

# Minimum Array Partition

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## 1 Problem description

Given a  $n \times n$  array  $A$  of non-negative integers, and a positive integer  $p$ , we need to choose  $p - 1$  horizontal dividers  $0 = h_0 < h_1 < \dots < h_p = n$  and  $p - 1$  vertical dividers  $0 = v_0 < v_1 < \dots < v_p = n$  partitioning  $A$  into  $p^2$  blocks, so as to minimize the following expression:

$$\max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}} \sum_{\substack{v_{i-1} \leq x \leq v_i \\ h_{j-1} \leq y \leq h_j}} A[x, y]$$

## 2 Methods

### 2.1 Brute Force (BF)

Firstly, we write a brute force algorithm for the purpose of finding optimal solutions for small test cases, which we will use to evaluate our optimisation algorithms.

BF generates all possible partitions of the matrix, and chooses the best one of them. This essentially means that BF will need an unreasonable amount of time to solve all but the smallest of testcases.

The complexity of this approach is  $O(n, p) = \binom{n}{p}^2 * p^2$ , indeed  $\binom{n}{p}^2$  is the total number of partitions, while for each of them we need to check  $p^2$  blocks. We find the sum of each block in constant time, by utilising a partial sum DP matrix.

#### 2.1.1 Cutting optimization

A small optimization of the previous approach involves abandoning some partitions early when it is clear that they will not yield a better solution than the current known best.

Consider this: we know the sum  $S$  of the entire matrix  $M$ , and we know there are  $p^2$  blocks in total. According to the Pigeonhole principle, at least one of said blocks will have a sum greater or equal to  $\frac{S}{p^2}$ .

Alternatively, if we have checked  $m < p^2$  blocks and their total sum is  $S_m < S$ , then the maximum value of the remaining blocks will be greater or equal to  $\frac{S - S_m}{m - p^2}$ .

If at any point this sum happens to become greater than the best solution we have found beforehand, we can safely skip checking the rest of the partition in question. In practice, this speeds up the algorithm 3 to 5 times.

## 2.2 Genetic Algorithm (GA)

A genetic algorithm (GA) is a p-metaheuristic inspired by the process of natural selection. Genetic algorithms are commonly used to generate high-quality solutions to optimization and search problems by relying on biologically inspired operators such as mutation, crossover and selection.

We use a nearly textbook GA implementation. Nevertheless, given the specificity of the problem, crossover and mutation implementation deserve a closer look.

### 2.2.1 Crossover

Crossover of two partitions can be done by an arithmetical approach. For simplicity, we will explain the process on a one-dimensional partition. This can be directly applied to two dimensions for the problem in question.

Consider two individuals,  $A$  and  $B$ , with partitions  $[1, 3, 9]$  and  $[2, 5, 7]$ . Their child  $C$  will try to take a middle path in the hopes of bettering their fitness, by taking the average of respective divider pairs:  $C = [\frac{1+2}{2}, \frac{3+5}{2}, \frac{9+7}{2}] = [1.5, 4, 8]$ . Since dividers have to be whole numbers, we fix  $C$  by either rounding the first element down, or rounding it up (chosen randomly).

### 2.2.2 Mutation

Mutation is achieved by trying to make small movements of individual dividers with a given probability to do so, considering there is an available place to move.

## 2.3 Simulated Annealing (SA)

Simulated Annealing is a probabilistic optimization method inspired by annealing in metallurgy. It's effective for combinatorial optimization problems with large search spaces. SA starts with an initial solution and uses a temperature schedule to accept worse solutions with a probability.

During the SA process, small changes are introduced through random movements of block borders, allowing for exploration of the solution space.

## 2.4 Variable Neighbourhood Search (VNS)

Variable Neighborhood Search (VNS) is an optimization technique that iteratively explores different neighborhoods of solutions. It begins with an initial solution and systematically varies the neighborhood structure to search for better solutions.

During the VNS process, changes are introduced through random movements of (multiple) block borders.

