4.2 Basis ortogonal dan prosedur Gram-Schmidt [J, sec. 4.2]

Contoh Basis standar $\{\bar{e}_1, \ldots, \bar{e}_n\}$ bagi $\mathbb{R}^n$ bersifat ortonormal thd HKD standar.

Contoh Himp $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, dengan

$$\bar{x}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \bar{x}_2 = \begin{pmatrix} 10 \\ -11 \\ -2 \end{pmatrix}, \quad \bar{x}_3 = \begin{pmatrix} 5 \\ 2 \\ 14 \end{pmatrix},$$

bersifat ortogonal thd HKD standar, karena

$$\langle \bar{x}_1, \bar{x}_2 \rangle = 2 \cdot 10 + 2 \cdot (-11) + (-1)(-2) = 0,$$

$$\langle \bar{x}_1, \bar{x}_3 \rangle = 2 \cdot 5 + 2 \cdot 2 + (-1) \cdot 14 = 0,$$

$$\langle \bar{x}_2, \bar{x}_3 \rangle = 10 \cdot 5 + (-11) \cdot 2 + (-2) \cdot 14 = 0,$$

shg himp tsb bbl. Karena himp tsb berkardinalitas $3 = \dim(\mathbb{R}^3)$, maka himp tsb merupakan basis ortogonal bagi $\mathbb{R}^3$. Basis ini tidak ortonormal, karena, misalnya, $\|\bar{x}_1\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3 \neq 1$. Namun, $\{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ merupakan basis ortonormal, di mana

$$\bar{y}_1 = \frac{\bar{x}_1}{\|\bar{x}_1\|} = \frac{1}{\sqrt{2^2+2^2+(-1)^2}} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix},$$

$$\bar{y}_2 = \frac{\bar{x}_2}{\|\bar{x}_2\|} = \frac{1}{\sqrt{10^2+(-11)^2+(-2)^2}} \begin{pmatrix} 10 \\ -11 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{11}{15} \\ -\frac{2}{15} \end{pmatrix},$$

$$\bar{y}_3 = \frac{\bar{x}_3}{\|\bar{x}_3\|} = \frac{1}{\sqrt{5^2+2^2+14^2}} \begin{pmatrix} 5 \\ 2 \\ 14 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{15} \\ \frac{14}{15} \end{pmatrix}.$$

Keuntungan penting dari penggunaan basis ortogonal adl koordinat dari setiap vektor dpt dinyatakan scr eksplisit dgn menggunakan HKD. Mis V RHKD dgn HKD $(\bar{x}, \bar{y}) \mapsto \langle \bar{x}, \bar{y} \rangle$.
Mis $\{\bar{x}_1, \dots, \bar{x}_n\}$ basis bagi V, maka utk setiap $\bar{y} \in V$ ada tunggal $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ shg

$$\bar{y} = \alpha_1 \bar{x}_1 + \dots + \alpha_n \bar{x}_n.$$

Utk mencari $\alpha_1, \dots, \alpha_n$ tsb, biasanya digunakan OBE. Tetapi jika basis $\{\bar{x}_1, \dots, \bar{x}_n\}$ ortogonal, maka utk setiap $i \in \{1, \dots, n\}$ berlaku

$$\begin{aligned}
\langle \bar{y}, \bar{x}_i \rangle &= \langle \alpha_1 \bar{x}_1 + \dots + \alpha_n \bar{x}_n, \bar{x}_i \rangle \\
&= \alpha_1 \langle \bar{x}_1, \bar{x}_i \rangle + \dots + \alpha_i \langle \bar{x}_i, \bar{x}_i \rangle + \dots + \alpha_n \langle \bar{x}_n, \bar{x}_i \rangle \\
&= \alpha_1 0 + \dots + \alpha_i \|\bar{x}_i\|^2 + \dots + \alpha_n 0 \\
&= \alpha_i \|\bar{x}_i\|^2,
\end{aligned}$$

shg

$$\alpha_i = \frac{\langle \bar{y}, \bar{x}_i \rangle}{\|\bar{x}_i\|^2}.$$

Jika $\{\bar{x}_1, \dots, \bar{x}_n\}$ ortonormal, maka

$$\alpha_i = \langle \bar{y}, \bar{x}_i \rangle.$$

Masalah (ortogonalisasi/ortonormalisasi)

Diberikan basis suatu ruang vektor. ubahlah basis tsb mjd basis ortogonal/ortonormal.

