EC504 ALGORITHMS AND DATA STRUCTURES
SPRING 2021 MONDAY & WEDNESDAY
2:30 PM - 4:15 PM

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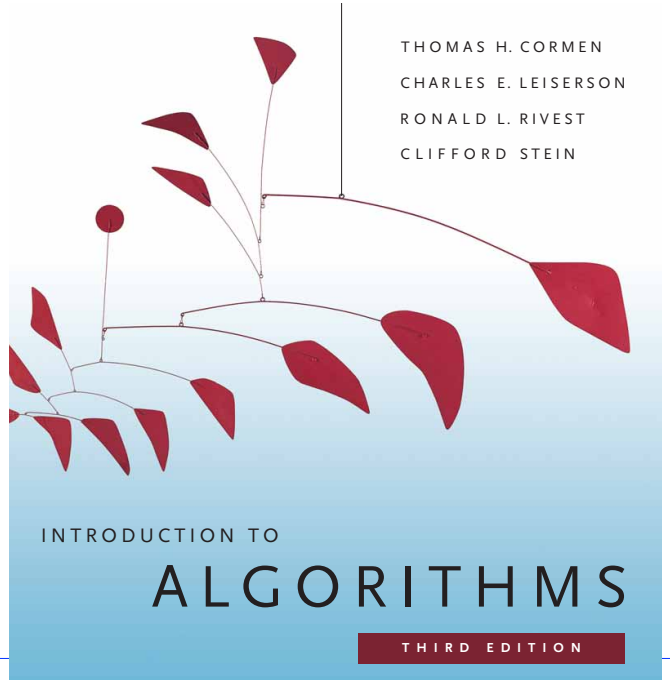
Course Organization

- **Text**:
  - Cormen, Leiserson, Rivest & Stein (CLRS), Fundamental text
    "Introduction to Algorithms" 3rd Edition MIT Pres
- **Reference**:

- Wikipedia
- Mark Allen Weiss "Data Structure and Algorithms in C++".
- Handouts on specific topics as needed

- **Grading**:

– HW (Written)	20%
– HW: Software	20%
– Project	20%
– Exams (2)	40%



Course Organization - 2

- Web Site in learn.bu.edu
- On-line classes
- Regular HW (not weekly; 8 or so during term)
- Office hours: TBD, or by appointment.
- Software: C++, using C++ 11 compatible
- Development: Accounts set up
- HW submission: On Blackboard, with code submitted separately
- Important: This is not a course to learn software. We use software to learn about data structures and algorithms, not the other way around…

Course Organization - 3

- Lectures are recorded, will be posted to web site within 5 hours
- In-class questions welcome
  - Unmute and ask
  - Or raise hand (less clear to instructor!)
- HW will have on-line component, hand-in component and software component
  - On-line short questions with instant feedback using Blackboard quiz
  - SW will provide main program, input files, and output files for debugging. You will develop needed functions.
- We will use Piazza for asynchronous question/answers at all hours
  - Enables all of you to benefit from answers

Course Organization - 4

- Course includes a project
  - Some topics will be suggested
  - Topics can be suggested by students
- Projects can be done in teams of up to 4 people
  - Or individually…Depends on how large a project one proposes
- Typical project: Develop a set of algorithms for solving a class of problems, implemented with different data structures, and evaluate the computation performance on realistic problem data.

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Course Organization - 5

- Policy on Cooperation
  - HW: Discussions with classmates, colleagues and googling on web are allowed in general
  - But you have to write down the solution yourself and fully understand what you write
  - Your code must be your own: base it on what you read on the web, but don't copy verbatim
  - Plagiarism tools will be used extensively. Any violations will result in course failure and referral to College Academic Council

Course Organization - 5

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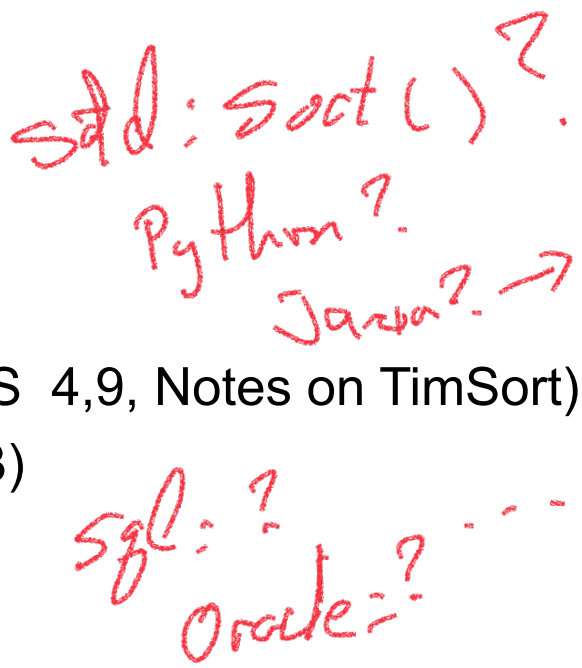
What is EC504 About?

