2 Lecture 2

2.1 Span

We want to build a vector space out of the smallest amount of vectors. Addition and scalar multiplication are the only operations we have in a vector space. If we have $x, y \in V$, then $ax + by$ for $a, b \in F$ can be built out of them. Given a set of vectors $v_1, \ldots, v_n$ what are all vectors that can be built from them? This is the idea of linear combination and span.

Definition 2.1. Let V be a vector space over F. A **linear combination** of $v_1, \ldots, v_n \in V$ is a vector of the form

$$a_1 v_1 + \cdots + a_n v_n$$

where $a_1, \ldots, a_n \in F$. The set of all linear combinations of $v_1, \ldots, v_n$ is denoted by $\operatorname{span}(v_1, \ldots, v_n)$. We use the convention that $\operatorname{span} \emptyset = \{0\}$.

Proposition 2.2. $\operatorname{span}(v_1, \ldots, v_n)$ *is a subspace of* V.

Proof. By Proposition 1.3 (ii), we have $0 = 0v_1 + \cdots + 0v_n \in \operatorname{span}(v_1, \ldots, v_n)$. If $x = a_1 v_1 + \cdots + a_n v_n$ and $y = b_1 v_1 + \cdots + b_n v_n$ are two linear combinations, then $x + y = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n$ and $cx = (ca_1)v_1 + \cdots + (ca_n)v_n$ are both linear combinations of $v_1, \ldots, v_n$. Thus $\operatorname{span}(v_1, \ldots, v_n)$ is a subspace. $\square$

Example 2.3. Let $v_1 = (1, 0, 0)^t$. Then $\operatorname{span}(v_1) = \{(a_1, 0, 0)^t : a_1 \in \mathbb{R}\}$ is the line through the origin in the direction of v_1.

Let $v_2 = (0, 1, 0)^t$, then $\operatorname{span}(v_1, v_2) = \{(a_1, a_2, 0)^t : a_1, a_2 \in \mathbb{R}\}$ is the plane determined by 0, v_1 and v_2.

Let $v_3 = (1, 1, 0)^t$, then $\operatorname{span}(v_1, v_2, v_3) = \{(a_1 + a_3, a_2 + a_3, 0)^t : a_1, a_2, a_3 \in \mathbb{R}\}$. This is the same as $\operatorname{span}(v_1, v_2)$ since it is just the set of points with $x_3 = 0$.

Let $v_4 = (0, 0, 1)^t$, then $\operatorname{span}(v_1, v_2, v_4) = \{(a_1, a_2, a_4)^t : a_1, a_2, a_4 \in \mathbb{R}\} = \mathbb{R}^3$ and $\operatorname{span}(v_1, v_2, v_3, v_4) = \{(a_1 + a_3, a_2 + a_3, a_4)^t : a_1, a_2, a_3 \in \mathbb{R}\} = V$.

Definition 2.4. $v_1, \ldots, v_n$ is a **spanning set** of V if $\operatorname{span}(v_1, \ldots, v_n) = V$.

If V has a spanning set, then V is called a **finite dimensional** vector space.

A vector space V is **infinite dimensional** if it is not finite dimensional.

2.2 Linear dependence and independence

We see in Example 2.3 that v_3 is redundant in building vectors i.e. adding v_3 to v_1, v_2 does not change their span. We want to get rid of the redundant vectors to find the smallest set of vectors that span a vector space. The following concepts are important.

Definition 2.5. $v_1, \ldots, v_n$ is **linearly dependent** if there exist $a_1, \ldots, a_n \in F$ not all zero such that $a_1 v_1 + \cdots + a_n v_n = 0$.

$v_1, \ldots, v_n$ is **linearly independent** if or it is not linearly dependent, in other words, for any $a_1, \ldots, a_n \in F$ such that $a_1 v_1 + \cdots + a_n v_n = 0$ we have $a_1 = \cdots = a_n = 0$.

