Answer Set 11

Problem 1

Matrix 1:

Matrix 2:

Matrix 3:

The eigenvalues are repeated (with multiplicity 2).

We check the smallest power :

Since , . Therefore, it must be the next power.

Matrix 4

Since is upper triangular, the determinant is the product of diagonal entries.

The minimal polynomial is of the form where .

- Check :
- Check :

Since the square of the matrix (shifted by its eigenvalue) results in the zero matrix, the minimal degree is 2.

Problem 2

Part (i): How many real matrices satisfy ?

This corresponds to solving for each eigenvalue.

Since , the equation has exactly two real solutions:
For to be a real matrix, given that is real, the diagonal matrix must be real. Thus, we must choose real values for .
For each of the eigenvalues, we have 2 independent choices (positive or negative root).

So, there are real matrices .

Part (ii): How many real matrices satisfy ?

This corresponds to solving .

The equation has three solutions in the complex plane. However, for , there is only one real solution:
As established in part (i), for to be a real matrix (given real eigenvectors), the eigenvalues must be real. If we chose a complex eigenvalue, would become complex.
We are forced to choose the single real cube root for every eigenvalue. There are no other choices.

So, there is 1 real matrix .

Part (iii): How many complex matrices satisfy ?

The equation has three distinct complex roots. If we let be the real root and , the roots are:
For each of the eigenvalues, we can independently choose any of these 3 roots to build the diagonal matrix .

There are complex matrices .

Problem 3

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is upper triangular

Saying that the matrix of is upper triangular in the basis is the same as saying:

In other words, when you apply to anything in the first basis vectors' span, the result stays inside that same subspace. It never creates components in .

But is invertible, so gives a bijection from onto itself. Therefore the inverse map must also send into itself for every .

That means that, with respect to the same basis, also never sends a lower-index vector "upwards" to higher-index basis vectors. This is exactly the condition that says the matrix of is upper triangular.

The diagonal are entries

For each basis vector , the upper triangular form of means:

Apply to both sides:

Because is linear, this is

Now write

Thus

which means the diagonal entry in the -th position of the matrix of is .

Problem 4

(i) Nilpotent Matrices

A:
If , then the only possible eigenvalue of is 0.

If , then applying gives

The vector is not zero, so the only way this works is .
So the characteristic polynomial of must be , because all eigenvalues are 0.
By the Cayley–Hamilton Theorem, a matrix satisfies its own characteristic polynomial:

So any nilpotent matrix automatically becomes zero by the time you raise it to the -th power.

B:
If a nilpotent matrix is diagonalizable, then all of its diagonal entries (which are eigenvalues) must be 0.

So any diagonal form would look like:

Which is exactly the zero matrix. So the only diagonalizable nilpotent matrix is the zero matrix itself.

(ii) Matrices satisfying

If

Then satisfies the polynomial:

If we factor the polynomial:

where .

All four roots are different. So any minimal polynomial that divides this also has only distinct linear factors.
A matrix whose minimal polynomial has no repeated roots is automatically diagonalizable.
Therefore, every matrix satisfying is diagonalizable.

Eigenvalues must be roots of the polynomial .
So the possible eigenvalues are:

where

(iii) Formula for the Inverse Using the Minimal Polynomial

A matrix is invertible exactly when zero is not an eigenvalue.
But the roots of the minimal polynomial are the eigenvalues.
So:

- If , that means , so 0 is a root 0 is an eigenvalue not invertible.
- If , then 0 is not a root   has no zero eigenvalues invertible.
  So:

Since , plug into the polynomial: