Problem set 7

Problem 1. Is the set of polynomials $p_1(x) = x^3 + 2x, p_2(x) = x^2 + x + 1, p_3(x) = x^3 + 5$ linear independent or dependent in $\mathcal{P}_3(\mathbb{R})$? Justify your answer.

Problem 2. Determine for which values of k the following matrix in $M_{3\times 3}(\mathbb{R})$ is invertible, and find the inverse when it exists:

$$\begin{pmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Problem 3. Let V be an n-dimensional vector space over F. The set V^m consists of elements the form $(v_1, \ldots, v_m)$ for $v_i \in V$, $1 \leq i \leq m$. (You can think of V^m the same as $M_{n\times m}(F)$ where each matrix $A = (v_1, \ldots, v_m) \in M_{n\times m}(F)$ is though of as m columns of vectors $v_i \in F^n$.)

An **alternating m-linear form** (or simply **m-form**) is a function $\omega : V^m \to F$ satisfying

1. ω is multilinear: If for some $1 \leq j \leq n$, $v_j = ax + by$ where $a, b \in F$ $x, y \in F^n$, then

$$\omega(v_1, \ldots, ax + by, \ldots, v_m) = a\omega(v_1, \ldots, x, \ldots, v_m) + b\omega(v_1, \ldots, y, \ldots, v_m).$$

2. ω alternative: If $v_i = v_j$ for $i \neq j$, then $\omega(v_1, \ldots, v_m) = 0$.

The set of all m-forms on V is denoted by $\bigwedge^m V^*$. It can be shown that this is a vector space over F under natural addition and scalar multiplication. Note that a 1-form is just a linear function on V and so $\bigwedge^1 V^* = V^*$. The determinant is an n-form on F^n.

(i) Show that if $\omega \in \bigwedge^m V^*$ then $\omega(v_1, \ldots, 0, \ldots, v_m) = 0$ and if $m \geq 2$ then for any $\lambda_i \in F$

$$\omega(v_1, \ldots, v_j + \sum_{i\neq j} \lambda_i v_i, \ldots, v_m) = \omega(v_1, \ldots, v_j, \ldots, v_m).$$

(ii) Show that if $\{v_1, \ldots, v_m\}$ are linearly dependent, then $\omega(v_1, \ldots, v_m) = 0$.

(iii) Show that if $m > n = \dim V \geq 1$, then for any $\omega \in \bigwedge^m V^*$ and $v_1, \ldots, v_m \in V$, we have $\omega(v_1, \ldots, v_m) = 0$. In other words, $\bigwedge^m V^* = \{0\}$.

Remark 1. We have seen in class that the determinant is the signed n-dimensional volume. Up to scaling, an m-form is computing the signed m-dimensional volume of the parallelepiped spanned by $(v_1, \ldots, v_m)$ projected to some m-dimensional subspace. For example, consider the dual basis $(\varphi_1, \ldots, \varphi_n)$ of a basis $(v_1, \ldots, v_n)$ of V. Each φ_i is a 1-form and $\varphi_i(v)$ is computing the signed 1-dimensional volume of v projected to v_i, i.e. the component of the projection of v to v_i.

Problem 4. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \qquad B = A^t = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

(i) Find all $x \in \mathbb{R}^3$ such that $Bx = 0$. (You can use the RREF computed in Homework 6 Problem 3.)

(ii) Does $Ax = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ has a solution? Justify your answer.

(iii) Find all $x, y \in \mathbb{R}^3$ such that $Ax = By$.

Hint: View $Ax - By = 0$ as $[A \mid B] \begin{pmatrix} x \\ -y \end{pmatrix} = 0$. Use Gauss elimination to find solutions $\begin{pmatrix} x \\ -y \end{pmatrix}$.

(iv) Find a basis of $\operatorname{im} A \cap \operatorname{im} B$.

Hint: $\operatorname{im} A \cap \operatorname{im} B$ consists of vectors $z \in \mathbb{R}^3$ such that $z = Ax = By$ for some $x, y \in \mathbb{R}^3$. That is $z = Ax$ for $\begin{pmatrix} x \\ -y \end{pmatrix}$ a solution to $Ax = By$.

(v) Find a basis of $L_A^{-1}(\operatorname{im} B)$.

Hint: $L_A^{-1}(\operatorname{im} B)$ consists of all $x \in \mathbb{R}^3$ such that $Ax = By$ for some $y \in \mathbb{R}^3$. That is all x such that $\begin{pmatrix} x \\ -y \end{pmatrix}$ is a solution to $Ax = By$.