- Data structures store data in manners that support efficient algorithm computation.
- Algorithms with appropriate data structures can improve problem solution efficiency by orders of magnitude.
- In this course, we study both advanced algorithms and appropriate data structures for solution of practical problems.
- We will focus on rigorous analysis, realistic applications and implementation.
- Software implementations of algorithms/data structures will be part of the course.

Course Outline

1. Review: Algorithm Analysis (CLRS 2-4)
   - Asymptotic complexity
   - Recursions
2. Sorting: Classical and modern approaches (CLRS 4,9, Notes on TimSort)
3. Review: Trees and their Properties (CLRS App. B)
4. Efficient Search in one-dimension
   - Hash Tables (CLRS 11)
   - Balanced Search Trees (Red-Black, Splay, B-Trees, VEB trees) (CLRS 12,13,18,20)
   - Priority queues (Binary, Binomial, Fibonacci Heaps, VEB trees) (CLRS 6,19,20)



Course Outline - 2

6. Graphs and Network Optimization (CLRS 22)
   - Minimum spanning trees: Greedy algorithms (CLRS 23)
   - Shortest Paths: Greedy algorithms, Dynamic Programming (CLRS 24,25)
   - Max-flow: Ford-Fulkerson, Preflow-push (CLRS 26)
   - Min-cost flow: Assignment problems, auction algorithms, successive shortest path algorithms (Notes)
   - Applications (Notes)
7. Complexity Theory
   - NP-Complete Problems - Definition and examples (CLRS 34)
   - Approximation Algorithms - Knapsack, Traveling Salesperson, … (CLRS 35)
8. Advanced Topics

What is an algorithm? An unambiguous list of steps (program) to transform some input into some output.

- Pick a Problem (set)
- Find method to solve
  1. Correct for all cases (elements of set)
  2. Each step is finite (Δt_{step} < max time)
  - Next step is unambiguous
  - Terminate in finite number of steps

◆ You know many examples:

GCD, Multiply 2 N bit integers, …

Abu Ja'far Muhammid ibn Musa Al-Khwarizmi
Bagdad (Iraq) 780-850

Growth of Algorithm Run Time with Size n: Big "Oh"

- O(g(n)) : Set of functions of n that grow no faster than g(n) as n gets large

$f(n) \in O(g(n))$ if and only if there exists $c > 0, N_0 > 0$ such that $|f(n)| \leq c|g(n)|$ whenever $n > N_0$

Alternative: $f(n) \in O(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$f(n) \in O(g(n))$, or $f(n)$ is $O(g(n))$, or $f(n) = O(g(n))$

$f(n) \in o(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

Growth of Algorithms: Other concepts

- o(g(n)) : grow strictly slower than g(n)

$$f(n) \in o(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

- Ω(g(n)) : grow no slower than g(n) as n gets large

$f(n) \in \Omega(g(n))$ if and only if there exists $c > 0, N_0 > 0$ such that $|f(n)| \geq c|g(n)|$ whenever $n > N_0$

- ω(g(n)) : grow strictly faster than g(n)

$$f(n) \in \omega(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

- Θ(g(n)) : grow similar to g(n): $f(n) \in O(g(n)), f(n) \in \Omega(g(n))$

Intuition

$f = O(g)$		$f \leq g$
$f = o(g)$		$f < g$
$f = \Omega(g)$	$\Leftrightarrow$	$f \geq g$
$f = \omega(g)$		$f > g$
$f = \Theta(g)$		$f = g$

$f \in O(g) \rightarrow$

$f \in \Theta(g) \rightarrow$

Rules of thumb

- For polynomials, only the largest term matters.

$$a_0 + a_1 N + a_2 N^2 + \cdots + a_k N^k \in O(N^k)$$

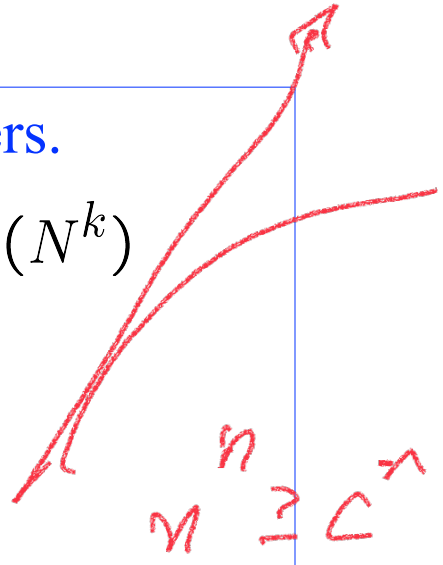
- log N is in o(N)

