Practice problem 1

October 11, 2025

Problem 1. If $\{v_1, \ldots, v_k\}$ is linearly independent and each $v_i \in \mathbb{R}^n$, what is the maximum possible value of k?

Problem 2. Suppose $v_1, v_2, v_3 \in \mathbb{R}^4$ are such that $\operatorname{span}(v_1, v_2, v_3) = \mathbb{R}^4$. Is this possible? Explain.

Problem 3. If $\{v_1, v_2, v_3\}$ is a linearly dependent set in a vector space V, what can you say about $\dim \operatorname{span}(v_1, v_2, v_3)$?

Problem 4. Suppose $v_1, \ldots, v_n$ is a basis for a vector space V. Let $w \in V$. Under what conditions is the set $\{v_1 + w, \ldots, v_n + w\}$ linearly dependent? Prove your claim.

Problem 5. (i) Show that $\sin x$ and $\sin 2x$ are linearly independent in $C(\mathbb{R})$.

(ii) Are the functions $\cos x$ and $\cos^2 x$ linearly independent in $C(\mathbb{R})$? Justify your answer.

Problem 6. Suppose V is finite-dimensional and $\dim \dim V > 1$. Prove that the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V, V)$.

Problem 7. If $v_1 + W, \ldots, v_m + W$ are linearly independent in V/W then $v_1, \ldots, v_m$ are linearly independent in V.

Problem 8. If $\varphi, \psi \in V^*$ are nonzero and $\ker \varphi = \ker \psi$. Show that φ, ψ are linearly dependent.

Problem 9. Show that $U+W$ is a direct sum if and only if there is an element $v \in U+W$ such that the decomposition $v = u + w$ for $u \in U$, $w \in W$ is unique.

Problem 10. Let V be finite dimensional and $\varphi_1, \ldots, \varphi_m \in V^*$. Show that the following are equivalent:

1. $\varphi_1, \ldots, \varphi_m$ are linearly independent
2. $\bigcap_{i=1}^{m} \ker \varphi_i$ has dimension $\dim V - m$
3. There exists a basis $v_1, \ldots, v_n$ of V such that $\varphi_i(v_j) = \delta_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$

Problem 11. Suppose that $v, x \in V$ and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Problem 12. Suppose $\dim \operatorname{span}(v_1, \ldots, v_n) \geq n$. Show that $v_1, \ldots, v_n$ is linearly independent.

Problem 13. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. If $k > n$, show that the set $v_1, \ldots, v_k$ is linearly dependent.

Problem 14. Let $T : V \to W$, $S : W \to U$ be linear transformations.
(i) Show that $\operatorname{im}(ST) \subset \operatorname{im} S$ and $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.
(ii)Show that $\operatorname{im}(T^t S^t) \subset \operatorname{im} T^t$ and $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$.
(iii) If $R : U \to X$ is linear. Show that $\operatorname{rank}(RST) \leq \operatorname{rank}(S)$.

Problem 15. We have seen in Lecture 1 that the set of all symmetric matrices $V = \{A \in M_{n\times n}(\mathbb{R}) : A^t = A\}$ and anti-symmetric matrices $W = \{A \in M_{n\times n}(\mathbb{R}) : A^t = -A\}$ are subspaces of $M_{n\times n}(\mathbb{R})$.
(i) Find a basis of V and $\dim V$.
(ii) Find a basis of W and $\dim W$.
(iii) Show that $M_{n\times n}(\mathbb{R}) = V \oplus W$. Hint: for any $A \in M_{n\times n}(\mathbb{R})$, write $A = \frac{A+A^t}{2} + \frac{A-A^t}{2}$.

Problem 16. Find a matrix representing the linear transformation
(i)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 \\ x_3 \end{pmatrix}.$$

(ii)

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{pmatrix}.$$

Problem 17. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$Tv_i = w_i \quad \text{for } i = 1, 2, 3,$$

where

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

(i) Find $T\begin{pmatrix}1\\0\\0\end{pmatrix}$.

