Regular Expressions

#fall2023 #COSC-455

Definition

> R is a regular expression if
>
> 1. for some a in alphabet $\Sigma \rightarrow$ every member of Σ is a regular expression
> 2. ϵ is a regular expression.
> $\epsilon \rightarrow \{\epsilon\}$
> 3. $\emptyset$ is a regular expression
> $\emptyset \rightarrow \{\}$
> 4. $R_1 \cup R_2$, where R_1 and R_2 are regular expressions
> 5. $R_1 \circ R_2$, where R_1 and R_2 are regular expressions
> 6. R_1^*, where R_1 is a regular expression

Arithmetic Expressions

Multiplication

- **Expression**: $3 \times 4 = 12$
  - **Operation**: The symbol $\times$ represents multiplication.

Division

- **Expression**: $10/2 = 5$
  - **Operation**: The symbol $/$ represents division.

Regular Expression Operations

1. Union (Set Union)

- **Definition**: Given two regular languages A and A, their union is $A \cup B = \{w | w \in A \vee w \in B\}$.
  - **Explanation**: A regular language is essentially a set of strings. The union operation combines the sets (A) and (B), including all unique elements from both.

2. Concatenation

- **Definition**: For sets A and B,

$$A \circ B = \{w_1 w_2 | w_1 \in A \wedge w_2 \in B\}$$

  - **Explanation**: Concatenation joins each string w_1 from set A with each string w_2 from set B.

3. Kleene Star

- **Definition**: For set A,

$$A^* = \{x_1 x_2 \ldots x_k | x_1 \in A \wedge k \geq 0\}$$

  - **Explanation**: The Kleene Star operation generates **all** possible strings by **concatenating** zero or more strings from set A.

Additional Notes

- **Value of a Regular Expression**: The value of a regular expression is the language represented by that expression.

Special Cases

- **Zero as a Regular Expression**: (0) is a regular expression representing the set ({0}).
- **One as a Regular Expression**: (1) is a regular expression representing the set $\{1\}$.
  - **Union of 0 and 1**: $0 \cup 1 = \{0, 1\}$

Example

- **Kleene Star of Zero**: $0^* = \{\epsilon, 0, 00, 000, 0000, \ldots\}$
  - **Explanation**: This represents all possible strings formed by concatenating zero or more 0's, including the empty string ϵ.

New Notation and Regular Expression Equivalence

Notation

- Σ^+ represents the set of all non-empty strings over the alphabet Σ.
- Σ^* represents the set of all strings over the alphabet Σ, including the empty string ϵ.

Regular Expression Equivalence

- $\Sigma^+ = \Sigma\Sigma^* \equiv \Sigma^*\Sigma$

This means that Σ^+ can be formed by taking a single symbol from Σ and then appending zero or more symbols from Σ (represented by Σ^*).

The equivalence indicates that you can either start with a single symbol and then append the rest, or start with a string (possibly empty) and then make sure to append at least one symbol from Σ.

Examples

1. Let $\Sigma = \{a, b\}$
    - $\Sigma^+ = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$
    - $\Sigma\Sigma^\star = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$
    - $\Sigma^\star\Sigma = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$
2. Let $\Sigma = \{0, 1\}$
    - $\Sigma^+ = \{0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$
    - $\Sigma\Sigma^\star = \{0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$
    - $\Sigma^\star\Sigma = \{0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$

Interpreting $(01^+)^\star$

The expression $(01^+)^\star$ can be broken down as follows:

- 01^+: This part means a string that starts with '0' and is followed by one or more '1's.
- Examples include '01', '011', '0111', etc.
- $(01^+)^\star$: This means zero or more occurrences of strings that match 01^+.
- Examples include ϵ (the empty string), '01', '011', '01 011 0111', etc.

So, $(01^+)^\star$ captures all the strings that are concatenations of zero or more instances of strings that start with '0' and are followed by one or more '1's.

