LINEAR ALGEBRA

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linear algebra

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To Our Families

contents

*Sections denoted by an asterisk are optional.

preface

The language and concepts of matrix theory and, more generally, of linear algebra have come into widespread usage in the social and natural sciences. In addition, linear algebra continues to be of great importance in modern treatments of geometry and analysis.

The primary purpose of this book is to present a careful treatment of the principal topics of linear algebra and to illustrate the power of the subject through a variety of applications. Although the only formal prerequisite for the book is a one-year course in calculus, the material in Chapters 6 and 7 requires the mathematical sophistication of typical college juniors and seniors (who may or may not have had some previous exposure to the subject).

The book is organized to permit a number of different courses (ranging from three to six semester hours in length) to be taught from it. The core material (vector spaces, linear transformations and matrices, systems of linear equations, determinants, and diagonalization) is found in Chapters 1 through 5. The remaining chapters, treating canonical forms and inner product spaces, are completely independent and may be studied in any order. In addition, throughout the book are a variety of applications to such areas as differential equations, economics, geometry, and physics. These applications are not central to the mathematical development, however, and may be excluded at the discretion of the instructor.

We have attempted to make it possible for many of the important topics of linear algebra to be covered in a one-semester course. This goal has led us to develop the major topics with fewer unnecessary preliminaries than in a traditional approach. (Our treatment of the Jordan canonical form, for instance, does not require any theory of polynomials.) The resulting economy permits us to cover most of the book (omitting many of the optional sections and a detailed discussion of determinants) in a one-semester four-hour course for students who have had some prior exposure to linear algebra.

Chapter 1 of the book presents the basic theory of finite-dimensional vector spaces—subspaces, linear combinations, linear dependence and independence, bases, and dimension. The chapter concludes with an optional section in which we prove the existence of a basis in infinite-dimensional vector spaces.

Linear transformations and their relationship to matrices are the subject of Chapter 2. We discuss there the null space and range of a linear transformation, matrix representations of a transformation, isomorphisms, and change of coordinates. Optional sections on dual spaces and homogeneous linear differential equations end the chapter.

The applications of vector space theory and linear transformations to systems of linear equations are found in Chapter 3. We have chosen to defer this important subject so that it can be presented as a consequence of the preceding material. This approach allows the familiar topic of linear systems to illuminate the abstract theory and permits us to avoid messy matrix computations in the presentation of Chapters 1 and 2. There will be occasional examples in these chapters, however, where we shall want to solve systems of linear equations. (Of course, these examples will not be a part of the theoretical development.) The necessary background is contained in Section 1.4.

Determinants, the subject of Chapter 4, are of much less importance than they once were. In a short course we prefer to treat determinants lightly so that more time may be devoted to the material in Chapters 5 through 7. Consequently we have presented two alternatives in Chapter 4—a complete development of the theory (Sections 4.1 through 4.4) and a summary of the important facts that are needed for the remaining chapters (Section 4.5).

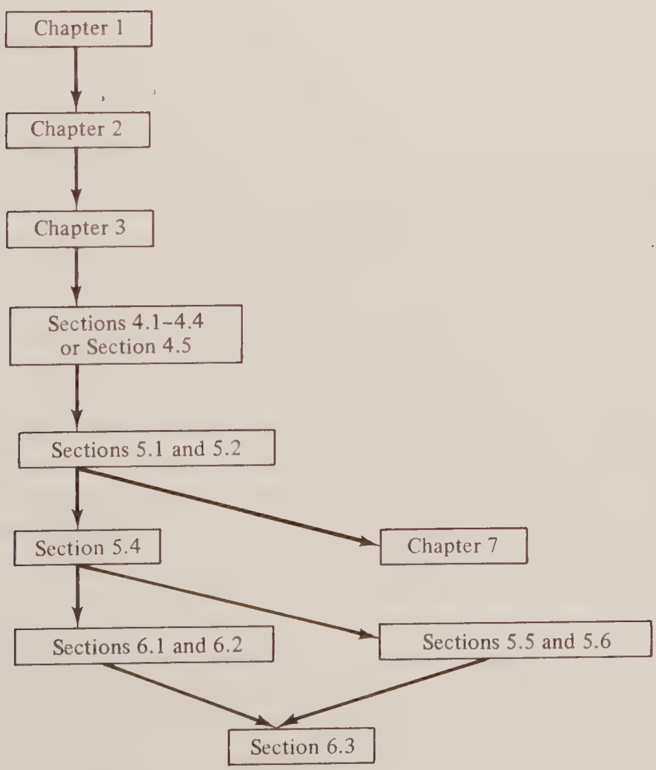
Chapter 5 discusses eigenvalues, eigenvectors, and diagonalization. One of the most important applications of this material occurs in computing matrix limits. We have therefore included an optional section on matrix limits and Markov chains in this chapter even though the most general statement of some of the results requires a knowledge of the Jordan canonical form. Sections 5.4, 5.5, and 5.6 contain material on invariant subspaces, the Cayley-Hamilton theorem, and the minimal polynomial, respectively.

Canonical forms are treated in Chapter 6. Sections 6.1 and 6.2 develop the Jordan form, and Section 6.3 presents the rational form.

Inner product spaces are the subject of Chapter 7. The basic mathematical theory (inner products; the Gram-Schmidt process; orthogonal complements; adjoint transformations; normal, self-adjoint, orthogonal, and unitary operators; orthogonal projections; and the spectral theorem) is contained in Sections 7.1, 7.2, 7.3, 7.5, 7.7, and 7.9. Sections 7.4, 7.6, 7.8, and 7.10 contain diverse applications of the rich inner product structure. The chapter ends with a discussion of bilinear and quadratic forms (Section 7.11).

There are five appendices. The first four, which discuss sets, functions, fields, and complex numbers, respectively, are intended to review basic ideas used throughout the book. Appendix E on polynomials is used primarily in Chapters 5 and 6, especially in Section 6.3. We prefer not to discuss the appendices independently but rather to refer to them as the need arises.

The following diagram illustrates the dependencies among the various chapters.



One final word is required about our notation. Sections denoted by an asterisk (*) are optional and may be omitted as the instructor sees fit. An exercise denoted by the dagger symbol (†) is not optional, however—we use this symbol to identify an exercise that will be cited at some later point of the text.

We are indebted to Douglas E. Cameron, *University of Akron*; Edward C. Ingraham, *Michigan State University*; David E. Kullman, *Miami University*; Carl D. Meyer, Jr., *North Carolina State University*; and Jean E. Rubin, *Purdue University*; who reviewed the entire manuscript, and to our colleagues and students for their suggestions and encouragement while the manuscript was in preparation. Special thanks are due to Jana Gehrke and Marilyn Parmantie for their help in typing the manuscript and to Harry Gaines, Ian List, and the staff of Prentice-Hall for their cooperation during the production process.

Normal, Illinois

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chapter 1

vector spaces

1.1 INTRODUCTION

Many familiar physical notions such as forces, velocities,† and accelerations involve both a magnitude (the amount of the force, velocity, or acceleration) and a direction. Any such entity involving both magnitude and direction is called a vector. Vectors are represented by arrows in which the length of the arrow denotes the magnitude of the vector and the direction of the arrow represents the direction of the vector. In most of the physical situations involving vectors, only the magnitude and direction of the vector are significant; consequently, we shall regard vectors with the same magnitude and direction as being equal irrespective of their positions.

In this section the geometry of vectors will be discussed. This geometry is derived from physical experiments that test the manner in which two vectors interact.

Familiar situations suggest that when two vectors act simultaneously at a point, the magnitude of the resultant vector (the vector obtained by

†The word "velocity" is being used here in its scientific sense—as an entity having both magnitude and direction. The magnitude of a velocity (without regard for the direction of motion) is called its *speed*.

adding the two original vectors) need not be the sum of the magnitudes of the original two. For example, a swimmer swimming upstream at the rate of 2 miles per hour against a current of 1 mile per hour will not progress at the rate of 3 miles per hour. For in this instance the motions of the swimmer and current oppose each other, and the rate of progress of the swimmer is only 1 mile per hour upstream. If, however, the swimmer were moving downstream (with the current), then his rate of progress would be 3 miles per hour downstream.

Experiments show that vectors add according to the following parallelogram law. (See Fig. 1.1.)

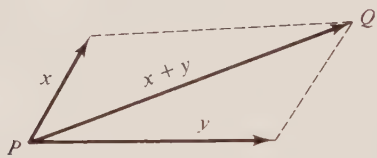


figure 1.1

Parallelogram Law for Vector Addition. *The sum of two vectors* x *and* y *that act at the same point* P *is the vector in the parallelogram having* x *and* y *as adjacent sides that is represented by the diagonal beginning at* P.

Since opposite sides of a parallelogram are parallel and of equal length, the endpoint Q of the arrow representing $x + y$ can also be obtained by allowing x to act at P and then allowing y to act at the endpoint of x. Likewise, the endpoint of the vector $x + y$ can be obtained by first permitting y to act at P and then allowing x to act at the endpoint of y. Thus two vectors x and y that both act at a point P may be added "tail-to-head"; that is, either x or y may be applied at P and a vector having the same magnitude and direction as the other may be applied to the endpoint of the first—the endpoint of this second vector is the endpoint of $x + y$.

The addition of vectors can be described algebraically with the use of analytic geometry. In the plane containing x and y, introduce a coordinate system with P at the origin. Let (a_1, a_2) denote the endpoint of x and (b_1, b_2) denote the endpoint of y. Then as Fig. 1.2 shows, the coordinates of Q, the endpoint of $x + y$, are $(a_1 + b_1, a_2 + b_2)$. Henceforth, when a reference is made to the coordinates of the endpoint of a vector, the vector should be assumed to emanate from the origin. Moreover, since a vector beginning at the origin is completely determined by its endpoint, we shall sometimes refer to *the point* x rather than *the endpoint of the vector* x if x is a vector emanating from the origin.

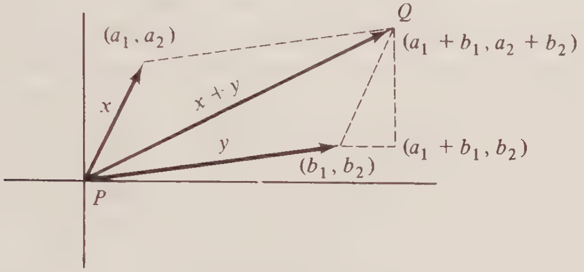


figure 1.2

Besides the operation of vector addition there is another natural operation that can be performed on vectors—the length of a vector may be magnified or contracted without changing the direction of the vector. This operation, called scalar multiplication, consists of multiplying a vector by a real number. If the vector x is represented by an arrow, then for any real number $t \geq 0$ the vector tx will be represented by an arrow having the same direction as the arrow representing x but having length t times the length of the arrow representing x. If $t < 0$, then the vector tx will be represented by an arrow having the opposite direction as x and having length $|t|$ times the length of the arrow representing x. Two non-zero vectors x and y are called *parallel* if $y = tx$ for some non-zero real number t. (Thus non-zero vectors having the same direction or opposite directions are parallel.)

To describe scalar multiplication algebraically, again introduce a coordinate system into a plane containing the vector x so that x emanates from the origin. If the endpoint of x has coordinates (a_1, a_2), then the coordinates of the endpoint of tx are easily shown to be (ta_1, ta_2). (See Exercise 5.)

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties for arbitrary vectors x, y, and z and arbitrary real numbers a and b:

1. $x + y = y + x$.
2. $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x.
4. For each vector x there is a vector y such that $x + y = 0$.
5. $1x = x$.
6. $(ab)x = a(bx)$.
7. $a(x + y) = ax + ay$.
8. $(a + b)x = ax + bx$.

Arguments similar to those given above show that these eight properties, as well as the geometric interpretations of vector addition and scalar multiplication, are true also for vectors acting in space rather than in a plane. We shall use these results to write equations of lines and planes in space.

Consider first the equation of a line in space that passes through two distinct points P and Q. Let O denote the origin of a coordinate system in space, and let u and v denote the vectors that begin at O and end at P and Q, respectively. If w denotes the vector beginning at P and ending at Q, then "tail-to-head" addition shows that $u + w = v$, and hence $w = v - u$, where $-u$ denotes the vector $(-1)u$. (See Fig. 1.3, in which quadrilateral

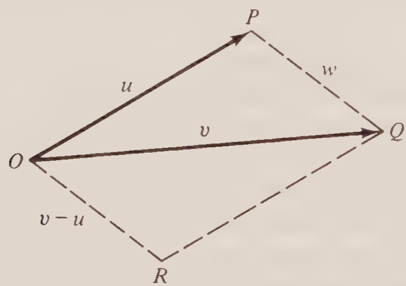


figure 1.3

$OPQR$ is a parallelogram.) Since a scalar multiple of w is parallel to w but possibly of a different length than w, any point on the line joining P and Q may be obtained as the endpoint of a vector that begins at P and has the form tw for some real number t. Conversely, the endpoint of every vector of the form tw that begins at P lies on the line joining P and Q. Thus an equation of the line through P and Q is $x = u + tw = u + t(v - u)$, where t is a real number and x denotes an arbitrary point on the line. Notice also that the endpoint R of the vector $v - u$ in Fig. 1.3 has coordinates equal to the difference of the coordinates of Q and P.

Example. We shall find the equation of the line through the points P and Q having coordinates $(-2, 0, 1)$ and $(4, 5, 3)$, respectively. The endpoint R of the vector emanating from the origin and having the same direction as the vector beginning at P and terminating at Q has coordinates $(4, 5, 3) - (-2, 0, 1) = (6, 5, 2)$. Hence the desired equation is

$$x = (-2, 0, 1) + t(6, 5, 2).$$

Now let P, Q, and R denote any three non-collinear points in space. These points determine a unique plane, whose equation can be found by use of our previous observations about vectors. Let u and v denote the vectors beginning at P and ending at Q and R, respectively. Observe that any point in the plane containing P, Q, and R is the endpoint S of a vector x beginning at P and having the form $t_1u + t_2v$ for some real numbers t_1 and t_2. The endpoint of t_1u will be the point of intersection of the line through P and Q with the line through S parallel to the line through P and R. (See Fig. 1.4.) A similar procedure will locate the endpoint of t_2v.

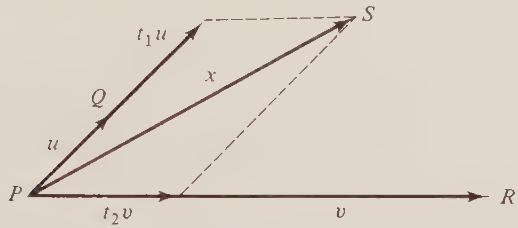


figure 1.4

Moreover, for any real numbers t_1 and t_2, $t_1u + t_2v$ is a vector lying in the plane containing P, Q, and R. It follows that an equation of the plane containing P, Q, and R is

$$x = P + t_1u + t_2v,$$

where t_1 and t_2 are arbitrary real numbers and x denotes an arbitrary point in the plane.

Example. Let P, Q, and R be the points having coordinates $(1, 0, 2)$, $(-3, -2, 4)$, and $(1, 8, -5)$, respectively. The endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at P and terminating at Q is $(-3, -2, 4) - (1, 0, 2) = (-4, -2, 2)$; likewise the endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at P and terminating at R is $(1, 8, -5) - (1, 0, 2) = (0, 8, -7)$. Hence the equation of the plane containing the three given points is

$$x = (1, 0, 2) + t_1(-4, -2, 2) + t_2(0, 8, -7).$$

Any mathematical structure possessing the eight properties on page 3 is called a "vector space." In the next section we shall formally define a vector space and consider many examples of vector spaces other than the ones mentioned above.

EXERCISES

1. Determine if the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) $(3, 1, 2)$ and $(6, 4, 2)$
(b) $(-3, 1, 7)$ and $(9, -3, -21)$
(c) $(5, -6, 7)$ and $(-5, 6, -7)$
(d) $(2, 0, -5)$ and $(5, 0, -2)$

2. Find the equations of the lines through the following pairs of points in space.
(a) $(3, -2, 4)$ and $(-5, 7, 1)$
(b) $(2, 4, 0)$ and $(-3, -6, 0)$
(c) $(3, 7, 2)$ and $(3, 7, -8)$
(d) $(-2, -1, 5)$ and $(3, 9, 7)$

3. Find the equations of the planes containing the following points in space.
(a) $(2, -5, -1)$, $(0, 4, 6)$, and $(-3, 7, 1)$
(b) $(3, -6, 7)$, $(-2, 0, -4)$, and $(5, -9, -2)$
(c) $(-8, 2, 0)$, $(1, 3, 0)$, and $(6, -5, 0)$
(d) $(1, 1, 1)$, $(5, 5, 5)$, and $(-6, 4, 2)$

4. What are the coordinates of the vector 0 in the Euclidean plane that satisfies condition 3 on page 3? Prove that this choice of coordinates does satisfy condition 3.

5. Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

6. Prove that the diagonals of a parallelogram bisect each other.

1.2 VECTOR SPACES

Because such diverse entities as the forces acting in a plane and the polynomials with real number coefficients both permit natural definitions of addition and scalar multiplication that possess properties 1 through 8 on page 3, it is natural to abstract these properties in the following definition.

Definition. *A* vector space (*or* linear space) V over a field† F *consists of a set in which two operations* (*called* addition *and* scalar multiplication, *respectively*) *are defined so that for each pair of elements* x, y *in* V *there is a unique element* x + y *in* V, *and for each element* a *in* F *and each element* x *in* V *there is a unique element* ax *in* V, *such that the following conditions hold:*

†See Appendix C. With few exceptions, however, the reader may interpret the word "field" to mean "field of real numbers" (which we denote by R) or "field of complex numbers" (which we denote by C).

(VS 1) *For all* x, y *in* V, $x + y = y + x$ (*commutativity of addition*).
(VS 2) *For all* x, y, z *in* V, $(x + y) + z = x + (y + z)$ (*associativity of addition*).
(VS 3) *There exists an element in* V *denoted by* 0 *such that* $x + 0 = x$ *for each* x *in* V.
(VS 4) *For each element* x *in* V *there exists an element* y *in* V *such that* $x + y = 0$.
(VS 5) *For each element* x *in* V, $1x = x$.
(VS 6) *For each pair* a, b *of elements in* F *and each element* x *in* V, $(ab)x = a(bx)$.
(VS 7) *For each element* a *in* F *and each pair of elements* x, y *in* V, $a(x + y) = ax + ay$.
(VS 8) *For each pair* a, b *of elements in* F *and each element* x *in* V, $(a + b)x = ax + bx$.

The elements $x + y$ *and* ax *are called the* sum *of* x *and* y *and the* product *of* a *and* x, *respectively.*

The elements of the field F are called *scalars* and the elements of the vector space V are called *vectors*. The reader should not confuse this use of the word "vector" with the physical entity discussed in Section 1.1; the word "vector" is now being used to describe any element of a vector space.

A vector space will frequently be discussed in the text without explicitly mentioning its field of scalars. The reader is cautioned to remember, however, that every vector space will be regarded as a vector space over a given field, which will be denoted by F.

In the remainder of this section we shall introduce several important examples of vector spaces that will be studied throughout the text. Observe that in describing a vector space it is necessary to specify not only the vectors but also the operations of addition and scalar multiplication.

An object of the form $(a_1, \ldots, a_n)$, where the entries a_i are elements of a field F, is called an *n-tuple* with entries from F. Two n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are defined to be equal if and only if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Example 1. The vector space F^n of n-tuples with entries from a field F.

The set of all n-tuples with entries from a field F forms a vector space, which we shall denote by F^n, under the operations of coordinatewise addition and multiplication; that is, if $x = (a_1, \ldots, a_n) \in \mathsf{F}^n$, $y = (b_1, \ldots, b_n) \in \mathsf{F}^n$, and $c \in F$, then

$$x + y = (a_1 + b_1, \ldots, a_n + b_n) \quad \text{and} \quad cx = (ca_1, \ldots, ca_n).$$

For example, in R^4

$$(3, -2, 0, 5) + (-1, 1, 4, 2) = (2, -1, 4, 7)$$

and

$$-5(1, -2, 0, 3) = (-5, 10, 0, -15).$$

Elements of F^n will often be written as *column vectors*:

$$\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$$

rather than as *row vectors* $(a_1, \ldots, a_n)$. Since a 1-tuple with entry from F may be regarded as an element of F, we shall write F rather than F^1 for the vector space of 1-tuples from F.

An $m \times n$ *matrix* with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where each entry a_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is an element of F. The entries $a_{i1}, a_{i2}, \ldots, a_{in}$ of the matrix above compose the i*th row* of the matrix and will often be regarded as a row vector in F^n, whereas the entries $a_{1j}, a_{2j}, \ldots, a_{mj}$ compose the j*th column* of the matrix and will often be regarded as a column vector in F^m. The $m \times n$ matrix having each entry equal to zero is called the *zero matrix* and will be denoted by O.

In this book we shall denote matrices by capital italic letters (e.g., A, B, and C) and shall denote the entry of a matrix A that lies in row i and column j by A_{ij}. In addition, if the number of rows and columns of a matrix are equal, then the matrix will be called *square*.

Two $m \times n$ matrices A and B are defined to be equal if and only if their corresponding entries are equal, that is, if and only if $A_{ij} = B_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 2. The vector space $\mathsf{M}_{m\times n}(F)$ of $m \times n$ matrices with entries from a field F.

The set of all $m \times n$ matrices with entries from a field F is a vector space, which we shall denote by $\mathsf{M}_{m\times n}(F)$, under the following operations of addition and scalar multiplication: for $A, B \in \mathsf{M}_{m\times n}(F)$ and $c \in F$,

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}.$$

For instance,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and

$$-3\begin{pmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{pmatrix}$$

in $\mathsf{M}_{2\times 3}(R)$.

Example 3. The vector space $\mathcal{F}(S, F)$ of all functions from a set S into a field F.

Let S be any non-empty set and F be any field, and let $\mathcal{F}(S, F)$ denote the set of all functions from S into F. Two elements f and g in $\mathcal{F}(S, F)$ are defined to be equal if $f(s) = g(s)$ for each $s \in S$. The set $\mathcal{F}(S, F)$ is a vector space under the operations of addition and scalar multiplication defined for $f, g, \in \mathcal{F}(S, F)$ and $c \in F$ by

$$(f + g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for each $s \in S$. Note that these are the familiar operations of addition and scalar multiplication used in calculus.

A *polynomial* with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a non-negative integer and $a_n, \ldots, a_0$ are elements of F. If $f(x) = 0$, that is, if $a_n = \cdots = a_0 = 0$, then $f(x)$ is called the *zero polynomial* and the degree of $f(x)$ is said to be -1; otherwise, the *degree* of a polynomial is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with a non-zero coefficient. Note that the polynomials of degree zero are of the form $f(x) = c$ for some non-zero scalar c. Two polynomials $f(x)$ and $g(x)$ are equal if and only if they have the same degree and their coefficients of like powers of x are equal.

When F is a field containing an infinite number of elements, we shall usually regard a polynomial with coefficients from F as a function from F into F. In this case the value of the function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

at $c \in F$ is the scalar

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Here we shall use either of the notations f or $f(x)$ for the polynomial function

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Example 4. The vector space $\mathsf{P}(F)$ of all polynomials with coefficients from a field F.

The set of all polynomials with coefficients from a field F is a vector space, which we shall denote by $\mathsf{P}(F)$, under the following operations: For

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and

$$g(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$$

in $\mathsf{P}(F)$ and $c \in F$,

$$(f + g)(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0)$$

and

$$(cf)(x) = ca_nx^n + ca_{n-1}x^{n-1} + \cdots + ca_0.$$

We shall see in Exercise 21 of Section 2.4 that the vector space defined in the example below is essentially the same as $\mathsf{P}(F)$.

Example 5. The space of all finitely non-zero sequences in a field F.

Let F be any field. A *sequence* in F is a function σ from the positive integers into F. As usual, the sequence σ such that $\sigma(n) = a_n$ will be denoted by $\{a_n\}$. The vector space V of all finitely non-zero sequences in F consists of all sequences $\{a_n\}$ in F that have only a finite number of non-zero terms a_n. If $\{a_n\}$ and $\{b_n\}$ are sequences in V and $t \in F$, then $\{a_n\} + \{b_n\}$ is that sequence $\{c_n\}$ in V such that $c_n = a_n + b_n$ $(n = 1, 2, \ldots)$ and $t\{a_n\}$ is that sequence $\{d_n\}$ in V such that $d_n = ta_n$ $(n = 1, 2, \ldots)$.

Our next two examples contain sets on which an addition and scalar multiplication are defined but which are not vector spaces.

Example 6. Let $S = \{(a_1, a_2)\colon\ a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Since (VS 1), (VS 2), and (VS 8) all fail to hold, S is not a vector space under these operations.

Example 7. Let S be as in Example 6 above. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0) \quad \text{and} \quad c(a_1, a_2) = (ca_1, 0).$$

Then S under these operations is not a vector space because (VS 3) (hence (VS 4)) and (VS 5) fail.

This section will conclude with a few of the elementary consequences of the definition of a vector space.

Theorem 1.1 (*Cancellation Law for Vector Addition*). *If* x, y, *and* z *are elements of a vector space* V *such that* $x + z = y + z$, *then* $x = y$.

PROOF. There exists an element v in V such that $z + v = 0$ (VS 4). Thus $x = x + 0 = x + (z + v) = (x + z) + v = (y + z) + v = y + (z + v) = y + 0 = y$ by (VS 2) and (VS 3). ∎

Corollary 1. *The vector* 0 *described in* (*VS* 3) *is unique.*

PROOF. Exercise.

Corollary 2. *The vector* y *described in* (*VS* 4) *is unique.*

PROOF. Exercise.

The vector 0 described in (VS 3) is called the *zero vector* of V, and the vector y described in (VS 4) (that is, the unique vector such that $x + y = 0$) is called the *additive inverse* of x and is denoted by $-x$.

The following result contains some of the elementary properties of scalar multiplication.

Theorem 1.2. *In any vector space* V *the following statements are true:*

(a) $0x = 0$ *for each* $x \in V$.
(b) $(-a)x = -(ax)$ *for each* $a \in F$ *and each* $x \in V$.
(c) $a0 = 0$ *for each* $a \in F$.

PROOF.

(a) By (VS 8), (VS 1), and (VS 3) it follows that

$$0x + 0x = (0 + 0)x = 0x = 0 + 0x.$$

Hence $0x = 0$ by Theorem 1.1.

(b) The element $-(ax)$ is the unique element of V such that $ax + [-(ax)] = 0$. Hence if $ax + (-a)x = 0$, Corollary 2 above will imply that $(-a)x = -(ax)$. But by (VS 8), $ax + (-a)x = [a + (-a)]x = 0x$, and so $ax + (-a)x = 0x = 0$ by (a). Thus $(-a)x = -(ax)$.

The proof of (c) is similar to the proof of (a). ∎

EXERCISES

1. Label the following statements as being true or false.

(a) Every vector space contains a zero vector.
(b) A vector space may have more than one zero vector.
(c) In any vector space $ax = bx$ implies that $a = b$.
(d) In any vector space $ax = ay$ implies that $x = y$.

(e) An element of F^n may be regarded as an element of $\mathsf{M}_{n\times 1}(F)$.
(f) An $m \times n$ matrix has m columns and n rows.
(g) In $\mathsf{P}(F)$ only polynomials of the same degree may be added.
(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.
(i) If f is a polynomial of degree n and c is a non-zero scalar, then cf is a polynomial of degree n.
(j) A non-zero element of F may be considered to be an element of $\mathsf{P}(F)$ having degree zero.
(k) Two functions in $\mathfrak{F}(S, F)$ are equal if and only if they have the same values at each point of S.

2. Write the zero vector of $\mathsf{M}_{3\times 4}(F)$.

3. If

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

what are M_{13}, M_{21}, and M_{22}?

4. Perform the indicated operations.

(a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix}$

(c) $4\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix}$

(d) $-5\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix}$

(e) $(2x^4 - 7x^3 + 4x + 3) + (8x^3 + 2x^2 - 6x + 7)$
(f) $(-3x^3 + 7x^2 + 8x - 6) + (2x^3 - 8x + 10)$
(g) $5(2x^7 - 6x^4 + 8x^2 - 3x)$
(h) $3(x^5 - 2x^3 + 4x + 2)$

Exercises 5 and 6 show why the definitions of matrix addition and scalar multiplication (as defined in Example 2) are the appropriate ones.

5. Richard Gard (Effects of Beaver on Trout in Sagehen Creek, California. *J. Wildlife Management*, **25**, 221–242) reports the following number of trout having crossed beaver dams in Sagehen Creek:

Upstream Crossings

	Fall	Spring	Summer
Brook trout	8	3	1
Rainbow trout	3	0	0
Brown trout	3	0	0

Downstream Crossings

	Fall	Spring	Summer
Brook trout	9	1	4
Rainbow trout	3	0	0
Brown trout	1	1	0

Record the upstream and downstream crossings as data in two 3×3 matrices and verify that the sum of these matrices gives the total number of crossings (both upstream and downstream) categorized by trout species and season.

6. At the end of May a furniture store had the following inventory:

	Early American	Spanish	Mediter-ranean	Danish
Living room suites	4	2	1	3
Bedroom suites	5	1	1	4
Dining room suites	3	1	2	6

Record this data as a 3×4 matrix M. In order to prepare for its June sale the store decided to double its inventory on each of the items above. Assuming that none of the present stock is sold until the additional furniture arrives, verify that the inventory on hand after the order is filled is described by the matrix $2M$. If the inventory at the end of June is described by the matrix

$$A = \begin{pmatrix} 5 & 3 & 1 & 2 \\ 6 & 2 & 1 & 5 \\ 1 & 0 & 3 & 3 \end{pmatrix},$$

interpret $2M - A$. How many suites were sold during the June sale?

7. Let $S = \{0, 1\}$ and $F = R$, the field of real numbers. In $\mathfrak{F}(S, R)$, show that $f = g$ and $f + g = h$, where $f(x) = 2x + 1$, $g(x) = 1 + 4x - 2x^2$, and $h(x) = 5^x + 1$.

8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in$ V and any $a, b \in F$.

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space under the operations of addition and scalar multiplication defined in Example 3.

11. Let V $= \{0\}$ consist of a single vector 0, and define $0 + 0 = 0$ and $c0 = 0$ for each c in F. Prove that V is a vector space over F. (V is called the *zero vector space.*)

12. A real-valued function defined on the real line is called an *even function* if $f(-x) = f(x)$ for each real number x. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and c is an element of F, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space under these operations? Verify your answer.

14. Let V $= \{(a_1, \ldots, a_n)\colon \ a_i \in C$ for $i = 1, 2, \ldots, n\}$. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

15. Let V $= \{(a_1, \ldots, a_n)\colon \ a_i \in R$ for $i = 1, 2, \ldots, n\}$. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

16. Let V $= \{(a_1, a_2)\colon \ a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in$ V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \dfrac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space under these operations? Justify your answer.

17. Let V $= \{(a_1, a_2)\colon \ a_1, a_2 \in C\}$. For $(a_1, a_2), (b_1, b_2) \in$ V and $c \in C$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space under these operations? Justify your answer.

18. Let V $= \{(a_1, a_2)\colon \ a_1, a_2 \in F\}$, where F is an arbitrary field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in$ V,

define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space under these operations? Justify your answer.

1.3 SUBSPACES

As usual, in the study of any algebraic structure it is of interest to examine subsets that possess the same structure as the set under consideration. The appropriate notion of substructure for vector spaces is introduced in this section.

Definition. A subset W *of a vector space* V *over a field* F *is called a* subspace *of* V *if* W *is a vector space over* F *under the operations of addition and scalar multiplication defined on* V.

In any vector space V, note that V and $\{0\}$ are subspaces. The latter is called the *zero subspace* of V.

Fortunately, it is not necessary to verify all the vector space conditions in order to prove that a subset W of a vector space V is in fact a subspace. Since conditions (VS 1), (VS 2), (VS 5), (VS 6), (VS 7), and (VS 8) are known to hold for elements of V, these conditions automatically hold for elements of a subset of V. Thus a subset W of V is a subspace of V if and only if the following four conditions hold:

1. $x + y \in$ W whenever $x \in$ W and $y \in$ W.
2. $ax \in$ W whenever $a \in F$ and $x \in$ W.
3. The zero vector of V belongs to W.
4. The additive inverse of each element of W belongs to W.

Actually, condition 4 is redundant, as the following theorem shows.

Theorem 1.3. Let V *be a vector space and* W *a subset of* V. *Then* W *is a subspace of* V *if and only if the following three conditions hold:*
(a) $0 \in$ W.
(b) $x + y \in$ W *whenever* $x \in$ W *and* $y \in$ W.
(c) $ax \in$ W *whenever* $a \in$ F *and* $x \in$ W.

PROOF. If W is a subspace of V, then W is a vector space under the operations of addition and scalar multiplication defined on V. Hence conditions (b) and (c) hold, and there exists an element $0' \in$ W such that $x + 0' = x$ for each $x \in$ W. But also $x + 0 = x$, and thus $0' = 0$ by Theorem 1.1. So condition (a) holds.

Conversely, if conditions (a), (b), and (c) hold, the discussion preceding this theorem shows that W will be a subspace of V if the additive inverse of each element of W belongs to W. But if $x \in$ W, then $(-1)x$ belongs to W by condition (c), and $-x = (-1)x$ by Theorem 1.2. Hence W is a subspace of V. ▮

The theorem above provides a simple method for determining whether or not a given subset of a vector space is in fact a subspace. Normally it is this result that is used to prove that a certain subset is a subspace.

The *transpose* M^t of an $m \times n$ matrix M is the $n \times m$ matrix obtained from M by interchanging the rows with the columns; that is, $(M^t)_{ij} = M_{ji}$. For example,

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}.$$

A *symmetric matrix* is a matrix M such that $M^t = M$. Clearly a symmetric matrix must be square. The set W of all symmetric matrices in $\mathsf{M}_{n\times n}(F)$ is a subspace of $\mathsf{M}_{n\times n}(F)$ since the conditions of Theorem 1.3 hold:

(a) The zero matrix is equal to its transpose and hence belongs to W.

It is easily proved that for any matrices A and B and any scalars a and b, $(aA + bB)^t = aA^t + bB^t$. (See Exercise 3.) Using this fact, we can easily establish conditions (b) and (c) of Theorem 1.3 as follows.

(b) If $A \in$ W and $B \in$ W, then $A = A^t$ and $B = B^t$. Now $(A + B)^t = A^t + B^t = A + B$, so that $A + B \in$ W.
(c) If $A \in$ W, then $A^t = A$. So for any $a \in F$, $(aA)^t = aA^t = aA$. Thus $aA \in$ W.

The following examples provide further illustrations of the concept of a subspace. The first three are particularly important.

Example 8. The diagonal matrices in $\mathsf{M}_{n\times n}(F)$.

Let M be an $n \times n$ matrix. The (*main*) *diagonal* of M consists of the entries $M_{11}, M_{22}, \ldots, M_{nn}$. An $n \times n$ matrix D is called a *diagonal matrix* if each entry not on the diagonal of D is zero, that is, if $D_{ij} = 0$ whenever $i \neq j$. The set of all diagonal matrices in $\mathsf{M}_{n\times n}(F)$ is a subspace of $\mathsf{M}_{n\times n}(F)$.

Example 9. The polynomials of degree less than or equal to n.

Let n be a non-negative integer, and let $\mathsf{P}_n(F)$ consist of all polynomials in $\mathsf{P}(F)$ having degree less than or equal to n. (Notice that the zero polynomial is an element of $\mathsf{P}_n(F)$ since its degree is -1.) Then $\mathsf{P}_n(F)$ is a subspace of $\mathsf{P}(F)$.

Example 10. The continuous real-valued functions defined on the real line R.

The set $\mathsf{C}(R)$ consisting of all the continuous real-valued functions defined on R is a subspace of $\mathcal{F}(R, R)$, where $\mathcal{F}(R, R)$ is as defined in Example 3.

Example 11. The *trace* of an $n \times n$ matrix M, denoted $\mathrm{tr}(M)$, is the sum of the entries of M lying on the diagonal; that is, $\mathrm{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}$. The set of all $n \times n$ matrices having trace equal to zero is a subspace of $\mathsf{M}_{n\times n}(F)$. (See Exercise 6.)

Example 12. The set of matrices in $\mathsf{M}_{m\times n}(F)$ having non-negative entries is not a subspace of $\mathsf{M}_{m\times n}(F)$ because condition (c) of Theorem 1.3 does not hold.

The next two theorems provide methods of forming subspaces from other subspaces.

Theorem 1.4. *Any intersection of subspaces of a vector space* V *is a subspace of* V.

PROOF. Let $\mathcal{C}$ be a collection of subspaces of V, and let W denote the intersection of all the subspaces in $\mathcal{C}$. Since every subspace contains the zero vector, $0 \in \mathsf{W}$. Let $a \in F$ and x, y be elements of W; then x and y are elements of each subspace in $\mathcal{C}$. Hence $x + y$ and ax are elements of each subspace in $\mathcal{C}$ (because the sum of vectors in a subspace and the product of a scalar and a vector from the subspace both belong to that subspace). Thus $x + y \in \mathsf{W}$ and $ax \in \mathsf{W}$, so that W is a subspace by Theorem 1.3. ∎

Having shown that the intersection of subspaces is a subspace, it is natural to consider the question of whether or not the union of subspaces is a subspace. It is easily seen that the union of subspaces must satisfy conditions (a) and (c) of Theorem 1.3 but that condition (b) need not hold. In fact, it can be readily shown (see Exercise 18) that the union of two subspaces is a subspace if and only if one of the subspaces is a subset of the other. It is natural, however, to expect that there should be a method of combining two subspaces W_1 and W_2 to obtain a larger subspace (that is, one that contains both W_1 and W_2). As we have suggested above, the key to finding such a subspace is condition (b) of Theorem 1.3. This observation suggests that we should consider the "sum" of two subspaces (as defined below).

Definition. *If* S_1 *and* S_2 *are non-empty subsets of a vector space* V, *then the* sum *of* S_1 *and* S_2, *denoted* $\mathsf{S}_1 + \mathsf{S}_2$, *is the set* $\{\mathsf{x} + \mathsf{y}\colon\ \mathsf{x} \in \mathsf{S}_1 \text{ and } \mathsf{y} \in \mathsf{S}_2\}$.

The sum *of any finite number of non-empty subsets* $S_1, \ldots, S_n$ *of* V *is defined analogously as the set*

$$S_1 + \cdots + S_n = \{x_1 + \cdots + x_n: \quad x_i \in S_i \textit{ for } i = 1, 2, \ldots, n\}.$$

Theorem 1.5. *If* W_1 *and* W_2 *are subspaces of a vector space* V, *then* $W_1 + W_2$ *is a subspace of* V.

PROOF. Let W_1 and W_2 be subspaces of V. Since $0 \in W_1$ and $0 \in W_2$, $0 = 0 + 0 \in W_1 + W_2$. Let $a \in F$ and $x, y \in W_1 + W_2$; then there exist $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$. Now

$$x + y = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2)$$

is an element of $W_1 + W_2$ because $x_1 + y_1 \in W_1$ and $x_2 + y_2 \in W_2$, and

$$ax = a(x_1 + x_2) = ax_1 + ax_2$$

is an element of $W_1 + W_2$ because $ax_1 \in W_1$ and $ax_2 \in W_2$. Thus $W_1 + W_2$ is a subspace of V by Theorem 1.3. ▮

Corollary. *The sum of any finite number of subspaces of* V *is a subspace of* V.

A special type of sum will play an important role in subsequent chapters. We shall introduce a special case of this concept in the definition below.

Definition. *A vector space* V *is said to be the* direct sum *of* W_1 *and* W_2, *denoted by* $V = W_1 \oplus W_2$, *if* W_1 *and* W_2 *are subspaces of* V *such that* $W_1 \cap W_2 = \{0\}$ *and* $W_1 + W_2 = V$.

Example 13. Let $W_1 = \{(a, 0): \quad a \in F\}$ and $W_2 = \{(0, b): \quad b \in F\}$. Then $F^2 = W_1 \oplus W_2$.

Example 14. A real-valued function g defined on R is called an *even function* if $g(-x) = g(x)$ for each $x \in R$ and is called an *odd function* if $g(-x) = -g(x)$ for each $x \in R$. Let W_1 and W_2 denote the sets of all even and odd functions, respectively, in $\mathfrak{F}(R, R)$.

We shall prove that $\mathfrak{F}(R, R) = W_1 \oplus W_2$. It is easily shown that W_1 and W_2 are subspaces of $\mathfrak{F}(R, R)$. (See Exercise 19.) Suppose that $g \in W_1 \cap W_2$; then g is both an even function and an odd function. So $g(-x) = g(x)$ and $g(-x) = -g(x)$ for each $x \in R$, and thus g is the zero function. Therefore $W_1 \cap W_2 = \{0\}$. Let $f \in \mathfrak{F}(R, R)$, and define $g, h \in \mathfrak{F}(R, R)$ as $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$. Then g is an even function and h is an odd function such that $f = g + h$. Hence $f \in W_1 + W_2$. Since f is an arbitrary element of $\mathfrak{F}(R, R)$, it follows that $\mathfrak{F}(R, R) = W_1 + W_2$. Therefore $\mathfrak{F}(R, R)$ is the direct sum of W_1 and W_2.

If W_1 and W_2 are subspaces of a vector space V such that $W_1 + W_2 = V$, then each element in V can be expressed as the sum of an element x_1 in W_1 and an element x_2 in W_2. It is possible that there may be many such representations, i.e., that x_1 and x_2 are not unique. For example, if

$$W_1 = \{(a_1, a_2, a_3) \in F^3: \quad a_3 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3) \in F^3: \quad a_1 = 0\},$$

then clearly $W_1 + W_2 = F^3$. In fact, for every $c \in F$, $(b_1, b_2, b_3) = (b_1, b_2 + c, 0) + (0, -c, b_3)$ is a representation of (b_1, b_2, b_3) as the sum of an element $(b_1, b_2 + c, 0)$ in W_1 and an element $(0, -c, b_3)$ in W_2. Thus in this instance the representation of elements of F^3 as sums of an element in W_1 and an element in W_2 is not unique. Our next result determines when this type of uniqueness exists.

Theorem 1.6. Let W_1 and W_2 be subspaces of a vector space V. Then V is the direct sum of W_1 and W_2 if and only if each element in V can be uniquely *written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.*

PROOF. Suppose that $V = W_1 \oplus W_2$. Since $V = W_1 + W_2$, each element in V can be expressed as the sum of vectors in W_1 and W_2. Suppose that some element z in V can be written as $z = x_1 + x_2$ and also as $z = y_1 + y_2$, where $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Then $x_1 + x_2 = y_1 + y_2$, so that $x_1 - y_1 = y_2 - x_2$. Now $x_1 - y_1 \in W_1$ since both x_1 and y_1 are elements of W_1, and likewise $y_2 - x_2 \in W_2$. But because $x_1 - y_1 = y_2 - x_2$, it follows that $x_1 - y_1 = y_2 - x_2 \in W_1 \cap W_2 = \{0\}$. Thus $x_1 - y_1 = y_2 - x_2 = 0$, and so $x_1 = y_1$ and $x_2 = y_2$. This proves the uniqueness of the representation of z as the sum of an element in W_1 and an element in W_2.

The proof of the converse is an exercise. ∎

EXERCISES

1. Label the following statements as being true or false:
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
   (b) The empty set is a subspace of every vector space.
   (c) If V is a vector space other than the zero vector space $\{0\}$, then V contains a subspace W such that $W \neq V$.
   (d) The sum of any two subsets of V is a subspace of V.
   (e) An $n \times n$ diagonal matrix can never have more than n non-zero entries.
   (f) The trace of a square matrix is the product of its entries on the diagonal.

2. Determine the transpose of each of the following matrices. In addition, if the matrix is square, compute its trace.

(a) $\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 8 & -6 \\ 3 & 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} -3 & 9 \\ 0 & -2 \\ 6 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

(e) $(1, \quad -1, \quad 3, \quad 5)$ (f) $\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ (h) $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathsf{M}_{m\times n}(F)$ and any $a, b \in F$.

4. Prove that $(A^t)^t = A$ for each $A \in \mathsf{M}_{m\times n}(F)$.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

6. Prove that $\operatorname{tr}(aA + bB) = a \operatorname{tr}(A) + b \operatorname{tr}(B)$ for any $A, B \in \mathsf{M}_{n\times n}(F)$.

7. Prove that diagonal matrices are symmetric matrices.

8. Verify that the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3.
(a) $\mathsf{W}_1 = \{(a_1, a_2, a_3) \in \mathsf{R}^3\colon \ a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(b) $\mathsf{W}_2 = \{(a_1, a_2, a_3) \in \mathsf{R}^3\colon \ 2a_1 - 7a_2 + a_3 = 0\}$
(c) $\mathsf{W}_3 = \{(a_1, a_2, a_3) \in \mathsf{R}^3\colon \ a_1 - 4a_2 - a_3 = 0\}$

9. Let W_1, W_2, and W_3 be as in Exercise 8. Describe $\mathsf{W}_1 \cap \mathsf{W}_2$, $\mathsf{W}_2 \cap \mathsf{W}_3$, and $\mathsf{W}_1 \cap \mathsf{W}_3$, and observe that each is a subspace of R^3.

10. Verify that $\mathsf{W}_1 = \{(a_1, \ldots, a_n) \in \mathsf{F}^n\colon \ a_1 + \cdots + a_n = 0\}$ is a subspace of F^n but that $\mathsf{W}_2 = \{(a_1, \ldots, a_n) \in \mathsf{F}^n\colon \ a_1 + \cdots + a_n = 1\}$ is not.

11. Is the set $\mathsf{W} = \{f \in \mathsf{P}(F)\colon \ f = 0 \text{ or } f \text{ has degree } n\}$ a subspace of $\mathsf{P}(F)$ if $n \geq 1$? Justify your answer.

12. An $m \times n$ matrix A is called *upper triangular* if all entries lying below the diagonal are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Verify that the upper triangular matrices form a subspace of $\mathsf{M}_{m\times n}(F)$.

13. Verify that for any $s_0 \in S$, $\mathsf{W} = \{f \in \mathcal{F}(S, F)\colon \ f(s_0) = 0\}$ is a subspace of $\mathcal{F}(S, F)$.

14. Is the set of all differentiable real-valued functions defined on R a subspace of $\mathsf{C}(R)$? Justify your answer.

15. Let $C^n(R)$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative (and hence continuous derivatives of orders $1, 2, \ldots, n$). Verify that $C^n(R)$ is a subspace of $\mathfrak{F}(R, R)$.

16. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \varnothing$ and $ax \in W$ and $x + y \in W$ whenever $a \in F$ and $x, y \in W$.

17. Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

18. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

19. Let F_1 and F_2 be fields. A function $g \in \mathfrak{F}(F_1, F_2)$ is called an *even function* if $g(-x) = g(x)$ for each $x \in F_1$ and is called an *odd function* if $g(-x) = -g(x)$ for each $x \in F_1$. Prove that the set of all even functions in $\mathfrak{F}(F_1, F_2)$ and the set of all odd functions in $\mathfrak{F}(F_1, F_2)$ are subspaces of $\mathfrak{F}(F_1, F_2)$.

20. Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, \ldots, a_n) \in F^n\colon \quad a_n = 0\}$$

and

$$W_2 = \{(a_1, \ldots, a_n) \in F^n\colon \quad a_1 = \cdots = a_{n-1} = 0\}.$$

21. Let W_1 denote the set of all polynomials f in $P(F)$ such that $f(x) = 0$ or, in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

the coefficients $a_0, a_2, a_4, \ldots$ of all even powers of x equal zero. Likewise, let W_2 denote the set of all polynomials g in $P(F)$ such that $g(x) = 0$ or, in the representation

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0,$$

the coefficients $b_1, b_3, b_5, \ldots$ of all odd powers of x equal zero. Prove that $P(F) = W_1 \oplus W_2$.

22. Let $W_1 = \{A \in M_{m\times n}(F)\colon \quad A_{ij} = 0 \quad \text{whenever} \quad i > j\}$ and $W_2 = \{A \in M_{m\times n}(F)\colon \quad A_{ij} = 0 \text{ whenever } i \leq j\}$. ($W_1$ is the set of upper triangular matrices defined in Exercise 12.) Show that $M_{m\times n}(F) = W_1 \oplus W_2$.

23. Let V denote the vector space consisting of all upper triangular $n \times n$ matrices (as defined in Exercise 12), and let W_1 denote the subspace of V consisting of all diagonal matrices. Show that $V = W_1 \oplus W_2$, where $W_2 = \{A \in V\colon \quad A_{ij} = 0 \text{ whenever } i > j\}$.

24.† Prove that if W is a subspace of V and $x_1, \ldots, x_n$ are elements of W, then $a_1 x_1 + \cdots + a_n x_n$ is an element of W for any scalars $a_1, \ldots, a_n$ in F.

†Exercises denoted by † will be referenced in other sections of the book.

25. A matrix M is called *skew-symmetric* if $M^t = -M$. Clearly a skew-symmetric matrix is square. Prove that the set of all skew-symmetric $n \times n$ matrices is a subspace W_1 of $\mathsf{M}_{n\times n}(R)$. Let W_2 be the subspace of $\mathsf{M}_{n\times n}(R)$ consisting of the symmetric $n \times n$ matrices. Prove that $\mathsf{M}_{n\times n}(R) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

26. Let $\mathsf{W}_1 = \{A \in \mathsf{M}_{n\times n}(F)\colon\ A_{ij} = 0 \text{ whenever } i \leq j\}$, and let W_2 denote the set of symmetric $n \times n$ matrices. Both W_1 and W_2 are subspaces of $\mathsf{M}_{n\times n}(F)$. Prove that $\mathsf{M}_{n\times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$. Compare Exercises 25 and 26.

27. Prove the corollary to Theorem 1.5.

28. Complete the proof of Theorem 1.6.

29. Let W be a subspace of a vector space V over a field F. For any $v \in \mathsf{V}$ the set $\{v\} + \mathsf{W} = \{v + w\colon\ w \in \mathsf{W}\}$ is called the *coset of* W *containing* v. It is customary to denote this coset by $v + \mathsf{W}$ rather than $\{v\} + \mathsf{W}$. Prove the following:
(a) $v + \mathsf{W}$ is a subspace of V if and only if $v \in \mathsf{W}$.
(b) $v_1 + \mathsf{W} = v_2 + \mathsf{W}$ if and only if $v_1 - v_2 \in \mathsf{W}$.
Addition and scalar multiplication by elements of F can be defined in the collection $S = \{v + \mathsf{W}\colon\ v \in \mathsf{V}\}$ of all cosets of W as follows:
$$(v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1 + v_2) + \mathsf{W}$$
for all $v_1, v_2 \in \mathsf{V}$ and
$$a(v + \mathsf{W}) = av + \mathsf{W}$$
for all $v \in \mathsf{V}$ and $a \in F$.
(c) Prove that the operations above are well-defined; i.e., show that if $v_1 + \mathsf{W} = v_1' + \mathsf{W}$ and $v_2 + \mathsf{W} = v_2' + \mathsf{W}$, then
$$(v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1' + \mathsf{W}) + (v_2' + \mathsf{W})$$
and
$$a(v_1 + \mathsf{W}) = a(v_1' + \mathsf{W})$$
for all $a \in F$.
(d) Prove that the set S is a vector space under the operations defined above. This vector space is called the *quotient space of* V *modulo* W and is denoted by V/W.

1.4 LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

In Section 1.1 it was shown that the equation of the plane through three non-collinear points P, Q, and R in space is $x = P + t_1u + t_2v$, where u and v denote the vectors beginning at P and ending at Q and R, respectively, and t_1 and t_2 denote arbitrary real numbers. An important special case occurs when P is the origin. In this case the equation of the plane simplifies to $x = t_1u + t_2v$, and the set of all points in this plane is a subspace of R^3. (This will be proved as Theorem 1.7 of this section.) Expressions of the form $t_1u + t_2v$, where t_1 and t_2 are scalars and u and v are vectors, play a central role in the theory of vector spaces. The appropriate generalization of such expressions is presented in the following definition.

Definition. *Let* V *be a vector space and* S *be a non-empty subset of* V. *A vector* x *in* V *is said to be a* linear combination of elements of S *if there exist a finite number of elements* $y_1, \ldots, y_n$ *in* S *and scalars* $a_1, \ldots, a_n$ *in* F *such that* $x = a_1y_1 + \cdots + a_ny_n$. *In this situation it is also customary to say that* x *is a* linear combination *of* $y_1, \ldots, y_n$.

Observe that in any vector space V, $0x = 0$ for each $x \in V$. Thus the zero vector is a linear combination of any non-empty subset of V.

Example 15. Table 1.1 shows the vitamin content of 100 grams of 12 foods with respect to vitamins A, B_1 (thiamine), B_2 (riboflavin), niacin, and C (ascorbic acid).

TABLE 1.1 Vitamin Content of 100 Grams of Certain Foods

	A (units)	B_1 (mg)	B_2 (mg)	Niacin (mg)	C (mg)
Apple butter	0	0.01	0.02	0.2	2
Raw, unpared apples (freshly harvested)	90	0.03	0.02	0.1	4
Chocolate-coated candy with coconut center	0	0.02	0.07	0.2	0
Clams (meat only)	100	0.10	0.18	1.3	10
Cupcake from mix (dry form)	0	0.05	0.06	0.3	0
Cooked farina (unenriched)	(0)†	0.01	0.01	0.1	(0)
Jams and preserves	10	0.01	0.03	0.2	2
Coconut custard pie (baked from mix)	0	0.02	0.02	0.4	0
Raw brown rice	(0)	0.34	0.05	4.7	(0)
Soy sauce	0	0.02	0.25	0.4	0
Cooked spaghetti (unenriched)	0	0.01	0.01	0.3	0
Raw wild rice	(0)	0.45	0.63	6.2	(0)

†Zeros in parentheses indicate that the amount of a vitamin present is either none or too small to measure.

SOURCE: *Composition of Foods* (Agriculture Handbook Number 8) by Bernice K. Watt and Annabel L. Merrill, Consumer and Food Economics Research Division, United States Department of Agriculture, 1963.

We shall record the vitamin content of 100 grams of each food as a column vector in R^5—for example, the vitamin vector for apple butter is

$$\begin{pmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{pmatrix}.$$

Considering the vitamin vectors for cupcake, coconut custard pie, brown rice, soy sauce, and wild rice, we see that

$$\begin{pmatrix} 0.00 \\ 0.05 \\ 0.06 \\ 0.30 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.02 \\ 0.02 \\ 0.40 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.34 \\ 0.05 \\ 4.70 \\ 0.00 \end{pmatrix} + 2\begin{pmatrix} 0.00 \\ 0.02 \\ 0.25 \\ 0.40 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.45 \\ 0.63 \\ 6.20 \\ 0.00 \end{pmatrix}.$$

Thus the vitamin vector for raw wild rice is a linear combination of the vitamin vectors for cupcake, coconut custard pie, raw brown rice, and soy sauce. So 100 grams of cupcake, 100 grams of coconut custard pie, 100 grams of raw brown rice, and 200 grams of soy sauce provide exactly the same amounts of the five vitamins as 100 grams of raw wild rice. Similarly, since

$$2\begin{pmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 90.00 \\ 0.03 \\ 0.02 \\ 0.10 \\ 4.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.02 \\ 0.07 \\ 0.20 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.01 \\ 0.01 \\ 0.10 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 10.00 \\ 0.01 \\ 0.03 \\ 0.20 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.01 \\ 0.01 \\ 0.30 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 100.00 \\ 0.10 \\ 0.18 \\ 1.30 \\ 10.00 \end{pmatrix},$$

200 grams of apple butter, 100 grams of apples, 100 grams of chocolate candy, 100 grams of farina, 100 grams of jam, and 100 grams of spaghetti provide exactly the same amounts of the five vitamins as 100 grams of clams.

Throughout Chapters 1 and 2 we shall encounter many different situations in which it will be necessary to determine if a vector can be expressed as a linear combination of other vectors, and if so, how. This question reduces to the problem of solving a system of linear equations. We shall illustrate this important technique by determining if the vector (8, 15, 15, 12) in R^4 can be written as a linear combination of $y_1 = (1, 2, 1, 2)$, $y_2 = (-2, -4, -2, -4)$, $y_3 = (1, 4, 2, 0)$, $y_4 = (2, 7, 5, 0)$, and $y_5 = (3, 7, 2, 6)$. Thus we must determine if there are scalars a_1, a_2, a_3, a_4, and a_5 such that

$$\begin{aligned}(8, 15, 15, 12) &= a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5\\ &= a_1(1, 2, 1, 2) + a_2(-2, -4, -2, -4) + a_3(1, 4, 2, 0)\\ &\quad + a_4(2, 7, 5, 0) + a_5(3, 7, 2, 6)\\ &= (a_1 - 2a_2 + a_3 + 2a_4 + 3a_5, 2a_1 - 4a_2 + 4a_3\\ &\quad + 7a_4 + 7a_5, a_1 - 2a_2 + 2a_3 + 5a_4 + 2a_5,\\ &\quad 2a_1 - 4a_2 + 6a_5).\end{aligned}$$

It is now easy to see that (8, 15, 15, 12) can be expressed as a linear combination of y_1, y_2, y_3, y_4, and y_5 if and only if there is a 5-tuple of scalars $(a_1, a_2, a_3, a_4, a_5)$ satisfying the system of linear equations

$$\begin{cases} a_1 - 2a_2 + a_3 + 2a_4 + 3a_5 = 8\\ 2a_1 - 4a_2 + 4a_3 + 7a_4 + 7a_5 = 15\\ a_1 - 2a_2 + 2a_3 + 5a_4 + 2a_5 = 15\\ 2a_1 - 4a_2 \qquad\qquad\quad + 6a_5 = 12 \end{cases} \tag{1}$$

obtained by equating corresponding coordinates of the equation above.

To solve the system in Eq. (1) we shall replace the system by another system which has the same solutions but which is easier to solve. The procedure that we shall use will express some of the unknowns in terms of the others by eliminating certain unknowns from all the equations but one. To begin, let us eliminate a_1 from the second, third, and fourth equations in Eq. (1). This elimination can be accomplished by adding -2 times the first equation to the second, -1 times the first equation to the third, and -2 times the first equation to the fourth; the result is the new system:

$$\begin{cases} a_1 - 2a_2 + a_3 + 2a_4 + 3a_5 = 8\\ \qquad\qquad 2a_3 + 3a_4 + a_5 = -1\\ \qquad\qquad a_3 + 3a_4 - a_5 = 7\\ \qquad\quad -2a_3 - 4a_4 \qquad = -4 \end{cases} \tag{2}$$

in which we have eliminated both a_1 and a_2 from each equation but the first. Continuing from the system in Eq. (2), we shall add multiples of the second equation to the others in order to eliminate a_3 from each equation but the second. In this case we shall have to add $-\frac{1}{2}$ times the second equation to the first in order to eliminate a_3 from the first equation. Notice, however, that if equations two and three are interchanged, the necessary computations are simplified. Thus we shall interchange the second and third equations of Eq. (2) to obtain

$$\begin{cases} a_1 - 2a_2 + a_3 + 2a_4 + 3a_5 = 8\\ \qquad\qquad a_3 + 3a_4 - a_5 = 7\\ \qquad\qquad 2a_3 + 3a_4 + a_5 = -1\\ \qquad\quad -2a_3 - 4a_4 \qquad = -4. \end{cases} \tag{3}$$

Now adding -1 times the second equation to the first, -2 times the second equation to the third, and 2 times the second equation to the fourth transforms Eq. (3) into

$$\begin{cases} a_1 - 2a_2 \qquad - a_4 + 4a_5 = 1 \\ \qquad a_3 + 3a_4 - a_5 = 7 \\ \qquad -3a_4 + 3a_5 = -15 \\ \qquad 2a_4 - 2a_5 = 10. \end{cases} \tag{4}$$

We must next add multiples of the third equation to the others in order to eliminate a_4 from each equation of Eq. (4) except the third. Once again the necessary computations will be made easier if we perform a preliminary operation—multiplying the third equation by $-\frac{1}{3}$. This yields

$$\begin{cases} a_1 - 2a_2 \qquad - a_4 + 4a_5 = 1 \\ \qquad a_3 + 3a_4 - a_5 = 7 \\ \qquad a_4 - a_5 = 5 \\ \qquad 2a_4 - 2a_5 = 10. \end{cases} \tag{5}$$

Finally, in (5), let us add 1 times the third equation to the first, -3 times the third equation to the second, and -2 times the third equation to the fourth to obtain

$$\begin{cases} a_1 - 2a_2 \qquad + 3a_5 = 6 \\ \qquad a_3 \quad + 2a_5 = -8 \\ \qquad a_4 - a_5 = 5 \\ \qquad 0 = 0. \end{cases} \tag{6}$$

The system in Eq. (6) is a system of the desired form: It is easy to solve for a_1, a_3, and a_4 (those unknowns that appear as the first unknowns present in one of the equations) in terms of the other unknowns (a_2 and a_5). Rewriting Eq. (6) in this way, we find

$$\begin{aligned} a_1 &= 2a_2 - 3a_5 + 6 \\ a_3 &= \qquad -2a_5 - 8 \\ a_4 &= \qquad a_5 + 5 \end{aligned}$$

Then for any choice of the scalars a_2 and a_5 a vector of the form

$$\begin{aligned} &(a_1, a_2, a_3, a_4, a_5) \\ &\quad = (2a_2 - 3a_5 + 6, a_2, -2a_5 - 8, a_5 + 5, a_5) \\ &\quad = a_2(2, 1, 0, 0, 0) + a_5(-3, 0, -2, 1, 1) + (6, 0, -8, 5, 0) \end{aligned}$$

will be a solution to the original system in Eq. (1). In particular, the vector $(6, 0, -8, 5, 0)$ obtained by setting $a_2 = 0$ and $a_5 = 0$ is a solution to Eq.

(1). Thus

$$(8, 15, 15, 12) = 6y_1 + 0y_2 - 8y_3 + 5y_4 + 0y_5,$$

so that (8, 15, 15, 12) is a linear combination of y_1, y_2, y_3, y_4, and y_5.

The procedure illustrated above can be used to solve any system of linear equations. Observe that three types of operations were used to simplify the original system:

1. Interchanging the order of any two equations in the system.
2. Multiplying any equation by a *non-zero* constant.
3. Adding any constant multiple of an equation to another equation.

These operations were employed until a system of equations with the following properties was obtained:

1. The first non-zero coefficient in each equation is one.
2. If an unknown is the first unknown with a non-zero coefficient in some equation, then that unknown occurs with a zero coefficient in each of the other equations.
3. The first unknown with a non-zero coefficient in any equation has a larger subscript than the first unknown with a non-zero coefficient in any preceding equation.

To help clarify the meaning of these properties, note that none of the following systems meet these requirements.

$$\begin{cases} x_1 + 3x_2 \phantom{{}+2x_3} + x_4 = 7 \\ \phantom{x_1 + 3x_2 + {}} 2x_3 - 5x_4 = -1 \end{cases} \tag{7}$$

$$\begin{cases} x_1 - 2x_2 + 3x_3 \phantom{{}+x_4} + x_5 = -5 \\ \phantom{x_1 - 2x_2 + {}} x_3 \phantom{{}+x_4} - 2x_5 = 9 \\ \phantom{x_1 - 2x_2 + 3x_3 + {}} x_4 + 3x_5 = 6 \end{cases} \tag{8}$$

$$\begin{cases} x_1 \phantom{{}+x_2} - 2x_3 \phantom{{}+x_4} + x_5 = 1 \\ \phantom{x_1 - 2x_3 + {}} x_4 - 6x_5 = 0 \\ \phantom{x_1 + {}} x_2 + 5x_3 \phantom{{}+x_4} - 3x_5 = 2 \end{cases} \tag{9}$$

Specifically, the system in Eq. (7) does not satisfy condition 1 because the first non-zero coefficient in the second equation is 2; the system in Eq. (8) does not satisfy condition 2 because x_3, the first unknown with a non-zero coefficient in the second equation, occurs with a non-zero coefficient in the first equation; and the system in Eq. (9) does not satisfy condition 3 because x_2, the first unknown with a non-zero coefficient in the third equation, does not have a larger subscript than x_4, the first unknown with a non-zero coefficient in the second equation.

Once a system with properties 1, 2, and 3 has been obtained, it is easy to solve for some of the unknowns in terms of the others (as in the example above). *If, however, in the course of using operations* 1, 2, *and* 3 *a system containing an equation of the form* $0 = c$, *where* c *is non-zero, is obtained, then the original system will have no solutions.* (See Example 16 below.)

We shall return to the study of systems of linear equations in Chapter 3. At that time we shall discuss the theoretical basis for this method of solving systems of linear equations and further simplify the method by use of matrices.

Example 16. We shall show that

$$2x^3 - 2x^2 + 12x - 6$$

is a linear combination of

$$x^3 - 2x^2 - 5x - 3 \quad \text{and} \quad 3x^3 - 5x^2 - 4x - 9$$

in $\mathsf{P}_3(R)$ but that

$$3x^3 - 2x^2 + 7x + 8$$

is not such a linear combination. In the first case we wish to find scalars a and b such that

$$\begin{aligned} 2x^3 - 2x^2 + 12x - 6 &= a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9) \\ &= (a + 3b)x^3 + (-2a - 5b)x^2 + (-5a - 4b)x + (-3a - 9b). \end{aligned}$$

Thus we are led to the following system of linear equations:

$$\begin{cases} a + 3b = 2 \\ -2a - 5b = -2 \\ -5a - 4b = 12 \\ -3a - 9b = -6. \end{cases}$$

Adding appropriate multiples of the first equation to the others in order to eliminate a, we find

$$\begin{cases} a + 3b = 2 \\ b = 2 \\ 11b = 22 \\ 0 = 0. \end{cases}$$

Now adding the appropriate multiples of the second equation to the others yields

$$\begin{cases} a = -4 \\ b = 2 \\ 0 = 0 \\ 0 = 0. \end{cases}$$

Hence

$$2x^3 - 2x^2 + 12x - 6$$
$$= -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9).$$

In the second case we wish to show that there are no scalars a and b for which

$$3x^3 - 2x^2 + 7x + 8 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9).$$

As above, we obtain a system of linear equations

$$\begin{cases} a + 3b = 3 \\ -2a - 5b = -2 \\ -5a - 4b = 7 \\ -3a - 9b = 8. \end{cases} \tag{10}$$

Eliminating a as before yields

$$\begin{cases} a + 3b = 3 \\ b = 4 \\ 11b = 22 \\ 0 = 17. \end{cases}$$

But the presence of the inconsistent equation $0 = 17$ indicates that the system in Eq. (10) has no solutions. Hence $3x^3 - 2x^2 + 7x + 8$ is not a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$.

The set of linear combinations of the elements of a non-empty subset of a vector space provides another example of a subspace, as the following result shows.

Theorem 1.7. *If* S *is a non-empty subset of a vector space* V, *then the set* W *consisting of all linear combinations of elements of* S *is a subspace of* V. *Moreover,* W *is the smallest subspace of* V *containing* S *in the sense that* W *is a subset of any subspace of* V *that contains* S.

PROOF. First, we shall use Theorem 1.3 to prove that W is a subspace of V. Since $S \neq \varnothing$, $0 \in$ W. If y and z are elements of W, then y and z are linear combinations of elements of S. So there exist elements $x_1, \ldots, x_n$ and $w_1, \ldots, w_m$ in S such that $y = a_1x_1 + \cdots + a_nx_n$ and $z = b_1w_1 + \cdots + b_mw_m$ for some choice of scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$. Now

$$y + z = a_1x_1 + \cdots + a_nx_n + b_1w_1 + \cdots + b_mw_m$$

and

$$cy = ca_1x_1 + \cdots + ca_nx_n$$

are linear combinations of elements of S; so $y + z$ and cy are elements of W for any scalar c. Thus W is a subspace of V.

Now let W′ denote any subspace of V that contains S. If y is an element of W, then y is a linear combination of elements of S—say $y = a_1x_1 + \cdots + a_nx_n$, where $a_1, \ldots, a_n \in F$ and $x_1, \ldots, x_n \in S$. Because $S \subseteq W'$, $x_1, \ldots, x_n \in W'$. Therefore $y = a_1x_1 + \cdots + a_nx_n$ is an element of W′ by Exercise 24 of Section 1.3. Since y, an arbitrary element of W, belongs to W′, W $\subseteq$ W′. This completes the proof. ∎

Definition. *The subspace* W *described in Theorem 1.7 is called the* span of S (*or the* subspace generated by the elements of S) *and is denoted* span(S). *For convenience we shall define* span($\varnothing$) = {0}.

Observe that Theorem 1.7 shows that x is a linear combination of elements of S if and only if x is an element of span(S). Thus, for instance, in R^3 span({(1, 0, 0), (0, 1, 0)}) is the xy-plane.

Definition. *A subset* S *of a vector space* V generates (*or* spans) V *if span*(S) = V. *In this situation we shall also say that the* elements of S generate (*or* span) V.

Example 17. The vectors (1, 1, 0), (1, 0, 1), and (0, 1, 1) generate R^3 since an arbitrary element (a_1, a_2, a_3) of R^3 is a linear combination of the three given vectors; in fact, the scalars r, s, and t for which

$$r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1) = (a_1, a_2, a_3)$$

are

$$r = \tfrac{1}{2}(a_1 + a_2 - a_3),\ s = \tfrac{1}{2}(a_1 - a_2 + a_3),\ \text{and}\ t = \tfrac{1}{2}(-a_1 + a_2 + a_3).$$

Example 18. The polynomials $x^2 + 3x - 2$, $2x^2 + 5x - 3$, and $-x^2 - 4x + 4$ generate $P_2(R)$ since each of the three given polynomials belongs to $P_2(R)$ and each polynomial $ax^2 + bx + c$ in $P_2(R)$ is a linear combination of these three; namely,

$$(-8a + 5b + 3c)(x^2 + 3x - 2) + (4a - 2b - c)(2x^2 + 5x - 3) + (-a + b + c)(-x^2 - 4x + 4) = ax^2 + bx + c.$$

Example 19. The matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

generate $M_{2\times 2}(R)$ since an arbitrary element of $M_{2\times 2}(R)$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

can be expressed as a linear combination of the four given matrices as follows:

$$\begin{aligned}
&\left(\tfrac{1}{3}a_{11}+\tfrac{1}{3}a_{12}+\tfrac{1}{3}a_{21}-\tfrac{2}{3}a_{22}\right)\begin{pmatrix}1&1\\1&0\end{pmatrix}\\
&+\left(\tfrac{1}{3}a_{11}+\tfrac{1}{3}a_{12}-\tfrac{2}{3}a_{21}+\tfrac{1}{3}a_{22}\right)\begin{pmatrix}1&1\\0&1\end{pmatrix}\\
&+\left(\tfrac{1}{3}a_{11}-\tfrac{2}{3}a_{12}+\tfrac{1}{3}a_{21}+\tfrac{1}{3}a_{22}\right)\begin{pmatrix}1&0\\1&1\end{pmatrix}\\
&+\left(-\tfrac{2}{3}a_{11}+\tfrac{1}{3}a_{12}+\tfrac{1}{3}a_{21}+\tfrac{1}{3}a_{22}\right)\begin{pmatrix}0&1\\1&1\end{pmatrix}\\
&=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}.
\end{aligned}$$

EXERCISES

1. Label the following statements as being true or false.
   (a) The zero vector is a linear combination of any non-empty set of vectors.
   (b) The span of $\varnothing$ is $\varnothing$.
   (c) If S is a subset of a vector space V, then span(S) equals the intersection of all subspaces of V that contain S.
   (d) In solving a system of linear equations it is permissible to multiply an equation by any constant.
   (e) In solving a system of linear equations it is permissible to add a multiple of one equation to another.
   (f) Every system of linear equations has a solution.
2. Solve the following systems of linear equations by the method introduced in this section.

   (a) $$\begin{cases}2x_1-2x_2-3x_3 &= -2\\ 3x_1-3x_2-2x_3+5x_4 = 7\\ x_1-x_2-2x_3-x_4 = -3\end{cases}$$

   (b) $$\begin{cases}3x_1-7x_2+4x_3=10\\ x_1-2x_2+x_3=3\\ 2x_1-x_2-2x_3=6\end{cases}$$

   (c) $$\begin{cases}x_1+2x_2-x_3+x_4=5\\ x_1+4x_2-3x_3-3x_4=6\\ 2x_1+3x_2-x_3+4x_4=8\end{cases}$$

(d) $\begin{cases} x_1 + 2x_2 + 2x_3 = 2 \\ x_1 + 8x_3 + 5x_4 = -6 \\ x_1 + x_2 + 5x_3 + 5x_4 = 3 \end{cases}$

(e) $\begin{cases} x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7 \\ -x_1 + 10x_3 - 3x_4 - 4x_5 = -16 \\ 2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 = 2 \\ 4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 = 7 \end{cases}$

(f) $\begin{cases} x_1 + 2x_2 + 6x_3 = -1 \\ 2x_1 + x_2 + x_3 = 8 \\ 3x_1 + x_2 - x_3 = 15 \\ x_1 + 3x_2 + 10x_3 = -5 \end{cases}$

3. For each of the following lists of vectors in R^3, determine whether or not the first vector can be expressed as a linear combination of the other two.
(a) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$
(b) $(1, 2, -3), (-3, 2, 1), (2, -1, -1)$
(c) $(3, 4, 1), (1, -2, 1), (-2, -1, 1)$
(d) $(2, -1, 0), (1, 2, -3), (1, -3, 2)$
(e) $(5, 1, -5), (1, -2, -3), (-2, 3, -4)$
(f) $(-2, 2, 2), (1, 2, -1), (-3, -3, 3)$

4. For each of the following lists of polynomials in $\mathsf{P}_3(R)$, determine whether or not the first polynomial can be expressed as a linear combination of the other two.
(a) $x^3 - 3x + 5,\ x^3 + 2x^2 - x + 1,\ x^3 + 3x^2 - 1$
(b) $4x^3 + 2x^2 - 6,\ x^3 - 2x^2 + 4x + 1,\ 3x^3 - 6x^2 + x + 4$
(c) $-2x^3 - 11x^2 + 3x + 2,\ x^3 - 2x^2 + 3x - 1,\ 2x^3 + x^2 + 3x - 2$
(d) $x^3 + x^2 + 2x + 13,\ 2x^3 - 3x^2 + 4x + 1,\ x^3 - x^2 + 2x + 3$
(e) $x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3$
(f) $6x^3 - 3x^2 + x + 2,\ x^3 - x^2 + 2x + 3,\ 2x^3 + x^2 - 3x + 1$

5. In F^n let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ generates F^n.

6. Show that $\mathsf{P}_n(F)$ is generated by $\{1, x, x^2, \ldots, x^n\}$.

7. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate $\mathsf{M}_{2\times 2}(F)$.

8. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

9.† For any element x in a vector space, prove that $\text{span}(\{x\}) = \{ax: \ a \in F\}$. Interpret this result geometrically in R^3.

10. Show that a subset W of a vector space V is a subspace of V if and only if span(W) = W.

11.† Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = \mathsf{V}$, deduce that $\text{span}(S_2) = \mathsf{V}$.

12.† Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

13. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and an example in which they are unequal.

1.5 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

At the beginning of Section 1.4 we remarked that the equation of a plane through three non-collinear points in space, one of which is the origin, is of the form $x = t_1u + t_2v$, where $u, v \in \mathsf{R}^3$ and t_1 and t_2 are scalars. Thus a vector x in R^3 is a linear combination of $u, v \in \mathsf{R}^3$ if and only if x lies in the plane containing u and v. (See Fig. 1.5.) We see, therefore, that in R^3 the span of two non-parallel vectors has a simple geometric interpretation. A similar interpretation can be given for the span of a single non-zero vector in R^3. (See Exercise 9 of Section 1.4.)

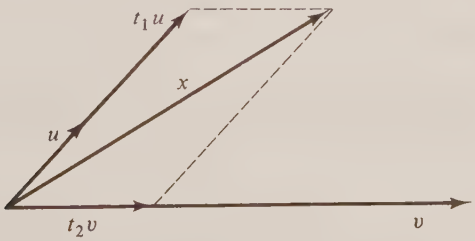


figure 1.5

In the equation $x = t_1u + t_2v$, x depends on u and v in the sense that x is a linear combination of u and v. A set in which at least one vector is a linear combination of the others is called a linearly dependent set. Consider, for example, the set $S = \{x_1, x_2, x_3, x_4\} \subseteq \mathsf{R}^3$, where $x_1 = (2, -1, 4)$, $x_2 = (1, -1, 3)$, $x_3 = (1, 1, -1)$, and $x_4 = (1, -2, -1)$. To see if S is linearly dependent, we must check whether or not there is a vector in S that is a linear combination of the others. Now the vector x_4 is a linear combination of x_1, x_2, and x_3 if and only if there are scalars a, b, and c such that

$$x_4 = ax_1 + bx_2 + cx_3,$$

i.e., if and only if

$$x_4 = (2a + b + c, -a - b + c, 4a + 3b - c).$$

Thus x_4 is a linear combination of x_1, x_2, and x_3 if and only if the system

$$\begin{cases} 2a + b + c = 1 \\ -a - b + c = -2 \\ 4a + 3b - c = -1 \end{cases}$$

has a solution. The reader should verify that no such solution exists. Notice, however, that this does not show that the set S is not linearly dependent, for we must now check whether or not x_1, x_2, or x_3 can be written as a linear combination of the other vectors in S. It can be shown, in fact, that x_3 is a linear combination of x_1, x_2, and x_4; namely, $x_3 = 2x_1 - 3x_2 + 0x_4$. So S is indeed linearly dependent.

We see from this example that the condition for linear dependence that we have given is inconvenient to use because not every vector in S need be a linear combination of the others even though S is linearly dependent. By reformulating the definition in the following way, we obtain a definition of dependence that is easier to use.

Definition. *A subset* S *of a vector space* V *is said to be* linearly dependent *if there exists a finite number of distinct vectors* $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ *in* S *and scalars* $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n$ *in* F, *not all zero, such that* $\mathsf{a}_1\mathsf{x}_1 + \mathsf{a}_2\mathsf{x}_2 + \cdots + \mathsf{a}_n\mathsf{x}_n = 0$. *We shall also describe this situation by saying that the elements of* S *are linearly dependent.*

To show that the subset S of R^3 defined above is linearly dependent using this definition, we must find scalars a_1, a_2, a_3, and a_4, not all zero, such that

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

i.e., such that

$$(2a_1 + a_2 + a_3 + a_4, -a_1 - a_2 + a_3 - 2a_4, 4a_1 + 3a_2 - a_3 - a_4) = (0, 0, 0).$$

Thus we must find a solution to the system

$$\begin{cases} 2a_1 + \ a_2 + a_3 + \ a_4 = 0 \\ -a_1 - \ a_2 + a_3 - 2a_4 = 0 \\ 4a_1 + 3a_2 - a_3 - \ a_4 = 0 \end{cases}$$

in which not all the unknowns are zero. Since we know that $x_3 = 2x_1 - 3x_2 + 0x_4$, it follows that $0 = 2x_1 - 3x_2 - x_3 + 0x_4$. Hence $a_1 = 2$, $a_2 = -3$, $a_3 = -1$, and $a_4 = 0$ is such a solution.

Therefore we see that the stated definition of linear dependence requires solving only one system of equations instead of two or more. The reader should verify that the two conditions for linear dependence that we have discussed are, in fact, equivalent. (See Exercise 10.)

It is easily seen that in any vector space a subset S that contains the zero vector must be linearly dependent. For since $1 \cdot 0 = 0$, the zero vector is a linear combination of elements of S in which some coefficient is non-zero.

Example 20. In R^4 the set $S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4)\}$ is linearly dependent because

$$4(1, 3, -4, 2) - 3(2, 2, -4, 0) + 2(1, -3, 2, -4) = (0, 0, 0, 0).$$

Likewise in $\mathsf{M}_{2\times 3}(R)$ the set

$$\left\{\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix}\right\}$$

is linearly dependent since

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Definition. A subset S *of a vector space that is not linearly dependent is said to be* linearly independent. *As before we shall often describe this situation by saying that the elements of* S *are linearly independent.*

Note that the empty set is linearly independent, for linearly dependent sets clearly must be non-empty. Furthermore, in any vector space a set consisting of a single non-zero vector is linearly independent. For if $\{x\}$ is linearly dependent, then $ax = 0$ for some non-zero scalar a. But then

$$x = a^{-1}(ax) = a^{-1}0 = 0.$$

Moreover, a set S is linearly independent if and only if the only linear combinations of elements of S that equal 0 are the trivial linear combinations in which all the scalars equal zero. This fact provides a very useful method for determining if a finite set is linearly independent. This technique is illustrated in the following example.

Example 21. Let x_k denote the vector in F^n whose first $k - 1$ coordinates are zero and whose last $n - k + 1$ coordinates are 1. Then $\{x_1, x_2, \ldots, x_n\}$ is linearly independent, for if $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, equating the corresponding coordinates of the left and right sides of this equality gives the following system of equations:

$$\begin{cases} a_1 & = 0 \\ a_1 + a_2 & = 0 \\ a_1 + a_2 + a_3 & = 0 \\ \quad\vdots & \\ a_1 + a_2 + a_3 + \cdots + a_n = 0. & \end{cases}$$

Clearly the only solution of this system is $a_1 = \cdots = a_n = 0$.

The following useful results are immediate consequences of the definitions of linear dependence and linear independence.

Theorem 1.8. Let V *be a vector space, and let* $S_1 \subseteq S_2 \subseteq \mathsf{V}$. *If* S_1 *is linearly dependent, then* S_2 *is linearly dependent.*

PROOF. Exercise.

Corollary. Let V *be a vector space, and let* $S_1 \subseteq S_2 \subseteq \mathsf{V}$. *If* S_2 *is linearly independent, then* S_1 *is linearly independent.*

PROOF. Exercise.

EXERCISES

1. Label the following statements as being true or false.
   (a) If S is a linearly dependent set, then each element of S is a linear combination of other elements of S.
   (b) Any set containing the zero vector is linearly dependent.
   (c) The empty set is linearly dependent.
   (d) Subsets of linearly dependent sets are linearly dependent.
   (e) Subsets of linearly independent sets are linearly independent.
   (f) If $x_1, x_2, \ldots, x_n$ are linearly independent and $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, then all the scalars a_i equal zero.
2. In F^n let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.
3. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathsf{P}_n(F)$.

4. Prove that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are linearly independent in $\mathsf{M}_{2\times 2}(F)$.

5. Find a set of linearly independent diagonal matrices that generate the vector space of 2×2 diagonal matrices.

6.† Show that $\{x, y\}$ is linearly dependent if and only if x or y is a multiple of the other.

7. Give an example of three linearly dependent vectors in R^2 such that none of the three is a multiple of another.

8. Prove Theorem 1.8 and its corollary.

9. (a) Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
   (b) Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

10. Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $y, x_1, x_2, \ldots, x_n$ in S such that y is a linear combination of $x_1, x_2, \ldots, x_n$.

11. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $x_1 = 0$ or $x_{k+1} \in \mathrm{span}(\{x_1, x_2, \ldots, x_k\})$ for some $k < n$.

12. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

13. Let M be a square upper triangular matrix (as defined in Exercise 12 of Section 1.3) having non-zero diagonal entries. Prove that the columns of M are linearly independent.

14. Let f and g be functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(R, R)$. *Hint:* Suppose that $ae^{rt} + be^{st} = 0$. Let $t = 0$ and obtain an equation involving a and b. Then differentiate $ae^{rt} + be^{st} = 0$, and let $t = 0$ to obtain a second equation involving a and b. Solve these equations for a and b.

1.6 BASES AND DIMENSION

A subset S of a vector space V that is linearly independent and generates V possesses a very useful property—every element of V can be expressed in one and only one way as a linear combination of elements of S. (This property will be proved in Theorem 1.9.) It is this result that makes linearly independent generating sets the building blocks of vector spaces.

Definition. *A basis β for a vector space V is a linearly independent subset of V that generates V. (If β is a basis for V, we shall often say that the elements of β form a basis for V.)*

Example 22. Recalling that span$(\varnothing) = \{0\}$, we see that $\varnothing$ is a basis for the vector space $\{0\}$.

Example 23. In F^n, let $e_1 = (1, 0, 0, \ldots, 0, 0)$, $e_2 = (0, 1, 0, \ldots, 0, 0)$, $\ldots, e_n = (0, 0, 0, \ldots, 0, 1,)$; $\{e_1, e_2, \ldots, e_n\}$ is readily seen to be a basis for F^n and is called the *standard basis for* F^n.

Example 24. In $\mathsf{M}_{m\times n}(F)$, let M^{ij} denote the matrix whose only non-zero entry is a 1 in the ith row and jth column. Then $\{M^{ij}\colon\ 1 \le i \le m, 1 \le j \le n\}$ is a basis for $\mathsf{M}_{m\times n}(F)$.

Example 25. In $\mathsf{P}_n(F)$ the set $\{1, x, x^2, \ldots, x^n\}$ is a basis.

Example 26. In $\mathsf{P}(F)$ the set $\{1, x, x^2, \ldots\}$ is a basis.

Observe that Example 26 shows that a basis need not be finite. In fact, we shall see later in this section that no basis for $\mathsf{P}(F)$ can be finite. Hence not every vector space has a finite basis.

The following theorem, which will be used frequently in Chapter 2, shows the most significant property of a basis.

Theorem 1.9. *Let V be a vector space and $\beta = \{x_1, \ldots, x_n\}$ be a subset of V. Then β is a basis for V if and only if each vector y in V can be uniquely expressed as a linear combination of vectors in β, i.e., can be expressed in the form*

$$y = a_1x_1 + \cdots + a_nx_n$$

for unique scalars $a_1, \ldots, a_n$.

PROOF. Let β be a basis for V. If $y \in$ V, then $y \in$ span(β) since span(β) = V. Thus y is a linear combination of the elements of β. Suppose that $y = a_1x_1 + \cdots + a_nx_n$ and $y = b_1x_1 + \cdots + b_nx_n$ are two such representations of y. Subtracting the second equality from the first gives

$$0 = (a_1 - b_1)x_1 + \cdots + (a_n - b_n)x_n.$$

Since β is linearly independent, it follows that $a_1 - b_1 = \cdots = a_n - b_n = 0$. Thus $a_1 = b_1, \ldots, a_n = b_n$, so that y is uniquely expressible as a linear combination of the elements of β.

The proof of the converse is an exercise. ∎

Theorem 1.9 shows that each vector v in a vector space V with basis $\beta = \{x_1, \ldots, x_n\}$ can be uniquely expressed in the form $v = a_1x_1 + \cdots$

$+ a_n x_n$ for appropriately chosen scalars $a_1, \ldots, a_n$. Thus v determines a unique n-tuple of scalars $(a_1, \ldots, a_n)$, and, conversely, each n-tuple of scalars determines a unique vector v by using the entries of the n-tuple as the coefficients of a linear combination of the vectors in β. This fact suggests that V is like the vector space F^n, where n is the number of vectors in a basis for V. We shall see in Section 2.4 that this is indeed the case.

Our next theorem will identify a large class of vector spaces having finite bases. First, however, we must prove a preliminary result.

Lemma. *Let* S *be a linearly independent subset of a vector space* V, *and let* x *be an element of* V *that is not in* S. *Then* $\mathrm{S} \cup \{\mathrm{x}\}$ *is linearly dependent if and only if* $\mathrm{x} \in$ *span*(S).

PROOF. If $S \cup \{x\}$ is linearly dependent, then there exist vectors $x_1, \ldots, x_n$ in $S \cup \{x\}$ and non-zero scalars $a_1, \ldots, a_n$ such that $a_1x_1 + \cdots + a_nx_n = 0$. Because S is linearly independent, one of the x_i, say x_1, equals x. Thus $a_1x + a_2x_2 + \cdots + a_nx_n = 0$, and so $x = a_1^{-1}(-a_2x_2 - \cdots - a_nx_n)$. Since x is a linear combination of $x_2, \ldots, x_n$, which are elements of S, $x \in \operatorname{span}(S)$.

Conversely, suppose that $x \in \operatorname{span}(S)$. Then there exist vectors $x_1, \ldots, x_n$ in S and scalars $a_1, \ldots, a_n$ such that $x = a_1x_1 + \cdots + a_nx_n$. So $0 = a_1x_1 + \cdots + a_nx_n + (-1)x$, and since $x \neq x_i$ for $i = 1, \ldots, n$, $\{x_1, \ldots, x_n, x\}$ is linearly dependent. Thus $S \cup \{x\}$ is linearly dependent by Theorem 1.8. ∎

Theorem 1.10. *If a vector space* V *is generated by a finite set* S_0, *then a subset of* S_0 *is a basis for* V. *Hence* V *has a finite basis.*

PROOF. If $S_0 = \varnothing$ or $S_0 = \{0\}$, then $\mathsf{V} = \{0\}$ and $\varnothing$ is a subset of S_0 that is a basis for V. Otherwise S_0 contains a non-zero element x_1. Recall that $\{x_1\}$ is a linearly independent set. Continue, if possible, choosing elements $x_2, \ldots, x_r$ in S_0 so that $\{x_1, x_2, \ldots, x_r\}$ is linearly independent. Since S_0 is a finite set, we shall eventually reach a stage at which $S = \{x_1, \ldots, x_r\}$ is a linearly independent subset of S_0 but adjoining to S any element of S_0 not in S produces a linearly dependent set. We shall show that S is a basis for V. Because S is linearly independent, it suffices to prove that $\operatorname{span}(S) = \mathsf{V}$. Since $\operatorname{span}(S_0) = \mathsf{V}$, it suffices by Theorem 1.7 to show that $S_0 \subseteq \operatorname{span}(S)$. Let $x \in S_0$. If $x \in S$, then clearly $x \in \operatorname{span}(S)$. Otherwise, if $x \notin S$, then the construction above shows that $S \cup \{x\}$ is linearly dependent. So $x \in \operatorname{span}(S)$ by the lemma. Thus $S_0 \subseteq \operatorname{span}(S)$. ∎

The method by which the basis S was obtained in the proof above is a useful way of obtaining bases. An example of this procedure is given on page 40.

Example 27. The elements $(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1)$, and $(7, 2, 0)$ generate R^3. We shall select a basis for R^3 from among these elements. To start, select any non-zero element from the generating set, say $(2, -3, 5)$, as one of the elements of the basis. Since $4(2, -3, 5) = (8, -12, 20)$, the set $\{(2, -3, 5), (8, -12, 20)\}$ is linearly dependent (Exercise 6, Section 1.5). Hence we do not include $(8, -12, 20)$ in our basis. Since $(1, 0, -2)$ is not a multiple of $(2, -3, 5)$ and vice versa, the set $\{(2, -3, 5), (1, 0, -2)\}$ is linearly independent. Hence we include $(1, 0, -2)$ in our basis. Proceeding to the next element in the generating set, we shall exclude from or include into our basis the element $(0, 2, -1)$ according to whether the set $\{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}$ is linearly dependent or linearly independent. An easy calculation shows that the set is linearly independent; so we include $(0, 2, -1)$ in our basis. The final element of the generating set $(7, 2, 0)$ will be excluded from or included into our basis according to whether $\{(2, -3, 5), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$ is linearly dependent or linearly independent. Since

$$2(2, -3, 5) + 3(1, 0, -2) + 4(0, 2, -1) - (7, 2, 0) = (0, 0, 0),$$

the set is linearly dependent and we exclude $(7, 2, 0)$ from the basis. So the set $\{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}$ is a basis for R^3.

The following theorem and its corollaries are perhaps the most significant results in Chapter 1.

Theorem 1.11. Let V *be a vector space having a basis* β *containing exactly* n *elements. Let* $S = \{y_1, \ldots, y_m\}$ *be a linearly independent subset of* V *containing exactly* m *elements, where* $m \leq n$*. Then there exists a subset* S_1 *of* β *containing exactly* $n - m$ *elements such that* $S \cup S_1$ *generates* V.

PROOF. The proof will be by induction on m. We shall begin the induction with $m = 0$, for in this case $S = \varnothing$, and so $S_1 = \beta$ clearly satisfies the conclusion of the theorem.

Now assume that the theorem is true for some m, where $m < n$. We shall prove that the theorem is true for $m + 1$. Let $S = \{y_1, \ldots, y_m, y_{m+1}\}$ be a linearly independent subset of V containing exactly $m + 1$ elements. Since $\{y_1, \ldots, y_m\}$ is linearly independent by the corollary to Theorem 1.8, we may apply the inductive hypothesis to conclude that there exists a subset $\{x_1, \ldots, x_{n-m}\}$ of β such that $\{y_1, \ldots, y_m\} \cup \{x_1, \ldots, x_{n-m}\}$ generates V. Thus there exist scalars $a_1, \ldots, a_m, b_1, b_2, \ldots, b_{n-m}$ such that

$$y_{m+1} = a_1 y_1 + \cdots + a_m y_m + b_1 x_1 + b_2 x_2 + \cdots + b_{n-m} x_{n-m}. \quad (11)$$

Observe that some b_i, say b_1, is non-zero, for otherwise Eq. (11) would imply that y_{m+1} is a linear combination of $y_1, \ldots, y_m$ in contradiction to the assumption that $\{y_1, \ldots, y_m, y_{m+1}\}$ is linearly independent. Solving

Eq. (11) for x_1 gives

$$x_1 = (-b_1^{-1}a_1)y_1 + \cdots + (-b_1^{-1}a_m)y_m - (-b_1^{-1})y_{m+1} + (-b_1^{-1}b_2)x_2 + \cdots + (-b_1^{-1}b_{n-m})x_{n-m}. \tag{12}$$

Hence $x_1 \in \text{span}(\{y_1, \ldots, y_m, y_{m+1}, x_2, \ldots, x_{n-m}\})$ by Eq. (12). But since $y_1, \ldots, y_m, x_2, \ldots, x_{n-m}$ are clearly elements of $\text{span}(\{y_1, \ldots, y_m, y_{m+1}, x_2, \ldots, x_{n-m}\})$, it follows that

$$\{y_1, \ldots, y_m, x_1, x_2, \ldots, x_{n-m}\} \subseteq \text{span}(\{y_1, \ldots, y_m, y_{m+1}, x_2, \ldots, x_{n-m}\}).$$

Thus Theorem 1.7 implies that

$$\text{span}(\{y_1, \ldots, y_m, y_{m+1}, x_2, \ldots, x_{n-m}\}) = \mathsf{V}.$$

So the choice of $S_1 = \{x_2, \ldots, x_{n-m}\}$ proves that the theorem is true for $m + 1$.

This completes the proof. ∎

To illustrate Theorem 1.11, note that $S = \{x^2 - 4, x + 6\}$ is a linearly independent subset of $\mathsf{P}_2(F)$. Since $\beta = \{1, x, x^2\}$ is a basis for $\mathsf{P}_2(F)$, there must be a subset S_1 of β containing $3 - 2 = 1$ element such that $S \cup S_1$ generates $\mathsf{P}_2(F)$. In this example any subset of β containing one element will suffice for S_1. Hence we see that the set S_1 in Theorem 1.11 need not be unique.

Corollary 1. *Let* V *be a vector space having a basis* β *containing exactly* n *elements. Then any linearly independent subset of* V *containing exactly* n *elements is a basis for* V.

PROOF. Let $S = \{y_1, \ldots, y_n\}$ be a linearly independent subset of V containing exactly n elements. Applying Theorem 1.11, we see that there exists a subset S_1 of β containing $n - n = 0$ elements such that $S \cup S_1$ generates V. Clearly $S_1 = \varnothing$; so S generates V. Since S is also linearly independent, S is a basis for V. ∎

Example 28. The vectors $(1, -3, 2)$, $(4, 1, 0)$, and $(0, 2, -1)$ form a basis for R^3, for if

$$a_1(1, -3, 2) + a_2(4, 1, 0) + a_3(0, 2, -1) = (0, 0, 0),$$

then a_1, a_2, and a_3 must satisfy the system of equations

$$\begin{cases} a_1 + 4a_2 \phantom{{}+2a_3} = 0 \\ -3a_1 + a_2 + 2a_3 = 0 \\ 2a_1 \phantom{{}+a_2} - a_3 = 0. \end{cases}$$

But it is easily seen that the only solution of this system is $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. Hence $(1, -3, 2)$, $(4, 1, 0)$, and $(0, 2, -1)$ are linearly independent and therefore form a basis for R^3 by Corollary 1.

Corollary 2. *Let* V *be a vector space having a basis* β *containing exactly* n *elements. Then any subset of* V *containing more than* n *elements is linearly dependent. Consequently, any linearly independent subset of* V *contains at most* n *elements.*

PROOF. Let S be a subset of V containing more than n elements. In order to reach a contradiction we shall assume that S is linearly independent. Let S_1 be any subset of S containing exactly n elements; then S_1 is a basis for V by the preceding corollary. Because S_1 is a proper subset of S, we can select an element x of S that is not an element of S_1. Since S_1 is a basis for V, $x \in \text{span}(S_1) = \mathsf{V}$. Thus the lemma to Theorem 1.10 implies that $S_1 \cup \{x\}$ is linearly dependent. But $S_1 \cup \{x\} \subseteq S$; so S is linearly dependent—a contradiction. We conclude therefore that S is linearly dependent. ∎

Example 29. Let $S = \{x^2 + 7, 8x^2 - 2x, 4x - 3, 7x + 2\}$. Although we can prove directly that S is a linearly dependent subset of $\mathsf{P}_2(F)$, this conclusion follows immediately from the preceding corollary since $\beta = \{1, x, x^2\}$ is a basis for $\mathsf{P}_2(F)$ containing fewer elements than S.

Corollary 3. *Let* V *be a vector space having a basis* β *containing exactly* n *elements. Then every basis for* V *contains exactly* n *elements.*

PROOF. Let S be a basis for V. Since S is linearly independent, S contains at most n elements by Corollary 2. Suppose that S contains exactly m elements; then $m \leq n$. But, moreover, S is a basis for V and β is a linearly independent subset of V. So Corollary 2 may be applied with the roles of β and S interchanged to yield $n \leq m$. Thus $m = n$. ∎

If a vector space has a basis containing a finite number of elements, then the corollary above asserts that the number of elements in each basis for the space is the same. This result makes the following definitions possible.

Definitions. *A vector space* V *is called* finite-dimensional *if it has a basis consisting of a finite number of elements; the unique number of elements in each basis for* V *is called the* dimension of V *and is denoted* dim(V). *If a vector space is not finite-dimensional, then it is called* infinite-dimensional.

The following results are consequences of Examples 22 through 26.

Example 30. The vector space $\{0\}$ has dimension zero.

Example 31. The vector space F^n has dimension n.

Example 32. The vector space $\mathsf{M}_{m\times n}(F)$ has dimension mn.

Example 33. The vector space $\mathsf{P}_n(F)$ has dimension $n + 1$.

Example 34. The vector space $\mathsf{P}(F)$ is infinite-dimensional.

The following two examples show that the dimension of a vector space depends on its field of scalars.

Example 35. The vector space of complex numbers has dimension 1 over the field of complex numbers. (A basis is $\{1\}$.)

Example 36. The vector space of complex numbers has dimension 2 over the field of real numbers. (A basis is $\{1, i\}$.)

Corollary 4. Let V *be a vector space having dimension* n, *and let* S *be a subset of* V *that generates* V *and contains at most* n *elements. Then* S *is a basis for* V *and hence contains exactly* n *elements.*

PROOF. There exists a subset S_1 of S such that S_1 is a basis for V (Theorem 1.10). By Corollary 3 above, S_1 contains exactly n elements. But $S_1 \subseteq S$ and S contains at most n elements. Hence $S = S_1$, and thus S is a basis for V. ∎

Example 37. It follows from Example 18 and Corollary 4 that $\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$ is a basis for $\mathsf{P}_2(R)$.

Example 38. It follows from Example 19 and Corollary 4 that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis for $\mathsf{M}_{2\times 2}(R)$.

Corollary 5. Let β be a basis for a vector space V *having dimension* n, *and let* S *be a linearly independent subset of* V *containing* m *elements. Then there exists a subset* S_1 *of β such that* $S \cup S_1$ *is a basis for* V.

PROOF. By Corollary 2 of Theorem 1.11 we know that $m \leq n$. Hence by Theorem 1.11 there exists a subset S_1 of β containing exactly $n - m$ elements such that $S \cup S_1$ generates V. Clearly $S \cup S_1$ contains at most n elements; so Corollary 4 above implies that $S \cup S_1$ is a basis for V. ∎

Theorem 1.10, Theorem 1.11 and its five corollaries, and Exercise 11 contain a wealth of information about the relationships among linearly independent sets, bases, and generating sets. For this reason we shall summarize here the main results of this section in order to put them into better perspective.

A basis for a vector space V is a linearly independent subset of V that generates V. If V has a finite basis, then every basis for V contains the same number of vectors. This number is called the dimension of V, and V is said to be finite-dimensional. Thus if the dimension of V is n, every basis for V contains exactly n vectors. Moreover, each linearly independent subset of V contains no more than n vectors and can be made into a basis for V by including appropriately chosen vectors. Also, each generating set for V contains at least n vectors and can be made into a basis for V by excluding appropriately chosen vectors. Figure 1.6 depicts these relationships. We shall see in Section 2.4 that every vector space over F of dimension n is essentially the space F^n.

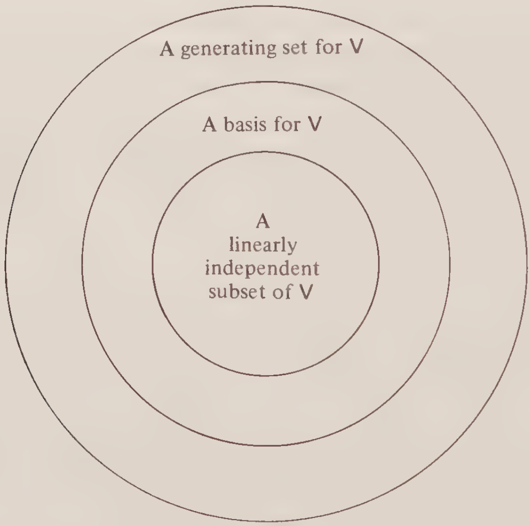


figure 1.6

The following example illustrates how these results may be used to obtain an important non-trivial conclusion.

Let $c_0, c_1, \ldots, c_n$ be distinct elements of an infinite field F. The polynomials $f_0(x), f_1(x), \ldots, f_n(x)$, where

$$\begin{aligned} f_i(x) &= \frac{(x-c_0)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_0)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)} \\ &= \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-c_j}{c_i-c_j} \end{aligned}$$

are called the *Lagrange polynomials* (*associated with* $c_0, c_1, \ldots, c_n$). Regarding $f_i(x)$ as a polynomial function $f_i\colon \quad F \longrightarrow F$, we see that

$$f_i(c_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{13}$$

We shall use this property of the Lagrange polynomials to show that $\beta = \{f_0, f_1, \ldots, f_n\}$ is a linearly independent subset of $\mathsf{P}_n(F)$. Since the dimension of $\mathsf{P}_n(F)$ is $n + 1$, it will follow from Corollary 1 of Theorem 1.11 that β is a basis for $\mathsf{P}_n(F)$. To show that β is linearly independent, suppose that

$$\sum_{i=0}^{n} a_i f_i = 0 \qquad \text{for some scalars } a_0, a_1, \ldots, a_n,$$

where 0 denotes the zero function. Then

$$\sum_{i=0}^{n} a_i f_i(c_j) = 0 \qquad \text{for } j = 0, 1, \ldots, n.$$

But also

$$\sum_{i=0}^{n} a_i f_i(c_j) = a_j$$

by Eq. (13). Hence $a_j = 0$ for $j = 0, 1, \ldots, n$, and so β is linearly independent.

Because β is a basis for $\mathsf{P}_n(F)$, every polynomial function g in $\mathsf{P}_n(F)$ is a linear combination of elements of β, say

$$g = \sum_{i=0}^{n} b_i f_i.$$

Then

$$g(c_j) = \sum_{i=0}^{n} b_i f_i(c_j) = b_j;$$

so

$$g = \sum_{i=0}^{n} g(c_i) f_i$$

is the unique representation of g as a linear combination of elements of β. This representation is called the *Lagrange interpolation formula*. Notice that the argument above shows that if $b_0, b_1, \ldots, b_n$ are any $n + 1$ elements of F (not necessarily distinct), then the polynomial function

$$g = \sum_{i=0}^{n} b_i f_i$$

is the unique element of $\mathsf{P}_n(F)$ such that $g(c_j) = b_j$. Thus we have found the unique polynomial of degree not exceeding n that has specified values b_j at given points c_j in its domain ($j = 0, 1, \ldots, n$). For example, let us construct the real polynomial g of degree at most 2 whose graph contains the points $(1, 8)$, $(2, 5)$, and $(3, -4)$. (Thus in the notation above $c_0 = 1$, $c_1 = 2$, $c_2 = 3$, $b_0 = 8$, $b_1 = 5$, and $b_2 = -4$.) The Lagrange polynomials associated with c_0, c_1, and c_2 are

$$f_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6),$$

$$f_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -1(x^2 - 4x + 3),$$

and

$$f_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2).$$

Hence the desired polynomial is

$$\begin{aligned} g(x) &= \sum_{i=0}^{2} b_i f_i(x) = 8f_0(x) + 5f_1(x) - 4f_2(x) \\ &= 4(x^2 - 5x + 6) - 5(x^2 - 4x + 3) - 2(x^2 - 3x + 2) \\ &= -3x^2 + 6x + 5. \end{aligned}$$

An important consequence of the Lagrange interpolation formula is the following result: If $f \in \mathsf{P}_n(F)$ and $f(c_j) = 0$ for $n + 1$ distinct elements $c_0, c_1, \ldots, c_n$ in F, then f is the zero function.

The next result relates the dimension of a subspace to the dimension of the vector space that contains it.

Theorem 1.12. *Let* W *be a subspace of a vector space* V *of dimension* n. *Then* W *is finite-dimensional and dim*$(\mathsf{W}) \leq$ n. *Moreover, if dim*$(\mathsf{W}) =$ n, *then* $\mathsf{W} = \mathsf{V}$.

PROOF. If $\mathsf{W} = \{0\}$, then W is finite-dimensional and $\dim(\mathsf{W}) = 0 \leq n$. Otherwise, there exists a non-zero element x_1 in W, and so $\{x_1\}$ is a linearly independent set. Continuing in this manner, choose elements $x_1, \ldots, x_k$ in W such that $\{x_1, \ldots, x_k\}$ is linearly independent. This process must stop at a stage where $\{x_1, \ldots, x_k\}$ is linearly independent but adjoining any other element of W produces a linearly dependent set (since no linearly independent subset of V can contain more than n elements). Thus W has a finite basis containing no more than n elements; that is, $\dim(\mathsf{W}) \leq n$.

If $\dim(\mathsf{W}) = n$, then a basis for W will be a linearly independent subset of V containing n elements. But Corollary 1 of Theorem 1.11 implies that the basis for W is also a basis for V, and hence $\mathsf{W} = \mathsf{V}$. ∎

Corollary. *If* W *is a subspace of a finite-dimensional space* V, *then* W *has a finite basis, and any basis for* W *is a subset of a basis for* V.

PROOF. The theorem shows that W has a finite basis S. If β is any basis for V, then there exists a subset S_1 of β such that $S \cup S_1$ is a basis for V (Theorem 1.11). Hence S is a subset of a basis for V. ∎

We can use Theorem 1.12 to analyze geometrically the subspaces of R^2 and R^3.

Since R^2 has dimension 2 over R, subspaces of R^2 can be of dimensions 0, 1, or 2 only. The only subspaces of dimensions 0 or 2 are $\{0\}$ and R^2, respectively. Any subspace of R^2 having dimension 1 consists of all scalar multiples of some non-zero vector in R^2 (Exercise 9 of Section 1.4).

If a point of R^2 is identified in the natural way with a point on the Euclidean plane, then it is possible to describe the subspaces of R^2 geometrically: A subspace of R^2 having dimension 0 consists of the origin of the Euclidean plane, a subspace of R^2 with dimension 1 consists of a line through the origin, and a subspace of R^2 having dimension 2 is the entire Euclidean plane.

As above, the subspaces of R^3 must have dimensions 0, 1, 2, or 3. Interpreting these possibilities geometrically, we see that a subspace of dimension zero must be the origin of Euclidean 3-space, a subspace of dimension 1 is a line through the origin, a subspace of dimension 2 is a plane through the origin, and a subspace of dimension 3 is Euclidean 3-space itself.

Example 39. Let $\mathsf{W} = \{(a_1, \ldots, a_5) \in \mathsf{F}^5 \colon a_1 + a_3 + a_5 = 0, a_2 = a_4\}$. Then W is a subspace of F^5 having $\{(1, 0, 0, 0, -1), (0, 0, 1, 0, -1), (0, 1, 0, 1, 0)\}$ as a basis. Thus the dimension of W is 3.

Example 40. The set of diagonal $n \times n$ matrices forms a subspace W of $\mathsf{M}_{n\times n}(F)$. (See Example 8.) A basis for W is $\{M^{11}, M^{22}, \ldots, M^{nn}\}$ where M^{ij} is the matrix defined in Example 24. Hence the dimension of W is n.

Example 41. We saw in Section 1.3 that the set of symmetric $n \times n$ matrices forms a subspace W of $\mathsf{M}_{n\times n}(F)$. A basis for W is $\{A^{ij}\colon 1 \leq i \leq j \leq n\}$, where A^{ij} is the $n \times n$ matrix having 1 in the ith row and jth column, 1 in the jth row and ith column, and 0 elsewhere. Therefore the dimension of W is $n + (n-1) + \cdots + 1 = \frac{1}{2}n(n+1)$.

Example 42. The set of polynomials of the form $a_{18}x^{18} + a_{16}x^{16} + \cdots + a_2x^2 + a_0$, where $a_0, a_2, \ldots, a_{16}, a_{18} \in F$, compose a subspace W of $\mathsf{P}_{19}(F)$ of dimension 10 since $\{1, x^2, x^4, \ldots, x^{18}\}$ is a basis for W.

If W_1 and W_2 are subspaces of a vector space V, we saw in Section 1.3 that so are $\mathsf{W}_1 \cap \mathsf{W}_2$ and $\mathsf{W}_1 + \mathsf{W}_2$. It is natural to ask if the dimensions of these subspaces can be computed directly from the dimensions of W_1 and W_2. Unfortunately this cannot be done. There is, however, a relationship among $\dim(W_1 + W_2)$, $\dim(W_1)$, and $\dim(W_2)$.

Theorem 1.13. *Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. Then $W_1 + W_2$ is finite-dimensional, and*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

PROOF. Since $W_1 \cap W_2$ is a subspace of a finite-dimensional space W_1, $W_1 \cap W_2$ has a finite basis $\beta_0 = \{x_1, \ldots, x_k\}$ (Theorem 1.12). Use the corollary to Theorem 1.12 to find $\beta_1 = \{y_1, \ldots, y_r\}$ and $\beta_2 = \{z_1, \ldots, z_m\}$ such that $\beta_0 \cup \beta_1$ is a basis for W_1 and $\beta_0 \cup \beta_2$ is a basis for W_2. We shall prove that $\beta_0 \cup \beta_1 \cup \beta_2 = \{x_1, \ldots, x_k, y_1, \ldots, y_r, z_1, \ldots, z_m\}$ is a basis for $W_1 + W_2$. It will follow that $W_1 + W_2$ is a finite-dimensional and that

$$\begin{aligned}\dim(W_1 + W_2) &= k + r + m = (k + r) + (k + m) - k\\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).\end{aligned}$$

To prove that $\beta_0 \cup \beta_1 \cup \beta_2$ is a basis for $W_1 + W_2$, we shall first prove that $\beta_0 \cup \beta_1 \cup \beta_2$ is linearly independent. Suppose that

$$a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_ry_r + c_1z_1 + \cdots + c_mz_m = 0$$

for some scalars $a_1, \ldots, a_k, b_1, \ldots, b_r, c_1, \ldots, c_m$. Let

$$v_0 = a_1x_1 + \cdots + a_kx_k, \qquad v_1 = b_1y_1 + \cdots + b_ry_r,$$

and

$$v_2 = c_1z_1 + \cdots + c_mz_m;$$

observe that $v_0 \in W_1 \cap W_2$, $v_1 \in W_1$, and $v_2 \in W_2$. The equality above may be written $v_0 + v_1 + v_2 = 0$; so $v_0 + v_1 = -v_2$. In this last equality the left side is an element of W_1 and the right side is an element of W_2. Thus $-v_2$ is an element of both W_1 and W_2, so that $-v_2 \in W_1 \cap W_2$. Since $\{x_1, \ldots, x_k\}$ is a basis for $W_1 \cap W_2$, there exist scalars $d_1, \ldots, d_k$ such that $-v_2 = d_1x_1 + \cdots + d_kx_k$. Now

$$\begin{aligned}0 &= v_0 + v_1 + v_2\\ &= (a_1x_1 + \cdots + a_kx_k) + (b_1y_1 + \cdots + b_ry_r)\\ &\quad + (-d_1x_1 - \cdots - d_kx_k)\\ &= (a_1 - d_1)x_1 + \cdots + (a_k - d_k)x_k + b_1y_1 + \cdots + b_ry_r.\end{aligned}$$

So a linear combination of elements of $\beta_0 \cup \beta_1$ equals the zero vector. But $\beta_0 \cup \beta_1$ is a linearly independent set, and thus $a_1 - d_1 = \cdots = a_k - d_k = b_1 = \cdots = b_r = 0$. Hence $v_1 = 0$. Thus

$$0 = v_0 + v_1 + v_2 = v_0 + v_2 = a_1x_1 + \cdots + a_kx_k + c_1z_1 + \cdots + c_mz_m,$$

so that a linear combination of elements of $\beta_0 \cup \beta_2$ equals the zero vector. As before, the fact that $\beta_0 \cup \beta_2$ is a linearly independent set implies that $a_1 = \cdots = a_k = c_1 = \cdots = c_m = 0$. Since $a_1 = \cdots = a_k = b_1 = \cdots = b_r = c_1 = \cdots = c_m = 0$, we have proved that $\beta_0 \cup \beta_1 \cup \beta_2$ is linearly independent.

It remains to prove that $\beta_0 \cup \beta_1 \cup \beta_2$ generates $\mathsf{W}_1 + \mathsf{W}_2$. Now $\text{span}(\beta_0 \cup \beta_1) = \mathsf{W}_1$ and $\text{span}(\beta_0 \cup \beta_2) = \mathsf{W}_2$ since $\beta_0 \cup \beta_1$ and $\beta_0 \cup \beta_2$ are bases for W_1 and W_2, respectively. But

$$\begin{aligned}\text{span}(\beta_0 \cup \beta_1 \cup \beta_2) &= \text{span}((\beta_0 \cup \beta_1) \cup (\beta_0 \cup \beta_2))\\ &= \text{span}(\beta_0 \cup \beta_1) + \text{span}(\beta_0 \cup \beta_2)\\ &= \mathsf{W}_1 + \mathsf{W}_2\end{aligned}$$

by Exercise 12 of Section 1.4. Hence $\beta_0 \cup \beta_1 \cup \beta_2$ generates $\mathsf{W}_1 + \mathsf{W}_2$. This completes the proof. ▌

As an immediate consequence of this result, we have the following useful corollary.

Corollary. *Let W_1 and W_2 be finite-dimensional subspaces of a vector space V such that $\mathsf{V} = \mathsf{W}_1 + \mathsf{W}_2$. Then V is the direct sum of W_1 and W_2 if and only if*

$$\dim(\mathsf{V}) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2).$$

Example 43. Let c be an element of an infinite field F, let W_1 denote the set of all constant functions in $\mathsf{P}_n(F)$, and define $\mathsf{W}_2 = \{f(x) \in \mathsf{P}_n(F): f(c) = 0\}$. It is easily seen that W_1 and W_2 are subspaces of $\mathsf{P}_n(F)$ and that $\mathsf{P}_n(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$. (Observe that for any $f(x) \in \mathsf{P}_n(F)$, $g(x) = f(c) \in \mathsf{W}_1$, $h(x) = f(x) - f(c) \in \mathsf{W}_2$, and $f(x) = g(x) + h(x)$.) Since the constant function $p(x) = 1$ clearly forms a basis for W_1, it follows from the corollary above that

$$\dim(\mathsf{W}_2) = \dim(\mathsf{P}_n(F)) - \dim(\mathsf{W}_1) = (n + 1) - 1 = n.$$

EXERCISES

1. Label the following statements as being true or false.
   (a) The zero vector space has no basis.
   (b) Every vector space that is generated by a finite set has a basis.
   (c) Every vector space has a finite basis.
   (d) A vector space cannot have more than one basis.
   (e) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(f) The dimension of $\mathsf{P}_n(F)$ is n.
(g) The dimension of $\mathsf{M}_{m\times n}(F)$ is $m + n$.
(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more elements than S_2.
(i) If S generates the vector space V, then every vector in V can be written as a linear combination of elements of S in only one way.
(j) Every subspace of a finite-dimensional space is finite-dimensional.
(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.
(l) If W_1 and W_2 are finite-dimensional subspaces of a vector space, then $\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2)$.

2. Determine which of the following sets are bases for R^3.
(a) $\{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$
(b) $\{(2, -4, 1), (0, 3, -1), (6, 0, -1)\}$
(c) $\{(1, 2, -1), (1, 0, 2), (2, 1, 1)\}$
(d) $\{(-1, 3, 1), (2, -4, -3), (-3, 8, 2)\}$
(e) $\{(1, -3, -2), (-3, 1, 3), (-2, -10, -2)\}$

3. Determine which of the following sets are bases for $\mathsf{P}_2(R)$.
(a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$
(b) $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$
(c) $\{1 + 4x - 2x^2, -2 + 3x - x^2, -3 - 12x + 6x^2\}$
(d) $\{-1 + 2x + 4x^2, 3 - 4x - 10x^2, -2 - 5x - 6x^2\}$
(e) $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$

4. Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$, and $3x - 2$ generate $\mathsf{P}_3(R)$? Justify your answer.

5. Is $\{(1, 4, -6), (1, 5, 8), (2, 1, 1), (0, 1, 0)\}$ a linearly independent subset of R^3? Justify your answer.

6. Give three different bases for F^2 and for $\mathsf{M}_{2\times 2}(F)$.

7. The vectors $x_1 = (2, -3, 1)$, $x_2 = (1, 4, -2)$, $x_3 = (-8, 12, -4)$, $x_4 = (1, 37, -17)$, and $x_5 = (-3, -5, 8)$ generate R^3. Find a subset of $\{x_1, x_2, x_3, x_4, x_5\}$ that is a basis for R^3.

8. Let V denote the vector space consisting of all vectors in R^5 for which the sum of the coordinates equals zero. The vectors

$$\begin{aligned}
x_1 &= (2, -3, 4, -5, 2), & x_2 &= (-6, 9, -12, 15, -6),\\
x_3 &= (3, -2, 7, -9, 1), & x_4 &= (2, -8, 2, -2, 6),\\
x_5 &= (-1, 1, 2, 1, -3), & x_6 &= (0, -3, -18, 9, 12),\\
x_7 &= (1, 0, -2, 3, -2), & x_8 &= (2, -1, 1, -9, 7)
\end{aligned}$$

generate V. Find a subset of $\{x_1, \ldots, x_8\}$ that is a basis for V.

9. The vectors $x_1 = (1, 1, 1, 1)$, $x_2 = (0, 1, 1, 1)$, $x_3 = (0, 0, 1, 1)$, and $x_4 = (0, 0, 0, 1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1, a_2, a_3, a_4) in F^4 as a linear combination of the vectors x_1, x_2, x_3, and x_4.

10. Let

$$\mathsf{V} = \mathsf{M}_{2\times 2}(F), \qquad \mathsf{W}_1 = \left\{\begin{pmatrix} a & b \\ c & a \end{pmatrix} \in \mathsf{V}\colon \ a, b, c \in F\right\}$$

and

$$\mathsf{W}_2 = \left\{\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in \mathsf{V}\colon \ a, b \in F\right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $\mathsf{W}_1 + \mathsf{W}_2$, and $\mathsf{W}_1 \cap \mathsf{W}_2$.

11.† Let V be a vector space having dimension n, and let S be a subset of V that generates V.
(a) Prove that S contains at least n elements.
(b) Prove that a subset of S is a basis for V. (Be careful not to assume that S is finite.)

12. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$. Prove that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n$ and $\dim(\mathsf{W}_1 + \mathsf{W}_2) \leq m + n$. Give examples of subspaces of R^3 in which each of the inequalities above holds as an equality.

13. Let $\{x, y\}$ be a basis for a vector space V. Show that both $\{x + y, x - y\}$ and $\{ax, by\}$ are bases for V, where a and b are arbitrary non-zero scalars.

14. Suppose that V is a vector space with a basis $\{x_1, x_2, x_3\}$. Show that $\{x_1 + x_2 + x_3, x_2 + x_3, x_3\}$ is also a basis for V.

15. The set of solutions to the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \end{cases}$$

is a subspace of R^3. Find a basis for this subspace.

16. Find bases for the following subspaces of F^5:

$$\mathsf{W}_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5\colon \ a_1 - a_3 - a_4 = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5\colon \ a_2 = a_3 = a_4, a_1 + a_5 = 0\}.$$

What are the dimensions of W_1 and W_2?

17. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $\mathsf{M}_{n\times n}(F)$. (See Example 11.) Find a basis for W. What is the dimension of W?

18. The set of all upper triangular $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$. (See Exercise 12 of Section 1.3.) Find a basis for W. What is the dimension of W?

19. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$. (See Exercise 25 of Section 1.3.) Find a basis for W. What is the dimension of W?

20. (a) Let W_1 and W_2 be subspaces of a vector space V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, prove that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.
(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

21. Complete the proof of Theorem 1.9.

22. Let W be a subspace of a finite-dimensional vector space V. Determine the dimension of the vector space V/W, the quotient space of V modulo W. (See Exercise 29 of Section 1.3.) Justify your answer.

23. Find a basis for the vector space of finitely non-zero sequences in a field F. (See Example 5.)

24. Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there is a subspace W_2 of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

25. Prove that a vector space is infinite-dimensional if and only if it contains an infinite linearly independent subset.

1.7* MAXIMAL LINEARLY INDEPENDENT SUBSETS

In this section several important results from Section 1.6 will be extended to include infinite-dimensional vector spaces. Our principal goal is to prove that every vector space has a basis. This result is of importance in the study of infinite-dimensional vector spaces because it is often extremely difficult to construct a basis for such a space explicitly.

The difficulty that arises in extending the theorems of the previous section to infinite-dimensional spaces is that the *principle of mathematical induction*, which played a crucial role in many of the proofs of Section 1.6, is no longer adequate. We shall use instead a more general result called the *maximal principle*, which will require the following terminology.

Definition. *Let $\mathfrak{F}$ be a family of sets. A member* M *of $\mathfrak{F}$ is called* maximal (with respect to set inclusion) *if no member of $\mathfrak{F}$ properly contains* M.

Example 44. Let $\mathfrak{F}$ be the family of all subsets of a non-empty set S. ($\mathfrak{F}$ is called the *power set of S*.) The set S is easily seen to be a maximal element of $\mathfrak{F}$.

Definition. A collection of sets $\mathcal{C}$ *is called a* chain (*or* nest *or* tower) *if, for each pair of sets* A *and* B *in* $\mathcal{C}$, *either* $A \subseteq B$ *or* $B \subseteq A$.

Example 45. Let A_n denote the set consisting of the integers $1, 2, \ldots, n$. Then $\mathcal{C} = \{A_n\colon \ n = 1, 2, 3, \ldots\}$ is a chain; in fact, $A_m \subseteq A_n$ if and only if $m \leq n$.

With this terminology we can now state the maximal principle.

Maximal Principle. Let $\mathfrak{F}$ *be a family of sets. If, for each chain* $\mathcal{C} \subseteq \mathfrak{F}$, *there exists a member of* $\mathfrak{F}$ *that contains each member of* $\mathcal{C}$, *then* $\mathfrak{F}$ *contains a maximal element.*

Because the maximal principle guarantees the existence of maximal elements in a family of sets, it will be useful to reformulate the definition of a basis in terms of a maximal property. We shall subsequently show that this reformulation is equivalent to the original definition of a basis.

Definition. Let S *be a subset of a vector space* V. *A* maximal linearly independent subset *of* S *is a subset* B *of* S *satisfying both of the following conditions:*

(a) B *is linearly independent.*
(b) *Any subset of* S *that properly contains* B *is linearly dependent.*

Example 46. Example 16 shows that $\{x^3 - 2x^2 - 5x - 3, 3x^3 - 5x^2 - 4x - 9\}$ is a maximal linearly independent subset of

$$S = \{2x^3 - 2x^2 + 12x - 6, x^3 - 2x^2 - 5x - 3, 3x^3 - 5x^2 - 4x - 9\}$$

in $P_3(R)$. In this case, however, any two-element subset of S is easily shown to be a maximal linearly independent subset of S. Hence maximal linearly independent subsets of a set need not be unique.

A basis β for a vector space V is a maximal linearly independent subset of V, for

(a) β is linearly independent by definition.
(b) If $x \in$ V and $x \notin \beta$, then $\beta \cup \{x\}$ is linearly dependent by the lemma to Theorem 1.10 since span$(\beta) =$ V.

Our next result shows that the converse of this statement is also true.

Theorem 1.14. *Let* S *be a subset of a vector space* V *such that* S *generates* V, *and let* β *be a maximal linearly independent subset of* S. *Then* β *is a basis for* V.

PROOF. Since β is linearly independent, it suffices to prove that β generates V. Suppose that $S \not\subseteq \text{span}(\beta)$; then there exists $x \in S$ such that $x \notin \text{span}(\beta)$. But the lemma to Theorem 1.10 then implies that $\beta \cup \{x\}$ is linearly independent—a contradiction of the maximality of β. So $S \subseteq \text{span}(\beta)$. Thus, since $\text{span}(S) = \text{V}$, it follows from Exercise 11 of Section 1.4 that $\text{span}(\beta) = \text{V}$. ∎

Corollary. *A subset* β *of a vector space* V *is a basis for* V *if and only if* β *is a maximal linearly independent subset of* V.

In view of the preceding corollary, we can accomplish our goal of proving that every vector space has a basis by proving that every vector space contains a maximal linearly independent subset. This result follows immediately from our next theorem.

Theorem 1.15. *Let* S *be a linearly independent subset of a vector space* V. *There exists a maximal linearly independent subset of* V *that contains* S.

PROOF. Let $\mathfrak{F}$ denote the family of all linearly independent subsets of V that contain S. We shall use the maximal principle to show that $\mathfrak{F}$ contains a maximal element. In order to apply the maximal principle, we must show that if $\mathcal{C}$ is a chain in $\mathfrak{F}$, then there exists a member U of $\mathfrak{F}$ that contains each member of $\mathcal{C}$. We shall show that U, the union of the members of $\mathcal{C}$, is the desired set. Since U clearly contains each member of $\mathcal{C}$, it suffices to prove that $U \in \mathfrak{F}$, i.e., that U is a linearly independent subset of V that contains S. Now each element of $\mathcal{C}$ is a subset of V containing S; hence $S \subseteq U \subseteq \text{V}$. To prove that U is linearly independent, let $u_1, \ldots, u_n$ be vectors in U and $c_1, \ldots, c_n$ be scalars such that $c_1u_1 + \cdots + c_nu_n = 0$. Because $u_i \in U$ for $i = 1, \ldots, n$, there exist sets A_i in $\mathcal{C}$ such that $u_i \in A_i$. But since $\mathcal{C}$ is a chain, one of the sets $A_1, \ldots, A_n$, say A_k, contains all the others. Thus $u_1, \ldots, u_n \in A_k$ for $i = 1, \ldots, n$. However A_k is a linearly independent set; so $c_1u_1 + \cdots + c_nu_n = 0$ implies that $c_1 = \cdots = c_n = 0$. Therefore U is linearly independent.

The maximal principle implies that $\mathfrak{F}$ contains a maximal element. This maximal element is easily seen to be a maximal linearly independent subset of V that contains S. ∎

Corollary. *Every vector space has a basis.*

It can be shown, analogously to Corollary 3 of Theorem 1.11, that every basis for an infinite-dimensional vector space has the same cardinality. (See, for example, N. Jacobson, *Lectures in Linear Algebra*, **III**, page 154, D. Van Nostrand Company, New York, 1964.)

Exercises 2 through 5 extend other results from Section 1.6 to include infinite-dimensional spaces.

EXERCISES

1. Label the following statements as being true or false.
   (a) Every family of sets contains a maximal element.
   (b) Every chain contains a maximal element.
   (c) If a family of sets has a maximal element, then that maximal element is unique.
   (d) If a chain of sets has a maximal element, then that maximal element is unique.
   (e) A basis for a vector space is a maximal linearly independent subset of that vector space.
   (f) A maximal linearly independent subset of a vector space is a basis for that vector space.
2. Let W be a subspace of a (not necessarily finite-dimensional) vector space V. Prove that any basis for W is a subset of a basis for V.
3. Prove the following infinite-dimensional version of Theorem 1.9: Let β be a subset of an infinite-dimensional vector space V. Then β is a basis for V if and only if for each non-zero vector y in V there exist unique vectors $x_1, \ldots, x_n$ in β and unique non-zero scalars $c_1, \ldots, c_n$ such that $y = c_1x_1 + \cdots + c_nx_n$.
4. Prove the following generalization of Theorem 1.10: Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subseteq S_2$. If S_1 is linearly independent and S_2 generates V, then there exists a basis β for V such that $S_1 \subseteq \beta \subseteq S_2$. *Hint:* Apply the maximal principle to the family of all linearly independent subsets of S_2 that contain S_1, and proceed as in the proof of Theorem 1.15.
5. Prove the following generalization of Theorem 1.11: Let β be a basis for a vector space V, and let S be a linearly independent subset of V. There exists a subset S_1 of β such that $S \cup S_1$ is a basis for V.

INDEX OF DEFINITIONS FOR CHAPTER 1

chapter 2

linear transformations and matrices

In Chapter 1 we developed the theory of abstract vector spaces in considerable detail. It is now natural to consider those functions defined on vector spaces that in some sense "preserve" the structure. These special functions are called "linear transformations" and they abound in both pure and applied mathematics. In calculus the operations of differentiation and integration provide us with two of the most important examples of linear transformations (see Examples 1 and 2). These two examples allow us to reformulate many of the problems in differential and integral equations in terms of linear transformations on particular vector spaces (see Sections 2.7 and 5.2).

In geometry, rotations, reflections, and projections (see Examples 5, 6, and 7) provide us with another class of linear transformations. Later we shall use these transformations to study the rigid motions in R^n (Section 7.8).

In the remaining chapters we shall see further examples of linear transformations occurring in both the physical and social sciences.

Throughout this chapter we shall assume that all vector spaces are over a common field F.

2.1 LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

In this section, we shall consider a number of examples of linear transformations. Many of these transformations will be studied in more detail in later sections.

Definition. *Let* V *and* W *be vector spaces* (*over* F). *A function* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *is called a* linear transformation from V into W *if for all* x, y $\in$ V *and* c $\in$ F *we have*

(a) $\mathsf{T}(x + y) = \mathsf{T}(x) + \mathsf{T}(y)$.
(b) $\mathsf{T}(cx) = c\mathsf{T}(x)$.

We shall often simply call T *linear*. The reader should verify the following facts about a function $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$.

1. If T is linear, then $\mathsf{T}(0) = 0$.
2. T is linear if and only if $\mathsf{T}(ax + y) = a\mathsf{T}(x) + \mathsf{T}(y)$ for all $x, y \in \mathsf{V}$ and $a \in F$.
3. T is linear if and only if for $x_1, \ldots, x_n \in \mathsf{V}$ and $a_1, \ldots, a_n \in F$ we have
$$\mathsf{T}\Big(\sum_{i=1}^{n} a_i x_i\Big) = \sum_{i=1}^{n} a_i \mathsf{T}(x_i).$$

We shall generally use property 2 to prove that a given transformation in linear.

Example 1. Let $\mathsf{V} = \mathsf{P}_n(R)$ and $\mathsf{W} = \mathsf{P}_{n-1}(R)$. Define $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ by $\mathsf{T}(f) = f'$, where f' denotes the derivative of f. To show that T is linear, let g and h be vectors in $\mathsf{P}_n(R)$ and $a \in R$. Now $\mathsf{T}(ag + h) = (ag + h)' = ag' + h' = a\mathsf{T}(g) + \mathsf{T}(h)$. So by property 2 above, T is linear.

Example 2. Let $\mathsf{V} = \mathsf{C}(R)$, the vector space of continuous real-valued functions on R. Let $a, b \in R$, $a < b$. Define $\mathsf{T}\colon\ \mathsf{V} \longrightarrow R$ by $\mathsf{T}(f) = \int_a^b f(t)\,dt$ for all $f \in \mathsf{V}$. Then T is a linear transformation by the elementary properties of the integral.

Two very important examples of linear transformations that will appear frequently in the remainder of the book and, therefore, deserve their own notation, are the identity and zero transformations.

For vector spaces V and W (over F) we define the *identity transformation* $\mathsf{I}_\mathsf{V}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{I}_\mathsf{V}(x) = x$ for all $x \in \mathsf{V}$ and the *zero transformation*

T_0: $V \longrightarrow W$ by $T_0(x) = 0$ for all $x \in V$. It is clear that both of these transformations are linear. We shall often write I instead of I_V.

We shall now look at some additional examples of linear transformations.

Example 3. Define

$$T: \quad R^2 \longrightarrow R^2 \text{ by } T(a_1, a_2) = (2a_1 + a_2, a_1).$$

To show that T is linear, let $c \in F$ and $x, y \in R^2$, *where*

$x = (b_1, b_2)$ and $y = (d_1, d_2)$.

Since

$$cx + y = (cb_1 + d_1, cb_2 + d_2),$$

we have

$$T(cx + y) = (2(cb_1 + d_1) + cb_2 + d_2, cb_1 + d_1).$$

Also

$$\begin{aligned} cT(x) + T(y) &= c(2b_1 + b_2, b_1) + (2d_1 + d_2, d_1) \\ &= (2cb_1 + cb_2 + 2d_1 + d_2, cb_1 + d_1) \\ &= (2(cb_1 + d_1) + cb_2 + d_2, cb_1 + d_1). \end{aligned}$$

So T is linear.

Example 4. Define T: $M_{m \times n}(F) \longrightarrow M_{n \times m}(F)$ by $T(A) = A^t$, where A^t is as defined in Section 1.3. Then T is a linear transformation by Exercise 3 of Section 1.3.

As we shall see in Sections 7.7 and 7.8, the applications of linear algebra to geometry are wide and varied. The main reason for this is that most of the important geometrical transformations are linear. Three particular transformations that we shall now consider are the rotation, reflection, and the projection. We leave the proofs of linearity to the reader.

Example 5. For $0 \leq \theta < 2\pi$ we define T_θ: $R^2 \longrightarrow R^2$ by

$$T_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta).$$

T_θ is called the *rotation by* θ. (See Fig. 2.1(a).)

Example 6. Define T: $R^2 \longrightarrow R^2$ by $T(a_1, a_2) = (a_1, -a_2)$. T is called the *reflection about the* x*-axis*. (See Fig. 2.1(b).)

Example 7. Define T: $R^2 \longrightarrow R^2$ by $T(a_1, a_2) = (a_1, 0)$. T is called a *projection on the* x*-axis*. (See Fig. 2.1(c).) Note that if we let $W_1 = \{(a, 0): \ a \in R\}$ and $W_2 = \{(0, a): \ a \in R\}$, then $R^2 = W_1 \oplus W_2$. So

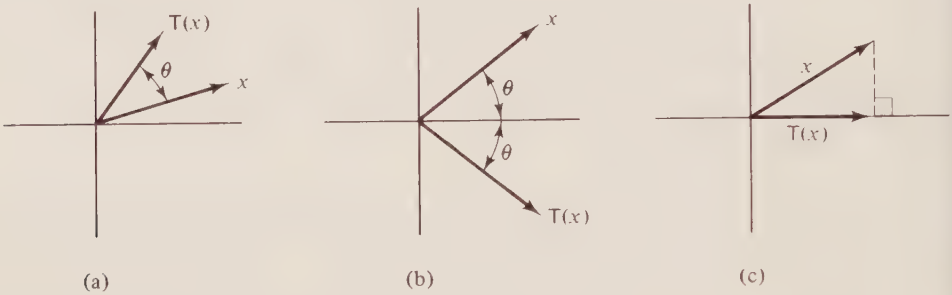


figure 2.1

for each $x \in \mathsf{R}^2$ there exist unique vectors $x_1 \in \mathsf{W}_1$ and $x_2 \in \mathsf{W}_2$ with $x = x_1 + x_2$ and $\mathsf{T}(x) = x_1$.

