Linear Algebra I: Practice Final solution

December 12, 2025

Problem 1. Compute the determinant of the following matrix (show your work):

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 4 & 4 \\ 1 & 0 & 3 & 0 & 5 \end{pmatrix}.$$

Proof. Expand along the first row, we have

$$\det(A) = 1 \cdot \det \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 4 & 4 \\ 0 & 3 & 0 & 5 \end{pmatrix}.$$

Expand along the first column we get

$$\det(A) = 1 \cdot 2 \cdot \det \begin{pmatrix} 3 & 0 & 0 \\ 3 & 4 & 4 \\ 3 & 0 & 5 \end{pmatrix}.$$

Expand along the first row, we have

$$\det(A) = 1 \cdot 2 \cdot 3 \cdot \det \begin{pmatrix} 4 & 4 \\ 0 & 5 \end{pmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

□

Problem 2. Let

$$A = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

Is A diagonalizable over $\mathbb{R}$? over $\mathbb{C}$? In any case, find the real canonical form of A. That is find $P \in M_{2\times 2}(\mathbb{R})$ invertible such that $P^{-1}AP$ is of one of the following form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Proof.

$$p_A(x) = \det(xI - A) = \det \begin{pmatrix} x-1 & -1 \\ 3 & x-1 \end{pmatrix} = (x-1)^2 + 3 = (x - 1 - \sqrt{3}i)(x - 1 + \sqrt{3}i).$$

$p_A(x)$ does not splits in $\mathbb{R}$, hence A is not diagonalizable over $\mathbb{R}$. $p_A(x)$ splits in $\mathbb{C}$ with distinct linear factors, A is diagonalizable over $\mathbb{C}$.

To find the real canonical form, we consider an eigenvalue of A $\lambda = 1 + \sqrt{3}i$. We solve

$$0 = (\lambda I - A)v = \begin{pmatrix} \sqrt{3}i & -1 \\ 3 & \sqrt{3}i \end{pmatrix} v = 0$$

to get $v = c\begin{pmatrix} 1 \\ \sqrt{3}i \end{pmatrix}$. Thus with $P = (\operatorname{Re} v, \operatorname{Im} v) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$ we have

$$P^{-1}AP = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

$\square$

Problem 3. Let

$$A = \frac{1}{2}\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix},$$

and

$$f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

(i) Find an invertible matrix P such that $P^{-1}AP = J$ gives the Jordan canonical form of A.

(ii) Find $f(A)$.

Proof. (i)

$$\begin{aligned} p_A(x) =& \det(xI - A) = \det \begin{pmatrix} x + \frac{1}{2} & 2 \\ -\frac{1}{2} & x - \frac{3}{2} \end{pmatrix} = (x + \tfrac{1}{2})(x - \tfrac{3}{2}) + 1 \\ =& x^2 - x + \frac{1}{4} = (x - \tfrac{1}{2})^2. \end{aligned}$$

We compute $\frac{1}{2}I - A = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{pmatrix}$. Since $\operatorname{rank}(\frac{1}{2}I - A) = 1$, $\dim\ker(\frac{1}{2}I - A) = 1 < 2$. Let $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $(\frac{1}{2}I - A)w = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$. We take $v = (A - \frac{1}{2}I)w = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$ and $P = (v, w) = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$. Then we have

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

(ii) We have $P^{-1} = (-2)\begin{pmatrix} 0 & -1 \\ -\frac{1}{2} & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$. Note that $f'(x) = \frac{1}{x-1}$. Then

$$\begin{aligned} f(A) =& Pf\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} P^{-1} \\ =& P\begin{pmatrix} \log\frac{1}{2} & -2 \\ 0 & \log\frac{1}{2} \end{pmatrix} P^{-1} \\ =& \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} -\log 2 & -2 \\ 0 & -\log 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \\ =& \begin{pmatrix} \log 2 & -\log 2 + 2 \\ -\frac{1}{2}\log 2 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \\ =& \begin{pmatrix} -\log 2 + 2 & 4 \\ -1 & -\log 2 - 2 \end{pmatrix} \end{aligned}$$

. $\square$

Problem 4. Let $A, B \in M_{n\times n}(F)$. If v is an eigenvector of AB with eigenvalue $\lambda \neq 0$, show that Bv is an eigenvector of BA with eigenvalue λ.

