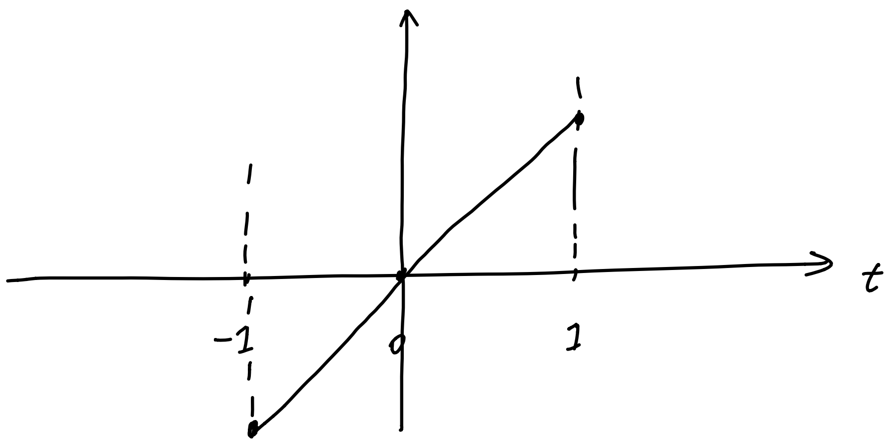
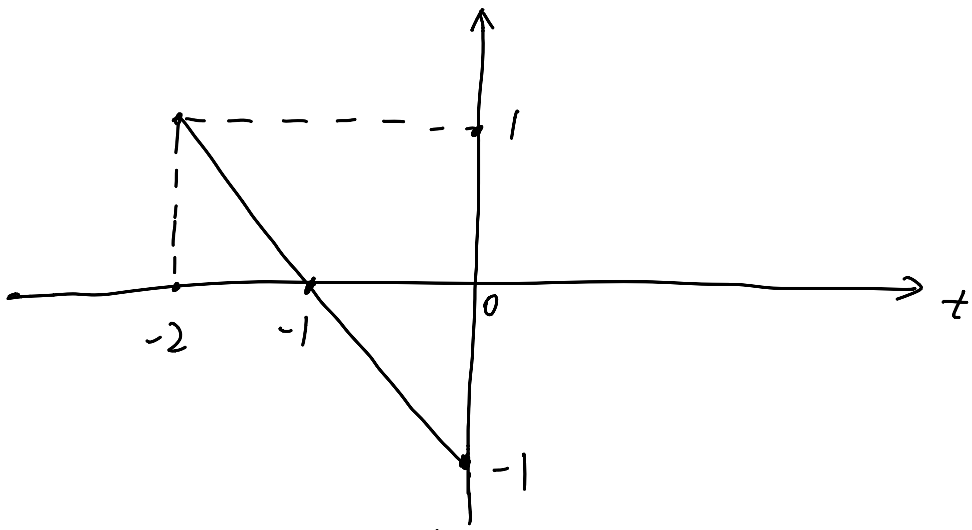
Final practice exam solutions

1. $x(t) = t\,[u(t+1) - u(t-1)]$:



$g(t) = x(-1-t) = x(-t-1) = x(t)$ first shift to right by 1 and then flipped around 0



OR $g(t) = x(-1-t) = (-1-t)\,\underline{[u(-t) - u(-t-2)]}$ ← a box function from -2 to 0.

2. By definition of convolution sum,

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] \,.$$

then $y[0] = \sum_{k=-\infty}^{+\infty} h[k]\, x[-k] = \sum_{k=-\infty}^{+\infty} \left(\frac{1}{2}\right)^{k} u[k] \cdot \left(\frac{1}{2}\right)^{-k} u[-k] = \sum_{k=0}^{0} 1 = 1$

