Homework 9 solution

1. (a) $y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} h(\tau)\, e^{j2000\pi(t-\tau)}\, d\tau$

$$= e^{j2000\pi t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j2000\pi\tau}\, d\tau$$

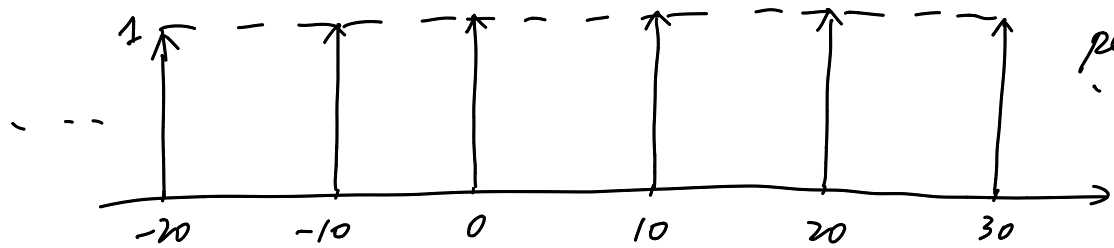
$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \Rightarrow H(j2000\pi) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j2000\pi\tau}\, d\tau$$

since $H(j2000\pi)$ is a complex number with amplitude $|H(j2000\pi)|$ and phase $\angle H(j2000\pi)$.

$$y(t) = H(j2000\pi)\cdot e^{j2000\pi t} = |H(j2000\pi)|\, e^{j2000\pi\left[t + \frac{\angle H(j2000\pi)}{2000\pi}\right]}$$

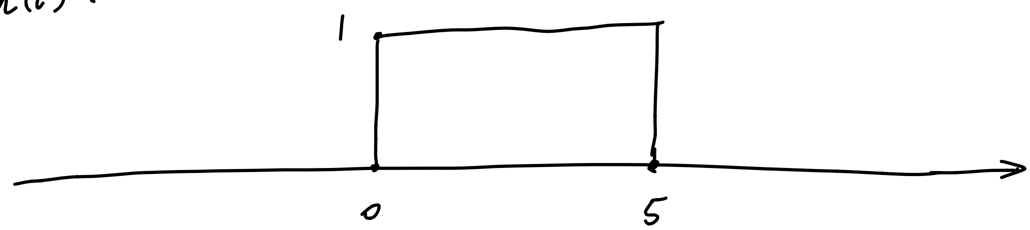
(b) By the sifting property of $\delta(t)$, $y(t) = \delta(t-2) * h(t) = h(t-2)$

(c) $x(t)$:



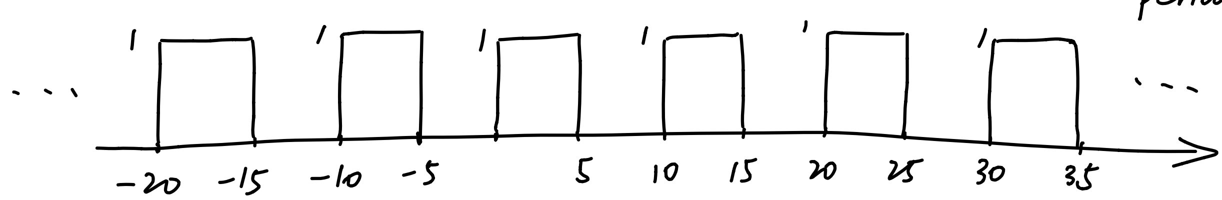
repeat with period 10 ...

