EC504 ALGORITHMS AND DATA STRUCTURES
FALL 2020 MONDAY & WEDNESDAY
2:30 PM - 4:15 PM

Prof: David Castañón, dac@bu.edu

GTF: Mert Toslali, toslali@bu.edu

Haoyang Wang: haoyangw@bu.edu

Christopher Liao: cliao25@bu.edu

Graphs

in the "real world"

- Networks are graphs
  - Information: WWW, citation, ...
  - Social: co-actor, dating, messenger, communities, ...
  - Technological: Internet, power grids, airline routes, …
  - Biological: Neural networks, food web, blood vessels, …
- Object hierarchies are graphs
- Circuit layouts are graphs
- Computer programs are graphs

Graph Traversals

- Traversals of graphs are also called *searches*
- We can use either breadth-first or depth-first traversals
  - Breadth-first requires a queue
  - Depth-first requires a stack
- We each case, we will have to track which vertices have been visited requiring $\Theta(|V|)$ memory
- The time complexity cannot be better than and should not be worse than $\Theta(|V| + |E|)$
  - Connected graphs simplify this to $\Theta(|E|)$
  - Worst case: $\Theta(|V|^2)$

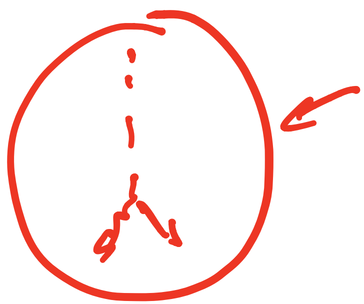
Applications

Applications of tree traversals include:

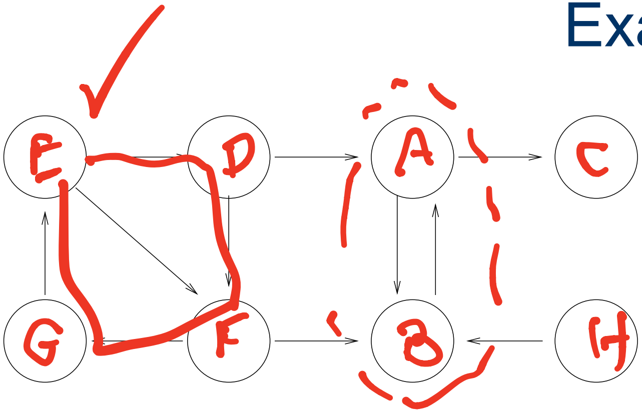
- Determining connectedness and finding connected sub-graphs
- Determining the path length from one vertex to all others
- Testing if a graph is bipartite
- Branch and bound search
- Topological Sort
- …

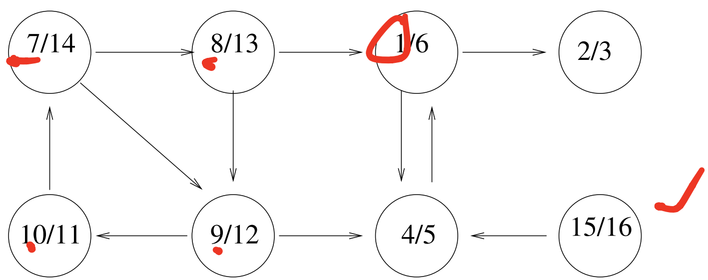
Strongly-Connected Components

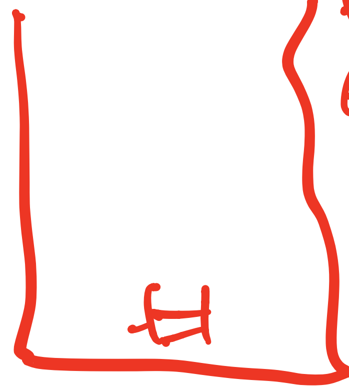
- Kosaraju's Algorithm
  - Perform DFS on graph G = (V, E), Directed
    - Number vertices according to their finishing time in DFS of G
  - Perform DFS on Gr = (V,Er), where Er are reverse of edges in E, selecting nodes in decreasing order of finishing time in previous DFS
  - Strongly connected components = reachable trees obtained in last DFS



