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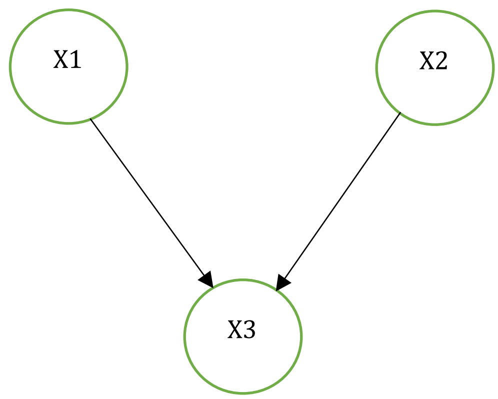
Q1. [30 pts] Bayesian Net: Conditional Independence

Consider the following random variables X_1 and X_2, which represent the outcomes of two independent coin tosses with bias (towards head, i.e. $X = 1$) 0.6. Let X_3 denotes the indicator function of the event that the outcomes are identical. That is, $X_3 = 1$ if $X_1 = X_2$ and $X_3 = 0$ otherwise.

(a) [15 pts] Specify a directed graphical model (Bayesian Network) that describes the joint probability distribution (i.e., draw the Bayesian Network and detail all conditional distributions)



P(X1=0) = 0.4, P(X1=1) = 0.6

P(X2=0) = 0.4, P(X2=1) = 0.6

we consider the conditional distribution of X3 given X1 and X2. If X1 and X2 have different values, then X3 must be 0. If X1 and X2 have the same value, then X3 must be 1. This can be expressed as:

P(X3=1 | X1=x1, X2=x2) = 1 if x1=x2

P(X3=1 | X1=x1, X2=x2) = 0 if x1!=x2

So,

P(X1=x1, X2=x2, X3=1) = P(X3=1 | X1=x1, X2=x2) * P(X1=x1) * P(X2=x2)

= 1(x1=x2) * 0.6 * 0.6

= 0.36(x1=x2)

P(X1=x1, X2=x2, X3=0) = P(X3=0 | X1=x1, X2=x2) * P(X1=x1) * P(X2=x2)

= 1(x1! = x2) * 0.6 * 0.4

= 0.24(x1! = x2)

P(X3=1 | X1=0, X2=0) = 1

P(X3=1 | X1=1, X2=1) = 1

P(X3=0 | X1=0, X2=1) = 1

P(X3=0 | X1=1, X2=0) = 1

P(X3=0 | X1=0, X2=0) = 0

P(X3=0 | X1=1, X2=1) = 0

P(X3=1 | X1=0, X2=1) = 0

P(X3=1 | X1=1, X2=0) = 0

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X1	X2	X3	P(X3| X1, X2)
H	H	1	1
H	T	1	0
T	H	1	0
T	T	1	1
H	H	0	0
H	T	0	1
T	H	0	1
T	T	0	0

(b) [15 pts] We mentioned in class that a distribution that factorizes according to the Bayesian Network (BN) will exhibit all conditional independences (CI) the BN implies. But conversely, the BN does not necessarily capture all CI implied numerically by the distribution. Show that this is case when we set the bias of the coin to 0.5.

when the bias of the coin is 0.5, both X1 and X2 are unbiased coin flips. In this case, the Bayesian Network that we drew in part (a) still applies, but the conditional probabilities change:

P(X1=1) = 0.5
P(X1=0) = 0.5
P(X2=1) = 0.5
P(X2=0) = 0.5
P(X3=1 | X1=x1, X2=x2) = 1(x1=x2)

P(X3=0 | X1=x1, X2=x2) = 1(x1!=x2)Using the definition of conditional independence, we can check whether the following holds:

P(X1, X2 | X3) = P(X1 | X3) * P(X2 | X3)

If this equation holds, then X1 and X2 are conditionally independent given X3. Otherwise, they are not.

