Homework 6 solution

1. (a) $x(t) = 2 = 2e^{j0}$, it's a constant function, has no fundamental period (infinitely small)

(b) $x(t) = -2 = 2e^{j\pi}$, it's a constant $\Rightarrow$ no fundamental period.

(c) $-1 = e^{j\pi} \Rightarrow x(t) = (-1)^t = e^{j\underbrace{\pi}_{\omega_0} t}$

By observation, the only complex exponential component is $e^{j\omega t}$, where $\omega = \pi$, so its fundamental period is $T = \frac{2\pi}{|\omega|} = 2$.

Note that $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$, $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$ for the rest 3 questions.

(d) $x(t) = \cos(\frac{\pi t}{4}) + \sin(\frac{\pi t}{2}) = \frac{e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}}{2} + \frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}$

$= \boxed{\frac{1}{2}e^{j\frac{\pi}{4}t} + \frac{1}{2}e^{-j\frac{\pi}{4}t}} \boxed{-\frac{j}{2}e^{j\frac{\pi}{2}t} + \frac{j}{2}e^{-j\frac{\pi}{2}t}}$

$T_1 = \frac{2\pi}{|\omega_1|} = \frac{2\pi}{\frac{\pi}{4}} = 8$ $\qquad T_2 = \frac{2\pi}{|\omega_2|} = \frac{2\pi}{\frac{\pi}{2}} = 4.$

Then the fundamental period is the least common period 8

(e) $x(t) = \cos(\frac{\pi}{4}t + \frac{\pi}{10}) + \sin(\frac{\pi}{2}t - \frac{\pi}{5})$

$= \frac{e^{j\frac{\pi}{10}}e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{10}}e^{-j\frac{\pi}{4}t}}{2} + \frac{e^{-j\frac{\pi}{5}}e^{j\frac{\pi}{2}t} - e^{j\frac{\pi}{5}}e^{-j\frac{\pi}{2}t}}{2j}$

$= \boxed{\frac{e^{j\frac{\pi}{10}}}{2}e^{j\frac{\pi}{4}t} + \frac{e^{-j\frac{\pi}{10}}}{2}e^{-j\frac{\pi}{4}t}} \boxed{-\frac{j}{2}e^{-j\frac{\pi}{5}}e^{j\frac{\pi}{2}t} + \frac{j}{2}e^{j\frac{\pi}{5}}e^{-j\frac{\pi}{2}t}}$

$T_1 = \frac{2\pi}{\frac{\pi}{4}} = 8$ $\qquad T_2 = \frac{2\pi}{\frac{\pi}{2}} = 4$

The fundamental period is 8.

(f) $x(t) = \cos(\frac{\pi}{4}t)\sin(\frac{\pi}{2}t) = \frac{e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}}{2} \cdot \frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}$

$= \frac{e^{j(\frac{\pi}{4}t + \frac{\pi}{2}t)} + e^{j(\frac{\pi}{2}t - \frac{\pi}{4}t)} - e^{j(\frac{\pi}{4}t - \frac{\pi}{2}t)} - e^{-j(\frac{\pi}{4}t + \frac{\pi}{2}t)}}{4j}$

$= -\frac{j}{4}\left(\boxed{e^{j\frac{3\pi}{4}t} - e^{-j\frac{3\pi}{4}t}} + \boxed{e^{j\frac{\pi}{4}t} - e^{-j\frac{\pi}{4}t}}\right)$

$T_1 = \frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3}$ $T_2 = \frac{2\pi}{\frac{\pi}{4}} = 8$

The fundamental period is the least common period 8.

2. (a) $x[n] = 5 = 5e^{j0}$, it's a constant $\Rightarrow$ no fundamental period

(b) $x[n] = 1 + (-1)^n = e^{j0} + (e^{j\pi})^n$

$= \boxed{e^{j0}} + \boxed{e^{j\pi n}}$

T_1 = whatever integer T_2 = least integer multiples of $(\frac{2\pi}{\pi} = 2) = 2$

Then fundamental period is just 2.

