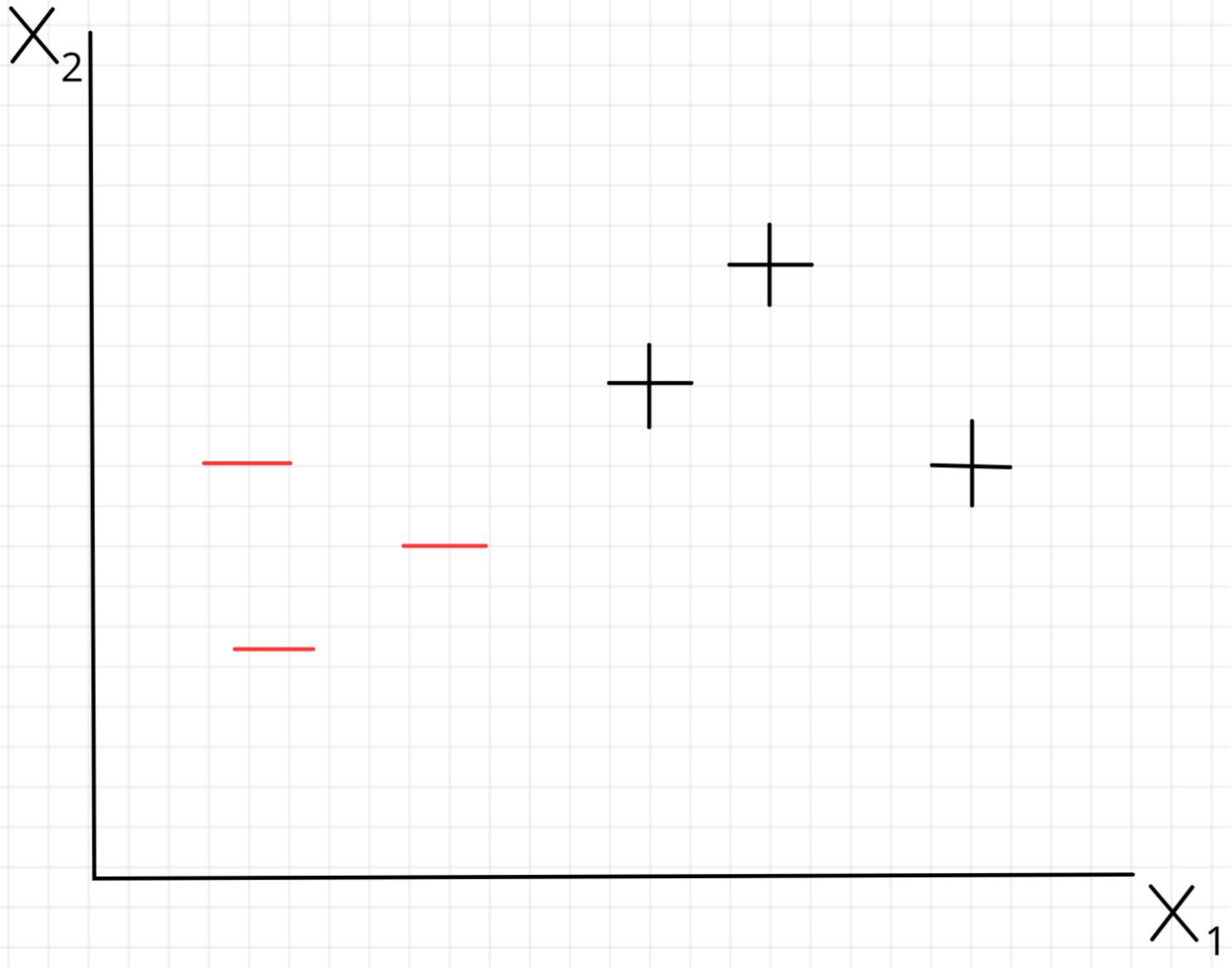
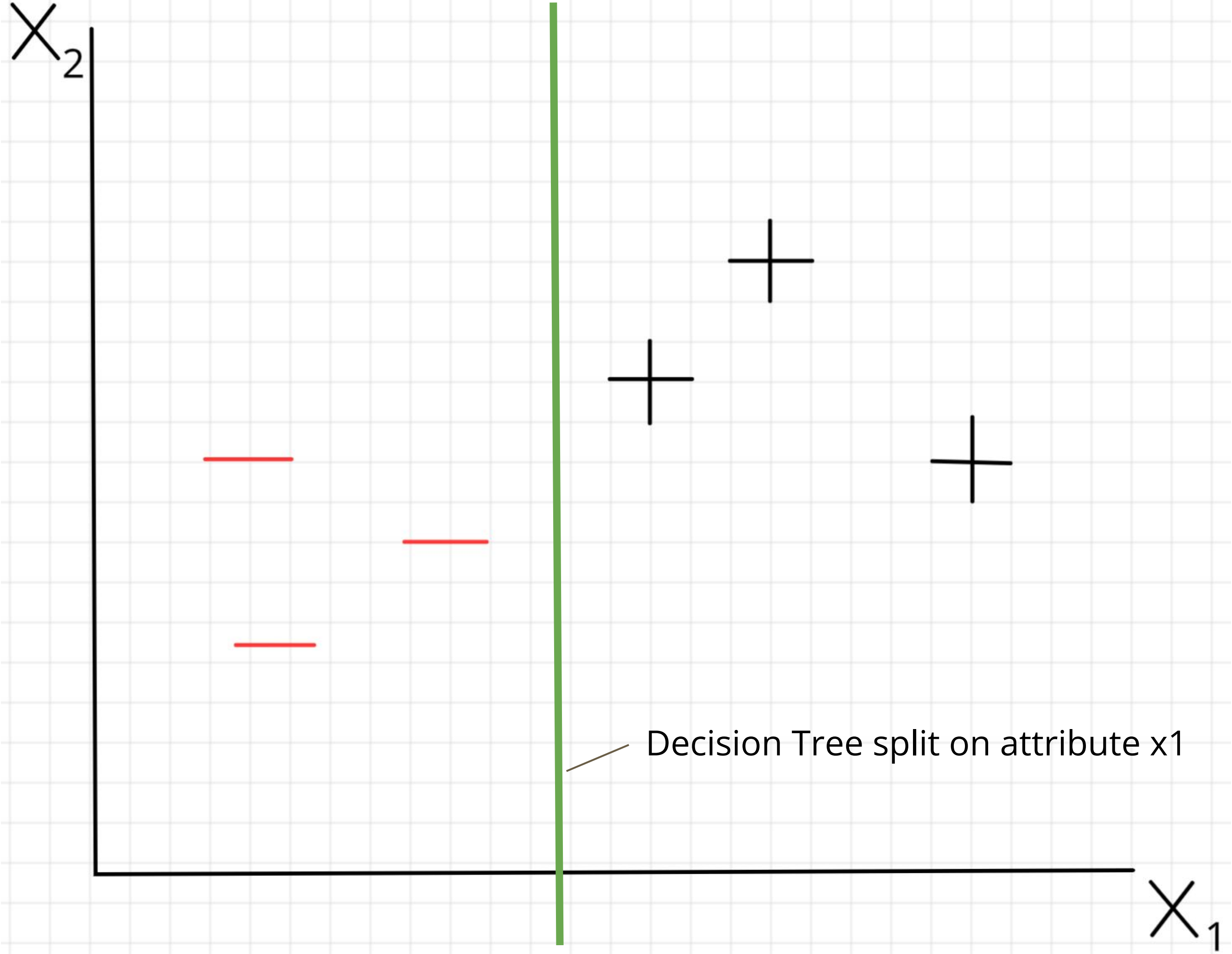
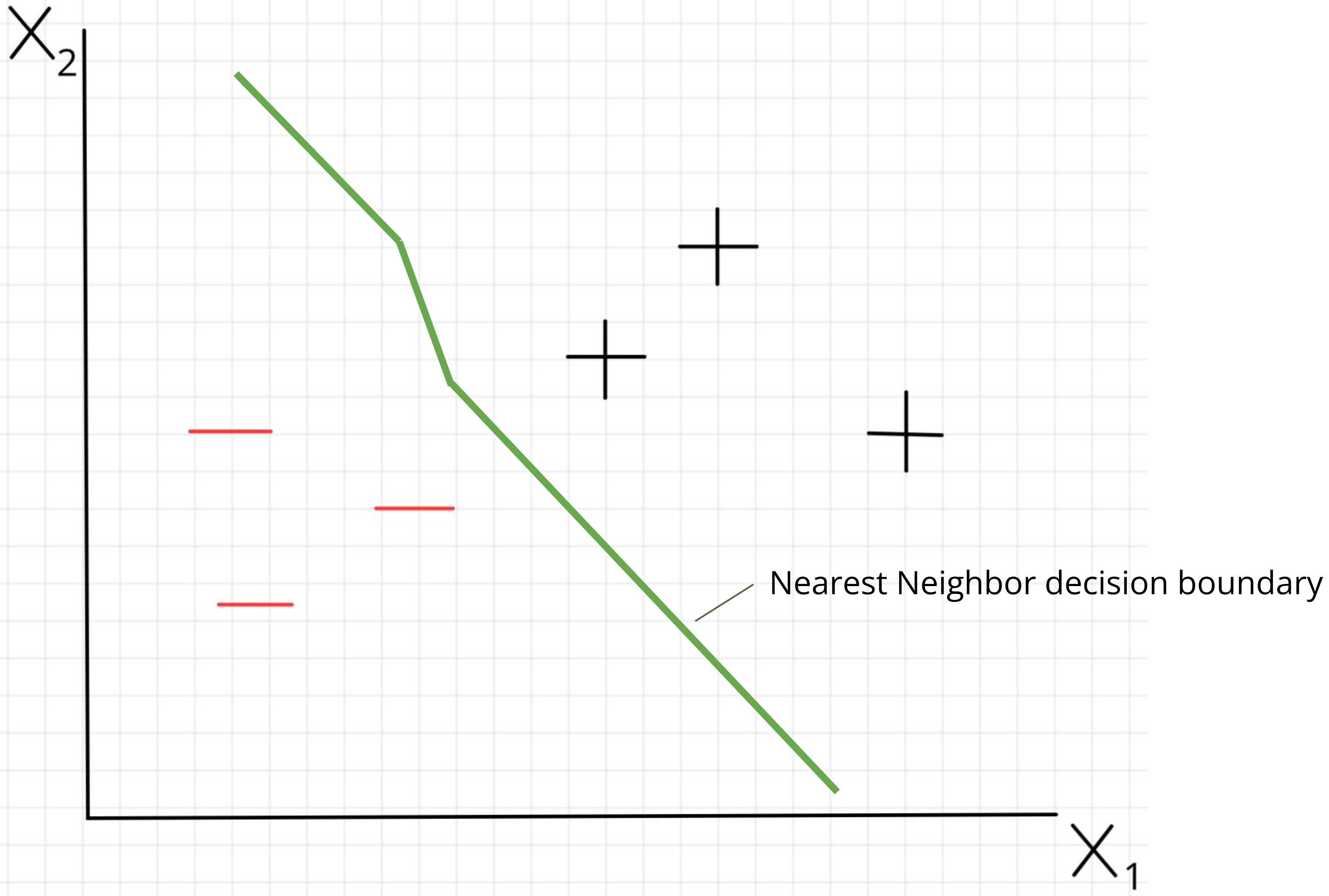
Support Vector Machines

