Homework 8 solutions

1. Note that the Fourier Series expansion of periodic signal $x[n]$ is

$$x[n] = \sum_{k=0}^{N-1} a_k \cdot e^{j\frac{2\pi k}{N}n},$$

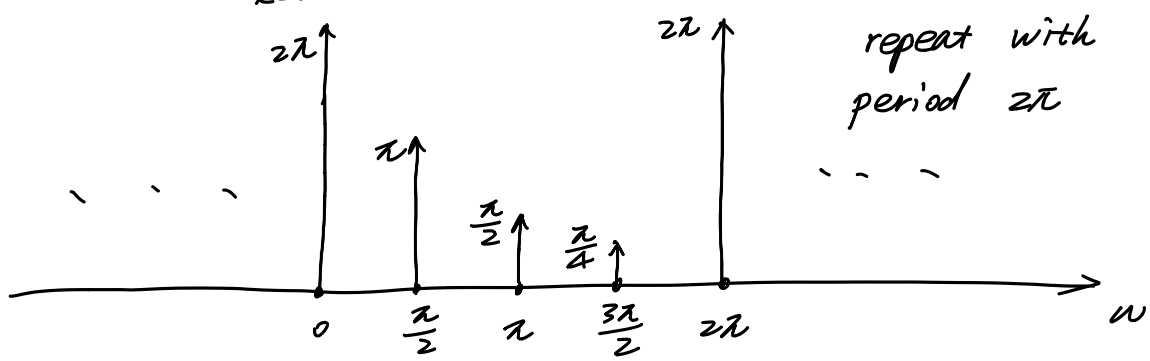
with that, the DTFT $X(e^{jw})$ of $x[n]$ is

$$X(e^{jw}) = \sum_{k=0}^{N-1} a_k \cdot 2\pi\,\delta\left(w - \frac{2\pi k}{N}\right) \qquad (0 \le w < 2\pi)$$

(a) $$x[n] = \sum_{k=0}^{3} \left(\frac{1}{2}\right)^k e^{j\frac{2\pi k}{4}n},$$ here $N = 4$,

$$a_k = \left(\frac{1}{2}\right)^k \quad (k = 0, 1, 2, 3)$$

$$\Rightarrow X(e^{jw}) = \sum_{k=0}^{3}\left[\left(\frac{1}{2}\right)^k \cdot 2\pi\,\delta\left(w - \frac{2\pi k}{4}\right)\right]$$



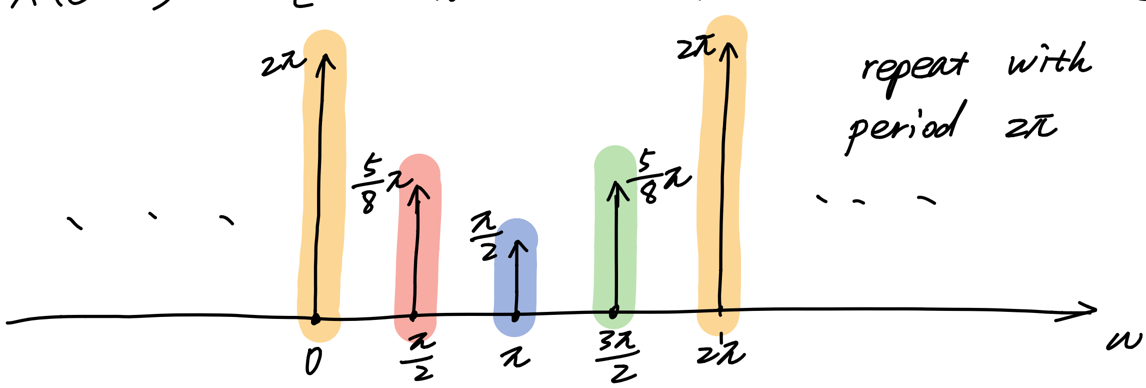
(b) $$\cos\left(\frac{2\pi kn}{4}\right) = \frac{1}{2}\left(e^{j\frac{2\pi kn}{4}} + e^{-j\frac{2\pi kn}{4}}\right)$$

$$x[n] = \sum_{k=0}^{3}\left(\frac{1}{2}\right)^{k+1}\left[e^{j\frac{2\pi kn}{4}} + e^{-j\frac{2\pi kn}{4}}\right]$$

$$= \frac{1}{2}e^{j\cdot 0} + \frac{1}{2}e^{j\cdot 0} + \frac{1}{4}e^{j\frac{2\pi n}{4}} + \frac{1}{4}e^{j\frac{6\pi n}{4}} + \frac{1}{8}e^{j\frac{4\pi n}{4}} + \frac{1}{8}e^{j\frac{4\pi n}{4}}$$