(c) $x[n] = (e^{j\pi})^n \cdot \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}$

$= \boxed{\frac{1}{2}e^{j\frac{5\pi}{4}n}} + \boxed{\frac{1}{2}e^{j\frac{3\pi}{4}n}}$

T_1 = least integer multiples of $(\frac{2\pi}{\frac{5\pi}{4}} = \frac{8}{5}) = 8$ T_2 = least integer multiples of $(\frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3}) = 8$

Then fundamental period is 8.

(d)

$$x[n] = e^{j0} + \frac{1}{2}\left(\boxed{e^{j\frac{\pi}{5}n} + e^{-j\frac{\pi}{5}n}} + \boxed{e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}} + \boxed{e^{j\frac{3\pi}{5}n} + e^{-j\frac{3\pi}{5}n}} + \boxed{e^{j\frac{4\pi}{5}n} + e^{-j\frac{4\pi}{5}n}}\right)$$

$$T_1 = \frac{2\pi}{\frac{\pi}{5}} = 10 \qquad T_2 = \frac{2\pi}{\frac{2\pi}{5}} = 5 \qquad T_3 = \frac{2\pi}{\frac{3\pi}{5}} \times 3 = 10 \qquad T_4 = \frac{2\pi}{\frac{4\pi}{5}} \times 2 = 5$$

The fundamental period is the least common period 10.

(e)

$$x[n] = -\frac{j}{2}\left(\boxed{e^{j\frac{\pi}{5}n} - e^{-j\frac{\pi}{5}n}} + \boxed{e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n}} + \boxed{e^{j\frac{3\pi}{5}n} - e^{-j\frac{3\pi}{5}n}} + \boxed{e^{j\frac{4\pi}{5}n} - e^{-j\frac{4\pi}{5}n}}\right)$$

$$T_1 = \frac{2\pi}{\frac{\pi}{5}} = 10 \qquad T_2 = \frac{2\pi}{\frac{2\pi}{5}} = 5 \qquad T_3 = \frac{2\pi}{\frac{3\pi}{5}} \times 3 = 10 \qquad T_4 = \frac{2\pi}{\frac{4\pi}{5}} \times 2 = 5$$

The fundamental period is 10.

(f)

$$x[n] = \frac{e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}}{2} \cdot \frac{e^{j\frac{4\pi}{5}n} - e^{-j\frac{4\pi}{5}n}}{2j}$$

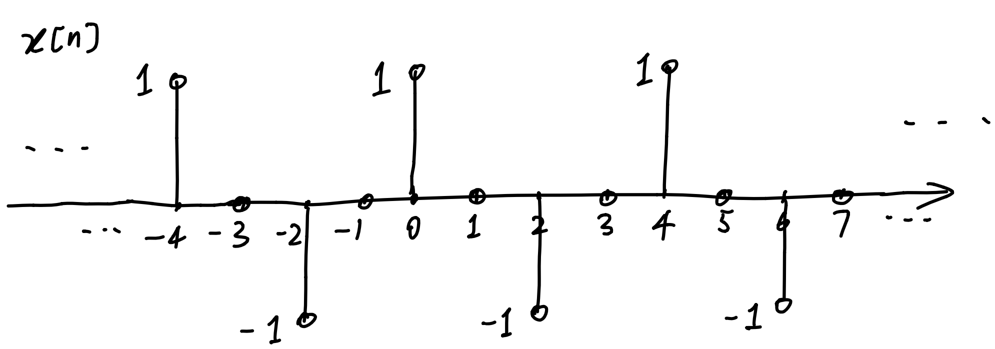
$$= -\frac{j}{4}\left(\boxed{e^{j\frac{6\pi}{5}n} - e^{-j\frac{6\pi}{5}n}} + \boxed{e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n}}\right)$$

$$T_1 = \frac{2\pi}{\frac{6\pi}{5}} \times 3 = 5 \qquad T_2 = \frac{2\pi}{\frac{2\pi}{5}} = 5$$

The fundamental period is 5.

