2.2 Vektor koordinat dan matriks representasi [F, sec. 2.2]

Pada bagian ini, pemetaan linear antara dua ruang vektor yg wujud anggotanya bisa bermacam² (vektor, matriks, polinomial, fungsi², dll.) akan kita "ubah" menjadi pemetaan linear $L_A : \mathbb{F}^n \to \mathbb{F}^m$ dengan $L_A(\bar{x}) = A\bar{x}$, utk suatu $A \in \mathbb{F}^{m \times n}$. Utk mencapai tujuan ini, terlebih dahulu kita harus bisa "mengubah" anggota² dari setiap ruang vektor tsb mjd anggota² $\mathbb{F}^n$ utk suatu $n \in \mathbb{N}$. Utk itu diperkenalkan konsep <u>vektor koordinat</u>.

Mis V ruang vektor atas lapangan $\mathbb{F}$, dan $\beta = \{\bar{x}_1, \dots, \bar{x}_n\} \subseteq V$ suatu basis (<u>terurut</u>, artinya urutannya spesifik) bagi V. Artinya, setiap $\bar{x} \in V$ dpt dituliskan <u>scr tunggal</u> sbg

$$\bar{x} = \alpha_1 \bar{x}_1 + \dots + \alpha_n \bar{x}_n.$$

Artinya, setiap vektor $\bar{x} \in V$ terkait dgn <u>tepat satu</u> vektor

$$[\bar{x}]_\beta := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n;$$

vektor ini disebut <u>vektor koordinat</u> dari $\bar{x}$ thd basis β.

<u>Contoh</u> Tentukan vektor koordinat dari $1 + 4x + 7x^2 \in \mathbb{R}[x]_{\leq 2}$ terhadap basis $\beta = \{1 + x^2, x + x^2, 1 + 2x + x^2\}$.

<u>Jawab</u>

Harus dicari $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ yg memenuhi

$$1 + 4x + 7x^2 = \alpha_1(1 + x^2) + \alpha_2(x + x^2) + \alpha_3(1 + 2x + x^2)$$

$$= (\alpha_1 + \alpha_3) + (\alpha_2 + 2\alpha_3)x + (\alpha_1 + \alpha_2 + \alpha_3)x^2,$$

yaitu yg memenuhi SPL

$$\begin{cases} \alpha_1 + \alpha_3 = 1, \\ \alpha_2 + 2\alpha_3 = 4, \\ \alpha_1 + \alpha_2 + \alpha_3 = 7. \end{cases}$$

Karena

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{array}\right) \overset{-R_1+R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 6 \end{array}\right)$$

$$\overset{R_2 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 1 & 2 & 4 \end{array}\right) \overset{-R_2+R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

$$\overset{\frac{1}{2}R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right) \overset{-R_3+R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

maka $\alpha_1 = 2$, $\alpha_2 = 6$, dan $\alpha_3 = -1$. Jadi, vektor koordinat dari $1+4x+7x^2$ thd basis β adl

$$[1+4x+7x^2]_\beta = \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix}.$$

Sekarang perhatikan pemetaan linear $T: V \to W$, dgn V, W ruang vektor atas lapangan $\mathbb{F}$ dengan basis msg² $\beta = \{\bar{x}_1, \ldots, \bar{x}_n\} \subseteq V$ dan $\gamma = \{\bar{y}_1, \ldots, \bar{y}_m\} \subseteq W$. Utk setiap $\bar{x} \in V$, apa hub antara $[\bar{x}]_\beta$ dgn $[T(\bar{x})]_\gamma$?

