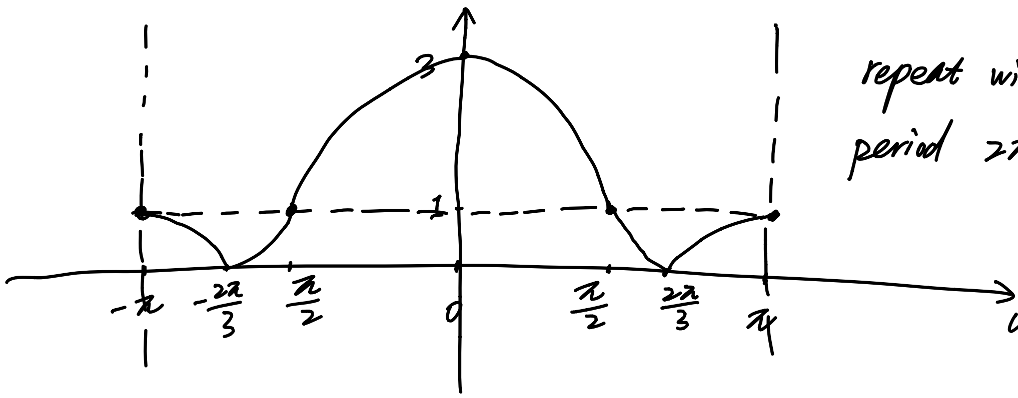
Homework 7 solution

1. The DTFT of $x[n]$ is $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

(a) $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} (u[n] - u[n-3]) e^{-j\omega n} = e^{-j\omega \cdot 0} + e^{-j\omega \cdot 1} + e^{-j\omega \cdot 2}$

$= e^{-j\omega} (e^{j\omega} + 1 + e^{-j\omega}) = e^{-j\omega} [1 + 2\cos(\omega)]$

$|X(e^{j\omega})| = |e^{-j\omega}| \cdot |1+2\cos(\omega)| = |1+2\cos(\omega)|$

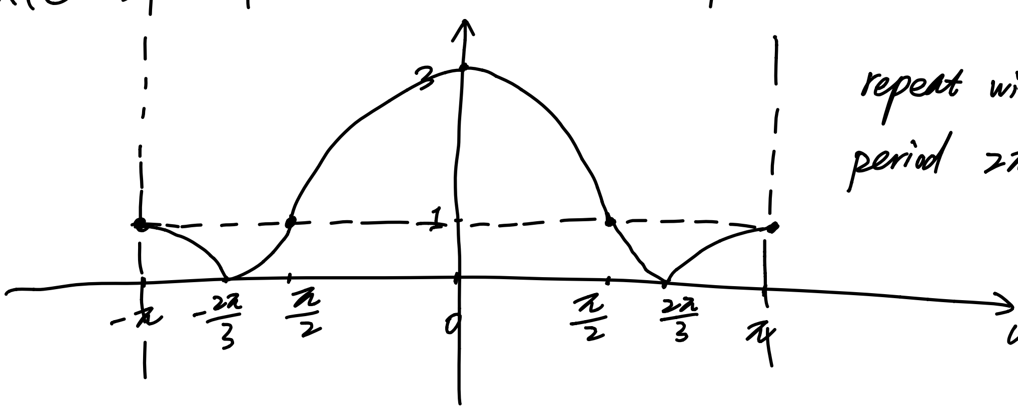


repeat with period 2π

(b) $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} (u[n-1] - u[n-4]) e^{-j\omega n} = e^{-j\omega \cdot 1} + e^{-j\omega 2} + e^{-j\omega 3}$

$= e^{-j2\omega} (e^{j\omega} + 1 + e^{-j\omega}) = e^{-j2\omega} [1 + 2\cos(\omega)]$

$|X(e^{j\omega})| = |e^{-j2\omega}| \cdot |1+2\cos(\omega)| = |1+2\cos(\omega)|$. same as (a)



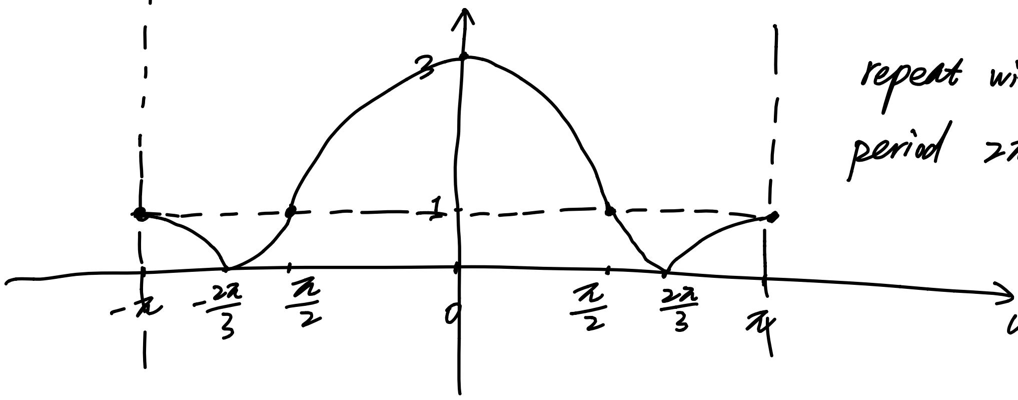
repeat with period 2π

(c)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} (u[n+2] - u[n-1]) e^{-jwn} = e^{jw\cdot 2} + e^{jw\cdot 1} + e^{jw\cdot 0}$$

$$= e^{jw}(e^{jw} + 1 + e^{-jw}) = e^{jw}[1 + 2\cos(w)]$$

$$|X(e^{jw})| = |e^{jw}| \cdot |1+2\cos(w)| = |1+2\cos(w)|. \text{ same as (a)}$$