3. (a) $x[n] = \cos(\frac{\pi}{2}n)$, fundamental period is $\frac{2\pi}{\frac{\pi}{2}} = 4$, only need to calculate the values within one period:

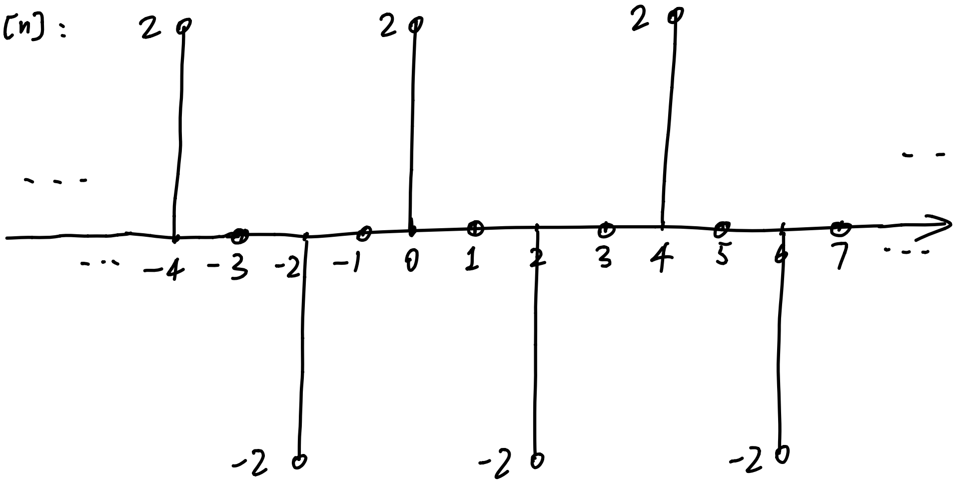
$x[0] = \cos(0) = 1$. $x[1] = \cos(\frac{\pi}{2}) = 0$ $x[2] = \cos(\pi) = -1$ $x[3] = \cos(\frac{3\pi}{2}) = 0$



(b)

$$y[n] = x[n] * h[n] = x[n] * (\delta[n] - \delta[n-2])$$
$$= x[n] * \delta[n] - x[n] * \delta[n-2] = x[n] - x[n-2]$$
$$= \cos(\tfrac{\pi}{2}n) - \cos(\tfrac{\pi}{2}n - \pi) = 2\cos(\tfrac{\pi}{2}n)$$

$y[n]$:



(c)

$$x[n] = \cos(\tfrac{\pi}{2}n) = \tfrac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})$$

The frequency response (DTFT of $h[n]$) of this system is:

$$H(e^{j\omega}) = \sum_{-\infty}^{+\infty} h[n]\, e^{-j\omega n} = \delta[n] e^{-j\omega n} - \delta[n-2] e^{-j\omega n} = 1 - e^{-j2\omega}$$

For the first complex exponential component $\frac{1}{2}e^{j\frac{\pi}{2}n}$, the frequency is $\frac{\pi}{2}$, the response is then $H(e^{j\frac{\pi}{2}}) \cdot \frac{1}{2}e^{j\frac{\pi}{2}n} = (1 - e^{-j\pi}) \cdot \frac{1}{2}e^{j\frac{\pi}{2}n} = e^{j\frac{\pi}{2}n}$.

For the second complex exponential, the frequency is $-\frac{\pi}{2}$, response is

$$H(e^{-j\frac{\pi}{2}}) \cdot \tfrac{1}{2}e^{-j\frac{\pi}{2}n} = (1 - e^{j\pi}) \cdot \tfrac{1}{2}e^{-j\frac{\pi}{2}n} = e^{-j\frac{\pi}{2}n}$$

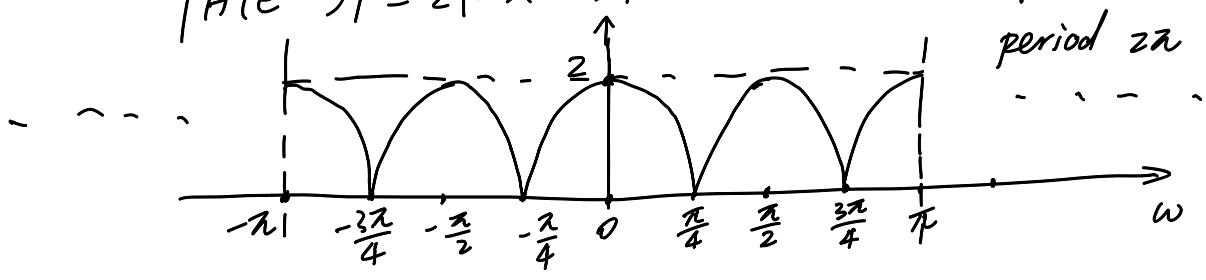
The output $y[n] = e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n} = 2\cos(\frac{\pi}{2}n)$