  Proof: As N ➔ 1 the ratio log(N)/N ➔ 0

- Some common functions in increasing order

1 log N √N N NlogN N^2 N^3 N^{100} 2^N 3^N N! N^N

- For r > 1, d > 0, $\Rightarrow n^d \in o(r^n)$
- For b > 1, r > 0, $\Rightarrow (\log(n))^b \in o(n^r)$



Some examples

- Which increases faster? What is the relation?
  - $(100n^2,\ 0.01 * 2^n)$
  - $(0.1 * \log n,\ 10n)$
  - $(10^{10}n,\ 10^{-10}n^2)$
  - $(0.01n^3 - 10,\ 100n^2)$
  - $(\ln(300n^4),\ \ln(n))$
  - $(n^4 + 3n^2 + 1,\ n^{0.01})$
  - $(2^n,\ 3^n)$
  - $(n^{10},\ n^n)$
  - $(n!,\ n^2)$ (Stirling's formula: $n! \approx (2\pi n)^{1/2}(\frac{n}{e})^n$)

2nd. $2^{n\ell}$ 1st (H)(.) $\log(300n^4) = 4\log n + \log(300)$

1st $\lim \frac{3^n}{2^n} = \lim (\frac{3}{2})^n = \infty$

$2^{n\ell}$ 2^{ℓ}

1st.

Why is big-O important?

input size \ time	N	10	20	30	40	50	60
	log n	3.3µsec	4.4µsec	5µsec	5.3µsec	5.6µsec	5.9µsec
	n	10µsec	20µsec	30µsec	40µsec	50µsec	60µsec
	n^2	100µsec	400µsec	900µsec	1.5msec	2.5msec	3.6msec
	n^5	0.1sec	3.2sec	24.3sec	1.7min	5.2min	13min
	3^n	59msec	48min	6.5yrs	385,500yrs	$2x10^8$ centuries...	
	n!	3sec	$7.8x10^8$ millennia ……				

(processor doing ~1,000,000 steps per second)

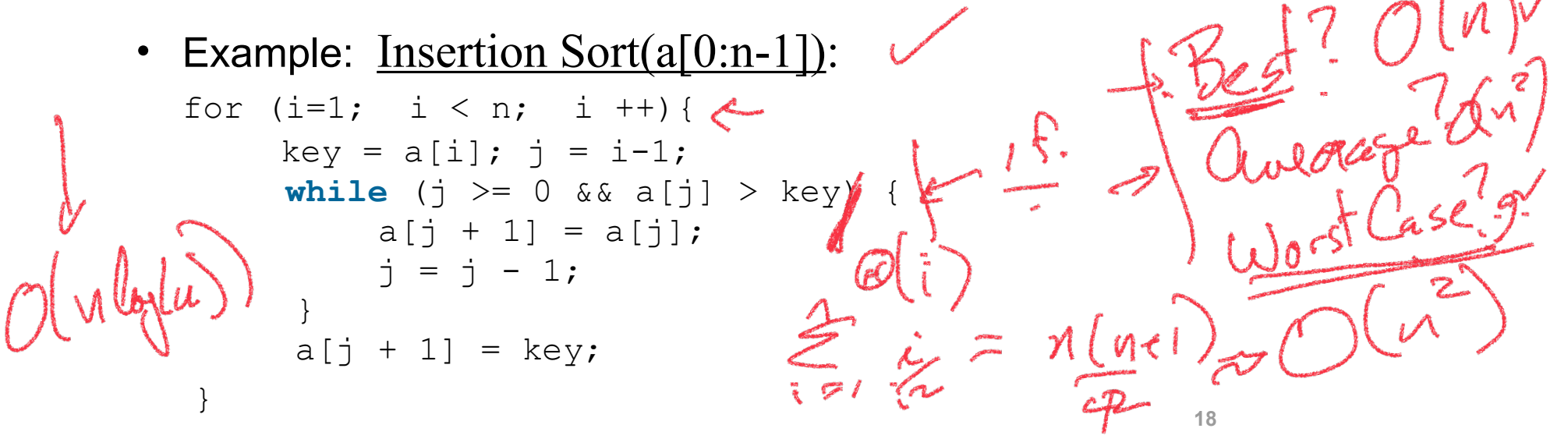
Non polynomial algorithms are terrible!
Logs are great!

