Synchronization on Complex Networks of Networks

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***Abstract*— In this paper, pinning synchronization on complex networks of networks is investigated, where there are many subnetworks with the interactions among them. The subnetworks and their connections can be regarded as the nodes and interactions of the networks, respectively, which form the networks of networks. In this new setting, the aim is to design pinning controllers on the chosen nodes of each subnetwork so as to reach synchronization behavior. Some synchronization criteria are established for reaching pinning control on networks of networks. Furthermore, the pinning scheme is designed, which shows that the nodes with very low degrees and large degrees are good candidates for applying pinning controllers. Then, the attack and robustness of the pinning scheme are discussed. Finally, a simulation example is presented to verify the theoretical analysis in this paper.**

***Index Terms*— Algebraic graph theory, complex networks, networks of networks, pinning control.**

I. Introduction

NETWORKS exist everywhere nowadays, such as biological neural networks, ecosystems, social network, metabolic pathways, the Internet, the WWW, and electrical power grids, to name just a few. Each network is typically composed of the nodes representing the individuals in the network and the edges representing the connections among them. In the last decade, small-world and scale-free complex network models were constructively proposed by Watts and Strogatz [33], and Barabási and Albert [2] after, random networks [8]. Thereafter, the study of various complex networks has received increasing attention from researchers in various disciplines, such as physics, mathematics, engineering, biology, and sociology.

Synchronization is one of the most typical collective behaviors in nature, which has received a lot of interests especially since the pioneering work of Pecora and Carroll [20] due to their potential applications in secure communications, chemical reactions, biological systems, and so on. For example, synchronization behavior of neuronal subnetworks are of great interest, where delay effect and transition were discussed [12], [27]. Recently, it is well known that there are large numbers of nodes in real-world complex networks compared with some simple coupled systems. Therefore, much attention has been paid to the study of synchronization in various large-scale complex networks with network topology [1], [7], [17], [19], [22], [28], [29], [34], [37], [38], [40]–[43], [45]. In [21], [28], and [29], local synchronization was investigated by the transverse stability to the synchronization manifold known as master stability function method, and a distance from the collective states to the synchronization manifold was defined in [34] and [37], based on which some results were obtained for global synchronization of complex networks [19], [38], [40], [41], [43], [45].

As we know, it is quite difficult to apply controllers on all the nodes if the network cannot be synchronized by itself. Recently, to save control cost, pinning control has been widely investigated in complex networks [6], [30], [38], [43]–[45]. Then, some controllers may be applied on a small fraction of all the nodes to force the network to synchronize, which is known as pinning control. However, there are several disadvantages for applying such a traditional pinning control, which are listed as follows.

1) Though the pinning controllers are applied on a small fraction of all nodes, which seems to be more reasonable than controlling all the nodes in the network, it is still difficult to control a fraction of nodes using the same information from the centralized leader in a very large-scale network.
2) The virtual leader is very abstract, which is different from the actual leader in the leader-follower or the master-slave setting. In many real systems, there are actually many leaders.
3) The designed pinning control framework is very fragile to the deliberate attacks, for example, if the virtual leader or even one node in the network is attacked, the whole network can be in a mess and cannot reach synchronization with a leader.

To overcome these shortcomings for the original framework in pinning control, in this paper, pinning control on networks of networks are established.

Nowadays, complex networks become very huge including tens of thousands of nodes. In a modern complex network, the network usually can form many clusters, which can be regraded as subnetworks of the network. This comes to the networks of networks where the subnetworks

Manuscript received March 19, 2013; revised December 10, 2013; accepted February 5, 2014. Date of publication February 26, 2014; date of current version October 15, 2014. This work was supported in part by the National Natural Science Foundation of China under Grant 61104145, Grant 61322302, and Grant 61025017, in part by the Natural Science Foundation of Jiangsu Province of China under Grant BK2011581, in part by the Research Fund for the Doctoral Program of Higher Education of China under Grant 20110092120024, in part by the Fundamental Research Funds for the Central Universities of China, in part by the National Science and Technology Major Project of China under Grant 2014ZX10004-001-014, and in part by the 973 Project under Grant 2014CB845302.

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Digital Object Identifier 10.1109/TNNLS.2014.2305443

and their connections can be regarded as the nodes and interactions of the networks, respectively. Recently, the studies in networks of networks have received increasing attention [9], [10]. In particular, phase synchronization of bursting neurons in clustered small-world networks [3] was previously discussed. Cluster synchronization of networks and pinning control for synchronization in networks of networks are different topics even though they have some relationships that both of them study synchronization in networks of networks. Cluster synchronization means that the nodes in each cluster subnetwork can reach synchronization, while pinning control for synchronization focuses on pinning a small fraction of nodes in each subnetwork for achieving synchronization with some several leaders. Some cooperative behaviors on networks of networks were proposed. For example, interdependent network reciprocity in evolutionary games [31], spreading of cooperative behavior across interdependent groups [18], optimal interdependence between the networks for the evolution of cooperation [32], and synchronization transitions in a neuronal network of subnetworks [27].

Based on the new concept networks of networks [9], [10] and the results for cooperative behaviors on networks of networks [18], [27], [31], [32], the main contribution of this paper is that a new framework for pinning synchronization on networks of networks is proposed, which has some advantages as illustrated: 1) there does not exist a centralized leader, from which a fraction of nodes in the whole network can receive the same information and 2) the new framework for pinning synchronization on networks of networks is very robust to the deliberate attacks. For example, even if some leaders and nodes in the subnetworks are attacked, the whole network may still reach synchronization or at least most of the nodes can achieve synchronization. In addition, the pinning scheme is designed and the robustness of the pinning scheme is also discussed.