Example

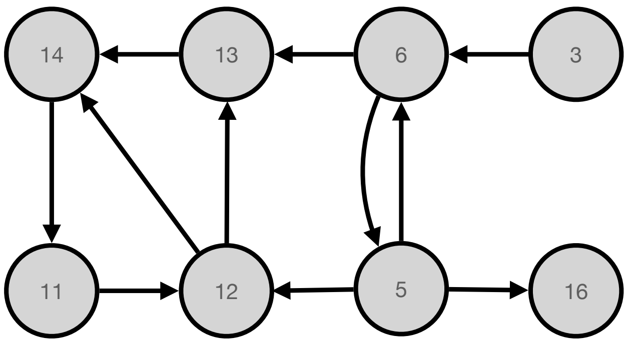




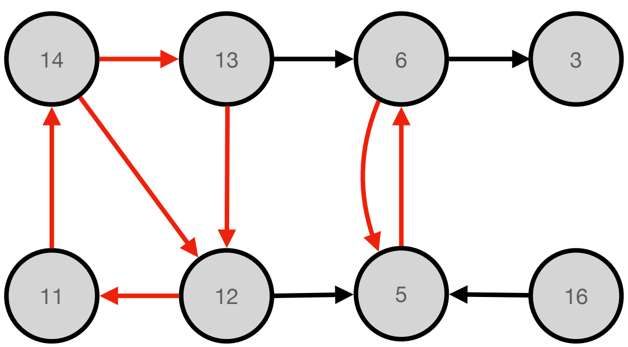


E=14
D=13
F=12
G=11
A=6
B=5
C=3
H=16

Example



Reverse graph with distance labels



Strongly Connected Components

First vertex in stack: 16 — No out neighbors, 16 pops, it is its own tree, so it is an isolated

Next vertex in stack: 14: adds 11: adds 12: adds 13; Pop 13, 12, 11, 14 as strongly connected component.

Next highest vertex into stack: 6: adds 5: Tries to add 16, but it has already been popped. Pop 5, 6: Strongly connected component.

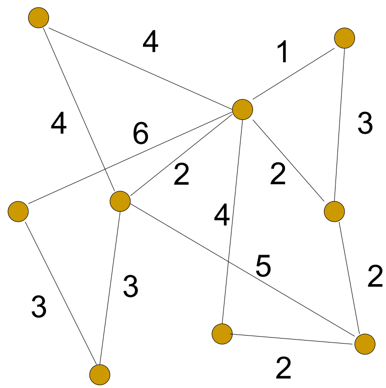
Add 3 to stack: tries to add 6, but it has popped already So 3 is its own connected component.

Strongly-Connected Components

- Correctness
  - If v and w are in a strongly-connected component
  - Then there is a path from v to w and a path from w to v
  - Therefore, there will also be a path between v and w in G and Gr
- Running time
  - Two executions of DFS
  - O(|E|+|V|)

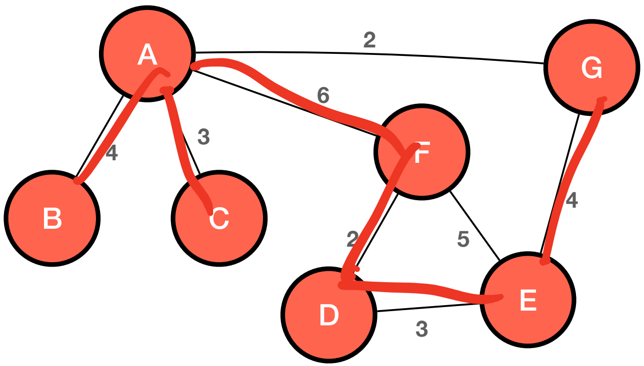
Weighted Graphs

- A weighted graph $G = (V, E)$ is a graph along with a weight function $w : E \rightarrow \Re$
- Weighted graphs can be directed or undirected