Analysis of Algorithms

- Algorithms don't have constant run time for a given input size
  - Run time depends on nature of input
  - Best-Case, Average Case, Worst Case
  - Typically care about worst case (other cases useful in practice)

- Example: Insertion Sort(a[0:n-1]):

```
for (i=1;  i < n;  i ++){
     key = a[i]; j = i-1;
     while (j >= 0 && a[j] > key) {
          a[j + 1] = a[j];
          j = j - 1;
     }
     a[j + 1] = key;
}
```



OUTER LOOP TRACE FOR INSERTION SORT:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	(Swaps)
6 |	5	2	8	3	4	7	1	(1)
5 ←	→ 6							
5	6 |	2	8	3	4	7	1	(2)
	2 ←	→ 6						
2 ←	→ 5							
2	5	6 |	8	3	4	7	1	(0)
2	5	6	8 |	3	4	7	1	(3)
2	3	5	6	8 |	4	7	1	(3)
2	3	4	5	6	8 |	7	1	(1)
2	3	4	5	6	7	8 |	1	(7)
1	2	3	4	5	6	7	8	(17 total swaps)

Analysis of Algorithms

- Example: Insertion Sort(a[0:n-1]):

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for (i=1;  i < n;  i ++){
     key = a[i]; j = i-1;
      while (j >= 0 && a[j] > key) {
          a[j + 1] = a[j];
          j = j - 1;
      }
      a[j + 1] = key;
}
```

- Analysis:
  - Best case: input is sorted, so while loop ends after 2 ops: $\Theta(n)$
  - Average case: while loop goes over 1/2 of list to find insert
  - Worst case: input is sorted in reverse, so while look ends at j = -1.

```
for ( i = 0; i< N; i++)
      for(j = i; j < N; j++){
         Sum = 0;
         for(k=i; k<j+1; k++)
              Sum += a[k];
         if(Sum > MaxSum)
              MaxSum = Sum;
     }
}
```

$$\int_0^N dx \int_x^y dy \int_x^z dz \rightarrow O(n^3)$$

```
for ( i = 0; i< N; i++){
       Sum = 0;
       for(j=i; j<N; j++){
           Sum += a[j];
           if(Sum > MaxSum)
                   MaxSum = Sum;
       }
}
```

$O(N^2)$

Different Algorithm

```
MergeSort(int *array, int l, int r):
  if(l < r) {
      int m = l+(r-l)/2;
      mergeSort(array, l, m);
      mergeSort(array, m+1, r);
      merge(array, l, m, r);
   }
```

Recursive equation algorithm:

T(n): run time for a list of size n. m(n): time to merge arrays of length n

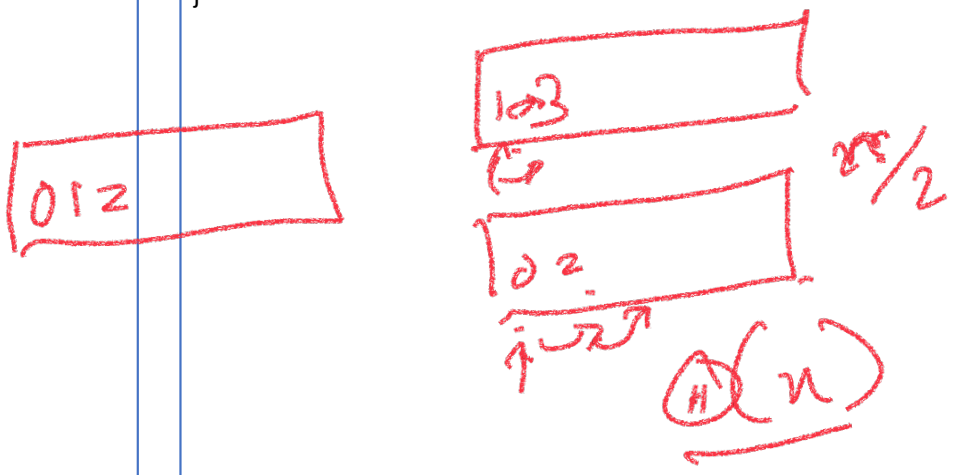
$$T(n) = 2T\left(\frac{n}{2}\right) + m(n)$$

$m(n) \in \Theta(n)$

Merge Algorithm

