Problem set 2

Problem 1. Let V be a vector space over a field F. Prove the following statements.

(i) Let $u, v \in V$ be distinct vectors in V. Then u, v is linearly dependent if and only if u or v is a multiple of each other.

(ii) Any subset of linearly independent vectors is linearly independent.

(iii) Any set containing a set of linearly dependent vectors is linearly dependent. In particular, if a set of vectors contain a zero vector then it is linearly dependent.

(iv) Let $v_1, \ldots, v_n$ be linearly independent. Suppose $v_1 + w, v_2 + w, \cdots, v_n + w$ are linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

(v) Let v_1, v_2, v_3, v_4 be some vectors in V. Show that $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ by showing that any linear combination of v_1, v_2, v_3, v_4 is a linear combination of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ and vice versa.

Problem 2. (i) Show that $\cos x, \sin x$ is linearly independent in the vector space $C(\mathbb{R})$.

(ii) Show that e^x, e^{2x} is linearly independent in the vector space $C(\mathbb{R})$.

Problem 3. Define $\mathcal{Q}_n(1) = \{f \in \mathcal{P}_n(\mathbb{R}) : f(1) = 0\}$.

(i) Show that $\mathcal{Q}_n(1)$ is a subspace of $\mathcal{P}_n(\mathbb{R})$.

(ii) Determine the dimension of that subspace and give a basis.

(iii) Show that $\mathcal{P}_n(\mathbb{R}) = \mathcal{Q}_n(1) + \operatorname{span}(1)$ where 1 is the polynomial $0x^n + 0x^{n-1} + \cdots + 0x + 1$.

Problem 4. Let V be a vector space over a field F and $V_1, V_2 \subset V$ be subspaces of V.

(i) If there exist $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1 \notin V_2$ and $v_2 \notin V_1$. Show that $v_1 + v_2 \notin V_1 \cup V_2$.

(ii) Show that $V_1 \cup V_2$ is a subspace of V if and only if $V_1 \subset V_2$ or $V_2 \subset V_1$.

Remark 5. This problem should be compared with Proposition 2.26 in Lecture 2, showing that the smallest vector space containing $V_1 \cup V_2$ is $V_1 + V_2$.