Using the joint probability distribution that we derived above, we can compute each side of the equation:

P(X1=1, X2=1 | X3=1) = 0.25 / 0.5 = 0.5
P(X1=1 | X3=1) * P(X2=1 | X3=1) = 0.5 * 0.5 = 0.25

P(x2=0, x3=0) = P(x2=0) * P(x1=0) * P(x3=0|x1=0,x2=0) + P(x2=0) * P(x1=1) * P(x3=0|x1=1,x2=0)
= 0.5 * 0.5 * 0 + 0.5 * 0.5 * 1
= 0.25
P(x2=1x3=0) = P(x2=1) * P(x1=0) * P(x3=0|x1=0,x2=1) + P(x2=1) * P(x1=1) * P(x3=0|x1=1,x2=1)
= 0.5 * 0.5 * 1 + 0.5 * 0.5 * 0
= 0.25
P(x1=0, x3=0) = P(x2=0) * P(x1=0) * P(x3=0|x1=0,x2=0) + P(x2=0) * P(x1=0) * P(x3=0|x1=0,x2=1)
= 0.5 * 0.5 * 0 + 0.5 * 0.5 *1
= 0.25
P(x1=1, x3=0) = P(x2=0) * P(x1=1) * P(x3=0|x1=0,x2=1) + P(x2=1) * P(x1=1) * P(x3=0|x1=1,x2=1)
= 0.5 * 0.5b* 1 + 0.5 * 0.5 * 0
= 0.25

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X2 ⊥ X3 | NULL
X1 ⊥ X3 | NULL

Q2. [20 pts] Bayesian Net: Conditional Independence

Consider the Bayesian Net shown in Fig 1, which depicts the relationships among variables associated with chest abnormality. Answer the following questions based on the graphical model.

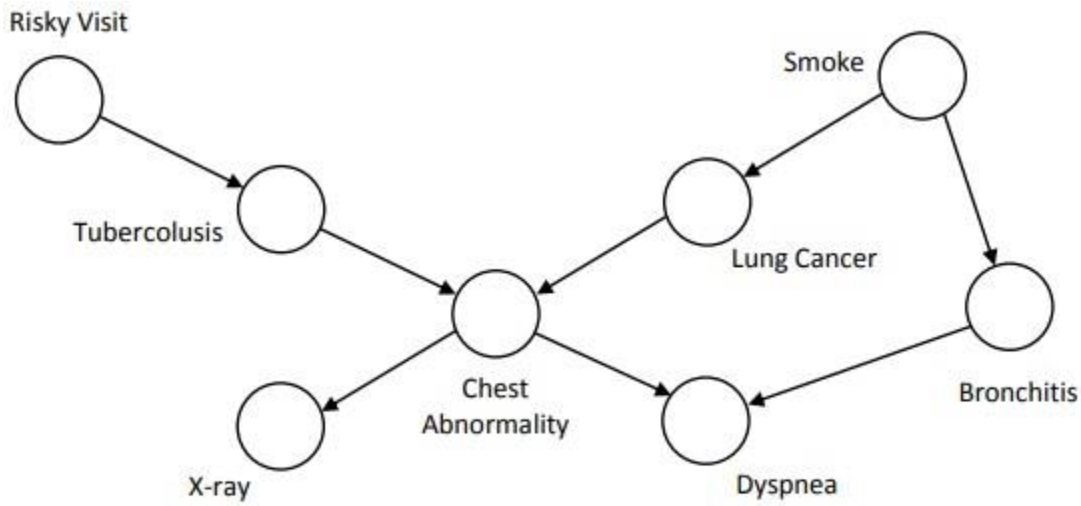

Figure 1: Bayesian Net for Chest Abnormality.

(a) [5 pts] Is (Smoke ⊥ Dyspnea | Bronchitis) True or False? Why?

Answer:
False. The statement (Smoke ⊥ Dyspnea | Bronchitis) means that given Bronchitis, Smoke and Dyspnea are independent of each other. However, in the Bayesian network, there is a direct path from Smoke to Dyspnea, which goes through Chest Abnormality. This means that even if we condition on Bronchitis, Smoke can still affect Dyspnea through Chest Abnormality. Therefore, (Smoke ⊥ Dyspnea | Bronchitis) is false.

(b) [5 pts] Is (Bronchitis ⊥ X − ray | Lung Cancer) True or False? Why?

Answer:
False. There is a direct causal path from Chest Abnormality to X-ray. This means that knowing the value of Chest Abnormality can provide information about X-ray, and thus Bronchitis and X-ray are not independent given Lung Cancer. Therefore, (Bronchitis ⊥ X-ray | Lung Cancer) is false in this case.

(c) [5 pts] Is Smoke ⊥ RiskyVisit | Dyspnea True or False? Why?

Answer:

False. Smoking and a Risky Visit can both lead to Chest Abnormalities, which can then lead to dyspnea. Furthermore, smoking is a known risk factor for lung cancer, which can also cause chest abnormalities

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and dyspnea. Therefore, smoking and a risky visit are not independent events in this context, and the statement Smoke ⊥ RiskyVisit | Dyspnea is false.