The rest of this paper is organized as follows. In Section II, some preliminaries are briefly outlined. Model formulation for pinning synchronization on networks of networks is proposed in Section III. In Section IV, synchronization criteria on networks of networks is established. The pinning scheme and the robustness of the designed pinning scheme are discussed in Sections V and VI, respectively. In Section VII, a simulation example is given to illustrate the theoretical analysis. The conclusion is finally drawn in Section VIII.

II. Preliminaries

In the literature, a complex dynamical network consisting of N identical nodes with linearly diffusive coupling [5], [19], [28], [29], [37], [40], [41] is described by

$$\dot{\xi}_i(t) = f(\xi_i(t)) + c \sum_{j=1, j\neq i}^{N} G_{ij}\Gamma(\xi_j(t) - \xi_i(t)), \quad i = 1, 2, \ldots, N \tag{1}$$

where $\xi_i(t) = (\xi_{i1}(t), \xi_{i2}(t), \ldots, \xi_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the ith node, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuously differentiable vector function, c is the coupling strength, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^{n\times n}$ is a positive semidefinite inner coupling matrix where $\gamma_j > 0$ if two nodes can communicate through the jth state, and $\gamma_j = 0$ otherwise, $G = (G_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network, where G_{ij} are defined as follows: if there exists a connection between node i and node j, then $G_{ij} = G_{ji} > 0$; otherwise, $G_{ij} = G_{ji} = 0$ $(j \neq i)$. In this paper, for simplicity, the undirected networks are considered. The Laplacian matrix $L = (L_{ij})_{N\times N}$ is defined by

$$L_{ii} = -\sum_{j=1, j\neq i}^{N} L_{ij} = k_i, \quad L_{ij} = -G_{ij}, \quad i \neq j \tag{2}$$

which ensures the property that $\sum_{j=1}^{N} L_{ij} = 0$. Equivalently, network (1) can be rewritten in a simpler form as follows:

$$\dot{\xi}_i(t) = f(\xi_i(t)) - c\sum_{j=1}^{N} L_{ij}\Gamma\xi_j(t), \quad i = 1, 2, \ldots, N. \tag{3}$$

Note that a solution $\eta(t) \in \mathbb{R}^n$ of an isolated node satisfies

$$\dot{\eta}(t) = f(\eta(t)). \tag{4}$$

Here, $\eta(t)$ may be an equilibrium point, a periodic orbit, or even a chaotic orbit.

It is well known that sometimes the network (3) may not reach synchronization by its own. To realize the synchronization of network (3), some selected controllers can be applied into a small fraction of the nodes, which is known as pinning control [6], [11], [15], [16], [23]–[26], [30], [36], [38], [39], [43], [46], [47]. Here, the pinning strategy is applied on a small fraction δ $(0 < \delta < 1)$ of the nodes in network (3). Suppose that the nodes $i_1, i_2, \ldots, i_l$ are selected, where $l = \llcorner \delta N \lrcorner$ represents the integer part of the real number δN. Without loss of generality, rearrange the order of the nodes in the network, and let the first l nodes be controlled. Thus, considering the virtual leader in (4), the pinning controlled network can be described by

$$\begin{aligned}
\dot{\xi}_i(t) &= f(\xi_i(t)) - c\sum_{j=1}^{N} L_{ij}\Gamma\xi_j(t) + u_i, \quad i = 1, 2, \ldots, l\\
\dot{\xi}_i(t) &= f(\xi_i(t)) - c\sum_{j=1}^{N} L_{ij}\Gamma\xi_j(t), \quad i = l+1, 2, \ldots, N
\end{aligned} \tag{5}$$

where

$$u_i = -cd_i\Gamma(\xi_i(t) - \eta(t)) \in \mathbb{R}^n, \quad i = 1, 2, \ldots, l \tag{6}$$

are n-dimensional linear feedback controllers with all the control gains $d_i > 0$.

Definition 1: Synchronization in network (5) is said to be reached if for any initial conditions

$$\lim_{t\to\infty} \|\xi_i(t) - \eta(t)\| = 0, \quad i = 1, 2, \ldots, N.$$

The objective of pinning control is to find some appropriate controllers (6) such that the states of the controlled network (5) synchronize with those in (4). When the controlled complex network (5) achieves synchronization, the coupling terms and control inputs will automatically vanish due to the diffusive condition $\sum_{j=1}^{N} L_{ij} = 0$. This indicates that any solution $\xi_i(t)$

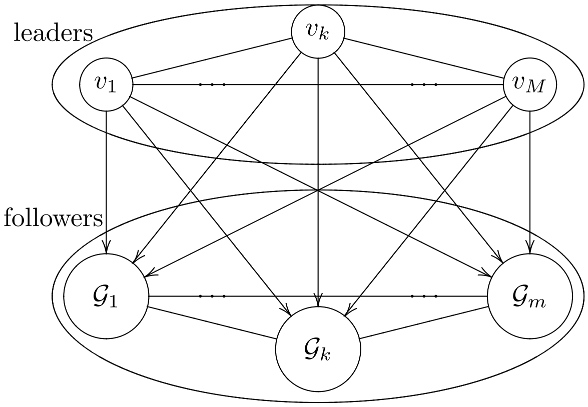

Fig. 1. Illustration figure for networks of networks with pinning control.

of any single node is also a solution of the synchronized coupled network.

Pinning synchronization of the network (5) with the controller (6) has been widely investigated recently [6], [11], [15], [16], [23]–[26], [30], [36], [38], [39], [43], [46], [47]. However, the general framework of (4)–(6) is very restrictive and hard to apply. In particular, some drawbacks are listed in the introduction. To overcome these disadvantages, a new result for pinning control on complex networks of networks is investigated where there are many leaders from which the nodes can get information and it is robust to the deliberate attack.