Boston University CS 506 - Lance Galletti

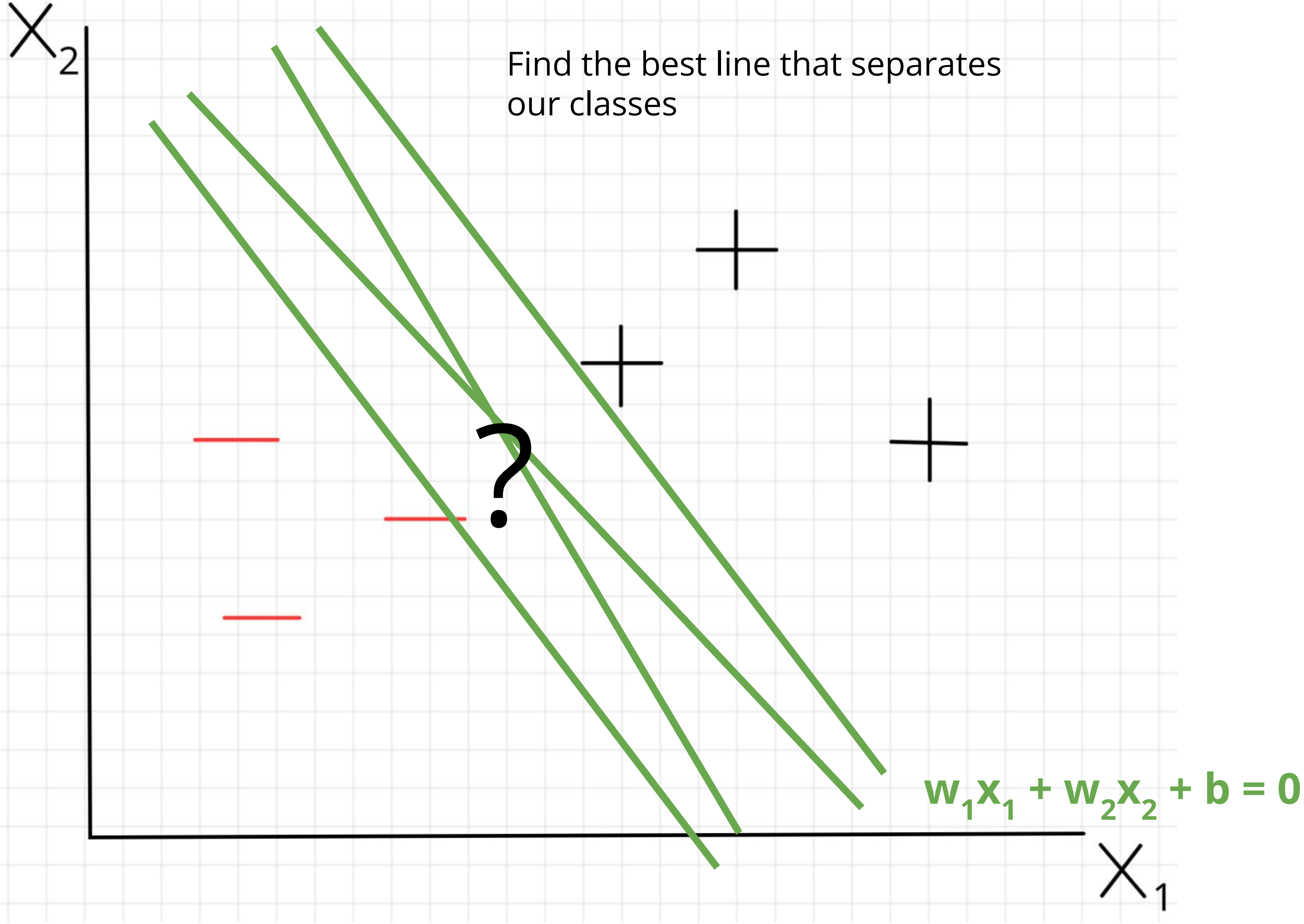


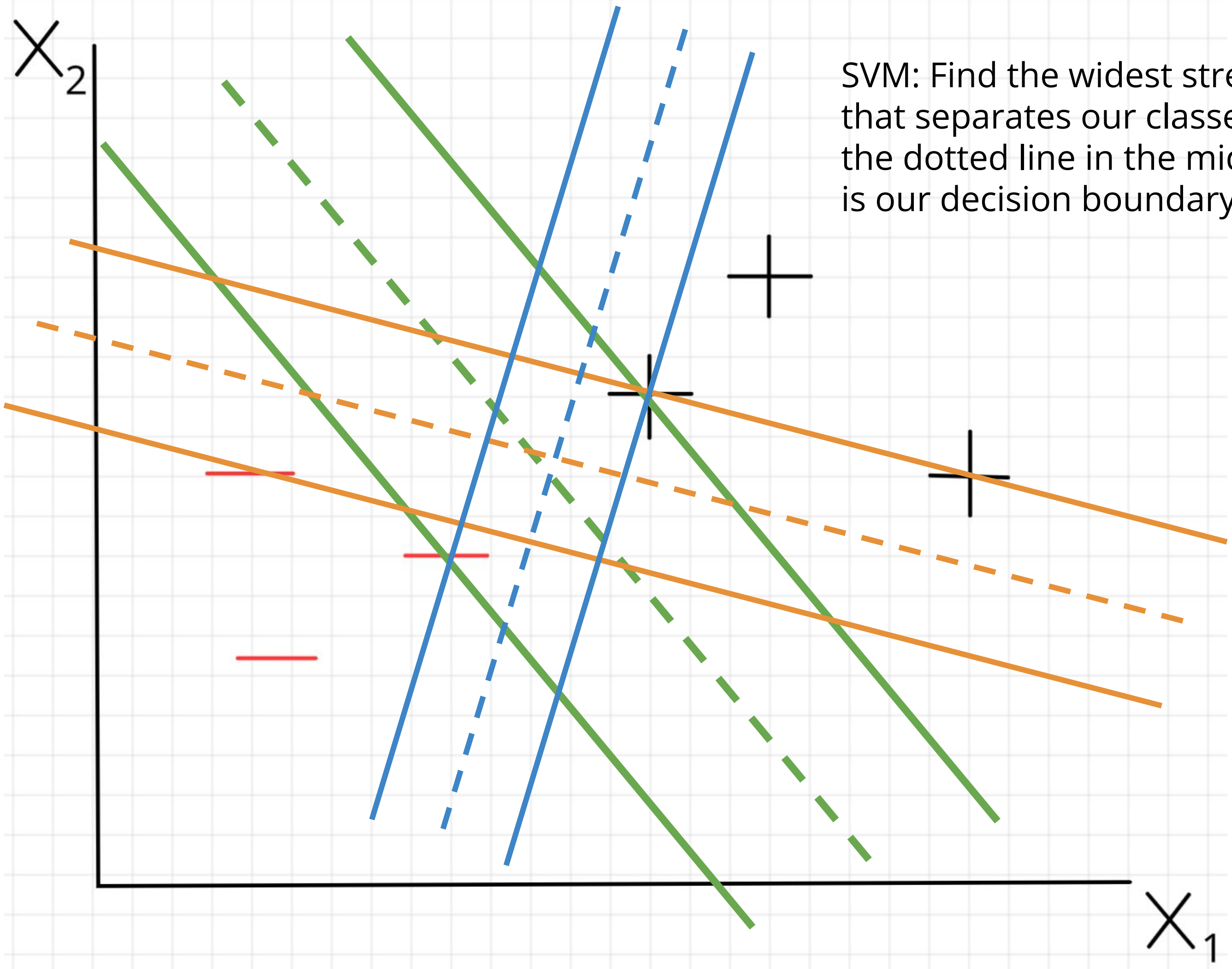
X_2
X_1


X2
Decision Tree split on attribute x1
X1


X_2
Nearest Neighbor decision boundary
X_1

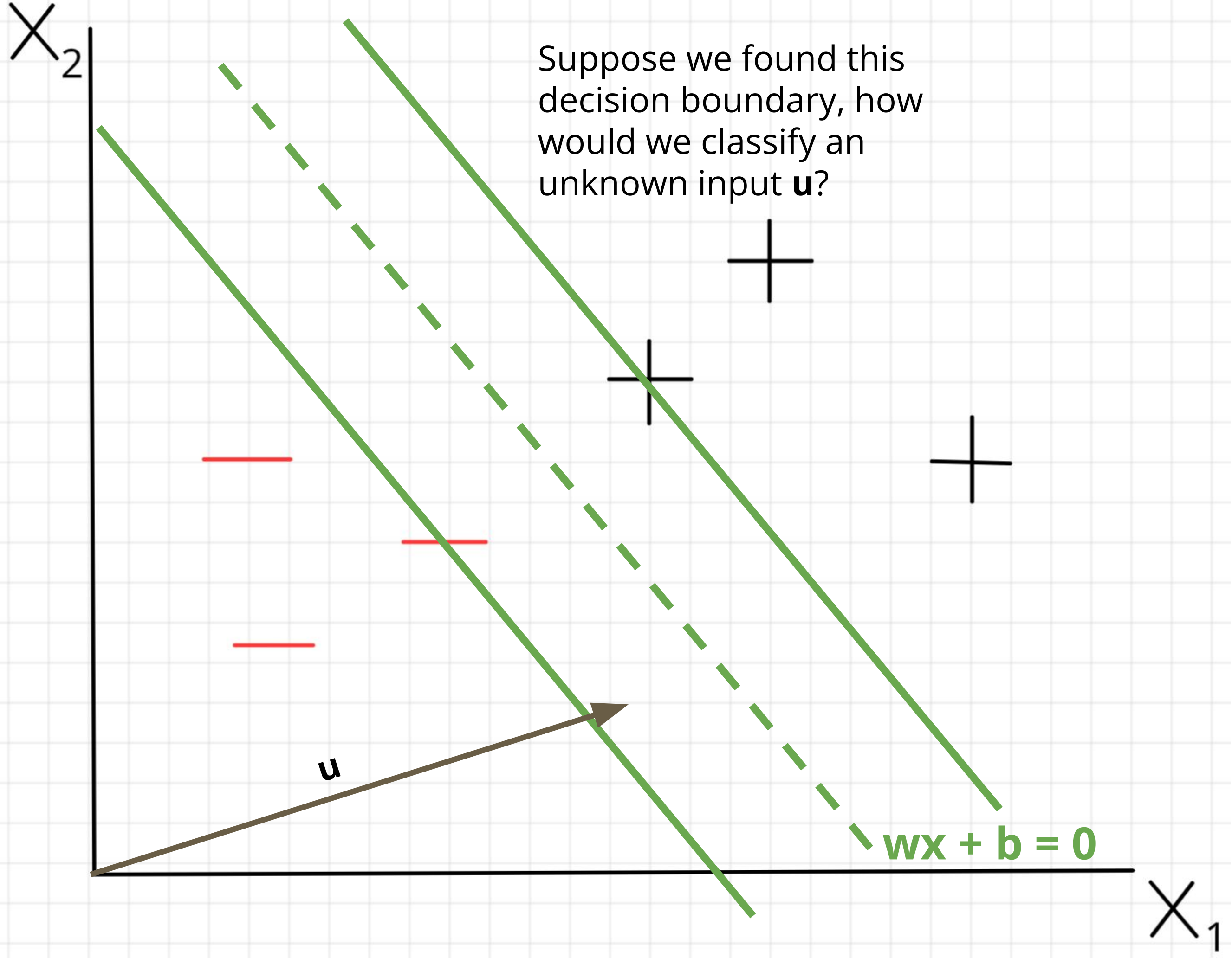
Find the best line that separates our classes