```
void merge(int *array, int l, int m, int r) {
   int i, j, k, nl, nr;
   nl = m-l+1; nr = r-m;
   int larr[nl], rarr[nr];
   for(i = 0; i<nl; i++)
      larr[i] = array[l+i];
   for(j = 0; j<nr; j++)
      rarr[j] = array[m+1+j];
   i = 0; j = 0; k = l;
   //marge temp arrays to real array
   while(i < nl && j<nr) {
      if(larr[i] <= rarr[j]) {
         array[k] = larr[i];
         i++;
      }else{
         array[k] = rarr[j];
         j++;
  }
  k++;
 }
```

```
  while(i<nl) {      //extra element in left array
    array[k] = larr[i];
    i++; k++;
  }
  while(j<nr) {     //extra element in right array
    array[k] = rarr[j];
    j++; k++;
  }
}
```



Mergesort Illustration

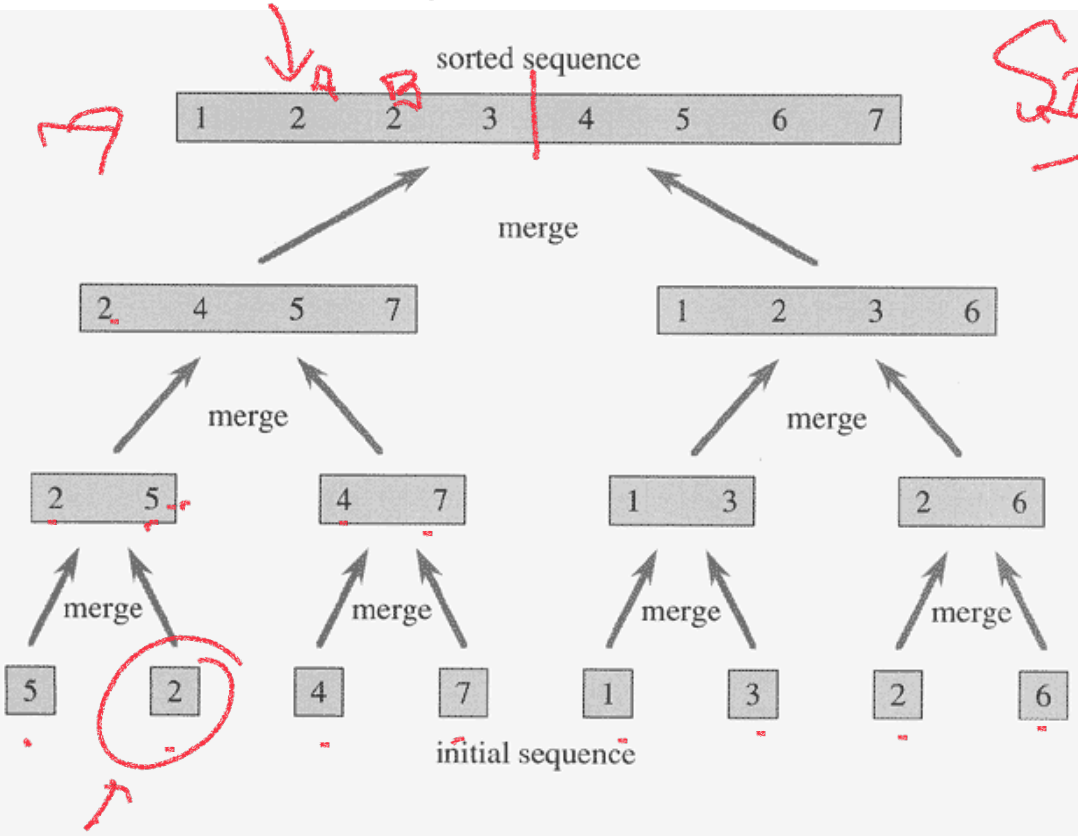


Figure 2.4 The operation of merge sort on the array $A = \langle 5, 2, 4, 7, 1, 3, 2, 6 \rangle$. The lengths of the sorted sequences being merged increase as the algorithm progresses from bottom to top.

MASTER EQUATION: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Theorem: The asymptotic Solution is:

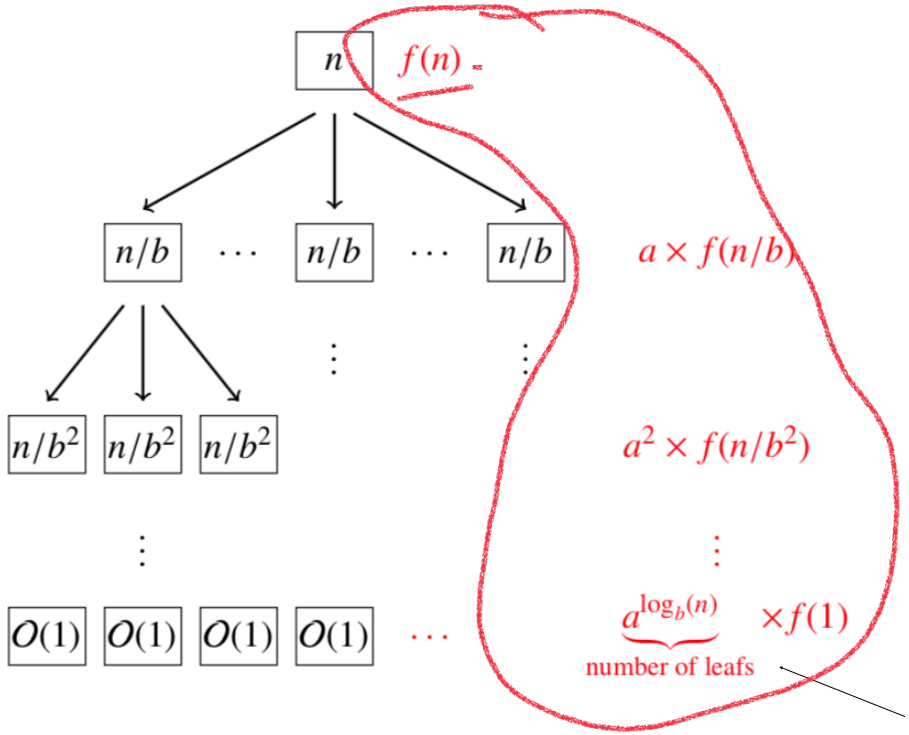
- Define $\gamma = \log_b(a) \rightarrow a = b^\gamma$