Ambil $\bar{x} \in V$. Tulis

$$\bar{x} = b_1\bar{x}_1 + \cdots + b_n\bar{x}_n \qquad \star$$

utk suatu $b_1, \ldots, b_n \in \mathbb{F}$. Dgn demikian, $[\bar{x}]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

Petakan kedua ruas ☆ oleh T, diperoleh

$$T(\bar{x}) = b_1 T(\bar{x}_1) + \cdots + b_n T(\bar{x}_n).$$

Karena $T(\bar{x}_1), \ldots, T(\bar{x}_n) \in W$, maka

$$T(\bar{x}_1) = a_{1,1}\bar{y}_1 + a_{2,1}\bar{y}_2 + \cdots + a_{m,1}\bar{y}_m,$$

$$T(\bar{x}_2) = a_{1,2}\bar{y}_1 + a_{2,2}\bar{y}_2 + \cdots + a_{m,2}\bar{y}_m,$$

$$\vdots$$

$$T(\bar{x}_n) = a_{1,n}\bar{y}_1 + a_{2,n}\bar{y}_2 + \cdots + a_{m,n}\bar{y}_m,$$

dengan semua $a_{i,j} \in \mathbb{F}$. Jadi,

$$T(\bar{x}) = b_1(a_{1,1}\bar{y}_1 + a_{2,1}\bar{y}_2 + \cdots + a_{m,1}\bar{y}_m) + b_2(a_{1,2}\bar{y}_1 + a_{2,2}\bar{y}_2 + \cdots + a_{m,2}\bar{y}_m)$$
$$+ \cdots + b_n(a_{1,n}\bar{y}_1 + a_{2,n}\bar{y}_2 + \cdots + a_{m,n}\bar{y}_m)$$

$$= (a_{1,1}b_1 + a_{1,2}b_2 + \cdots + a_{1,n}b_n)\bar{y}_1 + (a_{2,1}b_1 + a_{2,2}b_2 + \cdots + a_{2,n}b_n)\bar{y}_2$$
$$+ \cdots + (a_{m,1}b_1 + a_{m,2}b_2 + \cdots + a_{m,n}b_n)\bar{y}_m,$$

shg

$$[T(\bar{x})]_\gamma = \begin{pmatrix} a_{1,1}b_1 + a_{1,2}b_2 + \cdots + a_{1,n}b_n \\ a_{2,1}b_1 + a_{2,2}b_2 + \cdots + a_{2,n}b_n \\ \vdots \\ a_{m,1}b_1 + a_{m,2}b_2 + \cdots + a_{m,n}b_n \end{pmatrix}$$

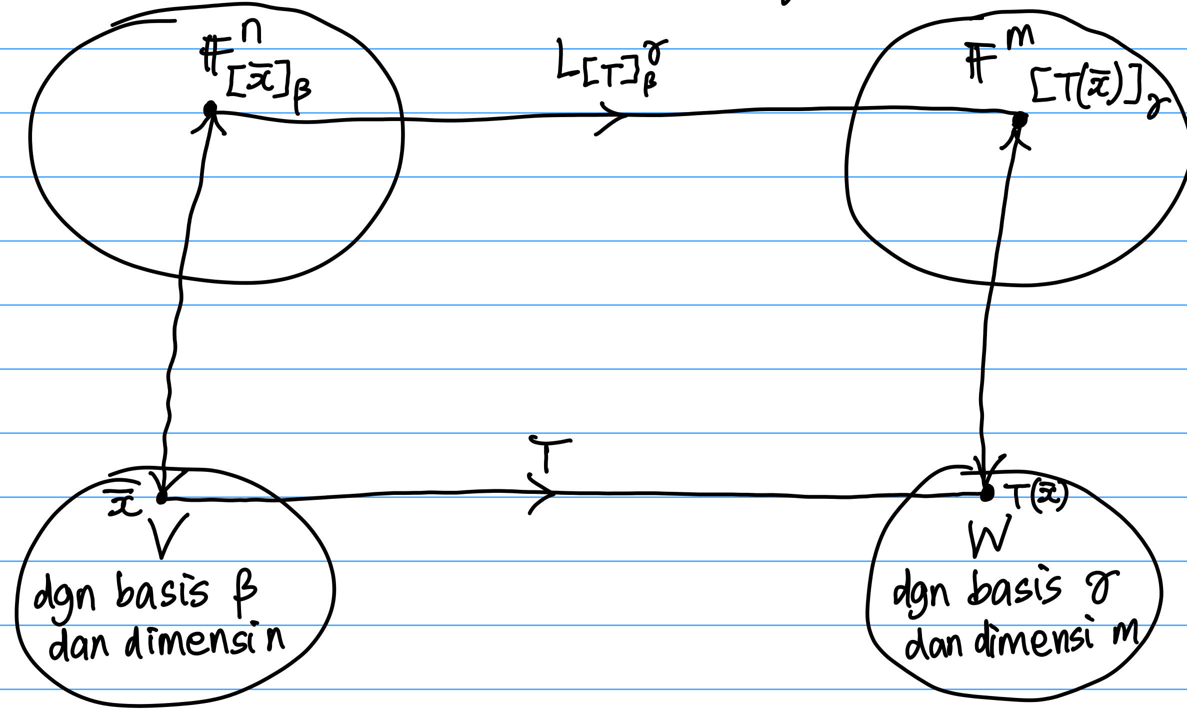
$$= \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \begin{pmatrix} [T(\bar{x}_1)]_\gamma & [T(\bar{x}_2)]_\gamma & \cdots & [T(\bar{x}_n)]_\gamma \end{pmatrix} [\bar{x}]_\beta.$$