SVM: Find the widest street that separates our classes - the dotted line in the middle is our decision boundary

Suppose we found this decision boundary, how would we classify an unknown input **u**?



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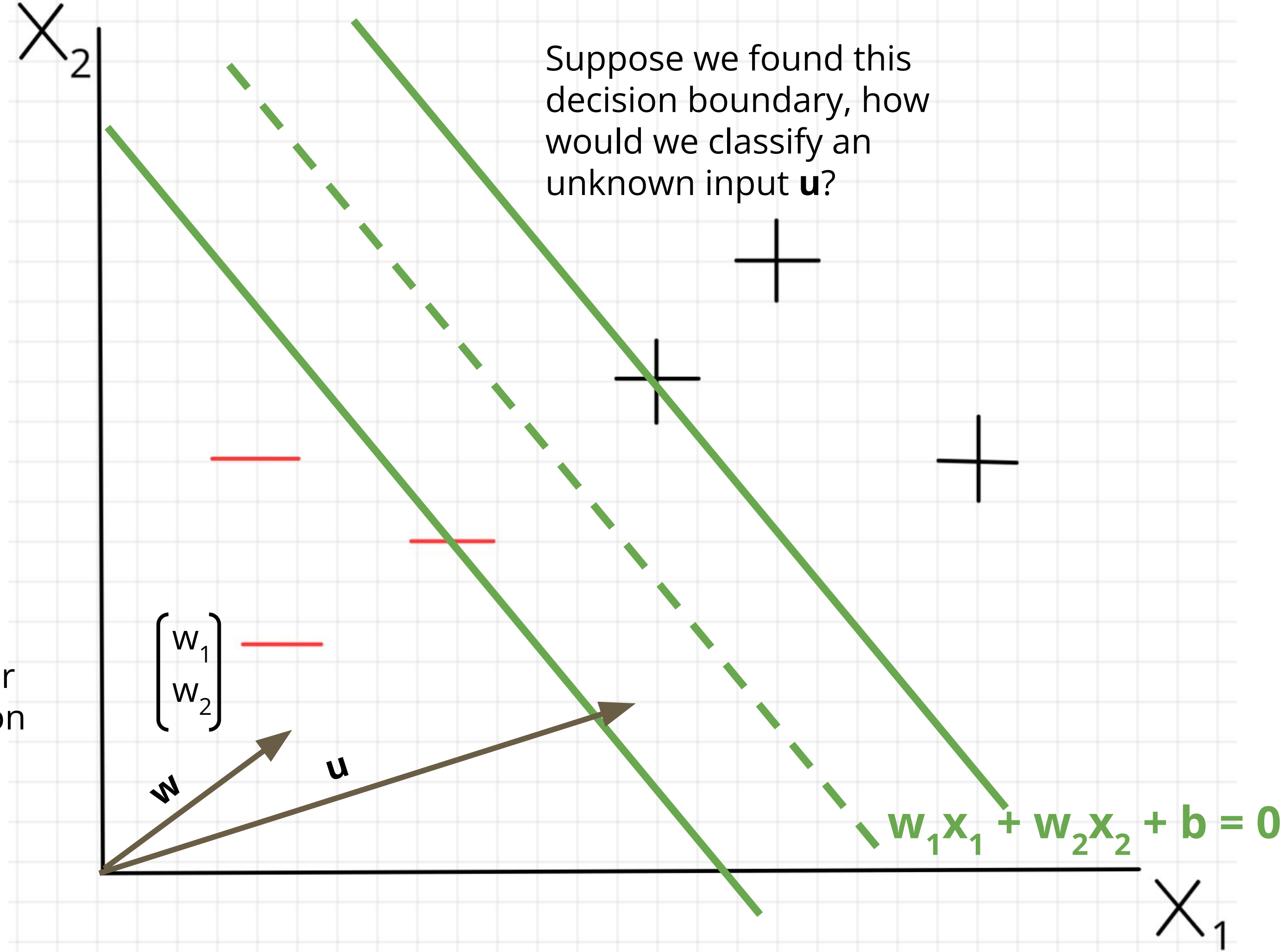
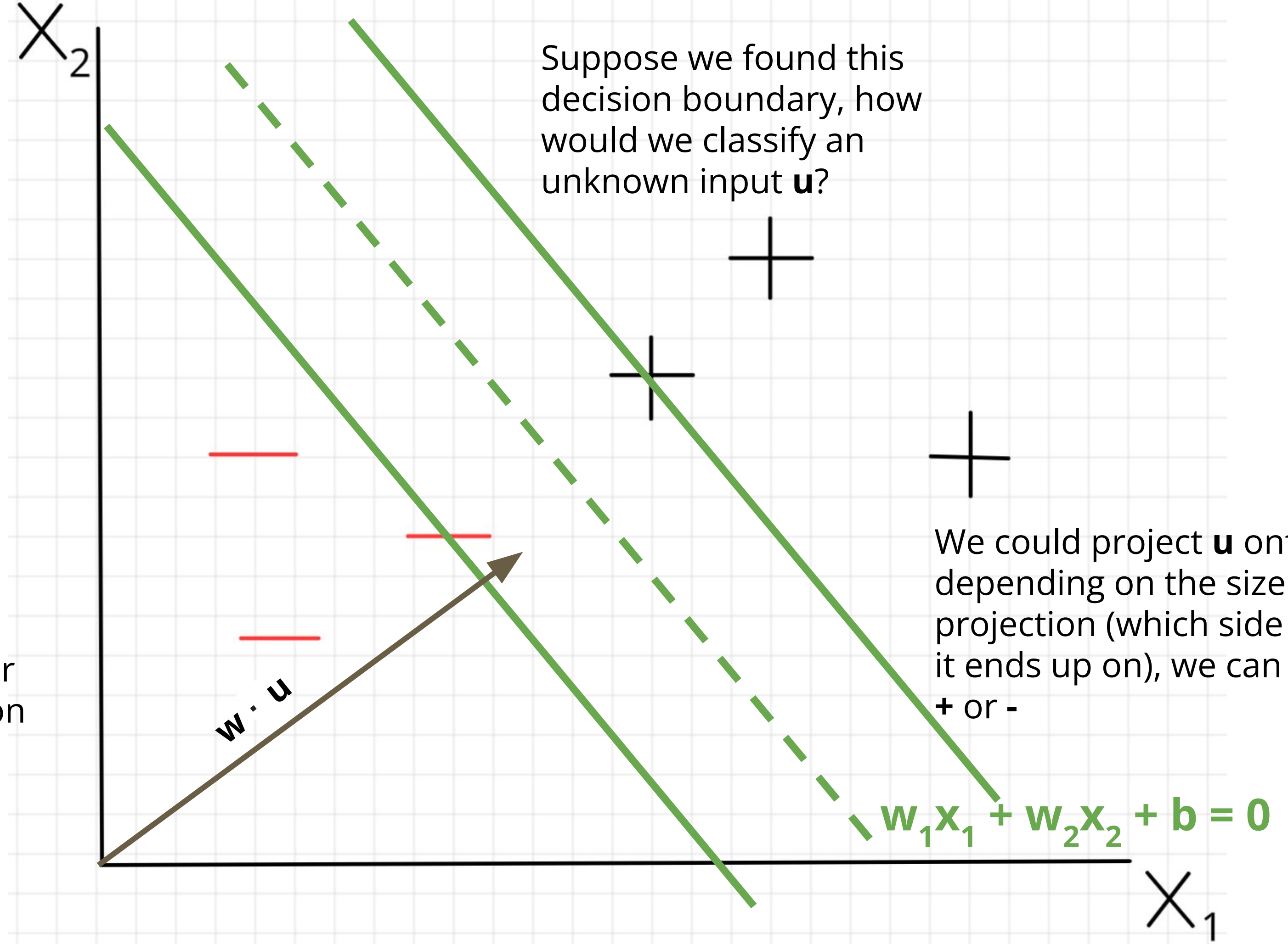