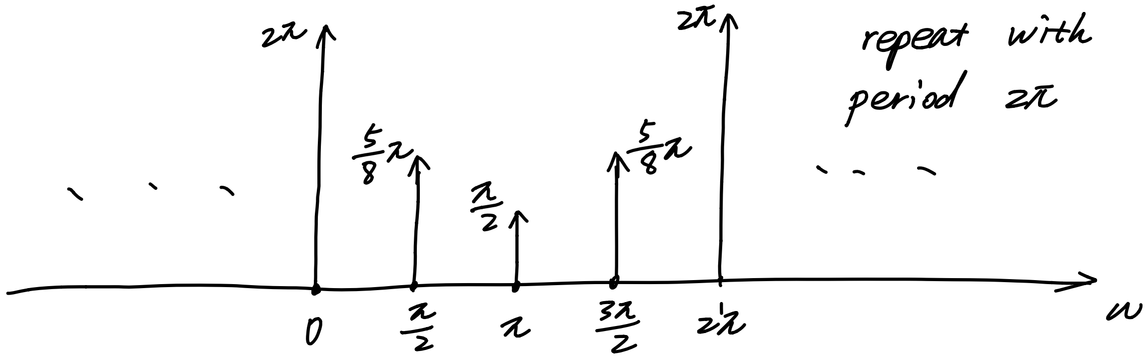
$$+ \frac{1}{16}e^{j\frac{6\pi n}{4}} + \frac{1}{16}e^{j\frac{2\pi n}{4}}$$

$$\Rightarrow X(e^{jw}) = 2\pi\left[\delta(w) + \frac{5}{16}\delta\left(w - \frac{\pi}{2}\right) + \frac{1}{4}\delta(w - \pi) + \frac{5}{16}\delta\left(w - \frac{3\pi}{2}\right)\right]$$



(c)

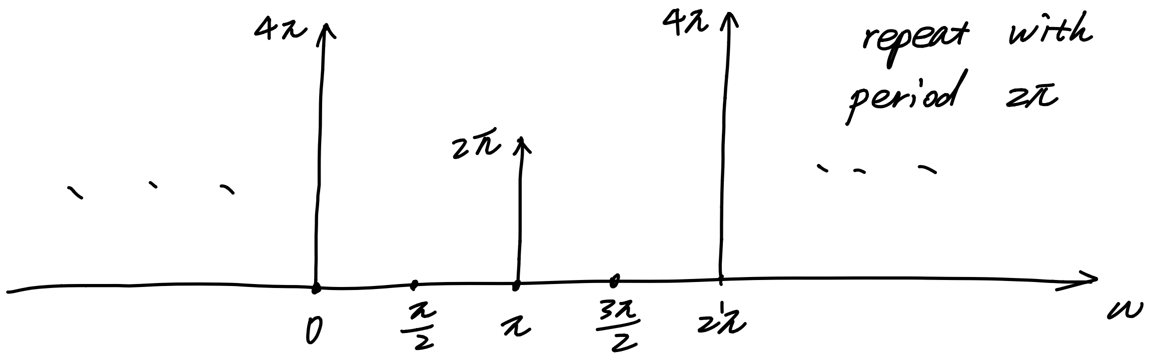
$$x[n] = \sum_{k=0}^{3} \left(\tfrac{1}{2}\right)^k \cos\left[\frac{2\pi k}{4}\left(n + \frac{2}{3}\right)\right],$$ it's a shifted version of (b), so the magnitude of the DTFT does not change.



