Problem set 4

Problem 1. For the following linear transformations $T : V \to W$, β a basis of V andγ a basis of W, find $[T]_\beta^\gamma$.

(i) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by $T(x_1, x_2, x_3, x_4)^t = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4)^t$ and $\beta = \gamma$ is the standard basis of $\mathbb{R}^4$.

(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane around the origin counter-clockwise by angle θ and $\beta = \gamma$ be the standard basis on $\mathbb{R}^2$.

(iii) Let $\mathrm{tr} : M_{2\times 2}(F) \to F$ be the **trace map** defined by $\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$, $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ and $\gamma = \{1\}$.

Problem 2. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be the linear transformation $Tf = f' - f$ where f' is the usual derivative.

(i) Find $[T]_\beta^\beta$ for $\beta = \{1, x, x^2, x^3\}$.

(ii) Find $\ker T$.

(iii) Show that $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ surjective. You can either use Corollary 4.15 or directly verify.

Remark 1. (ii) and (iii) means that for any polynomial p of degree ≤ 3, the ODE $f' - f = p$ always has a unique solution given by $f = T^{-1}(p)$ and f is also a polynomial of degree ≤ 3.

Problem 3. We assume all vector spaces to be finite dimensional.

(i) Let $T : V \to W$ be an isomorphism and $U \subset V$ be a subspace. We know from Homework 3 that $T(U)$ is a subspace of W. Show that $\dim T(U) = \dim U$.

(ii) Let $S, T : V \to V$ be linear transformations. Suppose ST is invertible. Show that S and T are both invertible.

(iii) Let $R, S, T : V \to V$ be linear transformations. Suppose $RST = \mathrm{id}_V$. Show that S is invertible and $S^{-1} = TR$. Hint: first show that R, T are invertible and then $S = R^{-1}T^{-1}$.

Problem 4. Let V be a finite-dimensional vector space and $U, W \subseteq V$ be subspaces. Define the map $\varphi : U \to (U + W)/W, \quad \varphi(u) = u + W$.

(i) Show that φ is a linear transformation.

(ii) Find $\ker \varphi$.

(iii) Prove that φ is surjective. Hint: show that any element in $(U + W)/W$ can be written as $u + W$ for some $u \in U$.

(iv) Show that $U/(U \cap W) \cong (U + W)/W$ by applying the First Isomorphism Theorem, Theorem 4.13 on φ.

(v) Show that $\dim U - \dim(U \cap W) = \dim(U + W) - \dim W$ using Theorem 3.16 proved in class.

Remark 2. (iv) is called the *Second Isomorphism Theorem* for vector spaces. It gives an alternative proof of Theorem 3.1 as shown in (v).