repeat with period 2π

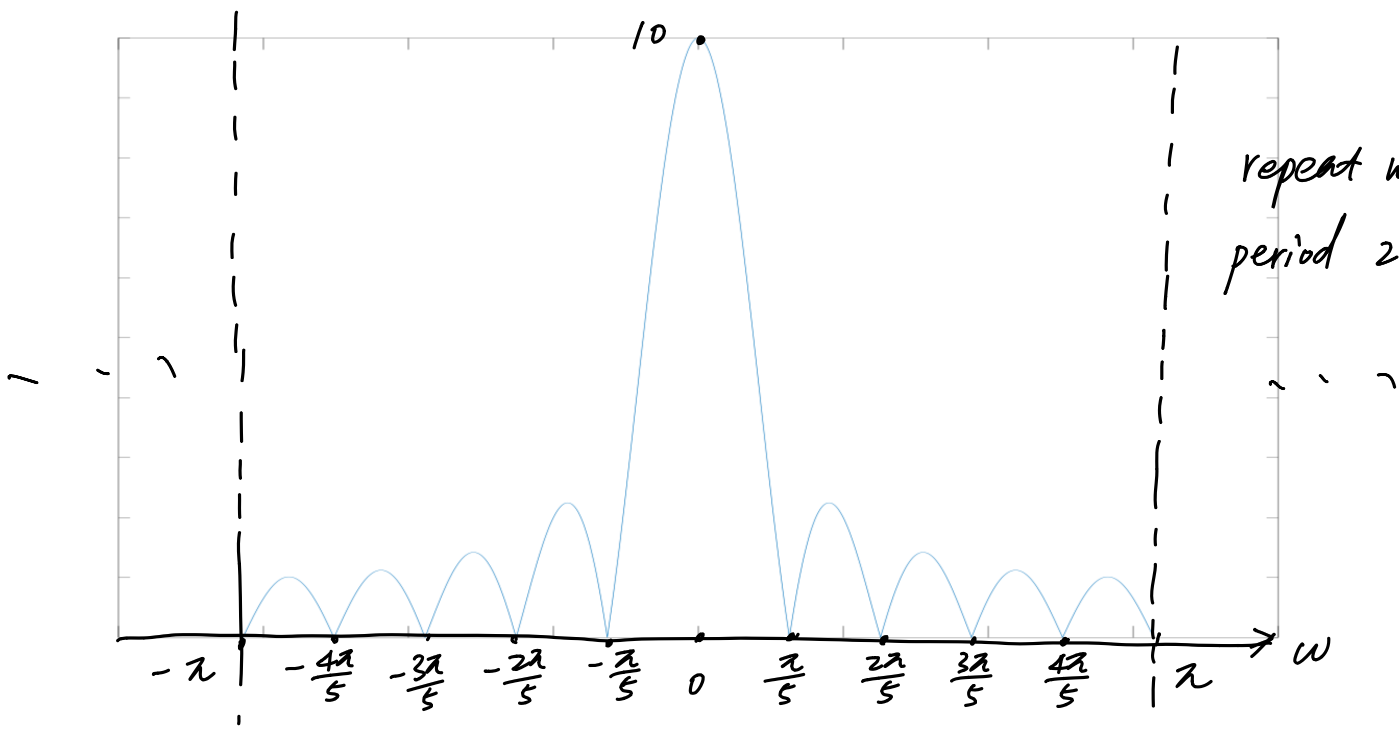
(d)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} (u[n] - u[n-10]) \cdot e^{-jwn} = \sum_{n=0}^{9} e^{-jwn}$$

$$= \frac{e^{-jw0}(1 - e^{-jw\cdot 10})}{1 - e^{-jw}} = \frac{e^{-jw5}(e^{jw5} - e^{-jw5})}{e^{-j\frac{w}{2}}(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}})}$$

$$= e^{-j\frac{9}{2}w} \cdot \frac{\sin(5w)}{\sin(\frac{w}{2})}$$

$$|X(e^{jw})| = |e^{-j\frac{9}{2}w}| \cdot \left|\frac{\sin(5w)}{\sin(\frac{w}{2})}\right| = \left|\frac{\sin(5w)}{\sin(\frac{w}{2})}\right| :$$



repeat with period 2π

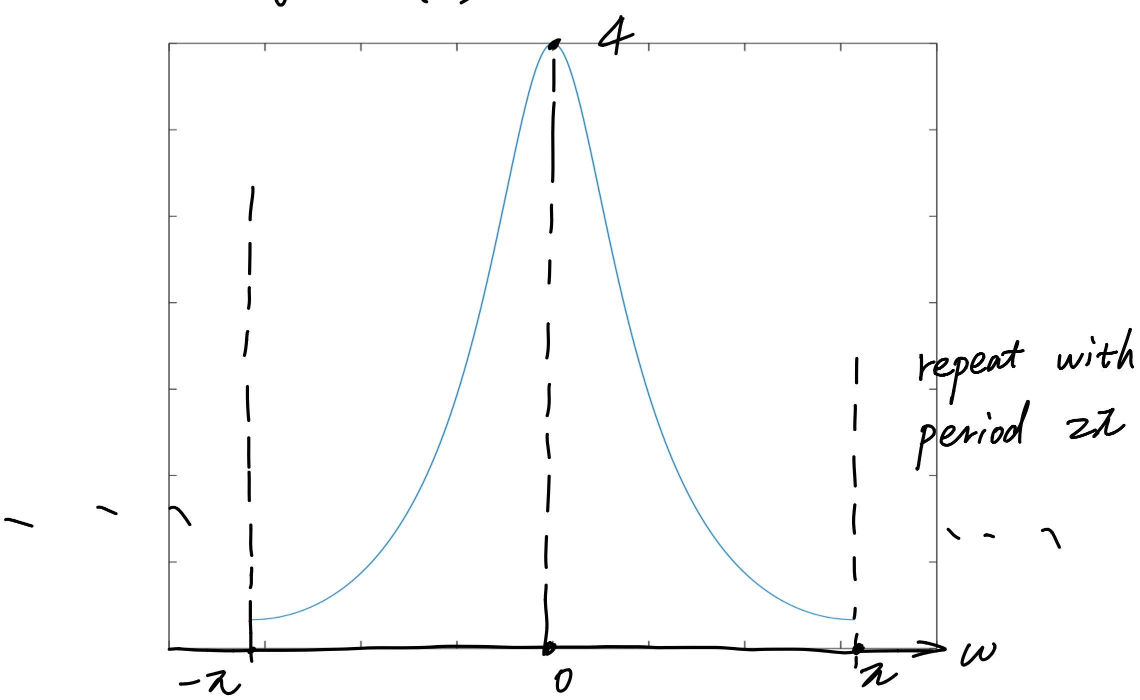
2. $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

(a) $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} (\frac{1}{2})^{n-1} u[n] e^{-j\omega n} = \sum_{n=0}^{+\infty} 2(\frac{e^{-j\omega}}{2})^n$

$= 2 \cdot \frac{(\frac{e^{-j\omega}}{2})^0}{1-\frac{1}{2}e^{-j\omega}} = \frac{2}{1-\frac{1}{2}e^{-j\omega}}$