4. By definition, the DTFT of $h[n]$ is $H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$

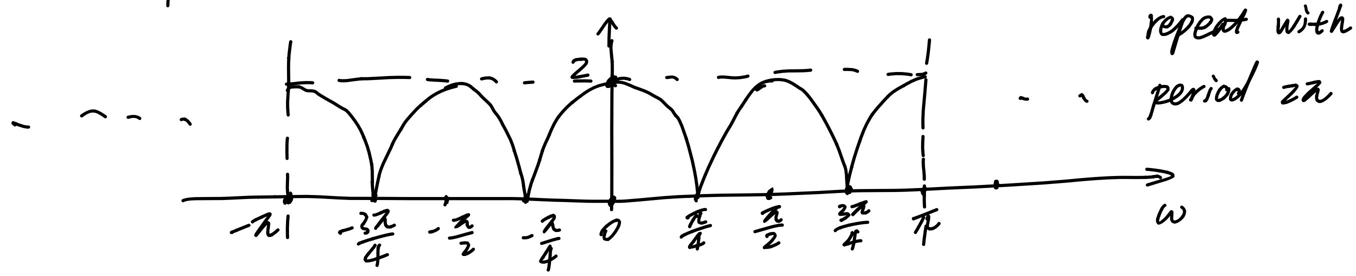
(a) $H(e^{j\omega}) = \delta[n+2] e^{-j\omega n} + \delta[n-2] e^{-j\omega n} = e^{j2\omega} + e^{-j2\omega} = 2\cos(2\omega)$

$|H(e^{j\omega})| = 2|\cos(2\omega)|$

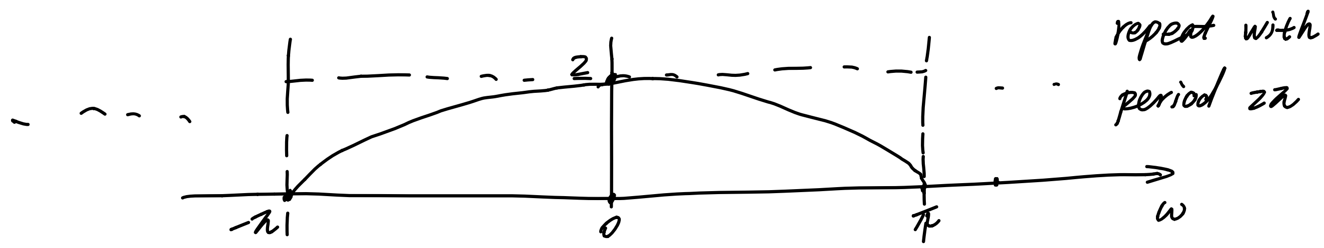
repeat with period 2π



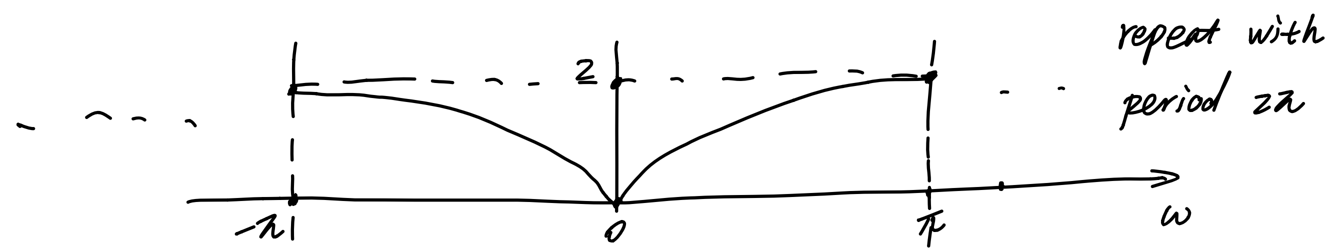
(b) $$H(e^{jw}) = \delta[n]e^{-jwn} + \delta[n-4]e^{-jwn} = 1 + e^{-j4w}$$
$$= e^{-j2w}(e^{j2w} + e^{-j2w}) = 2e^{-j2w}\cos(2w)$$
$$|H(e^{jw})| = 2|e^{-j2w}||\cos(2w)| = 2|\cos(2w)|$$



