Context Free Grammars

Context Free Grammars (CFGs)

Introduction

> A Context-Free Grammar (CFG) is a powerful descriptive tool that can define languages that Finite State Machines (FSMs) (or Finite Automata) and Regular Expressions can't express. CFGs are a way to represent the syntax of programming languages, natural languages, and more.

Basic Definitions

Context-Free Grammar

> A CFG is defined by a 4-tuple $G = (V, \Sigma, R, S)$ where:

1. V is the finite set of variables.
2. Σ is the finite set of terminal symbols, disjoint from V.
3. R is the finite set of rules. Each rule is defined as:
   - The left side is a variable.
   - The right side is a string consisting of variables and terminals.
4. S is the start variable, $S \in V$.

Example: Grammar G_1

The given example G_1 can be written as:

$$\begin{aligned} A &\rightarrow 0A1 \\ A &\rightarrow B \\ B &\rightarrow \# \end{aligned}$$

- $Sigma = \{0, 1, \#\}$
- Variables $= \{A, B\}$
- A is the start variable.

Derivations

Derivations demonstrate how strings are generated in the language. They show a sequence of rule applications, starting from the start variable, leading to a string of terminal symbols.

- $A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11$

Language Generated by G_1

The language $L(G_1)$ consists of strings like $\{\#, 00\#11, 000\#111, 0000\#1111, \ldots\}$.

Comparing with FSMs and Regular Expressions

1. **FSMs**: Finite State Machines can represent regular languages. All regular languages are also context-free, but not all context-free languages are regular.
2. **Regular Expressions**: These define regular languages and have similar limitations as FSMs.

Example: $L = \{a^n b^n \mid n \geq 1\}$ is not regular but is context-free.

Advanced Concepts

Parse Trees

Parse trees visually represent derivations in a hierarchical structure. A node labeled by a variable A is connected to children that represent the variables and terminals in the rule $A \rightarrow \alpha$.

Chomsky Normal Form (CNF)

A CFG is in Chomsky Normal Form if all its rules are of the form $A \rightarrow BC$ or $A \rightarrow a$, facilitating parsing and analysis.

Exercises

Context-Free Language L_2

$L_2 = \{a^n b^n \mid n \geq 0\}$
L_2 can be defined by a CFG:

$$S \rightarrow aSb \mid \epsilon$$

Arithmetic Expressions

$G_3 = (V, \Sigma, R, \mathrm{Expr})$

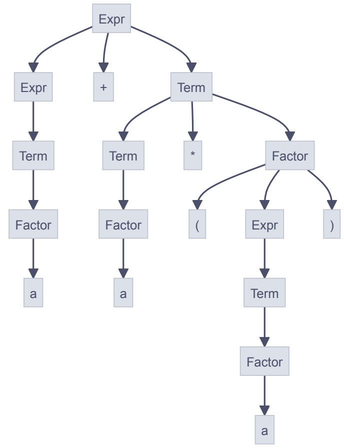
- $V = \{\mathrm{Expr}, \mathrm{Term}, \mathrm{Factor}\}$
- $\Sigma = \{+, \times, a, (,)\}$

Rules

```
Expr --> Expr+Term | Term
Term --> Term*Factor | Factor
Factor --> (Expr) | a
```

Reverse Exercise

$a + a \times (a)$ can be derived in G_3 and thus belongs in $L(G_3)$.



The parse tree represents the step-by-step breakdown of how the string $a + a \times (a)$ is derived from the initial start variable `Expr` in the context-free grammar G_3.

Here's what the tree is showing you:

1. **Start at `Expr1[Expr]`:** The entire expression $a + a \times (a)$ is an `Expr`.
2. **Split into `Expr2`, `+`, `Term1`:** The expression is split into another smaller expression `Expr2`, an addition operation `+`, and a term `Term1`. This corresponds to the rule $\text{Expr} \to \text{Expr} + \text{Term}$.
3. **Drill down `Expr2`:** It further boils down to a `Term` (`Term2`), which then simplifies to a `Factor` (`Factor2`), which then turns out to be the terminal `a`.
4. **Drill down `Term1`:** The term `Term1` also breaks down into a smaller term (`Term3`), a multiplication operation `*`, and another `Factor` (`Factor1`). This corresponds to the rule $\text{Term} \to \text{Term} \times \text{Factor}$.
5. **Drill down `Term3` and `Factor3`:** Both simplify to terminal `a`.
6. **Drill down `Factor1`:** This breaks down into an opening parenthesis `(`, another `Expr` (`Expr3`), and a closing parenthesis `)`. This corresponds to the rule $\text{Factor} \to (\text{Expr})$.
7. **Drill down `Expr3`:** It boils down to a `Term` (`Term4`), which then simplifies to a `Factor` (`Factor4`), which is the terminal `a`.

Each node represents a variable or a terminal from G_3, and each edge represents the application of a grammar rule from G_3. By following the tree from the root to the leaves, you can construct the original string $a + a \times (a)$, thereby proving that this string can be derived from G_3 and therefore belongs in $L(G_3)$.

$\text{input}, \text{stack}$
$a, b \to c$: when the machine reads an a from the input, it replaces the symbol b on the top of the stack with a c.

- if $a = \epsilon : \epsilon, b \to c$: the `machine` *reads* nothing from the `input`, *replaces* b on top of the `stack` with a c.
- if $b = \epsilon : a, \epsilon \to c$: *reads* a from the `input`, does not *pop* the `stack`, *pushes* c into the `stack`.
- if $c = \epsilon : a, b \to \epsilon$: *reads* a, if the top of the `stack` is b, *pops* b, *pushes* nothing