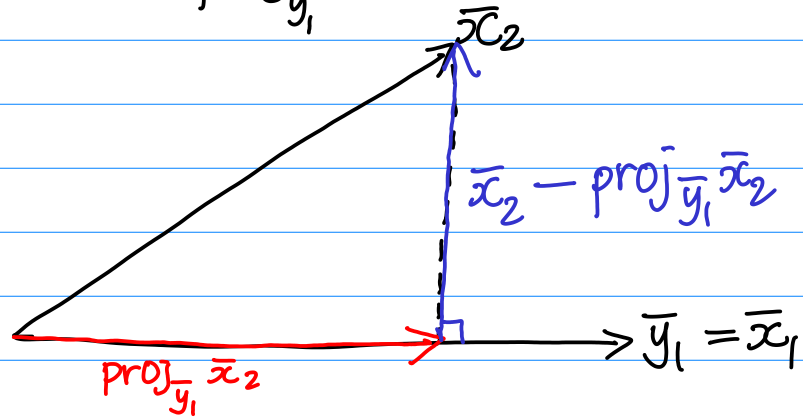
Ortogonalisasi di $\mathbb{R}^2$

Mis $\{\bar{x}_1, \bar{x}_2\} \subseteq \mathbb{R}^2$ suatu basis. Kita ingin mengubah basis ini mjd basis ortogonal

$\{\bar{y}_1, \bar{y}_2\} \subseteq \mathbb{R}^2$. Pertama, pilih

$$\bar{y}_1 = \bar{x}_1.$$

Utk memilih $\bar{y}_2$, proyeksikan $\bar{x}_2$ ke $\bar{y}_1$. Proyeksi $\bar{x}_2$ ke $\bar{y}_1$ ditulis $\mathrm{proj}_{\bar{y}_1} \bar{x}_2$.



Karena $\bar{y}_1$ searah dgn $\mathrm{proj}_{\bar{y}_1} \bar{x}_2$, maka keduanya memiliki vektor satuan yg sama:

$$\frac{\bar{y}_1}{\|\bar{y}_1\|} = \frac{\mathrm{proj}_{\bar{y}_1} \bar{x}_2}{\|\mathrm{proj}_{\bar{y}_1} \bar{x}_2\|},$$

shg

$$\mathrm{proj}_{\bar{y}_1} \bar{x}_2 = \frac{\|\mathrm{proj}_{\bar{y}_1} \bar{x}_2\|}{\|\bar{y}_1\|} \bar{y}_1$$

$$= \frac{\|\bar{x}_2\| \cos \measuredangle(\bar{x}_2, \bar{y}_1)}{\|\bar{y}_1\|} \bar{y}_1$$

$$= \frac{\|\bar{x}_2\| \dfrac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\|\bar{x}_2\| \, \|\bar{y}_1\|}}{\|\bar{y}_1\|} \bar{y}_1$$

$$= \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1.$$

Sekarang pilih

$$\bar{y}_2 = \bar{x}_2 - \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1.$$

Jadi, $\{\bar{y}_1, \bar{y}_2\} \subseteq \mathbb{R}^2$, dgn

$$\bar{y}_1 = \bar{x}_1,$$
$$\bar{y}_2 = \bar{x}_2 - \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1,$$

merupakan basis ortogonal.

Perumuman prosedur ortogonalisasi di atas disebut <u>prosedur Gram-Schmidt</u>.

