ASSIGNMENT: 6

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Q1. [40 pts] Temporal Reasoning

Consider the first-order Markov chain in Fig. 1 below. At each step *i*, the (random) state Z_i of the chain at step *i* can take any values in {0,1,2}. The probability of moving to $Z_{i+1} = v$ from $Z_i = u$ is detailed in the transition matrix in Fig. 1.

$Z_i \in \{0, 1, 2\}$

	0	1	2
0	0.3	0.2	0.5
1	0.1	0.7	0.2
2	0.3	0.6	0.1

Transition Matrix

$P(Z_{i+1} = v \mid Z_i = u) = T_{uv}$
where T_{uv} = the value at row u and column v of T

Figure 1: First-Order Markov Chain

(a) [4 pts] Prove that $P(Z_{i+1}) = \sum_{Z_i=0}^{2} P(Z_i)\ P(Z_{i+1} \mid Z_i)$ using the chain rule and principle of marginalization.

Answer:

To prove that P(Zi+1) = sum_(Zi=0)^2 P(Zi)P(Zi+1 | Zi), we can use the chain rule and principle of marginalization for probability distributions.

By the chain rule, we know that:

P(Zi+1, Zi) = P(Zi+1 | Zi) P(Zi)

We can marginalize over Zi to obtain P(Zi+1):

P(Zi+1) = SUM_(ZI=O)^2 P(Zi+1, Zi)

Substituting the expression for P(Zi+1, Zi) from the chain rule, we get:

P(Zi+1) = SUM_(ZI=O)^2 P(Zi+1 | Zi) P(Zi)

This shows that P(Zi+1) can be obtained by summing the probabilities of all possible pairs of states (Zi, Zi+1) weighted by the joint probability P(Zi, Zi+1) = P(Zi+1 | Zi) P(Zi). This is known as the principle of marginalization.

Therefore, we have shown that P(Zi+1) = SUM{ZI=O}^2 P(Zi) P(Zi+1 | Zi) using the chain rule and principle of marginalization for probability distributions.

P(Zi+1) = sum_(Zi=0)^2 P(Zi)P(Zi+1 | Zi)

(b) [9 pts] Given the prior probabilities $P(Z_0 = 0) = 0.2$ and $P(Z_0 = 1) = 0.4$, compute the marginal probabilities $P(Z_1 = 0)$, $P(Z_1 = 1)$ and $P(Z_1 = 2)$.

Answer:
P(Z0 = 0) = 0.2
P(Z0 = 1) = 0.4
P(Z0 = 2) = 1 – 0.2 – 0.4 = 0.4
P(Z1 = 0) = P(Z1 = 0 | Z0 = 0) * P(Z0 = 0) + P(Z1 = 0 | Z0 = 1) * P(Z0 = 1) + P(Z1 = 0 | Z0 = 2) * P(Z0 = 2)
= 0.3 * 0.2 + 0.1 * 0.4 + 0.3 * 0.4
= 0.22

P(Z1 = 1) = P(Z1 = 1 | Z0 = 0) * P(Z0 = 0) + P(Z1 = 1 | Z0 = 1) * P(Z0 = 1) + P(Z1 = 1 | Z0 = 2) * P(Z0 = 2)
= 0.2 * 0.2 + 0.7 * 0.4 + 0.6 * 0.4
= 0.56

P(Z1 = 2) = P(Z1 = 2 | Z0 = 0) * P(Z0 = 0) + P(Z1 = 2 | Z0 = 1) * P(Z0 = 1) + P(Z1 = 2 | Z0 = 2) * P(Z0 = 2)
= 0.5 * 0.2 + 0.2 * 0.4 + 0.1 * 0.4
= 0.22

(c) [9 pts] Given the same prior probabilities over Z_0 as in part (b), compute $P(Z_2 = 0)$, $P(Z_2 = 1)$ and $P(Z_2 = 2)$.

