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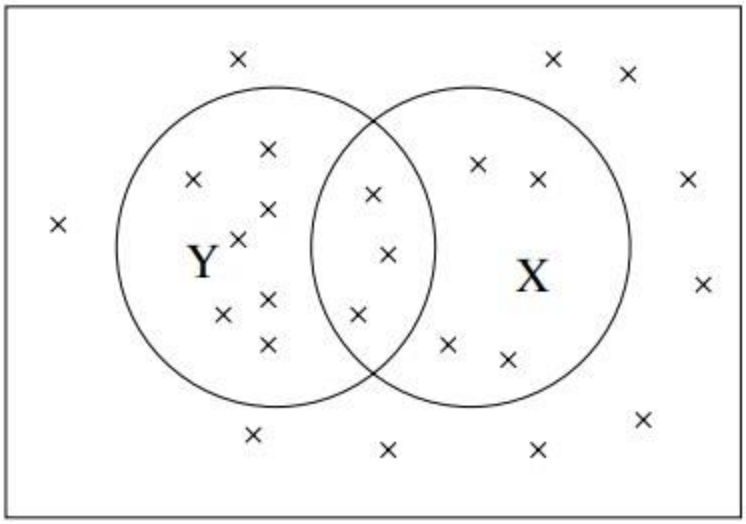
Q1. [20 pts] Probability

Based on the following Venn diagram, answer the following questions:



	X	$\neg X$
Y	$\frac{3}{24}$	
$\neg Y$		

Figure 1: Venn diagram

(a) [5 pts] Complete the joint probability distribution in the above table.

Answer:

	X	**¬*X***
Y	3/24	7/24
¬Y	4/24	10/24

(b) [12 pts] Based on the above joint probability distribution, find the following: $P(X)$, $P(Y)$, $P(\neg X)$, $P(\neg Y)$, $P(X|Y)$, $P(Y|X)$, $P(X|\neg Y)$, $P(Y|\neg X)$, $P(\neg Y|X)$, $P(\neg X|Y)$, $P(\neg X|\neg Y)$ and $P(\neg Y|\neg X)$.

Answer:

P(X) = 3/24 + 4/24 = 7/24

P(Y) = 3/24 + 7/24 = 10/24

P(¬X) = 7/24 + 10/24 = 17/24

P(¬Y) = 4/24 + 10/24 = 14/24

P(X|Y) = P(X∧Y)/P(Y) = (3/24) / (10/24) = 3/10

P(Y|X) = P(X∧Y)/P(X) = (3/24) / (7/24) = 3/7

P(X|¬Y) = P(X∧¬Y)/P(¬Y) = (4/24) / (14/24) = 2/7

P(Y|¬X) = P(¬X∧Y)/P(¬X) = (7/24) / (17/24) = 7/17

P(¬Y|X) = P(X∧¬Y)/P(X) = (4/24) / (7/24) = 4/7

P(¬X|Y) = P(¬X∧Y)/P(Y) = (7/24) / (10/24) = 7/10

P(¬X|¬Y) = P(¬X∧¬Y)/P(¬Y) = (10/24) / (14/24) = 5/7

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P(¬Y|¬X) = P(¬X∧¬Y)/P(¬X) = (10/24) / (17/24) = 10/17

(c) [3 pts] Prove the followings using only definition of conditional and marginal probability:

$$\begin{aligned} P(X|Y) &= 1 - P(\neg X|Y) \\ P(X|\neg Y) &= 1 - P(\neg X|\neg Y) \\ P(\neg Y|X) &= \frac{P(X|\neg Y)P(\neg Y)}{P(X|\neg Y)P(\neg Y) + P(X|Y)P(Y)} \end{aligned}$$

Answer:

