Problem set 6

Problem 1. (i) Let $T : V \to W$ be invertible. Show that T^t is invertible and $(T^t)^{-1} = (T^{-1})^t$.

(ii) Let $A \in M_{n\times n}(F)$ be an invertible matrix. Show that if A is symmetric then A^{-1} is also symmetric and if A is anti-symmetric, then A^{-1} is also anti-symmetric.

Problem 2. Let V be a finite-dimensional vector space over a field F and $W \subset V$ be a subspace .

(i) Let $\varphi \in W^*$ and $(w_1, \dots, w_m)$ be a basis for W. We extend $(w_1, \dots, w_m)$ to a basis $(w_1, \dots, w_m, v_{m+1}, \dots, v_n)$ of V. We define $\tilde{\varphi} : V \to F$ by specifying its values on the basis:

$$\tilde{\varphi}(w_i) = \varphi(w_i) \quad \text{for } i = 1, \dots, m$$

and

$$\tilde{\varphi}(v_j) = 0 \quad \text{for } j = m+1, \dots, n.$$

Show that $\tilde{\varphi}(w) = \varphi(w)$ for any $w \in W$.

(ii) Let $i : W^* \to V^*$ be the map defined by $i(\varphi) = \tilde{\varphi}$. Show that i is an injective linear map.

(iii) Let $\varphi_1, \dots, \varphi_n$ be the dual basis of $w_1, \dots, w_m, v_{m+1}, \dots, v_n$. Show that $i(W^*) = \operatorname{span}(\varphi_1, \dots, \varphi_m)$ and $W^\perp = \operatorname{span}(\varphi_{m+1}, \dots, \varphi_n)$.

(iv) Show that $V^* = i(W^*) \oplus W^\perp$.

Remark 1. $i(W^*)$ is like the mirror of W in V^* and $W^\perp$ is the complement of $i(W^*)$ in V^*.

Problem 3. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \qquad B = A^t = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Find the RREF of A and B.

Problem 4. Let $T : V \to W$ be a linear transformation and $X \subset W$ be a subspace.

(i) Show that $T^{-1}(X) = \{v \in V : T(v) \in X\}$ is a subspace of V.

(ii) Let $w_1, \dots, w_k \in X$ be a basis of $X \cap \operatorname{im}(T)$ and $v_1, \dots, v_k \in V$ be such that $T(v_i) = w_i$ for all $i = 1, \dots, k$. Show that $T^{-1}(X) = \operatorname{span}(v_1, \dots, v_k) \oplus \ker(T)$.

(iii) Show that $\dim T^{-1}(X) = \dim V + \dim X - \dim(X + \operatorname{im}(T))$.

Hint: You need to show that $v_1, \dots, v_k$ in (ii) are linearly independent.

(iv) The **codimension** of a subspace $X \subset W$ is defined to be $\dim W - \dim X$. A linear transformation T is **transversal** to X if $X + \operatorname{im}(T) = W$. Show that (iii) implies that if if T is transversal to X, then the codimension of X is equal to the codimension of $T^{-1}(X)$.