Example 2.6. $v_1 = (1,0,0)^t$, $v_2 = (0,1,0)^t$ are linearly independent since for any $a_1, a_2 \in F$ we have $0 = a_1v_1 + a_2v_2 = (a_1, a_2, 0)^t$ implies $a_1 = a_2 = 0$.

Example 2.7. $v_1 = (1,0,0)^t$, $v_2 = (0,1,0)^t$, $v_3 = (1,1,0)^t$ are linearly dependent since $v_1 + v_2 - v_3 = 0$.

We see in Example 2.7 that v_3 is a linear combination of v_1, v_2. In the following lemma, we see that this is always the case for linearly dependent vectors.

Lemma 2.8. *(i) $v_1, \dots, v_n$ is linearly dependent if and only if there exists $1 \le k \le n$ such that $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$.*

(ii) $v_1, \dots, v_n$ is linearly independent if and only if for any $1 \le k \le n$ we have $v_k \notin \operatorname{span}(v_1, \dots, v_{k-1})$.

Remark 2.9. If $k = 1$, we use the convention that $(v_1, \dots, v_{k-1})$ is the empty set and $\operatorname{span} \emptyset = \{0\}$.

Proof. Note that statement (ii) is logically the negation of statement (i), we only need to show (i).

"$\Longrightarrow$" Since $v_1, \dots, v_n$ is linearly dependent, there exist $a_1, \dots, a_n \in F$ not all zero such that $a_1v_1 + \cdots + a_nv_n = 0$. Let k be the largest index such that $a_k \ne 0$. Then we have $a_1v_1 + \cdots + a_kv_k = 0$. Thus if $k > 1$, then $v_k = -\frac{1}{a_k}(a_1v_1 + \cdots + a_{k-1}v_{k-1})$. If $k = 1$, then $a_1v_1 = 0$ and hence $v_1 = 0$ by Proposition 1.3 (iv).

"$\Longleftarrow$" If $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$, then there exists $a_1, \dots, a_{k-1} \in F$ such that $v_k = a_1v_1 + \cdots + a_{k-1}v_{k-1}$. In other words, $a_1v_1 + \cdots + a_{k-1}v_{k-1} - v_k = 0$. This shows $v_1, \dots, v_n$ are linearly dependent. $\square$

2.3 Basis

The previous examples shows that a linearly dependent set of vectors is not small enough i.e. it has vectors that are redundant. On the other hand, linearly independent vectors are not redundant. Thus our goal of finding a smallest set of vectors that builds the whole vector space is to find a basis:

Definition 2.10. $v_1, \dots, v_n$ is a **basis** of V if $\operatorname{span}(v_1, \dots, v_n) = V$ and $v_1, \dots, v_n$ are linearly independent.

We can reduce a spanning set to a basis by removing the "redundant" vectors as in the following lemma.

Lemma 2.11. *Every spanning set contains a basis.*

Proof. Let $v_1, \dots, v_n$ be a spanning set i.e. $\operatorname{span}(v_1, \dots, v_n) = V$. We remove vectors in $v_1, \dots, v_n$ through the following process.

Step 1: if $v_1 = 0$, then delete v_1. If $v_1 \ne 0$, leave it unchanged.

Step k: if $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$ then delete v_k. Otherwise leave v_k unchanged.

Stop the process after step n. We relabel the remaining vectors as $v_1, \dots, v_m$ preserving the original order. Since each time we discard a vector that is already in the span of previous vectors, we have $\mathrm{span}(v_1, \dots, v_m) = V$. By construction we also have $v_k \notin \mathrm{span}(v_1, \dots, v_{k-1})$ for any $1 \le k \le m$. By Lemma 2.8, we have $v_1, \dots, v_m$ is linearly independent. Thus $v_1, \dots, v_m$ is a basis of V. $\square$

We need the following Lemma for proving the main theorem of this lecture, Theorem 2.13.