Example 7 suggests the following definition.

Definition. *Let* V *be a vector space and* W_1 *a subspace of* V. *A function* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ *is called a* projection on W_1 *if*

(a) *There exists a subspace* W_2 *such that* $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

(b) *For* $x = x_1 + x_2$, *where* $x_1 \in \mathsf{W}_1$ *and* $x_2 \in \mathsf{W}_2$, *we have* $\mathsf{T}(x) = x_1$.

It is left to the reader to show that T is linear and that $\mathsf{W}_1 = \{x\colon\ \mathsf{T}(x) = x\}$.

Now suppose that there is a subspace $\mathsf{W}_2' \neq \mathsf{W}_2$ such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2'$. Define $\mathsf{U}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{U}(x) = x_1$ where $x = x_1 + x_2'$, $x_1 \in \mathsf{W}_1$, and $x_2 \in \mathsf{W}_2'$. Then U is another projection on W_1, and again $\mathsf{W}_1 = \{x\colon\ \mathsf{U}(x) = x\}$. For instance, in Example 7 let

$$\mathsf{W}_2' = \{(a, a)\colon\ a \in R\},$$

so that

$$(a_1, a_2) = (a_1 - a_2, 0) + (a_2, a_2) \quad \text{and} \quad \mathsf{U}(a_1, a_2) = (a_1 - a_2, 0).$$

Thus there are as many projections on W_1 as there are subspaces W_2 satisfying $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$. We shall see in Chapter 7 that the projection depicted in Fig. 2.1(c) is the "natural" one to study. This type of projection will be called an "orthogonal projection," and it is uniquely determined by the subspace W_1.

A characterization of projections will be provided in Exercise 14 of section 2.3; this will allow us to determine easily whether or not a given linear transformation is a projection.

We shall now turn our attention to two very important sets associated with linear transformations: the "range" and "null space." The determination of these sets will allow us to examine more closely the intrinsic properties of a linear transformation.

Definitions. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}: \mathsf{V} \longrightarrow \mathsf{W}$ *be linear. We define the* null space (*or* kernel) $\mathsf{N}(\mathsf{T})$ *of* T *to be the set of all vectors* x *in* V *such that* $\mathsf{T}(x) = 0$; *i.e.*, $\mathsf{N}(\mathsf{T}) = \{x \in \mathsf{V}: \mathsf{T}(x) = 0\}$.

We define the range (*or* image) $\mathsf{R}(\mathsf{T})$ *to be the subset of* W *consisting of all images* (*under* T) *of elements of* V; *i.e.*, $\mathsf{R}(\mathsf{T}) = \{\mathsf{T}(x): x \in \mathsf{V}\}$.

Example 8. Let V and W be vector spaces, and let $\mathsf{I}: \mathsf{V} \longrightarrow \mathsf{V}$ and $\mathsf{T}_0: \mathsf{V} \longrightarrow \mathsf{W}$ be the identity and zero transformations, respectively, as defined above. Then, $\mathsf{N}(\mathsf{I}) = \{0\}$, $\mathsf{R}(\mathsf{I}) = \mathsf{V}$, $\mathsf{N}(\mathsf{T}_0) = \mathsf{V}$, and $\mathsf{R}(\mathsf{T}_0) = \{0\}$.

Example 9. Let $\mathsf{T}: \mathsf{R}^3 \longrightarrow \mathsf{R}^2$ be defined by $\mathsf{T}(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. It is left as an exercise to verify that $\mathsf{N}(\mathsf{T}) = \{(a, a, 0): a \in R\}$ and $\mathsf{R}(\mathsf{T}) = \mathsf{R}^2$.

In Example 7 it is easily seen that the subspace $\mathsf{W}_1 = \mathsf{R}(\mathsf{T})$ and $\mathsf{W}_2 = \mathsf{N}(\mathsf{T})$. The next theorem tells us that this is the case for all projections.

Theorem 2.1. *Let* V *be a vector space and* W_1 *be a subspace of* V. *Let* T *be a projection on* W_1, *and let* W_2 *be as in the definition of a projection. Then*

$$\mathsf{W}_1 = \mathsf{R}(\mathsf{T}) \quad \textit{and} \quad \mathsf{W}_2 = \mathsf{N}(\mathsf{T}).$$

PROOF. As observed earlier, $\mathsf{W}_1 = \{x: \mathsf{T}(x) = x\}$. Hence $\mathsf{W}_1 \subseteq \mathsf{R}(\mathsf{T})$. If $x \in \mathsf{R}(\mathsf{T})$, then $x = \mathsf{T}(y)$ for some $y \in \mathsf{V}$. But $y = y_1 + y_2$, where $y_1 \in \mathsf{W}_1$ and $y_2 \in \mathsf{W}_2$, and so $x = y_1$. Therefore, $\mathsf{W}_1 = \mathsf{R}(\mathsf{T})$.

Since it is clear that $\mathsf{W}_2 \subseteq \mathsf{N}(\mathsf{T})$, we need only show that $\mathsf{N}(\mathsf{T}) \subseteq \mathsf{W}_2$. For this purpose, let $x \in \mathsf{N}(\mathsf{T})$. Then $x = x_1 + x_2$ with $x_1 \in \mathsf{W}_1$ and $x_2 \in \mathsf{W}_2$. So $0 = \mathsf{T}(x) = x_1$, and hence $x = x_2 \in \mathsf{W}_2$. ∎

This theorem tells us that W_2 is uniquely determined by the projection T on W_1. Furthermore, since T is a projection on its range, we shall simply use the term "projection" without mentioning the subspace W_1.

We have just observed in the case where T is a projection that $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$ are subspaces of V. This same result can be proved for any linear transformation.

Theorem 2.2. *Let* V *and* W *be vector spaces and* $\mathsf{T}: \mathsf{V} \longrightarrow \mathsf{W}$ *be linear. Then* $\mathsf{N}(\mathsf{T})$ *and* $\mathsf{R}(\mathsf{T})$ *are subspaces of* V *and* W, *respectively.*

PROOF. To clarify the notation, we shall use the symbols 0_V and 0_W to denote the zero vectors of V and W, respectively.

Since $\mathsf{T}(0_\mathsf{V}) = 0_\mathsf{W}$, we have that $0_\mathsf{V} \in \mathsf{N}(\mathsf{T})$. Let $x, y \in \mathsf{N}(\mathsf{T})$ and $c \in F$. Then $\mathsf{T}(x + y) = \mathsf{T}(x) + \mathsf{T}(y) = 0_\mathsf{W} + 0_\mathsf{W} = 0_\mathsf{W}$, and $\mathsf{T}(cx) = c\mathsf{T}(x) =$

$c0_W = 0_W$. Hence $x + y \in N(T)$ and $cx \in N(T)$, so that $N(T)$ is a subspace of V.

Because $T(0_V) = 0_W$, we have that $0_W \in R(T)$. Now let $x, y \in R(T)$ and $c \in F$. Then there exist v and w in V such that $T(v) = x$ and $T(w) = y$. So $T(v + w) = T(v) + T(w) = x + y$, and $T(cv) = cT(v) = cx$. Thus $x + y \in R(T)$ and $cx \in R(T)$, so that $R(T)$ is a subspace of W. ∎

As in Chapter 1, we shall measure the "size" of a subspace by its dimension. The two subspaces above are so important that we attach special names to their respective dimensions.

Definitions. *Let* V *and* W *be vector spaces, and let* $T\colon V \longrightarrow W$ *be linear. If* $N(T)$ *and* $R(T)$ *are finite-dimensional, then we define the* nullity of T, *denoted* nullity(T), *and the* rank of T, *denoted* rank(T), *to be the dimensions of* $N(T)$ *and* $R(T)$, *respectively.*

Reflecting upon the action of a linear transformation, we see intuitively that the larger the nullity, the smaller the rank. In other words, the more vectors that are carried into 0, the smaller the range. The same heuristic reasoning will tell us that the larger the rank, the smaller the nullity. This balance between the rank and the nullity is made precise in the next theorem.

Theorem 2.3. *Let* V *and* W *be vector spaces, and let* $T\colon V \longrightarrow W$ *be linear. If* V *is finite-dimensional, then* $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

PROOF. Suppose $\dim(V) = n$, and let $\{x_1, \ldots, x_k\}$ be a basis for $N(T)$. By the corollary to Theorem 1.12 we may extend $\{x_1, \ldots, x_k\}$ to a basis $\beta = \{x_1, \ldots, x_n\}$ for V. We shall show that the set $S = \{T(x_{k+1}), \ldots, T(x_n)\}$ is a basis for $R(T)$.

First, we shall prove that S generates $R(T)$. Let $y \in R(T)$. Then there exists $x \in V$ such that $y = T(x)$. Since β is a basis for V, we have

$$x = \sum_{i=1}^{n} a_i x_i \qquad \text{for some } a_1, \ldots, a_n \in F.$$

Since T is linear, it follows that

$$y = T(x) = \sum_{i=1}^{n} a_i T(x_i) = \sum_{i=k+1}^{n} a_i T(x_i) \in \text{span}(S).$$

The last equality follows since $x_1, \ldots, x_k \in N(T)$.

Now we shall prove that S is linearly independent. Suppose that

$$\sum_{i=k+1}^{n} b_i T(x_i) = 0 \qquad \text{for } b_{k+1}, \ldots, b_n \in F.$$

Again using the fact that T is linear, we have that

$$T\Big(\sum_{i=k+1}^{n} b_i x_i\Big) = 0.$$

So

$$\sum_{i=k+1}^{n} b_i x_i \in \mathsf{N}(\mathsf{T}).$$

Hence, there exist $c_1, \ldots, c_k \in F$ such that

$$\sum_{i=k+1}^{n} b_i x_i = \sum_{i=1}^{k} c_i x_i \quad \text{or} \quad \sum_{i=1}^{k} (-c_i) x_i + \sum_{i=k+1}^{n} b_i x_i = 0.$$

Since β is a basis for V, we have $b_i = 0$ for all i. Hence, S is linearly independent. ∎

The proof above yields the following corollary.

Corollary. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *be linear. If* V *has a basis* $\beta = \{x_1, \ldots, x_n\}$, *then* $\mathsf{R}(\mathsf{T}) = \mathit{span}(\mathsf{T}(\beta)) = \mathit{span}(\{\mathsf{T}(x_1), \ldots, \mathsf{T}(x_n)\})$.

This corollary tells us that the image of a basis for the domain of a linear transformation is a generating set for the range of the transformation. Hence this corollary provides a method for finding a basis for the range of a linear transformation. We shall employ this technique in the example below.

Example 10. Define the linear transformation $\mathsf{T}\colon \mathsf{P}_2(R) \longrightarrow \mathsf{M}_{2\times 2}(R)$ by

$$\mathsf{T}(f) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

Since $\beta = \{1, x, x^2\}$ is a basis for $\mathsf{P}_2(R)$, we have

$$\begin{aligned} \mathsf{R}(\mathsf{T}) &= \text{span}(\mathsf{T}(\beta)) = \text{span}(\{\mathsf{T}(1), \mathsf{T}(x), \mathsf{T}(x^2)\}) \\ &= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \\ &= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right). \end{aligned}$$

Thus we have found a basis for $\mathsf{R}(\mathsf{T})$, and we see that $\text{rank}(\mathsf{T}) = 2$. By Theorem 2.3 we have that $\text{nullity}(\mathsf{T}) + 2 = 3$, and so $\text{nullity}(\mathsf{T}) = 1$.

The reader should review the concepts of "one-to-one" and "onto" found in Appendix B. Interestingly, for a linear transformation both of these concepts are intimately connected with the rank and nullity of the transformation. This will be demonstrated in the next two theorems.

Theorem 2.4. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *be linear. Then* T *is one-to-one if and only if* $\mathsf{N}(\mathsf{T}) = \{0\}$.

PROOF. Suppose T is one-to-one and $x \in \mathsf{N}(\mathsf{T})$. Then $\mathsf{T}(x) = 0 = \mathsf{T}(0)$. Since T is one-to-one, we have $x = 0$. Hence $\mathsf{N}(\mathsf{T}) = \{0\}$.

Now assume that $\mathsf{N}(\mathsf{T}) = \{0\}$, and suppose that $\mathsf{T}(x) = \mathsf{T}(y)$. Then $0 = \mathsf{T}(x) - \mathsf{T}(y) = \mathsf{T}(x - y)$. Hence $x - y \in \mathsf{N}(\mathsf{T}) = \{0\}$. So $x - y = 0$, or $x = y$. This means that T is one-to-one. ∎

The reader should observe that Theorem 2.4 allows us to conclude that the transformation defined in Example 10 is not one-to-one.

Surprisingly, the conditions of one-to-one and onto are equivalent in an important special case.

Theorem 2.5. Let V *and* W *be vector spaces of equal (finite) dimension, and let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *be linear. Then* T *is one-to-one if and only if* T *is onto.*

PROOF. From Theorem 2.3 we have

$$\text{nullity}(\mathsf{T}) + \text{rank}(\mathsf{T}) = \dim(\mathsf{V}).$$

Now, with the use of Theorem 2.4, we have that T is one-to-one if and only if $\mathsf{N}(\mathsf{T}) = \{0\}$ if and only if $\text{nullity}(\mathsf{T}) = 0$ if and only if $\text{rank}(\mathsf{T}) = \dim(\mathsf{V})$ if and only if $\text{rank}(\mathsf{T}) = \dim(\mathsf{W})$ if and only if $\dim(\mathsf{R}(\mathsf{T})) = \dim(\mathsf{W})$. By Theorem 1.12 this equality is equivalent to $\mathsf{R}(\mathsf{T}) = \mathsf{W}$, the definition of T being onto. ∎

The linearity of T in Theorems 2.4 and 2.5 is essential, for it is easy to construct examples of functions from R into R that are not one-to-one but are onto and vice versa.

The following two examples make use of the theorems above in determining whether a given linear transformation is one-to-one or onto.

Example 11. Define

$$\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_3(R) \quad \text{by} \quad \mathsf{T}(f)(x) = 2f'(x) + \int_0^x 3f(t)\,dt.$$

Now

$$\mathsf{R}(\mathsf{T}) = \text{span}(\{\mathsf{T}(1), \mathsf{T}(x), \mathsf{T}(x^2)\}) = \text{span}(\{3x, 2 + \tfrac{3}{2}x^2, 4x + x^3\}).$$

Hence $\text{rank}(\mathsf{T}) = 3$. Since $\dim(\mathsf{P}_3(R)) = 4$, T is not onto. From Theorem 2.3, $\text{nullity}(\mathsf{T}) + 3 = 3$. So $\text{nullity}(\mathsf{T}) = 0$, and so $\mathsf{N}(\mathsf{T}) = \{0\}$. Thus by Theorem 2.4 T is one-to-one.

Example 12. Define

$$\mathsf{T}\colon\ \mathsf{F}^2 \longrightarrow \mathsf{F}^2 \quad \text{by} \quad \mathsf{T}(a_1, a_2) = (a_2 + a_1, a_1).$$

It is easy to see that $\mathsf{N}(\mathsf{T}) = \{0\}$; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

Our next theorem provides a characterization of one-to-one linear transformations as those linear transformations that preserve linear independence.

Theorem 2.6. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *be linear. Then* T *is one-to-one if and only if* T *carries linearly independent subsets of* V *onto linearly independent subsets of* W.

PROOF. Exercise.

Corollary. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *be a one-to-one linear transformation. Suppose that* S *is a subset of* V. *Then* S *is linearly independent if and only if* T(S) *is linearly independent.*

PROOF. Exercise.

Example 13. Define

$$\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{R}^3 \quad \text{by} \quad \mathsf{T}(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2).$$

Clearly T is one-to-one. Let $S = \{2 - x + 3x^2, x + x^2, 1 - 2x^2\}$. Then S is linearly independent in $\mathsf{P}_2(R)$ if and only if

$$\mathsf{T}(S) = \{(2, -1, 3), (0, 1, 1), (1, 0, -2)\}$$

is linearly independent in R^3.

In Example 13 we transferred a problem from the vector space of polynomials to a problem in the vector space of 3-tuples. This technique will be exploited more fully later.

One of the most important properties of linear transformations is that they are completely determined by their action on a basis. This result, which follows from the next theorem and corollary, will be used frequently throughout the book.

Theorem 2.7. *Let* V *and* W *be vector spaces, and suppose that* V *is a finite-dimensional vector space with basis* $\{x_1, \ldots, x_n\}$. *For any subset* $\{y_1, \ldots, y_n\}$ *of* W *there exists exactly one linear transformation* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *such that* $\mathsf{T}(x_i) = y_i$ *for* $i = 1, \ldots, n$.

PROOF. Let $x \in \mathsf{V}$. Then

$$x = \sum_{i=1}^{n} a_i x_i,$$

where $a_1, \ldots, a_n$ are unique scalars. Define

$$\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W} \qquad \text{by } \mathsf{T}(x) = \sum_{i=1}^{n} a_i y_i.$$

(a) T is linear: For suppose $u, v \in V$ and $d \in F$. Then we may write

$$u = \sum_{i=1}^{n} b_i x_i \quad \text{and} \quad v = \sum_{i=1}^{n} c_i x_i.$$

Now

$$du + v = \sum_{i=1}^{n} (db_i + c_i)x_i.$$

So

$$\mathsf{T}(du + v) = \sum_{i=1}^{n} (db_i + c_i)y_i = d\sum_{i=1}^{n} b_i y_i + \sum_{i=1}^{n} c_i y_i = d\mathsf{T}(u) + \mathsf{T}(v).$$

(b) Clearly

$$\mathsf{T}(x_i) = y_i \qquad \text{for } i = 1, \ldots, n.$$

(c) T is unique: For suppose $\mathsf{U}\colon \mathsf{V} \longrightarrow \mathsf{W}$ is linear and $\mathsf{U}(x_i) = y_i$ for $i = 1, \ldots, n$. Then for $x \in \mathsf{V}$ with

$$x = \sum_{i=1}^{n} a_i x_i$$

we have

$$\mathsf{U}(x) = \sum_{i=1}^{n} a_i \mathsf{U}(x_i) = \sum_{i=1}^{n} a_i y_i = \mathsf{T}(x).$$

Hence U = T. ∎

Corollary. *Let* V *and* W *be vector spaces, and suppose that* V *is finite-dimensional with basis* $\{x_1, \ldots, x_n\}$. *If* $\mathsf{U}, \mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *are linear and* $\mathsf{U}(x_i) = \mathsf{T}(x_i)$ *for* $i = 1, \ldots, n$, *then* U = T.

Example 14. Define $\mathsf{T}\colon \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ by $\mathsf{T}(a_1, a_2) = (2a_2 - a_1, 3a_1)$, and suppose $\mathsf{U}\colon \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ is linear. If we know that $\mathsf{U}(1, 2) = (3, 3)$ and $\mathsf{U}(1, 1) = (1, 3)$, then U = T. This follows from the corollary and from the fact that $\{(1, 2), (1, 1)\}$ is a basis for R^2.

EXERCISES

1. Label the following statements as being true or false. For the following, V and W are finite-dimensional vector spaces (over F) and T is a function from V into W.
   (a) If T is linear, then T preserves sums and scalar products.
   (b) If $\mathsf{T}(x + y) = \mathsf{T}(x) + \mathsf{T}(y)$, then T is linear.
   (c) T is one-to-one if and only if $\mathsf{N}(\mathsf{T}) = \{0\}$.
   (d) All projections must be linear.
   (e) If T is linear, then $\mathsf{T}(0_{\mathsf{V}}) = 0_{\mathsf{W}}$.
   (f) If T is linear, then nullity(T) + rank(T) = dim(W).

(g) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.
(h) If T, U: V $\longrightarrow$ W are both linear and agree on a basis of V, then T = U.
(i) Given $x_1, x_2 \in$ V and $y_1, y_2 \in$ W, there exists a linear transformation T: V $\longrightarrow$ W such that $\mathsf{T}(x_1) = y_1$ and $\mathsf{T}(x_2) = y_2$.

For Exercises 2 through 6, prove that T is a linear transformation and find bases for both N(T) and R(T). Then compute the nullity and rank of T and verify Theorem 2.3. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

2. T: $\mathsf{R}^3 \longrightarrow \mathsf{R}^2$; $\mathsf{T}(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

3. T: $\mathsf{R}^2 \longrightarrow \mathsf{R}^3$; $\mathsf{T}(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

4. T: $\mathsf{M}_{2\times 3}(F) \longrightarrow \mathsf{M}_{2\times 2}(F)$; $\mathsf{T}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$.

5. T: $\mathsf{P}_2(R) \longrightarrow \mathsf{P}_3(R)$; $\mathsf{T}(f(x)) = xf(x) + f'(x)$.

6. T: $\mathsf{M}_{n\times n} \longrightarrow F$; $\mathsf{T}(A) = \mathrm{tr}(A)$. Recall that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

7. Verify statements 1, 2, and 3 on page 58.

8. Verify that the transformations defined in Examples 5, 6, and 7 are linear.

9. For the following T: $\mathsf{R}^2 \longrightarrow \mathsf{R}^2$, state why T is *not* linear.
(a) $\mathsf{T}(a_1, a_2) = (1, a_2)$
(b) $\mathsf{T}(a_1, a_2) = (a_1, a_1^2)$
(c) $\mathsf{T}(a_1, a_2) = (\sin a_1, 0)$
(d) $\mathsf{T}(a_1, a_2) = (|a_1|, a_2)$
(e) $\mathsf{T}(a_1, a_2) = (a_1 + 1, a_2)$

10. Suppose T: $\mathsf{R}^2 \longrightarrow \mathsf{R}^2$ is linear and that $\mathsf{T}(1, 0) = (1, 4)$ and $\mathsf{T}(1, 1) = (2, 5)$. What is $\mathsf{T}(2, 3)$? Is T one-to-one?

11. Prove that there exists a linear transformation T: $\mathsf{R}^2 \longrightarrow \mathsf{R}^3$ such that $\mathsf{T}(1, 1) = (1, 0, 2)$ and $\mathsf{T}(2, 3) = (1, -1, 4)$. What is $\mathsf{T}(8, 11)$?

12. Is there a linear transformation T: $\mathsf{R}^3 \longrightarrow \mathsf{R}^2$ such that $\mathsf{T}(1, 0, 3) = (1, 1)$ and $\mathsf{T}(-2, 0, -6) = (2, 1)$?

13. Prove Theorem 2.6 and its corollary.

14. Suppose that T is a projection on a subspace W of a vector space V. Prove that $\mathsf{W} = \{x \in \mathsf{V}\colon\ \mathsf{T}(x) = x\}$.

15. Recall the definition of $\mathsf{P}(R)$ in Section 1.2. Define

$$\mathsf{T}\colon\ \mathsf{P}(R) \longrightarrow \mathsf{P}(R) \quad \text{by} \quad \mathsf{T}(f)(x) = \int_0^x f(t)\,dt.$$

Prove that T is one-to-one but not onto.

16. Let V and W be finite-dimensional vector spaces and $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be linear.
   (a) Prove that if $\dim(\mathsf{V}) < \dim(\mathsf{W})$, then T cannot be onto.
   (b) Prove that if $\dim(\mathsf{V}) > \dim(\mathsf{W})$, then T cannot be one-to-one.

17. Give an example of a linear transformation $\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ such that $\mathsf{N}(\mathsf{T}) = \mathsf{R}(\mathsf{T})$.

18. Give an example of distinct linear transformations T and U such that $\mathsf{N}(\mathsf{T}) = \mathsf{N}(\mathsf{U})$ and $\mathsf{R}(\mathsf{T}) = \mathsf{R}(\mathsf{U})$.

19. Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ is linear, prove that $\mathsf{T}(\mathsf{V}_1)$ is a subspace of W and $\{x \in \mathsf{V}\colon \mathsf{T}(x) \in \mathsf{W}_1\}$ is a subspace of V.

20. Let W be a subspace of a finite-dimensional vector space V. Prove that there exists a projection on W.

21. Let V and W be vector spaces and $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be linear. Let $\{y_1, \ldots, y_k\}$ be a linearly independent subset of $\mathsf{R}(\mathsf{T})$. If $S = \{x_1, \ldots, x_k\}$ is chosen so that $\mathsf{T}(x_i) = y_i$ for $i = 1, \ldots, k$, prove that S is linearly independent.

22. Let $\mathsf{T}\colon\ \mathsf{R}^3 \longrightarrow R$ be linear. Show that there exist scalars a, b, and c such that $\mathsf{T}(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathsf{R}^3$. Can you generalize this result for $\mathsf{T}\colon\ \mathsf{F}^n \longrightarrow F$? State and prove an analogous result for $\mathsf{T}\colon\ \mathsf{F}^n \longrightarrow \mathsf{F}^m$.

23. Let $\mathsf{T}\colon\ \mathsf{R}^3 \longrightarrow R$ be linear. Describe geometrically the possibilities for the null space of T. *Hint:* Use Exercise 22.

24. Let V be a vector space, and let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ be linear. A subspace W of V is said to be T-*invariant* if $\mathsf{T}(x) \in \mathsf{W}$ for every $x \in \mathsf{W}$, i.e., $\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W}$.
   (a) Prove that the subspaces $\{0\}$, V, $\mathsf{R}(\mathsf{T})$, and $\mathsf{N}(\mathsf{T})$ are all T-invariant.
   (b) If W is a T-invariant subspace of V, define $\mathsf{T}_\mathsf{W}\colon\ \mathsf{W} \longrightarrow \mathsf{W}$ by $\mathsf{T}_\mathsf{W}(x) = \mathsf{T}(x)$ for all $x \in \mathsf{W}$. Prove that T_W is linear.
   (c) If T is a projection on W, show that W is T-invariant and that $\mathsf{T}_\mathsf{W} = \mathsf{I}_\mathsf{W}$.
   (d) If $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{W}$ and W is T-invariant, prove that $\mathsf{W} \subseteq \mathsf{N}(\mathsf{T})$. Show that if V is also finite-dimensional, then $\mathsf{W} = \mathsf{N}(\mathsf{T})$.
   (e) Prove that $\mathsf{N}(\mathsf{T}_\mathsf{W}) = \mathsf{N}(\mathsf{T}) \cap \mathsf{W}$ and $\mathsf{R}(\mathsf{T}_\mathsf{W}) = \mathsf{T}(\mathsf{W})$.

25. Prove the following generalization of Theorem 2.7 to infinite-dimensional spaces: Let V and W be vector spaces and β be a basis for V. Then for any

function $f\colon \beta \longrightarrow \mathsf{W}$ there exists exactly one linear transformation $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ such that $\mathsf{T}(x) = f(x)$ for all $x \in \beta$.

26. A function $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ between vector spaces V and W is called *additive* if $\mathsf{T}(x + y) = \mathsf{T}(x) + \mathsf{T}(y)$ for all $x, y \in \mathsf{V}$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

27. Prove that there is an additive function $\mathsf{T}\colon \mathsf{R} \longrightarrow \mathsf{R}$ (as defined in Exercise 26) that is not linear. *Hint:* Regard R as a vector space over the field of rational numbers Q. By the corollary to Theorem 1.15 this vector space has a basis β. Let x and y be distinct elements of β, and define $f\colon \beta \longrightarrow \mathsf{R}$ by $f(x) = y, f(y) = x$, and $f(z) = z$ otherwise. By Exercise 26 there exists a linear transformation $\mathsf{T}\colon \mathsf{R} \longrightarrow \mathsf{R}$, where R is regarded as a vector space over Q, such that $\mathsf{T}(z) = f(z)$ for all $z \in \beta$. Then T is additive, but for $c = y/x$, $\mathsf{T}(cx) \neq c\mathsf{T}(x)$.

2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

Until now we have studied linear transformations by examining their ranges and null spaces. We shall now embark upon one of the most useful approaches to the analysis of a linear transformation on a finite-dimensional vector space, the representation of a linear transformation by a matrix. In fact, we shall develop a one-to-one correspondence between matrices and transformations that will allow us to utilize properties of one to study properties of the other.

We shall first need the concept of an "ordered basis" for a vector space.

Definition. *Let* V *be a finite-dimensional vector space. An* ordered basis for V *is a basis for* V *endowed with a specified order; i.e., an ordered basis for* V *is a finite sequence of linearly independent elements of* V *that generate* V.

Example 15. Let V have $\beta = \{x_1, x_2, x_3\}$ as an ordered basis. Then $\gamma = \{x_2, x_1, x_3\}$ is also an ordered basis, but $\beta \neq \gamma$ as ordered bases.

For the vector space F^n we shall call $\{e_1, e_2, \ldots, e_n\}$ *the standard ordered basis for* F^n.

Now that we have the concept of an ordered basis we shall be able to identify abstract vectors in an n-dimensional vector space with n-tuples. This identification will be provided through the use of "coordinate vectors" as introduced below.

Definition. *Let $\beta = \{x_1, \ldots, x_n\}$ be an ordered basis for a finite-dimensional vector space* V. *For* $x \in V$ *we define the* coordinate vector of x relative to β, *denoted* $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix},$$

where

$$x = \sum_{i=1}^{n} a_i x_i.$$

Notice that $[x_i]_\beta = e_i$ in the definition above. It is left as an exercise to show that the correspondence $x \longrightarrow [x]_\beta$ provides us with a linear transformation from V to F^n. We shall study this transformation in Section 2.4 in more detail.

Example 16. Let $V = P_2(R)$, and let $\beta = \{1, x, x^2\}$. If $f(x) = 4 + 6x - 7x^2$, then

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}.$$

Let us now proceed with the promised matrix representation of a linear transformation. Suppose that V and W are finite-dimensional vector spaces with ordered bases $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$, respectively. Let $T\colon V \longrightarrow W$ be linear. Then there exist unique scalars $a_{ij} \in F$ ($i = 1, \ldots, m$ and $j = 1, \ldots, n$) such that

$$T(x_j) = \sum_{i=1}^{m} a_{ij} y_i \qquad \text{for } 1 \leq j \leq n.$$

Definition. *Using the notation above, we call the* $m \times n$ *matrix* A *defined by* $A_{ij} = a_{ij}$ *the* matrix that represents T in the ordered bases β and γ *and will write* $A = [T]_\beta^\gamma$. *If* $V = W$ *and* $\beta = \gamma$, *we shall write* $A = [T]_\beta$.

Notice that the jth column of A is simply $[T(x_j)]_\gamma$. Also observe that from the corollary of Theorem 2.7 it follows that if $U\colon V \longrightarrow W$ is a linear transformation such that $[U]_\beta^\gamma = [T]_\beta^\gamma$, then $U = T$.

We shall illustrate the computation of $[T]_\beta^\gamma$ in the next several examples.

Example 17. Define

$$T\colon\ P_3(R) \longrightarrow P_2(R) \quad \text{by} \quad T(f) = f'.$$

Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ be ordered bases for $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$, respectively. Then

$$\begin{aligned} \mathsf{T}(1) &= 0\cdot 1 + 0\cdot x + 0\cdot x^2 \\ \mathsf{T}(x) &= 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ \mathsf{T}(x^2) &= 0\cdot 1 + 2\cdot x + 0\cdot x^2 \\ \mathsf{T}(x^3) &= 0\cdot 1 + 0\cdot x + 3\cdot x^2 \end{aligned}$$

So

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Note that the coefficients of $\mathsf{T}(x^i)$ when written as a linear combination of elements of γ give the entries of the ith column.

Example 18. Define

$$\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^3 \text{ by } \mathsf{T}(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Let β and γ be the standard ordered bases for R^2 and R^3, respectively. Now

$$\mathsf{T}(1, 0) = (1, 0, 2) = 1e_1 + 0e_2 + 2e_3$$

and

$$\mathsf{T}(0, 1) = (3, 0, -4) = 3e_1 + 0e_2 - 4e_3.$$

Hence

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}.$$

If we let $\gamma' = \{e_3, e_2, e_1\}$, then

$$[\mathsf{T}]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}.$$

Now that we have defined a procedure for associating matrices with linear transformations, we shall see shortly that this association "preserves" addition. To make this more explicit, we shall need some preliminary discussion about the addition of linear transformations.

Definition. *Let* $\mathsf{T}, \mathsf{U}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *be arbitrary functions, where* V *and* W *are vector spaces, and let* $a \in F$. *We define* $\mathsf{T} + \mathsf{U}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *by* $(\mathsf{T} + \mathsf{U})(x) = \mathsf{T}(x) + \mathsf{U}(x)$ *for all* $x \in \mathsf{V}$, *and* $a\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *by* $(a\mathsf{T})(x) = a\mathsf{T}(x)$ *for all* $x \in \mathsf{V}$.

Of course, this is just the usual definition of addition and scalar multiplication for functions. We are fortunate, however, to have the result that the sum of linear transformations is linear.

Theorem 2.8. *Let* V *and* W *be vector spaces, and let* T, U: V $\longrightarrow$ W *be linear. Then for all* a $\in$ F

(a) aT + U *is linear.*

(b) *Using the operations of addition and scalar multiplication as defined above, the collection of all linear transformations from* V *into* W*, denoted* $\mathcal{L}$(V, W)*, is a vector space over* F.

PROOF.

(a) Let $x, y \in$ V and $c \in F$. Then

$$\begin{aligned}(a\mathsf{T} + \mathsf{U})(cx + y) &= a\mathsf{T}(cx + y) + \mathsf{U}(cx + y)\\ &= a[c\mathsf{T}(x) + \mathsf{T}(y)] + c\mathsf{U}(x) + \mathsf{U}(y)\\ &= ac\mathsf{T}(x) + c\mathsf{U}(x) + a\mathsf{T}(y) + \mathsf{U}(y)\\ &= c[a\mathsf{T} + \mathsf{U}](x) + [a\mathsf{T} + \mathsf{U}](y).\end{aligned}$$

So $a\mathsf{T} + \mathsf{U}$ is linear.

(b) Noting that T_0, the zero transformation, plays the role of the zero element in $\mathcal{L}$(V, W), it is easy to show that $\mathcal{L}$(V, W) is a vector space over F. ∎

In the case where V = W we shall write $\mathcal{L}$(V) instead of $\mathcal{L}$(V, V).

In the next section we shall see a complete identification of $\mathcal{L}$(V, W) with the vector space $\mathsf{M}_{m\times n}(F)$, where n and m are the dimensions of V and W, respectively. This identification is easily established by use of the next theorem.

Theorem 2.9. *Let* V *and* W *be finite-dimensional vector spaces with ordered bases* β *and* γ*, respectively, and let* T, U: V $\longrightarrow$ W *be linear transformations. Then*

(a) $[\mathsf{T} + \mathsf{U}]_\beta^\gamma = [\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$.

(b) $[a\mathsf{T}]_\beta^\gamma = a[\mathsf{T}]_\beta^\gamma$ *for all* a $\in$ F.

PROOF. Let $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$. There exist unique scalars a_{ij} and b_{ij} in F $(1 \le i \le m, 1 \le j \le n)$ such that

$$\mathsf{T}(x_j) = \sum_{i=1}^m a_{ij}y_i \quad \text{and} \quad \mathsf{U}(x_j) = \sum_{i=1}^m b_{ij}y_i \qquad \text{for } 1 \le j \le n.$$

Hence

$$(\mathsf{T} + \mathsf{U})(x_j) = \sum_{i=1}^m (a_{ij} + b_{ij})y_i.$$

Thus

$$([\mathsf{T} + \mathsf{U}]_\beta^\gamma)_{ij} = a_{ij} + b_{ij} = ([\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma)_{ij}.$$

So (a) is proved, and the proof of (b) is similar. ∎

Example 19. Define

$$\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^3 \text{ by } \mathsf{T}(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1, -4a_2)$$

and

$$\mathsf{U}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^3 \text{ by } \mathsf{U}(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2).$$

Let β and γ be the standard ordered bases of R^2 and R^3, respectively. Then

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix},$$

(as computed in Example 18) and

$$[\mathsf{U}]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}.$$

If we now compute $\mathsf{T} + \mathsf{U}$ using the definitions above, we obtain

$$(\mathsf{T} + \mathsf{U})(a_1, a_2) = (2a_1 + 2a_2, 2a_1, 5a_1 - 2a_2).$$

So

$$[\mathsf{T} + \mathsf{U}]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix},$$

which is simply $[\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$, verifying Theorem 2.9.

EXERCISES

1. Label the following statements as being true or false. For the following we shall let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Assume $\mathsf{T}, \mathsf{U}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ are linear.
   (a) For any scalar a, $a\mathsf{T} + \mathsf{U}$ is a linear transformation from V into W.
   (b) $[\mathsf{T}]_\beta^\gamma = [\mathsf{U}]_\beta^\gamma$ implies $\mathsf{T} = \mathsf{U}$.
   (c) If $m = \dim(\mathsf{V})$ and $n = \dim(\mathsf{W})$, then $[\mathsf{T}]_\beta^\gamma$ is an $m \times n$ matrix.
   (d) $[\mathsf{T} + \mathsf{U}]_\beta^\gamma = [\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$.
   (e) $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is a vector space.
   (f) $\mathcal{L}(\mathsf{V}, \mathsf{W}) = \mathcal{L}(\mathsf{W}, \mathsf{V})$.

2. Let β and γ be the standard ordered bases for R^n and R^m, respectively. For the following transformations $\mathsf{T}\colon\ \mathsf{R}^n \longrightarrow \mathsf{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.
   (a) $\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.
   (b) $\mathsf{T}\colon\ \mathsf{R}^3 \longrightarrow \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.
   (c) $\mathsf{T}\colon\ \mathsf{R}^3 \longrightarrow R$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.

3. Let $\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^3$ be defined as $\mathsf{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for R^2 and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $[\mathsf{T}]_\beta^\gamma$. If $\alpha = \{(1, 2), (2, 3)\}$, compute $[\mathsf{T}]_\alpha^\gamma$.

4. Define

$$\mathsf{T}\colon\ \mathsf{M}_{2\times 2}(R) \longrightarrow \mathsf{P}_2(R) \quad \text{by} \quad \mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \quad \text{and} \quad \gamma = \{1, x, x^2\}.$$

Compute $[\mathsf{T}]_\beta^\gamma$.

5. For the following parts, let

$$\alpha = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

$$\beta = \{1, x, x^2\},$$

and

$$\gamma = \{1\}.$$

(a) Define $\mathsf{T}\colon\ \mathsf{M}_{2\times 2}(F) \longrightarrow \mathsf{M}_{2\times 2}(F)$ by $\mathsf{T}(A) = A^t$. Compute $[\mathsf{T}]_\alpha$.

(b) Define

$$\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{M}_{2\times 2}(R) \quad \text{by} \quad \mathsf{T}(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[\mathsf{T}]_\beta^\alpha$.

(c) Define $\mathsf{T}\colon\ \mathsf{M}_{2\times 2}(F) \longrightarrow F$ by $\mathsf{T}(A) = \mathrm{tr}(A)$. Compute $[\mathsf{T}]_\alpha^\gamma$.

(d) Define $\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow R$ by $\mathsf{T}(f) = f(2)$. Compute $[\mathsf{T}]_\beta^\gamma$.

(e) If

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix},$$

compute $[A]_\alpha$.

(f) If $f(x) = 3 - 6x + x^2$, compute $[f]_\beta$.

(g) For $a \in F$, compute $[a]_\gamma$.

6. Prove (b) of Theorem 2.9.

7.† Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.

8. Let V be the vector space of complex numbers over the field R. If $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ is defined by $\mathsf{T}(z) = \bar{z}$, where $\bar{z}$ is the complex conjugate of z, prove that T is linear, and compute $[\mathsf{T}]_\beta$, where $\beta = \{1, i\}$. Show that T is not linear if V is regarded as a vector space over the field C.

9. Let V be a vector space with the ordered basis $\beta = \{x_1, \ldots, x_n\}$. Define $x_0 = 0$. By Theorem 2.7 there exists a linear transformation $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ defined by $\mathsf{T}(x_j) = x_j + x_{j-1}$ for $j = 1, \ldots, n$. Compute $[\mathsf{T}]_\beta$.

10. Let V be an n-dimensional vector space, and let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see Exercise 24 of Section 2.1) having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form

$$\begin{bmatrix} A & B \\ O & C \end{bmatrix},$$

where A is a $k \times k$ matrix and O is an $(n - k) \times k$ zero matrix.

11. Let V be a vector space of finite dimension, and let T be a projection on a subspace W of V. Choose the appropriate ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

12. Let V and W be vector spaces, and let T and U be non-zero linear transformations from V into W. If $\mathsf{R}(\mathsf{T}) \cap \mathsf{R}(\mathsf{U}) = \{0\}$, prove that $\{\mathsf{T}, \mathsf{U}\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

13. Let $\mathsf{V} = \mathsf{P}(R)$, and for $j \geq 0$ define $\mathsf{T}_j\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{T}_j(f) = f^{(j)}$, where $f^{(j)}$ is the jth derivative of f. For any positive integer n, prove that $\{\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_n\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V})$.

14. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})\colon\ \mathsf{T}(x) = 0 \text{ for all } x \in S\}$. Prove
(a) S^0 is a subspace of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.
(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.
(c) If V_1 and V_2 are subspaces of V, then $(\mathsf{V}_1 + \mathsf{V}_2)^0 = \mathsf{V}_1^0 \cap \mathsf{V}_2^0$.

15. Let V and W be finite-dimensional vector spaces, and let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be linear. Assume that $\dim(\mathsf{V}) = \dim(\mathsf{W})$. Find ordered bases β and γ for V and W, respectively, such that $[\mathsf{T}]_\beta^\gamma$ is a diagonal matrix.

2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

In Section 2.2 we learned how to associate a matrix with a linear transformation in such a way that sums of matrices were associated with the corresponding sums of transformations. The question now arises as to how the matrix representation of a composition of linear transformations is related to the matrix representations of each of the associated linear transformations. The attempt to answer this question will lead to a definition of matrix multiplication. We shall use the notation UT for composition of linear transformations U and T as contrasted with $g \circ f$ for arbitrary functions g and f. Specifically, we have the following definition.

Definition. *Let* V, W, *and* Z *be vector spaces, and let* $\mathsf{T}: \mathsf{V} \longrightarrow \mathsf{W}$ *and* $\mathsf{U}: \mathsf{W} \longrightarrow \mathsf{Z}$ *be linear. We define* $\mathsf{UT}: \mathsf{V} \longrightarrow \mathsf{Z}$ *by* $(\mathsf{UT})(\mathsf{x}) = \mathsf{U}(\mathsf{T}(\mathsf{x}))$ *for all* $\mathsf{x} \in \mathsf{V}$.

Our first result shows that the composition of linear transformations is linear.

Theorem 2.10. *Let* V, W, *and* Z *be vector spaces and* $\mathsf{T}: \mathsf{V} \longrightarrow \mathsf{W}$ *and* $\mathsf{U}: \mathsf{W} \longrightarrow \mathsf{Z}$ be linear. Then $\mathsf{UT}: \mathsf{V} \longrightarrow \mathsf{Z}$ is linear.

PROOF. Let $x, y \in \mathsf{V}$ and $a \in F$. Then

$$\begin{aligned}\mathsf{UT}(ax + y) &= \mathsf{U}(\mathsf{T}(ax + y)) = \mathsf{U}(a\mathsf{T}(x) + \mathsf{T}(y)) \\ &= a\mathsf{U}(\mathsf{T}(x)) + \mathsf{U}(\mathsf{T}(y)) = a(\mathsf{UT})(x) + \mathsf{UT}(y). \quad \blacksquare\end{aligned}$$

The following theorem lists some properties of the composition of linear transformations.

Theorem 2.11. *Let* V *be a vector space. Let* $\mathsf{T}, \mathsf{U}_1, \mathsf{U}_2 \in \mathcal{L}(\mathsf{V})$. *Then*

(a) $\mathsf{T}(\mathsf{U}_1 + \mathsf{U}_2) = \mathsf{TU}_1 + \mathsf{TU}_2$ *and* $(\mathsf{U}_1 + \mathsf{U}_2)\mathsf{T} = \mathsf{U}_1\mathsf{T} + \mathsf{U}_2\mathsf{T}$.
(b) $\mathsf{T}(\mathsf{U}_1\mathsf{U}_2) = (\mathsf{TU}_1)\mathsf{U}_2$.
(c) $\mathsf{TI} = \mathsf{IT} = \mathsf{T}$.
(d) $\mathrm{a}(\mathsf{U}_1\mathsf{U}_2) = (\mathrm{a}\mathsf{U}_1)(\mathsf{U}_2) = \mathsf{U}_1(\mathrm{a}\mathsf{U}_2)$ *for all* $\mathrm{a} \in \mathrm{F}$.

PROOF. Exercise.

We are now in a position to define the product AB of two matrices A and B. Because of Theorem 2.9, it seems reasonable by analogy to require that if $A = [\mathsf{U}]_\beta^\gamma$ and $B = [\mathsf{T}]_\alpha^\beta$, where $\mathsf{T}: \mathsf{V} \longrightarrow \mathsf{W}$ and $\mathsf{U}: \mathsf{W} \longrightarrow \mathsf{Z}$, then $AB = [\mathsf{UT}]_\alpha^\gamma$.

Now let T, U, A, and B be as above, and let $\alpha = \{x_1, \ldots, x_n\}$, $\beta = \{y_1, \ldots, y_m\}$, and $\gamma = \{z_1, \ldots, z_p\}$ be ordered bases for V, W, and Z, respectively. For $1 \leq j \leq n$ we have

$$\begin{aligned}(\mathsf{UT})(x_j) = \mathsf{U}(\mathsf{T}(x_j)) &= \mathsf{U}\left(\sum_{k=1}^m B_{kj}y_k\right) = \sum_{k=1}^m B_{kj}\mathsf{U}(y_k) \\ &= \sum_{k=1}^m B_{kj}\left(\sum_{i=1}^p A_{ik}z_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m A_{ik}B_{kj}\right)z_i \\ &= \sum_{i=1}^p C_{ij}z_i,\end{aligned}$$

where

$$C_{ij} = \sum_{k=1}^m A_{ik}B_{kj}.$$

This computation suggests the following definition of matrix multiplication.

Definition. *Let* A *be an* m × n *matrix and* B *be an* n × p *matrix. We define the* product of A and B, *denoted* AB, *to be the* m × p *matrix such that*

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \qquad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq p.$$

Note that $(AB)_{ij}$ is the sum of products of corresponding elements from the ith row of A and the jth column of B.

At the end of this section the reader will see some interesting applications of this definition.

The reader should observe that in order for the product AB to be defined, there are restrictions regarding the relative sizes of A and B. The following symbolic device can be useful: "$(m \times n)\cdot(n \times p) = (m \times p)$"; that is, in order for the product AB to be defined, the two "inner" dimensions must be equal and the two "outer" dimensions yield the size of the product.

Example 20.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4+4+5 \\ 0+8-5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}$$

Notice again the symbolic relationship $(2 \times 3)\cdot(3 \times 1) = 2 \times 1$.

As in the case with composition of functions, we have that matrix multiplication is not commutative. Consider the following two products.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}.$$

Hence we see that even if both of the matrix products AB and BA are defined, it need not be true that $AB = BA$.

Recalling the definition of the transpose of a matrix from Section 1.3, we shall show that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then $(AB)^t = B^tA^t$. Since

$$(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki}$$

and

$$(B^tA^t)_{ij} = \sum_{k=1}^{n} (B^t)_{ik}(A^t)_{kj} = \sum_{k=1}^{n} B_{ki}A_{jk},$$

we are done. Hence the transpose of a product is the product of the transposes *in the opposite order*.

The following theorem is an immediate consequence of our definition of matrix multiplication.

Theorem 2.12. *Let* V, W, *and* Z *be finite-dimensional vector spaces with ordered bases* α, β, *and* γ, *respectively. Let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *and* $\mathsf{U}\colon \mathsf{W} \longrightarrow \mathsf{Z}$ *be linear transformations. Then*

$$[\mathsf{UT}]_\alpha^\gamma = [\mathsf{U}]_\beta^\gamma [\mathsf{T}]_\alpha^\beta.$$

Corollary. *Let* V *be a finite-dimensional vector space with an ordered basis* β. *Let* $\mathsf{T}, \mathsf{U} \in \mathcal{L}(\mathsf{V})$. *Then* $[\mathsf{UT}]_\beta = [\mathsf{U}]_\beta [\mathsf{T}]_\beta$.

We shall illustrate the above in the following example.

Example 21. Define

$$\mathsf{U}\colon\ \mathsf{P}_3(R) \longrightarrow \mathsf{P}_2(R) \quad \text{by} \quad \mathsf{U}(f) = f'$$

as in Example 17. Define

$$\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_3(R) \quad \text{by} \quad \mathsf{T}(f)(x) = \int_0^x f(t)\,dt.$$

Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2\}$. Clearly, $\mathsf{UT} = \mathsf{I}$. To illustrate Theorem 2.12, observe that

$$[\mathsf{UT}]_\beta = [\mathsf{U}]_\alpha^\beta [\mathsf{T}]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [\mathsf{I}]_\beta.$$

The 3×3 diagonal matrix above is called an "identity matrix" and is defined below along with a very useful notation, the "Kronecker delta".

Definitions. *We define the* Kronecker delta δ_{ij} *by* $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *if* $i \neq j$ *and the* $n \times n$ identity matrix I_n *by* $(I_n)_{ij} = \delta_{ij}$.

Thus

$$I_1 = (1), \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall see in the next theorem that the identity matrix acts as a unity element in $\mathsf{M}_{n \times n}(F)$. When the context is sufficiently clear, we shall sometimes omit the subscript n from I_n.

Theorem 2.13. *For any* n × n *matrix* A *we have* $I_nA = AI_n = A$. *Furthermore, if* V *is a finite-dimensional vector space of dimension* n *with an ordered basis* β, *then* $[I_V]_\beta = I_n$.

PROOF.

$$(I_nA)_{ij} = \sum_{k=1}^{n} (I_n)_{ik}A_{kj} = \sum_{k=1}^{n} \delta_{ik}A_{kj} = A_{ij}.$$

Hence $I_nA = A$. Similarly $AI_n = A$. Let $\beta = \{x_1, \ldots, x_n\}$. Then for each j we have

$$I_V(x_j) = x_j = \sum_{i=1}^{n} \delta_{ij}x_i.$$

Hence $[I_V]_\beta = I_n$. ∎

For an $n \times n$ matrix A we define $A^2 = AA$, $A^3 = A^2A$, and in general, $A^k = A^{k-1}A$ for $k = 2, 3, \ldots$. We define $A^0 = I_n$.

With this notation we see that if

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then $A^2 = O$ (the zero matrix) even though $A \neq O$. Thus the cancellation property for fields is not valid for matrices. The next theorem shows, however, that matrix multiplication does distribute over addition.

Theorem 2.14. *Let* A *be an* m × n *matrix, and let* B *and* C *be* n × p *matrices. Then*

$$A(B + C) = AB + AC,$$

and for any scalar a

$$a(AB) = (aA)B = A(aB).$$

PROOF.

$$\begin{aligned}[A(B + C)]_{ij} &= \sum_{k=1}^{n} A_{ik}(B + C)_{kj} = \sum_{k=1}^{n} A_{ik}(B_{kj} + C_{kj}) \\ &= \sum_{k=1}^{n} (A_{ik}B_{kj} + A_{ik}C_{kj}) = \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}C_{kj} \\ &= (AB)_{ij} + (AC)_{ij} = [AB + AC]_{ij}.\end{aligned}$$

The remainder of the proof is left as an exercise. ∎

Corollary. *Let* A *be an* m × n *matrix,* $B_1, \ldots, B_k$ *be* n × p *matrices, and* $a_1, \ldots, a_k \in F$. *Then*

$$A\left(\sum_{i=1}^{k} a_iB_i\right) = \sum_{i=1}^{k} a_iAB_i.$$

PROOF. Exercise.

If A is an $m \times n$ matrix, we shall sometimes write $A = (A^1, \ldots, A^n)$, where A^j is the jth column

$$\begin{pmatrix} A_{1j} \\ A_{2j} \\ \cdot \\ \cdot \\ \cdot \\ A_{mj} \end{pmatrix}$$

of the matrix A.

For the following theorem, e_j will denote the jth column of I_p.

Theorem 2.15. *Let* A *be an* $m \times n$ *matrix and* B *be an* $n \times p$ *matrix. Then*

(a) $(AB)^j = AB^j$.

(b) $B^j = Be_j$.

PROOF.

$$(AB)^j = \begin{pmatrix} (AB)_{1j} \\ \cdot \\ \cdot \\ \cdot \\ (AB)_{mj} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} A_{1k}B_{kj} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{k=1}^{n} A_{mk}B_{kj} \end{pmatrix} = A\begin{pmatrix} B_{1j} \\ \cdot \\ \cdot \\ \cdot \\ B_{nj} \end{pmatrix} = AB^j.$$

Hence (a) is proved. The proof of (b) is left as an exercise. ▮

The next result will justify much of our past work. It will utilize both the matrix representation of a linear transformation and matrix multiplication in order to evaluate the transformation at any given vector.

Theorem 2.16. *Let* V *and* W *be finite-dimensional vector spaces having ordered bases* β *and* γ, *respectively, and let* $T\colon\ V \longrightarrow W$ *be a linear transformation. Then for each* $x \in V$ *we have*

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta.$$

PROOF. Let $\beta = \{x_1, \ldots, x_n\}$, and let $A = [T]_\beta^\gamma$. Since

$$\left[T\left(\sum_{i=1}^{n} a_i x_i\right)\right]_\gamma = \left[\sum_{i=1}^{n} a_i T(x_i)\right]_\gamma = \sum_{i=1}^{n} a_i [T(x_i)]_\gamma$$

by Exercise 7 of Section 2.2, it suffices by the corollary to Theorem 2.14 to prove the theorem for $x = x_j$ $(1 \le j \le n)$. But this follows from the definition of A and Theorem 2.15 since

$$[T(x_j)]_\gamma = A^j = Ae_j = A[x_j]_\beta = [T]_\beta^\gamma [x_j]_\beta. \quad ▮$$

Example 22. Let $T\colon\ P_3(R) \longrightarrow P_2(R)$ be defined by $T(f) = f'$, and let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ be ordered bases for $P_3(R)$ and $P_2(R)$, respectively. If $A = [\mathsf{T}]_\beta^\gamma$, then we have from Example 17 that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

We shall illustrate Theorem 2.16 by verifying that $[\mathsf{T}(p)]_\gamma = [\mathsf{T}]_\beta^\gamma[p]_\beta$, where $p \in \mathsf{P}_3(R)$ is the polynomial $p(x) = 2 - 4x + x^2 + 3x^3$. Let $q = \mathsf{T}(p)$; then $q(x) = p'(x) = -4 + 2x + 9x^2$. Hence

$$[\mathsf{T}(p)]_\gamma = [q]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}.$$

But also

$$[\mathsf{T}]_\beta^\gamma[p]_\beta = A[p]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}.$$

We shall complete this section with the introduction of the "left-multiplication transformation" L_A, where A is an $m \times n$ matrix. This transformation is probably the most important tool for transferring properties about transformations to analogous properties about matrices and vice versa. For example, we shall use it to prove that matrix multiplication is associative.

Definition. *Let* A *be an* m $\times$ n *matrix with entries from a field* F. *We denote by* L_A *the mapping* $\mathsf{L}_\mathrm{A}\colon \ \mathsf{F}^\mathrm{n} \longrightarrow \mathsf{F}^\mathrm{m}$ *defined by* $\mathsf{L}_\mathrm{A}(\mathrm{x}) = \mathrm{Ax}$ (*the matrix product of* A *and* x) *for each column vector* $\mathrm{x} \in \mathsf{F}^\mathrm{n}$. *We call* L_A *a* left-multiplication transformation.

Example 23. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

If

$$x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix},$$

then

$$\mathsf{L}_A(x) = Ax = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

We shall see in the next theorem that not only is L_A linear, but, in fact, it has a great many other useful properties. These properties are all quite natural and so are easy to remember.

Theorem 2.17. *Let* A *be an* $m \times n$ *matrix with entries from* F. *Then the left-multiplication transformation* $\mathsf{L}_A\colon\ \mathsf{F}^n \longrightarrow \mathsf{F}^m$ *is linear. Furthermore, if* B *is any other* $m \times n$ *matrix (with entries from* F*), we have the following properties.*

(a) $[\mathsf{L}_A]_\beta^\gamma = A$, *where* β *and* γ *are the standard ordered bases for* F^n *and* F^m, *respectively.*
(b) $\mathsf{L}_A = \mathsf{L}_B$ *if and only if* $A = B$.
(c) $\mathsf{L}_{A+B} = \mathsf{L}_A + \mathsf{L}_B$ *and* $\mathsf{L}_{aA} = a\mathsf{L}_A$ *for all* $a \in F$.
(d) *If* $\mathsf{T}\colon\ \mathsf{F}^n \longrightarrow \mathsf{F}^m$ *is linear, then there exists a unique* $m \times n$ *matrix* C *such that* $\mathsf{T} = \mathsf{L}_C$.
(e) *If* E *is an* $n \times p$ *matrix, then* $\mathsf{L}_{AE} = \mathsf{L}_A\mathsf{L}_E$.
(f) *If* $m = n$, *then* $\mathsf{L}_{I_n} = \mathsf{I}_{\mathsf{F}^n}$.

PROOF. The fact that L_A is linear follows immediately from Theorem 2.14 and its corollary.

(a) The jth column of $[\mathsf{L}_A]_\beta^\gamma$ is equal to $\mathsf{L}_A(e_j)$. But $\mathsf{L}_A(e_j) = Ae_j = A^j$, and so $[\mathsf{L}_A]_\beta^\gamma = A$.

(b) If $\mathsf{L}_A = \mathsf{L}_B$, then we may use (a) to write $A = [\mathsf{L}_A]_\beta^\gamma = [\mathsf{L}_B]_\beta^\gamma = B$. Hence $A = B$. The proof of the converse is trivial.

The proof of (c) is left to the reader.

(d) Let $C = [\mathsf{T}]_\beta^\gamma$. By Theorem 2.16 we have $[\mathsf{T}(x)]_\gamma = [\mathsf{T}]_\beta^\gamma[x]_\beta$, or $\mathsf{T}(x) = Cx = \mathsf{L}_C(x)$ for all x. So $\mathsf{T} = \mathsf{L}_C$. The uniqueness of C follows from (b).

(e) For any j we have $\mathsf{L}_{AE}(e_j) = (AE)e_j = (AE)^j = AE^j = A(Ee_j) = \mathsf{L}_A(Ee_j) = \mathsf{L}_A(\mathsf{L}_E(e_j)) = (\mathsf{L}_A\mathsf{L}_E)(e_j)$. Hence $\mathsf{L}_{AE} = \mathsf{L}_A\mathsf{L}_E$ by Theorem 2.7.

The proof of (f) is left to the reader. ∎

We shall now use left-multiplication transformations to establish an important property about matrices.

Theorem 2.18. *Let* A, B, *and* C *be matrices such that* A(BC) *is defined. Then* (AB)C *is defined and* A(BC) = (AB)C; *that is, matrix multiplication is associative.*

PROOF. It is left to the reader to show that $(AB)C$ is defined. Using (e) of Theorem 2.17 and the associativity of functional composition, we have

$$\mathsf{L}_{A(BC)} = \mathsf{L}_A\mathsf{L}_{BC} = \mathsf{L}_A(\mathsf{L}_B\mathsf{L}_C) = (\mathsf{L}_A\mathsf{L}_B)\mathsf{L}_C = \mathsf{L}_{AB}\mathsf{L}_C = \mathsf{L}_{(AB)C}.$$

So from (b) of Theorem 2.17 we have $A(BC) = (AB)C$. ∎

Needless to say, this theorem could be proved directly from the definition of matrix multiplication. The proof above, however, provides a prototype of many other arguments that utilize the relationships between linear transformations and matrices.

An Application

A large and varied collection of interesting applications arises in connection with special matrices called "incidence matrices." An *incidence matrix* is a square matrix in which all the entries are either zero or one and, for convenience, all the diagonal entries are zero. If we have a relationship on a group of n objects that we shall denote by $1, 2, \ldots, n$, then we define the associated incidence matrix A by: $A_{ij} = 1$ if i is related to j, and $A_{ij} = 0$ otherwise.

To make things concrete, suppose that we have four people each of whom owns a communication device. If the relationship on this group is "can transmit to," then $A_{ij} = 1$ if i can send (a message) to j and $A_{ij} = 0$ otherwise. Suppose that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then since $A_{34} = 1$ and $A_{14} = 0$, we see that person 3 can send to 4 but 1 cannot send to 4.

We shall obtain an interesting interpretation of the entries of A^2. Consider, for instance,

$$(A^2)_{31} = A_{31}A_{11} + A_{32}A_{21} + A_{33}A_{31} + A_{34}A_{41}.$$

Note that any term $A_{3k}A_{k1}$ will equal 1 if and only if both A_{3k} and A_{k1} equal 1, that is, if and only if 3 can send to k and k can send to 1. Thus $(A^2)_{31}$ gives the number of ways in which 3 can send to 1 in two *stages* (or in one *relay*). Since

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

we see that 3 can send to 1 in two ways with one relay. In general, $(A + A^2 + \cdots + A^n)_{ij}$ is the number of ways in which i can send to j in at most n stages.

A maximal collection of three or more people with the property that any two can send to each other is called a *clique*. The problem of determining cliques seems at first to be quite difficult. However, if we define a new matrix B by: $B_{ij} = 1$ if i and j can send to each other, and $B_{ij} = 0$ otherwise, then it can be shown (see Exercise 16) that person i belongs to a clique if and only if $(B^3)_{ii} > 0$. For example, suppose that the incidence matrix associated with some relationship is

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

To determine which people belong to cliques, we form the matrix B as above and compute B^3. In this case

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B^3 = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{pmatrix}.$$

Since all the diagonal entries of B^3 are zero, we conclude that there are no cliques in this relationship.

Our final example of the use of incidence matrices is concerned with the concept of dominance. A relation among a group of people is called a *dominance relation* if the associated incidence matrix A has the property that $A_{ij} = 1$ if and only if $A_{ji} = 0$ for all i and j, that is, given any two people, exactly one of them dominates (or, using the terminology of our first example, can send a message to) the other. For such a relation, it can be shown (see Exercise 18) that the matrix $A + A^2$ has a row [column] containing positive entries in every position except on the diagonal. In other words, there is at least one person who dominates [is dominated by] all the others in one or two stages. In fact, it can be shown that any person who dominates [is dominated by] the greatest number of people in the first stage has this property. Consider, for example, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The reader should verify that this matrix corresponds to a dominance relation. Now

$$A + A^2 = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix}.$$

Thus persons 1, 3, 4, and 5 dominate (can send messages to) all the others in at most two stages, while persons 1, 2, 3, and 4 are dominated by (can receive messages from) all the others in at most two stages.

EXERCISES

1. Label the following statements as being true or false. For what occurs below, V, W, and Z will denote vector spaces with ordered (finite) bases α, β, and γ, respectively; T: V $\longrightarrow$ W and U: W $\longrightarrow$ Z will be linear; and A and B will denote matrices.

(a) $[\mathsf{UT}]_\alpha^\gamma = [\mathsf{U}]_\beta^\alpha [\mathsf{T}]_\gamma^\beta$.
(b) $[\mathsf{T}(x)]_\beta = [\mathsf{T}]_\alpha^\beta [x]_\alpha$ for all $x \in \mathsf{V}$.
(c) $[\mathsf{U}(y)]_\beta = [\mathsf{U}]_\alpha^\beta [y]_\beta$ for all $y \in \mathsf{W}$.
(d) $[\mathsf{I}_\mathsf{V}]_\alpha = I$.
(e) $[\mathsf{T}^2]_\alpha^\beta = ([\mathsf{T}]_\alpha^\beta)^2$.
(f) $A^2 = I$ implies $A = I$ or $A = -I$.
(g) $\mathsf{T} = \mathsf{L}_A$ for some matrix A.
(h) $A^2 = O$ implies $A = O$, where O denotes the zero matrix.
(i) $\mathsf{L}_{A+B} = \mathsf{L}_A + \mathsf{L}_B$.
(j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

2. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute $A(2B + 3C)$, $(AB)D$, and $A(BD)$.

3. Let $g(x) = 3 + x$. Define

$$\mathsf{T}\colon \; \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R) \quad \text{by} \quad \mathsf{T}(f) = f'g + 2f.$$

Define

$$\mathsf{U}\colon \; \mathsf{P}_2(R) \longrightarrow \mathsf{R}^3 \quad \text{by} \quad \mathsf{U}(a + bx + cx^2) = (a + b, c, a - b).$$

Let $\beta = \{1, x, x^2\}$ and $\gamma = \{e_1, e_2, e_3\}$.

(a) Compute $[\mathsf{U}]_\beta^\gamma$, $[\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly. Then use Theorem 2.12 to verify your result.
(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h]_\beta$ and $[\mathsf{U}(h)]_\gamma$. Then use $[\mathsf{U}]_\beta^\gamma$ from (a) and Theorem 2.16 to verify your result.

4. For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2. Use Theorem 2.16

to compute the following:

(a) $[\mathsf{T}(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.

(b) $[\mathsf{T}(f)]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.

(c) $[\mathsf{T}(A)]_\gamma$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(d) $[\mathsf{T}(f)]_\gamma$, where $f(x) = 6 - x + 2x^2$.

5. Complete the proof of Theorem 2.14 and its corollary.

6. Prove (b) of Theorem 2.15.

7. Prove Theorem 2.11.

8. Find linear transformations $\mathsf{U}, \mathsf{T}\colon \mathsf{F}^2 \longrightarrow \mathsf{F}^2$ such that $\mathsf{UT} = \mathsf{T}_0$ (the zero transformation) but $\mathsf{TU} \neq \mathsf{T}_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

9. Let A be an $n \times n$ matrix. Prove that A is a diagonal matrix if and only if $A_{ij} = \delta_{ij} A_{ii}$ for all i and j.

10. Let V be a vector space, and let $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ be linear. Prove that $\mathsf{T}^2 = \mathsf{T}_0$ if and only if $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})$.

11. Let V, W, and Z be vector spaces, and let $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ and $\mathsf{U}\colon \mathsf{W} \longrightarrow \mathsf{Z}$ be linear.

(a) If UT is one-to-one, prove that T is one-to-one. Must U also be one-to-one?

(b) If UT is onto, prove that U is onto. Must T also be onto?

(c) If U and T are one-to-one and onto, prove that UT is also.

12. Let A and B be $n \times n$ matrices. Recall that the trace of A, written $\text{tr}(A)$, equals

$$\sum_{i=1}^{n} A_{ii}.$$

Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$.

13. Let V be a finite-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ be linear.

(a) If $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$.

(b) Prove that there exists a positive integer k such that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$.

14.† Let V be a vector space. Determine all linear transformations $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ such that $\mathsf{T} = \mathsf{T}^2$. *Hint:* Note that $x = \mathsf{T}(x) + (x - \mathsf{T}(x))$ for every x in V, and show that $\mathsf{V} = \{y\colon \mathsf{T}(y) = y\} \oplus \mathsf{N}(\mathsf{T})$.

15. Using only the definition of matrix multiplication, prove that multiplication of matrices is associative.

16. For an incidence matrix A with the related matrix B defined by: $B_{ij} = 1$ if i is related to j and j is related to i, and $B_{ij} = 0$ otherwise, prove that i belongs to a clique if and only if $(B^3)_{ii} > 0$.

17. Use Exercise 16 to determine the cliques in the relations corresponding to the following incidence matrices.

(a) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

18. Let A be an incidence matrix that is associated with a dominance relation. Prove that the matrix $A + A^2$ has a row [column] that contains positive entries in all positions except on the diagonal.

19. Prove that the matrix A given below corresponds to a dominance relation, and use Exercise 18 to determine which person(s) dominate(s) [is dominated by] each of the others within two stages.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

20. Let A be an $n \times n$ incidence matrix that corresponds to a dominance relation. Determine the number of non-zero entries of A.

2.4 INVERTIBILITY AND ISOMORPHISMS

The concept of invertibility is introduced quite early in the study of functions. Fortunately, many of the intrinsic properties of functions are shared by their inverses. For example, in calculus we learned that the properties of being continuous or differentiable are generally retained by the inverse functions. We shall see in this section (Theorem 2.19) that the inverse of a linear transformation is also linear. This result will greatly aid us in the study of "inverses" of matrices. As one might expect from Section 2.3, the inverse of the left-multiplication transformation L_A (when it exists) can be used to determine properties of the inverse of the matrix A.

In the remainder of this section we shall apply many of the results about invertibility to the concept of "isomorphism." We shall see that finite-dimensional vector spaces (over F) of equal dimension may be identified. These ideas will be made more precise shortly.

The facts about inverse functions found in Appendix B are, of course, true for linear transformations. Nevertheless, we shall repeat some of these definitions for use in this section.

Definition. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *be linear.* T *has an* inverse $\mathsf{U}\colon \mathsf{W} \longrightarrow \mathsf{V}$ *if* $\mathsf{TU} = \mathsf{I_W}$ *and* $\mathsf{UT} = \mathsf{I_V}$. *As in Appendix B, inverses are unique, and we shall write* $\mathsf{U} = \mathsf{T}^{-1}$. *We say* T *is* invertible *if* T *has an inverse.*

The following facts hold for invertible functions T and U.

1. $(\mathsf{TU})^{-1} = \mathsf{U}^{-1}\mathsf{T}^{-1}$.
2. $(\mathsf{T}^{-1})^{-1} = \mathsf{T}$; in particular, T^{-1} is invertible.

We shall also use the fact that a function is invertible if and only if it is one-to-one and onto.

Example 24. Define $\mathsf{T}\colon \mathsf{P}_1(R) \longrightarrow \mathsf{R}^2$ by $\mathsf{T}(a + bx) = (a, a + b)$. The reader can verify directly that $\mathsf{T}^{-1}\colon \mathsf{R}^2 \longrightarrow \mathsf{P}_1(R)$ is defined by $\mathsf{T}^{-1}(c, d) = c + (d - c)x$. Observe that T^{-1} is also linear. As Theorem 2.19 demonstrates, this is true in general.

Theorem 2.19. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *be linear and invertible. Then* $\mathsf{T}^{-1}\colon \mathsf{W} \longrightarrow \mathsf{V}$ *is linear.*

PROOF. Let $y_1, y_2 \in \mathsf{W}$ and $c \in F$. Since T is onto and one-to-one, there exist unique vectors x_1 and x_2 such that $\mathsf{T}(x_1) = y_1$ and $\mathsf{T}(x_2) = y_2$. Thus $x_1 = \mathsf{T}^{-1}(y_1)$ and $x_2 = \mathsf{T}^{-1}(y_2)$, and so

$$\begin{aligned}\mathsf{T}^{-1}(cy_1 + y_2) &= \mathsf{T}^{-1}[c\mathsf{T}(x_1) + \mathsf{T}(x_2)] = \mathsf{T}^{-1}[\mathsf{T}(cx_1 + x_2)]\\ &= cx_1 + x_2 = c\mathsf{T}^{-1}(y_1) + \mathsf{T}^{-1}(y_2). \quad \blacksquare\end{aligned}$$

The next theorem follows immediately from Theorem 2.5.

Theorem 2.20. *Let* V *and* W *be finite-dimensional vector spaces of equal dimension, and let* $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$. *Then the following are equivalent.*

(a) T *is invertible.*
(b) T *is one-to-one.*
(c) T *is onto.*

We are now ready to define the inverse of a matrix. The reader should note the analogy with the inverse of a linear transformation.

Definition. *Let* A *be an* n $\times$ n *matrix. Then* A *is* invertible *if there exists an* n $\times$ n *matrix* B *such that* AB = BA = I.

The matrix B is unique and is called the *inverse* of A and written $B = A^{-1}$. (If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$.)

Example 25. The reader should verify that the inverse of

$$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}.$$

In Section 3.2 we shall learn a technique for actually computing the inverse of a matrix. At this point we would like to develop a number of results that relate the inverses of matrices with the inverses of linear transformations.

Theorem 2.21. *Let* V *and* W *be finite-dimensional vector spaces with ordered bases* β *and* γ, *respectively. Let* T: V $\longrightarrow$ W *be linear. Then* T *is invertible if and only if* $[\mathsf{T}]_\beta^\gamma$ *is invertible. Furthermore,* $[\mathsf{T}^{-1}]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^{-1}$.

PROOF. Suppose that T is invertible. Since T is one-to-one, Theorem 2.6 implies that $\mathsf{T}(\beta)$ is an independent subset of W. Since T is onto, the corollary to Theorem 2.3 implies that span($\mathsf{T}(\beta)$) = R(T) = W. So $\mathsf{T}(\beta)$ is a basis for W with dim(V) elements. Hence dim(V) = dim(W). Let n = dim(V). Then $[\mathsf{T}]_\beta^\gamma$ is an $n \times n$ matrix. Now T^{-1}: W $\longrightarrow$ V satisfies $\mathsf{TT}^{-1} = \mathsf{I_W}$ and $\mathsf{T}^{-1}\mathsf{T} = \mathsf{I_V}$. Thus

$$I_n = [\mathsf{I_V}]_\beta = [\mathsf{T}^{-1}\mathsf{T}]_\beta = [\mathsf{T}^{-1}]_\gamma^\beta[\mathsf{T}]_\beta^\gamma.$$

Similarly $[\mathsf{T}]_\beta^\gamma[\mathsf{T}^{-1}]_\gamma^\beta = I_n$, and hence $([\mathsf{T}]_\beta^\gamma)^{-1} = [\mathsf{T}^{-1}]_\gamma^\beta$.

Now let $A = [\mathsf{T}]_\beta^\gamma$ be invertible. Then there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Define

$$\mathsf{U}\colon\ \mathsf{W} \longrightarrow \mathsf{V} \qquad \text{by } \mathsf{U}(x_j) = \sum_{i=1}^{n} B_{ij} y_i,$$

where $\gamma = \{x_1, \ldots, x_n\}$ and $\beta = \{y_1, \ldots, y_n\}$. Then $[\mathsf{U}]_\gamma^\beta = B$. We shall show that $\mathsf{U} = \mathsf{T}^{-1}$. Observe that $[\mathsf{UT}]_\beta = [\mathsf{U}]_\gamma^\beta[\mathsf{T}]_\beta^\gamma = BA = I_n = [\mathsf{I_V}]_\beta$ by Theorem 2.12. So $\mathsf{UT} = \mathsf{I_V}$, and similarly $\mathsf{TU} = \mathsf{I_W}$. ∎

Example 26. For the vector spaces $\mathsf{P}_1(R)$ and R^2, choose the bases $\beta = \{1, x\}$ and $\gamma = \{e_1, e_2\}$, respectively. In the notation of Example 24, we have that

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad [\mathsf{T}^{-1}]_\gamma^\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It can be verified by matrix multiplication that each matrix is the inverse of the other.

Corollary 1. *Let* V *be a finite-dimensional vector space with an ordered basis* β, *and let* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ *be linear. Then* T *is invertible if and only if* $[\mathsf{T}]_\beta$ *is invertible. Furthermore,* $[\mathsf{T}^{-1}]_\beta = [\mathsf{T}]_\beta^{-1}$.

PROOF. Exercise.

Corollary 2. *Let* A *be an* $n \times n$ *matrix. Then* A *is invertible if and only if* L_A *is invertible. Furthermore,* $(\mathsf{L}_A)^{-1} = \mathsf{L}_{A^{-1}}$.

PROOF. Exercise.

The notion of invertibility may be used to formalize what may already have been observed by the reader, that is, that certain pairs of vector spaces strongly resemble one another except for the form of their elements. For example, in the case of $\mathsf{M}_{2\times 2}(F)$ and F^4, if we associate to each matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the 4-tuple (a, b, c, d), we see that sums and scalar products associate in a similar manner; that is, in terms of the vector space structure, these two vector spaces may be considered identical or "isomorphic."

Definition. *Let* V *and* W *be vector spaces. We say that* V *is* isomorphic *to* W *if there exists a linear transformation* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *that is invertible. Such a linear transformation is called an* isomorphism *from* V *onto* W.

We leave the proof of the fact that "is isomorphic to" is an equivalence relation as an exercise.

Example 27. Define $\mathsf{T}\colon\ \mathsf{F}^2 \longrightarrow \mathsf{P}_1(F)$ by $\mathsf{T}(a_1, a_2) = a_1 + a_2x$. Clearly T is invertible; so F^2 is isomorphic to $\mathsf{P}_1(F)$.

Example 28. Define

$$\mathsf{T}\colon\ \mathsf{P}_3(R) \longrightarrow \mathsf{M}_{2\times 2}(R) \qquad \text{by } \mathsf{T}(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}.$$

It is easily verified that T is linear. By use of the Lagrange interpolation formula in Section 1.6, it can be shown (compare with Exercise 20) that $\mathsf{T}(f) = O$ only when f is the zero polynomial. Thus T is one-to-one and so by Theorem 2.20 we have that T is invertible. We may now conclude that $\mathsf{P}_3(R)$ is isomorphic to $\mathsf{M}_{2\times 2}(R)$.

In each of the two examples above the reader may have observed that the isomorphic vector spaces have equal dimensions. As the next theorem shows, this is no coincidence.

Theorem 2.22. *Let* V *and* W *be finite-dimensional vector spaces (over the same field* F*). Then* V *is isomorphic to* W *if and only if* $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

PROOF. Suppose that V is isomorphic to W and that $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ is a one-to-one linear transformation from V onto W. Then as in the proof of Theorem 2.21 we have that $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

Now suppose that $\dim(\mathsf{V}) = \dim(\mathsf{W})$, and let $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_n\}$ be bases for V and W, respectively. By Theorem 2.7 there exists $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ such that T is linear and $\mathsf{T}(x_i) = y_i$ for $i = 1, \ldots, n$. Using the corollary to Theorem 2.3, we have $\mathsf{R}(\mathsf{T}) = \operatorname{span}(\mathsf{T}(\beta)) = \operatorname{span}(\gamma) = \mathsf{W}$. So T is onto. From Theorem 2.5 we have that T is also one-to-one. ∎

Corollary. *If* V *is a vector space of dimension* n, *then* V *is isomorphic to* F^n.

Until now we have associated transformations with their matrix representations. We are now in a position to prove that as a vector space the collection of all transformations between two given vector spaces may be identified with the appropriate vector space of $m \times n$ matrices.

Theorem 2.23. *Let* V *and* W *be finite-dimensional vector spaces of dimensions* n *and* m, *respectively, and let* β *and* γ *be ordered bases for* V *and* W, *respectively. Then the function* $\Phi\colon \mathcal{L}(\mathsf{V}, \mathsf{W}) \longrightarrow \mathsf{M}_{m\times n}(\mathsf{F})$, *defined by* $\Phi(\mathsf{T}) = [\mathsf{T}]_\beta^\gamma$ *for* $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$, *is an isomorphism.*

PROOF. Theorem 2.9 allows us to conclude that Φ is linear. Hence we must show Φ is one-to-one and onto. Let $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$.

(a) We shall first show that $\mathsf{N}(\Phi) = \{\mathsf{T}_0\}$, where T_0 is the zero transformation. This will imply that Φ is one-to-one. Suppose that $\Phi(\mathsf{T}) = O$. Then for each j we have $\mathsf{T}(x_j) = 0y_1 + \cdots + 0y_m = 0$. By the corollary to Theorem 2.7 we have $\mathsf{T} = \mathsf{T}_0$.

(b) We shall now show that Φ is onto. Let A be an $m \times n$ matrix. By Theorem 2.7 there exists $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$ such that

$$\mathsf{T}(x_j) = \sum_{i=1}^{m} A_{ij} y_i \qquad \text{for } 1 \leq j \leq n.$$

So $[\mathsf{T}]_\beta^\gamma = A$, and hence $\Phi(\mathsf{T}) = A$. Thus Φ is onto. ∎

Corollary. *Let* V *and* W *be finite-dimensional vector spaces of dimensions* n *and* m, *respectively. Then* $\mathcal{L}(\mathsf{V}, \mathsf{W})$ *is finite-dimensional of dimension* mn.

PROOF. The proof follows from Theorem 2.23, Theorem 2.22, and the fact that $\dim(\mathsf{M}_{m\times n}(F)) = mn$. ∎

We will conclude this section with a result that will allow us to see more clearly the relationship between linear transformations defined on abstract finite-dimensional vector spaces and linear transformations defined on F^n.

We shall begin by naming the transformation $x \longrightarrow [x]_\beta$ discussed in Section 2.2.

Definition. Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β *is the function ϕ_β: $\mathsf{V} \longrightarrow \mathsf{F}^n$ defined by $\phi_\beta(\mathsf{x}) = [\mathsf{x}]_\beta$ for each $\mathsf{x} \in \mathsf{V}$.*

Example 29. Let $\mathsf{V} = \mathsf{R}^2$, $\beta = \{(1, 0), (0, 1)\}$, and $\gamma = \{(1, 2), (3, 4)\}$. For $x = (1, -2)$ we have

$$\phi_\beta(x) = [x]_\beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \phi_\gamma(x) = [x]_\gamma = \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

We have observed earlier that ϕ_β is a linear transformation. The following theorem tells us much more.

Theorem 2.24. For any finite-dimensional vector space V with ordered basis β, ϕ_β is an isomorphism.

PROOF. Exercise.

This theorem provides us with an alternate proof that an n-dimensional vector space is isomorphic to F^n (see the corollary to Theorem 2.22).

We are now ready to use the standard representation of a vector space along with the matrix representation of a linear transformation to study the relationship between the linear transformation T: $\mathsf{V} \longrightarrow \mathsf{W}$, where V and W are abstract finite-dimensional vector spaces, and L_A: $\mathsf{F}^n \longrightarrow \mathsf{F}^m$, where $A = [\mathsf{T}]_\beta^\gamma$ and β and γ are arbitrary ordered bases of V and W, respectively.

Before stating Theorem 2.25 we shall consider Fig. 2.2. Notice that there are two compositions of linear transformations that will map V into F^m:

1. Map V into F^n with ϕ_β and follow this transformation with L_A; this yields the composition $\mathsf{L}_A\phi_\beta$.
2. Map V into W with T and follow it by ϕ_γ to obtain the composition $\phi_\gamma\mathsf{T}$.

These two compositions are depicted by the dashed arrows in the diagram. Theorem 2.25 asserts that both of the compositions give the same result; that is, the two compositions are equal.

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"'] \arrow[dr, phantom, "(1)\qquad\qquad(2)"] & \mathsf{W} \arrow[d, "\phi_\gamma"] \\
\mathsf{F}^n \arrow[r, "\mathsf{L}_A"'] & \mathsf{F}^m
\end{tikzcd}$$

figure 2.2

Heuristically, the theorem indicates that after V and W are identified with F^n and F^m via ϕ_β and ϕ_γ, respectively, we may "identify" T with L_A.

Theorem 2.25. *Let* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *be a linear transformation from an* n*-dimensional vector space* V *over* F *to an* m*-dimensional vector space* W *over* F. *Let* β *and* γ *be ordered bases for* V *and* W, *respectively, and let* $\mathsf{A} = [\mathsf{T}]_\beta^\gamma$. *Then* $\mathsf{L}_A\phi_\beta = \phi_\gamma\mathsf{T}$.

PROOF. The theorem is essentially a reformulation of Theorem 2.16. For, if $x \in \mathsf{V}$, then

$$\begin{aligned}(\mathsf{L}_A\phi_\beta)(x) &= \mathsf{L}_A(\phi_\beta(x)) = \mathsf{L}_A([x]_\beta) = A[x]_\beta = [\mathsf{T}]_\beta^\gamma[x]_\beta \\ &= [\mathsf{T}(x)]_\gamma = \phi_\gamma(\mathsf{T}(x)) = (\phi_\gamma\mathsf{T})(x). \quad \blacksquare\end{aligned}$$

Example 30. Recall the transformation $\mathsf{T}\colon\ \mathsf{P}_3(R) \longrightarrow \mathsf{P}_2(R)$ defined in Example 17. ($\mathsf{T}(f) = f'$.) Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ be ordered bases for $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$, respectively, and let $\phi_\beta\colon\ \mathsf{P}_3(R) \longrightarrow \mathsf{R}^4$ and $\phi_\gamma\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{R}^3$ be the standard representations of $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$ with respect to β and γ, respectively. Let $A = [\mathsf{T}]_\beta^\gamma$; then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

To illustrate Theorem 2.25, consider the polynomial $p(x) = 2 + x - 3x^2 + 5x^3$. We shall show that $\mathsf{L}_A\phi_\beta(p) = \phi_\gamma\mathsf{T}(p)$.

Now

$$\mathsf{L}_A\phi_\beta(p) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

But since

$$\mathsf{T}(p) = p' = 1 - 6x + 15x^2,$$

we have

$$\phi_\gamma \mathsf{T}(p) = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

So $\mathsf{L}_A \phi_\beta(p) = \phi_\gamma \mathsf{T}(p)$.

Try repeating this example with different polynomials $p(x)$.

EXERCISES

1. Label the following statements as being true or false. For the following, V and W will be vector spaces with ordered (finite) bases α and β, respectively, and $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ will be linear. A and B will denote matrices.
   (a) $([\mathsf{T}]_\alpha^\beta)^{-1} = [\mathsf{T}^{-1}]_\alpha^\beta$.
   (b) T is invertible if and only if T is one-to-one and onto.
   (c) $\mathsf{T} = \mathsf{L}_A$ where $A = [\mathsf{T}]_\alpha^\beta$.
   (d) $\mathsf{M}_{2\times 3}(F)$ is isomorphic to F^5.
   (e) $\mathsf{P}_n(F)$ is isomorphic to $\mathsf{P}_m(F)$ if and only if $n = m$.
   (f) $AB = I$ implies A and B are invertible.
   (g) $(A^{-1})^{-1} = A$.
   (h) A is invertible if and only if L_A is invertible.
   (i) A must be square in order to possess an inverse.

2.† Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

3.† Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

4. Prove that if A is invertible and $AB = O$, then $B = O$.

5. If $A^2 = O$, prove that A cannot be invertible.

6. Prove Corollaries 1 and 2 of Theorem 2.21.

7. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Show that, in general, this result is false if at least one of the matrices is not square.

8.† Let A and B be $n \times n$ matrices such that $AB = I_n$. Prove $A = B^{-1}$ (and hence $B = A^{-1}$). (We are in effect saying that for square matrices, a one-sided inverse is a two-sided inverse.)

9. Prove that the transformation defined in Example 28 is one-to-one.

10. Prove Theorem 2.24.

11. Let $\sim$ mean "is isomorphic to". Prove that $\sim$ is an equivalence relation on the class of vector spaces over F as defined in Appendix A.

12. Let
$$\mathsf{V} = \left\{\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : \ a, b, c \in F\right\}.$$
Construct an isomorphism from V onto F^3.

13. Let V and W be finite-dimensional vector spaces, and let T: V $\longrightarrow$ W be an isomorphism. If β is a basis for V, prove that T(β) is a basis for W.

14. Let B be an $n \times n$ invertible matrix. Define Φ: $\mathsf{M}_{n\times n}(F) \longrightarrow \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

15. Let V and W be finite-dimensional vector spaces and T: V $\longrightarrow$ W be an isomorphism. Let V_0 be a subspace of V.
   (a) Prove that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.
   (b) Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

16. Repeat Example 30 with the polynomial $p(x) = 1 + x + 2x^2 + x^3$.

17. Let $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, the four-dimensional vector space of 2×2 matrices having real entries. Recall from Example 4 that the mapping T: V $\longrightarrow$ V defined by $\mathsf{T}(A) = A^t$ for each $A \in \mathsf{V}$ is a linear transformation.
   (a) Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$, where E^{ij} is the 2×2 matrix having the i, jth entry equal to one and all other entries zero. Prove that β is an ordered basis for V.
   (b) Let $A = [\mathsf{T}]_\beta$. Compute A.
   (c) Let ϕ denote the standard representation of V with respect to β. Then $\mathsf{L}_A\phi = \phi\mathsf{T}$ by Theorem 2.25. Verify this equality for the matrix
$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};$$
   that is, prove that $\mathsf{L}_A\phi(M) = \phi\mathsf{T}(M)$.

18.† Let T: V $\longrightarrow$ W be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that rank(T) = rank(L_A) and that nullity(T) = nullity(L_A), where $A = [\mathsf{T}]_\beta^\gamma$. *Hint:* Use Theorem 2.25 and Exercise 15.

19. Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$, respectively. By Theorem 2.7 there exists a linear transformation T_{ij}: V $\longrightarrow$ W such that
$$\mathsf{T}_{ij}(x_k) = \begin{cases} y_i & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

First prove that $\{\mathsf{T}_{ij}\colon\ 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $\mathcal{L}(\mathsf{V}, \mathsf{W})$. Then let E^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[\mathsf{T}_{ij}]_\beta^\gamma = E^{ij}$. Again by Theorem 2.7 there exists a linear transformation $\Phi\colon\ \mathcal{L}(\mathsf{V}, \mathsf{W}) \longrightarrow \mathsf{M}_{m\times n}(F)$ such that $\Phi(\mathsf{T}_{ij}) = E^{ij}$. Prove that Φ is an isomorphism.

20. Let $c_0, c_1, \ldots, c_n$ be distinct elements of a field F. Define $\mathsf{T}\colon\ \mathsf{P}_n(F) \longrightarrow \mathsf{F}^{n+1}$ by $\mathsf{T}(f) = (f(c_0), \ldots, f(c_n))$. Prove that T is an isomorphism. *Hint:* Use the Lagrange polynomials associated with $c_0, \ldots, c_n$.

21. Let V denote the vector space of finitely non-zero sequences in F (defined in Example 5 of Section 1.2), and let $\mathsf{W} = \mathsf{P}(F)$. Define

$$\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W} \qquad \text{by } \mathsf{T}(\sigma) = \sum_{i=0}^{n} \sigma(i)x^i,$$

where n is the largest integer with a non-zero image. Prove that T is an isomorphism.

2.5 THE CHANGE OF COORDINATE MATRIX

In many areas of mathematics a change of variable is used to simplify the appearance of an expression. For example, in calculus an antiderivative of $2xe^{x^2}$ would be found by making the change of variable $u = x^2$. The resulting expression is of such a simple form that an antiderivative is easily recognized:

$$\int 2xe^{x^2}\,dx = \int e^u\,du = e^u = e^{x^2}.$$

Likewise, in plane geometry the change of variable

$$x = \frac{\sqrt{5}}{5}x' + \frac{2\sqrt{5}}{5}y' \quad \text{and} \quad y = \frac{-2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y'$$

can be used to transform the equation $2x^2 - 4xy + 5y^2 = 1$ into the simpler equation $6(x')^2 + (y')^2 = 1$, in which form it is easily seen to be the equation of an ellipse. (We shall see how this change of variable was determined in Section 7.7.) Geometrically, the change of variable

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}$$

amounts to a rotation of the coordinate axes so that the x-axis and y-axis coincide with the x'-axis and y'-axis, respectively, where

$$x' = \frac{\sqrt{5}}{5}x - \frac{2\sqrt{5}}{5}y \quad \text{and} \quad y' = \frac{2\sqrt{5}}{5}x + \frac{\sqrt{5}}{5}y.$$

(The values of x' and y' were found by solving the system

$$\begin{cases} \dfrac{\sqrt{5}}{5}x' + \dfrac{2\sqrt{5}}{5}y' = x \\ \dfrac{-2\sqrt{5}}{2}x' + \dfrac{\sqrt{5}}{5}y' = y \end{cases}$$

for x' and y' in terms of x and y.) The vectors obtained by subjecting the basis vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of the x, y-coordinate system to this rotation are

$$\begin{pmatrix} \dfrac{\sqrt{5}}{5} \\ \dfrac{2\sqrt{5}}{5} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dfrac{-2\sqrt{5}}{5} \\ \dfrac{\sqrt{5}}{5} \end{pmatrix},$$

respectively. These vectors are unit vectors lying on the x'-axis and y'-axis, respectively, and hence form a new basis

$$\beta' = \left\{ \begin{pmatrix} \dfrac{\sqrt{5}}{5} \\ \dfrac{2\sqrt{5}}{5} \end{pmatrix}, \begin{pmatrix} \dfrac{-2\sqrt{5}}{5} \\ \dfrac{\sqrt{5}}{5} \end{pmatrix} \right\}$$

for R^2.

A natural question arises: How can coordinate vectors with respect to one basis be changed into coordinate vectors relative to the other? The answer is provided by the relationship

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{5}}{5} & \dfrac{-2\sqrt{5}}{5} \\ \dfrac{2\sqrt{5}}{5} & \dfrac{\sqrt{5}}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that the matrix

$$Q = \begin{pmatrix} \dfrac{\sqrt{5}}{5} & \dfrac{-2\sqrt{5}}{5} \\ \dfrac{2\sqrt{5}}{5} & \dfrac{\sqrt{5}}{5} \end{pmatrix}$$

equals $[\mathsf{I}]_{\beta'}^{\beta}$, where I denotes the identity transformation on R^2. Thus by Theorem 2.16 $[v]_\beta = Q[v]_{\beta'}$ for all $v \in \mathsf{R}^2$. A similar result is true in general.

Theorem 2.26. Let β and β' be two ordered bases for a finite-dimensional vector space V, *and let* $Q = [\mathsf{I_V}]_{\beta'}^{\beta}$. *Then*

(a) Q *is invertible.*
(b) *For any* $v \in \mathsf{V}$, $[v]_\beta = Q[v]_{\beta'}$.

PROOF.
(a) Since $\mathsf{I_V}$ is invertible, Q is invertible by Theorem 2.21.
(b) For any $v \in \mathsf{V}$,

$$[v]_\beta = [\mathsf{I_V}(v)]_\beta = [\mathsf{I_V}]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$$

by Theorem 2.16. ∎

The matrix Q defined in Theorem 2.26 is called a *change of coordinate matrix*. Because of part (b) of the theorem we shall say that *Q changes β'-coordinates into β-coordinates*. Observe that if $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x_1', x_2', \ldots, x_n'\}$, then

$$x_j' = \sum_{i=1}^{n} Q_{ij}x_i$$

for $j = 1, 2, \ldots, n$; that is, the jth column of Q is $[x_j']_\beta$.

Example 31. Let $\mathsf{V} = \mathsf{R}^2$, $\beta = \{(1, 1), (1, -1)\}$, and $\beta' = \{(2, 4), (3, 1)\}$. Since $(2, 4) = 3(1, 1) - 1(1, -1)$ and $(3, 1) = 2(1, 1) + 1(1, -1)$, the matrix that changes β'-coordinates into β-coordinates is

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}.$$

Thus, for instance,

$$[(2, 4)]_\beta = Q[(2, 4)]_{\beta'} = Q\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Suppose now that $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ is a linear transformation between finite-dimensional vector spaces and that β and β' are ordered bases for V and γ and γ' are ordered bases for W. Then T can be represented by matrices relative to β and γ and relative to β' and γ'. What is the relationship between the matrices $[\mathsf{T}]_\beta^\gamma$ and $[\mathsf{T}]_{\beta'}^{\gamma'}$? The answer is easily seen from the equations $[\mathsf{T}(v)]_\gamma = [\mathsf{T}]_\beta^\gamma[v]_\beta$ and $[\mathsf{T}(v)]_{\gamma'} = [\mathsf{T}]_{\beta'}^{\gamma'}[v]_{\beta'}$ given by Theorem 2.16. For if Q and P are the change of coordinate matrices that change β'-coordinates into β-coordinates and γ'-coordinates into γ-coordinates, respectively, then from these equations it is clear that there are two methods for obtaining $[\mathsf{T}(v)]_\gamma$ from $[v]_{\beta'}$, as depicted in Fig. 2.3.

Since $[\mathsf{T}]_\beta^\gamma Q[v]_{\beta'} = P[\mathsf{T}]_{\beta'}^{\gamma'}[v]_{\beta'}$ for all $v \in \mathsf{V}$, Theorem 2.17(b) implies that $[\mathsf{T}]_\beta^\gamma Q = P[\mathsf{T}]_{\beta'}^{\gamma'}$. Because P is invertible (Theorem 2.26), this provides the following answer to the question posed above.

Theorem 2.27. *Let* $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ *be a linear transformation from a finite-dimensional vector space* V *to a finite-dimensional vector space* W, *and let* β *and* β' *be ordered bases for* V *and* γ *and* γ' *be ordered bases for* W. *Then* $[\mathsf{T}]_{\beta'}^{\gamma'} =$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\beta'}} \arrow[r, "\text{Multiply by } [\mathsf{T}]_{\beta'}^{\gamma'}."] \arrow[d, swap, "\begin{array}{c}\text{Multiply by } Q \text{ (to} \\ \text{change } \beta'\text{-coordinates} \\ \text{into } \beta\text{-coordinates).}\end{array}"] & {[\mathsf{T}(v)]_{\gamma'}} \arrow[d, "\begin{array}{c}\text{Multiply by } P \text{ (to change} \\ \gamma'\text{-coordinates into} \\ \gamma\text{-coordinates).}\end{array}"] \\
{[v]_{\beta}} \arrow[r, "\text{Multiply by } [\mathsf{T}]_{\beta}^{\gamma}."] & {[\mathsf{T}(v)]_{\gamma}}
\end{tikzcd}$$

figure 2.3

$\mathsf{P}^{-1}[\mathsf{T}]_\beta^\gamma\mathsf{Q}$, *where* Q *is the matrix which changes* β'*-coordinates into* β*-coordinates and* P *is the matrix which changes* γ'*-coordinates into* γ*-coordinates.*

Example 32. Let $\mathsf{V} = \mathsf{R}^3$, $\mathsf{W} = \mathsf{R}^2$, and $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2a_1 + a_2 \\ a_1 + a_2 - a_3 \end{pmatrix}.$$

Let β and γ be the standard ordered bases for R^3 and R^2, respectively, and let

$$\beta' = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \right\} \quad \text{and} \quad \gamma' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

(Observe that β' and γ' are ordered bases for R^3 and R^2, respectively.) We shall verify Theorem 2.27. Now

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad [\mathsf{T}]_{\beta'}^{\gamma'} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

and easy calculations show that the change of coordinate matrices changing β'-coordinates into β-coordinates and γ'-coordinates into γ-coordinates are

$$\underset{\beta'\to\beta}{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

respectively. In Section 3.2 a method for computing P^{-1} will be presented. Until then, check that

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

An easy multiplication then shows that $[\mathsf{T}]_{\beta'}^{\gamma'} = P^{-1}[\mathsf{T}]_\beta^\gamma Q$.

An important special case of the preceding result occurs when $V = W$. It is this situation that will be our primary concern in much of the remainder of the book. In this case the theorem takes the following form.

Corollary. *Let* $T\colon V \longrightarrow V$ *be a linear transformation on a finite-dimensional vector space* V *having ordered bases* β *and* β'. *Then* $[T]_{\beta'} = Q^{-1}[T]_\beta Q$, *where* Q *is the matrix changing* β'*-coordinates into* β*-coordinates.*

The relationship between the matrices $[T]_{\beta'}$ and $[T]_\beta$ in the corollary above will be the subject of further study in Chapters 5 and 6. At this time, however, we shall introduce the name for this relationship.

Definition. *Let* A *and* B *be* $n \times n$ *matrices with entries from the field* F. *We say that* B *is* similar *to* A *if there exists an invertible matrix* $Q \in M_{n\times n}(F)$ *such that* $B = Q^{-1}AQ$.

Observe that the relation of similarity is an equivalence relation. (See Exercise 7.)

Notice also that in this terminology the preceding corollary can be stated as follows: If $T\colon V \longrightarrow V$ is a linear transformation on a finite-dimensional vector space V, and if β and β' are any ordered bases for V, then $[T]_{\beta'}$ is similar to $[T]_\beta$.

EXERCISES

1. Label the following statements as being true or false.
   (a) If Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates, where $\beta' = \{x_1', \ldots, x_n'\}$ and $\beta = \{x_1, \ldots, x_n\}$ are ordered bases for a vector space, then the jth column of Q is $[x_j]_{\beta'}$.
   (b) Every change of coordinate matrix is invertible.
   (c) Let $T\colon V \longrightarrow W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W, and let β and β' be ordered bases for V and γ and γ' be ordered bases for W. Then $[T]_{\beta'}^{\gamma'} = P[T]_\beta^\gamma Q$, where Q and P are the change of coordinate matrices changing β'-coordinates to β-coordinates and γ-coordinates to γ'-coordinates, respectively.
   (d) The matrices $A, B \in M_{n\times n}(F)$ are called similar if $B = Q^tAQ$ for some $Q \in M_{n\times n}(F)$.
   (e) Let $T\colon V \longrightarrow V$ be a linear transformation on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$.

2. For each of the following pairs of ordered bases β and β' for R^2, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.
   (a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$
   (b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$
   (c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$
   (d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

3. For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.
   (a) $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$
   (b) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$
   (c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$
   (d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$
   (e) $\beta = \{x^2 - x, x^2 + 1, x - 1\}$ and $\beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$
   (f) $\beta = \{2x^2 - x + 1, x^2 + 3x - 2, -x^2 + 2x + 1\}$ and $\beta' = \{9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2\}$

4. Let $\mathsf{V} = \mathsf{R}^2$, $\mathsf{W} = \mathsf{R}^3$, and $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ be defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3a_1 - a_2 \\ 2a_1 + 4a_2 \\ -a_1 + a_2 \end{pmatrix}.$$

Let β and γ be the standard ordered bases for R^2 and R^3, respectively, and let

$$\beta' = \left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \quad \text{and} \quad \gamma' = \left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

   (a) Compute $A = [\mathsf{T}]_\beta^\gamma$ and $B = [\mathsf{T}]_{\beta'}^{\gamma'}$.
   (b) Compute Q, the change of coordinate matrix changing β'-coordinates into β-coordinates, and P, the change of coordinate matrix changing γ'-coordinates into γ-coordinates.
   (c) Verify that $B = P^{-1}AQ$. You should find that

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

5. Let $\mathsf{T}\colon \mathsf{P}_1(R) \longrightarrow \mathsf{P}_1(R)$ be defined by $\mathsf{T}(p) = p'$, the derivative of $p \in \mathsf{P}_1(R)$. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$.
   (a) Find the change of coordinate matrix Q that changes β'-coordinates into β-coordinates.
   (b) Find Q^{-1}. (See Example 32.)
   (c) Compute $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_{\beta'}$, and verify that $B = Q^{-1}AQ$.

6. Prove the corollary to Theorem 2.27.

7. Recall the definition of an equivalence relation from Appendix A. Prove that the relation "is similar to" is an equivalence relation on $\mathsf{M}_{n\times n}(F)$.

8. Prove that if A and B are similar $n \times n$ matrices, then $\operatorname{tr}(A) = \operatorname{tr}(B)$. *Hint:* Use Exercise 12 of Section 2.3.

9. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.
   (a) Prove that if Q and R are the change of coordinate matrices which change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix which changes α-coordinates into γ-coordinates.
   (b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

10. Let A be an $m \times n$ matrix with entries from a field F, and let β and γ be ordered bases for F^n and F^m, respectively. Let $B = [\mathsf{L}_A]_\beta^\gamma$. Prove that $B = P^{-1}AQ$, where P is the $m \times m$ matrix with jth column equal to the jth vector in γ and Q is the $n \times n$ matrix with the jth column equal to the jth vector in β.

11.† Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij}x_i \qquad \text{for } 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

12.† Prove the converse of Theorem 2.27: If A and B are each $m \times n$ matrices over a field F, and if there exist invertible $m \times m$ and $n \times n$ matrices P and Q, respectively, such that $B = PAQ$, then there exist an n-dimensional vector space V and an m-dimensional vector space W (both over F), ordered bases β and β' for V and γ and γ' for W, and a linear transformation $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ such that

$$A = [\mathsf{T}]_\beta^\gamma \quad \text{and} \quad B = [\mathsf{T}]_{\beta'}^{\gamma'}.$$

Hints: Let $V = F^n$, $W = F^m$, $T = L_A$, and β and γ be the standard ordered bases for F^n and F^m, respectively. Let β' be the ordered basis for V obtained from β via Q (according to the definition on page 98 and justified by Exercise 11), and let γ' be the basis for W obtained from γ via P^{-1}.

2.6* DUAL SPACES

In this section we shall be exclusively concerned with linear transformations from a vector space V into its field of scalars F, which is itself a vector space of dimension 1 over F. Such a linear transformation is called a *linear functional on* V. In calculus the definite integral provides us with one of the most important examples of a linear functional in mathematics. (See Example 33.) We shall generally use the letters f, g, h, . . . to denote linear functionals.

Example 33. Let V be the vector space of continuous complex- (or real-) valued functions on the interval $[a, b]$. The function f: $V \longrightarrow C$ (or R) defined by

$$\mathsf{f}(x) = \int_a^b x(t)\,dt$$

is a linear functional on V. If the interval is $[0, 2\pi]$ and n is an integer, the function defined by

$$\mathsf{h}_n(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-int}\,dt$$

is also a linear functional. In analysis texts the scalar $\mathsf{h}_n(x)$ is called the nth *Fourier coefficient of x.*

Example 34. Let $V = M_{n\times n}(F)$, and define f: $V \longrightarrow F$ by $\mathsf{f}(A) = \mathrm{tr}(A)$, the trace of A. By Exercise 6 of Section 1.3, we have that f is a linear functional.

Example 35. Let V be a finite-dimensional vector space with the ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$. For each $i = 1, \ldots, n$, define $\mathsf{f}_i(x) = a_i$, where

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$$

is the coordinate vector of x relative to β. Then f_i is a linear functional on V called the *ith coordinate function with respect to the basis* β. Note that $f_i(x_j) = \delta_{ij}$. These linear functionals will play a very important role in the theory of dual spaces. (See Theorem 2.28.)

Definition. *For a vector space* V *over* F, *we define the* dual space of V *to be the vector space* $\mathcal{L}(V, F)$, *denoted by* V^*.

Thus V^* is the vector space consisting of all linear functionals on V with the operations of addition and scalar multiplication as defined in Section 2.2. Note that if V is finite-dimensional, then $\dim(V^*) = \dim(\mathcal{L}(V, F)) = \dim(V)\cdot\dim(F) = \dim(V)$. Hence, by Theorem 2.22, V and V^* are isomorphic. We may also define the *double dual* V^{**} of V to be the dual of V^*. We shall show, in fact, that there is a natural identification of V and V^{**}.

Theorem 2.28. *Suppose that* V *is a finite-dimensional vector space with the ordered basis* $\beta = \{x_1, \ldots, x_n\}$. *Let* f_i $(1 \leq i \leq n)$ *be the coordinate functions with respect to* β *as defined above, and let* $\beta^* = \{f_1, \ldots, f_n\}$. *Then* β^* *is an ordered basis for* V^*, *and for any* $f \in V^*$ *we have that*

$$f = \sum_{i=1}^{n} f(x_i)f_i.$$

We call β^* *the* dual basis *of* β.

PROOF. Let $f \in V^*$. Since $\dim(V^*) = n$, we need only show that

$$f = \sum_{i=1}^{n} f(x_i)f_i,$$

for then β^* will generate V^*. Let

$$g = \sum_{i=1}^{n} f(x_i)f_i.$$

For $1 \leq j \leq n$, we have

$$\begin{aligned} g(x_j) &= \left(\sum_{i=1}^{n} f(x_i)f_i\right)(x_j) = \sum_{i=1}^{n} f(x_i)f_i(x_j) \\ &= \sum_{i=1}^{n} f(x_i)\delta_{ij} = f(x_j). \end{aligned}$$

Hence $g = f$ by the corollary to Theorem 2.7, and we are done. ∎

Example 36. Let $\beta = \{(2, 1), (3, 1)\}$ be an ordered basis for R^2. We shall explicitly determine the dual basis $\beta^* = \{f_1, f_2\}$ of β. We need to consider the equations:

$$1 = \mathsf{f}_1(2, 1) = \mathsf{f}_1(2e_1 + e_2) = 2\mathsf{f}_1(e_1) + \mathsf{f}_1(e_2)$$
$$0 = \mathsf{f}_1(3, 1) = \mathsf{f}_1(3e_1 + e_2) = 3\mathsf{f}_1(e_1) + \mathsf{f}_1(e_2).$$

Solving, we obtain that $\mathsf{f}_1(e_1) = -1$ and $\mathsf{f}_1(e_2) = 3$, i.e., that $\mathsf{f}_1(x, y) = -x + 3y$. Similarly it can be shown that $\mathsf{f}_2(x, y) = x - 2y$.

We shall now assume that V and W are finite-dimensional vector spaces over F with ordered bases β and γ, respectively. In Section 2.4 we proved that there exists a one-to-one correspondence between linear transformations $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ and $m \times n$ matrices (over F) via the correspondence $\mathsf{T} \longleftrightarrow [\mathsf{T}]_\beta^\gamma$. For a matrix of the form $A = [\mathsf{T}]_\beta^\gamma$, the question arises as to whether or not there exists a linear transformation U associated with T in some natural way such that U may be represented in some basis as A^t. Of course, if $m \neq n$, it would be impossible for U to be a linear transformation from V into W. We shall now answer this question by applying what we have already learned about dual spaces.

Theorem 2.29. *Let* V *and* W *be finite-dimensional vector spaces over* F *with ordered bases* β *and* γ*, respectively. For any linear transformation* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$*, the mapping* $\mathsf{T}^t\colon \mathsf{W}^* \longrightarrow \mathsf{V}^*$ *defined by* $\mathsf{T}^t(\mathsf{g}) = \mathsf{g} \circ \mathsf{T}$ *for all* $\mathsf{g} \in \mathsf{W}^*$ *is a linear transformation with the property that* $[\mathsf{T}^t]_{\gamma^*}^{\beta^*} = ([\mathsf{T}]_\beta^\gamma)^t$.

PROOF. For $\mathsf{g} \in \mathsf{W}^*$, it is clear that $\mathsf{T}^t(\mathsf{g}) = \mathsf{g} \circ \mathsf{T}$ is a linear functional on V and hence is an element of V^*. Thus T^t maps W^* into V^*. We shall leave the proof that T^t is linear to the reader.

To complete the proof, let $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$ with dual bases $\beta^* = \{\mathsf{f}_1, \ldots, \mathsf{f}_n\}$ and $\gamma^* = \{\mathsf{g}_1, \ldots, \mathsf{g}_m\}$, respectively. For convenience, let $A = [\mathsf{T}]_\beta^\gamma$ and $B = [\mathsf{T}^t]_{\gamma^*}^{\beta^*}$. Then

$$\mathsf{T}(x_i) = \sum_{k=1}^m A_{ki} y_k \qquad \text{for } 1 \leq i \leq n,$$

and

$$\mathsf{T}^t(\mathsf{g}_j) = \sum_{i=1}^n B_{ij} \mathsf{f}_i \qquad \text{for } 1 \leq j \leq m.$$

We must show that $B = A^t$. Theorem 2.28 shows that

$$\mathsf{T}^t(\mathsf{g}_j) = \mathsf{g}_j \circ \mathsf{T} = \sum_{i=1}^n (\mathsf{g}_j \circ \mathsf{T})(x_i)\mathsf{f}_i,$$

and so

$$B_{ij} = (\mathsf{g}_j \circ \mathsf{T})(x_i) = \mathsf{g}_j(\mathsf{T}(x_i)) = \mathsf{g}_j\Big(\sum_{k=1}^m A_{ki} y_k\Big)$$
$$= \sum_{k=1}^m A_{ki} \mathsf{g}_j(y_k) = \sum_{k=1}^m A_{ki} \delta_{jk} = A_{ji} = (A^t)_{ij}.$$

Hence $B = A^t$. ▮

The linear transformation T^t defined in Theorem 2.29 is called the *transpose of* T. It is clear that T^t is the unique linear transformation U such that $[\mathsf{U}]_{\gamma^*}^{\beta^*} = ([\mathsf{T}]_\beta^\gamma)^t$.

We shall now concern ourselves with demonstrating that any finite-dimensional vector space V may be identified in a very natural way with its double dual V^{**}. We shall, in fact, produce an isomorphism between V and V^{**} that will not depend on any choice of bases for the two vector spaces.

For a vector $x \in \mathsf{V}$ we define $\hat{x}\colon\ \mathsf{V}^* \longrightarrow F$ by $\hat{x}(\mathsf{f}) = \mathsf{f}(x)$ for every $\mathsf{f} \in \mathsf{V}^*$. It is easy to verify that $\hat{x}$ is a linear functional on V^*, and so $\hat{x} \in \mathsf{V}^{**}$. The correspondence $x \longleftrightarrow \hat{x}$ will allow us to define the desired isomorphism between V and V^{**}.

Lemma. *Let V be a finite-dimensional vector space, and let $\mathrm{x} \in \mathsf{V}$. If $\hat{\mathrm{x}}(\mathsf{f}) = 0$ for all $\mathsf{f} \in \mathsf{V}^*$, then $\mathrm{x} = 0$.*

PROOF. If $x \neq 0$, then we may choose an ordered basis $\beta = \{x_1, \ldots, x_n\}$ for V such that $x_1 = x$. Let $\{\mathsf{f}_1, \ldots, \mathsf{f}_n\}$ be the dual basis of β. Then $\mathsf{f}_1(x_1) = 1 \neq 0$, a contradiction.

Theorem 2.30. *Let V be a finite-dimensional vector space, and let $\psi\colon\ \mathsf{V} \longrightarrow \mathsf{V}^{**}$ be defined by $\psi(\mathrm{x}) = \hat{\mathrm{x}}$. Then ψ is an isomorphism.*

PROOF.

(a) ψ is linear: Let $x, y \in \mathsf{V}$ and $a \in F$. For $\mathsf{f} \in \mathsf{V}^*$ we have that

$$\begin{aligned}\psi(x + ay)(\mathsf{f}) &= \mathsf{f}(x + ay) = \mathsf{f}(x) + a\mathsf{f}(y) = \hat{x}(\mathsf{f}) + a\hat{y}(\mathsf{f}) \\ &= (\hat{x} + a\hat{y})(\mathsf{f}).\end{aligned}$$

Hence

$$\psi(x + ay) = \hat{x} + a\hat{y} = \psi(x) + a\psi(y).$$

(b) ψ is one-to-one: Suppose that $\psi(x)$ is the zero functional on V^* for some $x \in \mathsf{V}$. Then $\hat{x}(\mathsf{f}) = 0$ for every $\mathsf{f} \in \mathsf{V}^*$. By the previous lemma we may conclude that $x = 0$.

(c) ψ is an isomorphism: This follows from (b) and the fact that $\dim(\mathsf{V}) = \dim(\mathsf{V}^{**})$. ∎

Corollary. *Let V be a finite-dimensional vector space with dual space V^*. Then every ordered basis of V^* is the dual basis of some basis of V.*

PROOF. Let $\{\mathsf{f}_1, \ldots, \mathsf{f}_n\}$ be an ordered basis of V^*. We may combine Theorems 2.28 and 2.30 to conclude that for this basis of V^* there exists a dual basis $\{\hat{x}_1, \ldots, \hat{x}_n\}$ in V^{**}, that is, $\delta_{ij} = \hat{x}_i(\mathsf{f}_j) = \mathsf{f}_j(x_i)$. Thus $\{\mathsf{f}_1, \ldots, \mathsf{f}_n\}$ is the dual basis of $\{x_1, \ldots, x_n\}$. ∎

Although many of the ideas of this section can be extended to the case where V is not finite-dimensional, for example the existence of a dual space, only a finite-dimensional vector space is isomorphic to its double dual via the map $x \longrightarrow \hat{x}$. In fact, for infinite-dimensional vector spaces, V and V* are never isomorphic.

EXERCISES

1. Label the following as being true or false. Assume that all vector spaces are finite-dimensional.
   (a) Every linear transformation is a linear functional.
   (b) A linear functional defined on a field may be represented as a 1×1 matrix.
   (c) Every vector space is isomorphic to its dual space.
   (d) Every vector space is the dual of some other vector space.
   (e) If T is an isomorphism from V onto V* and β is a finite ordered basis of V, then $\mathsf{T}(\beta) = \beta^*$.
   (f) If T is a linear transformation from V into W, then the domain of $(\mathsf{T}^t)^t$ is V**.
   (g) If V is isomorphic to W, then V* is isomorphic to W*.
   (h) The derivative of a function may be considered as a linear functional on the vector space of differentiable functions.

2. For the following functions on a vector space V, determine which are linear functionals.
   (a) $\mathsf{V} = \mathsf{P}(R)$; $\mathsf{f}(p) = 2p'(0) + p''(1)$, where $'$ denotes differentiation
   (b) $\mathsf{V} = \mathsf{R}^2$; $\mathsf{f}(x, y) = (2x, 4y)$
   (c) $\mathsf{V} = \mathsf{M}_{2\times 2}(F)$; $\mathsf{f}(A) = \mathrm{tr}(A)$
   (d) $\mathsf{V} = \mathsf{R}^3$; $\mathsf{f}(x, y, z) = x^2 + y^2 + z^2$
   (e) $\mathsf{V} = \mathsf{P}(R)$; $\mathsf{f}(p) = \int_0^1 p(t)\,dt$
   (f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$; $\mathsf{f}(A) = A_{11}$

3. As in Example 36, for each vector space V and basis β below, find the dual basis β^* for V*.
   (a) $\mathsf{V} = \mathsf{R}^3$; $\beta = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$
   (b) $\mathsf{V} = \mathsf{P}_2(R)$; $\beta = \{1, x, x^2\}$

4. Let $\mathsf{V} = \mathsf{R}^3$ and define $\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3 \in \mathsf{V}^*$ by $\mathsf{f}_1(x, y, z) = x - 2y$, $\mathsf{f}_2(x, y, z) = x + y + z$, and $\mathsf{f}_3(x, y, z) = y - 3z$. Prove that $\{\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3\}$ is a basis for V*, and then find a basis for V for which it is the dual.

5. Let $\mathsf{V} = \mathsf{P}_1(R)$, and for $p \in \mathsf{V}$ define $\mathsf{f}_1, \mathsf{f}_2 \in \mathsf{V}^*$ by

$$\mathsf{f}_1(p) = \int_0^1 p(t)\,dt$$

and

$$f_2(p) = \int_0^2 p(t)\,dt.$$

Prove that $\{f_1, f_2\}$ is a basis for V^* and find a basis for V for which it is the dual.

6. Define $f \in (R^2)^*$ by $f(x, y) = 2x + y$ and $T\colon R^2 \longrightarrow R^2$ by $T(x, y) = (3x + 2y, x)$.
   (a) Compute $T^t(f)$.
   (b) Compute $[T^t]_{\beta^*}$, where β is the standard ordered basis for R^2 and $\beta^* = \{f_1, f_2\}$, by finding scalars a, b, c, and d such that $T^t(f_1) = af_1 + bf_2$ and $T^t(f_2) = cf_1 + df_2$.
   (c) Compute $[T]_\beta$ and $[T]^t_\beta$, and compare your results with part (b).
7. Let $V = P_1(R)$ and $W = R^2$ with respective ordered bases $\beta = \{1, x\}$ and $\gamma = \{e_1, e_2\}$. Define $T\colon V \longrightarrow W$ by $T(p) = (p(0) - 2p(1), p(0) + p'(0))$, where p' is the derivative of p.
   (a) If $f \in W^*$ is defined by
   $$f(a, b) = a - 2b,$$
   compute $T^t(f)$.
   (b) Compute $[T^t]_{\gamma^*}^{\beta^*}$ without appealing to Theorem 2.29.
   (c) Compute $[T]_\beta^\gamma$ and its transpose, and compare your result with part (b).
8. Show that every plane through the origin in R^3 may be identified with the null space of an element in $(R^3)^*$. State an analogous result in R^2.
9. Let T be a function from F^n into F^m. Prove that T is linear if and only if there exists $f_1, \ldots, f_m \in (F^n)^*$ such that $T(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in F^n$. *Hint:* If T is linear, define $f_i(x) = (g_i \circ T)(x)$ for $x \in F^n$, i.e., $f_i = T^t(g_i)$ for $1 \leq i \leq m$, where $\{g_1, \ldots, g_m\}$ is the dual basis of the standard ordered basis of F^m.
10. Let $V = P_n(F)$, and let $c_0, \ldots, c_n$ be distinct scalars in F.
   (a) For $0 \leq i \leq n$, define $f_i \in V^*$ by $f_i(p) = p(c_i)$. Prove that $\{f_0, \ldots, f_n\}$ is a basis of V^*. *Hint:* Apply any linear combination of this set that equals the zero transformation to $p(t) = (t - c_1)(t - c_2) \cdots (t - c_n)$ and deduce that the first coefficient is zero.
   (b) Use the corollary of Theorem 2.30 and part (a) to show that there exist unique polynomials $p_0, \ldots, p_n$ such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. These polynomials are the Lagrange polynomials defined in Section 1.6.
   (c) For any scalars $a_0, \ldots, a_n$ (not necessarily distinct), deduce that there exists a unique polynomial q of degree at most n such that $q(c_i) = a_i$ for $0 \leq i \leq n$. In fact,
   $$q = \sum_{i=0}^{n} a_i p_i.$$

(d) Deduce the Lagrange interpolation formula:

$$p = \sum_{i=0}^{n} p(c_i)p_i$$

for any $p \in \mathsf{V}$.

(e) Prove that

$$\int_a^b p(t)\,dt = \sum_{i=0}^{n} p(c_i)d_i,$$

where

$$d_i = \int_a^b p_i(t)\,dt.$$

Suppose now that

$$c_i = a + \frac{i(b-a)}{n} \quad \text{for } i = 0, \ldots, n.$$

For $n = 1$, the above yields the trapezoidal rule for polynomials. For $n = 2$, this result is Simpson's rule for polynomials.

11. Let V and W be finite-dimensional vector spaces over F, and let ψ_1 and ψ_2 be the isomorphisms between V and V** and W and W**, respectively, as defined in Theorem 2.30. Let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be linear, and define $\mathsf{T}^{tt} = (\mathsf{T}^t)^t$. Prove that the diagram depicted in Fig. 2.4 commutes, i.e., that $\psi_2\mathsf{T} = \mathsf{T}^{tt}\psi_1$.

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\psi_1"'] & \mathsf{W} \arrow[d, "\psi_2"] \\
\mathsf{V}^{**} \arrow[r, "\mathsf{T}^{tt}"'] & \mathsf{W}^{**}
\end{tikzcd}$$

figure 2.4

12. Let V be a finite-dimensional vector space with the ordered basis β. Prove that $\psi(\beta) = \beta^{**}$, where ψ is as defined in Theorem 2.30.

For problems 13 through 17, V will denote a finite-dimensional vector space over F. If S is a subset of V, we define the *annihilator* S^0 of S as $S^0 = \{\mathsf{f} \in \mathsf{V}^*\colon\ \mathsf{f}(x) = 0 \text{ for all } x \in S\}$.

13. (a) Prove that S^0 is a subspace of V*.
(b) If W is a subspace of V and $x \notin \mathsf{W}$, prove that there exists $\mathsf{f} \in \mathsf{W}^0$ such that $\mathsf{f}(x) \neq 0$.
(c) Prove that $S^{00} = \operatorname{span}(\psi(S))$, where ψ is as defined in Theorem 2.30.
(d) For subspaces W_1 and W_2, prove that $\mathsf{W}_1 = \mathsf{W}_2$ if and only if $\mathsf{W}_1^0 = \mathsf{W}_2^0$.
(e) For subspaces W_1 and W_2, show that $(\mathsf{W}_1 + \mathsf{W}_2)^0 = \mathsf{W}_1^0 \cap \mathsf{W}_2^0$.

14. If W is a subspace of V, prove that $\dim(W) + \dim(W^0) = \dim(V)$. *Hint:* Let $\{x_1, \ldots, x_k\}$ be an ordered basis of W and extend it to an ordered basis $\beta = \{x_1, \ldots, x_n\}$ of V. Let $\beta^* = \{f_1, \ldots, f_n\}$. Prove that $\{f_{k+1}, \ldots, f_n\}$ is a basis of W^0.

15. Suppose that W is a finite-dimensional vector space over F and that $T\colon V \longrightarrow W$ is linear. Prove that $N(T^t) = (R(T))^0$.

16. Use Exercises 14 and 15 to deduce that $\operatorname{rank}(L_{A^t}) = \operatorname{rank}(L_A)$ for any $A \in M_{m\times n}(F)$.

17. Let $T\colon V \longrightarrow V$ be a linear transformation and W be a subspace of V. Prove that W is T-invariant (as defined in Exercise 24 of Section 2.1) if and only if W^0 is T^t-invariant.

2.7* HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

As an introduction to this section, let us consider the following physical problem. A weight of mass m is attached to a vertically suspended spring that is allowed to stretch until the forces acting on the weight are in equilibrium. Let us suppose that the weight is now motionless and impose an XY-coordinate system with the weight at the origin and the spring lying on the upper part of the Y-axis. (See Fig. 2.5.)

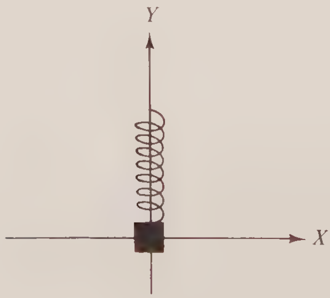


figure 2.5

Suppose that at a certain time, say $t = 0$, the weight is lowered a distance s along the Y-axis and released. The spring will then begin to oscillate.

Let us describe the motion of the spring. At any time $t \geq 0$, let $F(t)$ denote the force acting on the weight and $y(t)$ denote the coordinate of the weight along the Y-axis. For example, $y(0) = -s$. The second derivative of y with respect to time, $y''(t)$, is the acceleration of the weight at time t, and hence by Newton's second law

$$F(t) = my''(t). \tag{1}$$

It is reasonable to assume that the force acting on the weight is totally due to the tension of the spring and that this force satisfies Hooke's law: *The force acting on the weight is proportional to its displacement from the equilibrium position but acts in the opposite direction.* If $k > 0$ is the proportionality constant, then Hooke's law states that

$$F(t) = -ky(t). \tag{2}$$

Combining Eqs. (1) and (2), we obtain

$$my'' = -ky$$

or

$$y'' + \frac{k}{m}y = 0. \tag{3}$$

The expression in Eq. (3) is an example of a "differential equation." A *differential equation* in an unknown function $y = y(t)$ is an equation involving y, t, and derivatives of y. If the differential equation is of the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = f, \tag{4}$$

where $a_0, a_1, \ldots, a_n$ and f are functions of t and $y^{(k)}$ denotes the kth derivative of y, then the equation is said to be *linear*. The functions a_i are called the *coefficients* of the linear differential Eq. (4). Thus Eq. (3) is an example of a linear differential equation in which the coefficients are constants and the function f is identically zero. When the function f in Eq. (4) is identically zero, the linear differential equation is called *homogeneous*.

In this section we shall apply the linear algebra we have studied to solve homogeneous linear differential equations with constant coefficients. If $a_n \neq 0$, we say that the differential equation in Eq. (4) is of *order n*. In this case we may divide both sides by a_n to obtain a new, but equivalent, equation

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1 y^{(1)} + b_0 y = 0,$$

where $b_i = a_i/a_n$ for $i = 0, 1, \ldots, n-1$. Because of this observation we shall always assume that the leading coefficient a_n in Eq. (4) is 1.

A *solution* to Eq. (4) is a function that when substituted for y reduces Eq. (4) to an identity.

Example 37. The function $y(t) = \sin\sqrt{k/m}\,t$ is a solution to Eq. (3) since

$$y''(t) + \frac{k}{m}y(t) = -\frac{k}{m}\sin\sqrt{\frac{k}{m}}t + \frac{k}{m}\sin\sqrt{\frac{k}{m}}t = 0$$

for all t. Notice, however, that substituting $y(t) = t$ into Eq. (3) yields

$$y''(t) + \frac{k}{m}y(t) = \frac{k}{m}t,$$

which is not identically zero. Thus $y(t) = t$ is not a solution to Eq. (3).

While attempting to solve differential equations, we shall discover that it is useful to view solutions as complex-valued functions of a real variable even though the solutions that are meaningful to us in a physical sense are real-valued functions of a real variable. The convenience of this viewpoint will become clear later. Thus we shall be concerned with the vector space $\mathfrak{F}(R, C)$ (as defined in Example 3 of Section 1.2). In order to consider complex-valued functions of a real variable as solutions to differential equations we must define what it means to differentiate such functions. Given a complex-valued function $x \in \mathfrak{F}(R, C)$ of a real variable t, there exist unique real-valued functions x_1 and x_2 of t, such that

$$x(t) = x_1(t) + ix_2(t) \qquad \text{for } t \in R,$$

where i is the purely imaginary number such that $i^2 = -1$. We say that x_1 is the *real part* and x_2 is the *imaginary part* of x.

Definition. *Given a function* $x \in \mathfrak{F}(R, C)$ *with real part* x_1 *and imaginary part* x_2, *we say that* x *is* differentiable *if* x_1 *and* x_2 *are differentiable. If* x *is differentiable, we define the derivative of* x, x', *to be*

$$x' = x_1' + ix_2'.$$

Example 38. If $x(t) = \cos 2t + i \sin 2t$, then

$$x'(t) = -2 \sin 2t + i(2 \cos 2t).$$

We next find the real and imaginary parts of x^2. Since

$$\begin{aligned} x^2(t) &= (\cos 2t + i \sin 2t)^2 = (\cos^2 2t - \sin^2 2t) + i(2 \sin 2t \cos 2t) \\ &= \cos 4t + i \sin 4t, \end{aligned}$$

the real part of $x^2(t)$ is $\cos 4t$, and the imaginary part is $\sin 4t$.

The following theorem indicates that we may limit our investigations to a vector space considerably smaller than $\mathfrak{F}(R, C)$. Its proof, which is illustrated by Example 39, involves a simple induction argument, which we shall omit.

Theorem 2.31. *Any solution to a homogeneous linear differential equation with constant coefficients has derivatives of all orders; that is, if* x *is a solution to such an equation, then* $x^{(k)}$ *exists for every positive integer* k.

Example 39. As an illustration of Theorem 2.31, consider the equation

$$y^{(2)} + 4y = 0.$$

Clearly, to qualify as a solution, a function y must have two derivatives. If y is a solution, however, then

$$y^{(2)} = -4y.$$

Thus since $y^{(2)}$ is a constant multiple of a function that has two derivatives, namely y, $y^{(2)}$ must have two derivatives, and so $y^{(4)}$ exists. In fact

$$y^{(4)} = -4y^{(2)}.$$

Since $y^{(4)}$ is a constant multiple of a function that we have shown has at least two derivatives, it also has at least two derivatives, and hence $y^{(6)}$ exists. Continuing in this manner, we can show that any solution has derivatives of all orders.

Definition. *We shall use* C^∞ *to denote the set of all functions in* $\mathcal{F}(\mathrm{R}, \mathrm{C})$ *that have derivatives of all orders.*

It is a simple exercise to show that C^∞ is a subspace of $\mathcal{F}(R, C)$ and hence a vector space over C. In view of Theorem 2.31 it is this vector space that is of interest to us. For $x \in \mathsf{C}^\infty$ the derivative x' of x also lies in C^∞. We can use the derivative operation to define a mapping $\mathsf{D}\colon \mathsf{C}^\infty \longrightarrow \mathsf{C}^\infty$ by

$$\mathsf{D}(x) = x' \qquad \text{for } x \in \mathsf{C}^\infty.$$

It is easy to show that D is a linear transformation. More generally, consider any polynomial over C of the form

$$p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Then

$$p(\mathsf{D}) = a_n\mathsf{D}^n + a_{n-1}\mathsf{D}^{n-1} + \cdots + a_1\mathsf{D} + a_0\mathsf{I}$$

is a linear transformation. (See Appendix E.)

Definitions. *For any polynomial* p(t) *over* C, p(D) *is called a* differential operator. *The* order *of the differential operator* p(D) *is the degree of the polynomial* p(t).

Differential operators are useful since they provide us with a means of reformulating a differential equation in the context of linear algebra. Any homogeneous linear differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$$

can be rewritten by means of differential operators as

$$(\mathsf{D}^n + a_{n-1}\mathsf{D}^{n-1} + \cdots + a_1\mathsf{D} + a_0\mathsf{I})(y) = 0.$$

Definition. *Given the differential equation above, the complex polynomial*

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$$

is called the auxiliary polynomial *associated with the equation.*

For example, Eq. (3) has auxiliary polynomial

$$p(t) = t^2 + \frac{k}{m}.$$

Any homogeneous linear differential equation with constant coefficients can be rewritten as

$$p(\mathsf{D})(y) = 0,$$

where $p(t)$ is the auxiliary polynomial associated with the equation. Clearly this equation implies the following.

Theorem 2.32. *The set of all solutions to a homogeneous linear differential equation with constant coefficients coincides with the null space of* $p(\mathsf{D})$, *where* $p(t)$ *is the auxiliary polynomial associated with the equation.*

Corollary. *The set of all solutions to a homogeneous linear differential equation with constant coefficients is a subspace of* C^∞.

In view of the corollary above we shall call the set of solutions to a homogeneous linear differential equation with constant coefficients the *solution space* of the equation. A practical way of describing such a space is to find a basis for it. We shall examine a certain class of functions that will be of use in finding bases for these solution spaces.

For a real number s we are familiar with the real number e^s, where e is the unique number whose natural logarithm is 1 (that is, $\ln(e) = 1$). We know, for instance, certain properties of exponentiation:

$$e^{s+t} = e^se^t \quad \text{and} \quad e^{-t} = \frac{1}{e^t}$$

for any real numbers s and t. We shall now extend the definition of powers of e to include complex numbers in such a way that these properties remain true.

Definition. *Let* $c = a + ib$ *be any complex number with real part* a *and imaginary part* b. *Define*

$$e^c = e^a(\cos b + i \sin b).$$

For example, for $c = 2 + i(\pi/3)$,

$$e^c = e^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = e^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

Clearly if c is real ($b = 0$), we obtain the usual result: $e^c = e^a$. It can be shown with the use of trigonometric identities that

$$e^{c+d} = e^c e^d \quad \text{and} \quad e^{-c} = \frac{1}{e^c}$$

for any complex numbers c and d.

Definition. *Let* c *be any complex number. The function* f: R $\longrightarrow$ C *defined by* $f(t) = e^{ct}$ *for all* t *in* R *is called an* exponential function.

The derivative of an exponential function, as described in the following theorem, is as one would expect. The proof involves a straightforward but tedious computation, which we shall leave as an exercise.

Theorem 2.33. *For any exponential function* $f(t) = e^{ct}$, $f'(t) = ce^{ct}$.

We shall use exponential functions to describe all solutions of a homogeneous linear differential equation of order 1. Recall that the *order* of such an equation is the degree of its auxiliary polynomial. Thus an equation of order 1 is of the form

$$y' + a_0 y = 0. \tag{5}$$

Theorem 2.34. *The solution space for Eq. (5) is of dimension 1 and has* $\{e^{-a_0 t}\}$ *as a basis.*

PROOF. Clearly Eq. (5) has $e^{-a_0 t}$ as a solution. Suppose that $x(t)$ is any solution to Eq. (5). Then

$$x'(t) = -a_0 x(t) \qquad \text{for all } t \in R.$$

Define

$$z(t) = e^{a_0 t} x(t).$$

Differentiating z yields

$$z'(t) = (e^{a_0 t})' x(t) + e^{a_0 t} x'(t) = a_0 e^{a_0 t} x(t) - a_0 e^{a_0 t} x(t) = 0.$$

Notice that the familiar product rule for differentiation holds for complex-valued functions of a real variable. A justification involves a lengthy, although direct, computation.

Since z' is identically zero, z is a constant function. Again, this fact, well-known for real-valued functions of a real variable, is also true for complex-valued functions. The proof, which relies on the real case, involves looking separately at the real and imaginary parts of z. Thus there exists a complex number c such that

$$z(t) = e^{a_0 t} x(t) = c \qquad \text{for all } t \in R.$$

So

$$x(t) = ce^{-a_0 t}.$$

We conclude that any member of the solution space of Eq. (5) is a linear combination of $e^{-a_0 t}$. ∎

Another way of formulating Theorem 2.34 is as follows.