$h(t)$:



$$y(t) = x(t) * h(t) = \sum_{k=-\infty}^{+\infty} u(t-10k) - u(t-5-10k)$$

repeat with period 10

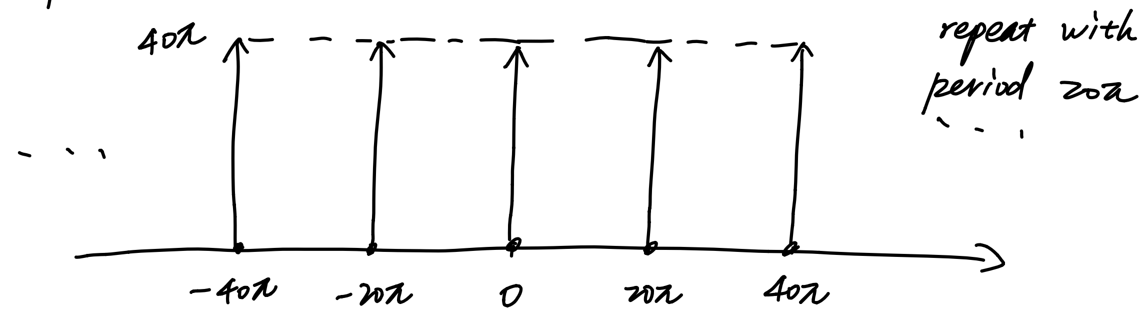


2. (i) $T = 0.1$, $\frac{2\pi}{T} = 20\pi$; Using the linearity of CTFT,

$$x(t) = \sum_{k=-\infty}^{+\infty} 2\delta(t - 0.1k) \longleftrightarrow X(j\omega) = 2 \cdot \sum_{k=-\infty}^{+\infty} 20\pi \cdot \delta(\omega - 20\pi k)$$

$$= 40\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 20\pi k)$$

$|X(j\omega)|$:



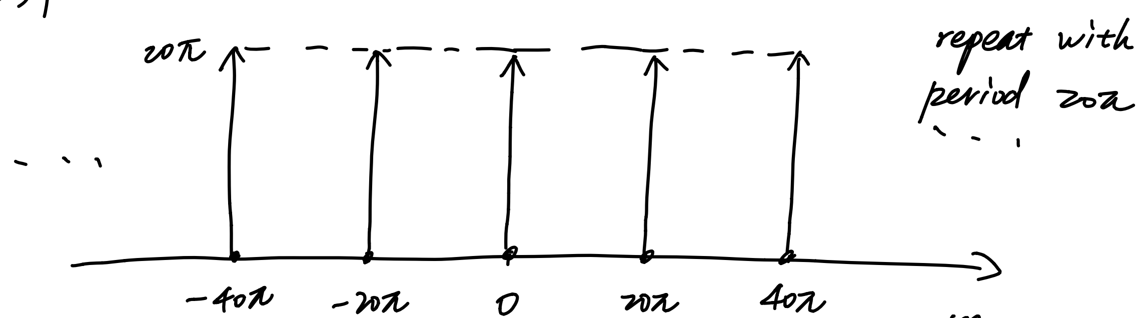
(ii)

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t + 0.1k) \xrightarrow[k \text{ by } -k]{\text{substituting}} \sum_{k=\infty}^{+\infty} \delta(t - 0.1k)$$

$$\updownarrow$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 20\pi \cdot \delta(\omega - 20\pi k) = 20\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 20\pi k)$$

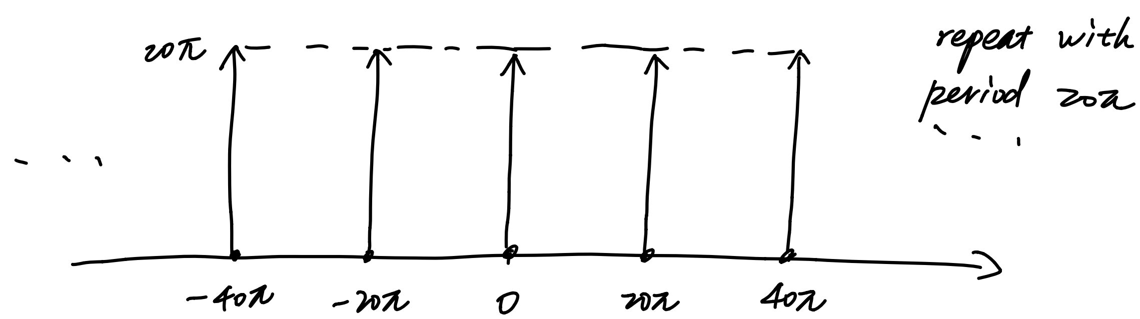
$|X(j\omega)|$:



(iii) Let $x_2(t)$ denote the signal in (ii), $x_2(t) = \sum_{k=-\infty}^{+\infty} \delta(t - 0.1k)$

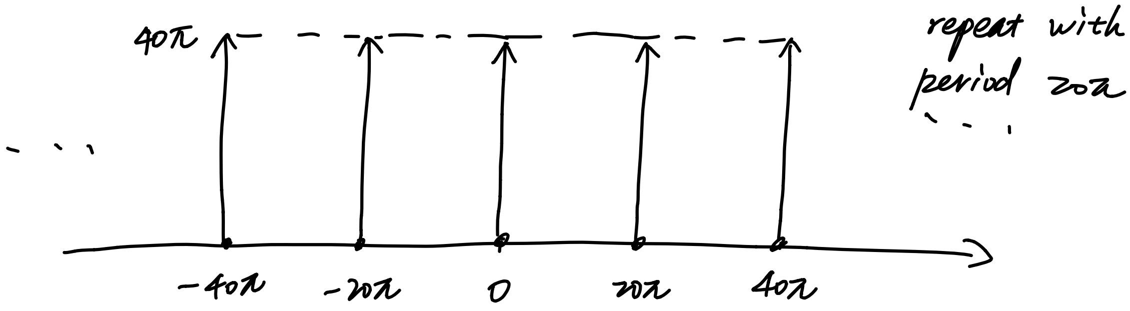
Here $x(t) = x_2(t-1) \Rightarrow X(j\omega) = e^{-j\omega} X_2(j\omega)$

$|X(j\omega)| = |X_2(j\omega)|$: same as (ii)



(iv) $x(t) = 2x_2(t-2) \Rightarrow X(j\omega) = 2e^{-j2\omega}X_2(j\omega)$

$|X(j\omega)| = 2|X_2(j\omega)|$:



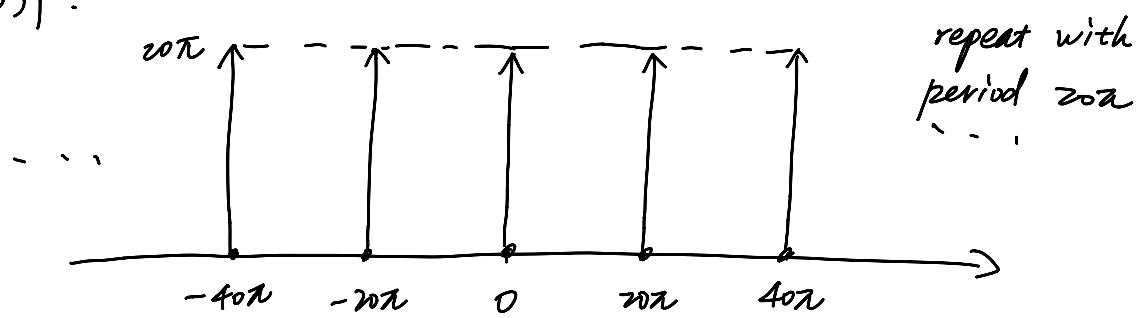
(v) $\delta(t-0.15) \longleftrightarrow e^{-j0.15\omega}$

$$\sum_{k=-\infty}^{+\infty} \delta(t-0.1k) \longleftrightarrow 20\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 20\pi k)$$