Let **w** be a vector perpendicular to the decision boundary

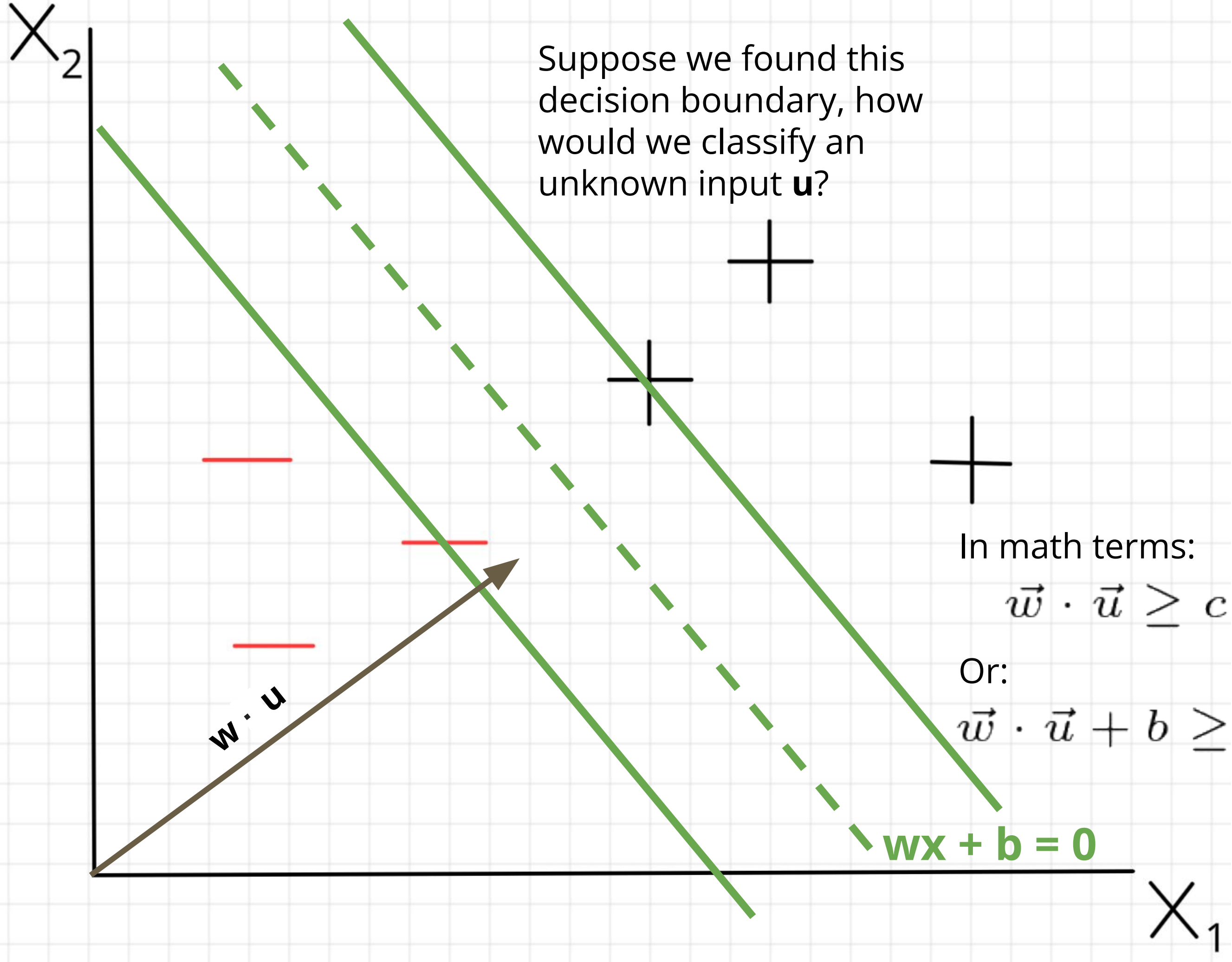
Suppose we found this decision boundary, how would we classify an unknown input **u**?



Let **w** be a vector perpendicular to the decision boundary

We could project **u** onto **w** and depending on the size of that projection (which side of the street it ends up on), we can classify it as **+** or **-**

Suppose we found this decision boundary, how would we classify an unknown input **u**?

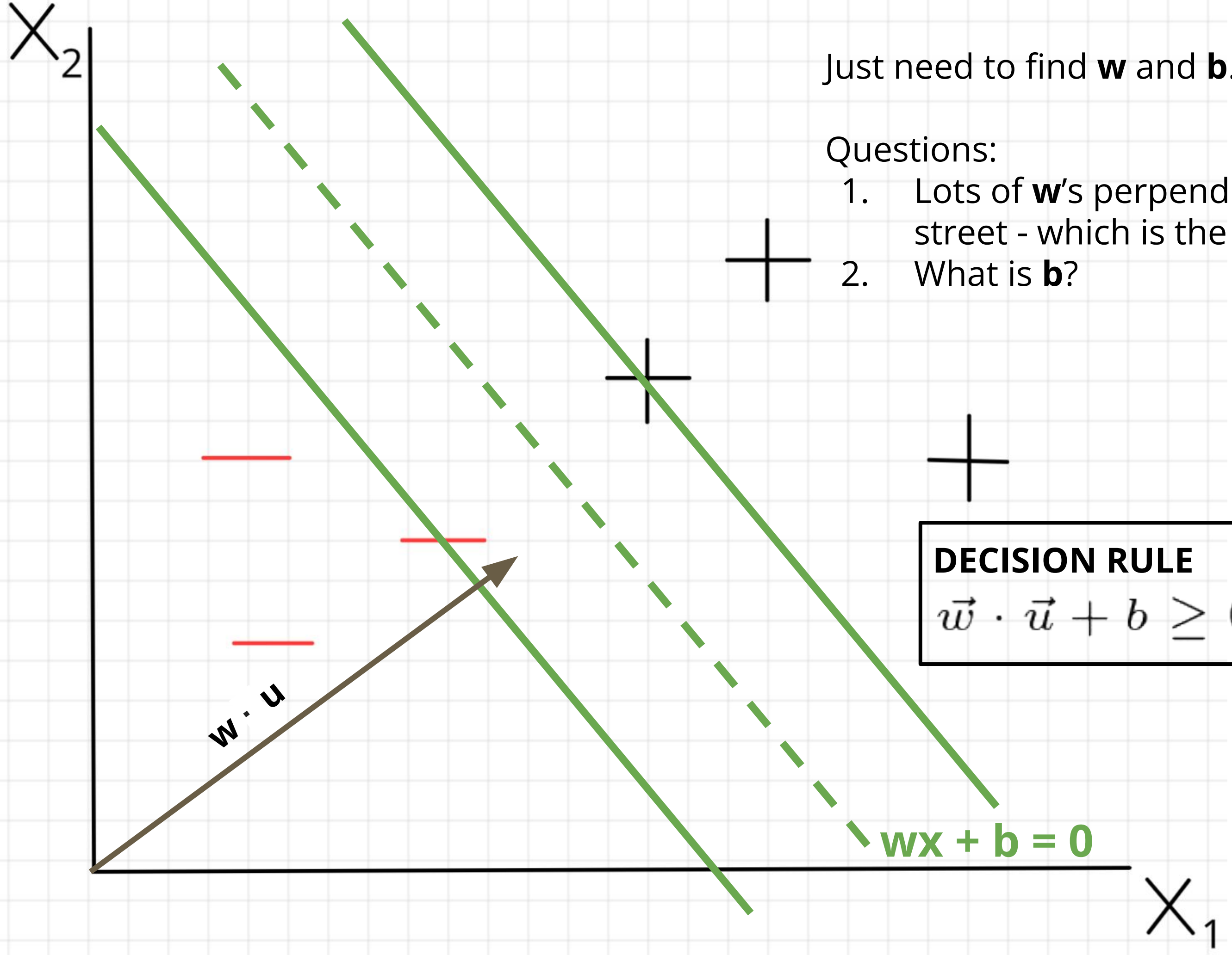


In math terms:

$$\vec{w} \cdot \vec{u} \geq c \quad \text{then } +$$

Or:

$$\vec{w} \cdot \vec{u} + b \geq 0 \quad \text{then } +$$



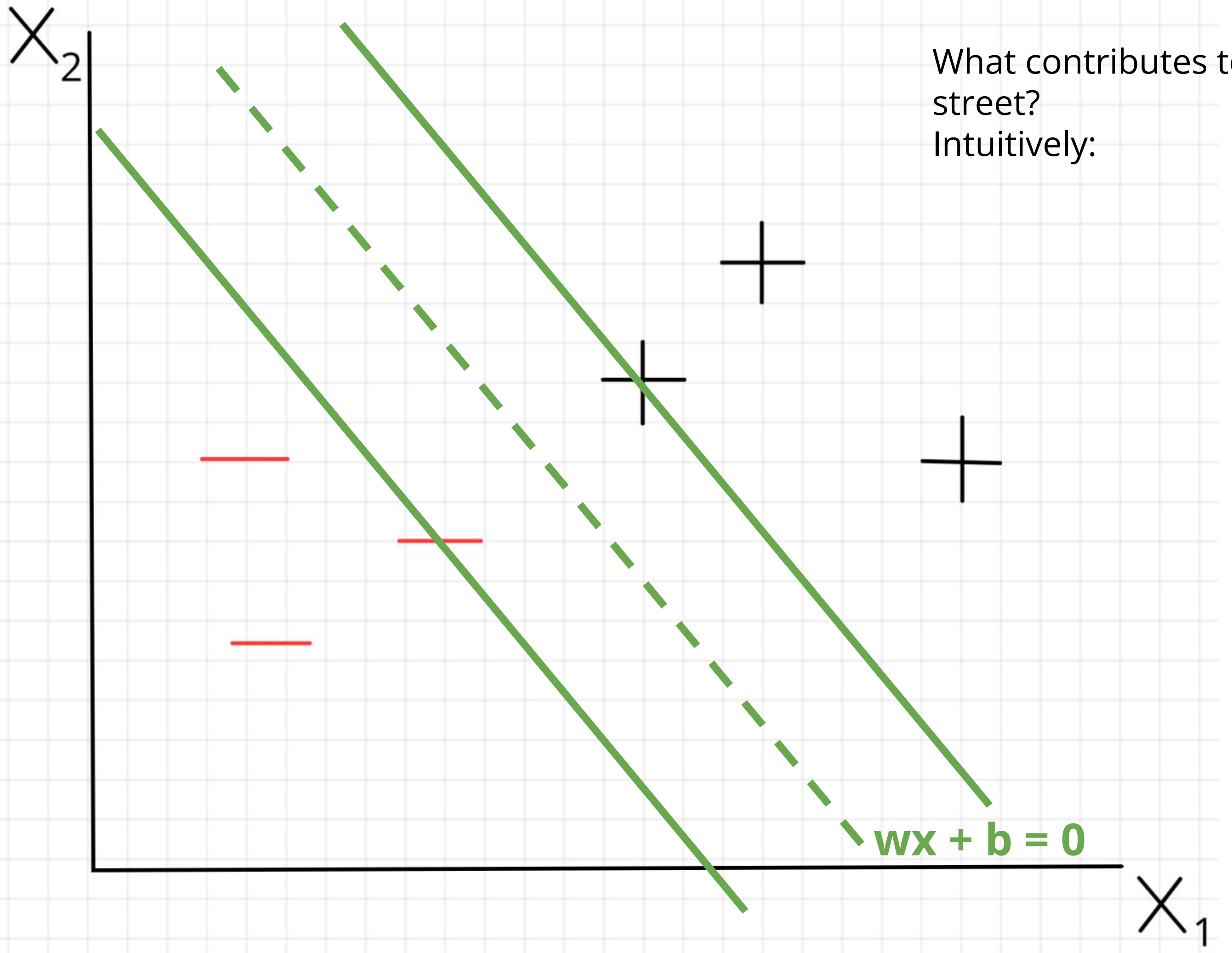
Just need to find **w** and **b**.