Answer:
P(Z2 = 0) = P(Z2 = 0 | Z1 = 0) * P(Z1 = 0) + P(Z2 = 0 | Z1 = 1) * P(Z1 = 1) + P(Z2 = 0 | Z1 = 2) * P(Z1 = 2)
= 0.3 * 0.22 + 0.1 * 0.56 + 0.3 * 0.22
= 0.188

P(Z2 = 1) = P(Z2 = 1 | Z1 = 0) * P(Z1 = 0) + P(Z2 = 1 | Z1 = 1) * P(Z1 = 1) + P(Z2 = 1 | Z1 = 2) * P(Z1 = 2)
= 0.2 * 0.22 + 0.7 * 0.56 + 0.6 * 0.22
= 0.568

P(Z2 = 2) = P(Z2 = 2 | Z1 = 0) * P(Z1 = 0) + P(Z2 = 2 | Z1 = 1) * P(Z1 = 1) + P(Z2 = 2 | Z1 = 2) * P(Z1 = 2)
= 0.5 * 0.22 + 0.2 * 0.56 + 0.1 * 0.22
= 0.244

(d) [9 pts] Given the same prior probabilities over Z_0 as in part (b), compute $P(Z_3 = 0)$, $P(Z_3 = 1)$ and $P(Z_3 = 2)$.

Answer:
P(Z3 = 0) = P(Z3 = 0 | Z2 = 0) * P(Z2 = 0) + P(Z3 = 0 | Z2 = 1) * P(Z2 = 1) + P(Z3 = 0 | Z2 = 2) * P(Z2 = 2)
= 0.3 * 0.188 + 0.1 * 0.568 + 0.3 * 0.244
= 0.1864

P(Z3 = 1) = P(Z3 = 1 | Z2 = 0) * P(Z2 = 0) + P(Z3 = 1 | Z2 = 1) * P(Z2 = 1) + P(Z3 = 1 | Z2 = 2) * P(Z2 = 2)
= 0.2 * 0.188 + 0.7 * 0.568 + 0.6 * 0.244
= 0.5816

P(Z3 = 2) = P(Z3 = 2 | Z2 = 0) * P(Z2 = 0) + P(Z3 = 2 | Z2 = 1) * P(Z2 = 1) + P(Z3 = 2 | Z2 = 2) * P(Z2 = 2)
= 0.5 * 0.188+ 0.2 * 0.568 + 0.1 * 0.244
= 0.232

(e) [9 pts] Suppose we receive an observation that $Z_1 = 2$, what are the probabilities of reaching $Z_3 = 0$, $Z_3 = 1$ and $Z_3 = 2$ now? That is, compute $P(Z_3 = 0 \mid Z_1 = 2)$, $P(Z_3 = 1 \mid Z_1 = 2)$ and $P(Z_3 = 2 \mid Z_1 = 2)$.

Answer:

P(Z3 = 0 | Z1 =2) = P(Z3 = 0 | Z2 = 0) * P(Z2 = 0 | Z1 = 2) + P(Z3 = 0 | Z2 = 1) * P(Z2 = 1 | Z1 = 2) + P(Z3 = 0 | Z2 = 2) * P(Z2 = 2 | Z1 = 2)
= 0.3 * 0.3 + 0.1 * 0.6 + 0.3 * 0.1
= 0.18

P(Z3 = 1 | Z1 =2) = P(Z3 = 1 | Z2 = 0) * P(Z2 = 0 | Z1 = 2) + P(Z3 = 1 | Z2 = 1) * P(Z2 = 1 | Z1 = 2) + P(Z3 = 1 | Z2 = 2) * P(Z2 = 2 | Z1 = 2)
= 0.2 * 0.3 + 0.7 * 0.6 + 0.6 * 0.1
= 0.54

P(Z3 = 2 | Z1 =2) = P(Z3 = 2 | Z2 = 0) * P(Z2 = 0 | Z1 = 2) + P(Z3 = 2 | Z2 = 1) * P(Z2 = 1 | Z1 = 2) + P(Z3 = 2 | Z2 = 2) * P(Z2 = 2 | Z1 = 2)
= 0.5 * 0.3 + 0.2 * 0.6 + 0.1 * 0.1
= 0.28

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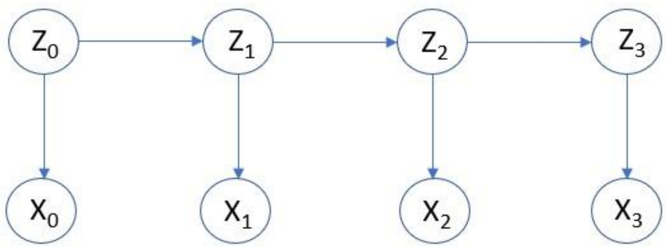
Q2. [30 pts] Filtering and Prediction

Consider the first-order hidden Markov model in Fig. 2 below. Note that for this problem, you need to show your step-by-step derivation. There is no partial credit to guessing work.