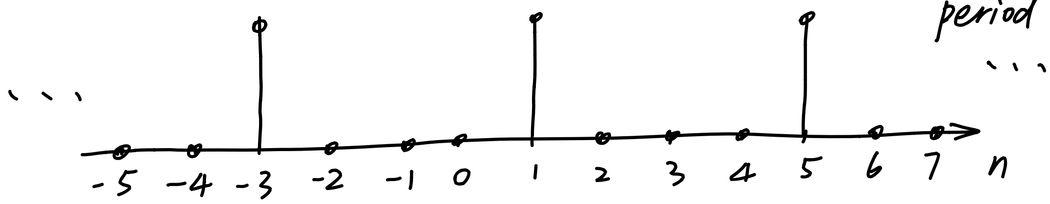
$$y[2] = \sum_{k=-\infty}^{+\infty} h[k]\, x[2-k] = \sum_{k=-\infty}^{+\infty} \left(\frac{1}{2}\right)^{k} u[k] \cdot \left(\frac{1}{2}\right)^{2-k} u[2-k]$$

$$= \sum_{k=0}^{2} \frac{1}{4} = 3 \times \frac{1}{4} = \frac{3}{4}$$

3. a) $x[n]$ is periodic since every 4 intgar we can find a new n such that $(n-1)$ is divisible by 4.

Inside $[0, 3]$, only $x[1] = 1$, $x[0] = x[2] = x[3] = 0$

$\Rightarrow x[n]$:



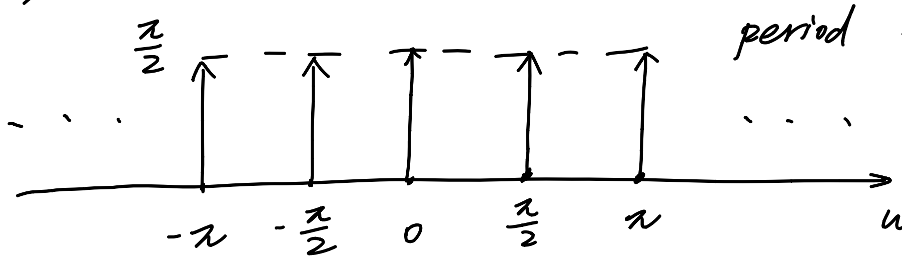
b) Note that the impulse train above can be expressed as

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-1-4k]$$ with period $N=4$ and time shift $+1$

$\Rightarrow$ The DTFT of $x[n]$ is another impulse train with phase shift

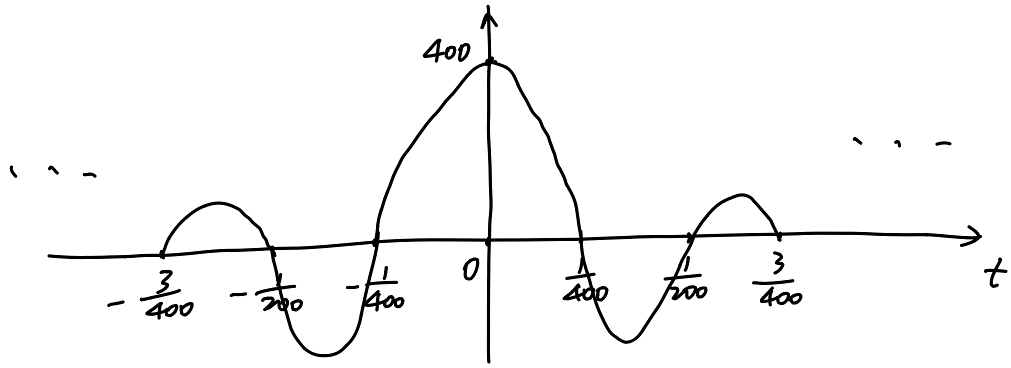
$$X(e^{jw}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(w - \frac{2\pi k}{N}\right) \cdot e^{-jw}$$

$|X(e^{jw})|$:



4. a) $y_1(t) = x_1(t) * h(t) = \delta(t) * h(t) = h(t) = \dfrac{\sin(400\pi t)}{\pi t}$