III. Model Formulation

In the recent studies of the new concept called networks of networks [9], [10], each node in the original network can be even a network itself and the edges between the nodes are the connections between these networks. In particular, pinning synchronization on networks of networks will be studied where there are some leaders coupled together in a global leaders' network and these leaders can communicate through local distributed protocols based on their neighboring nodes, as shown in Fig. 1. There are many subnetworks connected with each other, which can receive information form leaders' network while the leaders cannot receive information from these subnetworks in the followers' network.

Suppose that there is a network containing m subnetworks $\mathcal{G}_1, \ldots, \mathcal{G}_m$, with $N^{(k)}$ nodes in each subnetwork and M leaders $v_1, \ldots, v_M$, where the nodes in each subnetwork can receive information from some leaders, $k = 1, 2, \ldots, m$. The network structure is shown in Fig. 1. Let $N = N^{(1)} + N^{(2)} + \cdots + N^{(m)}$ represent the total nodes in the subnetworks.

Next, the pinning synchronization on networks of networks is investigated in this paper. Consider a complex dynamical network consisting of M identical leaders with linearly diffusive coupling, described by

$$\dot{s}_i(t) = f(s_i(t)) + c \sum_{j=1, j\neq i}^{M} G_{ij}^{(s)} \Gamma (s_j(t) - s_i(t)), \quad i = 1, 2, \ldots, M \tag{7}$$

where $s_i(t) = (s_{i1}(t), s_{i2}(t), \ldots, s_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the ith leader, $G^{(s)} = (G_{ij}^{(s)})_{M\times M}$ is the coupling configuration matrix representing the topological structure for the M leaders of the network, where if there exists a connection between the leaders i and j, then $G_{ij}^{(s)} = G_{ji}^{(s)} > 0$; otherwise, $G_{ij}^{(s)} = G_{ji}^{(s)} = 0$ $(j \neq i)$. The Laplacian matrix of the leader $L^{(s)} = (L_{ij}^{(s)})_{N\times N}$ is defined by

$$L_{ii}^{(s)} = -\sum_{j=1, j\neq i}^{M} L_{ij}^{(s)} = k_i, \quad L_{ij}^{(s)} = -G_{ij}^{(s)}, \quad i \neq j.$$

Without loss of generality, assume that the first $l^{(k)}$ nodes are controlled in the kth subnetwork, where $k = 1, 2, \ldots, m$. Considering the leaders in the network (7), the pinning controlled kth subnetwork can be described by

$$\begin{aligned}\dot{x}_i^{(k)}(t) &= f(x_i^{(k)}(t)) - c\sum_{j=1}^{N^{(k)}} L_{ij}^{(k)} \Gamma x_j^{(k)}(t) + u_i^{(k)}, \\ &\qquad i = 1, 2, \ldots, l^{(k)} \\ \dot{x}_i^{(k)}(t) &= f(x_i^{(k)}(t)) - c\sum_{j=1}^{N^{(k)}} L_{ij}^{(k)} \Gamma x_j^{(k)}(t), \\ &\qquad i = l^{(k)}+1, l^{(k)}+2, \ldots, N^{(k)}\end{aligned} \tag{8}$$

where $x_i^{(k)}(t) = (x_{i1}^{(k)}(t), x_{i2}^{(k)}(t), \ldots, x_{in}^{(k)}(t))^T \in \mathbb{R}^n$ is the state vector of the ith node in the kth subnetwork, $L^{(k)} = (L_{ij}^{(k)})_{N^{(k)}\times N^{(k)}}$ is the coupling Laplacian matrix representing the topological structure of the kth subnetwork, where if there exists a connection between node i and node j, then $L_{ij}^{(k)} = L_{ji}^{(k)} < 0$; otherwise, $L_{ij}^{(k)} = L_{ji}^{(k)} = 0$ $(j \neq i)$, $L_{ii}^{(k)} = -\sum_{j=1, j\neq i}^{N} L_{ij}^{(k)} = \deg^{(k)}(i)$ is the degree of the ith node in the kth subnetwork, and

$$u_i^{(k)} = -c\sum_{j=1}^{M} d_{ij}^{(k)} \Gamma (x_i^{(k)}(t) - s_j(t)) \in \mathbb{R}^n, \quad i=1, 2, \ldots, l^{(k)} \tag{9}$$

are n-dimensional linear feedback controllers with all the control gains $d_{ij}^{(k)} > 0$, $j = 1, 2, \ldots, M$.

Throughout the rest of this paper, the following assumption is needed.

Assumption 1: There exist a constant diagonal matrix $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n)$ and an $\varepsilon > 0$ such that

$$\begin{aligned}&(x-y)^T(f(x,t) - f(y,t)) - (x-y)^T \Delta\Gamma(x-y) \\ &\quad \le -\varepsilon (x-y)^T(x-y), \ \forall x, y \in \mathbb{R}^n \quad \forall t \in \mathbb{R}^+.\end{aligned} \tag{10}$$

Note that (10) is very mild. For example, all linear and piecewise linear functions satisfy this condition. In addition, if $\partial f_i/\partial x_j$ $(i, j = 1, 2, \ldots, n)$ are bounded and Γ is positive definite, the above condition is satisfied. Therefore, it includes many well-known systems, such as the Lorenz system, Chen system, Lü system, recurrent neural networks, Chua's circuit, and so on.

The following lemmas are needed to derive the main results.