Questions:

1. Lots of **w**'s perpendicular to the street - which is the best one?
2. What is **b**?

DECISION RULE

$$\vec{w} \cdot \vec{u} + b \geq 0 \quad \text{then } +$$



What contributes to the widest street?
Intuitively:

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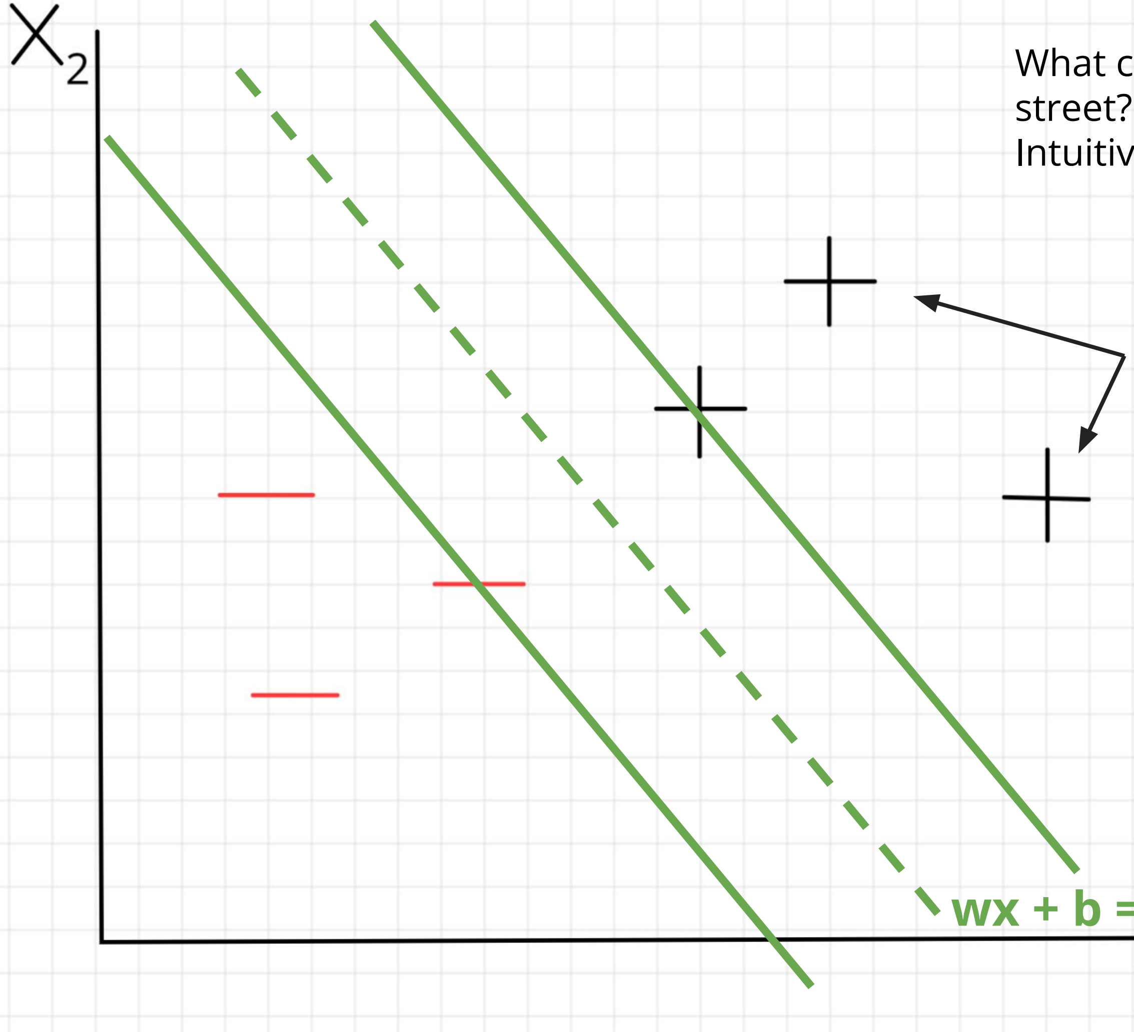
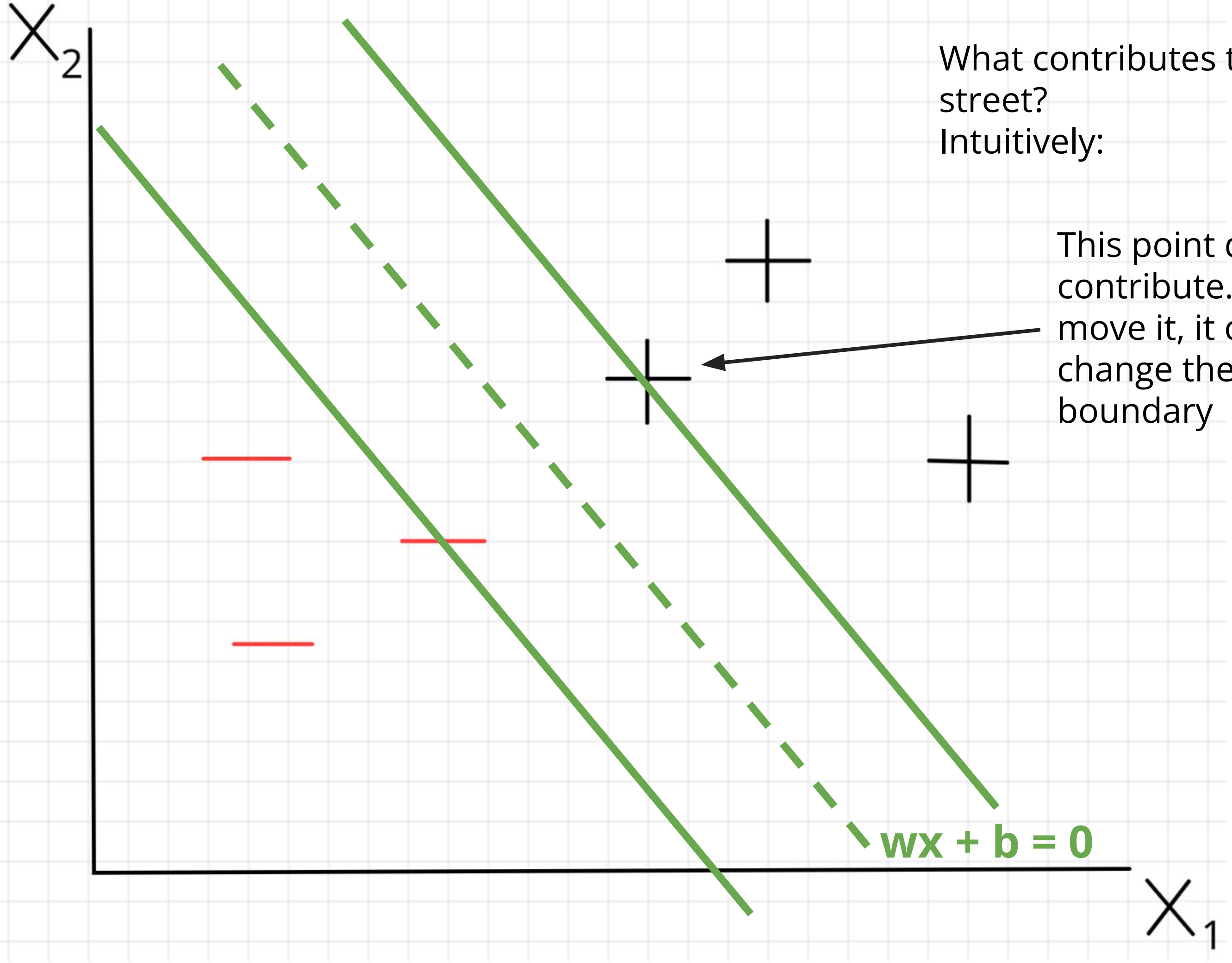