(c) $$H(e^{jw}) = \delta[n]e^{-jwn} + \delta[n-1]e^{-jwn} = 1 + e^{-jw}$$
$$= e^{-j\frac{w}{2}}(e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}) = 2e^{-j\frac{w}{2}}\cos(\tfrac{w}{2})$$
$$|H(e^{jw})| = 2|e^{-j\frac{w}{2}}||\cos(\tfrac{w}{2})| = 2|\cos(\tfrac{w}{2})|$$



(d) $$H(e^{jw}) = \delta[n]e^{-jwn} - \delta[n-1]e^{-jwn} = 1 - e^{-jw}$$
$$= e^{-j\frac{w}{2}}(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}) = 2je^{-j\frac{w}{2}}\sin(\tfrac{w}{2})$$
$$|H(e^{jw})| = 2|j|\cdot|e^{-j\frac{w}{2}}|\cdot|\sin(\tfrac{w}{2})| = 2|\sin(\tfrac{w}{2})|$$



(e) From the lecture we know directly applying DTFT to $h[n]$ is intractable. By the definition of inverse DTFT,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})\, e^{jwn}\, dw \qquad ①$$

$$= \cos\left(\frac{\pi}{4}n+\frac{\pi}{2}\right) = \frac{1}{2}e^{j\frac{\pi}{2}}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{2}}e^{-j\frac{\pi}{4}n} \qquad ②$$

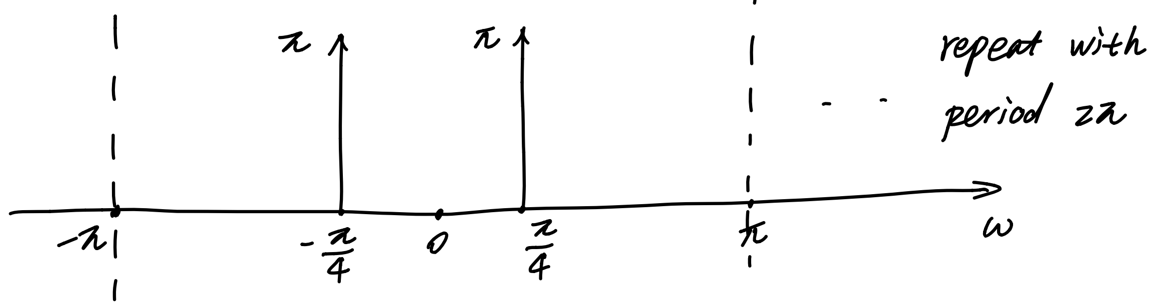
Comparing ① and ②, we can guess out

$$H(e^{jw}) = \pi e^{j2w}\left[\delta\left(w-\frac{\pi}{4}\right)+\delta\left(w+\frac{\pi}{4}\right)\right] \qquad ③$$

Then by validation and the uniqueness of DTFT, we know ③ is right.

$$|H(e^{jw})| = \pi\,|e^{j2w}|\,\left|\delta\left(w-\frac{\pi}{4}\right)+\delta\left(w+\frac{\pi}{4}\right)\right|$$

$$= \pi\left[\delta\left(w-\frac{\pi}{4}\right)+\delta\left(w+\frac{\pi}{4}\right)\right]$$