$Z_i \in \{0, 1, 2\}$
$X_i \in \{0, 1\}$

	0	1
0	0.4	0.6
1	0.8	0.2
2	0.1	0.9

Observation Matrix (**O**)

$P(X_i = v \mid Z_i = u) = O_{uv}$ where O_{uv} = the value at row u and column v of O

	0	1	2
0	0.3	0.2	0.5
1	0.1	0.7	0.2
2	0.3	0.6	0.1

Transition Matrix (**T**)

$P(Z_{i+1} = v \mid Z_i = u) = T_{uv}$ where T_{uv} = the value at row u and column v of T

Figure 2: First-Order Markov Model

(a) Suppose that before any observations, the prior probabilities P(Z0 = 0) = 0.3 and P(Z0 = 1) = 0.3. Now, if we observe that X0 = 0, what is the most likely value of Z0? That is, compute u = argmaxu P(Z0 = u | X0 = 0). Note: For any function g(u),argmaxu g(u) denotes the value of u such that g(u) is largest.

Answer:

P(X0 = 0) = ∑P(Z0) * P(X0 | Z0)
= 0.3 * P(X0 = 0 | Z0 = 0) + 0.3 * P(X0 = 0 | Z0 = 1) + 0.4 * P(X0 = 0 | Z0 = 2)
= 0.3 * 0.4 + 0.3 * 0.8 + 0.4 * 0.1
= 0.4

P(Z0 = 0 | X0 = 0) = P(X0 = 0 | Z0 = 0) * P(Z0 = 0) / P(X0 = 0)
= 0.4 * 0.3 /0.4
= 0.3

P(Z0 =1 | X0 = 0) = P(X0 = 0 | Z0 = 1) * P(Z0 = 1) / P(X0 = 0)
= 0.8 * 0.3 / 0.4
= 0.6

P(Z0 =2 | X0 = 0) = P(X0 = 0 | Z0 = 2) * P(Z0 = 2) / P(X0 = 0)
= 0.1 * 0.4 / 0.4
= 0.1

For u = 1 P(Z0 =1 | X0 = 0) gives highest value of 0.6.

(b) [10 pts] Given the same prior probabilities $P(Z_0 = 0) = 0.3$ and $P(Z_0 = 1) = 0.3$ as in part (a). Now, assume we have two observations $X_0 = 0, X_1 = 0$, what is the most likely value of Z_1? That is, compute $u = \text{argmax}_u$ $P(Z_1 = u \mid X_0 = 0, X_1 = 0)$.

Answer:

P(Z0 = 0) = 0.3
P(Z0 = 1) = 0.3
P(Z0 = 2) = 0.4

At time t=0:
P(Z0 = 0 | X0 = 0) = 0.3
P(Z0 = 1 | X0 = 0) = 0.6
P(Z0 = 2 | X0 = 0) = 0.1

P(Z1 = 0 | X0 = 0, X1 = 0) = alpha * P(X1 = 0 | Z1 = 0) * P(Z1 = 0 | X0 = 0)
= alpha * P(X1 = 0 | Z1 = 0) * [P(Z1 = 0 | Z0 =0) * P(Z0 = 0 | X0 = 0) + P(Z1 = 0| Z0 = 1) * P(Z0 = 1 | X0 = 0) + P(Z1 = 0 | Z0 = 2) * P(Z0 = 2 | X0 = 0)]
= alpha * 0.4 * [0.3 * 0.3 + 0.1 * 0.6 + 0.3 * 0.1]
= alpha * 0.4 * [0.18]
= alpha * 0.072