Corollary. *For any complex number* c *the null space of the differential operator* $\mathsf{D} - c\mathsf{I}$ *has* $\{e^{ct}\}$ *as a basis.*

We next concern ourselves with differential equations of order greater than one. Given an nth order homogeneous linear differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0,$$

its auxiliary polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

factors into a product of factors of degree 1:

$$p(t) = (t - c_1)(t - c_2) \cdots (t - c_n),$$

where $c_1, c_2, \ldots, c_n$ are (not necessarily distinct) complex numbers. (This follows from the fundamental theorem of algebra in Appendix D.) Thus

$$p(\mathsf{D}) = (D - c_1\mathsf{I})(\mathsf{D} - c_2\mathsf{I}) \cdots (\mathsf{D} - c_n\mathsf{I}).$$

Now the operators $\mathsf{D} - c_i\mathsf{I}$ commute, and so by Exercise 9 we have that

$$\mathsf{N}(\mathsf{D} - c_i\mathsf{I}) \subset \mathsf{N}(p(\mathsf{D})) \qquad \text{for all } i.$$

Since $\mathsf{N}(p(\mathsf{D}))$ coincides with the solution space of the given differential equation, we can conclude the following result by the corollary to Theorem 2.34.

Theorem 2.35. *Let* p(t) *be the auxiliary polynomial for a homogeneous linear differential equation with constant coefficients. For any complex number* c, *if* c *is a zero of* p(t), *then* e^{ct} *is a solution to the differential equation.*

Example 40. Given the differential equation

$$y'' - 3y' + 2y = 0,$$

its auxiliary polynomial $p(t) = t^2 - 3t + 2$ factors as

$$p(t) = (t - 1)(t - 2).$$

Hence by Theorem 2.35 e^t and e^{2t} are solutions to the equation above because $c = 1$ and $c = 2$ are zeros of $p(t)$. Since the solution space of the equation above is a subspace of C^∞, span($\{e^t, e^{2t}\}$) lies in the solution space. It is a simple matter to show that $\{e^t, e^{2t}\}$ is linearly independent. Thus if we could show that the solution space is two-dimensional, we would be able to conclude that $\{e^t, e^{2t}\}$ is a basis for the solution space. This result follows from the following theorem.

Theorem 2.36. *For any differential operator* $\mathsf{p(D)}$ *of order* n, *the null space of* $\mathsf{p(D)}$ *is an* n-*dimensional subspace of* C^∞.

As a preliminary to the proof of Theorem 2.36 we shall establish two lemmas.

Lemma 1. *The differential operator* $\mathsf{D} - c\mathsf{I}$: $\mathsf{C}^\infty \longrightarrow \mathsf{C}^\infty$ *is onto for any complex number* c.

PROOF. Let $x \in \mathsf{C}^\infty$. We wish to find a $y \in \mathsf{C}^\infty$ such that $(\mathsf{D} - c\mathsf{I})y = x$. Define a function w by $w(t) = x(t)e^{-ct}$ for $t \in R$.

Clearly $w \in \mathsf{C}^\infty$ because x and e^{-ct} lie in C^∞. Let w_1 and w_2 be the real and imaginary parts of w. Since $w \in \mathsf{C}^\infty$, w_1 and w_2 are differentiable and hence continuous. Thus they have antiderivatives, say W_1 and W_2, such that $W_1' = w_1$ and $W_2' = w_2$. Define W: $R \longrightarrow C$ by

$$W(t) = W_1(t) + iW_2(t) \qquad \text{for } t \in R.$$

Then $W \in \mathsf{C}^\infty$, and the real and imaginary parts of W are W_1 and W_2, respectively. Also $W' = w$. Finally, define y: $R \longrightarrow C$ by $y(t) = W(t)e^{ct}$ for $t \in R$.

Clearly $y \in \mathsf{C}^\infty$, and since

$$\begin{aligned}(\mathsf{D} - c\mathsf{I})y(t) &= y'(t) - cy(t)\\ &= W'(t)e^{ct} + W(t)ce^{ct} - cW(t)e^{ct}\\ &= w(t)e^{ct}\\ &= x(t)e^{-ct}e^{ct}\\ &= x(t),\end{aligned}$$

$(\mathsf{D} - c\mathsf{I})y = x$. ∎

Lemma 2. *Let* V *be a vector space, and suppose* T *and* U *are linear operators on* V *such that*

(a) U *is onto.*

(b) *The null spaces of* T *and* U *are finite-dimensional.*

Then the null space of TU *is finite-dimensional, and*

$$\dim(\mathsf{N(TU)}) = \dim(\mathsf{N(T)}) + \dim(\mathsf{N(U)}).$$

PROOF. Let $p = \dim(\mathsf{N(T)})$, $q = \dim(\mathsf{N(U)})$, and $\{u_1, u_2, \ldots, u_p\}$ and $\{v_1, v_2, \ldots, v_q\}$ be bases for $\mathsf{N(T)}$ and $\mathsf{N(U)}$, respectively. Since U is onto, we can choose for each i $(i = 1, \ldots, p)$ an element $w_i \in \mathsf{V}$ such that $\mathsf{U}(w_i) = u_i$. Thus we obtain a set of p elements $\{w_1, w_2, \ldots, w_p\}$. Note that for any i and j, $w_i \neq v_j$, for otherwise $u_i = \mathsf{U}(w_i) = \mathsf{U}(v_j) = 0$—a contradiction. Hence the set

$$\beta = \{w_1, w_2, \ldots, w_p, v_1, \ldots, v_q\}$$

contains $p + q$ distinct elements. To prove the lemma, it suffices to show that β is a basis for $\mathsf{N(TU)}$.

We shall first show that β generates $\mathsf{N(TU)}$. Since for any w_i and v_j in β

$$\mathsf{TU}(w_i) = \mathsf{T}(u_i) = 0 \quad \text{and} \quad \mathsf{TU}(v_j) = \mathsf{T}(0) = 0,$$

$$\beta \subseteq \mathsf{N(TU)}.$$

Now suppose $v \in \mathsf{N(TU)}$. Then

$$0 = \mathsf{TU}(v) = \mathsf{T}(\mathsf{U}(v)).$$

Thus $\mathsf{U}(v) \in \mathsf{N(T)}$. So there exist scalars $a_1, a_2, \ldots, a_p$ such that

$$\begin{aligned}\mathsf{U}(v) &= a_1u_1 + a_2u_2 + \cdots + a_pu_p \\ &= \mathsf{U}(a_1w_1 + a_2w_2 + \cdots + a_pw_p).\end{aligned}$$

Hence

$$\mathsf{U}(v - (a_1w_1 + a_2w_2 + \cdots + a_pw_p)) = 0.$$

We conclude that $v - (a_1w_1 + \cdots + a_pw_p)$ lies in $\mathsf{N(U)}$. It follows that there exist scalars $b_1, b_2, \ldots, b_q$ such that

$$v - (a_1w_1 + a_2w_2 + \cdots + a_pw_p) = b_1v_1 + b_2v_2 + \cdots + b_qv_q$$

or

$$v = a_1w_1 + a_2w_2 + \cdots + a_pw_p + b_1v_1 + b_2v_2 + \cdots + b_qv_q.$$

Therefore β spans $\mathsf{N(TU)}$.

We shall next show that β is linearly independent. Let $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q$ be any scalars such that

$$a_1w_1 + a_2w_2 + \cdots + a_pw_p + b_1v_1 + b_2v_2 + \cdots + b_qv_q = 0. \tag{6}$$

Applying U to both sides of Eq. (6), we obtain

$$a_1u_1 + a_2u_2 + \cdots + a_pu_p = 0.$$

Since $\{u_1, u_2, \ldots, u_p\}$ is linearly independent, the a_i's are all zero. Thus Eq. (6) reduces to

$$b_1v_1 + b_2v_2 + \cdots + b_qv_q = 0.$$

Again, the linear independence of $\{v_1, v_2, \ldots, v_q\}$ implies that the b_i's are all zero. We conclude that β is a basis for $\mathsf{N(TU)}$. Hence $\mathsf{N(TU)}$ is finite-dimensional and $\dim(\mathsf{N(TU)}) = p + q = \dim(\mathsf{N(T)}) + \dim(\mathsf{N(U)})$. ▮

PROOF OF THEOREM 2.36. The proof will use a mathematical induction argument on the order of the differential operator $p(\mathsf{D})$. The first-order case coincides with Theorem 2.34. For some integer $n > 1$ suppose Theorem 2.36 holds for any differential operator of order less than n, and suppose we are given a differential operator $p(\mathsf{D})$ of order n. The polynomial $p(t)$ can be factored into a product of two polynomials

$$p(t) = q(t)(t - c)$$

for some polynomial $q(t)$ of degree $n - 1$ and for some complex number c. Thus the given differential operator may be rewritten as

$$p(\mathsf{D}) = q(\mathsf{D})(\mathsf{D} - c\mathsf{I}).$$

By Lemma 1, $\mathsf{D} - c\mathsf{I}$ is onto, by the corollary to Theorem 2.34 $\dim(\mathsf{N}(\mathsf{D} - c\mathsf{I})) = 1$, and by the induction hypothesis $\dim(\mathsf{N}(q(\mathsf{D})) = n - 1$. Thus by applying Lemma 2 we conclude that

$$\begin{aligned}\dim(\mathsf{N}(p(\mathsf{D}))) &= \dim(\mathsf{N}(q(\mathsf{D}))) + \dim(\mathsf{N}(\mathsf{D} - c\mathsf{I}))\\ &= (n - 1) + 1 = n. \quad \blacksquare\end{aligned}$$

Corollary. *For any* n*th-order homogeneous linear differential equation with constant coefficients, the solution space is an* n*-dimensional subspace of* C^∞.

The corollary to Theorem 2.36 reduces the problem of finding all solutions of an nth-order homogeneous linear differential equation with constant coefficients to finding a set of n linearly independent solutions to the equation. By the results of Chapter 1 any such set must be a basis for the solution space. The following theorem enables us to find a basis quickly for many such equations. Hints for its proof are provided in the exercises.

Theorem 2.37. *Given* n *distinct complex numbers* $c_1, c_2, \ldots, c_n$, *the set of exponential functions* $\{e^{c_1 t}, e^{c_2 t}, \ldots, e^{c_n t}\}$ *is linearly independent.*

Corollary. *For any* n*th-order homogeneous linear differential equation with constant coefficients, if its auxiliary polynomial* $p(t)$ *has* n *distinct zeros* $c_1, c_2, \ldots, c_n$, *then the set* $\{e^{c_1 t}, e^{c_2 t}, \ldots, e^{c_n t}\}$ *is a basis for the solution space of the equation.*

PROOF. Exercise.

Example 41. We shall find all solutions to the differential equation

$$y'' + 5y' + 4y = 0.$$

Since the auxiliary polynomial $p(t)$ factors as $(t + 4)(t + 1)$, $p(t)$ has two distinct zeros: -1 and -4. Thus $\{e^{-t}, e^{-4t}\}$ is a basis for the solution space. So any solution to the given equation is of the form

$$y(t) = b_1e^{-t} + b_2e^{-4t} \qquad \text{for some constants } b_1 \text{ and } b_2.$$

Example 42. We shall find all solutions to the differential equation

$$y'' + 9y = 0.$$

The auxilary polynomial $p(t) = t^2 + 9$ factors as $p(t) = (t - 3i)(t + 3i)$ and hence has distinct zeros $c_1 = 3i$, $c_2 = -3i$. Thus $\{e^{3it}, e^{-3it}\}$ is a basis for the solution space. A more useful basis could be obtained by applying Exercise 7. Since

$$\cos 3t = \frac{1}{2}(e^{3it} + e^{-3it}) \quad \text{and} \quad \sin 3t = \frac{1}{2i}(e^{3it} - e^{-3it}),$$

it follows that $\{\cos 3t, \sin 3t\}$ is also a basis. This basis has an advantage over the original one in that it consists of the familiar sine and cosine functions and makes no reference to the imaginary number i.

Next consider the differential equation

$$y'' + 2y' + y = 0,$$

for which the auxiliary polynomial is $p(t) = (t + 1)^2$. By Theorem 2.35, e^{-t} is a solution to the equation above. By the corollary to Theorem 2.36 its solution space is two-dimensional. In order to find a basis for the solution space we need to find a solution that is linearly independent of e^{-t}. The reader can verify that te^{-t} will do. Thus $\{e^{-t}, te^{-t}\}$ is a basis for the solution space. This result can be generalized as follows.

Theorem 2.38. Let $\mathsf{p(t)} = (\mathsf{t} - \mathsf{c})^{\mathsf{n}}$, *where* c *is a complex number and* n *is a positive integer, be the auxiliary polynomial of a homogeneous linear differential equation with constant coefficients. The set*

$$\beta = \{\mathsf{e}^{\mathsf{ct}}, \mathsf{te}^{\mathsf{ct}}, \ldots, \mathsf{t}^{\mathsf{n}-1}\mathsf{e}^{\mathsf{ct}}\}$$

is a basis for the solution space.

PROOF. Since the solution space is n-dimensional, we need only show that β is linearly independent and lies in the solution space. First, observe that for any positive integer k

$$\begin{aligned}(\mathsf{D} - c\mathsf{I})(t^k e^{ct}) &= kt^{k-1}e^{ct} + ct^k e^{ct} - ct^k e^{ct} \\ &= kt^{k-1}e^{ct}.\end{aligned}$$

Hence for $k < n$,

$$(\mathsf{D} - c\mathsf{I})^n(t^k e^{ct}) = 0.$$

It follows that β is a subset of the solution space.

We shall next show that β is linearly independent. Consider any linear combination of β such that

$$b_1 t^{n-1} e^{ct} + b_2 t^{n-2} e^{ct} + \cdots + b_{n-1} t e^{ct} + b_n e^{ct} = 0 \tag{8}$$

for some scalars $b_1, \ldots, b_n$. Dividing Eq. (8) by e^{ct}, we obtain

$$b_1 t^{n-1} + b_2 t^{n-2} + \cdots + b_{n-1} t + b_n = 0. \tag{9}$$

Thus the left-hand side of Eq. (9) must be the zero polynomial function. We conclude that the coefficients $b_1, b_2, \ldots, b_n$ are all zero. Thus β is linearly independent and hence is a basis for the solution space. ∎

Example 43. Given the differential equation

$$y^{(4)} - 4y^{(3)} + 6y^{(2)} - 4y^{(1)} + y = 0,$$

we wish to find a basis for the solution space. Since its auxiliary polynomial is

$$p(t) = t^4 - 4t^3 + 6t^2 - 4t + 1 = (t-1)^4,$$

we can immediately conclude by Theorem 2.38 that $\{e^t, te^t, t^2e^t, t^3e^t\}$ is a basis for the solution space. So any solution to the given equation is of the form

$$y(t) = b_1 e^t + b_2 te^t + b_3 t^2 e^t + b_4 t^3 e^t$$

for some scalars b_1, b_2, b_3, and b_4.

The most general situation (whose proof we leave as an exercise) is stated in the following theorem.

Theorem 2.39. *For a homogeneous linear differential equation with constant coefficients whose auxiliary polynomial is*

$$p(t) = (t - c_1)^{n_1}(t - c_2)^{n_2} \cdots (t - c_k)^{n_k},$$

where $n_1, n_2, \ldots, n_k$ *are positive integers and* $c_1, c_2, \ldots, c_k$ *are distinct complex numbers, the following set is a basis for the solution space of the equation:*

$$\{e^{c_1t}, te^{c_1t}, \ldots, t^{n_1-1}e^{c_1t}, \cdots, e^{c_kt}, te^{c_kt}, \ldots, t^{n_k-1}e^{c_kt}\}.$$

Example 44. Consider the differential equation

$$y^{(3)} - 4y^{(2)} + 5y^{(1)} - 2y = 0.$$

We shall find a basis for its solution space. Since the auxiliary polynomial $p(t)$ factors as

$$p(t) = t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2),$$

we conclude that the solution space to the differential equation above has basis

$$\{e^t, te^t, e^{2t}\}.$$

Thus any solution of the given equation is of the form

$$y(t) = b_1e^t + b_2te^t + b_3e^{2t}$$

for some scalars b_1, b_2, and b_3.

EXERCISES

1. Label the following statements as being true or false.
   (a) The set of solutions to an nth-order homogeneous linear differential equation with constant coefficients is an n-dimensional subspace of C^∞.
   (b) The solution space of a homogeneous linear differential equation is the null space of a differential operator.
   (c) The auxiliary polynomial of a homogeneous linear differential equation with constant coefficients is a solution to the differential equation.
   (d) Any solution to a homogeneous linear differential equation with constant coefficients is of the form ae^{ct} or at^ke^{ct}, where a and c are complex numbers and k is a positive integer.
   (e) Any linear combination of solutions to a given homogeneous linear differential equation with constant coefficients is also a solution to the given equation.
   (f) For any homogeneous linear differential equation with constant coefficients having auxiliary polynomial $p(t)$, if $c_1, c_2, \ldots, c_k$ are the distinct roots of $p(t)$, then $\{e^{c_1t}, e^{c_2t}, \ldots, e^{c_kt}\}$ is a basis for the solution space of the given differential equation.
   (g) Given any polynomial $p(t) \in \mathsf{P}(C)$, there exists a homogeneous linear differential equation with constant coefficients whose auxiliary polynomial is $p(t)$.
2. For each of the following, determine whether the statement is true or false. Justify your claim with either a proof or counter-example, whichever is appropriate.
   (a) Any finite-dimensional subspace of C^∞ is the solution space of a homogeneous linear differential equation with constant coefficients.
   (b) There exists a homogeneous linear differential equation with constant coefficients whose solution space has $\{t, t^2\}$ as a basis.
   (c) For any homogeneous linear differential equation with constant coefficients, if x is a solution to the equation, so is its derivative x'.

Given two polynomials $p(t)$ and $g(t)$ in $\mathsf{P}(C)$, if $x \in \mathsf{N}(p(\mathsf{D}))$ and $y \in \mathsf{N}(q(\mathsf{D}))$, then

(d) $x + y \in \mathsf{N}(p(\mathsf{D})q(\mathsf{D}))$.

(e) $xy \in \mathsf{N}(p(\mathsf{D})q(\mathsf{D}))$.

3. Find bases for the solution spaces of the following differential equations.

(a) $y'' + 2y' + y = 0$

(b) $y''' = y'$

(c) $y^{(4)} - 2y^{(2)} + y = 0$

(d) $y'' + 2y' + y = 0$

(e) $y^{(3)} - y^{(2)} + 3y^{(1)} + 5y = 0$

4. Find bases for the following subspaces of C^∞.

(a) $\mathsf{N}(\mathsf{D}^2 - \mathsf{D} - \mathsf{I})$

(b) $\mathsf{N}(\mathsf{D}^3 - 3\mathsf{D}^2 + 3\mathsf{D} - \mathsf{I})$

(c) $\mathsf{N}(\mathsf{D}^3 + 6\mathsf{D}^2 + 8\mathsf{D})$

5. Show that C^∞ is a subspace of $\mathcal{F}(R, C)$.

6. (a) Show that $\mathsf{D}\colon\ \mathsf{C}^\infty \longrightarrow \mathsf{C}^\infty$ is a linear transformation.

(b) Show that any differential operator is a linear transformation on C^∞.

7. Prove that if $\{x, y\}$ is a basis for a vector space over C, then so is

$$\left\{\frac{1}{2}(x + y), \frac{1}{2i}(x - y)\right\}.$$

8. Given a second-order homogeneous linear differential equation with constant coefficients, suppose that the auxiliary polynomial has distinct conjugate complex roots $a + ib$ and $a - ib$, where $a, b \in R$. Show that $\{e^{at} \cos bt, e^{at} \sin bt\}$ is a basis for the solution space.

9. Given a collection of pairwise commutative linear transformations $\{\mathsf{U}_1, \mathsf{U}_2, \ldots, \mathsf{U}_n\}$ of a vector space V (i.e., transformations such that $\mathsf{U}_i\mathsf{U}_j = \mathsf{U}_j\mathsf{U}_i$ for all i, j), prove that for any $i = 1, 2, \ldots, n$

$$\mathsf{N}(\mathsf{U}_i) \subseteq \mathsf{N}(\mathsf{U}_1\mathsf{U}_2 \cdots \mathsf{U}_n).$$

10. Prove Theorem 2.37 and its corollary. *Hint:* Suppose that

$$b_1e^{c_1t} + b_2e^{c_2t} + \cdots + b_ne^{c_nt} = 0 \quad \text{(where the } c_i\text{'s are distinct).}$$

To show the b_i's are zero, apply mathematical induction on n. Verify the theorem for $n = 1$. Assuming the theorem is true for any $n - 1$ such functions, apply the operator $\mathsf{D} - c_n\mathsf{I}$ to both sides of the equation above to establish the theorem for n distinct exponential functions.

11. Prove Theorem 2.39. *Hint:* First verify that the alleged basis lies in the solution space. Then verify that this set is linearly independent by mathematical induction on k. The case $k = 1$ is Theorem 2.38. Assuming the theorem holds for $k - 1$ distinct c_i's, apply the operator $(\mathsf{D} - c_k\mathsf{I})^{n_k}$ to any linear combination of the alleged basis that equals 0.

12. Let V be the solution space of an nth order homogeneous linear differential equation with constant coefficients having auxiliary polynomial $p(t)$. Prove that if $p(t) = g(t)h(t)$, where $g(t)$ and $h(t)$ are polynomials of positive degree, then

$$\mathsf{N}(h(\mathsf{D})) = \mathsf{R}(g(\mathsf{D}_\mathsf{V})) = g(\mathsf{D})(\mathsf{V}),$$

where $\mathsf{D}_\mathsf{V}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ is defined by $\mathsf{D}_\mathsf{V}(x) = x'$ for $x \in \mathsf{V}$. *Hint:* First prove $g(\mathsf{D})(\mathsf{V}) \subseteq \mathsf{N}(h(\mathsf{D}))$. Then prove that the two spaces have the same finite dimension.

13. A differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y^{(1)} + a_0y = x$$

is called a *nonhomogeneous* linear differential equation with constant coefficients if the coefficients a_i are constant and the right-hand side of the equation, x, is a function that is not identically zero.

(a) Prove that for any $x \in \mathsf{C}^\infty$ there exists a $y \in \mathsf{C}^\infty$ such that y is a solution to the equation above. *Hint:* Use Lemma 1 to Theorem 2.36 to show that if

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0,$$

then $p(\mathsf{D})\colon\ \mathsf{C}^\infty \longrightarrow \mathsf{C}^\infty$ is onto.

(b) Let V be the solution space for the homogeneous linear equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y^{(1)} + a_0y = 0.$$

Prove that if z is any solution to the nonhomogeneous linear differential equation above, then the set of all solutions to the nonhomogeneous linear differential equation is

$$\{z + y\colon\ y \in \mathsf{V}\}.$$

14. Given any nth order homogeneous linear differential equation with constant coefficients, prove that for any solution x and any $t_0 \in R$ if $x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0$, then $x = 0$ (the zero function). *Hint:* Use mathematical induction on n. First prove the conclusion for the case $n = 1$. Next suppose it to be true for equations of order $n - 1$, and consider an nth-order equation with auxiliary polynomial $p(t)$. Factor $p(t)$ as $p(t) = q(t)(t - c)$ for some complex number c and polynomial $q(t)$ of degree $n - 1$. Let $z = q(\mathsf{D})x$. Show that $z(t_0) = 0$ and z is a solution to the equation $y' - cy = 0$. Conclude that $z = 0$. Now apply the induction hypothesis.

15. Let V be the solution space of an nth-order homogeneous linear differential equation with constant coefficients. Fix $t_0 \in R$, and define a mapping

$\Phi\colon \mathsf{V} \longrightarrow \mathsf{C}^n$ by

$$\Phi(x) = \begin{pmatrix} x(t_0) \\ x'(t_0) \\ \cdot \\ \cdot \\ \cdot \\ x^{(n+1)}(t_0) \end{pmatrix} \qquad \text{for each } x \in \mathsf{V}.$$

(a) Prove that Φ is linear and its null space is trivial. Deduce that Φ is an isomorphism. *Hint:* Use Exercise 14.

(b) Prove the following: For any nth-order homogeneous linear differential equation with constant coefficients, any $t_0 \in R$, and any complex numbers $c_0, c_1, \ldots, c_{n-1}$ (not necessarily distinct), there exists exactly one solution, x, to the given differential equation such that $x^{(k)}(t_0) = c_k$ for $k = 0, 1, \ldots, n-1$.

16. *Pendular Motion.* It is well-known that the motion of a pendulum is approximated by the differential equation

$$\theta'' + \frac{g}{l}\theta = 0,$$

where $\theta(t)$ is the angle in radians that the pendulum makes with a vertical line at time t (see Fig. 2.6) interpreted so that θ is positive if the pendulum

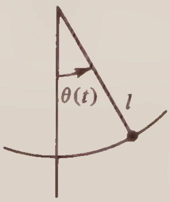


figure 2.6

is to the right and negative if the pendulum is to the left of the vertical line as viewed by the reader. Here l is the length of the pendulum and g is the magnitude of acceleration due to gravity. The variable t and constants l and g must be in compatable units, e.g., t in seconds, l in meters, and g in meters per second per second.

(a) Express an arbitrary solution to this equation as any linear combination of two fixed real-valued functions.

(b) Find the unique solution to the equation that satisfies the conditions

$$\theta(0) = \theta_0 > 0 \quad \text{and} \quad \theta'(0) = 0.$$

(The significance of the above is that at time $t = 0$ the pendulum is displaced from the vertical by θ_0 radians and has zero velocity.)

(c) Prove that it takes $2\pi\sqrt{\dfrac{l}{g}}$ units of time for the pendulum to make one circuit back and forth. (This time is called the *period* of the pendulum.)

17. *Periodic Motion of a Spring with Damping.* At the beginning of this section we discussed the motion of an oscillating spring under the assumption that the only force acting on the spring was the force due to the tension of the spring. We found in this case that Eq. (3) described the motion of the spring.

(a) Find the general form of all solutions of Eq. (3).

If we analyze the behavior of the general solution in part (a), we see that the solution is a periodic function. Hence Eq. (3) indicates that the spring will never stop oscillating. We know from experience, however, that the amplitude of the oscillation will decrease until motion finally ceases. The reason that the solutions in part (a) do not exhibit this behavior is that we ignored the effect of friction on the moving weight. At low speeds such as those under consideration the resistance of the air provides an example of viscous damping—the resistance is proportional to the velocity of the moving weight but opposite in direction. To correct for air resistance, we must add the term $-ry'$ to Eq. (2). The constant $r > 0$ depends on the medium in which the motion takes place (in this case, air), and the term $-ry'$ has a negative sign because the resistance is always opposite to the direction of the motion. Thus the differential equation of motion is $my'' = -ry' - ky$; i.e.,

$$my'' + ry' + ky = 0.$$

(b) Find the general solution of this equation.

(c) Find the unique solution in part (b) that satisfies the initial conditions $y(0) = 0$ and $y'(0) = v_0$.

(d) For $y(t)$ as in part (c), show that the amplitude of the oscillation decreases to zero; i.e., prove that $\lim_{t\to\infty} y(t) = 0$.

18. At the beginning of this section it was stated that it is useful to view solutions to differential equations as complex-valued functions of a real variable even though solutions that are meaningful to us in a physical sense are real-valued. Justify this point of view.

19. The following set of exercises do not involve linear algebra. We list them for the sake of completeness.

(a) Prove Theorem 2.31. *Hint:* Use mathematical induction on the number of derivatives possessed by a solution.

(b) For any $c, d \in C$, prove (i) $e^{c+d} = e^c e^d$.

(ii) $e^{-c} = \dfrac{1}{e^c}$

(c) Prove Theorem 2.33.

(d) Verify the product rule of differentiation for complex-valued functions of a real variable: For any differentiable functions x and y in $\mathfrak{F}(R, C)$ the product xy is differentiable and

$$(xy)' = x'y + xy'.$$

Hint: Find the real and imaginary parts of xy in terms of those of x and y; then differentiate.

(e) Prove that if $x \in \mathfrak{F}(R, C)$ and $x' = 0$, then x is a constant function.

INDEX OF DEFINITIONS FOR CHAPTER 2

chapter 3

elementary matrix operations and systems of linear equations

This chapter is devoted to two related objectives:

1. The study of certain "rank-preserving" operations on matrices.
2. The application of these operations and the theory of linear transformations to the solution of systems of linear equations.

As a consequence of objective 1 we shall obtain a simple method for computing the rank of a linear transformation between finite-dimensional vector spaces by applying these rank-preserving matrix operations to a matrix that represents that transformation.

The solution of systems of linear equations is probably the most important application of linear algebra. The familiar method of elimination for solving systems of linear equations, which was discussed in Section 1.4, involves the elimination of variables so that a simpler system can be obtained. The technique by which the variables are eliminated utilizes three types of operations:

1. Interchanging any two equations in the system.
2. Multiplying any equation in the system by a non-zero constant.
3. Adding a multiple of one equation to another equation.

We shall see in Section 3.3 that a system of linear equations can be expressed as a single matrix equation. In this representation of the system the three operations above are the "elementary row operations" for matrices. These operations will provide a convenient computational method for determining all solutions of a system of linear equations.

3.1 ELEMENTARY MATRIX OPERATIONS AND ELEMENTARY MATRICES

In this section we shall define the elementary matrix operations that will be used throughout the chapter. In subsequent sections we shall use these operations to obtain simple computational methods for determining the rank of a linear transformation and the solutions of a system of linear equations. There are two types of elementary matrix operations—row operations and column operations. As we shall see, the row operations are more useful. They arise from the three operations that can be used to eliminate variables in a system of linear equations.

Let A be an $m \times n$ matrix over a field F. Recall that A can be considered as an array of m rows,

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{pmatrix}$$

or as an array of n columns, $A = (A^1, A^2, \ldots, A^n)$.

Definitions. *Let* A *be an* m $\times$ n *matrix, as above. Any one of the following three operations on the rows* [*columns*] *of* A *is called an* elementary row [column] operation*:*

(a) *Interchanging any two rows* [*columns*] *of* A.

(b) *Multiplying any row* [*column*] *of* A *by a non-zero constant.*

(c) *Adding any constant multiple of a row* [*column*] *of* A *to another row* [*column*].

Any of the three operations above will be called elementary operations. *Elementary operations are either of* type 1, type 2, *or* type 3 *depending on whether they are obtained by* (*a*), (*b*), *or* (*c*).

Example 1. Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Interchanging A_2, the second row of A, with A_1, the first row of A, is an example of an elementary row operation of type 1. The resulting matrix is

$$B = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Again, multiplying A^2, the second column of A, by 3 is an example of an elementary column operation of type 2. The resulting matrix is

$$C = \begin{pmatrix} 1 & 6 & 3 & 4 \\ 2 & 3 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Finally, adding to A_1, the first row of A, four times A_3, the third row of A, is an example of an elementary row operation of type 3. The resulting matrix is

$$D = \begin{pmatrix} 17 & 2 & 7 & 12 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Definition. *An $n \times n$* elementary matrix *is a matrix obtained by performing an elementary operation on* $\mathrm{I_n}$. *The elementary matrix is said to be of* type 1, 2, *or* 3 *according to whether the elementary operation performed on* $\mathrm{I_n}$ *was a type 1, 2, or 3 operation, respectively.*

For example, interchanging the first two rows of I_3 produces the elementary matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that E can also be obtained by interchanging the first two columns of I_3. In fact *any elementary matrix can be obtained in at least two ways—* either by performing an elementary row operation on I_n or by performing an elementary column operation on I_n. Likewise

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an elementary matrix since it can be obtained from I_3 by an elementary column operation of type 3 (adding -2 times the first column of I_3 to the third column) or by an elementary row operation of type 3 (adding -2 times the third row to the first row).

Our first theorem shows that performing an elementary operation on a matrix is equivalent to multiplying the matrix by an elementary matrix.

Theorem 3.1. *Let* $A \in M_{m\times n}(F)$, *and suppose that* B *is obtained from* A *by performing an elementary row [column] operation. Then there exists an* $m \times m$ $[n \times n]$ *elementary matrix* E *such that* $B = EA$ $[B = AE]$. *In fact,* E *is obtained by performing the corresponding row [column] operation on* I_m $[I_n]$. *Conversely, if* E *is an elementary* $m \times m$ $[n \times n]$ *matrix, then* EA $[AE]$ *is a matrix that can be obtained by performing an elementary row [column] operation on* A.

Before considering a proof, we shall first consider an example to illustrate the meaning of the theorem.

Example 2. Consider the matrix B in Example 1. This matrix was obtained from A (in Example 1) by interchanging the first two rows of A. Performing this same operation on I_3, we obtain the elementary matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $EA = B$.

In the second part of Example 1, C is obtained from A by multiplying the second column of A by 3. Performing this same operation on I_4, we obtain the elementary matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that $AE = C$.

PROOF OF THEOREM 3.1. Suppose B is obtained from A by an elementary operation. We must consider six cases, one for each type of row operation and one for each type of column operation.

Suppose first that B is obtained by interchanging the pth and the qth rows of A $(p < q)$, an elementary row operation of type 1. Then

(a) $B_{ij} = A_{ij}$ for $i \neq p$ and $i \neq q$, and $j = 1, 2, \ldots, n$.
(b) $B_{pj} = A_{qj}$ for $j = 1, 2, \ldots, n$.
(c) $B_{qj} = A_{pj}$ for $j = 1, 2, \ldots, n$.

Let E be the $m \times m$ elementary matrix obtained from I_m by interchanging the pth and qth rows of I_m. Then for $i \neq p$ and $i \neq q$ and all j $(1 \leq j \leq m)$:

$$\begin{cases} E_{ij} = 0 & \text{if } i \neq j \\ E_{ij} = 1 & \text{if } i = j. \end{cases}$$

For $i = p$,

$$\begin{cases} E_{pj} = 0 & \text{if } j \neq q \\ E_{pq} = 1. \end{cases}$$

For $i = q$,

$$\begin{cases} E_{qj} = 0 & \text{if } j \neq p \\ E_{qp} = 1. \end{cases}$$

Since

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik}A_{kj},$$

for all j we have

$$\begin{aligned} (EA)_{ij} &= E_{ii}A_{ij} = A_{ij} \qquad \text{if } i \neq p \text{ or } i \neq q \\ (EA)_{pj} &= E_{pq}A_{qj} = A_{qj}, \\ (EA)_{qj} &= E_{qp}A_{pj} = A_{pj}. \end{aligned}$$

Hence

$$B_{ij} = (EA)_{ij} \qquad \text{for all } i \text{ and } j.$$

This establishes Case 1.

If B is obtained from A by an elementary row operation of type 2 or 3, then the proof is similar and will be left as an exercise.

Now suppose that B is obtained from A by performing an elementary column operation on A. Then by Exercise 5, B^t is obtainable from A^t by the corresponding elementary row operation on A. Thus the preceding parts of the proof show that the elementary $n \times n$ matrix M obtained by performing this same row operation on I_n has the property that $B^t = MA^t$. Observe that $E = M^t$ is an elementary matrix that can be obtained by performing the corresponding elementary column operation on I_n. Thus $B = (B^t)^t = (MA^t)^t = AM^t = AE$, establishing the result for column operations.

The proof of the converse is an exercise. ∎

It is a useful fact that the inverse of an elementary matrix is also an elementary matrix.

Theorem 3.2. *Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.*

PROOF. In view of the fact that any elementary $n \times n$ matrix can be obtained by an elementary row operation on I_n, we need consider only three cases—one for each type of operation.

Let E be an elementary $n \times n$ matrix.

CASE 1. Suppose E is obtained by interchanging the pth and the qth rows of I_n ($p \neq q$), an elementary row operation of type 1. It is easy to verify that $E^2 = I_n$. Hence E is invertible, and in fact $E = E^{-1}$. This establishes the first case.

CASE 2. Suppose E is obtained by multiplying the pth row of I_n by a non-zero constant c, an elementary row operation of type 2. Since $c \neq 0$, c has a multiplicative inverse. Let $\bar{E}$ be the elementary matrix obtained from I_n by the elementary row operation of multiplying the pth row of I_n by c^{-1}. It is easily shown that $E\bar{E} = \bar{E}E = I_n$. This establishes the second case.

CASE 3. Suppose E is obtained by adding to the pth row of I_n c times the qth row of I_n, where $p \neq q$ and c is any scalar. Thus E can be obtained from I_n by an elementary row operation of type 3.

Observe that I_n can be obtained from E via an elementary row operation of type 3—namely, by adding to the pth row of E $-c$ times the qth row of E. By Theorem 3.1 there is an elementary matrix $\bar{E}$ (of type 3) such that $\bar{E}E = I_n$. Thus by Exercise 8 of Section 2.4 E is invertible and $E^{-1} = \bar{E}$. ∎

EXERCISES

1. Label the following statements as being true or false.
   (a) An elementary matrix is always square.
   (b) The only entries of an elementary matrix are zeros and ones.
   (c) The $n \times n$ identity matrix is an elementary matrix.
   (d) The product of two $n \times n$ elementary matrices is an elementary matrix.
   (e) The inverse of an elementary matrix is an elementary matrix.
   (f) The sum of two $n \times n$ elementary matrices is an elementary matrix.
   (g) The transpose of an elementary matrix is an elementary matrix.
   (h) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.
   (i) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.
2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

Find an elementary operation which will transform A into B and an elementary operation which will transform B into C. By means of several additional elementary operations, transform C into I_3.

3. Prove the assertion made on page 131: Any elementary $n \times n$ matrix can be obtained in at least two ways—either by performing an elementary row operation on I_n or by performing an elementary column operation on I_n.

4. Prove that E is an elementary matrix if and only if E^t is.

5. Let A be an $m \times n$ matrix. Prove that if B can be obtained from A by an elementary row [column] operation, then B^t can be obtained from A^t by the corresponding elementary column [row] operation.

6. Complete the proof of Theorem 3.1.

7. Verify the assertion made in Case 1 of the proof of Theorem 3.2: If E is an elementary $n \times n$ matrix of type 1, then $E^2 = I_n$.

8. Verify that for the matrix $\bar{E}$ defined in the proof of Case 2 of Theorem 3.2 $E\bar{E} = \bar{E}E = I_n$.

9. Prove that any elementary row [column] operation of type 1 can be obtained by a succession of three elementary row [column] operations of type 3 followed by one elementary row [column] operation of type 2.

10. Prove that any row [column] operation of type 2 can be obtained by *dividing* some row [column] by a non-zero scalar.

11. Prove that any elementary row [column] operation of type 3 can be obtained by *subtracting* a multiple of some row [column] from another row [column].

3.2 THE RANK OF A MATRIX AND MATRIX INVERSES

In this section we shall define the rank of a matrix. We shall then use elementary operations to compute the rank of a matrix or a linear transformation. The section will conclude with a procedure for computing the inverse of an invertible matrix.

Definition. *If* $A \in M_{m \times n}(F)$, *we define the* rank *of* A, *denoted* rank(A), *to be the rank of the linear transformation* $L_A\colon\ F^n \longrightarrow F^m$.

Many results about the rank of matrices follow immediately from the corresponding facts about linear transformations. An important result of this type, which follows from Theorem 2.20 and Corollary 2 of Theorem 2.21, is that an $n \times n$ matrix is invertible if and only if its rank is n.

We would like the definition above to satisfy the condition that the rank of a linear transformation is equal to the rank of any matrix representing that transformation. Our first theorem shows that this condition is, in fact, fulfilled.

Theorem 3.3. *Let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ *be a linear transformation between finite-dimensional vector spaces, and let* β *and* γ *be ordered bases for* V *and* W, *respectively. Then rank*$(\mathsf{T}) =$ *rank*$([\mathsf{T}]_\beta^\gamma)$.

PROOF. This is a restatement of Exercise 18 of Section 2.4. ∎

Corollary 1. *Let* A *be an* m × n *matrix. If* P *and* Q *are any invertible* m × m *and* n × n *matrices, respectively, then rank*(PAQ) = *rank*(A). *In particular, rank*(PA) = *rank*(AQ) = *rank*(A).

PROOF. Let $B = PAQ$. By Exercise 12 of Section 2.5, there exist vector spaces V and W, bases β, β' for V and γ, γ' for W, and a linear transformation $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ such that $A = [\mathsf{T}]_\beta^\gamma$ and $B = [\mathsf{T}]_{\beta'}^{\gamma'}$. Then by Theorem 3.3

$$\text{rank}(PAQ) = \text{rank}(B) = \text{rank}(\mathsf{T}) = \text{rank}(A). \quad ∎$$

Corollary 2. *Elementary row and column operations on a matrix are rank-preserving.*

PROOF. If matrix B is obtained from matrix A by an elementary row operation, then there exists an elementary matrix E such that $B = EA$. By Theorem 3.2, E is invertible, and hence rank$(B) =$ rank(A) by Corollary 1. Thus elementary row operations are rank-preserving. The proof that elementary column operations are rank-preserving is left as an exercise. ∎

Theorem 3.3 intimately relates the rank of a linear transformation to the rank of a matrix. Since matrices are useful tools for studying linear transformations, it is important to develop a method for computing the rank of a matrix. This is our next task.

Theorem 3.4. *The rank of any matrix equals the maximum number of linearly independent columns of that matrix; that is, the rank of a matrix is the dimension of the subspace generated by the columns of that matrix.*

PROOF. For any $A \in \mathsf{M}_{m\times n}(F)$,

$$\text{rank}(A) = \text{rank}(\mathsf{L}_A) = \dim(\mathsf{R}(\mathsf{L}_A)).$$

Let $\beta = \{e_1, e_2, \ldots, e_n\}$ be the standard ordered basis for F^n. Then β spans F^n and hence

$$\mathsf{R}(\mathsf{L}_A) = \text{span}\{\mathsf{L}_A(e_1), \mathsf{L}_A(e_2), \ldots, \mathsf{L}_A(e_n)\}.$$

But we have seen that $\mathsf{L}_A(e_j) = A^j$, the jth column of A. Hence

$$\mathsf{R}(\mathsf{L}_A) = \operatorname{span}\{A^1, A^2, \ldots, A^n\}.$$

Thus

$$\operatorname{rank}(A) = \dim(\mathsf{R}(\mathsf{L}_A)) = \dim(\operatorname{span}\{A^1, A^2, \ldots, A^n\}). \quad \blacksquare$$

Example 3. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Observe that the first and second columns of A are linearly independent and that the third column is a linear combination of the first two. Thus

$$\operatorname{rank}(A) = \dim\left(\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}\right) = 2.$$

In order to compute the rank of a matrix A it is frequently useful to postpone the use of Theorem 3.4 until A has been suitably modified by means of appropriate elementary row and column operations so that the number of linearly independent columns is obvious. Corollary 2 of Theorem 3.3 guarantees that the rank of the modified matrix is the same as the rank of A. One such modification of A can be obtained by using elementary row and column operations to introduce zero entries. The following example illustrates this procedure.

Example 4. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

If we subtract the first row of A from rows 2 and 3 (type 3 elementary row operations), the result is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

If we now subtract twice the first column from the second and subtract the first column from the third (type 3 elementary column operations), we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

It is now obvious that the maximum number of linearly independent columns of this matrix is 2. Hence the rank of A is 2.

The next theorem uses this process of modifying a matrix by means of elementary row and column operations to transform it into a particularly simple form. The power of this theorem can be seen in its corollaries.

Theorem 3.5. *Let* A *be an* $m \times n$ *matrix of rank* r. *Then* $r \leq m$, $r \leq n$, *and by means of a finite number of elementary row and column operations* A *can be transformed into a matrix* D *such that*

(a) $D_{ij} = 0$ *for* $i \neq j$,
(b) $D_{ii} = 1$ *for* $i \leq r$,
(c) $D_{ii} = 0$ *for* $i > r$.

The theorem above and its corollaries are quite important. Its proof, though easy to understand, is tedious to read. As an aid in following the proof we shall first consider an example.

Example 5. Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}.$$

By means of a succession of elementary row and column operations we shall transform A into a matrix D as in Theorem 3.5. We shall list many of the intermediate matrices, but on several occasions a matrix will be transformed from the preceding one by means of several elementary operations. The number above the arrow will indicate how many operations are involved. Try to identify the nature of each operation (row or column and type).

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 4 & 4 & 4 & 8 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow{1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = D.$$

By Corollary 2 of Theorem 3.3, $\text{rank}(A) = \text{rank}(D)$. But clearly $\text{rank}(D) = 3$; so $\text{rank}(A) = 3$. Note that the first two elementary operations result in a 1 in the 1, 1 position, and the next several operations (type 3) result in 0's everywhere in the first row and first column except for the 1, 1 position. Subsequent elementary operations do not change the first row and first column. With this example in mind we shall proceed with the proof.

PROOF OF THEOREM 3.5. If A is the zero matrix, $r = 0$ by Exercise 3. In this case the conclusion follows with $D = A$.

Now suppose $A \neq O$ and $r = \text{rank}(A)$; then $r > 0$. The proof will be by mathematical induction on m, the number of rows of A.

Suppose that $m = 1$. By means of at most one type 1 column operation and at most one type 2 column operation, A can be transformed into a matrix with a 1 in the 1, 1 position. By means of at most $n - 1$ type 3 column operations this matrix, in turn, can be transformed into the matrix

$$D = (1, 0, 0, \ldots, 0).$$

Note that there is a maximum of one linearly independent column in D. So $\text{rank}(D) = \text{rank}(A) = 1$ by Corollary 2 of Theorem 3.3 and Theorem 3.4. Thus the theorem is established for $m = 1$.

Next assume the theorem holds for any matrix with at most $m - 1$ rows (for some $m > 1$). We shall prove that the theorem holds for any matrix with m rows.

Suppose A is any $m \times n$ matrix. If $n = 1$, Theorem 3.5 can be established in a manner analogous to that for $m = 1$. (See Exercise 10.)

We shall suppose that $n > 1$. Since $A \neq O$, $A_{ij} \neq 0$ for some i, j. By means of at most one row and one column operation (each of type 1) we can move the non-zero entry into the 1, 1 position (just as was done in Example 5). By means of at most one additional type 2 operation we can assure a 1 in the 1, 1 position. (Look at the second operation in Example 5.) By means of at most $m - 1$ type 3 row operations and $n - 1$ type 3 column operations we can eliminate all non-zero entries in the first row and the first column with the exception of the 1 in the 1, 1 position. (In Example 5 we used two row and three column operations to do this.)

Thus with a finite number of elementary operations A can be transformed into a matrix

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \cdot & & & \\ \cdot & & B' & \\ \cdot & & & \\ 0 & & & \end{array}\right),$$

where B' is an $(m-1) \times (n-1)$ matrix. In Example 5,

$$B' = \begin{pmatrix} 2 & 4 & 2 & 2 \\ -6 & -8 & -6 & 2 \\ -3 & -4 & -3 & 1 \end{pmatrix}.$$

By Exercise 11, B' has rank one less than B. Since $\operatorname{rank}(A) = \operatorname{rank}(B) = r$, $\operatorname{rank}(B') = r - 1$. By the induction hypothesis $r - 1 \leq n - 1$ and $r - 1 \leq m - 1$. Hence $r \leq m$ and $r \leq n$.

Also by the induction hypothesis B' can be transformed by means of a finite number of elementary row and column operations into an $(m-1) \times (n-1)$ matrix D' such that

$$\begin{aligned} (D')_{i,j} &= 0 \qquad \text{if } i \neq j, \\ (D')_{i,i} &= 1 \qquad \text{if } i \leq r - 1, \\ (D')_{i,i} &= 0 \qquad \text{if } i \geq r. \end{aligned}$$

That is, D' consists of all zeros except for ones in the first $r - 1$ positions of the main diagonal. Let

$$D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \cdot & & & \\ \cdot & & D' & \\ \cdot & & & \\ 0 & & & \end{array}\right).$$

We see that the theorem now follows once we show that D can be obtained from B by means of a finite number of elementary row and column operations. But this follows by the repeated application of Exercise 12.

Thus since A can be transformed into B and B can be transformed into D, each by a finite number of elementary operations, A can be transformed into D by a finite number of elementary operations.

Finally, since D' contains ones in its first $r - 1$ positions along the main diagonal, D contains ones in the first r positions along its main diagonal and zeros elsewhere. Thus $D_{ii} = 1$ if $i \leq r$, $D_{ii} = 0$ if $i > r$, and $D_{ij} = 0$ if $i \neq j$. This establishes the theorem. ∎

Corollary 1. *Let A be an m × n matrix of rank r. Then there exist invertible matrices B and C of dimensions m × m and n × n, respectively, such that D = BAC, where D is an m × n matrix satisfying*

(a) $\mathrm{D_{ij}} = 0$ *if* $\mathrm{i} \neq \mathrm{j}$,
(b) $\mathrm{D_{ii}} = 1$ *if* $\mathrm{i} \leq \mathrm{r}$,
(c) $\mathrm{D_{ii}} = 0$ *if* $\mathrm{i} > \mathrm{r}$.

PROOF. By Theorem 3.5, A can be transformed by means of a finite number of elementary row and column operations into the matrix D. We can appeal to Theorem 3.1 each time we perform an elementary operation. Thus there exist elementary $m \times m$ matrices $E_1, E_2, \ldots, E_p$ and elementary $n \times n$ matrices $G_1, G_2, \ldots, G_q$ such that

$$D = E_pE_{p-1} \cdots E_2E_1AG_1G_2 \cdots G_q.$$

By Theorem 3.2 each E_j and G_j is invertible. Let $B = E_pE_{p-1} \cdots E_1$ and $C = G_1 \cdots G_q$. Then B and C are invertible by Exercise 2 of Section 2.4, and $D = BAC$. ∎

Corollary 2. *Let A be any m × n matrix.*

(a) *rank*$(\mathrm{A^t})$ = *rank*(A).
(b) *The rank of any matrix equals the maximum number of linearly independent rows of that matrix; that is, the rank of a matrix is the dimension of the subspace generated by the rows of that matrix.*
(c) *The rows and columns of any matrix generate subspaces of the same dimension, numerically equal to the rank of the matrix.*

PROOF.

(a) By Corollary 1 there exist invertible matrices B and C such that $D = BAC$, where D satisfies the stated conditions of the corollary. Taking transposes, we have

$$D^t = C^tA^tB^t.$$

Since B and C are invertible, so are B^t and C^t by Exercise 3 of Section 2.4. Hence by Corollary 1 of Theorem 3.3

$$\mathrm{rank}(A^t) = \mathrm{rank}(C^tA^tB^t) = \mathrm{rank}(D^t).$$

Suppose $r = \mathrm{rank}(A)$. Then D^t is an $n \times m$ matrix satisfying the conditions of Corollary 1, and hence $\mathrm{rank}(D^t) = r$ by Theorem 3.4. Thus

$$\mathrm{rank}(A^t) = \mathrm{rank}(D^t) = r = \mathrm{rank}(A).$$

This establishes (a).

The proofs of (b) and (c) are left as exercises. ∎

Corollary 3. *Any invertible matrix is a product of elementary matrices.*

PROOF. If A is an invertible $n \times n$ matrix, then $\text{rank}(A) = n$. Hence by Corollary 1 there exist invertible matrices B and C such that $D = BAC$, where $D_{ij} = 0$ for $i \neq j$ and $D_{ii} = 1$ for $1 \leq i \leq n$. Thus $D = I_n$; that is, $I_n = BAC$.

In the proof of Corollary 1, note also that $B = E_p E_{p-1} \cdots E_1$ and $C = G_1 G_2 \cdots G_q$, where the E_i's and the G_i's are elementary matrices. Thus $A = B^{-1} I_n C^{-1} = B^{-1} C^{-1}$, so that $A = E_1^{-1} E_2^{-1} \cdots E_p^{-1} G_q^{-1} G_{q-1}^{-1} \cdots G_1^{-1}$. But the inverse of an elementary matrix is elementary, and hence A is the product of elementary matrices. ∎

We shall use Corollary 2 above to relate the rank of a matrix product to the rank of each factor. Notice how the proof exploits the relationship between the rank of a matrix and the rank of a linear transformation.

Theorem 3.6. *Let* $\mathsf{T}: \mathsf{V} \longrightarrow \mathsf{W}$ *and* $\mathsf{U}: \mathsf{W} \longrightarrow \mathsf{Z}$ *be linear transformations on finite-dimensional vector spaces* V, W, *and* Z, *and let* A *and* B *be matrices such that the product* AB *is defined. Then*

(a) *rank*$(\mathsf{UT}) \leq$ *rank*(U).
(b) *rank*$(\mathsf{UT}) \leq$ *rank*(T).
(c) *rank*$(\mathrm{AB}) \leq$ *rank*(A).
(d) *rank*$(\mathrm{AB}) \leq$ *rank*(B).

PROOF. Clearly $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{W}$. Hence

$$\mathsf{R}(\mathsf{UT}) = \mathsf{UT}(\mathsf{V}) = \mathsf{U}(\mathsf{R}(\mathsf{T})) \subseteq \mathsf{U}(\mathsf{W}) = \mathsf{R}(\mathsf{U}).$$

Thus

$$\text{rank}(\mathsf{UT}) = \dim(\mathsf{R}(\mathsf{UT})) \leq \dim(\mathsf{R}(\mathsf{U})) = \text{rank}(\mathsf{U}).$$

This establishes part (a).

By part (a)

$$\text{rank}(AB) = \text{rank}(\mathsf{L}_{AB}) = \text{rank}(\mathsf{L}_A \mathsf{L}_B) \leq \text{rank}(\mathsf{L}_A) = \text{rank}(A).$$

This establishes part (c).

By part (c) and Corollary 2 to Theorem 3.5

$$\text{rank}(AB) = \text{rank}((AB)^t) = \text{rank}(B^t A^t) \leq \text{rank}(B^t) = \text{rank}(B).$$

This establishes part (d).

Let α, β, and γ be ordered bases for V, W, and Z, respectively, and let $A' = [\mathsf{U}]_\beta^\gamma$ and $B' = [\mathsf{T}]_\alpha^\beta$. Then $A'B' = [\mathsf{UT}]_\alpha^\gamma$ by Theorem 2.12. Hence by Theorem 3.3 and part (d)

$$\text{rank}(\mathsf{UT}) = \text{rank}(A'B') \leq \text{rank}(B') = \text{rank}(\mathsf{T}).$$

This establishes part (b). ∎

We shall see later that it is important to be able to compute the rank of any matrix. We can use Corollary 2 of Theorem 3.3, Theorems 3.4 and 3.5, and Corollary 2 of Theorem 3.5 to accomplish this goal.

The object is to use elementary row and column operations on a matrix to "simplify" it (so that the transformed matrix has lots of zero entries) to the point where a simple observation enables us to determine how many linearly independent rows or columns the matrix has and thus to determine its rank.

Example 6.

(a) Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Note that the first and second rows of A are linearly independent since one is not a multiple of the other. Thus $\operatorname{rank}(A) = 2$.

(b) Let

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

In this case there are several ways to proceed. Suppose we begin with an elementary row operation to obtain a zero in the 2, 1 position. Subtracting the first row from the second row, we obtain

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

Now note that the third row is a multiple of the second row, and the first and second rows are linearly independent. Thus $\operatorname{rank}(A) = 2$.

As an alternate method, note that the first, third, and fourth columns of A are identical and that the first and second columns of A are linearly independent. Hence $\operatorname{rank}(A) = 2$.

(c) Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

Using various elementary row and column operations, we obtain the following sequence of matrices:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & -3 & -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -5 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

It is clear that the last matrix has three linearly independent rows and hence has rank 3.

In summary, perform row and column operations until the matrix is simplified enough so that the maximum number of linearly independent rows or columns is obvious.

The Inverse of a Matrix

We have remarked that an $n \times n$ matrix is invertible if and only if its rank is n. Since we know how to compute the rank of any matrix, we can always test a matrix to determine if it is invertible. We shall now provide a simple technique for computing the inverse of a matrix that will utilize elementary row operations.

Definition. *Let* A *and* B *be* m $\times$ n *and* m $\times$ p *matrices, respectively. By the* augmented matrix (A | B) *we shall mean the* m $\times$ (n + p) *matrix*

$$(A^1, \ldots, A^n, B^1, \ldots, B^p),$$

where A^i *and* B^j *denote the* i*th column of* A *and the* j*th column of* B, *respectively.*

Let A be an invertible $n \times n$ matrix, and consider the $n \times 2n$ augmented matrix $C = (A \,|\, I_n)$. By Exercise 15 we have

$$A^{-1}C = (A^{-1}A \,|\, A^{-1}I_n) = (I_n \,|\, A^{-1}). \tag{1}$$

By Corollary 3 of Theorem 3.5, A^{-1} is the product of elementary matrices, say $A^{-1} = E_pE_{p-1} \cdots E_1$. Thus Eq. (1) becomes

$$E_pE_{p-1} \cdots E_1(A \,|\, I_n) = A^{-1}C = (I_n \,|\, A^{-1}).$$

Because multiplying a matrix on the left by an elementary matrix transforms the matrix by an elementary row operation (Theorem 3.1), we have the following result: If A is an invertible $n \times n$ matrix, then it is possible to transform the matrix $(A \,|\, I_n)$ into the matrix $(I_n \,|\, A^{-1})$ by means of a finite number of elementary *row* operations.

Conversely, suppose that A is invertible and that the matrix $(A \,|\, I_n)$ can be transformed into the matrix $(I_n \,|\, B)$ by a finite number of elementary row operations. Let $E_1, E_2, \ldots, E_p$ be the elementary matrices associated with these elementary row operations as in Theorem 3.1; then

$$E_p \cdots E_2E_1(A \,|\, I_n) = (I_n \,|\, B). \tag{2}$$

Letting $M = E_p \cdots E_2E_1$, we have from Eq. (2) that

$$M(A\,|\,I_n) = (MA\,|\,M) = (I_n\,|\,B).$$

Hence $MA = I_n$ and $M = B$. It follows that $M = A^{-1}$. So $B = M = A^{-1}$. Thus we have the following result: *If* A *is an invertible* n $\times$ n *matrix and if the matrix* $(A\,|\,I_n)$ *is transformed into a matrix of the form* $(I_n\,|\,B)$ *by means of a finite number of elementary row operations, then* $B = A^{-1}$.

The following example demonstrates this procedure.

Example 7. We shall compute the inverse of the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}.$$

(The reader may wish to verify that rank$(A) = 3$ to be assured that A is invertible.) To compute A^{-1}, we must use elementary row operations to transform

$$(A\,|\,I) = \begin{pmatrix} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{pmatrix}$$

into $(I\,|\,A^{-1})$. The most efficient method for accomplishing this transformation is to change each column of A successively, beginning with the first column, into the corresponding column of I. Since we need a non-zero entry in the 1, 1 position, we shall begin by interchanging rows 1 and 2. The result is

$$\begin{pmatrix} 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

In order to place a 1 in the 1, 1 position, we must multiply the first row by $\frac{1}{2}$; this operation yields

$$\begin{pmatrix} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We now complete work in the first column by adding -3 times row 1 to row 3 to obtain

$$\begin{pmatrix} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{pmatrix}.$$

In order to change the second column of the matrix above into the second column of I, we shall multiply row 2 by $\frac{1}{2}$ to obtain a 1 in the 2, 2 position. This operation produces

$$\begin{pmatrix} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{pmatrix}.$$

We can now complete work on the second column by adding -2 times row 2 to row 1 and 3 times row 2 to row 3. The result is

$$\begin{pmatrix} 1 & 0 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 4 & \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}.$$

Only the third column remains to be changed. In order to place a 1 in the 3, 3 position, we multiply row 3 by $\frac{1}{4}$; this operation yields

$$\begin{pmatrix} 1 & 0 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix}.$$

Adding appropriate multiples of row 3 to rows 1 and 2 completes the process and gives

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix}.$$

Thus

$$A^{-1} = \begin{pmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix}.$$

Being able to compute the inverse of a matrix allows us to compute the inverse of a linear transformation. The following example demonstrates this technique.

Example 8. Let $\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ be defined by $\mathsf{T}(f) = f + f' + f''$, where f' and f'' denote the first and second derivatives of f. It is easily shown that $\mathsf{N}(\mathsf{T}) = \{0\}$, so that T is invertible. Taking $\beta = \{1, x, x^2\}$, we have

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now the inverse of this matrix is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

But $([\mathsf{T}]_\beta)^{-1} = [\mathsf{T}^{-1}]_\beta$ by Corollary 1 of Theorem 2.21. Hence by Theorem 2.16 we have

$$\mathsf{T}^{-1}(a_0 + a_1x + a_2x^2) = (a_0 - a_1) + (a_1 - 2a_2)x + a_2x^2.$$

EXERCISES

1. Label the following statements as being true or false.
   (a) The rank of a matrix is equal to the number of its non-zero columns.
   (b) The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.
   (c) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.
   (d) Elementary row operations preserve rank.
   (e) Elementary column operations do not necessarily preserve rank.
   (f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
   (g) The inverse of a matrix can be computed exclusively by means of elementary row operations.
   (h) An $n \times n$ matrix is of rank at most n.
   (i) An $n \times n$ matrix having rank n is invertible.

2. Find the rank of the following matrices:

   (a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

   (f) $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

3. Prove that for any $m \times n$ matrix A, $\operatorname{rank}(A) = 0$ if and only if A is the zero matrix.

4. Use elementary row and column operations to transform each of the following matrices into a matrix D satisfying the conditions of Theorem 3.5, and then determine the rank of each matrix.

(a) $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$

5. For each of the following matrices compute the rank and compute the inverse if it exists.

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$

6. For each of the following linear transformations T, determine if T is invertible and compute T^{-1} if it exists.
(a) T: $\mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ defined by $\mathsf{T}(f) = f'' + 2f' - f$
(b) T: $\mathsf{R}^3 \longrightarrow \mathsf{R}^3$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + 2a_2 + a_3, -a_1 + a_2 + 2a_3, a_1 + a_3)$$

(c) T: $\mathsf{R}^3 \longrightarrow \mathsf{P}_2(R)$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1x^2$$

7. Express the invertible matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

as a product of elementary matrices.

8. Let A be an $m \times n$ matrix. Prove that if c is any non-zero scalar, then $\operatorname{rank}(cA) = \operatorname{rank}(A)$.

9. Complete the proof of Corollary 2 of Theorem 3.3 by showing that elementary column operations preserve rank.

10. Prove Theorem 3.5 for the case that A is an $m \times 1$ matrix.

11. Let

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right),$$

where B' is an $m \times n$ submatrix. Prove that if $\text{rank}(B) = r$, then $\text{rank}(B') = r - 1$.

12. Let B' and D' be $m \times n$ matrices, and let B and D be the $(m + 1) \times (n + 1)$ matrices defined by

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right) \quad \text{and} \quad D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right).$$

Prove that if B' can be transformed into D' by an elementary row [column] operation, then B can be transformed into D by an elementary row [column] operation.

13. Prove parts (b) and (c) of Corollary 2 of Theorem 3.5.

14. Let $\mathsf{T}, \mathsf{U}\colon \ \mathsf{V} \longrightarrow \mathsf{W}$ be linear transformations. Prove that

(a) $\mathsf{R}(\mathsf{T} + \mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})$.

(b) If W is finite-dimensional, then $\text{rank}(\mathsf{T} + \mathsf{U}) \le \text{rank}(\mathsf{T}) + \text{rank}(\mathsf{U})$.

(c) Deduce from part (b) that, for any $m \times n$ matrices A and B, $\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B)$.

15. If A and B are matrices containing n rows, prove that $M(A \mid B) = (MA \mid MB)$ for any $m \times n$ matrix M.

16. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exist a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

3.3 SYSTEMS OF LINEAR EQUATIONS—THEORETICAL ASPECTS

This section and the next are devoted to the study of systems of linear equations, which arise naturally in both the physical and social sciences. In this section we shall apply results from Chapter 2 to describe the solution sets of systems of linear equations as subsets of a vector space. In Section

3.4, elementary row operations will be used to provide a computational method for finding all solutions to such systems.

The system of equations

$$(S)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

where a_{ij} and b_i ($1 \leq i \leq m$ and $1 \leq j \leq n$) are elements of a field F and $x_1, x_2, \ldots, x_n$ are n variables taking values in F, is called *a system of* m *linear equations in* n *unknowns over the field* F.

The $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

is called the *coefficient matrix* of the system (S).

If we let

$$X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix},$$

then the system (S) may be rewritten as a single matrix equation

$$AX = B.$$

To exploit the results that we have developed, we shall frequently consider a system of equations as a single matrix equation.

A *solution* to system (S) is an n-tuple

$$s = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix} \in \mathsf{F}^n$$

such that $As = B$. The set of all solutions to system (S) is called the *solution set* of the system.

Example 9.

(a) Consider the system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = 1. \end{cases}$$

By use of familiar techniques we can solve the system above and conclude that there is only one solution: $x_1 = 2$, $x_2 = 1$; i.e.,

$$s = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In matrix form the system can be written

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix};$$

so

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

(b) Consider

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = 6; \end{cases}$$

i.e.,

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

This system has many solutions such as

$$s = \begin{pmatrix} -6 \\ 2 \\ 7 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix}.$$

(c) Consider

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1; \end{cases}$$

i.e.,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is evident that this system has no solutions. Thus we see that a system of linear equations can have one, many, or no solutions.

We must be able to recognize when a system has solutions and then be able to describe all its solutions. This section and the next are devoted to this end.

We shall begin our study of systems of equations by examining the class of "homogeneous" systems of linear equations. As we shall see (Theorem 3.7), the set of solutions of a homogeneous system of m linear equations in n unknowns forms a subspace of F^n. We can then apply the theory of vector spaces to this set of solutions. For example, a basis for the solution space can be found, and any solution can be expressed as a linear combination of the basis vectors.

Definitions. *A system* $AX = B$ *of* m *equations and* n *unknowns is said to be* homogeneous *if* $B = 0$. *Otherwise the system is said to be* non-homogeneous.

Any homogeneous system has at least one solution, namely,

$$s = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

This solution is called the *trivial solution.* The next result gives further information about the set of solutions to a homogeneous system.

Theorem 3.7. *Let* $AX = 0$ *be a homogeneous system of* m *linear equations in* n *unknowns over a field* F. *Let* K *denote the set of all solutions to* $AX = 0$. *Then* $K = N(L_A)$; *hence* K *is a subspace of* F^n *of dimension* $n - \text{rank}(L_A) = n - \text{rank}(A)$.

PROOF. $K = \{s \in F^n\colon\ As = 0\} = N(L_A)$. The second part now follows from Theorem 2.3. ▮

Corollary. *If* $m < n$, *the system* $AX = 0$ *has a non-trivial solution.*

PROOF. Suppose $m < n$. Then $\text{rank}(A) = \text{rank}(L_A) \leq m$. Hence $\dim(K) = n - \text{rank}(L_A) \geq n - m > 0$. Since $\dim(K) > 0$, $K \neq \{0\}$. Thus there exists $s \in K$, $s \neq 0$. Then s is a non-trivial solution to $AX = 0$. ▮

Example 10.

(a) Consider the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0. \end{cases}$$

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

be the coefficient matrix. It is clear that $\text{rank}(A) = 2$. If K is the solution set of the system, then $\dim(K) = 3 - 2 = 1$. Thus any non-zero solution will constitute a basis for K. For example, since

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

is a solution,

$$\left\{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right\}$$

is a basis. Thus any element of K is of the form

$$t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ 3t \end{pmatrix},$$

where $t \in R$.

(b) Consider the system $x_1 - 2x_2 + x_3 = 0$ of one equation in three unknowns. If $A = (1, -2, 1)$ is the coefficient matrix, $\operatorname{rank}(A) = 1$. Hence if K is the solution set, $\dim(\mathsf{K}) = 3 - 1 = 2$. Note that

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

are linearly independent elements of K. Thus they constitute a basis for K, so that

$$\mathsf{K} = \left\{t_1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}: \quad t_1, t_2 \in R\right\}.$$

In Section 3.4 we shall discuss explicit computational methods for finding a basis for the solution set of a homogeneous system.

We shall now turn to the study of non-homogeneous systems. Our next result shows that the solution set of a non-homogeneous system $AX = B$ can be described in terms of the solution set of the homogeneous system $AX = 0$. We shall refer to the equation $AX = 0$ as the *homogeneous system corresponding to* $AX = B$.

Theorem 3.8. *Let* K *be the solution set of a system of linear equations* $AX = B$, *and let* K_H *be the solution set of the corresponding homogeneous system* $AX = 0$. *Then for any solution* s *of* $AX = B$

$$\mathsf{K} = \{s\} + \mathsf{K}_H = \{s + k: \quad k \in \mathsf{K}_H\}.$$

PROOF. Let s be any solution of $AX = B$. We must show that $K = \{s\} + \mathsf{K}_H$. If $w \in K$, then $Aw = B$. Hence $A(w - s) = Aw - As = B - B = 0$. So $w - s \in \mathsf{K}_H$. Thus there exists $k \in \mathsf{K}_H$ such that $w - s = k$. So $w = s + k \in \{s\} + \mathsf{K}_H$, and therefore

$$K \subseteq \{s\} + \mathsf{K}_H.$$

Conversely, suppose that $w \in \{s\} + \mathsf{K}_\mathsf{H}$; then $w = s + k$ for some $k \in \mathsf{K}_\mathsf{H}$. But then $Aw = A(s + k) = As + Ak = B + 0 = B$; so $w \in K$. Therefore $\{s\} + \mathsf{K}_\mathsf{H} \subseteq K$, and thus $K = \{s\} + \mathsf{K}_\mathsf{H}$. ▮

Example 11.

(a) Consider the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 7 \\ x_1 - x_2 - x_3 = -4. \end{cases}$$

The homogeneous system corresponding to the above is the system given in Example 10(a). It is easily verified that

$$s = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

is a solution to the non-homogeneous system above. So the solution set to the system is

$$K = \left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} : \quad t \in R \right\}$$

by Theorem 3.8.

(b) Consider the system $x_1 - 2x_2 + x_3 = 4$. The homogeneous system corresponding to this system is given in Example 10(b). Since

$$s = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

is a solution to this system, the entire solution set K can be written as

$$K = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} : \quad t_1, t_2 \in R \right\}.$$

Although Section 3.4 is reserved for computational methods, the following theorem does provide us with a means of computing solutions to certain systems of equations.

Theorem 3.9. *Let* $\mathrm{AX} = \mathrm{B}$ *be a system of* n *equations and* n *unknowns. If* A *is invertible, then the system has* exactly one *solution, namely* $\mathrm{A}^{-1}\mathrm{B}$. *Conversely, if the system has exactly one solution, then* A *is invertible.*

PROOF. Suppose that A is invertible. Substituting $A^{-1}B$ into the system, we have $A(A^{-1}B) = (AA^{-1})B = B$. Thus $A^{-1}B$ is a solution. If s is an arbitrary solution, then $As = B$. Multiplying both sides by A^{-1} gives

$s = A^{-1}B$. Thus the system has one and only one solution, namely $A^{-1}B$.

Conversely, suppose the system has exactly one solution s. Let $\mathsf{K_H}$ denote the solution set for the corresponding homogeneous system $AX = 0$. By Theorem 3.8, $\{s\} = \{s\} + \mathsf{K_H}$. But this can only occur if $\mathsf{K_H} = \{0\}$. Thus $\mathsf{N}(\mathsf{L}_A) = \{0\}$, and hence A is invertible. ∎

Example 12. Consider the system of three equations in three unknowns:

$$(S)\quad \begin{cases} \phantom{2x_1 + {}} 2x_2 + 4x_3 = 2 \\ 2x_1 + 4x_2 + 2x_3 = 3 \\ 3x_1 + 3x_2 + x_3 = 1. \end{cases}$$

In Example 7 we computed the inverse of the coefficient matrix A of this system. Thus (S) has exactly one solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}B = \begin{pmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{8} \\ \frac{5}{4} \\ -\frac{1}{8} \end{pmatrix}.$$

We shall use this technique for solving systems of linear equations having invertible coefficient matrices in the application that concludes this section.

In Example 9(c) we saw a system of linear equations that has no solutions. We shall now establish a criterion for determining when a system has solutions. This criterion involves the rank of the coefficient matrix of the system $AX = B$ and the rank of the matrix $(A \mid B)$. The matrix $(A \mid B)$ is called the *augmented matrix of the system* $\mathrm{AX} = \mathrm{B}$.

Theorem 3.10. *Let* $\mathrm{AX} = \mathrm{B}$ *be a system of linear equations. Then the system has at least one solution if and only if* $\mathit{rank}(\mathrm{A}) = \mathit{rank}(\mathrm{A} \mid \mathrm{B})$.

PROOF. To say that $AX = B$ has a solution is equivalent to saying that $B \in \mathsf{R}(\mathsf{L}_A)$. In the proof of Theorem 3.4 we saw that $\mathsf{R}(\mathsf{L}_A) = \operatorname{span}\{A^1, A^2, \ldots, A^n\}$, the span of the columns of A. Thus $AX = B$ has a solution if and only if $B \in \operatorname{span}\{A^1, A^2, \ldots, A^n\}$. But $B \in \operatorname{span}\{A^1, A^2, \ldots, A^n\}$ if and only if $\operatorname{span}\{A^1, A^2, \ldots, A^n\} = \operatorname{span}\{A^1, A^2, \ldots, A^n, B\}$. This last statement is equivalent to

$$\dim(\operatorname{span}\{A^1, A^2, \ldots, A^n\}) = \dim(\operatorname{span}\{A^1, A^2, \ldots, A^n, B\}).$$

So by Theorem 3.4 the equation above reduces to

$$\operatorname{rank}(A) = \operatorname{rank}(A \mid B). \quad ∎$$

Example 13. Recall the system of equations

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$$

in Example 9(c).

Since

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad (A\,|\,B) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$\operatorname{rank}(A) = 1$ and $\operatorname{rank}(A\,|\,B) = 2$. Because the two ranks are unequal, the system has no solutions.

Example 14. We shall use Theorem 3.10 to determine if (3, 3, 2) is in the range of the linear transformation $\mathsf{T}\colon\ \mathsf{R}^3 \longrightarrow \mathsf{R}^3$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + a_2 + a_3, a_1 - a_2 + a_3, a_1 + a_3).$$

Now $(3, 3, 2) \in \mathsf{R}(\mathsf{T})$ if and only if there exists a vector $s = (x_1, x_2, x_3) \in \mathsf{R}^3$ such that $\mathsf{T}(s) = (3, 3, 2)$. Such a vector s must be a solution of the system

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 - x_2 + x_3 = 3 \\ x_1 \phantom{{}+x_2} + x_3 = 2. \end{cases}$$

Since the ranks of the coefficient matrix and the augmented matrix of this system are 2 and 3, respectively, it follows that this system has no solutions. Hence $(3, 3, 2) \notin \mathsf{R}(\mathsf{T})$.

An Application

In 1973 Wassily Leontief won the Nobel Prize for Economics for his work in developing a mathematical model that may be used to describe various economic phenomena. We shall close this section by applying some of the ideas we have studied to illustrate two special cases of his work.

We shall begin by considering a simple society composed of three people (industries)—a farmer who grows all the food, a tailor who makes all the clothing, and a carpenter who builds all the housing. We shall assume that each person sells to and buys from a central pool and that everything produced is consumed. Since no commodities either enter or leave the system, this case is referred to as the *closed model.*

Each of the three individuals will consume all three of the commodities produced in the society. Suppose that the proportion of each of the commodities consumed by each person is as given in the table below. Notice that each of the columns of the table must sum to 1.

	Food	Clothing	Housing
Farmer	0.40	0.20	0.20
Tailor	0.10	0.70	0.20
Carpenter	0.50	0.10	0.60

Let p_1, p_2, and p_3 denote the incomes of the farmer, tailor, and carpenter, respectively. To assure that this society survives, we shall require that the consumption of each individual equals his income. In the case of the farmer, this requirement may be translated into the equation $0.40p_1 + 0.20p_2 + 0.20p_3 = p_1$. Thus we shall need to consider the system of linear equations

$$\begin{cases} 0.40p_1 + 0.20p_2 + 0.20p_3 = p_1 \\ 0.10p_1 + 0.70p_2 + 0.20p_3 = p_2 \\ 0.50p_1 + 0.10p_2 + 0.60p_3 = p_3 \end{cases}$$

or, equivalently, $AP = P$, where

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

and A is the coefficient matrix of the system. In this context A is called the *input-output* (or *consumption*) *matrix*, and $AP = P$ is called the *equilibrium condition.*

For matrices B and C of the same size we shall use the notation $B \geq C$ $[B > C]$ to mean $B_{ij} \geq C_{ij}$ $[B_{ij} > C_{ij}]$ for all i and j. B will be called *non-negative* [*positive*] if $B \geq O$ $[B > O]$, where O is the zero matrix.

At first it may seem reasonable to replace the equilibrium condition by the inequality $AP \leq P$, that is, the requirement that consumption not exceed production. But in fact $AP \leq P$ implies that $AP = P$ in the closed model. For otherwise there exists a k for which

$$p_k > \sum_j A_{kj}p_j.$$

Hence, since the columns of A sum to 1,

$$\sum_i p_i > \sum_i \sum_j A_{ij}p_j = \sum_j \left(\sum_i A_{ij}\right)p_j = \sum_j p_j,$$

which is a contradiction.

One solution of the homogeneous system $(I - A)X = 0$ equivalent to the equilibrium condition is

$$P = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}.$$

We may interpret this to mean that the society will survive if the farmer, tailor, and carpenter have incomes in the proportions 25: 35: 40 (or 5: 7: 8).

Notice that we are not simply interested in a non-trivial solution to the system but in one that is non-negative. Thus we must consider the question of whether or not the system $(I - A)X = 0$ has a non-negative solution, where A is a non-negative matrix whose columns sum to 1. A useful theorem in this direction [whose proof may be found in "Applications of Matrices to Economic Models and Social Science Relationships" by Ben Noble, *Proceedings of the Summer Conference for College Teachers on Applied Mathematics* (1971), CUPM, Berkeley, California] is stated below.

Theorem 3.11. *Let* A *be an* $n \times n$ *input-output matrix having the form*

$$\mathrm{A} = \begin{pmatrix} \mathrm{B} & \mathrm{C} \\ \mathrm{D} & \mathrm{E} \end{pmatrix},$$

where D *is a* $1 \times (n-1)$ *positive matrix and* C *is an* $(n-1) \times 1$ *positive matrix. Then* $(\mathrm{I} - \mathrm{A})\mathrm{X} = 0$ *has a one-dimensional solution set that is generated by a non-negative vector.*

Observe that any positive input-output matrix satisfies the hypothesis of this theorem. The matrix below does also.

$$\begin{pmatrix} 0.75 & 0.50 & 0.65 \\ 0 & 0.25 & 0.35 \\ 0.25 & 0.25 & 0 \end{pmatrix}$$

In the *open model* we assume that there is an outside demand for each of the commodities produced. Returning to our simple society, let x_1, x_2, and x_3 be the amounts of food, clothing, and housing produced with respective outside demands d_1, d_2, and d_3. Let A be the 3×3 matrix such that A_{ij} represents the proportion of commodity i consumed in producing commodity j. Then the surplus of food in the society is

$$x_1 - (A_{11}x_1 + A_{12}x_2 + A_{13}x_3),$$

that is, the amount of food produced minus the amount of food consumed in producing the three commodities. The assumption that everything produced is consumed gives us a similar equilibrium condition for the open model, namely, that the surplus of each of the three commodities must equal the corresponding outside demands. Hence

$$x_i - \sum_{j=1}^{3} A_{ij}x_j = d_i \qquad \text{for } j = 1, 2, \text{ and } 3.$$

In general, we must find a non-negative solution to the system $(I - A)X = D$, where A and D are non-negative matrices and the sum of the entries of each column of A does not exceed one. It is easy to see that if $(I - A)^{-1}$ exists and is non-negative, then the desired solution will be $(I - A)^{-1}D$.

Recall that for a real number a the series $1 + a + a^2 + \cdots$ converges to $(1 - a)^{-1}$ if $|a| < 1$. Likewise it can be shown (using the concept of convergence of matrices developed in Section 5.3) that the series $I +$

$A + A^2 + \cdots$ converges to $(I - A)^{-1}$ if A^n converges to the zero matrix. In this case $(I - A)^{-1}$ will be non-negative since the matrices $I, A, A^2, \ldots$ are non-negative.

To illustrate the open model, suppose that 30% of the food is used to produce food, 20% to produce clothing, and 30% to produce housing. Likewise suppose that 10% of the clothing is used to produce food, 40% to produce clothing, and 10% to produce housing. Finally suppose that 30% of the housing is used to produce food, 20% to produce clothing, and 30% to produce housing. Then the input-output matrix is

$$A = \begin{pmatrix} 0.30 & 0.20 & 0.30 \\ 0.10 & 0.40 & 0.10 \\ 0.30 & 0.20 & 0.30 \end{pmatrix},$$

and so

$$I - A = \begin{pmatrix} 0.70 & -0.20 & -0.30 \\ -0.10 & 0.60 & -0.10 \\ -0.30 & -0.20 & 0.70 \end{pmatrix} \quad \text{and} \quad (I - A)^{-1} = \begin{pmatrix} 2.0 & 1.0 & 1.0 \\ 0.5 & 2.0 & 0.5 \\ 1.0 & 1.0 & 2.0 \end{pmatrix}.$$

Since $(I - A)^{-1}$ is non-negative, we can find a (unique) non-negative solution to $(I - A)X = D$ for any demand D. For instance, if

$$D = \begin{pmatrix} 30 \\ 20 \\ 10 \end{pmatrix},$$

then

$$X = (I - A)^{-1}D = \begin{pmatrix} 90 \\ 60 \\ 70 \end{pmatrix}.$$

So a gross production of 90 units of food, 60 units of clothing, and 70 units of housing must be produced to meet a demand for 30 units of food, 20 units of clothing, and 10 units of housing.

EXERCISES

1. Label the following statements as being true or false.
   (a) Any system of linear equations has at least one solution.
   (b) Any system of linear equations has at most one solution.
   (c) Any homogeneous system of linear equations has at least one solution.
   (d) Any system of n linear equations in n unknowns has at most one solution.

(e) Any system of n linear equations in n unknowns has at least one solution.
(f) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.
(g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-trivial solutions.
(h) The solution set of any system of m linear equations in n unknowns is a subspace of F^n.

2. For each of the following systems of linear equations, find the dimension and a basis for the solution set.

(a) $\begin{cases} x_1 + x_2 - x_3 = 0 \\ 4x_1 + x_2 - 2x_3 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - 2x_3 = 0 \end{cases}$

(c) $x_1 + 2x_2 - 3x_3 + x_4 = 0$

(d) $\begin{cases} x_1 + 2x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$

3. Using the results of Exercise 2, find all solutions to the following systems.

(a) $\begin{cases} x_1 + x_2 - x_3 = 1 \\ 4x_1 + x_2 - 2x_3 = 3 \end{cases}$

(b) $\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ x_1 - x_2 + x_3 = 1 \\ x_1 + 2x_2 - 2x_3 = 4 \end{cases}$

(c) $x_1 + 2x_2 - 3x_3 + x_4 = 1$

(d) $\begin{cases} x_1 + 2x_2 = 5 \\ x_1 - x_2 = -1 \end{cases}$

4. Let A denote the coefficient matrix of

$$\begin{cases} x_1 + 2x_2 - x_3 = 5 \\ x_1 + x_2 + x_3 = 1 \\ 2x_1 - 2x_2 + x_3 = 4. \end{cases}$$

(a) Show that A is invertible.
(b) Compute A^{-1}.
(c) Use A^{-1} to solve the system.

5. Give an example of a system of n equations and n unknowns with infinitely many solutions.

6. Let $\mathsf{T}\colon \mathsf{R}^3 \longrightarrow \mathsf{R}^2$ be defined by $\mathsf{T}(a, b, c) = (a + b, 2a - c)$. Describe $\mathsf{T}^{-1}\{(1, 11)\}$.

7. Determine which of the following systems of linear equations has a solution.

(a) $\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 2 \\ x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + x_3 + 2x_4 = 4 \end{cases}$ (b) $\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 2 \end{cases}$

(c) $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ x_1 + x_2 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 3 \end{cases}$ (d) $\begin{cases} x_1 + x_2 + 3x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = 1 \\ 4x_1 + x_2 + 8x_3 - x_4 = 0 \end{cases}$

8. Prove that a system $AX = B$ of m linear equations in n unknowns has a solution if and only if $B \in \mathsf{R}(\mathsf{L}_A)$.
9. Prove or give a counter-example to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.
10. In the closed model of Leontief with food, clothing, and housing as the basic industries, suppose that the input-output matrix is

$$A = \begin{pmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

At what ratio must the farmer, tailor, and carpenter produce in order for equilibrium to be attained?
11. In the notation of the open model of Leontief, suppose that

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} \end{pmatrix}$$

and that the demand vector is $D = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. How much of each commodity must be produced to satisfy this demand?

3.4 SYSTEMS OF LINEAR EQUATIONS—COMPUTATIONAL ASPECTS

In Section 3.3 we obtained a necessary and sufficient condition for a system of linear equations to have solutions (Theorem 3.10) and learned how to express the solutions of a non-homogeneous system in terms of solutions to the corresponding homogeneous system (Theorem 3.8). The latter result enables us to determine all the solutions to a given system if we can find one solution of the given system and a basis for the solution set of the corresponding homogeneous system. In this section we shall use elementary row operations to accomplish these two objectives. The essence of this technique is to transform a given system of linear equations into a system having the same solutions but which is easier to solve (as in Section 1.4).

Definition. *Two systems of* m *linear equations in* n *unknowns are called* equivalent *if they have the same solution set.*

The following theorem and corollary give a useful method for obtaining equivalent systems.

Theorem 3.12. *Let* (S): $AX = B$ *be a system of* m *linear equations in* n *unknowns, and let* C *be any invertible* $m \times m$ *matrix. Then the system* (S'): $(CA)X = CB$ *is equivalent to* (S).

PROOF. Let K be the solution set for (S) and K' the solution set for (S'). If $w \in K$, then $Aw = B$. So $CAw = CB$, and hence $w \in K'$. Thus $K \subseteq K'$.

Conversely, if $w \in K'$, then $CAw = CB$. Hence $Aw = C^{-1}(CAw) = C^{-1}(CB) = B$, and so $w \in K$. Thus $K' \subseteq K$, and therefore $K = K'$. ∎

Corollary. *Let* $AX = B$ *be a system of* m *linear equations in* n *unknowns. If* $(A'|B')$ *is obtained from* $(A|B)$ *by a finite number of elementary row operations, then the system* $A'X = B'$ *is equivalent to the original system.*

PROOF. Suppose that $(A'|B')$ is obtained from $(A|B)$ by elementary row operations. These may be executed by multiplying by elementary $m \times m$ matrices $E_1, \ldots, E_p$. Let $C = E_p \cdots E_1$; then $(A'|B') = C(A|B) = (CA|CB)$. Since each E_i is invertible, so is C. Now $A' = CA$ and $B' = CB$. Thus by Theorem 3.12 the system $A'X = B'$ is equivalent to the system $AX = B$. ∎

Example 15. To find all solutions to

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 = 2 \\ x_1 + x_2 + x_3 \phantom{{}-2x_4} = 3 \\ 3x_1 + 2x_2 + 3x_3 - 2x_4 = 1, \end{cases}$$

construct the augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 2 & 3 & -2 & 1 \end{pmatrix}$$

of the system, and "simplify" it by a sequence of elementary row operations as follows:

(a) Put a 1 in the first row, first column. (This is already the case.)

(b) By means of type 3 operations, use the first row to obtain zeros in the remaining positions of the first column. The resulting matrix is

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -4 & 0 & 1 & -5 \end{pmatrix}.$$

For the remaining operations the first row is not used.

(c) Next (using the remaining rows) put a 1 in the second row and in the leftmost column possible—in this case the second coiumn. Then use type 3 row operations to obtain zeros in the remaining positions of this column. These operations give

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -3 & -9 \end{pmatrix}.$$

(d) Finally, using the remaining row only, put a 1 in the third row and in the leftmost column possible—in this case the fourth column. By means of type 3 operations, use this 1 to produce zeros in the remaining positions of the fourth column to obtain

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

This last matrix can be translated into a system of equations

$$\begin{cases} x_1 + \quad x_3 \quad = 1 \\ \quad x_2 \quad\quad = 2 \\ \quad\quad\quad x_4 = 3 \end{cases}$$

equivalent to the given system. Obviously $x_2 = 2$ and $x_4 = 3$. But x_1 and x_3 can have any values provided that their sum is 1. Letting $x_3 = t$, we then have $x_1 = 1 - t$. Thus an arbitrary solution is of the form

$$\begin{pmatrix} 1-t \\ 2 \\ t \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Observe that

$$\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

is a basis for the homogeneous system of equations corresponding to the given system.

In the example above we performed elementary row operations on the augmented matrix of the system until we obtained the augmented matrix of a system having properties 1, 2, and 3 on p. 27. Such a matrix has a special name.

Definition. *A matrix is said to be in* row echelon form *if the following three conditions are satisfied:*

(a) *Any row containing a non-zero entry precedes any row in which all the entries are zero (if any).*

(b) *The first non-zero entry in each row is the only non-zero entry in its column.*

(c) *The first non-zero entry in each row is* 1 *and it occurs in a column to the right of the leading* 1 *in any preceding row.*

Example 16.

(a) The first matrix in Example 15(d) is in row echelon form. Note that the first non-zero entry of each row is 1 and that the column containing this first entry has all zeros otherwise. Also note that each time we move downward to a new row, we must move to the right at least one (and possibly more) column(s) to find the first non-zero entry of the new row.

(b) The following are *not* in row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

because the first column contains more than one non-zero entry,

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

because the first non-zero entry of the second row is not to the right of the first non-zero entry of the first row, and

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

because the first non-zero entry of the first row is not 1.

The ease with which the final system of equations in Example 15 was solved is due to the fact that the augmented matrix of this system is in row echelon form. We shall soon present a procedure for solving any system of linear equations for which the augmented matrix is in row echelon form. First, however, we shall establish that every matrix can be transformed into a matrix in row echelon form by elementary row operations.

Theorem 3.13. *Any matrix can be transformed into a matrix in row echelon form by means of a finite number of elementary row operations.*

PROOF. The proof will be by induction on the number of columns in the matrix. We shall leave as an exercise the proof of the result for matrices having one column. Suppose then that the conclusion is valid for matrices containing n columns for some integer $n \geq 1$, and let A be an $m \times (n+1)$ matrix. Write A in the form $A = (A' \mid B)$, where B is the last column of A and A' is the $m \times n$ matrix obtained by deleting the last column of A. By the induction hypothesis, A' can be transformed into a matrix Q in row echelon form by means of a finite number of elementary row operations. Let C denote the product of the elementary matrices corresponding to these row operations. Then

$$CA = C(A' \mid B) = (CA' \mid CB) = (Q \mid B'),$$

where $B' = CB$. Clearly $(Q \mid B')$ is in row echelon form unless it contains a row of the form $(0 \quad \ldots \quad 0 \quad a)$, where $a \neq 0$. By multiplying such a row by a^{-1}, adding appropriate multiples of this row to the other rows, and performing the appropriate row interchange, for some j we can transform $(Q \mid B')$ into a matrix $(Q \mid e_j)$ in row echelon form by means of a finite number of elementary row operations. This completes the induction. ▌

It can be shown (see Exercise 9) that each matrix has a *unique* row echelon form; that is, if different sequences of elementary row operations transform a matrix into matrices Q and Q' in row echelon form, then $Q = Q'$.

We shall now describe a method for solving a system in which the augmented matrix is in row echelon form. To illustrate this procedure, we shall consider the system

$$\begin{cases} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 = 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 = 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 = 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 = 14 \end{cases}$$

for which the augmented matrix is

$$\begin{pmatrix} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{pmatrix}.$$

The following sequence of matrices obtained by row operations illustrates how the augmented matrix is transformed into row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system of equations (equivalent to the original one) associated with this last matrix (considered as an augmented matrix) is

$$\begin{cases} x_1 \qquad + 2x_3 \qquad - 2x_5 = 3 \\ \qquad x_2 - x_3 \qquad + x_5 = 1 \\ \qquad\qquad\qquad x_4 - 2x_5 = 2. \end{cases}$$

Notice that we have ignored the last row since it consists entirely of zeros.

To solve a system for which the augmented matrix is in row echelon form, divide the variables $x_1, x_2, \ldots, x_5$ into two sets. The first set consists of those variables that appear as leftmost variables in one of the equations of the system (in this case the set is $\{x_1, x_2, x_4\}$). The second set consists of all the remaining variables (in this case, $\{x_3, x_5\}$). To each variable in the second set, assign a parametric value $t_1, t_2, \ldots$ ($x_3 = t_1$, $x_5 = t_2$), and then solve for the variables of the first set in terms of those in the second set:

$$\begin{aligned} x_1 &= -2x_3 + 2x_5 + 3 = -2t_1 + 2t_2 + 3 \\ x_2 &= x_3 - x_5 + 1 = t_1 - t_2 + 1 \\ x_4 &= \qquad\quad 2x_5 + 2 = \qquad\quad 2t_2 + 2. \end{aligned}$$

Thus an arbitrary solution, s, is of the form

$$s = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2t_1 + 2t_2 + 3 \\ t_1 - t_2 + 1 \\ t_1 \\ 2t_2 + 2 \\ t_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

where $t_1, t_2 \in R$. Notice that

$$\left\{\begin{pmatrix}-2\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-1\\0\\2\\1\end{pmatrix}\right\}$$

forms a basis for the solution set of the corresponding homogeneous system of equations.

To use this process to solve a system of m equations in n unknowns $AX = B$, first test to see if $\text{rank}(A) = \text{rank}(A \mid B)$. If this equality does not hold, then the system has no solutions. Next (provided the system has solutions), use elementary row operations to transform the augmented matrix $(A \mid B)$ into row echelon form, $(A' \mid B')$. Discard the zero rows in $(A' \mid B')$, and rewrite the system of equations associated with $(A' \mid B')$. Solve the system as described above. You will finish with an arbitrary solution of the form

$$s = s_0 + t_1u_1 + t_2u_2 + \cdots + t_{n-m'}u_{n-m'},$$

where m' is the number of non-zero rows in A', $(m' \leq m)$. The equation above suggests that an arbitrary solution, s, can be expressed in terms of $n - m'$ parameters. The following theorem states that s cannot be expressed in fewer than $n - m'$ parameters.

Theorem 3.14. *Let* $\mathrm{AX} = \mathrm{B}$ *be a system of* m *non-zero equations in* n *unknowns. Suppose that rank*$(\mathrm{A}) =$ *rank*$(\mathrm{A} \mid \mathrm{B})$ *and that* $(\mathrm{A} \mid \mathrm{B})$ *is in row echelon form. Then*

(a) *rank*$(\mathrm{A}) = \mathrm{m}$.

(b) *If the general solution as obtained by the procedure above is of the form*

$$\mathrm{s} = \mathrm{s}_0 + \mathrm{t}_1\mathrm{u}_1 + \mathrm{t}_2\mathrm{u}_2 + \cdots + \mathrm{t}_{\mathrm{n-m}}\mathrm{u}_{\mathrm{n-m}},$$

then $\{\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_{\mathrm{n-m}}\}$ *is a basis for the solution set of the corresponding homogeneous system and* s_0 *is a solution of the original system.*

PROOF. Since $(A \mid B)$ is in row echelon form, $\text{rank}(A \mid B) = \text{rank}(A) = m$ by Exercises 5 and 6.

Let K be the solution set for $AX = B$, and let K_H be the solution set for $AX = 0$. Setting $t_1 = t_2 = \cdots = t_{n-m} = 0$, $s = s_0 \in K$. But by Theorem 3.8, $K = \{s_0\} + \mathsf{K}_\mathsf{H}$. Hence $\mathsf{K}_\mathsf{H} = K - \{s_0\} = \text{span}(\{u_1, u_2, \ldots, u_{n-m}\})$.

Since $\text{rank}(A) = m$, $\dim(\mathsf{K}_\mathsf{H}) = n - m$. Thus since $\dim(\mathsf{K}_\mathsf{H}) = n - m$ and K_H is generated by a set $\{u_1, u_2, \ldots, u_{n-m}\}$, containing at most $n - m$ elements, we conclude that the set above is a basis for K_H. ∎

EXERCISES

1. Label the following statements as being true or false.

(a) If $(A' \mid B')$ is obtained from $(A \mid B)$ by a finite sequence of elementary column operations, then the systems $AX = B$ and $A'X = B'$ are equivalent.

(b) If $(A' \mid B')$ is obtained from $(A \mid B)$ by a finite sequence of elementary row operations, then the systems $AX = B$ and $A'X = B'$ are equivalent.

(c) If A is an $n \times n$ matrix with rank n, then the row echelon form of A is I_n.

(d) Any matrix can be put in row echelon form by means of a finite sequence of elementary row operations.

(e) If $(A \mid B)$ is in row echelon form, then the system $AX = B$ must have a solution.

(f) Let $AX = B$ be a system of m linear equations in n unknowns for which the augmented matrix is in row echelon form. If this system has solutions, then the dimension of the solution set of $AX = 0$ is $n - m'$, where m' equals the number of non-zero rows in A.

(g) If a matrix A is transformed by elementary row operations into a matrix A' in row echelon form, then the number of non-zero rows in A' is equal to the rank of A.

2. Find all the solutions to the systems of equations in Exercises 2, 3, and 4 of Section 3.3 by the technique used in this section.

3. Suppose that the augmented matrix of the system $AX = B$ is transformed into a matrix $(A' \mid B')$ in row echelon form by a finite sequence of elementary row operations.

(a) Prove that $\text{rank}(A') \neq \text{rank}((A' \mid B'))$ if and only if $(A' \mid B')$ contains a row in which the only non-zero entry lies in the last column.

(b) Deduce that $AX = B$ has solutions if and only if $(A' \mid B')$ contains no row in which the only non-zero entry lies in the last column.

4. For each of the following systems, apply Exercise 3 to determine if the system has solutions. If there are solutions, find all of them. Finally, find a basis for the corresponding homogeneous system.

(a) $\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - 3x_3 + 2x_4 = 2 \end{cases}$ (b) $\begin{cases} x_1 + x_2 - 3x_3 + x_4 = -2 \\ x_1 + x_2 + x_3 - x_4 = 2 \\ x_1 + x_2 - x_3 = 0 \end{cases}$

(c) $\begin{cases} x_1 + x_2 - 3x_3 + x_4 = 1 \\ x_1 + x_2 + x_3 - x_4 = 2 \\ x_1 + x_2 - x_3 = 0 \end{cases}$

5. Prove that if A is a matrix in row echelon form, then rank(A) equals the number of non-zero rows in A.

6. If $(A \mid B)$ is in row echelon form, prove that A is also in row echelon form.

7. Prove Theorem 3.13 for matrices having only one column.

8. Prove Theorem 3.13 as follows. Let A be an $m \times n$ matrix with rank r. Since the result is immediate if $r = 0$, assume $r > 0$. Let β be the standard ordered basis for F^n, define $\mathsf{W}_k = \operatorname{span}\{\mathsf{L}_A(e_1), \mathsf{L}_A(e_2), \ldots, \mathsf{L}_A(e_k)\}$ for $1 \le k \le n$, and define $k_j = \min\{i\colon \dim(\mathsf{W}_i) = j\}$ for $1 \le j \le r$. Prove that $k_1 < k_2 < \cdots < k_r$ and $k_j \ge j$ for all j. Let $z_j = \mathsf{L}_A(e_{k_j})$, and prove that $\{z_1, z_2, \ldots, z_r\}$ is linearly independent. Extend this set to a basis β' for F^m. Let $B = [\mathsf{L}_A]_\beta^{\beta'}$, and prove the following:
   (a) $B = CA$ for some invertible $m \times m$ matrix C.
   (b) B is in row echelon form.
   (c) B can be obtained from A by a finite number of elementary row operations.

9. (a) Prove that if Q and Q' are $m \times n$ matrices each in row echelon form such that Q can be transformed into Q' by means of a finite number of elementary row operations, then $Q = Q'$. *Hint:* Use induction on n.
   (b) Deduce that if A is any matrix, then there is a unique matrix in row echelon form that can be obtained from A by a finite number of elementary row operations.

INDEX OF DEFINITIONS FOR CHAPTER 3

chapter 4

determinants

At one time determinants played a major role in the study of linear algebra. Now, however, they are of much less importance. We shall see, in fact, that virtually our only use of the determinant will be in the computation of "eigenvalues." For this reason the important facts about the determinant needed for later chapters are summarized in Section 4.5. The reader who is not interested in pursuing a development of the theory of determinants may proceed immediately to that section.

The determinant of a square matrix with entries from a field F is a scalar (element of F). Thus we may regard the determinant as a function having domain $\mathsf{M}_{n\times n}(F)$ and taking values in F. Although the determinant of a square matrix can be defined in terms of the entries of the matrix, the resulting definition is cumbersome to use for computations. Instead of defining the determinant in this manner, in Section 4.2 we shall define a determinant as a function $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ possessing three important properties. In that section we shall also verify that the familiar method of evaluating a determinant by expansion along a column is, in fact, a determinant in the sense of our definition. Section 4.3 contains further properties of a determinant and proves that there is a unique determinant on $\mathsf{M}_{n\times n}(F)$, i.e., that the three defining properties of a determinant are satisfied by one and only one function from $\mathsf{M}_{n\times n}(F)$ into F. Section 4.4 uses the determinant to find the inverse of an invertible matrix and to solve systems of linear equations having an invertible coefficient matrix by means of Cramer's rule.

The chapter begins with a discussion of the general theory in a simple setting. In this section we shall also investigate the geometric significance of the determinant in terms of area and orientation. Readers who have studied advanced calculus will recall that a change of coordinates in multiple integrals necessitated the use of a determinant called the Jacobian.

4.1 DETERMINANTS OF ORDER 2

Eventually we shall assign to each $n \times n$ matrix with entries from a field F a scalar called the "determinant" of the matrix, but first we shall consider an easy special case.

Definition. The determinant *of a* 2×2 *matrix* A *with entries from a field* F *is the scalar* $A_{11}A_{22} - A_{12}A_{21}$, *which we shall denote by* det(A).

Example 1. Consider the following element of $M_{2\times 2}(R)$:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Then

$$\det(A) = 1\cdot 4 - 2\cdot 3 = -2.$$

In the discussion that follows it will be convenient to represent a 2×2 matrix A in terms of its rows; as before, we shall write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

and denote its determinant by

$$\det\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

The determinant has the following important properties.

Theorem 4.1. The determinant of a 2×2 *matrix satisfies the following three conditions:*

(a) *The determinant is a linear function of each row when the other row is held fixed; that is,*

$$\det\begin{pmatrix} cA_1 + A_1' \\ A_2 \end{pmatrix} = c \det\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \det\begin{pmatrix} A_1' \\ A_2 \end{pmatrix}$$

and

$$det\begin{pmatrix} A_1 \\ cA_2 + A_2' \end{pmatrix} = c\, det\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + det\begin{pmatrix} A_1 \\ A_2' \end{pmatrix}$$

for all scalars c *in* F.

(b) *If* $A \in M_{2\times 2}(F)$ *has identical rows, then* $det(A) = 0$.
(c) *If* I *is the* 2×2 *identity matrix, then* $det(I) = 1$.

PROOF.
(a) Let $A_1 = (A_{11}\ \ A_{12})$, $A_1' = (A_{11}'\ \ A_{12}')$, and $A_2 = (A_{21}\ \ A_{22})$; then

$$\begin{aligned}
\det\begin{pmatrix} cA_1 + A_1' \\ A_2 \end{pmatrix} &= \det\begin{pmatrix} cA_{11} + A_{11}' & cA_{12} + A_{12}' \\ A_{21} & A_{22} \end{pmatrix} \\
&= (cA_{11} + A_{11}')A_{22} - (cA_{12} + A_{12}')A_{21} \\
&= c(A_{11}A_{22} - A_{12}A_{21}) + (A_{11}'A_{22} - A_{12}'A_{21}) \\
&= c\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \det\begin{pmatrix} A_{11}' & A_{12}' \\ A_{21} & A_{22} \end{pmatrix} \\
&= c\det\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \det\begin{pmatrix} A_1' \\ A_2 \end{pmatrix}.
\end{aligned}$$

A similar argument proves that the determinant is also a linear function of the second row.

(b) If the rows of A are identical, then A has the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{11} & A_{12} \end{pmatrix}.$$

So $\det(A) = A_{11}A_{12} - A_{12}A_{11} = 0$.

(c) Since

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\det(I) = 1\cdot 1 - 0\cdot 0 = 1. \quad \blacksquare$$

The next result shows that the three properties mentioned in Theorem 4.1 completely characterize the determinant as defined above.

Theorem 4.2. Let δ: $M_{2\times 2}(F) \longrightarrow F$ *be any function having the following three properties:*
(a) δ *is a linear function of each row when the other row is held fixed.*
(b) *If* $A \in M_{2\times 2}(F)$ *has identical rows, then* $\delta(A) = 0$.
(c) *If* I *is the* 2×2 *identity matrix, then* $\delta(I) = 1$.

Then $\delta = det$; *that is,* $\delta(A) = A_{11}A_{22} - A_{12}A_{21}$ *for each* $A \in M_{2\times 2}(F)$.

PROOF. Let I denote the 2×2 identity matrix, and let

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $\delta(M_1) = \delta(M_2) = 0$ by property (b). We shall first prove that $\delta(M_3) = -1$. Using properties (b) and (a), we have

$$\begin{aligned}
0 = \delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} &= \delta\begin{pmatrix} 1+0 & 0+1 \\ 1 & 1 \end{pmatrix} \\
&= \delta\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \delta\begin{pmatrix} 1 & 0 \\ 0+1 & 1+0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 0+1 & 1+0 \end{pmatrix} \\
&= \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \delta(I) + \delta(M_1) + \delta(M_2) + \delta(M_3) \\
&= 1 + 0 + 0 + \delta(M_3).
\end{aligned}$$

Thus $\delta(M_3) = -1$.

Now let A be an arbitrary element of $\mathsf{M}_{2\times 2}(F)$; then

$$\begin{aligned}
\delta(A) = \delta\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} &= \delta\begin{pmatrix} A_{11}+0 & 0+A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\
&= \delta\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} + \delta\begin{pmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\
&= \delta\begin{pmatrix} A_{11} & 0 \\ 0+A_{21} & A_{22}+0 \end{pmatrix} + \delta\begin{pmatrix} 0 & A_{12} \\ 0+A_{21} & A_{22}+0 \end{pmatrix} \\
&= \delta\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} + \delta\begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix} + \delta\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \\
&= A_{11}A_{22}\cdot\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A_{11}A_{21}\cdot\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + A_{12}A_{22}\cdot\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\
&\qquad + A_{12}A_{21}\cdot\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= A_{11}A_{22}\cdot\delta(I) + A_{11}A_{21}\cdot\delta(M_1) + A_{12}A_{22}\cdot\delta(M_2) + A_{12}A_{21}\cdot\delta(M_3) \\
&= A_{11}A_{22}(1) + A_{11}A_{21}(0) + A_{12}A_{22}(0) + A_{12}A_{21}(-1) \\
&= A_{11}A_{22} - A_{12}A_{21} = \det(A).
\end{aligned}$$

So $\delta = \det$. ∎

Motivated by this characterization of the determinant of a 2×2 matrix, in Section 4.2 we shall define a determinant on $M_{n \times n}(F)$ as a function possessing the three properties of Theorem 4.1. But first we shall use this uniqueness property to study the geometric significance of the determinant of a 2×2 matrix. In particular, we shall find that the sign of the determinant is of geometric importance in the study of orientation.

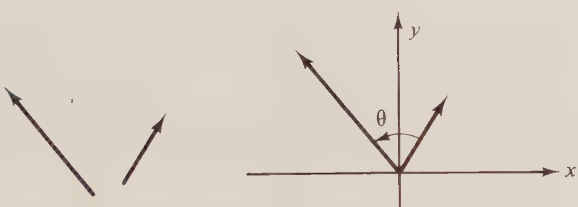
By the *angle between two vectors* in R^2 we mean the angle θ with measure such that $0 \leq \theta < \pi$ formed by the vectors having the same magnitude and direction as the given vectors but emanating from the origin. (See Fig. 4.1.) Given three vectors u, v, and w emanating from the same point, we shall say that v lies *between* u and w if the angle between u and w equals the sum of the angles between u and v and between v and w. (See Fig. 4.2.)

Given an ordered basis $\beta = \{u, v\}$ for R^2, where $u = (a_1, a_2)$ and $v = (b_1, b_2)$, we denote by

$$\det\begin{pmatrix} u \\ v \end{pmatrix}$$

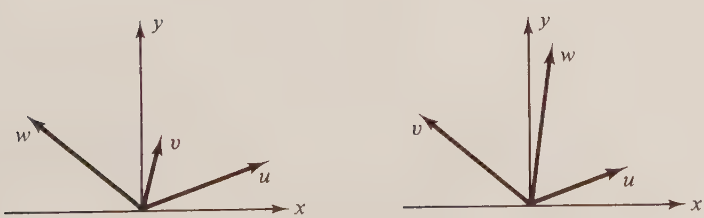
the scalar

$$\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$



Angle between two vectors in R^2

figure 4.1



v lies between u and w.

v does not lie between u and w. (Here w lies between u and v.)

figure 4.2

and define the *orientation* of β to be the real number

$$O\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}.$$

(It follows from Exercise 10 that the denominator is not zero.) Clearly

$$O\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1.$$

Notice that

$$O\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 1 \quad \text{and} \quad O\begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} = -1.$$

In general (see Exercise 11),

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

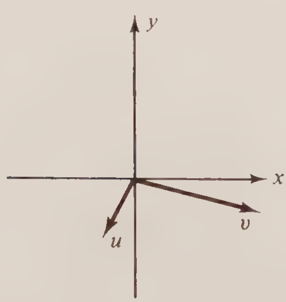
if and only if the ordered basis $\{u, v\}$ forms a right-handed coordinate system, and

$$O\begin{pmatrix} u \\ v \end{pmatrix} = -1$$

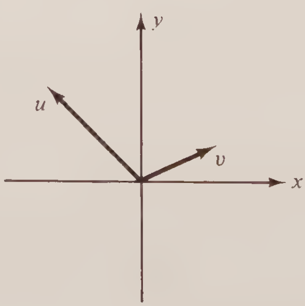
if and only if $\{u, v\}$ forms a left-handed coordinate system. (Recall that a coordinate system $\{u, v\}$ is right-handed if u can be rotated to coincide with v by rotating in a counterclockwise direction through an angle θ with measure such that $0 < \theta < \pi$; otherwise $\{u, v\}$ is a left-handed coordinate system. See Fig. 4.3.) For convenience, we define

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if $\{u, v\}$ is linearly dependent.



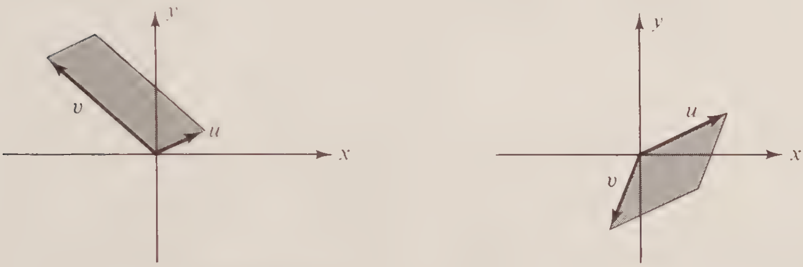
A right-handed coordinate system



A left-handed coordinate system

figure 4.3

Any ordered set $\{u, v\}$ in R^2 determines a parallelogram in the following manner. Regarding u and v as arrows emanating from the origin of R^2, we call the parallelogram having u and v as adjacent edges the *parallelogram determined by* u *and* v. (See Fig. 4.4.)



Parallelograms determined by u and v

figure 4.4

Observe that if the set $\{u, v\}$ is linearly dependent, i.e., if u and v are parallel, then the "parallelogram" determined by u and v is actually a line segment, which we shall consider to be a degenerate parallelogram having area zero.

There is an interesting relationship between

$$A\begin{pmatrix} u \\ v \end{pmatrix},$$

the area of the parallelogram determined by u and v, and

$$\det\begin{pmatrix} u \\ v \end{pmatrix},$$

which we shall now investigate. Observe first, however, that since

$$\det\begin{pmatrix} u \\ v \end{pmatrix}$$

may be negative, we cannot expect that

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

But we can prove that

$$A\begin{pmatrix} u \\ v \end{pmatrix} = O\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix},$$

from which it follows that

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|.$$

In arguing that

$$\mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix},$$

we shall employ a technique that, though somewhat indirect, can be generalized to R^n. First, since

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1,$$

we may multiply both sides of the desired equation by

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix}$$

to obtain the equivalent form

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

We shall establish this equation by verifying that the three conditions of Theorem 4.2 are satisfied by the function

$$\delta\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix}.$$

(a) We shall begin by showing that

$$\delta\begin{pmatrix} u \\ \lambda v \end{pmatrix} = \lambda \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}.$$

Observe that this conclusion is immediate if $\lambda = 0$ because

$$\delta\begin{pmatrix} u \\ \lambda v \end{pmatrix} = \mathrm{O}\begin{pmatrix} u \\ 0 \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ 0 \end{pmatrix} = 0.$$

So assume that $\lambda \neq 0$. Regarding λv as the base of the parallelogram determined by u and λv, we see that

$$\mathrm{A}\begin{pmatrix} u \\ \lambda v \end{pmatrix} = \text{base} \times \text{altitude} = |\lambda|(\text{length of } v)(\text{altitude}) = |\lambda| \cdot \mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix}$$

since the altitude h of the parallelogram determined by u and λv is the same as that in the parallelogram determined by u and v. (See Fig. 4.5.) Hence

$$\begin{aligned} \delta\begin{pmatrix} u \\ \lambda v \end{pmatrix} &= \mathrm{O}\begin{pmatrix} u \\ \lambda v \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ \lambda v \end{pmatrix} = \left[\frac{\lambda}{|\lambda|} \cdot \mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix}\right]\left[|\lambda| \cdot \mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix}\right] \\ &= \lambda \cdot \mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}. \end{aligned}$$

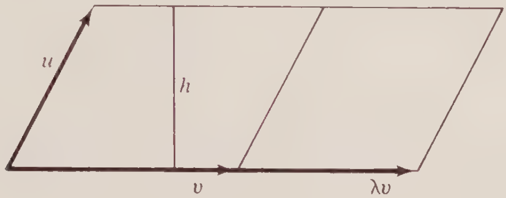


figure 4.5

A similar argument shows that

$$\delta\begin{pmatrix}\lambda u\\ u\end{pmatrix} = \lambda\cdot\delta\begin{pmatrix}u\\ v\end{pmatrix}.$$

We shall show next that

$$\delta\begin{pmatrix}u\\ au+bw\end{pmatrix} = b\cdot\delta\begin{pmatrix}u\\ w\end{pmatrix}$$

for any $u, w \in \mathsf{R}^2$ and any real numbers a and b. Observe that since the parallelograms determined by u and w and by u and $u + w$ have a common base u and the same altitude (see Fig. 4.6),

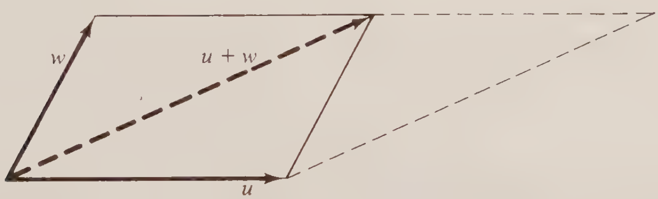


figure 4.6

$$\mathrm{A}\begin{pmatrix}u\\ w\end{pmatrix} = \mathrm{A}\begin{pmatrix}u\\ u+w\end{pmatrix}.$$

If $a = 0$, then

$$\delta\begin{pmatrix}u\\ au+bw\end{pmatrix} = \delta\begin{pmatrix}u\\ bw\end{pmatrix} = b\cdot\delta\begin{pmatrix}u\\ w\end{pmatrix}$$

by the first paragraph of part (a). Otherwise, if $a \neq 0$, then

$$\delta\begin{pmatrix}u\\ au+bw\end{pmatrix} = a\cdot\delta\begin{pmatrix}u\\ u+\dfrac{b}{a}w\end{pmatrix} = a\cdot\delta\begin{pmatrix}u\\ \dfrac{b}{a}w\end{pmatrix} = b\cdot\delta\begin{pmatrix}u\\ w\end{pmatrix}.$$

So the desired conclusion is obtained in either case.

We are now able to show that

$$\delta\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}$$

for all $u, v_1, v_2 \in \mathsf{R}^2$. Since the result is immediate if $u = 0$, we shall assume that $u \neq 0$. Choose any vector $w \in \mathsf{R}^2$ such that $\{u, w\}$ is linearly independent. Then for any vectors $v_1, v_2 \in \mathsf{R}^2$ there exist scalars a_i and b_i such that $v_i = a_i u + b_i w$ $(i = 1, 2)$. Thus

$$\delta\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ (a_1 + a_2)u + (b_1 + b_2)w \end{pmatrix} = (b_1 + b_2)\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

$$= \delta\begin{pmatrix} u \\ a_1 u + b_1 w \end{pmatrix} + \delta\begin{pmatrix} u \\ a_2 u + b_2 w \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}$$

A similar argument shows that

$$\delta\begin{pmatrix} u_1 + u_2 \\ v \end{pmatrix} = \delta\begin{pmatrix} u_1 \\ v \end{pmatrix} + \delta\begin{pmatrix} u_2 \\ v \end{pmatrix}$$

for all $u_1, u_2, v \in \mathsf{R}^2$.

(b) Since

$$\mathrm{A}\begin{pmatrix} u \\ u \end{pmatrix} = 0, \qquad \delta\begin{pmatrix} u \\ u \end{pmatrix} = 0 = \mathrm{O}\begin{pmatrix} u \\ u \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ u \end{pmatrix}$$

for any $u \in \mathsf{R}^2$.

(c) Because the parallelogram determined by e_1 and e_2 is the unit square,

$$\delta\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 1 = \mathrm{O}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Therefore δ satisfies the three conditions of Theorem 4.2, and hence $\delta = \det$. So the area of the parallelogram determined by u and v equals

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

Thus, for example, we see that the area of the parallelogram determined by $u = (-1, 5)$ and $v = (4, -2)$ is

$$\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\det\begin{pmatrix} -1 & 5 \\ 4 & -2 \end{pmatrix}\right| = 18.$$

EXERCISES

1. Label the following statements as being true or false.
   (a) The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.

(b) If I is the 2×2 identity matrix, then $\det(I) = 0$.
(c) If both rows of a 2×2 matrix A are identical, then $\det(A) = 0$.
(d) If u and v are vectors in R^2 emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is

$$\det\begin{pmatrix} u \\ v \end{pmatrix}.$$

(e) A coordinate system $\{u, v\}$ is right-handed if and only if its orientation equals 1.
(f) The determinant is a linear transformation from $\mathsf{M}_{2\times 2}(F)$ into F.

2. Compute the determinants of the following elements of $\mathsf{M}_{2\times 2}(R)$:

(a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

3. Compute the determinants of the following elements of $\mathsf{M}_{2\times 2}(C)$:

(a) $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$ (b) $\begin{pmatrix} 5-2i & 6+4i \\ -3+\ i & 7i \end{pmatrix}$ (c) $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

4. For each of the following pairs of vectors u and v in R^2, compute the area of the parallelogram determined by u and v.
(a) $u = (3, -2)$ and $v = (2, 5)$
(b) $u = (1, 3)$ and $v = (-3, 1)$
(c) $u = (4, -1)$ and $v = (-6, -2)$
(d) $u = (3, 4)$ and $v = (2, -6)$

5. Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.

6. Prove that for any $A \in \mathsf{M}_{2\times 2}(F)$, $\det(A^t) = \det(A)$.

7. Prove that if A is a triangular 2×2 matrix, then the determinant of A equals the product of the entries of A lying on the diagonal.

8. Prove that for any $A, B \in \mathsf{M}_{2\times 2}(F)$, $\det(AB) = \det(A)\cdot\det(B)$.

9. The *classical adjoint* of a 2×2 matrix A is the matrix

$$\operatorname{adj} A = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove that the classical adjoint of a matrix possesses the following properties:
(a) $(\operatorname{adj} A)A = A(\operatorname{adj} A) = [\det(A)]I$.
(b) $\det(\operatorname{adj} A) = \det(A)$.
(c) $\operatorname{adj} A^t = (\operatorname{adj} A)^t$.

10. Using Exercise 9(a), prove that a 2×2 matrix A is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = [\det(A)]^{-1}(\operatorname{adj} A)$.

11. Prove that

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if the ordered basis $\{u, v\}$ for R^2 forms a right-handed coordinate system. *Hint:* Recall the definition of a rotation as given in Example 5 of Section 2.1.

4.2 DETERMINANTS OF ORDER *n*

We have seen in Theorem 4.2 that the determinant of a 2×2 matrix is completely characterized by three properties. We shall soon define the determinant of an $n \times n$ matrix in terms of these properties, but first we shall require some preliminary results. To begin, we shall name the first of the conditions that characterized the determinant of a 2×2 matrix.

Definition. *A function* $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ *is said to be an n-*linear function *if* δ *is a linear function of each row of an* $n \times n$ *matrix when the remaining* $n - 1$ *rows are held fixed, that is, if*

$$\delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ cA_i + A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = c\cdot\delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} \qquad \text{for} \quad i = 1, 2, \ldots, n$$

whenever

$$\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ cA_i + A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}$$

is an element of $\mathsf{M}_{n\times n}(F)$.

Example 2. Theorem 4.1 shows that det: $\mathsf{M}_{2\times 2}(F) \longrightarrow F$ defined by $\det(A) = A_{11}A_{22} - A_{12}A_{21}$ is a 2-linear function.

Example 3. The function $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ defined by $\delta(A) = 0$ for each $A \in \mathsf{M}_{n\times n}(F)$ is an n-linear function.

Example 4. The function $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ defined by $\delta(A) = A_{1j}A_{2j} \cdots A_{nj}$ (that is, $\delta(A)$ equals the product of the entries of the jth column of A) is an n-linear function for each j $(1 \leq j \leq n)$ since

$$\delta\begin{pmatrix} A_1 \\ \vdots \\ cA_i + A_i' \\ \vdots \\ A_n \end{pmatrix} = A_{1j} \cdots A_{(i-1)j}(cA_{ij} + A_{ij}')A_{(i+1)j} \cdots A_{nj}$$

$$= c(A_{1j} \cdots A_{ij} \cdots A_{nj}) + (A_{1j} \cdots A_{(i-1)j}A_{ij}'A_{(i+1)j} \cdots A_{nj})$$

$$= c \cdot \delta\begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \vdots \\ A_i' \\ \vdots \\ A_n \end{pmatrix}.$$

Example 5. The function $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ defined by $\delta(A) = A_{11}A_{22} \cdots A_{nn}$ (that is, $\delta(A)$ equals the product of the entries of A lying on the diagonal) is an n-linear function.

Example 6. The function $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ defined by $\delta(A) = \mathrm{tr}(A)$ is not an n-linear function.

Our next result shows that n-linear functions may be combined to produce other n-linear functions.

Theorem 4.3. *A linear combination of two* n-*linear functions is an* n-*linear function (where the sum and scalar product are as defined in Example 3 of Section 1.2).*

PROOF. Let δ_1 and δ_2 be n-linear functions, and let a and b be scalars. If δ is the linear combination $\delta = a\delta_1 + b\delta_2$, then

$$\delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ cA_i + A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = a\cdot\delta_1\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ cA_i + A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + b\cdot\delta_2\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ cA_i + A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}$$

$$= a\left[c\cdot\delta_1\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta_1\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}\right] + b\left[c\cdot\delta_2\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta_2\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}\right]$$

$$= c\left[a\cdot\delta_1\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + b\cdot\delta_2\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}\right] + \left[a\cdot\delta_1\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + b\cdot\delta_2\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}\right]$$

$$= c\cdot\delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i' \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} \qquad \text{for each } i,\ 1 \le i \le n.$$

Thus δ is an n-linear function. ▮

Corollary. *Any linear combination of* n*-linear functions is an* n*-linear function.*

PROOF. Exercise.

The following definition names the second of the three properties that characterized the determinant of a 2×2 matrix.

Definition. *An* n*-linear function* δ *is said to be* alternating *if* $\delta(A) = 0$ *whenever two adjacent rows are identical.*

Example 7. Of the three n-linear functions given in Examples 3, 4, and 5, only the first is alternating.

The following result shows that the preceding definition is stronger than it first appears. In particular, there is no need for the rows to be assumed adjacent in the definition.

Theorem 4.4. *Let* $\delta\colon\ M_{n\times n}(F) \longrightarrow F$ *be an alternating* n*-linear function. Then both of the following are true:*

(a) *If* B *is obtained by interchanging any two rows of an* $n \times n$ *matrix* A, *then* $\delta(B) = -\delta(A)$.

(b) *If two rows of an* $n \times n$ *matrix are identical, then* $\delta(A) = 0$.

PROOF. We shall first prove that if B is obtained by interchanging any two adjacent rows of A, then $\delta(B) = -\delta(A)$. Suppose that B is obtained by interchanging rows i and $i + 1$ of

$$A = \begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ A_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}; \quad \text{thus} \quad B = \begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_{i+1} \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}.$$

Now

$$0 = \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i + A_{i+1} \\ A_i + A_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ A_i + A_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_{i+1} \\ A_i + A_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}$$

$$= \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ A_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_{i+1} \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_{i+1} \\ A_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}$$

$$= 0 + \delta(A) + \delta(B) + 0$$

since δ is an alternating n-linear function. Thus $\delta(B) = -\delta(A)$.

Now suppose that B is obtained from A by interchanging rows i and j, where $i < j$. Beginning with rows i and $i + 1$, successively interchange adjacent rows of A until the rows are in the order

$$A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_j, A_i, A_{j+1}, \ldots, A_n.$$

In all, $j - i$ interchanges are needed to produce this ordering. Now successively interchange A_j with the preceding row until the rows are in the order

$$A_1, \ldots, A_{i-1}, A_j, A_{i+1}, \ldots, A_{j-1}, A_i, A_{j+1}, \ldots, A_n.$$

This process requires $j - i - 1$ interchanges of adjacent rows and produces the matrix B. Hence by the first paragraph of the proof we see that

$$\delta(B) = (-1)^{j-i}(-1)^{j-i-1}\,\delta(A) = (-1)^{2(j-i)-1}\,\delta(A) = -\delta(A).$$

It remains to show that if two rows of A are identical, say rows i and j $(i < j)$, then $\delta(A) = 0$. If $j = i + 1$, then two adjacent rows of A are identical and $\delta(A) = 0$ by the hypothesis. If $j > i + 1$, interchange rows $i + 1$ and j to obtain a matrix B in which two adjacent rows are equal. Then $\delta(B) = 0$, but since $\delta(B) = -\delta(A)$ by the second paragraph of the proof, it follows that $\delta(A) = 0$. Hence δ satisfies conditions (a) and (b). ∎

We are now prepared to define a determinant on $\mathsf{M}_{n\times n}(F)$. Observe that the determinant is defined in terms of the three properties in Theorem 4.2 that characterize the determinant of a 2×2 matrix.

Definition. *A* determinant *on* $\mathsf{M}_{n\times n}(F)$ *is an alternating* n-*linear function* $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ *such that* $\delta(I) = 1$.

A simple example of a determinant can be given on $\mathsf{M}_{1\times 1}(F)$, for the function $\delta\colon\ \mathsf{M}_{1\times 1}(F) \longrightarrow F$ defined by $\delta(A) = A_{11}$ (the only entry of A) clearly satisfies the requirements of this definition. Moreover, Theorem 4.1 shows that by defining the determinant of a 2×2 matrix A as $A_{11}A_{22} - A_{12}A_{21}$ we obtain a determinant on $\mathsf{M}_{2\times 2}(F)$ in the sense of the definition above. Our next result enables us to define a determinant on $\mathsf{M}_{n\times n}(F)$ inductively for any $n \geq 3$.

Theorem 4.5. *Let δ be an alternating* n*-linear function on* $\mathsf{M}_{n\times n}(\mathrm{F})$. *For each* $(\mathrm{n}+1)\times(\mathrm{n}+1)$ *matrix* A *and each* j $(1 \leq \mathrm{j} \leq \mathrm{n}+1)$, *define*

$$\epsilon_{\mathrm{j}}(\mathrm{A}) = \sum_{\mathrm{i}=1}^{\mathrm{n}+1} (-1)^{\mathrm{i}+\mathrm{j}}\mathrm{A}_{\mathrm{ij}}\cdot\delta(\tilde{\mathrm{A}}_{\mathrm{ij}}),$$

where $\tilde{\mathrm{A}}_{\mathrm{ij}}$ *is the* $\mathrm{n}\times\mathrm{n}$ *matrix obtained from* A *by deleting the* i*th row and* j*th column. Then* ϵ_{j} *is an alternating* $(\mathrm{n}+1)$*-linear function on the* $(\mathrm{n}+1)\times(\mathrm{n}+1)$ *matrices with entries from* F.

PROOF. Since $\tilde{A}_{ij}$ is obtained from A by deleting the ith row and jth column, $\delta(\tilde{A}_{ij})$ is independent of the ith row of A. Thus, because δ is an n-linear function, $\delta(\tilde{A}_{ij})$ is a linear function of each row of A except row i. Hence $A_{ij}\cdot\delta(\tilde{A}_{ij})$ is an $(n+1)$-linear function on the $(n+1)\times(n+1)$ matrices with entries from F. Therefore, since

$$\epsilon_j(A) = \sum_{i=1}^{n+1} (-1)^{i+j}A_{ij}\cdot\delta(\tilde{A}_{ij})$$

is a linear combination of the $(n+1)$-linear functions $A_{ij}\cdot\delta(\tilde{A}_{ij})$, ϵ_j is an $(n+1)$-linear function by the corollary to Theorem 4.3.

We shall now prove that ϵ_j is alternating. If A is an $(n+1)\times(n+1)$ matrix in which rows k and $k+1$ are identical, then $\tilde{A}_{ij}$ has two identical rows whenever $i \neq k$ and $i \neq k+1$. Thus $\delta(\tilde{A}_{ij}) = 0$ whenever $i \neq k$ and $i \neq k+1$, and so

$$\epsilon_j(A) = (-1)^{k+j}A_{kj}\cdot\delta(\tilde{A}_{kj}) + (-1)^{(k+1)+j}A_{(k+1)j}\cdot\delta(\tilde{A}_{(k+1)j}).$$

But because rows k and $k+1$ of A are equal, $A_{kj} = A_{(k+1)j}$ and $\tilde{A}_{kj} = \tilde{A}_{(k+1)j}$. Hence $\epsilon_j(A) = 0$, proving that ϵ_j is alternating. ▮

Corollary 1. *Let δ and* ϵ_{j} *be as in the statement of Theorem 4.5. If δ is a determinant on* $\mathsf{M}_{n\times n}(\mathrm{F})$, *then* ϵ_{j} *is a determinant on the* $(\mathrm{n}+1)\times(\mathrm{n}+1)$ *matrices with entries from* F.

PROOF. Let I denote the $(n+1)\times(n+1)$ identity matrix, and let $\tilde{I}_{ij}$ denote the $n\times n$ matrix obtained from I by deleting row i and column j. Then $\tilde{I}_{jj}$ is the $n\times n$ identity matrix. Since $I_{ij} = 0$ if $i \neq j$ and $I_{jj} = 1$, we have

$$\epsilon_j(I) = \sum_{i=1}^{n+1} (-1)^{i+j}I_{ij}\cdot\delta(\tilde{I}_{ij}) = (-1)^{j+j}\cdot\delta(\tilde{I}_{jj})$$
$$= \delta(\tilde{I}_{jj}) = 1$$

because δ is a determinant on $\mathsf{M}_{n\times n}(F)$. Thus ϵ_j is a determinant on the $(n+1)\times(n+1)$ matrices with entries from F. ▮

Corollary 2. *There exists a determinant on* $\mathsf{M}_{n\times n}(F)$ *for any positive integer* n.

PROOF. The proof will be by induction on n. If $n = 1$, the function det: $\mathsf{M}_{1\times 1}(F) \longrightarrow F$ defined by $\det(A) = A_{11}$ is a determinant on $\mathsf{M}_{1\times 1}(F)$. Assume that there exists a determinant δ on $\mathsf{M}_{n\times n}(F)$. Then for any j $(1 \leq j \leq n + 1)$, the function ϵ_j defined in Theorem 4.5 is a determinant on the $(n + 1) \times (n + 1)$ matrices with entries from F. This completes the induction. ▮

Definitions. *If* δ *is a determinant on* $\mathsf{M}_{n\times n}(F)$, *then the determinant*

$$\epsilon_j(A) = \sum_{i=1}^{n+1} (-1)^{i+j} A_{ij} \cdot \delta(\tilde{A}_{ij})$$

defined in Theorem 4.5 is called the expansion of A along the jth column. *The scalar* $(-1)^{i+j} \cdot \delta(\tilde{A}_{ij})$ *is called the* cofactor of A_{ij} (*with respect to the determinant* δ).

Example 8. Let A denote the following element of $\mathsf{M}_{3\times 3}(F)$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The cofactors of A_{12}, A_{22}, and A_{32} are

$$(-1)^{1+2} \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = (-1)(4\cdot 9 - 6\cdot 7) = 6,$$

$$(-1)^{2+2} \det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = 1(1\cdot 9 - 3\cdot 7) = -12,$$

$$(-1)^{3+2} \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = (-1)(1\cdot 6 - 3\cdot 4) = 6,$$

respectively. Hence the expansion of A along the second column is

$$\begin{aligned} \epsilon_2(A) &= A_{12}(6) + A_{22}(-12) + A_{32}(6) \\ &= 2\cdot 6 + 5(-12) + 8\cdot 6 = 0. \end{aligned}$$

Likewise the cofactors of A_{13}, A_{23}, and A_{33} are

$$(-1)^{1+3} \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} = 1(4\cdot 8 - 5\cdot 7) = -3,$$

$$(-1)^{2+3} \det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = (-1)(1\cdot 8 - 2\cdot 7) = 6,$$

$$(-1)^{3+3} \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = 1(1\cdot 5 - 2\cdot 4) = -3,$$

respectively. Hence the expansion of A along the third column is

$$\epsilon_3(A) = A_{13}(-3) + A_{23}(6) + A_{33}(-3) = 3(-3) + 6(6) + 9(-3) = 0.$$

We shall see in Theorem 4.9 that the equality of $\epsilon_2(A)$ and $\epsilon_3(A)$ in Example 8 is not coincidental. In fact, we shall see that there is exactly one determinant on $\mathsf{M}_{n\times n}(F)$.

EXERCISES

1. Label the following statements as being true or false.
(a) A determinant on $\mathsf{M}_{n\times n}(F)$ is a linear function of each row of an $n \times n$ matrix with entries from F when the other $n - 1$ rows are held fixed.
(b) If δ is a determinant and any two rows of A are identical, then $\delta(A) = 0$.
(c) Let δ be a determinant. If B is a matrix obtained from A by interchanging any two rows, then $\delta(A) = \delta(B)$.
(d) The function $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ defined by $\delta(A) = 0$ for each $A \in \mathsf{M}_{n\times n}(F)$ is a determinant on $\mathsf{M}_{n\times n}(F)$.
(e) For any $n \geq 2$ there is a determinant on $\mathsf{M}_{n\times n}(F)$.
(f) Any determinant $\delta\colon\ \mathsf{M}_{n\times n}(F) \longrightarrow F$ is linear.

2. Verify that if A is the 3×3 matrix in Example 8, then the expansion of A along the first column equals zero.

3. Evaluate the determinant of each of the following matrices by expanding along the second column and also along the third column. (Each matrix is an element of $\mathsf{M}_{3\times 3}(C)$.)