- Then, there are three cases

1. $T(n) \in \Theta(n^\gamma)$ if $f(n) \in O(n^{\gamma-\epsilon})$ for some $\epsilon > 0$
2. $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\gamma+\epsilon})$ for some $\epsilon > 0$
3. $T(n) \in \Theta(n^\gamma \log(n))$ if $f(n) \in \Theta(n^\gamma)$

BUILD TREE TO SOLVE

$$T(n) = aT(n/b) + f(n)$$

$$n/b^h = 1 \implies h = \log_b(n)$$



Height

Breadth

$$\lim \frac{a^k f(n/b^k)}{n^\gamma} = ???$$

$h = \log_b(n)$

$$T(n) = f(n) + af(n/b) + \cdots + a^{log_b(n)-1} f(b^2) + a^h T(1)$$

LET'S ANALYZE FOR $n = b^h, f(n) \in O(n^k)$, so $f(n) = cn^k$

$$T(n) = f(n) + af(n/b) + \cdots + a^{log_b(n)-1} f(b^2) + a^h T(1)$$

$$T(n) = cn^k + ac(n/b)^k + a^2c(n/b^2)^k + \ldots + a^{h-1}c(b^2)^k + a^hT(1)$$

$$= cn^k + (\frac{a}{b^k})cn^k + (\frac{a}{b^k})^2cn^k + \ldots + (\frac{a}{b^k})^{h-1}cn^k + a^hT(1)$$

Note: $a^h = n^\gamma$

If $k < \gamma$, then $b^k < a$. Case 1

If $k > \gamma$, then $b^k > a$. Case 2

If $k = \gamma$, then $b^k = a$. Case 3

Back to Mergesort

$T(n) = 2T(\frac{n}{2}) + m(n)$

$m(n) \in \Theta(n)$

Therefore, $a = 2,\ b = 2,\ \gamma = 1 \Rightarrow T(n) \in \Theta(n \log(n))$