Lemma 2.12. *Let $v_1, \dots, v_n$ be a basis of V. If $v_1 = a_1w_1 + a_2v_2 + \cdots + a_nv_n$ where $a_i \in F$, $1 \le i \le n$ and $a_1 \ne 0$, then $w_1, v_2, \dots, v_n$ is also a basis.*

Proof. Since $v_1, \dots, v_n$ is a basis, for any $x \in V$, we have $x = x_1v_1 + \cdots + x_nv_n$ for some $x_i \in F$, $1 \le i \le n$. Since $v_1 = a_1w_1 + a_2v_2 + \cdots + a_nv_n$, we have $x = x_1(a_1w_1 + a_2v_2 + \cdots + a_nv_n) + x_2v_2 + \cdots + x_nv_n = a_1x_1w + (a_2x_1 + x_2)v_2 + \cdots + (a_nx_1 + x_n)v_n$. Thus $\mathrm{span}(w_1, v_2, \dots, v_n) = V$.

To show that $w_1, v_2, \dots, v_n$ is linearly independent, let $b_1w_1 + b_2v_2 + \cdots + b_nv_n = 0$. Then since $w_1 = \frac{1}{a_1}(v_1 - a_2v_2 - \cdots - a_nv_n)$, we have $b_1\frac{1}{a_1}(v_1 - a_2v_2 - \cdots - a_nv_n) + b_2v_2 + \cdots + b_nv_n = 0$ i.e. $\frac{b_1}{a_1}v_1 + (b_2 - \frac{b_1}{a_1}a_2)v_2 + \cdots + (b_n - \frac{b_1}{a_1}a_n)v_n = 0$. Since $v_1, \dots, v_n$ is a basis, we have $\frac{b_1}{a_1} = b_2 - \frac{b_1}{a_1}a_2 = \cdots = b_n - \frac{b_1}{a_1}a_n = 0$. Hence $b_1 = \cdots = b_n = 0$. $\square$

Theorem 2.13 (Replacement theorem)**.** *Let $v_1, \dots, v_n$ be a basis for V and $w_1, \dots, w_m$ be linearly independent. Then reordering $v_1, \dots, v_n$ if necessary, for each $1 \le k \le m$, we have $\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ is a basis of V. In particular $n \ge m$.*

Proof. We prove the theorem inductively on k.

We first prove that the conclusion is true for $k = 1$. Since $v_1, \dots, v_n$ is a basis, $w_1 = a_1v_1 + \cdots + a_nv_n$. Since $w_1, \dots, w_m$ are linearly independent, $w_1 \ne 0$. Thus there is $1 \le j \le n$ such that $a_j \ne 0$. We reorder v_i such that $a_1 \ne 0$. We replace v_1 by w_1. Then $v_1 = \frac{1}{a_1}(w_1 - a_2v_2 - \cdots - a_nv_n)$. By Lemma 2.12, $w_1, v_2, \dots, v_n$ is a basis of V.

Suppose the conclusion is true for $k-1$, we are going to show that the conclusion holds for k. By assumption, we have $w_1, \dots, w_{k-1}, v_k, \dots, v_n$ is a basis. Then there exist $a_1, \dots, a_n \in F$ such that $w_k = a_1w_1 + \cdots + a_{k-1}w_{k-1} + a_kv_k + \cdots + a_nv_n$. We claim that there exists $k \le j \le n$ such that $a_j \ne 0$. If this is not the case, then $a_k = \cdots = a_n = 0$ and we have $w_k = a_1w_1 + \cdots + a_{k-1}w_{k-1}$. This contradicts the fact that $w_1, \dots, w_m$ is linearly independent.

Now we reorder v_i such that $a_k \ne 0$ and we replace v_k by w_k. Then $v_k = \frac{1}{a_k}(w_k - a_1w_1 - \cdots - a_{k-1}w_{k-1} - \cdots - a_nv_n)$. Hence by Lemma 2.12 again, $w_1, \dots, w_k, v_{k+1}, \dots, v_n$ is a basis of V. This finishes the induction.