$|X(e^{j\omega})| = \frac{2}{|1-\frac{1}{2}e^{-j\omega}|} = \frac{2}{\sqrt{[1-\frac{1}{2}\cos(\omega)]^2 + [\frac{1}{2}\sin(\omega)]^2}}$

$= \frac{4}{\sqrt{5-4\cos(\omega)}}$

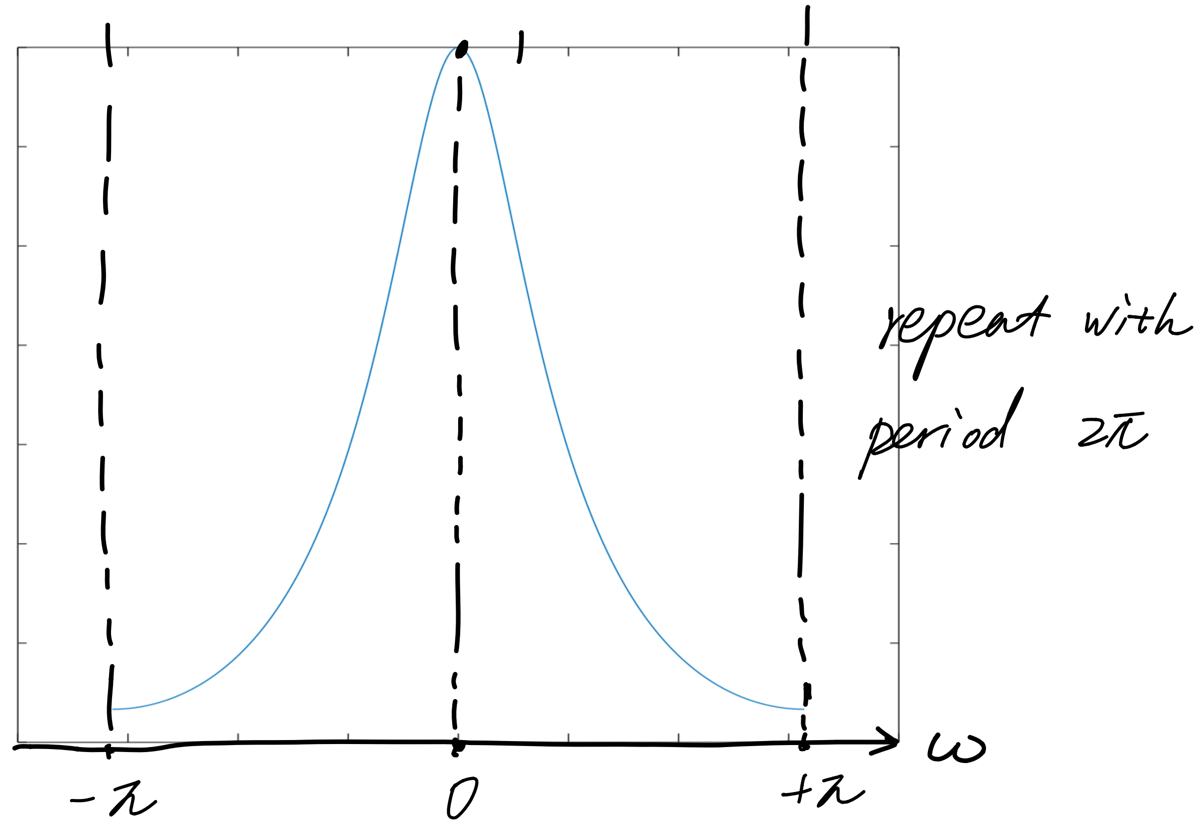


(b) $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} 2^n u[-n-1] e^{-j\omega n} = \sum_{n=-\infty}^{-1} (2e^{-j\omega})^n$

$= \sum_{n=1}^{+\infty} (\frac{1}{2}e^{j\omega})^n = \frac{\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}$

$|X(e^{j\omega})| = \frac{|\frac{1}{2}e^{j\omega}|}{|1-\frac{1}{2}e^{j\omega}|} = \frac{1}{|2-e^{j\omega}|} = \frac{1}{\sqrt{[2-\cos(\omega)]^2 + \sin^2(\omega)}}$

$= \frac{1}{\sqrt{5-4\cos(\omega)}}$



Note that $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\,\delta(\omega-\omega_0-2\pi k)$

is a common DTFT pair, and therefore

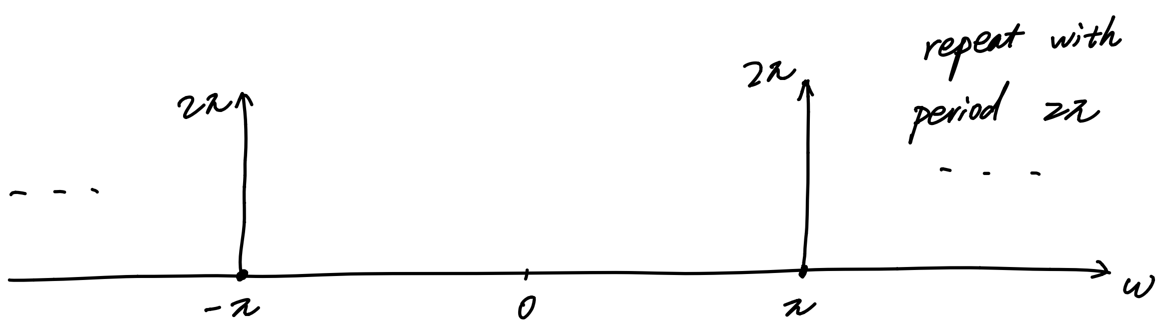
$$x[n] = \cos(\omega_0 n) = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} \pi\left[\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)\right]$$

$$x[n] = \sin(\omega_0 n) = \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} -j\pi\left[\delta(\omega-\omega_0-2\pi k)-\delta(\omega+\omega_0-2\pi k)\right]$$

(c) $X(e^{j\omega}) = (-1)^n = (e^{j\pi})^n = e^{j\pi n}$

$$\Updownarrow$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\,\delta(\omega-\pi-2\pi k), \quad |X(e^{j\omega})| = X(e^{j\omega})$$

