Math 104 Final Project Report

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1 Introduction

The wave equation is an example of a hyperbolic equation that represents the position of the wave in time t. Based on the complexity of the formula, the equation can represent 1-dimensional waves as well as 2 and 3-dimensional ones. In our case, the approximation, methods and results all deal with 1-dimensional wave on a bounded uniformly dense string. Thus, the problem that is approximated, in mathematical terms, is

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\frac{\partial^2}{\partial x^2}u(x,t) = 0,\ \Big[0 < x < l,\ 0 < t < T\Big] \tag{1}$$

$$u(x,0) = \sin(\pi x),\ \Big[0 \leq x \leq l\Big] \tag{2}$$

$$\frac{\partial}{\partial t}u(x,0) = 0,\ \Big[0 \leq x \leq l\Big] \tag{3}$$

$$u(0,t) = u(l,t) = 0,\ \Big[0 < t < T\Big] \tag{4}$$

where (1) is the wave equation with speed $c = \sqrt{\frac{T}{\rho}}$, with T being the tension of the string and ρ being the uniform density across the string. Furthermore, (2) is the initial position of the wave and (3) is the initial velocity of the wave. Finally, because algorithms require stopping conditions, (4) represents boundary conditions at $x = 0$ and l where the position of the wave is zero for any time t.

Therefore, the goal of the project is to find the position of the wave, $u(x,t)$ given any $t \in (0,T)$ and any $x \in [0,l]$. However, in order to find these values, certain numerical methods are required.
Namely, for approximation of derivatives, we first apply Taylor's expansion on f. Following, we use the second, centered, divided difference Δ^2 in order to compute

approximations in an iterative way. The details of the computations will be presented in the **Method** section.

2 Method

Theory behind the method

As mentioned above, this section is used to present the theory that serves as the foundation for the methods used to approximate

$$u_{tt} - c^2 u_{xx} = 0.$$

First, we show how we find the second derivatives with respect to t and x. First, define a step size $h = \frac{l}{m}$, where $m \geq 2$. Then consider points $x_0 + h$ and $x_0 - h$. Using, Taylor's Theorem about point x_0, it follows that we obtain

$f(x_0+h) = \sum_{n=0}^{3} \frac{f^{(n)}(x_0)}{n!}(x_0+h-x_0)^n = f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2}f^{(2)}(x_0)+\frac{h^3}{6}f^{(3)}(x_0)+\frac{h^4 f^{(4)}(\xi_1)}{24}$,

$f(x_0-h) = \sum_{n=0}^{3} \frac{f^{(n)}(x_0)}{n!}(x_0+h-x_0)^n = f(x_0)-hf^{(1)}(x_0)+\frac{h^2}{2}f^{(2)}(x_0)-\frac{h^3}{6}f^{(3)}(x_0)+\frac{h^4 f^{(4)}(\xi_{-1})}{24}$,

where $x_0 - h < \xi_{-1} < x_0 < \xi_1 < x_0 + h$. Note that $f(x) = \sin(\pi x)$ has derivatives of all orders and thus $f^{(n)}$ exists, for $n = 0, 1, 2, ...$ Then, adding terms $f(x_0 + h)$ and $f(x_0-h)$, we get $f(x_0+h)+f(x_0-h) = 2f(x_0)+h^2f^{(2)}(x_0)+\frac{h^4}{24}\Big[f^{(4)}(\xi_1)+f^{(4)}(\xi_{-1})\Big]$. Solving for $f^{(2)}(x_0)$ we obtain the formula,

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - \frac{h^2}{24}\Big[f^{(4)}(\xi_1) + f^{(4)}(\xi_{-1})\Big].$$

Because $f^{(4)}$ is continuous and $\frac{1}{2}\Big[f^{(4)}(\xi_1)+f^{(4)}(\xi_{-1})\Big]$ is in between ξ_{-1}, ξ_1(Not necessarily in that order), applying the Intermediate Value Theorem, we find ξ such that $f^{(4)}(\xi) = \frac{1}{2}\Big[f^{(4)}(\xi_1) + f^{(4)}(\xi_{-1})\Big]$. Thus, we obtain the expression

$$\boxed{f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - \frac{h^2}{12}f^{(4)}(\xi)}$$

Although this is not a partial derivative, the method still works. Holding t constant, and following the derivation we obtain the partial derivative of $\frac{\partial^2}{\partial x^2}$ of $u(x, t_o)$ for any x.