$y_1(0) = 400$, zero-crossings : $400\pi t = k\pi \Rightarrow t = \dfrac{k}{400}$ $(k \neq 0$, integar$)$



b) The CTFT of $h(t)$ (sinc function) is a box function

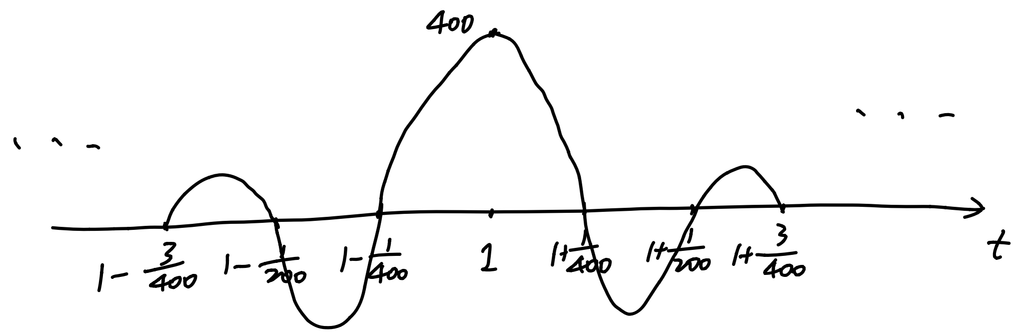
$$H(jw) = u(w+400\pi) - u(w-400\pi)$$

The CTFT of $x_2(t)$ (shifted sinc) is a complex scaled box function

$$X_2(jw) = e^{-jw}\left[u(w+4000\pi) - u(w-4000\pi)\right]$$

$$\Rightarrow Y_2(jw) = X_2(jw)\,H(jw) = e^{-jw}\left[u(w+400\pi) - u(w-400\pi)\right]$$

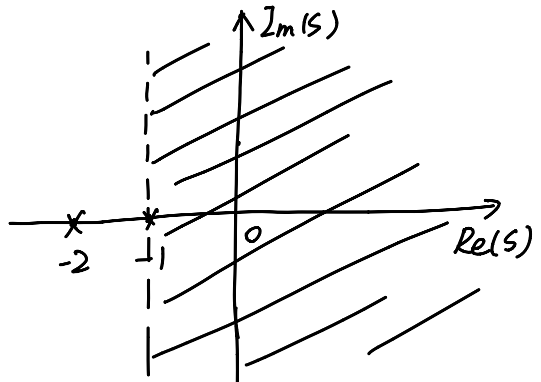
$$y_2(t) = h(t-1) = \frac{\sin[400\pi(t-1)]}{\pi(t-1)}$$: $h(t)$ shifted to right by 1



5. a) Taking Laplace transform of the differential equation on both sides:

$$s^2Y(s) + 3sY(s) + 2Y(s) = X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}$$, with 2 poles : $s_1=-1$, $s_2=-2$



The ROC is $Re\{s\} > -1$, which includes jw-axis

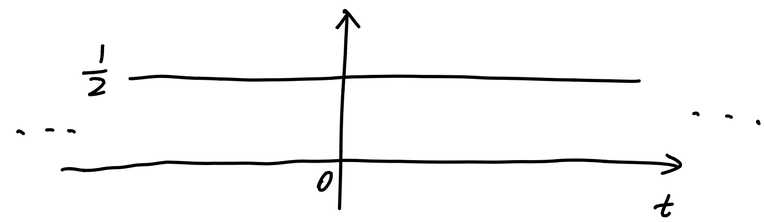
$\Rightarrow H$ is stable

b) The CTFT of $h(t)$: $H(jw) = H(s)|_{s=jw} = \frac{1}{(jw+1)(jw+2)}$

The CTFT of $x(t)$ $\rightarrow$ $X(jw) = 2\pi\delta(w)$

$$Y(jw) = X(jw)\,H(jw) = \frac{2\pi\,\delta(w)}{(jw+1)(jw+2)} = \frac{2\pi}{2}\,\delta(w) = \pi\,\delta(w)$$

$\Rightarrow y(t) = \frac{1}{2}$



6. 1) $h[n]$ is real $\Leftrightarrow |H(e^{jw})|$ is even

2) S is causal $\Leftrightarrow h[n] = 0$ for $n < 0$

3) $\left.\begin{array}{l} H(e^{j\frac{\pi}{2}}) = H(e^{j\pi}) = 0 \\ 1) \end{array}\right\} \Rightarrow H(e^{-j\frac{\pi}{2}}) = H(e^{-j\pi}) = 0$