To prove the last statement, we assume for contradiction that $n < m$. Then by the Theorem we just proved, $w_1, \dots, w_n$ is a basis. Hence $w_{n+1} = a_1w_1 + \cdots + a_nw_n$ for some $a_1, \dots, a_n \in F$. This contradicts linear independence of $w_1, \dots, w_m$ (Lemma 2.8). $\square$

Corollary 2.14. *Let V be a finite dimensional vector space. Then any basis of V has the same number of elements.*

Proof. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be two bases of V. Then using $v_1, \ldots, v_n$ is a basis and $w_1, \ldots, w_m$ is linearly independent, we have by Theorem 2.13, $n \leq m$. On the other hand, using $w_1, \ldots, w_m$ is a basis and $v_1, \ldots, v_n$ is linearly independent we have, by Theorem 2.13 again, $m \leq n$. Hence $m = n$. $\square$

Definition 2.15. Let V be a finite dimensional vector space. The number of elements in a basis is called the **dimension** of V, denoted by $\dim V$. We sometimes need to emphasize the field of the vector space and write $\dim_F V$.

Corollary 2.16. *Let V be a finite dimensional vector space. Then any set of linearly independent vectors can be extended to a basis of V.*

Proof. By Lemma 2.11, there is a basis $v_1, \ldots, v_n$ of V. Let $w_1, \ldots, w_m$ be linearly independent. By Theorem 2.13, we can reorder the v_i's so that $w_1, \ldots, w_m, v_{m+1}, \ldots, v_n$ is a basis. This finishes the proof. $\square$

Corollary 2.17. *Let V be a finite dimensional vector space and $W \subset V$ be a subspace. Then W is finite dimensional and $\dim W \leq \dim V$. If $\dim W = \dim V$ then $W = V$.*

Proof. Suppose $\dim V = n \geq 0$. We apply the following procedure. If $W = \{0\}$, then $\dim W = 0 \leq n$ and we are done. If $W \neq \{0\}$, choose $v_1 \in W$ such that $v_1 \neq 0$. If $W = \operatorname{span}(v_1)$ then since v_1 is linearly independent, $\dim W = 1 \leq \dim V$. If $W \neq \operatorname{span}(v_1)$, we choose $v_2 \notin \operatorname{span}(v_1)$. Then v_1, v_2 is linearly independent by Lemma 2.8. We continue this process to obtain linearly independent vectors $v_1, \ldots, v_m \in W$. By Theorem 2.13, $m \leq n$ and thus this process must stop at a stage where $m \leq n$ and $v_1, \ldots, v_m$ is linearly independent and for every vector $v \in W$, $v_1, \ldots, v_m, v$ is linearly dependent. Hence by Lemma 2.8, $v \in \operatorname{span}(v_1, \ldots, v_m)$. Since $v \in W$ is arbitrary, $W = \operatorname{span}(v_1, \ldots, v_m)$. Hence W is finite dimensional and $\dim W = m \leq n = \dim V$.

If $\dim W = \dim V = n$, then a basis of W consists of n vectors. By Theorem 2.12, any linearly independent n vectors in V is a basis. Thus a basis of W is also a basis of V hence $V = W$. $\square$

Example 2.18. The standard basis of F^n is

$$v_1 = (1, 0, \ldots, 0)^t,\ v_2 = (0, 1, \ldots, 0)^t, \ldots, v_n = (0, 0, \ldots, 1)^t.$$

Therefore, $\dim F^n = n$.

Example 2.19. $V = \{(a_1, a_2, a_3)^t \in F^3 : a_1 + a_2 + a_3 = 0\}$. It is an exercise to show that V is a subspace of F^3. To find a basis, we write $a_1 = -a_2 - a_3$. Set $a_2 = 1$, $a_3 = 0$ we get the first vector $(-1, 1, 0)^t$. Set $a_2 = 0$, $a_3 = 1$ we get $(-1, 0, 1)^t$. One can check that they are linearly independent and $(-a_2 - a_3, a_2, a_3)^t = a_2(-1, 1, 0)^t + a_3(-1, 0, 1)^t$. Thus $(-1, 1, 0)^t, (-1, 0, 1)^t$ is a basis and $\dim V = 2$. This is a baby version of finding all solutions to a system of linear equation. We will learn how to systematically do it later.

