Problem set 1

Problem 1. Let V be the set of positive continuous functions $f : \mathbb{R} \to (0, \infty)$ equipped with the addition:

$$(f+g)(x) = f(x)g(x) \quad \text{(multiplying the functions } f \text{ and } g\text{)}$$

and scalar multiplication

$$(cf)(x) = f(x)^c \quad (f \text{ raised to the } c \text{ power}).$$

Show that V is a vector space over $\mathbb{R}$ by finding the additive identity and additive inverse and checking the axioms.

Problem 2. Let S be the set of polynomials of degree exactly n i.e. $S = \{a_n t^n + \cdots + a_0 \in \mathcal{P}_n(F) : a_n \neq 0\}$. Is S a subspace of $\mathcal{P}_n(F)$? Justify your answer.

Problem 3. To make a distinction, we write the zero vector as $\vec{0}$ and the 0 scalar as 0. Use the axioms of vector spaces as well as *Proposition 1.3 (vii) proved in class* to show the following:

(i) The element $\vec{0} \in V$ is unique.

(ii) For any $x \in V$, $0x = \vec{0}$.

(iii) For any $a \in F$, $a\vec{0} = \vec{0}$.

(iv) If $a \in F$, $x \in V$ such that $ax = \vec{0}$, then either $a = 0$ (in F) or $x = \vec{0}$ (in V).

(v) For any $x \in V$, the element $-x$ is unique.

(vi) For any $x \in V$, $(-1)x = -x$.

Hint for (ii), (iii): write $0x = (0+0)x$ and $a\vec{0} = a(\vec{0} + \vec{0})$.

Problem 4. This problem is to introduce the set of complex numbers $\mathbb{C}$. A complex number is a number of the form $a + bi$ where $a, b \in \mathbb{R}$ and i is the imaginary number satisfying the property $i^2 = -1$. The number a is called the *real part* and b is called the *imaginary part*. A complex number $a + bi$ is 0 if and only if $a = b = 0$.[1] The addition of complex numbers is defined as

$$(a+bi) + (c+di) = (a+c) + (b+d)i.$$

The multiplication of two complex numbers is defined as

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i.$$

(continued on next page)

[1] this fact is needed in (iv).

This formula is obtained by expanding the product using distributive law and $i^2 = -1$.

(i) Show that if $a+bi \neq 0$, then $\frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$ i.e. $(a+bi)(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i) = 1$.

(ii) Calculate $3i + (5+2i)$, $(4+i) - 3$, $(2+i)(1-i)$ and $\frac{1-i}{1+i}$.

(iii) Note that a real number multiplying a complex number is given by $a(c+di) = ac + adi$. Show that $\mathbb{C}$ with the addition of complex number and multiplication by real numbers is a vector space over $\mathbb{R}$.

(iv) Viewing $\mathbb{C}$ as a vector space over $\mathbb{R}$, show that $1, i$ is linearly independent.