Problem set 3

Problem 1. Let $\mathcal{Q}_n(1) = \{f \in \mathcal{P}_n(\mathbb{R}) : f(1) = 0\}$ be as in Homework 2 Problem 3.

(i) Show that $\mathcal{P}_n(\mathbb{R}) = \mathcal{Q}_n(1) \oplus \operatorname{span}(1)$.

(ii) Describe the quotient space $\mathcal{P}_n(\mathbb{R})/\mathcal{Q}_n(1)$ (what does a typical coset look like?).

(iii) Determine $\dim(\mathcal{P}_n(\mathbb{R})/\mathcal{Q}_n(1))$.

You can use any results that were proved in Homework 2 Problem 3.

Problem 2. Let V be a vector space and V_1, V_2, U be subspaces such that $V = V_1 \oplus U = V_2 \oplus U$. Is it true that $V_1 = V_2$? If yes, prove it. If no, give an example.

Problem 3. Let $T : V \to W$ be a linear transformation between vector spaces V, W.

(i) Let $x \in \operatorname{span}(v_1, \ldots, v_n)$. Show that $T(x) \in \operatorname{span}(T(v_1), \ldots, T(v_n))$.

(ii) Let $U \subset V$ be a subspace. Show that $T(U)$ is a subspace of W.

(iii) If $v_1, \ldots, v_n \in V$ is linearly dependent, then $T(v_1), \ldots, T(v_n)$ is linearly dependent.

(iv) If $T(v_1), \ldots, T(v_n)$ is linearly independent, then $v_1, \ldots, v_n \in V$ is linearly independent.

(v) Give an example of T and $v_1, \ldots, v_n \in V$ where $v_1, \ldots, v_n$ is linearly independent but $T(v_1), \ldots, T(v_n)$ is linearly dependent.

Problem 4. Show that the following maps are linear transformations.

(i) Let $W \subset V$ be a subspace. The canonical projection $\pi : V \to V/W$ is given by $\pi(x) = x + W$.

(ii) Let $W \subset V$ be a subspace. The **inclusion map** $\iota : W \to V$ is given by $\iota(x) = x$ for $x \in W$.

(iii) Let V be the vector space in Homework 1 Problem 1. We define $T : V \to C(\mathbb{R})$ by $T(f)(x) = \log(f(x))$.

(iv) Let V be the vector space in Homework 1 Problem 1. We define $S : C(\mathbb{R}) \to V$ by $S(f)(x) = e^{f(x)}$.

(v) We define $T : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ by $T(p)(x) = p(x+1)$ which means we replace x by $x+1$ in the polynomial. For example $T(x^2+1) = (x+1)^2 + 1 = x^2 + 2x + 2$.