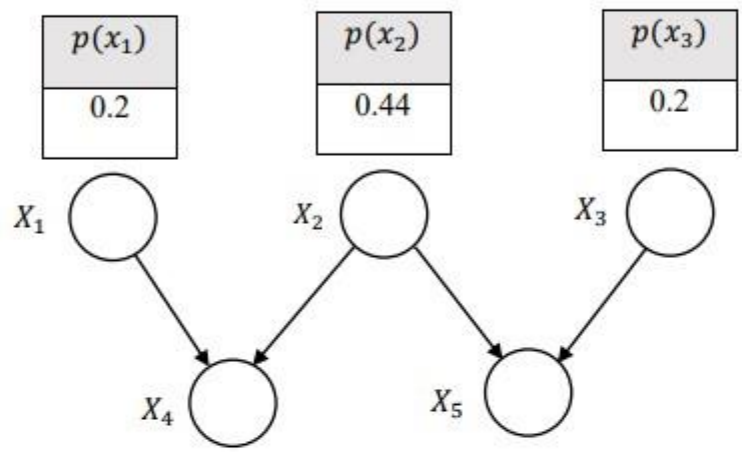
(d) [5 pts] Is X − ray ⊥ Smoke | {Cancer,Bronchitis} True or False? Why?

Answer:

True. X-ray and Smoke has no connection in between and cancer and bronchitis are both children of smoke. So X − ray ⊥ Smoke | {Cancer,Bronchitis} is false statement.

Q3. [25 pts] Bayesian Net: Inference

Given the Bayesian Net in Fig. 2, where each random variable is binary, i.e. $x_i \in \{True, False\}$. Compute the distribution tables for the following probabilities. Show step-by-step derivations clearly in each case. Provide answers accurate to 4 decimal places.



X_1	X_2	$p(x_4 \vert x_1, x_2)$
T	*T*	0.35
T	*F*	0.6
F	*T*	0.01
F	*F*	0.95

X_2	X_3	$p(x_5 \vert x_2, x_3)$
T	*T*	0.35
T	*F*	0.6
F	*T*	0.01
F	*F*	0.95

Figure 2: Bayesian Net with Conditional Probability Tables. All probabilities in the above 5 tables are given with respect to $X_1 = 1$, $X_2 = 1$, $X_3 = 1$, $X_4 = 1$ and $X_5 = 1$, respectively. Note that, the probabilities of being 0 and being 1 always sum up to 1. So, knowing the probability of being 1 is sufficient to derive the probability of being 0.

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(a) [5 pts] (a) $P(x_1 \mid x_5)$

Answer:
P(x1|x5) = P(x1)
P(x1) = 0.2
P(~x1) = 0.8

X1	X5	P(X1| X5)
T	T	0.2
T	F	0.2
F	T	0.8
F	F	0.8

(b) [5 pts] (b) $P(x_2 \mid x_4)$

Answer:
P(x4=1) = P(x5 = 1, x3 = 1, x2 =1) + P(x5=1, X3=1, X2=0) + P(x5=1,X3=0, X2=0) + P(x5=1,x3=0,x2=1)
P(x4=1) = (0.35 * 0.2 * 0.44) + (0.6 * 0.2 * 0.56) + (0.01 * 0.8 * 0.56) + (0.95*0.8*0.56)
P(x4=1) = 0.0308 + 0.0672 + 0.0045 + 0.4256
P(x4=1) = 0.5281
P(x4=0) = 1 – 0.5281
P(x4=0) = 0.4719

P(x2=0, x4=0) = P(x2=0) * P(x1=0) * P(x4=0|x1=0, x2=0) + P(x2=0) *P(x1=1) * P(x4=0|x1=1, x2=0)
P(x2=0, x4=0) = (0.56 * 0.8 * 0.05) + (0.56 * 0.2 * 0.4)
= 0.0224 + 0.0448
= 0.0672

P(x2=1, x4=0) = P(x2=1) * P(x1=0) * P(x4=0|x1=0, x2=1) + P(x2=1) *P(x1=1) * P(x4=0|x1=1, x2=1)
= (0.44 * 0.8 * 0.99) + (0.44 * 0.2 * 0.65)
= 0.3485 + 0.0572
= 0.0672

P(x2=0, x4=1) = P(x2=0) * P(x1=0) * P(x4=1|x1=0, x2=0) + P(x2=0) *P(x1=1) * P(x4=1|x1=1, x2=0)
= (0.56 * 0.8 * 0.95) + (0.56 * 0.2 * 0.6)
= 0.4256 + 0.0672
= 0.4928