(2) Find the 3×3 matrix $[T]_\beta^\beta$ with $\beta = \{v_1, v_2, v_3\}$.

Problem 18. True or False: There is a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^2$ with kernel

$$\ker T = \{(v_1, \dots, v_5)^t \in \mathbb{R}^5 : v_1 = 3v_2 \text{ and } v_3 = v_4 = v_5\}.$$

Justify your answer.

Problem 19. Let $D = \frac{d}{dt}$ be the derivative operator. Let V be the real vector space spanned by the basis

$$\mathcal{B} = \{1, t, e^t, e^{2t}, te^{2t}\}.$$

(1) Show that D is a linear map from V into itself.

(2) Find the matrix M associated with D relative to the basis $\mathcal{B}$.

(3) Determine the rank of D.

Problem 20. Let $T : V \to V$ be a linear transformation on a finite-dimensional vector space V. Let $\varphi \in V^*$.

(i) Define $i_\varphi : F \to V^*$ by $i_\varphi(c) = c\varphi$ for $c \in F$. Show that $\operatorname{im} i_\varphi = \operatorname{span}(\varphi)$.

(ii) We identify V with V^{**} by the natural isomorphism J and F^* with F by $\ell \mapsto \ell(1)$ for $\ell \in F^*$. Show that under the identification above $(i_\varphi)^t : V \to F$ is given by $(i_\varphi)^t(x) = \varphi(x)$ for $x \in V$.

(iii) If $\ker T^t = \operatorname{span}(\varphi)$, then $\operatorname{im} T = \ker \varphi$.

(iv) If $\operatorname{im} T^t = \operatorname{span}(\varphi)$, then $\ker T = \ker \varphi$.

Hint: For (iii) (iv), you can use (i) (ii) combined with Theorem 6.15 proved in class.

Problem 21. Let $V = \mathbb{R}^4$, and let

$$W = \operatorname{span}\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\} \subseteq V.$$

Find a basis for the annihilator $W^\perp \subseteq V^*$, the set of all linear functionals $\varphi : V \to \mathbb{R}$ such that $\varphi(w) = 0$ for all $w \in W$.

Problem 22. Consider the linear system

$$\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ c \\ 2 \end{pmatrix}.$$

(i) For which values of a, b, c does the system have a unique solution?

(ii) For those values of a, b, c for which it has a non-unique solution, describe all solutions.

Problem 23. Consider the system of linear equations:

$$\begin{cases} x + 3y + 2z + w = a, \\ 2x + 5y + 3z = a, \\ x - z = -b, \\ y + z = b. \end{cases}$$

(i) For what values of a and b does a solution of the system exist?
(ii) If a solution exists, what is its general form?

Problem 24. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \qquad B = A^t = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

(i) Find all $x \in \mathbb{R}^3$ such that $Bx = 0$.

(ii) Does $Ax = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ has a solution? Justify your answer.

(iii) Find all $x, y \in \mathbb{R}^3$ such that $Ax = By$.

Hint: View $Ax - By = 0$ as $[A \mid B] \begin{pmatrix} x \\ -y \end{pmatrix} = 0$. Use Gauss elimination to find solutions $\begin{pmatrix} x \\ -y \end{pmatrix}$.

(iv) Find a basis of $\operatorname{im} A \cap \operatorname{im} B$.

Hint: $\operatorname{im} A \cap \operatorname{im} B$ consists of vectors $z \in \mathbb{R}^3$ such that $z = Ax = By$ for some $x, y \in \mathbb{R}^3$. That is $z = Ax$ for $\begin{pmatrix} x \\ -y \end{pmatrix}$ a solution to $Ax = By$.

(v) Find a basis of $L_A^{-1}(\operatorname{im} B)$.

Hint: $L_A^{-1}(\operatorname{im} B)$ consists of all $x \in \mathbb{R}^3$ such that $Ax = By$ for some $y \in \mathbb{R}^3$. That is all x such that $\begin{pmatrix} x \\ -y \end{pmatrix}$ is a solution to $Ax = By$.