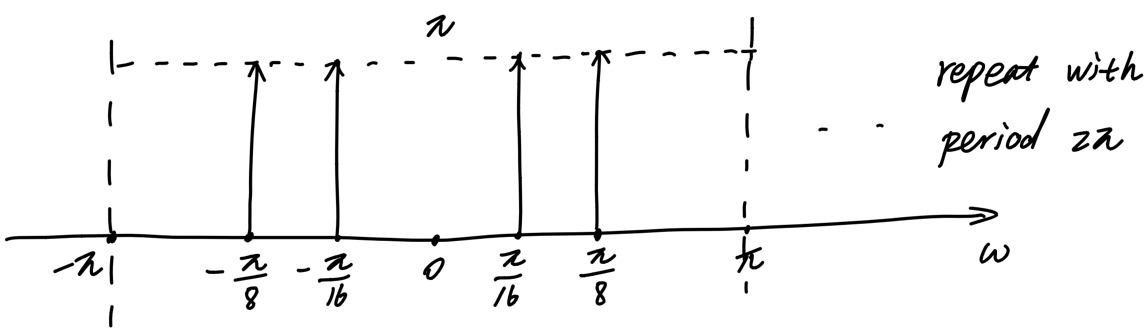
(f) $$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})\, e^{jwn}\, dw \qquad ①$$

$$= \sin\left(\frac{\pi}{8}n\right)+\cos\left(\frac{\pi}{16}n\right) = \frac{-j}{2}\left(e^{j\frac{\pi}{8}n}-e^{-j\frac{\pi}{8}n}\right)+\frac{1}{2}\left(e^{j\frac{\pi}{16}n}+e^{-j\frac{\pi}{16}n}\right) \qquad ②$$

Comparing ① and ② $\Rightarrow$

$$H(e^{jw}) = -j\pi\left[\delta\left(w-\frac{\pi}{8}\right)+\delta\left(w+\frac{\pi}{8}\right)\right]+\pi\left[\delta\left(w-\frac{\pi}{16}\right)+\delta\left(w+\frac{\pi}{16}\right)\right] \qquad ③$$

$$|H(e^{jw})| = \pi\left[\delta\left(w-\frac{\pi}{8}\right)+\delta\left(w+\frac{\pi}{8}\right)+\delta\left(w-\frac{\pi}{16}\right)+\delta\left(w+\frac{\pi}{16}\right)\right]$$



5\. The frequency response of this system is :

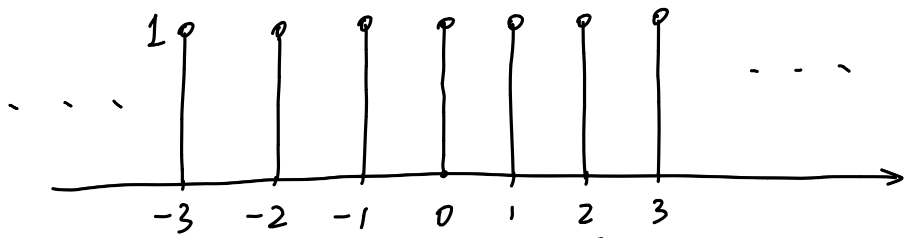
$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-jwn} = \sum_{n=-2}^{2} \frac{1}{5} e^{-jwn} = \frac{1}{5} \cdot \frac{e^{j2w}(1-e^{-j5w})}{1-e^{-jw}}$$

$$= \frac{1}{5} \cdot \frac{e^{j2w} \cdot e^{-j\frac{5w}{2}} (e^{j\frac{5w}{2}} - e^{-j\frac{5w}{2}})}{e^{-j\frac{w}{2}} (e^{j\frac{w}{2}} - e^{-j\frac{w}{2}})} = \frac{\sin(\frac{5w}{2})}{5\sin(\frac{w}{2})}$$

(a) The wave decomposition of $x[n]$ is

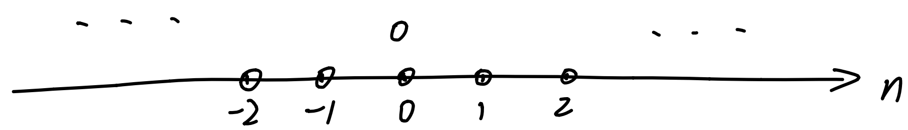
$x[n] = e^{j0} = e^{j0\cdot n}$, the frequency component is $w=0$, so the output