P(x2=1, x4=1) = P(x2=1) * P(x1=0) * P(x4=1|x1=0, x2=1) + P(x2=1) *P(x1=1) * P(x4=0|x1=1, x2=1)
= (0.44 * 0.8 * 0.01) + (0.44 * 0.2 * 0.35)
= 0.0035 + 0.0308
= 0.0343

P(x2=0|x4=0) = 0.0672 / 0.4719 = 0.1424
P(x2=1|x4=0) = 0.4057 / 0.4719 = 0.8597
P(x2=0|x4=1) = 0.4928 / 0.5281 = 0.9332
P(x2=1|x4=1) = 0.0343 / 0.5281 = 0.0649

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X3	X2	P(X3| X2)
T	T	0.0649
T	F	0.8597
F	T	0.9332
F	F	0.1424

(c) [5 pts] (c) $P(x_3 \mid x_2)$

Answer:
P(x3|x2) = P(x3) = 0.2

X3	X2	P(X3| X2)
T	T	0.2
T	F	0.2
F	T	0.8
F	F	0.8

(d) [5 pts] (d) $P(x_4 \mid x_3)$

Answer:
P(x4|x3) = P(x4) x4 and x3 are independent
P(x4=1) = P(x4=1, x2=1, x1=1) + P(x4=1, x2=1, x1=0) + P(x4=1, x2=0, x1=0) + P(x4=1, x2=0, x1=1)
= (0.35 * 0.2 * 0.44) + (0.6 * 0.2 * 0.56) +(0.01 * 0.8 * 0.56) + (0.95 * 0.8 * 0.56)
= 0.0308 + 0.0672 + 0.045 + 0.4256
= 0.5281
P(x4=0) = 1 – 0.5281
= 0.4719

X4	X3	P(X4| X3)
T	T	0.5281
T	F	0.5281
F	T	0.4719
F	F	0.4719

(e) [5 pts] (d) $P(x_5)$

Answer:
P(x5=1) = P(x5 = 1, x3 = 1, x2 =1) + P(x5=1, X3=1, X2=0) + P(x5=1,X3=0, X2=0) + P(x5=1,x3=0,x2=1)
P(x5=1) = (0.35 * 0.2 * 0.44) + (0.6 * 0.2 * 0.56) + (0.01 * 0.8 * 0.56) + (0.95*0.8*0.56)
P(x5=1) = 0.0308 + 0.0672 + 0.0045 + 0.4256
P(x5=1) = 0.5281
P(x5=0) = 1 – 0.5281
P(x5=0) = 0.4719

X1	X5	P(X1| X5)
T	T	0.2
T	F	0.2
F	T	0.8
F	F	0.8

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Q4. [25 pts] Bayesian Net: Inference

Fig. 3 shows a Bayesian Net characterizing the causal relationship between 5 random variables x_1, x_2, x_3, x_4 and x_5 where $x_1, x_2, x_4 \in \{0,1\}$ while $x_3, x_5 \in \{0,1,2\}$. Provide answers accurate to 4 decimal places.

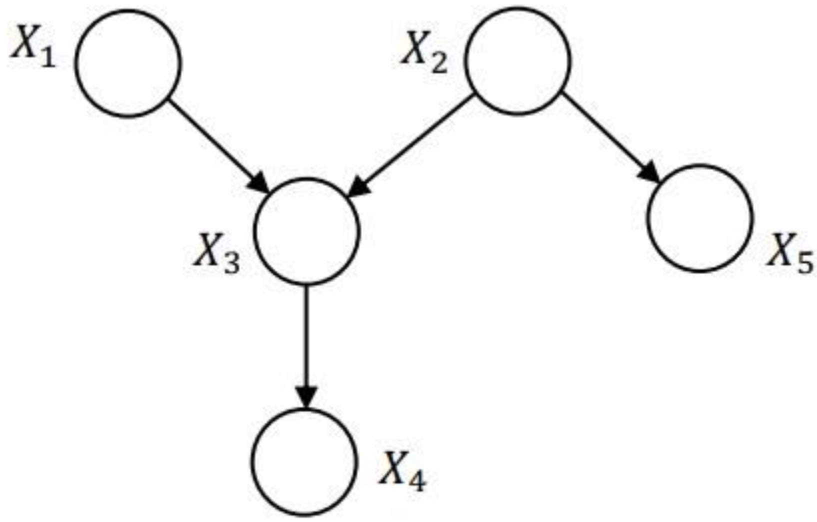

Figure 3: Bayesian Net.

