Problem set 5

Problem 1. Let $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ be the map defined by $T(B) = AB$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $[T]_\beta^\beta$ for $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

Note that in your answer, $[T]_\beta^\beta$ should be a 4×4 matrix.

Problem 2. Let $\beta = (1, x, x^2)$, $\gamma = (1, x+1, (x+1)^2)$ be two bases of $\mathcal{P}_2(\mathbb{R})$.

(i) Find the change of basis matrix P from γ to β and the change of basis matrix P^{-1} from β to γ.

(ii) Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be defined as $T(f) = (3+x)f'(x) + f(x)$. Find $[T]_\beta^\beta$ and $[T]_\gamma^\gamma$ directly using definitions.

(iii) Compute $P[T]_\beta^\beta P^{-1}$ and see that it is equal to $[T]_\gamma^\gamma$.

Problem 3. Define linear functionals $\varphi_i \in \mathcal{P}_n(\mathbb{R})^*$ by:

$$\varphi_i(f) = \frac{1}{i!} f^{(i)}(a), \quad \text{for } i = 0, 1, \ldots, n$$

where $a \in \mathbb{R}$ is fixed, and $f^{(i)}(a)$ denotes the i-th derivative of f evaluated at $x = a$.

(i) Show that $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ is the dual basis corresponding to the basis $\{1, (x-a), (x-a)^2, \ldots, (x-a)^n\}$ of $\mathcal{P}_n(\mathbb{R})$.

(ii) Use this dual basis to find the coordinate of $f(x) \in \mathcal{P}_n(\mathbb{R})$ under the basis $\{1, (x-a), (x-a)^2, \ldots, (x-a)^n\}$. That is, show that

$$f(x) = \sum_{i=0}^{n} \varphi_i(f)(x-a)^i.$$

(iii) Let $\ell_b \in \mathcal{P}_n(\mathbb{R})^*$ be the linear function defined by: $\ell_b(f) = f(b)$. Express ℓ_b as a linear combination of the dual basis $\{\varphi_0, \ldots, \varphi_n\}$. That is, find scalars $c_0, \ldots, c_n$ such that:

$$\ell_b = \sum_{i=0}^{n} c_i \varphi_i.$$

You need to compute the c_i explicitly in terms of a and b.

Remark 1. (ii) is proving the Taylor expansion for polynomials using linear algebra.

Problem 4. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be the differentiation map defined by $Tf = f'$ where f' is the usual derivative. Find two bases β, γ of $\mathcal{P}_3(\mathbb{R})$ such that

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hint: you can mimic the proof of Theorem 5.21 to find such bases.