Matriks

$$[T]_\beta^\gamma := \left([T(\bar{x}_1)]_\gamma \;\; [T(\bar{x}_2)]_\gamma \; \cdots \; [T(\bar{x}_n)]_\gamma \right)$$

berukuran $m \times n$ dan disebut matriks representasi dari pemetaan T terhadap basis β dan γ.



Contoh Tentukan matriks representasi dari pemetaan linear $T : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^{2\times 2}$ dengan

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

terhadap basis $\beta = \{1, x, x^2\} \subseteq \mathbb{R}[x]_{\leq 2}$ dan

$$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^{2\times 2}.$$

Jawab

Kita hitung

$$T(1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T(x^2) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

shg matriks representasi T thd β dan γ adl

$$[T]_\beta^\gamma = \left([T(1)]_\gamma \quad [T(x)]_\gamma \quad [T(x^2)]_\gamma \right)$$

$$= \begin{pmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Catatan

- Jika $V = W$ dan $\gamma = \beta$ maka kita akan menulis $[T]_\beta^\gamma =: [T]_\beta$.
- Mis $\beta = \{\bar{x}_1, \dots, \bar{x}_n\}$. Perhatikan bahwa

$$[I_V]_\beta = \left([I_V(\bar{x}_1)]_\beta \quad [I_V(\bar{x}_2)]_\beta \quad \cdots \quad [I_V(\bar{x}_n)]_\beta \right)$$

$$= \left([\bar{x}_1]_\beta \quad [\bar{x}_2]_\beta \quad \cdots \quad [\bar{x}_n]_\beta \right)$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

$$= I_n.$$

2.3 Komposisi pemetaan dan perkalian matriks [F, sec 2.2]

Mis V, W, Z ruang vektor atas lapangan $\mathbb{F}$.

Teorema Jika $T : V \to W$ dan $U : W \to Z$ pemetaan linear, maka $U \circ T : V \to Z$ pemetaan linear.