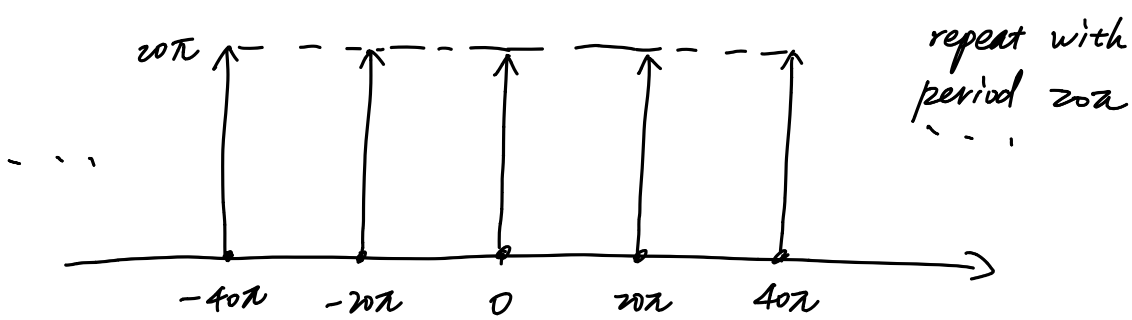
By convolution theorem,

$$x(t) = \delta(t-0.15) * \sum_{k=-\infty}^{+\infty} \delta(t-0.1k) \longleftrightarrow X(j\omega) = e^{-j0.15\omega}\, 20\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 20\pi k)$$

$|X(j\omega)|$:



(vi) Note that this is merely a rewriting of convolution in (v), the answer is the same:



3.

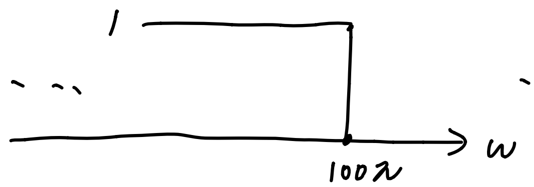
$$x(t) = u(t+T) - u(t-T) \longleftrightarrow X(j\omega) = \frac{2\sin(\omega T)}{\omega}$$

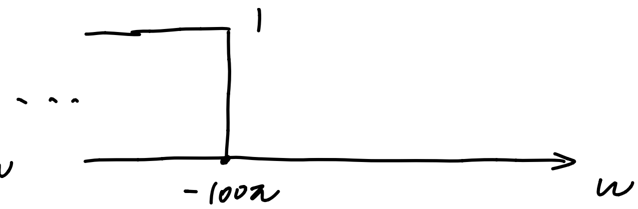
Using the duality property,

$$X(jt) = \frac{2\sin(Tt)}{t} \longleftrightarrow 2\pi x(-\omega) = 2\pi[u(-\omega+T) - u(-\omega-T)]$$

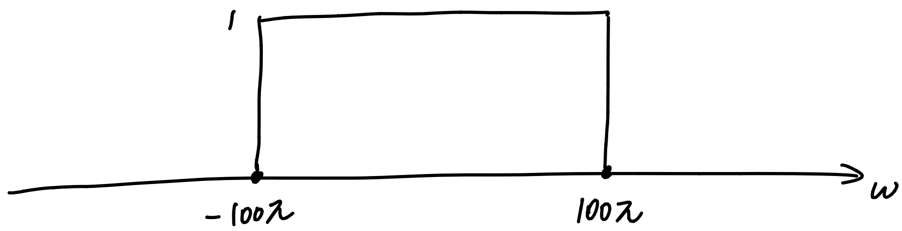
let $T = 100\pi$, using the linearity of CTFT (dividing both sides by 2π)

$$g(t) = \frac{\sin(100\pi t)}{\pi t} \longleftrightarrow u(-\omega+100\pi) - u(-\omega-100\pi)$$





$u(-\omega+100\pi) - u(-\omega-100\pi)$:



is equivalent to $u(\omega+100\pi) - u(\omega-100\pi)$

4. a) $\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}$, using the duality property,