<u>Teorema (Gram-Schmidt)</u>

Mis $i \geq 2$. Mis $\{\bar{y}_1, \ldots, \bar{y}_{i-1}\}$ himp ortogonal di V, dan $\bar{x}_i \in V$. Jika

$$\bar{y}_i = \bar{x}_i - \frac{\langle \bar{x}_i, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1 - \frac{\langle \bar{x}_i, \bar{y}_2 \rangle}{\|\bar{y}_2\|^2} \bar{y}_2 - \cdots - \frac{\langle \bar{x}_i, \bar{y}_{i-1} \rangle}{\|\bar{y}_{i-1}\|^2} \bar{y}_{i-1} \quad \star$$

maka $\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\}$ ortogonal dan

$$\operatorname{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{x}_i\} = \operatorname{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\}.$$

<u>Bukti</u>

Adb $\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\}$ ortogonal. Karena diketahui $\{\bar{y}_1, \ldots, \bar{y}_{i-1}\}$ ortogonal, maka cukup dibuktikan $\bar{y}_i$ ortogonal dgn $\bar{y}_1, \ldots, \bar{y}_{i-1}$. Utk setiap $j \in \{1, \ldots, i-1\}$ berlaku

$$\langle \bar{y}_i, \bar{y}_j \rangle = \left\langle \bar{x}_i - \frac{\langle \bar{x}_i, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1 - \frac{\langle \bar{x}_i, \bar{y}_2 \rangle}{\|\bar{y}_2\|^2} \bar{y}_2 - \cdots - \frac{\langle \bar{x}_i, \bar{y}_{i-1} \rangle}{\|\bar{y}_{i-1}\|^2} \bar{y}_{i-1}, \bar{y}_j \right\rangle$$

$$= \langle \bar{x}_i, \bar{y}_j \rangle - \frac{\langle \bar{x}_i, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \langle \bar{y}_1, \bar{y}_j \rangle - \cdots - \frac{\langle \bar{x}_i, \bar{y}_j \rangle}{\|\bar{y}_j\|^2} \langle \bar{y}_j, \bar{y}_j \rangle - \cdots$$
$$- \frac{\langle \bar{x}_i, \bar{y}_{i-1} \rangle}{\|\bar{y}_{i-1}\|^2} \langle \bar{y}_{i-1}, \bar{y}_j \rangle$$

$$= \langle \bar{x}_i, \bar{y}_j \rangle - \frac{\langle \bar{x}_i, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} 0 - \cdots - \frac{\langle \bar{x}_i, \bar{y}_j \rangle}{\|\bar{y}_j\|^2} \|\bar{y}_j\|^2 - \cdots - \frac{\langle \bar{x}_i, \bar{y}_{i-1} \rangle}{\|\bar{y}_{i-1}\|^2} 0$$

$$= \langle \bar{x}_i, \bar{y}_j \rangle - \langle \bar{x}_i, \bar{y}_j \rangle$$

$$= 0.$$

Sekarang tuliskan ☆ sbg

$$\bar{y}_i = \bar{x}_i + \beta_1 \bar{y}_1 + \beta_2 \bar{y}_2 + \cdots + \beta_{i-1} \bar{y}_{i-1},$$

dgn $\beta_1 := -\frac{\langle \bar{x}_i, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2}$, $\beta_2 := -\frac{\langle \bar{x}_i, \bar{y}_2 \rangle}{\|\bar{y}_2\|^2}$, ...,

$\beta_{i-1} := -\frac{\langle \bar{x}_i, \bar{y}_{i-1} \rangle}{\|\bar{y}_{i-1}\|^2}$.