### 3 Method evaluation

#### 3.1 Test generation

We wrote a helper script for generating testcases. The script takes 3 command line arguments:  $n$ ,  $p$ , and *type*; where *type* refers to how the matrix  $M$  is generated:

**Ones** fills  $M$  with ones, mainly used for debugging

**Random** fills the matrix with numbers from  $(1, n/2)$

**Linear** ,  $M_{i,j} = i + j + 1$

**Squared** ,  $M_{i,j} = i^2 + j^2 + 1$

For the purpose of evaluating our optimization algorithms, we created 3 sets (random, linear, and squared) of 4 testcases of different dimensions. The dimensions can be categorized as:

**Tiny**  $n = 50, p = 3$  (BF friendly)

**Small**  $n = 80, p = 6$

**Medium**  $n = 100, p = 4$

**Large**  $n = 500, p = 10$

Each optimization algorithm is run 5 times for every testcase, and we record the best, worst, and average results and execution times.

We only test BF on tiny testcases to compare the results from optimization algorithms. We run BF once per testcase, since there is no randomness involved.

#### 3.2 Results

Let us begin with testcases representing the *random* type:

```
Testcase: tests/random_n50_p3.in
BRUTE FORCE
  Result    : 3769
  Time (s)  : 2.46
GENETIC ALGORITHM (GA) [pop_size=75, num_iters=450, elitism_size=15]
  Results   : BEST=3769      WORST=3769      AVG=3769.00
  Time (s)  : BEST=2.11     WORST=2.28     AVG=2.20
SIMULATED ANNEALING (SA)
  Results   : BEST=3769      WORST=3769      AVG=3769.00
  Time (s)  : BEST=0.28     WORST=0.29     AVG=0.28
```

```
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=2500]
  Results   : BEST=3769      WORST=3769      AVG=3769.00
  Time (s)  : BEST=0.46      WORST=0.48      AVG=0.47
```

Testcase: tests/random\_n80\_p6.in

```
BRUTE FORCE - skipping due to speed
GENETIC ALGORITHM (GA) [pop_size=100, num_iters=600, elitism_size=20]
  Results   : BEST=3964      WORST=4072      AVG=3985.60
  Time (s)  : BEST=7.86      WORST=8.07      AVG=7.94
SIMULATED ANNEALING (SA)
  Results   : BEST=3964      WORST=3964      AVG=3964.00
  Time (s)  : BEST=2.18      WORST=2.27      AVG=2.21
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=6400]
  Results   : BEST=3964      WORST=3964      AVG=3964.00
  Time (s)  : BEST=8.26      WORST=8.45      AVG=8.35
```

Testcase: tests/random\_n100\_p4.in

```
BRUTE FORCE - skipping due to speed
GENETIC ALGORITHM (GA) [pop_size=100, num_iters=600, elitism_size=20]
  Results   : BEST=17061     WORST=17061     AVG=17061.00
  Time (s)  : BEST=5.06      WORST=5.22      AVG=5.11
SIMULATED ANNEALING (SA)
  Results   : BEST=17061     WORST=17061     AVG=17061.00
  Time (s)  : BEST=0.68      WORST=0.68      AVG=0.68
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=10000]
  Results   : BEST=17061     WORST=17061     AVG=17061.00
  Time (s)  : BEST=4.75      WORST=4.85      AVG=4.80
```

Testcase: tests/random\_n500\_p10.in

```
BRUTE FORCE - skipping due to speed
GENETIC ALGORITHM (GA) [pop_size=100, num_iters=600, elitism_size=20]
  Results   : BEST=326181    WORST=338514    AVG=330243.80
  Time (s)  : BEST=24.44     WORST=25.95     AVG=24.95
SIMULATED ANNEALING (SA)
  Results   : BEST=360022    WORST=552450    AVG=424930.40
  Time (s)  : BEST=9.21      WORST=10.39     AVG=9.89
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=10000]
  Results   : BEST=323192    WORST=417608    AVG=362644.00
  Time (s)  : BEST=24.83     WORST=28.67     AVG=26.29
```

For this first type of matrices, we can see that SA tends to find optimal (or close to optimal) solutions at a fraction of the time for tiny, small, and medium tests. However, SA falls behind on the large test case, displaying a big difference between its best (360022) and worst (552450) runs.