Repeating the same process for t, we get the formula

$$g''(t_0) = \frac{g(t_0+k) - 2g(t_0) + g(t_0-k)}{k^2} - \frac{k^2}{24}\Big[g^{(4)}(\xi_1) + g^{(4)}(\xi_{-1})\Big] \text{ and,}$$

$$\boxed{g''(t_0) = \frac{g(t_0+k) - 2g(t_0) + g(t_0-k)}{k^2} - \frac{k^2}{12}g^{(4)}(\xi)},$$

where $k = \frac{T}{N}$ is the time-step, $N \geq 2$. Truncating the errors and relabeling f and g, we get approximations,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}u(x_i, t_j) &= \frac{u(x_i+h, t_j) - 2u(x_i, t_j) + u(x_i-h, t_j)}{h^2} \\
\frac{\partial^2}{\partial t^2}u(x_i, t_j) &= \frac{u(x_i, t_j+k) - 2u(x_i, t_j) + u(x_i, t_j-k)}{k^2}.
\end{aligned}$$

where $x_i + h = x_{i+1},\ x_i - h = x_{i-1}$ and $t_j + k = t_{j+1},\ t_j - k = t_{j-1}$. Therefore, what we obtain is the centered divided differences for partials of x and t. Combining them, we get the wave equation in the form of the second, centered divided difference

$$\boxed{\frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - c^2\frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} = 0}.$$

Now, we want to compute the position of the wave for the most advanced time-step $u(x_i, t_{j+1})$ in order to accurately predict the position of the wave in the future time. Thus, using the boxed expression above and solving for $u(x_i, t_{j+1})$, we get

$$\tfrac{1}{k^2}\Big[u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})\Big] - \tfrac{c^2}{h^2}\Big[u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\Big] = 0,$$

$$\tfrac{1}{k^2}u(x_i, t_{j+1}) = 2(\tfrac{1}{k^2} - \tfrac{c^2}{h^2})u(x_i, t_j) - \tfrac{1}{k^2}u(x_i, t_{j-1}) + \tfrac{c^2}{h^2}\Big[u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\Big],$$

$$\boxed{u(x_i, t_{j+1}) = 2(1 - \frac{c^2k^2}{h^2})u(x_i, t_j) + \frac{c^2k^2}{h^2}\Big[u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\Big] - u(x_i, t_{j-1})}$$

which is the final form of the approximation of the wave for time t_{t+j} used in the algorithm. However, note that in the code itself this approach will fail for $u(x_0, t_j)$, $u(x_m, t_j)$, $u(x_i, t_0)$, and $u(x_m, t_1)$ because x_{-1}, x_{m+1}, t_{-1} and t_1 are not defined in this case. Therefore, we use our boundary and initial conditions to kick start the approximation above starting with $i = 1, 2, ..., m-1$ and $j = 1, 2, ...$ So, we set $u(x_0, t_j) = u(x_m, t_j) = 0$, $u(x_i, t_0) = f(x_i) = u(x, 0)$ and $u_t(x, 0) = u(x_i, t_1)$ where we have to use divided difference again to compute the derivative with respect to t.

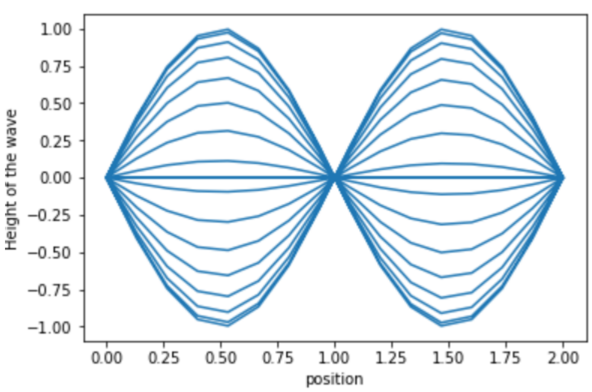
Rate of Convergence

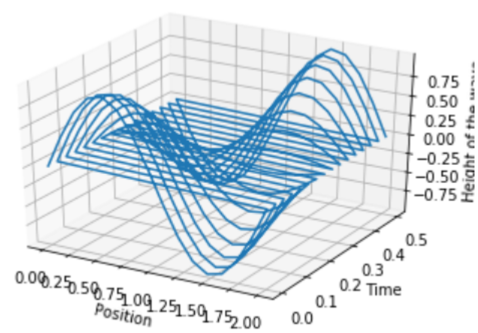
It is important to note that even though the algorithm produces approximations of