X_1	X_2	X_3	$p(x_3 \mid x_1, x_2)$
0	0	0	0.3
0	0	1	0.4
0	1	0	0.9
0	1	1	0.08
1	0	0	0.05
1	0	1	0.25
1	1	0	0.5
1	1	1	0.3

X_1	$p(x_1)$
0	0.6

X_2	$p(x_2)$
0	0.7

X_3	X_4	$p(x_4 \mid x_3)$
0	0	0.1
1	0	0.4
2	0	0.99

Figure 4: Conditional Probability Tables.

(a) [5 pts] Write down all the conditional independences given by the Bayesian Net.

Answer:

X1 ⊥ NULL | NULL

X1 ⊥ X2 | NULL

X1 ⊥ X5 | NULL

X1 ⊥ X4 | X5

X1 ⊥ X2 | {X3,X4,X5}

X1 ⊥ X5 | {X3,X4}

X2 ⊥ NULL | NULL

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X2 ⊥ X1 | NULL

X2 ⊥ X1 | {X3,X5,X4}

X2 ⊥ {X1,X3,X4,X5} | NULL

X3 ⊥ NULL | {X1,X2}

X3 ⊥ X5 | {X1,X4}

X3 ⊥ X4 | NULL

X4 ⊥ {X1,X2} | X3

X4 ⊥ X5 | {X1,X2,X3}

X4 ⊥ {X1,X2} | X5

X4 ⊥ X5 | NULL

X4 ⊥ NULL | X3

X5 ⊥ {X1,X3,X4,X5} | X2

X5 ⊥ X1 | NULL

X5 ⊥ X3 | NULL

X5 ⊥ X4 | NULL

(b) [5 pts] Write down the factorized expression of the joint probability given by the Bayesian Net.

Answer:

P(x1, x2, x3, x4, x5) = P(x1) * P(x2) * P(x3 | x1, x2) * P(x4 | x3) * P(x5 | x2)

(c) [15 pts] Fig. 4 gives the probability tables of the Bayesian Net. Find the conditional probability table for $p(x_1 \mid x_3 = 1, x_2)$. Show step-by-step derivation clearly.

Answer:

P(x1= 0, x3 = 1, x2 = 0) = 0.6 * 0.4 * 0.7
= 0.168

P(x1 = 0, x3 = 1, x2 = 1) = 0.6 * 0.08 * 0.3
= 0.0144

P(x1 = 1, x3 = 1, x2 = 0) = 0.4 * 0.25 * 0.7
= 0.07

P(x1 = 1, x3 = 1, x2 = 1) = 0.4 * 0.3 * 0.3
= 0.036

P(x3 = 1, x2 = 0) = P(x2=0) * P(x1=0) * P(x3=1|x1=0,x2=0) + P(x2=0) * P(x1=1) * P(x3=1|x1=1,x2=0)
= 0.7 * 0.6 * 0.4 + 0.7 * 0.4 * 0.25
= 0.168 + 0.07
= 0.238

P(x3 = 1, x2 = 1) = P(x2=1) * P(x1=0) * P(x3=1|x1=0,x2=1) + P(x2=1) * P(x1=1) * P(x3=1|x1=1,x2=1)
= 0.3 * 0.6 * 0.08 + 0.7 * 0.4 * 0.3

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= 0.0144 + 0.084
= 0.0984

P(x1= 0 | x3 = 1, x2 = 0) = P(x1= 0, x3 = 1, x2 = 0) / P(x3 = 1, x2 = 0)
= 0.168 / 0.238
= 0.7059
P(x1= 0 | x3 = 1, x2 = 1) = P(x1= 0, x3 = 1, x2 = 1) / P(x3 = 1, x2 = 1)
= 0.0144 / 0.0984
= 0.1463
P(x1= 1 | x3 = 1, x2 = 0) = P(x1= 1, x3 = 1, x2 = 0) / P(x3 = 1, x2 = 0)
= 0.07 / 0.238
= 0.2941
P(x1= 1 | x3 = 1, x2 = 1) = P(x1= 1, x3 = 1, x2 = 1) / P(x3 = 1, x2 = 1)
= 0.036 / 0.0984
= 0.3659

X1	X2	X3	P(X1| X3, X2)
0	0	1	0.7059
0	1	1	0.1463
1	0	1	0.2941
1	1	1	0.3659