Modeling and Implementing Small World Network

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***Abstract*— For the past few years, there has been a growing interest in using the small world concept for optimizing wireless network topology. Even though, there are numerous review articles on the small world based on complex network theory, one might find it too theoretical for application to wireless network optimization research. In this paper we present basic concepts needed to model the small world network and provide implementation issues for designing the network.**

Keywords: complex network, small world network

I. Introduction

The "small-world effect" or "six degrees of separation" principle was first studied by the social psychologist Milgram in [1]. The small-world concept in network based on graph theory was first studied by Watts and Strogatz in [2] where graphs were generated with high clustering coefficient and small path length. Since the discovery of the small world model, the small world model has been actively applied to the communications networks research due to resulting network topology with features such as smaller average transmission delay and more robust network connectivity [3, 4]. The first application of the small world model to the wireless network was done by Helmy in [5]. The small world concept was applied to the wireless network by using spatial graphs, where the distance between the nodes were modeled based on the radio range, rather than the relational graph used in [2]. Based on the study in [5], it was found that by using long range connections, the average path length is drastically reduced in wireless networks as in the original relational graph based small world network. In this paper, we review the basic concepts needed to model the small world network and provide implementation issues for designing the network to help readers interested in applying the small world concept for wireless network topology optimization. For readers who are interested in theoretical review of the complex networks including the small world network, we recommend [6, 7].

II. Basic Concepts

In this section, we present basic concepts needed for modeling small world networks. We briefly discuss very basics of graph theory, present important metrics used to analyze small world networks, and finally describe the algorithm for creating a small world networks.

A. Graph Theory

A complex networks such as computer networks, sensor networks, brain networks and social networks can be represented as a graph. A graph consists of vertices, nodes, or points connected by edges, links, or lines. Mathematically, a graph can be represented by an ordered paired sets $G = \{ V, E \}$, where V is a set of N vertices and E is a set of edges connecting elements of V. Furthermore, the degree K of a vertex is defined as the total number of edges connected to that vertex. Fig. 1 shows a graph consisting of 7 vertices and 7 edges. The graph in Fig. 1 can be described as a network with a set of vertices $V = \{ 1, 2, 3, 4, 5, 6, 7 \}$ and a set of edges $E = \{ (1, 2), (1, 3), (2, 4), (3, 4), (4, 5), (5, 6), (6, 7)\}$.

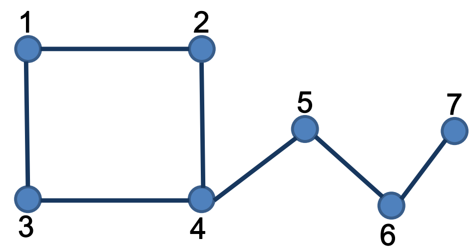

Figure 1. A graph example

B. Important Metrics

Two important metrics for evaluating a small world network are average path length and clustering coefficient. To evaluate the size of a network, the average path length is used and is defined as the average distance between two vertices, averaged over all possible pair of vertices. The distance between a pair of vertices is defined to be the number of minimum edges or hops connecting the two vertices. For example, the distance between the vertex 1 and 5 in the network in Fig.1 is equal to 3. The average path length is determined as follows:

$$L = \binom{N}{2}^{-1} \sum_{i \neq j} l_{ij} \ , \qquad (1)$$

where N is the total number of vertices in the network, l_{ij} is the distance between vertices i and j, and N choose 2 represents all possible number of pair of vertices. Fig. 2 shows a ring of vertices with $N = 4$ and $K = 3$. All possible pairs of vertices for $N = 4$ network are $E = \{ (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) \}$ with 6 pairs of possible edges. As shown in the figure, all the possible connections have been

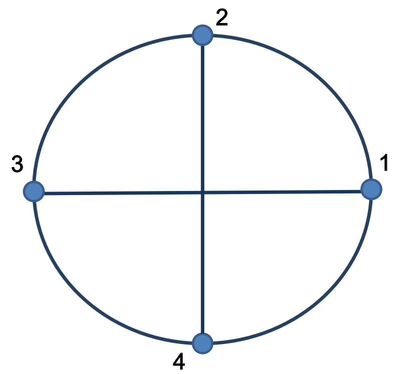

Figure 2. A ring of vertices

implemented in the network. Using (1), the average distance is calculated as $L = (6)^{-1} \times (1 + 1 + 1 + 1 + 1 + 1) = 1$.

The clustering coefficient is defined as the average fraction of pairs of neighbor vertices that are also neighbors of each other. The clustering coefficient measures the cliquishness of a typical friendship circle. The average clustering coefficient averaged over all vertices $i = 1 \ldots N$ is given by

$$C = \frac{1}{N}\sum_{i=1}^{N} C_i , \qquad (2)$$

where C_i is the clustering coefficient for vertex i defined as

$$C_i = \frac{2E_i}{k_i(k_i - 1)}, \qquad (3)$$

where E_i is the actual number of edges connecting the neighbors of vertex i, K_i is the total number of neighbor vertices connected to vertex i, and $K_i\ (K_i - 1)/2$ is the maximum number of possible connections between the neighbor vertices. For the network shown in Fig. 2, for vertex 1, $K_1 = 3$ and $E_1 = 3$, and the maximum number of connections between the neighbor vertices is equal to 3 (3 – 1) / 2 = 3.

C. *Algorithm*

The small world network is constructed by randomly rewiring the edges of a ring lattice with N nodes. The following procedure describes the basic steps of the small world network construction.

