FINAL EXAM FORMULA SHEET EC401 (SPRING 2020)

DT Unit Step (Switch)

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases}$$

DT Unit Impulse (Atom):

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

Discrete-Time Impulse decomposition of a Signal: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Even Signal: $x[n] = x[-n]$

Odd Signal: $x[n] = -x[-n]$

Even$\{x[n]\} = \frac{x[n]+x[-n]}{2}$

Odd$\{x[n]\} = \frac{x[n]-x[-n]}{2}$

Basic Signal Operations:
Shift: $y[n] = x[n-n_0]$
Flip: $y[n] = x[-n]$
Compress: $y[n] = x[Mn]$
Expand: $y[n] = \begin{cases} x\left[\frac{n}{M}\right] & \text{if } n > M \\ 0 & \text{otherwise} \end{cases}$

Multi-Signal Operations:
Linear Combination: $y[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$
Product: $y[n] = g[n]h[n]$

Bounded Signal: A signal $x[n]$ is bounded if and only if $|x[n]| \leq B$ for some finite (positive) number B.

Linear System:
Suppose S:$x_1[n] \rightarrow y_1[n]$ and S:$x_2[n] \rightarrow y_2[n]$. The system S is linear *if and only if* S:$\alpha_1 x_1[n] + \alpha_2 x_2[n] \rightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$ for all possible $x_1[n], x_2[n], \alpha_1$, and α_2.

Time-Invariant System:
Suppose S:$x_1[n] \rightarrow y_1[n]$. The system S is time-invariant if and only if S: $x_1[n-n_0] \rightarrow y[n-n_0]$ for all possible $x_1[n]$ and n_0.

Causal System:
A system S is causal *if and only if* the output at any given time is dependent only upon the input at the same time and/or past times.

Stable System:
A system S is stable *if and only* if bounded inputs always result in bounded outputs.

LTI System: S:$x[n] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ where S:$\delta[n] \rightarrow h[n]$

Sifting property of the impulse:

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Convolution Integral: $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$

Convolution Sum: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Properties of Convolution (also true for discrete-time):

$x(t) * h(t) = h(t) * x(t)$ (commutatitive)
$x(t) * \delta(t) = x(t)$ (identity element)
$x(t) * \big(g(t) * h(t)\big) = \big(x(t) * g(t)\big) * h(t)$ (associative)
$x(t) * \big(g(t) + h(t)\big) = x(t) * g(t) + x(t) * h(t)$ (distributive over addition)

Complex Exponentials, Cosines, and Sines:

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$\cos(\theta) = (1/2)(e^{j\theta} + e^{-j\theta})$

$\sin(\theta) = (1/2j)(e^{j\theta} - e^{-j\theta})$

The signal $x(t) = e^{j\omega_0 t}$ has fundamental period $T = 2\pi/|\omega_0|$

The signal $x[n] = e^{j2\pi na/b}$ has fundamental period $N = |b|$ provided the integers a and b don't have common factors.

Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad\qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Response of DT LTI systems to Complex Exponential Signals:

$$x[n] = e^{j\omega n} \xrightarrow{LTI} y[n] = H(e^{j\omega})e^{j\omega n};\ H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Basic DTFT Properties

$$x[n-n_0] \Leftrightarrow e^{-j\omega n_0} X(e^{j\omega}) \qquad e^{j\omega_0 n} x[n] \Leftrightarrow X(e^{j(\omega-\omega_0)}) \qquad x^*[n] \Leftrightarrow X^*(e^{-j\omega})$$

$$x[-n] \Leftrightarrow X(e^{-j\omega}) \qquad x[n] * h[n] \Leftrightarrow X(e^{j\omega}) H(e^{j\omega})$$

Common DTFT Pairs

$$e^{j\omega_0 n} \Leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \qquad \delta[n-n_0] \Leftrightarrow e^{-j\omega n_0}$$

$$u[n] - u[n-N] \Leftrightarrow \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2} \qquad \frac{\sin \omega_0 n}{\pi n} \Leftrightarrow \begin{cases} 1 & 0 \le |\omega| \le \omega_0 \\ 0 & \omega_0 < |\omega| \le \pi \end{cases}$$