Spanning Trees

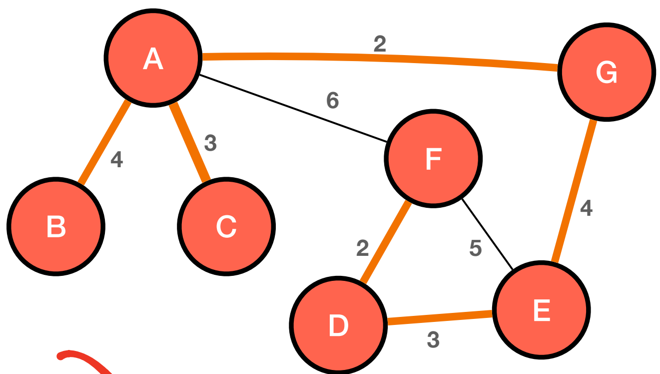
- A **spanning tree** of an undirected graph is
  - edge subset forming a tree that spans every vertex
  - #V - 1 edges



Minimum Spanning Trees

- A **minimum spanning tree** (MST) of an undirected weighted graph (V, E) with weights $w(\cdot)$:

  Connected subgraph (V, E') which is a tree and for which $\sum_{\{i,j\} \in E'} w(i,j)$ is minimized

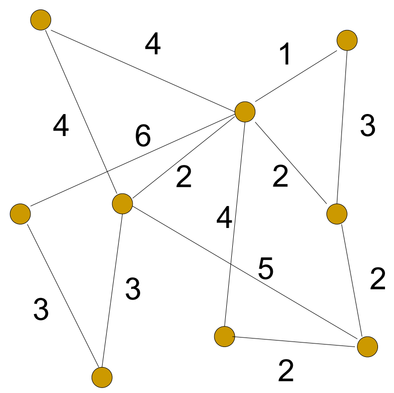


$+\log(n!)$

lok . . .

MST Problem

- Given a weighted graph G, we want a subgraph $G' = (V, E'),\ E' \subseteq E$, such that
  all vertices are connected on G'
  Subgraph $G' = (V, E'),\ E' \subseteq E$
  total weight $\sum_{(x,y)\in E'} w(x, y)$ is minimized

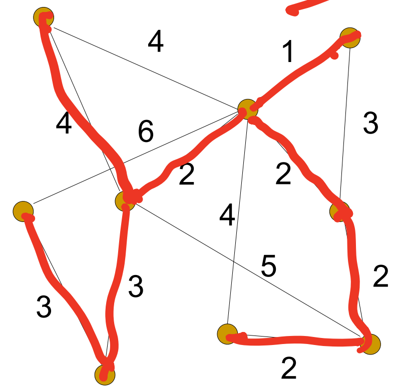


- Spanning tree: a tree containing all vertices in G
- Question: Find a spanning tree with minimum weight.
  - The problem is thus called Minimum Spanning Tree (MST)

MST: Problem and Motivation

MST!

- Suppose we have n computers, connected by wires as given in the graph
- Each wire has a renting cost
- We want to select some wires, such that all computers are connected (i.e. every two can communicate)
- Algorithmic question: How to select a subset of wires with the minimum renting cost?
- Answer to this graph?



#V - 1

Applications

- Networks
  - electric, computer, water, transportation
- Computer vision
  - Facial recognition
  - Handwriting recognition
  - Image segmentation
- Low-density parity check codes (LDPC)

Minimum Spanning Tree Algorithms

- Kruskal's algorithm (1956) ✓

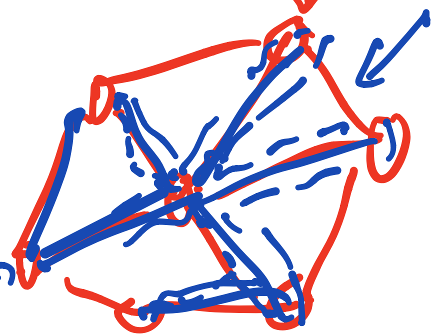
ON THE SHORTEST SPANNING SUBTREE OF A GRAPH AND THE TRAVELING SALESMAN PROBLEM

JOSEPH B. KRUSKAL, JR.

Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the *length* of the edge), and if these lengths are all distinct, then among the spanning[1] trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the "matrix of lengths" of the graph, that is, the matrix $\|a_{ij}\|$ where a_{ij} is the length of the edge connecting vertices i and j. Of course, it is assumed that $a_{ij}=a_{ji}$ and that $a_{ii}=0$ for all i and j.)

The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.

