Linear Algebra I: Practice Midterm Solution

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Problem 1. Suppose W is a subspace of V and $v_1 + W, \ldots, v_m + W$ is a basis of V/W. Let $w_1, \ldots, w_n$ be a basis of W. Show that $v_1, \ldots, v_m, w_1, \ldots, w_n$ is a basis of V.

Proof. We show that $v_1, \ldots, v_m, w_1, \ldots, w_n$ is a basis of V.

Spanning: Let $v \in V$. Since $v_1 + W, \ldots, v_m + W$ is a basis of V/W,

$$v + W = a_1(v_1 + W) + \cdots + a_m(v_m + W)$$

for some $a_1, \ldots, a_m \in F$. Then

$$v - (a_1 v_1 + \cdots + a_m v_m) \in W.$$

Let $u = v - (a_1 v_1 + \cdots + a_m v_m) \in W$. Since $w_1, \ldots, w_n$ is a basis of W,

$$u = b_1 w_1 + \cdots + b_n w_n$$

for some $b_j \in F$. Thus

$$v = a_1 v_1 + \cdots + a_m v_m + b_1 w_1 + \cdots + b_n w_n.$$

So the set spans V.

Linear independence: Suppose

$$a_1 v_1 + \cdots + a_m v_m + b_1 w_1 + \cdots + b_n w_n = 0.$$

Rewriting:

$$a_1 v_1 + \cdots + a_m v_m = -(b_1 w_1 + \cdots + b_n w_n) \in W.$$

Then in V/W,

$$a_1(v_1 + W) + \cdots + a_m(v_m + W) = 0 + W.$$

Since $v_1 + W, \ldots, v_m + W$ are linearly independent in V/W, $a_1 = \cdots = a_m = 0$. Then

$$b_1 w_1 + \cdots + b_n w_n = 0,$$

and since $w_1, \ldots, w_n$ is a basis of W, $b_1 = \cdots = b_n = 0$. So the set is linearly independent.

Hence $v_1, \ldots, v_m, w_1, \ldots, w_n$ is a basis of V. □

Problem 2. Let V be a finite dimensional vector space. Let $\varphi \in V^*$ be nonzero. Show that there exists $v \in V$ with $\varphi(v) = 1$ and $V = \operatorname{span}\{v\} \oplus \ker \varphi$.

Proof. Since $\varphi \neq 0$, pick $u \in V$ with $\varphi(u) \neq 0$. Define

$$v = \frac{u}{\varphi(u)}.$$

Then $\varphi(v) = 1$.

Now, for any $x \in V$, write

$$x = \varphi(x)v + \big(x - \varphi(x)v\big).$$

We have $\varphi(x)v \in \operatorname{span}\{v\}$, and

$$\varphi\big(x - \varphi(x)v\big) = \varphi(x) - \varphi(x) \cdot 1 = 0,$$

so $x - \varphi(x)v \in \ker \varphi$. Hence $V = \operatorname{span}\{v\} + \ker \varphi$.

To show the sum is direct: suppose $av \in \operatorname{span}\{v\} \cap \ker \varphi$. Then $\varphi(av) = a = 0$, so $a = 0$. Thus the intersection is trivial.

Therefore $V = \operatorname{span}\{v\} \oplus \ker \varphi$. □

Problem 3. Let V be a two-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear transformation. Suppose that $\beta = (v_1, v_2)$ and $\gamma = (w_1, w_2)$ are two bases in V such that

$$w_1 = v_1 + v_2, \quad w_2 = v_1 + 2v_2.$$

Find $[T]_\beta^\beta$ if

$$[T]_\gamma^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}.$$

Proof. By the assumption, we have $[w_1]_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $[w_2]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then $P = [\mathrm{id}]_\gamma^\beta = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

On the other hand, we observe that $v_2 = w_2 - w_1$, and $v_1 = w_1 - v_2 = 2w_1 - w_2$. Then $[v_1]_\gamma = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $[v_2]_\gamma = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then $P^{-1} = [\mathrm{id}]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. We have

$$\begin{aligned} [T]_\beta^\beta =& [\mathrm{id}]_\gamma^\beta [T]_\gamma^\gamma [\mathrm{id}]_\beta^\gamma = P[T]_\gamma^\gamma P^{-1} \\ =& \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ =& \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 5 & -2 \end{pmatrix} \\ =& \begin{pmatrix} 10 & -5 \\ 15 & -7 \end{pmatrix} \end{aligned}$$

. $\square$

Problem 4. Let V be the subspace of $C(\mathbb{R})$ given by $\operatorname{span}(e^{3x}\cos x, e^{3x}\sin x)$. Consider the linear map $L : V \to C(\mathbb{R})$ defined by $L(f) = f' - f$, where the prime denotes differentiation with respect to x.