(d) $x[n] = 2 \cdot e^{j\cdot 0\cdot n} + e^{j\pi n}$

$$2\cdot e^{j\cdot 0\cdot n} \longleftrightarrow 2\cdot 2\pi\cdot\delta(w) \qquad e^{j\pi n} \longleftrightarrow 2\pi\,\delta(w-\pi) \qquad (0 \le w < 2\pi)$$

$$\Rightarrow X(e^{jw}) = 4\pi\,\delta(w) + 2\pi\,\delta(w-\pi) \qquad (0 \le w < 2\pi)$$



2. (a) Directly applying the CTFT equation,

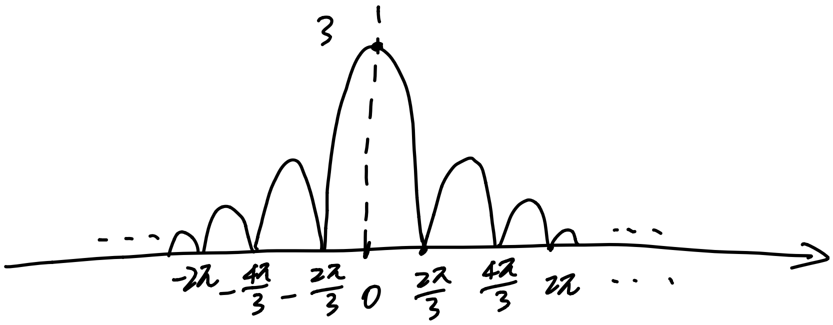
$$X(jw) = \int_{-\infty}^{+\infty} x(t)\, e^{-jwt}\, dt = \int_{-\infty}^{+\infty} [u(t) - u(t-3)]\, e^{-jwt}\, dt$$

$$= \int_0^3 e^{-jwt}\, dt = \frac{1}{-jw}\, e^{-jwt}\Big|_0^3 = \frac{1-e^{-j3w}}{jw}$$

$$= e^{-j\frac{3w}{2}} \cdot \frac{\left(e^{j\frac{3w}{2}} - e^{-j\frac{3w}{2}}\right)}{jw} = e^{-j\frac{3w}{2}}\left(\frac{\sin\left(\frac{3w}{2}\right)}{w/2}\right)$$

$$|X(jw)| = \left|\frac{\sin\left(\frac{3w}{2}\right)}{w/2}\right|,$$ zero-crossings: $\frac{3w}{2} = k\pi \Rightarrow w = \frac{2k\pi}{3}$ $(k \ne 0)$

$|X(j0)| = 3$, note CTFT is not periodic.

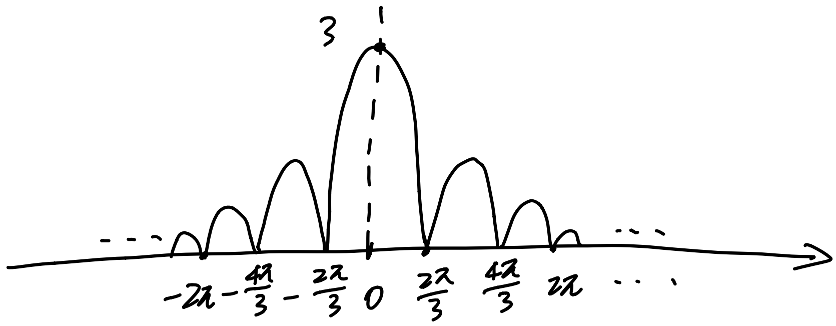


(b) Let $x_0(t) = u(t) - u(t-3)$ denote the signal in part (a)

Here $x(t) = u(t-1) - u(t-4) = x_0(t-1)$, then the CTFT

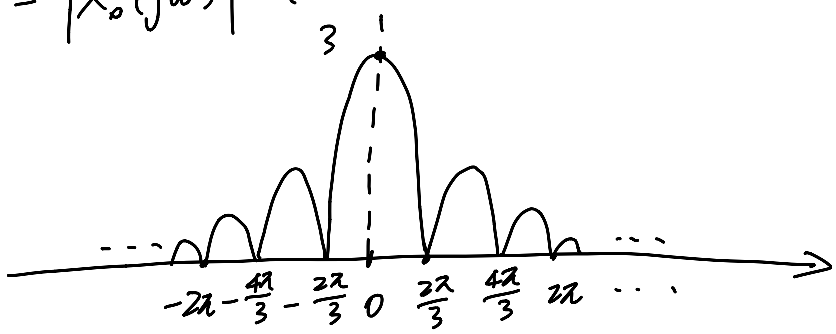
$X(j\omega) = e^{-j\omega} X_0(j\omega) \Rightarrow |X(j\omega)| = |X_0(j\omega)|$.

