an example of matrices in $M_{n\times n}(\mathbb{C})$ such that $A_k \to A$ where A_k are diagonalizable but A is not diagonalizable.

(ii) Find an example of matrices in $M_{n\times n}(\mathbb{C})$ such that $A_k \to A$ where A_k are not diagonalizable and A is diagonalizable.

(iii) Show that for any $A \in M_{n\times n}(\mathbb{C})$ there exists a sequence of diagonalizable matrices $A_k \in M_{n\times n}(\mathbb{C})$ such that $A_k \to A$. In other words, diagonalizable matrices are dense in $M_{n\times n}(\mathbb{C})$.

Hint: Try manipulating the Jordan canonical form directly. Corollary 10.12 is very useful here.

Optional exercises related to Lecture 12

Do NOT submit this with your homework.

The following problems are from Linear algebra done right.

Exercise 5E: 3,4,5

Exercise 8A: 2,13,16,17,23