Utk membuktikan $\text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{x}_i\} \subseteq \text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\}$, ambil $\bar{z} \in \text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{x}_i\}$, maka ada $\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \in \mathbb{R}$ shg

$$\begin{aligned}\bar{z} &= \alpha_1 \bar{y}_1 + \cdots + \alpha_{i-1} \bar{y}_{i-1} + \alpha_i \bar{x}_i \\ &= \alpha_1 \bar{y}_1 + \cdots + \alpha_{i-1} \bar{y}_{i-1} + \alpha_i (\bar{y}_i - \beta_1 \bar{y}_1 - \cdots - \beta_{i-1} \bar{y}_{i-1}) \\ &= (\alpha_1 - \alpha_i \beta_1) \bar{y}_1 + \cdots + (\alpha_{i-1} - \alpha_i \beta_{i-1}) \bar{y}_{i-1} + \alpha_i \bar{y}_i,\end{aligned}$$

artinya $\bar{z} \in \text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\}$.

Utk membuktikan $\text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\} \subseteq \text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{x}_i\}$, ambil $\bar{z} \in \text{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_i\}$, maka ada $\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \in \mathbb{R}$ shg

$$\begin{aligned}\bar{z} &= \alpha_1 \bar{y}_1 + \cdots + \alpha_{i-1} \bar{y}_{i-1} + \alpha_i \bar{y}_i \\ &= \alpha_1 \bar{y}_1 + \cdots + \alpha_{i-1} \bar{y}_{i-1} + \alpha_i (\bar{x}_i + \beta_1 \bar{y}_1 + \cdots + \beta_{i-1} \bar{y}_{i-1})\end{aligned}$$

$$= (\alpha_1 + \alpha_i \beta_1)\bar{y}_1 + \cdots + (\alpha_{i-1} + \alpha_i \beta_{i-1})\bar{y}_{i-1} + \alpha_i \bar{x}_i,$$

artinya $\bar{z} \in \operatorname{span}\{\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{x}_i\}$. $\blacksquare$

Prosedur Gram-Schmidt

Diberikan basis $\{\bar{x}_1, \ldots, \bar{x}_n\}$ bagi V. Pertama, pilih

$$\bar{y}_1 = \bar{x}_1.$$

Jelas $\operatorname{span}\{\bar{y}_1\} = \operatorname{span}\{\bar{x}_1\}$. Kemudian, pilih

$$\bar{y}_2 = \bar{x}_2 - \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2}\bar{y}_1.$$

Menurut teorema, $\{\bar{y}_1, \bar{y}_2\}$ ortogonal dan $\operatorname{span}\{\bar{y}_1, \bar{y}_2\} = \operatorname{span}\{\bar{y}_1, \bar{x}_2\} = \operatorname{span}\{\bar{x}_1, \bar{x}_2\}$.

Kemudian, pilih

$$\bar{y}_3 = \bar{x}_3 - \frac{\langle \bar{x}_3, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2}\bar{y}_1 - \frac{\langle \bar{x}_3, \bar{y}_2 \rangle}{\|\bar{y}_2\|^2}\bar{y}_2.$$

Menurut teorema, $\{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ ortogonal dan $\operatorname{span}\{\bar{y}_1, \bar{y}_2, \bar{y}_3\} = \operatorname{span}\{\bar{y}_1, \bar{y}_2, \bar{x}_3\} = \operatorname{span}\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$.

Dgn melanjutkan, diperoleh $\{\bar{y}_1, \ldots, \bar{y}_n\}$ ortogonal (shg bbl) dan $\operatorname{span}\{\bar{y}_1, \ldots, \bar{y}_n\} = \operatorname{span}\{\bar{x}_1, \ldots, \bar{x}_n\} = V$.

Jadi, $\{\bar{y}_1, \ldots, \bar{y}_n\}$ adl basis ortogonal bagi V.

Kesimpulannya:

Teorema Semua ruang vektor berdimensi berhingga memiliki basis ortogonal (shg juga memiliki basis ortonormal).

Contoh

Gunakan prosedur Gram-Schmidt utk mengubah basis $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ bagi $\mathbb{R}^3$ (dgn HKD standar), dgn

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \bar{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad \bar{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

menjadi: (a) basis ortogonal, (b) basis ortonormal.