Time shift doesn't change the magnitude of CTFT :



(c) Here $x(t) = x_0(t+2) \Rightarrow X(j\omega) = e^{j2\omega} X_0(j\omega)$

$|X(j\omega)| = |X_0(j\omega)|$: same as (a) and (b)

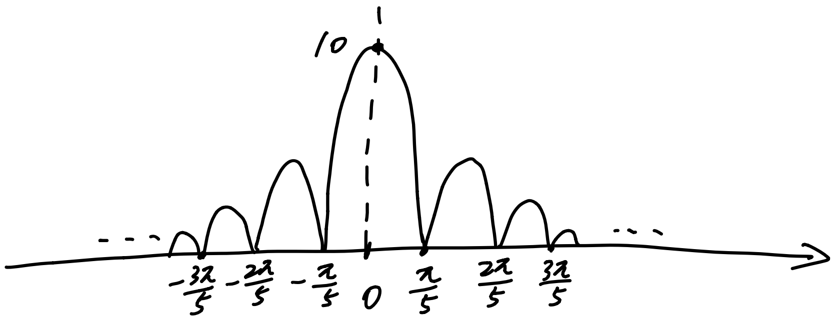


(d) $X(j\omega) = \int_{-\infty}^{+\infty} [u(t) - u(t-10)] e^{-j\omega t} dt = \int_0^{10} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_0^{10}$

$= \dfrac{1 - e^{-j10\omega}}{j\omega} = e^{-j5\omega} \cdot \dfrac{\sin(5\omega)}{\omega/2}$

$|X(j\omega)| = \left| \dfrac{\sin(5\omega)}{\omega/2} \right|$, $|X(j0)| = 10$

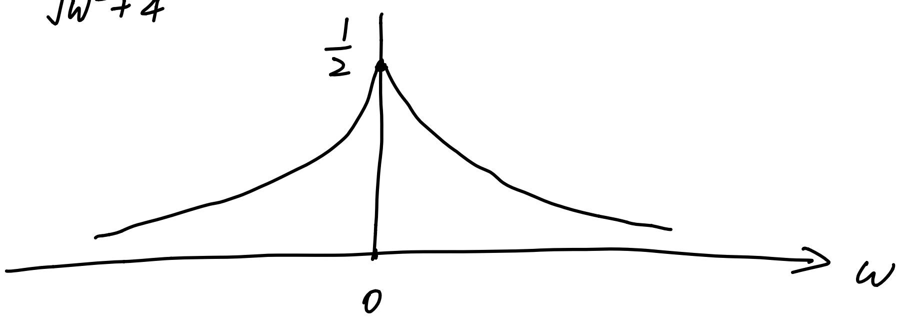
zero-crossings : $5\omega = k\pi \Rightarrow \omega = \frac{k\pi}{5}$ $(k \neq 0)$



3. (a) $X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt = \int_{0}^{+\infty} e^{-2t} e^{-jwt} dt$

$$= \frac{1}{-(jw+2)} e^{-(jw+2)t} \Big|_{0}^{+\infty} = \frac{0-1}{-(jw+2)} = \frac{1}{jw+2}$$

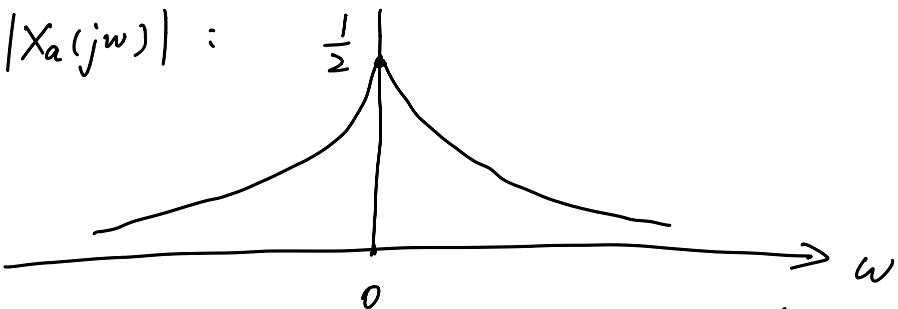
$|X(jw)| = \frac{1}{\sqrt{w^2+4}}$:



(b) let $x_a(t)$ denote signal in (a), $X_a(jw)$ is its CTFT.