Lemma 1 (Schur Complement [4]): The following linear matrix inequality:

$$\begin{pmatrix} \mathcal{Q}(x) & \mathcal{S}(x) \\ \mathcal{S}(x)^T & \mathcal{R}(x) \end{pmatrix} > 0$$

where $\mathcal{Q}(x) = \mathcal{Q}(x)^T, \mathcal{R}(x) = \mathcal{R}(x)^T$, is equivalent to one of the following conditions:

1) $\mathcal{Q}(x) > 0,\ \mathcal{R}(x) - \mathcal{S}(x)^T \mathcal{Q}(x)^{-1}\mathcal{S}(x) > 0$;
2) $\mathcal{R}(x) > 0,\ \mathcal{Q}(x) - \mathcal{S}(x)\mathcal{R}(x)^{-1}\mathcal{S}(x)^T > 0$.

Lemma 2 [6]: If L is the Laplacian matrix of a network, $L_{ij} = L_{ji} \le 0$ for $i \ne j$, and $\sum_{j=1}^{N} L_{ij} = 0$, for all $i = 1, 2, \ldots, N$, then all eigenvalues of the matrix

$$\begin{pmatrix} L_{11}+\varepsilon & L_{12} & \ldots & L_{1N} \\ L_{21} & L_{22} & \ldots & L_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \ldots & L_{NN} \end{pmatrix}$$

are positive for any positive constant ε.

Lemma 3 [14]: For matrices A, B, C, and D with appropriate dimensions, the Kronecker product $\otimes$ satisfies:

1) $(\phi A) \otimes B = A \otimes (\phi B)$, where ϕ is a constant;
2) $(A + B) \otimes C = A \otimes C + B \otimes C$;
3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;
4) $(A \otimes B)^T = A^T \otimes B^T$.

Lemma 4 [13]:

1) The Laplacian matrix L in an undirected graph $\mathcal{G}$ is semipositive definite. It has a simple zero eigenvalue and all the other eigenvalues are positive if and only if the graph $\mathcal{G}$ is connected.
2) The second smallest eigenvalue $\lambda_2(L)$ of the Laplacian matrix L in the undirected graph $\mathcal{G}$ satisfies $\lambda_2(L) = \min_{x^T 1_N = 0, x \ne 0_N} x^T Lx/x^T x$.

IV. Synchronization Criteria on Complex Networks of Networks

In this section, some pinning criteria are established to ensure the global synchronization on complex networks of networks. In the literature, synchronization problem in a system with dimension $\widetilde{N}$ can be transmitted to stability problem of a error system with dimension $\widetilde{N} - 1$ by introducing the error states [26]. However, it is different to study pinning synchronization as the previous works for introducing the error states since there are actually many leaders in complex networks of networks. Here, the average state of all the leaders' states can be regarded as the reference state and the aim is to analytically prove that all the states in the whole network can approach this reference state. Let $\bar{s}(t) = 1/M \Sigma_{j=1}^{M} s_j(t)$ be the average state of all the leaders' states, and $e_j = s_j(t) - \bar{s}(t)$ and $e_i^{(k)} = x_i^{(k)}(t) - \bar{s}(t)$ represent the relative states from the leaders and the nodes in the network to the average states of the leaders, respectively, where $i = 1, 2, \ldots, N^{(k)}$, $j = 1, 2, \ldots, M$, and $k = 1, 2, \ldots, m$.

The dynamics of $\bar{s}(t)$ are described by

$$\begin{aligned} \dot{\bar{s}}(t) &= \frac{1}{M}\sum_{i=1}^{M} f(s_i(t)) - c\frac{1}{M}\sum_{i=1}^{M}\sum_{j=1}^{M} L_{ij}^{(s)}\Gamma s_j(t) \\ &= \frac{1}{M}\sum_{i=1}^{M} f(s_i(t)) - c\frac{1}{M}\Gamma\sum_{j=1}^{M}\left(\sum_{i=1}^{M} L_{ij}^{(s)}\right)s_j(t) \\ &= \frac{1}{M}\sum_{i=1}^{M} f(s_i(t)) \end{aligned} \tag{11}$$

where the third inequality is satisfied since $\sum_{i=1}^{M} L_{ij}^{(s)} = 0$ for any $i, j = 1, 2, \ldots, M$.

Subtracting (11) from (7) yields the following error dynamical network:

$$\dot{e}_i(t) = f(s_i(t)) - \frac{1}{N}\sum_{i=1}^{M} f(s_i(t)) - c\sum_{j=1}^{M} L_{ij}^{(s)}\Gamma e_j(t), \quad i = 1, 2, \ldots, M \tag{12}$$

where $\sum_{j=1}^{M} L_{ij}^{(s)} = 0$ and $\sum_{j=1}^{N^{(k)}} L_{ij}^{(k)} = 0$ are used above.

Next, a theorem is established to derive the synchronization criteria for the complex networks of networks (12).

Theorem 1: Suppose that Assumption 1 holds. The controlled networks (7) and (8) are globally synchronized if the following conditions are satisfied:

$$\Delta_{\max} - c\lambda_2(L^{(s)}) < 0 \tag{13}$$

and

$$\Delta_{\max} I_{N^{(k)}} - c(L^{(k)} + D^{(k)}) < 0, \quad k = 1, 2, \ldots, m \tag{14}$$

where

$$D^{(k)} = \mathrm{diag}\left(\underbrace{\sum_{j=1}^{M} d_{1j}^{(k)}, \ldots, \sum_{j=1}^{M} d_{l^{(k)}j}^{(k)}}_{l^{(k)}}, \underbrace{0, \ldots, 0}_{N^{(k)}-l^{(k)}}\right)$$

$$\Delta_{\max} = \max\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$$

and $I_{N^{(k)}}$ is the $N^{(k)}$-dimensional identity matrix.