Jawab

(a) Pilih

$$\bar{y}_1 = \bar{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

$$\bar{y}_2 = \bar{x}_2 - \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{(-1)\cdot 1 + 0\cdot 2 + 2\cdot 2}{1^2 + 2^2 + 2^2} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix},$$

$$\bar{y}_3 = \bar{x}_3 - \frac{\langle \bar{x}_3, \bar{y}_1 \rangle}{\|\bar{y}_1\|^2} \bar{y}_1 - \frac{\langle \bar{x}_3, \bar{y}_2 \rangle}{\|\bar{y}_2\|^2} \bar{y}_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{0\cdot 1 + 0\cdot 2 + 1\cdot 2}{1^2 + 2^2 + 2^2} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \frac{0\left(-\frac{4}{3}\right) + 0\left(-\frac{2}{3}\right) + 1\left(\frac{4}{3}\right)}{\left(-\frac{4}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{9} \\ -\frac{2}{9} \\ \frac{1}{9} \end{pmatrix}.$$

Berdasarkan prosedur Gram-Schmidt, $\{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ merupakan basis ortogonal.

(b) Basis $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\}$ merupakan basis ortonormal, di mana

$$\bar{z}_1 = \frac{\bar{y}_1}{\|\bar{y}_1\|} = \frac{1}{\sqrt{1^2+2^2+2^2}}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{pmatrix},$$

$$\bar{z}_2 = \frac{\bar{y}_2}{\|\bar{y}_2\|} = \frac{1}{\sqrt{\left(-\frac{4}{3}\right)^2+\left(-\frac{2}{3}\right)^2+\left(\frac{4}{3}\right)^2}}\begin{pmatrix}-\frac{4}{3}\\ -\frac{2}{3}\\ \frac{4}{3}\end{pmatrix}$$

$$= \frac{\cancel{\frac{2}{3}}}{\cancel{\frac{2}{3}}\sqrt{(-2)^2+(-1)^2+2^2}}\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{2}{3}\\ -\frac{1}{3}\\ \frac{2}{3}\end{pmatrix},$$

$$\bar{z}_3 = \frac{\bar{y}_3}{\|\bar{y}_3\|} = \frac{1}{\sqrt{\left(\frac{2}{9}\right)^2+\left(-\frac{2}{9}\right)^2+\left(\frac{1}{9}\right)^2}}\begin{pmatrix}\frac{2}{9}\\ -\frac{2}{9}\\ \frac{1}{9}\end{pmatrix}$$

$$= \frac{\cancel{\frac{1}{9}}}{\cancel{\frac{1}{9}}\sqrt{2^2+(-2)^2+1^2}}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{3}\\ -\frac{2}{3}\\ \frac{1}{3}\end{pmatrix}.$$

Dari contoh di atas, misalkan

$$A = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

Kemudian, setiap $\bar{x}_i$ merupakan komlin dari basis ortonormal $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\}$ dgn koefisien^2 yg dpt dihitung dgn HKD:

$$\bar{x}_1 = \langle \bar{x}_1, \bar{z}_1 \rangle \bar{z}_1 + 0\,\bar{z}_2 + 0\,\bar{z}_3,$$

$$\bar{x}_2 = \langle \bar{x}_2, \bar{z}_1 \rangle \bar{z}_1 + \langle \bar{x}_2, \bar{z}_2 \rangle \bar{z}_2 + 0\,\bar{z}_3,$$

$$\bar{x}_3 = \langle \bar{x}_3, \bar{z}_1 \rangle \bar{z}_1 + \langle \bar{x}_3, \bar{z}_2 \rangle \bar{z}_2 + \langle \bar{x}_3, \bar{z}_3 \rangle \bar{z}_3.$$