P(Z1 = 1 | X0 = 0, X1 = 0) = alpha * P(X1 = 0 | Z1 = 1) * P(Z1 = 1 | X0 = 0)
= alpha * P(X1 = 0 | Z1 = 1) * [P(Z1 = 1 | Z0 =0) * P(Z0 = 0 | X0 = 0) + P(Z1 = 1| Z0 = 1) * P(Z0 = 1 | X0 = 0) + P(Z1 = 1 | Z0 = 2) * P(Z0 = 2 | X0 = 0)]
= alpha * 0.8 * [0.2 * 0.3 + 0.7 * 0.6 + 0.6 * 0.1]
= alpha * 0.8 * [0.54]
= alpha * 0.432

P(Z1 = 2 | X0 = 0, X1 = 0) = alpha * P(X1 = 0 | Z1 = 2) * P(Z1 = 2 | X0 = 0)
= alpha * P(X1 = 0 | Z1 = 2) * [P(Z1 = 2 | Z0 =0) * P(Z0 = 0 | X0 = 0) + P(Z1 = 2| Z0 = 1) * P(Z0 = 1 | X0 = 0) + P(Z1 = 2 | Z0 = 2) * P(Z0 = 2 | X0 = 0)]
= alpha * 0.1 * [0.5 * 0.3 + 0.2 * 0.6 + 0.1 * 0.1]
= alpha * 0.1 * [0.28]
= alpha * 0.028

We know,
P(Z1 = 0 | X0 = 0, X1 = 0) + P(Z1 = 1 | X0 = 0, X1 = 0) + P(Z1 = 2 | X0 = 0, X1 = 0) = 1
alpha * 0.072 + alpha * 0.432 + alpha * 0.028 = 1
alpha * (0.072 + 0.432 + 0.028) = 1
alpha * (0.532) = 1
alpha = 1 / 0.532
alpha = 1.87969924812

Putting value of alpha into equation
P(Z1 = 0 | X0 = 0, X1 = 0) = alpha * 0.072
= 1.87969924812 * 0.072
= 0.13533834586

P(Z1 = 1 | X0 = 0, X1 = 0) = alpha * 0.432
= 1.87969924812 * 0.432
= 0.81203007518

P(Z1 = 2 | X0 = 0, X1 = 0) = alpha * 0.028
= 1.87969924812 * 0.028
= 0.05263157894

(c) [10 pts] Given the same prior probabilities over Z_0 and the observations in part (b), compute $u = \mathrm{argmax}_u$ $P(Z_2 = u \mid X_0 = 0, X_1 = 0)$ and determine what is the most likely value of Z_2.

Answer:

P(Z2 = 0 | X0 = 0, X1 = 0) = P(Z2 = 0 | Z1 = 0) * P(Z1 = 0 | X1 = 0, X0 = 0) + P(Z2 = 0 | Z1 = 1) * P(Z1 = 1 | X1 = 0, X0 = 0) + P(Z2 = 0 | Z1 = 2) * P(Z1 = 2 | X1 = 0, X0 = 0)
= 0.3 * 0.13533834586 + 0.1 * 0.81203007518 + 0.3 * 0.05263157894
= 0.13759398495

P(Z2 = 1 | X0 = 0, X1 = 0) = P(Z2 = 1 | Z1 = 0) * P(Z1 = 0 | X1 = 0, X0 = 0) + P(Z2 = 1 | Z1 = 1) * P(Z1 = 1 | X1 = 0, X0 = 0) + P(Z2 = 1 | Z1 = 2) * P(Z1 = 2 | X1 = 0, X0 = 0)
= 0.2 * 0.13533834586 + 0.7 * 0.81203007518 + 0.6 * 0.05263157894
= 0.62706766916

P(Z2 = 2 | X0 = 0, X1 = 0) = P(Z2 = 2 | Z1 = 0) * P(Z1 = 0 | X1 = 0, X0 = 0) + P(Z2 = 2 | Z1 = 1) * P(Z1 = 1 | X1 = 0, X0 = 0) + P(Z2 = 2 | Z1 = 2) * P(Z1 = 2 | X1 = 0, X0 = 0)
= 0.5 * 0.13533834586 + 0.2 * 0.81203007518 + 0.1 * 0.05263157894
= 0.23533834586

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Q3. [30 pts] Decision Making

Consider the first-order Markov chain in Q1 (see Fig. 1) and suppose we have 3 actions to choose: (1) initializing $Z_0 = 0$; (2) initializing $Z_0 = 1$; and (3) initializing $Z_0 = 2$. Once a decision is made, the Markov chain will simulate forward 2 steps and stop at a certain (random) state Z_2.