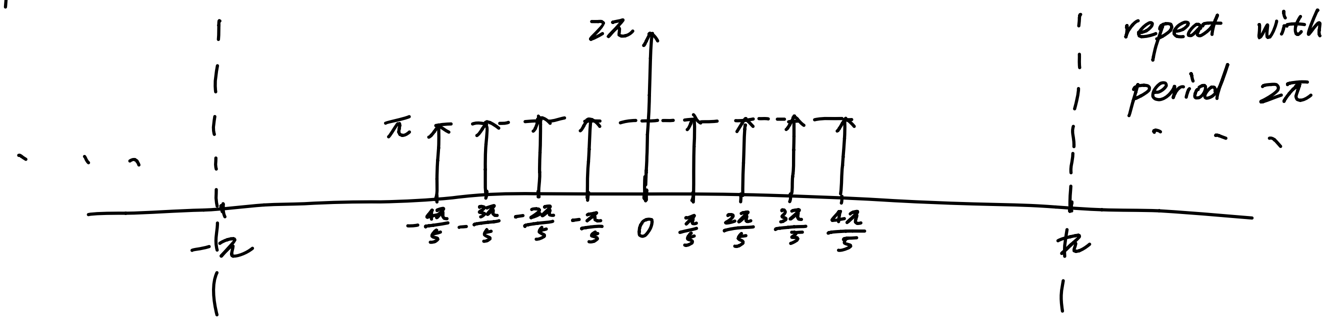


(d) $x[n] = e^{j0} + \cos(\frac{\pi}{5}n) + \cos(\frac{2\pi}{5}n) + \cos(\frac{3\pi}{5}n) + \cos(\frac{4\pi}{5}n)$

$$\Updownarrow$$

$$X(e^{j\omega}) = \boxed{\pi\left[2\delta(\omega) + \delta(\omega-\tfrac{\pi}{5}) + \delta(\omega+\tfrac{\pi}{5}) + \delta(\omega-\tfrac{2\pi}{5}) + \delta(\omega+\tfrac{2\pi}{5}) + \delta(\omega-\tfrac{3\pi}{5}) + \delta(\omega+\tfrac{3\pi}{5}) + \delta(\omega-\tfrac{4\pi}{5}) + \delta(\omega+\tfrac{4\pi}{5})\right] * \sum_{k=-\infty}^{+\infty} \delta(\omega-2\pi k)}$$

$|X(e^{j\omega})| = X(e^{j\omega})$:

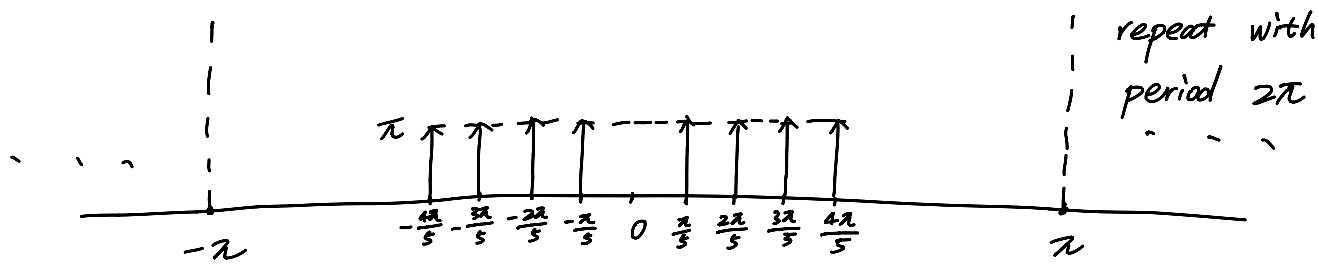


(e) $x[n] = \sin(\frac{\pi}{5}n) + \sin(\frac{2\pi}{5}n) + \sin(\frac{3\pi}{5}n) + \sin(\frac{4\pi}{5}n)$

$$\Updownarrow$$

$$X(e^{j\omega}) = -j\boxed{\pi\left[\delta(\omega-\tfrac{\pi}{5}) - \delta(\omega+\tfrac{\pi}{5}) + \delta(\omega-\tfrac{2\pi}{5}) - \delta(\omega-\tfrac{2\pi}{5}) + \delta(\omega-\tfrac{3\pi}{5}) - \delta(\omega+\tfrac{3\pi}{5}) + \delta(\omega-\tfrac{4\pi}{5}) - \delta(\omega+\tfrac{4\pi}{5})\right] * \sum_{k=-\infty}^{+\infty} \delta(\omega-2\pi k)}$$

$|X(e^{j\omega})|$:



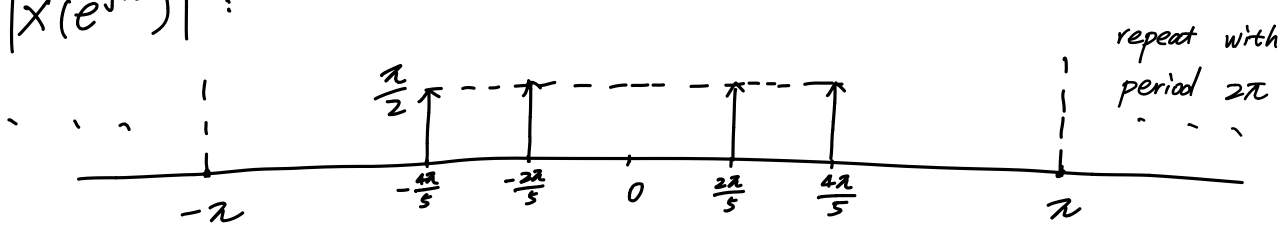
(f) $x[n] = \cos(\frac{2\pi n}{5})\sin(\frac{4\pi n}{5}) = \frac{1}{2}\left[\sin(\frac{2\pi n}{5} + \frac{4\pi n}{5}) - \sin(\frac{2\pi n}{5} - \frac{4\pi n}{5})\right]$

$= \underline{\underline{\frac{1}{2}\sin(\frac{6\pi}{5}n)}} + \frac{1}{2}\sin(\frac{2\pi}{5}n) = \underline{\underline{-\frac{1}{2}\sin(\frac{4\pi}{5}n)}} + \frac{1}{2}\sin(\frac{2\pi}{5}n)$

$$\Updownarrow$$

$$X(e^{j\omega}) = \left(\frac{j\pi}{2}\left[\delta(\omega-\tfrac{4\pi}{5}) - \delta(\omega+\tfrac{4\pi}{5})\right] - \frac{j\pi}{2}\left[\delta(\omega-\tfrac{2\pi}{5}) - \delta(\omega+\tfrac{2\pi}{5})\right]\right) * \sum_{k=-\infty}^{+\infty} \delta(\omega-2\pi k)$$