Here $x(t) = x_a(t-3) \Rightarrow X(jw) = e^{-j3w} X_a(jw)$

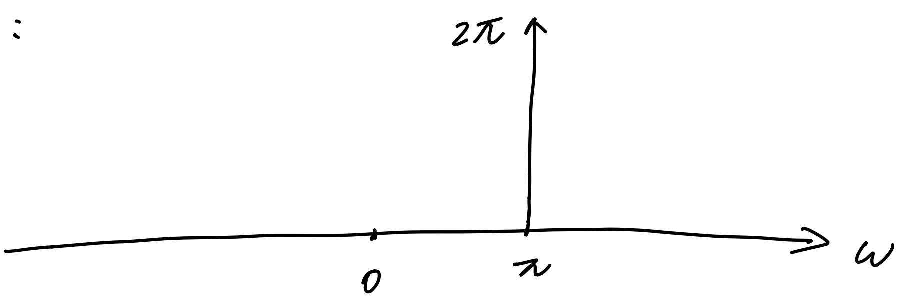
$|X(jw)| = |X_a(jw)|$:



(c) The CTFT of a continuous complex exponential is an impulse:

$x(t) = (-1)^t = e^{j\pi t} \longleftrightarrow X(jw) = 2\pi\delta(w-\pi)$

$|X(jw)|$:



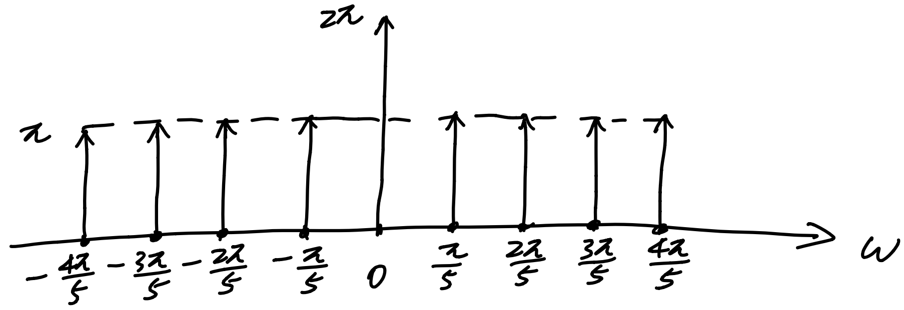
(d)

$$\cos(w_0 t) = \frac{1}{2}\left(e^{jw_0 t} + e^{-jw_0 t}\right) \longleftrightarrow \pi\left[\delta(w-w_0) + \delta(w+w_0)\right]$$

$x(t) = 1 + \cos(\frac{\pi}{5}t) + \cos(\frac{2\pi}{5}t) + \cos(\frac{3\pi}{5}t) + \cos(\frac{4\pi}{5}t)$, each component corresponds to $w_0 = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$. then

$$X(jw) = \pi\left[2\delta(w) + \delta(w-\tfrac{\pi}{5}) + \delta(w+\tfrac{\pi}{5}) + \delta(w-\tfrac{2\pi}{5}) + \delta(w+\tfrac{2\pi}{5}) + \delta(w-\tfrac{3\pi}{5}) + \delta(w+\tfrac{3\pi}{5}) + \delta(w-\tfrac{4\pi}{5}) + \delta(w+\tfrac{4\pi}{5})\right]$$

$|X(jw)|$:



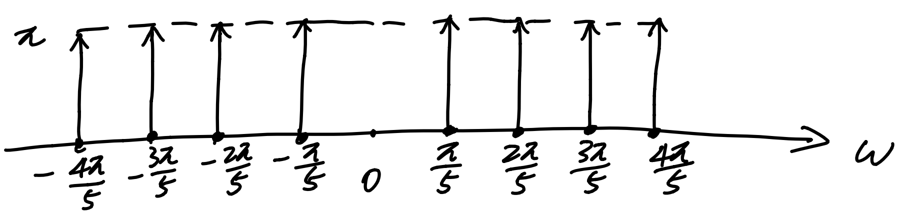
(e)

$$\sin(w_0 t) = \frac{1}{2j}\left(e^{jw_0 t} - e^{-jw_0 t}\right) \longleftrightarrow \frac{\pi}{j}\left[\delta(w-w_0) - \delta(w+w_0)\right]$$

Each component in $x(t)$'s expression corresponds to $w_0 = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$$X(jw) = \frac{\pi}{j}\left[\delta(w-\tfrac{\pi}{5}) - \delta(w+\tfrac{\pi}{5}) + \delta(w-\tfrac{2\pi}{5}) - \delta(w+\tfrac{2\pi}{5}) + \delta(w-\tfrac{3\pi}{5}) - \delta(w+\tfrac{3\pi}{5}) + \delta(w-\tfrac{4\pi}{5}) - \delta(w+\tfrac{4\pi}{5})\right]$$