(a) [15 pts] Suppose that we will be awarded with 5, 8 and 10 units if at the end of the Markov chain simulation, $Z_2 = 0$, $Z_2 = 1$ and $Z_2 = 2$ respectively. Compute the expected reward of each action above.

Answer:
(1) Z0 = 0,
(2) Z0 = 1,
(3) Z0 = 2

Let's pick (1) Z0 = 0:

P(Z2 = 0 | Z0 = 0) = P(Z2 = 0 | Z1 = 0) * P(Z1 = 0 | Z0 = 0) + P(Z2 = 0 | Z1 = 1) * P(Z1 = 1 | Z0 = 0) + P(Z2 = 0 | Z1 = 2) * P(Z1 = 2 | Z0 = 0)

= 0.3 * 0.3 + 0.1 * 0.2 + 0.3 * 0.5

= 0.26

P(Z2 = 1 | Z0 = 0) = P(Z2 = 1 | Z1 = 0) * P(Z1 = 0 | Z0 = 0) + P(Z2 = 1 | Z1 = 1) * P(Z1 = 1 | Z0 = 0) + P(Z2 = 1 | Z1 = 2) * P(Z1 = 2 | Z0 = 0)

= 0.2 * 0.3 + 0.7 * 0.2 + 0.6 * 0.5

= 0.5

P(Z2 = 2 | Z0 = 0) = P(Z2 = 2 | Z1 = 0) * P(Z1 = 0 | Z0 = 0) + P(Z2 = 2 | Z1 = 1) * P(Z1 = 1 | Z0 = 0) + P(Z2 = 2 | Z1 = 2) * P(Z1 = 2 | Z0 = 0)

= 0.5 * 0.3 + 0.2 * 0.2 + 0.1 * 0.5

= 0.24

(Z2 = 0, R = 5), (Z2 =1, R = 8), (Z2 = 2, R = 10)

Total reward for setting (1) Z0 = 0 is:

= P(Z2 = 0 | Z0 = 0) * 5 + P(Z2 = 1 | Z0 = 0) * 8 + P(Z2 = 2 | Z0 = 0) * 10

= 0.26 * 5 + 0.5 * 8 + 0.24 * 10

= 7.7

Let's pick (2) Z0 = 1:

P(Z2 = 0 | Z0 = 1) = P(Z2 = 0 | Z1 = 0) * P(Z1 = 0 | Z0 = 1) + P(Z2 = 0 | Z1 = 1) * P(Z1 = 1 | Z0 = 1) + P(Z2 = 0 | Z1 = 2) * P(Z1 = 2 | Z0 = 1)

= 0.3 * 0.1 + 0.1 * 0.7 + 0.3 * 0.2

= 0.16

P(Z2 = 1 | Z0 = 1) = P(Z2 = 1 | Z1 = 0) * P(Z1 = 0 | Z0 = 1) + P(Z2 = 1 | Z1 = 1) * P(Z1 = 1 | Z0 = 1) + P(Z2 = 1 | Z1 = 2) * P(Z1 = 2 | Z0 = 1)

= 0.2 * 0.1 + 0.7 * 0.7 + 0.6 * 0.2

= 0.63

P(Z2 = 2 | Z0 = 1) = P(Z2 = 2 | Z1 = 0) * P(Z1 = 0 | Z0 = 1) + P(Z2 = 2 | Z1 = 1) * P(Z1 = 1 | Z0 = 1) + P(Z2 = 2 | Z1 = 2) * P(Z1 = 2 | Z0 = 1)

= 0.5 * 0.1 + 0.2 * 0.7 + 0.1 * 0.2

= 0.21

(Z2 = 0, R = 5), (Z2 =1, R = 8), (Z2 = 2, R = 10)

Total reward for setting (2) Z0 = 1 is:

= P(Z2 = 0 | Z0 = 1) * 5 + P(Z2 = 1 | Z0 = 1) * 8 + P(Z2 = 2 | Z0 = 1) * 10

= 0.16 * 5 + 0.63 * 8 + 0.21 * 10

= 7.94

Let's pick (3) Z0 = 2:

P(Z2 = 0 | Z0 = 2) = P(Z2 = 0 | Z1 = 0) * P(Z1 = 0 | Z0 = 2) + P(Z2 = 0 | Z1 = 1) * P(Z1 = 1 | Z0 = 2) + P(Z2 = 0 | Z1 = 2) * P(Z1 = 2 | Z0 = 2)

= 0.3 * 0.3 + 0.1 * 0.6 + 0.3 * 0.1

= 0.18

P(Z2 = 1 | Z0 = 2) = P(Z2 = 1 | Z1 = 0) * P(Z1 = 0 | Z0 = 2) + P(Z2 = 1 | Z1 = 1) * P(Z1 = 1 | Z0 = 2) + P(Z2 = 1 | Z1 = 2) * P(Z1 = 2 | Z0 = 2)

= 0.2 * 0.3 + 0.7 * 0.6 + 0.6 * 0.1

= 0.54

P(Z2 = 2 | Z0 = 2) = P(Z2 = 2 | Z1 = 0) * P(Z1 = 0 | Z0 = 2) + P(Z2 = 2 | Z1 = 1) * P(Z1 = 1 | Z0 = 2) + P(Z2 = 2 | Z1 = 2) * P(Z1 = 2 | Z0 = 2)

= 0.5 * 0.3 + 0.2 * 0.6 + 0.1 * 0.1

= 0.28

(Z2 = 0, R = 5), (Z2 =1, R = 8), (Z2 = 2, R = 10)

Total reward for setting (3) Z0 = 2 is:

= P(Z2 = 0 | Z0 = 2) * 5 + P(Z2 = 1 | Z0 = 2) * 8 + P(Z2 = 2 | Z0 = 2) * 10

= 0.18 * 5 + 0.54 * 8 + 0.28 * 10

= 8.02

Therefore, the expected reward of each action is:

(1) initializing Z0 = 0: **7.7**

(2) initializing Z0 = 1: **7.94**

(3) initializing Z0 = 2: **8.02**

Using Z0 = 2, We get highest Rewards.

(b) [15 pts] Instead of the above 3 actions, suppose we are only given 2 actions **(A)** and **(B)**.

Executing **(A)** changes the prior probabilities over Z_0 to ($P(Z_0 = 0) = 0.1, P(Z_0 = 1) = 0.9$).

Executing **(B)** changes the prior probabilities over Z_0 to ($P(Z_0 = 0) = 0.3, P(Z_0 = 2) = 0.7$).

Given the above, compute the expected reward of **(A)** and **(B)**. Which one is better?

Answer:

(A) changes the prior probabilities over Z_0 to ($P(Z_0 = 0) = 0.1, P(Z_0 = 1) = 0.9$)

P(Z0) = [0.1, 0.9 , 0]

Reward for (Z0 = 0) = 7.7
Reward for (Z0 = 1) = 7.94
Reward for (Z0 = 2) = 8.02

For $P(Z_0 = 0) = 0.1$
= 0.1 * 7.7
= 0.77
For $P(Z_0 = 1) = 0.9$
= 0.9 * 7.94
= 7.146

The expected reward of **(A)**
= 0.1* 7.7 + 0.9 * 7.94
= 7.916

(B) changes the prior probabilities over Z_0 to ($P(Z_0 = 0) = 0.3, P(Z_0 = 2) = 0.7$)

P(Z0) = [0.3, 0, 0.7]

Reward for (Z0 = 0) = 7.7
Reward for (Z0 = 1) = 7.94
Reward for (Z0 = 2) = 8.02

For $P(Z_0 = 0) = 0.3$
= 0.3 * 7.7
= 2.31
For $P(Z_0 = 2) = 0.7$
= 0.7 * 8.02
= 5.614

The expected reward of **(A)**
= 0.3* 7.7 + 0.7 * 8.02
= 7.924

7.924 > 7.916 so (B) is better.