$e^{-jtt_0} \longleftrightarrow 2\pi\delta(-\omega-t_0)$, replace $-t$ by t, $-\omega$ by ω, t_0 by ω_0,

$\Rightarrow e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$

$$e^{j\omega_0 t}x(t) \longleftrightarrow \frac{1}{2\pi}[2\pi\delta(\omega-\omega_0) * X(j\omega)]$$

$$= X[j(\omega-\omega_0)]$$

b) From the conclusion in Q3,

$$\frac{\sin(500\pi t)}{\pi t} \longleftrightarrow u(w+500\pi) - u(w-500\pi)$$

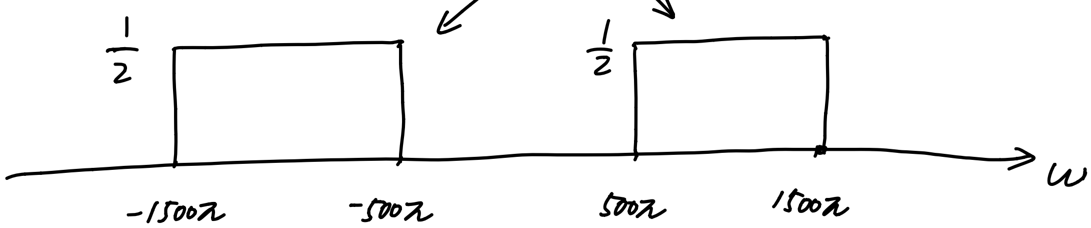
and knowing $\cos(1000\pi t) = \frac{1}{2}\left(e^{j1000\pi t} + e^{-j1000\pi t}\right)$

$$g(t) = \frac{1}{2}\left[e^{j1000\pi t} \cdot \frac{\sin(500\pi t)}{\pi t} + e^{-j1000\pi t} \cdot \frac{\sin(500\pi t)}{\pi t}\right]$$

$$\Updownarrow$$

$$G(jw) = \frac{1}{2}\left\{\left[u(w+500\pi-1000\pi) - u(w-500\pi-1000\pi)\right] + \left[u(w+500\pi+1000\pi) - u(w-500\pi+1000\pi)\right]\right\}$$

$$= \frac{1}{2}\left\{\underline{\left[u(w-500\pi) - u(w-1500\pi)\right]} + \underline{\left[u(w+1500\pi) - u(w+500\pi)\right]}\right\}$$