Complex Exponentials and LTI Systems:

If **S** is an LTI system with impulse response $h(t)$, then

$$e^{j\omega_0 t} \overset{S}{\Rightarrow} H(j\omega_0) e^{j\omega_0 t}$$

where $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$

CT Fourier Series (Periodic Signal)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k t/T}$$

Response of CT LTI systems to CT Periodic Signals:

If an LTI system has impulse response $h(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T} \xrightarrow{LTI} y(t) = \sum_{k=-\infty}^{\infty} a_k H(j2\pi k/T) e^{jk2\pi t/T}$$

where $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$

CT Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\varpi t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

CT Basic Fourier Transforms:

$$\delta(t-t_0) \Leftrightarrow e^{-j\omega t_0} \qquad e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0) \qquad u(t+T_1)-u(t-T_1) \Leftrightarrow \frac{2\sin(\omega T_1)}{\omega}$$

$$\frac{\sin(Wt)}{\pi t} \Leftrightarrow u(\omega+W)-u(\omega-W) \qquad e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}; \quad \mathrm{Re}\{a\}>0$$

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) \Leftrightarrow \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\omega-\frac{2\pi k}{T})$$

CT Fourier Transform Properties:

$$x(t) \Leftrightarrow X(j\omega) \qquad h(t) \Leftrightarrow H(j\omega) \qquad \alpha x(t)+\beta h(t) \Leftrightarrow \alpha X(j\omega)+\beta H(j\omega)$$

$$x(t-t_0) \Leftrightarrow e^{-j\omega t_0}X(j\omega) \qquad x(-t) \Leftrightarrow X(-j\omega) \qquad x^*(t) \Leftrightarrow X^*(-j\omega)$$

$$x(t)*h(t) \Leftrightarrow X(j\omega)\times H(j\omega) \qquad x(t)\times h(t) \Leftrightarrow (\tfrac{1}{2\pi})X(j\omega)*H(j\omega)$$

Duality: $X(jt) \leftrightarrow 2\pi x(-\omega)$

Sampling a General Continuous-Time Signal

$$x[n] = x(nT) \Leftrightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\frac{\omega}{T}-\frac{2\pi k}{T})$$

Causality of an LTI System

Continuous-time: $h(t) = 0$ for $t < 0$

Discrete-time: $h[n] = 0$ for $n < 0$

Stability of an LTI system

Continuous time: $\int_{-\infty}^{\infty} |h(t) dt < \infty$

Discrete-time: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Bilateral Laplace Transform

$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$ with region of convergence (ROC) *right of rightmost* pole if $h(t)$ is the impulse response of a causal system.

$s = \sigma + j\omega$

For absolutely integrable signals: $X(j\omega) = X(s)$ with s replaced by $j\omega$

Differentiation Property: $\frac{d}{dt} x(t) \leftrightarrow sX(s)$

Basic Laplace Transform: $e^{-at} u(t) \leftrightarrow \frac{1}{s+a}$ with ROC: $\sigma > -a$

Poles of the Laplace Transform are the values of s for which the polynomial in the denominator of the Laplace transform is zero.

LTI System is stable if and only if Laplace transform of impulse response *includes $j\omega$ axis in the ROC.*

Bilateral Z transform

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \text{ where } z = re^{j\omega}$$

Region of Convergence (ROC) is outside the outermost pole if $h[n]$ is the impulse response of a causal system.

For absolutely summable signals: $X\left(e^{j\omega}\right) = X(z)$ with z replaced by $e^{j\omega}$

Differencing Property: $x[n-n_0] \leftrightarrow z^{-n_0}X(z)$

Basic z-transform:

$$a^n u[n] \Leftrightarrow \frac{1}{1-az^{-1}} \quad for \quad |z|>|a|$$

Poles of the z Transform are the values of z for which the polynomial in the denominator of the Laplace transform is zero.

LTI System is stable if and only if z transform of impulse response *includes the unit circle in the ROC*

Finite Sum Formula

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} \qquad \alpha \neq 1$$

Infinite Sum Formula

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$