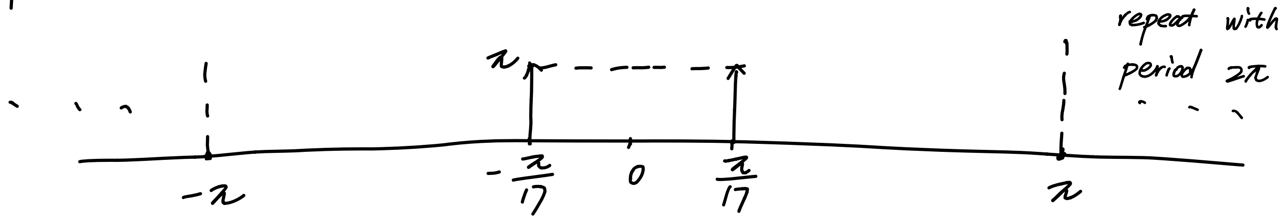
$|X(e^{j\omega})|$:



3. (a) From the conclusion in Q2, we know

$$x[n] = \cos\left(\frac{\pi n}{17}\right) \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} \pi\left[\delta\left(\omega - \frac{\pi}{17} - 2\pi k\right) + \delta\left(\omega + \frac{\pi}{17} - 2\pi k\right)\right]$$

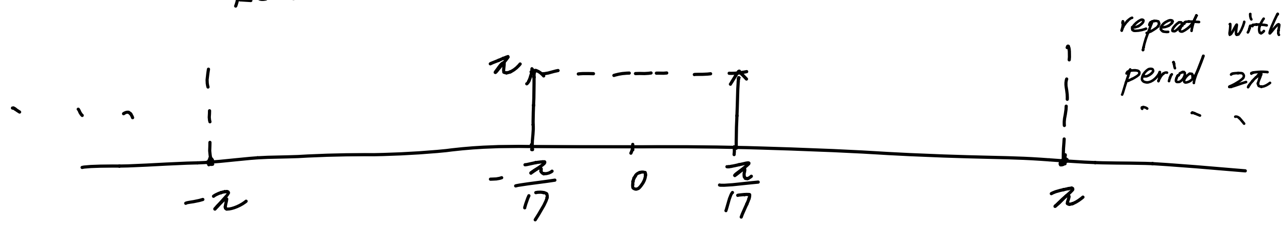
$$|X(e^{j\omega})| = X(e^{j\omega}) :$$



Obviously it's even.

(b) $$x[n] = \sin\left(\frac{\pi n}{17}\right) \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} -j\pi\left[\delta\left(\omega - \frac{\pi}{17} - 2\pi k\right) - \delta\left(\omega + \frac{\pi}{17} - 2\pi k\right)\right]$$

$$|X(e^{j\omega})| = \sum_{k=-\infty}^{+\infty} \pi\left[\delta\left(\omega - \frac{\pi}{17} - 2\pi k\right) + \delta\left(\omega + \frac{\pi}{17} - 2\pi k\right)\right] :$$



Obviously it's even.

(c) $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \quad X^*(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x^*[n]e^{j\omega n}$$

Then

$$|X(e^{j\omega})| = \sqrt{X(e^{j\omega})X^*(e^{j\omega})} = \sqrt{\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \sum_{n=-\infty}^{+\infty} x^*[n]e^{j\omega n}} \quad ①$$

$$|X(e^{-j\omega})| = \sqrt{X(e^{-j\omega})X^*(e^{-j\omega})} = \sqrt{\sum_{n=-\infty}^{+\infty} x[n]e^{j\omega n} \sum_{n=-\infty}^{+\infty} x^*[n]e^{-j\omega n}} \quad ②$$

Since $x[n]$ is real $\Rightarrow x[n] = x^*[n] \Rightarrow ① = ② \Rightarrow |X(e^{j\omega})|$ is even.

(d) $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \quad ①$$

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{j\omega n} \xrightarrow[n \text{ by } -n]{\text{substituting}} \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n} \quad ②$$

If $x[n]$ is even, $\Rightarrow x[n] = x[-n] \Rightarrow ① = ② \Rightarrow X(e^{j\omega})$ is even.

$$X^*(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x^*[n] e^{j\omega n} \xrightarrow[n \text{ by } -n]{\text{substituting}} \sum_{n=-\infty}^{+\infty} x^*[-n] e^{-j\omega n} \quad ③$$

If $x[n]$ is also real $\Rightarrow x^*[-n] = x[-n] = x[n] \Rightarrow ① = ③ \Rightarrow X(e^{j\omega})$ is real.

Then $X(e^{j\omega})$ is real and even.

(e) For example, $x[n] = \delta[n]$ is real and even;

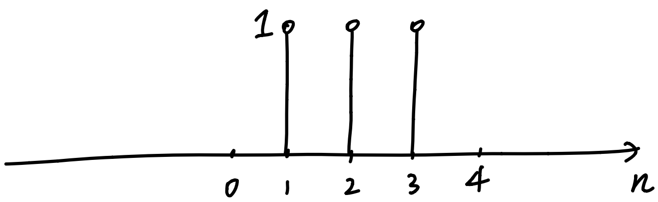
$X(e^{j\omega}) = 1$ is also real and even.

(f) Let $x'[n] = (-1)^n x[n] = e^{j\pi n} x[n]$, $X'(e^{j\omega})$ denote its DTFT,

$$X'(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x'[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\omega-\pi)n},$$

Since $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \Rightarrow X'(e^{j\omega}) = X\left(e^{j(\omega-\pi)}\right)$

4. $x[n] = h[n] = u[n-1] - u[n-4]$



(a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$

$h[n-k] = h[-(k-n)]$ = flipped and then shifted to right by n version of $h[k]$

when $n \le 1$, $h[n-k]$ will not overlap with $x[k]$, so $y[n] = 0$.

when $n = 2$, $y[2] = x[1] \cdot h[2-1] = 1$

when $n = 3$, $y[3] = x[1] \cdot h[3-1] + x[2] \cdot h[3-2] = 2$

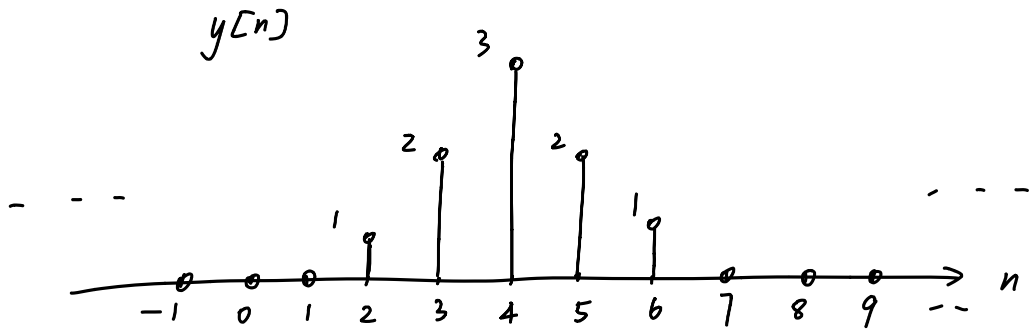
when $n = 4$, $y[4] = x[1] \cdot h[4-1] + x[2] \cdot h[4-2] + x[3] \cdot h[4-3] = 3$

when $n = 5$, $y[5] = x[2] \cdot h[5-2] + x[3] \cdot h[5-3] = 2$

when $n = 6$, $y[6] = x[3] \cdot h[6-3] = 1$

when $n \geq 7$, $h[n-k]$ will not overlap with $x[k]$, $y[n] = 0$

Then $y[n]$ look like:

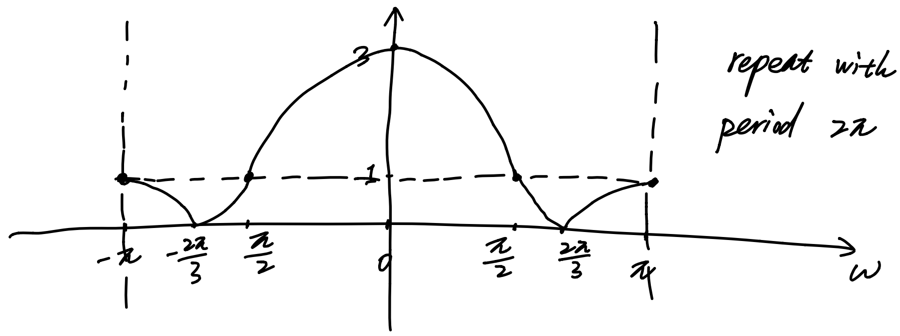


(b)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} (u[n-1] - u[n-4])\, e^{-jwn} = e^{-jw\cdot 1} + e^{-jw2} + e^{-jw3}$$

$$= e^{-j2w}(e^{jw} + 1 + e^{-jw}) = e^{-j2w}[1 + 2\cos(w)]$$

$$|X(e^{jw})| = |e^{-j2w}| \cdot |1 + 2\cos(w)| = |1 + 2\cos(w)|$$



repeat with period 2π

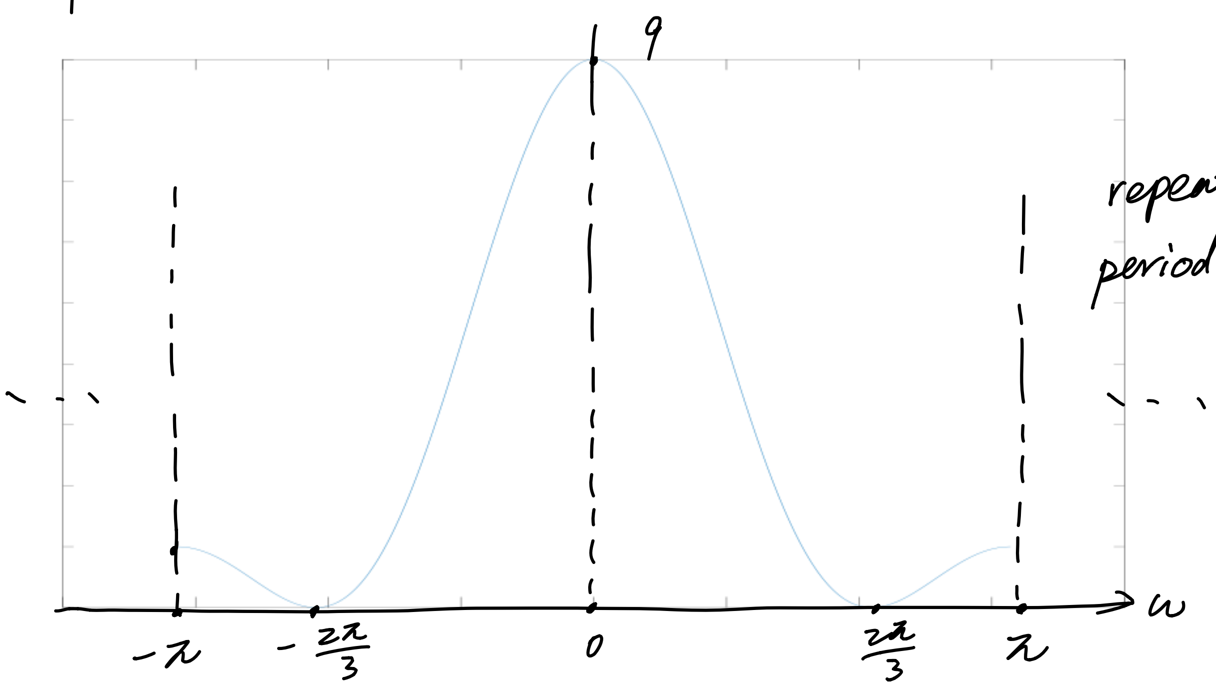
(c)

$$x[n] = h[n] \Rightarrow H(e^{jw}) = X(e^{jw}) = e^{-j2w}[1 + 2\cos(w)]$$

By convolution theorem,

$$Y(e^{jw}) = X(e^{jw}) H(e^{jw}) = e^{-j4w}[1 + 2\cos(w)]^2$$

$$|Y(e^{jw})| = |e^{-j4w}|\, [1 + 2\cos(w)]^2 = 1 + 4\cos(w) + 4\cos^2(w)$$



repeat with period 2π

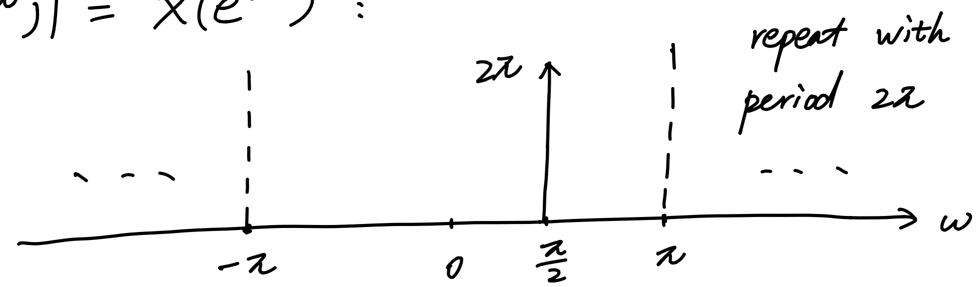
5. The main challenge here is to find an effective frequency component $e^{j\omega_0 n}$ to substitute $e^{j\omega n}$, where $\omega_0 \in [-\pi, \pi]$, by $e^{j\omega n} = e^{j(\omega + 2\pi k)n}$