Worst case for mergesort:

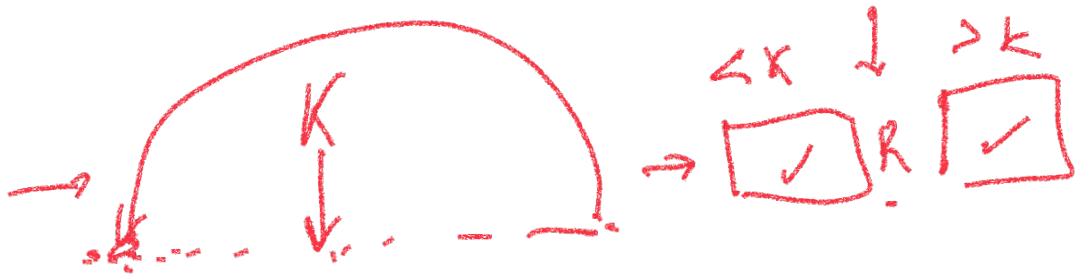
Average case for mergesort:

Best case for mergesort:

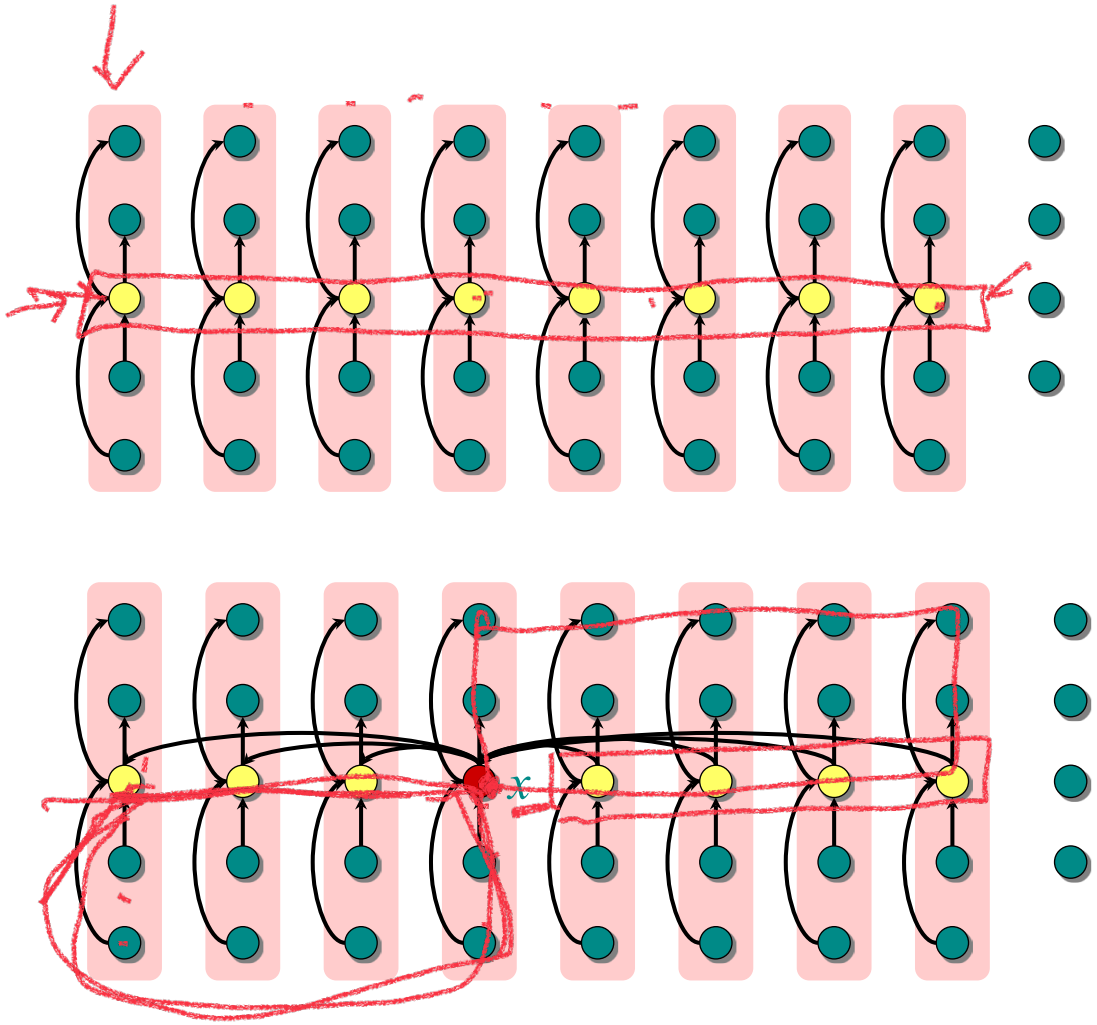
Order Statistics

- Select the i-th smallest of n elements (the element with rank i).
  - i = 1: minimum;
    i = n: maximum;
    i = $\lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: median.
- Naive algorithm: Sort and index ith element.
- Worst-case running time = $\Theta(n \lg n) + \Theta(1) = \Theta(n \lg n)$,
  - using merge sort or Timsort or …
- Can one do better?

Order Statistics

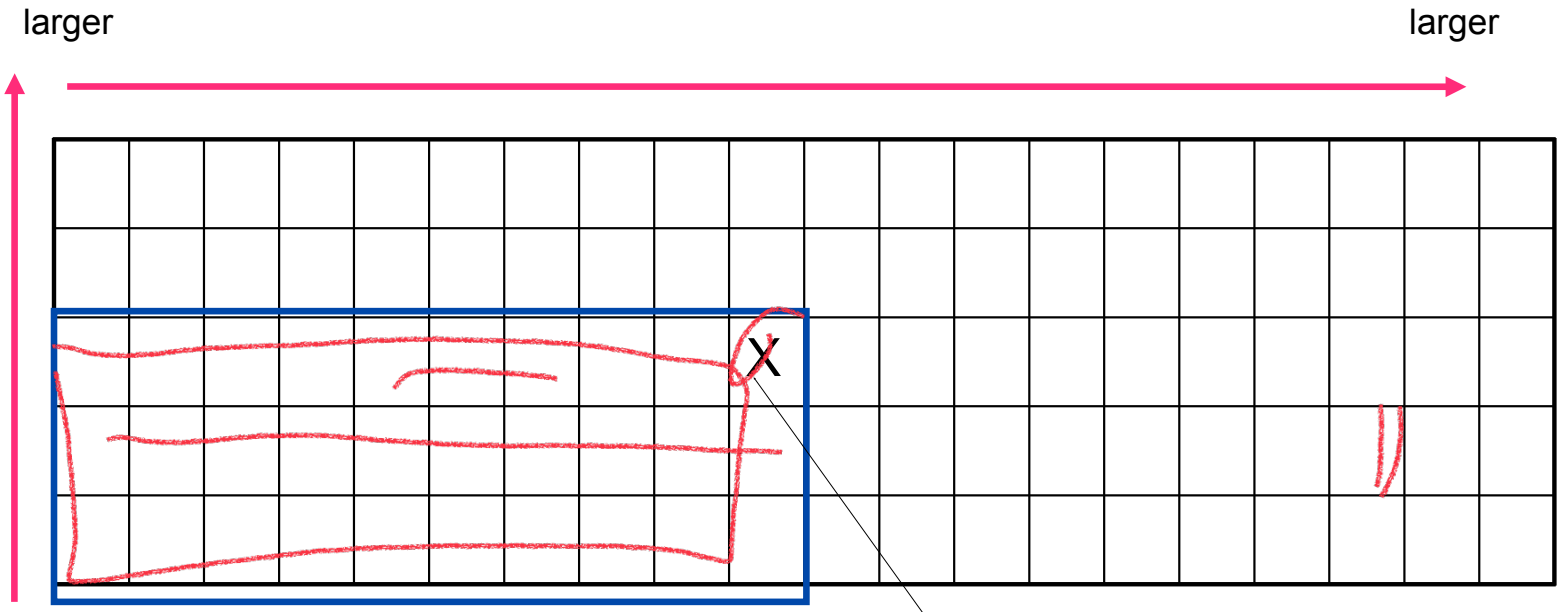


- SELECT(i, n)
  - Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
  - Recursively SELECT the median x of the ⌊n/5⌋ group medians to be the pivot.
  - Partition around the pivot x. Let k = rank(x).
  - if i=k then return x else if i < k
    - recursively SELECT the i-th smallest element in the lower part
  - else recursively SELECT the (i-k) th smallest element in the upper part



- At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians. Therefore, at least $3 \lfloor n/10 \rfloor$ elements are $\leq x$.
