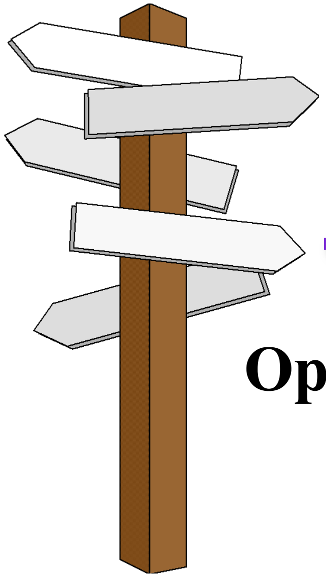
Amortized Analysis

Cost of an operation



Worst-case average behavior

Operations: $o_1, o_2, o_3, \ldots, o_n$

Average cost:

$$\frac{1}{n}\sum_{i=1}^{n}\text{cost}(o_i) = \text{AVG}$$

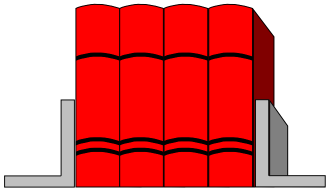
Worst-case Average:

$$\forall o_1, o_2, o_3, \ldots, o_n \qquad \frac{1}{n}\sum_{i=1}^{n}\text{cost}(o_i) = \text{AVG} \le \text{BOUND}$$