(a) $x[n] = e^{j\frac{5\pi}{2}n} = e^{j(\frac{5\pi}{2}n - 2\pi n)} = e^{j\frac{\pi}{2}n}$

By using the same conclusions in Q2,

$$x[n] = e^{j\frac{\pi}{2}n} \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\,\delta(\omega - \tfrac{\pi}{2} - 2\pi k)$$

$|X(e^{j\omega})| = X(e^{j\omega})$:

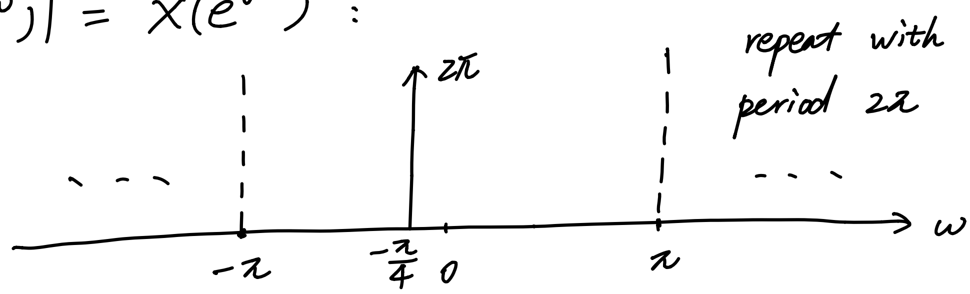


(b) $x[n] = e^{-j\frac{25}{4}\pi n} = e^{j(-\frac{25}{4}\pi n + 6\pi n)} = e^{-j\frac{\pi}{4}n}$

$$\updownarrow$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\,\delta(\omega + \tfrac{\pi}{4} - 2\pi k)$$

$|X(e^{j\omega})| = X(e^{j\omega})$:



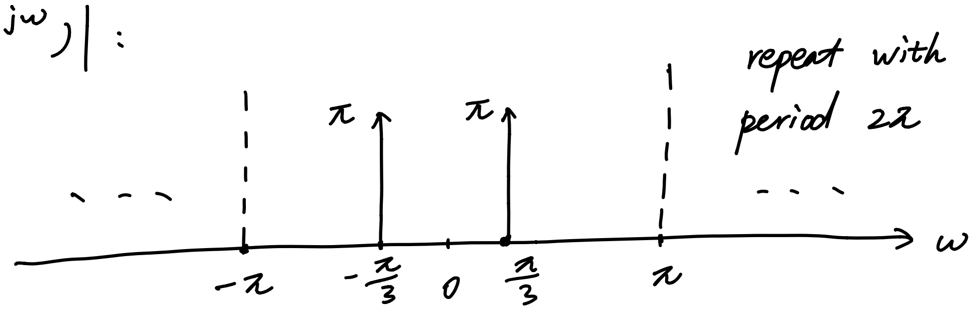
(c) $x[n] = \cos(\frac{17\pi}{3}n + \frac{\pi}{6}) = \cos(\frac{17\pi}{3}n - 6\pi n + \frac{\pi}{6}) = \cos(\frac{\pi}{3}n - \frac{\pi}{6})$

$= \frac{1}{2}[e^{j(\frac{\pi}{3}n - \frac{\pi}{6})} + e^{-j(\frac{\pi}{3}n - \frac{\pi}{6})}] = \frac{1}{2}e^{-j\frac{\pi}{6}} e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{j\frac{\pi}{6}} e^{-j\frac{\pi}{3}n}$

$$\updownarrow$$

$$X(e^{j\omega}) = [\pi e^{-j\frac{\pi}{6}}\delta(\omega - \tfrac{\pi}{3}) + \pi e^{j\frac{\pi}{6}}\delta(\omega + \tfrac{\pi}{3})] * \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

$|X(e^{j\omega})|$:



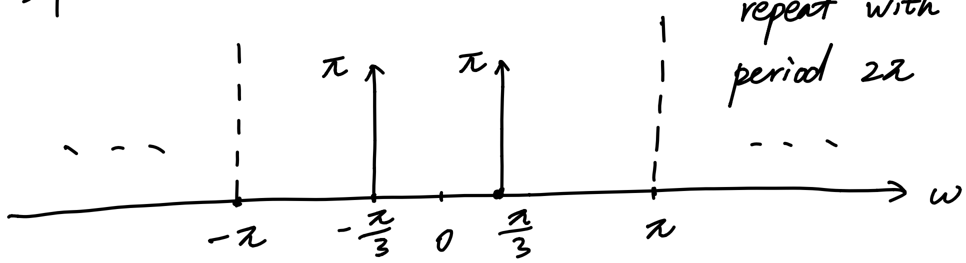
(d)

$$x[n] = \sin\left(\frac{17\pi}{3}n+\frac{\pi}{6}\right) = \sin\left(\frac{17\pi}{3}n - 6\pi n+\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}n-\frac{\pi}{6}\right)$$

$$= \frac{j}{2}\left[e^{j(\frac{\pi}{3}n-\frac{\pi}{6})} - e^{-j(\frac{\pi}{3}n-\frac{\pi}{6})}\right] = \frac{j}{2}e^{-j\frac{\pi}{6}}e^{j\frac{\pi}{3}n} - \frac{j}{2}e^{j\frac{\pi}{6}}e^{-j\frac{\pi}{3}n}$$

$$X(e^{jw}) = \left[\pi j e^{-j\frac{\pi}{6}}\delta\left(w-\frac{\pi}{3}\right) - \pi j e^{j\frac{\pi}{6}}\delta\left(w+\frac{\pi}{3}\right)\right] * \sum_{k=-\infty}^{+\infty}\delta(w-2\pi k)$$

$|X(e^{jw})|$:



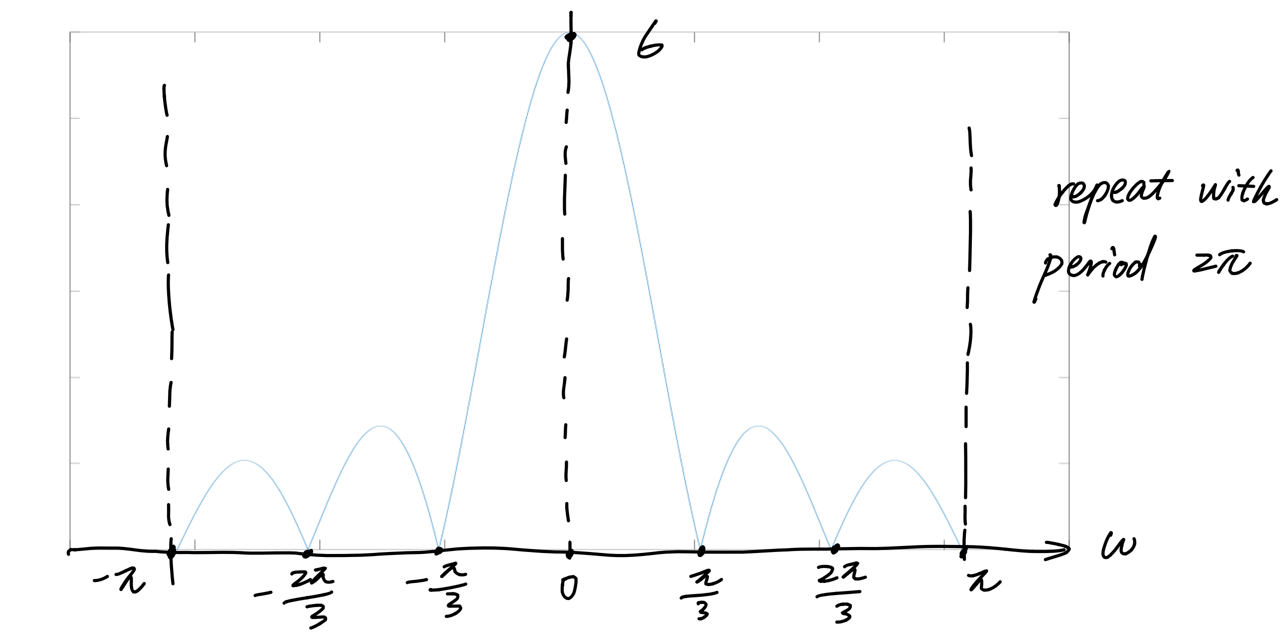
6. (a)

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty}(u[n]-u[n-6])e^{-jwn} = \sum_{n=0}^{5} e^{-jwn}$$

$$= \frac{e^{-jw\cdot 0}(1-e^{-jw6})}{1-e^{-jw}} = \frac{e^{-j3w}(e^{j3w}-e^{-j3w})}{e^{-j\frac{w}{2}}(e^{j\frac{w}{2}}-e^{-j\frac{w}{2}})}$$

$$= e^{-j\frac{5w}{2}}\cdot\frac{\sin(3w)}{\sin(\frac{w}{2})}$$

$$|H(e^{jw})| = \left|e^{-j\frac{5w}{2}}\right|\cdot\left|\frac{\sin(3w)}{\sin(\frac{w}{2})}\right| = \left|\frac{\sin(3w)}{\sin(\frac{w}{2})}\right|$$



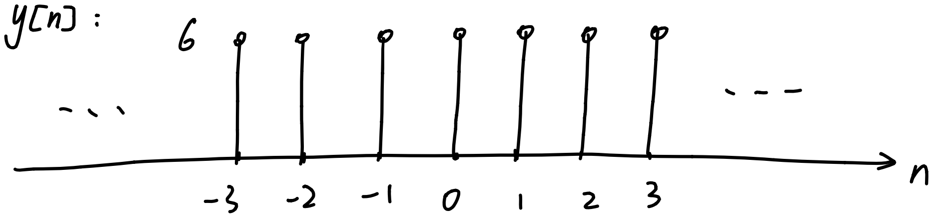
Next, perform wave decomposition for each of the input signals.

(b) $x[n] = 1 = e^{j\cdot 0\cdot n} \Rightarrow$ has frequency 0

$\Rightarrow$ frequency response $H(e^{j0}) = 6$,

$\Rightarrow$ output is $y[n] = H(e^{j0}) \cdot e^{j\cdot 0\cdot n} = 6$

$y[n]$:

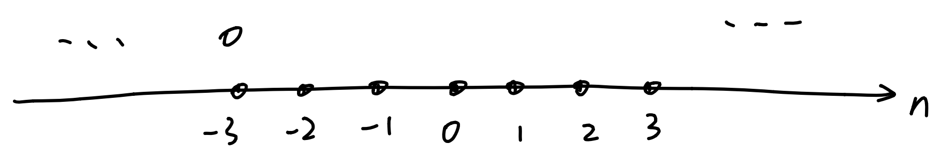


(c) $x[n] = (-1)^n = e^{j\pi n} \Rightarrow$ has frequency π

$\Rightarrow$ frequency response $H(e^{j\pi}) = 0$.

$\Rightarrow$ output is $y[n] = H(e^{j\pi}) e^{j\pi n} = 0$

$y[n]$:

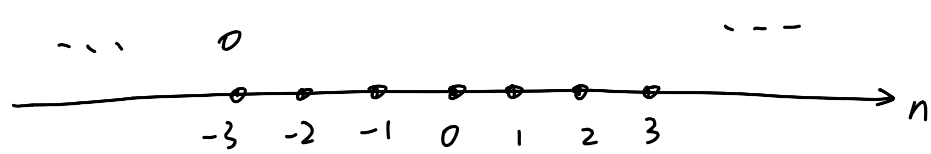


(d) $x[n] = \cos(\frac{\pi}{3}n) = \frac{1}{2}(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}) \Rightarrow$ has frequency $\frac{\pi}{3}$ and $-\frac{\pi}{3}$.

$\Rightarrow$ frequency response $H(e^{j\frac{\pi}{3}}) = H(e^{-j\frac{\pi}{3}}) = 0$

$\Rightarrow$ output is $y[n] = H(e^{j\frac{\pi}{3}}) \cdot \frac{1}{2} e^{j\frac{\pi}{3}n} + H(e^{-j\frac{\pi}{3}}) \cdot \frac{1}{2} e^{-j\frac{\pi}{3}n} = 0$

$y[n]$:



(e) $x[n] = e^{j\omega_0 n}$ has frequency ω_0, for output $y[n] = H(e^{j\omega_0}) x[n] = 0$,

need $H(e^{j\omega_0}) = 0$, $\Rightarrow \omega_0$ corresponds to zeros of $|H(e^{j\omega})|$ in

the range $-\pi < \omega_0 \le \pi$, from (a) we know

$\omega_0 = -\frac{2\pi}{3}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$