- Similarly, at least $3 \lfloor n/10 \rfloor$ elements are $\geq x$.
- Recursion:
  $T(n) \leq T(n/5) + T(7n/10) + cn$
- Result: $T(n) \in \Theta(n)$

5 row of N/5 Columns



3/5 * 1/2 N Boxes smaller than X

Exact Median of Middle Row

Generalization of Master's Equation

Let $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$ be such that $\sum_{j=1}^{k} \alpha_j < 1$. Then,

If $T(n) = T(\alpha_1 n) + T(\alpha_2 n) + \cdots + T(\alpha_k n) + cn$, then $T(n) \in \Theta(n)$

Other useful recursive solutions

$T(n) = aT(n-1) + bT(n-2)$ (Discrete difference equation, constant coefficient)

Can solve via z-transforms (if you know EC 401..) or follow method below:

Solution form: k^n.

Characteristic equation: Substitute into equation to get

$k^n = ak^{n-1} + k^{n-2} \Rightarrow k^2 - ak - b = 0$

Two solutions k_1, k_2 (or a repeated root).

If $k_1 \neq k_2$ then $T(n) = Ak_1^n + Bk_2^n$. Otherwise, $T(n) = Ak_1^n + Bnk_1^n$.

Example: Fibonacci numbers

$$F(n) = F(n-1) + F(n-2)$$

Characteristic equation:

Solutions:

Initial condition: F(0) = 0; F(1) = 1

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