entire waves at given times t, the rate of convergence is only applicable at each point in the mesh separately. The following results holds for both ∂_t^2 and ∂_x^2 and thus we will only show one.
Using the computation above we have that $u_{xx}(x_i, t_j) = \frac{u(x_i+h,t_j)-2f(x_i)+f(x_i-h,t_j)}{h^2}$, where t_j is fixed. Introducing the error again, we have that
$u_{xx}(x_i, t_j) - \frac{u(x_i+h,t_j)-2f(x_i)+f(x_i-h,t_j)}{h^2} = -\frac{h^2}{12}f^{(4)}(\xi)$. Applying the absolute value, it follows that
$|u_{xx}(x_i, t_j) - \frac{u(x_i+h,t_j)-2f(x_i)+f(x_i-h,t_j)}{h^2}| = |\frac{h^2}{12}f^{(4)}(\xi)|$. However, our initial condition dictates that $f(x) = \sin(\pi x)$ and thus $|f^{(4)}(\xi)| = |\sin(\pi\xi)| \leq 1$. Therefore, we have that
$|u_{xx}(x_i, t_j) - \frac{u(x_i+h,t_j)-2f(x_i)+f(x_i-h,t_j)}{h^2}| = |\frac{h^2}{12}f^{(4)}(\xi)| \leq \frac{h^2}{12}$. Using definition (1.19), we have that the behavior of the approximation of the second partial derivatives equals $O(h^2)$ and $O(k^2)$ for u_{tt} where h and k are sufficiently small and where $A = \frac{1}{12}$. Thus, the rate of convergence for our approximation is $O(h^2 + k^2)$ by the definition of the "big oh".

3 Results

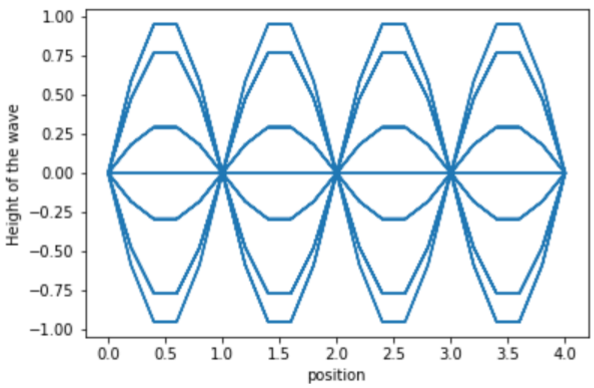
For initial conditions $u(x, 0) = \sin(\pi x)$ and $u_t(x, 0) = 0$ and with length $l = 2$,, time $T = 2$, $m = 15$, $N = 15$, and speed $c = 0.5$, we obtain the following graphs

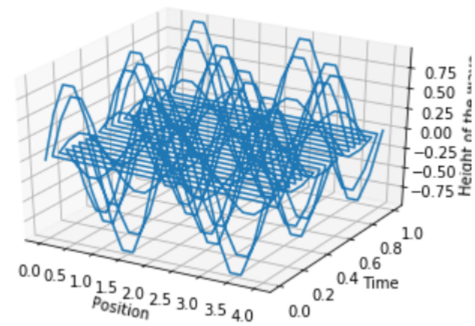




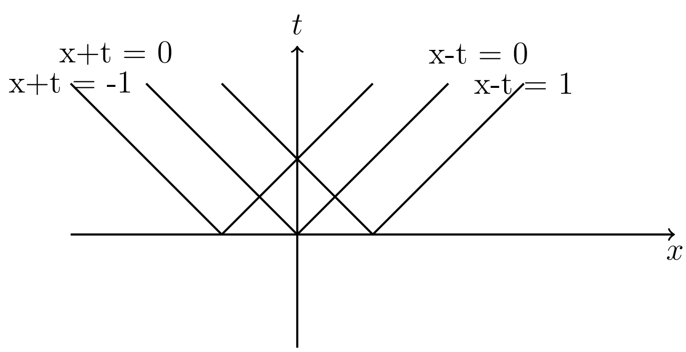
Although the 2D graph is a bit miss leading, what it shows is the line waves propagating through time. However, it appears that the wave is not moving in the x direction. The reason behind this is that we do not see the third axis that is time. Thus, we have to observe the different amplitudes depicted that represent the wave approaching the flat x-axis and then forming a negative wave. Now, if we look at the 3D graph, we can see the behaviour of the wave. We have to remind ourselves that the 1D wave is just a string thus the shape depicted is not one single wave. In fact the lines represent the same line wave in different times t.

The following example best shows the direction of the wave. The following graphs are plotted using parameters $l = 4$, $c = 1$, $T = 4$, $m = 20$, $N = 20$





It may seem like a complete mess but in fact the wave is behaving exactly like the theory predicts. Namely, if we look at the peaks of the lines represented, we notice that the peaks are following slanted lines. These lines are called characteristic lines and in our case are lines $x - t = a$, and $x + t = a$ where the wave is constant along them (peaks of the wave).



Conclusion

In conclusion, although the wave is traveling in both x and t directions, we can represent it as just traveling along a string stretched out along the x-axis. In doing so, we get the desired motion of the sin wave.
Although the algorithm produces an accurate approximation of the wave, i.e. up to $O(h^2 + k^2)$ it is still limited to a finite section of the 3D space. Moreover, if $\frac{c^2k^2}{h^2} > 1$, then the algorithm becomes unstable.
What we get in the end is an algorithm (submitted separately) that produces the position of the wave given points x and t in a bounded rectangle $0 \leq x \leq l$, $0 < t < T$.