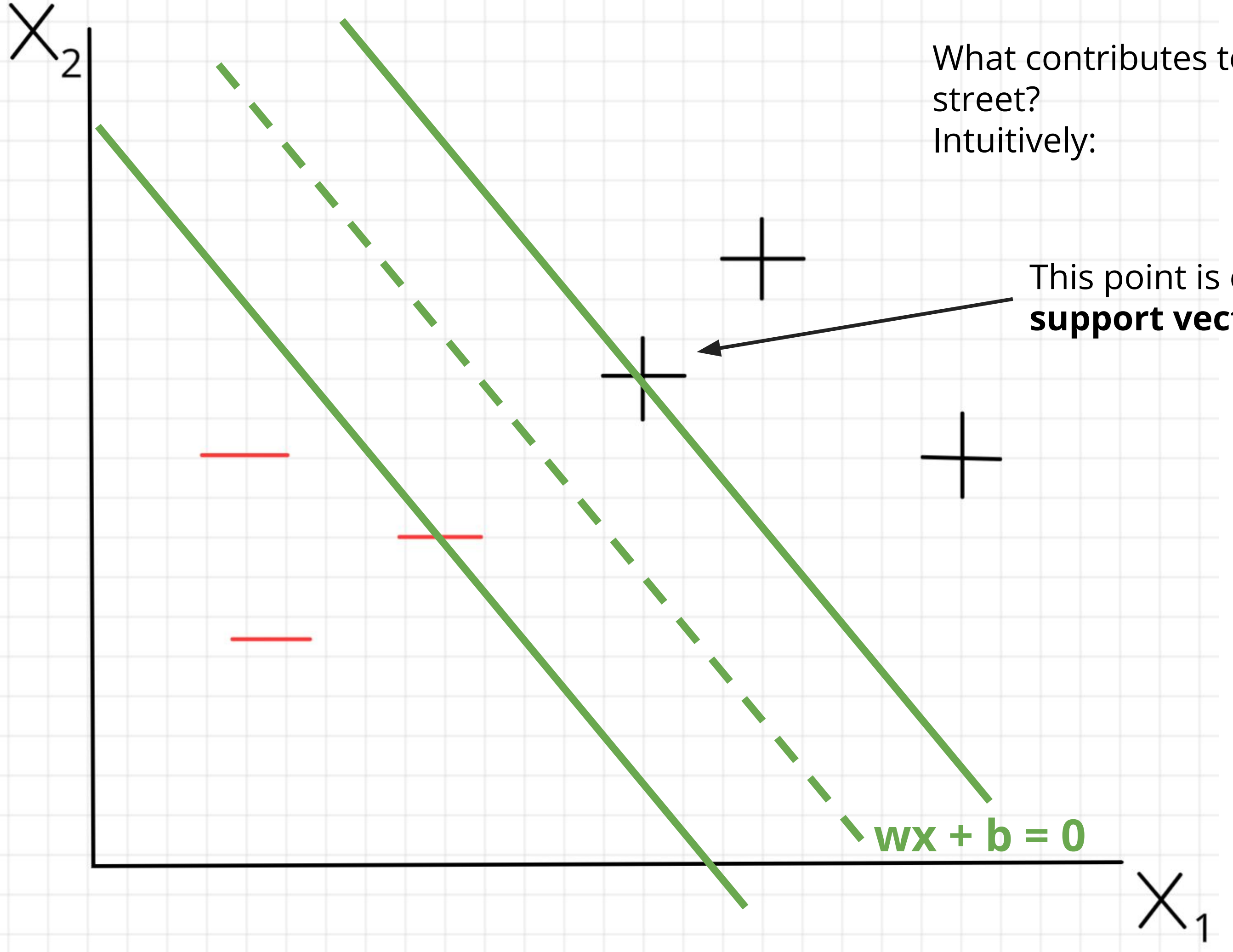
These points don't contribute. If we were to move them they would not affect the decision boundary



What contributes to the widest street?
Intuitively:

This point does contribute. If we were to move it, it could totally change the decision boundary

What contributes to the widest street?
Intuitively:



This point is called a **support vector**

How to find the widest street

We want our samples to lie beyond the street. That is:

$$\vec{w} \cdot \vec{x}_+ + b \geq 1$$

$$\vec{w} \cdot \vec{x}_- + b \leq -1$$

Note: for an unknown **u**, we can have

$$-1 < \vec{w} \cdot \vec{u} + b < 1$$

How to find the widest street

Let's introduce a variable

$$y_i = \begin{cases} +1 & \text{if } x_i \text{ is a } + \text{sample} \\ \\ -1 & \text{if } x_i \text{ is a } - \text{sample} \end{cases}$$

Note: this is effectively the class label of x_i

How to find the widest street

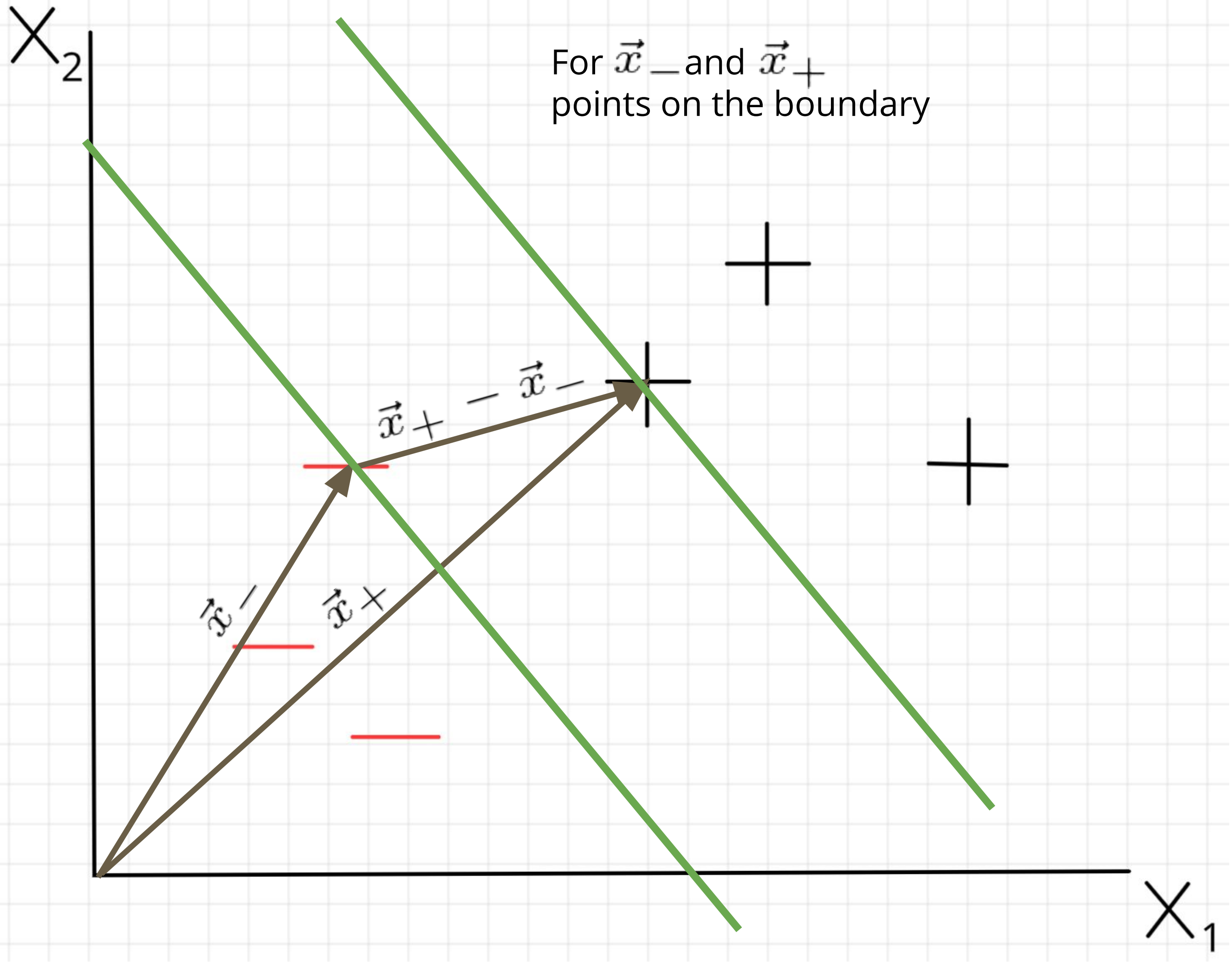
If we multiply our sample decision rules by this new variable:

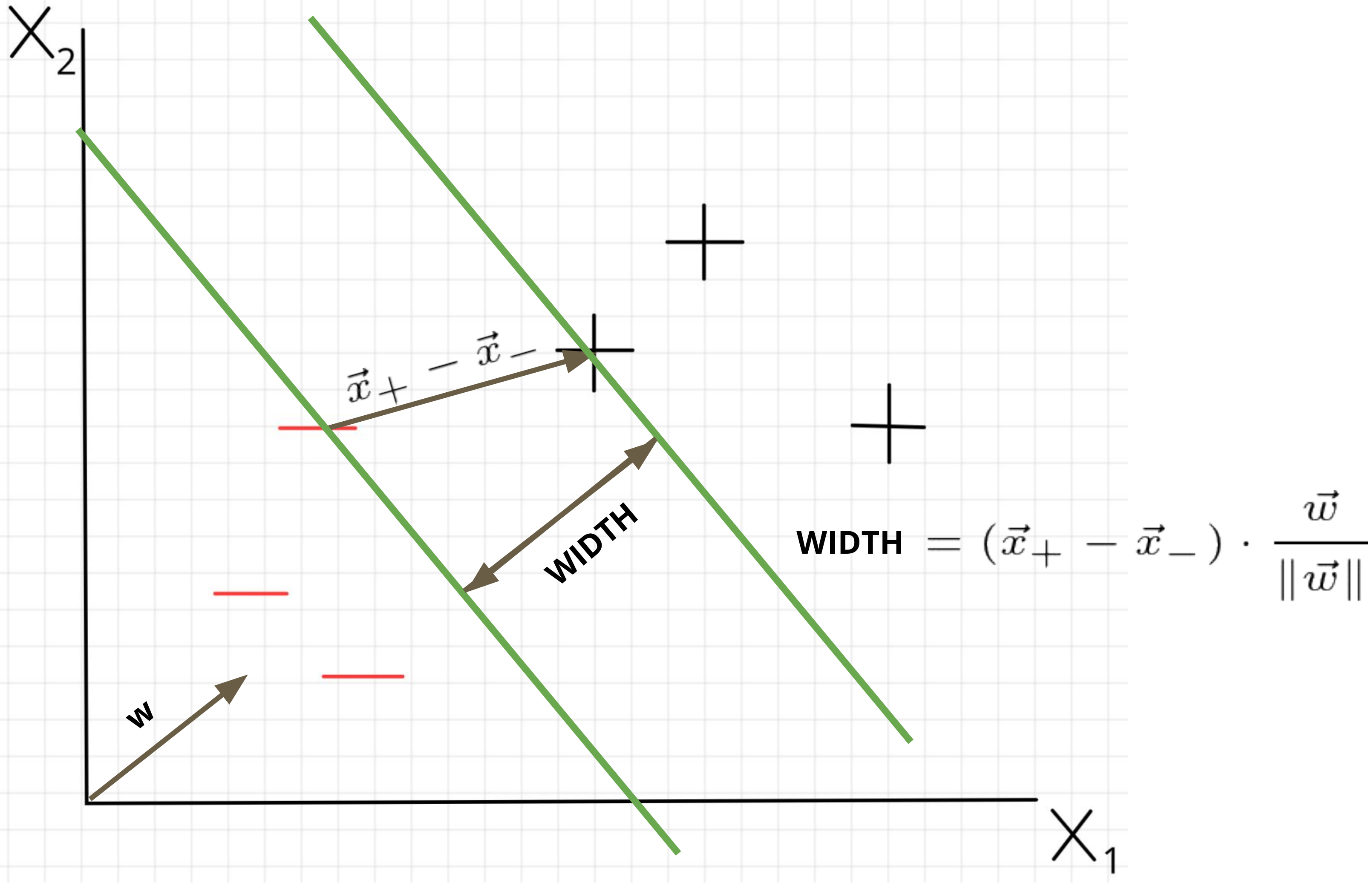
$$y_i(\vec{w} \cdot \vec{x}_i + b) \geq 1$$