$y[n] = H(e^{j0}) x[n] = 1 \cdot x[n] = 1$



(b) $x[n] = \sin(\frac{2\pi}{5}n) = -\frac{j}{2}(e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n})$, which contains 2 frequency components $w_1 = \frac{2\pi}{5}$, $w_2 = -\frac{2\pi}{5}$, then the output is

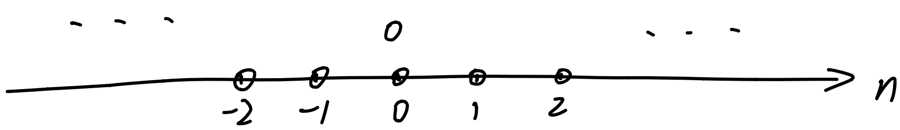
$$y[n] = -\frac{j}{2} H(e^{j\frac{2\pi}{5}}) e^{j\frac{2\pi}{5}n} + \frac{j}{2} H(e^{-j\frac{2\pi}{5}}) e^{-j\frac{2\pi}{5}n} = 0$$



(c) $x[n] = \cos(\frac{2\pi}{5}n) = \frac{1}{2}(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n})$, $w_1 = \frac{2\pi}{5}$, $w_2 = -\frac{2\pi}{5}$

$H(e^{jw_1}) = H(e^{jw_2}) = 0$, so the response is 0.

$$y[n] = \frac{1}{2} H(e^{j\frac{2\pi}{5}}) e^{j\frac{2\pi}{5}n} + \frac{1}{2} H(e^{-j\frac{2\pi}{5}}) e^{-j\frac{2\pi}{5}n} = 0$$



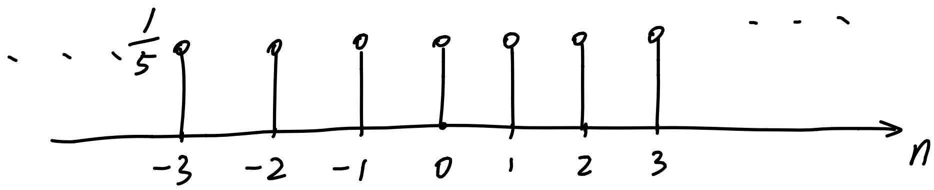
(d) It would be easier to look at them in time domain.

$$h[n] = \frac{1}{5}\left(u[n+2] - u[n-3]\right), \quad x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-5k]$$

$$y[n] = x[n] * h[n] = \frac{1}{5} \underbrace{\sum_{k=-\infty}^{+\infty} \left(u[n+2-5k] - u[n-3-5k]\right)}_{}$$

consecutive box functions $= 1$

$$= \frac{1}{5}$$



6. (a) By definition,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \text{ substituting } \omega \text{ by } (\omega - 2\pi k)$$

$$\Rightarrow X\left(e^{j(\omega-2\pi k)}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \boxed{e^{j2\pi k n}}$$

note $e^{j2\pi kn} = e^{j0} = 1$ since $e^{j\theta}$ is periodic with period 2π.

$$\Rightarrow X(e^{j\omega}) = X\left(e^{j(\omega-2\pi k)}\right)$$

(b)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \text{ substituting } \omega \text{ by } 0,$$

$$X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j0\cdot n} = \sum_{n=-\infty}^{+\infty} x[n]$$

(c)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \text{ substituting } \omega \text{ by } \pi,$$

$$X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\pi n}$$

$$= \sum_{k=-\infty}^{+\infty} x[2k] \overset{=1}{\boxed{e^{-j2\pi k}}} + \sum_{k=-\infty}^{+\infty} x[2k+1] \overset{=-1}{\boxed{e^{-j2\pi k} e^{-j\pi}}}$$

$$= \sum_{k=-\infty}^{+\infty} x[2k] - \sum_{k=-\infty}^{+\infty} x[2k+1]$$

(d) By the definition of inverse DTFT,

$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw$, substituting n by 0,

$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) dw \Rightarrow 2\pi x[0] = \int_{-\pi}^{\pi} X(e^{jw}) dw$