Problem set 10

Problem 1. (i) Show that $A \in M_{n \times n}(F)$ is invertible if and only if it has no 0 eigenvalue.

(ii) Let A be invertible and v be an eigenvector of A with eigenvalue λ. Show that v is an eigenvector of A^{-1} with eigenvalue λ^{-1}. Moreover, show that

$$p_{A^{-1}}(x) = \frac{(-1)^n}{\det(A)} x^n p_A(\frac{1}{x}).$$

(iii) Let $A \in M_{n \times n}(F)$ be invertible. Show that A is diagonalizable over F if and only if A^{-1} is diagonalizable over F.

Hint: For (ii) use the identity $xI - A^{-1} = -xA^{-1}(\frac{1}{x}I - A)$.

Problem 2. Let $A, B \in M_{n \times n}(\mathbb{C})$. We define the **commutator** as $[A, B] = AB - BA$.

Show that there is no $A, B \in M_{n \times n}(\mathbb{C})$ such that $[A, B] = I$.

Hint: What is $\mathrm{tr}([A, B])$?

Remark 1. Note that if P and Q are as in Example 3.27, then $[P, Q] = I$. The difference is that P, Q are linear transformations on infinite dimensional vector space. That $[P, Q] = I$ is one of the axioms of quantum mechanics. Since such operators do not exist on finite dimensional vector spaces, quantum mechanics studies operators on infinite dimensional spaces e.g. the space of square integrable functions $L^2(\mathbb{R}^3)$.

Problem 3. (i) Let $\lambda_1, \ldots, \lambda_n \in F$, $A = \mathrm{diag}(\lambda, \ldots, \lambda_n)$ and $f(x) \in F[x]$. Show that $f(A) = \mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))$.

(ii) Show that if $A \in M_{n \times n}(F)$ is diagonalizable over F, then $f(A)$ is also diagonalizable over F.

(iii) Let

$$A = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$$

Consider

$$f(x) = x^{2025} + x \in \mathbb{R}[x].$$

Compute $f(A)$, $\mathrm{tr}(f(A))$, and $\det(f(A))$. You can use the results from Homework 9 Problem 1. You do not need to evaluate large powers.

Problem 4. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ be distinct real numbers. Let V be the subpsace of $C(\mathbb{R})$ given by $\mathrm{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_m x})$. Let $D : V \to C(\mathbb{R})$ be the linear transformation given by $D(f) = f'$ where f' means the derivative of f.

(i) Show that $\mathrm{im}\, D \subset V$.

(ii) Show that $D(e^{\lambda_i x}) = \lambda_i e^{\lambda_i x}$ for $i = 1, \ldots, m$.

(iii) Show that $e^{\lambda_1 x}, \ldots, e^{\lambda_m x}$ are linearly independent.

Optional exercises related to Lecture 10

Do NOT submit this with your homework.

The following problems are from Linear algebra done right.

1. Exercise 5D: 3,4,5,6,8,9,12,21

2. Exercise 8D: 7,10

3. Exercise 9C: 8