Proof: Since $e_i(t)$ are independent of $e_j^{(k)}(t)$ as in (12), where $i = 1, 2, \ldots, M$, $k = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, N^{(k)}$, the synchronization in the leaders' network (12) is first established.

Consider the Lyapunov functional candidate for the error system (12)

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{M} e_i^T(t)e_i(t). \tag{15}$$

Since $\sum_{i=1}^{M} e_i^T(t) = 0$, one has that $\sum_{i=1}^{M} e_i^T(t)1/N \sum_{i=1}^{M} f(s_i(t)) = 0$ and $\sum_{i=1}^{M} e_i^T(t)1/N\sum_{i=1}^{M} f(\bar{s}(t)) = 0$.

The derivative of $V_1(t)$ along the trajectories of (12) gives

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{M} e_i^T(t)\dot{e}_i(t)\\
&= \sum_{i=1}^{M} e_i^T(t)\Bigg[f(s_i(t)) - \frac{1}{N}\sum_{i=1}^{M} f(s_i(t))\\
&\qquad - \sum_{j=1}^{M} L_{ij}^{(s)}\Gamma e_j(t)\Bigg]\\
&= \sum_{i=1}^{M} e_i^T(t)\Bigg[f(s_i(t)) - \frac{1}{N}\sum_{i=1}^{M} f(s_i(t))\Bigg]\\
&\qquad - c\sum_{i=1}^{N}\sum_{j=1}^{M} L_{ij}^{(s)} e_i^T(t)\Gamma e_j(t)\\
&= \sum_{i=1}^{M} e_i^T(t)\Bigg[f(s_i(t)) - f(\bar{s}(t)) + f(\bar{s}(t))\\
&\qquad - \frac{1}{N}\sum_{i=1}^{M} f(s_i(t))\Bigg] - c\sum_{i=1}^{N}\sum_{j=1}^{M} L_{ij}^{(s)} e_i^T(t)\Gamma e_j(t)\\
&\le \sum_{i=1}^{M} e_i^T(t)\Delta\Gamma e_i^T(t) - c e^{(L)T}(t)(L^{(s)}\otimes\Gamma)e^{(L)}(t)\\
&\quad - \varepsilon e^{(L)T}(t)e^{(L)}(t)\\
&= e^{(L)T}(t)\big[(I_N\otimes\Delta\Gamma) - c(L^{(s)}\otimes\Gamma)\big]e^{(L)}(t)\\
&\quad - \varepsilon e^{(L)T}(t)e^{(L)}(t)\\
&\le e^{(L)T}(t)\big[\Delta_{\max} - c\lambda_2(L^{(s)})\big]I_N\otimes\Gamma e^{(L)}(t)\\
&\quad - \varepsilon e^{(L)T}(t)e^{(L)}(t)
\end{aligned}\tag{16}$$

where $e^{(L)} = (e_1^T, e_2^T, \ldots, e_M^T)^T$, $\Delta_{\max} = \max\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$, and the last inequality holds using Lemma 2.

From (13), it is easy to observe that $e_i(t)$ exponentially approach to zero, and thus the dynamics of the final states satisfy

$$\dot{\bar{s}}(t) = f(\bar{s}(t)) + \mathcal{O}(e^{-\varepsilon t})\tag{17}$$

where $\mathcal{O}(e^{-\varepsilon t})$ is a higher order infinitesimal of the term $e^{-\varepsilon t}$.

Since $e_i^{(k)} = x_i^{(k)}(t) - \bar{s}(t)$, one can have the following error system between the nodes in the subnetworks and the leaders:

$$\begin{aligned}
\dot{e}_i^{(k)}(t) &= f(x_i^{(k)}(t)) - f(\bar{s}(t)) - c\sum_{j=1}^{N^{(k)}} L_{ij}^{(k)}\Gamma e_j^{(k)}(t)\\
&\qquad - c\sum_{j=1}^{M} d_{ij}\Gamma e_i^{(k)}(t) + \mathcal{O}(e^{-\varepsilon t}),\quad i = 1, 2, \ldots, l^{(k)}\\
\dot{e}_i^{(k)}(t) &= f(x_i^{(k)}(t)) - f(\bar{s}(t)) - c\sum_{j=1}^{N^{(k)}} L_{ij}^{(k)}\Gamma e_j^{(k)}(t)\\
&\qquad + \mathcal{O}(e^{-\varepsilon t}),\quad i = l^{(k)} + 1, 2, \ldots, N^{(k)}.
\end{aligned}\tag{18}$$

Consider the Lyapunov functional candidate for the above system

$$V_2(t) = \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)e_i^{(k)}(t).\tag{19}$$

The derivative of $V_2(t)$ along the trajectories of (18) gives

$$\begin{aligned}
\dot{V}_2(t) &= \sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)\dot{e}_i^{(k)}(t)\\
&= \sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)\Big[f(x_i^{(k)}(t)) - f(\bar{s}(t))\Big]\\
&\quad - c\sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}}\sum_{j=1}^{N^{(k)}} L_{ij}^{(k)} e_i^{(k)T}(t)\Gamma e_j^{(k)}(t)\\
&\quad - c\sum_{k=1}^{m}\sum_{i=1}^{l^{(k)}} e_i^{(k)T}(t)\sum_{j=1}^{M} d_{ij}^{(k)}\Gamma e_i^{(k)}(t)\\
&\quad + \sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)\mathcal{O}(e^{-\varepsilon t}).
\end{aligned}\tag{20}$$