Step 1: Start with a ring of N nodes.

Step 2: Connect K nearest nodes for all the nodes $i = 1 \ldots N$.

Step 3: Reconnect the edges to a randomly chosen node from node 1 to N with probability p.

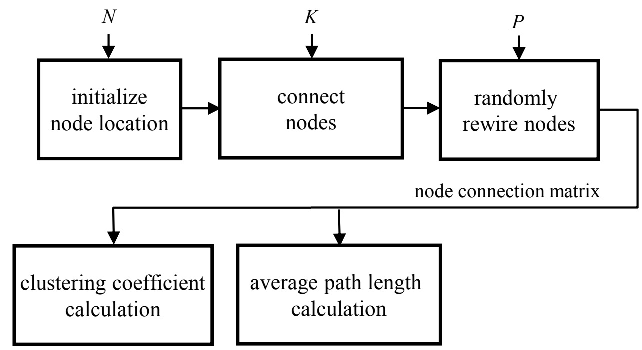

Figure 3. System model

By varying the rewiring probability p, one can analyze the transition of the network from a lattice structure to a random structure with $0 \le p \le 1$.

III. IMPLEMENTATION

In this section, we discuss the design and implementation issues related to modeling the small world networks through system model overview and metric calculation module overview.

A. *System Model*

Fig. 3 shows the system model for implementing a small world network. The system contains five major blocks: node initialization block, node connection block, rewiring block, average path length calculation block, and clustering coefficient calculation block. Parameters N, K, and P corresponds to total number of nodes, initial degree of all the nodes, and rewiring probability, respectively. Furthermore, the node connection matrix gives information about all the node connections after the completion of the rewiring process. Note that the small world network is modeled by a relational graph where the distance is based on edges or hops rather than the absolute distance used in spatial graphs.

B. *Metric Calculation Modules*

1) Average Path Length Calculation: Based on the node connectional matrix, which contains all the connections between the nodes, the number of hops needed to reach a node j from node i needs to be calculated. The first step in this module is to find all possible node pair index (i, j). For all the node pairs indices, we start by checking if there is a direct connection between node i and j. If there is no direct connection, we check for 2 hop connection where node j is connected through an intermediate node. We continue this process with increasing number of hops until all the number of connections for all the node pair indices has been found. Finally, all the number of hops found for all the node pairs are added and divided by the total number of node pairs.

2) Clustering Coefficient Calculation: Based on the node connectional matrix, the total number of neighbor nodes connected to node i is found. Using the number of neighbor nodes found, the maximum number of possible connections

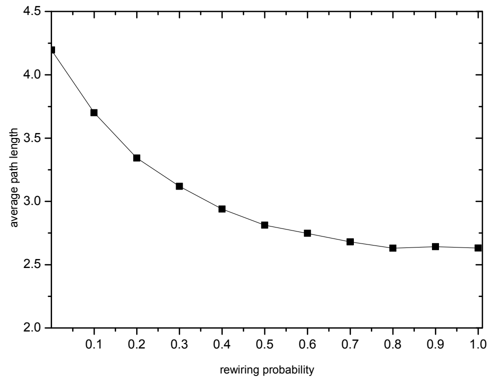

Figure 4. Average path length

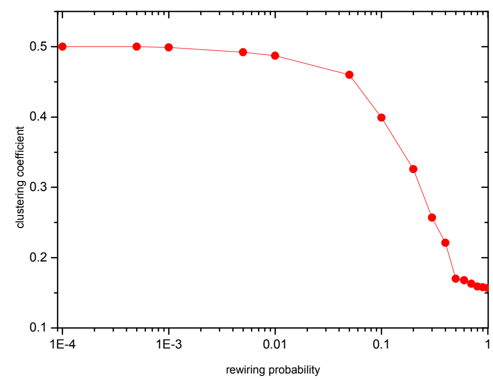

Figure 5. Clustering coefficient

is calculated by $K_i(K_i - 1)/2$. The next step is to find the actual number of edges connecting the neighbors of node i. We continue this process for all the N nodes. Finally, we use (3) to calculate the clustering coefficient for node i using the information found in the previous steps and get the final clustering coefficient using (2).

IV. SIMULATION RESULTS

In this section, we study the behavior of the small world network implemented based on the system architecture with metric calculation modules described in the previous section. We initially assumed a regular ring lattice model with N = 24 nodes and initial degree K = 4 all the nodes. Then the small world network was created according to the system architecture described previously with various rewiring probability p ranging from 0 to 1.

Fig. 4 shows the average path length of the implemented small world network. One could observe that the average path length is around 4.2 for p = 0 (without rewiring) and decreases to 2.6 for high rewiring probability p (random network). Even with small number of random rewiring, there is a drastic decrease in average path length. Note that theoretically the average path length for random network (p = 1) can be calculated as $L \sim ln(N)/ln(K)$.

Fig. 5 shows the clustering coefficient of the implemented small world network. One could observe that the clustering coefficient remains relatively constant with value around 0.5. However, there is rapid drop in the clustering coefficient for rewiring probability p greater than 0.1. Thus, we can observe that the small world network remains highly clustered like regular lattice for p less than 0.1.

From Fig. 4 and Fig. 5, we can conclude that the behavior of the small world network was fully confirmed, having highly clustered behavior as the regular lattice and small average path length as the random graphs, based on the proposed system architecture.

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