Meaning, for x_i on the decision boundary, we want:

$$y_i(\vec{w} \cdot \vec{x}_i + b) - 1 = 0$$

For $\vec{x}_-$ and $\vec{x}_+$
points on the boundary





X_2
$\vec{x}_+ - \vec{x}_-$
WIDTH
WIDTH $= (\vec{x}_+ - \vec{x}_-) \cdot \frac{\vec{w}}{\|\vec{w}\|}$
w
X_1

How to find the widest street

We know that **WIDTH** $= (\vec{x}_+ - \vec{x}_-) \cdot \dfrac{\vec{w}}{||\vec{w}||}$ for $\vec{x}_-$ and $\vec{x}_+$ points on the boundary

And, since they are on the boundary, we know that

$$y_i(\vec{w} \cdot \vec{x}_i + b) - 1 = 0$$

Hence, **WIDTH** $= \dfrac{2}{||\vec{w}||}$

(as an exercise, try to show this)

How to find the widest street

Goal is to maximize the width

$$\max\left(\frac{2}{||\vec{w}||}\right) = \min(||\vec{w}||)$$

$$= \min\left(\frac{1}{2}\,||\vec{w}||^2\right)$$

Subject to:

$$y_i(\vec{w} \cdot \vec{x}_i + b) - 1 = 0$$

How to find the widest street

Can use Lagrange multipliers to form a single expression to find the extremum of

$$L = \frac{1}{2}\|\vec{w}\|^2 - \sum_i \alpha_i \left[y_i(\vec{x}_i \cdot \vec{w} + b) - 1 \right]$$

where α_i is 0 for x_i not on the boundary.

Now we can take derivatives to find the extremum of **L**.

How to find the widest street

$$\frac{\partial L}{\partial \vec{w}} = \vec{w} - \sum_i \alpha_i y_i \vec{x}_i = 0$$

$$\implies \vec{w} = \sum_i \alpha_i y_i \vec{x}_i$$

Means **w** is a linear sum of vectors in our sample/training set!

$$\frac{\partial L}{\partial b} = -\sum_i \alpha_i y_i = 0$$

$$\implies \sum_i \alpha_i y_i = 0$$

How to find the widest street

Let's plug these values back into L to see what happens to L at its extremum

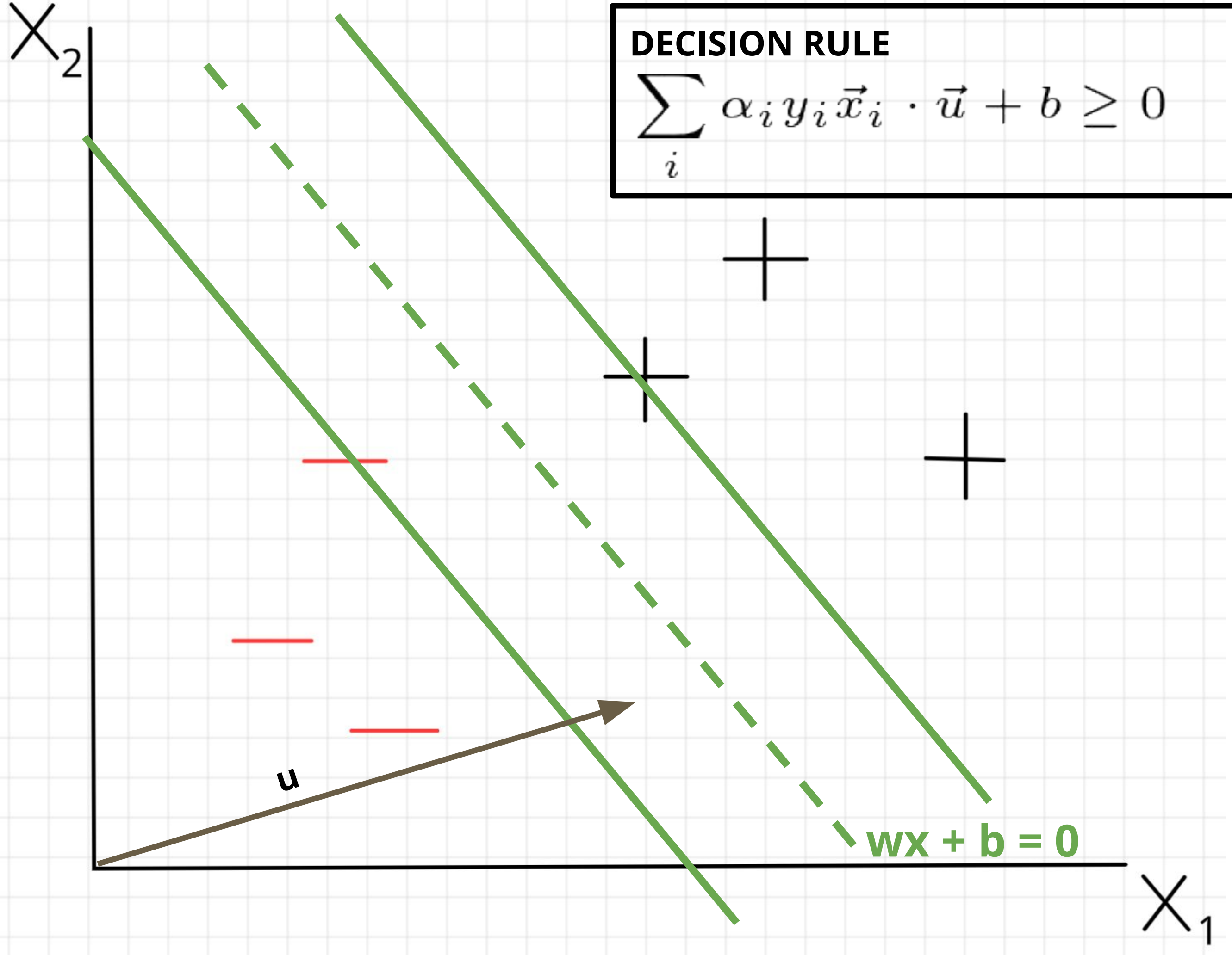
$$L = \frac{1}{2}\left(\sum_i \alpha_i y_i \vec{x}_i\right)\cdot\left(\sum_i \alpha_i y_i \vec{x}_i\right) - \left(\sum_i \alpha_i y_i \vec{x}_i\right)\cdot\left(\sum_i \alpha_i y_i \vec{x}_i\right) - \cancel{\sum_i \alpha_i y_i b} + \sum_i \alpha_i$$

Simplifying, we get:

$$\begin{aligned} L &= \sum_i \alpha_i - \frac{1}{2}\left(\sum_i \alpha_i y_i \vec{x}_i\right)\cdot\left(\sum_i \alpha_i y_i \vec{x}_i\right) \\ &= \sum_i \alpha_i - \frac{1}{2}\sum_i \sum_j \alpha_i \alpha_j y_i y_j \vec{x}_i \cdot \vec{x}_j \end{aligned}$$