$|X(jw)|$:

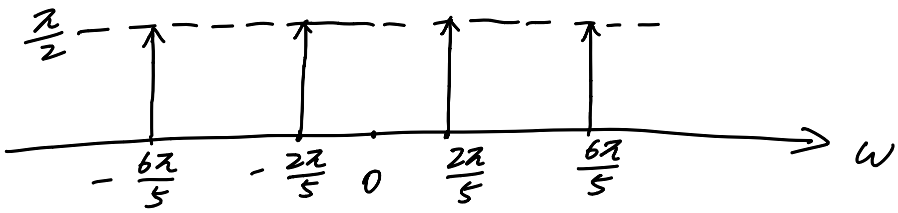


(f) $x(t) = \frac{1}{2}\left(e^{j\frac{2\pi}{5}t} + e^{-j\frac{2\pi}{5}t}\right) \cdot \frac{1}{2j}\left(e^{j\frac{4\pi}{5}t} - e^{-j\frac{4\pi}{5}t}\right)$

$$= \frac{1}{4j}\left(e^{j\frac{6\pi}{5}t} - e^{-j\frac{2\pi}{5}t} + e^{j\frac{2\pi}{5}t} - e^{-j\frac{6\pi}{5}t}\right)$$

$$\longleftrightarrow X(jw) = \frac{\pi}{2j}\left[\delta(w-\tfrac{6\pi}{5}) - \delta(w+\tfrac{2\pi}{5}) + \delta(w-\tfrac{2\pi}{5}) - \delta(w+\tfrac{6\pi}{5})\right]$$

$|X(jw)|$:



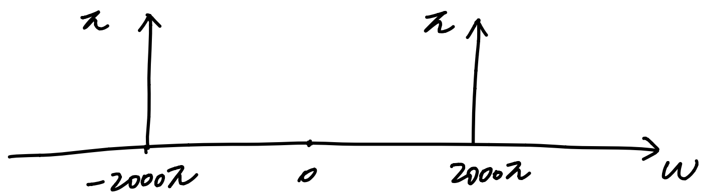
4.

(a)

$x(t)=\cos(2000\pi t)=\frac{1}{2}\left(e^{j2000\pi t}+e^{-j2000\pi t}\right)$ $|X(j\omega)|$:

$\Updownarrow$

$X(j\omega)=\pi[\delta(\omega-2000\pi)+\delta(\omega+2000\pi)]$



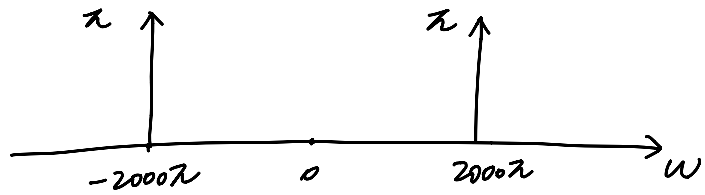
obviously it's symmetric about 0, hence even.

(b)

$x(t)=\sin(2000\pi t)=\frac{1}{2j}\left(e^{j2000\pi t}-e^{-j2000\pi t}\right)$ $|X(j\omega)|$:

$\Updownarrow$

$X(j\omega)=\frac{\pi}{j}[\delta(\omega-2000\pi)-\delta(\omega+2000\pi)]$



(c) $$|X(j\omega)|=\sqrt{X(j\omega)X^*(j\omega)}=\sqrt{\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt\cdot\int_{-\infty}^{+\infty}x^*(t)e^{j\omega t}dt}$$

$$|X(-j\omega)|=\sqrt{X(-j\omega)X^*(-j\omega)}=\sqrt{\int_{-\infty}^{+\infty}x(t)e^{j\omega t}dt\cdot\int_{-\infty}^{+\infty}x^*(t)e^{-j\omega t}dt}$$

$x(t)$ is real $\Rightarrow x(t)=x^*(t)\Rightarrow |X(j\omega)|=|X(-j\omega)|$

Then $|X(j\omega)|$ is even.