Bukti

Ambil $\alpha, \beta \in \mathbb{F}$ dan $\bar{x}, \bar{y} \in V$. Karena T dan U linear, maka

$$
\begin{aligned}
(U \circ T)(\alpha \bar{x} + \beta \bar{y}) &= U\left(T(\alpha \bar{x} + \beta \bar{y})\right) \\
&= U\left(\alpha T(\bar{x}) + \beta T(\bar{y})\right) \\
&= \alpha\, U(T(\bar{x})) + \beta\, U(T(\bar{y})) \\
&= \alpha (U \circ T)(\bar{x}) + \beta (U \circ T)(\bar{y}).
\end{aligned}
$$

Jadi, $U \circ T$ linear. $\blacksquare$

Mis α, β, γ basis bagi V, W, Z secara berturut-turut. Ambil $\bar{x} \in V$, maka

$$[T(\bar{x})]_\beta = [T]_\alpha^\beta [\bar{x}]_\alpha .$$

Kemudian,

$$[U(T(\bar{x}))]_\gamma = [U]_\beta^\gamma [T(\bar{x})]_\beta .$$

Jadi,

$$[(U \circ T)(\bar{x})]_\gamma = [U]_\beta^\gamma [T]_\alpha^\beta [\bar{x}]_\alpha .$$

Artinya, matriks representasi dari $U \circ T : V \to Z$ thd basis α dan γ adl

$$[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

<u>Contoh</u> Mis $U : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}[x]_{\leq 2}$ dan $T : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 3}$ pemetaan linear dengan

$$U(f(x)) = f'(x) \quad \text{dan} \quad T(f(x)) = \int_0^x f(t)\,dt.$$

Mis α dan β msg² basis terurut standar bagi $\mathbb{R}[x]_{\leq 3}$ dan $\mathbb{R}[x]_{\leq 2}$. Tentukan $[U]_\alpha^\beta$, $[T]_\beta^\alpha$, dan $[U \circ T]_\beta$.

<u>Jawab</u>

Diket $\alpha = \{1, x, x^2, x^3\}$ dan $\beta = \{1, x, x^2\}$.

Kita hitung

$$U(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2,$$
$$U(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2,$$
$$U(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2,$$
$$U(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2,$$

shg

$$[U]_\alpha^\beta = \begin{pmatrix} [U(1)]_\beta & [U(x)]_\beta & [U(x^2)]_\beta & [U(x^3)]_\beta \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Kita hitung

$$T(1) = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3,$$
$$T(x) = \tfrac{1}{2}x^2 = 0 \cdot 1 + 0 \cdot x + \tfrac{1}{2}x^2 + 0 \cdot x^3,$$

$$T(x^2) = \frac{1}{3}x^3 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \frac{1}{3}\cdot x^3,$$

shg

$$[T]_\beta^\alpha = \begin{pmatrix} [T(1)]_\alpha & [T(x)]_\alpha & [T(x^2)]_\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Jadi,

$$[U \circ T]_\beta = [U]_\alpha^\beta \, [T]_\beta^\alpha$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3.$$

2.4 Invertibilitas dan isomorfisma [F, sec. 2.4]

Mis V, W ruang vektor atas lapangan $\mathbb{F}$. Pemetaan linear $T: V \to W$ yg invertibel (memiliki invers / satu² dan pada) disebut <u>isomorfisma</u>.

<u>Teorema</u> Jika $T: V \to W$ isomorfisma, maka inversnya $T^{-1}: W \to V$ juga linear.

<u>Bukti</u>

Ambil $\alpha, \beta \in \mathbb{F}$ dan $\bar{x}, \bar{y} \in W$. Karena $T: V \to W$ isomorfisma, maka T pada, shg ada $\bar{x}', \bar{y}' \in V$ shg $T(\bar{x}') = \bar{x}$ dan $T(\bar{y}') = \bar{y}$. Perhatikan

$$\begin{aligned} T^{-1}(\alpha\bar{x} + \beta\bar{y}) &= T^{-1}(\alpha T(\bar{x}') + \beta T(\bar{y}')) \\ &= T^{-1}(T(\alpha\bar{x}' + \beta\bar{y}')) \\ &= (T^{-1} \circ T)(\alpha\bar{x}' + \beta\bar{y}') \\ &= \alpha\bar{x}' + \beta\bar{y}' \\ &= \alpha T^{-1}(\bar{x}) + \beta T^{-1}(\bar{y}). \end{aligned}$$

Jadi, T^{-1} linear. ▨

Jika ada isomorfisma dari V ke W, maka V dan W dikatakan bersifat <u>isomorfik</u>, dan ditulis $V \cong W$. Dua ruang vektor yg isomorfik berstruktur sama persis; perbedaan yg mungkin hanyalah perbedaan wujud anggota^{2}nya.