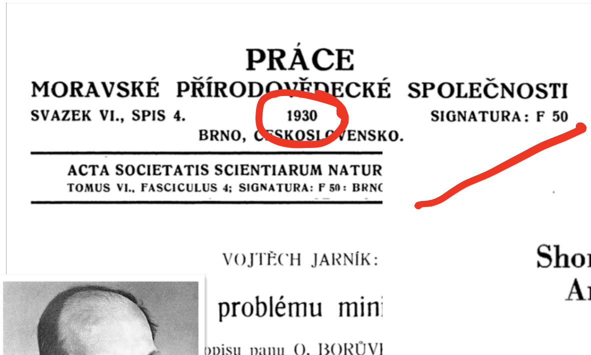
NP Complete



$\boxed{1.5}$ — ?

Minimum Spanning Tree Algorithms

- Prim-Jarnik Algorithm



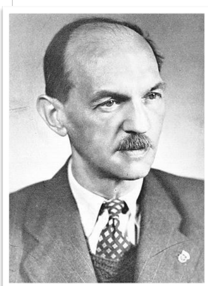
PRÁCE
MORAVSKÉ PŘÍRODOVĚDECKÉ SPOLEČNOSTI
SVAZEK VI., SPIS 4. 1930 SIGNATURA: F 50
BRNO, ČESKOSLOVENSKO.

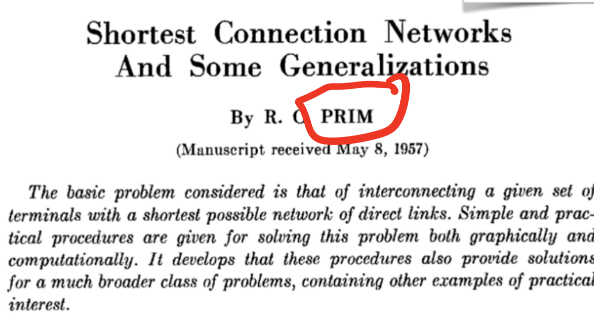
ACTA SOCIETATIS SCIENTIARUM NATUR
TOMUS VI., FASCICULUS 4; SIGNATURA: F 50: BRNO

VOJTĚCH JARNÍK:

problému mini

opisu panu O. BORŮVI





Shortest Connection Networks And Some Generalizations

By R. C. PRIM

(Manuscript received May 8, 1957)

The basic problem considered is that of interconnecting a given set of terminals with a shortest possible network of direct links. Simple and practical procedures are given for solving this problem both graphically and computationally. It develops that these procedures also provide solutions for a much broader class of problems, containing other examples of practical interest.

Preliminary ideas

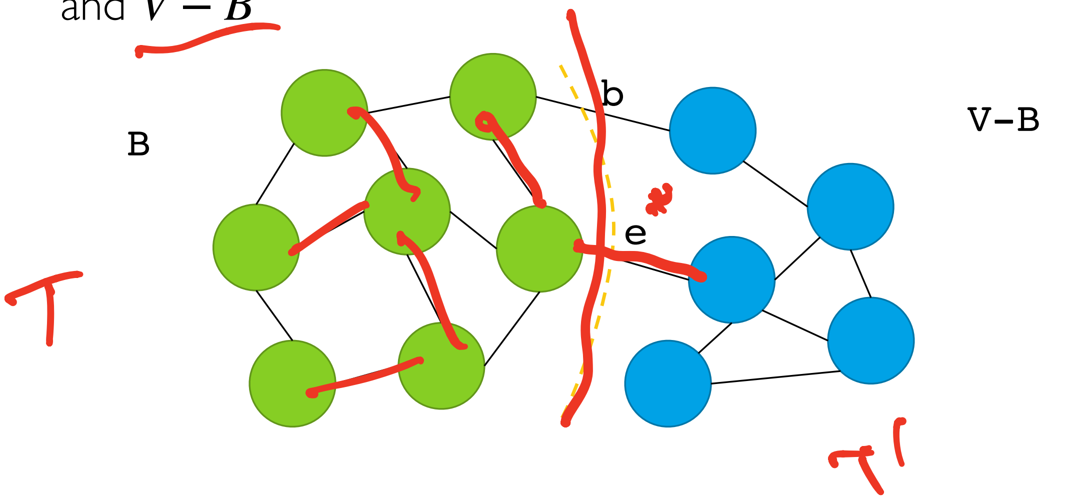
- Build minimal spanning trees incrementally
- Show by induction that algorithm is correct at each step

- Concept: A set of edges $T \subset E$ is **promising** if it is a subset of a ~~a~~ minimal spanning tree (V, E')
  - Let $B = \{v \in V : v \in e \text{ for some } e \in T\}$
  - Then, (B, T) is a subgraph of (V, E) which is a ~~tree~~ Forest

Greedy algo...