(d) $$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt,$$

$$X^*(j\omega)=\int_{-\infty}^{+\infty}x^*(t)e^{j\omega t}dt\equiv\int_{-\infty}^{+\infty}x^*(-t)e^{-j\omega t}dt$$

$x(t)$ is real and even $\Rightarrow x(t)=x^*(-t)\Rightarrow X(j\omega)=X^*(j\omega)$

$$X(-j\omega)=\int_{-\infty}^{+\infty}x(t)e^{j\omega t}dt=\int_{-\infty}^{+\infty}x(-t)e^{-j\omega t}dt$$

$x(t)$ is even $\Rightarrow x(t)=x(-t)\Rightarrow X(j\omega)=X(-j\omega)$

Then $X(j\omega)$ is real and even.

(e) $x(t) = \delta(t) \longleftrightarrow X(j\omega) = 1$,

$\delta(t)$ and 1 are both real and even

(f) The CTFT of $e^{j1000\pi t}x(t)$ is:

$$\int_{-\infty}^{+\infty} e^{j1000\pi t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} x(t) e^{-j(\omega-1000\pi)t} dt$$

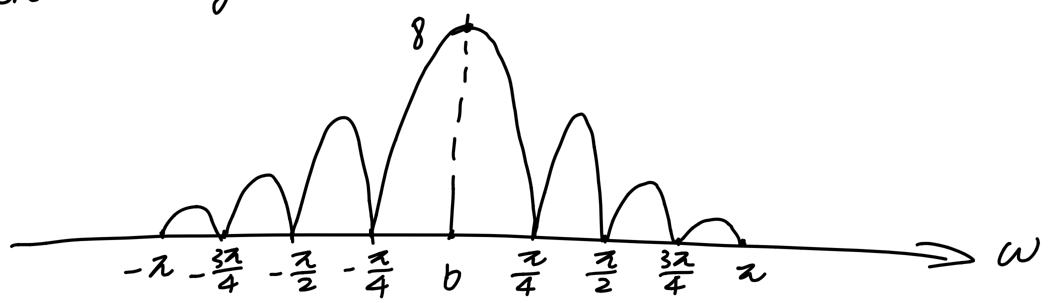
$$= X[j(\omega-1000\pi)]$$

5. (a) $$H(j\omega) = \int_{-\infty}^{+\infty} [u(t) - u(t-8)] e^{-j\omega t} dt = \int_{0}^{8} e^{-j\omega t} dt$$

$$= \frac{1}{j\omega}(1 - e^{-j8\omega}) = e^{-j4\omega} \cdot \frac{\sin(4\omega)}{\omega/2}$$

$$|H(j\omega)| = \left|\frac{\sin(4\omega)}{\omega/2}\right|, \quad |H(j0)| = 8$$

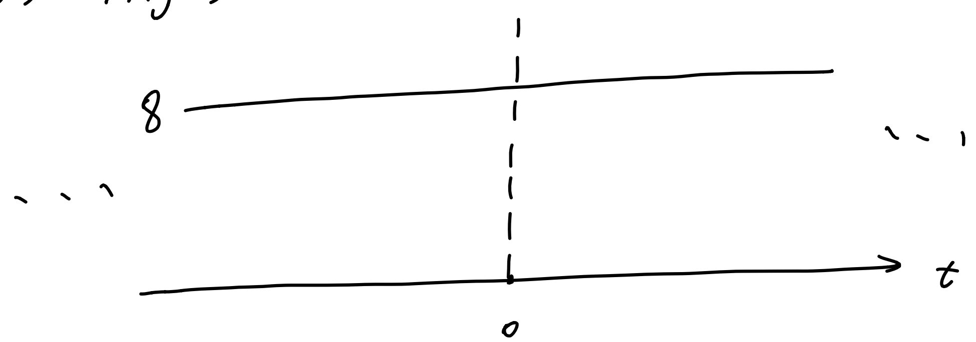
zero-crossings : $4\omega = k\pi \Rightarrow \omega = \frac{k\pi}{4}$ $(k \neq 0)$



(b) $x(t) = 1 = e^{j\cdot 0\cdot t} \Rightarrow$ frequency is 0

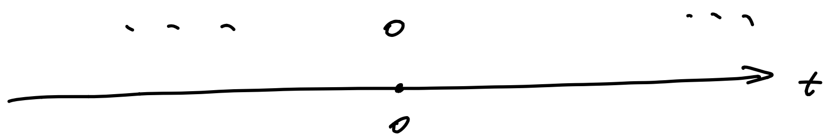
frequency response at 0 $H(j0) = 8$ $\Bigg\} \Rightarrow$