(a) $\begin{pmatrix} -3 & 2 & 5 \\ 1 & 0 & -1 \\ 4 & -6 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & -4 & 0 \\ 0 & 6 & -3 \\ -1 & 5 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -5 \\ 6 & -4 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1+i & -1 & 0 \\ 2 & 3i & 4i \\ 0 & 2-i & -1+2i \end{pmatrix}$

4. Which of the following functions $\delta\colon\ \mathsf{M}_{3\times 3}(F) \longrightarrow F$ are 3-linear functions? Justify each answer.
(a) $\delta(A) = c$, where c is any non-zero scalar
(b) $\delta(A) = A_{22}$
(c) $\delta(A) = A_{11}A_{23}A_{32}$
(d) $\delta(A) = A_{11}A_{21}A_{32}$

(e) $\delta(A) = A_{11}A_{31}A_{32}$
(f) $\delta(A) = A_{11}^2 A_{22}^2 A_{33}^2$
(g) $\delta(A) = A_{11}A_{22}A_{33} - A_{11}A_{21}A_{32}$

5. (a) Determine all 1-linear functions $\delta\colon\ \mathsf{M}_{1\times 1}(F) \longrightarrow F$.
(b) Determine all determinants on $\mathsf{M}_{1\times 1}(F)$.

6. Prove the equality of the three functions $\epsilon_j\colon\ \mathsf{M}_{3\times 3}(F) \longrightarrow F$ $(j = 1, 2, 3)$ defined in Theorem 4.5 for each $A \in \mathsf{M}_{3\times 3}(F)$ by

$$\epsilon_j(A) = \sum_{i=1}^{3} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}),$$

where $\tilde{A}_{ij}$ is the 2×2 matrix obtained from A by deleting the ith row and jth column and det denotes the unique determinant on $\mathsf{M}_{2\times 2}(F)$.

7. Prove that the unique determinant on $\mathsf{M}_{2\times 2}(F)$ is a 2-linear function of the columns of a 2×2 matrix and that the determinant of a 2×2 matrix in which both columns are identical is zero.

8. The proof of Theorem 4.2 shows that if δ is a 2-linear function $\delta\colon\ \mathsf{M}_{2\times 2}(F) \longrightarrow F$, then

$$\delta(A) = A_{11}A_{22}\cdot\delta(I) + A_{11}A_{21}\cdot\delta(M_1) + A_{12}A_{22}\cdot\delta(M_2) + A_{12}A_{21}\cdot\delta(M_3),$$

where I, M_1, M_2, and M_3 are as in the proof of the theorem. Prove that for any scalars $a, b, c, d \in F$ the function

$$\epsilon(A) = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d$$

is a 2-linear function. Thus $\delta'\colon\ \mathsf{M}_{2\times 2}(F) \longrightarrow F$ is a 2-linear function if and only if it is of the form above for some scalars a, b, c, and d.

9. Show that if F is not a field of characteristic two (as defined in Appendix D), then condition (a) of Theorem 4.4 implies condition (b) of that theorem. This result is not true in arbitrary fields, however.

10. Prove the corollary to Theorem 4.3.

4.3 PROPERTIES OF DETERMINANTS

There are several important properties that are quite useful in evaluating a determinant of a given matrix. These are summarized in the next theorem.

Theorem 4.6. *Any determinant δ on $\mathsf{M}_{n\times n}(F)$ has the following properties:*

(a) *If B is a matrix obtained from A by multiplying each entry of some row of A by a scalar c, then $\delta(B) = c\cdot\delta(A)$.*
(b) *If two rows of A are identical, then $\delta(A) = 0$.*
(c) *If B is a matrix obtained from A by interchanging two rows, then $\delta(B) = -\delta(A)$.*
(d) *If one row of A consists entirely of zero entries, then $\delta(A) = 0$.*

(e) *If* B *is a matrix obtained from* A *by adding a multiple of row* i *to row* j (i $\neq$ j), *then* $\delta(B) = \delta(A)$.

PROOF. Property (a) is a consequence of the fact that δ is an n-linear function, whereas properties (b) and (c) are consequences of Theorem 4.4.

(d) Suppose that A_i, the ith row of A, consists entirely of zero entries. Then

$$\delta(A) = \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ 0A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = 0\cdot\delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = 0.$$

(e) Let B be obtained from $A \in \mathsf{M}_{n\times n}(F)$ by adding c times row i to row j. Assume for the sake of argument that $i < j$. Thus if

$$A = \begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_j \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}, \quad \text{then} \quad B = \begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ cA_i + A_j \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}.$$

So

$$\delta(B) = c\cdot\delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} + \delta\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_j \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = c\cdot 0 + \delta(A) = \delta(A)$$

by the n-linearity of δ and property (b) above. ∎

Observe that properties (a), (c), and (e) of Theorem 4.6 show how the determinant of a matrix changes when an elementary row operation is performed on the matrix. We can reformulate these properties in terms of elementary matrices as follows.

Corollary. *Let* E_1, E_2, *and* E_3 *be elementary matrices in* $M_{n\times n}(F)$ *of types 1, 2, and 3, respectively. If* E_2 *is obtained by multiplying a row of* I *by the non-zero scalar* c, *then for any determinant* δ *on* $M_{n\times n}(F)$, $\delta(E_1) = -1$, $\delta(E_2) = c$, *and* $\delta(E_3) = 1$.

This corollary is one of the key ingredients of a proof of the uniqueness of a determinant on $M_{n\times n}(F)$. We shall now prove the remaining two theorems needed to establish this uniqueness. Our first result computes the determinant of any non-invertible matrix.

Theorem 4.7. *Let* δ *be a determinant on* $M_{n\times n}(F)$, *and let* A *be an element of* $M_{n\times n}(F)$ *having rank less than* n. *Then* $\delta(A) = 0$.

PROOF. Since $\operatorname{rank}(A) < n$, the rows of A are linearly dependent (Corollary 2 of Theorem 3.5). Hence there are scalars $c_1, \ldots, c_n$, not all zero, such that $c_1A_1 + c_2A_2 + \cdots + c_nA_n = 0$, where $A_1, A_2, \ldots, A_n$ are the rows of A. Assume for the sake of argument that $c_1 \neq 0$; then

$$A_1 + c_1^{-1}c_2A_2 + \cdots + c_1^{-1}c_nA_n = 0.$$

Let B be the matrix obtained from A by adding to the first row the multiple $c_1^{-1}c_iA_i$ of row i for each i ($i = 2, \ldots, n$). Then the first row of B consists entirely of zero entries, so that $\delta(B) = 0$. But $\delta(B) = \delta(A)$ by property (e) of Theorem 4.6. Therefore $\delta(A) = 0$. ∎

The next result establishes the final fact needed to prove the uniqueness of a determinant on $M_{n\times n}(F)$—that a determinant behaves well with respect to matrix multiplication. This theorem is of considerable importance in its own right, however. In particular, its second corollary, which provides a determinant test for invertibility of a matrix, will be frequently used in the following chapters.

Lemma. *If* E *is an* $n \times n$ *elementary matrix with entries from* F, *and if* δ *is a determinant on* $M_{n\times n}(F)$, *then* $\delta(EB) = \delta(E)\cdot\delta(B)$ *for any* $B \in M_{n\times n}(F)$.

PROOF. Suppose that multiplication on the left by E interchanges two rows of B. Then $\delta(EB) = -\delta(B)$ by Theorem 4.6(c). But $\delta(E) = -1$ by the corollary to Theorem 4.6; so $\delta(EB) = \delta(E)\cdot\delta(B)$. Similar proofs establish the result for multiplication of a row of B by a non-zero scalar or addition of a multiple of one row to another. ∎

Theorem 4.8. *Let δ be a determinant on $\mathsf{M}_{n\times n}(F)$, and let A and B be arbitrary elements of $\mathsf{M}_{n\times n}(F)$. Then $\delta(AB) = \delta(A)\cdot\delta(B)$.*

PROOF. If $\operatorname{rank}(A) < n$, then by Theorem 3.6 $\operatorname{rank}(AB) \leq \operatorname{rank}(A) < n$. Hence by Theorem 4.7 $\delta(AB) = 0$ and $\delta(A) = 0$. So in this case $\delta(AB) = \delta(A)\cdot\delta(B)$.

If $\operatorname{rank}(A) = n$, then A is invertible and hence is the product of elementary matrices (Corollary 3 of Theorem 3.5). Let $A = E_m \cdots E_1$, where each E_i is an elementary matrix. Then by the lemma we have

$$\begin{aligned}\delta(AB) &= \delta(E_m \cdots E_1B) = \delta(E_m)\cdot\delta(E_{m-1} \cdots E_1B) = \cdots \\ &= \delta(E_m)\cdot \cdots \cdot\delta(E_1)\cdot\delta(B) = \delta(E_m \cdots E_1)\cdot\delta(B) = \delta(A)\cdot\delta(B). \quad \blacksquare\end{aligned}$$

Corollary 1. *Let δ be a determinant on $\mathsf{M}_{n\times n}(F)$, and let $A \in \mathsf{M}_{n\times n}(F)$ be invertible. Then $\delta(A) \neq 0$, and $\delta(A^{-1}) = [\delta(A)]^{-1}$.*

PROOF. By Theorem 4.8 we have

$$\delta(A)\cdot\delta(A^{-1}) = \delta(AA^{-1}) = \delta(I_n) = 1.$$

So $\delta(A) \neq 0$, and $\delta(A^{-1}) = [\delta(A)]^{-1}$. $\blacksquare$

Corollary 2. *Let δ be a determinant on $\mathsf{M}_{n\times n}(F)$, and let $A \in \mathsf{M}_{n\times n}(F)$. Then the following conditions are equivalent:*

(a) $\delta(A) = 0$.

(b) A *is not invertible.*

(c) *rank*$(A) < n$.

PROOF. Corollary 1 above shows that if $\delta(A) = 0$, then A is not invertible. Hence condition (a) implies condition (b).

That condition (b) implies condition (c) follows from a previous remark on p. 135.

Finally, Theorem 4.7 shows that condition (c) implies condition (a). $\blacksquare$

It was proved in Theorem 4.1 and 4.2 that there is exactly one determinant on $\mathsf{M}_{2\times 2}(F)$. We are now able to prove a similar result for $\mathsf{M}_{n\times n}(F)$.

Theorem 4.9. *There is exactly one determinant on $\mathsf{M}_{n\times n}(F)$.*

PROOF. The existence of a determinant on $\mathsf{M}_{n\times n}(F)$ was proved in Corollary 2 of Theorem 4.5.

We shall complete the proof by showing that if δ_1 and δ_2 are both determinants on $\mathsf{M}_{n\times n}(F)$, then $\delta_1 = \delta_2$. Let A be an arbitrary $n \times n$ matrix with entries from F. If $\operatorname{rank}(A) < n$, then $\delta_1(A) = \delta_2(A) = 0$ by Corollary 2 of Theorem 4.8. If $\operatorname{rank}(A) = n$, then A is invertible and hence is the product of elementary matrices (Corollary 3 of Theorem 3.5). Let $A =$

$E_m \cdots E_1$, where each E_i is an elementary matrix. Since $\delta_1(E_i) = \delta_2(E_i)$ for each i $(1 \leq i \leq m)$ by the corollary to Theorem 4.6,

$$\begin{aligned}\delta_1(A) &= \delta_1(E_m \cdots E_1) = \delta_1(E_m) \cdots \delta_1(E_1) \\ &= \delta_2(E_m) \cdots \delta_2(E_1) = \delta_2(E_m \cdots E_1) = \delta_2(A)\end{aligned}$$

by Theorem 4.8. Hence $\delta_1 = \delta_2$. ▮

Henceforth we shall denote the unique determinant on $\mathsf{M}_{n\times n}(F)$ by *det*.

Corollary. *Let* $A \in \mathsf{M}_{n\times n}(F)$. *For any* j $(1 \leq j \leq n)$

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot det(\tilde{A}_{ij}),$$

where $\tilde{A}_{ij}$ *is the* $(n-1) \times (n-1)$ *matrix obtained from* A *by deleting row* i *and column* j.

Thus the determinant of an $n \times n$ matrix can be evaluated by expanding along any column; if $n > 2$, the resulting expansion will contain n determinants of matrices of size $(n-1) \times (n-1)$. The determinant of each of these $(n-1) \times (n-1)$ matrices can then be expanded along any column, and this process can be continued until an expansion involves only determinants of 2×2 matrices, which can be evaluated by the rule $\det(A) = A_{11}A_{22} - A_{12}A_{21}$.

Observe, however, that the evaluation of $\det(\tilde{A}_{ij})$ can be avoided whenever $A_{ij} = 0$, for the product $A_{ij} \cdot \det(\tilde{A}_{ij})$ will then be zero regardless of the value of the determinant. Therefore it is beneficial to expand along a column containing as many zero entries as possible. We shall illustrate this procedure with two examples.

Example 9. Let A denote the following element of $\mathsf{M}_{4\times 4}(F)$:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

To minimize the computation required to evaluate $\det(A)$, we shall expand along the second column. Then

$$\begin{aligned}\det(A) &= \sum_{i=1}^{4} (-1)^{i+2} A_{i2} \cdot \det(\tilde{A}_{i2}) \\ &= (-1)^{1+2} \cdot 1 \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + (-1)^{2+2} \cdot 0 \cdot \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\end{aligned}$$

$$+ (-1)^{3+2}\cdot 0\cdot \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + (-1)^{4+2}\cdot 1\cdot \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= (-1)\cdot 1\cdot 0 + 0 + 0 + 1\cdot 1\cdot \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(The first of the four 3×3 matrices has two identical rows, so that its determinant is zero.) We shall now evaluate the remaining determinant by expanding along the first column. Thus

$$\begin{aligned} \det(A) &= \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= (-1)^{1+1}\cdot 1\cdot \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-1)^{2+1}\cdot 1\cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\ &\quad + (-1)^{3+1}\cdot 0\cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\ &= 1\cdot 1\cdot 0 + (-1)\cdot 1\cdot (-1) + 0 = 1. \end{aligned}$$

Now let B denote the following element of $\mathsf{M}_{5\times 5}(R)$:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -2 & 0 & -3 & 1 & 6 \\ 5 & -4 & 0 & 2 & 0 \\ 0 & 3 & 0 & -1 & 4 \\ -9 & 8 & 0 & 0 & 0 \end{pmatrix}.$$

Expanding successively along the third, fourth, and third columns, we see that

$$\begin{aligned} \det(B) &= (-1)^{2+3}\cdot(-3)\cdot \det\begin{pmatrix} 1 & -1 & 0 & 0 \\ 5 & -4 & 2 & 0 \\ 0 & 3 & -1 & 4 \\ -9 & 8 & 0 & 0 \end{pmatrix} \\ &= 3(-1)^{3+4}\cdot 4\cdot \det\begin{pmatrix} 1 & -1 & 0 \\ 5 & -4 & 2 \\ -9 & 8 & 0 \end{pmatrix} \\ &= -12\cdot(-1)^{2+3}\cdot 2\cdot \det\begin{pmatrix} 1 & -1 \\ -9 & 8 \end{pmatrix} \\ &= 24[1\cdot 8 - (-1)(-9)] = 24(-1) = -24. \end{aligned}$$

As these examples suggest, the process of evaluating a determinant is often tedious even when there are zero entries present. Without zero entries the evaluation of a determinant by expanding along a column is quite inefficient. Instead of this procedure we can utilize property (e) of Theorem 4.6 to change a matrix A into a matrix B having the same determinant as A and having zero entries in one or more columns. This is essentially the same process that was used to reduce A to row echelon form. Examples of this technique follow.

Example 10. Let A denote the following element of $\mathsf{M}_{4\times 4}(R)$:

$$\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}.$$

Then

$$\det(A) = \det\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \det\begin{pmatrix} A_1 \\ 3A_1 + A_2 \\ -2A_1 + A_3 \\ 2A_1 + A_4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 7 & -4 \\ 0 & -3 & -7 & 10 \\ 0 & 4 & 0 & -1 \end{pmatrix} = (-1)^{1+1}\cdot 1\cdot\det\begin{pmatrix} 1 & 7 & -4 \\ -3 & -7 & 10 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 7 & -4 \\ -2 & 0 & 6 \\ 4 & 0 & -1 \end{pmatrix} = (-1)^{1+2}\cdot 7\cdot\det\begin{pmatrix} -2 & 6 \\ 4 & -1 \end{pmatrix}$$

$$= -7[(-2)(-1) - 6\cdot 4] = -7(-22) = 154.$$

Example 11. Let A denote the following element of $\mathsf{M}_{4\times 4}(R)$:

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{pmatrix}.$$

We shall introduce zero entries by use of Theorem 4.6(e) so that A is transformed into an upper triangular matrix having the same determinant as A. The determinant of the upper triangular matrix will then be evaluated by successive expansions along the first column.

$$\det(A) = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 9 & -6 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$$= 1\cdot\det\begin{pmatrix} 1 & -5 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 6 \end{pmatrix} = 1\cdot 1\cdot\det\begin{pmatrix} 3 & -4 \\ 0 & 6 \end{pmatrix}$$

$$= 1\cdot 1\cdot 3\cdot 6 = 18.$$

Until now, the roles played by the rows and columns of a matrix in the study of determinants have been quite different—a determinant was defined as a function on $\mathsf{M}_{n\times n}(F)$ that satisfies certain properties involving the rows of a matrix, whereas the evaluation of a determinant is accomplished by expanding along columns of a matrix. These roles are reversible, and we shall now verify this fact by showing that the determinants of A and A^t are equal. (Since the rows of A are columns of A^t and vice versa, this result will be sufficient to prove that the roles of rows and columns are interchangeable.)

Theorem 4.10. *For any* $n \times n$ *matrix* A, $det(A^t) = det(A)$.

PROOF. If A is not invertible, then $\operatorname{rank}(A) < n$. But since $\operatorname{rank}(A^t) = \operatorname{rank}(A)$ (Corollary 2 of Theorem 3.5), it follows that A^t is not invertible. So $\det(A) = 0 = \det(A^t)$ in this case.

If A is invertible, then $A = E_m \cdots E_1$, where $E_1, \ldots, E_m$ are elementary matrices. Since $\det(E_i^t) = \det(E_i)$ for each i (see Exercise 5), we have

$$\begin{aligned}\det(A^t) &= \det(E_1^t \cdots E_m^t) = \det(E_1^t) \cdots \det(E_m^t) \\ &= \det(E_1) \cdots \det(E_m) = \det(E_m) \cdots \det(E_1) \\ &= \det(E_m \cdots E_1) = \det(A). \quad \blacksquare\end{aligned}$$

Corollary. *Any statement about determinants that involves the rows of a matrix can be restated in terms of the columns of the matrix, and any statement about determinants that involves the columns of a matrix can be restated in terms of the rows of the matrix. In particular, if* A *is an* $n \times n$ *matrix,*

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij}\cdot det(\tilde{A}_{ij}),$$

where $\tilde{A}_{ij}$ *is the* $(n-1) \times (n-1)$ *matrix obtained from* A *by deleting row* i *and column* j.

Example 12. Let A denote the following element of $M_{4\times 4}(R)$:

$$\begin{pmatrix} 4 & -1 & -3 & 6 \\ -2 & 3 & 1 & 4 \\ 0 & 5 & 0 & 0 \\ 1 & 2 & 3 & -1 \end{pmatrix}.$$

In this case the computation required to evaluate $\det(A)$ can be minimized by expanding along the third row. Thus

$$\det(A) = -5\det\begin{pmatrix} 4 & -3 & 6 \\ -2 & 1 & 4 \\ 1 & 3 & -1 \end{pmatrix} = -5\det\begin{pmatrix} 0 & -15 & 10 \\ 0 & 7 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

$$= -5\det\begin{pmatrix} -15 & 10 \\ 7 & 2 \end{pmatrix} = -5[(-15)\cdot 2 - 10\cdot 7] = 500.$$

Our final result allows us to evaluate easily the determinant of a triangular matrix. This result makes the technique used in Example 11 a very efficient method for evaluating determinants.

Theorem 4.11. Suppose that A *is a triangular* $n \times n$ *matrix. Then* $\det(A) = A_{11}A_{22}\cdots A_{nn}$; *that is, the determinant of* A *is the product of the entries of* A *that lie on the diagonal.*

PROOF. Let A be an upper triangular $n \times n$ matrix. The proof is by induction on n. If $n = 2$, then A has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

and so $\det(A) = A_{11}A_{22} - A_{12}\cdot 0 = A_{11}A_{22}$, proving the theorem for upper triangular matrices if $n = 2$.

Assume that the theorem is true for upper triangular $(n-1) \times (n-1)$ matrices, and let A be an upper triangular $n \times n$ matrix. Then A has the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1(n-1)} & A_{1n} \\ 0 & A_{22} & \cdots & A_{2(n-1)} & A_{2n} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & A_{nn} \end{pmatrix}.$$

Expanding along the first column, we see that

$$\det(A) = A_{11}\cdot\det\begin{pmatrix} A_{22} & \cdots & A_{2(n-1)} & A_{2n} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix}$$

$$= A_{11}\cdot(A_{22} \cdots A_{nn})$$

by the induction hypothesis. This completes the induction and proves the theorem for upper triangular matrices.

If A is a lower triangular matrix, then A^t is an upper triangular matrix. Hence the first part of this proof and Theorem 4.10 imply that

$$\det(A) = \det(A^t) = (A^t)_{11} \cdots (A^t)_{nn} = A_{11} \cdots A_{nn}. \quad \blacksquare$$

As in Section 4.1, it is possible to interpret the determinant of an element A in $\mathsf{M}_{n\times n}(R)$ geometrically. Letting $A_1, \ldots, A_n$ denote the n rows of A, we can interpret

$$\left|\det\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}\right|$$

as the n-dimensional volume (the generalization of area in R^2 and volume in R^3) of the parallelepiped having the vectors $A_1, \ldots, A_n$ as adjacent sides. (For a proof of this result, see Serge Lang, *Analysis I*, Addison-Wesley, 1968, pp. 413–418.)

In our earlier discussion of the geometric significance of the determinant formed from the vectors in an ordered basis for R^2, we also saw that this determinant is positive if and only if the basis induces a right-handed coordinate system. A similar statement is true in R^n. Specifically, if γ is any ordered basis for R^n and β is the standard ordered basis for R^n, then γ induces a right-handed coordinate system if and only if $\det(Q) > 0$, where Q is the change of coordinate matrix changing γ-coordinates into β-coordinates. Thus, for instance,

$$\gamma = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

induces a left-handed coordinate system in R^3 since

$$\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -2 < 0,$$

whereas

$$\gamma' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

induces a right-handed coordinate system in R^3 since

$$\det\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5 > 0.$$

More generally, if β and γ are any two ordered bases for R^n, then the coordinate systems induced by β and γ have the same orientation (both right-handed or both left-handed) if and only if $\det(Q) > 0$, where Q is the change of coordinate matrix changing γ-coordinates into β-coordinates.

EXERCISES

1. Label the following as being true or false.

(a) If two rows of A are identical, then $\det(A) = 0$.

(b) If B is a matrix obtained from A by interchanging two rows, then $\det(B) = -\det(A)$.

(c) If B is a matrix obtained from A by multiplying a row of A by a scalar c, then $\det(A) = \det(B)$.

(d) If B is a matrix obtained from A by adding a scalar multiple of row i to row j ($i \neq j$), then $\det(B) = \det(A)$.

(e) If E is an elementary matrix, then $\det(E) = \pm 1$.

(f) If $A, B \in \mathsf{M}_{n \times n}(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.

(g) A matrix M is invertible if and only if $\det(M) = 0$.

(h) A matrix $M \in \mathsf{M}_{n \times n}(F)$ has rank n if and only if $\det(M) \neq 0$.

(i) The determinant of a matrix may be evaluated by expanding along any row or column.

(j) $\det(A^t) = -\det(A)$.

(k) The determinant of a diagonal matrix is the product of its entries on the diagonal.

2. Evaluate each determinant in the manner indicated.

(a) Expand

$$\begin{pmatrix} 2 & -1 & 4 \\ 3 & 6 & 1 \\ -1 & 2 & 3 \end{pmatrix}$$

along the second column.

(b) Expand

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -1 & 4 \\ 5 & 6 & 1 \end{pmatrix}$$

along the first row.

(c) Expand

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

along the third column.

(d) Expand

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

along the fourth row.

3. Evaluate the determinants of the matrices below by any legitimate method. In each case C is the field of scalars.

(a) $\begin{pmatrix} 4 & -7 & 3 \\ 1 & 2 & -1 \\ -3 & 4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 0 & 0 \\ 4 & 8 & 0 \\ 3 & 2 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -5 & 2 \\ 2 & 8 & 1 \\ 6 & -1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} -2+i & -1 & 5i \\ 3 & 3+2i & -2i \\ 4i & 0 & 1+i \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 & -1 & -1 \\ -3 & 0 & 2 & 1 \\ 2 & -1 & 5 & 4 \\ -1 & 6 & 3 & 3 \end{pmatrix}$

(f) $\begin{pmatrix} 2 & 0 & -1 & 3 \\ -4 & 3 & 5 & 1 \\ 1 & 6 & 0 & 2 \\ 0 & -5 & 3 & 7 \end{pmatrix}$ (g) $\begin{pmatrix} -1+3i & 2i & 6 & 0 \\ 4 & 0 & 3+i & 4i \\ 0 & 1-2i & 0 & 2-i \\ 2i & 5 & 0 & 1+i \end{pmatrix}$

4. Prove that a triangular $n \times n$ matrix is invertible if and only if zero does not lie on the diagonal.

5. Complete the proof of Theorem 4.10 by proving that if E is an elementary matrix, then $\det(E^t) = \det(E)$. *Hint:* E^t is an elementary matrix of the same type as E.

6. Prove that if $A \in \mathsf{M}_{n\times n}(F)$, then $\det(cA) = c^n \det(A)$ for any scalar c.

7. (a) A matrix B in $\mathsf{M}_{n\times n}(R)$ is called *orthogonal* if $BB^t = I$. Prove that if B is orthogonal, then $\det(B) = \pm 1$.
   (b) A matrix B in $\mathsf{M}_{n\times n}(C)$ is called *unitary* if $BB^* = I$, where $(B^*)_{ij} = \overline{B_{ji}}$, the complex conjugate of B_{ji}. Prove that if B is unitary, then $|\det(B)| = 1$. *Hint:* First prove that $\det(\overline{B}) = \overline{\det(B)}$.

8. A matrix B in $\mathsf{M}_{n\times n}(C)$ is called *skew-symmetric* if $B^t = -B$. Prove that if $B \in \mathsf{M}_{n\times n}(C)$ is skew-symmetric and n is odd, then $\det(B) = 0$.

9.† Let $A \in \mathsf{M}_{n\times n}(F)$, satisfy $A_{ij} = 0$ for $m + 1 \leq i$ and $j \leq m$, where m is a positive integer such that $1 \leq m \leq n - 1$, Let

$$B_1 = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} A_{1(m+1)} & \cdots & A_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{m(m+1)} & \cdots & A_{mn} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} A_{(m+1)(m+1)} & \cdots & A_{(m+1)n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{n(m+1)} & \cdots & A_{nn} \end{pmatrix},$$

and O denote the $(n - m) \times m$ zero matrix. Symbolically

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}.$$

Prove that $\det(A) = \det(B_1)\cdot\det(B_3)$.

10. Let $\beta = \{x_1, \ldots, x_n\}$ be a subset of F^n containing n distinct vectors, and let B denote the element of $\mathsf{M}_{n\times n}(F)$ whose jth column is the vector x_j. Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

11. Complete the proof of the lemma to Theorem 4.8.

12. Recall the linear transformation $\mathsf{T}\colon \mathsf{P}_n(F) \longrightarrow \mathsf{F}^{n+1}$ defined in Exercise 20 of Section 2.4 by $\mathsf{T}(f) = (f(c_0), \ldots, f(c_n))$, where $c_0, \ldots, c_n$ are distinct elements of an infinite field F. Let $\beta = \{1, x, x^2, \ldots, x^n\}$ be an ordered basis for $\mathsf{P}_n(F)$ and γ be the standard ordered basis for F^{n+1}.
    (a) Compute $M = [\mathsf{T}]_\beta^\gamma$. A matrix having the form of M is called a *Vandermonde matrix.*
    (b) Show that $\det(M) \neq 0$ by using Exercise 20 of Section 2.4.
    (c) Prove that

$$\det(M) = \prod_{0 \leq i < j \leq n} (c_j - c_i),$$

the product of all terms of the form $c_j - c_i$ for $0 \leq i < j \leq n$.

13. Let $A \in \mathsf{M}_{n\times n}(F)$ be non-zero. For any m $(1 \leq m \leq n)$, an $m \times m$ submatrix of A is obtained by deleting any $n - m$ rows and any $n - m$ columns from A. Let k $(1 \leq k \leq n)$ denote the largest integer such that some

$k \times k$ submatrix of A has a non-zero determinant. Prove that $\text{rank}(A) = k$.

14. Use the results of this section to prove Exercise 8 of Section 2.4: If A and B are $n \times n$ matrices such that $AB = I_n$, then A is invertible (and hence $B = A^{-1}$).

15. Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.

4.4 THE CLASSICAL ADJOINT AND CRAMER'S RULE

In this section we shall define the classical adjoint of an $n \times n$ matrix and use it to compute the inverse of a matrix. We shall also derive Cramer's rule, which allows us to use determinants to solve a system of linear equations having an invertible coefficient matrix. Our principal tool will be the theorem below, which shows the consequences of an expansion of a matrix by entries of one column and cofactors of another column.

Theorem 4.12. *Let* A *be an* $n \times n$ *matrix, and let* c_{ij} *denote the cofactor of* A_{ij} $(1 \leq i, j \leq n)$. *Then*

$$\sum_{i=1}^{n} A_{ij} \cdot c_{ik} = \delta_{jk} \cdot det(A),$$

where δ_{jk} *is the Kronecker delta.*

PROOF. If $j = k$, the equation follows from the corollary to Theorem 4.9. Suppose then that $j \neq k$, and let B denote the matrix having all its columns identical to the corresponding columns of A except for the kth column, B^k, which is identical to the jth column of A. So $B^k = B^j = A^j$, and the cofactor of B_{ik} is c_{ik}. Now $\det(B) = 0$ because two columns of B are identical. But, expanding B along the kth column, we also have

$$\det(B) = \sum_{i=1}^{n} B_{ik} c_{ik} = \sum_{i=1}^{n} A_{ij} c_{ik}.$$

Hence

$$\sum_{i=1}^{n} A_{ij} c_{ik} = 0 \qquad \text{if } j \neq k. \quad \blacksquare$$

Corollary 1. *If* A *and* c_{ij} *are as in the statement of Theorem 4.12, then*

$$\sum_{j=1}^{n} A_{ij} c_{kj} = \delta_{ik} \cdot det(A).$$

PROOF. Exercise.

Definition. *Let* A *be an* $n \times n$ *matrix. The* $n \times n$ *matrix* adj A *whose entry in the* i*th row and* j*th column is the cofactor of* A_{ji} *is called the* classical adjoint *of* A. *(Thus adj* $A = C^t$, *where* C_{ij} *is the cofactor of* A_{ij}.*)*

Example 13. Let A and B denote the following elements of $\mathsf{M}_{3\times 3}(R)$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & -1 \\ -1 & -4 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Then

$$\text{adj } A = \begin{pmatrix} -1 & 2 & 1 \\ 5 & -4 & 1 \\ 2 & 2 & -2 \end{pmatrix}^t = \begin{pmatrix} -1 & 5 & 2 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{pmatrix},$$

and

$$\text{adj } B = \begin{pmatrix} -4 & 5 & 8 \\ -3 & 3 & 6 \\ 2 & -1 & -1 \end{pmatrix}^t = \begin{pmatrix} -4 & -3 & 2 \\ 5 & 3 & -1 \\ 8 & 6 & -1 \end{pmatrix}.$$

Corollary 2. *For any* $n \times n$ *matrix* A, $(adj\ A)A = [det(A)]I$.

PROOF. Let $A \in \mathsf{M}_{n\times n}(F)$, and let c_{ij} denote the cofactor of A_{ij}. Then $(\text{adj } A)_{ji} = c_{ij}$, and hence the entry of $(\text{adj } A)A$ in the jth row and kth column is

$$\sum_{i=1}^{n} (\text{adj } A)_{ji} A_{ik} = \sum_{i=1}^{n} c_{ij}A_{ik} = \delta_{jk}\cdot\det(A)$$

by Corollary 1. Thus $(\text{adj } A)A = [\det(A)]I$. ∎

Example 14. Let A be as in Example 13. By expanding A along the second row, we see that

$$\det(A) = (-1)\cdot 1\cdot[2(-1) - 3\cdot 1] + (-1)\cdot 1\cdot(1\cdot 1 - 2\cdot 1) = 6.$$

And

$$(\text{adj } A)A = \begin{pmatrix} -1 & 5 & 2 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$
$$= [\det(A)]I.$$

Similarly, if B is as in Example 13, then by expanding B along the third row, we see that

$$\det(B) = 2[3\cdot 2 - (-1)(-4)] + 1[1(-4) - 3(-1)] = 3.$$

Moreover,

$$(\text{adj } B)B = \begin{pmatrix} -4 & -3 & 2 \\ 5 & 3 & -1 \\ 8 & 6 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -1 \\ -1 & -4 & 2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
$$= [\det(B)]I.$$

Corollary 3. *If* A *is an invertible matrix, then*

$$A^{-1} = [\det(A)]^{-1}(\operatorname{adj} A).$$

PROOF. If A is invertible, then $\det(A) \neq 0$ (Corollary 1 of Theorem 4.8). Now $(\operatorname{adj} A)A = [\det(A)]I$ by Corollary 2 above, and hence $[\det(A)]^{-1}(\operatorname{adj} A)A = I$. So $[\det(A)]^{-1}(\operatorname{adj} A) = A^{-1}$. ∎

Example 15. Continuing from Example 14, we have

$$A^{-1} = [\det(A)]^{-1}(\operatorname{adj} A) = \frac{1}{6}\begin{pmatrix} -1 & 5 & 2 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$$

and

$$B^{-1} = [\det(B)]^{-1}(\operatorname{adj} B) = \frac{1}{3}\begin{pmatrix} -4 & -3 & 2 \\ 5 & 3 & -1 \\ 8 & 6 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & -1 & \frac{2}{3} \\ \frac{5}{3} & 1 & -\frac{1}{3} \\ \frac{8}{3} & 2 & -\frac{1}{3} \end{pmatrix}.$$

We shall conclude this section with a discussion of Cramer's rule, which provides an interesting method for solving matrix equations of the form $AX = B$, where A is an invertible matrix. This method, however, is extremely inefficient, for if A is an $n \times n$ matrix, the solution of the system $AX = B$ by Cramer's rule requires the evaluation of $n + 1$ determinants of $n \times n$ matrices. (By comparison, the method of solution presented in Section 3.4 is a more efficient way of solving such systems. Thus Cramer's rule is of theoretical and aesthetic interest, rather than of practical concern.)

Theorem 4.13 (*Cramer's rule*). *Let* $AX = B$ *be the matrix equation of a system of* n *linear equations in* n *unknowns, where* $X = (x_1, \ldots, x_n)^t$ *and* $B = (b_1, \ldots, b_n)^t$. *If* $\det(A) \neq 0$, *then the system has a unique solution, and for each* i $(1 \leq i \leq n)$

$$x_i = [\det(A)]^{-1} \cdot \det(M_i),$$

where M_i *is the* $n \times n$ *matrix obtained from* A *by replacing the* ith *column of* A *by* B.

PROOF. By Corollary 2 of Theorem 4.8, $\det(A) \neq 0$ implies that A is invertible. Hence by Theorem 3.9 the matrix equation $AX = B$ has a unique solution. Multiplying this equation on the left by adj A and using Corollary 2 of Theorem 4.12 gives

$$[\det(A)]IX = (\operatorname{adj} A)AX = (\operatorname{adj} A)B.$$

Examining the ith coordinates of the column vectors $[\det(A)]X = (\operatorname{adj} A)B$, we see that

$$[\det(A)]x_i = \sum_{j=1}^{n} (\operatorname{adj} A)_{ij} b_j = \sum_{j=1}^{n} c_{ji} b_j,$$

where c_{ji} is the cofactor of A_{ji}. But

$$\sum_{j=1}^{n} c_{ji}b_j$$

is the expansion of M_i along the ith column; so

$$[\det(A)]x_i = \sum_{j=1}^{n} c_{ji}b_j = \det(M_i),$$

and thus

$$x_i = [\det(A)]^{-1}\cdot\det(M_i) \qquad \text{for } 1 \leq i \leq n. \quad \blacksquare$$

Example 16. We shall use Cramer's rule to solve the matrix equation $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

First, as we have seen in Example 14, $\det(A) = 6$, so that Cramer's rule applies. Letting M_i denote the matrix obtained from A by replacing the ith column of A by B, we have

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det\begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{15}{6} = \frac{5}{2},$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{-6}{6} = -1,$$

and

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}}{\det(A)} = \frac{3}{6} = \frac{1}{2}.$$

EXERCISES

1. Label the following as being true or false.
   (a) If a matrix is expanded by the entries of one column and cofactors of a different column, the result is the determinant of the matrix.
   (b) If $A \in \mathsf{M}_{n\times n}(F)$, then $(\text{adj } A)A = I$.

(c) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

(d) Let $AX = B$ be the matrix form of a system of n linear equations in n unknowns, where $X = (x_1, \ldots, x_n)^t$. If $\det(A) \neq 0$ and if M_i is the matrix obtained from A by replacing the ith row of A by B^t, then

$$x_i = [\det(A)]^{-1} \cdot \det(M_i).$$

2. Find the classical adjoint of the following matrices.

(a) $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

(b) $\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1-i & 0 & 0 \\ 4 & 3i & 0 \\ 2i & 1+4i & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 7 & 1 & 4 \\ 6 & -3 & 0 \\ -3 & 5 & -2 \end{pmatrix}$

(f) $\begin{pmatrix} 3 & 2+i & 0 \\ -1+i & 0 & i \\ 0 & 1 & 3-2i \end{pmatrix}$

(g) $\begin{pmatrix} -1 & 2 & 5 \\ 8 & 0 & -3 \\ 4 & 6 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

3. Solve the following systems by Cramer's rule.

(a) $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2, \end{cases}$
where $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

(b) $\begin{cases} 2x_1 + x_2 - 3x_3 = 5 \\ x_1 - 2x_2 + x_3 = 10 \\ 3x_1 + 4x_2 - 2x_3 = 0 \end{cases}$

(c) $\begin{cases} 2x_1 + x_2 - 3x_3 = 1 \\ x_1 - 2x_2 + x_3 = 0 \\ 3x_1 + 4x_2 - 2x_3 = -5 \end{cases}$

(d) $\begin{cases} x_1 - x_2 + 4x_3 = -4 \\ -8x_1 + 3x_2 + x_3 = 8 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}$

(e) $\begin{cases} x_1 - x_2 + 4x_3 = -2 \\ -8x_1 + 3x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 = 6 \end{cases}$

(f) $\begin{cases} 3x_1 + x_2 + x_3 = 4 \\ -2x_1 - x_2 = 12 \\ x_1 + 2x_2 + x_3 = -8 \end{cases}$

4. Prove that, for any $A \in \mathsf{M}_{n\times n}(F)$, $\det(\operatorname{adj} A) = [\det(A)]^{n-1}$.

5. Let A be an invertible upper triangular $n \times n$ matrix. Prove that adj A is upper triangular and hence that A^{-1} is upper triangular. Show that similar results are true if A is lower triangular.

6. Prove Corollary 1 of Theorem 4.12.

7. Prove that $\operatorname{adj} A^t = (\operatorname{adj} A)^t$.

4.5 SUMMARY—IMPORTANT FACTS ABOUT DETERMINANTS

In this section we shall summarize the important properties of the determinant needed for the remainder of the text. The results contained in this section have been derived in Sections 4.2 and 4.3; consequently the facts presented here will be stated without proofs.

The *determinant* of an $n \times n$ matrix A having entries from a field F is an element of F denoted $\det(A)$, which can be computed in the following manner:

1. If A is 1×1, then $\det(A) = A_{11}$, the single entry of A.
2. If A is 2×2, then $\det(A) = A_{11}A_{22} - A_{12}A_{21}$. Thus, for example,

$$\det\begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = (-1)(3) - (2)(5) = -13.$$

3. If A is $n \times n$ for $n > 2$, then the determinant of A can be expressed as the sum of products of each entry of some row or column of A multiplied by ± 1 times the determinant of an $(n-1) \times (n-1)$ matrix obtained by deleting from A the row and column containing the entry in question. The precise formula is

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

(if the determinant is evaluated by the entries of row i of A) or

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

(if the determinant is evaluated by the entries of column j of A), where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

In the formulas above the scalar $(-1)^{i+j}\det(\tilde{A}_{ij})$ is called the *cofactor* of the entry A_{ij}. In this language the determinant of A is evaluated as the sum of products of each entry of some row or column of A multiplied by the cofactor of that entry. Thus $\det(A)$ is expressed in terms of n determinants of $(n-1) \times (n-1)$ matrices. These determinants are then evaluated in terms of determinants of $(n-2) \times (n-2)$ matrices, and so forth, until 2×2 matrices are obtained. The determinants of the 2×2 matrices are then evaluated as in item 2 above.

Let us consider several examples of this technique in evaluating the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix}.$$

First we shall evaluate the determinant of A by expanding along the fourth row. This will require knowing the cofactors of each entry of that row. The cofactor of $A_{41} = 3$ is

$$(-1)^{4+1} \det\begin{pmatrix} 1 & 1 & 5 \\ 1 & -4 & -1 \\ 0 & -3 & 1 \end{pmatrix}.$$

Let us evaluate the determinant above by expanding along the first column. Then

$$\begin{aligned}
\det\begin{pmatrix} 1 & 1 & 5 \\ 1 & -4 & -1 \\ 0 & -3 & 1 \end{pmatrix} &= (-1)^{1+1}(1)\det\begin{pmatrix} -4 & -1 \\ -3 & 1 \end{pmatrix} + (-1)^{2+1}(1)\det\begin{pmatrix} 1 & 5 \\ -3 & 1 \end{pmatrix} \\
&\quad + (-1)^{3+1}(0)\det\begin{pmatrix} 1 & 5 \\ -4 & -1 \end{pmatrix} \\
&= 1(1)[(-4)(1) - (-1)(-3)] \\
&\quad + (-1)(1)[(1)(1) - (5)(-3)] + 0 \\
&= -7 - 16 + 0 = -23.
\end{aligned}$$

Thus the cofactor of A_{41} is $(-1)^5(-23) = 23$. Likewise the cofactor of $A_{42} = 6$ is

$$(-1)^{4+2} \det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 2 & -3 & 1 \end{pmatrix}.$$

Evaluating this determinant along the second row gives

$$\begin{aligned}
\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 2 & -3 & 1 \end{pmatrix} &= (-1)^{2+1}(1)\det\begin{pmatrix} 1 & 5 \\ -3 & 1 \end{pmatrix} + (-1)^{2+2}(-4)\det\begin{pmatrix} 2 & 5 \\ 2 & 1 \end{pmatrix} \\
&\quad + (-1)^{2+3}(-1)\det\begin{pmatrix} 2 & 1 \\ 2 & -3 \end{pmatrix} \\
&= (-1)(1)[(1)(1) - (5)(-3)] + (1)(-4)[(2)(1) - (5)(2)] \\
&\quad + (-1)(-1)[(2)(-3) - (1)(2)] \\
&= -16 + 32 - 8 = 8.
\end{aligned}$$

So the cofactor of A_{42} is $(-1)^6(8) = 8$. The cofactor of $A_{43} = 1$ is

$$(-1)^{4+3}\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

If we evaluate this determinant by expanding along the third row, we find

$$\begin{aligned}\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} &= (-1)^{3+1}(2)\det\begin{pmatrix} 1 & 5 \\ 1 & -1 \end{pmatrix} + (-1)^{3+2}(0)\det\begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix} \\ &\quad + (-1)^{3+3}(1)\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= 1(2)[(1)(-1) - (5)(1)] + 0 + 1(1)[(2)(1) - (1)(1)] \\ &= -12 + 0 + 1 = -11.\end{aligned}$$

Hence the cofactor of A_{43} is $(-1)^7(-11) = 11$. Finally, the cofactor of $A_{44} = 2$ is

$$(-1)^{4+4}\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -4 \\ 2 & 0 & -3 \end{pmatrix}.$$

Computing this determinant by expanding along the second column, we obtain

$$\begin{aligned}\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -4 \\ 2 & 0 & -3 \end{pmatrix} &= (-1)^{1+1}(1)\det\begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} + (-1)^{2+2}(1)\det\begin{pmatrix} 2 & 1 \\ 2 & -3 \end{pmatrix} \\ &\quad + (-1)^{3+2}(0)\det\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} \\ &= (-1)(1)[(1)(-3) - (-4)(2)] + 1(1)[(2)(-3) - (1)(2)] \\ &\quad + 0 \\ &= -5 - 8 + 0 = -13.\end{aligned}$$

Therefore the cofactor of A_{44} is $(-1)^8(-13) = -13$. We can now evaluate the determinant of A by multiplying each entry of the fourth row by its cofactor; this gives

$$\det(A) = 3(23) + 6(8) + 1(11) + 2(-13) = 102.$$

For the sake of comparison we shall also compute the determinant of

A by expanding along the second column. The reader should verify that the cofactors of A_{12}, A_{22}, and A_{42} are 14, 40, and 8, respectively. Thus

$$\det(A) = (-1)^{1+2}(1)\det\begin{pmatrix} 1 & -4 & -1 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} + (-1)^{2+2}(1)\det\begin{pmatrix} 2 & 1 & 5 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$+ (-1)^{3+2}(0)\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 3 & 1 & 2 \end{pmatrix} + (-1)^{4+2}(6)\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 2 & -3 & 1 \end{pmatrix}$$

$$= 14 + 40 + 0 + 48 = 102.$$

Of course, the fact that the value 102 is obtained again is no surprise since the value of the determinant of A is independent of the choice of row or column used in the expansion.

Observe that the computation of $\det(A)$ is easier when expanded along the second column than when expanded along the fourth row. The difference is the presence of a zero in the second column, which made it unnecessary to evaluate one of the cofactors (the cofactor of A_{32}). For this reason it is beneficial to evaluate the determinant of a matrix by expanding along a row or column of the matrix that contains the largest number of zero entries. In fact, it is often helpful to introduce zeros into the matrix by means of elementary row operations before computing the determinant. This technique utilizes the first three properties of the determinant.

Properties of the Determinant

1. If B is a matrix obtained by interchanging two rows or two columns of A, then $\det(B) = -\det(A)$.
2. If B is a matrix obtained by multiplying each entry of some row or column of A by a scalar c, then $\det(B) = c\det(A)$.
3. If B is a matrix obtained from A by adding a multiple of row i to row j or a multiple of column i to column j, where $i \neq j$, then $\det(B) = \det(A)$.

We shall illustrate the use of these three properties in evaluating determinants by computing the determinant of the 4×4 matrix A considered previously. Our procedure will be to introduce zeros into the second column of A by employing property 3 and then expanding along that column. (The elementary row operations used here consist of adding multiples of row 1 to rows 2 and 4.) This procedure yields

$$\det(A) = \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ -1 & 0 & -5 & -6 \\ 2 & 0 & -3 & 1 \\ -9 & 0 & -5 & -28 \end{pmatrix}$$

$$= 1(-1)^{1+2}\det\begin{pmatrix} -1 & -5 & -6 \\ 2 & -3 & 1 \\ -9 & -5 & -28 \end{pmatrix}.$$

The resulting determinant of a 3×3 matrix can be evaluated in the same manner. We shall use type 3 elementary row operations to introduce two zeros into the first column and then expand along that column. Continuing from above, we have

$$\det(A) = (-1)\cdot\det\begin{pmatrix} -1 & -5 & -6 \\ 2 & -3 & 1 \\ -9 & -5 & -28 \end{pmatrix} = (-1)\cdot\det\begin{pmatrix} -1 & -5 & -6 \\ 0 & -13 & -11 \\ 0 & 40 & 26 \end{pmatrix}$$

$$= (-1)\left[(-1)^{1+1}(-1)\det\begin{pmatrix} -13 & -11 \\ 40 & 26 \end{pmatrix}\right]$$

$$= (-13)(26) - (-11)(40) = 102.$$

The reader should compare this calculation of $\det(A)$ with the preceding ones to see how much less work was required when properties 1, 2, and 3 were employed.

In the following chapters we shall often have to evaluate the determinant of matrices having special forms. The next three properties of the determinant are useful in this regard.

4. $\det(I) = 1$.
5. If two rows (or columns) of a matrix are identical, then the determinant of the matrix is zero.
6. The determinant of a triangular matrix is the product of the entries on the diagonal.

As an illustration of property 6, notice that

$$\det\begin{pmatrix} -3 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 5 & -6 \end{pmatrix} = (-3)(4)(-6) = 72.$$

The remaining four properties of the determinant will be frequently used in later chapters. Indeed, perhaps the most significant property of the determinant is that it provides a simple characterization of invertible matrices. (See property 10.)

7. For any A, $\det(A) = \det(A^t)$.
8. For any $A, B \in \mathsf{M}_{n\times n}(F)$, $\det(AB) = \det(A)\cdot\det(B)$.
9. If Q is an invertible matrix, then $\det(Q^{-1}) = [\det(Q)]^{-1}$.
10. A matrix Q is invertible if and only if $\det(Q) \neq 0$.

EXERCISES

1. Label the following as being true or false.
   (a) The determinant of a square matrix may be computed by expanding the matrix along any row or column.
   (b) In evaluating the determinant of a matrix, it is wise to expand along a row or column containing the largest number of zero entries.
   (c) If two rows or columns of A are identical, then $\det(A) = 0$.
   (d) If B is a matrix obtained by interchanging two rows or two columns of A, then $\det(B) = \det(A)$.
   (e) If B is a matrix obtained by multiplying each entry of some row or column of A by a scalar, then $\det(B) = \det(A)$.
   (f) If B is a matrix obtained from A by adding a multiple of some row to a different row (or a multiple of some column to a different column), then $\det(B) = \det(A)$.
   (g) The determinant of a triangular $n \times n$ matrix is the product of its diagonal entries.
   (h) $\det(A^t) = -\det(A)$.
   (i) If $A, B \in \mathsf{M}_{n\times n}(F)$, then $\det(AB) = \det(A)\cdot\det(B)$.
   (j) If Q is an invertible matrix, then $\det(Q^{-1}) = [\det(Q)]^{-1}$.
   (k) A matrix Q is invertible if and only if $\det(Q) \neq 0$.
2. Evaluate the determinant of the following 2×2 matrices.
   (a) $\begin{pmatrix} 4 & -5 \\ 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 7 \\ 3 & 8 \end{pmatrix}$
   (c) $\begin{pmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 4i \\ -6i & 2i \end{pmatrix}$
3. Evaluate the determinant of the following matrices in the manner indicated.
   (a) Expand
   $$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$$
   along the second column.

(b) Expand

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

along the third row.

(c) Expand

$$\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$$

along the first column.

(d) Expand

$$\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$$

along the first row.

(e) Expand

$$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$$

along the fourth column.

4. Evaluate the determinant of the following matrices by any legitimate method.

(a) $\begin{pmatrix} 2 & 5 & 0 \\ -6 & 1 & 3 \\ 0 & -4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -1 & 1 \\ 2 & 2 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$ (e) $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 4 & 2+i & 2i & 5+2i \\ 0 & 1-i & 1 & 3-4i \\ 0 & 0 & 3i & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

5.† Work Exercise 9 of Section 4.3.

INDEX OF DEFINITIONS FOR CHAPTER 4

chapter 5

diagonalization

This chapter is concerned with the so-called "diagonalization problem." Given a linear transformation $T: V \longrightarrow V$, where V is a finite-dimensional vector space, we shall seek answers to the following questions:

1. Does there exist an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix?
2. If such a basis exists, how can it be found?

Since computations involving diagonal matrices are simple, an affirmative answer to question 1 will lead us to a clearer understanding of how the transformation T acts on V, and an answer to question 2 will enable us to obtain easy solutions to many practical problems that can be formulated in a linear algebra context. We shall consider some of these problems and their solutions in this chapter—see, for example, Section 5.3.

A solution to the diagonalization problem leads naturally to the concepts of "eigenvalue" and "eigenvector." Aside from the important role that these concepts play in the diagonalization problem, they will also prove to be useful tools in the study of many non-diagonalizable transformations, as we shall see in Chapter 6.

Since the diagonalization problem involves the study of a linear transformation that maps a vector space into itself, it is useful to name such a transformation. Accordingly, we shall call a linear transformation $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ on a vector space V a *linear operator* on V.

For a given linear operator T on a finite-dimensional vector space V, we shall be concerned with the matrices that represent T relative to various ordered bases for V.

Throughout this chapter, we shall usually omit the adjective "ordered" from the expression "ordered basis."

Consider a linear operator T on a finite-dimensional vector space V and any two bases β and β' for V. Recall from the corollary to Theorem 2.27 that the matrices $[\mathsf{T}]_\beta$ and $[\mathsf{T}]_{\beta'}$ are related by

$$[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q,$$

where Q is the change of coordinate matrix changing β'-coordinates into β-coordinates. In Section 2.5 we defined such matrices to be *similar*. A useful special case of this relationship is proved in the following theorem.

Theorem 5.1. *Let* $A \in \mathsf{M}_{n\times n}(F)$, *and let* $\gamma = \{x_1, x_2, \ldots, x_n\}$ *be any basis for* F^n. *Then* $[\mathsf{L}_A]_\gamma = Q^{-1}AQ$, *where* Q *is the* $n \times n$ *matrix in which the* jth *column is* x_j $(j = 1, 2, \ldots, n)$.

PROOF. Let β be the standard basis for F^n. It is easily seen that the matrix Q is the change of coordinate matrix changing γ-coordinates into β-coordinates. Hence

$$[\mathsf{L}_A]_\gamma = Q^{-1}[\mathsf{L}_A]_\beta Q = Q^{-1}AQ. \quad \blacksquare$$

Example 1. To illustrate Theorem 5.1, let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R) \quad \text{and} \quad \gamma = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

It is a simple matter to check that if

$$Q = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix},$$

then

$$Q^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$$

and

$$[\mathsf{L}_A]_\gamma = Q^{-1}AQ = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -11 & -8 \\ 18 & 13 \end{pmatrix}.$$

As mentioned above, matrices that represent the same linear operator relative to different bases are similar. We shall now establish the converse of this result.

Theorem 5.2. *Let* T *be a linear operator on an* n*-dimensional vector space* V, *and let* β *be a basis for* V. *If* B *is any* $n \times n$ *matrix similar to* $[T]_\beta$, *then there exists a basis* β' *for* V *such that* $B = [T]_{\beta'}$.

PROOF. If B is similar to $[T]_\beta$, then there exists an invertible matrix Q such that $B = Q^{-1}[T]_\beta Q$. Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$, and define

$$x'_j = \sum_{i=1}^{n} Q_{ij}x_i \qquad \text{for } 1 \leq j \leq n.$$

Then $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ is a basis for V such that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates (Exercise 11 of Section 2.5). Hence

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = B$$

by the corollary to Theorem 2.27. ∎

The concept of similarity is useful in studying the diagonalization problem since it can be used to reformulate the problem in the context of matrices. We shall now introduce the definitions of diagonalizability.

Definitions. *A linear operator* T *on a finite-dimensional vector space* V *is said to be* diagonalizable *if there exists a basis* β *for* V *such that* $[T]_\beta$ *is a diagonal matrix.*

A square matrix A *is said to be* diagonalizable *if* A *is similar to a diagonal matrix.*

The following theorem relates these two concepts and leads to a reformulation of the diagonalization problem in the context of matrices.

Theorem 5.3. *Let* T *be a linear operator on a finite-dimensional vector space* V. *The following are equivalent:*

(a) T *is diagonalizable.*

(b) *There exists a basis* β *for* V *such that the matrix* $[T]_\beta$ *is diagonalizable.*

(c) *The matrix* $[T]_\gamma$ *is diagonalizable for any basis* γ *for* V.

PROOF. If T is diagonalizable, then there exists a basis β for V such that $[T]_\beta$ is a diagonal matrix. Thus $[T]_\beta$ is trivially diagonalizable. So (a) implies (b).

Let β be a basis for V such that $[T]_\beta$ is diagonalizable, and let γ be any basis for V. Then $[T]_\beta$ and $[T]_\gamma$ are similar. So if $[T]_\beta$ is similar to a diagonal matrix, then so is $[T]_\gamma$ by the transitivity of the similarity relation. Hence $[T]_\gamma$ is diagonalizable, proving that (b) implies (c).

Finally, if $[\mathsf{T}]_\gamma$ is diagonalizable, then there is a diagonal matrix D similar to $[\mathsf{T}]_\gamma$. So by Theorem 5.2 there exists a basis β' for V such that $[\mathsf{T}]_{\beta'} = D$. Thus T is diagonalizable, and so (c) implies (a). ∎

As an immediate consequence of this theorem we have the following useful result.

Corollary. *A matrix A is diagonalizable if and only if L_A is diagonalizable.*

Because of Theorem 5.3 we can reformulate the diagonalization problem as follows.

1. Is a given square matrix A diagonalizable?
2. If A is diagonalizable, how can an invertible matrix Q be determined so that $Q^{-1}AQ$ is a diagonal matrix?

We shall now present the first of several results leading to a solution of the diagonalization problem.

Theorem 5.4. *A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists a basis $\beta = \{x_1, \ldots, x_n\}$ for V and scalars $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) such that $\mathsf{T}(x_j) = \lambda_j x_j$, for $1 \leq j \leq n$. Under these circumstances*

$$[\mathsf{T}]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

PROOF. Suppose T is diagonalizable. Then there is a basis β for V such that $[\mathsf{T}]_\beta = D$ is a diagonal matrix. Let $\lambda_j = D_{jj}$ and $\beta = \{x_1, \ldots, x_n\}$. Then for each j,

$$\mathsf{T}(x_j) = \sum_{i=1}^{n} D_{ij}x_i = D_{jj}x_j = \lambda_j x_j.$$

Conversely, suppose there exists a basis $\beta = \{x_1, \ldots, x_n\}$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $\mathsf{T}(x_j) = \lambda_j x_j$. Then clearly

$$[\mathsf{T}]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \quad ∎$$

Theorem 5.4 motivates the following definitions.

Definitions. *Let* T *be a linear operator on a vector space* V. *A non-zero element* x ∈ V *is called an* eigenvector *of* T *if there exists a scalar* λ *such that* T(x) = λx. *The scalar* λ *is called the* eigenvalue *corresponding to the eigenvector* x.

Similarly, if A *is an* n × n *matrix over a field* F, *a non-zero element* x ∈ F^n *is called an* eigenvector *of the matrix* A *if* x *is an eigenvector of* L_A. *As above, the scalar* λ *is called an* eigenvalue *of* A *corresponding to the eigenvector* x.

The words *characteristic vector* and *proper vector* are often used in place of *eigenvector*. The corresponding terms for an eigenvalue are *characteristic value* and *proper value*.

In this terminology we see that in Theorem 5.4 the basis β consists of eigenvectors of T and that the diagonal entries of $[\mathsf{T}]_\beta$ are eigenvalues of T. Thus Theorem 5.4 can be restated as follows: *A linear operator* T *on a finite-dimensional vector space* V *is diagonalizable if and only if there exists a basis* β *for* V *consisting of eigenvectors of* T. *Furthermore, if* T *is diagonalizable,* $\beta = \{x_1, x_2, \ldots, x_n\}$ *is a basis of eigenvectors of* T, *and* $D = [\mathsf{T}]_\beta$, *then* D *is a diagonal matrix and* D_{ii} *is the eigenvalue corresponding to* x_i $(i = 1, 2, \ldots, n)$.

Before continuing our examination of the diagonalization problem, we shall consider two examples involving eigenvectors and eigenvalues.

Example 2. Let $C^\infty(R)$ denote the set of all functions $f\colon\ R \longrightarrow R$ having derivatives of all orders. (Thus $C^\infty(R)$ includes all polynomial functions, the sine and cosine functions, the exponential functions, etc.) It is easy to see that $C^\infty(R)$ is a subspace of the vector space $\mathfrak{F}(R, R)$ of all functions from R to R as defined in Section 1.2. Define $\mathsf{T}\colon\ C^\infty(R) \longrightarrow C^\infty(R)$ by $\mathsf{T}(y) = y'$, where y' denotes the derivative of y. It is easily verified that T is a linear operator on $C^\infty(R)$. We shall determine the eigenvalues and eigenvectors of T.

If λ is an eigenvalue of T, then there is an eigenvector $y \in C^\infty(R)$ such that $y' = \mathsf{T}(y) = \lambda y$. This is a first-order differential equation whose solutions are of the form $y(t) = ce^{\lambda t}$ for some constant c. Consequently every real number λ is an eigenvalue of T, and the corresponding eigenvectors are of the form $ce^{\lambda t}$ for $c \neq 0$. (Note that if $\lambda = 0$, the eigenvectors are the non-zero constant functions.)

Example 3. Let

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since

$$\mathsf{L}_A(x_1) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2x_1,$$

x_1 is an eigenvector of L_A (and hence of A). Also $\lambda_1 = -2$ is the eigenvalue associated with x_1. Moreover,

$$\mathsf{L}_A(x_2) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5x_2.$$

Thus x_2 is an eigenvector of L_A (and of A) with $\lambda_2 = 5$ as the associated eigenvalue. Note that $\beta = \{x_1, x_2\}$ is a basis for R^2, and hence by Theorem 5.4

$$[\mathsf{L}_A]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}.$$

Finally, if

$$Q = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix},$$

then

$$Q^{-1}AQ = [\mathsf{L}_A]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

by Theorem 5.1.

The preceding example demonstrates a technique for diagonalizing an $n \times n$ matrix A: If $\beta = \{x_1, x_2, \ldots, x_n\}$ is a basis for F^n consisting of eigenvectors of A and Q is the $n \times n$ matrix whose jth column is the eigenvector x_j ($j = 1, 2, \ldots, n$), then $Q^{-1}AQ$ is a diagonal matrix. In order to use this procedure we need a method for determining the eigenvectors of a matrix or operator. As we shall see, eigenvectors are easily determined once the eigenvalues are known. For this reason we shall begin by discussing a method for computing eigenvalues. As an aid in this computation we shall utilize the following theorem to introduce the "determinant" of a linear operator.

Theorem 5.5. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* β *and* β' *be any two bases for* V. *Then* $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_{\beta'})$.

PROOF. Let $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_{\beta'}$. Since A and B are similar, there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Thus

$$\begin{aligned}\det(B) &= \det(Q^{-1}AQ) = \det(Q^{-1})\cdot\det(A)\cdot\det(Q) \\ &= [\det(Q)]^{-1}\cdot[\det(A)]\cdot[\det(Q)] = \det(A). \quad \blacksquare\end{aligned}$$

This result motivates the following definition.

Definition. *Let* T *be a linear operator on a finite-dimensional vector space* V. *We define the* determinant *of* T, *denoted* $\det(\mathsf{T})$, *as follows: Choose any* *basis β for* V, *and define* det(T) $=$ *det*($[\mathsf{T}]_\beta$). *Note that by Theorem 5.5 det*(T) *is well-defined, i.e., is independent of the choice of basis β.*

Example 4. Let T: $\mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ be defined by $\mathsf{T}(f) = f'$, the derivative of f. To compute det(T), let $\beta = \{1, x, x^2\}$. Then β is a basis for $\mathsf{P}_2(R)$ and

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $\det(\mathsf{T}) = \det([\mathsf{T}]_\beta) = 0$.

Our next result establishes some important properties of the determinant of a linear operator. Note the similarity of these properties to those proved for the determinant of a matrix in Chapter 4.

Theorem 5.6. *Let* T *be a linear operator on a finite-dimensional vector space* V. *Then*

(a) T *is invertible if and only if det*(T) $\neq 0$.
(b) *If* T *is invertible, then det*$(\mathsf{T}^{-1}) = [\det(\mathsf{T})]^{-1}$.
(c) *If* U: V $\longrightarrow$ V *is linear, then det*(TU) $=$ *det*(T)$\cdot$*det*(U).
(d) *If λ is any scalar and β any basis for* V, *then*

$$\det(\mathsf{T} - \lambda \mathsf{I}_\mathsf{V}) = \det(A - \lambda I),$$

where $A = [\mathsf{T}]_\beta$.

PROOF. The proofs of (a), (b), and (c) are exercises. To prove (d), suppose that λ is a scalar, β is a basis for V, and $A = [\mathsf{T}]_\beta$. Then $[\mathsf{I}_\mathsf{V}]_\beta = I$, and hence $[\mathsf{T} - \lambda \mathsf{I}_\mathsf{V}]_\beta = A - \lambda I$. Thus by definition we have $\det(\mathsf{T} - \lambda \mathsf{I}_\mathsf{V}) = \det(A - \lambda I)$. ∎

The following theorem provides us with a method for computing eigenvalues.

Theorem 5.7. *Let* T *be a linear operator on a finite-dimensional vector space* V *over a field* F. *A scalar $\lambda \in$* F *is an eigenvalue of* T *if and only if det*$(\mathsf{T} - \lambda \mathsf{I}) = 0$.

PROOF. Suppose that λ is an eigenvalue of T. Then there exists an eigenvector $x \in$ V ($x \neq 0$) such that $\mathsf{T}(x) = \lambda x$. Thus $0 = \mathsf{T}(x) - \lambda x = (\mathsf{T} - \lambda \mathsf{I})(x)$. Since $x \neq 0$, $\mathsf{T} - \lambda \mathsf{I}$ is not invertible. So, by Theorem 5.6, $\det(\mathsf{T} - \lambda \mathsf{I}) = 0$.

Conversely, suppose that $\det(\mathsf{T} - \lambda \mathsf{I}) = 0$. Then, again by Theorem 5.6, $\mathsf{T} - \lambda \mathsf{I}$ is not invertible. So there exists a non-zero vector $x \in$ V such that $x \in \mathsf{N}(\mathsf{T} - \lambda\mathsf{I})$. Thus $(\mathsf{T} - \lambda\mathsf{I})(x) = 0$, and so $\mathsf{T}(x) = \lambda x$. Hence x is an eigenvector (with λ as the associated eigenvalue) of T. ∎

Corollary 1. *Let* A *be an* $n \times n$ *matrix over a field* F. *Then a scalar* $\lambda \in F$ is an *eigenvalue of* A *if and only if* $\det(A - \lambda I) = 0$.

PROOF. Exercise.

Example 5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R).$$

Since

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1),$$

the only eigenvalues of A are 3 and -1.

Example 6. Let $\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ be the linear operator defined by $\mathsf{T}(f(x)) = f(x) + xf'(x) + f'(x)$, and let $\beta = \{1, x, x^2\}$. Then β is a basis for $\mathsf{P}_2(R)$ and

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Since

$$\begin{aligned} \det(\mathsf{T} - \lambda\mathsf{I}) = \det([\mathsf{T}]_\beta - \lambda I) &= \det\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \\ &= (1-\lambda)(2-\lambda)(3-\lambda) \\ &= -(\lambda-1)(\lambda-2)(\lambda-3), \end{aligned}$$

λ is an eigenvalue of T if and only if $\lambda = 1, 2,$ or 3.

Example 6 makes use of the following obvious consequence of Theorem 5.6.

Corollary 2. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* β *be a basis for* V. *Then* λ *is an eigenvalue of* T *if and only if* λ *is an eigenvalue of* $[\mathsf{T}]_\beta$.

In Examples 5 and 6 the reader may have observed that if A is an $n \times n$ matrix, then $\det(A - \lambda I_n)$ is a polynomial in λ of degree n with leading coefficient $(-1)^n$. The eigenvalues of A are simply the zeros of this polynomial. Thus the following definition is appropriate.

Definition. *If* $A \in M_{n\times n}(F)$, *the polynomial* $det(A - tI_n)$ *in the indeterminate* t *is called the* characteristic polynomial *of* A.†

It is easily shown that similar matrices have the same characteristic polynomial (see Exercise 12). This fact permits the following definition.

Definition. *Let* T *be a linear operator on a finite-dimensional vector space* V *with basis* β. *We define the* characteristic polynomial $f(t)$ *of* T *to be the characteristic polynomial of* $A = [T]_\beta$; *that is,*

$$f(t) = det(A - tI).$$

The remark preceding the definition shows that this definition is independent of the choice of the basis β. We shall often denote the characteristic polynomial of an operator T by $\det(T - tI)$.

The next result confirms our observations about Examples 5 and 6. It can be proved by a straightforward induction argument.

Theorem 5.8. *The characteristic polynomial of* $A \in M_{n\times n}(F)$ *is a polynomial of degree* n *with leading coefficient* $(-1)^n$.

The following consequences of Theorem 5.8 are immediate. (See also Corollary 2 of Theorem E.2.)

Corollary 1. *Let* A *be any* $n \times n$ *matrix, and let* $f(t)$ *be the characteristic polynomial of* A. *Then*

(a) *A scalar* λ *is an eigenvalue of* A *if and only if* λ *is a zero of the polynomial* $f(t)$ (*i.e., if and only if* $f(\lambda) = 0$).

(b) A *has at most* n *distinct eigenvalues.*

Corollary 2. *Let* T *be a linear operator on an* n*-dimensional vector space* V *with characteristic polynomial* $f(t)$. *Then*

(a) *A scalar* λ *is an eigenvalue of* T *if and only if* λ *is a zero of the polynomial* $f(t)$ (*i.e., if and only if* $f(\lambda) = 0$).

(b) T *has at most* n *distinct eigenvalues.*

The two corollaries above provide us with a method for determining all the eigenvalues of a matrix or an operator. Our next result gives us a

†The observant reader may have noticed that the entries of the matrix $A - tI_n$ are not elements of the field F. They are, however, elements of another field $F(t)$. (The field $F(t)$ is the field of quotients of the polynomial ring $F[t]$. It is usually studied in abstract algebra courses.) Consequently the results proved about determinants in Chapter 4 remain true in this context.

procedure for determining the eigenvectors corresponding to a given eigenvalue.

Theorem 5.9. *Let* T *be a linear operator on a vector space* V, *and let* λ *be an eigenvalue of* T. *A vector* $x \in V$ *is an eigenvector of* T *corresponding to* λ *if and only if* $x \neq 0$ *and* $x \in N(T - \lambda I)$.

PROOF. Exercise.

Example 7. To find all the eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

in Example 5, recall that A has two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = --1$. We begin by finding all the eigenvectors corresponding to $\lambda_1 = 3$. Let

$$B = A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathsf{R}^2$$

is an eigenvector corresponding to $\lambda_1 = 3$ if and only if $x \neq 0$ and $x \in$ $N(L_B)$, i.e., $x \neq 0$ and

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Clearly the set of all solutions to the equation above is

$$\left\{t\begin{pmatrix} 1 \\ 2 \end{pmatrix} : \quad t \in R\right\}.$$

Hence x is an eigenvector corresponding to $\lambda_1 = 3$ if and only if

$$x = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{for some } t \neq 0.$$

Now suppose that x is an eigenvector of A corresponding to $\lambda_2 = -1$. Let

$$B = A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix};$$

then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N(L_B)$$

if and only if x is a solution to the system

$$\begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0. \end{cases}$$

Hence

$$\mathsf{N}(\mathsf{L}_B) = \left\{ t\begin{pmatrix} 1 \\ -2 \end{pmatrix} : \quad t \in R \right\}.$$

Thus x is an eigenvector corresponding to $\lambda_2 = -1$ if and only if

$$x = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{for some } t \neq 0.$$

Observe that

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

is a basis for R^2 consisting of eigenvectors of A. Thus, by Theorem 5.4, L_A, and hence A, is diagonalizable. In fact, if

$$Q = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix},$$

then Theorem 5.1 implies that

$$Q^{-1}AQ = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

In Example 6 we saw that the linear operator T on $\mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = f(x) + xf'(x) + f'(x)$ has eigenvalues of 1, 2, and 3. We shall now compute the eigenvectors of T.

$$\begin{tikzcd} \mathsf{P}_2(R) \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"'] & \mathsf{P}_2(R) \arrow[d, "\phi_\beta"] \\ \mathsf{R}^3 \arrow[r, "\mathsf{L}_A"] & \mathsf{R}^3 \end{tikzcd}$$

figure 5.1

Recall the diagram in Fig. 5.1 from Section 2.4, where $\beta = \{1, x, x^2\}$ and

$$A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

We shall show that $v \in \mathsf{P}_2(R)$ is an eigenvector of T corresponding to λ if and only if $\phi_\beta(v)$ is an eigenvector of A corresponding to λ. (This argument is valid for any operator on a finite-dimensional vector space.) If v is an eigenvector of T corresponding to λ, then $\mathsf{T}(v) = \lambda v$. Hence

$$\mathsf{L}_A\phi_\beta(v) = \phi_\beta\mathsf{T}(v) = \phi_\beta(\lambda v) = \lambda\phi_\beta(v).$$

Now $\phi_\beta(v) \neq 0$ since ϕ_β is an isomorphism. Thus $\phi_\beta(v)$ is an eigenvector of L_A (and hence of A) corresponding to λ. Since the argument above is reversible, we can establish similarly that if $\phi_\beta(v)$ is an eigenvector of A corresponding to λ, then v is an eigenvector of T corresponding to λ.

An equivalent formulation of the result proved in the preceding paragraph is that for any eigenvalue λ of A (and hence of T), a vector $y \in \mathsf{R}^3$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ. This fact allows us to compute eigenvectors of T as we did in Example 7.

Let $\lambda_1 = 1$, and define

$$B = A - \lambda_1 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

It is easily shown that

$$\mathsf{N}(\mathsf{L}_B) = \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \quad a \in R \right\}.$$

Thus the eigenvectors of A corresponding to λ_1 are of the form

$$a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

for some $a \neq 0$. Consequently the eigenvectors of T corresponding to $\lambda_1 = 1$ are of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = a\phi_\beta^{-1}(e_1) = a$$

for some $a \neq 0$. Hence the non-zero constant polynomials are the eigenvectors of T corresponding to λ_1.

Next let $\lambda_2 = 2$, and define

$$B = A - \lambda_2 I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again it is easily verified that

$$\mathsf{N}(\mathsf{L}_B) = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : \quad a \in R \right\}.$$

Thus the eigenvectors of T corresponding to λ_2 are of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) = a\phi_\beta^{-1}(e_1 + e_2) = a(1 + x) = a + ax$$

for some $a \neq 0$.

Finally, consider $\lambda_3 = 3$ and

$$B = A - \lambda_3 I = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathsf{N}(\mathsf{L}_B) = \left\{ a\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} : \quad a \in R \right\},$$

any eigenvector of T corresponding to $\lambda_3 = 3$ is of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) = a\phi_\beta^{-1}(e_1 + 2e_2 + e_3) = a(1 + 2x + x^2) = a + 2ax + ax^2$$

for some $a \neq 0$.

Note also that $\gamma = \{1, 1 + x, 1 + 2x + x^2\}$ is a basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T. Thus T is diagonalizable and

$$[\mathsf{T}]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We shall close this section by analyzing eigenvectors and eigenvalues from a geometric viewpoint. If x is an eigenvector of the linear operator T on V, then $\mathsf{T}(x) = \lambda x$ for some scalar λ. Let $\mathsf{W} = \text{span}(\{x\})$ be the one-dimensional subspace of V spanned by x. If $y \in \mathsf{W}$, then $y = cx$ for some scalar c. So

$$\mathsf{T}(y) = \mathsf{T}(cx) = c\mathsf{T}(x) = c\lambda x = \lambda y \in \mathsf{W}.$$

Thus T maps W into itself. If V is a vector space over the field of real numbers, then W can be regarded as a line passing through the origin of

V (i.e., through 0). The operator T acts on elements of W by multiplying each element by the scalar λ. There are several possibilities for the action of T depending on the value of λ (see Fig. 5.2).

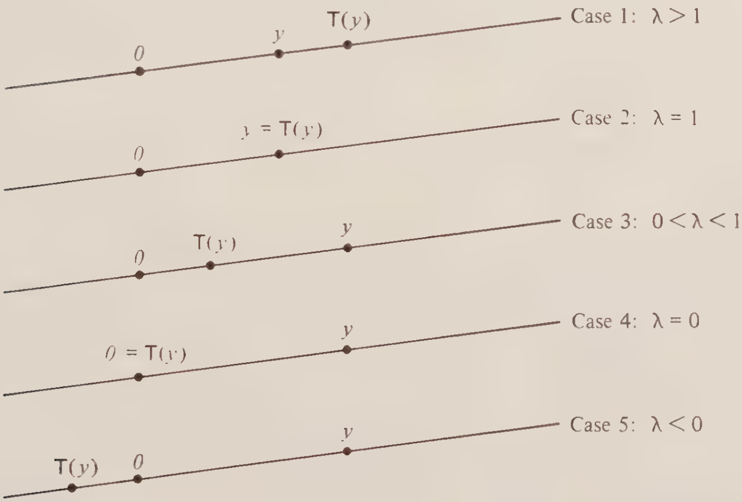
CASE 1. If $\lambda > 1$, then T moves elements of W to points farther from 0 by a factor of λ.

CASE 2. If $\lambda = 1$, then T acts as the identity transformation on W.

CASE 3. If $0 < \lambda < 1$, T moves elements of W to points closer to 0 by a factor of λ.

CASE 4. If $\lambda = 0$, then T acts as the zero transformation on W.

CASE 5. If $\lambda < 0$, then T reverses the orientation of W; that is, T moves points of W from one side of 0 to the other.



The action of T on W = span($\{x\}$) when x is an eigenvector of T.

figure 5.2

To illustrate these ideas, consider the linear operators introduced in Examples 6, 7, and 5 of Section 2.1. Recall that the operator T: $\mathsf{R}^2 \longrightarrow \mathsf{R}^2$ defined by $\mathsf{T}(x_1, x_2) = (x_1, -x_2)$ is a reflection about the x-axis. It is easily seen that T maps both axes onto themselves; thus e_1 and e_2 are eigenvectors of T (corresponding to the eigenvalues 1 and -1, respectively). Observe that T acts as the identity on the x-axis and reverses the orientation of the y-axis. Next consider the projection on the x-axis defined by $\mathsf{U}(x_1, x_2) = (x_1, 0)$. Again it is geometrically clear that U acts as the identity on the x-axis and acts as the zero transformation on the y-axis. This behavior is a consequence of the fact that e_1 and e_2 are eigenvectors of U corresponding to the eigenvalues 1 and 0, respectively. Finally, recall that the rotation through the angle θ is the operator T_θ: $\mathsf{R}^2 \longrightarrow \mathsf{R}^2$ defined by $\mathsf{T}_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$. If $0 < \theta < \pi$, then it is geometrically clear that T_θ maps no one-dimensional subspace of R^2 into itself. This observation implies that T_θ has no eigenvectors (and hence no eigenvalues). To confirm this conclusion using Corollary 2 of Theorem 5.8, we note that the characteristic polynomial of T_θ is

$$\det(\mathsf{T}_\theta - t\mathsf{I}) = \det\begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix} = t^2 - (2\cos\theta)t - 1,$$

which has no real zeros since the discriminant $4\cos^2\theta - 4$ is negative for $0 < \theta < \pi$. Thus there exist operators (and hence matrices) with no eigenvalues or eigenvectors. Of course, such operators and matrices are not diagonalizable.

EXERCISES

1. Label the following statements as being true or false.
(a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.
(b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
(c) There exists a square matrix with no eigenvectors.
(d) Eigenvalues must be non-zero scalars.
(e) Any two eigenvectors are linearly independent.
(f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
(g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.
(h) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.
(i) Similar matrices always have the same eigenvalues.
(j) Similar matrices always have the same eigenvectors.
(k) The sum of two eigenvectors of an operator T is always an eigenvector of T.

2. For each matrix A and basis β find $[\mathsf{L}_A]_\beta$. Also find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(b) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

3. For each of the following matrices $A \in \mathsf{M}_{n \times n}(F)$
(i) Determine all the eigenvalues of A.
(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
(iii) If possible, find a basis for F^n consisting of eigenvectors of A.
(iv) If successful in finding a basis in (iii), determine a matrix Q such that $Q^{-1}AQ$ is a diagonal matrix and compute $Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = R$

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = R$

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = C$

4. Let $\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ be defined by $\mathsf{T}((fx)) = f(x) + xf'(x)$. Find all the eigenvalues of T, and find a basis β for $\mathsf{P}_2(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

5. Prove parts (a), (b), and (c) of Theorem 5.6.

6. Prove Corollaries 1 and 2 of Theorem 5.7.

7. Prove Theorem 5.9.

8. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.
(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

9. Prove that the eigenvalues of a triangular matrix M are the diagonal entries of M.

10. Let V be a finite-dimensional vector space and λ be any scalar.
(a) For any basis β for V prove that $[\lambda \mathsf{I}_\mathsf{V}]_\beta = \lambda I$.
(b) Compute the characteristic polynomial of $\lambda \mathsf{I}_\mathsf{V}$.
(c) Show that $\lambda \mathsf{I}_\mathsf{V}$ is diagonalizable and has only one eigenvalue.

11. A *scalar matrix* is a square matrix of the form λI for some scalar λ; i.e., a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Conclude that the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable.

12. (a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

13. Prove the following assertions made on p. 227.

(a) If $v \in \mathsf{P}_2(R)$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to the eigenvalue λ.

(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{R}^3$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

14.† For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

15.† (a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

(b) State and prove the result for matrices that is analogous to that in part (a).

16. (a) Prove that similar matrices have the same trace. *Hint:* Use Exercise 12 of Section 2.3.

(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

17. Let T: $\mathsf{M}_{n\times n}(F) \longrightarrow \mathsf{M}_{n\times n}(F)$ be the mapping defined by $\mathsf{T}(A) = A^t$, the transpose of A.

(a) Verify that T is a linear operator on $\mathsf{M}_{n\times n}(F)$.

(b) Show that ± 1 are the only eigenvalues of T.

(c) Describe the matrices that are eigenvectors corresponding to the eigenvalues 1 and -1, respectively.

18. Show that for any $A, B \in \mathsf{M}_{n\times n}(C)$ such that B is invertible, there exists a scalar $c \in C$ such that $A + cB$ is not invertible. *Hint:* Examine $\det(A + cB)$.

19.† Let A and B be similar $n \times n$ matrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and bases β and γ for V such that $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_\gamma$. *Hint:* Use Exercise 12 of Section 2.5.

20. Let A be an $n \times n$ matrix with characteristic polynomial
$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$
Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

21. Let A and $f(t)$ be as in Exercise 20.
(a) Prove that $f(t) = (A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t) + q(t)$, where $q(t)$ is polynomial in t of degree at most $n - 2$.
(b) Show that $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

22.† Let T be a linear operator on a finite-dimensional vector space V over the field F. Prove that if $g(t) \in \mathsf{P}(F)$ and x is an eigenvector of T corresponding to eigenvalue λ, then $g(\mathsf{T})(x) = g(\lambda)x$.

5.2 DIAGONALIZABILITY

In Section 5.1 we presented the diagonalization problem and observed that not all linear operators or matrices are diagonalizable. Although we were able to diagonalize certain operators and matrices and even obtained a necessary and sufficient condition for diagonalizability (Theorem 5.4), we have not solved the diagonalization problem. What is still needed is a simple test to determine if an operator or a matrix can be diagonalized and, if so, an algorithm for obtaining a basis of eigenvectors. In this section we shall develop such a test and an algorithm.

In Example 7 in Section 5.1 we obtained a basis of eigenvectors by choosing one eigenvector corresponding to each eigenvalue. In general such a procedure will not yield a basis, but the following theorem shows that any set constructed in this manner must be linearly independent.

Theorem 5.10. *Let* T *be a linear operator on* V, *and let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be distinct eigenvalues of* T. *If* $x_1, x_2, \ldots, x_k$ *are eigenvectors of* T *such that* λ_j *corresponds to* x_j $(1 \leq j \leq k)$, *then* $\{x_1, x_2, \ldots, x_k\}$ *is linearly independent.*

PROOF. We shall use mathematical induction on the number k. Suppose that $k = 1$. Then $x_1 \neq 0$ since x_1 is an eigenvector, and hence $\{x_1\}$ is linearly independent. Assume that the theorem always holds for $k-1$ eigenvectors, where $k-1 \geq 1$, and that we have k eigenvectors $x_1, \ldots, x_k$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. We wish to show that $\{x_1, \ldots, x_k\}$ is linearly independent. Suppose that there are scalars $a_1, \ldots, a_k$ such that

$$a_1x_1 + \cdots + a_kx_k = 0. \tag{1}$$

Applying T to both sides of Eq. (1), we obtain

$$a_1\mathsf{T}(x_1) + \cdots + a_k\mathsf{T}(x_k) = a_1\lambda_1x_1 + \cdots + a_k\lambda_kx_k = 0. \tag{2}$$

Now multiply both sides of Eq. (1) by λ_k to obtain

$$a_1\lambda_kx_1 + \cdots + a_k\lambda_kx_k = 0. \tag{3}$$

Then, subtracting Eq. (3) from Eq. (2), we have

$$a_1(\lambda_1 - \lambda_k)x_1 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)x_{k-1} = 0.$$

By the induction hypothesis $\{x_1, \ldots, x_{k-1}\}$ is linearly independent; hence

$$a_1(\lambda_1 - \lambda_k) = \cdots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$$

Since $\lambda_1, \ldots, \lambda_k$ are distinct, it follows that $\lambda_i - \lambda_k \neq 0$ for $1 \leq i \leq k-1$. So $a_1 = \cdots = a_{k-1} = 0$. Thus Eq. (1) reduces to $a_kx_k = 0$. Since $x_k \neq 0$, $a_k = 0$. Therefore $a_1 = \cdots = a_k = 0$, and hence $\{x_1, \ldots, x_k\}$ is linearly independent. ▮

Corollary. *Let* T *be a linear operator on* V, *a finite-dimensional vector space of dimension* n. *If* T *has* n *distinct eigenvalues, then* T *is diagonalizable.*

PROOF. Let T have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and let $x_1, \ldots, x_n$ be eigenvectors of T such that λ_j corresponds to x_j for $1 \leq j \leq n$. By Theorem 5.10 $\{x_1, \ldots, x_n\}$ is linearly independent, and since $\dim(\mathsf{V}) = n$, this set is a basis for V. Thus, by Theorem 5.4, T is diagonalizable. ▮

Example 8. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R).$$

The characteristic polynomial of A (and hence of L_A) is

$$\det(A - tI) = \det\begin{pmatrix} 1-t & 1 \\ 1 & 1-t \end{pmatrix} = t(t-2),$$

and thus the eigenvalues of L_A are 0 and 2. Since L_A is a linear operator on the two-dimensional vector space R^2, we conclude from the corollary above that L_A (and hence A) is diagonalizable.

Although the corollary to Theorem 5.10 provides a sufficient condition for diagonalizability, this condition is not necessary. In fact the identity operator is diagonalizable but has only one eigenvalue, namely $\lambda = 1$.

We have seen that the existence of eigenvalues is a necessary condition for diagonalizability. The next result tells us more.

Theorem 5.11. *Let* T *be a diagonalizable linear operator on an* n*-dimensional vector space* V, *and let* f(t) *be the characteristic polynomial of* T. *Then* f(t) *factors into a product of* n *factors each of degree* 1*; that is, there exist scalars* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *(not necessarily distinct) such that*

$$f(t) = (-1)^n(t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

PROOF. Suppose that T is diagonalizable. Then there exists a basis β for V such that $[T]_\beta = D$ is a diagonal matrix. If

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

then

$$f(t) = \det(D - tI) = \det\begin{pmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n - t \end{pmatrix}$$

$$= (\lambda_1 - t)(\lambda_2 - t) \cdots (\lambda_n - t) = (-1)^n(t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

∎

From this theorem it is clear that if T is a diagonalizable linear operator on an n-dimensional vector space that fails to have n distinct eigenvalues, then the characteristic polynomial of T must have repeated zeros. This observation leads us to the following definition.

Definition. *Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial* f(t). *The* (algebraic) multiplicity *of λ is the largest positive integer* k *for which* $(t - \lambda)^k$ *is a factor of* f(t).

Example 9. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

If $f(t)$ is the characteristic polynomial of A, then $f(t) = -(t-1)^2(t-2)$. Hence $\lambda = 1$ is an eigenvalue of A with multiplicity 2, and $\lambda = 2$ is an eigenvalue of A with multiplicity 1.

If T is a diagonalizable linear operator on a finite-dimensional vector space V, then there is a basis β for V consisting of eigenvectors of T. We know from Theorem 5.4 that $[\mathsf{T}]_\beta$ is a diagonal matrix in which the diagonal entries are the eigenvalues of T. Since the characteristic polynomial of T is $\det([\mathsf{T}]_\beta - tI)$, it is easily seen that each eigenvalue of T must occur as a diagonal entry of $[\mathsf{T}]_\beta$ exactly as many times as its multiplicity. Hence β contains as many (linearly independent) eigenvectors corresponding to an eigenvalue as the multiplicity of that eigenvalue. Thus we see that the number of linearly independent eigenvectors corresponding to a given eigenvalue is of great interest in determining when an operator can be diagonalized. Recalling from Theorem 5.9 that the eigenvectors of T corresponding to the eigenvalue λ are the non-zero vectors in the null space of $\mathsf{T} - \lambda\mathsf{I}$, we are led naturally to the study of this set.

Definition. *Let* T *be a linear operator on a vector space* V, *and let* λ *be an eigenvalue of* T. *Define* $\mathsf{E}_\lambda = \{x \in \mathsf{V}\colon\ \mathsf{T}(x) = \lambda x\} = \mathsf{N}(\mathsf{T} - \lambda\mathsf{I}_\mathsf{V})$. *The set* E_λ *is called the* eigenspace of T corresponding to the eigenvalue λ. *As expected, by an* eigenspace *of a matrix* A, *we shall mean the corresponding eigenspace of the operator* L_A.

Clearly E_λ is a subspace of V consisting of the zero vector and the eigenvectors of T corresponding to the eigenvalue λ. The number of linearly independent eigenvectors of T corresponding to the eigenvalue λ is therefore the dimension of E_λ. Our next result relates this dimension to the multiplicity of λ.

Theorem 5.12. *Let* T *be a linear operator on a finite-dimensional vector space* V. *If* λ *is an eigenvalue of* T *having multiplicity* m, *then* $1 \le \dim(\mathsf{E}_\lambda) \le m$.

PROOF. Pick a basis $\{x_1, \ldots, x_p\}$ for E_λ, and extend it to a basis $\beta = \{x_1, \ldots, x_p, x_{p+1}, \ldots, x_n\}$ for V. Observe that x_i $(1 \le i \le p)$ is an eigenvector of T corresponding to λ, and let $A = [\mathsf{T}]_\beta$. Then

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

where $B_1 = \lambda I_p$ and O is a zero matrix.

By Exercise 9 of Section 4.3 the characteristic polynomial of T is

$$f(t) = \det(A - tI_n) = \det\begin{pmatrix} B_1 - tI_p & B_2 \\ O & B_3 - tI_{n-p} \end{pmatrix}$$
$$= \det(B_1 - tI_p)\cdot\det(B_3 - tI_{n-p}).$$

Let $g(t) = \det(B_3 - tI_{n-p})$, the characteristic polynomial of B_3. It is easily seen that $\det(B_1 - tI_p) = (\lambda - t)^p = (-1)^p(t - \lambda)^p$. Hence $f(t) = (-1)^p(t - \lambda)^p g(t)$, so that the multiplicity of λ is at least p. But $\dim(\mathsf{E}_\lambda) = p$; so $\dim(\mathsf{E}_\lambda) \leq m$. ▌

Example 10. Let $\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ be the linear operator defined by $\mathsf{T}(f) = f'$, the derivative of f. The matrix of T with respect to the basis $\beta = \{1, x, x^2\}$ for $\mathsf{P}_2(R)$ is

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently the characteristic polynomial of T is

$$\det([\mathsf{T}]_\beta - tI) = \det\begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{pmatrix} = -t^3.$$

Thus T has only one eigenvalue ($\lambda = 0$) with multiplicity 3. So $\mathsf{E}_\lambda = \mathsf{N}(\mathsf{T} - \lambda\mathsf{I}) = \mathsf{N}(\mathsf{T})$. Hence E_λ is the subspace of $\mathsf{P}_2(R)$ containing the constant polynomials. So in this case $\{1\}$ is a basis for E_λ, and $\dim(\mathsf{E}_\lambda) = 1$. Consequently there is no basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T, so that T is not diagonalizable.

Example 11. Let T be the linear operator on R^3 defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{pmatrix}.$$

We shall determine the eigenspace of T corresponding to each eigenvalue. If β is the standard basis for R^3, then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}.$$

Hence the characteristic polynomial of T is

$$\det([\mathsf{T}]_\beta - tI) = \det\begin{pmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{pmatrix} = -(t-5)(t-3)^2.$$

So the eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively.

Since

$$\mathsf{E}_{\lambda_1} = \mathsf{N}(\mathsf{T} - \lambda_1 \mathsf{I}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

E_{λ_1} is the solution space of the system of equations

$$\begin{cases} -x_1 \qquad\quad + \; x_3 = 0 \\ 2x_1 - 2x_2 + 2x_3 = 0 \\ \;\; x_1 \qquad\quad - \; x_3 = 0. \end{cases}$$

It is easily seen (using the techniques of Chapter 3) that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is a basis for E_{λ_1}. Hence $\dim(\mathsf{E}_{\lambda_1}) = 1$.

Likewise $\mathsf{E}_{\lambda_2} = \mathsf{N}(\mathsf{T} - \lambda_2 \mathsf{I})$ is the solution space of the system

$$\begin{cases} x_1 + x_3 = 0 \\ 2x_1 + 2x_3 = 0 \\ x_1 + x_3 = 0. \end{cases}$$

So

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for E_{λ_2}, and $\dim(\mathsf{E}_{\lambda_2}) = 2$.

In this case the multiplicity of each eigenvalue λ_i equals the dimension of the corresponding eigenspace E_{λ_i}. Observe that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for R^3 consisting of eigenvectors of T. Consequently T is diagonalizable.

Examples 10 and 11 suggest the following conjecture: If T is a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T factors into a product of factors of degree 1, then T is diagonalizable if and only if the multiplicity of each eigenvalue equals the dimension of the eigenspace of T corresponding to that eigenvalue. This conjecture is, in fact, true, but its proof involves a complication that we are not yet able to resolve. The difficulty is that we do not yet know in general that the union of bases for each of the eigenspaces will be a basis for V. (This fact was clear in the context of Example 11 but has not been proved in general.) Notice that Theorem 5.10 is of no help here unless each eigenspace is of dimension 1. Thus we must digress from the diagonalization problem to establish this fact, which will require generalizing the concept of a direct sum (as defined in Section 1.3). For this purpose it will be convenient to denote a (not necessarily direct) sum of subspaces $W_1, W_2, \ldots, W_k$ as

$$\sum_{i=1}^{k} W_i.$$

Definition. *Let* $W_1, W_2, \ldots, W_k$ *be subspaces of a vector space* V. *We shall write* $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ *and call* V *the* direct sum *of* $W_1, W_2, \ldots, W_k$ *if*

$$V = \sum_{i=1}^{k} W_i$$

and

$$W_i \cap \left(\sum_{j \neq i} W_j\right) = \{0\} \qquad \textit{for each } i \ (1 \leq i \leq k).$$

Example 12. Let $V = R^4$, and let $W_1 = \{(a, b, 0, 0)\colon\ a, b \in R\}$, $W_2 = \{(0, 0, c, 0)\colon\ c \in R\}$, and $W_3 = \{(0, 0, 0, d)\colon\ d \in R\}$. For any element (a, b, c, d) of V

$$(a, b, c, d) = (a, b, 0, 0) + (0, 0, c, 0) + (0, 0, 0, d) \in W_1 + W_2 + W_3.$$

Thus

$$V = \sum_{i=1}^{3} W_i.$$

To show that V is the direct sum of W_1, W_2, and W_3, we must prove that $W_1 \cap (W_2 + W_3) = \{0\}$, $W_2 \cap (W_1 + W_3) = \{0\}$, and $W_3 \cap (W_1 + W_2) = \{0\}$. But these equalities are obvious; so $V = W_1 \oplus W_2 \oplus W_3$.

Our next result contains several conditions that are equivalent to the definition of a direct sum. Notice that this theorem contains Theorem 1.6 as a special case.

Theorem 5.13. *Let* $W_1, W_2, \ldots, W_k$ *be subspaces of a finite-dimensional vector space* V. *The following conditions are equivalent:*

(a) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

(b) $V = \sum_{i=1}^{k} W_i$ *and, for any vectors* $x_1, x_2, \ldots, x_k$ *such that* $x_i \in W_i$ $(i = 1, 2, \ldots, k)$, *if* $x_1 + x_2 + \cdots + x_k = 0$, *then* $x_i = 0$ $(i = 1, 2, \ldots, k)$.

(c) *Each vector* v *in* V *can be uniquely written in the form* $v = x_1 + x_2 + \cdots + x_k$, *where* $x_i \in W_i$ $(i = 1, 2, \ldots, k)$.

(d) *If, for each* $i = 1, 2, \ldots, k$, γ_i *is any ordered basis for* W_i, *then* $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$† *is an ordered basis for* V.

(e) *For each* $i = 1, 2, \ldots, k$ *there exists an ordered basis* γ_i *for* W_i $(i = 1, 2, \ldots, k)$ *such that* $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ *is an ordered basis for* V.

PROOF. If (a) is true, then

$$V = \sum_{i=1}^{k} W_i$$

by definition. Suppose that $x_1, x_2, \ldots, x_k$ are vectors such that $x_i \in W_i$ $(i = 1, 2, \ldots, k)$ and $x_1 + x_2 + \cdots + x_k = 0$. Then for any i

$$-x_i = \sum_{j \neq i} x_j \in \sum_{j \neq i} W_j.$$

But also

$$-x_i \in W_i, \text{ and so } -x_i \in W_i \cap \left(\sum_{j \neq i} W_j\right) = \{0\}.$$

Hence $x_i = 0$, proving (b).

We shall next prove that (b) implies (c). Since

$$V = \sum_{i=1}^{k} W_i$$

by (b), any vector $v \in V$ can be represented in the form $v = x_1 + x_2 + \cdots + x_k$ for some elements $x_i \in W_i$ $(i = 1, 2, \ldots, k)$. We must show that this representation is unique. Suppose therefore that $v = y_1 + y_2 + \cdots + y_k$, where $y_i \in W_i$ $(i = 1, 2, \ldots, k)$. Then

$$(x_1 - y_1) + (x_2 - y_2) + \cdots + (x_k - y_k) = 0.$$

But since $x_i - y_i \in W_i$, it follows from (b) that $x_i - y_i = 0$ $(i = 1, 2 \ldots, k)$. Thus $x_i = y_i$ for each i, proving the uniqueness of the representation.

To show that (c) implies (d), let γ_i be a basis for W_i $(i = 1, 2, \ldots, k)$. Since

$$V = \sum_{i=1}^{k} W_i$$

by (c), it is clear that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ generates V. Suppose that there are vectors $x_{ij} \in \gamma_i$ $(j = 1, 2, \ldots, m_i$ and $i = 1, 2, \ldots, k)$ and scalars a_{ij} such that

$$\sum_{i,j} a_{ij} x_{ij} = 0.$$

†We shall regard $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ as an ordered basis in the natural way—the vectors in γ_1 are listed first (in the same order as in γ_1), then the vectors in γ_2 (in the same order as in γ_2), etc.

Set

$$y_i = \sum_{j=1}^{m_i} a_{ij}x_{ij};$$

then $y_i \in \text{span}(\gamma_i) = \mathsf{W}_i$ and

$$y_1 + y_2 + \cdots + y_k = \sum_{i,j} a_{ij}x_{ij} = 0.$$

Since $0 \in \mathsf{W}_i$ for each i and $0 + 0 + \cdots + 0 = y_1 + y_2 + \cdots + y_k$, condition (c) implies that $y_i = 0$ for each i. Thus

$$0 = y_i = \sum_{j=1}^{m_i} a_{ij}x_{ij}$$

for each i. But since γ_i is linearly independent, it follows that $a_{ij} = 0$ for $j = 1, 2, \ldots, m_i$ and each i. Hence $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is linearly independent and therefore is a basis for V.

It is immediate that (d) implies (e).

Finally, we shall show that (e) implies (a). If γ_i is a basis for W_i ($i = 1, 2, \ldots, k$) such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V, then

$$\begin{aligned}\mathsf{V} &= \text{span}(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k)\\ &= \text{span}(\gamma_1) + \text{span}(\gamma_2) + \cdots + \text{span}(\gamma_k) = \sum_{i=1}^{k} \mathsf{W}_i\end{aligned}$$

by repeated applications of Exercise 12 of Section 1.4. Fix an index i and suppose that

$$0 \neq v \in \mathsf{W}_i \cap (\sum_{j\neq i} \mathsf{W}_j).$$

Then

$$v \in \mathsf{W}_i = \text{span}(\gamma_i) \quad \text{and} \quad v \in \sum_{j\neq i} \mathsf{W}_j = \text{span}(\bigcup_{j\neq i} \gamma_j).$$

Hence v is a non-trivial linear combination of both γ_i and $\bigcup_{j\neq i} \gamma_j$, so that v can be expressed as a linear combination of $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ in more than one way. But these representations contradict Theorem 1.9, and so we conclude that

$$\mathsf{W}_i \cap (\sum_{j\neq i} \mathsf{W}_j) = \{0\},$$

proving (a). ∎

Our reason for discussing direct sums is to enable us to show that if T is a diagonalizable linear operator on a finite-dimensional vector space V, then the union of bases for each of the eigenspaces of T is a basis for V. The preceding theorem shows that this condition is equivalent to proving that if T is diagonalizable, then V is the direct sum of its eigenspaces. We shall now establish this important result.

Theorem 5.14. *Let* T *be a linear operator on an* n*-dimensional vector space* V. *Suppose that the characteristic polynomial of* T *factors into a product of factors of degree* 1, *and let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the distinct eigenvalues of* T. *Then the following are equivalent:*

(a) T *is diagonalizable.*
(b) $\mathsf{V} = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}$.
(c) *If* $\mathrm{d_j} = \dim(\mathsf{E}_{\lambda_j})$ *for* $1 \leq \mathrm{j} \leq \mathrm{k}$, *then* $\mathrm{d_1} + \mathrm{d_2} + \cdots + \mathrm{d_k} = \mathrm{n}$.
(d) *If* $\mathrm{m_j}$ *is the multiplicity of* λ_j *for each* j $(1 \leq \mathrm{j} \leq \mathrm{k})$, *then* $\dim(\mathsf{E}_{\lambda_j}) = \mathrm{m_j}$ $(\mathrm{j} = 1, 2, \ldots, \mathrm{k})$.

PROOF. First we shall prove that (a) implies (b). If T is diagonalizable, then V has a basis consisting of eigenvectors of T. It follows easily that

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{E}_{\lambda_i}.$$

Let $x_i \in \mathsf{E}_{\lambda_i}$ $(i = 1, 2, \ldots, k)$ be vectors such that $x_1 + x_2 + \cdots + x_k = 0$. Now each x_i is either the zero vector or an eigenvector of T corresponding to λ_i. Since the set of these non-zero vectors x_i is linearly independent by Theorem 5.10, $x_1 + x_2 + \cdots + x_k = 0$ implies that $x_1 = x_2 = \cdots = x_k = 0$. Thus, by Theorem 5.13, $\mathsf{V} = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}$.

If $\mathsf{V} = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}$, then it follows from Exercise 5 that

$$n = \dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{E}_{\lambda_i}) = d_1 + d_2 + \cdots + d_k.$$

Thus (b) implies (c).

Next we shall prove that (c) implies (d). Suppose that

$$\sum_{i=1}^{k} d_i = n.$$

By Theorem 5.12, $d_j \leq m_j$ for all j, and hence

$$n = \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i.$$

But

$$\sum_{i=1}^{k} m_i = n$$

since the characteristic polynomial factors into a product of factors of degree 1. Thus since

$$\sum_{i=1}^{k} (m_i - d_i) = 0 \quad \text{and} \quad m_i - d_i \geq 0$$

for each i, we conclude that $d_i = m_i$ for each i.

Finally, we shall prove that (d) implies (a). Suppose that $d_j = \dim(\mathsf{E}_{\lambda_j}) = m_j$ for all j, and let

$$\mathsf{W} = \sum_{i=1}^{k} \mathsf{E}_{\lambda_i}.$$

An argument similar to that in the first paragraph of this proof shows that $W = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$. If β_i is an ordered basis for E_{λ_i} ($i = 1, 2, \ldots, k$), then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a basis for W by Theorem 5.13. But $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ contains

$$\sum_{i=1}^{k} \dim(E_{\lambda_i}) = \sum_{i=1}^{k} m_i = n$$

vectors, and hence $W = V$. So V has a basis $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ consisting of eigenvectors of T. Thus T is diagonalizable, proving (a). ▮

This theorem completes our study of the diagonalization problem. We shall summarize some of our previous results in the following test and algorithm.

A Test for Diagonalizability

Let T be a linear operator on an n-dimensional vector space. Then T is diagonalizable if and only if both of the following conditions hold.

1. The characteristic polynomial of T factors into a product of factors of degree 1.
2. The multiplicity of λ equals $n - \operatorname{rank}(T - \lambda I)$ for each eigenvalue λ of T.

Observe that condition 2 above is automatically satisfied for eigenvalues having multiplicity 1 (Theorem 5.12). Thus condition 2 need only be checked for those eigenvalues having multiplicity greater than 1.

An Algorithm for Diagonalization

Let T be a diagonalizable linear operator on a finite-dimensional vector space V, and let $\lambda_1, \ldots, \lambda_k$ denote the distinct eigenvalues of T. For each j, let β_j be a basis for $E_{\lambda_j} = N(T - \lambda_j I)$ and $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. Then β is a basis for V, and $[T]_\beta$ is a diagonal matrix.

Example 13. We shall test the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(R)$$

for diagonalizability. Since the test above is stated for linear operators rather than for matrices, we shall apply the test to the operator L_A.

The characteristic polynomial of L_A is $\det(A - tI) = -(t-4)(t-3)^2$. Hence L_A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively. Clearly condition 1 of the test for diagonalizability is satisfied, and since λ_1 has multiplicity 1, condition 2 is satisfied for λ_1. Thus we need only check condition 2 of the test for λ_2. Since

$$B = A - \lambda_2 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has rank 2, $3 - \text{rank}(B) = 1$. Thus condition 2 of the test fails for λ_2, and consequently L_A (and hence A) is not diagonalizable.

Example 14. Let $\mathsf{T}\colon\ \mathsf{R}^3 \longrightarrow \mathsf{R}^3$ be defined by

$$\mathsf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2b - 3c \\ a + 3b + 3c \\ c \end{pmatrix}.$$

We shall test T for diagonalizability. Letting γ denote the standard basis for R^3, we have

$$[\mathsf{T}]_\gamma = \begin{pmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of T is $-(t-1)^2(t-2)$. Thus T has two eigenvalues: $\lambda_1 = 1$ with multiplicity 2 and $\lambda_2 = 2$ with multiplicity 1. Note that condition 1 of the test for diagonalizability is satisfied. We shall now consider condition 2.

For $\lambda_1 = 1$,

$$3 - \text{rank}(\mathsf{T} - \lambda_1 \mathsf{I}) = 3 - \text{rank}\begin{pmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} = 3 - 1 = 2.$$

Thus the dimension of E_{λ_1} is the same as the multiplicity of λ_1. Since λ_2 has multiplicity 1, the dimension of E_{λ_2} is equal to the multiplicity of λ_2. Hence T is diagonalizable.

We shall now find a basis β for R^3 such that $[\mathsf{T}]_\beta$ is a diagonal matrix. Since

$$\mathsf{E}_{\lambda_1} = \mathsf{N}(\mathsf{T} - \lambda_1 \mathsf{I}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3\colon \begin{pmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \right\},$$

E_{λ_1} is the solution set of

$$\begin{cases} -x_1 - 2x_2 - 3x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0, \end{cases}$$

which has

$$\beta_1 = \left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$$

as a basis. Also

$$\mathsf{E}_{\lambda_2} = \mathsf{N}(\mathsf{T} - \lambda_2 \mathsf{I}) = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} -2 & -2 & -3 \\ 1 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0\right\}.$$

Thus E_{λ_2} is the solution set of

$$\begin{cases} -2x_1 - 2x_2 - 3x_3 = 0 \\ \quad x_1 + \ x_2 + 3x_3 = 0 \\ \qquad\qquad\ \ - \ x_3 = 0, \end{cases}$$

which has

$$\beta_2 = \left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

as a basis. Let $\beta = \beta_1 \cup \beta_2$; then β is a basis for V and

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Our next example is an application of diagonalization that will be of interest in Section 5.3.

Example 15. Let

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}.$$

We shall show that A is diagonalizable and find a 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. This information will then be used to compute A^n for any positive integer n.

Recall that A is diagonalizable if and only if L_A is diagonalizable. Now the characteristic polynomial of L_A is $(t-1)(t-2)$. Thus L_A has two distinct eigenvalues, and so L_A (and hence A) is diagonalizable. To find a basis β for R^2 such that $[\mathsf{L}_A]_\beta$ is a diagonal matrix, note that L_A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. It is easily seen that

$$\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\}$$

is a basis for E_{λ_1} and that

$$\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

is a basis for E_{λ_2}. Thus for the basis

$$\beta = \left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

we have

$$[\mathsf{L}_A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Moreover, if

$$Q = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix},$$

then

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

by Theorem 5.1.

Finally, since

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A = Q\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}Q^{-1}.$$

Thus

$$\begin{aligned} A^n &= \left[Q\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}Q^{-1}\right]^n \\ &= Q\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}Q^{-1}Q\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}Q^{-1}\cdots Q\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}Q^{-1} \\ &= Q\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^n Q^{-1} \\ &= Q\begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix}Q^{-1} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2-2^n & 2-2^{n+1} \\ -1+2^n & -1+2^{n+1} \end{pmatrix}. \end{aligned}$$

We shall conclude this section with an application that uses diagonalization to solve a system of differential equations.

Example 16. Consider the system of differential equations

$$\begin{cases} x_1' = 3x_1 + x_2 + x_3 \\ x_2' = 2x_1 + 4x_2 + 2x_3 \\ x_3' = -x_1 - x_2 + x_3, \end{cases}$$

where, for each i, $x_i = x_i(t)$ is a differentiable real-valued function of the real variable t. Clearly this system has a solution, namely the solution in which each $x_i(t)$ is the zero function. We shall determine all the solutions of this system.

Let $X\colon \ R \longrightarrow \mathsf{R}^3$ be the function defined by

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

The derivative of X is defined as the function X', where

$$X'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}.$$

Letting

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

be the coefficient matrix of the given system, we can rewrite the given system in the matrix form $X' = AX$, where AX is the matrix product of A and X.

The reader should verify that A is diagonalizable and that if

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & -1 \end{pmatrix},$$

then

$$Q^{-1}AQ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Set

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

and substitute $A = QDQ^{-1}$ into $X' = AX$ to find $X' = QDQ^{-1}X$ or, equivalently, $Q^{-1}X' = DQ^{-1}X$. Define $Y\colon\ R \longrightarrow \mathsf{R}^3$ by $Y(t) = Q^{-1}X(t)$. It can be shown that Y is a differentiable function and, in fact, $Y' = Q^{-1}X'$. Hence the original system can be written as $Y' = DY$.

Since D is a diagonal matrix, the system $Y' = DY$ is easy to solve. For if

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix},$$

then $Y' = DY$ can be written

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} 2y_1(t) \\ 2y_2(t) \\ 4y_3(t) \end{pmatrix}.$$

The three equations

$$\begin{aligned} y_1'(t) &= 2y_1(t) \\ y_2'(t) &= 2y_2(t) \\ y_3'(t) &= 4y_3(t) \end{aligned}$$

are independent of each other and thus can be solved individually. It is easily seen (as in Example 2 in Section 5.1) that the general solution of these equations is $y_1(t) = c_1e^{2t}$, $y_2(t) = c_2e^{2t}$, and $y_3(t) = c_3e^{4t}$, where c_1, c_2, and c_3 are arbitrary scalars. Finally

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = X(t) = QY(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} c_1e^{2t} \qquad\qquad + \ c_3e^{4t} \\ c_2e^{2t} + 2c_3e^{4t} \\ -c_1e^{2t} - c_2e^{2t} - \ c_3e^{4t} \end{pmatrix}$$

yields the general solution of the original system. Note that this solution can be written as

$$X(t) = e^{2t}\left[c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right] + e^{4t}\left[c_3\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right].$$

The expressions in brackets are simply arbitrary elements of E_{λ_1} and E_{λ_2}, respectively, where $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus the general solution of the original system is $X(t) = e^{2t}z_1 + e^{4t}z_2$, where $z_1 \in \mathsf{E}_{\lambda_1}$ and $z_2 \in \mathsf{E}_{\lambda_2}$.

EXERCISES

1. Label the following statements as being true or false.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) Eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(c) If a vector space is the direct sum of subspaces $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$, then $\mathsf{W}_i \cap \mathsf{W}_j = \{0\}$ for $i \neq j$.

(d) If

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \quad \text{and} \quad \mathsf{W}_i \cap \mathsf{W}_j = \{0\} \qquad \text{for } i \neq j,$$

then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

(e) If λ is an eigenvalue of a linear operator T, then each element of E_λ is an eigenvector of T.

(f) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $\mathsf{E}_{\lambda_1} \cap \mathsf{E}_{\lambda_2} = \{0\}$.

(g) Let $A \in \mathsf{M}_{n\times n}(F)$ and $\beta = \{x_1, \ldots, x_n\}$ be a basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose ith column is x_i $(i = 1, 2, \ldots, n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(h) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(i) Every diagonalizable linear operator has at least one eigenvalue.

2. For each of the following matrices A in $\mathsf{M}_{n\times n}(R)$, test A for diagonalizability and if A is diagonalizable, find a matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T, test T for diagonalizability and if T is diagonalizable, find a basis β such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) T: $\mathsf{P}_3(R) \longrightarrow \mathsf{P}_3(R)$ defined by $\mathsf{T}(f) = f' + f''$, where f' and f'' are the first and second derivatives of f, respectively
(b) T: $\mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$
(c) T: $\mathsf{R}^3 \longrightarrow \mathsf{R}^3$ defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$$

4. Prove the matrix version of the corollary to Theorem 5.10: If $A \in \mathsf{M}_{n \times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

5. Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} \mathsf{W}_i = \mathsf{V}.$$

Prove that V is the direct sum of $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ if and only if

$$\dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{W}_i).$$

6. Let V be a finite-dimensional vector space with basis $\beta = \{x_1, x_2, \ldots, x_n\}$, and let $\beta_1, \beta_2, \ldots, \beta_k$ be a partition of β (that is, $\beta_1, \beta_2, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \varnothing$ if $i \neq j$). Prove that $\mathsf{V} = \operatorname{span}(\beta_1) \oplus \operatorname{span}(\beta_2) \oplus \cdots \oplus \operatorname{span}(\beta_k)$.

7. State and prove the matrix version of Theorem 5.11.

8. (a) Justify the test for diagonalizability and the algorithm for diagonalization stated in this section.
(b) Formulate part (a) for matrices.

9. If

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

find A^n for any positive integer n.

10. Let $A \in \mathsf{M}_{n \times n}(F)$ have two distinct eigenvalues λ_1 and λ_2. If $\dim(\mathsf{E}_{\lambda_1}) = n - 1$, prove that A is diagonalizable.

11. Let T be a linear operator on a finite-dimensional vector space V for which the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$\operatorname{span}(\{x \in \mathsf{V}\colon\ x \text{ is an eigenvector of T}\}) = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

12. Let T be a linear operator on a finite-dimensional vector space V for which the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ occur with multiplicities $m_1, m_2, \ldots, m_k$, respectively. If β is a basis for V such that $[\mathsf{T}]_\beta$ is a triangular

matrix, prove that the diagonal entries of $[\mathsf{T}]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_j occurs m_j times $(j = 1, 2, \ldots, k)$.

13. Suppose that A is an $n \times n$ matrix whose characteristic polynomial factors into a product of factors of degree 1 and that the distinct eigenvalues of A are $\lambda_1, \ldots, \lambda_k$. For each j, let m_j denote the multiplicity of λ_j. Prove that

$$\operatorname{tr}(A) = \sum_{j=1}^{k} m_j \lambda_j.$$

14. Let T be an invertible linear operator on a finite-dimensional vector space. Prove that T is diagonalizable if and only if T^{-1} is diagonalizable.

15. Let $A \in \mathsf{M}_{n\times n}(F)$. Show that A is diagonalizable if and only if A^t is diagonalizable.

16. Find the general solution of the system of differential equations

$$\begin{cases} x_1' = 8x_1 + 10x_2 \\ x_2' = -5x_1 - 7x_2. \end{cases}$$

17. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be the coefficient matrix of the system of differential equations

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{cases}$$

Suppose that A is diagonalizable and that the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that a differentiable function $X\colon\ R \longrightarrow \mathsf{R}^n$ is a solution to the system if and only if X is of the form

$$X(t) = e^{\lambda_1 t}z_1 + e^{\lambda_2 t}z_2 + \cdots + e^{\lambda_k t}z_k,$$

where $z_i \in \mathsf{E}_{\lambda_i}$ for $i = 1, 2, \ldots, k$. Conclude that the set of solutions to the system is an n-dimensional real vector space.

Exercises 18–20 will be concerned with simultaneous diagonalization.

Definitions. *Two linear operators* T *and* U *on the same finite-dimensional vector space* V *are called* simultaneously diagonalizable *if there exists a basis* β *for* V *such that both* $[\mathsf{T}]_\beta$ *and* $[\mathsf{U}]_\beta$ *are diagonal matrices. Likewise* $A, B \in$

$M_{n\times n}(F)$ *are called* simultaneously diagonalizable *if there exists an invertible matrix* $Q \in M_{n\times n}(F)$ *such that both* $Q^{-1}AQ$ *and* $Q^{-1}BQ$ *are diagonal matrices.*

18. (a) If T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, prove that $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable matrices for any basis β.
(b) Show that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable operators.

19. (a) Show that if T and U are simultaneously diagonalizable operators, then T and U commute: i.e., TU = UT.
(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

The converse of (a) and (b) will be established in Exercise 11 of Section 5.4.

20. Let T be a diagonalizable linear operator on a finite-dimensional vector space, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

21. Let $W_1, W_2, K_1, K_2, \ldots, K_p, M_1, M_2, \ldots, M_q$ be subspaces of a vector space V such that $W_1 = K_1 \oplus K_2 \oplus \cdots \oplus K_p$ and $W_2 = M_1 \oplus M_2 \oplus \cdots \oplus M_q$. Prove that if $W_1 \cap W_2 = \{0\}$, then $W_1 + W_2 = W_1 \oplus W_2 = K_1 \oplus K_2 \oplus \cdots \oplus K_p \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_q$.

5.3* MATRIX LIMITS AND MARKOV CHAINS

If A is a square matrix having complex entries, then for any positive integer m, A^m is a square matrix of the same size that also has complex entries. In many of the life and natural sciences there are important practical applications that require determining the "limit" (if one exists) of the sequence of matrices $A, A^2, A^3, \ldots$. In this section we shall consider such limits and examine one important situation in which this type of limit arises.

Definitions. Let $L, A_1, A_2, A_3, \ldots$ *be* $n \times p$ *matrices having complex entries. The sequence* $A_1, A_2, A_3, \ldots$ *is said to* converge *to the matrix* L, *called the* limit *of the sequence, if for each* i $(1 \le i \le n)$ *and* j $(1 \le j \le p)$ *the sequence of complex numbers* $(A_1)_{ij}, (A_2)_{ij}, (A_3)_{ij}, \ldots$ *converges to* L_{ij}. *(The limit of a sequence of complex numbers* $\{z_m\colon\ m = 1, 2, \ldots\}$, *where* $z_m = r_m + is_m$ *with* r_m *and* s_m *being real numbers, is defined in terms of the limits of the sequence of real and imaginary parts as*

$$\lim_{m\to\infty} z_m = (\lim_{m\to\infty} r_m) + i(\lim_{m\to\infty} s_m).)$$

To denote the fact that the sequence $A_1, A_2, A_3, \ldots$ *converges to* L, *we shall write* $\lim_{m\to\infty} A_m = L$.

Example 17. If

$$A_m = \begin{pmatrix} 1 - \dfrac{1}{m} & \left(-\dfrac{3}{4}\right)^m & \dfrac{3m^2}{m^2+1} + i\left(\dfrac{2m+1}{m-1}\right) \\ \left(\dfrac{i}{2}\right)^m & 2 & \left(1 + \dfrac{1}{m}\right)^m \end{pmatrix},$$

then

$$\lim_{m\to\infty} A_m = \begin{pmatrix} 1 & 0 & 3+2i \\ 0 & 2 & e \end{pmatrix},$$

where e is the base of the natural logarithm.

A simple but important property of matrix limits is contained in the next theorem. Note the analogy with the familiar property of limits of sequences of real numbers which asserts that if $\lim_{m\to\infty} a_m$ exists, then

$$\lim_{m\to\infty} ca_m = c(\lim_{m\to\infty} a_m).$$

Theorem 5.15. *Let* $A_1, A_2, A_3, \ldots$ *be a sequence of* $n \times p$ *matrices having complex entries such that*

$$\lim_{m\to\infty} A_m = L \in M_{n\times p}(C).$$

Then for any $P \in M_{r\times n}(C)$ *and* $Q \in M_{p\times s}(C)$,

$$\lim_{m\to\infty} PA_m = PL \quad \text{and} \quad \lim_{m\to\infty} A_mQ = LQ.$$

PROOF. For any i $(1 \le i \le r)$ and j $(1 \le j \le p)$,

$$\lim_{m\to\infty} [(PA_m)_{ij}] = \lim_{m\to\infty} \left[\sum_{k=1}^{n} P_{ik}(A_m)_{kj}\right]$$

$$= \sum_{k=1}^{n} P_{ik}\{\lim_{m\to\infty} [(A_m)_{kj}]\} = \sum_{k=1}^{n} P_{ik}L_{kj} = (PL)_{ij}.$$

Hence $\lim_{m\to\infty} PA_m = PL$. The proof that $\lim_{m\to\infty} A_mQ = LQ$ is similar. ∎

Corollary. *Let* $A \in M_{n\times n}(C)$, *and let* $\lim_{m\to\infty} A^m = L$. *Then for any invertible matrix* $Q \in M_{n\times n}(C)$,

$$\lim_{m\to\infty} (QAQ^{-1})^m = QLQ^{-1}.$$

PROOF. Since

$$(QAQ^{-1})^m = (QAQ^{-1})(QAQ^{-1}) \cdots (QAQ^{-1}) = QA^mQ^{-1},$$

we have

$$\lim_{m\to\infty} [(QAQ^{-1})^m] = \lim_{m\to\infty} (QA^mQ^{-1}) = Q(\lim_{m\to\infty} A^m)Q^{-1} = QLQ^{-1}$$

by Theorem 5.15. ∎

The following important result gives necessary and sufficient conditions for the existence of the type of limit under consideration.

Theorem 5.16. *Let* A *be a square matrix having complex entries. Then* $\lim_{m\to\infty} A^m$ *exists if and only if the following conditions hold:*

(a) *If λ is an eigenvalue of* A, *then* $|\lambda| \leq 1$.

(b) *If λ is an eigenvalue of* A *such that* $|\lambda| = 1$, *then λ is the real number* 1.

(c) *If* 1 *is an eigenvalue of* A, *then the dimension of the eigenspace corresponding to* 1 *equals the multiplicity of* 1 *as an eigenvalue of* A.

Unfortunately, it will not be possible to prove the sufficiency of these conditions or the necessity of condition (c) until we study the Jordan canonical form. For this reason the proof of the theorem will be deferred until Section 6.2 (Exercise 18). The necessity of the first two conditions, however, follows easily from the fact that $\lim_{m\to\infty} \lambda^m$ exists if and only if $\lambda = 1$ or $|\lambda| < 1$. (This fact, which the reader undoubtedly knows for real numbers λ, can be shown to hold also for complex numbers.) Suppose then that λ is an eigenvalue of A for which condition (a) or (b) fails, i.e., such that $|\lambda| > 1$ or such that $|\lambda| = 1$ but $\lambda \neq 1$. Let x be an eigenvector of A corresponding to λ. Regarding x as an $n \times 1$ matrix, we see that

$$\lim_{m\to\infty} (A^m x) = (\lim_{m\to\infty} A^m)x = Lx$$

by Theorem 5.15, where $L = \lim_{m\to\infty} A^m$. But $\lim_{m\to\infty} (A^m x) = \lim_{m\to\infty} (\lambda^m x)$ diverges since $\lim_{m\to\infty} \lambda^m$ does not exist. Hence if $\lim_{m\to\infty} A^m$ exists, then conditions (a) and (b) of Theorem 5.16 must hold. Although we are unable to prove the necessity of the third condition at this time, let us consider an example for which this condition fails. Observe that for the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the eigenvalue $\lambda = 1$ has multiplicity 2, whereas $\dim(\mathsf{E}_\lambda) = 1$. But

$$B^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

by an easy induction, and hence $\lim_{m\to\infty} B^m$ does not exist. (We shall see later that if A is a matrix for which condition (c) fails, then the Jordan canonical form of A can be chosen so that its upper left 2×2 submatrix is precisely this matrix B.)

In most of the applications involving this type of limit, however, the matrix A is diagonalizable. When condition (c) of Theorem 5.16 is replaced by the stronger condition that A is diagonalizable (see Theorem 5.14), then the existence of the limit is easily shown.

Theorem 5.17. *Let* $A \in M_{n\times n}(C)$ *be such that the following conditions hold:*

(a) *If* λ *is an eigenvalue of* A, *then* $|\lambda| \leq 1$.

(b) *If* λ *is an eigenvalue of* A *such that* $|\lambda| = 1$, *then* λ *is the real number 1.*

(c) A *is diagonalizable.*

Then $\lim_{m\to\infty} A^m$ *exists.*

PROOF. Since A is diagonalizable, there exists an invertible matrix Q such that $Q^{-1}AQ = D$, a diagonal matrix. Let

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Because $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, conditions (a) and (b) show that either $\lambda_i = 1$ or $|\lambda_i| < 1$ for $1 \leq i \leq n$. Thus

$$\lim_{m\to\infty} \lambda_i^m = \begin{cases} 1 & \text{if } \lambda_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But since

$$D^m = \begin{pmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \lambda_n^m \end{pmatrix},$$

the sequence $D, D^2, D^3, \ldots$ converges to a limit L. Hence

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = QLQ^{-1}$$

by the corollary to Theorem 5.15. ∎

The technique for computing $\lim_{m\to\infty} A^m$ that was used in the proof of the preceding theorem is quite useful. We shall employ this method to compute $\lim_{m\to\infty} A^m$ for the matrix

$$A = \begin{pmatrix} \frac{7}{4} & -\frac{9}{4} & -\frac{15}{4} \\ \frac{3}{4} & \frac{7}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{9}{4} & -\frac{11}{4} \end{pmatrix}.$$

If

$$Q = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix},$$

then

$$D = Q^{-1}(AQ) = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{4} \\ -3 & 1 & \frac{1}{4} \\ 2 & -\frac{3}{2} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Hence

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty}(QDQ^{-1})^m = \lim_{m\to\infty}(QD^mQ^{-1}) = Q(\lim_{m\to\infty} D^m)Q^{-1}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix}\left[\lim_{m\to\infty}\begin{pmatrix} 1 & 0 & 0 \\ 0 & (-\frac{1}{2})^m & 0 \\ 0 & 0 & (\frac{1}{4})^m \end{pmatrix}\right]\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ -2 & 0 & 2 \end{pmatrix}.$$

Let us now consider a simple example in which the limit of powers of a matrix occurs. Suppose that the population of a certain metropolitan area remains constant but that there is a continual movement of people between the city and the suburbs. Specifically, let the entries of the matrix A below represent the probabilities that someone living in the city or in the suburbs on January 1 will be living in each region on January 1 of the next year.

$$\begin{array}{lcc} & \text{Presently Living in the City} & \text{Presently Living in the Suburbs} \\ \text{Living Next Year in the City} & 0.90 & 0.02 \\ \text{Living Next Year in the Suburbs} & 0.10 & 0.98 \end{array} = A$$

Thus, for instance, the probability that someone living in the city (on January 1) will be living in the suburbs next year (on January 1) is 0.10. Notice that since the entries of each column of A represent probabilities of residing in each of the two locations, the entries of A are non-negative. Moreover, the assumption of a constant population in the metropolitan area requires that the sum of the entries of each column of A be 1. Any matrix having these two properties (that the entries are non-negative and that the sum of the entries in each column is 1) is called a *transition matrix* (or a *stochastic matrix*). For an arbitrary $n \times n$ transition matrix M, the rows and columns correspond to n *states*, and the entry M_{ij} represents the probability of moving from state j into state i in one *stage*. In our example, there are two states (residing in the city and residing in the suburbs), and A_{21} represents the probability of moving from the city to the suburbs in one stage (year).

Let us now determine the probability that a city resident will be living in the suburbs after 2 years. Observe first that there are two different ways in which such a move can be made—either by remaining in the city for 1 year and then moving to the suburbs or by moving to the suburbs during the first year and remaining there the second (see Fig. 5.3). The probability

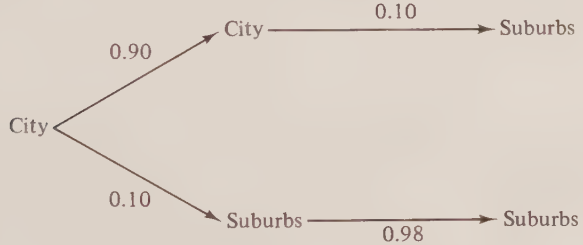


figure 5.3

that a city dweller will stay in the city during the next year is 0.90, and the probability that a city dweller will move to the suburbs during the following year is 0.10. Hence the probability that a city resident will stay in the city for 1 year and will move to the suburbs during the next is 0.90(0.10). Likewise the probability of a city dweller moving to the suburbs during the first year and remaining there the next is 0.10(0.98). Thus the probability that a city resident will be living in the suburbs after 2 years is $0.90(0.10) + 0.10(0.98) = 0.188$. Observe that this number is obtained by the same calculation as that which produces $(A^2)_{21}$—hence $(A^2)_{21}$ represents the probability that a city dweller will be residing in the suburbs after 2 years. In general, for any transition matrix M, $(M^m)_{ij}$ represents the probability of moving from state j to state i in m stages.

Suppose additionally that 70% of the 1970 population of the metropolitan area lived in the city and 30% lived in the suburbs. Let us record this data as a column vector:

$$\begin{array}{l} \text{Proportion of City Dwellers} \\ \text{Proportion of Suburb Residents} \end{array} \quad \begin{pmatrix} 0.70 \\ 0.30 \end{pmatrix} = P.$$

Notice that the rows of P correspond to the states of residing in the city and residing in the suburbs, respectively—the same order as the states are listed in the transition matrix A. Observe also that P is a column vector containing non-negative entries whose sum is 1; such a vector is called a *probability vector.* In this terminology each column of a transition matrix is a probability vector.

Let us now consider the significance of the vector AP. The first coordinate of this vector is formed by the calculation $0.90(0.70) + 0.02(0.30)$. The term $0.90(0.70)$ represents the proportion of the 1970 metropolitan population that remained in the city during the next year, and the term $0.02(0.30)$ represents the proportion of the 1970 metropolitan population that moved into the city during the next year. Hence the first coordinate of AP represents the proportion of the metropolitan population that was living in the city 1 year after 1970. Likewise the second coordinate of

$$AP = \begin{pmatrix} 0.636 \\ 0.364 \end{pmatrix}$$

represents the proportion of the metropolitan population that was living in the suburbs in 1971. This argument can be easily extended to show that the coordinates of

$$A^2P = A(AP) = \begin{pmatrix} 0.57968 \\ 0.42032 \end{pmatrix}$$

represent the proportions of the metropolitan population that were living in each location in 1972. In general, the coordinates of A^mP represent the proportion of the metropolitan population that will be living in the city and suburbs, respectively, after m stages (m years after 1970).

Will the city eventually be depleted if this trend continues? In view of the preceding discussion it is natural to define the eventual proportion of city dwellers and suburbanites to be the first and second coordinates, respectively, of $\lim_{m\to\infty} A^mP$. Let us now compute this limit. With the previous notation

$$D = Q^{-1}AQ = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & -\frac{1}{6} \end{pmatrix}\begin{pmatrix} 0.90 & 0.02 \\ 0.10 & 0.98 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix};$$

thus

$$L = \lim_{m\to\infty} A^m = \lim_{m\to\infty} (QD^mQ^{-1}) = Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{pmatrix}.$$

Hence

$$\lim_{m\to\infty} A^mP = LP = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix};$$

so eventually $\frac{1}{6}$ of the population will live in the city and $\frac{5}{6}$ will live in the suburbs. It is easy to show that in this example

$$LP = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$$

for any probability vector P. Hence in this example the eventual proportions of city dwellers and suburbanites are independent of the initial proportions (as given by the vector P)!

In analyzing the city-suburb problem we gave probabilistic interpretations of A^2 and AP, showing that A^2 is a transition matrix and AP is a probability vector. Analogous arguments can be used to show that the product of two transition matrices is a transition matrix and that the product of a transition matrix and a probability vector is a probability vector. An alternate proof of these results can be based on the following theorem, which characterizes transition matrices and probability vectors.

Theorem 5.18. *Let* M *be an* $n \times n$ *matrix having (real) non-negative entries,* x *be a column vector in* R^n *having non-negative coordinates, and* $u \in R^n$ *be the column vector in which each coordinate equals* 1. *Then*

(a) M *is a transition matrix if and only if* $M^t u = u$.

(b) x *is a probability vector if and only if* $u^t x = (1)$.

PROOF. Exercise.

Corollary.

(a) *The product of two* $n \times n$ *transition matrices is an* $n \times n$ *transition matrix. In particular, any power of a transition matrix is a transition matrix.*

(b) *The product of a transition matrix and a probability vector is a probability vector.*

PROOF. Exercise.

A *stochastic process* is concerned with predicting the state of an object that is constrained to be in exactly one of a number of possible states at any given time but that changes states in some random manner. Normally, the probability that the object is in some particular state at a given time will depend on such factors as

1. The state in question.
2. The time in question.
3. Some or all of the previous states in which the object has been.
4. The states that other objects are in or have been in.

For instance, the object could be an American voter and the state of the object could be his preference of political party, or the object could be a molecule of H_2O and the states could be the physical states in which H_2O can exist (the solid, liquid, and gaseous states). In these examples all four of the factors mentioned above will influence the probability that the objects are in a particular state at a particular time.

If, however, the probability that an object in one state will change to a different state depends only on the two states (and not on the time, earlier states, or other factors), then the stochastic process is called a *Markov process*. If, in addition, the number of possible states is finite, then the Markov process is called a *Markov chain*. The preceding example of the movement of population between the city and suburbs is a two-state Markov chain.

Let us consider another Markov chain. A certain junior college would like to obtain information about the likelihood that various categories of presently enrolled students will graduate. The school classifies a student as a sophomore or a freshman depending on the number of credits that the student has earned. Data from the school indicates that from one fall semester to the next 40% of the sophomores will graduate, 30% will remain sophomores, and 30% will quit permanently. For freshmen the data shows that 10% will graduate by next fall, 50% will become sophomores, 20% will remain freshmen, and 20% will quit permanently. During the present year 50% of the students at the school are sophomores and 50% are freshmen. Assuming that the trend indicated by the data continues indefinitely, the school would like to know

1. The percentage of the present students who will graduate, the percentage who will be sophomores, the percentage who will be freshmen, and the percentage who will quit school permanently by next fall.
2. The same percentages as in item 1 for the fall semester 2 years hence.
3. The percentage of its present students who will eventually graduate.

The preceding paragraph describes a four-state Markov chain with states

1. Having graduated.
2. Being a sophomore.
3. Being a freshman.
4. Having quit permanently.

The data cited above provides us with the transition matrix

$$A = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix}$$

of the Markov chain. (Notice that students who have graduated or have quit permanently are assumed to remain indefinitely in those respective states. Thus a freshman who quits the school and returns during a later semester is not regarded as having changed states—the student is assumed to have remained in the state of being a freshman during the time he was not enrolled.) Moreover, we are told that the present distribution of students is half in each of states 2 and 3 and none in states 1 and 4. The vector

$$P = \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

that describes the initial probability of being in each state is called the *initial probability vector* for the Markov chain.