Graph Cuts

- A cut is any partition of the vertices into two groups, B and $V - B$



- with edges **b** and **e** joining the partitions

Useful Property

- **Lemma**: Let $T \subset E$, and $B \subset V, B \neq V$. Assume T is a promising set of edges for the MST problem in (V,E), and no edge in T leaves B (no edge across cut (B, V-B)). Let $e*$ be the smallest weight edge in E such that $e* = \{i, j\}, i \in B, j \notin B$. Then, $T' = T \cup \{e*\}$ is a promising set.

- Allows us to grow a promising set! (Induction)

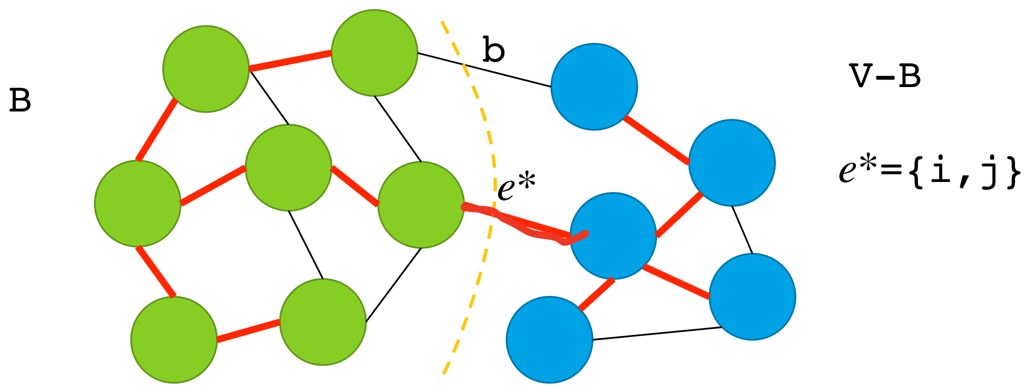
Proof.

(B, V-B) form a partition of V: a cut

Edge $e* = \{i, j\}, i \in B, j \notin B$ has the smallest weight among all edges that cross the cut.

Case 1: MST includes e^*

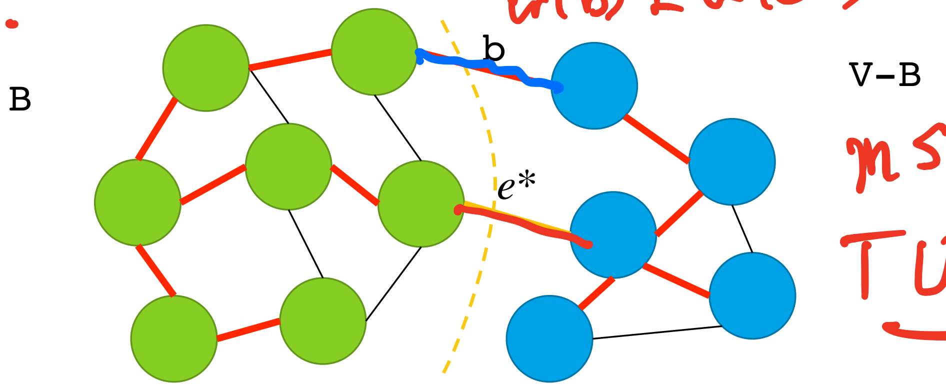
- Let MST (G, E') where $T \subset E'$ be such that $e^* \in E'$
- Simple: then $T \cup \{e^*\} \subset E'$ is promising



Case2: MST does not includes e

- Let MST (G, E') where $T \subset E'$ be such that $e^* \notin E'$
- Hard: Look at cycle including e. Swap b for e^* in E', total weight must not increase, hence have MST $(V, E' \cup \{e^*\} - \{b\})$

$w(b) \geq w(e^*)$



MST

$\underline{T \cup \{e^*\}}$

Proof of Correctness

- **Lemma**: Proof (cont)

Since T is promising, let MT = minimum spanning tree $(V, E'), T \subset E'$.