Alphabet = {0, 1, 2, ..., 9}
$(+ \cup - \cup \epsilon)(D^+ \cup D \circ D^\star \cup D^\star \circ D^+)$
$\circ =$ Decimal point

The expression $(+ \cup - \cup \epsilon)(D^+ \cup D \circ D^\star \cup D^\star \circ D^+)$ is a regular expression that describes a set of strings.
Let's break it down:

Components

- (+\cup-\cup\epsilon): This part represents either a plus sign (+), a minus sign (-), or an empty string (\epsilon).
- (D^+): This represents one or more digits.
- (D\circ D^{\star}): This represents a decimal number less than 1, starting with a digit followed by a decimal point (\circ) and zero or more digits. Examples: (0.1, 0.12, 0.)
- (D^{\star}\circ D^+): This represents a decimal number greater than or equal to 1, starting with zero or more digits, followed by a decimal point (\circ) and one or more digits. Examples: (1.1, .12, 123.)

Combined Expression:

The entire expression $(+ \cup - \cup \epsilon)(D^+ \cup D \circ D^\star \cup D^\star \circ D^+)$ can generate strings that are:

1. Prefixed by $+$, $-$, or nothing (ϵ).
2. Followed by either:
    - One or more digits D^+ (e.g., (+123, -123, 123))
    - A decimal number less than 1 $D \circ D^\star$ (e.g., (+0.1, -0.1, 0.1))
    - A decimal number greater than or equal to 1 $D^\star \circ D^+$ (e.g., $+1.1, -1.1, 1.1$)

Examples

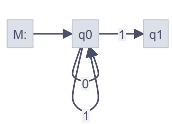
- (+123)
- (-0.12)
- (0.1)
- (123)
- (.12)
- (+1.1)
- (-.12)

This regular expression captures a wide range of numerical values, including positive and negative integers and decimals, with or without a leading sign.

Regular expressions represent regular languages. Regular expressions and finite state machines are equivalent

> R.E $\equiv$ FSM

$\Sigma = \{0, 1\}$



$$L(M) = \text{everything that ends with a 1.}$$

$$\Sigma^\star 1$$

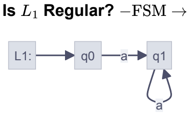
$\Sigma = \{a, b\}$

$$a^\star a \equiv a^+$$

$$L_1 = \{\downarrow a^n | n \geq 1\} = \{a, aa, aaa, \ldots\}$$

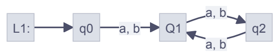
$$\text{a is a sequence of a's only with length at least 1}$$

Is L_1 Regular? $-\text{FSM} \rightarrow$



$\Sigma = \{a, b\}$
$L = \{a^1 b^{2j} | i \geq 1, j \geq 1\}$
$a^+(bb)^+$

Odd



$$\Sigma(\Sigma\Sigma)^\star$$

Design a finite state machine for this language:
$\Sigma = \{a, b\}$
$L = \{a^n b^n | n \geq 1\}$
Is L regular? if no why not.

- The language $L = \{a^n b^n | n \geq 1\}$ is not a regular language. This is because it requires counting the number of a's and ensuring that there are an equal number of b's following them.
- Finite state machines (FSMs) do not have the capability to count or remember past states, so they can't verify that the number of a's and b's are equal.
- To formally prove that L is not regular, you can use the Pumping Lemma for regular languages.
- The Pumping Lemma states that for any regular language L, there exists a constant p such that any string s in L of length at least p can be divided into three parts, $s = xyz$, satisfying the following conditions:
  1. For each $i \geq 0$, the string xy^iz is in L.
  2. $|y| > 0$
  3. $|xy| \leq p$
- For the language L, you can choose a string $s = a^p b^p$.
- According to the Pumping Lemma, s should be able to be split into xyz such that all three conditions hold. However, you'll find that no matter how you split s, you can't satisfy all three conditions, thus proving that L is not a regular language.
- Since L is not regular, it can't be represented by a finite state machine.