Problem set 8

Problem 1.

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & a_{n-2} \\ 0 & \cdots & 0 & 0 & -1 & a_{n-1} \end{pmatrix}$$

Compute $\det(A + tI)$. You should get a nice expression involving $a_0, a_1, \ldots, a_{n-1}$ and t. Row expansion and induction is probably the best way to go.

Problem 2. (i) Suppose $A \in M_{n\times n}(F)$ is **nilpotent** i.e. there is an integer $m > 0$ such that $A^m = 0$. Show that $\det(A) = 0$.

(ii) Show that if $A \in M_{n\times n}(F)$ and $c \in F$, then $\det(cA) = c^n \det(A)$.

(iii) Show that if $A \in M_{(2n+1)\times(2n+1)}(\mathbb{R})$ is anti-symmetric then $\det(A) = 0$.

(iv) Suppose $A \in M_{n\times n}(\mathbb{R})$ is **orthogonal** i.e. $AA^t = I$. Show that $\det(A) = \pm 1$.

(v) Suppose $A \in M_{3\times 3}(\mathbb{R})$ is orthogonal and $\det(A) = 1$. Show that $\det(I - A) = 0$.

Hint: Show that $A^t - I = (I - A)A^t$ and observe that $A^t - I = -(I - A)^t$.

Remark 1. (v) is saying that any rotation in $\mathbb{R}^3$ has an axis which is a nonzero vector in $\ker(I - A)$ or an eigenvector of A with eigenvalue 1.

Problem 3. Let $A \in M_{n\times n}(\mathbb{R})$ be an invertible matrix with all entries being integers. Show that the entries of its inverse A^{-1} are all integers if and only if $\det(A) = \pm 1$.

Hint: Use the cofactor formula for A^{-1}.

Problem 4. Suppose $A \in M_{n\times n}(F)$ is invertible and take $u, v \in F^n$.

(i) If $A = I_n$, $u = v = (1, \ldots, 1)^t$. Compute $A + uv^t \in M_{n\times n}(F)$ and $v^t A^{-1} u \in F$.

(ii) Use row operations by blocks to show that $\det \begin{pmatrix} 1 & -v^t \\ u & A \end{pmatrix} = \det \begin{pmatrix} 1 & -v^t \\ 0 & A + uv^t \end{pmatrix}$

(iii) Use column operations by blocks to show that $\det \begin{pmatrix} 1 & -v^t \\ u & A \end{pmatrix} = \det \begin{pmatrix} 1 + v^t A^{-1} u & -v^t \\ 0 & A \end{pmatrix}$

(iii) Conclude that $\det\left(A + uv^t\right) = \det(A)(1 + v^t A^{-1} u)$.

(iv) Compute $\det \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}$.

(v) The so-called Sherman-Morrison formula states that

$$\left(A + uv^t\right)^{-1} = A^{-1} - \frac{A^{-1} u v^t A^{-1}}{1 + v^t A^{-1} u}.$$

Verify the Sherman-Morrison formula. Can you think of a way to derive the Sherman-Morrison formula (without knowing it) via the above elementary operations by blocks?