1. P(X|Y) = P(X) ....(1)
   1 – P(¬X|Y) = 1 – P(¬X) = P(X) ....(2)
   From (1) and (2) **P(X|Y) = 1 – P(¬X|Y)**

2. P(X|¬Y) = P(X) ....(3)
   1 – P(¬X|¬Y) = 1 – P(¬X) = P(X) ....(4)
   From (3) and (4) **P(X|¬Y) = 1 – P(¬X|¬Y)**

3. P(¬Y|X) = P(¬Y) ....(5)
   P(X|¬Y) P(¬Y) / P(X|¬Y) P(¬Y) + P(X|Y) P(Y) = P(X) P(¬Y) / P(X) P(¬Y) + P(X) P(Y)
   = P(X) P(¬Y) / P(X) [P(¬Y) + P(Y)]
   = P(¬Y) / [1]
   = P(¬Y) ....(6)
   From (5) and (6) **P(¬Y|X) = P(X|¬Y) P(¬Y) / P(X|¬Y) P(¬Y) + P(X|Y) P(Y)**

Q2. [20 pts] Probabilistic Reasoning

Assume that 2% of the population in a country carry a particular virus. A test kit developed by a pharmaceutical firm is able to detect the presence of the virus from a patient's blood sample. The firm claims that the test kit has a high accuracy of detection in terms of the following conditional probabilities obtained from their quality test:

- P(the kit shows positive | the patient is a carrier) = 0.998
- P(the kit shows negative | the patient is not a carrier) = 0.996

(a) [10 pts] Given that a patient is tested to be positive using this kit, what is the probability that he is not a carrier? Give your answer to 3 decimal places.

Answer:

X represent the test kit shows positive and ¬X represent the test kit shows negative. Y and ¬Y represent the patient is a carrier and not a carrier, respectively.

P(Y) = 0.02

So, P(¬Y) = 1 − P(Y) = 1−0.02 = 0.98

P(X|Y) = 0.998

P(¬X|¬Y) = 0.996

So, P(X|¬Y) = 1 − P(¬X|¬Y) = 1 − 0.996 = 0.004

Applying Bayes' Rule,

P(¬Y|X) = [P(X|¬Y) P(¬Y)] / [P(X|¬Y) P(¬Y) + P(X|Y) P(Y)]

= [0.004 ×0.98] / [0.004 × 0.98 +0.998×0.02]

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= 0.00392 / 0.02388

= 0.164

(b) [10 pts] Suppose that the patient does not entirely trust the result offered by the first kit (perhaps because it has expired) and decides to use another test kit. If the patient is again tested to be positive using this second kit, what is the (updated) likelihood that he is not a carrier? You can assume conditional independence between results of different test kits given the patient's state of virus contraction. Give your answer to 4 decimal places.

Answer: $P(\neg Y, \neg Y|X) = P(\neg Y|X) * P(\neg Y|X)$
$= 0.164 * 0.164$
$= 0.026896$

Q3. [25 pts] Conditional Probability

Prove the following:

(a) [10 pts] (a) $P(a|a \land b) = 1$ from using only the definition of conditional independence.

Answer:
$P(a|a \land b) = P(a \land a \land b) / P(a \land b)$
$= P(a \land b) / P(a \land b)$
$= 1$

(b) [15 pts] (b) Show that (1) $P(a|b) = P(a)$, (2) $P(b|a) = P(b)$ and (3) $P(a \land b) = P(a)P(b)$ are equivalent. That is, any two of these statements are equivalent.

Answer:

1) $P(a \mid b) = P(a)$
$P(a \mid b) = P(a \land b) / P(b)$
$= P(a)P(b) / P(b)$
$= P(a)$

2) $P(b \mid a) = P(b)$
$P(b \mid a) = P(a \land b) / P(a)$
$= P(a)P(b) / P(a)$
$= P(b)$

3) $P(a \land b) = P(a)P(b)$
$P(a \land b) = P(b \mid a)P(a)$
$= P(b) P(a)$
$= P(a)P(b)$

OR

$P(a \land b) = P(a \mid b)P(b)$
$= P(a) P(b)$

Q4. [35 pts] Bayesian Network

Given the following conditional table:

$P(WetGrass \mid Sprinkler \land Rain)$	0.95
$P(WetGrass \mid Sprinkler \land \neg Rain)$	0.9
$P(WetGrass \mid \neg Sprinkler \land Rain)$	0.8
$P(WetGrass \mid \neg Sprinkler \land \neg Rain)$	0.1
$P(Sprinkler \mid RainySeason)$	0.0
$P(Sprinkler \mid \neg RainySeason)$	1.0

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$P(Rain \mid RainySeason)$	0.9
$P(Rain \mid \neg RainySeason)$	0.1
$P(RainySeason)$	0.7

(a) [5 pts] Show that $P(S) = P(\neg RS)$ or in other words, $S \equiv \neg RS$.