<u>Contoh</u> Definisikan $T: \mathbb{R}^2 \to \mathbb{R}[x]_{\leq 1}$ dengan

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2 x.$$

<u>Adb T linear:</u>

Ambil $\alpha, \beta \in \mathbb{R}$ dan $\bar{x}, \bar{y} \in \mathbb{R}^2$. Tulis $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ dan $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ untuk suatu $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Perhatikan bahwa

$$T(\alpha \bar{x} + \beta \bar{y}) = T\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$$

$$= T\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}$$

$$= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2)x$$

$$= \alpha(x_1 + x_2 x) + \beta(y_1 + y_2 x)$$

$$= \alpha T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \alpha T(\bar{x}) + \beta T(\bar{y}).$$

Jadi, T linear.

Adb T satu-satu :

Karena

$$\mathrm{Ker}(T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \; ; \; T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 + 0x \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \; ; \; x_1 + x_2 x = 0 + 0x \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \; ; \; x_1 = 0 \text{ dan } x_2 = 0 \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

maka $\mathrm{null}(T) = 0$, shg T satu-satu.

Adb T pada;

Menurut Teorema Rank-Nulitas,

$$\mathrm{rank}(T) = \dim(\mathbb{R}^2) - \mathrm{null}(T) = 2 - 0 = 2 = \dim(\mathbb{R}[x]_{\leq 1}),$$

shg T pada.

Jadi, T isomorfisma. Artinya $\mathbb{R}^2 \cong \mathbb{R}[x]_{\leq 1}$.

Lebih lanjut, dapat dibuktikan bahwa ruang² vektor berikut isomorfik:

$$\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\},$$

$$\mathbb{R}[x]_{\leq 1} = \{x_1 + x_2 x : x_1, x_2 \in \mathbb{R}\},$$

$$\mathbb{C} = \{x_1 + x_2 i : x_1, x_2 \in \mathbb{R}\},$$

$$\left\{\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R}\right\},$$

$$\left\{\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R}\right\},$$

dsb. Bahkan, semua ruang vektor berdimensi sama isomorfik:

<u>Teorema</u> Ruang² vektor V, W isomorfik jika dan hanya jika $\dim(V) = \dim(W)$.

<u>Bukti</u>

($\Rightarrow$) Mis V, W isomorfik, artinya ada isomorfisma $T: V \to W$. Menurut Teorema Rank-Nulitas,

$$\begin{aligned}\dim(V) &= \operatorname{rank}(T) + \operatorname{null}(T) \\ &= \dim(W) + 0 \\ &= \dim(W).\end{aligned}$$

($\Leftarrow$) Mis $\dim(V) = \dim(W) = n$. Adb $V \cong W$.

Mis $\{\bar{x}_1, \dots, \bar{x}_n\}$ basis bagi V dan $\{\bar{y}_1, \dots, \bar{y}_n\}$ basis bagi W, maka ada tunggal pemetaan linear $T : V \to W$ yg memenuhi $T(\bar{x}_i) = \bar{y}_i$ utk setiap $i \in \{1, \dots, n\}$. Karena $V = \text{span}\{\bar{x}_1, \dots, \bar{x}_n\}$ maka

$$\begin{aligned} \text{Im}(T) &= \text{span}\{T(\bar{x}_1), \dots, T(\bar{x}_n)\} \\ &= \text{span}\{\bar{y}_1, \dots, \bar{y}_n\} \\ &= W, \end{aligned}$$

shg T pada dan $\text{rank}(T) = \dim(W) = n$.