In order to answer question 1 we must determine the probability that a present student will be in each state by next fall. As we have seen, these probabilities are the coordinates of the vector

$$AP = \begin{pmatrix} 0.25 \\ 0.40 \\ 0.10 \\ 0.25 \end{pmatrix}.$$

Hence by next fall 25% of the present students will graduate, 40% will be sophomores, 10% will be freshmen, and 25% will quit the school. Likewise

$$A^2P = A(AP) = \begin{pmatrix} 0.42 \\ 0.17 \\ 0.02 \\ 0.39 \end{pmatrix}$$

provides the information needed to answer question 2: within 2 years 42% of the present students will graduate, 17% will be sophomores, 2% will be freshmen, and 39% will quit the school.

Finally, the answer to question 3 is provided by the vector LP, where $L = \lim_{m\to\infty} A^m$. The reader should verify that if

$$Q = \begin{pmatrix} 1 & -4 & 19 & 0 \\ 0 & 7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & -3 & 13 & 1 \end{pmatrix},$$

then

$$D = Q^{-1}AQ = \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & \frac{1}{7} & \frac{5}{7} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 & 19 & 0 \\ 0 & 7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & -3 & 13 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$L = \lim_{m\to\infty} A^m = Q(\lim_{m\to\infty} D^m)Q^{-1}$$

$$= \begin{pmatrix} 1 & -4 & 19 & 0 \\ 0 & 7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & -3 & 13 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & \frac{1}{7} & \frac{5}{7} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix}.$$

So

$$LP = \begin{pmatrix} \frac{59}{112} \\ 0 \\ 0 \\ \frac{53}{112} \end{pmatrix},$$

and hence the probability that one of the present students will graduate is $\frac{59}{112}$.

In the two previous examples we have seen that $\lim_{m\to\infty} A^mP$, where A is the transition matrix and P is the initial probability vector of the Markov chain, gives the eventual proportions in each state. In general, however, the limit of powers of a transition matrix need not exist. For example, if

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\lim_{m\to\infty} M^m$ clearly does not exist. (Odd powers of M equal M and even powers of M equal I.) The reason that the limit fails to exist is that condition (b) of Theorem 5.16 does not hold for M (-1 is an eigenvalue). In fact, it can be shown (see Exercise 20 of Section 6.2) that the only transition matrices A such that $\lim_{m\to\infty} A^m$ does not exist are precisely those matrices for which condition (b) of Theorem 5.16 fails to hold.

But even if the limit of powers of the transition matrix exists, the computation of the limit may be quite difficult. (The reader is encouraged to work Exercise 6 to appreciate the truth of the last sentence.) Fortunately, there is a large and important class of transition matrices for which this limit exists and is easily computed—this is the class of "regular" transition matrices.

Definition. *If some power of a transition matrix contains only positive entries, then the matrix is called a* regular *transition matrix.*

Example 18. The transition matrix

$$\begin{pmatrix} 0.90 & 0.02 \\ 0.10 & 0.98 \end{pmatrix}$$

of the Markov chain describing the movement of population between the city and suburbs is clearly regular since each entry is positive. On the other hand, the transition matrix

$$A = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix}$$

of the Markov chain describing the junior college enrollments is not regular. (It is easy to show that the first column of A^m is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for any m. Hence $(A^m)_{41}$, for instance, is never positive.)

Observe that a regular transition matrix may contain zero entries; for example,

$$M = \begin{pmatrix} 0.9 & 0.5 & 0 \\ 0 & 0.5 & 0.4 \\ 0.1 & 0 & 0.6 \end{pmatrix}$$

is regular since every entry of M^2 is positive.

In the remainder of this section we shall be concerned primarily with proving that if A is a regular transition matrix, then $L = \lim_{m\to\infty} A^m$ exists and the columns of L are identical. (Recall the appearance of L in the city-suburb problem.) From this fact it will be easy to compute the limit. In the course of proving this result we shall obtain some interesting theorems about the magnitude of the eigenvalues of any square matrix. These bounds will be given in terms of the sum of the absolute values of the entries of the rows and columns of the matrix. The necessary terminology is introduced in the definition below.

Definitions. *Let* $A \in M_{n\times n}(C)$. *Define* $\rho_i(A)$ *to be the sum of the absolute values of the entries of row* i *of* A and $\nu_j(A)$ *to be the sum of the absolute values of the entries of column* j *of* A. *Thus*

$$\rho_i(A) = \sum_{j=1}^{n} |A_{ij}| \qquad \textit{for } i = 1, 2, \ldots, n$$

and

$$\nu_j(A) = \sum_{i=1}^{n} |A_{ij}| \qquad \textit{for } j = 1, 2, \ldots, n.$$

The row sum *of* A, *denoted* $\rho(A)$, and *the* column sum *of* A, *denoted* $\nu(A)$, *are defined as*

$$\rho(A) = \max\{\rho_i(A)\colon\ 1 \le i \le n\} \ \textit{and}\ \ \nu(A) = \max\{\nu_j(A)\colon\ 1 \le j \le n\}.$$

Example 19. For the matrix

$$A = \begin{pmatrix} 1 & -1 & 5 \\ -4 & 0 & 6 \\ 3 & 2 & -1 \end{pmatrix},$$

$\rho_1(A) = 7$, $\rho_2(A) = 10$, $\rho_3(A) = 6$, $\nu_1(A) = 8$, $\nu_2(A) = 3$, and $\nu_3(A) = 12$. Hence $\rho(A) = 10$ and $\nu(A) = 12$.

Our next results show that the smaller of $\rho(A)$ and $\nu(A)$ is an upper bound for the absolute value of the eigenvalues of A. In the preceding example, for instance, A has no eigenvalue having absolute value greater than 10.

Theorem 5.19. *Let* λ *be an eigenvalue of* $A \in M_{n\times n}(C)$. *Then* $|\lambda| \le \rho(A)$.

PROOF. Let

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

be an eigenvector of A for which λ is the corresponding eigenvalue. Then x satisfies the matrix equation $Ax = \lambda x$, which can be written as the system of linear equations

$$\sum_{j=1}^{n} A_{ij}x_j = \lambda x_i \qquad (i = 1, 2, \ldots, n). \tag{4}$$

Suppose that x_k is the coordinate of x having the largest absolute value, and let $b = |x_k|$. From the kth equation in Eq. (4) we have

$$\begin{aligned} |\lambda| b = |\lambda||x_k| = |\lambda x_k| &= \left|\sum_{j=1}^{n} A_{kj}x_j\right| \leq \sum_{j=1}^{n} |A_{kj}x_j| \\ &= \sum_{j=1}^{n} |A_{kj}||x_j| \leq \sum_{j=1}^{n} |A_{kj}| b = \rho_k(A)b \leq \rho(A)b. \end{aligned} \tag{5}$$

But $b \neq 0$ since x, an eigenvector, is non-zero. Thus dividing both sides of Eq. (5) by b gives $|\lambda| \leq \rho(A)$. ∎

Corollary 1. *Let λ be an eigenvalue of* $A \in M_{n\times n}(C)$. *Then* $|\lambda| \leq \min\{\rho(A), \nu(A)\}$.

PROOF. Since $|\lambda| \leq \rho(A)$ by Theorem 5.19, it suffices to prove that $|\lambda| \leq \nu(A)$.

Exercise 14 of Section 5.1 shows that λ is an eigenvalue of A^t. Hence $|\lambda| \leq \rho(A^t)$ by Theorem 5.19. But the rows of A^t are the columns of A. Thus $\rho(A^t) = \nu(A)$. So $|\lambda| \leq \nu(A)$. ∎

Since the column sum of a transition matrix is 1, the following conclusion is immediate from Corollary 1.

Corollary 2. *If λ is an eigenvalue of a transition matrix, then $|\lambda| \leq 1$.*

The next result shows that the upper bound in Corollary 2 above is attained.

Theorem 5.20. *Every transition matrix has* 1 *as an eigenvalue.*

PROOF. Let A be an $n \times n$ transition matrix, and let $u \in R^n$ be the column vector in which each coordinate is 1. Then $A^t u = u$ by Theorem 5.18, and hence u is an eigenvector of A^t corresponding to the eigenvalue 1. But since A and A^t have the same eigenvalues, it follows that 1 is also an eigenvalue of A. ∎

Suppose now that A is a transition matrix for which some eigenvector corresponding to the eigenvalue 1 has only non-negative coordinates. Then some multiple of this vector will be a probability vector P as well as an eigenvector of A corresponding to the eigenvalue 1. It is interesting to observe that if P is the initial probability vector of a Markov chain having

A as its transition matrix, then the Markov chain is completely static. For in this situation $A^m P = P$ for every positive integer m, and hence the probability of being in each state never changes. Consider, for instance, the city-suburb problem with

$$P = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

Theorem 5.21. *Let* $A \in M_{n \times n}(C)$ *be a matrix in which each entry is positive, and let* λ *be an eigenvalue of* A *such that* $|\lambda| = \rho(A)$. *Then* $\lambda = \rho(A)$, *and* $\{u\}$ *is a basis for* E_λ, *where*

$$u = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

PROOF. Since $|\lambda| = \rho(A)$, the three inequalities of Eq. (5) in the proof of Theorem 5.19 are actually equalities; that is,

(a) $\left|\sum_{j=1}^{n} A_{kj}x_j\right| = \sum_{j=1}^{n} |A_{kj}x_j|$,

(b) $\sum_{j=1}^{n} |A_{kj}||x_j| = \sum_{j=1}^{n} |A_{kj}|b$,

(c) $\rho_k(A) = \rho(A)$,

where x, b, and k are as defined in the proof of Theorem 5.19.

We shall see in Exercise 15(b) of Section 7.1 that (a) holds if and only if all the terms $A_{kj}x_j$ $(j = 1, 2, \ldots, n)$ are non-negative multiples of some non-zero complex number z. Without loss of generality we shall assume that $|z| = 1$. Thus there exist non-negative real numbers $c_1, \ldots, c_n$ such that

$$A_{kj}x_j = c_j z. \tag{6}$$

Clearly (b) holds if and only if for each j we have $A_{kj} = 0$ or $|x_j| = b$. Since each entry of A is assumed to be positive, we conclude that (b) holds if and only if

$$|x_j| = b \qquad \text{for } j = 1, 2, \ldots, n. \tag{7}$$

Thus Eq. (5), and hence (c) above, is valid for $k = 1, 2, \ldots, n$.

From Eq. (6) we see that

$$x_j = \frac{c_j}{A_{kj}} z \qquad (j = 1, 2, \ldots, n),$$

and hence

$$b = |x_j| = \left|\frac{c_j}{A_{kj}} z\right| = \frac{c_j}{A_{kj}} \qquad (j = 1, 2, \ldots, n)$$

by Eq. (7). Therefore $x_j = bz$ for $j = 1, 2, \ldots, n$. So

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} bz \\ \cdot \\ \cdot \\ \cdot \\ bz \end{pmatrix} = bz \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix},$$

and hence $\{u\}$ is a basis for E_λ.

Clearly u is an eigenvector of A corresponding to eigenvalue $\rho(A)$ since

$$Au = A\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} A_{1j} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=1}^{n} A_{nj} \end{pmatrix} = \begin{pmatrix} \rho_1(A) \\ \cdot \\ \cdot \\ \cdot \\ \rho_n(A) \end{pmatrix} = \begin{pmatrix} \rho(A) \\ \cdot \\ \cdot \\ \cdot \\ \rho(A) \end{pmatrix} = \rho(A)\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = \rho(A)u$$

by (c) above. But the previous paragraph shows that if λ is any eigenvalue of A such that $|\lambda| = \rho(A)$, then u is an eigenvector to which λ corresponds. Hence $\lambda = \rho(A)$. ▮

Corollary 1. *Let* $A \in \mathsf{M}_{n\times n}(C)$ *be a matrix in which each entry is positive, and let* λ *be an eigenvalue of* A *such that* $|\lambda| = \nu(A)$. *Then* $\lambda = \nu(A)$, *and the dimension of* E_λ *is* 1.

PROOF. Exercise.

Corollary 2. *Let* $A \in \mathsf{M}_{n\times n}(C)$ *be a transition matrix in which each entry is positive, and let* λ *denote any eigenvalue of* A *other than* 1. *Then* $|\lambda| < 1$. *Moreover, the dimension of the eigenspace corresponding to the eigenvalue* 1 *is* 1.

PROOF. Exercise.

Our next result generalizes the preceding corollary to regular transition matrices and thus shows that regular transition matrices satisfy the first two conditions of Theorems 5.16 and 5.17.

Theorem 5.22. *Let* A *be a regular transition matrix.*

(a) *If* λ *is an eigenvalue of* A, *then* $|\lambda| \leq 1$.
(b) *If* λ *is an eigenvalue of* A *such that* $|\lambda| = 1$, *then* λ *is the real number* 1 *and* $\dim(\mathsf{E}_\lambda) = 1$.

In other words, $\lambda = 1$ *is the only eigenvalue of* A *having absolute value* 1, *and* $\dim(\mathsf{E}_\lambda) = 1$. *All other eigenvalues of* A *have absolute value less than* 1.

PROOF. Statement (a) was proved as Corollary 2 of Theorem 5.19.

Since A is regular, there exists a positive integer s such that A^s has only positive entries. Because A is a transition matrix and the entries of A^s are positive, the entries of $A^{s+1} = A^s(A)$ are positive. Let λ be an eigenvalue of A having absolute value 1. Then λ^s and λ^{s+1} are eigenvalues of A^s and A^{s+1}, respectively, having absolute value 1. So by Corollary 2 of Theorem 5.21, $\lambda^s = \lambda^{s+1} = 1$. Thus $\lambda = 1$. Let E_λ and E'_λ denote the eigenspaces of A and A^s, respectively, corresponding to eigenvalue $\lambda = 1$. Then $\mathsf{E}_\lambda \subseteq \mathsf{E}'_\lambda$, but E'_λ has dimension 1 (Corollary 2 of Theorem 5.21). Hence $\mathsf{E}_\lambda = \mathsf{E}'_\lambda$, and $\dim(\mathsf{E}_\lambda) = 1$. ∎

Corollary. *Let* A *be a regular transition matrix that is diagonalizable. Then* $\lim_{m\to\infty} A^m$ *exists.*

The preceding corollary, which follows immediately from Theorems 5.22 and 5.17, is not the best possible result. In fact, it can be shown that if A is a regular transition matrix, then the multiplicity of 1 as an eigenvalue of A is 1. Thus by Theorem 5.12 the third condition of Theorem 5.16 is satisfied. So if A is a regular transition matrix, $\lim_{m\to\infty} A^m$ exists whether A is diagonalizable or not. As with Theorem 5.16, however, the fact that the multiplicity of 1 as an eigenvalue of A is 1 cannot be proved at this time. Nevertheless, we shall state this result here (leaving the proof until Exercise 20 of Section 6.2) and deduce further facts about $\lim_{m\to\infty} A^m$ when A is a regular transition matrix.

Theorem 5.23. *Let* A *be an* $n \times n$ *regular transition matrix. Then*

(a) *The multiplicity of* 1 *as an eigenvalue of* A *is* 1.

(b) $\lim_{m\to\infty} A^m$ *exists.*

(c) $L = \lim_{m\to\infty} A^m$ *is a transition matrix.*

(d) $AL = LA = L$.

(e) *The columns of* L *are identical. In fact, each column of* L *is equal to the unique probability vector* v *that is also an eigenvector corresponding to the eigenvalue* 1 *of* A.

(f) *For any probability vector* x, $\lim_{m\to\infty} (A^m x) = v$.

PROOF.

(a) See Exercise 20 of Section 6.2.

(b) The proof that $\lim_{m\to\infty} A^m$ exists follows from part (a) and Theorems 5.22 and 5.16.

(c) Since A^m is a transition matrix by the corollary to Theorem 5.18, each entry of A^m is non-negative ($m = 1, 2, 3, \ldots$). Hence

$$L_{ij} = \lim_{m\to\infty} (A^m)_{ij} \geq 0 \qquad \text{for } 1 \leq i, j \leq n.$$

Moreover,

$$\sum_{i=1}^{n} L_{ij} = \sum_{i=1}^{n} [\lim_{m\to\infty} (A^m)_{ij}] = \lim_{m\to\infty}\left[\sum_{i=1}^{n} (A^m)_{ij}\right] = \lim_{m\to\infty} (1) = 1 \quad \text{for } 1 \leq j \leq n.$$

Thus L is a transition matrix.

(d) By Theorem 5.15,

$$AL = A(\lim_{m\to\infty} A^m) = \lim_{m\to\infty} (AA^m) = \lim_{m\to\infty} A^{m+1} = L.$$

Similarly $LA = L$.

(e) Since $AL = L$ by part (d), each column of L is an eigenvector of A corresponding to the eigenvalue 1. Moreover, by part (c), each column of L is a probability vector. Thus by part (a) each column of L is equal to the unique probability vector v corresponding to the eigenvalue 1 of A.

(f) Let x be any probability vector, and set $y = Lx$. Then y is a probability vector (corollary to Theorem 5.18), and $Ay = ALx = Lx = y$ by part (d). Hence y is also an eigenvector corresponding to the eigenvalue 1 of A. So $y = v$ by part (e). ∎

Definition. *The vector* v *in part* (e) *of the preceding theorem is called the* fixed probability vector (*or* stationary vector) *of the regular transition matrix* A.

We shall now use Theorem 5.23 to deduce information about the eventual percentage in each state of a Markov chain having a regular transition matrix.

Example 20. A survey in ancient Persia showed that on a particular day 50% of the Persians preferred a loaf of bread, 30% preferred a jug of wine, and 20% preferred thou beside me in the wilderness. A subsequent survey 1 month later yielded the following data: Of those who preferred a loaf of bread on the first survey, 40% continued to prefer a loaf of bread, 10% now preferred a jug of wine, and 50% preferred thou; of those who preferred a jug of wine on the first survey, 20% now preferred a loaf of bread, 70% continued to prefer a jug of wine, and 10% now preferred thou; of those who preferred thou on the first survey, 20% now preferred a loaf of bread, 20% now preferred a jug of wine, and 60% continued to prefer thou.

The situation described in the preceding paragraph is a three-state Markov chain in which the states are the three possible preferences.

Assuming that the trend described above continues, we can predict the percentage of Persians in each state for each month following the original survey. Letting the first, second, and third states be preference for bread, wine, and thou, respectively, we see that the probability vector that gives the initial probability of being in each state is

$$P = \begin{pmatrix} 0.50 \\ 0.30 \\ 0.20 \end{pmatrix},$$

and the transition matrix is

$$A = \begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.50 & 0.10 & 0.60 \end{pmatrix}.$$

The probabilities of being in each state m months after the original survey are the coordinates of the vector A^mP. The reader may check that

$$AP = \begin{pmatrix} 0.30 \\ 0.30 \\ 0.40 \end{pmatrix}, \qquad A^2P = A(AP) = \begin{pmatrix} 0.26 \\ 0.32 \\ 0.42 \end{pmatrix}, \qquad A^3P = A(A^2P) = \begin{pmatrix} 0.252 \\ 0.334 \\ 0.414 \end{pmatrix},$$

and

$$A^4P = A(A^3P) = \begin{pmatrix} 0.2504 \\ 0.3418 \\ 0.4078 \end{pmatrix}.$$

Note the seeming convergence of A^mP.

Since A is regular, the long-range prediction concerning the Persians' preferences can be found by computing the fixed probability vector for A. This vector is the unique probability vector v such that $(A - I)v = 0$. Letting

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

we see that the matrix equation $(A - I)v = 0$ yields the following system of linear equations:

$$\begin{cases} -0.60v_1 + 0.20v_2 + 0.20v_3 = 0 \\ 0.10v_1 - 0.30v_2 + 0.20v_3 = 0 \\ 0.50v_1 + 0.10v_2 - 0.40v_3 = 0. \end{cases}$$

It is easily shown that

$$\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}$$

is a basis for the solution space of this system. Hence the unique fixed probability vector for A is

$$\begin{pmatrix} \dfrac{5}{5+7+8} \\ \dfrac{7}{5+7+8} \\ \dfrac{8}{5+7+8} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}.$$

Thus in the long run 25% of the Persians will prefer a loaf of bread, 35% will prefer a jug of wine, and 40% will prefer thou beside me in the wilderness.

Note that if

$$Q = \begin{pmatrix} 5 & 0 & 3 \\ 7 & 1 & 1 \\ 8 & -1 & -4 \end{pmatrix},$$

then

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

So

$$\lim_{m\to\infty} A^m = Q\left[\lim_{m\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}^m\right] Q^{-1} = Q\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} 0.25 & 0.25 & 0.25 \\ 0.35 & 0.35 & 0.35 \\ 0.40 & 0.40 & 0.40 \end{pmatrix}.$$

Example 21. Farmers in Lamron plant one crop per year—either corn, soybeans, or wheat. Because they believe in the necessity of rotating their crops, these farmers will not plant the same crop in successive years. In fact, of the total acreage on which a particular crop is planted, exactly half will be planted with each of the other two crops during the succeeding year. This year 300 acres of corn were planted, 200 acres of soybeans were planted, and 100 acres of wheat were planted.

The situation described in the paragraph above is another three-state Markov chain in which the three states correspond to the planting of corn, soybeans, and wheat, respectively. In this problem, however, the amount of land devoted to each crop, rather than the percentage of the total acreage (600 acres), was given. By converting these amounts into fractions of the total acreage, we see that the transition matrix A and the initial probability vector P of the Markov chain are

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \frac{300}{600} \\ \frac{200}{600} \\ \frac{100}{600} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}.$$

Then the fraction of the total acreage devoted to each crop in m years will be given by the coordinates of A^mP, and the eventual proportions of the total acreage to be used for each crop are the coordinates of $\lim_{m\to\infty} A^mP$. Thus the eventual amounts of land to be devoted to each crop are found by multiplying this limit by the total acreage; i.e., the eventual amounts of land to be used for each crop are the coordinates of $600(\lim_{m\to\infty} A^mP)$.

Since A is a regular transition matrix, Theorem 5.23 shows that $\lim_{m\to\infty} A^m$ is a matrix L in which each column equals the unique fixed probability vector for A. It is easily seen that the fixed probability vector for A is

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Hence

$$L = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

and so

$$600(\lim_{m\to\infty} A^mP) = 600LP = \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}.$$

Thus in the long run we expect 200 acres of each crop to be planted each year. (For a direct computation of $600(\lim_{m\to\infty} A^mP)$, see Exercise 14.)

In this section we have concentrated primarily on the theory of regular transition matrices. There is another interesting class of transition matrices that can be represented in the form

$$\begin{pmatrix} I & B \\ O & C \end{pmatrix},$$

where I is an identity matrix and O is a zero matrix. (Such transition matrices are not regular since the lower left block remains O in any power of the matrix.) The states corresponding to the identity submatrix are called *absorbing states* because such a state is never left once it is entered. A Markov chain is called an *absorbing Markov chain* if it is possible to go from an arbitrary state into an absorbing state in a finite number of stages. Observe that the Markov chain that described the enrollment pattern in a junior college is an absorbing Markov chain with states 1 and 4 as its absorbing states. Readers interested in learning more about absorbing Markov chains are referred to *Introduction to Finite Mathematics* (third edition) by J. Kemeny, J. Snell, and G. Thompson, Prentice-Hall, Inc., 1974, or *Discrete Mathematical Models* by Fred S. Roberts, Prentice-Hall, Inc., 1976.

An Application

In species that reproduce sexually, the characteristics of an offspring with respect to a particular genetic trait are determined by a pair of genes, one inherited from each parent. The genes for a particular trait are of two types, which we shall denote by G and g. The gene G represents the dominant characteristic and g represents the recessive characteristic. Offspring with genotypes GG or Gg exhibit the dominant characteristic, whereas offspring with genotype gg exhibit the recessive characteristic. For example, in humans, brown eyes are a dominant characteristic and blue eyes are the corresponding recessive characteristic; thus offspring with genotypes GG or Gg will be brown-eyed, whereas those of type gg will be blue-eyed.

Let us consider the probability of offspring of each genotype for a male parent of genotype Gg. (We shall assume that the population under consideration is large, that mating is random with respect to genotype, and that the distribution of each genotype within the population is independent of sex and life expectancy.) Let

$$P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

denote the proportion of the adult population with genotypes GG, Gg, and gg, respectively, at the start of the experiment. This experiment describes a three-state Markov chain with transition matrix

$$\begin{array}{ll} & \text{Genotype of Female Parent} \\ & \begin{array}{ccc} \text{GG} & \text{Gg} & \text{gg} \end{array} \\ \begin{array}{l} \text{Genotype} \\ \text{of} \\ \text{Offspring} \end{array} \begin{array}{l} \text{GG} \\ \text{Gg} \\ \text{gg} \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = B. \end{array}$$

It is easily checked that B^2 contains only positive entries; so B is regular. Thus by permitting only males of genotype Gg to reproduce, the proportion of offspring in the population having a certain genotype will stabilize at the fixed probability vector for B, which is

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

Now suppose that similar experiments are to be performed with males of genotypes GG and gg. As above, these experiments are three-state Markov chains with transition matrices

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix},$$

respectively. In order to consider the case where all male genotypes are permitted to reproduce, we must form the transition matrix $M = pA + qB + rC$, which is a linear combination of A, B, and C weighted by the proportion of males of each genotype. Thus

$$M = \begin{pmatrix} p + \frac{1}{2}q & \frac{1}{2}p + \frac{1}{4}q & 0 \\ \frac{1}{2}q + r & \frac{1}{2}p + \frac{1}{2}q + \frac{1}{2}r & p + \frac{1}{2}q \\ 0 & \frac{1}{4}q + \frac{1}{2}r & \frac{1}{2}q + r \end{pmatrix}.$$

To simplify the notation, let $a = p + \frac{1}{2}q$ and $b = \frac{1}{2}q + r$. (The numbers a and b represent the proportion of G and g genes, respectively, in the population.) Then

$$M = \begin{pmatrix} a & \frac{1}{2}a & 0 \\ b & \frac{1}{2} & a \\ 0 & \frac{1}{2}b & b \end{pmatrix},$$

where $a + b = p + q + r = 1$.

Let p', q', and r' denote the proportion of the first-generation offspring having genotypes GG, Gg, and gg, respectively. Then

$$\begin{pmatrix} p' \\ q' \\ r' \end{pmatrix} = MP = \begin{pmatrix} ap + \frac{1}{2}aq \\ bp + \frac{1}{2}q + ar \\ \frac{1}{2}bq + br \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix}.$$

In order to consider the effects of unrestricted matings among the first-generation offspring, a new transition matrix $\tilde{M}$ must be determined based upon the distribution of first-generation genotypes. As before, we find that

$$\tilde{M} = \begin{pmatrix} p' + \frac{1}{2}q' & \frac{1}{2}p' + \frac{1}{4}q' & 0 \\ \frac{1}{2}q' + r' & \frac{1}{2}p' + \frac{1}{2}q' + \frac{1}{2}r' & p' + \frac{1}{2}q' \\ 0 & \frac{1}{4}q' + \frac{1}{2}r' & \frac{1}{2}q' + r' \end{pmatrix} = \begin{pmatrix} a' & \frac{1}{2}a' & 0 \\ b' & \frac{1}{2} & a' \\ 0 & \frac{1}{2}b' & b' \end{pmatrix},$$

where $a' = p' + \frac{1}{2}q'$ and $b' = \frac{1}{2}q' + r'$. But

$$a' = a^2 + \tfrac{1}{2}(2ab) = a(a + b) = a$$

and

$$b' = \tfrac{1}{2}(2ab) + b^2 = b(a + b) = b.$$

Thus $\tilde{M} = M$, and so the distribution of second-generation offspring among the three genotypes is

$$\begin{aligned} \tilde{M}(MP) = M^2P &= \begin{pmatrix} a^3 + a^2b \\ a^2b + ab + ab^2 \\ ab^2 + b^3 \end{pmatrix} = \begin{pmatrix} a^2(a + b) \\ ab(a + 1 + b) \\ b^2(a + b) \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix} \\ &= MP, \end{aligned}$$

the same as the first-generation offspring. In other words, MP is the fixed probability vector for M, and genetic equilibrium is achieved in the population after only one generation. (This result is called the *Hardy-Weinberg law.*) Notice that in the important special case where $a = b$ (or equivalently, where $p = r$), the distribution at equilibrium is

$$MP = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix} = \begin{pmatrix} a^2 \\ 2a^2 \\ a^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

EXERCISES

1. Label the following statements as being true or false.
   (a) If $A \in \mathsf{M}_{n\times n}(C)$ and $\lim_{m\to\infty} A^m = L$, then, for any invertible matrix $Q \in \mathsf{M}_{n\times n}(C)$, $\lim_{m\to\infty} QA^mQ^{-1} = QLQ^{-1}$.
   (b) If 2 is an eigenvalue of $A \in \mathsf{M}_{n\times n}(C)$, then $\lim_{m\to\infty} A^m$ does not exist.

(c) A vector

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in \mathsf{R}^n$$

such that $x_1 + \cdots + x_n = 1$ is a probability vector.

(d) The sum of the entries of each row of a transition matrix equals 1.

(e) The product of a transition matrix and a probability vector is a probability vector.

(f) The matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

does not have 3 as an eigenvalue.

(g) Every transition matrix has 1 as an eigenvalue.

(h) No transition matrix can have -1 as an eigenvalue.

(i) If A is a transition matrix, then $\lim_{m\to\infty} A^m$ exists.

(j) If A is a regular transition matrix, then $\lim_{m\to\infty} A^m$ exists and has rank 1.

2. Determine whether or not $\lim_{m\to\infty} A^m$ exists for the following matrices A. If the limit exists, compute it.

(a) $\begin{pmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{pmatrix}$ (b) $\begin{pmatrix} 0.50 & 2 \\ 0.75 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & -1 \\ 4 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} -0.5 & 0.5 \\ 0.9 & -0.1 \end{pmatrix}$

(g) $\begin{pmatrix} -1.8 & 0 & -1.4 \\ -5.6 & 1 & -2.8 \\ 2.8 & 0 & 2.4 \end{pmatrix}$ (h) $\begin{pmatrix} -2.5 & 4.5 & 7.5 \\ -1.5 & -2.5 & -1.5 \\ -1.5 & 4.5 & 6.5 \end{pmatrix}$

(i) $\begin{pmatrix} -\frac{1}{2} - 2i & 4i & \frac{1}{2} + 5i \\ 1 + 2i & -3i & -1 - 4i \\ -1 - 2i & 4i & 1 + 5i \end{pmatrix}$

(j) $\begin{pmatrix} \frac{-26}{3} + \frac{i}{3} & \frac{-28}{3} - \frac{4i}{3} & 28 \\ \frac{-7}{3} + \frac{2i}{3} & \frac{-5}{3} + \frac{i}{3} & 7 - 2i \\ \frac{-13}{6} + i & \frac{-5}{6} + i & \frac{35}{6} - \frac{10i}{3} \end{pmatrix}$

3. Prove that if $A_1, A_2, A_3, \ldots$ is a sequence of $n \times p$ matrices with complex number entries such that $\lim_{m\to\infty} A_m = L$, then $\lim_{m\to\infty} A_m^t = L^t$.

4. Prove that if $A \in \mathsf{M}_{n\times n}(C)$ is diagonalizable and $L = \lim_{m\to\infty} A^m$ exists, then either $L = I_n$ or $\text{rank}(L) < n$.

5. Find 2×2 matrices A and B having real number entries such that $\lim_{m\to\infty} A^m$, $\lim_{m\to\infty} B^m$, and $\lim_{m\to\infty} (AB)^m$ all exist but $\lim_{m\to\infty} (AB)^m \neq (\lim_{m\to\infty} A^m)(\lim_{m\to\infty} B^m)$.

6. A hospital trauma unit has determined that 30% of its patients are ambulatory and 70% are bedridden at the time of arrival at the hospital. A month after arrival, 60% of the ambulatory patients have recovered, 20% remain ambulatory, and 20% have become bedridden. After the same time, 10% of the bedridden patients have recovered, 20% have become ambulatory, 50% remain bedridden, and 20% have died. Determine the percentage of patients who have recovered, are ambulatory, are bedridden, and have died 1 month after arrival. Also determine the eventual percentage of patients of each type.

7. A player begins a game of chance by placing a marker in box 2 (marked *start*). (See Fig. 5.4.) A die is rolled, and the marker is moved one square to the left if a 1 or 2 is rolled and one square to the right if a 3, 4, 5, or 6 is rolled. This process continues until the marker lands in square 1 (in which case the player wins the game) or square 4 (in which case the player loses the game). What is the probability of winning this game?

Win	Start		Lose
1	2	3	4

figure 5.4

8. Which of the following are regular transition matrices?

(a) $\begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0.5 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0.5 & 0 & 1 \\ 0.5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0.2 \\ 0 & 0.3 & 0.8 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 1 \end{pmatrix}$

9. Compute $\lim_{m\to\infty} A^m$, if it exists, for each of the matrices A in Exercise 8.

10. Each of the following matrices is a regular transition matrix for a three-state Markov chain. In each case the initial probability vector is

$$P = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}.$$

For each transition matrix, compute the proportion of objects in each state after two stages and the eventual proportion of objects in each state by determining the fixed probability vector.

(a) $\begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0 & 0.7 \end{pmatrix}$ (b) $\begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$ (c) $\begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.1 \\ 0 & 0.3 & 0.8 \end{pmatrix}$

(d) $\begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{pmatrix}$ (e) $\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$ (f) $\begin{pmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0.8 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix}$

11. In 1940 a county land-use survey showed that 10% of the county land was urban, 50% was unused, and 40% was agricultural. Five years later a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentage of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages.

12. A diaper liner is placed in each diaper worn by a baby. If, after a diaper change, the liner is soiled, then it is discarded. Otherwise the liner is washed with the diapers and reused, except that each liner is discarded after its third use (even if it has never been soiled). The probability that the baby will soil any diaper liner is one-third. If there are only new diapers at first, eventually what proportion of the diaper liners being used will be new, once-used, and twice-used?

13. In 1965 the automobile industry determined that 40% of American car owners drove large cars, 20% drove intermediate-sized cars, and 40% drove small cars. A second survey in 1975 showed that 70% of the large car owners in 1965 still owned large cars in 1975, but 30% had changed to an intermediate-sized car. Of those who owned intermediate-sized cars in 1965, 10% had switched to large cars, 70% continued to drive intermedi-

ate-sized cars, and 20% had changed to small cars in 1975. Finally, of the small car owners in 1965, 10% owned intermediate-sized cars and 90% owned small cars in 1975. Assuming that these trends continue, determine the percentage of Americans who will own cars of each size in 1985 and the corresponding eventual percentages.

14. Show that if A is as in Example 21, then

$$A^m = \begin{pmatrix} r_m & r_{m+1} & r_{m+1} \\ r_{m+1} & r_m & r_{m+1} \\ r_{m+1} & r_{m+1} & r_m \end{pmatrix},$$

where

$$r_m = \frac{1}{3}\left[1 + \frac{(-1)^m}{2^{m-1}}\right].$$

Deduce that

$$600(A^m P) = A^m \begin{pmatrix} 300 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200 + \dfrac{(-1)^m}{2^m}(100) \\ 200 \\ 200 + \dfrac{(-1)^{m+1}}{2^m}(100) \end{pmatrix}.$$

15. Prove Theorem 5.18 and its corollary.

16. Prove the two corollaries of Theorem 5.21.

17. Prove the corollary to Theorem 5.22.

Definition. *If* $A \in M_{n \times n}(C)$, *define* $e^A = \lim_{m \to \infty} B_m$, *where*

$$B_m = I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}.$$

Thus e^A *is the sum of the infinite series*

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots,$$

and B_m *is the* m*th partial sum of this series. Note the analogy with the power series*

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots,$$

which is valid for all complex numbers a.

18. Compute e^O and e^I, where O and I denote the $n \times n$ zero and identity matrices, respectively.

19. Suppose that $P^{-1}AP$ is a diagonal matrix D. Prove that $e^A = P^{-1}e^D P$.

20. Let $A \in \mathsf{M}_{n\times n}(C)$ be diagonalizable. Use the result of Exercise 19 to show that e^A exists. (Exercise 21 of Section 6.2 will show that e^B exists for each $B \in \mathsf{M}_{n\times n}(C)$.)

21. Find $A, B \in \mathsf{M}_{2\times 2}(R)$ such that $e^A e^B \neq e^{A+B}$.

22. Prove that a differentiable function $X\colon\ R \longrightarrow \mathsf{R}^n$ is a solution to the system of differential equations defined in Exercise 17 of Section 5.2 if and only if X is of the form $X(t) = e^{tA}v$ for some $v \in \mathsf{R}^n$, where A is as defined in that exercise.

5.4† INVARIANT SUBSPACES

In Section 5.1 we observed that if x is an eigenvector of a linear operator T, then T maps the span of $\{x\}$ into itself. Subspaces that are mapped into themselves are of great importance in the study of linear operators.

Definition. *Let* T *be a linear operator on a vector space* V. *A subspace* W *of* V *is called a* T-invariant subspace *of* V *if* $\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W}$, *i.e.*, $\mathsf{T}(x) \in \mathsf{W}$ *for all* $x \in \mathsf{W}$.

For any linear operator T on V the subspaces $\{0\}$ and V are T-invariant. These two subspaces are called *improper* T-invariant subspaces; all others are called *proper* T-invariant subspaces.

It is desirable to decompose a finite-dimensional vector space V into a direct sum of as many proper T-invariant subspaces as possible, for the behavior of T can then be inferred from its behavior on each of the direct summands. If T is diagonalizable, then V can be decomposed into a direct sum of one-dimensional T-invariant subspaces, namely the spans of the vectors in a basis consisting of eigenvectors of T. (See Exercise 7.) In general, such a decomposition does not exist. In Chapter 6 we shall consider ways of decomposing V into a direct sum of T-invariant subspaces when T is not diagonalizable. In this section we shall study two basic properties of direct sums of T-invariant subspaces.

For a linear operator T on a vector space V, the restriction of T to a T-invariant subspace W is a mapping of W into itself. (See Appendix B.) It is a simple matter to show that this mapping T_W is a linear operator on W. (See Exercise 4.) Our first result relates the characteristic polynomial of T_W to that of T.

†This section is not needed for Chapter 7.

Theorem 5.24. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* W *be a* T*-invariant subspace of* V. *Then the characteristic polynomial of* $\mathsf{T_W}$ *divides the characteristic polynomial of* T.

PROOF. Extend a basis $\gamma = \{x_1, \ldots, x_k\}$ for W to a basis $\beta = \{x_1, \ldots, x_k, \ldots, x_n\}$ for V. Let $A = [\mathsf{T}]_\beta$ and $B_1 = [\mathsf{T_W}]_\beta$. Then by Exercise 5

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

where O is an $(n-k) \times k$ zero matrix. If $f(t)$ is the characteristic polynomial of T and $g(t)$ is the characteristic polynomial of $\mathsf{T_W}$, then

$$f(t) = \det(A - tI_n) = \det\begin{pmatrix} B_1 - tI_k & B_2 \\ O & B_3 - tI_{n-k} \end{pmatrix} = g(t)\cdot\det(B_3 - tI_{n-k})$$

by Exercise 9 of Section 4.3. Thus $g(t)$ divides $f(t)$. ∎

Example 22. Let $\mathsf{T}\colon\ \mathsf{R}^4 \longrightarrow \mathsf{R}^4$ be defined by

$$\mathsf{T}(a, b, c, d) = (a + b + 2c - d, b + d, 2c - d, c + d),$$

and let $\mathsf{W} = \{(t, s, 0, 0)\colon\ t, s \in R\}$. Observe that W is a T-invariant subspace of R^4, for

$$\mathsf{T}(a, b, 0, 0) = (a + b, b, 0, 0) \in \mathsf{W}.$$

Let $\gamma = \{e_1, e_2\}$, and note that γ is a basis for W. Extend γ to the standard basis β for R^4. Then

$$B_1 = [\mathsf{T_W}]_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

in the notation of Theorem 5.24. Thus if $f(t)$ is the characteristic polynomial of T and $g(t)$ is the characteristic polynomial of $\mathsf{T_W}$, then

$$f(t) = \det(A - tI_4) = \det\begin{pmatrix} 1-t & 1 & 2 & -1 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 2-t & -1 \\ 0 & 0 & 1 & 1-t \end{pmatrix}$$

$$= \det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix}\cdot\det\begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix} = g(t)\cdot\det\begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix}.$$

The next theorem shows that if V is a direct sum of T-invariant subspaces, then the characteristic polynomial of T is completely determined by the characteristic polynomials of the restrictions of T to each of the direct summands.

Theorem 5.25. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and suppose that* $V = W_1 \oplus \cdots \oplus W_k$, *where* W_i *is a* T*-invariant subspace of* V *for each* i $(1 \leq i \leq k)$. *If* f(t) *denotes the characteristic polynomial of* T *and* $f_i(t)$ *denotes the characteristic polynomial of* T_{W_i} $(1 \leq i \leq k)$, *then*

$$f(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_k(t).$$

PROOF. The proof will be by induction on k. Suppose first that $k = 2$. Let β_1 be a basis for W_1, β_2 be a basis for W_2, and $\beta = \beta_1 \cup \beta_2$. Then β is a basis for V. Let $A = [T]_\beta$, $B_1 = [T_{W_1}]_{\beta_1}$, and $B_2 = [T_{W_2}]_{\beta_2}$. It is easily seen by Exercise 5 that

$$A = \begin{pmatrix} B_1 & O \\ O' & B_2 \end{pmatrix},$$

where O and O' are zero matrices. Thus

$$f(t) = \det(A - tI) = \det(B_1 - tI) \cdot \det(B_2 - tI) = f_1(t) \cdot f_2(t)$$

by Exercise 9 of Section 4.3, proving the result if $k = 2$.

Now assume that the theorem is true for $k - 1$ summands, where $k - 1$ is some integer greater than or equal to 1. Suppose that V is a direct sum of k summands,

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

and define $W = W_1 + W_2 + \cdots + W_{k-1}$. It is easily verified that $V = W \oplus W_k$. So by the case for $k = 2$, $f(t) = g(t) \cdot f_k(t)$, where $g(t)$ is the characteristic polynomial of T_W. Clearly $W = W_1 \oplus W_2 \oplus \cdots \oplus W_{k-1}$. Hence by the induction hypothesis we have $g(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_{k-1}(t)$. Thus $f(t) = g(t) \cdot f_k(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_k(t)$. ∎

If T is a diagonalizable linear operator on an n-dimensional vector space V for which the distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$ by Theorem 5.14. It is easily seen (see Exercise 3) that each E_{λ_i} is T-invariant and the restriction of T to E_{λ_i} has characteristic polynomial $(\lambda_i - t)^{m_i}$, where m_i is the multiplicity of λ_i. Hence in this context the preceding theorem yields the obvious conclusion that the characteristic polynomial of T is $(\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \cdots (\lambda_k - t)^{m_k} = (-1)^n(t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$.

The following application of Theorem 5.25 will suggest another result about direct sums.

Example 23. Let $T\colon \ R^4 \longrightarrow R^4$ be defined by

$$T(a, b, c, d) = (2a - b, a + b, c - d, c + d),$$

and let $W_1 = \{(s, t, 0, 0)\colon\ s, t \in R\}$ and $W_2 = \{(0, 0, s, t)\colon\ s, t \in R\}$. Notice that W_1 and W_2 are each T-invariant and that $R^4 = W_1 \oplus W_2$. Let $\beta_1 = \{e_1, e_2\}$, $\beta_2 = \{e_3, e_4\}$, and $\beta = \{e_1, e_2, e_3, e_4\}$. Then β_1 is a basis for W_1, β_2 is a basis for W_2, and β is a basis for R^4. If $B_1 = [T_{W_1}]_{\beta_1}$, $B_2 = [T_{W_2}]_{\beta_2}$, and $A = [T]_\beta$, then

$$B_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Moreover, if $f(t)$ denotes the characteristic polynomial of T, $f_1(t)$ the characteristic polynomial of T_{W_1}, and $f_2(t)$ the characteristic polynomial of T_{W_2}, then

$$f(t) = \det(A - tI) = \det(B_1 - tI)\cdot\det(B_2 - tI) = f_1(t)\cdot f_2(t).$$

The matrix A in the example above can be obtained by joining the matrices B_1 and B_2 in the manner explained in the following definition.

Definition. *Let* B_1 *and* B_2 *be square matrices (not necessarily of the same size) having entries from the same field. If* B_1 *is an* $m \times m$ *matrix and* B_2 *is an* $n \times n$ *matrix, then the* direct sum *of* B_1 *and* B_2, *denoted* $B_1 \oplus B_2$, is the $(m + n) \times (m + n)$ *matrix* A *such that*

$$A_{ij} = \begin{cases} (B_1)_{ij} & \text{for } 1 \le i, j \le m \\ (B_2)_{(i-m),(j-m)} & \text{for } m + 1 \le i, j \le n + m \\ 0 & \text{otherwise.} \end{cases}$$

If $B_1, B_2, \ldots, B_k$ *are square matrices with entries from the same field, then we define the direct sum of* $B_1, B_2, \ldots, B_k$ *recursively by* $B_1 \oplus B_2 \oplus \cdots \oplus B_k = (B_1 \oplus B_2 \oplus \cdots \oplus B_{k-1}) \oplus B_k$. If $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, *then we shall often write*

$$A = \begin{pmatrix} B_1 & O & \cdots & O \\ O & B_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & B_k \end{pmatrix}.$$

Example 24. Let

$$B_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad B_2 = (3), \quad \text{and} \quad B_3 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$B_1 \oplus B_2 \oplus B_3 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The final result of this section relates direct sums of matrices to direct sums of invariant subspaces. It states the general case of the relationship among the matrices A, B_1, and B_2 in Example 23.

Theorem 5.26. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* $W_1, W_2, \ldots, W_k$ *be* T*-invariant subspaces of* V *such that* $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. *For each* i, *let* β_i *be a basis for* W_i *and* $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. *If* $A = [T]_\beta$ *and* $A_i = [T_{W_i}]_{\beta_i}$ *for* $i = 1, 2, \ldots, k$, *then* $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$.

PROOF. Exercise.

EXERCISES

1. Label the following statements as being true or false.
   (a) There exist linear operators T having no T-invariant subspaces.
   (b) If T is a linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.
   (c) If T is a linear operator on a finite-dimensional vector space V, and if V is a direct sum of T-invariant subspaces, then there is a basis β for V such that $[T]_\beta$ is a direct sum of matrices.

2. For each of the following linear operators T, determine if the given subspace W is a T-invariant subspace of V.
   (a) Let T be the operator on $V = P_3(R)$ defined by $T(f) = f'$, the derivative of f, and $W = P_2(R)$.
   (b) Let T be the operator on $V = P(R)$ defined by $T(f(x)) = xf(x)$ and $W = P_2(R)$.

(c) Let T be the operator on $V = R^3$ defined by

$$T(a_1, a_2, a_3) = (a_1 + a_2 + a_3, a_1 + a_2 + a_3, a_1 + a_2 + a_3)$$

and

$$W = \{(t, t, t): \quad t \in R\}.$$

(d) Let T be the operator on the vector space V of continuous real-valued functions on [0, 1] defined by

$$T(f)(t) = \left[\int_0^1 f(x)\,dx\right]t \quad \text{and} \quad W = \{f \in V: \; f(t) = at + b \quad \text{for}$$

some $a, b \in R$ and all $0 \leq t \leq 1\}$.

3. Let T be a linear operator on a finite-dimensional vector space V.
   (a) Prove that $\{0\}$ and V are T-invariant subspaces of V.
   (b) Prove that N(T) and R(T) are T-invariant subspaces of V.
   (c) Prove that if λ is an eigenvalue of T, then E_λ is a T-invariant subspace of V and the restriction of T to E_λ is λI.
   (d) If $W_1, W_2, \ldots, W_k$ are T-invariant subspaces of V, prove that

$$\sum_{i=1}^{k} W_i \quad \text{and} \quad \bigcap_{i=1}^{k} W_i$$

   are T-invariant subspaces of V.

4. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V.
   (a) Prove that T_W is a linear operator on W.
   (b) Show that if λ is a eigenvalue of T_W, then λ is an eigenvalue of T.
   (c) Prove that if x is an eigenvector of T_W, then x is an eigenvector of T.

5. Verify that

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$$

in the proof of Theorem 5.24 and that

$$A = \begin{pmatrix} B_1 & O \\ O' & B_2 \end{pmatrix}$$

in the proof of Theorem 5.25.

6. Prove Theorem 5.26.

7.† Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is a direct sum of one-dimensional T-invariant subspaces.

8. Let T be a linear operator on a finite dimensional vector space V, and let $W_1, W_2, \ldots, W_k$ be proper T-invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Prove that $\det(T) = \det(T_{W_1}) \cdot \det(T_{W_2}) \cdot \cdots \cdot \det(T_{W_k})$.

9. Let T be a linear operator on a finite-dimensional vector space V, and suppose that $W_1, W_2, \ldots, W_k$ are T-invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Prove that if T_{W_i} is diagonalizable for each i, then T is diagonalizable.

10. Let T be a linear operator on a finite-dimensional vector space V.
(a) Prove that if the characteristic polynomial of T factors as a product of factors of degree 1, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.
(b) Deduce that if the characteristic polynomial of T factors as a product of factors of degree 1, then any non-zero T-invariant subspace of V contains an eigenvector of T.
(c) Let W be any T-invariant subspace of V. Show that if λ is an eigenvalue of T_W, then the eigenspace of T_W corresponding to λ is $E_\lambda \cap W$, where E_λ denotes the eigenspace of T corresponding to λ.
(d) Let W be any T-invariant subspace of V. Prove that if T is diagonalizable, then so is T_W. *Hint:* Use part (a), part (c), and Theorem 5.14.

11. (a) Prove a converse of Exercise 19(a) in Section 5.2: If T and U are diagonalizable linear operators on a finite-dimensional vector space such that UT = TU, then T and U are simultaneously diagonalizable. *Hint:* Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T, and let E_{λ_i} $(i = 1, 2, \cdots, k)$ denote the corresponding eigenspaces of T. Prove that each E_{λ_i} is U-invariant, and apply Exercise 10(d) to obtain a basis for E_{λ_i} consisting of eigenvectors of U.
(b) State and prove the matrix version of part (a).

12. (a) The result of Exercise 11(a) can be generalized as follows: Let V be a finite-dimensional vector space. A collection $\mathcal{C}$ of diagonalizable linear operators on V is said to be *simultaneously diagonalizable* if there exists a basis β for V such that $[T]_\beta$ is a diagonal matrix for each $T \in \mathcal{C}$.

Prove that a collection $\mathcal{C}$ of diagonalizable linear operators on a finite-dimensional vector space V is simultaneously diagonalizable if and only if UT = TU for all T, U $\in \mathcal{C}$. *Hint:* In the case that UT = TU for all T, U $\in \mathcal{C}$, first establish the result when each operator in $\mathcal{C}$ has only one eigenvalue. Then establish the general result by induction on dim(V) using the fact that V can be expressed as the direct sum of the eigenspaces for some operator in $\mathcal{C}$.
(b) State and prove the matrix version of part (a).

Exercises 13 and 14 require familiarity with Exercise 29 of Section 1.3.

13.† Let T be a linear operator on V, and let W be a T-invariant subspace of V. Define

$$\bar{T}\colon\ V/W \longrightarrow V/W \quad \text{by} \quad \bar{T}(v + W) = T(v) + W.$$

(a) Show that $\bar{T}$ is well-defined; that is, show that $\bar{T}(v + W)$ is independent of the choice of v in the coset of $v + W$.
(b) Prove that $\bar{T}$ is a linear operator on V/W.
(c) Define

$$\eta\colon\ V \longrightarrow V/W \quad \text{by} \quad \eta(v) = v + W.$$

Prove that η is a linear transformation with null space W and range V/W.
(d) Show that the diagram in Fig. 5.5 is commutative; that is, $\eta T = \bar{T}\eta$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\eta"'] & V \arrow[d, "\eta"] \\ V/W \arrow[r, "\bar{T}"] & V/W \end{tikzcd}$$

figure 5.5

14. (a) Let T be a linear operator on a finite-dimensional vector space V, and let W be a proper T-invariant subspace of V. Let $f(t)$, $g(t)$, and $h(t)$ be the characteristic polynomials of T, T_W, and $\bar{T}$ (as defined in Exercise 13), respectively. Prove that $f(t) = g(t)h(t)$. *Hint:* Extend a basis $\gamma = \{x_1, \ldots, x_k\}$ for W to a basis $\beta = \{x_1, \ldots, x_k, \ldots, x_n\}$ for V. Show that $[T]_\beta$ is of the form

$$\begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

that $\bar{\beta} = \{x_{k+1} + W, \ldots, x_n + W\}$ is a basis for V/W, and that $[\bar{T}]_{\bar{\beta}} = B_3$.
(b) Use Exercise 13 to prove that if T is diagonalizable, then so is $\bar{T}$.

5.5† THE CAYLEY-HAMILTON THEOREM

In Section 5.4 we mentioned that if T is a linear operator on a finite-dimensional vector space V, then it is desirable to decompose V into a direct sum of as many T-invariant subspaces as possible. When the characteristic polynomial of T factors into a product of factors of degree 1, we shall show in Section 6.1 that V can always be decomposed into a direct sum of the "generalized eigenspaces" of T. In fact, when T is diagonalizable, this is precisely the decomposition given in Theorem 5.14. If, however, the

†The material in this section is needed for Sections 5.6 and 6.3 but not for Sections 6.1 and 6.2.

characteristic polynomial of T does not factor into a product of factors of degree 1, then T may even fail to have eigenvalues. In this case we cannot expect to decompose V into a direct sum as before. Nevertheless, a direct sum decomposition of V into T-invariant subspaces is still possible. In this section we shall define the special type of T-invariant subspaces needed for this decomposition and shall use them to prove one of the most famous theorems of linear algebra, the Cayley-Hamilton theorem.

Definitions. *Let* T *be a linear operator on a vector space* V. *A subspace* W *of* V *is called a* T-cyclic subspace *if there exists an element* $x \in W$ *such that*

$$W = span(\{x, T(x), T^2(x), \ldots\}).$$

In this case we say that W *is* generated *by* x. *The* T*-cyclic subspace generated by* x *will be denoted* C_x.

Example 25. Let $T\colon R^3 \longrightarrow R^3$ be defined by

$$T(a, b, c) = (-b + c, a + c, 3c).$$

We shall determine the T-cyclic subspace generated by $e_1 = (1, 0, 0)$. Since $T(e_1) = T(1, 0, 0) = (0, 1, 0) = e_2$ and $T^2(e_1) = T(T(e_1)) = T(e_2) = (-1, 0, 0) = -e_1$, $C_{e_1} = \text{span}(\{e_1, T(e_1), T^2(e_1), \ldots\}) = \text{span}(\{e_1, e_2\}) = \{(s, t, 0)\colon\ s, t \in R\}$. Notice that for this transformation $C_{e_1} = C_{e_2}$.

Example 26. Let T be the linear operator on $P(R)$ defined by $T(f) = f'$. Then $C_{x^2} = \text{span}(\{x^2, 2x, 2\}) = P_2(R)$.

It is easily seen that T-cyclic subspaces are T-invariant. Our next result establishes some additional properties of T-cyclic subspaces.

Theorem 5.27. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* W *denote the* T*-cyclic subspace of* V *generated by* $x \in V$. *Suppose that* $dim(W) = k \geq 1$ (*and hence* $x \neq 0$). *Then*

(a) $\{x, T(x), T^2(x), \ldots, T^{k-1}(x)\}$ *is a basis for* W.

(b) *If* $-a_0, -a_1, \ldots, -a_{k-1}$ *are the scalars given by part* (a) *such that* $T^k(x) = -a_0x - a_1T(x) - \cdots - a_{k-1}T^{k-1}(x)$, *then* $f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$ *is the characteristic polynomial of* T_W.

PROOF. Let j be the smallest integer for which $\{x, T(x), \ldots, T^j(x)\}$ is linearly dependent. (Such a j must exist since W is finite-dimensional.) Since $x \neq 0$, $j \geq 1$. Thus $\{x, T(x), \ldots, T^{j-1}(x)\}$ is linearly independent and $T^j(x) \in \text{span}(\{x, T(x), \ldots T^{j-1}(x)\})$ by the lemma to Theorem 1.10. We shall show by mathematical induction that $T^s(x)$ lies in this span for any non-negative integer s. This is clear for $0 \leq s \leq j$. Suppose that

$\mathsf{T}^m(x) \in \text{span}(\{x, \mathsf{T}(x), \ldots, \mathsf{T}^{j-1}(x)\})$ for some $m \geq j$. Then there exist scalars $b_0, b_1, \ldots, b_{j-1}$ such that

$$\mathsf{T}^m(x) = b_0 x + b_1 \mathsf{T}(x) + \cdots + b_{j-1}\mathsf{T}^{j-1}(x).$$

Applying T to both sides of the preceding equality, we obtain

$$\mathsf{T}^{m+1}(x) = b_0 \mathsf{T}(x) + b_1 \mathsf{T}^2(x) + \cdots + b_{j-1}\mathsf{T}^{j}(x).$$

So $\mathsf{T}^{m+1}(x)$ is a linear combination of $\mathsf{T}(x), \mathsf{T}^2(x), \ldots, \mathsf{T}^j(x)$, each of which lies in $\text{span}(\{x, \mathsf{T}(x), \ldots, \mathsf{T}^{j-1}(x)\})$. Thus $\mathsf{T}^{m+1}(x)$ lies in this span, completing the induction. Hence

$$\mathsf{W} = \text{span}(\{x, \mathsf{T}(x), \mathsf{T}^2(x), \ldots\}) \subseteq \text{span}(\{x, \mathsf{T}(x), \ldots, \mathsf{T}^{j-1}(x)\}).$$

But clearly the reverse inclusion is also true, and therefore $\{x, \mathsf{T}(x), \ldots, \mathsf{T}^{j-1}(x)\}$ spans W. Since this set is also linearly independent, it is a basis for W. But $\dim(\mathsf{W}) = k$; so this set must contain k elements. Therefore $j = k$, and thus $\{x, \mathsf{T}(x), \ldots, \mathsf{T}^{k-1}(x)\}$ is a basis for W, proving (a).

To prove (b), let $\beta = \{x, \mathsf{T}(x), \ldots, \mathsf{T}^{k-1}(x)\}$ be the basis from part (a), and let $-a_0, -a_1, \ldots, -a_{k-1}$ be the scalars such that $\mathsf{T}^k(x) = -a_0 x - a_1\mathsf{T}(x) - \cdots - a_{k-1}\mathsf{T}^{k-1}(x)$. Observe that

$$[\mathsf{T}_\mathsf{W}]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

and so the characteristic polynomial of $[\mathsf{T}_\mathsf{W}]_\beta$ is

$$f(t) = (-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$$

by Exercise 10. Thus $f(t)$ is the characteristic polynomial of T_W, proving (b). ∎

Definition. *The matrix*

$$[\mathsf{T}_\mathsf{W}]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

in the proof of Theorem 5.27 is called the companion matrix *of the polynomial*

$$f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k).$$

We can now prove the celebrated Cayley-Hamilton theorem. The reader should refer to Appendix E for the definition of $f(\mathsf{T})$ when T is a linear operator and $f(t)$ is a polynomial.

Theorem 5.28 (*Cayley-Hamilton*). *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* $f(t)$ *be the characteristic polynomial of* T. *Then* $f(\mathsf{T}) = \mathsf{T}_0$ (*the zero transformation*)*; i.e.,* T *satisfies its characteristic polynomial.*

PROOF. We must show that $f(\mathsf{T})(x) = 0$ for all $x \in \mathsf{V}$. If $x = 0$, then $f(\mathsf{T})(x) = 0$ since $f(\mathsf{T})$ is a linear transformation. Suppose then that $x \neq 0$, and let $\mathsf{W} = \mathsf{C}_x$. If $\dim(\mathsf{W}) = k$, then by Theorem 5.27 there exist scalars $-a_0, -a_1, \ldots, -a_{k-1}$ such that

$$\mathsf{T}^k(x) = -a_0x - a_1\mathsf{T}(x) - \cdots - a_{k-1}\mathsf{T}^{k-1}(x).$$

Hence Theorem 5.27 implies that

$$g(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$$

is the characteristic polynomial of T_W. Combining these two equations yields

$$g(\mathsf{T})(x) = (-1)^k(a_0\mathsf{I} + a_1\mathsf{T} + \cdots + a_{k-1}\mathsf{T}^{k-1} + \mathsf{T}^k)(x) = 0.$$

By Theorem 5.24, $g(t)$ divides $f(t)$; hence there exists a polynomial $q(t)$ such that $f(t) = q(t)g(t)$. So

$$f(\mathsf{T})(x) = q(\mathsf{T})g(\mathsf{T})(x) = q(\mathsf{T})(g(\mathsf{T})(x)) = q(\mathsf{T})(0) = 0. \quad \blacksquare$$

Example 27. Define $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^2$ by $\mathsf{T}(a, b) = (a + 2b, -2a + b)$, and let $\beta = \{e_1, e_2\}$. Then

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

where $A = [\mathsf{T}]_\beta$. The characteristic polynomial of T is therefore

$$f(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 2 \\ -2 & 1-t \end{pmatrix} = t^2 - 2t + 5.$$

It is easily verified that $\mathsf{T}_0 = f(\mathsf{T}) = \mathsf{T}^2 - 2\mathsf{T} + 5\mathsf{I}$. Likewise

$$f(A) = A^2 - 2A + 5I = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 28. Let T be the linear operator on $\mathsf{P}_2(R)$ defined by $\mathsf{T}(f) = f' + f$. The characteristic polynomial of T is easily seen to be $g(t) = (1 - t)^3 = -t^3 + 3t^2 - 3t + 1$. Now

$$\mathsf{T}^2(f) = \mathsf{T}(f' + f) = f'' + 2f' + f$$

and

$$\mathsf{T}^3(f) = f''' + 3f'' + 3f' + f.$$

Hence

$$g(\mathsf{T})(f) = -\mathsf{T}^3(f) + 3\mathsf{T}^2(f) - 3\mathsf{T}(f) + \mathsf{I}(f) = -f'''.$$

But for $f \in \mathsf{P}_2(R)$, $f''' = 0$. Hence $g(\mathsf{T}) = \mathsf{T}_0$.

EXERCISES

1. Label the following statements as being true or false.
   (a) Let C_x and C_y be T-cyclic subspaces of a linear operator T on a finite-dimensional vector space V. If $\mathsf{C}_x = \mathsf{C}_y$, then $x = y$.
   (b) If T is a linear operator on a finite-dimensional vector space, then $\mathsf{C}_x = \mathsf{C}_{\mathsf{T}(x)}$.
   (c) Let T be a linear operator on an n-dimensional vector space. There exists a polynomial $g(t)$ of degree n such that $g(\mathsf{T}) = \mathsf{T}_0$.
   (d) The characteristic polynomial of the companion matrix of $g(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$ is $g(t)$.
   (e) A polynomial of the form $(-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$ is the characteristic polynomial of some linear operator.

2. Find a basis for the T-cyclic subspace C_z in each of the following.
   (a) $\mathsf{T}\colon \mathsf{R}^4 \longrightarrow \mathsf{R}^4$ defined by $\mathsf{T}(a_1, a_2, a_3, a_4) = (a_1 + a_2, a_2 - a_3, a_1 + a_3, a_1 + a_4)$ and $z = e_1$
   (b) $\mathsf{T}\colon \mathsf{P}_3(R) \longrightarrow \mathsf{P}_3(R)$ defined by $\mathsf{T}(f) = f''$ and $z = x^3$
   (c) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \longrightarrow \mathsf{M}_{2\times 2}(R)$ defined by

$$\mathsf{T}(A) = A^t \quad \text{and} \quad z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   (d) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \longrightarrow \mathsf{M}_{2\times 2}(R)$ defined by

$$\mathsf{T}(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. For each of the linear operators T in Exercise 2
   (i) Compute the characteristic polynomial of $\mathsf{T}_{\mathsf{C}_z}$.
   (ii) Compute the characteristic polynomial of T.
   (iii) Verify the Cayley-Hamilton theorem for T.

4. Let $\mathsf{T}\colon\ \mathsf{V} \to \mathsf{V}$ be a linear operator. Prove that for any $x \in \mathsf{V}$, C_x is T-invariant.

5. Prove the Cayley-Hamilton theorem for matrices: If A is an $n \times n$ matrix with characteristic polynomial $f(t)$, then $f(A) = O$, the $n \times n$ zero matrix.

6. Let V be a two-dimensional vector space and $\mathsf{T}\colon\ \mathsf{V} \to \mathsf{V}$ be a linear operator. Prove that either V is a T-cyclic subspace of itself or $\mathsf{T} = \lambda\mathsf{I}$ for some scalar λ.

7. Let T be a linear operator on a finite-dimensional vector space V, and let $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of T. Prove that
   (a) T is invertible if and only if $a_0 \neq 0$.
   (b) If T is invertible, then

$$\mathsf{T}^{-1} = \frac{(-1)^{n+1}}{a_0}\mathsf{T}^{n-1} - \frac{a_{n-1}}{a_0}\mathsf{T}^{n-2} - \cdots - \frac{a_1}{a_0}\mathsf{I}.$$

8. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that for any polynomial $g(t)$, W is $g(\mathsf{T})$-invariant.

9. Let T be a linear operator on a vector space V. For any $x \in \mathsf{V}$, prove that the T-cyclic subspace C_x is the smallest T-invariant subspace of V containing x; that is, for any T-invariant subspace W containing x, $\mathsf{C}_x \subseteq \mathsf{W}$.

10. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Hint: Use induction on k.

11. Use Exercise 22 of Section 5.1 to obtain an easy proof of the Cayley-Hamilton theorem for diagonalizable operators.

12. Let T be a linear operator on an n-dimensional vector space V. Prove that $\text{span}(\{\mathsf{I}, \mathsf{T}, \mathsf{T}^2, \ldots\})$ is a subspace of $\mathcal{L}(\mathsf{V})$ having dimension not exceeding n.

5.6† THE MINIMAL POLYNOMIAL

For a given operator T on a finite-dimensional vector space V the Cayley-Hamilton theorem shows that there is a polynomial $f(t)$ for which $f(\mathsf{T}) = \mathsf{T}_0$, namely the characteristic polynomial of T. There are many other polynomials having this property. One of the most important of these, the minimal polynomial, provides another means for studying linear operators.

Definition. *Let* T *be a linear operator on a vector space* V. *A polynomial* $p(t)$ *is called a* minimal polynomial *for* T *if* $p(t)$ *is a monic polynomial of least positive degree for which* $p(\mathsf{T}) = \mathsf{T}_0$. (*Recall from Appendix E that a monic polynomial is one in which the leading coefficient is* 1.)

It is easy to see that any linear operator T on a finite-dimensional vector space has a minimal polynomial. Note that if $g(t)$ is a polynomial of degree k with leading coefficient a such that $g(\mathsf{T}) = \mathsf{T}_0$, then $h(t) = (1/a)g(t)$ is a monic polynomial of degree k for which $h(\mathsf{T}) = \mathsf{T}_0$. Hence the Cayley-Hamilton theorem shows that the degree of a minimal polynomial for T is at most the dimension of the vector space on which T is defined. The next result shows that the requirement that a minimal polynomial be monic guarantees that it is unique.

Theorem 5.29. *Let* $p(t)$ *be a minimal polynomial for a linear operator* T *on a finite-dimensional vector space* V.

(a) *If* $g(t)$ *is any polynomial for which* $g(\mathsf{T}) = \mathsf{T}_0$, *then* $p(t)$ *divides* $g(t)$. *In particular,* $p(t)$ *divides the characteristic polynomial of* T.

(b) *There is only one minimal polynomial for* T; *i.e.,* $p(t)$ *is unique.*

PROOF.

(a) Let $g(t)$ be any polynomial for which $g(\mathsf{T}) = \mathsf{T}_0$. The division algorithm for polynomials (see Appendix E) implies that there exist polynomials $q(t)$ and $r(t)$ such that

$$g(t) = q(t)p(t) + r(t), \tag{8}$$

where $r(t)$ has degree less than that of $p(t)$. Substituting T in Eq. (8) and using that $g(\mathsf{T}) = p(\mathsf{T}) = \mathsf{T}_0$, we have $r(\mathsf{T}) = \mathsf{T}_0$. Since $r(t)$ has degree less than $p(t)$ and $p(t)$ is a minimal polynomial, $r(t)$ must be the zero polynomial. Thus Eq. (8) simplifies to $g(t) = q(t)p(t)$, proving (a).

(b) Suppose that $p_1(t)$ and $p_2(t)$ are each minimal polynomials for T. Then $p_1(t)$ divides $p_2(t)$ by part (a). But since $p_1(t)$ and $p_2(t)$ have the same

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†This section is needed only for Section 6.3.

non-negative degree, we must have $p_1(t) = cp_2(t)$ for some non-zero scalar c. Moreover, since $p_1(t)$ and $p_2(t)$ are monic, $c = 1$. Thus $p_1(t) = p_2(t)$. ∎

Before continuing our study of the minimal polynomial for an operator, we shall introduce the minimal polynomial for a matrix.

Definition. *The* minimal polynomial $\mathrm{p(t)}$ *for* $\mathrm{A} \in \mathsf{M}_{n\times n}(\mathrm{F})$ *is the monic polynomial of least positive degree for which* $\mathrm{p(A)}$ *equals the zero matrix.*

Throughout the book, statements about linear transformations have been translated into statements about matrices and vice versa. The following theorem and its corollary are of this type.

Theorem 5.30. *Let* T *be a linear operator on a finite-dimensional vector space* V*, and let* β *be a basis for* V*. Then the minimal polynomial for* T *is the same as the minimal polynomial for* $[\mathsf{T}]_\beta$.

PROOF. Exercise.

Corollary. *For any* $\mathrm{A} \in \mathsf{M}_{n\times n}(\mathrm{F})$*, the minimal polynomial for* A *is the same as the minimal polynomial for* L_A.

PROOF. Exercise.

As a consequence of the preceding theorem and corollary, subsequent theorems of this section that are stated for operators are also true for matrices.

In the remainder of this section we shall study primarily minimal polynomials for operators whose characteristic polynomials factor into a product of factors of degree 1. A more general treatment of minimal polynomials will be given in Section 6.3.

Theorem 5.31. *Let* T *be a linear operator on a finite-dimensional vector space* V*, and let* $\mathrm{p(t)}$ *be the minimal polynomial for* T*. A scalar* λ *is an eigenvalue of* T *if and only if* $\mathrm{p}(\lambda) = 0$*. Hence the characteristic polynomial and the minimal polynomial for* T *have the same zeros.*

PROOF. Let $f(t)$ be the characteristic polynomial of T. Since $p(t)$ divides $f(t)$, $f(t) = q(t)p(t)$ for some polynomial $q(t)$. Let λ be a zero of $p(t)$. Then

$$f(\lambda) = q(\lambda)p(\lambda) = q(\lambda)\cdot 0 = 0.$$

So λ is also a zero for $f(t)$; that is, λ is an eigenvalue of T.

Conversely, suppose that λ is an eigenvalue of T, and let $x \in \mathsf{V}$ be an eigenvector corresponding to λ. Then by Exercise 22 of Section 5.1 we have

$$0 = \mathsf{T}_0(x) = p(\mathsf{T})(x) = p(\lambda)x.$$

Since $x \neq 0$, $p(\lambda) = 0$, and so λ is a zero for $p(t)$. ∎

As an immediate consequence of the preceding result we have the following corollary.

Corollary. *Let* T *be a linear operator on a finite-dimensional vector space* V *with minimal polynomial* $p(t)$ *and characteristic polynomial* $f(t)$. *Suppose that* $f(t)$ *factors as*

$$f(t) = (\lambda_1 - t)^{n_1}(\lambda_2 - t)^{n_2} \cdots (\lambda_k - t)^{n_k},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are the distinct eigenvalues of* T. *Then there exist integers* $m_1, m_2, \ldots, m_k$ *such that* $1 \leq m_i \leq n_i$ *for all* i *and*

$$p(t) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}.$$

Example 29. We shall compute the minimal polynomial for the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

Since A has characteristic polynomial

$$f(t) = \det\begin{pmatrix} 3-t & -1 & 0 \\ 0 & 2-t & 0 \\ 1 & -1 & 2-t \end{pmatrix} = -(t-2)^2(t-3),$$

the minimal polynomial for A must be either $(t-2)(t-3)$ or $(t-2)^2(t-3)$ by the corollary to Theorem 5.31. Substituting A into $p(t) = (t-2)(t-3)$ shows that $p(A)$ is the zero matrix. Thus $p(t)$ is the minimal polynomial for A.

Example 30. Let $\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ be defined by

$$\mathsf{T}(a, b) = (2a + 5b, 6a + b).$$

If β is the standard basis for R^2, then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 2 & 5 \\ 6 & 1 \end{pmatrix}.$$

So the characteristic polynomial of $[\mathsf{T}]_\beta$, and hence of T, is

$$f(t) = \det\begin{pmatrix} 2-t & 5 \\ 6 & 1-t \end{pmatrix} = (t-7)(t+4).$$

Thus the minimal polynomial for T must be $(t-7)(t+4)$.

Example 31. Let $\mathsf{D}\colon \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ be the differentiation operator defined by $\mathsf{D}(f) = f'$. We shall compute the minimal polynomial for D. For the basis $\beta = \{1, t, t^2\}$ we have

$$[\mathsf{D}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial of D is $-t^3$. So the corollary to Theorem 5.31 shows that the minimal polynomial for D is t, t^2, or t^3. Since $\mathsf{D}^2(t^2) = 2 \neq 0$, $\mathsf{D}^2 \neq \mathsf{T}_0$. Thus the minimal polynomial for D must be t^3.

In the preceding example it is easy to verify that $\mathsf{P}_2(R)$ is a D-cyclic subspace (of itself). In this example we saw that the minimal and characteristic polynomials were of the same degree. This is no coincidence.

Theorem 5.32. Let T be a linear operator on an n-dimensional vector space V. If V is a T-cyclic subspace, i.e., if $\mathsf{V} = \mathsf{C}_x$ for some $x \in \mathsf{V}$, then the characteristic polynomial $f(t)$ and the minimal polynomial $p(t)$ for T are of the same degree. Hence $f(t) = (-1)^n p(t)$.

PROOF. If V is a T-cyclic subspace, then there exists an element $x \in \mathsf{V}$ such that

$$\beta = \{x, \mathsf{T}(x), \ldots, \mathsf{T}^{n-1}(x)\}$$

is a basis for V (Theorem 5.27). Let

$$g(t) = a_0 + a_1 t + \cdots + a_k t^k,$$

where $a_k \neq 0$ and $0 \leq k < n$. Then

$$g(\mathsf{T})(x) = a_0 x + a_1 \mathsf{T}(x) + \cdots + a_k \mathsf{T}^k(x)$$

is a linear combination of elements of β having at least one non-zero coefficient, namely a_k. Since β is linearly independent, $g(\mathsf{T})(x) \neq 0$, and hence $g(\mathsf{T}) \neq \mathsf{T}_0$. Therefore the minimal polynomial for T is of degree n, which is also the degree of the characteristic polynomial of T. ∎

Theorem 5.32 states a condition under which the degree of the minimal polynomial for an operator is as large as possible. We shall now investigate when the degree of the minimal polynomial is as small as possible. It follows from Theorem 5.31 that if the characteristic polynomial of an operator with k distinct eigenvalues factors as a product of factors of degree 1, then the minimal polynomial must be of degree at least k. The next theorem shows that the operators for which the degree of the minimal polynomial is as small as possible are precisely the diagonalizable operators.

Theorem 5.33. *Let* T *be a linear operator on a finite-dimensional vector space* V. *Then* T *is diagonalizable if and only if the minimal polynomial for* T *is of the form*

$$\mathrm{p}(\mathrm{t}) = (\mathrm{t} - \lambda_1)(\mathrm{t} - \lambda_2) \cdots (\mathrm{t} - \lambda_k),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are distinct scalars. (Note that* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are necessarily the distinct eigenvalues of* T.)

PROOF. Suppose that T is diagonalizable, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T with corresponding eigenspaces $\mathsf{E}_{\lambda_1}, \mathsf{E}_{\lambda_2}, \ldots, \mathsf{E}_{\lambda_k}$. If $\dim(\mathsf{E}_{\lambda_i}) = n_i$ $(i = 1, 2, \ldots, k)$, then the characteristic polynomial of T, $f(t)$, is

$$f(t) = (\lambda_1 - t)^{n_1}(\lambda_2 - t)^{n_2} \cdots (\lambda_k - t)^{n_k}.$$

Let $p(t)$ be the minimal polynomial for T, and define

$$q(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k).$$

We shall show that $q(\mathsf{T}) = \mathsf{T}_0$. Since $q(t)$ divides $p(t)$ by Theorem 5.31, it will follow that $p(t) = q(t)$. Recall that for any i $(1 \leq i \leq k)$, $x \in \mathsf{E}_{\lambda_i}$ if and only if $(\mathsf{T} - \lambda_i\mathsf{I})(x) = 0$. Hence for any $x \in \mathsf{E}_{\lambda_i}$

$$q(\mathsf{T})(x) = (\mathsf{T} - \lambda_1\mathsf{I})(\mathsf{T} - \lambda_2\mathsf{I}) \cdots (\mathsf{T} - \lambda_k\mathsf{I})(x) = 0. \tag{9}$$

Because T is diagonalizable, $\mathsf{V} = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}$ by Theorem 5.14. So $q(\mathsf{T})(x) = 0$ for all $x \in \mathsf{V}$ by Eq. (9), and thus $q(\mathsf{T}) = \mathsf{T}_0$.

Conversely, suppose that there exist distinct scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ (necessarily eigenvalues of T) such that the minimal polynomial, $p(t)$, for T factors as

$$p(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k).$$

If $k = 1$, then $(\mathsf{T} - \lambda_1\mathsf{I})(x) = 0$ for all $x \in \mathsf{V}$. Hence $\mathsf{T} = \lambda_1\mathsf{I}$, which is clearly diagonalizable. Suppose then that $k > 1$. Let $\{p_j(t)\colon\ j = 1, 2, \ldots, k\}$ denote the Lagrange polynomials associated with $\lambda_1, \lambda_2, \ldots, \lambda_k$ (as defined in Section 1.6). The Lagrange interpolation formula shows that

$$\sum_{j=1}^{k} p_j(t) = 1,$$

where the right side of the equality is the constant polynomial 1. Hence

$$\sum_{j=1}^{k} p_j(\mathsf{T})(x) = \mathsf{I}(x) = x \tag{10}$$

for all $x \in \mathsf{V}$. Moreover, the definition of the Lagrange polynomials shows that $(t - \lambda_j)p_j(t) = cp(t)$, where c is a scalar. Hence

$$(\mathsf{T} - \lambda_j\mathsf{I})p_j(\mathsf{T})(x) = cp(\mathsf{T})(x) = c\mathsf{T}_0(x) = 0,$$

so that $p_j(\mathsf{T})(x) \in \mathsf{E}_{\lambda_j}$. Thus by Eq. (10)

$$\mathsf{V} = \mathsf{E}_{\lambda_1} + \mathsf{E}_{\lambda_2} + \cdots + \mathsf{E}_{\lambda_k},$$

and so V is the span of its set of eigenvectors. Therefore V has a basis of eigenvectors (Exercise 11 of Section 1.6), and so T is diagonalizable by Theorem 5.4. ∎

Example 32. We shall determine all matrices $A \in \mathsf{M}_{2\times 2}(R)$ for which $A^2 - 3A + 2I = O$, where O is the 2×2 zero matrix. Define $g(t) = t^2 - 3t + 2 = (t-1)(t-2)$. Since $g(A) = O$, the minimal polynomial $p(t)$ for A divides $g(t)$. Hence the only possible candidates for $p(t)$ are $t-1$, $t-2$, or $(t-1)(t-2)$. Note that in any of these cases A is diagonalizable by Theorem 5.33. If $p(t) = t - 1$ or $p(t) = t - 2$, then $A = I$ or $A = 2I$. If $p(t) = (t-1)(t-2)$, then A is similar to

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Example 33. We shall prove that if A is a real $n \times n$ matrix such that $A^3 = A$, then A is diagonalizable. Note that if $g(t) = t^3 - t = t(t+1)(t-1)$, then $g(A) = O$, where O is the $n \times n$ zero matrix. Hence the minimal polynomial $p(t)$ for A divides $g(t)$. Since $g(t)$ has no repeated factors, neither does $p(t)$. Thus A is diagonalizable by Theorem 5.33.

Example 34. In Example 31 we saw that the minimal polynomial for the differentiation operator $\mathsf{D}\colon \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ is t^3. Hence D is not diagonalizable (Theorem 5.33).

EXERCISES

1. Label the following statements as being true or false. Assume in what follows that all vector spaces are finite-dimensional.
   (a) Every linear operator T has a polynomial $p(t)$ of largest degree for which $p(\mathsf{T}) = \mathsf{T}_0$.
   (b) Every linear operator has a unique minimal polynomial.
   (c) The characteristic polynomial of a linear operator divides the minimal polynomial for that operator.
   (d) The minimal and characteristic polynomials of any diagonalizable operator are identical.
   (e) Let T be a linear operator on an n-dimensional vector space V, $p(t)$ be the minimal polynomial for T, and $f(t)$ be the characteristic polynomial of T. If $f(t)$ factors into a product of factors of degree 1, then $f(t)$ divides $[p(t)]^n$.
   (f) The minimal polynomial for a linear operator always has the same degree as the characteristic polynomial of the operator.

(g) A linear operator is diagonalizable if its minimal polynomial factors into a product of factors of degree 1.
(h) Let T be a linear operator on V. If V is a T-cyclic subspace, then the degree of the minimal polynomial for T equals dim(V).

2. Compute the minimal polynomials for each of the following matrices.

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -14 & 5 \\ 1 & -4 & 2 \\ 1 & -6 & 4 \end{pmatrix}$

3. Compute the minimal polynomial for each of the following linear operators.
(a) T: $\mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$, where $\mathsf{T}(f) = f' + 2f$
(b) T: $\mathsf{R}^2 \longrightarrow \mathsf{R}^2$, where $\mathsf{T}(a, b) = (a + b, a - b)$
(c) T: $\mathsf{M}_{n \times n}(R) \longrightarrow \mathsf{M}_{n \times n}(R)$, where $\mathsf{T}(A) = A^t$ *Hint:* Note that $\mathsf{T}^2 = \mathsf{I}$.

4. Determine which of the matrices and operators in Exercises 2 and 3 are diagonalizable.

5. Describe all linear operators T on R^2 such that T is diagonalizable and $\mathsf{T}^3 - 2\mathsf{T}^2 + \mathsf{T} = \mathsf{T}_0$.

6. Prove Theorem 5.30 and its corollary.

7. Prove the corollary to Theorem 5.31.

8. Let T be a linear operator on a finite-dimensional vector space. Prove that if $g(t)$ is the minimal polynomial of T, then
(a) T is invertible if and only if $g(0) \neq 0$.
(b) If T is invertible and $g(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, then

$$\mathsf{T}^{-1} = -\left(\frac{1}{a_0}\mathsf{T}^{n-1} + \frac{a_{n-1}}{a_0}\mathsf{T}^{n-2} + \cdots + \frac{a_1}{a_0}\mathsf{I}\right).$$

9. Let T be a diagonalizable linear operator on a finite-dimensional vector space V. Prove that V is a T-cyclic subspace if and only if each of the eigenspaces of T is one-dimensional.

10. Let $g(t)$ be the auxiliary polynomial of a homogeneous linear differential equation with constant coefficients (as defined in Section 2.7), and let V denote the solution space of the differential equation. Show that
(a) V is a D-invariant subspace, where D: $\mathsf{C}^\infty \longrightarrow \mathsf{C}^\infty$ is the differentiation operator.
(b) The minimal polynomial for D_V (the restriction of D to V) is $g(t)$.
(c) If the degree of $g(t)$ is n, then the characteristic polynomial of D: $\mathsf{V} \longrightarrow \mathsf{V}$ is $(-1)^n g(t)$.

Hint: For (b) and (c), use Theorem 2.36.

11. Let $D\colon\ P(R) \longrightarrow P(R)$ be the differentiation operator on the space of all polynomials over R. Prove that there exists no polynomial $g(t)$ for which $g(D) = T_0$. Hence $D\colon\ P(R) \longrightarrow P(R)$ has no minimal polynomial.

12. Let V be a finite-dimensional vector space and T be a linear operator on V. Suppose that W_1 and W_2 are T-invariant subspaces of V such that $V = W_1 \oplus W_2$, and let $p_1(t)$ and $p_2(t)$ denote the minimal polynomials for T_{W_1} and T_{W_2}, the restrictions of T to W_1 and W_2, respectively. Prove or disprove that $p_1(t)p_2(t)$ is the minimal polynomial for T.

13.† Let T be a linear operator on a finite-dimensional vector space V, and let W_1 be a T-invariant subspace of V. If $x \in V$ and $x \notin W_1$, prove the following:

(a) There exists a unique monic polynomial $g_1(t)$ of least positive degree such that $g_1(T)(x) \in W_1$.

(b) If $h(t)$ is a polynomial for which $h(T)(x) \in W_1$, then $g_1(t)$ divides $h(t)$.

(c) Let W_2 be a T-invariant subspace of V such that $W_2 \subseteq W_1$. Prove that if $g_2(t)$ is the unique monic polynomial of least positive degree such that $g_2(T)(x) \in W_2$, then $g_1(t)$ divides $g_2(t)$. Deduce that $g_1(t)$ divides the minimal and characteristic polynomials of T.

Definition. *Let* $T\colon\ V \longrightarrow V$ *be a linear operator on a finite-dimensional vector space* V. *For each non-zero* x *in* V *the* T-annihilator of x *is the monic polynomial* $p_x(t)$ *of least positive degree for which* $p_x(T)(x) = 0$. *Observe that by Exercise 13(a) above (with* $W_1 = \{0\}$*) every non-zero* x *in* V *has a unique* T-*annihilator.*

14.† Let T be a linear operator on a finite-dimensional vector space V, and let x be a non-zero element of V.

(a) Show that if $q(t)$ is any polynomial such that $q(T)(x) = 0$, then $p_x(t)$, the T-annihilator of x, divides $q(t)$.

(b) Let $W = C_x$, the T-cyclic subspace of V generated by x. Prove that the minimal polynomial for T_W is $p_x(t)$, and hence the dimension of C_x equals the degree of the T-annihilator of x by Theorem 5.32.

(c) Prove that $p_x(t)$ is of degree 1 if and only if x is an eigenvector of T.

INDEX OF DEFINITIONS FOR CHAPTER 5

chapter 6

canonical forms

For a linear operator T on a finite-dimensional vector space V, we saw in Chapter 5 that it is beneficial to decompose V into a direct sum of as many proper T-invariant subspaces as possible. Exercise 7 of Section 5.4 shows that V can be decomposed into a direct sum of one-dimensional T-invariant subspaces if and only if T is diagonalizable. In this chapter we shall be concerned with obtaining a decomposition of V into proper T-invariant subspaces when T is not diagonalizable. Sections 6.1 and 6.2 are concerned with operators whose characteristic polynomials factor into a product of factors of degree 1, and Section 6.3 is concerned with linear operators whose characteristic polynomials do not factor in this manner. These decompositions will lead to simple (canonical) representations of such operators.

6.1 GENERALIZED EIGENVECTORS

In the first two sections of this chapter we shall consider linear operators on finite-dimensional vector spaces for which the characteristic polynomials factor into a product of factors of degree 1. (In particular, if V is a finite-dimensional vector space over an algebraically closed field, every linear operator on V satisfies this condition.) Such operators have at least one eigenvalue. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) are all the eigenvalues of $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$, recall from Theorem 5.4 that T is diagonalizable if and only if there is an ordered basis for V consisting of eigenvectors of T. If $\beta = \{x_1, x_2, \ldots, x_n\}$ is such a basis in which x_j is an eigenvector corresponding to the eigenvalue λ_j, then

$$[\mathsf{T}]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Although not every linear operator T on V is diagonalizable, we shall prove that for any linear operator whose characteristic polynomial factors into a product of factors of degree 1 there exists an ordered basis β for V such that

$$[\mathsf{T}]_\beta = J_1 \oplus J_2 \oplus \cdots \oplus J_k = \begin{pmatrix} J_1 & O & \cdots & O \\ O & J_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & J_k \end{pmatrix}.$$

where J_i is a square matrix of the form (λ_j) or the form

$$\begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{pmatrix}$$

for some eigenvalue λ_j of T. Such a matrix J_i will be called a *Jordan block* corresponding to λ_j, and the matrix $[\mathsf{T}]_\beta = J_1 \oplus J_2 \oplus \cdots \oplus J_k$ will be called a *Jordan canonical form* of T. We shall also say that the ordered basis β is a *Jordan canonical basis* for T. Observe that each Jordan block J_i is "almost" a diagonal matrix—in fact, $[\mathsf{T}]_\beta$ is a diagonal matrix if and only if each J_i is of the form (λ_j).

For example, the 8×8 matrix

$$J = J_1 \oplus J_2 \oplus J_3 \oplus J_4 = \left(\begin{array}{ccc|c|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

is a Jordan canonical form of a linear operator $\mathsf{T}\colon\ \mathsf{C}^8 \longrightarrow \mathsf{C}^8$; that is, there exists a basis $\beta = \{x_1, x_2, \ldots, x_8\}$ for C^8 such that $[\mathsf{T}]_\beta = J$. Notice that the characteristic polynomial for T and J is $\det(J - tI) = (t-2)^4(t-3)^2t^2$, and so the multiplicity of each eigenvalue is the number of times that eigenvalue appears on the diagonal of J. Also observe that of the vectors $x_1, x_2, \ldots, x_8$, only x_1, x_4, x_5, and x_7 (the basis vectors corresponding to the first column of each of the Jordan blocks J_1, J_2, J_3, and J_4) are eigenvectors of T.

Although it will be proved that every operator whose characteristic polynomial factors into a product of factors of degree 1 has a unique Jordan canonical form (up to the order of the Jordan blocks), it is not the case that the Jordan canonical form is completely determined by the characteristic polynomial of the transformation. For example, the characteristic polynomial of

$$J' = \left(\begin{array}{cc|cc|c|c|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

also is $(t-2)^4(t-3)^2t^2$.

Consider again the matrix J above. We have seen that x_1 and x_4 are both eigenvectors corresponding to the eigenvalue $\lambda_1 = 2$, but neither x_2 nor x_3 is an eigenvector. Hence $(\mathsf{T} - 2\mathsf{I})(x_1) = (\mathsf{T} - 2\mathsf{I})(x_4) = \mathit{0}$, whereas $(\mathsf{T} - 2\mathsf{I})(x_2) \neq \mathit{0}$ and $(\mathsf{T} - 2\mathsf{I})(x_3) \neq \mathit{0}$. But since $[\mathsf{T}]_\beta = J$, $\mathsf{T}(x_2) =$

$x_1 + 2x_2$ and $\mathsf{T}(x_3) = x_2 + 2x_3$. Thus

$$(\mathsf{T} - 2\mathsf{I})^2(x_2) = (\mathsf{T} - 2\mathsf{I})(\mathsf{T}(x_2) - 2x_2) = (\mathsf{T} - 2\mathsf{I})(x_1) = 0,$$

and, similarly,

$$(\mathsf{T} - 2\mathsf{I})^3(x_3) = (\mathsf{T} - 2\mathsf{I})^2(\mathsf{T}(x_3) - 2x_3) = (\mathsf{T} - 2\mathsf{I})^2(x_2) = 0.$$

So although $(\mathsf{T} - 2\mathsf{I})(x_2) \neq 0$ and $(\mathsf{T} - 2\mathsf{I})(x_3) \neq 0$, $(\mathsf{T} - 2\mathsf{I})^p(x_2) = (\mathsf{T} - 2\mathsf{I})^p(x_3) = 0$ if $p \geq 3$. This observation motivates the following definition:

Definition. *Let* T *be a linear operator on a finite-dimensional vector space* V. *A non-zero element* $\mathrm{x} \in \mathsf{V}$ *is called a* generalized eigenvector *of* T *if there exists a scalar* λ *such that* $(\mathsf{T} - \lambda\mathsf{I})^{\mathrm{p}}(\mathrm{x}) = 0$ *for some positive integer* p. *We shall say that* x *is a generalized eigenvector corresponding to* λ.

Observe that if x is a generalized eigenvector of T corresponding to λ, then λ is an eigenvalue of T. For if p is the smallest positive integer such that $(\mathsf{T} - \lambda\mathsf{I})^p(x) = 0$, then $y = (\mathsf{T} - \lambda\mathsf{I})^{p-1}(x)$ is an eigenvector of T corresponding to the eigenvalue λ.

It is easily seen that if β is a Jordan canonical basis for an operator T on a finite-dimensional vector space V, then β consists of generalized eigenvectors of T. Theorem 6.4 will show that a Jordan canonical basis exists for every operator on V whose characteristic polynomial factors into a product of factors of degree 1. The proof of this theorem will require some additional terminology, which we shall now introduce.

Definitions. *Let* T *be a linear operator on a vector space* V, *and let* x *be a generalized eigenvector of* T *corresponding to the eigenvalue* λ. *If* p *denotes the smallest positive integer such that* $(\mathsf{T} - \lambda\mathsf{I})^{\mathrm{p}}(\mathrm{x}) = 0$, *then the ordered set*

$$\{(\mathsf{T} - \lambda\mathsf{I})^{\mathrm{p}-1}(\mathrm{x}), (\mathsf{T} - \lambda\mathsf{I})^{\mathrm{p}-2}(\mathrm{x}), \ldots, (\mathsf{T} - \lambda\mathsf{I})(\mathrm{x}), \mathrm{x}\}$$

is called a cycle *of generalized eigenvectors of* T *corresponding to* λ. *The elements* $(\mathsf{T} - \lambda\mathsf{I})^{\mathrm{p}-1}(\mathrm{x})$ *and* x *are called the* initial vector *and the* end vector *of the cycle, respectively. We shall also say that the* length *of the cycle is* p.

Recalling the matrix J on p. 304, we see that $\beta_1 = \{x_1, x_2, x_3\}$, $\beta_2 = \{x_4\}$, $\beta_3 = \{x_5, x_6\}$, and $\beta_4 = \{x_7, x_8\}$ are cycles of generalized eigenvectors of T corresponding to the eigenvalues 2, 2, 3, and 0, respectively. Let $\mathsf{W}_i = \mathrm{span}(\beta_i)$ for $1 \leq i \leq 4$. Since $\mathsf{T}(x_1) = 2x_1$, $\mathsf{T}(x_2) = x_1 + 2x_2$, and $\mathsf{T}(x_3) = x_2 + 2x_3$, W_1 is a T-invariant subspace. Likewise W_2, W_3, and W_4 are T-invariant subspaces. It is easily seen that $[\mathsf{T}_{\mathsf{W}_i}]_{\beta_i} = J_i$ $(1 \leq i \leq 4)$.

Our first result contains several useful facts about cycles.

Theorem 6.1. *Let* T *be a linear operator on* V, *and let* γ *be a cycle of generalized eigenvectors of* T *corresponding to the eigenvalue* λ.

(a) *The initial vector of* γ *is an eigenvector of* T *corresponding to the eigenvalue* λ, *and no other member of* γ *is an eigenvector of* T.

(b) γ *is linearly independent.*

(c) *Let* β *be an ordered basis for* V. *Then* β *is a Jordan canonical basis for* V *if and only if* β *is a disjoint union of cycles of generalized eigenvectors of* T.

PROOF. We shall prove only (b); the proofs of (a) and (c) are left as exercises. The proof will be by induction on the length of the cycle γ. If γ has length 1, then $\gamma = \{x_1\}$ is linearly independent since x_1, a generalized eigenvector, is a non-zero vector. Now assume that cycles of length $k - 1$ are linearly independent for some integer $k - 1 \geq 1$. Suppose that $\gamma = \{x_1, x_2, \ldots, x_k\}$ is a cycle of generalized eigenvectors corresponding to the eigenvalue λ and that

$$\sum_{i=1}^{k} a_i x_i = 0$$

for some scalars $a_1, a_2, \ldots, a_k$. Applying $\mathsf{T} - \lambda\mathsf{I}$ to the equation above gives

$$\sum_{i=2}^{k} a_i x_{i-1} = 0.$$

But the sum in the preceding equality is a linear combination of elements from a cycle $\{x_1, x_2, \ldots, x_{k-1}\}$ of length $k - 1$. Hence $a_i = 0$ for $i = 2, 3, \ldots, k$. Thus

$$\sum_{i=1}^{k} a_i x_i = 0$$

reduces to $a_1 x_1 = 0$. But since $x_1 \neq 0$, it follows that $a_1 = 0$. So $a_1 = a_2 = \cdots = a_k = 0$, proving that γ is linearly independent. This completes the induction. ∎

Recall that if T is a diagonalizable linear operator on V, then V is the direct sum of the eigenspaces of T (Theorem 5.14). One of the principal results of this section (Theorem 6.5) will show that if T is any linear operator on V whose characteristic polynomial factors into a product of factors of degree 1, then V is the direct sum of the "generalized eigenspaces" of T (defined below). Hence as the eigenspaces of a diagonalizable operator yielded a basis for V consisting of eigenvectors, the generalized eigenspaces of an operator will yield a Jordan canonical basis for V.

Definition. *Let* λ *be an eigenvalue of a linear operator* T *on* V. *The* generalized eigenspace *of* T *corresponding to* λ, *denoted* K_λ, *is the set*

$$\mathsf{K}_\lambda = \{x \in \mathsf{V}\colon\ (\mathsf{T} - \lambda\mathsf{I}_\mathsf{V})^p(x) = 0 \text{ for some positive integer } p\}.$$

Thus K_λ *consists of the zero vector and all the generalized eigenvectors corresponding to* λ.

Our next theorem contains two simple facts about generalized eigenspaces.

Theorem 6.2. *Let* λ *be an eigenvalue of a linear operator* T *on* V. *Then* K_λ *is a subspace of* V *containing* E_λ *(the eigenspace of* T *corresponding to* λ*).*

PROOF. Clearly $0 \in \mathsf{K}_\lambda$. Suppose that $x, y \in \mathsf{K}_\lambda$; then there exist positive integers p and q such that $(\mathsf{T}-\lambda\mathsf{I})^p(x) = 0$ and $(\mathsf{T}-\lambda\mathsf{I})^q(y) = 0$. Now

$$\begin{aligned}(\mathsf{T}-\lambda\mathsf{I})^{p+q}(x+y) &= (\mathsf{T}-\lambda\mathsf{I})^{p+q}(x) + (\mathsf{T}-\lambda\mathsf{I})^{p+q}(y)\\ &= (\mathsf{T}-\lambda\mathsf{I})^{q}(\mathsf{T}-\lambda\mathsf{I})^{p}(x) + (\mathsf{T}-\lambda\mathsf{I})^{p}(\mathsf{T}-\lambda\mathsf{I})^{q}(y)\\ &= (\mathsf{T}-\lambda\mathsf{I})^{q}(0) + (\mathsf{T}-\lambda\mathsf{I})^{p}(0)\\ &= 0 + 0 = 0,\end{aligned}$$

and so $x + y \in \mathsf{K}_\lambda$. Finally, for any scalar c,

$$(\mathsf{T}-\lambda\mathsf{I})^p(cx) = c(\mathsf{T}-\lambda\mathsf{I})^p(x) = c0 = 0,$$

so that $cx \in \mathsf{K}_\lambda$. Hence K_λ is a subspace of V.

Clearly $\mathsf{E}_\lambda = \mathsf{N}(\mathsf{T}-\lambda\mathsf{I}) \subseteq \mathsf{K}_\lambda$. ∎

We shall see in Theorem 6.5 that K_λ is actually a T-invariant subspace and that V is the direct sum of the generalized eigenspaces of T. The next theorem proves part of the latter result.

Theorem 6.3. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be distinct eigenvalues of a linear operator* T *on* V. *Then*

$$\mathsf{K}_{\lambda_i} \cap \Big(\sum_{j\neq i} \mathsf{K}_{\lambda_j}\Big) = \{0\} \qquad \textit{for } i = 1, 2, \ldots, k.$$

PROOF. For convenience of notation we shall assume without loss of generality that $i = 1$. Suppose that

$$a_2x_2 + \cdots + a_kx_k = x_1 \tag{1}$$

where $x_j \in \mathsf{K}_{\lambda_j}$ for $1 \le j \le k$. Let p_j $(1 \le j \le k)$ be the smallest positive integer such that $(\mathsf{T}-\lambda_j\mathsf{I})^{p_j}(x_j) = 0$. Assume that $x_1 \neq 0$; then $(\mathsf{T}-\lambda_1\mathsf{I})^{p_1-1}(x_1)$ is an eigenvector corresponding to the eigenvalue λ_1. Applying $(\mathsf{T}-\lambda_1\mathsf{I})^{p_1-1}(\mathsf{T}-\lambda_2\mathsf{I})^{p_2}\cdots(\mathsf{T}-\lambda_k\mathsf{I})^{p_k}$ to both sides of Eq. (1) yields

$$\begin{aligned}0 &= (\mathsf{T}-\lambda_1\mathsf{I})^{p_1-1}(\mathsf{T}-\lambda_2\mathsf{I})^{p_2}\cdots(\mathsf{T}-\lambda_k\mathsf{I})^{p_k}(x_1)\\ &= (\mathsf{T}-\lambda_2\mathsf{I})^{p_2}\cdots(\mathsf{T}-\lambda_k\mathsf{I})^{p_k}((\mathsf{T}-\lambda_1\mathsf{I})^{p_1-1}(x_1))\\ &= (\lambda_1-\lambda_2)^{p_2}\cdots(\lambda_1-\lambda_k)^{p_k}(\mathsf{T}-\lambda_1\mathsf{I})^{p_1-1}(x_1)\end{aligned}$$

by Exercise 22 of Section 5.1. Hence, since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, $(\mathsf{T} - \lambda_1 \mathsf{I})^{p_1-1}(x_1) = 0$, contradicting that $(\mathsf{T} - \lambda_1 \mathsf{I})^{p_1-1}(x_1)$ is an eigenvector. We conclude that $x_1 = 0$. ∎

Corollary. *No vector can be a generalized eigenvector corresponding to different eigenvalues of the same operator.*

We are now prepared to prove the existence of a Jordan canonical form for every linear operator on a finite-dimensional vector space whose characteristic polynomial factors into a product of factors of degree 1.

Lemma. *Let* T *be a linear operator on a finite-dimensional vector space* V. *Let* S_i $(1 \leq i \leq k)$ *be a cycle of generalized eigenvectors of* T *corresponding to the eigenvalue* λ, *and let* p_i *and* y_i *denote the length and the initial vector of* S_i, *respectively. If* $\{y_1, y_2, \ldots, y_k\}$ *is a linearly independent set containing* k *elements, then*

$$\bigcup_{i=1}^{k} S_i$$

is a linearly independent set containing

$$\sum_{i=1}^{k} p_i$$

elements.

PROOF. Suppose without loss of generality that $p_1 \geq p_2 \geq \cdots \geq p_k$. The proof will be by induction on p_1. If $p_1 = 1$, then $p_1 = \cdots = p_k = 1$. Hence each cycle S_i contains only a single element, so that

$$\bigcup_{i=1}^{k} S_i = \{y_1, y_2, \ldots, y_k\}$$

is a linearly independent set containing

$$\sum_{i=1}^{k} p_i = k$$

elements by hypothesis.

Now assume that the theorem is true whenever $p_1 < n$, and let S_i $(1 \leq i \leq k)$ be a cycle of generalized eigenvectors of T corresponding to the eigenvalue λ with length p_i and initial vector y_i. Suppose that $n = p_1 \geq p_2 \geq \cdots \geq p_k$, and let r $(1 < r \leq k)$ denote the largest subscript such that $p_r > 1$. Let

$$S = \bigcup_{i=1}^{k} S_i,$$

and let S_i' $(1 \leq i \leq r)$ denote the cycle obtained by deleting the end vector x_i from S_i. Then S_i' is a cycle of generalized eigenvectors corresponding to the eigenvalue λ with length $p_i - 1$ and initial vector y_i. Since

$\{y_1, \ldots, y_r\}$ is linearly independent, it follows from the induction hypothesis that

$$S' = \bigcup_{i=1}^{r} S_i'$$

is a linearly independent set containing

$$\sum_{i=1}^{r} (p_i - 1)$$

elements. Clearly

$$\bigcup_{i=1}^{k} S_i = S' \cup \{x_1, \ldots, x_k\}$$

is a disjoint union; thus

$$\bigcup_{i=1}^{k} S_i$$

contains

$$\sum_{i=1}^{r} (p_i - 1) + k = \sum_{i=1}^{k} (p_i - 1) + k = \sum_{i=1}^{k} p_i$$

elements.

We need only show that

$$S = \bigcup_{i=1}^{k} S_i$$

is a linearly independent set. Suppose that for some scalars a_z

$$\sum_{z \in S} a_z z = 0. \tag{2}$$

Since, by Theorem 6.1(a), y_i is an eigenvector of T corresponding to the eigenvalue λ, $(\mathsf{T} - \lambda\mathsf{I})(y_i) = 0$ for $1 \leq i \leq k$. Hence by applying $\mathsf{T} - \lambda\mathsf{I}$ to both sides of Eq. (2), we obtain

$$0 = \sum_{z \in S} a_z(\mathsf{T} - \lambda\mathsf{I})(z) = \sum_{z \in Z} a_z(\mathsf{T} - \lambda\mathsf{I})(z), \tag{3}$$

where $Z = \{v \in S\colon v \neq y_i \text{ for } 1 \leq i \leq k\}$. But the final sum in Eq. (3) is a linear combination of elements of S'; thus since S' is linearly independent, it follows that $a_z = 0$ if $z \in Z$. So Eq. (2) reduces to a linear combination of $\{y_1, y_2, \ldots, y_k\}$, which is linearly independent by hypothesis. Therefore all the coefficients a_z in Eq. (2) equal zero, proving that S is linearly independent. ∎

Example 1. Let $\mathsf{T}\colon\ \mathsf{C}^{11} \to \mathsf{C}^{11}$ be a linear operator whose characteristic polynomial factors into a product of factors of degree 1. This example and Exercise 8 illustrate the manner in which a Jordan canonical basis γ for the restriction of T to $\mathsf{R}(\mathsf{T})$ is extended to a Jordan canonical basis β for T in

Case 1 of Theorem 6.4. Suppose that $\gamma = \{y_1, w_1, y_2, v_1, v_2, v_3\}$ and that

$$[\mathsf{T}_1]_\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

where T_1 denotes the restriction of T to $\mathsf{R}(\mathsf{T})$. Thus $\{y_1, w_1\}$, $\{y_2\}$, $\{v_1, v_2\}$, and $\{v_3\}$, the cycles that compose γ, correspond to the eigenvalues 0, 0, 2, and 3, respectively. In the notation of Theorem 6.4 γ_0, the union of the cycles corresponding to zero, equals $\{y_1, w_1\} \cup \{y_2\}$. So $Y = \{y_1, y_2\}$ (the set containing the eigenvectors in γ corresponding to zero) is a linearly independent subset of $\mathsf{N}(\mathsf{T})$ that can be extended to a basis $Y \cup Z = \{y_1, y_2, z_1, z_2, z_3\}$ for $\mathsf{N}(\mathsf{T})$. Finally, choose x_1 and x_2 so that $\mathsf{T}(x_1) = w_1$ and $\mathsf{T}(x_2) = y_2$. Then $\{y_1, w_1, x_1\}$, $\{y_2, x_2\}$, $\{z_1\}$, $\{z_2\}$, and $\{z_3\}$ are cycles of generalized eigenvectors of T corresponding to the eigenvalue zero and $\beta = \{y_1, w_1, x_1, y_2, x_2, v_1, v_2, v_3, z_1, z_2, z_3\}$ is an ordered basis for C^{11}. Observe that

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 6.4. *Let* T *be a linear operator on an* n*-dimensional vector space* V *such that the characteristic polynomial of* T *factors into a product of factors of degree* 1. *Then there exists a Jordan canonical basis for* T*; i.e., there exists an ordered basis* β *for* V *that is a disjoint union of cycles of generalized eigenvectors of* T.

PROOF. The proof will be by induction on n. Clearly the result is true for $n = 1$ since every 1×1 matrix is a Jordan canonical form. Assume that the conclusion is true for vector spaces of dimension less than n, and suppose that $\dim(\mathsf{V}) = n$. We shall consider two cases.

CASE 1. $\operatorname{rank}(\mathsf{T}) < n$. Since $\mathsf{R}(\mathsf{T})$ is a T-invariant subspace of V, we may define $\mathsf{T}_1\colon\ \mathsf{R}(\mathsf{T}) \longrightarrow \mathsf{R}(\mathsf{T})$ to be the restriction of T to R(T). The assumption of Case 1 allows us to apply the induction hypothesis to T_1 to conclude that there exists a Jordan canonical basis γ for T_1 containing r elements ($r < n$). Hence by Theorem 6.1(c) γ is a disjoint union of cycles of generalized eigenvectors of T_1 (and hence of T). Let $S_1, S_2, \ldots, S_k$ denote all those cycles in γ that correspond to the eigenvalue zero, and let y_i and w_i denote the initial and end vectors of S_i, respectively. Since $w_i \in \gamma \subseteq \mathsf{R}(\mathsf{T})$, there exists $x_i \in \mathsf{V}$ such that $\mathsf{T}(x_i) = w_i$. Define $Y = \{y_1, y_2, \ldots, y_k\}$, $X = \{x_1, x_2, \ldots, x_k\}$, and

$$\gamma_0 = \bigcup_{i=1}^{k} S_i.$$

Assume that γ_0 contains p elements ($p \le r$). Recall from Theorem 6.1(a) that each y_i is an eigenvector corresponding to the eigenvalue zero; hence $Y \subseteq \mathsf{N}(\mathsf{T})$. Thus, since Y is linearly independent (it is a subset of γ), it may be extended to a basis $Y \cup Z$ for N(T). Observe that Z must contain $n - r - k$ elements since $\operatorname{nullity}(\mathsf{T}) = n - \operatorname{rank}(\mathsf{T}) = n - r$. Let $S_i' = S_i \cup \{x_i\}$; then S_i' is a cycle of generalized eigenvectors of T corresponding to the eigenvalue zero and having initial vector y_i. Moreover, if $z \in Z$, then $\{z\}$ is a cycle corresponding to the eigenvalue zero having length 1. Hence the lemma implies that

$$\left(\bigcup_{i=1}^{k} S_i'\right) \cup Z = (\gamma_0 \cup X) \cup Z$$

is a linearly independent set containing $p + k + (n - r - k) = n - (r - p)$ elements since $Y \cup Z$ is a linearly independent set of initial vectors for cycles corresponding to the eigenvalue zero.

We shall prove that $\beta = \gamma \cup X \cup Z$ is the desired basis. First, observe that if $\gamma_0 = \gamma$ (so that $p = r$), then $\beta = \gamma_0 \cup X \cup Z$ is a linearly independent set containing $n - (r - p) = n$ elements. Thus β is a basis for V. Otherwise, if $\gamma_0 \neq \gamma$, then $\gamma_1 = \{v \in \gamma\colon\ v \notin \gamma_0\}$ is a non-empty union of disjoint cycles of generalized eigenvectors corresponding to non-zero eigenvalues $\lambda_2, \ldots, \lambda_m$ of T. Suppose that

$$0 = \sum_{v \in \beta} a_v v = \sum_{v \in \gamma_0 \cup X \cup Z} a_v v + \sum_{v \in \gamma_1} a_v v.$$

Then

$$\sum_{v \in \gamma_0 \cup X \cup Z} (-a_v) v = \sum_{v \in \gamma_1} a_v v.$$

But the left side is an element of K_{λ_1}, where $\lambda_1 = 0$, and the right side is an element of $\mathsf{K}_{\lambda_2} + \cdots + \mathsf{K}_{\lambda_m}$. Hence by Theorem 6.3 both sides of the previous equality equal 0. Thus, since both $\gamma_0 \cup X \cup Z$ and γ_1 are linearly independent sets, it follows that $a_v = 0$ for each $v \in \beta$. Therefore β is linearly independent, and since $\beta = \gamma \cup X \cup Z$ contains $r + k + (n - r - k) = n$ elements, β is a basis for V. But β is clearly a disjoint union of cycles of generalized eigenvectors of T, so that β is a Jordan canonical basis for T by Theorem 6.1.

CASE 2. $\operatorname{rank}(\mathsf{T}) = n$. Since the characteristic polynomial of T factors as a product of factors of degree 1, T has an eigenvalue λ. Apply Case 1 to the non-invertible operator $\mathsf{T} - \lambda\mathsf{I}$ on V to obtain an ordered basis β for V such that $[\mathsf{T} - \lambda\mathsf{I}]_\beta = J$ is a Jordan canonical form for $\mathsf{T} - \lambda\mathsf{I}$. But then $[\mathsf{T}]_\beta = J + \lambda I_n$ is a Jordan canonical form for T. ∎

Having established the existence of a Jordan canonical form, we can now derive several important properties of generalized eigenspaces.

Theorem 6.5. *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T factors into a product of factors of degree 1. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T and that the multiplicity of λ_i is m_i $(1 \le i \le k)$, and let β be a Jordan canonical basis for T. Define $\beta_i = \beta \cap \mathsf{K}_{\lambda_i}$ $(1 \le i \le k)$. Then*

(a) $\mathsf{V} = \mathsf{K}_{\lambda_1} \oplus \mathsf{K}_{\lambda_2} \oplus \cdots \oplus \mathsf{K}_{\lambda_k}$.

(b) *β_i is a basis for K_{λ_i}. Conversely, for each i if γ_i is a union of cycles of generalized eigenvectors corresponding to λ_i that forms a basis for K_{λ_i}, then*

$$\bigcup_{i=1}^{k} \gamma_i$$

is a Jordan canonical basis for T.

(c) *K_{λ_i} $(1 \le i \le k)$ is a T-invariant subspace of V.*

(d) *For each i $(1 \le i \le k)$, $\dim(\mathsf{K}_{\lambda_i}) = m_i$.*

(e) *For each i $(1 \le i \le k)$, $\mathsf{K}_{\lambda_i} = \mathsf{N}((\mathsf{T} - \lambda_i\mathsf{I})^{m_i})$.*

(f) *T is diagonalizable if and only if $\mathsf{E}_{\lambda_i} = \mathsf{K}_{\lambda_i}$ for each i $(1 \le i \le k)$.*

PROOF.

(a) Clearly $\operatorname{span}(\beta_i) \subseteq \mathsf{K}_{\lambda_i}$. But since

$$\operatorname{span}\left(\bigcup_{i=1}^{k} \beta_i\right) = \operatorname{span}(\beta) = \mathsf{V},$$

it follows that $\mathsf{K}_{\lambda_1} + \mathsf{K}_{\lambda_2} + \cdots + \mathsf{K}_{\lambda_k} = \mathsf{V}$. Part (a) now follows from Theorem 6.3.

(b) Define $\mathsf{W}_i = \operatorname{span}(\beta_i)$ for $1 \le i \le k$. Then $\mathsf{W}_i \subseteq \mathsf{K}_{\lambda_i}$, and hence $\dim(\mathsf{W}_i) \le \dim(\mathsf{K}_{\lambda_i})$. But since β is the disjoint union of $\beta_1, \beta_2, \ldots, \beta_k$ and $\operatorname{span}(\beta) = \mathsf{V}$, it follows that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Thus by part (a) we have

$$\dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{W}_i) \leq \sum_{i=1}^{k} \dim(\mathsf{K}_{\lambda_i}) = \dim(\mathsf{V}).$$

Therefore $\dim(\mathsf{W}_i) = \dim(\mathsf{K}_{\lambda_i})$ for $1 \leq i \leq k$. Because $\mathsf{W}_i \subseteq \mathsf{K}_{\lambda_i}$ and $\dim(\mathsf{W}_i) = \dim(\mathsf{K}_{\lambda_i})$, it follows that $\text{span}(\beta_i) = \mathsf{W}_i = \mathsf{K}_{\lambda_i}$. Since β_i is linearly independent (it is a subset of β), β_i is a basis for K_{λ_i}.

The converse follows from part (a) and Theorems 5.13 and 6.1(c).

(c) Recall that K_{λ_i} is a subspace of V (Theorem 6.2). Now β_i is a basis for K_{λ_i} consisting of cycles of generalized eigenvectors corresponding to λ_i. But the image under T of any vector in a cycle is obviously a linear combination of vectors in that cycle and hence is an element of K_{λ_i}. Thus $\mathsf{T}(\beta_i) \subseteq \mathsf{K}_{\lambda_i}$, proving that K_{λ_i} is T-invariant.

(d) Define T_i $(1 \leq i \leq k)$ to be the restriction of T to K_{λ_i}. Then by part (b) $A_i = [\mathsf{T}_i]_{\beta_i}$ is a Jordan canonical form for T_i and $[\mathsf{T}]_\beta = A_1 \oplus A_2 \oplus \cdots \oplus A_k$. If $n_i = \dim(\mathsf{K}_{\lambda_i})$, then the characteristic polynomial of T_i is $\det(A_i - tI_{n_i}) = (\lambda_i - t)^{n_i}$ since $A_i - tI_{n_i}$ is an upper triangular matrix having $\lambda_i - t$ in each diagonal position. If $f(t)$ is the characteristic polynomial of T, then

$$\begin{aligned} f(t) &= \det(A_1 - tI_{n_1}) \cdot \det(A_2 - tI_{n_2}) \cdot \cdots \cdot \det(A_k - tI_{n_k}) \\ &= (\lambda_1 - t)^{n_1}(\lambda_2 - t)^{n_2} \cdots (\lambda_k - t)^{n_k}. \end{aligned}$$

So the multiplicity of λ_i is n_i; i.e., $m_i = n_i = \dim(\mathsf{K}_{\lambda_i})$.

(e) Clearly $\mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^{m_i}) \subseteq \mathsf{K}_{\lambda_i}$. Suppose that $x \in \mathsf{K}_{\lambda_i}$. Then the cycle S with end vector x is a linearly independent subset of K_{λ_i} by Theorem 6.1. Since $\dim(\mathsf{K}_{\lambda_i}) = m_i$, it follows that the length of S cannot exceed m_i; that is, $(\mathsf{T} - \lambda_i \mathsf{I})^p(x) = 0$ for some positive integer $p \leq m_i$. Hence $x \in \mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^{m_i})$, proving that $\mathsf{K}_{\lambda_i} \subseteq \mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^{m_i})$.

(f) If $\mathsf{E}_{\lambda_i} = \mathsf{K}_{\lambda_i}$ for $1 \leq i \leq k$, then

$$\mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k} = \mathsf{K}_{\lambda_1} \oplus \mathsf{K}_{\lambda_2} \oplus \cdots \oplus \mathsf{K}_{\lambda_k} = \mathsf{V}$$

by part (a). So T is diagonalizable by Theorem 5.14.

Conversely, if T is diagonalizable, then $\dim(\mathsf{E}_{\lambda_i}) = m_i$ by Theorem 5.14. But since E_{λ_i} is a subspace of K_{λ_i} and $\dim(\mathsf{K}_{\lambda_i}) = m_i$ by part (d), it follows that $\mathsf{E}_{\lambda_i} = \mathsf{K}_{\lambda_i}$ for $1 \leq i \leq k$. ∎

Example 2. Let $\mathsf{T}\colon\ \mathsf{C}^3 \longrightarrow \mathsf{C}^3$ be defined by $\mathsf{T} = \mathsf{L}_A$, where

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}.$$

We shall find a basis for each eigenspace and each generalized eigenspace of T.

The characteristic polynomial of T is

$$f(t) = \det(A - tI) = -(t-3)(t-2)^2.$$

Hence $\lambda_1 = 3$ and $\lambda_2 = 2$ are the eigenvalues of T having multiplicities 1 and 2, respectively. By Theorem 6.5, K_{λ_1} has dimension 1, K_{λ_2} has dimension 2, $\mathsf{K}_{\lambda_1} = \mathsf{N}(\mathsf{T} - 3\mathsf{I})$, and $\mathsf{K}_{\lambda_2} = \mathsf{N}((\mathsf{T} - 2\mathsf{I})^2)$. Now $\mathsf{E}_{\lambda_1} = \mathsf{N}(\mathsf{T} - 3\mathsf{I})$ and $\mathsf{E}_{\lambda_2} = \mathsf{N}(\mathsf{T} - 2\mathsf{I})$. Hence $\mathsf{E}_{\lambda_1} = \mathsf{K}_{\lambda_1}$. Since

$$A - 3I = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathsf{E}_{\lambda_1} = \mathsf{K}_{\lambda_1}$$

if and only if

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or, equivalently, if and only if

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is a solution to the system

$$\begin{cases} \quad\quad\;\; b - 2c = 0 \\ -a - 3b + 5c = 0 \\ -a - \;\; b + \;\; c = 0. \end{cases}$$

Because the solution set to the system above has

$$\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

as a basis,

$$\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

is a basis for $\mathsf{E}_{\lambda_1} = \mathsf{K}_{\lambda_1}$.

Similarly, since

$$A - 2I = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathsf{E}_{\lambda_2}$$

if and only if

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, equivalently, if and only if

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is a solution to the system

$$\begin{cases} a + b - 2c = 0 \\ -a - 2b + 5c = 0 \\ -a - b + 2c = 0. \end{cases}$$

A basis for the solution set of this system, and hence for E_{λ_2}, is

$$\left\{\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}\right\}.$$

Since

$$(A - 2I)^2 = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ -2 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathsf{K}_{\lambda_2}$$

if and only if

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is a solution of the system

$$\begin{cases} 2a + b - c = 0 \\ -4a - 2b + 2c = 0 \\ -2a - b + c = 0. \end{cases}$$

A basis for the solution set of this system, and hence a basis for K_{λ_2}, is

$$\left\{\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right\}.$$

Observe that this basis is a cycle of generalized eigenvectors corresponding to λ_2. Hence by Theorem 6.5(b)

$$\beta = \left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right\}$$

is a basis for C^3, and

$$[\mathsf{T}]_\beta = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is a Jordan canonical form for T.

Example 3. Let $T: P_2(C) \longrightarrow P_2(C)$ be defined by $T(f) = -f - f'$. Again we shall find a basis for each eigenspace and generalized eigenspace of T. If $\beta = \{1, x, x^2\}$, then β is an ordered basis for $P_2(C)$ and

$$A = [T]_\beta = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence the characteristic polynomial of T is $f(t) = \det(A - tI) = -(t+1)^3$. Thus $\lambda = -1$ is the only eigenvalue of T, and hence $K_\lambda = P_2(C)$ by Theorem 6.5. So any basis for $P_2(C)$, for example, β, is a basis for K_λ.

Now $E_\lambda = N(T - \lambda I) = N(T + I)$. Thus if $f(x) = a + bx + cx^2 \in P_2(C)$, then $f(x) \in E_\lambda$ if and only if

$$\begin{aligned} 0 &= T(f(x)) + f(x) \\ &= [-(a + bx + cx^2) - (b + 2cx)] + (a + bx + cx^2) \\ &= -(b + 2cx). \end{aligned}$$

But $-(b + 2cx) = 0$ if and only if $b = c = 0$. Hence $f(x) \in E_\lambda$ if and only if $f(x) = a$ for some $a \in C$, and so $\{1\}$ is a basis for E_λ.

Since $K_\lambda = P_2(C)$, there must be cycles of generalized eigenvectors corresponding to λ that form a Jordan canonical basis for T. In fact, it follows from Exercise 4 that a single cycle (of length 3), rather than a union of two cycles (one of length 2 and the other of length 1) or a union of three cycles (all of length 1), will form a basis for $P_2(C)$. Such a cycle is $\gamma = \{2, -2x, x^2\}$, and

$$[T]_\gamma = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

is a Jordan canonical form for T. We shall see how to find such a Jordan canonical basis in the next section.

EXERCISES

1. Label the following statements as being true or false.
   (a) Eigenvectors of a linear operator T are also generalized eigenvectors of T.
   (b) It is possible for a generalized eigenvector of a linear operator T to be associated with a scalar that is not an eigenvalue of T.
   (c) Any linear operator on a finite-dimensional vector space has a Jordan canonical form.

(d) Cycles of generalized eigenvectors are linearly independent.
(e) There exists exactly one cycle of generalized eigenvectors corresponding to each eigenvalue of a linear operator on a finite-dimensional vector space.
(f) Let T be a linear operator on a finite-dimensional vector space whose characteristic polynomial factors into a product of factors of degree 1, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. If, for each i, β_i is any basis for K_{λ_i}, then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a Jordan canonical basis for T.
(g) For any Jordan block J, L_J has Jordan canonical form J.
(h) Let T be a linear operator on an n-dimensional vector space whose characteristic polynomial factors as a product of factors of degree 1. For any eigenvalue λ of T, $\mathsf{K}_\lambda = \mathsf{N}((\mathsf{T} - \lambda\mathsf{I})^n)$.

2. For each of the following linear operators T, find a basis for each eigenspace and each generalized eigenspace.
(a) $\mathsf{T} = \mathsf{L}_A$, where

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

(b) $\mathsf{T} = \mathsf{L}_A$, where

$$A = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}$$

(c) $\mathsf{T}\colon\ \mathsf{P}_2(C) \longrightarrow \mathsf{P}_2(C)$ defined by $\mathsf{T}(f) = 2f - f'$

3.† Let S be a cycle of generalized eigenvectors of a linear operator T on V that corresponds to the eigenvalue λ. Prove that span(S) is a T-invariant subspace of V.

4. Let β be a Jordan canonical basis for a linear operator T on V, and let $[\mathsf{T}]_\beta = J_1 \oplus J_2 \oplus \cdots \oplus J_k$, where each J_i $(1 \le i \le k)$ is a Jordan block. Let λ be an eigenvalue of T, and let m denote the number of Jordan blocks having λ in each diagonal position. Prove that $1 \le m \le \dim(\mathsf{E}_\lambda)$. We shall see later that $m = \dim(\mathsf{E}_\lambda)$.

5. Let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be a linear transformation. Prove the following:
(a) $\mathsf{N}(\mathsf{T}) = \mathsf{N}(-\mathsf{T})$.
(b) $\mathsf{N}(\mathsf{T}^k) = \mathsf{N}((-\mathsf{T})^k)$ for any positive integer k.
(c) If W = V (so that T is a linear operator on V) and λ is an eigenvalue of T, then for any positive integer k

$$\mathsf{N}((\mathsf{T} - \lambda\mathsf{I}_\mathsf{V})^k) = \mathsf{N}((\lambda\mathsf{I}_\mathsf{V} - \mathsf{T})^k).$$

6. Let U be a linear operator on a finite-dimensional vector space V. Prove the following:

(a) $N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$.
(b) If $\operatorname{rank}(U^m) = \operatorname{rank}(U^{m+1})$ for some positive integer m, then $\operatorname{rank}(U^m) = \operatorname{rank}(U^k)$ for any positive integer $k \geq m$.
(c) If $\operatorname{rank}(U^m) = \operatorname{rank}(U^{m+1})$ for some positive integer m, then $N(U^m) = N(U^k)$ for any positive integer $k \geq m$.
(d) Let T be a linear operator, and let λ be an eigenvalue of T. Prove that if $\operatorname{rank}((T - \lambda I)^m) = \operatorname{rank}((T - \lambda I)^{m+1})$ for some integer m, then $K_\lambda = N((T - \lambda I)^m)$.
(e) *Second Test for Diagonalizability.* Let T be a linear operator whose characteristic polynomial factors into a product of factors of degree 1. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T. Then T is diagonalizable if and only if $\operatorname{rank}(T - \lambda_i I) = \operatorname{rank}((T - \lambda_i I)^2)$ for $1 \leq i \leq k$.
(f) Use part (e) to obtain a simpler proof of Exercise 10(d) of Section 5.4: If T is a diagonalizable linear operator on a finite-dimensional vector space V and W is any T-invariant subspace of V, then T_W is diagonalizable.

7. Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial $f(t)$ of T factors into a product of factors of degree 1. Prove that $f(T) = T_0$; i.e., prove that T satisfies its characteristic polynomial. (This is a special case of the Cayley-Hamilton theorem.) *Hint:* Show that if β is a Jordan canonical basis for T, then $f(T)(x) = 0$ for each $x \in \beta$.

8. This exercise is intended to illustrate the proof of Case 1 of Theorem 6.4 for a particular linear transformation T: $C^{11} \longrightarrow C^{11}$. (See also Example 1.)

Let T: $C^{11} \longrightarrow C^{11}$ be defined by

$$T(u) = (a_1 + 2a_2 - a_3, -a_1 - 5a_2 + 3a_3, -2a_1 - 7a_2 + 4a_3, \\ 6a_4 - 9a_5, 4a_4 - 6a_5, a_6 + a_7, -a_6 + 3a_7, 3a_8, 0, 0, 0),$$

where $u = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11})$.
(a) Let $\gamma = \{y_1, w_1, y_2, v_1, v_2, v_3\}$, where $y_1 = e_1 - 2e_2 - 3e_3$, $w_1 = 2e_1 - 3e_2 - 5e_3$, $y_2 = 3e_4 + 2e_5$, $v_1 = e_6 + e_7$, $v_2 = 3e_6 + 4e_7$, and $v_3 = e_8$. Prove that γ is an ordered basis for R(T). *Hint:* For u as above,

$$T(u) = (-a_1 + 4a_2 - 3a_3)y_1 + (a_1 - a_2 + a_3)w_1 + (2a_4 - 3a_5)y_2 \\ + (7a_6 - 5a_7)v_1 + (-2a_6 + 2a_7)v_2 + (3a_8)v_3.$$

(b) Deduce that r, the rank of T, equals 6 and that the nullity of T is 5.
(c) Let T_1 denote the restriction of T to R(T). Prove that γ is a Jordan canonical basis for T_1.

(d) Show that $S_1 = \{y_1, w_1\}$, $S_2 = \{y_2\}$, $S_3 = \{v_1, v_2\}$, and $S_4 = \{v_3\}$ are cycles of generalized eigenvectors of T_1 corresponding to the eigenvalues 0, 0, 2, and 3, respectively. (Hence, in the notation of the proof of Theorem 6.4, $k = 2$, $\gamma_0 = \{y_1, w_1, y_2\}$, and $p = 3$.)

(e) Let $x_1 = -e_1 + 5e_2 + 7e_3$ and $x_2 = 8e_4 + 5e_5$. Prove that $\mathsf{T}(x_i) = w_i$ for $i = 1, 2$. Let $X = \{x_1, x_2\}$.

(f) Observe that in the notation of the proof of Theorem 6.4 the vectors y_2 and w_2 are equal, and let $Y = \{y_1, y_2\}$. Define $z_1 = e_9$, $z_2 = e_{10}$, and $z_3 = e_{11}$. Show that $Z = \{z_1, z_2, z_3\}$ is a set (containing $n - r - k$ elements) such that $Y \cup Z$ is a basis for $\mathsf{N}(\mathsf{T})$.

(g) Define $S_1' = \{y_1, w_1, x_1\}$ and $S_2' = \{w_2, x_2\}$. Prove that S_1' and S_2' are cycles of generalized eigenvectors of T corresponding to the eigenvalue zero. Thus by the lemma to Theorem 6.4

$$\left(\bigcup_{i=1}^{k} S_i'\right) \cup Z = (\gamma_0 \cup X) \cup Z$$

is a linearly independent set containing $n - (r - p) = 8$ elements.

(h) The proof of Theorem 6.4 shows that $\beta = \gamma \cup X \cup Z$ is a linearly independent set. Assuming this fact, deduce that β is a basis for C^{11}.

(i) Finally, show that β is a Jordan canonical basis for T by computing $[\mathsf{T}]_\beta$.

9. Prove parts (a) and (c) of Theorem 6.1.

10. Let T be a linear operator on a finite-dimensional vector space. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T and that the largest Jordan block corresponding to λ_j in a Jordan canonical form of T is of size $p_j \times p_j$. Prove that the minimal polynomial of T is

$$(t - \lambda_1)^{p_1}(t - \lambda_2)^{p_2} \cdots (t - \lambda_k)^{p_k}.$$

6.2 JORDAN CANONICAL FORM

For the purposes of this section we shall fix a linear operator T on an n-dimensional vector space V such that the characteristic polynomial of T factors into a product of factors of degree 1. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the distinct eigenvalues of T.

Theorem 6.4 assures the existence of a Jordan canonical basis β for T; that is, $J = [\mathsf{T}]_\beta$ is a Jordan canonical form for T. Let us briefly summarize results from Section 6.1. For each $i = 1, 2, \ldots, k$ there exists a basis β_i for K_{λ_i} such that β_i is a disjoint union of cycles corresponding to the eigenvalue λ_i and

$$\beta = \bigcup_{i=1}^{k} \beta_i.$$

Let T_i denote the restriction of T to K_{λ_i}. Then $A_i = [T_i]_{\beta_i}$ is a Jordan canonical form for T_i, and

$$J = [T]_\beta = A_1 \oplus A_2 \oplus \cdots \oplus A_k$$

is a Jordan canonical form for T.

In this section we shall compute the matrices A_i and the bases β_i, thereby computing J and β also. While developing a method for finding J, it will become evident that in some sense the matrices A_i are unique. What we mean by "in some sense" will become clear as we proceed.

To aid in formulating a uniqueness theorem for J, we shall adopt the following convention: The basis β_i for K_{λ_i} will henceforth be ordered in such a way that the cycles appear in order of decreasing length. That is, if β_i is a disjoint union of cycles $S_1, S_2, \ldots, S_{k_i}$, and if the length of the cycle S_j is p_j, we shall index the cycles so that $p_1 \geq p_2 \geq \cdots \geq p_{k_i}$. This ordering of the cycles determines an ordering for β_i and hence determines the matrix A_i. It is in this sense that A_i is unique. It then follows that the Jordan canonical form for T is unique up to an ordering of the eigenvalues of T. As we shall also see, there is no comparable uniqueness theorem for the bases β_i or for β. Specifically, what will be shown is that the number, k_i, of cycles that form β_i and the length, p_j $(j = 1, 2, \ldots, k_i)$, of each cycle is completely determined by T.

Example 4. To illustrate that the matrix A_i is entirely determined by the numbers $k_i, p_1, p_2, \ldots, p_{k_i}$, suppose that $k_i = 4$ (i.e., there are four cycles), $p_1 = 3, p_2 = 3, p_3 = 2$, and $p_4 = 1$. Then

$$A_i = \left(\begin{array}{ccc|ccc|cc|c}
\lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_i & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_i & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i
\end{array}\right);$$

that is, A_i is a direct sum of the form $J_1 \oplus J_2 \oplus J_3 \oplus J_4$.

As an aid in computing A_i and β_i, we shall now introduce an array of dots, called a *dot diagram*, to help us visualize the form of the matrix A_i and the basis β_i. Suppose as above that β_i is a disjoint union of cycles $S_1, S_2, \ldots, S_{k_i}$ with lengths $p_1 \geq p_2 \geq \cdots \geq p_{k_i}$, respectively. The dot diagram contains one dot for each member of β_i and is constructed according to the following rules.

1. The array consists of k_i columns (one column for each cycle).
2. Counting from left to right, the jth column consists of p_j dots that correspond to the members of S_j in the following manner: If x_j is the end vector of S_j, then the top dot corresponds to $(\mathsf{T} - \lambda_i \mathsf{I})^{p_j - 1}(x_j)$; the second dot, to $(\mathsf{T} - \lambda_i \mathsf{I})^{p_j - 2}(x_j)$; etc. Hence the final (lowermost) dot of the column corresponds to x_j.

Thus the dot diagram associated with β_i may be depicted as

$$\begin{array}{llll}
\bullet(\mathsf{T} - \lambda_i\mathsf{I})^{p_1-1}(x_1) & \bullet(\mathsf{T} - \lambda_i\mathsf{I})^{p_2-1}(x_2) & \cdots & \bullet(\mathsf{T} - \lambda_i\mathsf{I})^{p_{k_i}-1}(x_{k_i}) \\
\bullet(\mathsf{T} - \lambda_i\mathsf{I})^{p_1-2}(x_1) & \bullet(\mathsf{T} - \lambda_i\mathsf{I})^{p_2-2}(x_2) & & \bullet(\mathsf{T} - \lambda_i\mathsf{I})^{p_{k_i}-2}(x_{k_i}) \\
\quad\vdots & \quad\vdots & & \quad\vdots \\
 & & & \bullet(\mathsf{T} - \lambda_i\mathsf{I})(x_{k_i}) \\
\quad\vdots & \bullet(\mathsf{T} - \lambda_i\mathsf{I})(x_2) & & \bullet x_{k_i} \\
\bullet(\mathsf{T} - \lambda_i\mathsf{I})(x_1) & \bullet x_2 & & \\
\bullet x_1 & & &
\end{array}$$

In the diagram above we have labeled each dot with the member of β_i to which it corresponds.