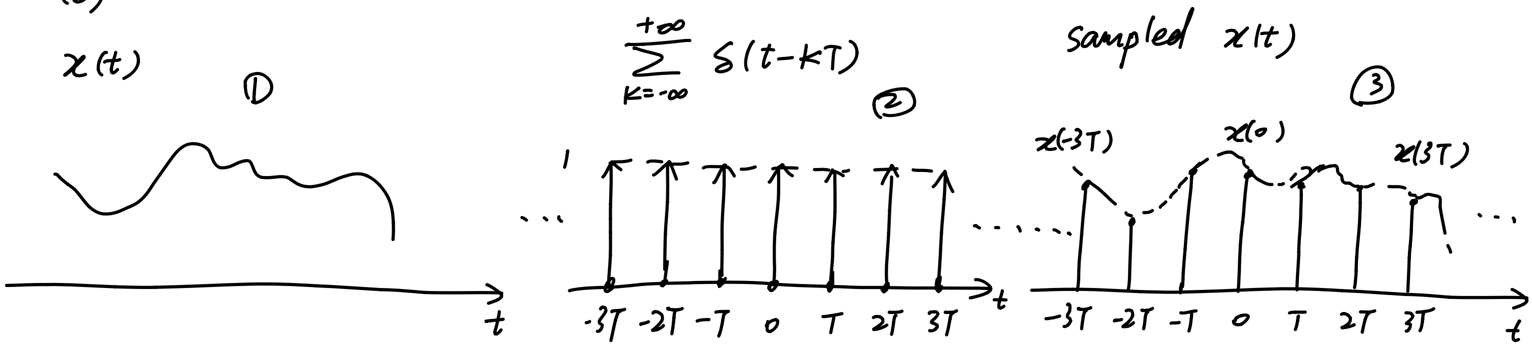


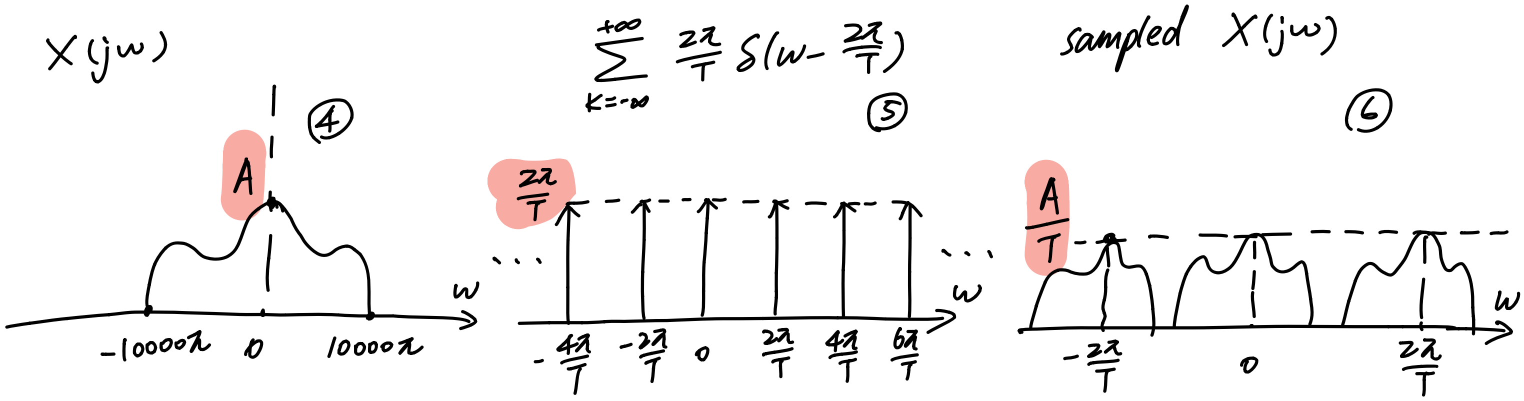
5. (a) From last homework, we know the CTFT magnitude of a real signal is even. Then $X(jw) = 0$ for $w \le -10000\pi$.

Using the Nyquist theorem, $\frac{2\pi}{T} \geq 2 \cdot 10000\pi$

$\Rightarrow T \le 10^{-4}$

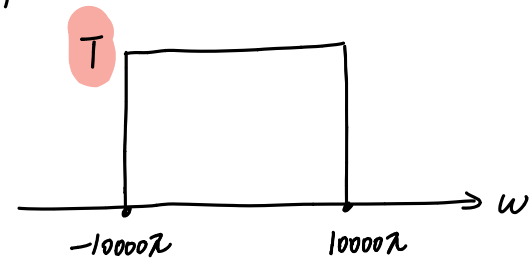
(b)





To transfer ⑥ back to ④, need to apply $H(j\omega)$:

note there is a scaling factor of T here.



6. (c) $x_c(t) = \cos(800\pi t) \longleftrightarrow X_c(j\omega) = \pi[\delta(\omega-800\pi)+\delta(\omega+800\pi)]$