$$\sum_{i}\text{cost}(o_i) \le n \cdot \text{BOUND}$$

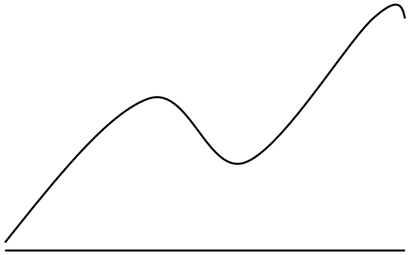
Types of Amortized Analysis

- Aggregate

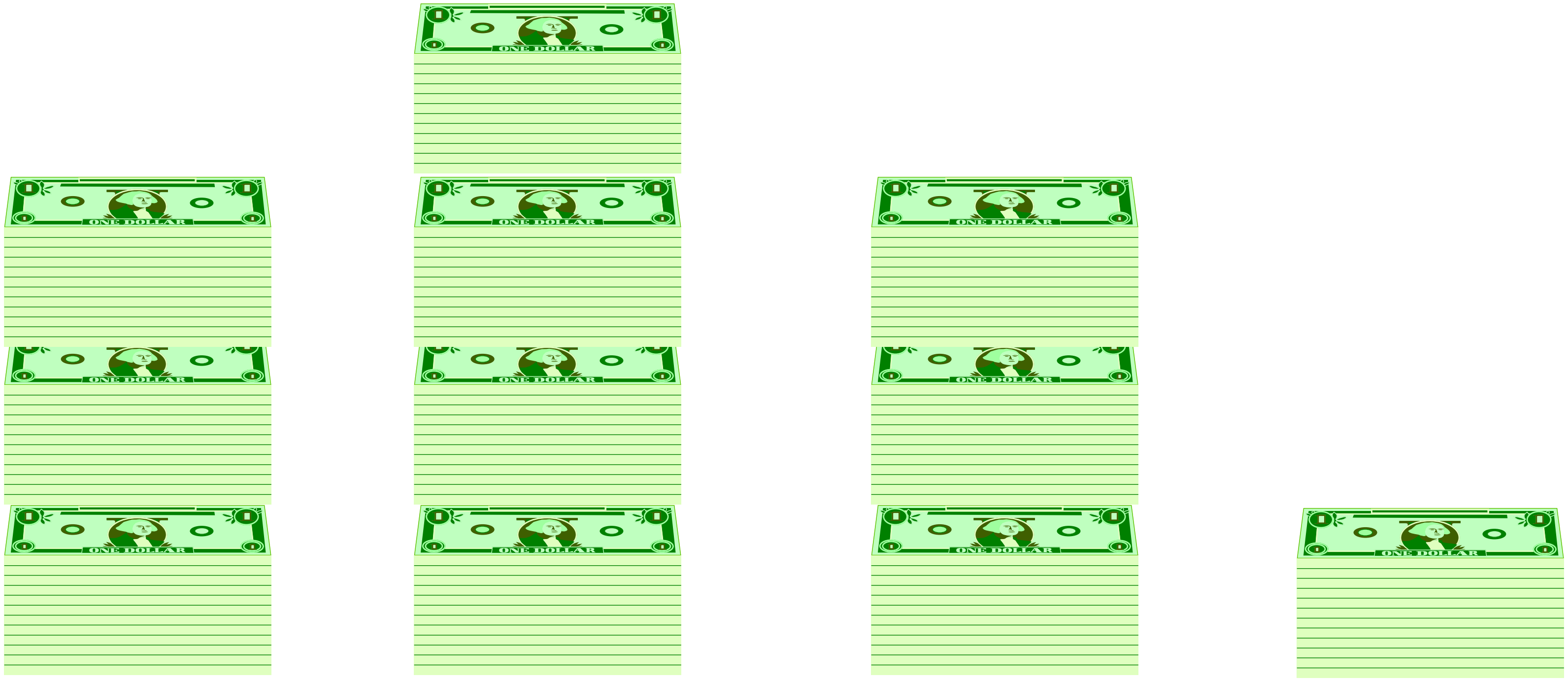


- Accounting

- Potential Functions



Aggregate



The Stack

push: $O(1)$

pop: $O(1)$

multi-pop: $\Theta(k)$

Worst Case: $n \times \Theta(n) = \Theta(n^2)$

Aggregate: $\Theta(n)$

Aggregate (binary counter)

Decimal	Binary
1	000001
2	000010
3	000011
4	000100
5	000101
	. . .
n	fedcba

k bits

Basic operation: flip a bit

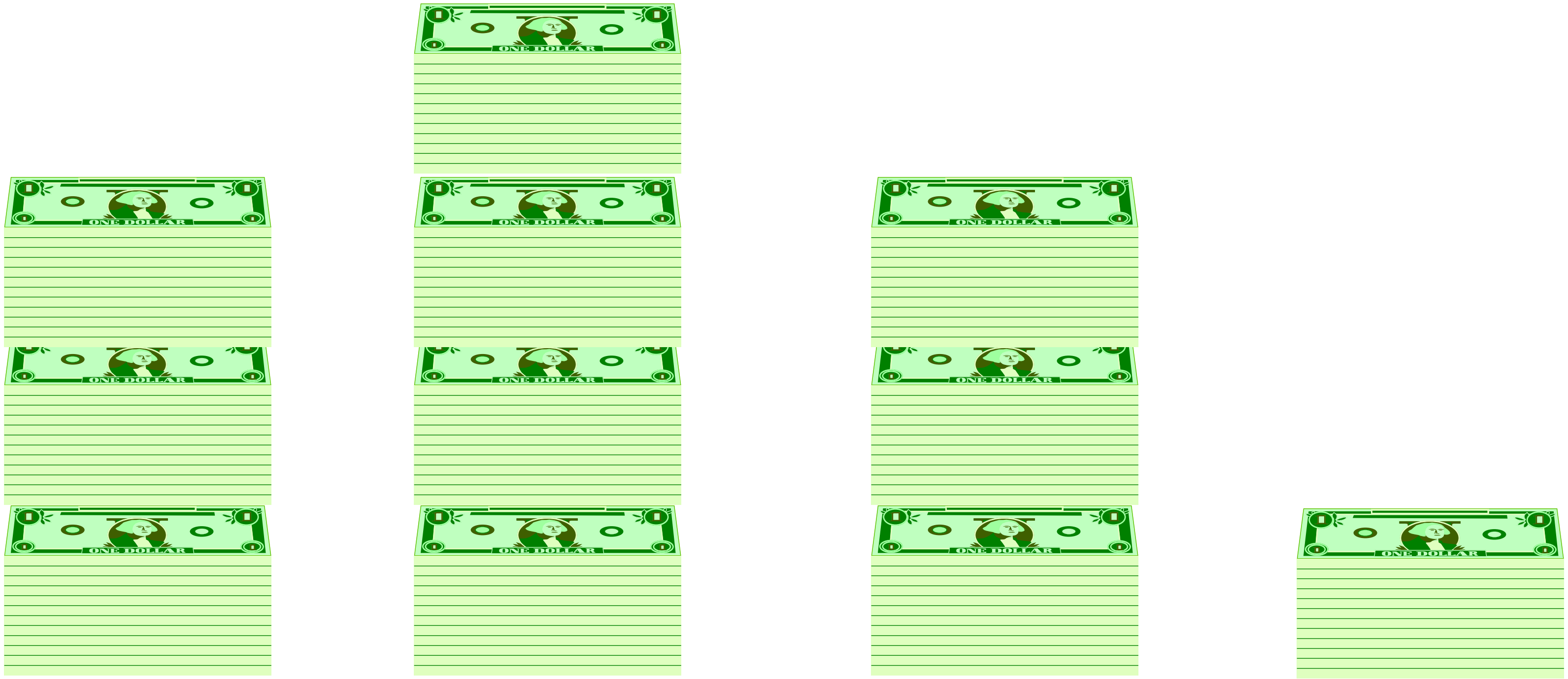
Worst Case:

$$\Theta(nk)$$

Aggregate:

$$\Theta(n)$$

Accounting



The Stack

push:
cost=$2

pop:
cost=$0

multi-pop:
cost=$0

n operations cost $(2n); O(1) amortized cost

Accounting(counter)

Decimal	Binary
1	000001
2	000010
3	000011
4	000100
5	000101
	. . .
n	fedcba

Menu

0-->1 $2

1-->0 FREE

Potential Functions

actual cost

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

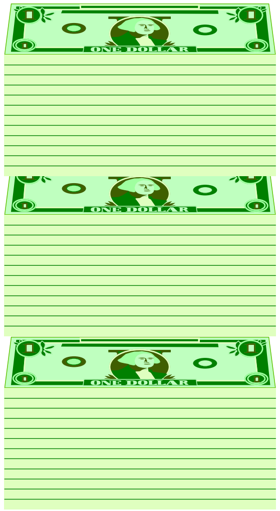
amortized cost

potential after
i-1 operations

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

$$\text{actual cost} = \text{amortized cost} - \Phi(D_n) + \Phi(D_0)$$

Potential Functions (examples)



The Stack

$$\Phi(\text{stack}) = \#\ \text{of items on stack}$$

$$\Phi(D_0) = 0$$

$$\Phi(D_n) \geq \Phi(D_0) \geq 0$$

amort(pop)=actual(pop)+#(after)-#(before)

amort(push)=actual(push)+#(after)-#(before)

Potential Functions (examples 2)

$$\Phi(\text{counter}) = \#\ \text{of 1 bits}$$

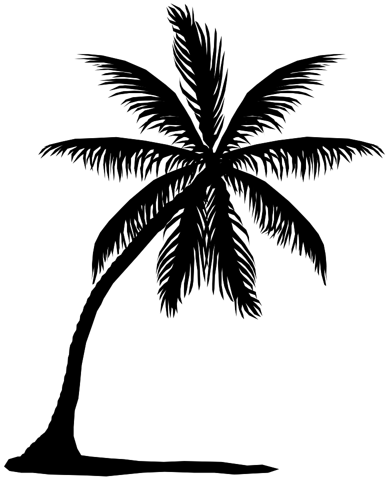
$$\Phi(D_0) = 0$$

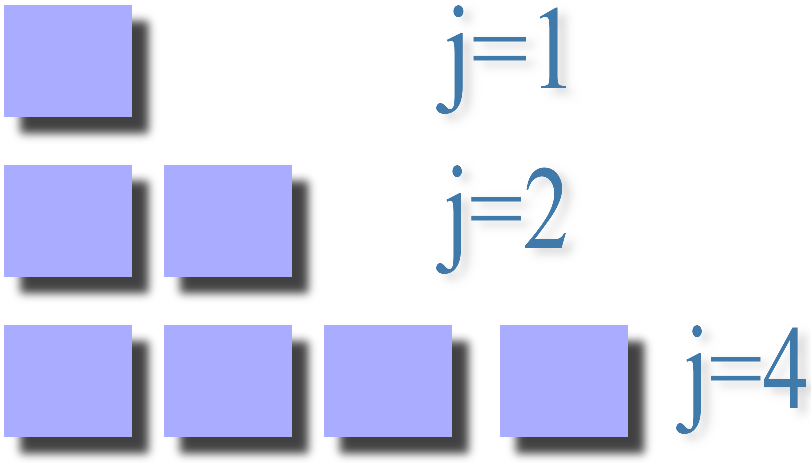
$$\Phi(D_n) \geq 0$$

amort(0->1)=actual(0->1)+#(after)-#(before)

amort(1->0)=actual(1->0)+#(after)-#(before)

Hash Tables





insert: O(1)
insert/double: O(size)

Aggregate

$$n + \sum_{j=1}^{\lfloor \lg n \rfloor} 2^{j-1} \le n + 2n = 3n$$

Accounting

$3/insertion

Potential

$\Phi(T) =$ 2*(# of items in table)-
(size of the table)