## Problem set 9

**Problem 1.** Are the following matrices diagonalizable over $\mathbb{R}$? Over $\mathbb{C}$? If yes, find the invertible matrix $P \in M_{n\times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$ such that $P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. If no, give the reason.

$\begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}.$

**Problem 2.** Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and $-4$, $5$, and $\sqrt{7}$ are all eigenvalues of $T$. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})^t$.

*Hint*: The right hand side $(-4, 5, \sqrt{7})^t$ is a distraction. In fact you would only need to show that $T - 9I$ is invertible.

**Problem 3.** (i) If $z, w \in \mathbb{C}$ show that $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$, $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \bar{z}\,\bar{w}$.

(ii) If $A, C \in M_{m\times n}(\mathbb{C})$ and $B \in M_{n\times p}(\mathbb{C})$, show that $\operatorname{Re} A = \frac{1}{2}(A + \bar{A})$, $\operatorname{Im} A = \frac{1}{2i}(A - \bar{A})$, $\overline{A + C} = \bar{A} + \bar{C}$ and $\overline{AB} = \bar{A}\bar{B}$.

(iii) If $A \in M_{n\times n}(\mathbb{C})$, show that $\det(\bar{A}) = \overline{\det(A)}$.

**Problem 4.** Let $\lambda_0, \dots, \lambda_n$ be distinct points in $\mathbb{R}$. Let $p_0, \dots, p_n$ be the polynomials in $\mathcal{P}_n(\mathbb{R})$ defined uniquely by the relations

$$p_i(\lambda_j) = \delta_{ij}, \quad \forall i, j \in \{0, \dots, n\}, \tag{1}$$

where $\delta_{ij}$ is the Kronecker delta.

(i) Show that

$$p_i(x) = \frac{(x - \lambda_0)\cdots(x - \lambda_{i-1})(x - \lambda_{i+1})\cdots(x - \lambda_n)}{(\lambda_i - \lambda_0)\cdots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\cdots(\lambda_i - \lambda_n)}.$$

*Hint*: Observe that $\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n$ are roots of $p_i$ and use Proposition 9.10 and degree to show that $p_i(x) = c(x - \lambda_0)\cdots(x - \lambda_{i-1})(x - \lambda_{i+1})\cdots(x - \lambda_n)$ for some $c \in \mathbb{R}$. Determine the value of $c$ using $p_i(\lambda_i) = 1$.

(ii) Show that $p_0, \dots, p_n$ are linearly independent in $\mathcal{P}_n(\mathbb{R})$.

*Hint*: It's easier to evaluate the linear combination of $p_0, \dots, p_n$ at $\lambda_i$ using (1) than trying to find the coordinates of $p_i$ under the standard basis.

(iii) We define $\varphi_i : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}$ given by $\varphi_i(f) = f(\lambda_i)$. From Homework 5 Problem 4, we know that $\varphi_i$ is a linear function i.e. $\varphi_i \in V^*$. Conclude that $\varphi_0, \dots, \varphi_n$ is the dual basis of $p_0, \dots, p_n$.

(iv) Show that for any $p(x) \in \mathcal{P}_n(\mathbb{R})$, $p(x) = \sum_{i=0}^n p(\lambda_i)p_i$. This is the **Lagrange interpolation formula**.

(v) Show that the change of basis matrix from $p_0, \dots, p_n$ to the standard basis of $\mathcal{P}_n(\mathbb{R})$ is

$$\begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^n \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^n \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^n \end{pmatrix}.$$

This is called the **Vandermonde matrix**.

## Optional exercises related to Lecture 9

Do NOT submit this with your homework.

The following problems are from Linear algebra done right.

1. Exercise 1B: 8, Exercise 3B: 33 Exercise 5A: 17, 18

   *Remark* 1. Exercise 1B: 8 and 3B: 33 answer the question: we can view $A \in M_{n\times n}(\mathbb{R})$ as a matrix $A \in M_{n\times n}(\mathbb{C})$ easily but what is the vector space over $\mathbb{C}$ that the linear transformation acting on? In other words, if $T : V \to V$ is a linear transformation on a vector space $V$ over $\mathbb{R}$, how do we view $T$ as a linear transformation on some vector space over $\mathbb{C}$ where $T$ has the same matrix representation under a basis? Exercises 5A: 17,18 are simply restatements of the results we proved in class using matrices, now expressed in the language of linear transformations.

2. Exercise 4: 8,11,14.

   *Remark* 2. Exercise 4: 8 is a very useful test whether a polynomial has repeated roots. Exercise 4: 14 is an interesting proof of the Bezout identity.

3. Exercise 5A: 9, 19, 20

   *Remark* 3. In infinite dimensional space, characteristic polynomials are not available. One can only use the definition to find eigenvalues and eigenvectors. There could be infinitely many eigenvalues or no eigenvalues at all regardless of the choice of the field.