The output CTFT is

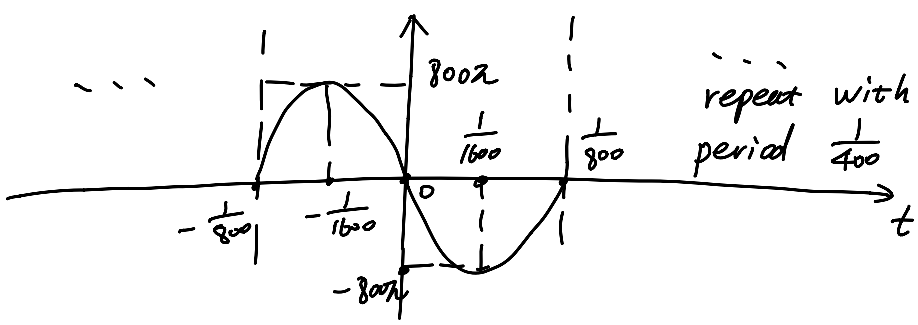
$$Y_c(j\omega) = X_c(j\omega)H(j\omega) = \pi[j800\pi\,\delta(\omega-800\pi) - j800\pi\,\delta(\omega+800\pi)]$$

$$= j800\pi^2[\delta(\omega-800\pi)-\delta(\omega+800\pi)]$$

$\updownarrow$

$$y_c(t) = j800\pi^2\,\frac{(e^{j800\pi t}-e^{-j800\pi t})}{2\pi} = j800\pi^2\cdot\frac{j}{\pi}\cdot\sin(800\pi t)$$

$$= -800\pi\sin(800\pi t) \qquad T = \frac{2\pi}{800\pi} = \frac{1}{400}$$



(b) $x_b(t) = \sum_{k=-\infty}^{+\infty} \delta\left(t - \frac{k}{400}\right) \longleftrightarrow X_b(jw) = 800\pi \sum_{k=-\infty}^{+\infty} \delta(w - 800\pi k)$

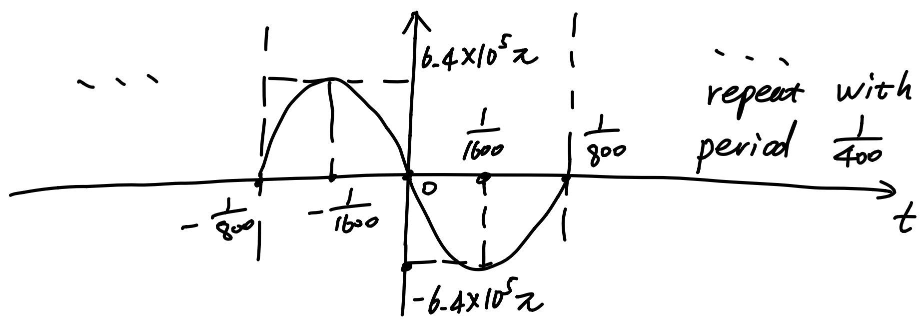
$$Y_b(jw) = X_b(jw) H(jw) = 800\pi \left[j\,800\pi\, \delta(w-800\pi) + j \cdot 0 \cdot \delta(w) - j\,800\pi\, \delta(w+800\pi) \right]$$

$$= (800\pi)^2 \left[j\, \delta(w-800\pi) - j\, \delta(w+800\pi) \right]$$

$$\updownarrow$$

$$y_b(t) = (800\pi)^2 \left[j \cdot \frac{e^{j800\pi t}}{2\pi} - j \frac{e^{-j800\pi t}}{2\pi} \right] = 3.2\times10^5 \pi \left[j \cdot 2j \sin(800\pi t) \right]$$

$$= -6.4\times10^5 \pi \sin(800\pi t)$$



(a) $x_a(t)$ is odd $\Rightarrow$ $x_a(t) = -x_a(-t) \longleftrightarrow X_a(jw) = -X_a(-jw)$

Note $H(jw)$ is also odd, $H(jw) = -H(-jw)$

Then $Y_a(jw) = X_a(jw) H(jw)$,

$Y_a(-jw) = X_a(-jw) H(-jw) = X_a(jw) H(jw)$

$\Rightarrow Y_a(jw) = Y_a(-jw) \longleftrightarrow y_a(t) = y_a(-t)$

$\Rightarrow y_a(t)$ is even.