(i) Show that $e^{3x}\cos x, e^{3x}\sin x$ are linearly independent.

(ii) Show that the image of L is in V, that is $\operatorname{im} L \subset V$.

(iii) Let $\beta = (e^{3x}\cos x,\ e^{3x}\sin x)$, find $[L]_\beta^\beta$.

(iv) Find $\ker L$ and $\operatorname{im} L$.

(v) Find a solution to the differential equation $f' - f = 2e^{3x}\cos x$.

Proof. (i) Suppose $ae^{3x}\cos x + be^{3x}\sin x = 0$ for all x. Dividing by e^{3x}, we get $a\cos x + b\sin x = 0$. Let $x = 0$ we have $a = 0$. Let $x = \pi/2$ we have $b = 0$. So they are linearly independent.

(ii) Let $f = Ae^{3x}\cos x + Be^{3x}\sin x$. Then

$$f' = e^{3x}[(3A + B)\cos x + (-A + 3B)\sin x].$$

So

$$L(f) = f' - f = e^{3x}[(2A + B)\cos x + (-A + 2B)\sin x] \in V.$$

(iii) Let $u_1 = e^{3x}\cos x, u_2 = e^{3x}\sin x$. By the formula in (ii) we have

$$L(u_1) = 2u_1 - u_2, \quad L(u_2) = u_1 + 2u_2.$$

So

$$[L]_\beta^\beta = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

(iv) Let $f = Ae^{3x}\cos x + Be^{3x}\sin x \in V$. And suppose

$$L(f) = e^{3x}\left[(2A + B)\cos x + (-A + 2B)\sin x = 0\right].$$

Hence by linear independence

$$2A + B = 0, \quad -A + 2B = 0.$$

From the first, $B = -2A$. Sub into second: $-A - 4A = -5A = 0 \Rightarrow A = 0 \Rightarrow B = 0$. So $\ker L = \{0\}$.

Since $L : V \to V$ is linear and injective, we know from class it is surjective and we have $\operatorname{im} L = V$.

(v) Solve $L(f) = 2e^{3x}\cos x$. Let $[f]_\beta = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Solving: $a = \frac{4}{5}, b = \frac{2}{5}$. So

$$f(x) = \frac{4}{5}e^{3x}\cos x + \frac{2}{5}e^{3x}\sin x.$$

$\square$

Problem 5. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & 1 \\ -3 & -6 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

(i) Find all $x \in \mathbb{R}^3$ such that $Ax = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

(ii) Let $V \subset \mathbb{R}^3$ be the set of vectors $b \in \mathbb{R}^3$ such that the system $Ax = b$ is solvable. Find a basis for V.

Proof. (i) Solve $Ax = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 4 & 1 & 3 \\ -3 & -6 & 2 & -1 \\ 1 & 2 & 1 & 2 \end{array}\right)$$

Step 1: Swap $R1 \leftrightarrow R3$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -6 & 2 & -1 \\ 2 & 4 & 1 & 3 \end{array}\right)$$

Step 2: $R2 \leftarrow R2 + 3R1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 5 & 5 \\ 2 & 4 & 1 & 3 \end{array}\right)$$

Step 3: $R3 \leftarrow R3 - 2R1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

Step 4: $R2 \leftarrow \frac{1}{5}R2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

Step 5: $R3 \leftarrow R3 + R2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Step 6: $R1 \leftarrow R1 - R2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From RREF: $x_1 + 2x_2 = 1$, $x_3 = 1$, x_2 free.
Let $x_2 = t$, then $x_1 = 1 - 2t$, $x_3 = 1$.

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

(ii) $V = \operatorname{im}(A)$. RREF of A:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Pivot columns 1 and 3 of A form a basis:

$$\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

$\square$