Both GA and VNS seem clearly better than SA, however our genetic approach has the better average result across all 5 runs, indicating it is much less reliant on chance to find a good

solution.

Next, let us compare the results achieved on the *linear* test type.

Testcase: tests/linear\_n50\_p3.in

BRUTE FORCE

Result : 15080

Time (s) : 3.18

GENETIC ALGORITHM (GA) [pop\_size=75, num\_iters=450, elitism\_size=15]

Results : BEST=15080 WORST=15318 AVG=15127.60

Time (s) : BEST=2.15 WORST=2.24 AVG=2.19

SIMULATED ANNEALING (SA)

Results : BEST=15080 WORST=15080 AVG=15080.00

Time (s) : BEST=0.28 WORST=0.30 AVG=0.29

VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num\_iters=2500]

Results : BEST=15080 WORST=15318 AVG=15270.40

Time (s) : BEST=0.40 WORST=0.46 AVG=0.42

Testcase: tests/linear\_n80\_p6.in

BRUTE FORCE - skipping due to speed

GENETIC ALGORITHM (GA) [pop\_size=100, num\_iters=600, elitism\_size=20]

Results : BEST=16055 WORST=16055 AVG=16055.00

Time (s) : BEST=7.88 WORST=8.10 AVG=7.98

SIMULATED ANNEALING (SA)

Results : BEST=15936 WORST=16055 AVG=15983.60

Time (s) : BEST=2.23 WORST=2.42 AVG=2.33

VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num\_iters=6400]

Results : BEST=16055 WORST=16055 AVG=16055.00

Time (s) : BEST=8.13 WORST=8.32 AVG=8.24

Testcase: tests/linear\_n100\_p4.in

BRUTE FORCE - skipping due to speed

GENETIC ALGORITHM (GA) [pop\_size=100, num\_iters=600, elitism\_size=20]

Results : BEST=67760 WORST=68355 AVG=67879.00

Time (s) : BEST=5.16 WORST=5.34 AVG=5.23

SIMULATED ANNEALING (SA)

Results : BEST=67760 WORST=68355 AVG=67879.00

Time (s) : BEST=0.71 WORST=0.71 AVG=0.71

VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num\_iters=10000]

Results : BEST=67760 WORST=72500 AVG=69065.00

Time (s) : BEST=4.03 WORST=4.87 AVG=4.38

Testcase: tests/linear\_n500\_p10.in

BRUTE FORCE - skipping due to speed

GENETIC ALGORITHM (GA) [pop\_size=100, num\_iters=600, elitism\_size=20]

Results : BEST=1371552 WORST=1621573 AVG=1454183.80

Time (s) : BEST=23.79 WORST=26.55 AVG=24.81

```

SIMULATED ANNEALING (SA)
  Results   : BEST=1383025   WORST=2073344   AVG=1565182.40
  Time (s)  : BEST=9.15     WORST=11.14    AVG=9.76
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=10000]
  Results   : BEST=1378620   WORST=2168270   AVG=1566251.40
  Time (s)  : BEST=21.42    WORST=29.76    AVG=25.80

```

Here the story is very similar to what we observed with the previous test type on smaller test cases. Once again SA is able to find a competitive solution in a fraction of the time. In fact, it even beats GA and VNS on the small test, albeit only by a small margin.

Once again, SA has a big difference between its best and worst runs on the large test, similarly to VNS. GA displays the best performance, and the lowest dependance on chance.