Example 2.20. Let $E^{ij} \in M_{m\times n}(F)$ be the matrix whose (i,j) entry is 1 and other entries are 0. Then $\{E^{ij}\}_{1\le i\le m, 1\le j\le n}$ is a basis of $M_{m\times n}(F)$ and hence $\dim M_{m\times n}(F) = mn$. For example, $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ is a basis of $M_{2\times 2}(F)$.

Example 2.21. The standard basis of $\mathcal{P}_n(F)$ is

$$p_0(x) = 1,\ p_1(x) = x, \dots, p_n(x) = x^n.$$

Therefore, $\dim \mathcal{P}_n(F) = n+1$.

Example 2.22. $V = \{f \in C^2(\mathbb{R}) : f'' + f = 0\} = \{c_1 \cos x + c_2 \sin x : c_1, c_2 \in \mathbb{R}\}$. $\cos x$ and $\sin x$ is a basis of V (homework). Therefore $\dim V = 2$.

2.4 Intersection, sum and direct sum of subspaces

In this section, we introduce several constructions of new vector spaces out of old spaces.

Proposition 2.23. *Let V be a vector space over F and $V_1, \dots, V_m \subset V$ be subspaces, then $V_1 \cap \cdots \cap V_m$ is a subspace.*

Proof. We check the definition of a subspace. Since V_i is a subspace for any $1 \le i \le m$, we have $0 \in V_i$, if $x, y \in V_i$ then $x + y \in V_i$ and if $a \in F$, $x \in V_i$, we have $ax \in V_i$. Then we have $0 \in V_1 \cap \cdots \cap V_m$. If $x, y \in V_1 \cap \cdots \cap V_m$, then $x, y \in V_i$ for any i and hence $x + y \in V_i$ for any i i.e. $x + y \in V_1 \cap \cdots \cap V_m$. Similarly if $a \in F$, then $ax \in V_i$ for any i i.e. $ax \in V_1 \cap \cdots \cap V_m$. □

Example 2.24. Let $V_1 = \mathrm{span}(v_1, v_2) = \{(a_1, a_2, 0)^t : a_1, a_2 \in \mathbb{R}\}$ and $V_2 = \mathrm{span}(v_3, v_4) = \{(a_3, a_3, a_4)^t : a_3, a_4 \in \mathbb{R}\}$. Then $V_1 \cap V_2 = \{(a_3, a_3, 0)^t : a_3 \in \mathbb{R}\} = \mathrm{span}(v_3)$.

Definition 2.25. Let V be a vector space over F and $V_1, \dots, V_m \subset V$ be subspaces. The **sum** of $V_1, \dots, V_m$ is denoted by $V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_i \in V_i, 1 \le i \le m\}$.

Proposition 2.26. *$V_1 + \cdots + V_m$ is the smallest subspace of V containing all of $V_1, \dots, V_m$ in the following sense*

(i) $V_1 + \cdots + V_m$ is a subspace.

(ii) If U is a subspace of V containing $V_1, \dots, V_m$, then U contains $V_1 + \cdots + V_m$

Proof. (i) This is proved by directly checking the definition.

(ii) Let $v_i \in V_i$ for $1 \le i \le m$. Then $v_i \in U$. Since U is a subspace, $v_1 + \cdots + v_m \in U$. Thus $V_1 + \cdots + V_m \subset U$. □

Example 2.27. Let v_1-v_4 be as in Example 2.3 and V_1, V_2 be as in Example 2.24. Then we have $\mathrm{span}(v_1) + \mathrm{span}(v_2) = V_1$ and $V_1 + V_2 = F^3$. If $V_3 = \mathrm{span}(v_4)$, then $V_1 + V_3 = F^3$.

Theorem 2.28. $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.