Notice that the dot diagram for β_i has k_i columns (one for each cycle) and p_1 rows. Observe also that since $p_1 \geq p_2 \geq \cdots \geq p_{k_i}$, the columns of the dot diagram become shorter (or at least not longer) as we move from left to right.

You might also observe that if r_j denotes the number of dots in the jth row of the array, then $r_1 \geq r_2 \geq \cdots \geq r_{p_1}$. Since the proof of this fact is combinatorial in nature, it will be left to the exercises.

Returning to Example 4, where $k_i = 4$, $p_1 = 3$, $p_2 = 3$, $p_3 = 2$, and $p_4 = 1$, we see that the dot diagram for β_i is

$$\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & &
\end{array}$$

We shall devise a method for computing the dot diagram for β_i in terms of T alone. Hence the dot diagram is uniquely determined by T. It is important to understand, however, that when we say that the dot diagram is uniquely determined by T, we are making no assertions about the uniqueness of β_i. Indeed, as we shall see, the basis β_i is not unique. By the uniqueness of the dot diagram we mean that if β_i and β_i' are two Jordan canonical bases for K_{λ_i}, then the dot diagrams for β_i and β_i' are identical. Thus, if β_i' is a disjoint union of k_i' cycles of lengths $p_1' \geq p_2' \geq \cdots \geq p_{k_i'}'$, then $k_i' = k_i$ and $p_1' = p_1, p_2' = p_2, \ldots, p_{k_i}' = p_{k_i}$.

To establish this uniqueness result, we shall use the following combinatorial fact: Any dot diagram is completely determined by the number of its rows and the number of dots in each row. (See Exercise 7.) Thus if these numbers could be computed from properties intrinsic to the transformation T (for example, as the ranks of $(\mathsf{T} - \lambda_i \mathsf{I})^j$ for various values of j), the dot diagram could be reconstructed and the uniqueness of the numbers $k_i, p_1, p_2, \ldots, p_{k_i}$ would be proved. The following results provide the desired method for computing these numbers.

Theorem 6.6. *For any positive integer* r *the basis vectors in* β_i *that are associated with dots in the first* r *rows of a dot diagram for* β_i *form a basis for* $\mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^r)$. *Hence the number of dots in the first* r *rows of a dot diagram for* β_i *is the nullity of* $(\mathsf{T} - \lambda_i \mathsf{I})^r$.

PROOF. The basis vectors in β_i which are associated with dots in the first r rows of a dot diagram for β_i are the first r elements of the cycles S_j $(j = 1, 2, \ldots, k_i)$ which form β_i. Hence these basis vectors are elements of $\mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^r)$. Moreover, these vectors are linearly independent since they form a subset of β_i. So it suffices to prove that these basis vectors generate $\mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^r)$.

For each j $(j = 1, 2, \ldots, k_i)$, let $\mathsf{W}_j = \operatorname{span}(S_j)$. Since W_j is T-invariant by Exercise 3 of Section 6.1, it is also $(\mathsf{T} - \lambda_i \mathsf{I})^r$-invariant. Moreover, $\mathsf{K}_{\lambda_i} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_{k_i}$ by Theorem 5.13. If $x \in \mathsf{N}((\mathsf{T} - \lambda_i \mathsf{I})^r)$, then $x \in \mathsf{K}_{\lambda_i}$ by definition. Thus there exist unique elements $w_j \in \mathsf{W}_j$ $(j = 1, 2, \ldots, k_i)$ such that $x = w_1 + w_2 + \cdots + w_{k_i}$. Hence

$$\begin{aligned} 0 &= (\mathsf{T} - \lambda_i \mathsf{I})^r(x) \\ &= (\mathsf{T} - \lambda_i \mathsf{I})^r(w_1) + (\mathsf{T} - \lambda_i \mathsf{I})^r(w_2) + \cdots + (\mathsf{T} - \lambda_i \mathsf{I})^r(w_{k_i}). \end{aligned}$$

It follows that

$$(\mathsf{T} - \lambda_i \mathsf{I})^r(w_j) = 0 \qquad \text{for } j = 1, 2, \ldots, k_i.$$

Suppose for each j that

$$S_j = \{(\mathsf{T} - \lambda_i \mathsf{I})^{p_j - 1}(x_j), (\mathsf{T} - \lambda_i \mathsf{I})^{p_j - 2}(x_j), \ldots, (\mathsf{T} - \lambda_i \mathsf{I})(x_j), x_j\}.$$

Then since

$$w_j = a_{p_j - 1}(\mathsf{T} - \lambda_i \mathsf{I})^{p_j - 1}(x_j) + \cdots + a_1(\mathsf{T} - \lambda_i \mathsf{I})(x_j) + a_0 x_j$$

for some scalars $a_{p_j - 1}, \ldots, a_1, a_0$,

$$0 = (\mathsf{T} - \lambda_i \mathsf{I})^r(w_j) = a_{p_j - r - 1}(\mathsf{T} - \lambda_i \mathsf{I})^{p_j - 1}(x_j) + \cdots + a_0(\mathsf{T} - \lambda_i \mathsf{I})^r(x_j).$$

Since S_j is linearly independent, it follows that $a_{p_j-r-1} = \cdots = a_0 = 0$. Therefore

$$w_j = a_{p_j-1}(\mathsf{T} - \lambda_i\mathsf{I})^{p_j-1}(x_j) + a_{p_j-2}(\mathsf{T} - \lambda_i\mathsf{I})^{p_j-2}(x_j)$$
$$+ \cdots + a_{p_j-r}(\mathsf{T} - \lambda_i\mathsf{I})^{p_j-r}(x_j).$$

So w_j is a linear combination of the basis vectors in β_i that are associated with dots in the first r rows of the jth column of a dot diagram for β_i. Hence $x = w_1 + w_2 + \cdots + w_{k_i}$ is a linear combination of members of β_i associated with dots in the first r rows of a dot diagram for β_i. We conclude that these vectors form a basis for $\mathsf{N}((\mathsf{T} - \lambda_i\mathsf{I})^r)$. ∎

In the case that $r = 1$, Theorem 6.6 yields the following corollary.

Corollary. *Let β_i be a Jordan canonical basis for the restriction of* T *to* K_{λ_i}, *and suppose that β_i is the disjoint union of* k_i *cycles of generalized eigenvectors corresponding to λ_i. Then the dimension of* E_{λ_i} *equals* k_i. *Hence in a Jordan canonical form for* T *the number of Jordan blocks corresponding to the eigenvalue λ_i equals the dimension of* E_{λ_i}.

We are now able to formulate a procedure for computing the dot diagram for β_i directly from T.

Theorem 6.7. *Let* r_j *denote the number of dots in the* j*th row of a dot diagram for β_i. Then*

(a) $r_1 = \dim(\mathsf{V}) - \operatorname{rank}(\mathsf{T} - \lambda_i\mathsf{I})$.

(b) $r_j = \operatorname{rank}((\mathsf{T} - \lambda_i\mathsf{I})^{j-1}) - \operatorname{rank}((\mathsf{T} - \lambda_i\mathsf{I})^{j})$ *if* $j > 1$.

PROOF. By Theorem 6.6,

$$\begin{aligned} r_1 + r_2 + \cdots + r_j &= \text{nullity}((\mathsf{T} - \lambda_i\mathsf{I})^j) \\ &= \dim(\mathsf{V}) - \text{rank}((\mathsf{T} - \lambda_i\mathsf{I})^j) \quad \text{for any } j \geq 1. \end{aligned}$$

Hence

$$r_1 = \dim(\mathsf{V}) - \text{rank}((\mathsf{T} - \lambda_i\mathsf{I})^1)$$

and

$$\begin{aligned} r_j &= (r_1 + r_2 + \cdots + r_j) - (r_1 + r_2 + \cdots + r_{j-1}) \\ &= (\dim(\mathsf{V}) - \text{rank}((\mathsf{T} - \lambda_i\mathsf{I})^j)) - (\dim(\mathsf{V}) - \text{rank}((\mathsf{T} - \lambda_i\mathsf{I})^{j-1})) \\ &= \text{rank}((\mathsf{T} - \lambda_i\mathsf{I})^{j-1}) - \text{rank}((\mathsf{T} - \lambda_i\mathsf{I})^j) \quad \text{for } j > 1. \end{aligned}$$ ∎

This theorem shows that a dot diagram for β_i is completely determined by T. Hence we have proved the following uniqueness result.

Corollary. *For any eigenvalue λ_i of* T *the dot diagram for β_i is unique. Thus, subject to the convention that cycles are listed in order of decreasing length, the Jordan canonical form of a linear operator is unique up to the ordering of its eigenvalues.*

Before giving some examples of the use of Theorem 6.7, we shall define the Jordan canonical form of a matrix in the obvious manner.

Definition. *Let* A *be an* $n \times n$ *matrix with entries from* F *such that the characteristic polynomial of* A *(and hence* L_A*) factors into a product of factors of degree* 1. *Then the* Jordan canonical form *of* A *is defined to be the Jordan canonical form of the linear operator* L_A *on* F^n.

Observe that if J is the Jordan canonical form of a matrix A, then J and A are similar. In fact, if $\beta = \{z_1, z_2, \ldots, z_n\}$ is a Jordan canonical basis for L_A and Q is the $n \times n$ matrix having z_j as its jth column, then $J = Q^{-1}AQ$ by Theorem 5.1.

In the three examples that follow we shall compute the Jordan canonical form of two matrices and a linear operator.

Example 5. Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

We shall find the Jordan canonical form of A and a Jordan canonical basis for the linear transformation L_A. The characteristic polynomial of A is

$$\det(A - tI) = (t-2)^3(t-3).$$

Thus A has two distinct eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 3$ with multiplicities 3 and 1, respectively.

Let β_1 be a Jordan canonical basis for the restriction of L_A to K_{λ_1}. Since λ_1 has multiplicity 3, $\dim(\mathsf{K}_{\lambda_1}) = 3$ by Theorem 6.5. Thus the dot diagram for β_1 contains 3 dots. As above, let r_j denote the number of dots in the jth row of this dot diagram. Applying Theorem 6.7, we have

$$r_1 = 4 - \operatorname{rank}(A - 2I) = 4 - \operatorname{rank}\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 4 - 2 = 2$$

and

$$r_2 = \operatorname{rank}(A - 2I) - \operatorname{rank}((A - 2I)^2) = 2 - 1 = 1.$$

(Actually, the computation of r_2 is unnecessary in this case. We could deduce that $r_2 = 1$ from the facts that $r_1 = 2$ and that the dot diagram consists of three dots.) Hence the dot diagram associated with β_1 is

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array}$$

So if T_i denotes the restriction of L_A to K_{λ_i} $(i = 1, 2)$, we must have

$$A_1 = [\mathsf{T}_1]_{\beta_1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $\dim(\mathsf{K}_{\lambda_2}) = 1$, any basis β_2 for K_{λ_2} will consist of a single eigenvector corresponding to $\lambda_2 = 3$. Thus

$$A_2 = [\mathsf{T}_2]_{\beta_2} = (3).$$

Setting $\beta = \beta_1 \cup \beta_2$, we have

$$J = [\mathsf{L}_A]_\beta = A_1 \oplus A_2 = \left(\begin{array}{ccc:c} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hdashline 0 & 0 & 0 & 3 \end{array}\right),$$

and so J is the Jordan canonical form of A.

We now seek a Jordan canonical basis for $\mathsf{T} = \mathsf{L}_A$. First we must find a Jordan canonical basis β_1 for T_1. We know from the preceding computations that the dot diagram corresponding to β_1 must be

$$\begin{array}{ll} \bullet(\mathsf{T} - \lambda_i \mathsf{I})(x_1) & \quad \bullet x_2 \\ \bullet x_1 & \end{array}$$

From this diagram we see that we must choose x_1 so that $x_1 \in \mathsf{N}((\mathsf{T} - \lambda_1 \mathsf{I})^2)$ but $x_1 \notin \mathsf{N}((\mathsf{T} - \lambda_1 \mathsf{I})^1)$. Since

$$A - 2I = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \qquad (A - 2I)^2 = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix}.$$

It is now easily seen that

$$\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}\right\}$$

is a basis for $\mathsf{N}((\mathsf{T}-\lambda_1\mathsf{I})^2) = \mathsf{K}_{\lambda_1}$. Of these basis vectors,

$$\begin{pmatrix}0\\1\\2\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\\0\\2\end{pmatrix}$$

satisfy the condition of not belonging to $\mathsf{N}((\mathsf{T}-\lambda_1\mathsf{I})^1)$. Hence we may select x_1 to be either of these vectors, say

$$x_1 = \begin{pmatrix}0\\1\\2\\0\end{pmatrix}.$$

Then

$$(\mathsf{T}-\lambda_1\mathsf{I})(x_1) = (A-2I)(x_1) = \begin{pmatrix}0 & -1 & 0 & 1\\0 & 1 & -1 & 0\\0 & 1 & -1 & 0\\0 & -1 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\2\\0\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}.$$

Now simply choose x_2 to be an element of E_{λ_1} that is linearly independent of

$$(\mathsf{T}-\lambda_1\mathsf{I})(x_1) = \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix};$$

for example, select

$$x_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Thus we have associated the Jordan canonical basis

$$\beta_1 = \left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

with the dot diagram in the following manner:

$$\begin{array}{ll} \cdot\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} & \cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \cdot\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} & \end{array}$$

The reader might be concerned that the linear independence of β_1 was not verified. Be assured, however, that this verification is not necessary because of the lemma to Theorem 6.4. Since x_2 was chosen to be linearly independent of the initial vector $(\mathsf{T} - \lambda_1\mathsf{I})(x_1)$ of the cycle $\{(\mathsf{T} - \lambda_1\mathsf{I})(x_1), x_1\}$, it follows from this lemma that β_1 is linearly independent.

Any eigenvector of L_A corresponding to the eigenvalue $\lambda_2 = 3$ will form the desired basis β_2 for K_{λ_2}—for example,

$$\beta_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus

$$\beta = \beta_1 \cup \beta_2 = \left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a Jordan canonical basis for L_A.

Notice that if

$$Q = \begin{pmatrix} -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

then $J = Q^{-1}AQ$.

Example 6. Let

$$A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

Again we shall find a Jordan canonical form J for A and a matrix Q such that $J = Q^{-1}AQ$.

The characteristic polynomial of A is $\det(A - tI) = (t - 2)^2(t - 4)^2$. Let $\lambda_1 = 2, \lambda_2 = 4$, and β_i be the Jordan canonical basis for T_i, the restriction of L_A to K_{λ_i}, for $i = 1, 2$.

We begin by computing the dot diagram for β_1. Let r_1 denote the number of dots in the first row of this diagram; then $r_1 = 4 - \operatorname{rank}(A - 2I) = 4 - 2 = 2$. So the dot diagram for β_1 is

$$\bullet \quad \bullet$$

Thus

$$A_1 = [\mathsf{T}_1]_{\beta_1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Next we compute the dot diagram for β_2. Since $\operatorname{rank}(A - 4I) = 3$, there is only $4 - 3 = 1$ dot in the first row of the diagram. Since K_{λ_2} has dimension 2 (Theorem 6.5), the dot diagram for β_2 must be

$$\begin{matrix} \bullet \\ \bullet \end{matrix}$$

Thus

$$A_2 = [\mathsf{T}_2]_{\beta_2} = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

So if $\beta = \beta_1 \cup \beta_2$, then the Jordan canonical form of L_A is

$$J = [\mathsf{L}_A]_\beta = \left(\begin{array}{cc:cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hdashline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right).$$

In order to find a matrix Q such that $Q^{-1}AQ = J$, we must first find a Jordan canonical basis β for T. The dot diagram for β_1 indicates that β_1 can be chosen to be any linearly independent set of eigenvectors of A corresponding to $\lambda_1 = 2$. For example,

$$\beta_1 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

will suffice. For β_2 we must find an element $x_1 \in \mathsf{K}_{\lambda_2} = \mathsf{N}((\mathsf{L}_A - \lambda_2\mathsf{I})^2)$ such that $x_1 \notin \mathsf{N}((\mathsf{L}_A - \lambda_2\mathsf{I})^1)$. One way of finding such an element was used to select the vector x_1 in Example 5. In this example we shall illustrate another method for obtaining such a vector. A simple calculation shows that a basis for the null space of $\mathsf{L}_A - \lambda_2\mathsf{I}$ is

$$\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Let

$$(A - 4I)(x_1) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

and choose x_1 to be any pre-image of

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

To do this, we must find a solution to the matrix equation

$$(A - 4I)\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix};$$

i.e.,

$$\begin{pmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

It is easily verified that

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

is a solution; so select

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

Thus

$$\beta_2 = \{(\mathsf{L}_A - \lambda_2 \mathsf{I})(x_1), x_1\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Hence

$$\beta = \beta_1 \cup \beta_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

is a Jordan canonical basis for L_A.

So if

$$Q = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix},$$

then $J = Q^{-1}AQ$.

Example 7. Let V denote the vector space of polynomial functions over R in two variables x and y of degree at most 2. (A basis for V is $\alpha = \{1, x, y, x^2, y^2, xy\}$.) Consider the mapping $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ defined by

$$\mathsf{T}(f) = \frac{\partial}{\partial x} f.$$

For example, if $f(x, y) = x + 2x^2 - 3xy + y$, then

$$\mathsf{T}(f) = \frac{\partial}{\partial x} f(x, y) = 1 + 4x - 3y.$$

We shall find a Jordan canonical basis for T.

First, observe that if $A = [\mathsf{T}]_\alpha$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of T is

$$\det(A - tI) = \det\begin{pmatrix} -t & 1 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 2 & 0 & 0 \\ 0 & 0 & -t & 0 & 0 & 1 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & -t \end{pmatrix} = t^6.$$

Hence T has only one eigenvalue ($\lambda = 0$), and $\mathsf{K}_\lambda = \mathsf{V}$. Let β denote any Jordan canonical basis for T. If r_i denotes the number of dots in the ith row of the dot diagram for β, then $r_1 = 6 - \text{rank}(A) = 6 - 3 = 3$. Since

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$r_2 = \text{rank}(A) - \text{rank}(A^2) = 3 - 1 = 2$. Thus because $r_1 = 3$, $r_2 = 2$, and there are six dots in the dot diagram, it follows that $r_3 = 1$. So the dot diagram for β is

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

We conclude that the Jordan canonical form J of T is

$$J = \left(\begin{array}{ccc:cc:c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We now seek a Jordan canonical basis for T. Since the first column of the dot diagram for β consists of three dots, we must find a vector x_1 such that

$$\frac{\partial^2}{\partial x^2}(x_1) \neq 0.$$

Examining the basis $\alpha = \{1, x, y, x^2, y^2, xy\}$ for K_λ, we see that x^2 is a candidate for x_1. Letting $x_1 = x^2$, we find that

$$(\mathsf{T} - \lambda\mathsf{I})(x_1) = \mathsf{T}(x_1) = \frac{\partial}{\partial x}(x_1) = 2x \quad \text{and}$$

$$(\mathsf{T} - \lambda\mathsf{I})^2(x_1) = \mathsf{T}^2(x_1) = \frac{\partial^2}{\partial x^2}(x_1) = 2.$$

Likewise, since the second column of the dot diagram for β consists of two dots, we must find a vector x_2 such that

$$\frac{\partial}{\partial x}(x_2) \neq 0.$$

Examining α with 1, x, and x^2 eliminated from consideration (because they lie in the span of the cycle $\{2, 2x, x^2\}$), we see that we may select $x_2 = xy$. Thus

$$(\mathsf{T} - \lambda\mathsf{I})(x_2) = \mathsf{T}(x_2) = \frac{\partial}{\partial x}(xy) = y.$$

Finally, choose $x_3 = y^2$. Then we have identified the following basis with the dot diagram:

$$\begin{array}{lll} \cdot 2 & \cdot y & \cdot y^2 \\ \cdot 2x & \cdot xy & \\ \cdot x^2 & & \end{array}$$

Thus $\beta = \{2, 2x, x^2, y, xy, y^2\}$ is a Jordan canonical basis for T.

In the three preceding examples we relied upon our ingenuity and the context of the problem to find a Jordan canonical basis. The reader will be able to do the same in the exercises. We are successful in these cases because the dimensions of the generalized eigenspaces under consideration are small. We shall not attempt, however, to develop a general algorithm for computing a Jordan canonical basis although one could be formulated by following the steps in the proof of the existence of such a basis (Theorem 6.4).

The following result may be thought of as a corollary to Theorem 6.7.

Theorem 6.8. *Let* A *and* B *be two square matrices of the same size, each having Jordan canonical forms computed according to the conventions of this section. Then* A *and* B *are similar if and only if they have (up to a permutation of their eigenvalues) the same Jordan canonical form.*

PROOF. If A and B have the same Jordan canonical form J, then A and B are each similar to J and hence are similar to each other.

Conversely, suppose that A and B are similar. Then A and B must have the same eigenvalues with the same multiplicities. Let J_A and J_B denote the Jordan canonical forms of A and B, respectively, for some fixed ordering of their eigenvalues. Then since A is similar to J_A and B is similar to J_B, the hypothesis implies that J_A and J_B are similar. Hence by Exercise 19 of Section 5.1 there exists a linear operator T on a finite-dimensional vector space V and bases β and γ for V such that $[\mathsf{T}]_\beta = J_A$ and $[\mathsf{T}]_\gamma = J_B$. Thus J_A and J_B are Jordan canonical forms of the same linear operator. Hence, since the eigenvalues of A and B are ordered in the same way, the corollary to Theorem 6.7 implies that $J_A = J_B$. ∎

Example 8. We shall determine which of the matrices

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

are similar. Observe that A, B, and C have the same characteristic polynomial $-(t-1)(t-2)^2$, whereas D has $-t(t-1)(t-2)$ as its characteristic polynomial. Thus, because similar matrices have the same characteristic polynomials, D cannot be similar to A, B, or C. Now each of the matrices A, B, and C has the same eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicities 1 and 2, respectively. If J_A, J_B, and J_C denote the Jordan canonical forms of A, B, and C, respectively, with respect to this ordering of their eigenvalues, then

$$J_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad J_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad J_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $J_A = J_C$, A is similar to C, while B is similar to neither A nor C.

The reader should observe that any diagonal matrix is a Jordan canonical form. Thus T is diagonalizable if and only if its Jordan canonical form is a diagonal matrix. Hence if T is a diagonalizable operator on V, any Jordan canonical basis for T is a basis for V consisting of eigenvectors of T.

EXERCISES

1. Label the following statements as being true or false.
   (a) The Jordan canonical form of a diagonal matrix is the matrix itself.
   (b) Let T be a linear operator on a finite-dimensional vector space V that has a Jordan canonical form J. If β is any basis for V, then the Jordan canonical form of $[\mathsf{T}]_\beta$ is J.

(c) Linear operators having the same characteristic polynomial are similar.
(d) Matrices having the same Jordan canonical form are similar.
(e) Every matrix is similar to its Jordan canonical form.
(f) Let T be a linear operator on a finite-dimensional vector space with characteristic polynomial $(-1)^n(t-\lambda)^n$. Subject to the convention that the Jordan blocks are ordered by decreasing size, T has a unique Jordan canonical form.
(g) If an operator has a Jordan canonical form, then there is a unique Jordan canonical basis for that operator.
(h) The dot diagram of any linear operator having a Jordan canonical form is unique.

2. Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T factors into a product of factors of degree 1. Let $\lambda_1 = 2$, $\lambda_2 = 4$, and $\lambda_3 = -3$ be the distinct eigenvalues of T, and suppose that the dot diagrams for the restriction of $\mathsf{T} - \lambda_i\mathsf{I}$ to K_{λ_i} $(i = 1, 2, 3)$ are as follows:

$\lambda_1 = 2$			$\lambda_2 = 4$		$\lambda_3 = -3$	
•	•	•	•	•	•	•
•	•		•			
•			•			

Find the Jordan canonical form of T.

3. Let T be a linear operator on a finite-dimensional vector space V such that the Jordan canonical form of T is

$$\left(\begin{array}{ccc|cc|cc}
2 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 2 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 3
\end{array}\right)$$

(a) Find the characteristic polynomial of T.
(b) Find the dot diagram corresponding to each eigenvalue of T.
(c) For which eigenvalues λ_i, if any, does $\mathsf{E}_{\lambda_i} = \mathsf{K}_{\lambda_i}$?
(d) For each eigenvalue λ_i, find the smallest positive integer p_i for which $\mathsf{K}_{\lambda_i} = \mathsf{N}((\mathsf{T} - \lambda_i\mathsf{I})^{p_i})$.
(e) Let U_i denote the restriction of $\mathsf{T} - \lambda_i\mathsf{I}$ to K_{λ_i} for each i. Compute the following for each i:

(i) $\text{rank}(\mathsf{U}_i)$
(ii) $\text{rank}(\mathsf{U}_i^2)$
(iii) $\text{nullity}(\mathsf{U}_i)$
(iv) $\text{nullity}(\mathsf{U}_i^2)$

4. For each of the following matrices A, find a Jordan canonical form J and a matrix Q such that $J = Q^{-1}AQ$. Notice that the matrices in parts (a), (b) and (c) are matrices used in Example 8.

(a) $A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}$

5. Let A be an $n \times n$ matrix whose characteristic polynomial factors into a product of factors of degree 1. Prove that A and A^t have the same Jordan canonical form, and conclude that A and A^t are similar. *Hint:* For any eigenvalue λ of A and A^t and any positive integer r, show that $\text{rank}((A - \lambda I)^r) = \text{rank}((A^t - \lambda I)^r)$.

6. Let V denote the vector space of functions that are linear combinations of e^x, xe^x, x^2e^x, and e^{2x}. Define $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{T}(f) = f'$ (the derivative of f). Find both a Jordan canonical form and a Jordan canonical basis for T.

7. Suppose that an array of dots (such as a dot diagram) has k columns and m rows and that the ith column of the array contains p_i dots and the ith row of the array contains r_i dots. If $p_1 \geq p_2 \geq \cdots \geq p_k$, prove the following:
(a) $m = p_1$ and $k = r_1$.
(b) $p_i = \max\{j\colon\ r_j \geq i\}$ for $1 \leq i \leq k$ and $r_i = \max\{j\colon\ p_j \geq i\}$ for $1 \leq i \leq m$. *Hint:* Use induction on m.
(c) $r_1 \geq r_2 \geq \cdots \geq r_m$.
(d) Conclude that the number of dots in each column of a dot diagram is completely determined if the number of dots in each row is known.

Definition. *A linear operator* T *on* V *is called* nilpotent *if* $\mathsf{T}^p = \mathsf{T}_0$ *for some positive integer* p.

8. Prove that if T is a nilpotent operator on an n-dimensional vector space V, then the characteristic polynomial of T is $(-1)^n t^n$. Hence the characteristic polynomial of T factors into a product of factors of degree 1, and T has only one eigenvalue (zero) of multiplicity n. *Hint:* Use induction on n. In the general step, assume that the conclusion is true for all vector spaces of dimension less than n, and follow the steps below.

(a) Prove that T has at least one eigenvector corresponding to $\lambda = 0$. Thus $\dim(\mathsf{R}(\mathsf{T})) < \dim(\mathsf{V}) = n$.

(b) Apply the induction hypothesis to the T-invariant subspace R(T).

(c) Extend a basis $\{x_1, x_2, \ldots, x_k\}$ for R(T) to a basis $\beta = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ for V.

(d) Show that

$$[\mathsf{T}]_\beta = \begin{pmatrix} B_1 & B_2 \\ O & O' \end{pmatrix},$$

where O and O' are $(n-k) \times k$ and $(n-k) \times (n-k)$ zero matrices, respectively.

(e) Deduce that $\det(\mathsf{T} - t\mathsf{I}) = (-1)^n t^n$.

9. Prove the converse of Exercise 8: If T is a linear operator on V having characteristic polynomial $(-1)^n t^n$, then T is nilpotent.

10. Give an example of a linear operator T such that T is not nilpotent but zero is the only eigenvalue of T. Characterize all such transformations.

Definition. *An* $n \times n$ *matrix* A *is called* nilpotent *if* A^p *equals the* $n \times n$ *zero matrix for some positive integer* p.

11. Let $A \in \mathsf{M}_{n\times n}(F)$. Prove that $A^p = O$, where O denotes the $n \times n$ zero matrix, if and only if $(\mathsf{L}_A)^p = \mathsf{T}_0$. Conclude that A is nilpotent if and only if L_A is nilpotent.

12. Prove that any square triangular matrix with each diagonal entry equal to zero is nilpotent.

13. Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T factors into a product of factors of degree 1. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the distinct eigenvalues of T. Since $\mathsf{V} = \mathsf{K}_{\lambda_1} \oplus \mathsf{K}_{\lambda_2} \oplus \cdots \oplus \mathsf{K}_{\lambda_k}$, we may define a mapping $\mathsf{U}\colon \mathsf{V} \to \mathsf{V}$ as follows: For $x \in \mathsf{V}$, where $x = x_1 + x_2 + \cdots + x_k$ with $x_i \in \mathsf{K}_{\lambda_i}$, define

$$\mathsf{U}(x) = (\mathsf{T} - \lambda_1\mathsf{I})(x_1) + (\mathsf{T} - \lambda_2\mathsf{I})(x_2) + \cdots + (\mathsf{T} - \lambda_k\mathsf{I})(x_k).$$

Prove that

(a) U is a linear operator.

(b) U is nilpotent.

(c) UT = TU.

14. Let T and U be as in Exercise 13. Suppose that β_i is a Jordan canonical basis for the restriction of T to K_{λ_i}, and let J_i denote the Jordan canonical form of this restriction. Then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a Jordan canonical basis for T. Let $J = [\mathsf{T}]_\beta$ and $\mathsf{S} = \mathsf{T} - \mathsf{U}$. Prove the following:

(a) $[\mathsf{S}]_\beta$ is a diagonal matrix whose diagonal entries are identical to the diagonal entries of J; that is, if $D = [\mathsf{S}]_\beta$, then

$$D_{ij} = \begin{cases} J_{ij} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $M = [\mathsf{U}]_\beta$, then

$$M_{ij} = \begin{cases} J_{ij} & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) $J = D + M$.

(d) $MD = DM$.

(e) As a consequence of parts (c) and (d) there is a binomial expansion for J. Let p be the smallest positive integer for which M^p equals the zero matrix. Then

$$J^r = D^r + rD^{r-1}M + \frac{r(r-1)}{2!}D^{r-2}M^2 + \cdots + rDM^{r-1} + M^r \qquad \text{if } r < p,$$

and

$$J^r = D^r + rD^{r-1}M + \frac{r(r-1)}{2!}D^{r-2}M^2 + \cdots + \frac{r!}{(r-p+1)!(p-1)!}D^{r-p+1}M^{p-1} \qquad \text{if } r \geq p.$$

(f) If $\mathsf{T} = \mathsf{L}_A$, then there exists a matrix Q such that $A = QJQ^{-1}$.

(g) For the matrix Q above and any positive integer r, $A^r = QJ^rQ^{-1}$.

15. Let T be a nilpotent linear operator on a finite-dimensional vector space V. Recall from Exercise 8 that $\lambda = 0$ is the only eigenvalue of T; hence $\mathsf{V} = \mathsf{K}_\lambda$. Let β be a Jordan canonical basis for T. Prove that for any positive integer i if we delete from β the vectors corresponding to the last i dots in each column of a dot diagram for β, the resulting set is a basis for $\mathsf{R}(\mathsf{T}^i)$. (If a column of the dot diagram contains fewer than i dots, all the vectors associated with that column are removed from β.)

16. Find a linear operator on a finite-dimensional vector space having two distinct Jordan canonical bases.

17. Let T be a linear operator, and let λ be an eigenvalue of T.

(a) Prove that $\dim(\mathsf{K}_\lambda)$ is the sum of the lengths of all the blocks corresponding to λ in the Jordan canonical form of T.

(b) Deduce that $\mathsf{E}_\lambda = \mathsf{K}_\lambda$ if and only if all the Jordan blocks corresponding to λ are 1×1.

18. (a) Let J be the Jordan block corresponding to the eigenvalue λ of a matrix; thus

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Suppose that J is $m \times m$, and let $N = J - \lambda I_m$. Prove that N^m is the zero matrix.

(b) Observe as in Exercise 14 that for any $r \geq m$

$$J^r = \lambda^r I_m + r\lambda^{r-1}N + \frac{r(r-1)}{2!}\lambda^{r-2}N^2 + \cdots + \frac{r(r-1)\cdots(r-m+2)}{(m-1)!}\lambda^{r-m+1}N^{m-1}$$

$$= \begin{pmatrix} \lambda^r & r\lambda^{r-1} & \frac{r(r-1)}{2!}\lambda^{r-2} & \cdots & \frac{r(r-1)\cdots(r-m+2)}{(m-1)!}\lambda^{r-m+1} \\ 0 & \lambda^r & r\lambda^{r-1} & \cdots & \frac{r(r-1)\cdots(r-m+3)}{(m-2)!}\lambda^{r-m+2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & \lambda^r \end{pmatrix}.$$

Prove that $\lim_{r\to\infty} J^r$ exists if and only if one of the following holds:

(i) $|\lambda| < 1$.

(ii) $\lambda = 1$ and $m = 1$.

Furthermore, show that $\lim_{r\to\infty} J^r$ is the zero matrix if (i) holds and is the matrix (1) if (ii) holds.

(c) Prove Theorem 5.16.

19. For any $A \in \mathsf{M}_{n\times n}(C)$, define $\|A\| = \max\{|A_{ij}|: \ 1 \leq i, j \leq n\}$. Prove the following results for arbitrary $A, B \in \mathsf{M}_{n\times n}(C)$ and $c \in C$:

(a) $\|A\| \geq 0$, and $\|A\| = 0$ if and only if A is the zero matrix.

(b) $\|cA\| = |c|\cdot\|A\|$.

(c) $\|A + B\| \leq \|A\| + \|B\|$.

(d) $\|AB\| \leq n\|A\|\cdot\|B\|$.

20. Let $A \in \mathsf{M}_{n\times n}(R)$ be a transition matrix, and $P^{-1}AP = J$ be the Jordan canonical form of A. Let $\|\cdot\|$ be as defined in Exercise 19.

(a) Show that for every positive integer m, $\|A^m\| \leq 1$.
(b) Deduce that $\{\|J^m\|: \quad m = 1, 2, \ldots\}$ is bounded.
(c) Using part (b) above and Exercise 18(b), prove that each Jordan block corresponding to the eigenvalue $\lambda = 1$ of A is 1×1.
(d) Use part (c), Theorem 5.16, and Exercise 18(b) to show that $\lim_{m\to\infty} A^m$ exists if and only if A has the property that whenever λ is an eigenvalue of A with $|\lambda| = 1$, then $\lambda = 1$.
(e) Prove Theorem 5.23(a) using part (c) and Theorem 5.22.

21. (This exercise requires knowledge of absolutely convergent series.) Recall from p. 279 that if $A \in \mathsf{M}_{n\times n}(C)$, then e^A is defined as $\lim_{m\to\infty} B_m$, where

$$B_m = I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}.$$

Use Exercise 19(d) to show that e^A exists for every $A \in \mathsf{M}_{n\times n}(C)$.

6.3* RATIONAL CANONICAL FORM

Throughout Chapters 5 and 6 we have been using eigenvalues and eigenvectors in our analysis of linear operators on a finite-dimensional vector space. As we have seen, these are useful tools provided that the characteristic polynomial of the linear operator factors into a product of factors of degree 1. There are linear operators for which this is not the case, however. Indeed, there are linear operators with no eigenvalues. What must be done in these cases is to generalize the concepts of eigenvalue and eigenvector in order to obtain structure theorems to replace those found in previous sections.

Given a linear operator T on a finite-dimensional vector space V with characteristic polynomial $f(t)$, we can always factor $f(t)$ uniquely as a product of powers of distinct irreducible monic polynomials times $(-1)^n$, where $n = \dim(\mathsf{V})$. Thus

$$f(t) = (-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2} \cdots (\phi_k(t))^{n_k},$$

where $\phi_i(t)$ is an irreducible monic polynomial of positive degree, n_i is a positive integer ($i = 1, 2, \ldots, k$), and $\phi_i(t) \neq \phi_j(t)$ for $i \neq j$. This follows easily from the unique factorization theorem in Appendix E. In the case that $f(t)$ factors into a product of factors of degree 1, $\phi_i(t) = t - \lambda_i$ for some eigenvalue λ_i ($i = 1, 2, \ldots, k$). In this case there is a one-to-one correspondence between the set of distinct eigenvalues and the set of distinct irreducible monic factors of the characteristic polynomial. In the general case eigenvalues may not exist, but the irreducible monic factors always do. It seems reasonable, therefore, to seek structure theorems based on monic irreducible factors of the characteristic polynomial rather than on eigenvalues and eigenvectors.

In this section we shall consider several structure theorems pertaining to this most general situation. For any linear operator T on a finite-dimensional vector space V, we shall see that V can be decomposed as a direct sum of T-cyclic subspaces. Furthermore, by imposing certain additional requirements relating the T-cyclic subspaces in the sum to the irreducible monic factors of the characteristic polynomial of T, we shall obtain a uniqueness theorem involving certain properties of these T-cyclic subspaces. Consequently it will be possible to choose a basis β for V in order to obtain a matrix $[\mathsf{T}]_\beta$ that is unique for T in the same way that the Jordan canonical form of an operator is unique for that operator. This matrix will be called the "rational canonical form" of T. It will serve in place of the Jordan canonical form in the case that the characteristic polynomial of T does not factor into a product of factors of degree 1.

At this point it would be of help to the reader to review the definitions and techniques employed in Sections 5.4 and 5.5. In particular, the reader should study cyclic subspaces and companion matrices and note the relationship between them. This relationship is so important to our development that we reiterate it here: Given a linear operator T on a finite-dimensional vector space V and a non-zero vector $x \in \mathsf{V}$, suppose that the T-cyclic subspace of V generated by x, C_x, has dimension $d > 0$. From Section 5.4 it follows that $\beta = \{x, \mathsf{T}(x), \ldots, \mathsf{T}^{d-1}(x)\}$ is an ordered basis for C_x. Thus $\mathsf{T}^d(x)$ is a linear combination of β, say

$$\mathsf{T}^d(x) = -a_0x - a_1\mathsf{T}(x) - \cdots - a_{d-1}\mathsf{T}^{d-1}(x)$$

for unique scalars $-a_0, -a_1, \ldots, -a_{d-1}$. (We have used $-a_i$ instead of a_i as a notational convenience.) Since C_x is a T-invariant subspace of V, we can consider the restriction of T to C_x, $\mathsf{T}_{\mathsf{C}_x}$. As we have seen,

$$[\mathsf{T}_{\mathsf{C}_x}]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix}.$$

This matrix has characteristic polynomial

$$f(t) = (-1)^d(a_0 + a_1t + \cdots + a_{d-1}t^{d-1} + t^d)$$

and is called the *companion matrix* of $f(t)$. As a consequence of Theorem 5.32 the polynomial

$$p(t) = a_0 + a_1t + \cdots + a_{d-1}t^{d-1} + t^d$$

is the minimal polynomial of $\mathsf{T}_{\mathsf{C}_x}$. Moreover, by Exercise 14 of Section 5.6 the polynomial $p(t)$ is the T-annihilator of x. (Incidentally, it is essential for the reader to work out Exercises 13 and 14 of Section 5.6 since they will be needed to establish some of the results of this section.)

Consider again the linear operator T above. Suppose that V is decomposed into a direct sum of T-cyclic subspaces

$$\mathsf{V} = \mathsf{C}_{x_1} \oplus \mathsf{C}_{x_2} \oplus \cdots \oplus \mathsf{C}_{x_k}$$

for some non-zero vectors $x_1, x_2, \ldots, x_k$ in V, where $\dim(\mathsf{C}_{x_i}) = d_i$ for each i. If $\beta_i = \{x_i, \mathsf{T}(x_i), \ldots, \mathsf{T}^{d_i-1}(x_i)\}$ and $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$, then β is a basis for V and

$$[\mathsf{T}]_\beta = [\mathsf{T}_{\mathsf{C}_{x_1}}]_{\beta_1} \oplus [\mathsf{T}_{\mathsf{C}_{x_2}}]_{\beta_2} \oplus \cdots \oplus [\mathsf{T}_{\mathsf{C}_{x_k}}]_{\beta_k}$$

by Theorem 5.26. Notice that $[\mathsf{T}]_\beta$ is a direct sum of companion matrices. We can summarize the above as follows.

Theorem 6.9. Let T *be a linear operator on a finite-dimensional vector space* V. *If* V *is decomposable as a direct sum of* T*-cyclic subspaces, then there exists a basis β for* V *such that $[\mathsf{T}]_\beta$ is a direct sum of companion matrices.*

As we have stated earlier, the object of this section is to show that V can always be decomposed into a direct sum of T-cyclic subspaces. Furthermore, it will be shown that it is always possible to choose each such T-cyclic subspace C_x so that the T-annihilator of x is of the form $(\phi(t))^m$, where $\phi(t)$ is an irreducible monic factor of the characteristic polynomial of T and m is a positive integer. The advantage of this decomposition is that the matrix corresponding to a basis chosen as in Theorem 6.9 is essentially unique (subject to certain conventions involving the ordering of the cyclic subspaces).

The discussion above leads us to the following definition.

Definition. Let T *be a linear operator on a finite-dimensional vector space* V, *and let β be an ordered basis for* V. *The matrix $[\mathsf{T}]_\beta$ will be called a* rational canonical form *for* T *if*

$$[\mathsf{T}]_\beta = \mathrm{C}_1 \oplus \mathrm{C}_2 \oplus \cdots \oplus \mathrm{C}_k,$$

where each C_i is the companion matrix of a polynomial of the form $(-1)^{\mathrm{md}}(\phi(\mathrm{t}))^{\mathrm{m}}$, $\phi(\mathrm{t})$ is an irreducible monic factor of the characteristic polynomial of T, d *is the degree of $\phi(\mathrm{t})$, and* m *is a positive integer.*

The following result is simply a restatement of Theorem 6.9 in the terminology above.

Corollary. *Let* T *be a linear operator on a finite-dimensional vector space* V. *If* V *is decomposable as a direct sum of* T*-cyclic subspaces*

$$V = C_{x_1} \oplus C_{x_2} \oplus \cdots \oplus C_{x_k}$$

such that, for each i, x_i *has* T*-annihilator* $(\phi_i(t))^{m_i}$, *where* $\phi_i(t)$ *is an irreducible monic factor of the characteristic polynomial of* T *and* m_i *is a positive integer, then* T *has a rational canonical form.*

In Theorem 6.5 we saw that if the characteristic polynomial of a linear operator T on a finite dimensional vector space V factors into a product of factors of degree 1, then

$$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T. Our next result will be the analog of Theorem 6.5 in the case that the characteristic polynomial of T does not factor into factors of degree 1. First, however, we shall introduce the analog of the generalized eigenspaces of T.

Definition. *Let* T *be a linear operator on a finite-dimensional vector space* V *with characteristic polynomial*

$$f(t) = (-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2} \cdots (\phi_r(t))^{n_r},$$

where $\phi_1(t), \phi_2(t), \ldots, \phi_r(t)$ *are distinct irreducible monic polynomials and* $n, n_1, n_2, \ldots, n_r$ *are positive integers. For each* $i = 1, 2, \ldots, r$, *define*

$$K_{\phi_i} = N((\phi_i(T))^{n_i}).$$

Observe that for any polynomial $g(t)$, T commutes with $g(T)$. Hence each K_{ϕ_i} is T-invariant. Moreover, if $p(t)$ is the minimal polynomial of T, then by Theorem 5.29

$$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \cdots (\phi_r(t))^{m_r}$$

for some integers m_i such that $0 \leq m_i \leq n_i$ for $1 \leq i \leq r$. We shall see later that, in fact, $m_i \geq 1$ for each i.

Theorem 6.10 (*Primary Decomposition Theorem*). *Let* T *be a linear operator on a finite-dimensional vector space* V *with minimal polynomial* $p(t) = (\phi_1(t))^{m_1} \cdots (\phi_r(t))^{m_r}$, *where* $\phi_1(t), \ldots, \phi_r(t)$ *are the distinct monic irreducible factors of* p(t). *Then*

(a) $V = K_{\phi_1} \oplus \cdots \oplus K_{\phi_r}$.

(b) *For* $1 \leq i \leq r$ *the minimal polynomial of the restriction of* T *to* K_{ϕ_i} *is* $(\phi_i(t))^{m_i}$.

PROOF. The proof is by induction on r. If $r = 1$, then the conclusion is immediate. So suppose that the theorem has been proved for operators with minimal polynomials having $r - 1$ distinct monic irreducible factors for some integer $r > 1$.

Let $g(t) = (\phi_1(t))^{m_1}$ and $h(t) = (\phi_2(t))^{m_2} \cdots (\phi_r(t))^{m_r}$. Then $g(t)$ and $h(t)$ are relatively prime. We shall prove that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$, where $\mathsf{W}_1 = \mathsf{N}(g(\mathsf{T}))$ and $\mathsf{W}_2 = \mathsf{N}(h(\mathsf{T}))$. Since $g(t)$ and $h(t)$ are relatively prime, there exist polynomials $q(t)$ and $r(t)$ such that $q(t)g(t) + r(t)h(t) = 1$, where 1 denotes the constant polynomial. (See Appendix E.) Substituting T into this equation gives

$$q(\mathsf{T})h(\mathsf{T}) + r(\mathsf{T})g(\mathsf{T}) = \mathsf{I}. \tag{4}$$

Thus $v = q(\mathsf{T})h(\mathsf{T})(v) + r(\mathsf{T})g(\mathsf{T})(v)$ for each $v \in \mathsf{V}$. But

$$g(\mathsf{T})q(\mathsf{T})h(\mathsf{T})(v) = q(\mathsf{T})h(\mathsf{T})g(\mathsf{T})(v) = q(\mathsf{T})p(\mathsf{T})(v) = q(\mathsf{T})\mathsf{T}_0(v) = 0;$$

so $q(\mathsf{T})h(\mathsf{T})(v) \in \mathsf{W}_1$. Likewise $r(\mathsf{T})g(\mathsf{T})(v) \in \mathsf{W}_2$. Thus $\mathsf{V} = \mathsf{W}_1 + \mathsf{W}_2$. Finally, if $w \in \mathsf{W}_1 \cap \mathsf{W}_2$, then by Eq. (4)

$$w = \mathsf{I}(w) = q(\mathsf{T})h(\mathsf{T})(w) + r(\mathsf{T})g(\mathsf{T})(w) = 0 + 0 = 0;$$

so $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

Observe that since T commutes with $g(\mathsf{T})$ and $h(\mathsf{T})$, W_1 and W_2 are T-invariant. Let $p_1(t)$ and $p_2(t)$ denote the minimal polynomials of T_1 and T_2, the restrictions of T to W_1 and W_2, respectively. We shall now prove that $p_1(t) = g(t)$ and $p_2(t) = h(t)$. The definitions of W_1 and W_2 show that $g(\mathsf{T}_1)$ and $h(\mathsf{T}_2)$ are both zero operators. Thus

$$p_1(t) \text{ divides } g(t), \quad \text{and} \quad p_2(t) \text{ divides } h(t). \tag{5}$$

So $p_1(t)p_2(t)$ divides $g(t)h(t) = p(t)$. But for any $v \in \mathsf{V}$, $v = w_1 + w_2$ for some $w_1 \in \mathsf{W}_1$ and $w_2 \in \mathsf{W}_2$. Hence

$$p_1(\mathsf{T})p_2(\mathsf{T})(v) = p_2(\mathsf{T})p_1(\mathsf{T})(w_1) + p_1(\mathsf{T})p_2(\mathsf{T})(w_2) = 0 + 0 = 0.$$

Thus $p(t)$, the minimal polynomial of T, also divides $p_1(t)p_2(t)$. Since $p(t)$, $p_1(t)$, and $p_2(t)$ are all monic, it follows that $p(t) = p_1(t)p_2(t)$. Finally the equation $g(t)h(t) = p_1(t)p_2(t)$, Eq. (5), and the fact that all four polynomials are monic imply that $p_1(t) = g(t)$ and $p_2(t) = h(t)$.

Applying the inductive hypothesis to T_2 and W_2, we have that

$$\mathsf{W}_2 = \mathsf{K}'_{\phi_2} \oplus \cdots \oplus \mathsf{K}'_{\phi_r}, \qquad \text{where } \mathsf{K}'_{\phi_i} = \mathsf{N}((\phi_i(\mathsf{T}_2))^{m_i}),$$

and that $(\phi_i(t))^{m_i}$ is the minimal polynomial of the restriction of T_2 to K'_{ϕ_i}. But since $\mathsf{N}((\phi_i(\mathsf{T}))^{m_i}) \subseteq \mathsf{W}_2$ for $i = 2, \ldots, r$, it follows that

$$\mathsf{K}_{\phi_i} = \mathsf{N}((\phi_i(\mathsf{T}))^{m_i}) = \mathsf{N}((\phi_i(\mathsf{T}_2))^{m_i}) = \mathsf{K}'_{\phi_i} \qquad \text{for } i = 2, \ldots, r.$$

Moreover, the restriction of T to K_{ϕ_i} is the same as the restriction of T_2 to K_{ϕ_i} $(i = 2, \ldots, r)$, and thus $(\phi_i(t))^{m_i}$ is the minimal polynomial of the restriction of T to K_{ϕ_i}. Thus since

$$\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{K}_{\phi_1} \oplus \mathsf{K}'_{\phi_2} \oplus \cdots \oplus \mathsf{K}'_{\phi_r} = \mathsf{K}_{\phi_1} \oplus \mathsf{K}_{\phi_2} \oplus \cdots \oplus \mathsf{K}_{\phi_r},$$

the proof is complete. ∎

We shall now begin the process of showing that every linear operator T on a finite-dimensional vector space V has, subject to certain conventions, a unique rational canonical form. We shall begin with the special case that the characteristic polynomial of T is of the form $\pm(\phi(t))^n$, where $\phi(t)$ is irreducible and n is a positive integer. In this case the minimal polynomial of T is of the form $(\phi(t))^m$ for some positive integer $m \leq n$, and we can prove that V has a decomposition into a direct sum of T-cyclic subspaces.

Theorem 6.11. *Let* T *be a linear operator on a finite-dimensional vector space* V *with minimal polynomial* $p(t) = (\phi(t))^m$, *where* $\phi(t)$ *is an irreducible monic polynomial and* m *is a positive integer. Then there exist non-zero vectors* $x_1, x_2, \ldots, x_k$ *in* V *and positive integers* $n_1, n_2, \ldots, n_k$ *with* $n_i \leq m$ *for each* i *such that*

(a) $V = C_{x_1} \oplus C_{x_2} \oplus \cdots \oplus C_{x_k}$, *where* C_{x_i} *is the* T*-cyclic subspace generated by* x_i.

(b) $(\phi(t))^{n_i}$ *is the* T*-annihilator of* x_i *for* $i = 1, 2, \ldots, k$.

PROOF. The proof is by induction on the dimension of V. If $\dim(V) = 1$, the result is trivial. Suppose then that the theorem holds for all vector spaces of dimension less than n, where $n > 1$ is an integer, and let $\dim(V) = n$.

Since the minimal polynomial of T is $p(t) = (\phi(t))^m$, there exists a non-zero vector x_1 in V such that $(\phi(T))^{m-1}(x_1) \neq 0$. Thus the T-annihilator of x_1 is $p(t)$. Let $W = C_{x_1}$, and recall that W is T-invariant. Let

$$\bar{T}\colon\ V/W \longrightarrow V/W$$

be the linear operator induced by T on the quotient space V/W. (See Exercise 13 of Section 5.4.) It is easily shown that for any polynomial $g(t)$ the operator induced by $g(T)$ on V/W is $g(\bar{T})$. Hence if $g(T) = T_0$, then $g(\bar{T}) = \bar{T}_0$. So the minimal polynomial of $\bar{T}$ divides $p(t)$, and therefore the induction hypothesis applies to $\bar{T}$ and V/W. Thus there exist $\bar{T}$-cyclic subspaces $C_2, \ldots, C_k$ of V/W such that

$$V/W = C_2 \oplus \cdots \oplus C_k$$

and such that for $2 \leq i \leq k$ the $\bar{T}$-annihilator of the generator of C_i is $(\phi(t))^{n_i}$ for some positive integer $n_i \leq m$.

We shall show that for $2 \leq i \leq k$ there exists a vector x_i in the generator of C_i such that the T-annhilator of x_i is $(\phi(t))^{n_i}$. Let y be an element of the generator of C_i; then $(\phi(T))^{n_i}(y) \in W = C_{x_1}$. So there is a polynomial $h(t)$ such that

$$(\phi(T))^{n_i}(y) = h(T)(x_1). \tag{6}$$

Because $(\phi(t))^m$ is the minimal polynomial of T, it follows from Eq. (6) that

$$0 = (\phi(\mathsf{T}))^m(y) = (\phi(\mathsf{T}))^{m-n_i}h(\mathsf{T})(x_1).$$

Now $(\phi(t))^m$ is the T-annihilator of x_1. Therefore $(\phi(t))^m$ divides $(\phi(t))^{m-n_i}h(t)$, and hence $(\phi(t))^{n_i}$ divides $h(t)$. Thus $(\phi(t))^{n_i}q(t) = h(t)$ for some polynomial $q(t)$. Define $x_i = y - q(\mathsf{T})(x_1)$. Then $y - x_i = q(\mathsf{T})(x_1) \in \mathsf{C}_{x_1} = \mathsf{W}$, and so x_i lies in the generator of C_i. It follows that the $\bar{\mathsf{T}}$-annihilator of the generator of C_i divides the T-annihilator of x_i. But also, by Eq. (6),

$$\begin{aligned}(\phi(\mathsf{T}))^{n_i}(x_i) &= (\phi(\mathsf{T}))^{n_i}(y - q(\mathsf{T})(x_1)) \\ &= (\phi(\mathsf{T}))^{n_i}(y) - h(\mathsf{T})(x_1) = 0.\end{aligned}$$

So the T-annihilator of x_i equals $(\phi(t))^{n_i}$.

If the degree of $\phi(t)$ is d, then $(\phi(t))^{n_i}$ has degree dn_i. Thus, since $(\phi(t))^{n_i}$ is both the T-annihilator of x_i and the $\bar{\mathsf{T}}$-annihilator of the generator of C_i, Theorem 5.27 and Exercise 14 of Section 5.6 show that

$$\beta_i = \{x_i, \mathsf{T}(x_i), \ldots, \mathsf{T}^{dn_i-1}(x_i)\}$$

and

$$\gamma_i = \{x_i + \mathsf{W}, \bar{\mathsf{T}}(x_i + \mathsf{W}), \ldots, \bar{\mathsf{T}}^{dn_i-1}(x_i + \mathsf{W})\}$$

are bases for C_{x_i} and C_i, respectively. But since $\mathsf{V}/\mathsf{W} = \mathsf{C}_2 \oplus \cdots \oplus \mathsf{C}_k$, $\gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V/W. It follows that $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a basis for V. Hence $\mathsf{V} = \mathsf{C}_{x_1} \oplus \mathsf{C}_{x_2} \oplus \cdots \oplus \mathsf{C}_{x_k}$. ∎

The following result is immediate from the preceding theorem and the corollary to Theorem 6.9.

Corollary 1. *Let* T *be a linear operator on a finite-dimensional vector space. If the characteristic or minimal polynomial of* T *is of the form* $\pm(\phi(t))^m$ *for some irreducible monic polynomial* $\phi(t)$ *and positive integer* m, *then* T *has a rational canonical form.*

Corollary 2. *Let* T *be a linear operator on a finite-dimensional vector space* V *with characteristic polynomial*

$$f(t) = (-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2} \cdots (\phi_k(t))^{n_k},$$

where $\phi_1(t), \phi_2(t), \ldots, \phi_k(t)$ *are the distinct monic irreducible factors of* f(t). *Then, for each* i, $(\phi_i(t))^{n_i}$ *is the characteristic polynomial of* T_i, *the restriction of* T *to* K_{ϕ_i}. *Hence, for each* i, K_{ϕ_i} *is non-zero and* $\phi_i(t)$ *is a factor of the minimal polynomial of* T.

PROOF. By renumbering $\phi_1(t), \phi_2(t), \ldots, \phi_k(t)$ if necessary, we may assume that the minimal polynomial of T is

$$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \cdots (\phi_r(t))^{m_r},$$

where $r \leq k$ and $1 \leq m_i \leq n_i$ for each $i = 1, 2, \ldots, r$. Let $f_i(t)$ denote the characteristic polynomial of T_i. Since $\mathsf{V} = \mathsf{K}_{\phi_1} \oplus \mathsf{K}_{\phi_2} \oplus \cdots \oplus \mathsf{K}_{\phi_r}$ by the primary decomposition theorem, it then follows that $f(t) = f_1(t)f_2(t) \cdots f_r(t)$ by Theorem 5.25.

Consider any i, $1 \leq i \leq r$, and let the degree of $\phi_i(t)$ be d. Since the minimal polynomial of T_i is $(\phi_i(t))^{m_i}$ by the primary decomposition theorem, we conclude from Corollary 1 of Theorem 6.11 that there is a basis β for K_{ϕ_i} such that

$$[\mathsf{T}_i]_\beta = C_1 \oplus C_2 \oplus \cdots \oplus C_s,$$

where C_j is the companion matrix of $(-1)^{q_j d}(\phi_i(t))^{q_j}$ for certain positive integers $q_1, q_2, \ldots, q_s$. Thus, if $d_i = q_1 + q_2 + \cdots + q_s$, we have

$$f_i(t) = \det(C_1 - tI)\cdot\det(C_2 - tI)\cdot \cdots \cdot\det(C_s - tI) = (-1)^{dd_i}(\phi_i(t))^{d_i}.$$

So

$$f(t) = f_1(t)f_2(t) \cdots f_r(t) = \epsilon(\phi_1(t))^{d_1}(\phi_2(t))^{d_2} \cdots (\phi_r(t))^{d_r},$$

where $\epsilon = \pm 1$. Hence the unique factorization theorem implies that $r = k$ and $d_i = n_i$ for all i. In particular, $f_i(t) = \pm(\phi_i(t))^{n_i}$, $\mathsf{K}_{\phi_i} \neq \{0\}$, and $\phi_i(t)$ is a factor of the minimal polynomial of T for $i = 1, 2, \ldots, k$. ∎

Continuing with the special case that the minimal polynomial of T has the form $(\phi(t))^m$ for some irreducible monic polynomial $\phi(t)$ of degree d, we shall now formulate a uniqueness theorem for the rational canonical form of T. In order to formulate this result we shall henceforth adopt the convention that the vectors $x_1, x_2, \ldots, x_k$ in Theorem 6.11 will always be indexed so that $n_1 \geq n_2 \geq \cdots \geq n_k$. Subject to this convention, we shall show that the integers $n_1, n_2, \ldots, n_k$ are unique. Indeed, we shall actually provide a method for computing these integers. At this point the reader should observe that the uniqueness of the integers $n_1, n_2, \ldots, n_k$ will imply the uniqueness of the rational canonical form of T. In fact, it follows that the rational canonical form of T is

$$C_1 \oplus C_2 \oplus \cdots \oplus C_k,$$

where C_i is the companion matrix of $(-1)^{dn_i}(\phi(t))^{n_i}$.

To aid us in computing the integers $n_1, n_2, \ldots, n_k$ in Theorem 6.11 (and thus establishing their uniqueness), we shall introduce a new dot diagram corresponding to the decomposition of V as a direct sum of cyclic subspaces. Unlike the dot diagrams of Section 6.2 the diagrams that we shall now consider do not correspond to bases for V. Let T be a linear operator on a finite-dimensional vector space V having minimal polynomial $(\phi(t))^m$ for some irreducible monic polynomial $\phi(t)$ and positive integer m. Suppose also, as in Theorem 6.11, that $\mathsf{V} = \mathsf{C}_{x_1} \oplus \cdots \oplus \mathsf{C}_{x_k}$ for some non-zero vectors $x_1, x_2, \ldots, x_k$ in V and that, for each i, x_i has annihilator $(\phi(t))^{n_i}$ for some positive integer n_i. We assume that the x_i's are indexed so that $n_1 \geq n_2 \geq \cdots \geq n_k$. The *dot diagram* associated with the decomposition above is defined to be the array of dots consisting of k columns having n_i dots in the ith column and arranged so that the ith column begins at the top and terminates after n_i dots. Thus if $k = 3$, $n_1 = 4$, $n_2 = 2$, and $n_3 = 2$, the dot diagram would appear as

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \\ \bullet & & \end{array}$$

If we define r_i to be the number of dots in the ith row of the dot diagram, we see that the numbers r_i are determined by the formula given in Exercise 7 of Section 6.2. Furthermore, knowing the numbers r_i for all i enables us to compute the integers $n_1, n_2, \ldots, n_k$.

The following theorem tells us that the r_i's are expressible in terms of the ranks of certain operators. It follows that the n_i's are unique, and thus the theorem provides an algorithm for computing them.

Theorem 6.12. *Let* T *and* $\mathrm{r_i}$ *be as above. Then*

$$\mathrm{r}_1 = \frac{1}{\mathrm{d}}[\mathit{dim}(\mathsf{V}) - \mathit{rank}(\phi(\mathsf{T}))]$$

and

$$\mathrm{r_i} = \frac{1}{\mathrm{d}}[\mathit{rank}((\phi(\mathsf{T}))^{\mathrm{i}-1}) - \mathit{rank}((\phi(\mathsf{T}))^{\mathrm{i}})] \qquad \textit{for } \mathrm{i} > 1,$$

where d *is the degree of* $\phi(\mathrm{t})$.

PROOF. The following is an outline of the proof. The reader should justify each step.

We can establish both equations simultaneously by adopting the convention that

$$(\phi(\mathsf{T}))^i = \mathsf{I} \qquad \text{if } i = 0.$$

Then for any $i \geq 0$

$$\mathsf{R}((\phi(\mathsf{T}))^i) = \mathsf{C}_{(\phi(\mathsf{T}))^i(x_1)} \oplus \mathsf{C}_{(\phi(\mathsf{T}))^i(x_2)} \oplus \cdots \oplus \mathsf{C}_{(\phi(\mathsf{T}))^i(x_k)},$$

and hence (by Exercise 14 of Section 5.6 applied to $\phi^{n_j - i}(t)$)

$$\dim(\mathsf{R}((\phi(\mathsf{T}))^i)) = \sum_{n_j \geq i} d(n_j - i).$$

Thus for $i \geq 1$

$$\begin{aligned} \mathrm{rank}((\phi(\mathsf{T}))^{i-1}) - \mathrm{rank}((\phi(\mathsf{T}))^i) &= d\Big[\sum_{n_j \geq i-1} (n_j - (i-1)) - \sum_{n_j \geq i} (n_j - i)\Big] \\ &= d \sum_{n_j \geq i} [(n_j - (i-1)) - (n_j - i)] \\ &= d \sum_{n_j \geq i} 1 = d(\max\{j\colon\ n_j \geq i\}) = dr_i \end{aligned}$$

by Exercise 7 of Section 6.2. ∎

Corollary 1. *The integers* $n_1, n_2, \ldots, n_k$ *of Theorem 6.11 are unique. That is, with the notation as in Theorem 6.11, if there exist non-zero vectors* $x'_1, x'_2, \ldots, x'_r$ *in* V *and positive integers* $n'_1, n'_2, \ldots, n'_r$ *such that*

$$V = C_{x'_1} \oplus C_{x'_2} \oplus \cdots \oplus C_{x'_r},$$

where x'_i *has annihilator* $(\phi(t))^{n'_i}$ *for* $i = 1, 2, \ldots, r$ *and* $n'_1 \geq n'_2 \geq \cdots \geq n'_r$, *then* $k = r$, $n_1 = n'_1$, $n_2 = n'_2, \ldots, n_k = n'_r$.

Corollary 2. *Let* T *be as in Theorem 6.11. Assuming that a basis* β *is chosen for* V *as in Theorem 6.9, then the rational canonical form of* T, $[T]_\beta$, *is unique. In fact,*

$$[T]_\beta = C_1 \oplus C_2 \oplus \cdots \oplus C_k,$$

where C_i *is the companion matrix of* $(-1)^{n_i d}(\phi(t))^{n_i}$ $(i = 1, 2, \ldots, k)$.

We now define the rational canonical form of a matrix in the natural way.

Definition. *The* rational canonical form *of* $A \in M_{n\times n}(F)$ *is defined to be the rational canonical form of the linear operator* $L_A\colon\ F^n \longrightarrow F^n$.

Example 9. Consider the real 4×4 matrix A defined by

$$A = \begin{pmatrix} 0 & -1 & 5 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 5 & -3 \end{pmatrix}.$$

We shall compute the rational canonical form of A. The characteristic polynomial of A is

$$f(t) = \det\begin{pmatrix} -t & -1 & 5 & -3 \\ 1 & -t & 0 & -1 \\ 0 & 0 & 3-t & -2 \\ 0 & 0 & 5 & -3-t \end{pmatrix} = (t^2 + 1)^2.$$

Thus in our previous notation $\phi(t) = t^2 + 1$ and $d = 2$. In the dot diagram for A we have

$$r_1 = \tfrac{1}{2}[\dim(R^4) - \operatorname{rank}(\phi(A))] = \tfrac{1}{2}(4 - 0) = 2$$

and

$$r_i = \tfrac{1}{2}[\operatorname{rank}((\phi(A))^{i-1}) - \operatorname{rank}((\phi(A))^i)] = \tfrac{1}{2}(0 - 0) = 0$$

for $i > 1$. Thus the dot diagram for A is

$$\cdot \quad \cdot$$

We conclude that $n_1 = 1$ and $n_2 = 1$. Observe that the number of dots in the dot diagram is $(1/d)\dim(\mathsf{V})$. (In this case $\dim(\mathsf{V}) = 4$ and $d = 2$.) Hence there exist vectors x_1 and x_2 in R^4 such that $\mathsf{R}^4 = \mathsf{C}_{x_1} \oplus \mathsf{C}_{x_2}$ and x_1 and x_2 each have annihilator $\phi(t) = t^2 + 1$. Since the companion matrix corresponding to $t^2 + 1$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we conclude that the rational canonical form of A is

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Example 10. Let A denote the real 4×4 matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Again we shall compute the rational canonical form of A. Note that A is a Jordan canonical form. As we shall see, the rational canonical form of A differs from its Jordan canonical form. It is easily seen that the characteristic polynomial $f(t)$ of A is $f(t) = (t - 2)^4$. Thus $\phi(t) = t - 2$ and $d = 1$. Now

$$r_1 = 4 - \operatorname{rank}(\phi(A)) = 4 - 2 = 2,$$

$$r_2 = \operatorname{rank}(\phi(A)) - \operatorname{rank}((\phi(A))^2) = 2 - 1 = 1,$$

and

$$r_3 = \operatorname{rank}((\phi(A))^2) - \operatorname{rank}((\phi(A))^3) = 1 - 0 = 1.$$

Since there are $4 = \dim(\mathsf{V})/d$ dots in the dot diagram, we may terminate the computation with r_3. Thus the dot diagram for A is

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array}$$

We conclude that $n_1 = 3$ and $n_2 = 1$. So there exist elements x_1 and x_2 in R^4 such that $\mathsf{R}^4 = \mathsf{C}_{x_1} \oplus \mathsf{C}_{x_2}$, x_1 has annihilator $(t - 2)^3$, and x_2 has annihilator $t - 2$. Since the companion matrix of $(-1)^3(t - 2)^3$ is

$$C_1 = \begin{pmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 1 & 6 \end{pmatrix}$$

and the companion matrix of $(-1)(t-2)$ is $C_2 = (2)$, the rational canonical form of A is

$$C = C_1 \oplus C_2 = \left(\begin{array}{ccc|c} 0 & 0 & 8 & 0 \\ 1 & 0 & -12 & 0 \\ 0 & 1 & 6 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right).$$

Example 11. For the matrices A and C in Example 10, we shall find a matrix Q such that $Q^{-1}AQ = C$.

Observe that A and C are similar by the corollary to Theorem 2.27. Hence we need only find an ordered basis β for R^4 such that $[\mathsf{L}_A]_\beta = C$ and then take Q to be the matrix whose columns are the members of β. To find such a basis β, we need to find non-zero vectors x_1 and x_2 in R^4 such that x_1 has annihilator $(t-2)^3$, x_2 has annihilator $(t-2)$, and $\{x_1, \mathsf{L}_A(x_1), \mathsf{L}_{A^2}(x_1), x_2\}$ is linearly independent. To begin, let us find an element of R^4 with annihilator $(t-2)^3$, i.e., an element x_1 such that $(\mathsf{L}_A - 2\mathsf{I})^3(x_1) = 0$ but $(\mathsf{L}_A - 2\mathsf{I})^2(x_1) \neq 0$. If we methodically consider the members of the standard basis $\{e_1, e_2, e_3, e_4\}$, we see that e_3 has this property. Setting

$$x_1 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

we find that

$$\mathsf{L}_A(x_1) = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathsf{L}_{A^2}(x_1) = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 0 \end{pmatrix}.$$

Next, choose an element $x_2 \in \mathsf{R}^4$ linearly independent of $\{x_1, \mathsf{L}_A(x_1), \mathsf{L}_{A^2}(x_1)\}$ and having annihilator $t-2$. Clearly e_4 satisfies this condition. Thus

$$\beta = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for R^4 such that $[\mathsf{L}_A]_\beta = C$. Hence if

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $Q^{-1}AQ = C$.

We shall now consider the general case of a linear operator T on a finite-dimensional vector space for which the characteristic polynomial contains more than one distinct irreducible factor. By combining Theorems 6.11 and 5.26, we can easily show that T has a rational canonical form.

Theorem 6.13. *Let* T *be any linear operator on a finite-dimensional vector space* V. *Then* T *has a rational canonical form.*

PROOF. Suppose the characteristic polynomial of T is

$$(-1)^n(\phi_1(t))^{m_1}(\phi_2(t))^{m_2}\cdots(\phi_r(t))^{m_r},$$

where $m_i \geq 1$ and the $\phi_i(t)$'s are distinct irreducible monic polynomials. If $r = 1$, the result follows from Corollary 1 of Theorem 6.11. Otherwise, for each $i = 1, 2, \ldots, r$, T_i, the restriction of T to K_{ϕ_i}, has characteristic polynomial $\pm(\phi_i(t))^{m_i}$ by Corollary 2 of Theorem 6.11. Hence by Corollary 1 of Theorem 6.11 there exists a basis β_i such that $[\mathsf{T}_i]_{\beta_i} = D_i$ is a rational canonical form for T_i. Setting $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_r$, it is clear by Theorem 6.10 that β is a basis for V. Thus by Theorem 5.26

$$[\mathsf{T}]_\beta = D_1 \oplus D_2 \oplus \cdots \oplus D_r.$$

So $[\mathsf{T}]_\beta$ is a rational canonical form for T. ∎

The proof of the theorem above involves a choice of basis for V that guarantees a rational canonical form for T. In the context of this result we can make the following statement: If $D = [\mathsf{T}]_\beta$ is the rational canonical form constructed above, then $D = D_1 \oplus D_2 \oplus \cdots \oplus D_r$, and for each $i = 1, 2, \ldots, r$ there exists a sequence of integers $n_{i1} \geq n_{i2} \geq \cdots \geq n_{ik_i} \geq 1$ such that

$$D_i = C_{i1} \oplus C_{i2} \oplus \cdots \oplus C_{ik_i},$$

where C_{ij} is the companion matrix of $(-1)^{n_{ij}d_j}(\phi_i(t))^{n_{ij}}$ and d_j is the degree of $\phi_j(t)$.

The following theorem guarantees the uniqueness of the rational canonical form of an operator provided that it satisfies the description above.

Theorem 6.14. *Let* T *be a linear operator on a finite-dimensional vector space* V *with characteristic polynomial*

$$f(t) = (-1)^n(\phi_1(t))^{m_1}(\phi_2(t))^{m_2}\cdots(\phi_r(t))^{m_r},$$

where the $\phi_i(t)$*'s are distinct irreducible monic polynomials and* $m_i \geq 1$ *for all* i. *Suppose* D *is a rational canonical form for* T *such that* $D = D_1 \oplus D_2 \oplus \cdots \oplus D_r$ *and for each* $i = 1, 2, \ldots, r$ *there exists a sequence of integers* $n_{i1} \geq n_{i2} \geq \cdots \geq n_{ik_i} \geq 1$ *such that* $D_i = C_{i1} \oplus C_{i2} \oplus \cdots \oplus C_{ik_i}$, *where* C_{ij} *is the companion matrix of* $(-1)^{n_{ij}d_j}(\phi_i(t))^{n_{ij}}$ *and* d_j *is the degree of* $\phi_j(t)$. *Then* D *is unique in the sense that if* D' *is any other rational canonical form for* T *satisfying the description above for sequences of integers* $n'_{i1} \geq n'_{i2} \geq \cdots \geq n'_{ik_i} \geq 1$, *then* $D = D'$.

PROOF. Given $D = D_1 \oplus D_2 \oplus \cdots \oplus D_r$, let β be a basis for V such that $[\mathsf{T}]_\beta = D$. Suppose for each $i = 1, 2, \ldots, r$ that D_i is a $p_i \times p_i$ matrix. Let β_1 be the ordered set consisting of the first p_1 members of β, β_2 be the ordered set consisting of the next p_2 members of β, and so on. For each $i = 1, 2, \ldots, r$ define $\mathsf{W}_i = \text{span}(\beta_i)$. By virtue of the fact that D is a direct sum of the D_i's, W_i is a T-invariant subspace of V and $[\mathsf{T}_{\mathsf{W}_i}]_{\beta_i} = D_i$. Since the characteristic polynomial of D_i is a product of characteristic polynomials of the companion matrices C_{ij}, D_i must have characteristic polynomial $\pm(\phi_i(t))^{m_i}$. Hence so does $\mathsf{T}_{\mathsf{W}_i}$. Thus by the Cayley-Hamilton theorem $(\phi_i(\mathsf{T}))^{m_i}(x) = 0$ for all $x \in \mathsf{W}_i$. So $\mathsf{W}_i \subseteq \mathsf{K}_{\phi_i}$ and therefore

$$\dim(\mathsf{W}_i) \leq \dim(\mathsf{K}_{\phi_i}) \qquad \text{for } i = 1, 2, \ldots, r. \tag{7}$$

Since

$$\beta = \bigcup_{i=1}^{r} \beta_i \quad \text{and} \quad \beta_i \cap \beta_j = \varnothing \qquad \text{for } i \neq j,$$

we have $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_r$. Hence

$$\dim(\mathsf{V}) = \sum_{i=1}^{r} \dim(\mathsf{W}_i). \tag{8}$$

But by the primary decomposition theorem we also have

$$\dim(\mathsf{V}) = \sum_{i=1}^{r} \dim(\mathsf{K}_{\phi_i}). \tag{9}$$

Thus by Eqs. (7), (8), and (9) we conclude that $\dim(\mathsf{W}_i) = \dim(\mathsf{K}_{\phi_i})$ for all i. So $\mathsf{W}_i = \mathsf{K}_{\phi_i}$ for each i, and therefore β_i is a basis for K_{ϕ_i} $(i = 1, 2, \ldots, r)$. Hence

$$D_i = [\mathsf{T}_i]_{\beta_i}$$

is a rational canonical form for T_i, the restriction of T to K_{ϕ_i}. But D_i is unique by Corollary 2 of Theorem 6.12, and the uniqueness of $D = D_1 \oplus D_2 \oplus \cdots \oplus D_r$ follows. ▌

Example 12. We shall find the rational canonical form of the real matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix}.$$

If $f(t)$ denotes the characteristic polynomial of A, then it can be shown that $f(t) = -(t^2 + 2)^2(t - 2)$.

Thus $\phi_1(t) = t^2 + 2$ and $\phi_2(t) = t - 2$ are the distinct irreducible monic factors of $f(t)$. Let $\mathsf{T} = \mathsf{L}_A$ and T_i denote the restriction of T to K_{ϕ_i}. Then the characteristic polynomials of T_1 and T_2 are $(t^2 + 2)^2$ and $-(t - 2)$, respectively, by Corollary 2 of Theorem 6.11. So $\dim(\mathsf{K}_{\phi_1}) = 4$

and $\dim(\mathsf{K}_{\phi_2}) = 1$. Since the rational canonical form of T is the direct sum of the rational canonical forms of T_1 and T_2, we must compute each of these.

To find the rational canonical form of T_1, we must apply Theorem 6.12 to T_1. But by Exercise 13 we may instead apply Theorem 6.12 directly to $\mathsf{T}\colon\ \mathsf{R}^5 \longrightarrow \mathsf{R}^5$. First, however, observe that the number of dots in the dot diagram for T_1 is $(1/d)\dim(\mathsf{K}_{\phi_1}) = \frac{1}{2}(4) = 2$, where d is the degree of $\phi_1(t)$. Letting r_1 denote the number of dots in the first row of the dot diagram for T_1, Exercise 13 shows that

$$\begin{aligned} r_1 &= \tfrac{1}{2}[\dim(\mathsf{R}^5) - \operatorname{rank}(\phi_1(\mathsf{T}))] \\ &= \tfrac{1}{2}[5 - \operatorname{rank}(A^2 + 2I)] \\ &= \tfrac{1}{2}\left[5 - \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 12 & -24 & 12 \end{pmatrix}\right] = \tfrac{1}{2}(5-1) = 2. \end{aligned}$$

Thus the first row contains all the dots in the dot diagram for T_1; i.e., the dot diagram for T_1 is

$$\bullet \quad \bullet$$

We conclude that $n_1 = n_2 = 1$. So if D_1 denotes the rational canonical form of T_1, then

$$D_1 = C_{11} \oplus C_{12},$$

where

$$C_{11} = C_{12} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$D_1 = \left(\begin{array}{cc|cc} 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

The situation for T_2 is trivial. Since $\dim(\mathsf{K}_{\phi_2}) = 1$, the dot diagram contains only one dot. Thus if D_2 is the rational canonical form for T_2, then $D_2 = (2)$. So the rational canonical form of A is

$$D = D_1 \oplus D_2 = \left(\begin{array}{cc|cc|c} 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right).$$

The reader should note that if we had written $f(t) = -(t-2)(t^2+2)^2$ and set $\phi_1(t) = (t-2)$ and $\phi_2(t) = t^2+2$, then we would have computed the rational canonical form of A to be

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Either form of D is acceptable. Notice that except for the permutation of $\phi_1(t)$ and $\phi_2(t)$, D is unique.

EXERCISES

1. Label the following statements as being true or false.
   (a) The rational canonical form of a linear operator is the direct sum of companion matrices.
   (b) If T is a linear operator on a finite-dimensional vector space V, and if β is a basis for V such that $[\mathsf{T}]_\beta$ is the direct sum of companion matrices, then $[\mathsf{T}]_\beta$ is a rational canonical form for T.
   (c) There exist square matrices having no rational canonical form.
   (d) A square matrix is similar to its rational canonical form.
   (e) The Jordan canonical form and rational canonical form of any linear operator are the same.
   (f) For any linear operator T on a finite-dimensional vector space V, any irreducible factor of the characteristic polynomial of T divides the minimal polynomial of T.
   (g) Let $\phi(t)$ be an irreducible monic divisor of the characteristic polynomial of a linear operator T. The dots in the dot diagram used to compute the rational canonical form of $\mathsf{T}_{\mathsf{K}_\phi}$ are in one-to-one correspondence with the vectors in a basis for K_ϕ.
2. For each of the following, find the rational canonical form.
   (a) The real matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

   (b) The real matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

(c) The complex matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

(d) The real matrix

$$A = \begin{pmatrix} 0 & -7 & 14 & -6 \\ 1 & -4 & 6 & -3 \\ 0 & -4 & 9 & -4 \\ 0 & -4 & 11 & -5 \end{pmatrix}$$

(e) The real matrix

$$A = \begin{pmatrix} 0 & -4 & 12 & -7 \\ 1 & -1 & 3 & -3 \\ 0 & -1 & 6 & -4 \\ 0 & -1 & 8 & -5 \end{pmatrix}$$

3. Prove that if T is a linear operator on a finite-dimensional vector space V with minimal polynomial $(\phi(t))^m$ for some positive integer m, then $\mathsf{N}((\phi(\mathsf{T}))^{m-1})$ is a proper T-invariant subspace of V.

4. Let T be a linear operator on a finite-dimensional vector space V with characteristic polynomial $f(t) = (-1)^n(\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \cdots (\phi_r(t))^{m_r}$, where the $\phi_i(t)$'s are distinct irreducible monic polynomials, m_i is a positive integer for each i, and $n = \dim(\mathsf{V})$. Prove that for any $i = 1, 2, \ldots, r$, if d_i is the degree of $\phi_i(t)$, then $\dim(\mathsf{K}_{\phi_i}) = m_i d_i$.

5. Let T be as in Exercise 4. Consider any i and j such that $i \neq j$. Prove that the restriction of $\phi_j(\mathsf{T})$ to K_{ϕ_i} is one-to-one and onto.

6. Let T be a linear operator on a finite-dimensional vector space V with minimal polynomial $(\phi(t))^m$ for some irreducible monic polynomial $\phi(t)$ and some positive integer m. Prove that the restriction of T to $\mathsf{R}(\phi(\mathsf{T}))$ has minimal polynomial $(\phi(t))^{m-1}$.

7. Let T be a linear operator on a finite-dimensional vector space V. Prove that the rational canonical form of T is a diagonal matrix if and only if T is diagonalizable.

8. Let T be a linear operator on a finite-dimensional vector space V with characteristic polynomial $f(t) = (-1)^n\phi_1(t)\phi_2(t)$, where $\phi_1(t)$ and $\phi_2(t)$ are distinct irreducible monic polynomials and $n = \dim(\mathsf{V})$.
   (a) Prove that there exist elements x_1 and x_2 in V such that x_1 has T-annihilator $\phi_1(t)$, x_2 has T-annihilator $\phi_2(t)$, and
   $$\mathsf{V} = \mathsf{C}_{x_1} \oplus \mathsf{C}_{x_2}.$$
   (b) Prove that there exists an element x_3 in V with T-annihilator $\phi_1(t)\phi_2(t)$ for which $\mathsf{V} = \mathsf{C}_{x_3}$.

Thus to assure that the decomposition of V into a direct sum of cyclic subspaces is unique, we must require that the generators of the cyclic subspaces in the sum have powers of irreducible monic factors of the characteristic polynomial as their T-annihilators.

9. In the notation of Theorem 6.11, prove that if the x_i's are indexed so that $n_1 \geq n_2 \geq \cdots \geq n_k$, then $n_1 = m$.

10. Let T be a linear operator on a finite-dimensional vector space V. Assuming that the notation is as in the statement of Theorem 6.14, prove that the minimal polynomial $p(t)$ of T is

$$p(t) = (\phi_1(t))^{n_{11}}(\phi_2(t))^{n_{21}} \cdots (\phi_r(t))^{n_{r1}}.$$

11. Let T be a linear operator on a finite-dimensional vector space V. Prove that for any irreducible polynomial $\phi(t)$, if $\phi(\mathsf{T})$ is not one-to-one on V, then $\phi(t)$ divides the characteristic polynomial of T. *Hint:* Apply Exercise 14 of Section 5.6.

12. Justify the following observation made in Example 9: If T is a linear operator on a finite-dimensional vector space T with minimal polynomial $(\phi(t))^m$, where $\phi(t)$ is irreducible, monic, and of degree d, then the number of dots in the dot diagram for T is $\dim(\mathsf{V})/d$.

13. Justify the application of Theorem 6.12 in Example 12; that is, prove the following result: Let T be a linear operator on a finite-dimensional vector space V with characteristic polynomial

$$f(t) = (-1)^n(\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \cdots (\phi_r(t))^{m_r},$$

where the $\phi_i(t)$'s are the distinct irreducible monic factors of $f(t)$, m_i is a positive integer $(i = 1, 2, \ldots, r)$, and $n = \dim(\mathsf{V})$. Then for any $i = 1, 2, \ldots, r$

$$\dim(\mathsf{V}) - \operatorname{rank}(\phi_i(\mathsf{T})) = \dim(\mathsf{K}_{\phi_i}) - \operatorname{rank}(\phi(\mathsf{T}_{\mathsf{K}_{\phi_i}})),$$

and for any integer $j > 1$

$$\operatorname{rank}((\phi_i(\mathsf{T}))^{j-1}) - \operatorname{rank}((\phi_i(\mathsf{T}))^j) = \operatorname{rank}((\phi_i(\mathsf{T}_{\mathsf{K}_{\phi_i}}))^{j-1}) - \operatorname{rank}((\phi_i(\mathsf{T}_{\mathsf{K}_{\phi_i}}))^j).$$

Thus if r_j is the number of elements in the jth row of the dot diagram for $\mathsf{T}_{\mathsf{K}_{\phi_i}}$, then

$$r_1 = \frac{1}{d}[\dim(\mathsf{V}) - \operatorname{rank}(\phi_i(\mathsf{T}))],$$

and

$$r_j = \frac{1}{d}[\operatorname{rank}((\phi_i(\mathsf{T}))^{j-1}) - \operatorname{rank}((\phi_i(\mathsf{T}))^j)] \qquad \text{for } j > 1,$$

where d is the degree of $\phi_i(t)$.

INDEX OF DEFINITIONS FOR CHAPTER 6

chapter 7

inner product spaces

Most applications of mathematics are involved with the concept of measurement and hence of the magnitude or relative size of various quantities. So it is not surprising that the fields of real and complex numbers that have a built-in notion of distance should play a special role. In this chapter, we shall assume that all our vector spaces are over the field F, where F denotes either R or C.

We shall introduce the idea of distance or length into vector spaces, obtaining a much richer structure, the so-called "inner product space" structure. This added structure will provide applications to geometry (Section 7.8), physics (Section 7.4), conditioning in systems of equations (Section 7.6), least squares applications (Section 7.10), and quadratic forms (Section 7.11).

7.1 INNER PRODUCTS AND NORMS

Many of the geometric notions such as angle, length, and perpendicularity in R^2 and R^3 may be extended to more general real and complex vector spaces. All of these ideas are related to the concept of "inner product."

Definition. *Let* V *be a vector space over* F. *An* inner product *on* V *is a function that assigns to every ordered pair of vectors* x *and* y *in* V *a scalar in* F, *denoted* (x, y), *such that for all* x, y, *and* z *in* V *and all* c *in* F *we have*

(a) $(x + z, y) = (x, y) + (z, y)$.
(b) $(cx, y) = c(x, y)$.
(c) $\overline{(x, y)} = (y, x)$, *where the bar denotes complex conjugation.*
(d) $(x, x) > 0$ *if* $x \neq 0$.

Note that (c) reduces to $(x, y) = (y, x)$ if $F = R$. Conditions (a) and (b) simply require that the inner product be linear in the first component.

It is easily shown that if $a_1, \ldots, a_n \in F$ and $y, x_1, x_2, \ldots, x_n \in V$, then

$$\left(\sum_{i=1}^{n} a_i x_i, y\right) = \sum_{i=1}^{n} a_i (x_i, y).$$

Example 1. Let $V = F^n$. For $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$, define

$$(x, y) = \sum_{i=1}^{n} a_i \overline{b_i}.$$

$(\cdot, \cdot)$ satisfies conditions (a) through (d) and is called the *standard inner product* on F^n. (In elementary courses in linear algebra, this is called the *dot product.*)

The verification of (a) through (d) is easy. For example, if $z = (c_1, \ldots, c_n)$, we have for (a)

$$(x + z, y) = \sum_{i=1}^{n} (a_i + c_i)\overline{b_i} = \sum_{i=1}^{n} a_i \overline{b_i} + \sum_{i=1}^{n} c_i \overline{b_i}$$
$$= (x, y) + (z, y).$$

Thus for $x = (1 + i, 4)$ and $y = (2 - 3i, 4 + 5i)$ in C^2 we have $(x, y) = (1 + i)(2 + 3i) + 4(4 - 5i) = 15 - 15i$.

Example 2. If (x, y) is any inner product on a vector space V and $r > 0$, we may define another inner product by the rule $(x, y)' = r(x, y)$. If $r < 0$, then (d) would not hold.

Example 3. Let $V = C([0, 1])$, the vector space of real-valued continuous functions on [0, 1]. For $f, g \in V$, define $(f, g) = \int_0^1 f(t)g(t)\,dt$. Since the integral above is linear in f, (a) and (b) are immediate, and (c) is trivial. If $f \neq 0$, then the graph of f^2 lies above the x-axis on some subinterval of [0, 1] (continuity is used here), and hence $(f, f) = \int_0^1 [f(t)]^2\,dt > 0$.

Definition. *Let* A *be an* m × n *matrix with entries from* F. *We define the* conjugate transpose (*or* adjoint) of A *to be the* n × m *matrix* A* *such that* $(A^*)_{ij} = \overline{A_{ji}}$.

Example 4. Let

$$A = \begin{pmatrix} i & 1+2i \\ 2 & 3+4i \end{pmatrix}.$$

Then

$$A^* = \begin{pmatrix} -i & 2 \\ 1-2i & 3-4i \end{pmatrix}.$$

The conjugate transpose of a matrix will play a very important role in the remainder of this chapter. Note that if A has real entries, then A^* is simply the transpose of A.

Example 5. Let $\mathsf{V} = \mathsf{M}_{n\times n}(F)$, and define $(A, B) = \mathrm{tr}(B^*A)$ for $A, B \in \mathsf{V}$. (Recall that the trace of a matrix A is defined by $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$.) We shall verify that (a) and (d) of the definition of inner product hold and leave (b) and (c) to the reader. For this purpose, let $A, B, C \in \mathsf{V}$. Then (using Exercise 6 of Section 1.3) $(A + B, C) = \mathrm{tr}(C^*(A + B)) = \mathrm{tr}(C^*A + C^*B) = \mathrm{tr}(C^*A) + \mathrm{tr}(C^*B) = (A, C) + (B, C)$. Also

$$(A, A) = \mathrm{tr}(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} (A^*)_{ik}A_{ki}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{A_{ki}}A_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2.$$

Hence if $A \neq O$, then $A_{ki} \neq 0$ for some k and i. So $(A, A) > 0$.

A vector space V over F endowed with a specific inner product is called an *inner product space*. If $F = C$, we call V a *complex inner product space*, whereas if $F = R$, we call V a *real inner product space*.

Thus Examples 1, 3, and 5 also provide examples of inner product spaces. *For the remainder of this chapter* F^n *will denote the inner product space with the inner product given in Example* 1.

The reader should be cautioned that two distinct inner products on a given vector space yield two distinct inner product spaces.

A very important inner product space that resembles $\mathsf{C}([0, 1])$ is the space H of continuous complex-valued functions defined on the interval $[0, 2\pi]$ with the inner product

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

The reason for the constant $1/2\pi$ will become evident later. This inner product space, which arises often in the context of physical situations, will be examined more closely in later sections.

At this point we shall mention a few facts about integration of complex-valued functions. First, the imaginary number i can be treated as a constant under the integral sign. Second, every complex-valued function f may be written as $f = f_1 + if_2$, where f_1 and f_2 are real-valued functions. Thus we have that

$$\int f = \int f_1 + i \int f_2 \quad \text{and} \quad \overline{\int f} = \int \bar{f}.$$

From these properties, as well as the assumption of continuity, it follows that H is an inner product space.

Some properties that follow easily from the definition of an inner product are contained in the next theorem.

Theorem 7.1. *Let* V *be an inner product space. Then for* x, y, z ∈ V *and* c ∈ F

(a) $(x, y + z) = (x, y) + (x, z)$.
(b) $(x, cy) = \bar{c}(x, y)$.
(c) $(x, x) = 0$ *if and only if* $x = 0$.
(d) *If* $(x, y) = (x, z)$ *for all* $x \in V$, *then* $y = z$.

PROOF.

(a) $$(x, y + z) = \overline{(y + z, x)} = \overline{(y, x) + (z, x)}$$
$$= \overline{(y, x)} + \overline{(z, x)} = (x, y) + (x, z).$$

The proofs of (b), (c), and (d) are left as exercises. ▮

The reader should observe that (a) and (b) of Theorem 7.1 show that the inner product is *conjugate linear* in the second component.

In order to generalize the notion of length in R^3 to arbitrary inner product spaces, we need only observe that the length of $x = (a, b, c) \in R^3$ is given by $\sqrt{a^2 + b^2 + c^2} = \sqrt{(x, x)}$. Hence we make the following definition.

Definition. *Let* V *be an inner product space. For* x ∈ V *we define the* norm (*or* length) *of* x *by* $\|x\| = \sqrt{(x, x)}$.

Example 6. Let $V = F^n$. Then

$$\|(a_1, \ldots, a_n)\| = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$$

is the Euclidean definition of length. Note that if $n = 1$, we have $\|a\| = |a|$.

As we might expect, the well-known properties of length in R^3 hold in general, as shown below.

Theorem 7.2. *Let* V *be an inner product space. Then for all* x, y ∈ V *and* c ∈ F *we have*

(a) $\|cx\| = |c|\cdot\|x\|$.
(b) $\|x\| = 0$ *if and only if* $x = 0$. *In any case,* $\|x\| \geq 0$.
(c) (*Cauchy-Schwarz Inequality*) $|(x, y)| \leq \|x\|\cdot\|y\|$.
(d) (*Triangle Inequality*) $\|x + y\| \leq \|x\| + \|y\|$.

PROOF. We shall leave the proof of (a) and (b) as exercises.

(c) If $y = 0$, then the result is immediate. So assume that $y \neq 0$. Then for any $c \in F$, we have

$$0 \leq \|x - cy\|^2 = (x - cy, x - cy) = (x, x - cy) - c(y, x - cy)$$
$$= (x, x) - \bar{c}(x, y) - c(y, x) + c\bar{c}(y, y).$$

Setting

$$c = \frac{(x, y)}{(y, y)},$$

the inequality above becomes

$$0 \leq (x, x) - \frac{|(x, y)|^2}{(y, y)} = \|x\|^2 - \frac{|(x, y)|^2}{\|y\|^2},$$

from which (c) follows.

(d)
$$\|x + y\|^2 = (x + y, x + y) = (x, x) + (y, x) + (x, y) + (y, y)$$
$$= \|x\|^2 + 2\,\mathrm{Re}(x, y) + \|y\|^2$$
$$\leq \|x\|^2 + 2|(x, y)| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2,$$

where $\mathrm{Re}(x, y)$ denotes the real part of the complex number (x, y). Note that we used (c) to prove (d). ∎

The case when equality results in (c) and (d) is considered in Exercise 15.

Example 7. For $V = F^n$ we may apply (c) and (d) above to the standard inner product to obtain the following well-known inequalities:

$$\left|\sum_{i=1}^{n} a_i\bar{b}_i\right| \leq \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}\left[\sum_{i=1}^{n} |b_i|^2\right]^{1/2}$$

and

$$\left[\sum_{i=1}^{n} |a_i + b_i|^2\right]^{1/2} \leq \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2} + \left[\sum_{i=1}^{n} |b_i|^2\right]^{1/2}.$$

The reader may recall from earlier courses that for $V = R^3$ or R^2 we have that $(x, y) = \|x\|\cdot\|y\|\cos\theta$, where θ denotes the angle $(0 \leq \theta \leq \pi)$ between x and y. This equation implies (c) immediately since $|\cos\theta| \leq 1$.

Notice also that x and y are perpendicular if and only if $\cos\theta = 0$, that is, if and only if $(x, y) = 0$.

We are now at the point where we can generalize the notion of perpendicularity to arbitrary inner product spaces.

Definitions. *Let* V *be an inner product space. A vector* x *in* V *is a* unit vector *if* $\|x\| = 1$. *Vectors* x *and* y *in* V *are* orthogonal (perpendicular) *if* $(x, y) = 0$. *A subset* S *of* V *is* orthogonal *if any two distinct elements of* S *are orthogonal. Finally, a subset* S *of* V *is* orthonormal *if* S *is orthogonal and consists entirely of unit vectors.*

Note that if $S = \{x_1, x_2, \ldots, x_n\}$, then S is orthonormal if and only if $(x_i, x_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Also observe that for any non-zero vector x, $(1/\|x\|)x$ is a unit vector.

Example 8. In F^2 the set $S = \{(1, 1), (1, -1)\}$ is orthogonal but not orthonormal; however,

$$S = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$$

is orthonormal.

Example 9. Recall H (defined on p. 360). We shall produce a very important example of an orthonormal subset of H to which we shall return in later examples. Define $S = \{e^{ijx}\colon\ j \text{ is an integer}\}$, where i is the imaginary number $\sqrt{-1}$. Clearly S is a subset of H. (Recall that $e^{ijx} = \cos jx + i \sin jx$). Using the property that $\overline{e^{it}} = e^{-it}$ for every real number t, we have for $j \neq k$ that

$$(e^{ijx}, e^{ikx}) = \frac{1}{2\pi}\int_0^{2\pi} e^{ijt}\overline{e^{ikt}}\, dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(j-k)t}\, dt$$
$$= \frac{1}{2\pi i(j-k)} e^{i(j-k)t}\Big|_0^{2\pi} = 0.$$

Also

$$(e^{ijx}, e^{ijx}) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(j-j)t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} dt = 1.$$

In other words $(e^{ijx}, e^{ikx}) = \delta_{jk}$.

If we consider the spaces R^2 and R^3, it is geometrically obvious that orthogonal sets of non-zero vectors are linearly independent. The next theorem tells us that this is true in every inner product space.

Theorem 7.3. *Let V be an inner product space, and let S be an orthogonal set consisting of non-zero vectors. Then S is linearly independent.*

PROOF. Let $x_1, \ldots, x_n$ be distinct elements in S, and suppose that

$$0 = \sum_{i=1}^{n} a_i x_i.$$

Then for any j, $1 \leq j \leq n$,

$$0 = (0, x_j) = \left(\sum_{i=1}^{n} a_i x_i, x_j\right) = \sum_{i=1}^{n} a_i (x_i, x_j) = a_j \|x_j\|^2$$

since $(x_i, x_j) = 0$ for $i \neq j$. Since $x_j \neq 0$, we have that $a_j = 0$. Hence S is linearly independent. ∎

This theorem tells us, for example, that the vector space H in Example 9 contains an infinite independent set and hence is not a finite-dimensional vector space.

EXERCISES

1. Label the following statements as being true or false.
   (a) An inner product is a scalar-valued function on the set of ordered pairs of vectors.
   (b) An inner product space must be over the field of real or complex numbers.
   (c) An inner product is linear in both components.
   (d) There is exactly one inner product on the vector space R^n.
   (e) The triangle inequality only holds in finite-dimensional inner product spaces.
   (f) Every orthogonal set is linearly independent.
   (g) Every orthonormal set is linearly independent.
   (h) Only square matrices have a conjugate-transpose.
   (i) If $(x, y) = 0$ for all x in an inner product space, then $y = 0$.
2. Let $\mathsf{V} = \mathsf{C}^3$ with the standard inner product. Let $x = (2, 1 + i, i)$ and $y = (2 - i, 2, 1 + 2i)$. Compute (x, y), $\|x\|$, $\|y\|$, and $\|x + y\|^2$. Then verify both Cauchy's inequality and the triangle inequality.
3. In $\mathsf{C}([0, 1])$, let $f(t) = t$ and $g(t) = e^t$. Compute (f, g) (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f + g\|$. Then verify both Cauchy's inequality and the triangle inequality.
4. Let $\mathsf{V} = \mathsf{M}_{n \times n}(F)$ with $(A, B) = \operatorname{tr}(B^*A)$. Complete the proof in Example 5 that $(\cdot, \cdot)$ is an inner product. If $n = 2$ and

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix},$$

compute $\|A\|$, $\|B\|$, and (A, B).

5. On C^2, show that $(x, y) = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Compute (x, y) for $x = (1 - i, 2 + 3i)$ and $y = (2 + i, 3 - 2i)$.

6. Complete the proof of Theorem 7.1.

7. Complete the proof of Theorem 7.2.

8. Provide reasons why each of the following are *not* inner products on the given vector spaces.
(a) $((a, b), (c, d)) = ac - bd$ on R^2
(b) $(A, B) = \operatorname{tr}(A + B)$ on $\mathsf{M}_{2\times 2}(R)$
(c) $(f, g) = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation

9. Let β be a basis for a finite-dimensional inner product space. Prove that if $(x, y) = 0$ for all $x \in \beta$, then $y = 0$.

10.† Let V be an inner product space, and suppose that x and y are orthogonal elements of V. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in R^2.

11. Prove the *parallelogram law* on an inner product space V; that is, show that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \text{for all } x, y \in \mathsf{V}.$$

What does this equation state about parallelograms in R^2?

12.† Let $\{x_1, \ldots, x_k\}$ be an orthogonal set in V, and let $a_1, \ldots, a_k \in F$. Prove that

$$\left\| \sum_{i=1}^{k} a_i x_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|x_i\|^2.$$

13. Suppose $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ are two inner products on a vector space V. Prove that $(\cdot, \cdot) = (\cdot, \cdot)_1 + (\cdot, \cdot)_2$ is another inner product on V.

14. Let A and B be $n \times n$ matrices, and let $c \in F$. Prove that $(A + cB)^* = A^* + \bar{c}B^*$.

15. (a) Prove that if V is an inner product space, then $|(x, y)| = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a multiple of the other. *Hint:* If $y \neq 0$, let

$$a = \frac{(x, y)}{\|y\|^2}.$$

Then $x = ay + z$, where $(y, z) = 0$. By assumption

$$|a| = \frac{\|x\|}{\|y\|}.$$

Apply Exercise 10 to $\|x\|^2 = \|ay + z\|^2$ and obtain $\|z\| = 0$.
(b) Derive a similar result for the equality $\|x + y\| = \|x\| + \|y\|$, and generalize it to the case of n vectors.

16. Let $V = C([0, 1])$, and define

$$(f, g) = \int_0^{1/2} f(t)g(t)\,dt.$$

Is this an inner product on V?

17. Let V be an inner product space, and suppose that $T\colon V \longrightarrow V$ is linear and that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

18. Let V be a vector space over F, where $F = R$ or $F = C$, and let W be an inner product space over F with inner product $(\cdot, \cdot)$. If $T\colon V \longrightarrow W$ is linear, prove that $(x, y)' = (T(x), T(y))$ defines an inner product on V if and only if T is one-to-one.

19. Let V be an inner product space; prove that
(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\,\mathrm{Re}(x, y) + \|y\|^2$ for all $x, y \in V$, where $\mathrm{Re}(x, y)$ denotes the real part of the complex number (x, y).
(b) $\big|\|x\| - \|y\|\big| \leq \|x - y\|$ for all $x, y \in V$.

20. Let V be an inner product space over F. Verify the *polar identities*: For all $x, y \in V$
(a) $(x, y) = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = R$.
(b) $(x, y) = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2$ if $F = C$,
where $i = \sqrt{-1}$.

21. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad\text{and}\quad A_2 = \frac{1}{2i}(A - A^*).$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?
(b) Let A be an $n \times n$ matrix. Prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

22. Let V be a vector space over F, where F is either R or C. Whether or not V is an inner product space, we may still define a "norm" $\|\cdot\|$ as a real-valued function on V satisfying the following conditions for all $x, y \in V$ and $a \in F$:
(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|ax\| = |a|\cdot\|x\|$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$.
Prove that the following are norms on the given vector spaces V.

(a) $V = M_{m\times n}(F)$; $\|A\| = \max_{i,j} |A_{ij}|$ for all $A \in V$
(b) $V = C([0, 1])$; $\|f\| = \max_{t\in[0,1]} |f(t)|$ for all $f \in V$
(c) $V = C([0, 1])$; $\|f\| = \int_0^1 |f(t)|\,dt$ for all $f \in V$
(d) $V = R^2$; $\|(a, b)\| = \max\{|a|, |b|\}$ for all (a, b) in V

Use Exercise 20 to show that there is no inner product $(\cdot, \cdot)$ on R^2 such that $\|x\|^2 = (x, x)$ for all $x \in R^2$ if $(\cdot, \cdot)$ is defined as in (d).

23. Let V be an inner product space, and define for each ordered pair of vectors the scalar $d(x, y) = \|x - y\|$, called the *distance* between x and y. Prove for all $x, y, z \in V$ that
(a) $d(x, y) \geq 0$.
(b) $d(x, y) = d(y, x)$.
(c) $d(x, y) \leq d(x, z) + d(z, y)$.
(d) $d(x, x) = 0$.
(e) $d(x, y) \neq 0$ if $x \neq y$.

24. Let V be a real or complex vector space (possibly infinite-dimensional), and let β be a basis of V. For $x, y \in V$ there exist $x_1, \ldots, x_n \in \beta$ such that

$$x = \sum_{i=1}^{n} a_i x_i \quad \text{and} \quad y = \sum_{i=1}^{n} b_i x_i.$$

Define

$$(x, y) = \sum_{i=1}^{n} a_i \bar{b}_i.$$

Prove that $(\cdot, \cdot)$ is an inner product on V. Thus every real or complex vector space may be regarded as an inner product space.

Prove that if $V = R^n$ or C^n and β is the standard ordered basis, then the inner product defined above is the standard inner product.

7.2 THE GRAM-SCHMIDT ORTHOGONALIZATION PROCESS AND ORTHOGONAL COMPLEMENTS

In previous chapters we have seen the special role of the standard ordered basis in R^n. The special properties of this basis stem from the fact that the basis vectors form an orthonormal set. Just as bases are the building blocks of vector spaces, bases that are also orthonormal sets are the building blocks of inner product spaces. We shall now name such bases.

Definition. *Let* V *be an inner product space. A subset* β *of* V *is an* orthonormal basis *for* V *if* β *is an ordered basis that is orthonormal.*

Example 10. If $V = F^n$, then the standard ordered basis is an orthonormal basis for V.

Example 11.

$$\left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\right\}$$

is an orthonormal basis for R^2.

Of course, we have not yet shown that every finite-dimensional inner product space possesses an orthonormal basis. The next theorem takes us most of the way in obtaining this result. It tells us how to construct an orthogonal set from a linearly independent set of vectors in such a way that both sets generate the same subspace.

Before stating this theorem, let us consider a simple case. Suppose that $\{y_1, y_2\}$ is a linearly independent subset of an inner product space (and hence a basis for some two-dimensional subspace). We would like to construct an orthogonal set from $\{y_1, y_2\}$ that spans the same subspace. Figure 7.1 shown below suggests that the set $\{x_1, x_2\}$ where $x_1 = y_1$ and $x_2 = y_2 - cy_1$, will work if c is properly chosen.

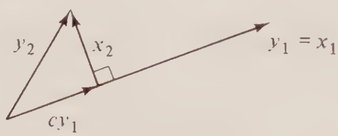

figure 7.1

To find c we need only to solve the following equation.

$$0 = (x_2, y_1) = (y_2 - cy_1, y_1) = (y_2, y_1) - c(y_1, y_1)$$

So

$$c = \frac{(y_2, y_1)}{\|y_1\|^2}.$$

Thus

$$x_2 = y_2 - \frac{(y_2, y_1)}{\|y_1\|^2} y_1.$$

This process can be extended to any finite linearly independent subset.

Theorem 7.4. *Let V be an inner product space, and let $S = \{y_1, \ldots, y_n\}$ be a linearly independent subset of V. Define $S' = \{x_1, \ldots, x_n\}$, where $x_1 = y_1$, and*

$$x_k = y_k - \sum_{j=1}^{k-1} \frac{(y_k, x_j)}{\|x_j\|^2} x_j \qquad \text{for } 2 \le k \le n. \tag{1}$$

Then S′ is an orthogonal set of non-zero vectors such that span(S′) = span(S).

PROOF. The proof will be by induction on n. Let $S_n = \{y_1, \ldots, y_n\}$. If $n = 1$, then the theorem is proved by taking $S'_1 = S_1$; i.e., $x_1 = y_1 \neq 0$. Assume then that the set $S'_k = \{x_1, \ldots, x_k\}$ has been constructed by the use of Eq. (1) with the desired properties. We shall show that the set $S'_{k+1} = \{x_1, \ldots, x_k, x_{k+1}\}$ also has the desired properties, where

$$x_{k+1} = y_{k+1} - \sum_{j=1}^{k} \frac{(y_{k+1}, x_j)}{||x_j||^2} x_j. \tag{2}$$

If $x_{k+1} = 0$, then Eq. (2) would imply that $y_{k+1} \in \text{span}(S'_k) = \text{span}(S_k)$, which contradicts the assumption that S_{k+1} is linearly independent.

For $1 \leq i \leq k$ we have from Eq. (2) that

$$\begin{aligned}(x_{k+1}, x_i) &= (y_{k+1}, x_i) - \sum_{j=1}^{k} \frac{(y_{k+1}, x_j)}{||x_j||^2}(x_j, x_i) \\ &= (y_{k+1}, x_i) - \frac{(y_{k+1}, x_i)}{||x_i||^2}||x_i||^2 = 0,\end{aligned}$$

since $(x_j, x_i) = 0$ if $i \neq j$ by the inductive assumption that S'_k is orthogonal. Hence S'_{k+1} is orthogonal. Now by Eq. (2) we have that $\text{span}(S'_{k+1}) \subseteq \text{span}(S_{k+1})$. But by Theorem 7.3, S'_{k+1} is linearly independent; so $\dim(\text{span}(S'_{k+1})) = k + 1 = \dim(\text{span}(S_{k+1}))$. Hence $\text{span}(S'_{k+1}) = \text{span}(S_{k+1})$. ∎

The construction of $\{x_1, \ldots, x_n\}$ by the use of Eq. (1) is called the *Gram-Schmidt orthogonalization process.*

Example 12. Let $V = R^3$, and let $y_1 = (1, 1, 0)$, $y_2 = (2, 0, 1)$, and $y_3 = (2, 2, 1)$. Then $\{y_1, y_2, y_3\}$ is linearly independent. We shall use Eq. (1) above to compute the orthogonal vectors x_1, x_2, and x_3. Take $x_1 = (1, 1, 0)$. Then $||x_1||^2 = 2$, and so

$$\begin{aligned}x_2 &= y_2 - \frac{(y_2, x_1)}{||x_1||^2}x_1 \\ &= (2, 0, 1) - \frac{2}{2}(1, 1, 0) \\ &= (1, -1, 1).\end{aligned}$$

Finally,

$$\begin{aligned}x_3 &= y_3 - \frac{(y_3, x_1)}{||x_1||^2}x_1 - \frac{(y_3, x_2)}{||x_2||^2}x_2 \\ &= (2, 2, 1) - \frac{4}{2}(1, 1, 0) - \frac{1}{3}(1, -1, 1) \\ &= \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right).\end{aligned}$$

Theorem 7.5. *Let* V *be a finite-dimensional inner product space. Then* V *has an orthonormal basis* β. *Furthermore, if* $\beta = \{x_1, x_2, \ldots, x_n\}$ *and* $x \in V$, *then*

$$x = \sum_{i=1}^{n} (x, x_i)x_i.$$

PROOF. Let β_0 be an ordered basis for V. Apply Theorem 7.4 to obtain an orthogonal set β' of non-zero vectors with $\text{span}(\beta') = \text{span}(\beta_0) = V$. By dividing each vector in β' by its length, we obtain an orthonormal set β that generates V. By Theorem 7.3, β is linearly independent, and therefore β is an orthonormal basis for V.

Let $\beta = \{x_1, \ldots, x_n\}$, and let $x \in V$. Then

$$x = \sum_{i=1}^{n} a_i x_i$$

for some scalars a_i. For $1 \leq j \leq n$ we have

$$\begin{aligned}(x, x_j) &= \left(\sum_{i=1}^{n} a_i x_i, x_j\right) = \sum_{i=1}^{n} a_i(x_i, x_j)\\ &= \sum_{i=1}^{n} a_i \delta_{ij} = a_j. \quad \blacksquare\end{aligned}$$

Example 13. Using the orthogonal set obtained in Example 12, we may obtain the orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{3}}(1, -1, 1), \frac{1}{\sqrt{6}}(-1, 1, 2)\right\}.$$

Let $x = (2, 1, 3)$. We shall compute the "coefficients" of x as given by Theorem 7.5:

$$a_1 = \frac{1}{\sqrt{2}}(2 + 1) = \frac{3}{\sqrt{2}}, \qquad a_2 = \frac{1}{\sqrt{3}}(2 - 1 + 3) = \frac{4}{\sqrt{3}},$$

and

$$a_3 = \frac{1}{\sqrt{6}}(-2 + 1 + 6) = \frac{5}{\sqrt{6}}.$$

As a check, we have

$$(2, 1, 3) = \frac{3}{2}(1, 1, 0) + \frac{4}{3}(1, -1, 1) + \frac{5}{6}(-1, 1, 2).$$

Thus we have a very simple way of computing the coefficients of a given vector when expressed as a linear combination of vectors in an orthonormal basis.

The same theorem provides an easy method for computing the matrix representation of a linear operator.

Corollary. *Let* V *be a finite-dimensional inner product space with an orthonormal basis* $\beta = \{x_1, \ldots, x_n\}$. *Let* T *be a linear operator on* V, *and let* $A = [T]_\beta$. *Then* $A_{ij} = (T(x_j), x_i)$.

PROOF. From Theorem 7.5 we have

$$T(x_j) = \sum_{i=1}^{n} (T(x_j), x_i)x_i.$$

Hence $A_{ij} = (T(x_j), x_i)$. ▮

The scalars (x, x_i) associated with x have been studied extensively for certain special vector spaces. Although the vectors $x_1, \ldots, x_n$ were chosen from an orthonormal basis, we shall consider more general sets β for the definition of the scalars (x, x_i).

Definition. *Let* β *be an orthonormal subset (possibly infinite) of an inner product space* V, *and let* $x \in V$. *We define the* Fourier coefficients of x relative to β *to be the scalars* (x, y), *where* $y \in \beta$.

In the nineteenth century the French mathematician Jean Baptiste Fourier was associated with the study of the coefficients

$$\int_0^{2\pi} f(t) \sin nt \, dt \quad \text{and} \quad \int_0^{2\pi} f(t) \cos nt \, dt,$$

or more generally,

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt,$$

of a function f. In the context of Example 9, we see that $c_n = (f, e^{inx})$; that is, c_n is the nth Fourier coefficient of a continuous function $f \in$ H relative to S. These coefficients are the "classical" Fourier coefficients of a function, and the literature concerning the behavior of these coefficients is extensive. We shall learn more about these Fourier coefficients in the remainder of this chapter.

Example 14. In H define $f(x) = x$. We shall compute the Fourier coefficients of f relative to the orthonormal set S of Example 9. Using integration by parts, we have, for $n \neq 0$,

$$(f, e^{inx}) = \frac{1}{2\pi} \int_0^{2\pi} t\overline{e^{int}} \, dt = \frac{1}{2\pi} \int_0^{2\pi} te^{-int} \, dt = \frac{-1}{in}.$$

And, for $n = 0$,

$$(f, 1) = \frac{1}{2\pi} \int_0^{2\pi} t(1) \, dt = \pi.$$

Now by Exercise 14 we have that

$$\|f\|^2 \geq \sum_{n=1}^{k} |(f, e^{inx})|^2$$

for every k. Thus, using the fact that $\|f\|^2 = \frac{4}{3}\pi^2$, we have

$$\frac{4}{3}\pi^2 \geq \sum_{n=1}^{k} \left|\frac{-1}{in}\right|^2 = \sum_{n=1}^{k} \frac{1}{n^2}.$$

Since this inequality is true for all k, we have by the appropriate use of limits that

$$\frac{4}{3}\pi^2 \geq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Other results may be obtained similarly by using other functions.

We are now ready to proceed with the concept of an "orthogonal complement."

Definition. *Let* V *be an inner product space, and let* S *be a subset of* V. *We define* $S^\perp$ (*read* "S perp") *to be the set of all those vectors in* V *that are orthogonal to every vector in* S; *that is,* $S^\perp = \{x \in V\colon\ (x, y) = 0$ *for all* $y \in S\}$. $S^\perp$ *is called the* orthogonal complement of S.

It is easy to show that $S^\perp$ is a subspace of V for any subset S of V.

Example 15. The reader should verify that $\{0\}^\perp = V$ and $V^\perp = \{0\}$.

Example 16. If $V = R^3$ and $S = \{x\}$, then $S^\perp$ is simply the set of all vectors that are perpendicular to x. (See Exercise 5.)

Exercise 16 provides an interesting example of an orthogonal complement in the case that V is infinite-dimensional.

The use of the word "complement" becomes clearer with the next theorem.

Theorem 7.6. *Let* W *be a finite-dimensional subspace of an inner product space* V. *Then* $V = W \oplus W^\perp$.

PROOF. We may choose an orthonormal basis $\{x_1, \ldots, x_k\}$ for W by Theorem 7.5. For $y \in V$, define

$$y_1 = \sum_{i=1}^{k} (y, x_i)x_i \quad \text{and} \quad y_2 = y - y_1.$$

Clearly $y = y_1 + y_2$ and $y_1 \in W$. In order to show that $V = W + W^\perp$, we must prove that $y_2 \in W^\perp$. It is sufficient to prove that $(y_2, x_j) = 0$ for

$j = 1, \ldots, k$. Now

$$(y_2, x_j) = (y - y_1, x_j) = (y, x_j) - (y_1, x_j).$$

But

$$(y_1, x_j) = \left(\sum_{i=1}^{k} (y, x_i)x_i, x_j\right) = \sum_{i=1}^{k} (y, x_i)(x_i, x_j) = \sum_{i=1}^{k} (y, x_i)\delta_{ij} = (y, x_j).$$

Hence $(y_2, x_j) = 0$.

To complete the proof, we must show that $\mathsf{W} \cap \mathsf{W}^\perp = \{0\}$. But if $x \in \mathsf{W} \cap \mathsf{W}^\perp$, then $(x, x) = 0$. Thus $x = 0$. ▌

The following result is an immediate consequence of the proof of Theorem 7.6.

Corollary 1. *Under the hypotheses of Theorem 7.6 if $\{x_1, \ldots, x_k\}$ is an orthonormal basis for* W *and if $y \in \mathsf{V}$, then*

$$y = \sum_{i=1}^{k} (y, x_i)x_i + z,$$

where $z \in \mathsf{W}^\perp$.

Corollary 2. *Let* V *be a finite-dimensional inner product space, and let* W *be a subspace of* V. *Then $\dim(\mathsf{W}) + \dim(\mathsf{W}^\perp) = \dim(\mathsf{V})$.*

Example 17. Let $\mathsf{V} = \mathsf{F}^3$ and $\mathsf{W} = \text{span}(\{e_1, e_2\})$. Then $x = (a, b, c) \in \mathsf{W}^\perp$ if and only if $0 = (x, e_1) = a$ and $0 = (x, e_2) = b$. So $x = (0, 0, c)$, and therefore $\mathsf{W}^\perp = \text{span}(\{e_3\})$. One could deduce the same result by simply noting from Corollary 2 that $\dim(\mathsf{W}^\perp) = 1$ and that the vector e_3 is orthogonal to both e_1 and e_2.

EXERCISES

1. Label the following statements as being true or false.
   (a) The Gram-Schmidt orthogonalization process allows us to construct an orthonormal set from an arbitrary set of vectors.
   (b) Every finite-dimensional inner product space possesses an orthonormal basis.
   (c) The orthogonal complement of any set is a subspace.
   (d) If $\beta = \{x_1, \ldots, x_n\}$ is a basis for an inner product space V, then for any $x \in \mathsf{V}$ the scalars (x, x_i) $(i = 1, \ldots, n)$ are the Fourier coefficients of x.
   (e) For any subspace W of a finite-dimensional inner product space V, we have $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^\perp$.
   (f) An orthonormal basis must be an ordered basis.

2. In each of the following parts, apply the Gram-Schmidt process to the given subset S of the inner product space V. Then find an orthonormal basis β for V and compute the Fourier coefficients of the given vector relative to β. Finally, use Theorem 7.5 to verify your result.
   (a) $\mathsf{V} = \mathsf{R}^3$, $S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$, and $x = (1, 1, 2)$
   (b) $\mathsf{V} = \mathsf{R}^3$, $S = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$, and $x = (1, 0, 1)$
   (c) $\mathsf{V} = \mathsf{P}_2(R)$ with the inner product $(f, g) = \int_0^1 f(t)g(t)\,dt$, $S = \{1, x, x^2\}$, and $f(x) = 1 + x$
   (d) $\mathsf{V} = \mathsf{C}^3$, $S = \{(1, i, 0), (1 - i, 2, 4i)\}$, and $x = \{(i, 2 + 3i, 1)\}$
3. Let $\mathsf{V} = \mathsf{R}^2$ and let
$$\beta = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}.$$
   Find the Fourier coefficients of (3, 4) relative to β.
4. Let $\mathsf{V} = \mathsf{C}^3$, and let $S = \{(1, 0, i), (1, 2, 1)\}$. Compute $S^\perp$.
5. Let $\mathsf{V} = \mathsf{R}^3$, and let $S = \{x_0\}$, where $x_0 \neq 0$. Describe $S^\perp$ geometrically. If $\{x_1, x_2\} = S_0$ is linearly independent, describe $S_0^\perp$ geometrically.
6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin \mathsf{W}$, prove that there exists $y \in \mathsf{V}$ such that $y \in \mathsf{W}^\perp$ but $(x, y) \neq 0$. *Hint:* Use Corollary 1 of Theorem 7.6.
7. Prove that if $\{y_1, \ldots, y_n\}$ is an orthogonal set of non-zero vectors, then the vectors $\{x_1, \ldots, x_n\}$ derived from the Gram-Schmidt process satisfy $x_i = y_i$ for $i = 1, \ldots, n$. *Hint:* Use induction.
8. Let $\mathsf{V} = \mathsf{C}^3$ with the standard inner product, and let $\mathsf{W} = \text{span}(\{(i, 0, 1)\})$. Find orthonormal bases for W and $\mathsf{W}^\perp$.
9. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W such that $\mathsf{N}(\mathsf{T}) = \mathsf{W}^\perp$. In addition, prove that $\|\mathsf{T}(x)\| \leq \|x\|$ for all $x \in \mathsf{V}$. *Hint:* Use Exercise 10 of Section 7.1.
10. Let A be an $n \times n$ matrix with complex entries such that the rows of A form an orthonormal set. Prove that $AA^* = I$.
11. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(\mathsf{W}_1 + \mathsf{W}_2)^\perp = \mathsf{W}_1^\perp \cap \mathsf{W}_2^\perp$ and $(\mathsf{W}_1 \cap \mathsf{W}_2)^\perp = \mathsf{W}_1^\perp + \mathsf{W}_2^\perp$.
12.† Let V be an inner product space, and let S and S_0 be subsets of V. Prove the following:
   (a) $S_0 \subseteq S$ implies $S^\perp \subseteq S_0^\perp$.
   (b) $S \subseteq (S^\perp)^\perp$, and so $\text{span}(S) \subseteq (S^\perp)^\perp$.
   (c) If W is a finite-dimensional subspace of V, then $\mathsf{W} = (\mathsf{W}^\perp)^\perp$. *Hint:* Use Exercise 6.

13. *Parseval's Identity.* Let $\{x_1, \ldots, x_n\}$ be an orthonormal basis for V. For any $x, y \in$ V prove that

$$(x, y) = \sum_{i=1}^{n} (x, x_i)\overline{(y, x_i)}.$$

14. Let V be an inner product space, and let $S = \{x_1, \ldots, x_n\}$ be any orthonormal subset of V. Prove that for any x in V we have

$$\|x\|^2 \geq \sum_{i=1}^{n} |(x, x_i)|^2.$$

This inequality is called *Bessel's inequality. Hint:* Apply Corollary 1 of Theorem 7.6 to $x \in$ V and W = span(S). Then use Exercise 10 of Section 7.1.

15. Let T be a linear operator on a finite-dimensional inner product space V. If $(\mathsf{T}(x), y) = 0$ for all $x, y \in$ V, prove that $\mathsf{T} = \mathsf{T}_0$. In fact, prove this result if the equality holds for all x and y in some basis for V.

16. Let V = C([−1, 1]). Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $\mathsf{W}_e^\perp = \mathsf{W}_o$ if the inner product on V is

$$(f, g) = \int_{-1}^{1} f(t)g(t)\,dt.$$

7.3 THE ADJOINT OF A LINEAR OPERATOR

In Section 7.1 we defined the conjugate transpose A^* of a matrix A. For a linear operator T on an inner product space V, we shall now define a related linear operator on V called the "adjoint" of T, whose matrix is $[\mathsf{T}]_\beta^*$, where β is any orthonormal basis for V. The analogy between complex conjugation of complex numbers and adjoints of linear operators will become apparent. We shall first need a preliminary result, however.

Let V be an inner product space, and let $y \in$ V. The function $\mathsf{g}\colon \mathsf{V} \longrightarrow F$ defined by $\mathsf{g}(x) = (x, y)$ for all $x \in$ V is clearly linear. More interesting is the fact that if V is finite-dimensional, every linear transformation from V into F is of this form.

Theorem 7.7. *Let* V *be a finite-dimensional inner product space over* F, *and let* $\mathsf{g}\colon \mathsf{V} \longrightarrow \mathsf{F}$ *be a linear transformation. Then there exists a unique vector* $\mathsf{y} \in \mathsf{V}$ *such that* $\mathsf{g(x) = (x, y)}$ *for all* $\mathsf{x} \in \mathsf{V}$.

PROOF. Let β be an orthonormal basis for V, say $\beta = \{x_1, \ldots, x_n\}$, and let

$$y = \sum_{i=1}^{n} \overline{\mathsf{g}(x_i)}x_i.$$

If we define $h: V \longrightarrow F$ by $h(x) = (x, y)$, then h is clearly linear. Now for $1 \leq j \leq n$ we have

$$h(x_j) = (x_j, y) = \left(x_j, \sum_{i=1}^{n} \overline{g(x_i)} x_i\right) = \sum_{i=1}^{n} g(x_i)(x_j, x_i)$$

$$= \sum_{i=1}^{n} g(x_i)\delta_{ji} = g(x_j).$$

Since g and h both agree on β, we have g = h by the corollary to Theorem 2.7.

To show that y is unique, suppose that $g(x) = (x, y')$ for all x. Then $(x, y) = (x, y')$ for all x, and so by Theorem 7.1 we have $y = y'$. ∎

Example 18. Define $g: R^2 \longrightarrow R$ by $g(a_1, a_2) = 2a_1 + a_2$; clearly g is a linear transformation. Let $\beta = \{e_1, e_2\}$, and, as in the proof of Theorem 7.7, let $y = g(e_1)e_1 + g(e_2)e_2 = 2e_1 + e_2 = (2, 1)$. Then $g(a_1, a_2) = ((a_1, a_2), (2, 1)) = 2a_1 + a_2$.

Theorem 7.8. Let V *be a finite-dimensional inner product space, and let* T *be a linear operator on* V*. Then there exists a unique function* $T^*: V \rightarrow V$ *such that* $(T(x), y) = (x, T^*(y))$ *for all* $x, y \in V$. Moreover T^* is linear.

PROOF. Let $y \in V$. Define $g: V \longrightarrow F$ by $g(x) = (T(x), y)$ for all $x \in V$. We shall first show that g is linear. Let $x_1, x_2 \in V$ and $c \in F$. Then $g(cx_1 + x_2) = (T(cx_1 + x_2), y) = (cT(x_1) + T(x_2), y) = c(T(x_1), y) + (T(x_2), y) = cg(x_1) + g(x_2)$. Hence g is linear.

Now we may apply Theorem 7.7 to obtain a unique vector $y' \in V$ such that $g(x) = (x, y')$; i.e., $(T(x), y) = (x, y')$, for all $x \in V$. Defining $T^*: V \longrightarrow V$ by $T^*(y) = y'$, we have $(T(x), y) = (x, T^*(y))$.

To show that T^* is linear, let $y_1, y_2 \in V$ and $c \in F$. Then for any $x \in V$, we have

$$\begin{aligned}(x, T^*(cy_1 + y_2)) &= (T(x), cy_1 + y_2)\\ &= \bar{c}(T(x), y_1) + (T(x), y_2)\\ &= \bar{c}(x, T^*(y_1)) + (x, T^*(y_2))\\ &= (x, cT^*(y_1) + T^*(y_2)).\end{aligned}$$

Since x is arbitrary, we have $T^*(cy_1 + y_2) = cT^*(y_1) + T^*(y_2)$ by Theorem 7.1(d).

Finally, we need only show that T^* is unique. Suppose that $U: V \longrightarrow V$ is linear and satisfies $(T(x), y) = (x, U(y))$ for all $x, y \in V$. Then $(x, T^*(y)) = (x, U(y))$ for all $x, y \in V$, and so $T^* = U$. ∎

The linear operator T^* described in Theorem 7.8 is called the *adjoint* of the operator T. The symbol T^* is read "T star."

Thus T^* is the unique operator on V satisfying $(\mathsf{T}(x), y) = (x, \mathsf{T}^*(y))$ for all $x, y \in \mathsf{V}$. Note that we also have

$$(x, \mathsf{T}(y)) = \overline{(\mathsf{T}(y), x)} = \overline{(y, \mathsf{T}^*(x))} = (\mathsf{T}^*(x), y),$$

and so $(x, \mathsf{T}(y)) = (\mathsf{T}^*(x), y)$ for all $x, y \in \mathsf{V}$. We may view these equations symbolically as adding a * to T when we shift its position inside the inner product symbol.

In the infinite-dimensional case the adjoint of a linear operator on T may be defined to be the function T^* such that $(\mathsf{T}(x), y) = (x, \mathsf{T}^*(y))$ for all $x, y \in \mathsf{V}$. The uniqueness and linearity of T^* will follow as before. However, the existence of an adjoint is not guaranteed. The reader should observe the necessity of the hypothesis of finite-dimensionality in the proof of Theorem 7.7. Many of the theorems we shall prove about adjoints, nevertheless, are independent of the dimension of V. *Thus for the remainder of this chapter we shall adopt the convention for the exercises that a reference to the adjoint of a linear operator on an infinite-dimensional inner product space assumes its existence.*

A useful result for computing adjoints is Theorem 7.9 below.

Theorem 7.9. *Let* V *be a finite-dimensional inner product space, and let* β *be an orthonormal basis for* V. *If* T *is a linear operator on* V, *then*

$$[\mathsf{T}^*]_\beta = [\mathsf{T}]_\beta^*.$$

PROOF. Let $A = [\mathsf{T}]_\beta$, $B = [\mathsf{T}^*]_\beta$, and $\beta = \{x_1, \ldots, x_n\}$. Then from the corollary to Theorem 7.5 we have

$$\begin{aligned} B_{ij} &= (\mathsf{T}^*(x_j), x_i) = \overline{(x_i, \mathsf{T}^*(x_j))} \\ &= \overline{(\mathsf{T}(x_i), x_j)} = \overline{A_{ji}} = (A^*)_{ij}. \end{aligned}$$

Hence $B = A^*$. ∎

Corollary. *Let* A *be an* $n \times n$ *matrix. Then* $\mathsf{L}_{A^*} = (\mathsf{L}_A)^*$.

PROOF. If β is the standard ordered basis for F^n, then by Theorem 2.17 we have that $[\mathsf{L}_A]_\beta = A$. Hence $[(\mathsf{L}_A)^*]_\beta = [\mathsf{L}_A]_\beta^* = A^* = [\mathsf{L}_{A^*}]_\beta$, and so $(\mathsf{L}_A)^* = \mathsf{L}_{A^*}$. ∎

As an application of the above, we shall compute the adjoint of a specific linear operator.

Example 19. Define $\mathsf{T}\colon \mathsf{C}^2 \to \mathsf{C}^2$ by $\mathsf{T}(a_1, a_2) = (2ia_1 + 3a_2, a_1 - a_2)$. If β is the standard ordered basis for C^2, then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 2i & 3 \\ 1 & -1 \end{pmatrix}.$$

So

$$[\mathsf{T}^*]_\beta = [\mathsf{T}]_\beta^* = \begin{pmatrix} -2i & 1 \\ 3 & -1 \end{pmatrix}.$$

Hence

$$\mathsf{T}^*(a_1, a_2) = (-2ia_1 + a_2, 3a_1 - a_2).$$

The following theorem demonstrates the analogy between complex conjugates of complex numbers and the adjoints of linear operators.

Theorem 7.10. *Let* V *be a finite-dimensional inner product space, and let* T *and* U *be linear operators on* V. *Then*

(a) $(\mathsf{T} + \mathsf{U})^* = \mathsf{T}^* + \mathsf{U}^*$.
(b) $(c\mathsf{T})^* = \bar{c}\mathsf{T}^*$ *for any* $c \in F$.
(c) $(\mathsf{TU})^* = \mathsf{U}^*\mathsf{T}^*$.
(d) $\mathsf{T}^{**} = \mathsf{T}$.
(e) $\mathsf{I}^* = \mathsf{I}$.

PROOF. We shall prove (a) and (d); the rest are proved similarly. Let $x, y \in \mathsf{V}$.

Since

$$\begin{aligned}(x, (\mathsf{T} + \mathsf{U})^*(y)) &= ((\mathsf{T} + \mathsf{U})(x), y) = (\mathsf{T}(x) + \mathsf{U}(x), y) \\ &= (\mathsf{T}(x), y) + (\mathsf{U}(x), y) = (x, \mathsf{T}^*(y)) + (x, \mathsf{U}^*(y)) \\ &= (x, \mathsf{T}^*(y) + \mathsf{U}^*(y)) = (x, (\mathsf{T}^* + \mathsf{U}^*)(y)),\end{aligned}$$

(a) follows.

Likewise since

$$\begin{aligned}(x, \mathsf{T}(y)) &= (\mathsf{T}^*(x), y) \\ &= (x, \mathsf{T}^{**}(y)),\end{aligned}$$

(d) follows. ∎

The same proof works in the infinite-dimensional case provided that the existence of T^* and U^* is assumed.

Corollary. *Let* A *and* B *be* $n \times n$ *matrices. Then*

(a) $(A + B)^* = A^* + B^*$.
(b) $(cA)^* = \bar{c}A^*$ *for all* $c \in F$.
(c) $(AB)^* = B^*A^*$.
(d) $A^{**} = A$.
(e) $I^* = I$.

PROOF. We shall prove only (c); the remaining parts can be proved similarly.

Since $\mathsf{L}_{(AB)^*} = (\mathsf{L}_{AB})^* = (\mathsf{L}_A\mathsf{L}_B)^* = (\mathsf{L}_B)^*(\mathsf{L}_A)^* = \mathsf{L}_{B^*}\mathsf{L}_{A^*} = \mathsf{L}_{B^*A^*}$, we have $(AB)^* = B^*A^*$. ∎

In the proof above we relied on the corollary to Theorem 7.9. An alternate proof could be given by appealing directly to the definition of the conjugate transpose. (See Exercise 5.)

EXERCISES

1. Label the following statements as being true or false. Assume that the underlying inner product spaces are finite-dimensional.
(a) Every linear operator has an adjoint.
(b) Every linear operator on V has the form $x \longrightarrow (x, y)$ for some $y \in \mathsf{V}$.
(c) For every linear operator T on V and every basis β of V, $[\mathsf{T}^*]_\beta = ([\mathsf{T}]_\beta)^*$.
(d) The adjoint of a linear operator is always unique.
(e) For any operators T and U and scalars a and b,

$$(a\mathsf{T} + b\mathsf{U})^* = a\mathsf{T}^* + b\mathsf{U}^*.$$

(f) For any $n \times n$ matrix A, $(\mathsf{L}_A)^* = \mathsf{L}_{A^*}$.
(g) For any operator T, $(\mathsf{T}^*)^* = \mathsf{T}$.