The last group of test cases belong to the *squared* group:

```

Testcase: tests/squared_n50_p3.in
BRUTE FORCE
  Result    : 520720
  Time (s)  : 4.03
GENETIC ALGORITHM (GA) [pop_size=75, num_iters=450, elitism_size=15]
  Results   : BEST=520720   WORST=520720   AVG=520720.00
  Time (s)  : BEST=2.13     WORST=2.22     AVG=2.17
SIMULATED ANNEALING (SA)
  Results   : BEST=520720   WORST=520720   AVG=520720.00
  Time (s)  : BEST=0.28     WORST=0.29     AVG=0.28
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=2500]
  Results   : BEST=520720   WORST=525369   AVG=521649.80
  Time (s)  : BEST=0.47     WORST=0.48     AVG=0.47

```

```

Testcase: tests/squared_n80_p6.in
BRUTE FORCE - skipping due to speed
GENETIC ALGORITHM (GA) [pop_size=100, num_iters=600, elitism_size=20]
  Results   : BEST=902840   WORST=968898   AVG=916051.60
  Time (s)  : BEST=8.06     WORST=8.19     AVG=8.12
SIMULATED ANNEALING (SA)
  Results   : BEST=901215   WORST=922944   AVG=905560.80
  Time (s)  : BEST=2.25     WORST=2.35     AVG=2.30
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=6400]
  Results   : BEST=902840   WORST=902840   AVG=902840.00
  Time (s)  : BEST=7.94     WORST=8.32     AVG=8.13

```

```

Testcase: tests/squared_n100_p4.in
BRUTE FORCE - skipping due to speed
GENETIC ALGORITHM (GA) [pop_size=100, num_iters=600, elitism_size=20]
  Results   : BEST=4808345   WORST=4808345   AVG=4808345.00
  Time (s)  : BEST=5.10     WORST=5.16     AVG=5.14
SIMULATED ANNEALING (SA)

```

```

Results   : BEST=4807152    WORST=4807152    AVG=4807152.00
Time (s)  : BEST=0.65      WORST=0.70      AVG=0.67
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=10000]
Results   : BEST=4808345    WORST=5033479    AVG=4853371.80
Time (s)  : BEST=4.16      WORST=4.76      AVG=4.61

Testcase: tests/squared_n500_p10.in
BRUTE FORCE - skipping due to speed
GENETIC ALGORITHM (GA) [pop_size=100, num_iters=600, elitism_size=20]
Results   : BEST=522518392  WORST=619537980  AVG=571390189.00
Time (s)  : BEST=23.39     WORST=26.13     AVG=25.28
SIMULATED ANNEALING (SA)
Results   : BEST=533205270  WORST=1038957839 AVG=680165623.40
Time (s)  : BEST=9.39      WORST=11.87     AVG=10.50
VARIABLE NEIGHBOURHOOD SEARCH (VNS) [num_iters=10000]
Results   : BEST=551863221  WORST=851774417  AVG=660668487.00
Time (s)  : BEST=19.00     WORST=25.25     AVG=22.11

```

We find the results for *squared* test the most interesting.

The results are very similar to the previous group when we look at the smaller test dimensions.

Interestingly, GA had one particularly bad run on the small squared test (968898 compared to its 902840 best). Looking at the average (916051.60), we conclude that only 1 out of 5 runs exhibited such behavior. Despite displaying the smallest reliance on chance before, it is important to note that all optimization algorithms do still depend on random number generation, and as such can be prone to individual bad performances.

The largest test once again exhibits the results we have observed beforehand, with GA having the smallest result interval, and the lowest average by a significant margin.

## 4 Failed ideas

### 4.1 Linear Programming

We initially had an idea to solve the problem using linear programming, as it seemed easy to define all the conditions that needed to hold true. The problem arose when defining the cost function, as it required block borders as matrix indices, and it appears that it's not possible to index the array with LP variables.

## 5 Conclusion

In conclusion, the Minimum Array Partition problem has been established as NP-hard, making it challenging to find optimal solutions efficiently. While there exist polynomial algorithms that can provide approximations within a constant factor, their implementation complexity often outweighs the benefits (our approximations factor is better).

As demonstrated in our study, optimization algorithms can be successfully used to solve the Minimum Array Partition problem, with genetic approach displaying the most consistent results, followed closely by VNS. SA remains as a good alternative for when we need to find a decent solution quickly.

These methods strike a balance between computational efficiency and solution quality, making them valuable tools for addressing the Minimum Array Partition problem.

## 6 References

Efficient array partitioning - Sanjeev Khanna, S. Muthukrishnan & Steven Skiena  
On the complexity of the generalized block distribution - Michelangelo Grigni & Fredrik Manne