Let $e^{(k)} = (e_1^{(k)T}, e_2^{(k)T}, \ldots, e_{N^{(k)}}^{(k)T})^T$. It follows that:

$$\begin{aligned}
\dot{V} &\le \sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)\Delta\Gamma e_i^{(k)}(t)\\
&\quad - c\sum_{k=1}^{m} e^{(k)T}(t)((L^{(k)} + D^{(k)})\otimes\Gamma)e^{(k)}(t)\\
&\quad + \sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)\mathcal{O}(e^{-\varepsilon t}) - \varepsilon\sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)e_i^{(k)}(t)\\
&\le \sum_{k=1}^{m} e^{(k)T}(t)\big(\Delta_{\max} I_{N^{(k)}} - c(L^{(k)} + D^{(k)})\big)\\
&\quad \otimes\Gamma e^{(k)}(t) - \varepsilon\sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)e_i^{(k)}(t)\\
&\quad + \sum_{k=1}^{m}\sum_{i=1}^{N^{(k)}} e_i^{(k)T}(t)\mathcal{O}(e^{-\varepsilon t})
\end{aligned}\tag{21}$$

where

$$D^{(k)} = \mathrm{diag}\left(\underbrace{\sum_{j=1}^{M} d_{1j}^{(k)}, \ldots, \sum_{j=1}^{M} d_{l^{(k)}j}^{(k)}}_{l^{(k)}}, \underbrace{0, \ldots, 0}_{N^{(k)} - l^{(k)}}\right)$$

and $\sum_{j=1}^{M} d_{ij}$ is the total weights of the ith node in the kth subnetwork receiving from the leaders.

From the conditions in (13) and (14), it is easy to observe that the network (7) and (8) are globally synchronized under the given linear feedback pinning controllers. The proof is completed.

Remark 1: Theorem 1 is very intuitive. The coupling strength c plays a key role in conditions (13) and (14). If it can be designed, one knows that the larger the coupling strength is, the easier these conditions are satisfied. Actually, the condition (13) in Theorem 1 is the synchronization criteria in leaders' network. When the leaders coordinate to reach synchronization, the condition (14) in Theorem 1 provides the control design for synchronization in each subnetworks. In particular, if $M = 1$, the present framework is the original pinning synchronization setting. It has some advantages as illustrated: 1) there does not exist a centralized leader, from which a fraction of nodes in the whole network can receive the same information and 2) the new framework for pinning synchronization on networks of networks are very robust to the deliberate attacks. For example, even if some leaders and nodes in the subnetworks are attacked, the whole network may still reach synchronization or at least most of the nodes can achieve synchronization.

The condition (14) in Theorem 1 is very important for pinning scheme design. Next, the aim is to design some useful pinning schemes for synchronization on complex networks of networks.

V. Pinning Scheme on Complex Networks of Networks

To design the pinning scheme in this section, some notations are introduced for simplicity. Let

$$\Xi^{(k)} = \Delta_{\max} I_{N^{(k)}} - c(L^{(k)} + D^{(k)}) = \begin{pmatrix} A^{(k)} - \widetilde{D}^{(k)} & B^{(k)} \\ B^{(k)T} & C^{(k)} \end{pmatrix}$$

where $\widetilde{D}^{(k)} = \mathrm{diag}(\sum_{j=1}^{M} d_{1j}^{(k)}, \ldots, \sum_{j=1}^{M} d_{l^{(k)}j}^{(k)})$ and $C^{(k)}$ is obtained by removing the $1, 2, \ldots, l^{(k)}$ row–column pairs of matrix $\Xi^{(k)}$.

Next, to show the design method for pinning controllers, some decompositions on (14) are given to simplify the this condition and to clearly reveal how the network characters can affect the pining synchronization criterion.

Theorem 2 (Necessary Condition): Suppose that Assumption 1 holds. To satisfy condition (14), it is necessary that

$$\Xi_{ii}^{(k)} = \Delta_{\max} - c\deg^{(k)}(i) - c\sum_{j=1}^{M} d_{ij} < 0, \quad 1 \le i \le l^{(k)}$$

$$\Xi_{ii}^{(k)} = \Delta_{\max} - c\deg^{(k)}(i) < 0, \quad l^{(k)} + 1 \le i \le N^{(k)} \quad (22)$$

for all $k = 1, 2, \ldots, m$.

Proof: From the condition (14) that $\Xi^{(k)} = \Delta_{\max} I_{N^{(k)}} - c(L^{(k)} + D^{(k)}) < 0$, it is necessary that $\Xi_{ii}^{(k)} < 0$. This completes the proof.

Theorem 3 (Sufficient Condition): Suppose that Assumption 1 holds. By appropriately choosing the control gains, the condition (14) $\Xi^{(k)} = \Delta_{\max} I_{N^{(k)}} - c(L^{(k)} + D^{(k)}) < 0$ in Theorem 1 is equivalent to the following condition:

$$C^{(k)} < 0. \quad (23)$$

For simplicity, one can select $\widetilde{D}^{(k)} > A^{(k)} - B^{(k)} (C^{(k)})^{-1} B^{(k)T}$.

Proof: This can be proved by the following two steps.

1) If $\Xi^{(k)} < 0$ then $C^{(k)} < 0$.
2) If $C^{(k)} < 0$, one can choose $\widetilde{D}^{(k)} > A^{(k)} - B^{(k)}(C^{(k)})^{-1} B^{(k)T}$. Then, by Lemma 1, one indeed has $\Xi^{(k)} < 0$. The proof is completed.