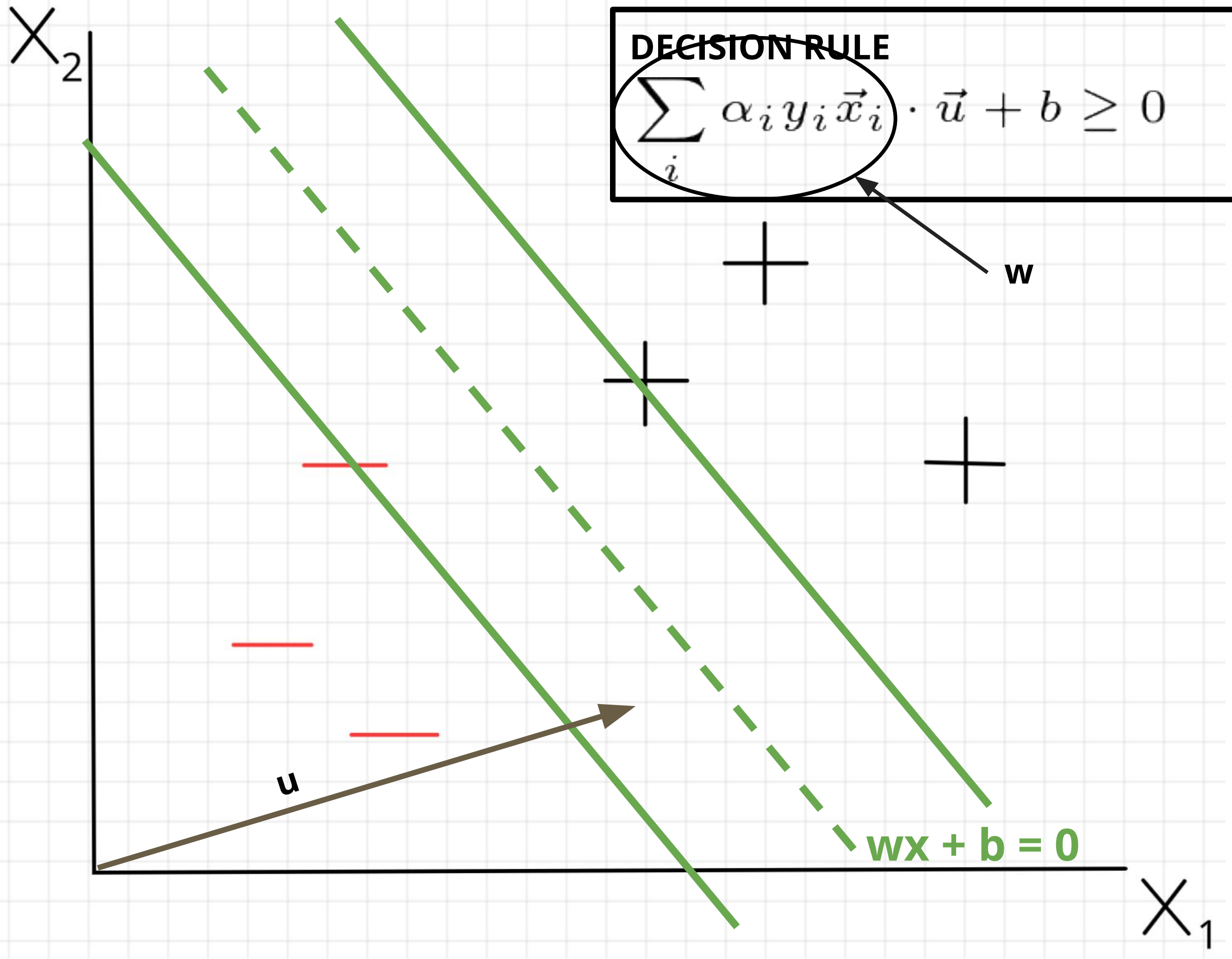
DECISION RULE

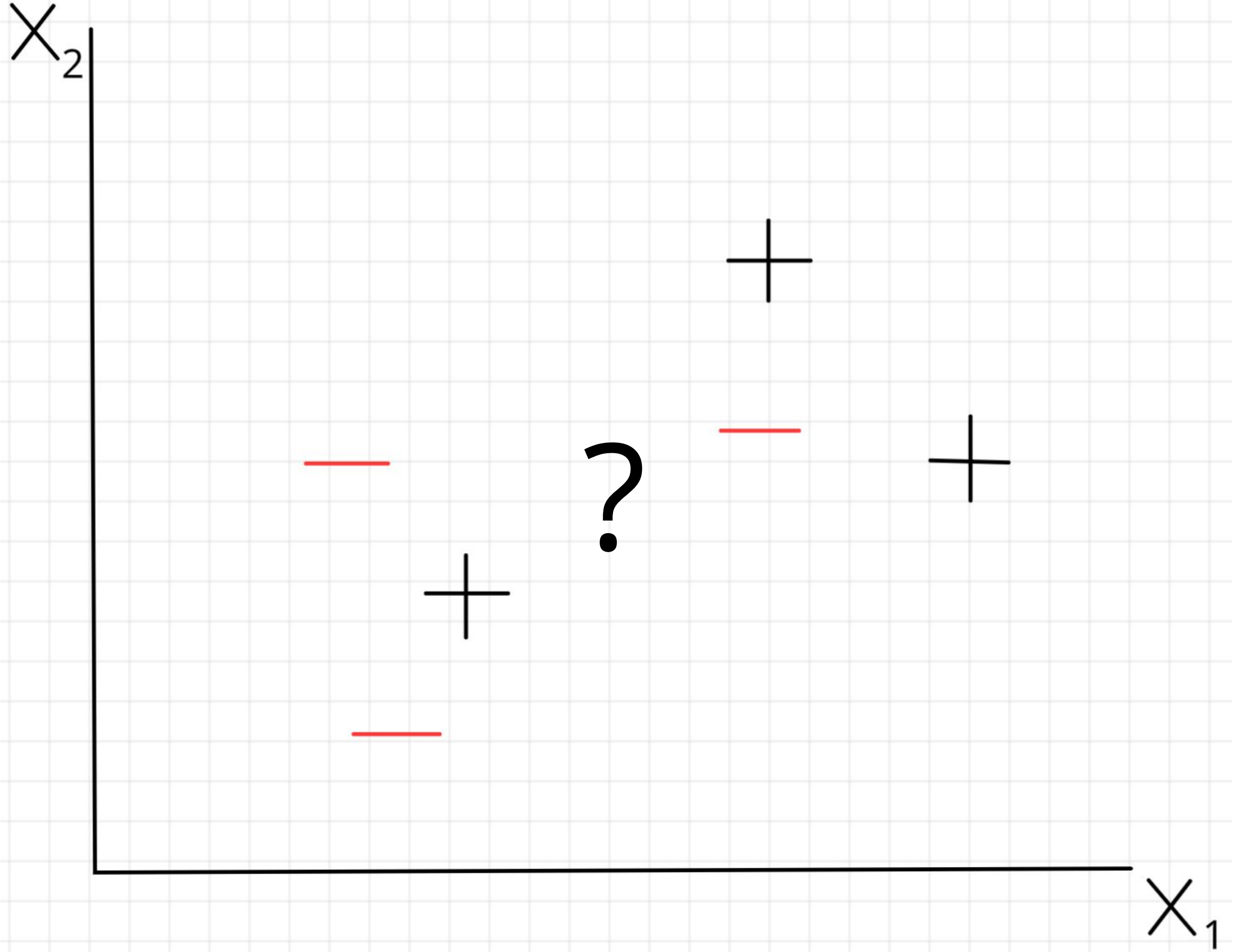
$$\sum_i \alpha_i y_i \vec{x}_i \cdot \vec{u} + b \geq 0 \quad \text{then } +$$



DECISION RULE

$$\sum_i \alpha_i y_i \vec{x}_i \cdot \vec{u} + b \geq 0 \quad \text{then } +$$

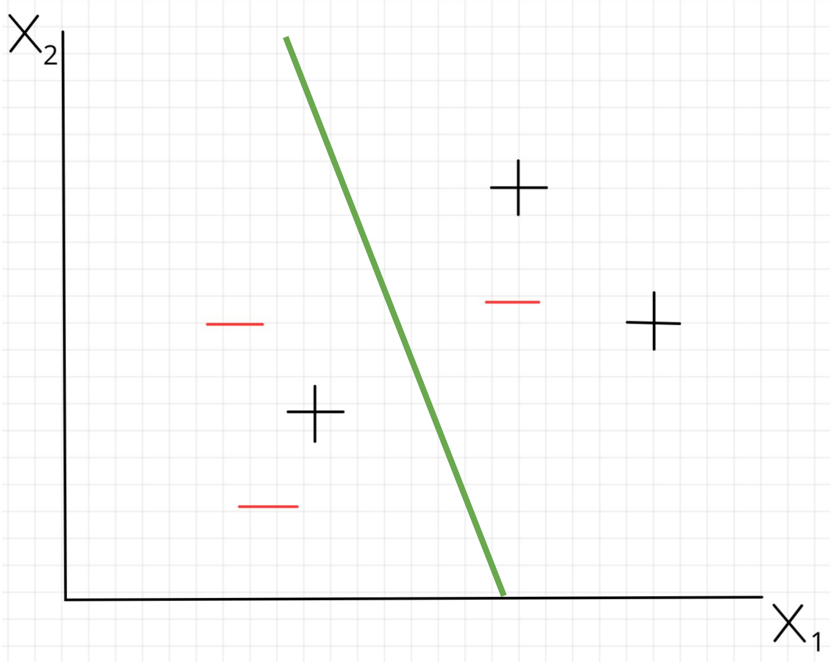




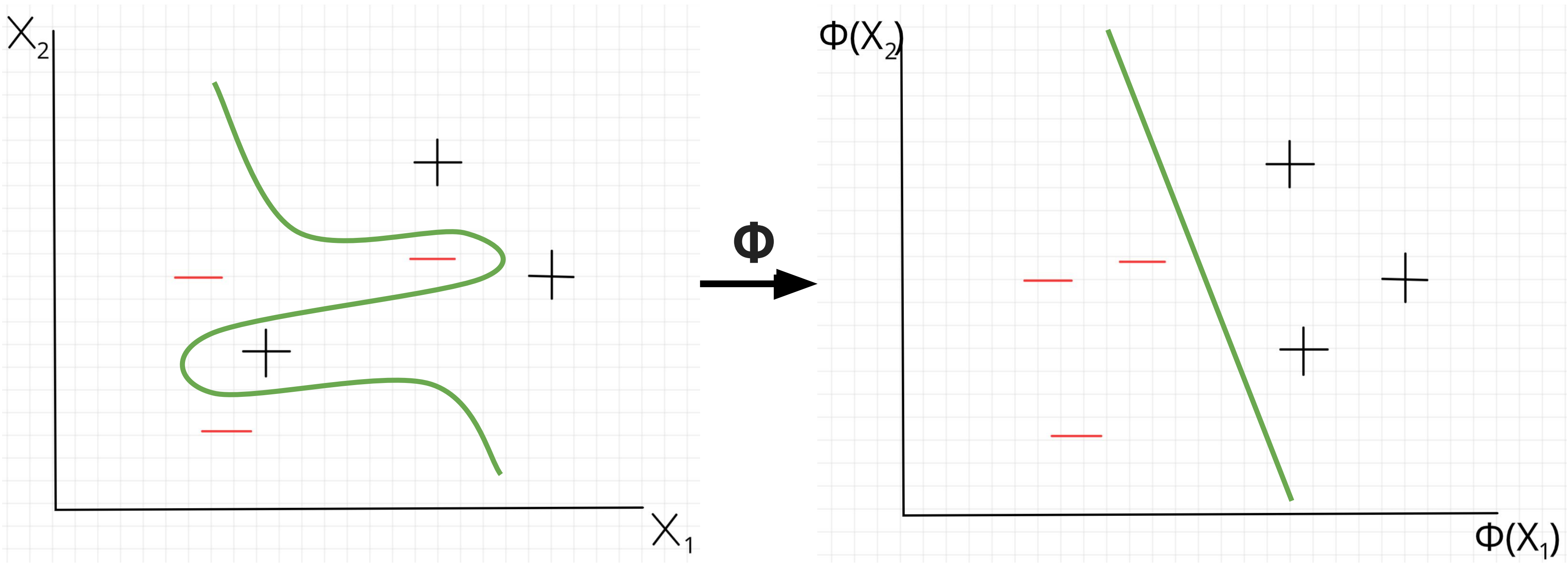
X_2
?
X_1

Option 1: Soft Margins

Can allow for some points in the dataset to be misclassified.



Option 2: Change perspective



But how to find Φ?

Turns out we don't need to find or define a transformation Φ!

Looking back at L, since **it depends only on the dot product of our input**, we only need to define the dot product in our transformed space.

i.e. we only need to define

$$K(\vec{x}_i, \vec{x}_j) = \phi(\vec{x}_i) \cdot \phi(\vec{x}_j)$$

Called a Kernel function. This is often referred to as the "kernel trick".

Kernel Function (intuition)

- The inner product of a space describes how close / similar points are
- Kernel Functions allow for specifying the closeness / similarity of points in a hypothetical transformed space
- The hope is that with that new notion of closeness, points in the dataset are linearly separable.

Example Kernel Functions

Polynomial Kernel

$$K(\vec{x}_i, \vec{x}_j) = (\vec{x}_i \cdot \vec{x}_j + 1)^n$$

Radial Basis Function Kernel

$$K(\vec{x}_i, \vec{x}_j) = e^{\frac{||\vec{x}_i - \vec{x}_j||}{\sigma}}$$

Demo