2. For each of the following inner product spaces V (over F) and linear transformations $\mathsf{g}\colon \mathsf{V} \longrightarrow F$, find a vector y such that $\mathsf{g}(x) = (x, y)$ for all $x \in \mathsf{V}$.
(a) $\mathsf{V} = \mathsf{R}^3$, $\quad \mathsf{g}(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$
(b) $\mathsf{V} = \mathsf{C}^2$, $\quad \mathsf{g}(z_1, z_2) = z_1 - 2z_2$
(c) $\mathsf{V} = \mathsf{P}_2(R)$ with $(f, h) = \int_0^1 f(t)h(t)\,dt$, $\quad \mathsf{g}(f) = f(0) + f'(1)$

3. For each of the following inner product spaces V and linear operators T on V, evaluate T* at the given element of V.
(a) $\mathsf{V} = \mathsf{R}^2$, $\quad \mathsf{T}(a, b) = (2a + b, a - 3b)$, $\quad x = (3, 5)$
(b) $\mathsf{V} = \mathsf{C}^2$, $\quad \mathsf{T}(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1)$,

$$x = (3 - i, 1 + 2i)$$

(c) $\mathsf{V} = \mathsf{P}_2(R)$ with

$$(f, g) = \int_0^1 f(t)g(t)\,dt, \qquad \mathsf{T}(f) = f' + 3f,$$

$$f(x) = 4 - x + 3x^2$$

4. Complete the proof of Theorem 7.10.

5. Complete the proof of the corollary to Theorem 7.10 in two ways. First use Theorem 7.10 as in the proof of (c). Then use the matrix definition of A^*.

6. Let T be a linear operator on an inner product space V. Let $\mathsf{U}_1 = \mathsf{T} + \mathsf{T}^*$ and $\mathsf{U}_2 = \mathsf{T}\mathsf{T}^*$. Prove that $\mathsf{U}_1 = \mathsf{U}_1^*$ and $\mathsf{U}_2 = \mathsf{U}_2^*$.

7. Give an example of a linear operator T on an inner product space V such that $N(T) \neq N(T^*)$.

8. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

9. Prove that if $V = W \oplus W^{\perp}$ and T is the projection on W with $N(T) = W^{\perp}$, then $T = T^*$.

10. Let T be a linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $(T(x), T(y)) = (x, y)$ for all $x, y \in V$. *Hint:* Use Exercise 20 of Section 7.1.

11. For a linear operator T on an inner product space V, prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

12.† Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that $R(T^*) = N(T)^{\perp}$. *Hint:* Prove that $R(T^*)^{\perp} = N(T)$, and then use Exercise 12(c) of Section 7.2.

13. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following.
   (a) $N(T^*T) = N(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.
   (b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce from (a) that $\text{rank}(TT^*) = \text{rank}(T)$.
   (c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

14. Let V be an inner product space, and let $y, z \in V$. Define $T: V \longrightarrow V$ by $T(x) = (x, y)z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and define it explicitly.

15. Let $T: V \longrightarrow W$ be a linear transformation between finite-dimensional inner product spaces V and W.
   (a) Prove that there exists a unique linear transformation $T^*: W \longrightarrow V$ such that $(T(x), y) = (x, T^*(y))$ for all $x \in V$ and $y \in W$.
   (b) Let β and γ be orthonormal bases for V and W, respectively. Prove that $[T^*]_{\gamma}^{\beta} = ([T]_{\beta}^{\gamma})^*$.

16. Let A be an $n \times n$ matrix. Prove that $\det(A^*) = \overline{\det(A)}$.

7.4* EINSTEIN'S SPECIAL THEORY OF RELATIVITY

As a result of physical experiments performed in the latter half of the nineteenth century (most notably the Michelson-Morley experiment of 1887) physicists concluded that *the results obtained in measuring the speed of light are independent of the velocity of the instrument used to measure the speed of light.* For example, suppose that while on earth an experimenter measures the speed of light emitted from the sun and finds it to be 186,000 miles per second. Now suppose that the experimenter places the measuring equipment in a spaceship and leaves the earth traveling at 100,000 miles per second in a direction away from the sun. A repetition of the same experiment from the spaceship would yield the same result: Light is traveling at 186,000 miles per second relative to the spaceship rather than 86,000 miles per second as one might expect!

This revelation led to a new way of relating coordinate systems used to locate events in space-time. The result was Albert Einstein's *special theory of relativity*. We shall develop via a linear algebra viewpoint the essence of Einstein's theory.

The basic problem is to compare two different non-accelerating coordinate systems that are in motion relative to each other under the assumption that the speed of light is the same when measured in either system. Suppose we are given two inertial (non-accelerating) coordinate systems S and S' in three-space (R^3) such that S' moves at a constant velocity in relation to S as measured from S (see Fig. 7.2). To simplify matters, let us suppose that

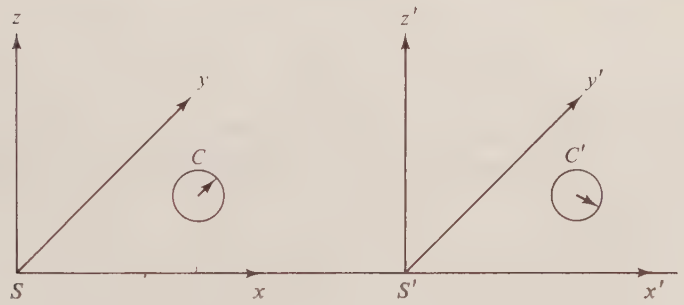

figure 7.2

1. The corresponding axes of S and S' (x and x', y and y', z and z') are parallel, and the origin of S' moves in the positive direction of the x-axis of S at a constant velocity $v > 0$ relative to S.
2. Two clocks C and C' are placed in space—the first stationary relative to coordinate system S, and the second stationary relative to S'. These clocks are designed to give as readings real numbers in units of seconds. The clocks are calibrated so that at the instant the origins of S and S' coincide both clocks give the reading zero.
3. Our unit of length shall be the *light second* (the distance light travels in one second) and our unit of time shall be the second. Note that with respect to these units the speed of light is 1 light second per second.

Given any event (something whose position and time of occurrence can be described), we may assign a set of "space-time coordinates" to it. For example, if p is an event that occurs at position

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

relative to S and at time t as read on clock C, we can assign to p the set of coordinates

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

This ordered 4-tuple is called *the space-time coordinates* of p relative to S and C. Likewise p has a set of space-time coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}$$

relative to S' and C'.

We can define a mapping $\mathsf{T}_v\colon\ \mathsf{R}^4 \longrightarrow \mathsf{R}^4$ (which depends on the velocity v) as a consequence of the above such that, for any set of space-time coordinates

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

measuring an event with respect to S and C,

$$\mathsf{T}_v\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}$$

is the set of space-time coordinates of this event with respect to S' and C'. Intuitively T_v is one-to-one and onto.

Einstein made certain assumptions about T_v that led to his special theory of relativity. We shall formulate an equivalent set of assumptions.

Axioms of the Special Theory of Relativity

R_1: The speed of any light beam, when measured in either coordinate system using a clock stationary relative to that coordinate system, is 1.

R_2: The mapping T_v: $R^4 \longrightarrow R^4$ is an isomorphism.

R_3: For any

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in R^4,$$

if

$$T_v \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix},$$

then $y' = y$ and $z' = z$.

R_4: For

$$T_v \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix},$$

x' and t' are independent of y and z; that is, if

$$T_v \begin{pmatrix} x \\ y_1 \\ z_1 \\ t \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \quad \text{and} \quad T_v \begin{pmatrix} x \\ y_2 \\ z_2 \\ t \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \\ z'' \\ t'' \end{pmatrix},$$

then $x'' = x'$ and $t'' = t'$.

R_5: The origin of S moves in the negative direction of the X' axis of S' at the constant velocity $-v < 0$ as measured from S'.

As we shall see, these five axioms (R_1, R_2, R_3, R_4, and R_5) completely characterize T_v. The operator T_v is called the *Lorentz transformation in direction* x. We intend to compute T_v and use it to study the curious phenomena of time contraction.

Theorem 7.11. On R^4

(a) $T_v(e_i) = e_i$ *for* $i = 2, 3$.

(b) *span*($\{e_2, e_3\}$) *is* T_v*-invariant.*

(c) *span*($\{e_1, e_4\}$) *is* T_v*-invariant.*

(d) *Both span*($\{e_2, e_3\}$) *and span*($\{e_1, e_4\}$) *are* T_v^**-invariant.*

(e) $T_v^*(e_i) = e_i$ *for* $i = 2, 3$.

PROOF.

(a) By axiom R_2

$$\mathsf{T}_v\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

and hence by axiom R_4 the first and fourth coordinates of

$$\mathsf{T}_v\begin{pmatrix}0\\a\\b\\0\end{pmatrix}$$

are both zero for any $a, b \in R$. Thus by axiom R_3

$$\mathsf{T}_v\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad\text{and}\quad \mathsf{T}_v\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

The proofs of (b), (c), and (d) are left as exercises.

(e) For any $j \neq 2$, $(\mathsf{T}_v^*(e_2), e_j) = (e_2, \mathsf{T}_v(e_j)) = 0$ by (a) and (c); for $j = 2$, $(\mathsf{T}_v^*(e_2), e_j) = (e_2, \mathsf{T}_v(e_2)) = (e_2, e_2) = 1$ by (a). We conclude that $\mathsf{T}_v^*(e_2)$ is a multiple of e_2, i.e., that $\mathsf{T}_v^*(e_2) = \lambda e_2$ for some $\lambda \in R$. Thus $1 = (e_2, e_2) = (e_2, \mathsf{T}_v(e_2)) = (\mathsf{T}_v^*(e_2), e_2) = (\lambda e_2, e_2) = \lambda$, and hence $\mathsf{T}_v^*(e_2) = e_2$. Similarly $\mathsf{T}_v^*(e_3) = e_3$. ∎

Suppose that at the instant the origins of S and S' coincide a light flash is emitted from their common origin. The event of the light flash when measured either relative to S and C or relative to S' and C' has space-time coordinates

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Let P be the set of all events whose space-time coordinates

$$\begin{pmatrix}x\\y\\z\\t\end{pmatrix}$$

relative to S and C are such that the flash is observable from the point with coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(as measured relative to S) at the time t (as measured on C). Let us characterize P in terms of x, y, z and t. Since the speed of light is 1, at any time $t \geq 0$ the light flash is observable from any point whose distance to the origin of S (as measured on S) is $t \cdot 1 = t$. These are precisely the points that lie on the sphere of radius t with center at the origin. The coordinates (relative to S) of such points satisfy the equation $x^2 + y^2 + z^2 = t^2$. Hence an event lies in P if and only if relative to S and C its space-time coordinates

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad (t \geq 0)$$

satisfy the equation $x^2 + y^2 + z^2 - t^2 = 0$. By virtue of axiom R_1 we can characterize P in terms of the space-time coordinates relative to S' and C' similarly: An event lies in P if and only if relative to S' and C' its space-time coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \qquad (t' \geq 0)$$

satisfy the equation $(x')^2 + (y')^2 + (z')^2 - (t')^2 = 0$.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Theorem 7.12. For any $\mathrm{w} \in \mathsf{R}^4$ *if* $(\mathsf{L}_\mathrm{A}(\mathrm{w}), \mathrm{w}) = 0$, *then* $(\mathsf{T}_\mathrm{v}^*\mathsf{L}_\mathrm{A}\mathsf{T}_\mathrm{v}(\mathrm{w}), \mathrm{w}) = 0$.

PROOF. Let

$$w = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathsf{R}^4,$$

and suppose that $(\mathsf{L}_A(w), w) = 0$.

CASE 1. $t \geq 0$. Since $(\mathsf{L}_A(w), w) = x^2 + y^2 + z^2 - t^2$, w is the set of coordinates of an event in P relative to S and C. Because

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}$$

are the space-time coordinates of the same event relative to S' and C', the discussion preceding Theorem 7.12 yields

$$(x')^2 + (y')^2 + (z')^2 - (t')^2 = 0.$$

Thus $(\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w), w) = (\mathsf{L}_A\mathsf{T}_v(w), \mathsf{T}_v(w)) = (x')^2 + (y')^2 + (z')^2 - (t')^2 = 0$, and the conclusion follows.

CASE 2. $t < 0$. The proof follows by applying Case 1 to $-w$. ▮

We now proceed to deduce information about T_v. Let

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

By Exercise 3, $\{w_1, w_2\}$ is an orthogonal basis for $\text{span}(\{e_1, e_4\})$, and $\text{span}(\{e_1, e_4\})$ is $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v$-invariant. The next result tells us even more.

Theorem 7.13. *There exist non-zero scalars* a *and* b *such that*

(a) $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_1) = aw_2$,

(b) $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_2) = bw_1$.

PROOF.

(a) Because $(\mathsf{L}_A(w_1), w_1) = 0$, $(\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_1), w_1) = 0$ by Theorem 7.12. Thus $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_1)$ is orthogonal to w_1. Since $\text{span}(\{e_1, e_4\}) = \text{span}(\{w_1, w_2\})$ is $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v$-invariant, $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_1)$ must lie in this set. But $\{w_1, w_2\}$ is an orthogonal basis for this subspace, and so $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_1)$ must be a multiple of w_2. Thus $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v(w_1) = aw_2$ for some scalar a. Since T_v and A are invertible, so is $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v$. Thus $a \neq 0$, proving (a). The proof of (b) is similar. ▮

Corollary. *Let* $\mathsf{B}_v = [\mathsf{T}_v]_\beta$, *where* β *is the standard ordered basis for* R^4. *Then*

(a) $\mathsf{B}_v^*A\mathsf{B}_v = A$.

(b) $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v = \mathsf{L}_A$.

We shall leave the proof of the corollary as an exercise. For hints, see Exercise 4.

Now consider the situation one second after the origins of S and S' have coincided as measured by the clock C. Since the origin of S' is moving along the x-axis at a velocity v as measured in S, its space-time coordinates relative to S and C are

$$\begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Likewise, the space-time coordinates for the origin of S' relative to S' and C' must be

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix}$$

for some $t' > 0$. Thus we have

$$\mathsf{T}_v \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix} \qquad \text{for some } t' > 0. \tag{3}$$

By the corollary to Theorem 7.13,

$$\left(\mathsf{T}_v^* \mathsf{L}_A \mathsf{T}_v \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = \left(\mathsf{L}_A \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = v^2 - 1. \tag{4}$$

But also

$$\left(\mathsf{T}_v^* \mathsf{L}_A \mathsf{T}_v \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = \left(\mathsf{L}_A \mathsf{T}_v \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix}, \mathsf{T}_v \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = \left(\mathsf{L}_A \begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix} \right) = -(t')^2. \tag{5}$$

Combining Eqs. (4) and (5), we conclude that

$$v^2 - 1 = -(t')^2, \quad \text{or} \quad t' = \sqrt{1 - v^2}. \tag{6}$$

Thus, from Eqs. (3) and (6), we obtain

$$\mathsf{T}_v \begin{pmatrix} v \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{1 - v^2} \end{pmatrix}. \tag{7}$$

Next recall that the origin of S moves in the negative direction of the x'-axis of S' at the constant velocity $-v < 0$ as measured from S'. (This fact is axiom R_5.) Consequently one second after the origins of S and S' have coincided as measured on clock C, there exists a time $t' > 0$ as measured on clock C' such that

$$\mathsf{T}_v\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-vt'\\0\\0\\t'\end{pmatrix}. \tag{8}$$

From Eq. (8) it follows in a manner similar to the derivation of Eq. (7) that

$$t' = \frac{1}{\sqrt{1-v^2}}, \tag{9}$$

and hence from Eqs. (8) and (9)

$$\mathsf{T}_v\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\dfrac{-v}{\sqrt{1-v^2}}\\0\\0\\\dfrac{1}{\sqrt{1-v^2}}\end{pmatrix}. \tag{10}$$

The following result is now easily proved using Eqs. (7) and (10) and Theorem 7.11.

Theorem 7.14. *Let β be the standard ordered basis for* R^4. *Then*

$$[\mathsf{T}_{\mathrm{v}}]_\beta = \mathbf{B}_{\mathrm{v}} = \begin{pmatrix}\dfrac{1}{\sqrt{1-\mathrm{v}^2}} & 0 & 0 & \dfrac{-\mathrm{v}}{\sqrt{1-\mathrm{v}^2}}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \dfrac{-\mathrm{v}}{\sqrt{1-\mathrm{v}^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-\mathrm{v}^2}}\end{pmatrix}.$$

Time Contraction

A most curious and paradoxical conclusion follows if we accept Einstein's theory, that of time contraction. Suppose that an astronaut leaves our solar system in a space vehicle traveling at a fixed velocity v as measured relative to our solar system. It follows from Einstein's theory that at the end of time t as measured on earth the time that will have passed on the space vehicle is only $t\sqrt{1-v^2}$. To establish this result, consider the coordinate systems S and S' and clocks C and C' that we studied above. Suppose that the origin of S' coincides with the space vehicle and the origin of S coincides with a point in the solar system (stationary relative to the sun) so that the origin of S and S' coincide and clocks C and C' read zero at the moment the astronaut embarks on his trip.

As viewed from S, the space-time coordinates of the vehicle at any time $t > 0$ as measured by C are

$$\begin{pmatrix} vt \\ 0 \\ 0 \\ t \end{pmatrix},$$

whereas as viewed from S' the space-time coordinates of the vehicle at any time $t' > 0$ as measured by C' are

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix}.$$

But if two sets of space-time coordinates

$$\begin{pmatrix} vt \\ 0 \\ 0 \\ t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix}$$

are to describe the same event, it must follow that

$$\mathsf{T}_v \begin{pmatrix} vt \\ 0 \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \dfrac{1}{\sqrt{1-v^2}} & 0 & 0 & \dfrac{-v}{\sqrt{1-v^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{-v}{\sqrt{1-v^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} vt \\ 0 \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ t' \end{pmatrix}.$$

From the equation above it follows that

$$\frac{-v^2 t}{\sqrt{1-v^2}} + \frac{t}{\sqrt{1-v^2}} = t' \quad \text{or} \quad t' = t\sqrt{1-v^2}. \tag{11}$$

This is the desired result.

A dramatic consequence of time contraction is provided in Exercise 9 at the end of this section.

Let us make one additional point. Suppose we consider units of distance and time more commonly used than the light second and second, such as the mile and the hour or the kilometer and the second. Let c denote the speed of light relative to our chosen units of distance and time. It is easily seen that if an object travels at a velocity v relative to a set of units, then it is traveling at a velocity v/c in units of light seconds per second. Thus for an arbitrary set of units of distance and time, Eq. (11) becomes

$$t' = t\sqrt{1 - \frac{v^2}{c^2}}.$$

EXERCISES

1. Prove (b), (c), and (d) of Theorem 7.11.
2. Complete the proof of Theorem 7.12 for the case $t < 0$.
3. For

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

show that

(a) $\{w_1, w_2\}$ is an orthogonal basis for span($\{e_1, e_4\}$).

(b) span($\{e_1, e_4\}$) is $\mathsf{T}_v^*\mathsf{L}_A\mathsf{T}_v$-invariant.

4. Prove the corollary to Theorem 7.13.

Hints:

(a) Prove that

$$B_v^* A B_v = \begin{pmatrix} p & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q & 0 & 0 & -p \end{pmatrix},$$

where

$$p = \frac{a+b}{2} \quad \text{and} \quad q = \frac{a-b}{2}.$$

(b) Show that $q = 0$ by using the fact that $B_v^* A B_v$ is self-adjoint.
(c) Apply Theorem 7.12 to

$$w = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

to show that $p = 1$.

5. Prove that

$$\mathsf{T}_v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{-v}{\sqrt{1-v^2}} \\ 0 \\ 0 \\ \dfrac{1}{\sqrt{1-v^2}} \end{pmatrix} \qquad \text{[Eq. (10)].}$$

Hint: Use a technique similar to the derivation of Eq. (7).

6. Given three coordinate systems, S, S', and S'' with corresponding axes (x, x', x''; y, y', y''; and z, z', z'') parallel and such that the x-, x'-, and x''-axes coincide. Suppose S' is moving past S at a velocity $v_1 > 0$ (as measured on S), S'' is moving past S' at a velocity $v_2 > 0$ (as measured on S'), and S'' is moving past S at a velocity $v_3 > 0$ (as measured on S), and that there are three clocks C, C', and C'' such that C is stationary relative to S, C' is stationary relative to S', and C'' is stationary relative to S''. Suppose that when measured on any of the three clocks all the origins of S, S', and S'' coincide at time 0. Assuming that $\mathsf{T}_{v_3} = \mathsf{T}_{v_2}\mathsf{T}_{v_1}$ (i.e., $B_{v_3} = B_{v_2}B_{v_1}$), prove that

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}.$$

Note that substituting $v_2 = 1$ in the equation above yields $v_3 = 1$. This tells us that the speed of light as measured in either S or S' is the same. Why would we be surprised if this were not the case?

7. Compute $(B_v)^{-1}$. Show $(B_v)^{-1} = B_{(-v)}$. Conclude that if S' moves at a negative velocity v relative to S, then $[\mathsf{T}_v]_\beta = B_v$, where B_v is of the form given in Theorem 7.14.

8. Suppose that an astronaut left earth in the year 1776 and traveled to a star 99 light years away from the earth at 99% of the speed of light and that upon reaching the star he immediately turned around and returned to earth at the same speed. Assuming Einstein's special theory of relativity, show that if the astronaut was 20 years old at the time of departure, then

he would return to the earth at age 48.2 in the year 1976. Explain the use of Exercise 7 in solving this problem.

9. Recall the moving space vehicle considered in the study of time contraction. Suppose the vehicle is moving toward a fixed star located on the x-axis of S at a distance b units from the origin of S. If the space vehicle moves toward the star at velocity v, earthlings (who remain "almost" stationary relative to S) will compute the time it takes for the vehicle to reach the star as $t = b/v$. Due to the phenomina of time contraction the astronaut will perceive a time span of $t' = t\sqrt{1 - v^2} = (b/v)\sqrt{1 - v^2}$. A paradox appears in that the astronaut perceives a time span inconsistent with a distance of b and a velocity of v. The paradox is resolved by observing that the distance from the solar system to the star as measured by the astronaut is less than b.

Assuming that the coordinate systems S and S' and clocks C and C' are as in the discussion of time contraction,

(a) Argue that at time t (as measured on C) the space-time coordinates of the star relative to S and C are

$$\begin{pmatrix} b \\ 0 \\ 0 \\ t \end{pmatrix}.$$

(b) Show that at time t (as measured on C) the space-time coordinates of the star relative to S' and C' are

$$\begin{pmatrix} \dfrac{b - vt}{\sqrt{1 - v^2}} \\ 0 \\ 0 \\ \dfrac{t - bv}{\sqrt{1 - v^2}} \end{pmatrix}.$$

(c) Setting

$$x' = \frac{b - tv}{\sqrt{1 - v^2}} \quad \text{and} \quad t' = \frac{t - bv}{\sqrt{1 - v^2}},$$

show that $x' = b\sqrt{1 - v^2} - t'v$.

This result may be interpreted to mean that at time t' as measured by the astronaut, the distance from the astronaut to the star as measured by the astronaut (see Fig. 7.3) is

$$b\sqrt{1 - v^2} - t'v.$$

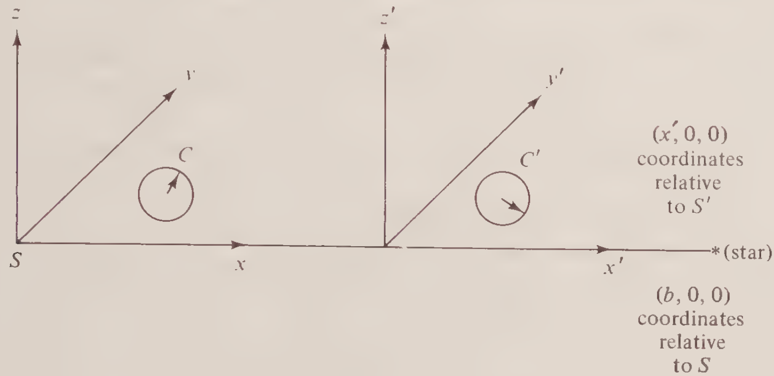

figure 7.3

(d) Conclude from this that
  (i) The speed of the space vehicle relative to the star as measured by the astronaut is v.
  (ii) The distance from earth to the star as measured by the astronaut is $b\sqrt{1 - v^2}$.

Thus distances along the line of motion of the space vehicle appear to be contracted by a factor of $\sqrt{1 - v^2}$.

7.5 NORMAL AND SELF-ADJOINT OPERATORS

In this section we shall prove that on any finite-dimensional inner product space there exists a class of linear operators that are determined entirely by their eigenvectors. Specifically, for such a linear operator T, there exists an orthonormal basis β of eigenvectors. Since the matrices that represent T and T* in the ordered basis β are diagonal matrices, they commute. Hence the condition TT* = T*T becomes necessary for the existence of such a β. We shall show that for complex inner product spaces this condition is also sufficient.

Definitions. *Let* V *be an inner product space, and let* T *be a linear operator on* V. *We say that* T *is* normal *if* TT* = T*T. *An* n × n *matrix* A *is* normal *if* AA* = A*A.

Note that if β is a finite orthonormal basis consisting of eigenvectors of T, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal. Of course, any diagonal matrix is normal.

If V is not finite-dimensional, then for T to be normal it is necessary that T* exist.

In order to construct an orthonormal basis of eigenvectors for a normal operator T, we must first prove that every such operator has at least one eigenvector. For this result we shall need the following theorem.

Theorem 7.15. *Let* V *be an inner product space, and let* T *be a normal operator on* V. *Then*

(a) $\|\mathsf{T}(\mathrm{x})\| = \|\mathsf{T}^*(\mathrm{x})\|$ *for all* $\mathrm{x} \in \mathsf{V}$.
(b) $\mathsf{T} - c\mathsf{I}$ *is normal for every* $c \in \mathsf{F}$.
(c) *If* λ *is an eigenvalue of* T, *then* $\bar{\lambda}$ *is an eigenvalue of* T^*. *In fact,* $\mathsf{T}(\mathrm{x}) = \lambda\mathrm{x}$ *implies that* $\mathsf{T}^*(\mathrm{x}) = \bar{\lambda}\mathrm{x}$.
(d) *If* λ_1 *and* λ_2 *are distinct eigenvalues of* T *with corresponding eigenvectors* x_1 *and* x_2, *then* x_1 *and* x_2 *are orthogonal.*

PROOF.

(a) For any $x \in \mathsf{V}$, we have

$$\begin{aligned}\|\mathsf{T}(x)\|^2 &= (\mathsf{T}(x), \mathsf{T}(x)) = (\mathsf{T}^*\mathsf{T}(x), x)\\ &= (\mathsf{T}\mathsf{T}^*(x), x) = (\mathsf{T}^*(x), \mathsf{T}^*(x)) = \|\mathsf{T}^*(x)\|^2.\end{aligned}$$

The proof of (b) is left as an exercise.

(c) Let $\mathsf{U} = \mathsf{T} - \lambda\mathsf{I}$, and suppose that $\mathsf{T}(x) = \lambda x$ for some $x \in \mathsf{V}$. Then $\mathsf{U}(x) = 0$, and by (a) and (b) we have

$$0 = \|\mathsf{U}(x)\| = \|\mathsf{U}^*(x)\| = \|(\mathsf{T}^* - \bar{\lambda}\mathsf{I})(x)\| = \|\mathsf{T}^*(x) - \bar{\lambda}x\|.$$

Hence $\mathsf{T}^*(x) = \bar{\lambda}x$.

(d) Let λ_1 and λ_2 be distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2. Then, using (c), we have

$$\begin{aligned}\lambda_1(x_1, x_2) &= (\lambda_1 x_1, x_2) = (\mathsf{T}(x_1), x_2)\\ &= (x_1, \mathsf{T}^*(x_2)) = (x_1, \bar{\lambda}_2 x_2) = \lambda_2(x_1, x_2).\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, we conclude that $(x_1, x_2) = 0$. ∎

Corollary 1. *Let* T *be a normal operator on an inner product space* V, *and let* β *be an orthonormal basis for* V. *Then* β *consists of eigenvectors of* T *if and only if* β *consists of eigenvectors of* T^*.

As we mentioned earlier, there is a strong parallel between the complex conjugate of a complex number and the adjoint of a linear operator. (See, for example, Theorem 7.10.) Real numbers may be characterized as those complex numbers that are equal to their complex conjugates. If we consider the condition $\mathsf{T} = \mathsf{T}^*$ for a linear operator, we shall see that many of the same properties of real numbers carry over to such operators. In fact, we shall see (in Exercise 5) that every operator may be written in the form $\mathsf{T}_1 + i\mathsf{T}_2$, where T_1 and T_2 satisfy the condition above. Also, all such operators have only real eigenvalues.

Definitions. *Let* V *be an inner product space, and let* T *be a linear operator on* V. T *is called a* self-adjoint (*or* Hermitian) *operator if* $\mathsf{T} = \mathsf{T}^*$. *An* $n \times n$ *matrix* A *is* self-adjoint (*or* Hermitian) *if* $A = A^*$.

So for real matrices, being self-adjoint is equivalent to being symmetric.

It is easy to see that if β is a finite orthonormal basis for V, then T is self-adjoint if and only if $[\mathsf{T}]_\beta$ is self-adjoint.

Note also that any diagonal matrix that has at least one non-real entry is normal but not self-adjoint. Self-adjoint matrices are, of course, normal.

Example 20. Let $\mathsf{V} = \mathsf{R}^2$; then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$, where $\mathsf{W}_1 = \text{span}(\{(1, 1)\})$ and $\mathsf{W}_2 = \text{span}(\{(0, 1)\})$. Let T be the projection on W_1 such that $\mathsf{N}(\mathsf{T}) = \mathsf{W}_2$; that is, $\mathsf{T}(a, b) = (a, a)$. If $\beta = \{e_1, e_2\}$, then

$$A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since

$$A^* = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

an easy computation shows that $AA^* \neq A^*A$. So T is neither self-adjoint nor normal.

We shall see in Section 7.5 that a special type of projection T, called an "orthogonal projection" (one having the property that $\mathsf{R}(\mathsf{T}) = \mathsf{N}(\mathsf{T})^\perp$), is always self-adjoint.

Corollary 2. *Let* T *be a self-adjoint linear operator on an inner product space* V. *If* λ *is an eigenvalue of* T, *then* λ *is a real number.*

PROOF. Let x be an eigenvector corresponding to the eigenvalue λ. By (c) of the preceding theorem we have

$$\lambda x = \mathsf{T}(x) = \mathsf{T}^*(x) = \bar{\lambda} x.$$

Since $x \neq 0$, we have that $\lambda = \bar{\lambda}$; so λ is real. ∎

In Theorem 7.16 below we shall prove that a certain type of linear operator always possesses an eigenvalue.

The reader will observe that the hypotheses of this theorem are divided into two cases, $F = R$ and $F = C$. The reason for this will be evident at that point in the proof where we want to obtain a zero of the characteristic polynomial. For, although many real-valued polynomials do not possess (real) zeros, the fundamental theorem of algebra (Appendix D) guarantees this result for complex-valued polynomials.

Theorem 7.16. *Let* V *be a finite-dimensional vector space over* F, *and let* T *be a linear operator on* V.

(a) *If* V *is a complex vector space (i.e., if* F = C), *then* T *has an eigenvalue.*

(b) *If* V *is a real inner product space (i.e., if* F = R) *and* T *is self-adjoint, then* T *has a (real) eigenvalue.*

PROOF. Assume that dim(V) = n, and let f denote the characteristic polynomial of T.

(a) If $F = C$, then the fundamental theorem of algebra guarantees that f has a zero. So T has an eigenvalue.

(b) If V is a real inner product space, let β be an orthonormal basis for V. Then $A = [\mathsf{T}]_\beta$ is self-adjoint and has real entries.

Define T_A: $\mathsf{C}^n \longrightarrow \mathsf{C}^n$ by $\mathsf{T}_A(x) = Ax$. From (a) it follows that T_A has an eigenvalue λ. Since the matrix of T_A in the standard ordered basis for C^n is A, we have that T_A is self-adjoint, and hence by Corollary 2 of Theorem 7.15 that λ is real. Thus the polynomial $f(t) = \det(A - tI)$ has the real zero λ, and so T has the eigenvalue λ. ∎

Theorem 7.17(C). *Let* V *be a finite-dimensional complex inner product space, and let* T *be a linear operator on* V. *Then* T *is normal if and only if* V *has an orthonormal basis consisting of eigenvectors of* T.

Theorem 7.17(R). *Let* V *be a finite-dimensional real inner product space, and let* T *be a linear operator on* V. *Then* T *is self-adjoint if and only if* V *has an orthonormal basis consisting of eigenvectors of* T.

PROOF. We shall first assume that T is either normal or self-adjoint and then produce the appropriate orthonormal basis. The proof will be by induction on $n = \dim(\mathsf{V})$.

If $n = 1$, then $\mathsf{V} = \text{span}(\{x\})$ for some $x \neq 0$. Clearly in this case $\{(1/\|x\|)x\}$ is an orthonormal basis consisting of an eigenvector of T.

Now assume that the result is true for normal [self-adjoint] operators on inner product spaces of dimension $n - 1$. We shall show that the result is true for the operator T on V.

By Theorem 7.16, T has an eigenvalue λ_1; let x_1 be an associated eigenvector. We may assume that $\|x_1\| = 1$. Let $\mathsf{W} = \text{span}(\{x_1\})$. By Theorem 7.15, x_1 is also an eigenvector of T*, so that W is clearly both T- and T*-invariant. By Exercise 6, $\mathsf{W}^\perp$ is also both T- and T*-invariant; hence by the same exercise, $\mathsf{T}_{\mathsf{W}^\perp}$ is normal [self-adjoint] since T is. From Corollary 2 of Theorem 7.6 we have $\dim(\mathsf{W}^\perp) = n - 1$. Thus we may apply the induction hypothesis to $\mathsf{T}_{\mathsf{W}^\perp}$ to produce an orthonormal basis $\{x_2, \ldots, x_n\}$ for $\mathsf{W}^\perp$ consisting of eigenvectors of $\mathsf{T}_{\mathsf{W}^\perp}$ and hence of T. It now follows easily that $\{x_1, \ldots, x_n\}$ is the desired orthonormal basis of V.

The first and most difficult part of the proof is complete. For the converse assume that β is an orthonormal basis for V consisting of eigenvectors of T. Then $[\mathsf{T}]_\beta$ is a diagonal matrix and hence by Theorem 7.9

$$[\mathsf{T}^*]_\beta = ([\mathsf{T}]_\beta)^*$$

is also a diagonal matrix. Now

$$[\mathsf{T}^*\mathsf{T}]_\beta = [\mathsf{T}^*]_\beta [\mathsf{T}]_\beta = [\mathsf{T}]_\beta [\mathsf{T}^*]_\beta = [\mathsf{T}\mathsf{T}^*]_\beta$$

since diagonal matrices commute. Thus $\mathsf{T}^*\mathsf{T} = \mathsf{T}\mathsf{T}^*$; i.e., T is normal.

If V is a real inner product space, then

$$[\mathsf{T}^*]_\beta = ([\mathsf{T}]_\beta)^* = ([\mathsf{T}]_\beta)^t = [\mathsf{T}]_\beta$$

since $[\mathsf{T}]_\beta$ is a diagonal matrix. Thus $\mathsf{T}^* = \mathsf{T}$; i.e., T is self-adjoint. ∎

In order to see why the condition of normality on a real inner product space is not sufficient to guarantee even one eigenvector, we need only consider rotations. Let $0 < \theta < \pi$, and define $\mathsf{T}\colon \ \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ to be the rotation by θ. The matrix of T in the standard ordered basis is

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It is easy to see that $AA^* = I = A^*A$ but that $A \neq A^*$. It is geometrically obvious that such a rotation has no eigenvectors.

We shall conclude this section with an example of a normal operator on a complex inner product space that has no eigenvectors! Thus the hypothesis that V be finite-dimensional is crucial in Theorem 7.17(C) as well as Theorem 7.16.

Example 21. Consider the inner product space H defined earlier, and let $x_k = e^{ikx}$. Suppose that $\mathsf{V} = \operatorname{span}(\{x_k\colon \ k \text{ is an integer}\})$. Clearly $\beta = \{x_k\colon \ k \text{ is an integer}\}$ is an orthonormal basis of V. Now choose linear operators T and U on V such that $\mathsf{T}(x_k) = x_{k+1}$ and $\mathsf{U}(x_k) = x_{k-1}$ for all integers k. Then

$$\begin{aligned}(\mathsf{T}(x_i), x_j) &= (x_{i+1}, x_j) = \delta_{(i+1)j} = \delta_{i(j-1)} \\ &= (x_i, x_{j-1}) = (x_i, \mathsf{U}(x_j)).\end{aligned}$$

It follows that $\mathsf{U} = \mathsf{T}^*$. Furthermore $\mathsf{T}\mathsf{T}^* = \mathsf{I} = \mathsf{T}^*\mathsf{T}$, and so T is normal. For any element $x \in \mathsf{V}$ we have that

$$x = \sum_{i=-k}^{k} a_i x_i$$

for some k and scalars a_i, and so

$$\mathsf{T}(x) = \sum_{i=-k}^{k} a_i x_{i+1}.$$

Since β is independent, it follows that T has no eigenvectors.

The condition that $\mathsf{TT}^* = \mathsf{I} = \mathsf{T}^*\mathsf{T}$, which appeared in the last two examples, will be considered in the next section.

EXERCISES

1. Label the following statements as being true or false. Assume that the underlying inner product spaces are finite-dimensional.
   (a) Every self-adjoint operator is normal.
   (b) Operators and their adjoints have the same eigenvectors.
   (c) If T is an operator on an inner product space V, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal, where β is any ordered basis for V.
   (d) A matrix A is normal if and only if L_A is normal.
   (e) The eigenvalues of a self-adjoint operator must all be real.
   (f) The identity and zero operators are self-adjoint.
   (g) Every normal operator is diagonalizable.
   (h) Every self-adjoint operator is diagonalizable.

2. For each of the linear operators below, determine whether they are normal, self-adjoint, or neither.
   (a) $\mathsf{T}\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ defined by $\mathsf{T}(a, b) = (2a - 2b, -2a + 5b)$
   (b) $\mathsf{T}\colon\ \mathsf{C}^2 \longrightarrow \mathsf{C}^2$ defined by $\mathsf{T}(a, b) = (2a + ib, a + 2b)$
   (c) $\mathsf{T}\colon\ \mathsf{P}_2(R) \longrightarrow \mathsf{P}_2(R)$ defined by $\mathsf{T}(f) = f'$
   For part (a), find an orthonormal basis for R^2 consisting of eigenvectors of T.

3. Let T and U be self-adjoint operators on an inner product space. Prove that TU is self-adjoint if and only if $\mathsf{TU} = \mathsf{UT}$.

4. Prove (b) of Theorem 7.15.

5. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$\mathsf{T}_1 = \frac{1}{2}(\mathsf{T} + \mathsf{T}^*) \quad \text{and} \quad \mathsf{T}_2 = \frac{1}{2i}(\mathsf{T} - \mathsf{T}^*).$$

   (a) Prove that T_1 and T_2 are self-adjoint and that $\mathsf{T} = \mathsf{T}_1 + i\mathsf{T}_2$.
   (b) Suppose also that $\mathsf{T} = \mathsf{U}_1 + i\mathsf{U}_2$, where U_1 and U_2 are self-adjoint. Prove that $\mathsf{U}_1 = \mathsf{T}_1$ and $\mathsf{U}_2 = \mathsf{T}_2$.
   (c) Prove that T is normal if and only if $\mathsf{T}_1\mathsf{T}_2 = \mathsf{T}_2\mathsf{T}_1$.

6. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following.
(a) If T is self-adjoint, then T_W is self-adjoint.
(b) $W^\perp$ is T^*-invariant.
(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

7. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant. *Hint:* Use Exercise 10(d) of Section 5.4.

8. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$. *Hint:* Use Theorem 7.15 and Exercise 12 of Section 7.3.

9. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

Deduce that $(T - iI)$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

10. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove
(a) If T is self-adjoint, then $(T(x), x)$ is real for all $x \in V$.
(b) If T satisfies $(T(x), x) = 0$ for all $x \in V$, then $T = T_0$.
Hint: Replace x by $x + y$ and then by $x + iy$ and expand the resulting inner products.
(c) If $(T(x), x)$ is real for all $x \in V$, then $T = T^*$.

11. Let A be an $n \times n$ real matrix. A is said to be a *Gramian* matrix if there exists a real (square) matrix B such that $A = B^tB$. Prove that A is a Gramian matrix if and only if A is symmetric and all of its eigenvalues are non-negative. *Hint:* Apply Theorem 7.17(R) to L_A to obtain an orthonormal basis $\{x_1, \ldots, x_n\}$ of eigenvectors with the associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Define the linear operator U by $U(x_i) = \sqrt{\lambda_i}x_i$ and complete the proof.

12. Let T be a self-adjoint operator on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. T is said to be *positive definite* [*semidefinite*] if $(T(x), x) > 0$ for all $x \neq 0$ [$(T(x), x) \geq 0$ for all x]. Prove
(a) T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [non-negative].
(b) T is positive definite [semidefinite] if and only if L_A is also.
(c) T is positive definite if and only if

$$\sum_{i,j} A_{ij}a_i\overline{a_j} > 0 \qquad \text{for all non-zero } n\text{-tuples } (a_1, \ldots, a_n).$$

(d) T is positive semidefinite if and only if A is a Gramian matrix (as defined in Exercise 11).

Is the composition of two positive definite operators positive definite?

13. *Simultaneous Diagonalization.*

(a) Let V be a finite-dimensional real inner product space, and let U and T be self-adjoint operators on V such that UT = TU. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both U and T. (The complex version of this result appears as Exercise 10 in Section 7.9.) *Hint:* For any eigenspace W = E_λ of T we have that W is both T- and U-invariant. By Exercise 6 we have that $\mathsf{W}^\perp$ is both T- and U-invariant. Apply Theorem 7.17(R) and Theorem 7.6.

(b) State and prove the analagous result about commuting symmetric (real) matrices.

14. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following.

(a) If $\beta = \{x_1, \ldots, x_n\}$ is an ordered basis for V, then $[\mathsf{T}]_\beta$ is upper triangular if and only if $\mathsf{T}(x_j) \in \text{span}(\{x_1, \ldots, x_j\})$ for $j = 1, \ldots, n$.

(b) If V is a complex inner product space, then there exists an orthonormal basis γ for V such that $[\mathsf{T}]_\gamma$ is upper triangular. *Hint:* Use induction on $n = \dim(\mathsf{V})$. Choose an eigenvector x of T*, and let W = span($\{x\}$). Apply the induction hypothesis to $\mathsf{W}^\perp$, which is T-invariant by Exercise 6(b).

(c) Every complex matrix A is similar to an upper triangular matrix.

15. Prove the *Cayley-Hamilton theorem* for a complex $n \times n$ matrix A; that is, if $f(t)$ is the characteristic polynomial of A, prove that $f(A) = O$. *Hint:* By (c) of Exercise 14, show that you may assume that A is upper triangular, in which case

$$f(t) = \prod_{i=1}^{n} (A_{ii} - t).$$

Now if $\mathsf{T} = \mathsf{L}_A$, we have $(A_{jj}\mathsf{I} - \mathsf{T})(x_j) \in \text{span}(\{x_1, \ldots, x_{j-1}\})$ for $j \geq 2$, where $\{x_1, \ldots, x_n\}$ is the standard ordered basis of C^n.

7.6* CONDITIONING AND THE RAYLEIGH QUOTIENT

In Section 3.4 we studied specific techniques that allowed us to solve systems of linear equations in the form $AX = b$, where A is an $m \times n$ matrix and b is an $m \times 1$ vector. Such systems often arise through applications to the real world. The coefficients in the system are frequently obtained from experimental data, and in many cases both m and n are so large that a computer must be used in the calculation of the solution. Thus two types of errors must be considered. First, experimental errors arise in the collection of data since no instruments can provide completely accurate measurements. Second, computers will introduce roundoff errors. One might intuitively feel that small relative changes in the coefficients in the system will cause small relative errors in the solution. A system that has this property is called *well-conditioned*; otherwise, the system is called *ill-conditioned.*

We shall now consider several examples of these types of errors, concentrating primarily on changes in b rather than in changes in the entries of A. In addition, we shall assume that A is a square, complex (or real), invertible matrix, since this is the case most frequently encountered in applications.

Example 22. Consider the system

$$\begin{cases} x_1 + x_2 = 5 \\ x_1 - x_2 = 1. \end{cases}$$

The solution of this system is

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Now suppose we change the system somewhat and consider the new system

$$\begin{cases} x_1 + x_2 = 5 \\ x_1 - x_2 = 1.0001. \end{cases}$$

This modified system has the solution

$$\begin{pmatrix} 3.00005 \\ 1.99995 \end{pmatrix}.$$

We see that a change of 10^{-4} in one coefficient has caused a change of less than 10^{-4} in each coordinate of the new solution. More generally, the system

$$\begin{cases} x_1 + x_2 = 5 \\ x_1 - x_2 = 1 + \delta \end{cases}$$

has the solution

$$\begin{pmatrix} 3 + \dfrac{\delta}{2} \\ 2 - \dfrac{\delta}{2} \end{pmatrix}.$$

Hence small changes in b introduce small changes in the solution. Of course, we are really interested in "relative changes" since a change in the solution of, say, 10 is considered large if the original solution is of the order of 10^{-2} but small if the original solution is of the order of 10^6.

We shall introduce the notation δb to denote the vector $b' - b$, where b is the vector in the original system and b' is the vector in the modified system. Thus, in Example 22, we have

$$\delta b = \begin{pmatrix} 5 \\ 1+\delta \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}.$$

We now define the *relative change* in b to be the scalar $\|\delta b\|/\|b\|$, where $\|\cdot\|$ denotes the standard norm in C^n (or R^n); i.e. $\|b\| = \sqrt{(b, b)}$. Most of what follows, however, is true for any norm. Similar definitions hold for the *relative change* in x. So in Example 22

$$\frac{\|\delta b\|}{\|b\|} = \frac{|\delta|}{\sqrt{26}}, \quad \text{and} \quad \frac{\|\delta x\|}{\|x\|} = \frac{\left\|\begin{pmatrix} 3 + (\delta/2) \\ 2 - (\delta/2) \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\|}{\left\|\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\|} = \frac{|\delta|}{\sqrt{26}}.$$

Thus the relative change in x equals, coincidently, the relative change in b, and so the system is well-conditioned.

Example 23. Consider the system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 + 1.00001x_2 = 3.00001, \end{cases}$$

which has

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

as its solution. The solution for the related system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 + 1.00001x_2 = 3.00001 + \delta \end{cases}$$

is

$$\begin{pmatrix} 2 - (10^5)\delta \\ 1 + (10^5)\delta \end{pmatrix}.$$

Hence

$$\frac{\|\delta x\|}{\|x\|} = 10^5\sqrt{\tfrac{2}{5}}\,|\delta| \geq 10^4|\delta|,$$

while

$$\frac{\|\delta b\|}{\|b\|} \approx \frac{|\delta|}{5}.$$

Thus the relative change in x is at least 10^4 times the relative change in b! This system is very ill-conditioned. Observe that the lines defined by the two equations in this system are nearly coincident. So a small change in either line could greatly alter the point of intersection, i.e., the solution of the system.

To apply the full strength of the theory of self-adjoint matrices to the study of conditioning, we shall need the notion of the norm of a matrix (See Exercise 22 of Section 7.1 for further results about norms.)

Definition. *Let* A *be a complex* (*or real*) n × n *matrix. Define the* (Euclidean) norm *of* A *by*

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

where $x \in \mathsf{C}^n$ (*or* R^n).

We see intuitively that $\|A\|$ represents the maximum "magnification" of a vector by the matrix A.

The question of whether or not this maximum exists, as well as the problem of how to compute it, will be answered by use of the so-called "Rayleigh quotient."

Definition. *Let* B *be an* n × n *self-adjoint matrix. The* Rayleigh quotient *for* $x \neq 0$ *is defined to be the scalar* $R(x) = (Bx, x)/\|x\|^2$.

Theorem 7.18. *For a self-adjoint matrix* B *we have that* $\max_{x \neq 0} R(x)$ *is the largest eigenvalue of* B *and* $\min_{x \neq 0} R(x)$ *is the smallest eigenvalue of* B.

PROOF. By Theorem 7.17 we may choose an orthonormal basis $\{x_1, \ldots, x_n\}$ of eigenvectors of B such that $Bx_i = \lambda_i x_i$, $1 \leq i \leq n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. (Recall that by Corollary 2 of Theorem 7.15 the eigenvalues of B are real.) Now for $x \in \mathsf{C}^n$ (or R^n) there exist scalars $a_1, \ldots, a_n$ such that

$$x = \sum_{i=1}^{n} a_i x_i;$$

hence

$$R(x) = \frac{(Bx, x)}{\|x\|^2} = \frac{\left(\sum_{i=1}^{n} a_i \lambda_i x_i, \sum_{j=1}^{n} a_j x_j\right)}{\|x\|^2} = \frac{\sum_{i=1}^{n} \lambda_i |a_i|^2}{\|x\|^2} \leq \frac{\lambda_1 \sum_{i=1}^{n} |a_i|^2}{\|x\|^2} = \lambda_1.$$

It is easy to see that $R(x_1) = \lambda_1$, and so we have demonstrated the first half of the theorem. The second half is proved similarly. ∎

Corollary 1. *For any square matrix* A, $\|A\|$ *is finite and, in fact, equals* $\sqrt{\lambda}$, *where* λ *is the largest eigenvalue of* A^*A.

PROOF. Let B be the self-adjoint matrix A^*A, and let λ be the largest eigenvalue of B. Since, for $x \neq 0$,

$$0 \leq \frac{\|Ax\|^2}{\|x\|^2} = \frac{(Ax, Ax)}{\|x\|^2} = \frac{(A^*Ax, x)}{\|x\|^2} = \frac{(Bx, x)}{\|x\|^2} = R(x),$$

we have from Theorem 7.18 that $\|A\|^2 = \lambda$. ▮

Observe that the proof of Corollary 1 shows that all the eigenvalues of A^*A are non-negative. For the next corollary we shall need the following lemma.

Lemma. *For any square matrix* A, λ *is an eigenvalue of* A^*A *if and only if* λ *is an eigenvalue of* AA^*.

PROOF. Let λ be an eigenvalue of A^*A. If $\lambda = 0$, then A^*A is not invertible. Hence A (and A^*) is not invertible, so that λ is also an eigenvalue of AA^*. The proof of the converse is similar.

Suppose now that $\lambda \neq 0$. Then there exists $x \neq 0$ such that $A^*Ax = \lambda x$. Apply A to both sides to obtain $(AA^*)(Ax) = \lambda(Ax)$. Since $Ax \neq 0$ (lest $\lambda x = 0$), we have that λ is an eigenvalue of AA^*. The converse is left as an exercise. ▮

Corollary 2. *Let* A *be an invertible matrix. Then* $\|A^{-1}\| = 1/\sqrt{\lambda}$, *where* λ *is the smallest eigenvalue of* A^*A.

PROOF. We shall make use of the observation that λ is an eigenvalue of an invertible matrix if and only if λ^{-1} is an eigenvalue of its inverse.

Now let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A^*A, which by the lemma are the eigenvalues of AA^*. Then $\|A^{-1}\|^2$ equals the largest eigenvalue of $(A^{-1})^*A^{-1} = (AA^*)^{-1}$, which equals $1/\lambda_n$. ▮

For many applications it is only the largest and smallest eigenvalues that are of interest. For example, in the case of vibration problems, the smallest eigenvalue represents the lowest frequency at which vibrations can occur.

We shall see the role of both these eigenvalues in our study of conditioning.

Example 24. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then

$$B = A^*A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of B are 3, 3, and 0. Therefore $\|A\| = \sqrt{3}$. For any

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0,$$

we may compute $R(x)$ for the matrix B as

$$3 \geq R(x) = \frac{(Bx, x)}{\|x\|^2} = \frac{2(a^2 + b^2 + c^2 - ab + ac + bc)}{a^2 + b^2 + c^2}$$

for all $a, b, c \in R$.

Now that we know $\|A\|$ exists for every square matrix, we shall make use of the inequality $\|Ax\| \leq \|A\| \cdot \|x\|$, which holds for every x.

Assume in what follows that A is invertible, $b \neq 0$, and $Ax = b$. For a given δb, let δx be the vector that satisfies $A(x + \delta x) = b + \delta b$. Then $A(\delta x) = \delta b$, and so $\delta x = A^{-1}(\delta b)$. Hence

$$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\| \quad \text{and} \quad \|\delta x\| = \|A^{-1}(\delta b)\| \leq \|A^{-1}\| \cdot \|\delta b\|.$$

Thus

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\| \cdot \|\delta b\| \cdot \|A\|}{\|b\|} = \|A\| \cdot \|A^{-1}\| \cdot \left(\frac{\|\delta b\|}{\|b\|}\right).$$

Similarly

$$\frac{1}{\|A\| \cdot \|A^{-1}\|} \left(\frac{\|\delta b\|}{\|b\|}\right) \leq \frac{\|\delta x\|}{\|x\|}.$$

The number $\|A\| \cdot \|A^{-1}\|$ is called the *condition number* of A and is denoted *cond*(A). It should be noted that the definition of cond(A) depends on how we define the norm of A. There are many reasonable ways of defining the norm of a matrix. In fact, the only property we used to establish the inequalities above was that $\|Ax\| \leq \|A\| \cdot \|x\|$ for all x. We summarize the above in the following theorem.

Theorem 7.19. *For the system* AX = b, *where* A *is invertible and* b $\neq$ 0, *we have the following two results:*

(a) $$\frac{1}{cond(\mathrm{A})} \frac{\|\delta \mathrm{b}\|}{\|\mathrm{b}\|} \leq \frac{\|\delta \mathrm{x}\|}{\|\mathrm{x}\|} \leq cond(\mathrm{A}) \frac{\|\delta \mathrm{b}\|}{\|\mathrm{b}\|} \qquad (\textit{for any norm } \|\cdot\|).$$

(b) $cond(\mathrm{A}) = \sqrt{\dfrac{\lambda_1}{\lambda_n}}$, *where* λ_1 *and* λ_n *are the largest and smallest eigenvalues, respectively, of* A*A. (*In this part, we assume that* $\|\cdot\|$ *is the Euclidean norm defined in this section.*)

PROOF. Statement (a) follows from the previous inequalities, and (b) follows from Corollaries 1 and 2 of Theorem 7.18. ∎

It is clear from Theorem 7.19 that cond(A) ≥ 1. It is left as an exercise to prove that cond(A) $= 1$ if and only if A is a scalar multiple of a "unitary" or "orthogonal" matrix as defined in Section 7.7. Moreover, it can be shown with some work that equality can be obtained in (a) by an appropriate choice of b and δb.

We can see immediately from (a) that if cond(A) is close to 1, then we are sure that a small relative error in b forces a small relative error in x. If cond(A) is large, however, then the relative error in x may be small even though the relative error in b is large, or the relative error in x may be large even though the relative error in b is small! In short, cond(A) merely indicates the potential for large relative errors.

We have so far considered only errors in the vector b. If there is an error δA in the coefficient matrix of the system $AX = b$, the situation is more complicated. For example, $A + \delta A$ may fail to be invertible. But under appropriate assumptions it can be shown that a bound for the relative error in x can be given in terms of cond(A). For example, if $A + \delta A$ is invertible, then Forsythe and Moler (Forsythe, George, and Moler, Cleve B., *Computer Solution of Linear Algebraic Systems*, Prentice-Hall, Inc., 1976, p. 23) show that

$$\frac{||\delta x||}{||x + \delta x||} \leq \text{cond}(A) \frac{||\delta A||}{||A||}.$$

It should be mentioned that, in practice, one almost never knows cond(A), for it would be an unnecessary waste of time to compute A^{-1} merely to determine its norm. In fact, if a computer is used to find A^{-1}, the computed inverse of A will in all likelihood only approximate A^{-1}, and the error in the computed inverse will be affected by the size of cond(A). So we are caught in a vicious circle! There are, however, some situations in which a usable approximation of cond(A) can be found. Thus, in most cases, the estimate of the relative error in x is based on an estimate of cond(A).

EXERCISES

1. Label the following statements as being true or false.
   (a) If $AX = b$ is well-conditioned, then cond(A) is small.
   (b) If cond(A) is large, then $AX = b$ is ill-conditioned.
   (c) If cond(A) is small, then $AX = b$ is well-conditioned.
   (d) The norm of A equals the Rayleigh quotient.
   (e) The norm of A is always equal to the largest eigenvalue of A.

2. Compute the norms of the following matrices.

   (a) $\begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 3 \\ -3 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & \dfrac{-2}{\sqrt{3}} & 0 \\ 0 & \dfrac{-2}{\sqrt{3}} & 1 \\ 0 & \dfrac{2}{\sqrt{3}} & 1 \end{pmatrix}$

3. Prove that if B is symmetric, then $\|B\|$ is the largest eigenvalue of B.

4. Let A and A^{-1} be as follows:

$$A = \begin{pmatrix} 6 & 13 & -17 \\ 13 & 29 & -38 \\ -17 & -38 & 50 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 6 & -4 & 1 \\ -4 & 11 & 7 \\ -1 & 7 & 5 \end{pmatrix}.$$

The eigenvalues of A are approximately 84.74, 0.2007, and 0.0588.

(a) Approximate $\|A\|$, $\|A^{-1}\|$, and cond(A). (Note Exercise 3 above.)

(b) Suppose that we have vectors x and $\tilde{x}$ such that $Ax = b$ and $\|b - A\tilde{x}\| \leq 0.001$. Use (a) to determine upper bounds for $\|\tilde{x} - A^{-1}b\|$ (the absolute error) and $\|\tilde{x} - A^{-1}b\|/\|A^{-1}b\|$ (the relative error).

5. Suppose that x is the actual solution of $AX = b$ and that a computer arrives at an approximate solution $\tilde{x}$. If cond(A) $= 100$, $\|b\| = 1$, and $\|b - A\tilde{x}\| = 0.1$, obtain upper and lower bounds for $\|x - \tilde{x}\|/\|x\|$.

6. Let

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Compute

$$R\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \qquad \|B\|, \quad \text{and} \quad \text{cond}(B).$$

7. Let B be a symmetric matrix. Prove that $\min_{x \neq 0} R(x)$ equals the smallest eigenvalue of B.

8. Prove that if λ is an eigenvalue of AA^*, then λ is an eigenvalue of A^*A. This completes the proof of the lemma to Corollary 2 of Theorem 7.18.

9. Prove the left inequality of (a) in Theorem 7.19.

10. Prove that cond(A) $= 1$ if and only if A is a scalar multiple of a unitary or orthogonal matrix as defined in Section 7.7.

11. (a) Let A and B be square matrices that are unitarily equivalent as defined in Section 7.7. Prove that $\|A\| = \|B\|$.

(b) Let V be a finite-dimensional inner product space, and let T be a linear operator on T. Define

$$||\mathsf{T}|| = \max_{x \neq 0} \frac{||\mathsf{T}(x)||}{||x||}.$$

Prove that $||\mathsf{T}|| = ||[\mathsf{T}]_\beta||$, where β is any orthonormal basis of V.

(c) Let V be an infinite-dimensional inner product space with an orthonormal basis $\{x_1, x_2, \ldots\}$. Let T be the linear operator on V such that $\mathsf{T}(x_k) = kx_k$. Prove that $||\mathsf{T}||$ (defined in (b)) does not exist.

7.7 UNITARY AND ORTHOGONAL OPERATORS AND THEIR MATRICES

In this section we shall continue our analogy between complex numbers and linear operators. Recall that the adjoint of a linear operator acts similarly to the complex conjugate of a complex number. (See, for example, Theorem 7.10.) A complex number z has length 1 if $z\bar{z} = 1$. In this section we shall study those linear operators T on a vector space V such that $\mathsf{TT}^* = \mathsf{T}^*\mathsf{T} = \mathsf{I}$. We shall see that these are precisely the linear operators that "preserve length" in the sense that $||\mathsf{T}(x)|| = ||x||$ for all $x \in \mathsf{V}$. As another characterization, we shall prove that on a finite-dimensional complex inner product space these are the normal operators whose eigenvalues all have absolute value 1.

In past chapters we were interested in studying those functions that preserve the structure of the underlying space. In particular, linear operators preserve the operations of vector addition and scalar multiplication, and isomorphisms preserve all the vector space structure. It is now natural to consider those linear operators T on an inner product space that preserve length, i.e., $||\mathsf{T}(x)|| = ||x||$ for all x. We shall see that this condition guarantees, in fact, that T preserves the inner product.

Definitions. *Let* V *be an inner product space* (*over* F), *and let* T *be a linear operator on* V. *If* $||\mathsf{T}(\mathrm{x})|| = ||\mathrm{x}||$ *for all* $\mathrm{x} \in \mathsf{V}$, *we call* T *a* unitary operator *if* F = C *and an* orthogonal operator *if* F = R.

Clearly any rotation or reflection in R^2 preserves length and hence is an orthogonal operator. We shall study these operators in much more detail in the next section.

Example 25. Let V = H, and let $h \in \mathsf{V}$ with $|h(x)| = 1$ for all x. Define $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{T}(f) = hf$. Then

$$||\mathsf{T}(f)||^2 = ||hf||^2 = \frac{1}{2\pi}\int_0^{2\pi} h(t)f(t)\overline{h(t)f(t)}\,dt = ||f||^2$$

since $|h(t)|^2 = 1$ for all t. So T is a unitary operator.

Theorem 7.20. *Let* V *be a finite-dimensional inner product space, and let* T *be a linear operator on* V. *Then the following are equivalent:*

(a) $TT^* = T^*T = I$.

(b) $(T(x), T(y)) = (x, y)$ *for all* $x, y \in V$.

(c) *If* β *is an orthonormal basis for* V, *then* $T(\beta)$ *is an orthonormal basis for* V.

(d) *There exists an orthonormal basis* β *for* V *such that* $T(\beta)$ *is an orthonormal basis for* V.

(e) $\|T(x)\| = \|x\|$ *for all* $x \in V$.

Thus, all the conditions above are equivalent to the definition of a unitary or orthogonal operator. From (a) it follows that unitary or orthogonal operators are normal.

Before proving the theorem, we shall first prove the following lemma. Compare this lemma to Exercise 10(b) of Section 7.5.

Lemma. *Let* V *be a finite-dimensional inner product space, and let* U *be a self-adjoint operator on* V. *If* $(x, U(x)) = 0$ *for all* $x \in V$, *then* $U = T_0$.

PROOF. By Theorem 7.17 we may choose an orthonormal basis β of eigenvectors of U. If $x \in \beta$, then $U(x) = \lambda x$ for some λ. Thus

$$0 = (x, U(x)) = (x, \lambda x) = \bar{\lambda}(x, x);$$

so $\bar{\lambda} = 0$. Hence $U(x) = 0$ for all $x \in \beta$, and thus $U = T_0$. ∎

PROOF OF THEOREM 7.20. We shall first prove that (a) implies (b).

Let $x, y \in V$. Then $(x, y) = ((T^*T)(x), y) = (T(x), T(y))$.

Secondly, we shall prove that (b) implies (c). Let $\beta = \{x_1, \ldots, x_n\}$ be an orthonormal basis for V. Then $T(\beta) = \{T(x_1), \ldots, T(x_n)\}$. Now $(T(x_i), T(x_j)) = (x_i, x_j) = \delta_{ij}$. So $T(\beta)$ is an orthonormal basis of V.

That (c) implies (d) is obvious.

Next we shall prove that (d) implies (e). Let $x \in V$, and let $\beta = \{x_1, \ldots, x_n\}$. Now

$$x = \sum_{i=1}^{n} a_i x_i$$

for some scalars a_i, and so

$$\|x\|^2 = \left(\sum_{i=1}^{n} a_i x_i, \sum_{j=1}^{n} a_j x_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{a_j}(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{a_j}\delta_{ij} = \sum_{i=1}^{n} |a_i|^2$$

since β is orthonormal.

Applying the same manipulations to

$$T(x) = \sum_{i=1}^{n} a_i T(x_i)$$

and using the fact that $T(\beta)$ is also orthonormal, we obtain

$$\|T(x)\|^2 = \sum_{i=1}^{n} |a_i|^2.$$

Hence $\|T(x)\| = \|x\|$.

Finally, we shall prove that (e) implies (a). For any $x \in V$ we have

$$(x, x) = \|x\|^2 = \|T(x)\|^2 = (T(x), T(x)) = (x, (T^*T)(x)).$$

So $(x, (I - T^*T)(x)) = 0$ for all $x \in V$. Let $U = I - T^*T$; then U is self-adjoint, and $(x, U(x)) = 0$ for $x \in V$. So by the lemma we have $T_0 = U = I - T^*T$, and hence $T^*T = I$. Thus since V is finite-dimensional, $T^* = T^{-1}$, and so $TT^* = I$. ∎

It follows immediately from the definition that every eigenvalue of a unitary or orthogonal operator has absolute value 1. In fact, even more is true.

Corollary 1. *Let* T *be a linear operator on a finite-dimensional real inner product space* V. V *has an orthonormal basis of eigenvectors of* T *with corresponding eigenvalues of absolute value* 1 *if and only if* T *is both self-adjoint and orthogonal.*

PROOF. Suppose that V has an orthonormal basis $\{x_1, \ldots, x_n\}$ such that $T(x_i) = \lambda_i x_i$ and $|\lambda_i| = 1$ for all i. By Theorem 7.17(*R*) T is self-adjoint. Thus $(TT^*)(x_i) = T(\lambda_i x_i) = \lambda_i \lambda_i x_i = \lambda_i^2 x_i = x_i$ for each i. So $TT^* = I$, and by (a) of Theorem 7.20, T is orthogonal.

If T is self-adjoint, then by Theorem 7.17(*R*) we have that V possesses an orthonormal basis $\{x_1, \ldots, x_n\}$ such that $T(x_i) = \lambda_i x_i$ for all i. If T is also orthogonal, we have $|\lambda_i| \cdot \|x_i\| = \|\lambda_i x_i\| = \|T(x_i)\| = \|x_i\|$, and so $|\lambda_i| = 1$ for every i. ∎

Corollary 2. *Let* T *be a linear operator on a finite-dimensional complex inner product space* V. *Then* V *has an orthonormal basis of eigenvectors of* T *with corresponding eigenvalues of absolute value* 1 *if and only if* T *is unitary.*

PROOF. The proof is similar to the proof of Corollary 1. ∎

Example 26. Let $T\colon R^2 \longrightarrow R^2$ be a rotation by θ, where $0 < \theta < \pi$. It is clear geometrically that T "preserves length," i.e., that $\|T(x)\| = \|x\|$ for all $x \in R^2$. The fact that rotations by a fixed angle preserve perpendicularity not only can be seen geometrically but now follows from (b) of Theorem 7.20. Perhaps the fact that such a transformation preserves the inner product is not so obvious geometrically; however, we obtain this fact from (b) also. Finally, an inspection of the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

reveals that T is not self-adjoint for the given restriction on θ. As we mentioned earlier, this fact also follows from the geometric observation that T has no eigenvectors and from Theorem 7.16. It can be seen easily from the matrix above that T^* is a rotation by $-\theta$.

We shall now examine the matrices that represent unitary and orthogonal transformations.

Definitions. *Suppose that* A *is an* $n \times n$ *matrix that satisfies* $AA^* = A^*A = I$. *We call* A *a* unitary matrix *if* A *has complex entries, and we call* A *an* orthogonal matrix *if* A *has real entries.*

Note that the condition $AA^* = I$ is equivalent to the statement that the rows $A_1, \ldots, A_n$ of A form an orthonormal set in F^n, for

$$\delta_{ij} = I_{ij} = (AA^*)_{ij} = \sum_{k=1}^{n} A_{ik}(A^*)_{kj} = \sum_{k=1}^{n} A_{ik}\overline{A_{jk}} = (A_i, A_j).$$

A similar remark can be made about the columns of A and the condition $A^*A = I$.

It also follows from the definition above that if V is an inner product space and T is a linear operator on V, then T is unitary [orthogonal] if and only if $[T]_\beta$ is unitary [orthogonal] for some orthonormal basis β of V.

Example 27. From Example 26 the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is clearly orthogonal. One can easily see that the rows of the matrix form an orthonormal set in R^2.

We know that for a complex normal [real self-adjoint] matrix A there is an orthonormal basis β for F^n consisting of eigenvectors of A. Hence A is similar to a diagonal matrix D. By Theorem 5.1 the matrix Q whose columns are the vectors in β is such that $D = Q^{-1}AQ$. But since the columns of Q are an orthonormal basis for F^n, it follows that Q is unitary [orthogonal]. In this case we say that A is *unitarily equivalent* [*orthogonally equivalent*] to D. It is easily seen (see Exercise 17) that this relation is an equivalence relation on $M_{n\times n}(C)$ [$M_{n\times n}(R)$]. More generally, A and B are unitarily equivalent [orthogonally equivalent] if and only if there exists a unitary [orthogonal] matrix P such that $A = P^*BP$.

The preceding paragraph has proved half of each of the following pair of theorems.

Theorem 7.21(C). *Let* A *be a complex* $n \times n$ *matrix. Then* A *is normal if and only if* A *is unitarily equivalent to a diagonal matrix.*

Theorem 7.21(R). *Let A be a real n × n matrix. Then A is self-adjoint if and only if A is orthogonally equivalent to a real diagonal matrix.*

PROOF. By the remarks above we need only prove that if A is unitarily [orthogonally] equivalent to a diagonal matrix, then A is normal [self-adjoint].

Suppose that $A = P^*DP$, where P is a unitary matrix and D is a diagonal matrix. Then $AA^* = (P^*DP)(P^*DP)^* = (P^*DP)(P^*D^*P) = P^*DID^*P = P^*DD^*P$.

Similarly, $A^*A = P^*D^*DP$. Since D is a diagonal matrix, however, we have $DD^* = D^*D$. Thus $AA^* = A^*A$.

The remainder of the proof is left to the reader. ▌

Example 28. Let

$$A = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix};$$

then $A = A^*$. The eigenvalues of L_A are $\lambda_1 = 5$ and $\lambda_2 = -5$. Corresponding to each of these eigenvalues are the eigenvectors $y_1 = (-2, 1)$ and $y_2 = (1, 2)$. As expected, y_1 and y_2 are orthogonal. Let

$$x_1 = \frac{1}{\sqrt{5}}(-2, 1), \quad x_2 = \frac{1}{\sqrt{5}}(1, 2), \quad \text{and } \beta = \{x_1, x_2\}.$$

Then β is an orthonormal basis of eigenvectors of L_A. As in the paragraph preceding Theorem 7.21, let

$$P = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}.$$

An easy computation shows that $P^*AP = D$.

An Application (Conic Sections)

As an application of Theorem 7.21, we would like to consider the quadratic equation

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0. \tag{12}$$

For special choices of the coefficients in Eq. (12), we obtain the various conic sections. For example, if $a = c = 1$, $b = d = e = 0$, and $f = -1$, we obtain the circle $x^2 + y^2 = 1$ with center at the origin. The remaining conic sections, namely, the ellipse, parabola, and hyperbola, are obtained by other choices of the coefficients. The absence of the xy term allows easy graphing of these conics by the method of completing the square. For example, $x^2 + 2x + y^2 + 4y + 2 = 0$ may be rewritten as $(x + 1)^2 + (y + 2)^2 = 3$, a circle with center at $(-1, -2)$ in the x, y-coordinate system and radius $\sqrt{3}$. If we consider the transformation of coordinates $(x, y) \longrightarrow (x', y')$, where $x' = x + 1$ and $y' = y + 2$, then our equation simplifies to $(x')^2 + (y')^2 = 3$. This type of transformation (called a *translation*) allows us to eliminate the x and y terms.

We shall now concentrate solely on the elimination of the xy term. To accomplish this we shall consider the expression

$$ax^2 + 2bxy + cy^2, \tag{13}$$

which is called the *associated quadratic form* of Eq. (12). Quadratic forms will be studied in more generality in Section 7.11.

If we let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ y \end{pmatrix},$$

then Eq. (13) may be rewritten as $X^tAX = (AX, X)$. For example, the quadratic form $3x^2 + 4xy + 6y$ may be written as

$$X^t\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}X.$$

The fact that A is self-adjoint is crucial in our discussion. For, by Theorem 7.21, we may choose an orthogonal matrix P and a diagonal matrix D with real diagonal entries λ_1 and λ_2 such that $P^tAP = D$. Now define

$$X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

by $X' = P^tX$ or, equivalently, by $PX' = PP^tX = X$. Then

$$X^tAX = (PX')^tA(PX') = X'^t(P^tAP)X' = X'^tDX' = \lambda_1(x')^2 + \lambda_2(y')^2.$$

Thus the transformation $(x, y) \longrightarrow (x', y')$ allows us to eliminate the xy term in Eq. (13) and hence in Eq. (12).

Furthermore, since P is orthogonal, we have by Exercise 20(c) that $\det(P) = \pm 1$. If $\det(P) = -1$, we may replace P by $Q = PE$, where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then Q is orthogonal, $\det(Q) = 1$, and

$$Q^tAQ = E^tP^tAPE = E^tDE = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}.$$

Hence we may as well assume that $\det(P) = 1$. By Exercise 20, it follows that P (or P^t) geometrically represents a rotation.

In summary, the xy term in Eq. (12) may be eliminated by a rotation of the x-axis and y-axis to new axes x' and y' given by $X = PX'$, where P is an orthogonal matrix and $\det(P) = 1$. Furthermore the coefficients of $(x')^2$ and $(y')^2$ are the eigenvalues of

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

This result is a restatement of the *principal axis theorem* for R^2. The arguments above, of course, are easily extended to quadratic equations in n variables. For example, in the case $n = 3$, by special choices of the coefficients we obtain the quadric surfaces—the elliptic cone, the ellipsoid, the hyperbolic parabolid, etc.

As an example, consider the quadratic equation $2x^2 - 4xy + 5y^2 - 36 = 0$, for which the associated quadratic form is $2x^2 - 4xy + 5y^2$. In the notation above

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix},$$

so that the eigenvalues of A are 6 and 1 with associated eigenvectors

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

As expected (from Theorem 7.15), these vectors are orthogonal. The corresponding orthonormal basis of eigenvectors is

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\right\}.$$

Hence if

$$P = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

then $P^tAP = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$. Under the transformation $X = PX'$ or

$$x = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y', \qquad y = \frac{-2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y',$$

we have the new quadratic form $6(x')^2 + (y')^2$. Thus the original equation $2x^2 - 4xy + 5y^2 - 36 = 0$ may be written in the form $6(x')^2 + (y')^2 =$

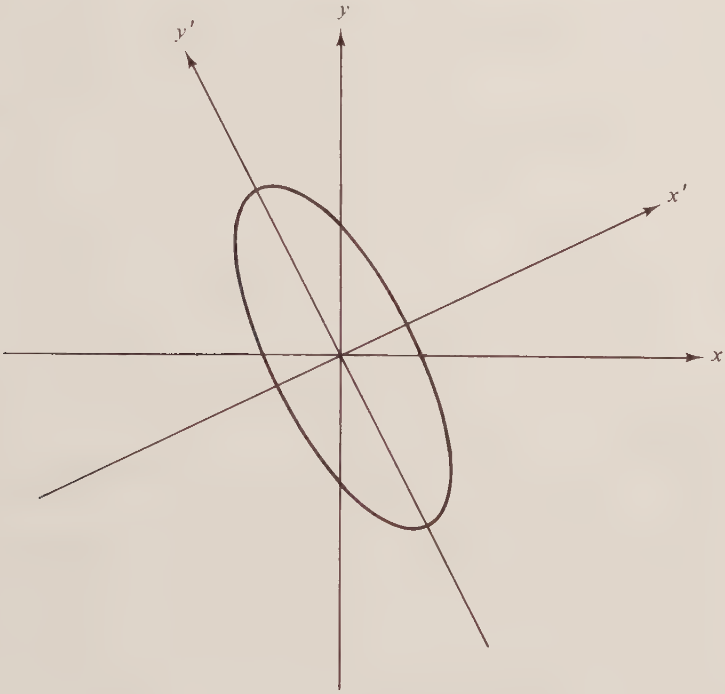

figure 7.4

36, in which form it is easily seen to be the equation of an ellipse. (See Fig. 7.4.)

EXERCISES

1. Label the following statements as being true or false. Assume that the underlying inner product spaces are finite-dimensional.
   (a) Every unitary operator is normal.
   (b) Every orthogonal operator is diagonalizable.
   (c) A matrix is unitary if and only if it is invertible.
   (d) If two matrices are unitarily equivalent, then they are also similar.
   (e) The sum of unitary matrices is unitary.
   (f) The adjoint of a unitary operator is unitary.
   (g) If T is an orthogonal operator on V, then $[\mathsf{T}]_\beta$ is an orthogonal matrix for any ordered basis β for V.

(h) If all the eigenvalues of an operator are 1, then the operator must be unitary or orthogonal.
(i) An operator may preserve the norm but not the inner product.

2. For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $A = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ (e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

3. Prove that the composition of unitary [orthogonal] operators is unitary [orthogonal].

4. For $z \in C$ define $\mathsf{T}_z\colon\ C \longrightarrow C$ by $\mathsf{T}_z(u) = zu$. Characterize those z for which T_z is normal, self-adjoint, or unitary.

5. Which of the following pairs of matrices are unitarily equivalent?

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$

6. Let V be the inner product space of complex-valued continuous functions on $[0, 1]$ with the inner product

$$(f, g) = \int_0^1 f(t)\overline{g(t)}\, dt.$$

Let $h \in \mathsf{V}$, and define $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{T}(f) = hf$. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for $0 \le t \le 1$.

7. Prove that if T is a unitary operator on a finite-dimensional inner product space, then T has a "square root"; i.e., there exists a unitary operator U such that $\mathsf{T} = \mathsf{U}^2$.

8. Let V be an inner product space, and let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{V}$ be self-adjoint. If $\mathsf{U} = (\mathsf{T} + i\mathsf{I})(\mathsf{T} - i\mathsf{I})^{-1}$, prove, using Exercise 9 of Section 7.5, that U is unitary.

9. Let U be a linear operator on a finite-dimensional inner product space V. If $\|U(x)\| = \|x\|$ for all x in some orthonormal basis for V, must U be unitary? Prove or give a counter-example.

10. Let A be a complex normal or real symmetric $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Prove that

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \operatorname{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2.$$

11. Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

12. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\det(A) = \prod_{i=1}^{n} \lambda_i,$$

where the λ_i's are the (not necessarily distinct) eigenvalues of A.

13. Suppose that A and B are diagonalizable matrices. Prove or disprove that A is similar to B if and only if A and B are unitarily equivalent.

14. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove
(a) $U(W) = W$.
(b) $W^\perp$ is U-invariant.
Contrast (b) with Exercise 15.

15. Find an example of a unitary operator U on an inner product space and a U-invariant subspace W such that $W^\perp$ is not U-invariant.

16. Prove that a matrix that is both unitary and upper triangular must be a diagonal matrix.

17. Show that "is unitarily equivalent to" is an equivalence relation on $M_{n\times n}(C)$.

18. Let W be a finite-dimensional subspace of an inner product space V. By Theorem 7.6, $V = W \oplus W^\perp$. Define $U\colon V \to V$ by $U(x_1 + x_2) = x_1 - x_2$, where $x_1 \in W$ and $x_2 \in W^\perp$. Prove that U is a self-adjoint unitary operator.

19. Let V be a finite-dimensional inner product space. A linear operator U on V is called a *partial isometry* if there exists a subspace W of V such that $\|U(x)\| = \|x\|$ for all $x \in W$ and $U(x) = 0$ for all $x \in W^\perp$. Observe that W need *not* be U-invariant. Suppose that U is such an operator and $\{x_1, \ldots, x_k\}$ is an orthonormal basis of W. Prove the following.
(a) $(U(x), U(y)) = (x, y)$ for all $x, y \in W$. *Hint:* Use Exercise 20 of Section 7.1.
(b) $\{U(x_1), \ldots, U(x_k)\}$ is an orthonormal basis for $R(U)$.

(c) There exists an orthonormal basis γ for V such that the first k columns of $[\mathsf{U}]_\gamma$ form an orthonormal set and the remaining columns are zero.
(d) Let $\{y_1, \ldots, y_j\}$ be an orthonormal basis for $\mathsf{R}(\mathsf{U})^\perp$. Let $\beta = \{\mathsf{U}(x_1), \ldots, \mathsf{U}(x_k), y_1, \ldots, y_j\}$. Then β is an orthonormal basis for V.
(e) Define T to be the linear operator on V that satisfies $\mathsf{T}(\mathsf{U}(x_i)) = x_i$ $(1 \leq i \leq k)$ and $\mathsf{T}(y_i) = 0$ $(1 \leq i \leq j)$. Prove that T is well-defined and that $\mathsf{T} = \mathsf{U}^*$. *Hint:* Show that $(\mathsf{U}(x), y) = (x, \mathsf{T}(y))$ for all $x, y \in \beta$. There are four cases.
(f) Prove that U^* is a partial isometry.

This exercise is continued in Exercise 9 of Section 7.9.

A Geometric Application

The purpose of the next exercise is to use the knowledge thus far obtained in this chapter to characterize the so-called "rigid motions" of R^2. One may think intuitively of such a motion as a transformation that does not affect the shape of a figure under its action, hence the name "rigid." For example, reflections, rotations, and translations $(x \longrightarrow x + x_0)$ are examples of rigid motions. We shall see, in fact, that every rigid motion is a composition of these three transformations. The general situation in R^n will be handled in Section 7.8 and will use the results of this exercise.

20.† Let V be a real inner product space. A function $f\colon$ $\mathsf{V} \longrightarrow \mathsf{V}$ is called a *rigid motion* if

$$||f(x) - f(y)|| = ||x - y|| \qquad \text{for all } x, y \in \mathsf{V}.$$

For such a function f, define $\mathsf{T}\colon$ $\mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{T}(x) = f(x) - f(0)$.

(a) Show that T is linear by proving the following four parts.
   (i) $||\mathsf{T}(x)|| = ||x||$ for all $x \in \mathsf{V}$.
   (ii) $||\mathsf{T}(x) - \mathsf{T}(y)|| = ||x - y||$ for all $x, y \in \mathsf{V}$.
   (iii) $(\mathsf{T}(x), \mathsf{T}(y)) = (x, y)$ for all $x, y \in \mathsf{V}$. *Hint:* Expand both sides of (ii) using the properties of inner products and then equate the results.
   (iv) $||\mathsf{T}(x + ay) - \mathsf{T}(x) - a\mathsf{T}(y)|| = 0$ for all $x, y \in \mathsf{V}$ and $a \in R$.
(b) Use (a) to deduce that every rigid motion is an orthogonal operator followed by a translation.
(c) Prove $\det(\mathsf{T}) = \pm 1$.
(d) Let $\mathsf{V} = \mathsf{R}^2$, and let β be the standard ordered basis for R^2. Show that there exists an angle θ $(0 \leq \theta < 2\pi)$ such that

$$[\mathsf{T}]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{if } \det(\mathsf{T}) = 1$$

and

$$[\mathsf{T}]_\beta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \text{if } \det(\mathsf{T}) = -1.$$

Hint: Use the fact that the columns of $[\mathsf{T}]_\beta$ form an orthonormal subset of R^2.

(e) Use (d) to deduce that every rigid motion in R^2 is either a rotation (about the origin) followed by a translation or a reflection (about the x-axis) followed by a rotation (about the origin) followed by a translation. *Hint:* Observe that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

21. Let $y_1, \ldots, y_n$ be linearly independent vectors in F^n, and let $x_1, \ldots, x_n$ be the orthogonal vectors obtained from $y_1, \ldots, y_n$ by the Gram-Schmidt orthogonalization process. Let $z_1, \ldots, z_n$ be the orthonormal basis obtained by defining

$$z_k = \frac{x_k}{||x_k||}.$$

(a) By solving Eq. (1) in Section 7.2 for y_k in terms of z_k, show that

$$y_k = ||x_k||z_k + \sum_{j=1}^{k-1} (y_k, z_j)z_j \qquad (1 \le k \le n).$$

(b) Let A and Q denote the $n \times n$ matrices in which the kth columns are y_k and z_k, respectively. Define $R \in \mathsf{M}_{n\times n}(F)$ by

$$R_{jk} = \begin{cases} ||x_j|| & \text{if } j = k \\ (y_k, z_j) & \text{if } j < k \\ 0 & \text{if } j > k. \end{cases}$$

Prove that $A = QR$.

(c) Compute Q and R as in part (b) for the 3×3 matrix whose columns are the vectors y_1, y_2, and y_3, respectively, in Example 12 in Section 7.2.

(d) Since Q is unitary [orthogonal] and R is upper triangular in part (b), we have shown that every invertible matrix is the product of a unitary [orthogonal] matrix and an upper triangular matrix. Suppose that $A \in \mathsf{M}_{n\times n}(F)$ is invertible and $A = Q_1R_1 = Q_2R_2$, where $Q_1, Q_2 \in \mathsf{M}_{n\times n}(F)$ are unitary and $R_1, R_2 \in \mathsf{M}_{n\times n}(F)$ are upper triangular. Prove that $D = R_2R_1^{-1}$ is a unitary diagonal matrix. *Hint:* Use Exercise 16.

(e) The decomposition described in part (b) provides an orthogonalization method for solving a linear system $AX = B$ when A is invertible: Decompose A to QR, by the Gram-Schmidt process or other means, where Q is unitary and R is upper triangular. Then $QRX = B$, and hence $RX = Q^*B$. This last system can be easily solved since R is upper triangular.

At one time, because of its great stability, this method for solving large systems of linear equations with a computer was being advocated as a better method than Gaussian elimination even though it requires about three times as much work. (Later, however, J. H. Wilkinson showed that if Gaussian elimination is done properly, then it is nearly as stable as the orthogonalization method.)

Use the orthogonalization method and part (c) to solve the system

$$\begin{cases} x_1 + 2x_2 + 2x_3 = 1 \\ x_1 \qquad\quad + 2x_3 = 11 \\ \qquad\; x_2 + x_3 = -1. \end{cases}$$

22. Find new coordinates x', y' so that the following quadratic forms can be written as $\lambda_1(x')^2 + \lambda_2(y')^2$.
(a) $x^2 + 4xy + y^2$
(b) $2x^2 + 2xy + 2y^2$

23. Consider the expression X^tAX, where $X^t = (x, y, z)$ and A is as defined in Exercise 2(e). Find a change of coordinates x', y', z' so that the expression above can be written in the form $\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2$.

7.8* THE GEOMETRY OF ORTHOGONAL OPERATORS

Exercise 20 of Section 7.7 establishes that any rigid motion on a real inner product space is the composition of an orthogonal operator followed by a translation. Thus to understand the geometry of rigid motions thoroughly, we must analyze the structure of orthogonal operators. Such is the aim of this section. As we shall discover, an orthogonal operator on a finite-dimensional real inner product space is the composition of rotations and reflections. We begin our investigation with definitions of these terms.

Definitions. *Let* T *be a linear operator on a finite-dimensional real inner product space* V. *The operator* T *is called a* rotation *if* T *is the identity on* V *or if there exists a two-dimensional subspace* W *of* V, *an orthonormal basis* $\beta = \{x_1, x_2\}$ *for* W, *and a real number* θ *such that*

$$T(x_1) = x_1 \cos\theta + x_2 \sin\theta, \qquad T(x_2) = -x_1 \sin\theta + x_2 \cos\theta,$$

and $T(y) = y$ *for all* $y \in W^\perp$. *In this context* T *is called a* rotation of W about $W^\perp$. *The subspace* $W^\perp$ *is called the* axis of rotation.

Rotations were defined in Section 2.1 for the special case that $V = R^2$.

Definitions. *Let* T *be a linear operator on a finite-dimensional real inner product space* V. *The operator* T *is called a* reflection *if there exists a one-dimensional subspace* W *of* V *such that* $T(x) = -x$ *for all* $x \in W$ *and* $T(y) = y$ *for all* $y \in W^{\perp}$. *In this context* T *is called a* reflection of V about $W^{\perp}$.

It should be noted that rotations and reflections (or compositions thereof) are orthogonal operators. (See Exercise 2.) The principal aim of this section is to establish that the converse is also true, i.e., that any orthogonal operator on a finite-dimensional real inner product space is the composition of rotations and reflections.

Example 29. Characterization of orthogonal operators on a one-dimensional real inner product space.

Let T be an orthogonal operator on a one-dimensional inner product space V. Choose any non-zero vector x in V. Then $V = \text{span}(\{x\})$, and so $T(x) = \lambda x$ for some $\lambda \in R$. Since T is orthogonal and λ is an eigenvalue of T, $\lambda = \pm 1$. If $\lambda = 1$, then T is the identity on V, and hence T is a rotation. If $\lambda = -1$, then $T(x) = -x$ for all $x \in V$, and hence T is a reflection of V about $V^{\perp} = \{0\}$. Thus T is either a rotation or a reflection. Note that in the first case $\det(T) = 1$, and in the second case $\det(T) = -1$.

Example 30. Some typical reflections.

(a) Let $T\colon R^2 \longrightarrow R^2$ be defined by $T(a, b) = (-a, b)$. If $W = \text{span}(\{e_1\})$, then $T(x) = -x$ for all $x \in W$ and $T(y) = y$ for all $y \in W^{\perp}$. Thus T is a reflection of R^2 about $W^{\perp} = \text{span}(\{e_2\})$, the y-axis.

(b) Let $T\colon R^3 \longrightarrow R^3$ be defined by $T(a, b, c) = (a, b, -c)$. If $W = \text{span}(\{e_3\})$, then $T(x) = -x$ for all $x \in W$ and $T(y) = y$ for all $y \in W^{\perp} = \text{span}(\{e_1, e_2\})$, the xy-plane.

In Example 29 we characterized all orthogonal operators on a one-dimensional real inner product space. The following theorem characterizes all orthogonal operators on a two-dimensional real inner product space. The proof of this result follows easily from Exercise 20 of Section 7.7 since a reflection about the x-axis followed by a rotation by θ is a reflection about the line through the origin with slope $\tan \frac{1}{2}\theta$.

Theorem 7.22. *Let* T *be an orthogonal operator on a two-dimensional real inner product space* V. *Then* T *is either a rotation or a reflection. Furthermore,* T *is a rotation if and only if* $\det(T) = 1$, *and* T *is a reflection if and only if* $\det(T) = -1$.

It is immediate from the definition that any reflection on R^2 has eigenvalues of 1 and -1 and that any two eigenvectors corresponding to

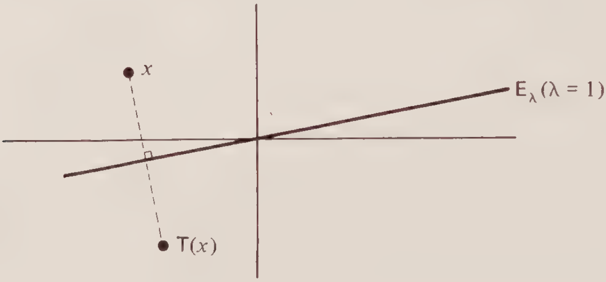

figure 7.5

these eigenvalues are orthogonal. Moreover, the eigenspace of T corresponding to $\lambda = 1$ is one-dimensional and hence can be described as a line passing through the origin. Geometrically, T reflects points in R^2 about this line. (See Fig. 7.5.) For example, if

$$A = \begin{pmatrix} \dfrac{1}{\sqrt{5}} & \dfrac{2}{\sqrt{5}} \\ \dfrac{2}{\sqrt{5}} & \dfrac{-1}{\sqrt{5}} \end{pmatrix},$$

it is clear that L_A is an orthogonal operator on R^2 and that $\det(L_A) = \det(A) = -1$. Hence L_A is a reflection by Theorem 7.22. To find the subspace about which L_A reflects, it suffices to find an eigenvector of L_A corresponding to eigenvalue $\lambda = 1$. One such eigenvector is

$$x = \begin{pmatrix} \dfrac{1}{\sqrt{5}} + 1 \\ \dfrac{2}{\sqrt{5}} \end{pmatrix}.$$

Consequently the subspace about which L_A reflects is the line

$$\left\{ t \begin{pmatrix} \dfrac{1}{\sqrt{5}} + 1 \\ \dfrac{2}{\sqrt{5}} \end{pmatrix} : \quad t \in R \right\}.$$

Corollary. *Let* V *be a two-dimensional real inner product space. The composition of a reflection and a rotation on* V *is a reflection on* V.

PROOF. If T_1 is a reflection on V and T_2 is a rotation on V, then by Theorem 7.22 $\det(T_1) = 1$ and $\det(T_2) = -1$. Let $T = T_2T_1$ be the composition. Since T_2 and T_1 are orthogonal, so is T. Moreover $\det(T) =$

$\det(\mathsf{T}_2)\cdot\det(\mathsf{T}_1) = -1$. Thus, by Theorem 7.22, T is a reflection. The proof for $\mathsf{T}_1\mathsf{T}_2$ is similar. ▮

We shall now study orthogonal operators on spaces of higher dimension.

Lemma. *If* T *is a linear operator on a non-zero finite-dimensional real inner product space* V, *then there exists a* T*-invariant subspace* W *of* V *such that* $1 \leq \dim(\mathsf{W}) \leq 2$.

PROOF. Fix an ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$ for V, and let $A = [\mathsf{T}]_\beta$. Let $\phi_\beta\colon \mathsf{V} \longrightarrow \mathsf{R}^n$ be the linear transformation defined by $\phi_\beta(x_i) = e_i$ for $i = 1, 2, \ldots, n$. Then ϕ_β is an isomorphism, and we have seen in Section 2.4 that the diagram in Fig. 7.6 is commutative, i.e., that $\mathsf{L}_A\phi_\beta = \phi_\beta\mathsf{T}$. As a consequence it suffices to show that there exists an L_A-invariant subspace Z of R^n such that $1 \leq \dim(\mathsf{Z}) \leq 2$. If we then define $\mathsf{W} = \phi_\beta^{-1}(\mathsf{Z})$, it will follow that W satisfies the conclusion of the theorem. (See Exercise 12.)

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"] & \mathsf{V} \arrow[d, "\phi_\beta"] \\
\mathsf{R}^n \arrow[r, "\mathsf{L}_A"] & \mathsf{R}^n
\end{tikzcd}$$

figure 7.6

The matrix A can be considered as an $n \times n$ matrix over C and as such can be used to define a linear operator U on C^n by $\mathsf{U}(x) = Ax$ for all column vectors x in C^n. Since U is an operator on a finite-dimensional vector space over C, it has an eigenvalue $\lambda \in C$. Let $x \in \mathsf{C}^n$ be an eigenvector corresponding to λ. We may write $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real, and

$$x = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n + ib_n \end{pmatrix},$$

where the a_i's and b_i's are real. Thus, setting

$$x_1 = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix},$$

we have $x = x_1 + ix_2$, where x_1 and x_2 are n-tuples consisting of real entries. Note that at least one of x_1 or x_2 is non-zero since $x \neq 0$. Hence

$$\begin{aligned} \mathsf{U}(x) &= \lambda x = (\lambda_1 + i\lambda_2)(x_1 + ix_2) \\ &= (\lambda_1 x_1 - \lambda_2 x_2) + i(\lambda_1 x_2 + \lambda_2 x_1). \end{aligned} \tag{14}$$

Likewise

$$\mathsf{U}(x) = A(x_1 + ix_2) = Ax_1 + iAx_2. \tag{15}$$

Comparing the real and imaginary parts of Eqs. (14) and (15), we conclude that

$$Ax_1 = \lambda_1 x_1 - \lambda_2 x_2 \quad \text{and} \quad Ax_2 = \lambda_1 x_1 + \lambda_2 x_1. \tag{16}$$

Finally, let $\mathsf{Z} = \mathrm{span}(\{x_1, x_2\})$, the span being taken as a subspace of R^n. Since $x_1 \neq 0$ or $x_2 \neq 0$, Z is non-zero. Thus $1 \leq \dim(\mathsf{Z}) \leq 2$, and by Eq. (16), Z is L_A-invariant. ∎

Theorem 7.23. Let T *be an orthogonal operator on a non-zero finite-dimensional real inner product space* V. *Then there exists a collection of pairwise orthogonal* T*-invariant subspaces* $\{\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_m\}$ *of* V *such that*

(a) $1 \leq \dim(\mathsf{W}_i) \leq 2$ *for* $i = 1, 2, \ldots, m$.

(b) $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_m$.

PROOF. The proof is by induction on dim(V). If $\dim(\mathsf{V}) = 1$, the result is obvious. So assume that the result is true whenever $\dim(\mathsf{V}) < n$ for some fixed integer $n > 1$.

Suppose $\dim(\mathsf{V}) = n$. By the lemma there is a T-invariant subspace W_1 of V such that $1 \leq \dim(\mathsf{W}_1) \leq 2$. If $\mathsf{W}_1 = \mathsf{V}$, the result is established. Otherwise, $\mathsf{W}_1^\perp \neq \{0\}$. By Exercise 13 $\mathsf{W}_1^\perp$ is T-invariant and the restriction of T to $\mathsf{W}_1^\perp$ is orthogonal. Since $\dim(\mathsf{W}_1^\perp) < n$, we may apply the induction hypothesis to $\mathsf{T}_{\mathsf{W}_1^\perp}$ and conclude that there exists a collection of pairwise orthogonal T-invariant subspaces $\{\mathsf{W}_2, \mathsf{W}_3, \ldots, \mathsf{W}_m\}$ of $\mathsf{W}_1^\perp$ such that $1 \leq \dim(\mathsf{W}_i) \leq 2$ for $i = 2, 3, \ldots, m$ and $\mathsf{W}_1^\perp = \mathsf{W}_2 \oplus \mathsf{W}_3 \oplus \cdots \oplus \mathsf{W}_m$. Thus $\{\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_m\}$ is pairwise orthogonal and

$$\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_1^\perp = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_m. \quad ∎$$

Applying Example 29 and Theorem 7.22 in the context of Theorem 7.23, we can conclude that the restriction of T to W_i is either a rotation or a reflection for each $i = 1, 2, \ldots, m$. Thus in some sense T is made up of rotations and reflections. Unfortunately, very little can be said about the decomposition of V in Theorem 7.23 in terms of uniqueness. For example, the W_i's, the number m of W_i's, and the number of W_i's for which T_{W_i} is a reflection are not unique. Although the number of W_i's for which T_{W_i} is a reflection is not unique, whether this number is even or odd is an intrinsic property of T. Moreover, we can always decompose V so that T_{W_i} is a reflection for at most one W_i. These facts are established in the following result.

Theorem 7.24. *Let* $T, V, W_1, \ldots, W_m$ *be as in Theorem 7.23.*

(a) *The number of* i*'s for which* T_{W_i} *is a reflection is even or odd according to whether* $det(T) = 1$ *or* $det(T) = -1$.

(b) *It is always possible to decompose* V *as in Theorem 7.23 so that the number of* i*'s for which* T_{W_i} *is a reflection is zero or one according to whether* $det(T) = 1$ *or* $det(T) = -1$. *Furthermore, if* T_{W_i} *is a reflection, then* $dim(W_i) = 1$.

PROOF.

(a) Let r denote the number of W_i's in the decomposition for which T_{W_i} is a reflection. Then by Exercise 14

$$\det(T) = \det(T_{W_1})\cdot\det(T_{W_2})\cdot\cdots\cdot\det(T_{W_m}) = (-1)^r,$$

proving (a).

(b) Let $E = \{x \in V: \; T(x) = -x\}$; then E is a T-invariant subspace of V. If $W = E^{\perp}$, then W is T-invariant. So by applying Theorem 7.23 to T_W, we obtain a collection of pairwise orthogonal T-invariant subspaces $\{W_1, W_2, \ldots, W_k\}$ of W such that $1 \leq \dim(W_i) \leq 2$ for $1 \leq i \leq k$ and $W = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Observe that, for each $i = 1, 2, \ldots, k$, T_{W_i} is a rotation. For otherwise, if T_{W_i} is a reflection, there exists a non-zero $x \in W_i$ for which $T(x) = -x$. But then $x \in W_i \cap E \subseteq E^{\perp} \cap E = \{0\}$, a contradiction. If $E = \{0\}$, the result follows. Otherwise choose an orthonormal basis β for E containing p elements ($p > 0$). It is possible to decompose β into a pairwise disjoint union $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_r$ such that each β_i contains exactly two elements for $i < r$, and β_r contains two elements if p is even and one element if p is odd. For each $i = 1, 2, \ldots, r$, let $W_{k+i} = \text{span}(\beta_i)$. Then clearly $\{W_1, W_2, \ldots, W_k, \ldots, W_{k+r}\}$ is pairwise orthogonal and

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k \oplus \cdots \oplus W_{k+r}. \tag{17}$$

Moreover, if any β_i contains two elements, then

$$\det(T_{W_{k+i}}) = \det([T_{W_{k+i}}]_{\beta_i}) = \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1.$$

So $T_{W_{k+i}}$ is a rotation, and hence T_{W_j} is a rotation for $j < k + r$. If β_r consists of one element, then $\dim(W_{k+r}) = 1$ and $\det(T_{W_{k+r}}) = \det([T_{W_{k+r}}]_{\beta_r}) = \det(-1) = -1$. Thus $T_{W_{k+r}}$ is a reflection by Theorem 7.23, and we conclude that the decomposition in Eq. (17) satisfies the condition of part (b). ∎

As a consequence of the preceding theorem, an orthogonal operator can be factored as a product of rotations and reflections.

Corollary. *Let* T *be an orthogonal operator on a finite-dimensional real inner product space* V*. Then there exists a collection* $\{T_1, T_2, \ldots, T_m\}$ *of orthogonal operators on* V *such that*

(a) *For each* i, T_i *is either a reflection or a rotation.*
(b) *For at most one* i, T_i *is a reflection.*
(c) $T_iT_j = T_jT_i$ *for all* i *and* j.
(d) $T = T_1T_2 \cdots T_m$.
(e) $\det(T) = \begin{cases} 1 & \text{if } T_i \text{ is a rotation for each } i \\ -1 & \text{otherwise.} \end{cases}$

PROOF. As in the proof of part (b) of Theorem 7.24 we can write $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$, where T_{W_i} is a rotation for $i < m$. For each $i = 1, 2, \ldots, m$, define $T_i\colon\ V \to V$ by

$$T_i(x_1 + \cdots + x_m) = x_1 + \cdots + x_{i-1} + T(x_i) + x_{i+1} + \cdots + x_m,$$

where $x_j \in W_j$ for all j. It is easily shown that each T_i is an orthogonal operator on V. In fact, T_i is a rotation or a reflection according to whether T_{W_i} is a rotation or a reflection. This establishes (a) and (b). The proofs of (c), (d), and (e) are left as an exercise. (See Exercise 15.) ∎

Example 31. Orthogonal operators on a three-dimensional real inner product space.

Let T be an orthogonal operator on a three-dimensional real inner product space V. We shall show that T can be decomposed into the composition of a rotation and at most one reflection. Let $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ be a decomposition as in Theorem 7.24(b). Clearly $m = 2$ or $m = 3$.

If $m = 2$, then $V = W_1 \oplus W_2$. Without loss of generality, suppose that $\dim(W_1) = 1$ and $\dim(W_2) = 2$. Thus T_{W_1} is a reflection or the identity on W_1, and T_{W_2} is a rotation. Defining T_1 and T_2 as in the proof of the corollary to Theorem 7.24, we have that $T = T_1T_2$ is the composition of a rotation and at most one reflection. (Note that if T_{W_1} is not a reflection, then T_1 is the identity on V and $T = T_2$.)

If $m = 3$, then $V = W_1 \oplus W_2 \oplus W_3$ and $\dim(W_i) = 1$ for all i. For each i, let T_i be as in the proof of the corollary to Theorem 7.24. If T_{W_i} is not a reflection, then T_i is the identity on W_i. Otherwise T_i is a reflection. Since $\mathsf{T}_{\mathsf{W}_i}$ is a reflection for at most one i, we conclude that T is either a single reflection or the identity (a rotation).

EXERCISES

1. Label the following statements as being true or false. Assume for the following that the underlying vector spaces are finite-dimensional real inner product spaces.
   (a) Any orthogonal operator is either a rotation or a reflection.
   (b) The composition of any two rotations on a two-dimensional space is a rotation.
   (c) The composition of any two rotations on a three-dimensional space is a rotation.
   (d) The composition of any two rotations on a four-dimensional space is a rotation.
   (e) The identity operator is a rotation.
   (f) The composition of two reflections is a reflection.
   (g) Any orthogonal operator is a composition of rotations.
   (h) For any orthogonal operator T, if $\det(\mathsf{T}) = -1$, then T is a reflection.
   (i) Reflections always have eigenvalues.
   (j) Rotations always have eigenvalues.

2. Prove that rotations, reflections, and compositions of rotations and reflections are orthogonal operators.

3. Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Prove that $\mathsf{L}_A\colon\ \mathsf{R}^2 \longrightarrow \mathsf{R}^2$ is a reflection.
   (b) Find the axis in R^2 about which L_A reflects, i.e., the subspace of R^2 on which L_A acts as the identity.
   (c) Prove that L_{AB} and L_{BA} are rotations.

4. For any real number ϕ, let

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

   (a) Prove that L_A is a reflection.
   (b) Find the axis in R^2 about which L_A reflects.

5. For any real number ϕ, define $\mathsf{T}_\phi = \mathsf{L}_A$, where
$$A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$
   (a) Prove that any rotation of R^2 is of the form T_ϕ for some ϕ.
   (b) Prove that $\mathsf{T}_\phi \mathsf{T}_\psi = \mathsf{T}_{(\phi+\psi)}$ for any $\phi, \psi \in R$.
   (c) Deduce that any two rotations of R^2 commute.

6. Prove that the composition of any two rotations of R^3 is a rotation of R^3.

7. Given real numbers ϕ and ψ, define matrices
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
   (a) Prove that L_A and L_B are rotations.
   (b) Prove that L_{AB} is a rotation.
   (c) Find the axis of rotation for L_{AB}.

8. Prove that no orthogonal operator can be both a rotation and a reflection.

9. Prove that if V is a two- or three-dimensional real inner product space, then the composition of two reflections on V is a rotation of V.

10. Give an example of an orthogonal operator that is neither a reflection nor a rotation.

11. Let V be a finite-dimensional real inner product space. Define $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ by $\mathsf{T}(x) = -x$. Prove that T is a product of rotations if and only if dim(V) is even.

12. Complete the proof of the lemma to Theorem 7.23 by showing that $\mathsf{W} = \phi_\beta^{-1}(\mathsf{Z})$ satisfies the required conditions.

13. Let T be an orthogonal [unitary] operator on a finite-dimensional real [complex] inner product space V. If W is a T-invariant subspace of V, prove that
   (a) T_W is an orthogonal [unitary] operator on W.
   (b) $\mathsf{W}^\perp$ is a T-invariant subspace of V.
   Hint: Use the fact that T_W is one-to-one and onto to conclude that, for any $y \in \mathsf{W}$, $\mathsf{T}^*(y) = \mathsf{T}^{-1}(y) \in \mathsf{W}$.
   (c) $\mathsf{T}_{\mathsf{W}^\perp}$ is an orthogonal [unitary] operator on W.

14. Let T be a linear operator on a finite-dimensional vector space V. Suppose that V is a direct sum of T-invariant subspaces $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that $\det(\mathsf{T}) = \det(\mathsf{T}_{\mathsf{W}_1}) \cdot \det(\mathsf{T}_{\mathsf{W}_2}) \cdot \cdots \cdot \det(\mathsf{T}_{\mathsf{W}_k})$.

15. Complete the proof of the corollary to Theorem 7.24.

16. Let T be an orthogonal operator on an n-dimensional real inner product space V. Suppose that T is not the identity. Prove that
(a) If n is odd, then T can be expressed as the composite of at most one reflection and at most $\frac{1}{2}(n-1)$ rotations.
(b) If n is even, then T can be expressed as the composite of at most $\frac{1}{2}n$ rotations or as the composite of one reflection and at most $\frac{1}{2}(n-2)$ rotations.

17. Let V be a real inner product space of dimension 2. For any $x, y \in \mathsf{V}$ such that $x \neq y$ and $\|x\| = \|y\| = 1$, show that there exists a unique rotation T on V such that $\mathsf{T}(x) = y$.

7.9 ORTHOGONAL PROJECTIONS AND THE SPECTRAL THEOREM

In this section we shall rely heavily upon Theorem 7.17 to develop an elegant representation of a normal operator T on a complex finite-dimensional inner product space. We shall prove that such an operator can be written in the form $\lambda_1\mathsf{T}_1 + \cdots + \lambda_k\mathsf{T}_k$, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of T and $\mathsf{T}_1, \ldots, \mathsf{T}_k$ are "orthogonal projections." We first must develop some results about these special projections.

The reader will recall from Section 2.1 that a linear transformation $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ is a projection (on its range R(T)) if $\mathsf{V} = \mathsf{R(T)} \oplus \mathsf{N(T)}$. In fact, T is a projection if and only if $\mathsf{T} = \mathsf{T}^2$ (see Exercise 14 of Section 2.3).

Definition. *Let* V *be an inner product space, and let* $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{V}$ *be a projection. We say that* T *is an* orthogonal projection *if* $\mathsf{R(T)}^\perp = \mathsf{N(T)}$ *and* $\mathsf{N(T)}^\perp = \mathsf{R(T)}$.

Note that by Exercise 12(c) in Section 7.2 if V is finite-dimensional, we need only assume that one of the conditions above hold. For example, if $\mathsf{R(T)}^\perp = \mathsf{N(T)}$, then $\mathsf{R(T)} = \mathsf{R(T)}^{\perp\perp} = \mathsf{N(T)}^\perp$.

Now assume that W is a finite-dimensional subspace of an inner product space V. Theorem 7.6 guarantees that there exists an orthogonal projection on W. We can say even more—there exists exactly one orthogonal projection on W. For if T and U are orthogonal projections on W, then $\mathsf{R(T)} = \mathsf{W} = \mathsf{R(U)}$. Hence $\mathsf{N(T)} = \mathsf{R(T)}^\perp = \mathsf{R(U)}^\perp = \mathsf{N(U)}$, and since all projections are uniquely determined by their range and null space, we have that $\mathsf{T} = \mathsf{U}$. We call T the *orthogonal projection on* W. To understand the geometric difference between an arbitrary projection on W and the orthogonal projection on W, let $\mathsf{V} = \mathsf{R}^2$ and $\mathsf{W} = \operatorname{span}\{(1, 1)\}$. Define U and T as in Fig. 7.7, where $\mathsf{T}(v)$ is the foot of a perpendicular from v on the line $y = x$ and $\mathsf{U}(a_1, a_2) = (a_1, a_1)$. Then T is the orthogonal

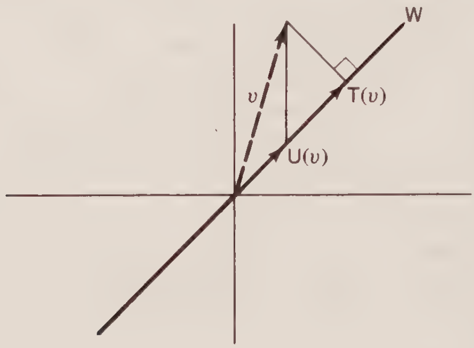

figure 7.7

projection on W, and U is a projection on W that is not orthogonal. Note that $v - \mathsf{T}(v) \in \mathsf{W}^{\perp}$, whereas $v - \mathsf{U}(v) \notin \mathsf{W}^{\perp}$.

From Fig. 7.7 we see that $\mathsf{T}(v)$ is the "best approximation in W to v"; i.e., if $w \in \mathsf{W}$, then $\|w - v\| \geq \|\mathsf{T}(v) - v\|$. This approximation property characterizes T. In fact, many authors define orthogonal projections in terms of this property.

Theorem 7.25. Let W *be a finite-dimensional subspace of an inner product space* V, *and let* T *be the orthogonal projection on* W. *Then for any* $\mathrm{v} \in \mathsf{V}$ *the vector* $\mathsf{T}(\mathrm{v})$ *is the unique element of* W *that is closest to* v; *that is,* $\|\mathrm{v} - \mathsf{T}(\mathrm{v})\| \leq \|\mathrm{v} - \mathrm{w}\|$ *for every* $\mathrm{w} \in \mathsf{W}$.

PROOF. Let $v \in \mathsf{V}$. Since T is an orthogonal projection, we may write $v = \mathsf{T}(v) + (v - \mathsf{T}(v))$, where $v - \mathsf{T}(v) \in \mathsf{N}(\mathsf{T}) = \mathsf{W}^{\perp}$. Let $w \in \mathsf{W}$. By Exercise 10 of Section 7.1 we have

$$\begin{aligned}\|v - w\|^2 &= \|\mathsf{T}(v) - w + (v - \mathsf{T}(v))\|^2\\ &= \|\mathsf{T}(v) - w\|^2 + \|v - \mathsf{T}(v)\|^2 \geq \|v - \mathsf{T}(v)\|^2,\end{aligned}$$

proving the inequality above. If for some $w \in \mathsf{W}$ we have $\|v - w\| = \|v - \mathsf{T}(v)\|$, then we see from the computation above that $\|\mathsf{T}(v) - w\|^2 = 0$, i.e., that $w = \mathsf{T}(v)$. ∎

In Section 7.10 we shall see a very important application of Theorem 7.25 to the topic of least squares approximation that occurs frequently in statistics.

For now we shall apply Theorem 7.25 to obtain a well-known result in Fourier analysis. Recall the inner product space H of continuous functions on the interval $[0, 2\pi]$ introduced in Section 7.1. Define a *trigonometric* *polynomial of degree n* to be a function $g \in \mathsf{H}$ of the form

$$g(x) = \sum_{j=-n}^{n} a_j e^{ijx},$$

where a_n or a_{-n} is non-zero.

Let $f \in \mathsf{H}$. We shall show that the best approximation to f by a trigonometric polynomial of degree less than or equal to n is the polynomial whose coefficients are the Fourier coefficients of f relative to the orthonormal set $\{e^{ijx}\colon\ j \text{ is an integer}\}$.

For this result, let $\mathsf{W} = \operatorname{span}(\{e^{ijx}\colon\ |j| \le n\})$, and let T be the orthogonal projection on W. Theorem 7.25 tells us that

$$\mathsf{T}(f) = \sum_{j=-n}^{n} (f, e^{ijx})e^{ijx}$$

is the best approximation to f in H. (See also Corollary 1 of Theorem 7.6.)

An algebraic characterization of orthogonal projections follows in the next theorem.

Theorem 7.26. *Let V be an inner product space, and let T be a linear operator on V. Then T is an orthogonal projection if and only if $\mathsf{T}^2 = \mathsf{T} = \mathsf{T}^*$.*

PROOF. Suppose that T is an orthogonal projection. Since $\mathsf{T} = \mathsf{T}^2$ because T is a projection, we need only show that $\mathsf{T} = \mathsf{T}^*$. Now $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})^\perp = \mathsf{N}(\mathsf{T})$. If $x, y \in \mathsf{V}$, then $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in \mathsf{R}(\mathsf{T})$ and $x_2, y_2 \in \mathsf{N}(\mathsf{T})$. Hence

$$(x, \mathsf{T}(y)) = (x_1 + x_2, y_1) = (x_1, y_1) + (x_2, y_1) = (x_1, y_1)$$

and

$$(x, \mathsf{T}^*(y)) = (\mathsf{T}(x), y) = (x_1, y_1 + y_2) = (x_1, y_1) + (x_1, y_2) = (x_1, y_1).$$

So $(x, \mathsf{T}(y)) = (x, \mathsf{T}^*(y))$ for all $x, y \in \mathsf{V}$, and thus $\mathsf{T} = \mathsf{T}^*$.

Now suppose that $\mathsf{T} = \mathsf{T}^2 = \mathsf{T}^*$. Then T is a projection by Exercise 14 of Section 2.3, and hence we must show that $\mathsf{R}(\mathsf{T}) = \mathsf{N}(\mathsf{T})^\perp$ and $\mathsf{R}(\mathsf{T})^\perp = \mathsf{N}(\mathsf{T})$. Let $x \in \mathsf{R}(\mathsf{T})$ and $y \in \mathsf{N}(\mathsf{T})$. Then $x = \mathsf{T}(x) = \mathsf{T}^*(x)$, and so $(x, y) = (\mathsf{T}^*(x), y) = (x, \mathsf{T}(y)) = (x, 0) = 0$. Therefore $x \in \mathsf{N}(\mathsf{T})^\perp$, from which it follows that $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})^\perp$.

Let $y \in \mathsf{N}(\mathsf{T})^\perp$. We must show that $y \in \mathsf{R}(\mathsf{T})$, that is, that $\mathsf{T}(y) = y$. Now

$$\begin{aligned}\|y - \mathsf{T}(y)\|^2 &= (y - \mathsf{T}(y), y - \mathsf{T}(y))\\ &= (y, y - \mathsf{T}(y)) - (\mathsf{T}(y), y - \mathsf{T}(y)).\end{aligned}$$

Since $y - \mathsf{T}(y) \in \mathsf{N}(\mathsf{T})$, the first term is zero. But also $(\mathsf{T}(y), y - \mathsf{T}(y)) = (y, \mathsf{T}^*(y - \mathsf{T}(y))) = (y, \mathsf{T}(y - \mathsf{T}(y))) = (y, 0) = 0$. Thus $y - \mathsf{T}(y) = 0$; i.e., $y = \mathsf{T}(y) \in \mathsf{R}(\mathsf{T})$. Hence $\mathsf{R}(\mathsf{T}) = \mathsf{N}(\mathsf{T})^\perp$.

Using the above, we have that $R(T)^\perp = N(T)^{\perp\perp} \supseteq N(T)$ (by Exercise 12(b) of Section 7.2). We need only show that if $x \in R(T)^\perp$, then $x \in N(T)$. For any $y \in V$, we have $(T(x), y) = (x, T^*(y)) = (x, T(y)) = 0$. So $T(x) = 0$, and thus $x \in N(T)$. ∎

Let V be a finite-dimensional inner product space, W be a subspace of V, and T be the orthogonal projection on W. We may choose an orthonormal basis $\beta = \{x_1, \ldots, x_n\}$ for V such that $\{x_1, \ldots, x_k\}$ is a basis for W. Then $[T]_\beta$ is a diagonal matrix with ones along the first k diagonal entries and zeros elsewhere. In fact, $[T]_\beta$ has the form

$$\begin{pmatrix} I_k & O_1 \\ O_2 & O_3 \end{pmatrix},$$

where O_1, O_2, and O_3 denote zero matrices.

If U is any projection on W, we may choose a basis γ for V such that $[U]_\gamma$ has the form above; however, γ will not necessarily be orthonormal.

Theorem 7.27 (*The Spectral Theorem*). *Suppose that* T *is a linear operator on a finite-dimensional inner product space* V *over* F. *Assume that* T *is normal if* $F = C$ *and that* T *is self-adjoint if* $F = R$. *If* $\lambda_1, \ldots, \lambda_k$ *are the distinct eigenvalues of* T, *let* $W_i = E_{\lambda_i} = \{x \in V\colon\ T(x) = \lambda_i x\}$ *be the eigenspace of* T *corresponding to the eigenvalue* λ_i $(1 \le i \le k)$, *and let* T_i *be the orthogonal projection on* W_i $(1 \le i \le k)$. *Then*

(a) $V = W_1 \oplus \cdots \oplus W_k$.

(b) *If* W_i' *denotes the direct sum of the subspaces of* W_j, $j \neq i$, *then* $W_i^\perp = W_i'$.

(c) $T_i T_j = \delta_{ij} T_i$ *for* $1 \le i, j \le k$.

(d) $I = T_1 + \cdots + T_k$.

(e) $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$.

PROOF.

(a) By Theorem 7.17, T is diagonalizable, and so $V = W_1 \oplus \cdots \oplus W_k$ by Theorem 5.14.

(b) If $x \in W_i$ and $y \in W_j$ for some i and j, then $(x, y) = 0$ by Theorem 7.15. It follows easily from this that $W_i' \subseteq W_i^\perp$. Now from (a) we have that

$$\dim(W_i') = \sum_{j \neq i} \dim(W_j) = \dim(V) - \dim(W_i).$$

On the other hand, by Corollary 2 of Theorem 7.6 we have that $\dim(W_i^\perp) = \dim(V) - \dim(W_i)$. Hence $W_i' = W_i^\perp$, proving (b).

The proof of part (c) is left as an exercise.

Since T_i is the orthogonal projection on W_i, we have from (b) that $N(T_i) = R(T_i)^\perp = W_i^\perp = W_i'$. Hence for $x \in V$ we have that $x = x_1 + \cdots + x_k$, where $x_j \in W_j$ and $T_i(x) = x_i$, proving (d).

(e) For $x \in \mathsf{V}$, write $x = x_1 + \cdots + x_k$, where $x_j \in \mathsf{W}_j$ $(1 \leq j \leq k)$. Then $\mathsf{T}(x) = \mathsf{T}(x_1) + \cdots + \mathsf{T}(x_k) = \lambda_1 x_1 + \cdots + \lambda_k x_k = \lambda_1 \mathsf{T}_1(x) + \cdots + \lambda_k \mathsf{T}_k(x) = (\lambda_1 \mathsf{T}_1 + \cdots + \lambda_k \mathsf{T}_k)(x)$. ∎

The set $\{\lambda_1, \ldots, \lambda_k\}$ of eigenvalues of T is called the *spectrum of* T, the sum $\mathsf{I} = \mathsf{T}_1 + \cdots + \mathsf{T}_k$ in (d) is called the *resolution of the identity operator induced by* T, and the sum $\mathsf{T} = \lambda_1 \mathsf{T}_1 + \cdots + \lambda_k \mathsf{T}_k$ in (e) is called the *spectral decomposition of* T. Since the distinct eigenvalues of T are uniquely determined (up to order) by the subspaces W_i (and hence by the orthogonal projections T_i), the spectral decomposition of T is unique.

With the notation above, let β be the union of orthonormal bases of the W_i's and let $m_i = \dim(\mathsf{W}_i)$. (Thus m_i is the multiplicity of λ_i.) Then $[\mathsf{T}]_\beta$ has the form

$$\begin{pmatrix} \lambda_1 I_{m_1} & O & \cdots & O \\ O & \lambda_2 I_{m_2} & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & \lambda_k I_{m_k} \end{pmatrix};$$

that is, $[\mathsf{T}]_\beta$ is a diagonal matrix in which the diagonal entries are the eigenvalues λ_i of T, and each λ_i is repeated m_i times. If $\mathsf{T} = \lambda_1 \mathsf{T}_1 + \cdots + \lambda_k \mathsf{T}_k$ as in (e) of the spectral theorem, then it follows (from Exercise 7) that $g(\mathsf{T}) = g(\lambda_1)\mathsf{T}_1 + \cdots + g(\lambda_k)\mathsf{T}_k$ for any polynomial g. This fact will be used below.

We shall now list several interesting corollaries of the spectral theorem; many more results are found in the exercises. For what follows we shall assume that V is a finite-dimensional inner product space over F and that T is a linear operator on V.

Corollary 1. *If* F = C, *then* T *is normal if and only if* $\mathsf{T}^* = g(\mathsf{T})$ *for some polynomial* g.

PROOF. Suppose first that T is normal. Let $\mathsf{T} = \lambda_1 \mathsf{T}_1 + \cdots + \lambda_k \mathsf{T}_k$ be the spectral decomposition of T. Taking the adjoint of both sides of the equation above we have $\mathsf{T}^* = \bar{\lambda}_1 \mathsf{T}_1 + \cdots + \bar{\lambda}_k \mathsf{T}_k$ since each T_i is self-adjoint. Using the Lagrange interpolation formula (see p. 45), we may choose a polynomial g such that $g(\lambda_i) = \bar{\lambda}_i$ for $1 \leq i \leq k$. Then

$$\begin{aligned} g(\mathsf{T}) &= g(\lambda_1)\mathsf{T}_1 + \cdots + g(\lambda_k)\mathsf{T}_k \\ &= \bar{\lambda}_1 \mathsf{T}_1 + \cdots + \bar{\lambda}_k \mathsf{T}_k \\ &= \mathsf{T}^*. \end{aligned}$$

Conversely, if $\mathsf{T}^* = g(\mathsf{T})$ for some polynomial g, then T commutes with T^* since T commutes with every polynomial in T. ∎

Corollary 2. *If* $\mathsf{F} = \mathsf{C}$, *then* T *is unitary if and only if* T *is normal and* $|\lambda| = 1$ *for every eigenvalue* λ *of* T.

PROOF. Suppose first that T is unitary and hence normal. Then if $\mathsf{T}(x) = \lambda x$, we have $|\lambda|\cdot\|x\| = \|\lambda x\| = \|\mathsf{T}(x)\| = \|x\|$, and hence $|\lambda| = 1$ if $x \neq 0$.

Now suppose that $|\lambda| = 1$ for every eigenvalue λ of T, and let $\mathsf{T} = \lambda_1\mathsf{T}_1 + \cdots + \lambda_k\mathsf{T}_k$ be the spectral decomposition of T. Then by (c) of Theorem 7.27

$$\begin{aligned}
\mathsf{TT}^* &= (\lambda_1\mathsf{T}_1 + \cdots + \lambda_k\mathsf{T}_k)(\bar{\lambda}_1\mathsf{T}_1 + \cdots + \bar{\lambda}_k\mathsf{T}_k)\\
&= |\lambda_1|^2\mathsf{T}_1 + \cdots + |\lambda_k|^2\mathsf{T}_k\\
&= \mathsf{T}_1 + \cdots + \mathsf{T}_k\\
&= \mathsf{I}.
\end{aligned}$$

Hence T is unitary. ∎

Corollary 3. *If* $\mathsf{F} = \mathsf{C}$ *and* T *is normal, then* T *is self-adjoint if and only if every eigenvalue of* T *is real.*

PROOF. Let $\mathsf{T} = \lambda_1\mathsf{T}_1 + \cdots + \lambda_k\mathsf{T}_k$ be the spectral decomposition of T. Suppose that every eigenvalue of T is real. Then $\mathsf{T}^* = \bar{\lambda}_1\mathsf{T}_1 + \cdots + \bar{\lambda}_k\mathsf{T}_k = \lambda_1\mathsf{T}_1 + \cdots + \lambda_k\mathsf{T}_k = \mathsf{T}$.

The converse has been proved in Corollary 2 of Theorem 7.15. ∎

Corollary 4. *Let* T *be as in the spectral theorem with spectral decomposition* $\mathsf{T} = \lambda_1\mathsf{T}_1 + \cdots + \lambda_k\mathsf{T}_k$. *Then each* T_i *is a polynomial in* T.

PROOF. Choose a polynomial g_j $(1 \leq j \leq k)$ such that $g_j(\lambda_i) = \delta_{ij}$. Then $g_j(\mathsf{T}) = g_j(\lambda_1)\mathsf{T}_1 + \cdots + g_j(\lambda_k)\mathsf{T}_k = \delta_{1j}\mathsf{T}_1 + \cdots + \delta_{kj}\mathsf{T}_k = \mathsf{T}_j$. ∎

EXERCISES

1. Label the following statements as being true or false. Assume that the underlying inner product spaces are finite-dimensional.
   (a) All projections are self-adjoint.
   (b) An orthogonal projection is uniquely determined by its range.
   (c) Every self-adjoint operator is a linear combination of orthogonal projections.
   (d) If an operator possesses a spectral decomposition, then so does its adjoint.
   (e) If T is a projection on W, then $\mathsf{T}(x)$ is the vector in W that is closest to x.
   (f) Every orthogonal projection is a unitary operator.

2. Let $V = R^2$, $W = \text{span}(\{(1, 2)\})$, and β be the standard ordered basis for V. Compute $[T]_\beta$, where T is the orthogonal projection on W. Do the same for $V = R^3$ and $W = \text{span}(\{(1, 0, 1)\})$.

3. For each of the matrices A in Exercise 2 of Section 7.7
(i) Prove that L_A possesses a spectral decomposition.
(ii) Explicitly define each of the orthogonal projections on the eigenspaces of L_A.
(iii) Verify your results using the spectral theorem.

4. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection on W, then $I - T$ is the orthogonal projection on $W^\perp$.

5. Let V be a finite-dimensional inner product space, and let $T\colon V \longrightarrow V$ be a projection.
(a) If T is an orthogonal projection, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. Give an example of a projection T for which this inequality does not hold. If equality holds, what can be concluded about T?
(b) If T is also normal and V is complex, prove that T must be an orthogonal projection.

6. Let T be a projection on a finite-dimensional inner product space V. That Prove that $\|T(x)\| \leq \| x \|$ for all $x \in V$, then T is an orthogonal projection.

7. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \cdots + \lambda_k T_k$ of T to prove the following.
(a) If g is a polynomial, then
$$g(T) = \sum_{i=1}^{k} g(\lambda_i)T_i.$$
(b) If $T^n = T_0$ for some n, then $T = T_0$.
(c) $U\colon V \longrightarrow V$ commutes with T if and only if U commutes with each T_i.
(d) If $U\colon V \longrightarrow V$ is normal and commutes with T, then $U = \mu_1 T_1 + \cdots + \mu_r T_r$, where $\mu_1, \ldots, \mu_r$ are the (not necessarily distinct) eigenvalues of U. *Hint:* Show that the eigenspaces of T are invariant under U.
(e) There exists a normal operator U on V such that $U^2 = T$.
(f) T is invertible if $\lambda_i \neq 0$ for $1 \leq i \leq k$.
(g) T is a projection if and only if every eigenvalue of T is 1 or 0.
(h) $T = -T^*$ (such a T is called *skew-symmetric*) if and only if every λ_i is an imaginary number.

8. Use Corollary 1 of the spectral theorem to show that if T is a normal operator on a complex finite-dimensional inner product space and U commutes with T, then U commutes with T^*.

9. Referring to Exercise 19 of Section 7.7, prove the following facts about U.
   (a) U*U is an orthogonal projection on W.
   (b) UU*U = U.

10. *Simultaneous Diagonalization.* Let V be a finite-dimensional complex inner product space, and let U, T: V ⟶ V be normal operators such that TU = UT. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both T and U. *Hint:* Use the hint of Exercise 13 of Section 7.5 along with Exercise 8.

11. Prove part (c) of the spectral theorem.

7.10* LEAST SQUARES APPROXIMATION

Consider the following problem: An experimenter collects data by taking measurements $y_1, y_2, \ldots, y_m$ at times $t_1, t_2, \ldots, t_m$, respectively. For example, he may be measuring unemployment at various times during some period. Suppose that he plots the data $(t_1, y_1), \ldots, (t_m, y_m)$ as points in the plane. (See Fig. 7.8.) From this distribution, he feels that there exists an essentially linear relationship between y and t, say, $y = ct + d$. He would like to find the constants c and d so that the line $y = ct + d$ represents the best possible "fit" to the data that he has collected. One

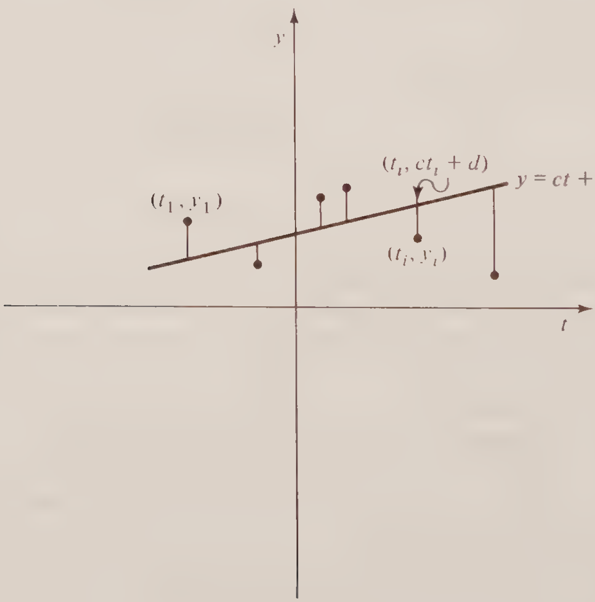

figure 7.8

such estimate of fit is to calculate the error E that represents the sum of the squares of the vertical distances from the points to the line; i.e.,

$$E = \sum_{i=1}^{m} (y_i - ct_i - d)^2.$$

Thus his problem is to find the constants c and d that minimize E. (For this reason the line $y = ct + d$ is called the *least squares line*.) This leads him to consider the following system of equations:

$$\begin{cases} t_1c + d = y_1 \\ t_2c + d = y_2 \\ \vdots \\ t_mc + d = y_m, \end{cases}$$

or $AX = y$, where

$$A = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_m & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} c \\ d \end{pmatrix}, \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

Note that $E = \|y - AX\|^2$.

Of course, it would be unrealistic to assume that such a system has a solution since in practice the number of equations greatly exceeds the number of unknowns. We shall now develop a general method using the theory of orthogonal projections to find an explicit vector $x_0 \in \mathsf{F}^n$ that minimizes E; that is, given an $m \times n$ matrix A, we shall find $x_0 \in \mathsf{F}^n$ such that $\|y - Ax_0\| \leq \|y - Ax\|$ for all vectors $x \in \mathsf{F}^n$. This method will not only allow us to find the linear function which best fits the data but also the polynomial of any fixed degree which best fits the data.

We shall first need some notation and two simple lemmas.

For $x, y \in \mathsf{F}^n$, we let $(x, y)_n$ denote the standard inner product of x and y in F^n.

Lemma 1. *Let* A *be an* $m \times n$ *matrix over* F, $x \in \mathsf{F}^n$, *and* $y \in \mathsf{F}^m$. *Then*

$$(Ax, y)_m = (x, A^*y)_n.$$

PROOF. We shall prove the result for x and y contained in the standard ordered bases for F^n and F^m, respectively. We leave the general case for the reader. Let $x = e_i$ and $y = e'_j$ be such elements. Using Theorem 2.15, we have

$$(Ae_i, e'_j)_m = (A^i, e'_j)_m = A_{ji} \quad \text{and} \quad (e_i, A^*e'_j)_n = (e_i, (A^*)^j)_n = \overline{(A^*)_{ij}} = A_{ji},$$

where A^i and $(A^*)^j$ denote the ith column of A and the jth column of A^*, respectively. ∎

Lemma 2. *Let* A *be an* m $\times$ n *matrix over* F. *Then rank*(A*A) = *rank*(A).

PROOF. We need only show that for $x \in \mathsf{F}^n$ we have that $A^*Ax = 0$ if and only if $Ax = 0$. Clearly $Ax = 0$ implies that $A^*Ax = 0$. So assume that $A^*Ax = 0$. Then $0 = (A^*Ax, x)_n = (Ax, A^{**}x)_m = (Ax, Ax)_m$, so that $Ax = 0$. ∎

Corollary. *If* A *is an* m $\times$ n *matrix such that rank*(A) = n (*i.e.*, A *has "full rank"*), *then* A*A *is invertible.*

Now consider the system $AX = y$, where A is an $m \times n$ matrix and $y \in \mathsf{F}^m$. Define $\mathsf{W} = \{Ax: x \in \mathsf{F}^n\}$, i.e., $\mathsf{W} = \mathsf{R}(\mathsf{L}_A)$. Letting T be the orthogonal projection on W, choose $x_0 \in \mathsf{F}^n$ such that $\mathsf{T}(y) = Ax_0$. Then, by Theorem 7.25, $\|\mathsf{T}(y) - y\| \leq \|u - y\|$ for all $u \in \mathsf{W}$; that is, $\|Ax_0 - y\| \leq \|Ax - y\|$ for all $x \in \mathsf{F}^n$.

To develop a practical method of finding such an x_0, we observe that since T is an orthogonal projection, $Ax_0 - y = \mathsf{T}(y) - y \in \mathsf{W}^\perp$, and so $(Ax, Ax_0 - y)_m = 0$ for all $x \in \mathsf{F}^n$. Thus by Lemma 1 we have that $(x, A^*(Ax_0 - y))_n = 0$ for all $x \in \mathsf{F}^n$; that is, $A^*(Ax_0 - y) = 0$. So we need only find a solution to $A^*AX = A^*y$. If, in addition, we assume that rank$(A) = n$, then by Lemma 2 we have $x_0 = (A^*A)^{-1}A^*y$. We may summarize this discussion in the following theorem.