Actually, Theorems 2 and 3 reveal some very interesting phenomena, which provide some schemes for choosing pinning controlled nodes on complex networks of networks.

Remark 2: From (22), for the nodes without of control, the degree of these nodes must be greater than a critical value, namely, $\deg^{(k)}(i) > \Delta_{\max}/c$, which indicates that the nodes with low degrees should be first controlled and otherwise, (22) is not satisfied. This is consistent with the common intuition that the nodes with very low degrees can receive little information from other nodes, which cannot force them to follow the other nodes in the whole network.

Remark 3: From (23), one may aim to find a submatrix of $\Xi^{(k)}$ such that $C^{(k)} < 0$ where $C^{(k)}$ is obtained by removing the $1, 2, \ldots, l^{(k)}$ row–column pairs of matrix $\Xi^{(k)}$. The key problem is to find such a $C^{(k)}$. Note that $(B^{(k)T}\ C^{(k)})$ is a $\Delta_{\max}$-row-sum matrix, which means that the sums of the elements in each row is $\Delta_{\max}$. From the definition of $\Xi^{(k)}$, each element in $B^{(k)T} = B^{(k)T}$ is positive, which represent the connections between the controlled and uncontrolled nodes. If there are more connections in $B^{(k)}$, the matrix $C^{(k)}$ will more likely to be a negative definite, which indicates that the nodes with large degrees should be controlled since these nodes can affect many connected nodes and thus $C^{(k)}$ can be negative.

From Theorems 2 and 3, one can observe that the nodes with very low and large degrees are good candidates for applying pinning controllers [43].

VI. Attack and Robustness of Pinning Synchronization on Complex Networks of Networks

A serious big drawback for the original pinning synchronization scheme is that the network is fragile to the deliberate attacks, where the attack simply means that some nodes or edges are removed from the network. Here, if the edges are removed, one only needs to check whether the conditions in this paper are satisfied. Thus, the node attack is discussed in this section. If the virtual leader or even one node in the network is attacked, the whole network can be in a mess and cannot reach synchronization with a leader. However, this problem can be fully solved using the new framework for pinning synchronization on complex networks of networks. There are three kinds of nodes in the network, namely, the leaders, the pinning controlled nodes and uncontrolled nodes, and the attack can be made on these three kinds of nodes as follows.

1) Attack on the leaders. This is very important for the current framework. Though the whole network may not reach synchronization if (13) is not satisfied under the attack, it improves the original pinning framework where synchronization must not be achieved under the attack on the virtual leader. Therefore, it is desirable to design some protocols such that (13) in Theorem 1 can be satisfied if some of the leaders are attacked in the network.

2) Attack on the controlled nodes and uncontrolled nodes. If the controlled and uncontrolled nodes are attacked, there are at several subnetworks, which may not reach synchronization. For the other networks without attack, synchronization can still be achieved, which cannot cause a cascading failure in the whole network.

Remark 4: If $M = 1$, pinning control in complex networks of networks can be reduced to the original pinning control in complex networks [6], [30], [38], [43], [45]. It should be noted that synchronization cannot be reached if one attacks the virtual leader node, from which many nodes can receive information.

Therefore, the present framework for pinning synchronization on complex networks of networks are very robust to the deliberate attacks, which could be very reasonable and useful for practical applications.

VII. Simulation Examples

In this section, some examples are simulated to verify the theoretical analysis in this paper. Consider two different cases that there are totally five leaders with two leaders' network. One is fully connected and the other is a tree, that is

$$L^{(s)} = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

$$L^{(s)} = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Γ is an identity matrix. There are also five subnetworks forming a scale-free network structure with 100 nodes [2]. The nonlinear function f is described by Chua's circuit

$$f(x_i(t), v_i(t), t) = \begin{pmatrix} \varsigma(-v_{i1} + v_{i2} - l(v_{i1})) \\ v_{i1} - v_{i2} + v_{i3} \\ -\varrho v_{i2} \end{pmatrix} \qquad (24)$$

with $l(v_{i1}) = bv_{i1} + 0.5(a-b)(|v_{i1}+1| - |v_{i1}-1|)$. System (24) is chaotic when $\varsigma = 10$, $\varrho = 18$, $a = -4/3$, and $b = -3/4$, as shown in [41]. It is easy to check that Assumption 1 is satisfied, where $\Delta_{\max} = 4.3871$.

Here, 20% of all the nodes with very low and large degrees are chosen as the candidates for applying pinning controllers. From Theorem 1, the controlled networks are globally synchronized. The states of the errors between the nodes and the average leaders $\bar{s}(t)$ with two different leaders' networks in (11) are shown in Figs. 2 and 3, which verify the theoretical analysis in this paper. From the illustrations of these figures, one can observe that the errors converge faster under a fully connected leader network than a tree-style one.

Synchronization can still be reached if one attacks on any leader of the fully connected leader network since the rest four leaders are also fully connected. The error states are shown in Fig. 4, which is robust to the deliberate attack. However, if one attacks the first leader in the tree-style leader network making it disconnected, synchronization cannot be ensured between the leaders networks and subnetworks, which are shown in Fig. 5 where the first leader is removed from the network.

From the simulations in this section, one can observe that the leaders network structure is very important. If one attacks the leaders network and disconnects it, then synchronization may not be reached. However, for a general connected network, synchronization can still be ensured even if one attacks some leaders under a connected topology.