Proof. We have $BA(Bv) = B((AB)v) = B\lambda v = \lambda(Bv)$.

To show that Bv is an eigenvector, we show that $Bv \neq 0$. We have $A(Bv) = (AB)v = \lambda v$ for $\lambda \neq 0$. Since v is an eigenvector, we have $v \neq 0$ and hence $A(Bv) = \lambda v \neq 0$. Thus $Bv \neq 0$ (since otherwise $A(Bv) = 0$). □

Problem 5. Suppose that $A \in M_{2\times 2}(\mathbb{R})$ is symmetric. Prove that A is diagonalizable over $\mathbb{R}$.

Proof. Since A is symmetric, we write $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for $a, b, c \in \mathbb{R}$. Then $p_A(x) = \det \begin{pmatrix} x-a & -b \\ -b & x-c \end{pmatrix} = (x-a)(x-c) - b^2 = x^2 - (a+c)x + ac - b^2 = (x-\lambda_1)(x-\lambda_2)$ where with $(a+c)^2 - 4(ac-b^2) = a^2 - 2ac + c^2 + 4b^2 = (a-c)^2 + 4b^2$ we have

$$\lambda_1 = \frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}, \quad \lambda_2 = \frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}.$$

If $b = 0$, then $A = \operatorname{diag}(a, c)$ is diagonalizable over $\mathbb{R}$.

If $b \neq 0$, then $(a-c)^2 + 4b^2 > 0$ and λ_1, λ_2 are distinct real numbers. Then $p_A(x)$ splits as distint linear factors in $\mathbb{R}$. Hence A is diagonalizable over $\mathbb{R}$. $\square$

Problem 6. Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 = -I$.

(i) Show that $n = 2m$ is even and $\det(A) = \pm 1$.

(ii) Show that i and $-i$ are both eigenvalues of A in $\mathbb{C}$.

(iii) Show that A is diagonalizable over $\mathbb{C}$.

(iv) Show that $\operatorname{tr}(A) = 0$ and that $\dim E(i, A) = \dim E(-i, A) = m$.

(v) Find the characteristic polynomial $p_A(x)$.

(vi) Show that $\det(A) = 1$.

(vii) Find $\det(I - A)$.

Proof. (i) $(\det(A))^2 = \det(A^2) = \det(-I) = (-1)^n$. Since $A \in M_{n\times n}(\mathbb{R})$ we have $(-1)^n = (\det(A))^2 > 0$ and thus $n = 2m$ is even. Since $(\det(A))^2 = 1$, we have $\det(A) = \pm 1$.

(ii) Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then $\lambda^2 = -1$. Thus $\lambda = \pm i$. Since $A \in M_{n\times n}(\mathbb{R})$, if λ is an eigenvalue of A, so is $\bar{\lambda}$. Thus both i and $-i$ are eigenvalues of A.

(iii) Let $f(x) = x^2 + 1$. Since $f(A) = A^2 + I = 0$, we have $m_A(x) \mid f(x)$. Since $f(x) = (x - i)(x + i)$ splits in $\mathbb{C}$ as distinct linear factors, so is $m_A(x)$. Thus A is diagonalizable over $\mathbb{C}$.

(iv) We set $\dim E(i, A) = k$. Since A is diagonalizable, $\dim E(-i, A) = n - k$ and there is $P \in M_{n\times n}(\mathbb{C})$ invertible such that

$$P^{-1}AP = \operatorname{diag}(\underbrace{i, \ldots, i}_{k \text{ times}}, \underbrace{-i, \ldots, -i}_{n-k \text{ times}}).$$

We have $\operatorname{tr}(A) = ki - (n - k)i = (n - 2k)i$. Since $A \in M_{n\times n}(\mathbb{R})$, we have $\operatorname{tr}(A) \in \mathbb{R}$. Thus we must have $n - 2k = 0$ i.e. $k = m$.

(v) $p_A(x) = (x - i)^m(x + i)^m = (x^2 + 1)^m$.

(vi) $\det(A) = (-1)^n \det(A) = p_A(0) = 1$.

(vii) $\det(I - A) = p_A(1) = 2^m$. $\square$