Linear Algebra I: Practice Midterm

October 10, 2025

Problem 1. Suppose W is a subspace of V and $v_1 + W, \dots, v_m + W$ is a basis of V/W. Let $w_1, \dots, w_n$ be a basis of W. Show that $v_1, \dots, v_m, w_1, \dots, w_n$ is a basis of V.

Problem 2. Let V be a finite dimensional vector space. Let $\varphi \in V^*$ be nonzero. Show that there exists $v \in V$ with $\varphi(v) = 1$ and $V = \operatorname{span}\{v\} \oplus \ker \varphi$.

Problem 3. Let V be a two-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear transformation. Suppose that $\beta = (v_1, v_2)$ and $\gamma = (w_1, w_2)$ are two bases in V such that

$$w_1 = v_1 + v_2, \quad w_2 = v_1 + 2v_2.$$

Find $[T]_\beta^\beta$ if

$$[T]_\gamma^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}.$$

Problem 4. Let V be the subspace of $C(\mathbb{R})$ given by $\operatorname{span}(e^{3x}\cos x, e^{3x}\sin x)$. Consider the linear map $L : V \to C(\mathbb{R})$ defined by $L(f) = f' - f$, where the prime denotes differentiation with respect to x.

(i) Show that $e^{3x}\cos x, e^{3x}\sin x$ are linearly independent.

(ii) Show that the image of L is in V, that is $\operatorname{im} L \subset V$.

(iii) Let $\beta = (e^{3x}\cos x,\ e^{3x}\sin x)$, find $[L]_\beta^\beta$.

(iv) Find $\ker L$ and $\operatorname{im} L$.

(v) Find a solution to the differential equation $f' - f = 2e^{3x}\cos x$.

Problem 5. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & 1 \\ -3 & -6 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

(i) Find all $x \in \mathbb{R}^3$ such that $Ax = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

(ii) Let $V \subset \mathbb{R}^3$ be the set of vectors $b \in \mathbb{R}^3$ such that the system $Ax = b$ is solvable. Find a basis for V.