4) $e^{j\frac{3w}{2}} H(e^{jw})$ is real $\Rightarrow H(e^{jw}) = R(w) e^{-j\frac{3w}{2}}$, where $R(w)$ is some real function of w

5) $\sum_{k=-\infty}^{+\infty} h[n] = 8 \Rightarrow H(e^{j0}) = 8$

A (periodic) sinc $H(e^{jw})$ satisfies all above

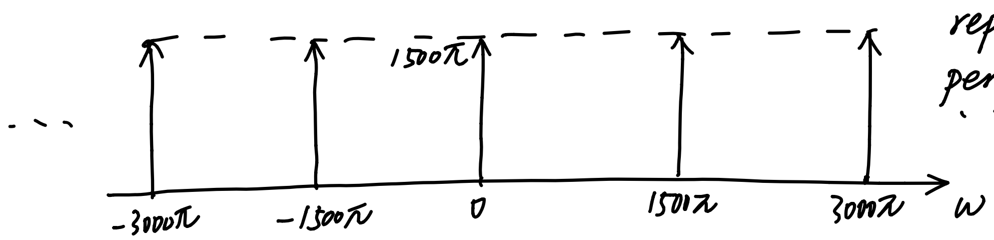
$$H(e^{jw}) = 2 \frac{\sin(2w)}{\sin(w/2)} e^{-j\frac{3w}{2}}$$

$$\Updownarrow$$

$$h[n] = 2\left(u[n] - u[n-4]\right)$$

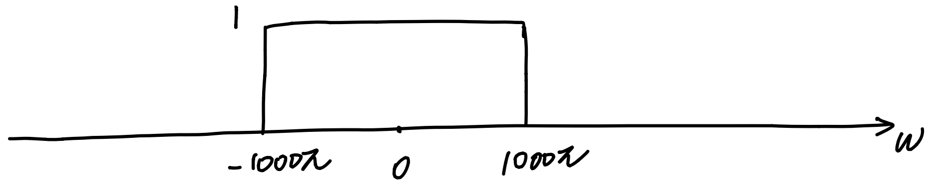
7. Denote $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - \frac{1}{750}k)$, which is an impulse train with $T = \frac{1}{750}$

$\Leftrightarrow X(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi}{T}k) = 1500\pi \sum_{k=-\infty}^{+\infty} \delta(w - 1500\pi k)$



repeat with period 1500π

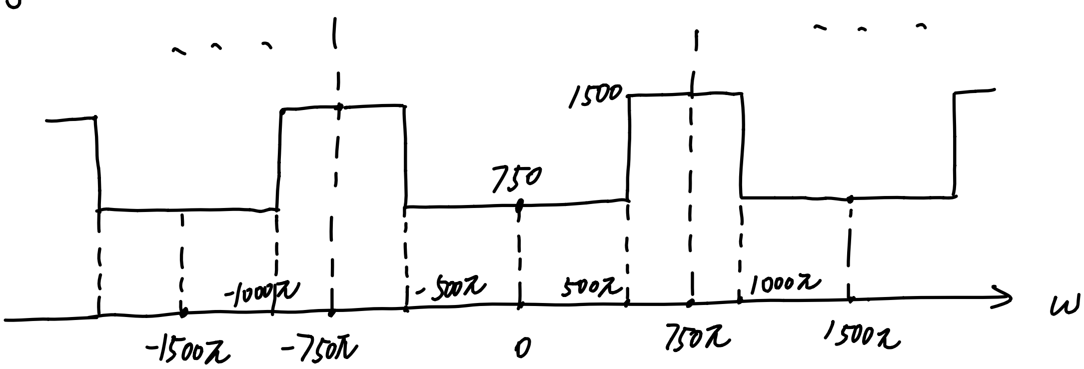
$$h(t) = \frac{\sin(1000\pi t)}{\pi t} \Longleftrightarrow H(j\omega) = u(\omega+1000\pi) - u(\omega-1000\pi)$$



and $y(t) = x(t)\,h(t) \Longleftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * H(j\omega)$, which is periodically replicating $H(j\omega)$ with frequency aliasing

$Y(j\omega)$:

repeat with period 1500π



$$\Rightarrow A = \int_{-\infty}^{+\infty} y(t)\,dt = Y(j0) = 750$$