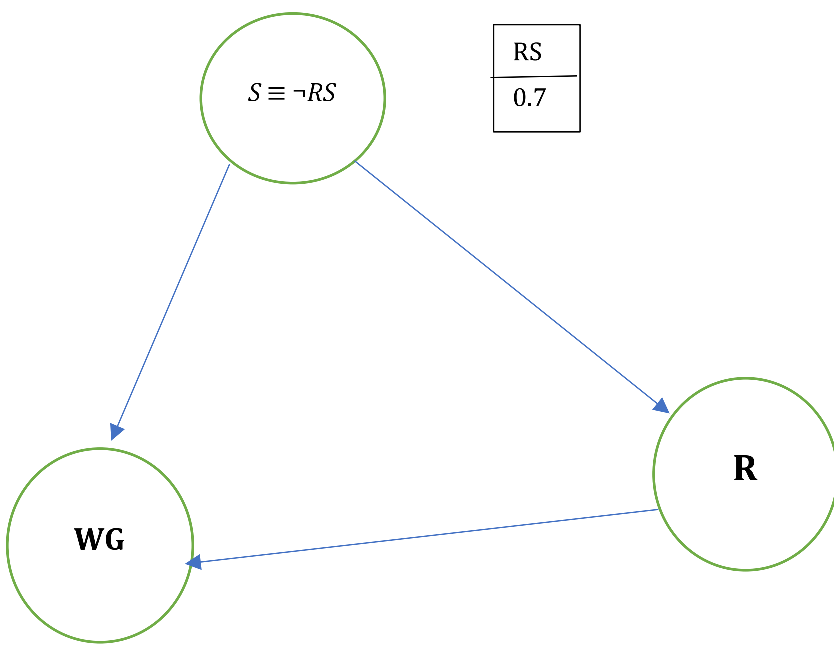
Answer:
P(RainySeason) = P(RS) = 0.7
P(¬RS) = 1 – 0.7 = 0.3

P(S|¬RS) P(¬RS) / P(S|¬RS) P(¬RS) + P(S|RS) P(RS) = P(¬RS)
P(S) P(¬RS) / P(S) P(¬RS) + P(S) P(RS) = P(¬RS)
P(S) (0.3)/P(S) (0.3) + P(S) (0.7) = 0.7
P(S) = 0.3

So, S ≡ ¬RS

(b) [20 pts] Construct a Bayesian Network (BN) with as few parameters as possible (hint: use result of part (a)). You will only get 50% credit for this question if the BN is not optimal in the number of parameters.

Answer:



RS
0.7

RS	P(R)
T	0.9
F	0.1

RS	R	PR
T	T	0.8
T	F	0.1
F	T	0.95
F	F	0.9

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(c) [10 pts] Compute the following probability:

$P(WetGrass \wedge RainySeason \wedge \neg Rain \wedge \neg Sprinkler)$
Answer:

P(WetGrass ∧ RainySeason ∧ ¬Rain ∧ ¬Sprinkler)
= P(WetGrass | RainySeason ∧ ¬Rain ∧ ¬Sprinkler) P(RainySeason ∧ ¬Rain ∧ ¬Sprinkler)
= P(WetGrass | RainySeason ∧ ¬Rain ∧ ¬Sprinkler) P(¬Rain | RainySeason ∧ ¬Sprinkler) P(RainySeason ∧ ¬Sprinkler)
= P(WetGrass | RainySeason ∧ ¬Rain ∧ ¬Sprinkler) P(¬Rain | RainySeason ∧ ¬Sprinkler) P(¬Sprinkler | RainySeason) P(RainySeason)
= 0.1 (1 – 0.9) (1 -0.0) 0.7
= 0.007