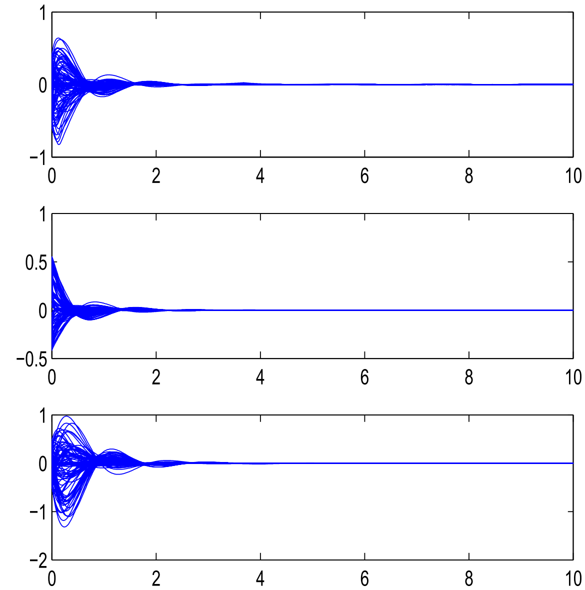

Fig. 2. States of errors between the nodes x_i^k and the average leader $\bar{s}$ in the network with fully connected leaders.

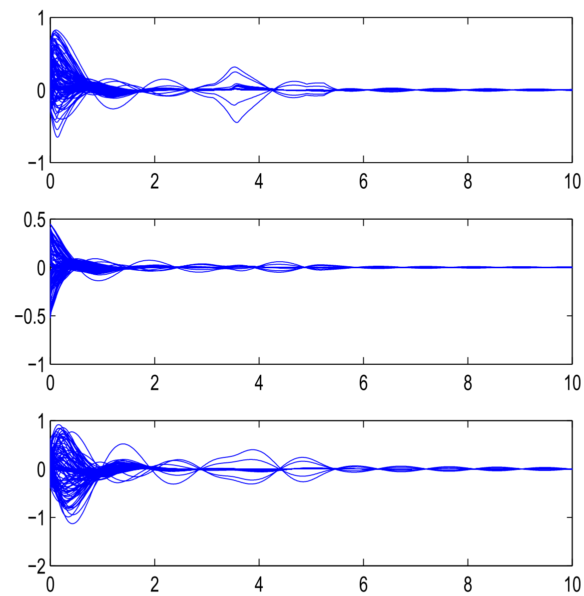

Fig. 3. States of errors between the nodes x_i^k and the average leader $\bar{s}$ in the network with tree-style leaders.

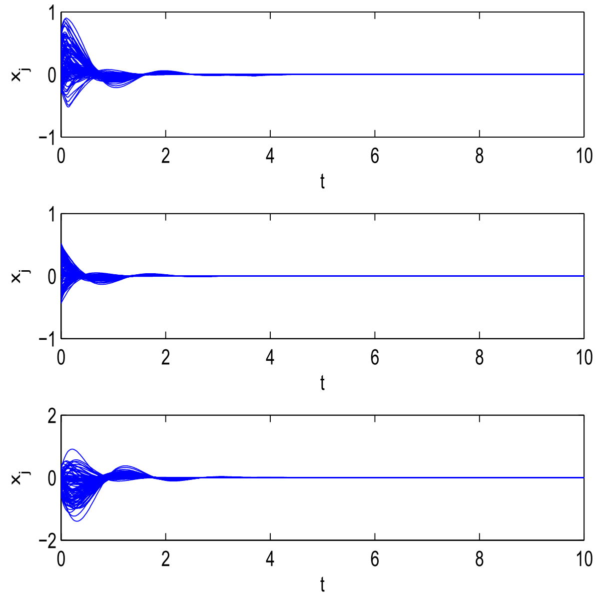

Fig. 4. States of errors between the nodes x_i^k and the average leader $\bar{s}$ in the network with fully connected leaders after removing one leader.

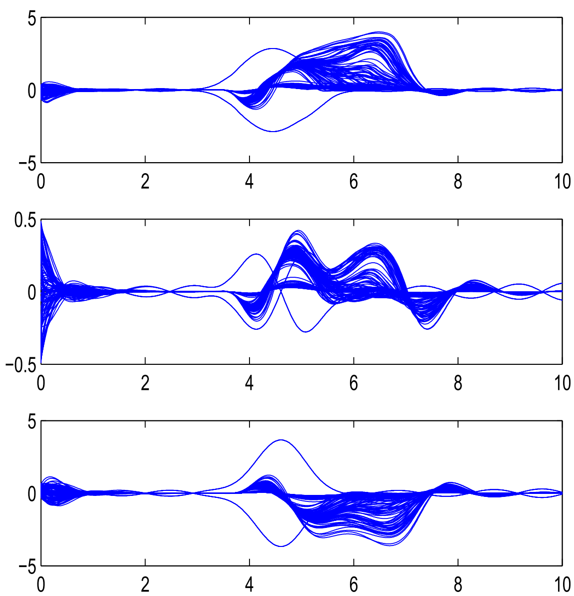

Fig. 5. States of errors between the nodes x_i^k and the average leader $\bar{s}$ in the network with tree-style leaders after removing the first leader.

VIII. Conclusion

In this paper, pinning synchronization on networks of networks has been established, where the subnetworks and the leaders form the networks of networks. Some pinning controllers have been designed on the chosen nodes of each subnetwork so as to reach synchronization behavior on networks of networks. It has been found that pinning control under this new framework has many advantages compared to the original pinning control method. Furthermore, the attack and robustness of the pinning scheme have been discussed.

Though the framework is simple, it is of great interest to investigate theoretical analysis and practical applications for dynamic analysis and control in complex network and networks. In the future, we will investigate some other works on complex networks of networks, such as the synchronization criteria, hybrid control, finite-time control, disturbance rejection control, adaptive laws, time-varying networks structures, and so on.

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