Menurut Teorema Rank-Nulitas,

$$\text{null}(T) = \dim(V) - \text{rank}(T) = n - n = 0,$$

shg T satu-satu. Jadi, T isomorfisma.

Dengan demikian, $V \cong W$. $\blacksquare$

Jadi, utk setiap $n \in \mathbb{N}$, semua ruang vektor berdimensi n isomorfik dengan $\mathbb{F}^n$.

<u>Teorema</u> Pemetaan linear $T : V \to W$ invertibel jika dan hanya jika matriks $[T]_\beta^\gamma$ invertibel utk suatu basis β bagi V dan γ bagi W; dalam hal itu berlaku

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1},$$

yaitu matriks representasi dari T^{-1} adl invers dari matriks representasi dari T.

Bukti

($\Rightarrow$) Mis $T: V \to W$ invertibel. Ambil basis β bagi V dan γ bagi W. Karena T invertibel, maka berdasarkan teorema terakhir, $|\beta| = |\gamma| = n$ utk suatu $n \in \mathbb{N}$. Dengan demikian,

$$I_n = [I_V]_\beta = [T^{-1} \circ T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma.$$

Artinya, $[T]_\beta^\gamma$ invertibel, dan berlaku

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}.$$

($\Leftarrow$) Mis $A := [T]_\beta^\gamma$ invertibel utk suatu basis β bagi V dan γ bagi W. Artinya, A adalah matriks persegi berorde $|\beta| = |\gamma| = n$, utk suatu $n \in \mathbb{N}$, dan ada $B = (b_{i,j}) \in \mathbb{F}^{n \times n}$ shg $AB = BA = I_n$. Mis $\beta = \{\bar{x}_1, \ldots, \bar{x}_n\}$ dan $\gamma = \{\bar{y}_1, \ldots, \bar{y}_n\}$. Utk membuktikan $T: V \to W$ invertibel, harus dicari pemetaan linear $U: W \to V$ yg memenuhi $U \circ T = I_V$ dan $T \circ U = I_W$. Mis

$$\bar{z}_1 := b_{1,1}\bar{x}_1 + b_{2,1}\bar{x}_2 + \cdots + b_{n,1}\bar{x}_n,$$

$$\bar{z}_2 := b_{1,2}\bar{x}_1 + b_{2,2}\bar{x}_2 + \cdots + b_{n,2}\bar{x}_n,$$

$$\vdots$$

$$\bar{z}_n := b_{1,n}\bar{x}_1 + b_{2,n}\bar{x}_2 + \cdots + b_{n,n}\bar{x}_n.$$

Karena $\bar{z}_1, \ldots, \bar{z}_n \in V$ dan $\gamma = \{\bar{y}_1, \ldots, \bar{y}_n\}$, maka ada tunggal pemetaan linear $U: W \to V$ yg memenuhi $U(\bar{y}_i) = \bar{z}_i$ utk setiap $i \in \{1, \ldots, n\}$.

Matriks representasi U thd γ dan β adl

$$
\begin{aligned}
[U]_\gamma^\beta &= \left([U(\bar{y}_1)]_\beta \quad [U(\bar{y}_2)]_\beta \quad \cdots \quad [U(\bar{y}_n)]_\beta \right) \\
&= \left([\bar{z}_1]_\beta \quad [\bar{z}_2]_\beta \quad \cdots \quad [\bar{z}_n]_\beta \right) \\
&= \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix} \\
&= B.
\end{aligned}
$$

Perhatikan bahwa

$$[U \circ T]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$$

dan

$$[T \circ U]_\gamma = [T]_\beta^\gamma [U]_\gamma^\beta = AB = I_n = [I_W]_\gamma.$$

Jadi, $U \circ T = I_V$ dan $T \circ U = I_W$. Artinya, T invertibel. ∎