Jadi,

$$A = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \langle \bar{x}_1, \bar{z}_1 \rangle & \langle \bar{x}_2, \bar{z}_1 \rangle & \langle \bar{x}_3, \bar{z}_1 \rangle \\ 0 & \langle \bar{x}_2, \bar{z}_2 \rangle & \langle \bar{x}_3, \bar{z}_2 \rangle \\ 0 & 0 & \langle \bar{x}_3, \bar{z}_3 \rangle \end{pmatrix}}_{R}$$

$$= QR,$$

dengan

$$Q = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{pmatrix}$$

adl suatu matriks yg vektor^2 kolomnya ortonormal (disebut <u>matriks ortogonal</u> dan memenuhi $Q^TQ = I$) dan

$$R = \begin{pmatrix} \langle \bar{x}_1, \bar{z}_1 \rangle & \langle \bar{x}_2, \bar{z}_1 \rangle & \langle \bar{x}_3, \bar{z}_1 \rangle \\ 0 & \langle \bar{x}_2, \bar{z}_2 \rangle & \langle \bar{x}_3, \bar{z}_2 \rangle \\ 0 & 0 & \langle \bar{x}_3, \bar{z}_3 \rangle \end{pmatrix}$$

adl suatu matriks segitiga atas. Penulisan $A=QR$ tsb disebut faktorisasi/dekomposisi QR dari matriks A.

Lebih lanjut, karena vektor² kolom dari A membentuk basis, maka $A\bar{x}=\bar{b}$ memiliki solusi tunggal; solusi tsb dpt dicari tanpa OBE maupun invers:

$$(QR)\bar{x}=\bar{b} \Rightarrow (Q^TQ)R\bar{x}=Q^T\bar{b} \Rightarrow R\bar{x}=Q^T\bar{b}.$$

Karena R matriks segitiga atas, maka $\bar{x}$ dpt dicari dgn substitusi balik. Fakta ini dpt dimanfaatkan dlm komputasi.

Contoh Diket $R[x]_{\leq 2}$ dgn HKD

$$\langle f(x), g(x)\rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$

Gunakan prosedur Gram-Schmidt utk mengortogonalisasi basis $\{f_1(x), f_2(x), f_3(x)\}$ dgn $f_1(x)=1$, $f_2(x)=x$, dan $f_3(x)=x^2$.

Jawab

Pilih

$$g_1(x)=f_1(x)=1,$$

$$g_2(x)=f_2(x)-\frac{\langle f_2(x), g_1(x)\rangle}{\|g_1(x)\|^2}g_1(x)$$

$$=x-\frac{\int_{-1}^{1}x\,dx}{\int_{-1}^{1}1\,dx}\,1$$

$$=x-\frac{\left[\frac{x^2}{2}\right]_{-1}^{1}}{[x]_{-1}^{1}}$$

$$=x,$$

$$g_3(x) = f_3(x) - \frac{\langle f_3(x), g_1(x) \rangle}{\|g_1(x)\|^2} g_1(x) - \frac{\langle f_3(x), g_2(x) \rangle}{\|g_2(x)\|^2} g_2(x)$$

$$= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} 1 - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} x$$

$$= x^2 - \frac{\left[\frac{x^3}{3}\right]_{-1}^{1}}{[x]_{-1}^{1}} - \frac{\left[\frac{x^4}{4}\right]_{-1}^{1}}{\left[\frac{x^3}{3}\right]_{-1}^{1}} x$$

$$= x^2 - \frac{\frac{2}{3}}{2}$$

$$= x^2 - \frac{1}{3}.$$

Berdasarkan prosedur Gram-Schmidt,

$$\{g_1(x), g_2(x), g_3(x)\} = \{1, x, x^2 - \tfrac{1}{3}\}$$

merupakan basis ortogonal.

Ortogonalisasi pd contoh terakhir bila dilanjutkan utk $f_4(x) = x^3$, $f_5(x) = x^4$, dst., hasilnya adl

$$g_4(x) = x^3 - \frac{3}{5}x,$$

$$g_5(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35},$$

$$g_6(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x,$$

dst. Polinomial² $g_1(x), g_2(x), \ldots$ disebut <u>polinomial² Legendre</u> dan dipakai dlm metode numerik.