$y(t) = H(j0)\cdot x(t) = 8\,x(t) = 8$



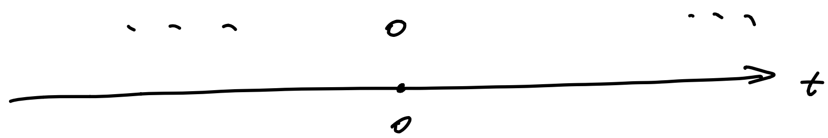
(c) $x(t) = \cos(\frac{\pi}{4}t) = \frac{1}{2}(e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t})$, $H(j\frac{\pi}{4}) = H(-j\frac{\pi}{4}) = 0$

$$\Rightarrow y(t) = H(j\tfrac{\pi}{4})\cdot\tfrac{1}{2}e^{j\frac{\pi}{4}t} + H(-j\tfrac{\pi}{4})\cdot\tfrac{1}{2}e^{-j\frac{\pi}{4}t} = 0$$



(d) $x(t) = \sin(1000\pi t) = \frac{1}{2j}(e^{j1000\pi t} - e^{-j1000\pi t})$,

$H(j1000\pi) = H(-j1000\pi) = 0 \quad \Rightarrow y(t) = 0$



(e) From the magnitude of $H(j\omega)$, ω_0 should not equal to the zero-crossings of $H(j\omega) \Rightarrow \omega_0 \neq \frac{k\pi}{4}$ ($k \neq 0$, is integer)

(f) $x(t) = h(t-5)$. is a shifted version of $h(t)$

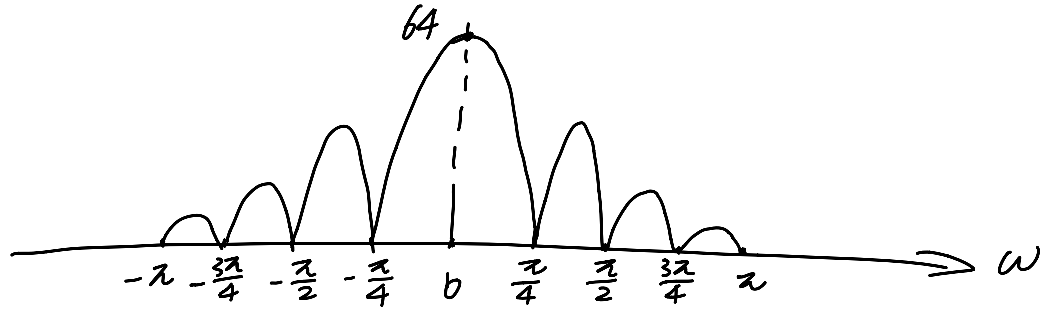
$$\Rightarrow X(j\omega) = e^{-j5\omega}H(j\omega)$$

The CTFT of output $y(t)$ is:

$$Y(j\omega) = X(j\omega)H(j\omega) = e^{-j5\omega}H^2(j\omega)$$

$$|Y(j\omega)| = |H(j\omega)|^2 = \frac{\sin^2(4\omega)}{(\omega/2)^2}, \quad Y(j0) = 8^2 = 64.$$

zero-crossings would be the same as $|H(j\omega)|$



6. $$H(jw) = \int_{-\infty}^{+\infty} e^{-2t} u(t-3) e^{-jwt} dt = \int_{3}^{+\infty} e^{-(jw+2)t} dt$$

$$= \frac{1}{-(jw+2)} e^{-(jw+2)t} \Big|_{3}^{+\infty} = \frac{e^{-3(jw+2)}}{jw+2}$$

$$X(jw) = \int_{-\infty}^{+\infty} [u(t-2) - u(t-6)] e^{-jwt} dt = \int_{2}^{6} e^{-jwt} dt$$

$$= \frac{1}{-jw} e^{-jwt} \Big|_{2}^{6} = e^{-j4w} \frac{\sin(2w)}{w/2}$$

$Y(jw) = X(jw) H(jw)$, note $|H(jw)| = e^{-6} \cdot \frac{1}{\sqrt{w^2+4}} > 0$

$Y(jw_0) = 0 \iff X(jw_0) = 0$.

zero-crossings of $X(jw_0)$ is $2w_0 = k\pi$ ($k \neq 0$, integar)

$\Rightarrow w_0 = \frac{k\pi}{2}$ ($k \neq 0$, integar)