Theorem 7.28. *Let* $\mathrm{A} \in \mathsf{M}_{m\times n}(\mathrm{F})$ *and* $\mathrm{y} \in \mathsf{F}^m$. *Then there exists* $\mathrm{x}_0 \in \mathsf{F}^n$ *such that* $(\mathrm{A}^*\mathrm{A})\mathrm{x}_0 = \mathrm{A}^*\mathrm{y}$ *and* $\|\mathrm{Ax}_0 - \mathrm{y}\| \leq \|\mathrm{Ax} - \mathrm{y}\|$ *for all* $\mathrm{x} \in \mathsf{F}^n$. *Furthermore, if rank*(A) = n, *then* $\mathrm{x}_0 = (\mathrm{A}^*\mathrm{A})^{-1}\mathrm{A}^*\mathrm{y}$.

To return to our experimenter, let us suppose that he collects the data (1, 2), (2, 3), (3, 5), and (4, 7). Then

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix};$$

hence

$$A^*A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix},$$

and so

$$(A^*A)^{-1} = \frac{1}{20}\begin{pmatrix} 4 & -10 \\ -10 & 30 \end{pmatrix}.$$

Therefore

$$x_0 = \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 4 & -10 \\ -10 & 30 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1.7 \\ 0 \end{pmatrix}.$$

Thus the line $y = 1.7t$ is the least squares line. The error E may be computed directly as $||Ax_0 - y||^2 = 0.3$.

The method above may also be applied if the experimenter wants to fit a parabola $y = ct^2 + dt + e$ to the data. In this case, he would use

$$\begin{pmatrix} t_1^2 & t_1 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ t_m^2 & t_m & 1 \end{pmatrix}$$

as the matrix A.

Finally, suppose in the linear case that the experimenter chose his times, t_i, to satisfy

$$\sum_{i=1}^{m} t_i = 0.$$

Then the two columns of A would be orthogonal, and so A^*A would be a diagonal matrix. (See Exercise 1.) This, of course, would greatly simplify the computations.

Minimal Solutions

In the preceding discussion we showed that if $\operatorname{rank}(A) = n$, then there exists a unique $x_0 \in \mathsf{F}^n$ such that Ax_0 is the point in W that is closest to y. Of course, if $\operatorname{rank}(A) < n$, there will be infinitely many such vectors. It is often desirable to find such a vector of minimal norm. For what follows, we shall let $b = Ax_0$; that is, $b = \mathsf{T}(y)$, where T is the orthogonal projection on W. Then the system $AX = b$ has at least one solution. A solution s is called a *minimal solution* if $||s|| \leq ||u||$ for all other solutions u of $AX = b$.

Theorem 7.29. *Let $A \in \mathsf{M}_{m\times n}(F)$ and $b \in \mathsf{F}^m$. Suppose that $AX = b$ has at least one solution. Then*

(a) *There exists exactly one minimal solution s of $AX = b$, and $s \in \mathsf{R}(\mathsf{L}_{A^*})$.*

(b) *s is the only solution of $AX = b$ that lies in $\mathsf{R}(\mathsf{L}_{A^*})$; i.e., if u is a solution of $(AA^*)X = b$, then $s = A^*u$.*

PROOF. For simplicity of notation, we shall let $\mathsf{N}(A) = \mathsf{N}(\mathsf{L}_A)$ and $\mathsf{R}(A^*) = \mathsf{R}(\mathsf{L}_{A^*})$. By Theorem 7.6 and Exercise 12 of Section 7.3 we have that $\mathsf{F}^n = \mathsf{N}(A)^\perp \oplus \mathsf{N}(A) = \mathsf{R}(A^*) \oplus \mathsf{N}(A)$. Let x be any solution of $AX = b$. Then, by the above, $x = s + y$, where $s \in \mathsf{R}(A^*)$ and $y \in \mathsf{N}(A)$. Note that $b = Ax = As + Ay = As$, so that s is a solution of $AX = b$ that lies in $\mathsf{R}(A^*)$. To prove (a), we need only show that s is the unique minimal solution. Let v be any solution of $AX = b$. By Theorem 3.8 we have that $v = s + u$, where $u \in \mathsf{N}(A)$. Since $s \in \mathsf{R}(A^*) = \mathsf{N}(A)^\perp$, we have by Exercise 10 of Section 7.1 that $||v||^2 = ||s + u||^2 = ||s||^2 + ||u||^2 \geq ||s||^2$. Thus s is a minimal solution. We also can see from the calculation above that if $||v|| = ||s||$, then $u = 0$ and $v = s$. Hence s is the unique minimal solution of $AX = b$.

In order to prove (b) we shall assume that v is also a solution of $AX = b$ that lies in $\mathsf{R}(A^*)$. Then $v - s \in \mathsf{R}(A^*) \cap \mathsf{N}(A) = \{0\}$, and so $v = s$. ∎

Example 32. Consider the system

$$\begin{cases} x + 2y + z = 4 \\ x - y + 2z = -11 \\ x + 5y = 19. \end{cases}$$

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 4 \\ -11 \\ 19 \end{pmatrix}.$$

To find the minimal solution of this system, we must find a solution of $AA^*X = b$. Now

$$AA^* = \begin{pmatrix} 6 & 1 & 11 \\ 1 & 6 & -4 \\ 11 & -4 & 26 \end{pmatrix};$$

so we consider the system

$$\begin{cases} 6x + y + 11z = 4 \\ x + 6y - 4z = -11 \\ 11x - 4y + 26z = 19, \end{cases}$$

for which a solution is

$$u = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

(Any solution will suffice.) Hence

$$s = A^*u = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$$

is the minimal solution of the given system.

EXERCISES

1. Prove that if A is an $m \times n$ matrix whose columns are orthogonal, then A^*A is a diagonal matrix.
2. Given the data $(-3, 9)$, $(-2, 6)$, $(0, 2)$, and $(1, 1)$, find the parabola that provides the least squares fit. Compute E.
3. Find the minimal solution of

$$\begin{cases} x + 2y - z = 1 \\ 2x + 3y + z = 2 \\ 4x + 7y - z = 4. \end{cases}$$

4. Let A be an $m \times n$ matrix. Prove $(Ax, y)_m = (x, A^*y)_n$ for $x \in \mathsf{F}^n$ and $y \in \mathsf{F}^m$, thus completing the proof of Lemma 1.
5. For the least squares line $y = ct + d$ corresponding to the m observations $(t_1, y_1), \ldots, (t_m, y_m)$, use Theorem 7.28 to derive the *normal equations*:

$$\left(\sum_{i=1}^{m} t_i^2\right)c + \left(\sum_{i=1}^{m} t_i\right)d = \sum_{i=1}^{m} t_i y_i$$

and

$$\left(\sum_{i=1}^{m} t_i\right)c + md = \sum_{i=1}^{m} y_i.$$

These equations may also be obtained by setting each of the partial derivatives of the error E to zero.

7.11* BILINEAR AND QUADRATIC FORMS

There is a certain class of scalar-valued functions of two variables defined on a vector space that is often considered in the study of such diverse subjects as geometry and multivariable calculus. This is the class of "bilinear forms." We shall study the basic properties of this class with a special emphasis on symmetric bilinear forms and shall consider some of its applications to quadratic surfaces and multivariable calculus.

Throughout this section all bases should be regarded as ordered bases.

Definition. *Let* V *be a vector space over a field* F. *A function* H *from the set* $V \times V$ *of ordered pairs of vectors in* V *to* F *is called a* bilinear form *on* V *if* H *is linear in each variable when the other variable is held fixed, that is, if*

(a) $H(ax_1 + x_2, y) = aH(x_1, y) + H(x_2, y)$ *for all* $x_1, x_2, y \in V$ *and* $a \in F$.

(b) $H(x, ay_1 + y_2) = aH(x, y_1) + H(x, y_2)$ *for all* $x, y_1, y_2 \in V$ *and* $a \in F$.

We shall denote the set of all bilinear forms on V by $\mathcal{B}(V)$. Observe that an inner product on a real space V is a bilinear form.

Example 33. Define a function $H\colon\ R^2 \times R^2 \longrightarrow R$ by

$$H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = 2a_1b_1 + 3a_1b_2 + 4a_2b_1 - a_2b_2 \quad \text{for } \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in R^2.$$

We may verify directly that H is a bilinear form on R^2. It will prove more enlightening and less tedious, however, to observe that if

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \text{and} \quad y = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

then

$$H(x, y) = x^t A y.$$

The bilinearity of H now follows directly from the distributive property of matrix multiplication over matrix addition.

The bilinear form above is a special case of the following more general situation.

Example 34. Let $V = F^n$, the vector space of all column vectors of length n over a field F. For any $n \times n$ matrix A with entries from F define $H\colon\ V \times V \longrightarrow F$ by

$$H(x, y) = x^t A y \qquad \text{for } x, y \in V.$$

Notice that since x and y are $n \times 1$ matrices and A is an $n \times n$ matrix, $H(x, y)$ is a 1×1 matrix for any $x, y \in V$. We identify this matrix with its single entry. As in Example 33 the bilinearity of H follows from the distributive property of matrix multiplication over matrix addition. For example, if $a \in F$, and $x_1, x_2, y \in V$, then

$$\begin{aligned} H(ax_1 + x_2, y) &= (ax_1 + x_2)^t A y = (ax_1^t + x_2^t) A y \\ &= ax_1^t A y + x_2^t A y \\ &= aH(x_1, y) + H(x_2, y). \end{aligned}$$

We shall now list several properties possessed by all bilinear forms. Their proofs are left to the reader. (See Exercise 2.)

For any bilinear form H on a vector space V over a field F:

1. If, for any $x \in \mathsf{V}$, functions $\mathsf{L}_x, \mathsf{R}_x\colon \ \mathsf{V} \longrightarrow F$ are defined by $\mathsf{L}_x(y) = H(x, y)$ and $\mathsf{R}_x(y) = H(y, x)$ for all $y \in \mathsf{V}$, then L_x and R_x are linear.
2. $H(0, x) = H(x, 0) = 0$ for all $x \in \mathsf{V}$.
3. If $x, y, z, w \in \mathsf{V}$, then
$$H(x + y, z + w) = H(x, z) + H(x, w) + H(y, z) + H(y, w).$$
4. If $J\colon \ \mathsf{V} \times \mathsf{V} \longrightarrow F$ is defined by $J(x, y) = H(y, x)$, then J is a bilinear form.

For a vector space V, $H_1, H_2 \in \mathcal{B}(\mathsf{V})$, and any scalar a, we define the *sum* $H_1 + H_2$ and the *product* aH_1 by the equations

$$(H_1 + H_2)(x, y) = H_1(x, y) + H_2(x, y)$$

and

$$(aH_1)(x, y) = a(H_1(x, y)) \qquad \text{for all } x, y \in \mathsf{V}.$$

It is a simple exercise to verify that $H_1 + H_2$ and aH_1 are again bilinear forms. It is not surprising that with respect to these operations $\mathcal{B}(\mathsf{V})$ is a vector space.

Theorem 7.30. For any vector space V, $\mathcal{B}(\mathsf{V})$ *is a vector space with respect to the definitions of sum and product above.*

PROOF. Exercise.

Let V be an n-dimensional vector space with basis $\beta = \{x_1, x_2, \ldots, x_n\}$. For any bilinear form $H \in \mathcal{B}(\mathsf{V})$ we can associate with H an $n \times n$ matrix A whose entry in row i column j is defined by

$$A_{ij} = H(x_i, x_j) \qquad \text{for all } i, j = 1, 2, \ldots, n.$$

Definition. The matrix A *above will be called the* matrix representation of H with respect to the basis β.

We can therefore define a mapping ψ_β from $\mathcal{B}(\mathsf{V})$ to $\mathsf{M}_{n\times n}(F)$, where F is the field of scalars for V, such that for any $H \in \mathcal{B}(\mathsf{V})$, $\psi_\beta(H) = A$, where A is the matrix representation of H with respect to β.

Example 35. Consider the bilinear form H of Example 33. Let

$$\beta = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \quad \text{and} \quad B = \psi_\beta(H).$$

Then

$$B_{11} = H\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 2 + 3 + 4 - 1 = 8,$$

$$B_{12} = H\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = 2 - 3 + 4 + 1 = 4,$$

$$B_{21} = H\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 2 + 3 - 4 + 1 = 2,$$

$$B_{22} = H\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = 2 - 3 - 4 - 1 = -6.$$

So

$$\psi_\beta(H) = \begin{pmatrix}8 & 4\\2 & -6\end{pmatrix}.$$

If γ is the standard basis for R^2, the reader can verify that

$$\psi_\gamma(H) = \begin{pmatrix}2 & 3\\4 & -1\end{pmatrix}.$$

Theorem 7.31. *For any* n*-dimensional vector space* V *over a field* F *and any basis* β *for* V, ψ_β *is a vector space isomorphism from* $\mathcal{B}(\mathsf{V})$ *onto* $\mathsf{M}_{n\times n}(\mathrm{F})$.

PROOF. We shall leave it to the reader to verify that ψ_β is a linear transformation.

To show that ψ_β is one-to-one, suppose that $H \in \mathcal{B}(\mathsf{V})$ and $\psi_\beta(H) = O$, the zero matrix. We wish to show that H is trivial, i.e., $H(x, y) = 0$ for all $x, y \in \mathsf{V}$. Fix an $x_i \in \beta$, and recall the function $\mathsf{L}_{x_i}\colon\ \mathsf{V} \longrightarrow F$ defined by $\mathsf{L}_{x_i}(x) = H(x_i, x)$ for all $x \in \mathsf{V}$. By property 1 on p. 443, L_{x_i} is linear; by hypothesis, $\mathsf{L}_{x_i}(x_j) = H(x_i, x_j) = 0$ for all $x_j \in \beta$. Hence L_{x_i} is the zero transformation from V to F. So

$$H(x_i, x) = \mathsf{L}_{x_i}(x) = 0 \qquad \text{for all } x \in \mathsf{V} \text{ and } x_i \in \beta. \tag{18}$$

Next fix an arbitrary $y \in \mathsf{V}$, and recall the mapping $\mathsf{R}_y\colon\ \mathsf{V} \longrightarrow F$ defined by $\mathsf{R}_y(x) = H(x, y)$ for all $x \in \mathsf{V}$. Again R_y is linear. But by Eq. (18) $\mathsf{R}_y(x_i) = H(x_i, y) = 0$ for any $x_i \in \beta$. Thus R_y is trivial, and we conclude that $H(x, y) = \mathsf{R}_y(x) = 0$ for all $x, y \in \mathsf{V}$. So H is trivial, and ψ_β is therefore one-to-one.

To show that ψ_β is onto, let $A \in \mathsf{M}_{n\times n}(F)$. Recall the isomorphism $\phi_\beta\colon\ \mathsf{V} \longrightarrow \mathsf{F}^n$ as defined in Section 2.4. For $x \in \mathsf{V}$ we shall view $\phi_\beta(x) \in \mathsf{F}^n$ as a column vector. Define a mapping $H\colon\ \mathsf{V} \times \mathsf{V}\colon\ \longrightarrow F$ by

$$H(x, y) = [\phi_\beta(x)]^t A[\phi_\beta(y)] \qquad \text{for all } x, y \in \mathsf{V}.$$

By Example 34, $H \in \mathcal{B}(\mathsf{V})$. We shall show that $\psi_\beta(H) = A$. If $x_i, x_j \in \beta$, then $\phi_\beta(x_i) = e_i$ and $\phi_\beta(x_j) = e_j$. Consequently for any i and j

$$H(x_i, x_j) = [\phi_\beta(x_i)]^t A[\phi_\beta(x_j)] = e_i^t A e_j = A_{ij}.$$

We conclude that $\psi_\beta(H) = A$, and thus ψ_β is onto. ▮

Corollary 1. *For any* n*-dimensional vector space* V, $\mathcal{B}(V)$ *is of dimension* n^2.

PROOF. Exercise.

The following corollary is easily established by reviewing the proof of Theorem 7.31.

Corollary 2. *Let* V *be an* n*-dimensional vector space over the field* F *with basis* β. *If* $H \in \mathcal{B}(V)$ *and* $A \in M_{n \times n}(F)$, *then* $\psi_\beta(H) = A$ *if and only if* $H(x, y) = [\phi_\beta(x)]^t A[\phi_\beta(y)]$ *for all* $x, y \in V$.

The following is now an immediate consequence of Corollary 2.

Corollary 3. *For any field* F, *positive integer* n, *and* $H \in \mathcal{B}(F^n)$, *there exists a unique matrix* $A \in M_{n \times n}(F)$, *namely* $A = \psi_\beta(H)$, *such that*

$$H(x, y) = x^t A y \qquad \textit{for all } x, y \in F^n,$$

where β *is the standard basis for* F^n.

There appears to be an analogy between bilinear forms and linear operators in that each is associated with a unique square matrix and in that this correspondence depends on the choice of a basis for the vector space. As in the case of operators one can pose the question: How is the matrix corresponding to a fixed bilinear form modified when the basis is changed? As we have seen, when this question was posed for linear operators, it led to the study of the similarity relation on square matrices. In the case of bilinear forms we are led to the study of another relation on square matrices, the "congruence" relation.

Definition. *Two matrices* $A, B \in M_{n \times n}(F)$ *are said to be* congruent *if there exists an invertible matrix* $Q \in M_{n \times n}(F)$ *such that*

$$Q^t A Q = B.$$

It is easily seen that congruence is an equivalence relation. (See Exercise 11.)

The following theorem relates congruence to the matrix representation of a bilinear form.

Theorem 7.32. *Let* V *be a finite-dimensional vector space with bases* $\beta = \{x_1, x_2, \ldots, x_n\}$ *and* $\gamma = \{y_1, y_2, \ldots, y_n\}$, *and let* Q *be the change of coordi-*

nate matrix changing γ-coordinates to β-coordinates. Then, for any $\mathrm{H} \in \mathcal{B}(\mathrm{V})$, $\psi_\gamma(\mathrm{H}) = \mathrm{Q}^t\psi_\beta(\mathrm{H})\mathrm{Q}$. *In particular,* $\psi_\gamma(\mathrm{H})$ *and* $\psi_\beta(\mathrm{H})$ *are congruent.*

PROOF. There are essentially two proofs of this theorem. One involves a direct computation, while the other follows immediately from a certain clever observation. We shall present the former proof here and leave the latter proof as an exercise. (See Exercise 12.)

Suppose $A = \psi_\beta(H)$ and $B = \psi_\gamma(H)$. Then for any i and j such that $1 \leq i, j \leq n$

$$y_i = \sum_{k=1}^{n} Q_{ki}x_k \quad \text{and} \quad y_j = \sum_{r=1}^{n} Q_{rj}x_r.$$

Thus

$$\begin{aligned} B_{ij} = H(y_i, y_j) &= H\left(\sum_{k=1}^{n} Q_{ki}x_k, y_j\right) \\ &= \sum_{k=1}^{n} Q_{ki}H(x_k, y_j) \\ &= \sum_{k=1}^{n} Q_{ki}H\left(x_k, \sum_{r=1}^{n} Q_{rj}x_r\right) \\ &= \sum_{k=1}^{n} Q_{ki}\sum_{r=1}^{n} Q_{rj}H(x_k, x_r) \\ &= \sum_{k=1}^{n} Q_{ki}\sum_{r=1}^{n} Q_{rj}A_{kr} \\ &= \sum_{k=1}^{n} Q_{ki}\sum_{r=1}^{n} A_{kr}Q_{rj} \\ &= \sum_{k=1}^{n} Q_{ki}(AQ)_{kj} \\ &= \sum_{k=1}^{n} Q^t_{ik}(AQ)_{kj} = (Q^tAQ)_{ij}. \end{aligned}$$

Thus $B = Q^tAQ$. ∎

The following is a converse to Theorem 7.32.

Corollary. *Let* $\mathrm{A}, \mathrm{B} \in \mathsf{M}_{n\times n}(\mathrm{F})$. *If* A *and* B *are congruent, then there exists an* n*-dimensional vector space* V *over* F, *bases* β *and* γ *for* V, *and a bilinear form* H *on* V *such that*

$$\psi_\beta(\mathrm{H}) = \mathrm{A} \quad \textit{and} \quad \psi_\gamma(\mathrm{H}) = \mathrm{B}.$$

PROOF. Suppose Q is an invertible matrix for which $B = Q^tAQ$. Let $\mathsf{V} = \mathsf{F}^n$, $\beta = \{e_1, e_2, \ldots, e_n\}$ be the standard basis for F^n, and H be the pre-image of A under ψ_β. Let $\gamma = \{Q^1, Q^2, \ldots, Q^n\}$, the set of columns of Q. Then γ is a basis for F^n, and Q is the change of coordinate matrix changing γ-coordinates to β-coordinates. Thus, by Theorem 7.32, $B = Q^tAQ = Q^t\psi_\beta(H)Q = \psi_\gamma(H)$. ∎

Like the diagonalization problem for linear operators, there is an analogous "diagonalization" problem for bilinear forms, namely the problem of determining those bilinear forms for which there are diagonal matrix representations. As we shall see, the "diagonalizable" bilinear forms are those that are "symmetric."

Definition. *A bilinear form* H *over a vector space* V *is called* symmetric *if* $\mathrm{H}(\mathrm{x}, \mathrm{y}) = \mathrm{H}(\mathrm{y}, \mathrm{x})$ *for all* $\mathrm{x}, \mathrm{y} \in \mathrm{V}$.

As the name suggests, symmetric bilinear forms correspond to symmetric matrices.

Theorem 7.33. *Let* V *be a finite-dimensional vector space. For* $\mathrm{H} \in \mathcal{B}(\mathrm{V})$ *the following are equivalent:*

(a) H *is symmetric.*
(b) *For any basis* γ *for* V, $\psi_\gamma(\mathrm{H})$ *is a symmetric matrix.*
(c) *There exists a basis* β *for* V *such that* $\psi_\beta(\mathrm{H})$ *is a symmetric matrix.*

PROOF. First we shall prove that (a) implies (b). Suppose H is symmetric. Let $\gamma = \{y_1, y_2, \ldots, y_n\}$ be a basis for V, and let $B = \psi_\gamma(H)$. Then, for any i and j, $B_{ij} = H(y_i, y_j) = H(y_j, y_i) = B_{ji}$. Thus B is a symmetric matrix, proving (b).

Clearly (b) implies (c).

Finally, we shall prove that (c) implies (a). Suppose that, for some basis $\beta = \{x_1, x_2, \ldots, x_n\}$, $\psi_\beta(H) = A$ is a symmetric matrix. Define $J\colon\ \mathrm{V} \times \mathrm{V} \longrightarrow F$, where F is the field of scalars for V, by $J(x, y) = H(y, x)$ for all $x, y \in \mathrm{V}$. By property 4 on p. 443, $J \in \mathcal{B}(\mathrm{V})$. Let $C = \psi_\beta(J)$. Then for any i and j

$$C_{ij} = J(x_i, x_j) = H(x_j, x_i) = A_{ji} = A_{ij}.$$

Thus $C = A$. Since ψ_β is one-to-one, we conclude that $J = H$. Hence $H(y, x) = J(x, y) = H(x, y)$ for all $x, y \in \mathrm{V}$, and so H is symmetric, proving (a). ∎

Definition. *A bilinear form* H *on a finite-dimensional vector space* V *is called* diagonalizable *if there exists a basis* β *for* V *such that* $\psi_\beta(\mathrm{H})$ *is a diagonal matrix.*

Corollary. *Let* V *be a finite-dimensional vector space. For any* $\mathrm{H} \in \mathcal{B}(\mathrm{V})$, *if* H *is diagonalizable, then* H *is symmetric.*

PROOF. Suppose H is diagonalizable. Then there exists a basis β for V such that $\psi_\beta(H) = D$, a diagonal matrix. Trivially D is a symmetric matrix. So, by Theorem 7.33, H is symmetric. ∎

Unfortunately the converse is not true, as illustrated by the following example.

Example 36. Let $F = Z_2$ (see Appendix C), and let $\mathsf{V} = \mathsf{F}^2$. Define $H\colon \mathsf{V} \times \mathsf{V} \to F$ by

$$H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = a_1 b_2 + a_2 b_1.$$

Clearly H is symmetric. In fact, if β is the standard basis for F^2, then

$$A = \psi_\beta(H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

a symmetric matrix. We shall show that assuming H is diagonalizable leads to a contradiction.

Suppose H is diagonalizable. Then there exists a basis γ for F^2 such that $B = \psi_\gamma(H)$ is a diagonal matrix. Thus by Theorem 7.32 there exists an invertible matrix Q such that $B = Q^t A Q$. Since Q is invertible, $\operatorname{rank}(B) = \operatorname{rank}(A) = 2$. So B is a diagonal matrix whose diagonal entries are non-zero. Since the only non-zero element of F is 1,

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B = Q^t A Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac + ac & bc + ad \\ bc + ad & bd + bd \end{pmatrix}.$$

But $p + p = 0$ for all $p \in F$, and so $ac + ac = 0$. Thus, comparing the upper left-hand entries of the matrices in the equation above, we conclude that $1 = 0$, a contradiction. Consequently H is not diagonalizable.

The bilinear form of Example 36 is an anomaly. Its failure to be diagonalizable stems from the fact that the scalar field Z_2 is of characteristic two. If F is not of characteristic two, then $1 + 1$ is invertible. Under these circumstances we shall denote "$1 + 1$" by "2" and its multiplicative inverse by $\frac{1}{2}$.

Prior to proving the converse of the corollary to Theorem 7.33 for scalar fields other than those of characteristic two, we must establish the following lemma.

Lemma. *Let* H *be a non-trivial symmetric bilinear form on a vector space* V *over a field* F *not of characteristic two. Then there exists an element* $x \in V$ *such that* $H(x, x) \neq 0$.

PROOF. Suppose that for some $v, w \in V$, $H(v, w) \neq 0$. If $H(v, v) \neq 0$ or $H(w, w) \neq 0$, there is nothing to prove. Otherwise, suppose that $H(v, v) = H(w, w) = 0$. Setting $x = v + w$, we have

$$\begin{aligned} H(x, x) &= H(v, v) + H(v, w) + H(w, v) + H(w, w) \\ &= 2H(v, w) \neq 0 \end{aligned}$$

since $2 \neq 0$ and $H(v, w) \neq 0$. ∎

Theorem 7.34. *Let* V *be a finite-dimensional vector space over a field* F *not of characteristic two. Then every symmetric bilinear form on* V *is diagonalizable.*

PROOF. We shall use mathematical induction on $n = \dim(V)$. If $n = 1$, then every member of $\mathcal{B}(V)$ is diagonalizable. Suppose that the theorem is valid for vector spaces of dimension less than n for some fixed integer $n > 1$. If H is the trivial bilinear form, then certainly H is diagonalizable. Suppose then that H is non-trivial and symmetric. Then by the lemma there exists an element $x \in V$ (necessarily non-zero) such that $H(x, x) \neq 0$. Define $L\colon V \longrightarrow F$ by $L(z) = H(x, z)$ for all $z \in V$. Then L is linear, and since $L(x) = H(x, x) \neq 0$, L is non-trivial. Consequently $\operatorname{rank}(L) = 1$, and hence $\dim(N(L)) = n - 1$. The restriction of H to N(L) is obviously a symmetric bilinear form on a vector space of dimension $n - 1$. Thus by the induction hypothesis there exists a basis $\{x_1, x_2, \ldots, x_{n-1}\}$ for N(L) such that $H(x_i, x_j) = 0$ for $i \neq j$ $(1 \leq i, j \leq n - 1)$. Set $x_n = x$. Then $x_n \notin N(L)$, and hence $\beta = \{x_1, x_2, \ldots, x_n\}$ is a basis for V. Also $H(x_i, x_n) = H(x_n, x_i) = 0$ for $i = 1, 2, \ldots, n - 1$. We conclude that $\psi_\beta(H)$ is a diagonal matrix, and so H is diagonalizable. ∎

Corollary. *Let* F *be a field that is not of characteristic two. If* $A \in M_{n\times n}(F)$ *is a symmetric matrix, then* A *is congruent to a diagonal matrix.*

PROOF. Exercise.

Let A be a symmetric $n \times n$ matrix with entries from a field not of characteristic two. By the corollary to Theorem 7.34, A is congruent to a diagonal matrix. We shall show how to find a diagonal matrix D and an invertible matrix Q such that $Q^tAQ = D$. The reader may wish to review Section 3.1 to recall the relationship between elementary matrices and the elementary matrix operations.

If E is an elementary $n \times n$ matrix, then AE is obtained from A by means of a certain elementary column operation on A. By Exercise 20, E^tA is obtained from A by means of the same operation performed on the rows rather than on the columns of A. Thus E^tAE is obtained from A by performing an elementary operation on the columns of A and then performing the same operation on the rows of the matrix AE. (Note that the order of the operations can be reversed.) Now suppose that Q is an invertible matrix and D is a diagonal matrix such that $Q^tAQ = D$. By Corollary 3 to Theorem 3.5, Q is a product of elementary matrices, $Q = E_1E_2 \cdots E_k$. Thus $D = Q^tAQ = E_k^tE_{k-1}^t \cdots E_1^tAE_1E_2 \cdots E_k$.

On the basis of the equation above, we conclude that by means of several elementary column operations and the corresponding row operations A can be transformed into a diagonal matrix D. Furthermore, if $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding to the elementary column operations (indexed in the order performed) and if $Q = E_1E_2 \cdots E_k$, then $Q^tAQ = D$.

The statement above provides the key for finding D and Q for a given A.

Example 37. Suppose that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

We begin by using elementary column operations to insert a zero in the first row, second column; in this case we must subtract twice the first column of A from the second column of A. The corresponding row operation would involve subtracting twice the first row from the second row. Let E_1 be the elementary matrix corresponding to the elementary column operation above. Then

$$E_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix};$$

consequently

$$AE_1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad E_1^tAE = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that since E_1 produced a zero in row 1, column 2, E_1^t produced a zero in row 2, column 1. Thus for

$$Q = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q^tAQ = D.$$

Next consider

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 4 \end{pmatrix}.$$

It is desirable to have ± 1 in the row 1, column 1 position and use it to eliminate all other entries in the first row and first column of A. Thus we begin by interchanging the first and second columns of A. The elementary matrix corresponding to this column operation is

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly

$$E_1^t A E_1 = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{pmatrix}.$$

This matrix is obtained by interchanging the first two columns of A to obtain AE and then interchanging the first two rows of AE. Next we produce a zero in the first row, second column and in the second row, first column of $E_1^t A E_1$ by adding the first column of $E_1^t A E_1$ to the second column of $E_1^t A E_1$ and following this operation with the corresponding row operation. Finally we add three times the first column to the third column and follow with the corresponding row operation. Note that the column operations can be performed in succession prior to performing the row operations. Thus if

$$E_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_3^t E_2^t (E_1^t A E_1) E_2 E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 5 & 13 \end{pmatrix}.$$

The reader can now easily see that by setting

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$E_4^t (E_3^t E_2^t E_1^t A E_1 E_2 E_3) E_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{pmatrix}.$$

Thus with $Q = E_1 E_2 E_3 E_4$ and

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{pmatrix},$$

$Q^t A Q = D$.

The reader should justify the following method (similar to that introduced in Section 3.2 to compute the inverse of a matrix) for computing Q^t (and hence Q) without recording each elementary matrix separately: Use a sequence of elementary column operations followed by the corresponding elementary row operations to change the augmented matrix $(A \mid I)$ into the form $(D \mid B)$, where D is a diagonal matrix. Then $B = Q^t$.

In the preceding example this method would produce the following sequence of matrices:

$$(A \mid I) = \left(\begin{array}{rrr|rrr} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{rrr|rrr} -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{rrr|rrr} -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 5 & 4 & 0 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{rrr|rrr} -1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 3 & 5 & 4 & 0 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{rrr|rrr} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 3 & 5 & 13 & 0 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{rrr|rrr} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 0 & 5 & 13 & 0 & 3 & 1 \end{array}\right), \quad \left(\begin{array}{rrr|rrr} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 5 & -12 & 0 & 3 & 1 \end{array}\right),$$

and

$$\left(\begin{array}{rrr|rrr} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -12 & -5 & -2 & 1 \end{array}\right) = (D \mid Q^t).$$

Hence

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{pmatrix}, \quad Q^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -5 & -2 & 1 \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} 0 & 1 & -5 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Quadratic Forms

Associated with symmetric bilinear forms are functions called "quadratic forms."

Definition. *Let* V *be a vector space over a field* F. *A function* K: V $\longrightarrow$ F *is called a* quadratic form *if there exists a symmetric bilinear form* H $\in \mathcal{B}$(V) *such that*

$$\mathrm{K}(\mathrm{x}) = \mathrm{H}(\mathrm{x}, \mathrm{x}) \textit{ for all } \mathrm{x} \in \mathrm{V}. \tag{19}$$

If the field F is not of characteristic two, there is a one-to-one correspondence between symmetric bilinear forms and quadratic forms given by Eq. (19). In fact, if K is a quadratic form on a vector space V over a field F not of characteristic two, and if $K(x) = H(x, x)$ for some symmetric bilinear form on V, then

$$H(x, y) = \tfrac{1}{2}[K(x + y) - K(x) - K(y)]. \tag{20}$$

See Exercise 15 for details.

Example 38. Certainly the classical example of a quadratic form is the homogeneous second-degree polynomial of several variables. Given variables $t_1, t_2, \ldots, t_n$ that take values in a field F not of characteristic two and (not necessarily distinct) scalars a_{ij} $(1 \leq i \leq j \leq n)$, define the polynomial

$$f(t_1, t_2, \ldots, t_n) = \sum_{i \leq j} a_{ij} t_i t_j.$$

Let $K\colon\ \mathsf{F}^n \longrightarrow F$ be the quadratic form defined by $K(c_1, c_2, \ldots, c_n) = f(c_1, c_2, \ldots, c_n)$.

Any polynomial of the form above is called a *homogeneous polynomial of the second degree in* n *variables*. In fact, if β is the standard basis for F^n, then the symmetric bilinear form corresponding to the quadratic form above is H, where $\psi_\beta(H) = A$ and

$$A_{ij} = A_{ji} = \begin{cases} a_{ii} & \text{if } i = j \\ \tfrac{1}{2}a_{ij} & \text{if } i \neq j. \end{cases}$$

To see this, simply apply Eq. (20) to obtain $H(e_i, e_j) = A_{ij}$ from the quadratic form K, and verify that $f(t_1, \ldots, t_n)$ is computable from H by means of Eq. (19).

In particular, given the polynomial

$$f(t_1, t_2, t_3) = 2t_1^2 - t_2^2 + 6t_1t_2 - 4t_2t_3$$

with real coefficients, let

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 0 \end{pmatrix}.$$

Setting $H(x, y) = x^t A y$ for all $x, y \in \mathsf{R}^3$, we see that

$$f(t_1, t_2, t_3) = (t_1, t_2, t_3)A\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \quad \text{for } \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in \mathsf{R}^3.$$

Quadratic Forms over the Field R

Since symmetric matrices over R are "orthogonally diagonalizable" (see Theorem 7.21), the theory of symmetric bilinear forms and quadratic forms on finite-dimensional vector spaces over R is especially nice. The following theorem and its corollary are certainly among the most useful results in the theory of bilinear and quadratic forms.

Theorem 7.35. *Let* V *be a finite-dimensional real inner product space, and let* H *be a symmetric bilinear form on* V. *Then there exists an orthonormal basis* β *for* V *such that* $\psi_\beta(\mathrm{H})$ *is a diagonal matrix.*

PROOF. Choose any orthonormal basis $\gamma = \{x_1, x_2, \ldots, x_n\}$ for V. Let $A = \psi_\gamma(H)$. Since A is a symmetric matrix, it is self-adjoint with respect to the inner product on V. By applying Theorem 7.21 we can find an orthogonal matrix Q such that $D = Q^tAQ$ is a diagonal matrix. For $j = 1, 2, \ldots, n$, define

$$y_j = \sum_{i=1}^{n} Q_{ij}x_i.$$

It is a simple matter to verify that $\beta = \{y_1, y_2, \ldots, y_n\}$ is an orthonormal basis for V. Furthermore, due to the manner in which β is defined, Q is the change of coordinate matrix changing β-coordinates to γ-coordinates. Consequently by Theorem 7.32

$$\psi_\beta(H) = Q^t\psi_\gamma(H)Q = Q^tAQ = D,$$

a diagonal matrix. ∎

Corollary. *Let* K *be a quadratic form on a finite-dimensional real inner product space* V. *There exists an orthonormal basis* $\beta = \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n\}$ *for* V *and scalars* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *(not necessarily distinct) such that if* $\mathrm{x} \in \mathrm{V}$ *and*

$$\mathrm{x} = \sum_{i=1}^{n} \mathrm{s}_i\mathrm{x}_i, \qquad \mathrm{s}_i \in \mathrm{R},$$

then

$$\mathrm{K}(\mathrm{x}) = \sum_{i=1}^{n} \lambda_i \mathrm{s}_i^2.$$

In fact, if H *is the symmetric bilinear form determined by* K, *then* β *can be chosen to be any orthonormal basis for* V *for which* $\psi_\beta(\mathrm{H})$ *is a diagonal matrix.*

PROOF. Let H be the symmetric bilinear form for which $K(x) = H(x, x)$ for all $x \in$ V. By Theorem 7.35 there exists an orthonormal basis $\beta = \{x_1, x_2, \ldots, x_n\}$ for V for which

$$\psi_\beta(H) = D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Let $x \in \mathsf{V}$ and suppose that

$$x = \sum_{i=1}^{n} s_i x_i.$$

Then

$$K(x) = H(x, x) = [\phi_\beta(x)]^t D[\phi_\beta(x)] = (s_1, \ldots, s_n)D\begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i s_i^2. \quad \blacksquare$$

Example 39. For the homogeneous real polynomial of degree 2

$$f(t_1, t_2) = 5t_1^2 + 2t_2^2 + 4t_1t_2, \tag{21}$$

we shall find an orthonormal basis $\beta = \{x_1, x_2\}$ for R^2 and scalars λ_1 and λ_2 such that if

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathsf{R}^2 \quad \text{and} \quad \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = s_1x_1 + s_2x_2,$$

then $f(t_1, t_2) = \lambda_1 s_1^2 + \lambda_2 s_2^2$. We may think of s_1 and s_2 as the coordinates of

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

relative to β. Thus the polynomial $f(t_1, t_2)$, as an expression involving the coordinates of a point with respect to the standard basis for R^2, is transformed into a new polynomial $g(s_1, s_2) = \lambda_1 s_1^2 + \lambda_2 s_2^2$ interpreted as an expression involving the coordinates of a point relative to the new basis β.

Let H denote the symmetric bilinear form corresponding to the quadratic form defined by Eq. (21). If γ is the standard basis for R^2, then

$$\psi_\gamma(H) = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

Next we shall find an orthogonal matrix Q for which Q^tAQ is a diagonal matrix. As in Section 7.7 we begin by computing an orthonormal basis of eigenvectors of L_A. The characteristic polynomial $h(t)$ of A is

$$h(t) = \det\begin{pmatrix} 5-t & 2 \\ 2 & 2-t \end{pmatrix} = (t-6)(t-1).$$

Thus $\lambda_1 = 6$ and $\lambda_2 = 1$ are the eigenvalues of A, and each has multiplicity 1. A simple computation yields corresponding eigenvectors of norm one,

$$x_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Since x_1 and x_2 are orthogonal, $\beta = \{x_1, x_2\}$ is an orthonormal basis for R^2. Setting

$$Q = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

we see that Q is an orthogonal matrix and

$$Q^t A Q = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly Q is also a change of coordinate matrix. Consequently

$$\psi_\beta(H) = Q^t \psi_\alpha(H) Q = Q^t A Q = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus by the corollary to Theorem 7.35, for any $x = s_1 x_1 + s_2 x_2 \in \mathsf{R}^2$,

$$K(x) = 6s_1^2 + 1s_2^2.$$

So $g(s_1, s_2) = 6s_1^2 + 1s_2^2$.

The following example illustrates how the theory of quadratic forms can be applied to the problem of describing quadratic surfaces in R^3.

Example 40. Consider the surface $\mathcal{S}$ in R^3 defined by the equation

$$2t_1^2 + 6t_1t_2 + 5t_2^2 - 2t_2t_3 + 2t_3^2 + 3t_1 - 2t_2 - t_3 + 14 = 0; \quad (22)$$

i.e., $\mathcal{S}$ is the set of all elements

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in \mathsf{R}^3$$

that satisfy Eq. (22). If γ is the standard basis for R^3, then Eq. (22) is an equation involving the coordinates of points in $\mathcal{S}$ relative to γ. We would like to select a new orthonormal basis β for R^3 such that the equation describing the coordinates of any point of $\mathcal{S}$ relative to β is considerably simpler than Eq. (22).

We begin with the observation that the terms of second degree on the left-hand side of Eq. (22) add to form a quadratic form on R^3:

$$K\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 2t_1^2 + 6t_1t_2 + 5t_2^2 - 2t_2t_3 + 2t_3^2.$$

Next we diagonalize K. If H is the symmetric bilinear form corresponding to K and $A = \psi_\gamma(H)$, then

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Now the characteristic polynomial of A is

$$h(t) = \det\begin{pmatrix} 2-t & 3 & 0 \\ 3 & 5-t & -1 \\ 0 & -1 & 2-t \end{pmatrix} = -1(t-2)(t-7)t,$$

and consequently A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 7$, and $\lambda_3 = 0$. A simple calculation yields eigenvectors

$$x_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \qquad x_2 = \frac{1}{\sqrt{35}}\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \quad \text{and} \quad x_3 = \frac{1}{\sqrt{14}}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

of norm 1 corresponding to the respective eigenvalues.

Now set $\beta = \{x_1, x_2, x_3\}$ and

$$Q = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & -\frac{3}{\sqrt{14}} \\ 0 & \frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \end{pmatrix}.$$

As in Example 39, Q is the change of coordinate matrix changing β-coordinates to γ-coordinates and

$$\psi_\beta(H) = Q^t\psi_\gamma(H)Q = Q^tAQ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the corollary to Theorem 7.35 if $x = s_1x_1 + s_2x_2 + s_3x_3$, then

$$K(x) = 2s_1^2 + 7s_2^2. \tag{23}$$

We are now ready to transform Eq. (22) into an equation involving coordinates relative to β.

Let

$$x = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in \mathsf{R}^3.$$

If $x = s_1x_1 + s_2x_2 + s_3x_3$, we have

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = s_1\begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix} + s_2\begin{pmatrix} \frac{3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ \frac{-1}{\sqrt{35}} \end{pmatrix} + s_3\begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}.$$

Thus

$$t_1 = \frac{s_1}{\sqrt{10}} + \frac{3s_2}{\sqrt{35}} - \frac{3s_3}{\sqrt{14}},$$
$$t_2 = \frac{5s_2}{\sqrt{35}} + \frac{2s_3}{\sqrt{14}},$$
$$t_3 = \frac{3s_1}{\sqrt{10}} - \frac{s_2}{\sqrt{35}} + \frac{s_3}{\sqrt{14}}.$$

Therefore

$$3t_1 - 2t_2 - t_3 = -\frac{14s_3}{\sqrt{14}} = -\sqrt{14}s_3. \tag{24}$$

Combining Eqs. (22), (23), and (24), we conclude that if $x \in \mathsf{R}^3$ and $x = s_1x_1 + s_2x_2 + s_3x_3$, then $x \in \mathcal{S}$ if and only if

$$2s_1^2 + 7s_2^2 - \sqrt{14}s_3 + 14 = 0 \quad \text{or} \quad s_3 = \frac{\sqrt{14}}{7}s_1^2 + \frac{\sqrt{14}}{2}s_2^2 + \sqrt{14}. \tag{25}$$

Consequently if we draw new axes x', y', and z' in the directions of x_1, x_2, and x_3, respectively, the graph of Eq. (25) rewritten as

$$z' = \frac{\sqrt{14}}{7}(x')^2 + \frac{\sqrt{14}}{2}(y')^2 + \sqrt{14}$$

will coincide with the surface $\mathcal{S}$. Thus we recognize $\mathcal{S}$ to be an elliptic paraboloid. See Fig. 7.9.

We shall conclude this section with an application of the theory of quadratic forms to multivariable calculus—the derivation of the second derivative test for local extrema of a function of several variables. We shall assume an acquaintance with the calculus of functions of several variables to the extent of Taylor's theorem. The reader is undoubtedly familiar with the one-variable version of Taylor's theorem. For a statement and proof of the multivariable version, consult for example, *Advanced Calculus* by Avner Friedman, Holt, Rinehart and Winston, Inc., 1971.

Let $z = f(t_1, t_2, \ldots, t_n)$ be a real-valued function of n real variables for which all third-order partial derivatives exist and are continuous. The function f is said to have a *local maximum* at a point $p \in \mathsf{R}^n$ if there exists a positive number δ for which $f(p) \geq f(x)$ whenever $\|x - p\| < \delta$. Likewise f is said to have a *local minimum* at $p \in \mathsf{R}^n$ if, for some $\delta > 0$, $f(p) \leq f(x)$ whenever $\|x - p\| < \delta$. If f has either a local maximum or a local minimum at p, we say that f has a *local extremum* at p. A point $p \in \mathsf{R}^n$ is called a *critical point* of f if $\partial f(p)/\partial t_i = 0$ for $i = 1, 2, \ldots, n$. It is a well-known fact that if f has a local extremum at a point $p \in \mathsf{R}^n$, then p is a critical point of f. For, if f has a local extremum at p, then for any $i = 1, 2, \ldots, n$ we may define a real-valued function ϕ_i of one variable by

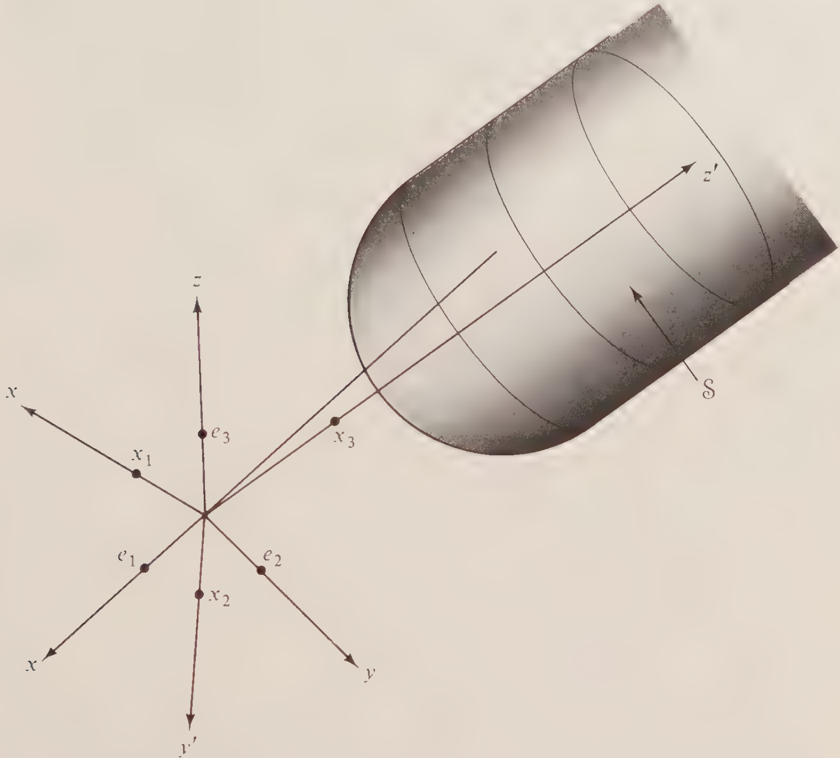

figure 7.9

$\phi_i(t) = f(p_1, p_2, \ldots, p_{i-1}, t, p_{i+1}, \ldots, p_n)$, where p_j is the jth coordinate of p for each j. Obviously ϕ_i has a local extremum at $t = p_i$. So by ordinary one-variable calculus arguments

$$\frac{d\phi_i(p_i)}{dt} = \frac{\partial f(p)}{\partial t_i} = 0.$$

So p is a critical point of f. Unfortunately critical points are not necessarily local extrema. The second derivative test gives us additional conditions under which critical points are local extrema.

Theorem 7.36 (*The Second Derivative Test*). *Let* $f(t_1, t_2, \ldots, t_n)$ *be a real-valued function of* n *real variables for which all the third-order partial derivatives exist and are continuous. Let* p *be a critical point of* f, *and let* A *denote the* $n \times n$ *matrix whose entries are given by*

$$A_{ij} = \frac{\partial^2 f(p)}{(\partial t_i)(\partial t_j)}.$$

(Note that A *is a symmetric matrix and therefore has real eigenvalues.) Then*

(a) *If all of the eigenvalues of* A *are positive, then* f *has a local minimum at* p.

(b) *If all the eigenvalues of* A *are negative, then* f *has a local maximum at* p.

(c) *If* A *has at least one positive and at least one negative eigenvalue, then* f *has no local extremum at* p *(i.e.,* p *is a saddle-point of* f*).*

(d) *If rank*(A) $<$ n *and* A *does not have both positive and negative eigenvalues, then the second derivative test is inconclusive.*

PROOF. If $p \neq 0$, we may define a function $g\colon\ \mathsf{R}^n \longrightarrow R$ by

$$g(t_1, t_2, \ldots, t_n) = f(t_1 + p_1, t_2 + p_2, \ldots, t_n + p_n) - f(p_1, p_2, \ldots, p_n).$$

The following observations are easily verified:

1. The function f has a local maximum [minimum] at p if and only if g has a local maximum [minimum] at $0 = (0, 0, \ldots, 0)$.
2. The partial derivatives of g at 0 coincide with the corresponding partial derivatives of f at p.
3. 0 is a critical point of g.
4. $A_{ij} = \dfrac{\partial^2 g(0)}{(\partial t_i)(\partial t_j)}$ for all i and j.

In view of the above we may suppose without loss of generality that $p = 0$ and $f(p) = 0$.

We next apply Taylor's theorem to f at 0 and conclude that there exists a real-valued function S on R^n such that

$$\lim_{x \to 0} \frac{S(x)}{\|x\|^2} = \lim_{(t_1, \ldots, t_n) \to 0} \frac{S(t_1, \ldots, t_n)}{t_1^2 + \cdots + t_n^2} = 0, \tag{26}$$

and

$$f(t_1, \ldots, t_n) = f(0) + \sum_{i=1}^{n} \frac{\partial f(0)}{\partial t_i} t_i + \frac{1}{2}\left[\sum_{i,j=1}^{n} \frac{\partial^2 f(0)}{(\partial t_i)(\partial t_j)} t_i t_j\right] + S(t_1, \ldots, t_n). \tag{27}$$

Under the hypotheses that 0 is a critical point and $f(0) = 0$, Eq. (27) reduces to

$$f(t_1, \ldots, t_n) = \frac{1}{2}\left[\sum_{i,j=1}^{n} \frac{\partial^2 f(0)}{(\partial t_i)(\partial t_j)} t_i t_j\right] + S(t_1, \ldots, t_n). \tag{28}$$

Let us define a quadratic form $K\colon\ \mathsf{R}^n \longrightarrow R$ by

$$K\begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f(0)}{(\partial t_i)(\partial t_j)} t_i t_j. \tag{29}$$

Let H be the symmetric bilinear form corresponding to K, and let γ be the standard basis for R^n. It is a simple matter to verify that $\psi_\gamma(H) = \frac{1}{2}A$. Since A is self-adjoint, Theorem 7.21 shows that there exists an orthogonal matrix Q such that

$$Q^t AQ = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is a diagonal matrix whose diagonal entries are the eigenvalues of A. Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be the orthonormal basis for R^n whose ith member is Q^i, the ith column of Q. Then Q is the change of coordinate matrix changing β-coordinates to γ-coordinates, and by Theorem 7.32

$$\psi_\beta(H) = Q^t\psi_\gamma(H)Q = \frac{1}{2}Q^tAQ = \begin{pmatrix} \frac{\lambda_1}{2} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_2}{2} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \frac{\lambda_n}{2} \end{pmatrix}.$$

Suppose that A is not the zero matrix. Then A has non-zero eigenvalues. Pick a positive number ϵ such that $\epsilon < |\lambda_i|/2$ for all $\lambda_i \neq 0$. By Eq. (26) there exists a positive number δ such that if $x \in \mathsf{R}^n$ and $\|x\| < \delta$, then $|S(x)| < \epsilon\|x\|^2$. Now pick any $x \in \mathsf{R}^n$ for which $\|x\| < \delta$. Then by Eqs. (28) and (29)

$$|f(x) - K(x)| = |S(x)| < \epsilon\|x\|^2$$

or

$$K(x) - \epsilon\|x\|^2 < f(x) < K(x) + \epsilon\|x\|^2. \tag{30}$$

If

$$x = \sum_{i=1}^{n} s_i x_i,$$

then

$$\|x\|^2 = \sum_{i=1}^{n} s_i^2 \quad \text{and} \quad K(x) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i s_i^2. \tag{31}$$

Thus by Eqs. (30) and (31)

$$\sum_{i=1}^{n}\left(\frac{1}{2}\lambda_i - \epsilon\right)s_i^2 < f(x) < \sum_{i=1}^{n}\left(\frac{1}{2}\lambda_i + \epsilon\right)s_i^2. \tag{32}$$

Now suppose that all the eigenvalues of A are positive. Then $\frac{1}{2}\lambda_i - \epsilon > 0$ for all i, and hence by the left inequality in Eq. (32)

$$f(0) = 0 \leq \sum_{i=1}^{n} \left(\frac{1}{2}\lambda_i - \epsilon\right) s_i^2 < f(x).$$

Thus for $\|x\| < \delta$, $f(0) \leq f(x)$. We conclude that f has a local minimum at 0. Similarly by an argument involving the right inequality in Eq. (32) we conclude that if all the eigenvalues of A are negative, then f has a local maximum at 0. This establishes parts (a) and (b) of the theorem.

Next suppose A has both a positive and a negative eigenvalue, say $\lambda_i > 0$ and $\lambda_j < 0$ for some i and j. Then $\frac{1}{2}\lambda_i - \epsilon > 0$ and $\frac{1}{2}\lambda_j + \epsilon < 0$. Let s be any real number such that $|s| < \delta$. Then by Eq. (32)

$$f(0) = 0 \leq (\tfrac{1}{2}\lambda_i - \epsilon)s^2 < f(sx_i) \quad \text{and} \quad f(sx_j) < (\tfrac{1}{2}\lambda_j + \epsilon)s^2 \leq 0 = f(0).$$

Since $\|sx_i\| = \|sx_j\| = |s|$, we conclude that f attains positive and negative values arbitrarily close to 0. Consequently f has neither a local maximum nor a local minimum at 0. This establishes (c).

To illustrate that the second derivative test is inconclusive under the conditions stated in (d) of the theorem, consider the functions

$$f(t_1, t_2) = t_1^2 - t_2^4 \quad \text{and} \quad f(t_1, t_2) = t_1^2 + t_2^4$$

at $p = 0$. In either case

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

but in the former case f does not have a local extremum at 0, while in the latter case f has a local minimum at 0. ∎

EXERCISES

1. Label the following statements as being true or false.
   (a) Every quadratic form is a bilinear form.
   (b) If two matrices are congruent, they have the same eigenvalues.
   (c) Symmetric bilinear forms have symmetric matrix representations.
   (d) Any symmetric matrix is congruent to a diagonal matrix.
   (e) The sum of two symmetric bilinear forms is a symmetric bilinear form.
   (f) Two symmetric matrices with the same characteristic polynomial are matrix representations of the same bilinear form.
   (g) There exists a bilinear form H such that $H(x, y) \neq 0$ for all x and y.
   (h) If V is a vector space of dimension n, then $\dim(\mathcal{B}(V)) = 2n$.
   (i) Let H be a bilinear form on a finite-dimensional vector space V. For any $x \in V$ there exists a $y \in V$ such that $y \neq 0$ but $H(x, y) = 0$.

(j) If H is any bilinear form on a finite-dimensional real inner product space V, there exists a basis β for V such that $\psi_\beta(H)$ is a diagonal matrix.

2. Prove properties 1, 2, 3, and 4 on p. 443.

3. (a) Verify that the sum of two bilinear forms is a bilinear form.
(b) Verify that the product of a scalar and a bilinear form is a bilinear form.
(c) Prove Theorem 7.30.

4. Determine which of the following are bilinear forms.
(a) Let $\mathsf{V} = \mathsf{C}[0, 1]$ be the space of continuous real-valued functions on the closed interval $[0, 1]$. For $f, g \in \mathsf{V}$, define

$$H(f, g) = \int_0^1 f(t)g(t)\, dt.$$

(b) Let V be a vector space over a field F, and let $J \in \mathcal{B}(\mathsf{V})$ be non-trivial. Define $H\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow F$ by

$$H(x, y) = [J(x, y)]^2 \qquad \text{for all } x, y \in \mathsf{V}.$$

(c) Define $H\colon\ R \times R \longrightarrow R$ by $H(t_1, t_2) = t_1 + 2t_2$.
(d) Consider the members of R^2 as column vectors. Define $H\colon\ \mathsf{R}^2 \longrightarrow R$ for $x, y \in \mathsf{R}^2$ by $H(x, y) = \det(x, y)$, where $\det(x, y)$ denotes the determinant of the 2×2 matrix with x as its first column and y as its second column.
(e) Let V be a real inner product space. Define $H\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow R$ by $H(x, y) = (x, y)$ for $x, y \in \mathsf{V}$.
(f) Let V be a complex inner product space. Define $H\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow C$ by $H(x, y) = (x, y)$ for $x, y \in \mathsf{V}$.

5. Verify that each of the given mappings is a bilinear form. Then compute the matrix representation of H with respect to the given basis.
(a) $H\colon\ \mathsf{R}^3 \times \mathsf{R}^3 \longrightarrow R$, where

$$H\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right) = a_1b_1 - 2a_1b_2 + a_2b_1 - a_3b_3$$

with

$$\beta = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

(b) Let $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ with basis

$$\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

Define $H\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow R$ by $H(A, B) = \operatorname{tr}(A)\cdot\operatorname{tr}(B)$.

(c) Let $\beta = \{\cos t, \sin t, \cos 2t, \sin 2t\}$ and $\mathsf{V} = \text{span}(\beta)$. In the space of all continuous functions on R, V is a four-dimensional subspace with basis β. Define $H\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow R$ by $H(f, g) = f'(0)\cdot g''(0)$.

6. Let V and W be vector spaces over the same field, and let $\mathsf{T}\colon\ \mathsf{V} \longrightarrow \mathsf{W}$ be a linear transformation. For any $H \in \mathcal{B}(\mathsf{W})$, define $\hat{\mathsf{T}}(H)\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow F$ by $\hat{\mathsf{T}}(H)(x, y) = H(\mathsf{T}(x), \mathsf{T}(y))$ for all $x, y \in \mathsf{V}$. Prove that

(a) For $H \in \mathcal{B}(\mathsf{W})$, $\hat{\mathsf{T}}(H) \in \mathcal{B}(\mathsf{V})$.

(b) $\hat{\mathsf{T}}\colon\ \mathcal{B}(\mathsf{W}) \longrightarrow \mathcal{B}(\mathsf{V})$ is a linear transformation.

(c) If T is an isomorphism, then so is $\hat{\mathsf{T}}$.

7. In the proof of Theorem 7.31

(a) Prove that for any basis β, ψ_β is linear.

(b) Let V be an n-dimensional vector space over a field F with basis β, and let $\phi_\beta\colon\ \mathsf{V} \longrightarrow \mathsf{F}^n$ be the standard representation of V with respect to β. Let $A \in \mathsf{M}_{n\times n}(F)$. Define $H\colon\ \mathsf{V} \times \mathsf{V} \longrightarrow F$ by $H(x, y) = [\phi_\beta(x)]^t A[\phi_\beta(y)]$. Prove that $H \in \mathcal{B}(\mathsf{V})$. Can you establish this as a corollary to Exercise 6?

8. (a) Prove Corollary 1 to Theorem 7.31.

(b) For a finite-dimensional vector space V, describe a method for finding a basis for $\mathcal{B}(\mathsf{V})$.

9. Prove Corollary 2 of Theorem 7.31.

10. Prove Corollary 3 of Theorem 7.31.

11. Prove that the relation of congruence is an equivalence relation.

12. The following outline provides an alternate proof to Theorem 7.32.

(a) If β and γ are bases for a finite-dimensional vector space V, and if Q is the change of coordinate matrix changing γ-coordinates into β-coordinates, prove that $\phi_\beta = \mathsf{L}_Q\phi_\gamma$, where ϕ_β and ϕ_γ are the standard representations of V with respect to β and γ, respectively.

(b) Apply Corollary 2 of Theorem 7.31 to (a) to obtain an alternate proof of Theorem 7.32.

13. Let V be a finite-dimensional vector space and $H \in \mathcal{B}(\mathsf{V})$. Prove that, for any bases β and γ of V, $\text{rank}(\psi_\beta(H)) = \text{rank}(\psi_\gamma(H))$.

14. Prove the following.

(a) Any square diagonal matrix is symmetric.

(b) Any matrix congruent to a diagonal matrix is symmetric.

(c) Prove the corollary to Theorem 7.34.

15. Let V be a vector space over a field F not of characteristic two, and let H be a symmetric bilinear form on V. Prove that if $K(x) = H(x, x)$ is the quadratic form associated with H, then

$$H(x, y) = \tfrac{1}{2}[K(x + y) - K(x) - K(y)]$$

for all $x, y \in \mathsf{V}$.

16. For the following quadratic forms K over a real inner product space V, find a symmetric bilinear form H such that $K(x) = H(x, x)$ for all $x \in \mathsf{V}$. Then find an orthonormal basis β for V such that $\psi_\beta(H)$ is a diagonal matrix.

(a) K: $\mathsf{R}^2 \longrightarrow R$ defined by

$$K\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -2t_1^2 + 4t_1t_2 + t_2^2$$

(b) K: $\mathsf{R}^2 \longrightarrow R$ defined by

$$K\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = 7t_1^2 - 8t_1t_2 + t_2^2$$

(c) K: $\mathsf{R}^3 \longrightarrow R$ defined by

$$K\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 3t_1^2 + 3t_2^2 + 3t_3^2 - 2t_1t_3$$

17. Let $\mathcal{S}$ be the set of all

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in \mathsf{R}^3$$

such that

$$3t_1^2 + 3t_2^2 + 3t_3^2 - 2t_1t_3 + 2\sqrt{2}(t_1 + t_3) + 1 = 0.$$

Find an orthonormal basis β for R^3 such that the equation relating the coordinates of points of $\mathcal{S}$ relative to β is simplified. Describe $\mathcal{S}$ geometrically.

18. Prove the following refinement of part (d) of Theorem 7.36.

(a) If $0 < \operatorname{rank}(A) < n$ and A has no negative eigenvalues, then f has no local minimum at p.

(b) If $0 < \operatorname{rank}(A) < n$ and A has no positive eigenvalues, then f has no local maximum at p.

19. Prove the following variation of the second derivative test for the case $n = 2$. Define

$$D = \left[\frac{\partial^2 f(p)}{(\partial t_1)^2}\right]\left[\frac{\partial^2 f(p)}{(\partial t_2)^2}\right] - \left[\frac{\partial^2 f(p)}{(\partial t_1)(\partial t_2)}\right]^2.$$

(a) If $D > 0$ and $\partial^2 f(p)/(\partial t_1)^2 > 0$, then f has a local minimum at p.
(b) If $D > 0$ and $\partial^2 f(p)/(\partial t_2)^2 < 0$, then f has a local maximum at p.
(c) If $D < 0$, then f has no local extremum at p.
(d) If $D = 0$, then the test is inconclusive.
Hint: Observe that $D = \det(A) = \lambda_1\lambda_2$, where λ_1 and λ_2 are the eigenvalues of A and A is as in Theorem 7.36.

20. Let A be an $n \times n$ matrix over a field F, and let E be an elementary $n \times n$ matrix over F. In Section 3.1 it was shown that AE can be obtained from A by means of an elementary column operation. Prove that E^tA can be obtained from A by means of the same elementary operation but performed on the rows rather than on the columns of A. *Hint:* Note that $E^tA = (A^tE)^t$.

21. For each of the following matrices A with entries from the field of rational numbers, find a diagonal matrix D and an invertible matrix Q such that $Q^tAQ = D$.

(a) $$A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

(b) $$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hint: Use an elementary operation other than that of interchanging columns.

(c) $$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

INDEX OF DEFINITIONS FOR CHAPTER 7

appendices

APPENDIX A SETS

A *set* is a collection of objects, called *elements* of the set. If x is an element of the set A, then we write $x \in A$; if x is not an element of A, then we write $x \notin A$. For example, if Z is the set of integers, then $3 \in Z$ and $\frac{1}{2} \notin Z$.

Two sets A and B are called *equal*, denoted $A = B$, f they contain exactly the same elements. Sets may be described in one of two ways:

1. By listing the elements of the set between set braces $\{\quad\}$.
2. By describing the elements of the set in terms of some characteristic property.

For example, the set consisting of the elements 1, 2, 3, and 4 can be written as $\{1, 2, 3, 4\}$ or as

$$\{x: \quad x \text{ is a positive integer less than } 5\}.$$

Note that the order in which the elements of a set are listed is immaterial; hence

$$\{1, 2, 3, 4\} = \{3, 1, 2, 4\} = \{1, 3, 1, 4, 2\}.$$

Example 1. Let A denote the set of real numbers between 1 and 2. Then A may be written as

$$A = \{x\colon \quad x \text{ is a real number and } 1 < x < 2\}$$

or, if R is the set of real numbers, as

$$A = \{x \in R\colon \quad 1 < x < 2\}.$$

A set B is said to be a *subset* of a set A, written $B \subseteq A$ or $A \supseteq B$, if every element of B is an element of A. For example, $\{1, 2, 6\} \subseteq \{2, 8, 7, 6, 1\}$. Observe that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, a fact that is often used to prove that two sets are equal.

The *empty set*, denoted by $\varnothing$, is the set containing no element . The empty set is a subset of every set.

Sets may be combined to form other sets in two basic ways. The *union* of two sets A and B, denoted $A \cup B$, is the set of elements that are in A, or B, or both; that is,

$$A \cup B = \{x\colon \quad x \in A \text{ or } x \in B\}.$$

The *intersection* of two sets A and B, denoted $A \cap B$, is the set of elements that are in both A and B; that is,

$$A \cap B = \{x\colon \quad x \in A \text{ and } x \in B\}.$$

Two sets are called *disjoint* if their intersection is the empty set.

Example 2. Let $A = \{1, 3, 5\}$ and $B = \{1, 5, 7, 8\}$. Then

$$A \cup B = \{1, 3, 5, 7, 8\} \quad \text{and} \quad A \cap B = \{1, 5\}.$$

Likewise, if $X = \{1, 2, 8\}$ and $Y = \{3, 4, 5\}$, then

$$X \cup Y = \{1, 2, 3, 4, 5, 8\} \quad \text{and} \quad X \cap Y = \varnothing.$$

Thus X and Y are disjoint sets.

The union and intersection of more than two sets can be defined analogously. Specifically, if $A_1, A_2, \ldots, A_n$ are sets, then the union and intersection of these sets are defined as

$$\bigcup_{i=1}^{n} A_i = \{x\colon \quad x \in A_i \text{ for some } i = 1, 2, \ldots, n\}$$

and

$$\bigcap_{i=1}^{n} A_i = \{x\colon \quad x \in A_i \text{ for all } i = 1, 2, \ldots, n\}.$$

Likewise, if Λ is an index set and $\{A_\alpha\colon \quad \alpha \in \Lambda\}$ is a collection of sets, the union and intersection of these sets are defined by

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x\colon \quad x \in A_\alpha \text{ for some } \alpha \in \Lambda\}$$

and

$$\bigcap_{\alpha \in \Lambda} A_\alpha = \{x\colon \quad x \in A_\alpha \text{ for all } \alpha \in \Lambda\}.$$

Example 3. Le $\Lambda = \{\alpha \in R\colon \alpha > 1\}$, and let

$$A_\alpha = \left\{x \in R\colon \frac{-1}{\alpha} \leq x \leq 1 + \alpha\right\}$$

for each $\alpha \in \Lambda$, where R denotes the set of real numbers. Then

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x \in R\colon \quad x > -1\} \quad \text{and} \quad \bigcap_{\alpha \in \Lambda} A_\alpha = \{x \in R\colon \quad 0 \leq x \leq 2\}.$$

By a relation on a set A we mean a rule for determining whether or not, for any elements x and y in A, x stands in a given relationship to y. More precisely, a *relation* on A is a set S of ordered pairs of elements of A such that $(x, y) \in S$ if and only if x stands in the given relationship to y. On the set of real numbers, for instance, "is equal to," "is less than," and "is greater than or equal to" are familiar relations. A relation S on a set A is called an *equivalence relation* on A if the following three conditions hold:

1. For each $x \in A$, $(x, x) \in S$ (*reflexivity*).
2. If $(x, y) \in S$, then $(y, x) \in S$ (*symmetry*).
3. If $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$ (*transitivity*).

If S is an equivalence relation on a set A, we shall usually write $x \sim y$ in place of $(x, y) \in S$. For example, if we define $x \sim y$ to mean that $x - y$ is divisible by a fixed integer n, then $\sim$ is an equivalence relation on the set of integers.

APPENDIX B FUNCTIONS

If A and B are sets, then a *function* f from A into B, written $f\colon \; A \longrightarrow B$, is a rule that associates to each element x in A a unique element denoted $f(x)$ in B. The element $f(x)$ is called the *image of* x (*under* f) and x is called a *pre-image of* f(x) (*under* f). If $f\colon \; A \longrightarrow B$, then A is called the *domain* of f, and the set $\{f(x)\colon \; x \in A\}$ of all images of elements in A is called the *range* of f. Note that the range of f is a subset of B. If $S \subseteq A$, we shall denote by $f(S)$ the set $\{f(x)\colon \; x \in S\}$ of all images of elements of S. Likewise, if $T \subseteq B$, we shall denote by $f^{-1}(T)$ the set $\{x \in A\colon \; f(x) \in T\}$ of all pre-images of elements in T. Finally, two functions $f\colon \; A \longrightarrow B$ and $g\colon \; A \longrightarrow B$ are *equal* if $f(x) = g(x)$ for all $x \in A$.

Example 1. Suppose that $A = [-10, 10]$ and $B = R$, the set of real numbers. Let $f\colon \; A \longrightarrow B$ be the function that assigns to each element x in

A the element $x^2 + 1$ in B; that is, f is defined by $f(x) = x^2 + 1$. Then A is the domain of f and $[1, 101]$ is the range of f. Since $f(2) = 5$, the image of 2 is 5 and 2 is a pre-image of 5. Notice that -2 is another pre-image of 5. Moreover, if $S = [1, 2]$ and $T = [82, 101]$, then $f(S) = [2, 5]$ and $f^{-1}(T) = [-10, -9] \cup [9, 10]$.

As the example above shows, the pre-image of an element in the range need not be unique. Functions such that each element of the range has a unique pre-image are called *one-to-one*; that is, $f\colon\ A \longrightarrow B$ is one-to-one if $f(x) = f(y)$ implies $x = y$ or, equivalently, if $x \neq y$ implies $f(x) \neq f(y)$.

If $f\colon\ A \longrightarrow B$ is a function with range B, i.e., if $f(A) = B$, then f is called *onto*.

Suppose that $f\colon\ A \longrightarrow B$ is a function and $S \subseteq A$. Then a function $f_S\colon\ S \longrightarrow B$ called the *restriction of* f *to* S can be formed by defining $f_S(x) = f(x)$ for each $x \in S$.

The following example illustrates these concepts.

Example 2. Let $f\colon\ [-1, 1] \longrightarrow [0, 1]$ be defined by $f(x) = x^2$. This function is onto but not one-to-one since $f(-1) = f(1) = 1$. Note that if $S = [0, 1]$, then f_S is both onto and one-to-one. Finally, if $T = [\frac{1}{2}, 1]$, then f_T is one-to-one but not onto.

Let A, B, and C be sets and $f\colon\ A \longrightarrow B$ and $g\colon\ B \longrightarrow C$ be functions. By following f with g we obtain a function $g \circ f\colon\ A \longrightarrow C$ called the *composite* of g and f. Thus $(g \circ f)(x) = g(f(x))$ for all $x \in A$. For example, let $A = B = C = R$ (the set of real numbers), $f(x) = \sin x$, and $g(x) = x^2 + 3$. Then $(g \circ f)(x) = g(f(x)) = \sin^2 x + 3$, whereas $(f \circ g)(x) = f(g(x)) = \sin (x^2 + 3)$. Hence $g \circ f \neq f \circ g$. Functional composition is associative, however; that is, if $h\colon\ C \longrightarrow D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

A function $f\colon\ A \longrightarrow B$ is said to be *invertible* if there exists a function $g\colon\ B \longrightarrow A$ such that $(f \circ g)(y) = y$ for all $y \in B$ and $(g \circ f)(x) = x$ for all $x \in A$. If such a function g exists, then it is unique and is called the *inverse* of f. We shall denote the inverse of f (when it exists) by f^{-1}. It can be shown that f is invertible if and only if f is both one-to-one and onto.

Example 3. The function $f\colon\ R \longrightarrow R$ defined by $f(x) = 3x + 1$ is one-to-one and onto; hence f is invertible. The inverse of f is the function $f^{-1}\colon\ R \longrightarrow R$ defined by $f^{-1}(x) = (x - 1)/3$.

The following facts about invertible functions are easily proved:

1. If $f\colon\ A \longrightarrow B$ is invertible, then f^{-1} is invertible and $(f^{-1})^{-1} = f$.
2. If $f\colon\ A \longrightarrow B$ and $g\colon\ B \longrightarrow C$ are invertible, then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

APPENDIX C FIELDS

The set of real numbers is an example of an algebraic structure called a "field." Basically, a field is a set in which four operations (called addition, multiplication, subtraction, and division) can be defined so that, with exception of division by zero, the sum, product, difference, and quotient of any two elements in the set is an element of the set. More precisely, a field is defined as follows.

Definitions. *A* field F *is a set in which two operations* $+$ *and* $\cdot$ (*called* addition *and* multiplication, *respectively*) *are defined so that for each pair of elements* a, b *in* F *there are unique elements* $a + b$ *and* $a \cdot b$ *in* F *and such that the following conditions hold for all elements* a, b, c *in* F:

(F 1) $a + b = b + a$ *and* $a \cdot b = b \cdot a$
(*commutativity of addition and multiplication*).

(F 2) $(a + b) + c = a + (b + c)$ *and* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(*associativity of addition and multiplication*).

(F 3) *There exist distinct elements* 0 *and* 1 *in* F *such that*

$$0 + a = a \quad \textit{and} \quad 1 \cdot a = a$$

(*existence of identity elements for addition and multiplication*).

(F 4) *For each element* a *in* F *and each non-zero element* b *in* F *there exist elements* c *and* d *in* F *such that*

$$a + c = 0 \quad \textit{and} \quad b \cdot d = 1$$

(*existence of inverses for addition and multiplication*).

(F 5) $a \cdot (b + c) = a \cdot b + a \cdot c$
(*distributivity of multiplication over addition*).

The elements $a + b$ *and* $a \cdot b$ *are called the* sum *and* product, *respectively, of* a *and* b. *The elements* 0 (*read* "zero") *and* 1 (*read* "one") *mentioned in* (F 3) *are called* identity elements *for addition and multiplication, respectively, and the elements* c *and* d *referred to in* (F 4) *are called an* additive inverse *for* a *and a* multiplicative inverse *for* b, *respectively.*

Example 1. The set of real numbers with the usual definitions of addition and multiplication is a field, which will be denoted by R.

Example 2. The set of rational numbers with the usual definitions of addition and multiplication is a field.

Example 3. The set of all real numbers of the form $a + b\sqrt{2}$, where

a and b are rational numbers, with addition and multiplication as in R is a field.

Example 4. The field Z_2 consists of two elements 0 and 1 with the operations of addition and multiplication defined by the equations

$$0+0=0, \quad 0+1=1+0=1, \quad 1+1=0,$$
$$0\cdot 0=0, \quad 0\cdot 1=1\cdot 0=0, \quad \text{and} \quad 1\cdot 1=1.$$

Example 5. Neither the set of positive integers nor the set of integers with the usual definitions of addition and multiplication is a field, for in either case (F 4) does not hold.

The elements of a field whose existence is guaranteed by (F 3) and (F 4) are unique; this is a consequence of the following theorem.

Theorem C.1 (*Cancellation Laws*). *Let* a, b, *and* c *be arbitrary elements of a field* F.

(a) *If* $a+b=c+b$, *then* $a=c$.
(b) *If* $a\cdot b=c\cdot b$ *and* $b\neq 0$, *then* $a=c$.

PROOF. The proofs of (a) and (b) are similar; so only (b) will be proved.

If $b\neq 0$, then (F 4) guarantees the existence of an element d in F such that $b\cdot d=1$. Multiply both sides of the equality $a\cdot b=c\cdot b$ by d to obtain $(a\cdot b)\cdot d=(c\cdot b)\cdot d$. Consider the left side of this equality: by (F 2) and (F 3) we have

$$(a\cdot b)\cdot d=a\cdot(b\cdot d)=a\cdot 1=a.$$

Similarly, the right side of the equality reduces to c. Thus

$$a=(a\cdot b)\cdot d=(c\cdot b)\cdot d=c. \quad \blacksquare$$

Corollary. The elements 0 *and* 1 *mentioned in* (F 3) *and the elements* c *and* d *mentioned in* (F 4) *are unique.*

PROOF. Suppose that $0'\in F$ satisfies $0'+a=a$ for each $a\in F$. Since $0+a=a$ for each $a\in F$, we have $0'+a=0+a$ for each $a\in F$. Thus $0'=0$ by Theorem C.1.

The proofs of the remaining parts are similar. ∎

Thus each element b in a field has a unique additive inverse and, if $b\neq 0$, a unique multiplicative inverse. (It will be shown in the corollary to Theorem C.2 that 0 has no multiplicative inverse.) The additive inverse and the multiplicative inverse of b are denoted by $-b$ and b^{-1}, respectively. Notice that $-(-b)=b$ and that $(b^{-1})^{-1}=b$.

Subtraction and division can be defined in terms of addition and multiplication by using the additive and multiplicative inverses. Specifically, subtraction of b is defined to be addition of $-b$ and division by $b \neq 0$ is defined to be multiplication by b^{-1}; that is,

$$a - b = a + (-b) \quad \text{and} \quad a/b = a \cdot b^{-1}.$$

Division by zero is undefined, but with this exception the sum, product, difference, and quotient of any two elements of a field are defined.

Many of the familiar properties of multiplication of real numbers are true in any field, as the following theorem shows.

Theorem C.2. *Let* a *and* b *be arbitrary elements of a field. Then each of the following are true.*

(a) $a \cdot 0 = 0$.

(b) $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.

(c) $(-a) \cdot (-b) = a \cdot b$.

PROOF.

(a) Since $0 + 0 = 0$, (F 5) shows that

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$$

Thus $0 + a \cdot 0 = a \cdot 0 + a \cdot 0$, and cancellation of $a \cdot 0$ by Theorem C.1 gives $0 = a \cdot 0$.

(b) By definition $-(a \cdot b)$ is the unique element of F such that $a \cdot b + [-(a \cdot b)] = 0$. So in order to prove that $(-a) \cdot b = -(a \cdot b)$ it suffices to show that $a \cdot b + (-a) \cdot b = 0$. But $-a$ is the element of F such that $a + (-a) = 0$, and so

$$a \cdot b + (-a) \cdot b = [a + (-a)] \cdot b = 0 \cdot b = b \cdot 0 = 0$$

by (F 5) and part (a). Thus $(-a) \cdot b = -(a \cdot b)$. The proof that $a \cdot (-b) = -(a \cdot b)$ is similar.

(c) By twice applying part (b), we find

$$(-a) \cdot (-b) = -[a \cdot (-b)] = -[-(a \cdot b)] = a \cdot b. \quad \blacksquare$$

Corollary. *The additive identity of a field has no multiplicative inverse.*

In an arbitrary field F it may happen that a sum $1 + 1 + \cdots + 1$ (p summands) equals 0 for some positive integer p. For example, in the field Z_2 (defined in Example 4), $1 + 1 = 0$. In this case the smallest positive integer p for which a sum of p 1's equals 0 is called the *characteristic* of F; if no such positive integer exists, then F is said to have *characteristic zero*. Thus Z_2 has characteristic two, and R has characteristic zero. Observe that if F is a field of characteristic $p \neq 0$, then $x + x + \cdots + x$ (p summands) equals 0 for all $x \in F$. In a field having finite characteristic (especially characteristic two), many unnatural problems arise. For this reason some of the results about vector spaces stated in this book require that the field over which the vector space is defined be of characteristic zero (or, at least, of some characteristic other than two).

Finally, note that in other sections of this book the product of two elements a and b in a field is denoted ab rather than $a \cdot b$.

APPENDIX D COMPLEX NUMBERS

For the purposes of algebra the field of real numbers is not sufficient, for there are polynomials of non-zero degree with real number coefficients that have no zeros in the field of real numbers (for example, $x^2 + 1$). It is often desirable to have a field in which any polynomial of non-zero degree with coefficients from that field has a zero in the field. For this reason we shall "enlarge" the field of real numbers to obtain such a field.

Definitions. *A* complex number *is an expression of the form* $\mathrm{z} = \mathrm{a} + \mathrm{bi}$, *where* a *and* b *are real numbers called the* real part *and the* imaginary part *of* z, *respectively.*

The sum *and* product *of two complex numbers* $\mathrm{z} = \mathrm{a} + \mathrm{bi}$ *and* $\mathrm{w} = \mathrm{c} + \mathrm{di}$ (*where* a, b, c, *and* d *are real numbers*) *are defined as follows:*

$$\mathrm{z} + \mathrm{w} = (\mathrm{a} + \mathrm{bi}) + (\mathrm{c} + \mathrm{di}) = (\mathrm{a} + \mathrm{c}) + (\mathrm{b} + \mathrm{d})\mathrm{i}$$

and

$$\mathrm{zw} = (\mathrm{a} + \mathrm{bi})(\mathrm{c} + \mathrm{di}) = (\mathrm{ac} - \mathrm{bd}) + (\mathrm{bc} + \mathrm{ad})\mathrm{i}.$$

Example 1. The sum and product of $z = 3 - 5i$ and $w = 9 + 7i$ are

$$z + w = (3 - 5i) + (9 + 7i) = (3 + 9) + [(-5) + 7]i = 12 + 2i$$

and

$$\begin{aligned} zw &= (3 - 5i)(9 + 7i) = [3 \cdot 9 - (-5) \cdot 7] + [(-5) \cdot 9 + 3 \cdot 7]i \\ &= 62 - 24i. \end{aligned}$$

Any real number c may be regarded as a complex number by associating c with the complex number $c + 0i$. Observe that this correspondence preserves sums and products; that is,

$$(c + 0i) + (d + 0i) = (c + d) + 0i, \quad \text{and} \quad (c + 0i)(d + 0i) = cd + 0i.$$

Any complex number of the form $bi = 0 + bi$, where b is a non-zero real number, is called *imaginary*. The product of two imaginary numbers is real since

$$\begin{aligned} (bi)(di) &= (0 + bi)(0 + di) = (0 - bd) + (b \cdot 0 + 0 \cdot d)i \\ &= -bd. \end{aligned}$$

In particular, for $i = 0 + 1i$, we have $i \cdot i = -1$.

The observation that $i^2 = i \cdot i = -1$ provides an easy way to remember the definition of multiplication of complex numbers: simply multiply two complex numbers as you would any two algebraic expressions and replace i^2 by -1. Example 2 illustrates this technique.

Example 2. The product of $-5 + 2i$ and $1 - 3i$ is

$$\begin{aligned}(-5 + 2i)(1 - 3i) &= -5(1 - 3i) + 2i(1 - 3i)\\ &= -5 + 15i + 2i - 6i^2\\ &= -5 + 15i + 2i - 6(-1)\\ &= 1 + 17i.\end{aligned}$$

The real number 0, regarded as a complex number, is an additive identity element for the set of complex numbers since

$$\begin{aligned}(a + bi) + 0 = (a + bi) + (0 + 0i) &= (a + 0) + (b + 0)i\\ &= a + bi.\end{aligned}$$

Likewise the real number 1, regarded as a complex number, is a multiplicative identity element for the set of complex numbers since

$$\begin{aligned}(a + bi)\cdot 1 = (a + bi)(1 + 0i) &= (a\cdot 1 - b\cdot 0) + (b\cdot 1 - a\cdot 0)i\\ &= a + bi.\end{aligned}$$

Clearly each complex number $a + bi$ has an additive inverse, namely $(-a) + (-b)i$. But also each complex number except 0 has a multiplicative inverse. In fact,

$$(a + bi)^{-1} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a^2 + b^2}\right)i.$$

In view of the preceding statements the following result is not surprising.

Theorem D.1. The set of complex numbers with the operations of addition and multiplication defined above is a field.

We shall denote the field of complex numbers by C.

Definition. The (complex) conjugate *of a complex number* $a + bi$ *is the complex number* $a - bi$. *We shall denote the conjugate of the complex number* z *by* $\bar{z}$.

Example 3. The conjugates of $-3 + 2i$, $4 - 7i$, and 6 are as follows:

$$\overline{-3 + 2i} = -3 - 2i, \qquad \overline{4 - 7i} = 4 + 7i,$$

and

$$\bar{6} = \overline{6 + 0i} = 6 - 0i = 6.$$

The following result is an easy consequence of the definition of the complex conjugate.

Theorem D.2. *A complex number* z *is a real number if and only if* $z = \bar{z}$.

For any complex number $z = a + bi$, $z\bar{z}$ is real and non-negative, for

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2.$$

This fact can be used to define the absolute value of a complex number.

Definition. *The* absolute value (*or* modulus) *of a complex number* $z = a + bi$ *is the real number* $\sqrt{a^2 + b^2}$. *We shall denote the absolute value of* z *by* $|z|$. *Observe that* $z\bar{z} = |z|^2$.

The fact that the product of a complex number and its conjugate is real provides an easy method for determining the quotient of two complex numbers; for if $c + di \neq 0$, then

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

Example 4. We shall illustrate the procedure described above by computing the quotient $(1 + 4i)/(3 - 2i)$:

$$\frac{1 + 4i}{3 - 2i} = \frac{1 + 4i}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} = \frac{-5 + 14i}{9 + 4} = -\frac{5}{13} + \frac{14}{13}i.$$

The absolute value of a complex number has the familiar properties of the absolute value of a real number, as the following result shows.

Theorem D.3. *Let* z *and* w *denote any two complex numbers. Then*

(a) $|z + w| \leq |z| + |w|$.

(b) $|zw| = |z| \cdot |w|$.

(c) $|z| - |w| \leq |z + w|$.

PROOF. Let $z = a + bi$ and $w = c + di$, where a, b, c, and d are real numbers.

(a) Observe first that

$$0 \leq (ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2,$$

so $2abcd \leq a^2d^2 + b^2c^2$. Adding $a^2c^2 + b^2d^2$ to both sides of the inequality gives

$$(ac + bd)^2 = a^2c^2 + 2abcd + b^2d^2$$
$$\leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 = (a^2 + b^2)(c^2 + d^2).$$

By taking square roots, we obtain

$$ac + bd \leq \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}.$$

Now

$$\begin{aligned} |z + w|^2 &= |(a + c) + (b + d)i|^2 \\ &= (a + c)^2 + (b + d)^2 \\ &= a^2 + c^2 + b^2 + d^2 + 2(ac + bd) \\ &\leq a^2 + c^2 + b^2 + d^2 + 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\ &= (\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2})^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

By taking square roots, we obtain (a).

(b) From the definition of absolute value we see that

$$\begin{aligned} |zw| &= |(a + bi)(c + di)| = |(ac - bd) + (bc + ad)i| \\ &= \sqrt{(ac - bd)^2 + (bc + ad)^2} = \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} = |a + bi| \cdot |c + di| = |z| \cdot |w|. \end{aligned}$$

(c) From (a) and (b) it follows that

$$|z| = |(z + w) - w| \leq |z + w| + |-w| = |z + w| + |w|.$$

So

$$|z| - |w| \leq |z + w|. \quad \blacksquare$$

Our motivation for enlarging the set of real numbers to the set of complex numbers was to obtain a field such that every polynomial with nonzero degree having coefficients in that field has a zero. Our next result guarantees that the field of complex numbers has this property.

Theorem D.4 (*The Fundamental Theorem of Algebra*). *Let* $a_0, \ldots, a_n$ $(n \geq 1)$ *be complex numbers such that* $a_n \neq 0$. *Then*

$$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$$

has a zero in the field of complex numbers.

For a proof, see Walter Rudin's *Principles of Mathematical Analysis*, McGraw-Hill Book Company, 1964.

The following important corollary follows from Theorem D.4 and the division algorithm for polynomials (Theorem E.1).

Corollary. *If* $p(z) = a_nz^n + \cdots + a_1z + a_0$ *is a polynomial of degree* $n \geq 1$ *with complex coefficients, then there exist complex numbers* $c_1, \ldots, c_n$ (*not necessarily distinct*) *such that*

$$p(z) = a_n(z - c_1) \cdots (z - c_n).$$

A field is called *algebraically closed* if it has the property that every polynomial with coefficients from that field factors as a product of factors of degree 1. Thus the corollary above shows that the field of complex numbers is algebraically closed.

APPENDIX E POLYNOMIALS

In this appendix, we shall discuss some basic properties of polynomials needed for Chapters 5 and 6. For the definition of a polynomial, refer to Section 1.2.

Definition. *A polynomial* f(x) divides *a polynomial* g(x) *if there exists a polynomial* q(x) *such that* g(x) = f(x)q(x).

Our first result shows that the familiar long division process for polynomials with real coefficients is valid for polynomials with coefficients from an arbitrary field.

Theorem E.1 (*The Division Algorithm for Polynomials*). *Let* $f_1(x)$ *be a polynomial of degree* n, *and let* $f_2(x)$ *be a polynomial of degree* $m \geq 0$. *Then there exist polynomials* q(x) *and* r(x) *such that*

(a) *The degree of* r(x) *is less than* m.

(b) $f_1(x) = q(x)f_2(x) + r(x)$.

(c) q(x) *and* r(x) *are unique with respect to conditions* (a) *and* (b).

PROOF. We shall begin by establishing the existence of $q(x)$ and $r(x)$ that will satisfy conditions (a) and (b). If $n < m$, we can take $q(x) = 0$ and $r(x) = f_1(x)$ to satisfy (a) and (b).

Assume, therefore, that $m \leq n$. In this case we shall establish the existence of $q(x)$ and $r(x)$ by induction on n. Suppose first that $n = 0$; then $m \leq n$ implies that $m = 0$. Thus $f_1(x)$ and $f_2(x)$ are non-zero constants. Hence we may take $q(x) = f_1(x)f_2^{-1}(x)$ and $r(x) = 0$ to satisfy (a) and (b).

Now suppose that the theorem is true whenever $f_1(x)$ has degree less than $n > 0$. Let

$$f_1(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

and

$$f_2(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0,$$

where $m \leq n$. Define a polynomial $h(x)$ by

$$\begin{aligned} h(x) &= f_1(x) - a_nb_m^{-1}x^{n-m}f_2(x) \qquad (1)\\ &= (a_{n-1} - a_nb_m^{-1}b_{m-1})x^{n-1} + (a_{n-2} - a_nb_m^{-1}b_{m-2})x^{n-2} \\ &\qquad + \cdots + (a_0 - a_nb_m^{-1}b_0). \end{aligned}$$

Then $h(x)$ is a polynomial of degree less than n. We shall consider two cases.

CASE 1. $h(x)$ is of degree less than m. In this case, let $q(x) = a_n b_m^{-1} x^{n-m}$ and $r(x) = h(x)$. Then by Eq. (1) we obtain

$$f_1(x) = q(x)f_2(x) + r(x),$$

and $r(x)$ has degree less than m.

CASE 2. $h(x)$ has degree at least m. Since $h(x)$ has degree less than n, we may apply the induction hypothesis to obtain polynomials $q_1(x)$ and $r(x)$ such that $r(x)$ has degree less than m, and

$$h(x) = q_1(x)f_2(x) + r(x). \tag{2}$$

Combining Eqs. (1) and (2) and solving for $f_1(x)$, we have

$$f_1(x) = [a_n b_m^{-1} x^{n-m} + q_1(x)]f_2(x) + r(x).$$

In this case, let $q(x) = a_n b_m^{-1} x^{n-m} + q_1(x)$, so that $f_1(x) = q(x)f_2(x) + r(x)$, where $r(x)$ has degree less than m. This proves the existence of $q(x)$ and $r(x)$.

We now show the uniqueness of q and r. Suppose that $q_1(x)$, $q_2(x)$, $r_1(x)$, and $r_2(x)$ exist such that $r_1(x)$ and $r_2(x)$ each has degree less than m and

$$f_1(x) = q_1(x)f_2(x) + r_1(x) = q_2(x)f_2(x) + r_2(x).$$

Then

$$[q_1(x) - q_2(x)]f_2(x) = r_2(x) - r_1(x). \tag{3}$$

The right-hand side of Eq. (3) is a polynomial of degree less than m. Since $f_2(x)$ has degree m, it must follow that $q_1(x) - q_2(x)$ is the zero polynomial. Hence $q_1(x) = q_2(x)$; thus by Eq. (3) $r_1(x) = r_2(x)$. ▮

In the context of Theorem E.1 we call $q(x)$ and $r(x)$ the *quotient* and *remainder*, respectively, for the division of $f_1(x)$ by $f_2(x)$. For example, the quotient and remainder for the division of the complex polynomial

$$f_1(x) = (3 + i)x^5 - (1 - i)x^4 + 6x^3 + (-6 + 2i)x^2 + (2 + i)x + 1$$

by the complex polynomial

$$f_2(x) = (3 + i)x^2 - 2ix + 4$$

are

$$q(x) = x^3 + ix^2 - 2 \quad \text{and} \quad r(x) = (2 - 3i)x + 9.$$

Corollary 1. Let f(x) *be a polynomial of degree at least* 1, *and let* a ∈ F. *Then* f(a) = 0 *if and only if* x − a *divides* f(x).

PROOF. Suppose that $x - a$ divides $f(x)$. Then there exists a polynomial $q(x)$ such that $f(x) = (x - a)q(x)$. Thus $f(a) = (a - a)q(a) = 0 \cdot q(a) = 0$.

Conversely, suppose that $f(a) = 0$. By Theorem E.1 there exist polynomials $q(x)$ and $r(x)$ such that $r(x)$ has degree less than one and

$$f(x) = q(x)(x - a) + r(x).$$

Substituting a for x in the above we obtain $r(a) = 0$. Since $r(x)$ has degree less than 1, it must be the constant polynomial $r(x) = 0$. Thus $f(x) = q(x)(x - a)$. ∎

For any polynomial $f(x)$ with coefficients from a field F, an element $a \in F$ is called a *zero* of $f(x)$ if $f(a) = 0$. With this terminology the corollary above states that a is a zero of $f(x)$ if and only if $x - a$ divides $f(x)$.

Corollary 2. *Any polynomial of degree* $n \geq 1$ *has at most* n *distinct zeros.*

PROOF. The proof is by induction on n. The result is obvious if $n = 1$. Suppose therefore that the result is true for some positive integer n, and let $f(x)$ be a polynomial of degree $n + 1$. If $f(x)$ has no zeros, then there is nothing to prove. Otherwise, if a is a zero of $f(x)$, then by Corollary 1 we may write $f(x) = (x - a)q(x)$ for some polynomial $q(x)$. Note that $q(x)$ must be of degree n; therefore by the induction hypothesis $q(x)$ can have at most n distinct zeros. Thus, since any zero of $f(x)$ distinct from a is also a zero of $q(x)$, $f(x)$ can have at most $n + 1$ distinct zeros. ∎

Polynomials having no common divisors arise naturally in the study of canonical forms.

Definition. *Let* f(x) *and* g(x) *be polynomials each of degree greater than* 0. *These polynomials are said to be* relatively prime *if there exists no polynomial of positive degree that divides both of them.*

For example, the polynomials $f(x) = x^2(x - 1)$ and $h(x) = (x - 1)(x - 2)$ are not relatively prime since $x - 1$ divides both $f(x)$ and $h(x)$. The polynomials $f(x)$ and $g(x) = (x - 2)(x - 3)$ are relatively prime, however, since they have no common factors of positive degree.

The following theorem establishes that a combination of relatively prime polynomials equals the constant polynomial 1.

Theorem E.2. *If* $f_1(x)$ *and* $f_2(x)$ *are relatively prime polynomials, there exist polynomials* $q_1(x)$ *and* $q_2(x)$ *such that* $q_1(x)f_1(x) + q_2(x)f_2(x) = 1$, *the constant polynomial of degree zero with value* 1.

PROOF. Without loss of generality, suppose that the degree of $f_1(x)$ is greater than or equal to the degree of $f_2(x)$. We shall use mathematical induction on the degree of $f_2(x)$. Suppose $f_2(x)$ has degree 1. By Theorem E.1 there exist polynomials $q(x)$ and $r(x)$ such that $r(x)$ has degree less than 1 and such that

$$f_1(x) = q(x)f_2(x) + r(x). \tag{4}$$

Notice that $r(x)$ cannot be the zero polynomial since $f_1(x)$ and $f_2(x)$ are relatively prime. Thus $r(x)$ is a non-zero constant c. Then Eq. (4) can be rewritten as

$$(c^{-1})f_1(x) + (-c)^{-1}q(x)f_2(x) = 1. \tag{5}$$

So the conclusion holds with $q_1(x) = c^{-1}$ and $q_2(x) = (-c)^{-1}q(x)$. Now suppose the theorem holds whenever $f_2(x)$ has degree less than n for some integer $n \geq 2$, and suppose $f_2(x)$ has degree n. By Theorem E.1 there exist polynomials $q(x)$ and $r(x)$ such that $r(x)$ has degree less than n and

$$f_1(x) = q(x)f_2(x) + r(x). \tag{6}$$

Since $f_1(x)$ and $f_2(x)$ are relatively prime, $r(x)$ is not the zero polynomial. If $r(x)$ has degree 0, then $r(x)$ is a non-zero constant, c, and we obtain Eq. (5) as before. Suppose then that $r(x)$ has degree greater than zero. Since $r(x)$ has degree less than n, we may apply the induction hypothesis to $f_2(x)$ and $r(x)$ provided that we can show these polynomials to be relatively prime. Suppose otherwise; then there exists a non-zero polynomial $g(x)$ that divides both $f_2(x)$ and $r(x)$. So there exist polynomials $h_1(x)$ and $h_2(x)$ such that

$$r(x) = g(x)h_1(x) \quad \text{and} \quad f_2(x) = g(x)h_2(x). \tag{7}$$

Combining Eqs. (6) and (7), we obtain

$$f_1(x) = [q(x)h_2(x) + h_1(x)]g(x),$$

and so $g(x)$ divides $f_1(x)$. But $g(x)$ divides $f_2(x)$, contradicting the fact that $f_1(x)$ and $f_2(x)$ are relatively prime. Thus $r(x)$ and $f_2(x)$ are relatively prime. Hence by the induction hypothesis there exist $g_1(x)$ and $g_2(x)$ such that

$$g_1(x)f_2(x) + g_2(x)r(x) = 1. \tag{8}$$

Combining Eqs. (6) and (8), we obtain

$$g_1(x)f_2(x) + g_2(x)[f_1(x) - q(x)f_2(x)] = 1.$$

Thus

$$g_2(x)f_1(x) + [g_1(x) - g_2(x)q(x)]f_2(x) = 1.$$

Setting $q_1(x) = g_2(x)$ and $q_2(x) = g_1(x) - g_2(x)q(x)$, we obtain the desired conclusion. ∎

Example 1. For the relatively prime polynomials $f_1(x) = x^2(x - 1)$ and $f_2(x) = (x - 2)(x - 3)$, it is easily verified that

$$q_1(x)f_1(x) + q_2(x)f_2(x) = 1,$$

where

$$q_1(x) = \tfrac{1}{36}(-7x + 23) \quad \text{and} \quad q_2(x) = \tfrac{1}{36}(x^2 + 5x + 6).$$

Throughout Chapters 5, 6, and 7 we shall consider linear operators that are polynomials in some particular operator T and matrices that are polynomials in a particular matrix A. For these operators and matrices the following notation is convenient.

Definitions. *Let*

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

be a polynomial with coefficients from a field F. *If* T *is a linear operator on* V, *a vector space over* F, *we define* f(T) *by*

$$f(T) = a_0I + a_1T + \cdots + a_nT^n.$$

Similarly, if A *is an* $n \times n$ *matrix with entries from* F, *we define* f(A) *by*

$$f(A) = a_0I + a_1A + \cdots + a_nA^n.$$

Example 2. Let T be the linear operator on R^2 defined by $T(a, b) = (2a + b, a - b)$, and let $f(x) = x^2 + 2x - 3$. Since $T^2(a, b) = (5a + b, a + 2b)$,

$$\begin{aligned} f(T)(a, b) &= (T^2 + 2T - 3I)(a, b) \\ &= (5a + b, a + 2b) + (4a + 2b, 2a - 2b) - 3(a, b) \\ &= (6a + 3b, 3a - 3b). \end{aligned}$$

Likewise, if

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix},$$

then

$$f(A) = A^2 + 2A - 3I = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix} + 2\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & -3 \end{pmatrix}.$$

The following three theorems utilize this notation.

Theorem E.3. *Let* f(x) *be a polynomial with coefficients from a field* F, *and let* T *be a linear operator on* V, *where* V *is a vector space over* F. *Then*

(a) f(T) *is a linear operator on* V.

(b) *If* β *is a finite ordered basis for* V *and* $A = [T]_\beta$, *then* $[f(T)]_\beta = f(A)$.

PROOF. Exercise.

Theorem E.4. *Let* T *be a linear operator on a vector space* V *over* F, *and let* A *be a square matrix with entries from* F. *Then for any polynomials* $f_1(x)$ *and* $f_2(x)$ *with coefficients from* F

(a) $f_1(T)f_2(T) = f_2(T)f_1(T)$.

(b) $f_1(A)f_2(A) = f_2(A)f_1(A)$.

PROOF. Exercise.

Theorem E.5. *Let* T *be a linear operator on a vector space* V *over a field* F, *and let* A *be an* $n \times n$ *matrix with entries from* F. *If* $f_1(x)$ *and* $f_2(x)$ *are relatively prime polynomials with entries from* F, *then there exist polynomials* $q_1(x)$ *and* $q_2(x)$ *with entries from* F *such that*

(a) $q_1(T)f_1(T) + q_2(T)f_2(T) = I$.
(b) $q_1(A)f_1(A) + q_2(A)f_2(A) = I$.

PROOF. Exercise.

In Chapters 5 and 6 we are concerned with determining when a linear operator T on a finite-dimensional vector space can be "diagonalized" and with finding a simple (canonical) representation of T. Both of these problems are affected by the factorization of a certain polynomial determined by T (the "characteristic polynomial" of T). In this setting certain types of polynomials play an important role.

Definitions. *A polynomial* $f(x)$ *with coefficients from a field* F *is called* monic *if its leading coefficient is* 1. *If* $f(x)$ *has positive degree and cannot be expressed as a product of polynomials with coefficients from* F *each having positive degree, then* $f(x)$ *is called* irreducible.

Observe that whether or not a polynomial is irreducible depends on the field from which its coefficients come. For example, $f(x) = x^2 + 1$ is irreducible over the field of real numbers but not irreducible over the field of complex numbers since $x^2 + 1 = (x + i)(x - i)$.

Clearly any polynomial of degree 1 is irreducible. Moreover, for polynomials with coefficients from an algebraically closed field, the polynomials of degree 1 are the only irreducible polynomials.

The following facts are easily established.

Theorem E.6. *Let* $\phi(x)$ *and* $f(x)$ *be polynomials with coefficients from a field* F. *If* $\phi(x)$ *is irreducible and* $\phi(x)$ *does not divide* $f(x)$, *then* $\phi(x)$ *and* $f(x)$ *are relatively prime.*

PROOF. Exercise.

Theorem E.7. *Any two distinct irreducible monic polynomials are relatively prime.*

PROOF. Exercise.

We shall now establish a result that will lead to a proof of the unique factorization theorem for polynomials which states that every polynomial of positive degree is uniquely expressible as a constant times a product of irreducible monic polynomials.

Theorem E.8. *Let* f(x), g(x), *and* ϕ(x) *be polynomials with coefficients from the same field. If* ϕ(x) *is irreducible and divides the product* f(x)g(x), *then* ϕ(x) *divides* f(x) *or* ϕ(x) *divides* g(x).

PROOF. Suppose that $\phi(x)$ does not divide $f(x)$. Then $\phi(x)$ and $f(x)$ are relatively prime by Theorem E.6, and so there exist polynomials $q_1(x)$ and $q_2(x)$ such that

$$1 = q_1(x)\phi(x) + q_2(x)f(x).$$

Multiplying both sides of this equation by $g(x)$ yields

$$g(x) = q_1(x)\phi(x)g(x) + q_2(x)f(x)g(x). \tag{9}$$

Since $\phi(x)$ divides $f(x)g(x)$, there exists a polynomial $h(x)$ such that $f(x)g(x) = \phi(x)h(x)$. Thus Eq. (9) becomes

$$g(x) = q_1(x)\phi(x)g(x) + q_2(x)\phi(x)h(x) = \phi(x)[q_1(x)g(x) + q_2(x)h(x)].$$

So $\phi(x)$ divides $g(x)$. ▮

Corollary. *Let* ϕ(x), ϕ_1(x), ϕ_2(x), . . . , ϕ_n(x) *be irreducible monic polynomials with coefficients from the same field. If* ϕ(x) *divides the product* ϕ_1(x)ϕ_2(x) $\cdots$ ϕ_n(x), *then* ϕ(x) = ϕ_i(x) *for some* i (i = 1, 2, . . . , n).

PROOF. We shall prove the coro'lary by induction on n. For $n = 1$ the result is an immediate consequence of Theorem E.7. Suppose then that the corollary is true for any $n - 1$ irreducible monic polynomials and that we are given n irreducible monic polynomials $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$. If $\phi(x)$ divides the product

$$\phi_1(x)\phi_2(x) \cdots \phi_n(x) = [\phi_1(x)\phi_2(x) \cdots \phi_{n-1}(x)]\phi_n(x),$$

then $\phi(x)$ divides the product $\phi_1(x)\phi_2(x) \cdots \phi_{n-1}(x)$ or $\phi(x)$ divides $\phi_n(x)$ by Theorem E.8. In the first case, $\phi(x) = \phi_i(x)$ for some i ($i = 1, 2, \ldots, n - 1$) by the induction hypothesis; in the second case, $\phi(x) = \phi_n(x)$ by Theorem E.7. ▮

We are now able to establish the unique factorization theorem, which is used throughout Chapters 5 and 6.

Theorem E.9 (*Unique Factorization Theorem for Polynomials*). *For any polynomial* f(x) *of positive degree, there exists a unique constant* c, *unique distinct irreducible monic polynomials* ϕ_1(x), ϕ_2(x), . . . , ϕ_k(x), *and unique positive integers* $n_1, n_2, \ldots, n_k$ *such that*

$$f(x) = c[\phi_1(x)]^{n_1}[\phi_2(x)]^{n_2} \cdots [\phi_k(x)]^{n_k}.$$

PROOF. We shall begin by showing the existence of such a factorization by using induction on the degree of $f(x)$. If $f(x)$ is of degree 1, then $f(x) = ax + b$ for some constants a and b with $a \neq 0$. Setting $\phi(x) = x + b/a$, we have $f(x) = a\phi(x)$. Since $\phi(x)$ is an irreducible monic polynomial, the result is proved in this case. Now suppose that the conclusion is true for any polynomial with positive degree less than some integer $n > 1$, and let $f(x)$ be a polynomial of degree n. Then

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

for some scalars a_i with $a_n \neq 0$. If $f(x)$ is irreducible, then

$$f(x) = a_n\left(x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_1}{a_n}x + \frac{a_0}{a_n}\right)$$

is a representation of $f(x)$ as a product of a_n times a monic irreducible polynomial. If $f(x)$ is not irreducible, then $f(x) = g(x)h(x)$ for some polynomials $g(x)$ and $h(x)$ each of positive degree less than n. The induction hypothesis guarantees that both $g(x)$ and $h(x)$ factor as products of a constant times powers of distinct irreducible monic polynomials. Consequently $f(x) = g(x)h(x)$ also factors in this way. Thus in either case $f(x)$ can be factored as a product of a constant times powers of distinct irreducible monic polynomials.

It remains to establish the uniqueness of such a factorization. Suppose that

$$\begin{aligned} f(x) &= c[\phi_1(x)]^{n_1}[\phi_2(x)]^{n_2} \cdots [\phi_k(x)]^{n_k} \\ &= d[\psi_1(x)]^{m_1}[\psi_2(x)]^{m_2} \cdots [\psi_r(x)]^{m_r}, \end{aligned} \tag{10}$$

where c and d are constants, $\phi_i(x)$ and $\psi_j(x)$ are irreducible monic polynomials, and n_i and m_j are positive integers ($i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, r$). Clearly both c and d must be the leading coefficient of $f(x)$; hence $c = d$. By dividing, Eq. (10) becomes

$$[\phi_1(x)]^{n_1}[\phi_2(x)]^{n_2} \cdots [\phi_k(x)]^{n_k} = [\psi_1(x)]^{m_1}[\psi_2(x)]^{m_2} \cdots [\psi_r(x)]^{m_r}. \tag{11}$$

So $\phi_i(x)$ divides the right-hand side of Eq. (11) for $i = 1, 2, \ldots, k$. Consequently, by the corollary to Theorem E.8, for each i ($i = 1, 2, \ldots, k$) $\phi_i(x) = \psi_j(x)$ for some $j = 1, 2, \ldots, r$, and for any j ($j = 1, 2, \ldots, r$) $\psi_j(x) = \phi_i(x)$ for some $i = 1, 2, \ldots, k$. We conclude that $r = k$ and that by renumbering if necessary, $\phi_i(x) = \psi_i(x)$ for $i = 1, 2, \ldots, k$. Suppose that $n_i \neq m_i$ for some i. Without loss of generality we may suppose that $i = 1$ and $n_1 > m_1$. Then by canceling $[\phi_1(x)]^{m_1}$ from both sides of Eq. (11), we obtain

$$[\phi_1(x)]^{n_1-m_1}[\phi_2(x)]^{n_2} \cdots [\phi_k(x)]^{n_k} = [\phi_2(x)]^{m_2} \cdots [\phi_k(x)]^{m_k}. \tag{12}$$

Since $n_1 - m_1 > 0$, $\phi_1(x)$ divides the left-hand side of Eq. (12) and hence divides the right-hand side also. So $\phi_1(x) = \phi_i(x)$ for some $i = 2, \ldots, k$ by the corollary to Theorem E.8. But this contradicts that $\phi_1(x), \phi_2(x), \ldots, \phi_k(x)$ are distinct. Hence the factorizations of $f(x)$ in Eq. (10) are the same. ∎

It is often useful to regard a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with coefficients from a field F as a function $f\colon\ F \to F$. In this case the value of f at $c \in F$ is $f(c) = a_n c^n + \cdots + a_1 c + a_0$. Unfortunately, for arbitrary fields F, there is not a one-to-one correspondence between polynomials and polynomial functions. For example, if $f(x) = x^2$ and $g(x) = x$ are two polynomials from the field Z_2 (as defined in Example 4 of Appendix C), then $f(x)$ and $g(x)$ have different degrees and hence are not equal as polynomials. But $f(a) = g(a)$ for all $a \in Z_2$, so that f and g are equal polynomial functions. Our final result shows that this anomaly cannot occur if F is an infinite field.

Theorem E.10. *Let* f(x) *and* g(x) *be polynomials with coefficients from an infinite field* F. *If* f(a) = g(a) *for all* a $\in$ F, *then* f(x) *and* g(x) *are equal.*

PROOF. Suppose that $f(a) = g(a)$ for all $a \in F$. Define $h(x) = f(x) - g(x)$, and suppose that $h(x)$ is of degree $n \geq 1$. It follows from the corollary to Theorem E.9 that $h(x)$ can have at most n zeros. But $h(a) = f(a) - g(a) = 0$ for any $a \in F$, contradicting the assumption that $h(x)$ has positive degree. Thus $h(x)$ is a constant polynomial, and since $h(a) = 0$ for each $a \in F$, it follows that $h(x)$ is the zero polynomial. Hence $f(x) = g(x)$. ∎

answers to selected exercises

SECTION 1.1

1. Only the pairs in parts (b) and (c) are parallel.
2. (a) $(3, -2, 4) + t(-8, 9, -3)$
   (c) $(3, 7, 2) + t(0, 0, -10)$
3. (a) $(2, -5, -1) + t_1(-2, 9, 7) + t_2(-5, 12, 2)$
   (c) $(-8, 2, 0) + t_1(9, 1, 0) + t_2(14, -7, 0)$

SECTION 1.2

1. (a) T (b) F (c) F (d) F (e) T (f) F
   (g) F (h) F (i) T (j) T (k) T
3. $M_{13} = 3$, $M_{21} = 4$, and $M_{22} = 5$
4. (a) $\begin{pmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 8 & 20 & -12 \\ 4 & 0 & 28 \end{pmatrix}$
   (e) $2x^4 + x^3 + 2x^2 - 2x + 10$
   (g) $10x^7 - 30x^4 + 40x^2 - 15x$

13. No, (VS 4) fails.
14. Yes.
15. No.

SECTION 1.3

1. (a) F (b) F (c) T (d) F (e) T (f) F
2. (a) $\begin{pmatrix} -4 & 5 \\ 2 & -1 \end{pmatrix}$; the trace is -5.

   (c) $\begin{pmatrix} -3 & 0 & 6 \\ 9 & -2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 \\ -1 \\ 3 \\ 5 \end{pmatrix}$

11. No, the set is not closed under addition.
14. Yes.

SECTION 1.4

1. (a) T (b) F (c) T (d) F (e) T (f) F
2. (a) $\{x_2(1, 1, 0, 0) + x_4(-3, 0, -2, 1) + (5, 0, 4, 0)\colon\ x_2, x_4 \in R\}$

   (c) There are no solutions.

   (e) $\{x_3(10, -3, 1, 0, 0) + x_4(-3, 2, 0, 1, 0) + (-4, 3, 0, 0, 5)\colon\ x_3, x_4 \in R\}$
3. (a) Yes. (c) No. (e) No.
4. (a) Yes. (c) Yes. (e) No.

SECTION 1.5

1. (a) F (b) T (c) F (d) F (e) T (f) T
5. $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

SECTION 1.6

1. (a) F (b) T (c) F (d) F (e) T (f) F
   (g) F (h) T (i) F (j) T (k) T (l) F
2. (a) Yes. (c) Yes. (e) No.

3. (a) No. (c) No. (e) No.
4. No.
5. No.
8. $\{x_1, x_3, x_5, x_7\}$
9. $(a_1, a_2, a_3, a_4) = a_1x_1 + (a_2 - a_1)x_2 + (a_3 - a_2)x_3 + (a_4 - a_3)x_4$
10. $\dim(\mathsf{W}_1) = 3$, $\dim(\mathsf{W}_2) = 2$, $\dim(\mathsf{W}_1 + \mathsf{W}_2) = 4$, and $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = 1$
17. $n^2 - 1$
19. $\frac{1}{2}n(n - 1)$

SECTION 1.7

1. (a) F (b) F (c) F (d) T (e) T (f) T

SECTION 2.1

1. (a) T (b) F (c) F (d) T (e) T (f) F
   (g) F (h) T (i) F
2. The nullity is 1, and the rank is 2. T is not one-to-one but is onto.
4. The nullity is 4, and the rank is 2. T is neither one-to-one nor onto.
5. The nullity is 0, and the rank is 3. T is one-to-one but not onto.
10. $\mathsf{T}(2, 3) = (5, 11)$. T is one-to-one.
12. No.

SECTION 2.2

1. (a) T (b) T (c) F (d) T (e) T (f) F
2. (a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$
3. $$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix} \quad \text{and} \quad [\mathsf{T}]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$$
5. (a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

SECTION 2.3

1. (a) F (b) T (c) F (d) T (e) F (f) F
(g) F (h) F (i) T (j) T

2. $A(2B + 3C) = \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}$ and $A(BD) = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

3. $[\mathsf{T}]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$, $[\mathsf{U}]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, and $[\mathsf{UT}]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

4. (a) $\begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$ (c) (5)

11. (a) No. (b) No.

SECTION 2.4

1. (a) F (b) T (c) F (d) F (e) T (f) F
(g) T (h) T (i) T

17. (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

SECTION 2.5

1. (a) F (b) T (c) T (d) F (e) T

2. (a) $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

3. (a) $\begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

4. (a)
$$A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & -4 \\ 7 & -2 \\ -1 & 2 \end{pmatrix}$$
(b)
$$Q = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

SECTION 2.6

1. (a) F (b) T (c) T (d) T (e) F (f) T
(g) T (h) F
2. The functions in parts (a), (c), (e), and (f) are linear functionals.
3. (a) $\mathsf{f}_1(x, y, z) = x - \frac{1}{2}y$, $\mathsf{f}_2(x, y, z) = \frac{1}{2}y$, and $\mathsf{f}_3(x, y, z) = -x + z$
5. The basis for V is $\{p_1(x), p_2(x)\}$, where $p_1(x) = 2 - 2x$ and $p_2(x) = -\frac{1}{2} + x$.
7. (a) $\mathsf{T}^t(\mathsf{f}) = \mathsf{g}$, where $\mathsf{g}(a + bx) = -3a - 4b$
(b) $[\mathsf{T}^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$ (c) $[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

SECTION 2.7

1. (a) T (b) T (c) F (d) F (e) T (f) F
(g) T
2. (a) F (b) F (c) T (d) T (e) F
3. (a) $\{e^{-t}, te^{-t}\}$ (c) $\{e^{-t}, te^{-t}, e^t, te^t\}$
(e) $\{e^{-t}, e^t \cos 2t, e^t \sin 2t\}$
4. (a) $\{e^{(1+\sqrt{5})t/2}, e^{(1-\sqrt{5})t/2}\}$ (c) $\{1, e^{-4t}, e^{-2t}\}$

SECTION 3.1

1. (a) T (b) F (c) T (d) F (e) T (f) F
(g) T (h) F (i) T
2. Adding -2 times column 1 to column 2 transforms A into B.

SECTION 3.2

1. (a) F (b) F (c) T (d) T (e) F (f) T
(g) T (h) T (i) T
2. (a) 2 (c) 2 (e) 3 (g) 1
4. (a)
$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \text{ the rank is 2.}$$
5. (a) The rank is 2, and the inverse is $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$.
(c)
The rank is 3, and the inverse is $\begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$.
(e) The rank is 3; so no inverse exists.
6. (a) $\mathsf{T}^{-1}(ax^2 + bx + c) = -ax^2 - (4a + b)x - (10a + 2b + c)$
(c) $\mathsf{T}^{-1}(ax^2 + bx + c) = (a, -\frac{1}{2}b + \frac{1}{2}c, -a + \frac{1}{2}b + \frac{1}{2}c)$
7.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SECTION 3.3

1. (a) F (b) F (c) T (d) F (e) F (f) F
(g) T (h) F
2. (a)
$$\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}$$
(c)
$$\left\{\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\right\}$$
3. (a)
$$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : \quad t \in R\right\}$$
(c)
$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_3\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} : \quad t_1, t_2, t_3 \in R\right\}$$

4. (b) $A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{pmatrix}$ (c) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$

6. $$\mathsf{T}^{-1}\{(1, 11)\} = \left\{ \begin{pmatrix} \frac{11}{2} \\ -\frac{9}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} : \quad t \in R \right\}$$

7. The systems in parts (b), (c), and (d) have solutions.
10. The farmer, tailor, and carpenter must have incomes in the proportions 4: 3: 4.
11. There must be 7.8 units of the first commodity and 9.5 units of the second.

SECTION 3.4

1. (a) F (b) T (c) T (d) T (e) F (f) T (g) T
4. (a) $$\left\{ \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} : \quad t \in R \right\}$$
   (c) There are no solutions.

SECTION 4.1

1. (a) T (b) F (c) T (d) F (e) T (f) F
2. (a) 30 (c) -8
3. (a) $-10 + 15i$ (c) -24
4. (a) 19 (c) 14

SECTION 4.2

1. (a) T (b) T (c) F (d) F (e) T (f) F
3. (a) -34 (c) -49
4. The functions in parts (c), (d), and (g) are 3-linear.

SECTION 4.3

1. (a) T (b) T (c) F (d) T (e) F (f) T
   (g) F (h) T (i) T (j) F (k) T
2. (a) 90 (c) 0
3. (a) 100 (c) 0 (e) 86 (g) $-180 + 40i$

SECTION 4.4

1. (a) F (b) F (c) F (d) F
2. (a) $\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$ (c) $\begin{pmatrix} -3i & 0 & 0 \\ 4 & -1+i & 0 \\ 10+16i & -5-3i & 3+3i \end{pmatrix}$

   (e) $\begin{pmatrix} 6 & 22 & 12 \\ 12 & -2 & 24 \\ 21 & -38 & -27 \end{pmatrix}$ (g) $\begin{pmatrix} 18 & 28 & -6 \\ -20 & -21 & 37 \\ 48 & 14 & -16 \end{pmatrix}$
3. (a) $x_1 = \dfrac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \; x_2 = \dfrac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$

   (c) $x_1 = -1, x_2 = -1.2, x_3 = -1.4$

   (e) $x_1 = -43, x_2 = -109, x_3 = -17$

SECTION 4.5

1. (a) T (b) T (c) T (d) F (e) F (f) T
   (g) T (h) F (i) T (j) T (k) T
2. (a) 22 (c) $2 - 4i$
3. (a) -12 (c) 22 (e) -3
4. (a) 88 (c) -6 (e) $17 - 3i$ (g) $24 + 24i$

SECTION 5.1

1. (a) F (b) T (c) T (d) F (e) F (f) F
(g) F (h) T (i) T (j) F (k) F

2. (a) $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $[\mathsf{L}_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$

3. (a) The eigenvalues are 4 and -1, and a basis of eigenvectors is
$$\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}. \qquad Q = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}.$$
(c) The eigenvalues are 1 and -1, and a basis of eigenvectors is
$$\left\{\begin{pmatrix} 1 \\ 1-i \end{pmatrix}, \begin{pmatrix} 1 \\ -1-i \end{pmatrix}\right\}. \qquad Q = \begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}.$$

4. The eigenvalues are 1, 2, and 3, and a basis of eigenvectors is $\{1, x, x^2\}$.

SECTION 5.2

1. (a) F (b) F (c) T (d) F (e) F (f) T
(g) T (h) F (i) T

2. (a) Not diagonalizable. (c) $Q = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}$
(e) Not diagonalizable. (g) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

3. (a) Not diagonalizable. (c) Not diagonalizable.

9. $$A^n = \begin{pmatrix} \dfrac{5^n}{3} + \dfrac{2(-1)^n}{3} & \dfrac{2(5^n)}{3} - \dfrac{2(-1)^n}{3} \\[2ex] \dfrac{5^n}{3} - \dfrac{(-1)^n}{3} & \dfrac{2(5^n)}{3} + \dfrac{(-1)^n}{3} \end{pmatrix}$$

16. $X(t) = c_1 e^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

SECTION 5.3

1. (a) T (b) T (c) F (d) F (e) T (f) T
(g) T (h) F (i) F (j) T

2. (a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{7}{13} & \frac{7}{13} \\ \frac{6}{13} & \frac{6}{13} \end{pmatrix}$ (e) No limit exists.

(g) $\begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -2 \\ 2 & 0 & 2 \end{pmatrix}$ (i) No limit exists.

6. One month after arrival 25% of the patients have recovered, 20% are ambulatory, 41% are bedridden, and 14% have died; eventually $\frac{59}{90}$ recover and $\frac{31}{90}$ die.

7. $\frac{4}{7}$

8. Only the matrices in parts (a) and (b) are regular transition matrices.

9. (a) $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ (c) No limit exists.

(e) $\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix}$

10. (a) $\begin{pmatrix} 0.225 \\ 0.441 \\ 0.334 \end{pmatrix}$ after two stages and $\begin{pmatrix} 0.20 \\ 0.60 \\ 0.20 \end{pmatrix}$ eventually.

(c) $\begin{pmatrix} 0.368 \\ 0.350 \\ 0.282 \end{pmatrix}$ after two stages and $\begin{pmatrix} 0.50 \\ 0.20 \\ 0.30 \end{pmatrix}$ eventually.

(e) $\begin{pmatrix} 0.329 \\ 0.334 \\ 0.337 \end{pmatrix}$ after two stages and $\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$ eventually.

12. $\frac{9}{19}$ new, $\frac{6}{19}$ once-used, and $\frac{4}{19}$ twice-used.

13. In 1985, 24% will own large cars, 34% will own intermediate-sized cars, and 42% will own small cars; the corresponding eventual proportions are 0.10, 0.30, and 0.60.

18. $e^O = I$ and $e^I = eI$.

SECTION 5.4

1. (a) F (b) T (c) T

2. The subspaces in parts (a), (c), and (d) are T-invariant.

SECTION 5.5

1. (a) F (b) F (c) T (d) T (e) T
2. (a) $\{e_1, e_2, e_3, e_4\}$ (c) $\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$
3. (a) (i) $-t(1-t)(t^2-3t+3)$ (ii) $-t(1-t)(t^2-3t+3)$
   (c) (i) $1-t$ (ii) $(t-1)^3(t+1)$

SECTION 5.6

1. (a) F (b) T (c) F (d) F (e) T (f) F
   (g) F (h) T
2. (a) $(t-1)(t-3)$ (c) $(t-1)^2(t-2)$
3. (a) $(t-2)^3$ (c) $(t-1)(t+1)$
5. The diagonalizable operators on R^2 satisfying $\mathsf{T}^3 - 2\mathsf{T}^2 + \mathsf{T} = \mathsf{T}_0$ are T_0, I, and those operators having both 0 and 1 as eigenvalues.

SECTION 6.1

1. (a) T (b) F (c) F (d) T (e) F (f) F
   (g) T (h) T
2. (a) For $\lambda = 2$,

$$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

is a basis for E_λ; any basis for R^2 is a basis for K_λ.

(b) For $\lambda = -1$,

$$\left\{\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}\right\}$$

is a basis for both E_λ and K_λ.

For $\lambda = 2$,

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

is a basis for E_λ and

$$\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

is a basis for K_λ.

SECTION 6.2

1. (a) T (b) T (c) F (d) T (e) T (f) T
(g) F (h) T

2. $$\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}\oplus\begin{pmatrix}2&1\\0&2\end{pmatrix}\oplus(2)\oplus\begin{pmatrix}4&1&0\\0&4&1\\0&0&4\end{pmatrix}\oplus(4)\oplus(-3)\oplus(-3)$$

3. (a) $-(t-2)^5(t-3)^2$
(b) For $\lambda_1 = 2$ For $\lambda_2 = 3$

$$\begin{array}{cc}\bullet&\bullet\\\bullet&\bullet\\\bullet&\end{array}\qquad\qquad\begin{array}{cc}\bullet&\bullet\end{array}$$

(c) $\lambda_2 = 3$
(d) $p_1 = 3$ and $p_2 = 1$
(e) (i) $\operatorname{rank}(\mathsf{U}_1) = 3$ and $\operatorname{rank}(\mathsf{U}_2) = 0$
(ii) $\operatorname{rank}(\mathsf{U}_1^2) = 1$ and $\operatorname{rank}(\mathsf{U}_2^2) = 0$
(iii) $\operatorname{nullity}(\mathsf{U}_1) = 2$ and $\operatorname{nullity}(\mathsf{U}_2) = 2$
(iv) $\operatorname{nullity}(\mathsf{U}_1^2) = 4$ and $\operatorname{nullity}(\mathsf{U}_2^2) = 2$

4. (a)
$$J=\left(\begin{array}{c|cc}1&0&0\\\hline 0&2&1\\0&0&2\end{array}\right)\quad\text{and}\quad Q=\begin{pmatrix}1&1&1\\2&1&2\\1&-1&0\end{pmatrix}$$

(d)
$$J=\left(\begin{array}{cc|cc}0&1&0&0\\0&0&0&0\\\hline 0&0&2&0\\0&0&0&2\end{array}\right)\quad\text{and}\quad Q=\begin{pmatrix}1&0&1&-1\\1&-1&0&1\\1&-2&0&1\\1&0&1&0\end{pmatrix}$$

6. The Jordan canonical form is

$$\left(\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&0&1&0\\\hline 0&0&0&2\end{array}\right),$$

and a Jordan canonical basis is $\{2e^x, 2xe^x, x^2e^x, e^{2x}\}$.

SECTION 6.3

1. (a) T (b) F (c) F (d) T (e) F (f) T
   (g) F
2. (a) $\begin{pmatrix} 0 & 0 & 27 \\ 1 & 0 & -27 \\ 0 & 1 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

   (c) $\begin{pmatrix} \frac{1}{2}(-1+i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(-1-i\sqrt{3}) \end{pmatrix}$ (e) $\left(\begin{array}{cc|cc} 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right)$

SECTION 7.1

1. (a) T (b) T (c) F (d) F (e) F (f) F
   (g) T (h) F (i) T
2. $(x, y) = 8 + 5i$, $\|x\| = \sqrt{7}$, $\|y\| = \sqrt{14}$, and $\|x + y\|^2 = 37$.
3. $(f, g) = 1$, $\|f\| = \dfrac{\sqrt{3}}{3}$, $\|g\| = \sqrt{\dfrac{e^2 - 1}{2}}$,

   and $\|f + g\| = \sqrt{\dfrac{11 + 3e^2}{6}}$.

SECTION 7.2

1. (a) F (b) T (c) T (d) F (e) T (f) T
2. (b) The orthonormal basis is

$$\left\{\frac{\sqrt{3}}{3}(1, 1, 1), \frac{\sqrt{6}}{6}(-2, 1, 1), \frac{\sqrt{2}}{2}(0, -1, 1)\right\}.$$

   The Fourier coefficients are $2\sqrt{3}/3$, $-\sqrt{6}/6$, and $\sqrt{2}/2$.

   (c) The orthonormal basis is $\{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\}$.
   The Fourier coefficients are $3/2$, $\sqrt{3}/6$, and 0.
4. $S^\perp = \operatorname{span}\{(i, \frac{1}{2}(1 - i), -1)\}$
5. In the first case, $S^\perp$ is the plane through the origin that is perpendicular to x_0; in the second case, $S^\perp$ is the line through the origin that is perpendicular to the plane containing x_1 and x_2.

SECTION 7.3

1. (a) T (b) F (c) F (d) T (e) F (f) T
   (g) T
2. (a) $y = (1, -2, 4)$ (c) $y = 210x^2 - 204x + 33$
3. (a) $\mathsf{T}^*(x) = (11, -12)$ (c) $\mathsf{T}(f(x)) = 69x^2 - 9x - 5$
14. $\mathsf{T}^*(x) = (x, z)y$

SECTION 7.5

1. (a) T (b) F (c) F (d) T (e) T (f) T
   (g) F (h) T
2. (a) T is self-adjoint; the orthonormal basis is $\left\{\frac{1}{\sqrt{5}}(1, -2), \frac{1}{\sqrt{5}}(2, 1)\right\}$.
   (b) T is normal but not self-adjoint.
   (c) T is not normal.

SECTION 7.6

1. (a) F (b) T (c) T (d) F (e) F
2. (a) $\sqrt{18}$ (c) 2
4. (a) $\|A\| \approx 84.74$, $\|A^{-1}\| \approx 17.01$, and $\operatorname{cond}(A) \approx 1441$
   (b) $\|\tilde{x} - A^{-1}b\| \leq \|A^{-1}\| \cdot \|A\tilde{x} - b\| \approx .17$ and
$$\frac{\|\tilde{x} - A^{-1}b\|}{\|A^{-1}b\|} \leq \operatorname{cond}(A)\frac{\|b - A\tilde{x}\|}{\|b\|} \approx \frac{14.41}{\|b\|}$$
5. $0.001 \leq \dfrac{\|x - \tilde{x}\|}{\|x\|} \leq 10$
6. $$R\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \frac{9}{7}, \qquad \|B\| = 2, \quad \text{and} \quad \operatorname{cond}(B) = 2$$

SECTION 7.7

1. (a) T (b) F (c) F (d) T (e) F (f) T
   (g) F (h) F (i) F
2. (a) $$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

(d)
$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

4. T_z is normal for all $z \in C$; T_z is self-adjoint if and only if $z \in R$; T_z is unitary if and only if $|z| = 1$.
5. Only the pair of matrices in part (d) is unitarily equivalent.

21. (c)
$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\sqrt{6}}{6} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{6} \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{3} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{2}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{pmatrix}.$$

(e) $x_1 = 3, x_2 = -5, x_3 = 4$

22. (a) $x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$ and $y = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$

The new quadratic form is $3(x')^2 - (y')^2$.

SECTION 7.8

1. (a) F (b) T (c) T (d) F (e) T (f) F
(g) F (h) T (i) T (j) F

3. $\left\{t\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}: \quad t \in R\right\}$

4. (b) $\left\{t\begin{pmatrix} 1 \\ 0 \end{pmatrix}: \quad t \in R\right\}$ if $\phi = 0$ and $\left\{t\begin{pmatrix} \cos\phi - 1 \\ \sin\phi \end{pmatrix}: \quad t \in R\right\}$ if $\phi \neq 0$

7. (c) There are six possibilities:
(i) Any line through the origin if $\phi = \psi = 0$.
(ii) $\left\{t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}: \quad t \in R\right\}$ if $\phi = 0$ and $\psi = \pi$.
(iii) $\left\{t\begin{pmatrix} \cos\psi + 1 \\ -\sin\psi \\ 0 \end{pmatrix}: \quad t \in R\right\}$ if $\phi = \pi$ and $\psi \neq \pi$.

(iv) $\left\{t\begin{pmatrix}0\\ \cos\phi - 1\\ \sin\phi\end{pmatrix}:\ t \in R\right\}$ if $\psi = \pi$ and $\phi \neq \pi$.

(v) $\left\{t\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}:\ t \in R\right\}$ if $\phi = \psi = \pi$.

(vi) $\left\{t\begin{pmatrix}\sin\phi\,(\cos\psi + 1)\\ -\sin\phi\sin\psi\\ \sin\psi\,(\cos\phi + 1)\end{pmatrix}:\ t \in R\right\}$ otherwise.

SECTION 7.9

1. (a) F (b) T (c) T (d) T (e) T (f) F
2. For $\mathsf{W} = \text{span}(\{(1, 2)\})$, $[\mathsf{T}]_\beta = \begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{4}{5}\end{pmatrix}$.
3. (ii) (a) $\mathsf{T}_1(a, b) = \frac{1}{2}(a + b, a + b)$ and $\mathsf{T}_2(a, b) = \frac{1}{2}(a - b, -a + b)$
   (d) $\mathsf{T}_1(a, b, c) = \frac{1}{3}(2a - b - c, -a + 2b - c, -a - b + 2c)$ and $\mathsf{T}_2(a, b, c) = \frac{1}{3}(a + b + c, a + b + c, a + b + c)$

SECTION 7.10

2. The parabola is $y = \dfrac{t^2}{3} - \dfrac{4t}{3} + 2$, and the error is 0.
3. $x = \frac{2}{7}$, $y = \frac{3}{7}$, $z = \frac{1}{7}$

SECTION 7.11

1. (a) F (b) F (c) T (d) F (e) T (f) F
   (g) F (h) F (i) T (j) F
4. (a) Yes. (b) No. (c) No. (d) Yes. (e) Yes.
   (f) No.
5. (a) $\begin{pmatrix}0 & 2 & -2\\ 2 & 0 & -2\\ 1 & 1 & 0\end{pmatrix}$ (b) $\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}$ (c) $\begin{pmatrix}0 & 0 & 0 & 0\\ -1 & 0 & -4 & 0\\ 0 & 0 & 0 & 0\\ -2 & 0 & -8 & 0\end{pmatrix}$

22. (a) $\left\{\begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\right\}$

(b) Same as (a).

(c) $\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right\}$

23. Same as 22(c).

33. (a) $Q = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -7 \end{pmatrix}$

(b) $Q = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

(c) $Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -0.25 \\ 1 & 0 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6.75 \end{pmatrix}$

list of frequently used symbols

index of theorems

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