If $e^* \in E'$, then $T' = T \cup \{e^*\} \subset E'$ has no cycles, and is thus a promising set of edges, proving the theorem.

If $\{i, j\} \notin E'$, then $E' \cup \{e^*\}$ has one cycle that includes e^*, as it has #V edges and a tree must have at most #V-1 edges

In that cycle, there exists edge b that leaves B, hence $b \notin T$

Note $w(b) \geq w(e^*)$ by how we selected e^*

Note: $E' \cup \{e^*\} - \{b\}$ leaves graph connected, and has number of edges = #V-1, and its total weight is no greater than the weight of E'

Hence, (V, $E' \cup \{e^*\} - \{b\}$) is also MST, and $T' = T \cup \{e^*\} \subset E' \cup \{e^*\} - \{b\}$ so T' is a promising subset

Kruskal's Algorithm

- Sort edges by weight in ascending order
- Start with empty set T (note: it is promising)
- For each edge e in sorted list
  - If adding edge e to T does not create cycle in $(V, T \cup e)$
  - ...add it to ~~MST~~: $T = T \cup \{e\}$
  - Claim: T is now promising set with one more edge
- Stop when you have $\#V - 1$ edges in T

Proof of Correctness ✓

- At any stage in algorithm, T is a forest (no cycles can be created)
- At any stage in algorithm, T is a promising set .
  - True initially when $T = \varnothing$ ✓
  - If at a stage in algorithm, ~~vertex~~ edge e^* is the lowest weight edge remaining, and it does not form a cycle with T, then:
    - Let {a,b} = e^*; let B be vertices in T connected to a
    - Then, no edge in T leaves B and e^* is minimum weight edge among edges that leave B
    - By lemma, $T \cup \{e^*\}$ is promising set
- Thus, algorithm converges to MST .

Kruskal

$O(1)$? $O(\log^*(n))$

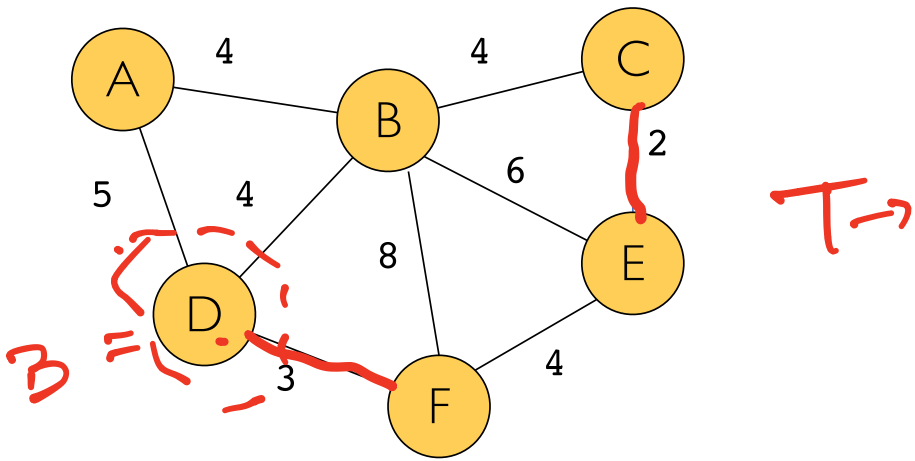
- How can we tell if adding edge will create cycle?
- Start by giving each vertex its own "cloud", which consists of all connected vertices in current T (Disjoint Sets) ✓
- If both ends of lowest-cost edge are in same cloud
  - we know that adding the edge will create a cycle!
- When edge is added to MST T
  - merge clouds of the endpoints

$\to \{a, b\} \to\ !$

Kruskal Pseudo-Code

```
function kruskal(G):
   // Input: undirected, weighted graph G
   // Output: list of edges in MST
   for vertices v in G:
      makeCloud(v) // put every vertex into it own set
   MST = []
   Sort all edges
   for all edges (u,v) in G sorted by weight:
      if u and v are not in same cloud:
         add (u,v) to MST
         if size(MST) = |V| - 1:
            break
         merge clouds containing u and v
   return MST
```

Example



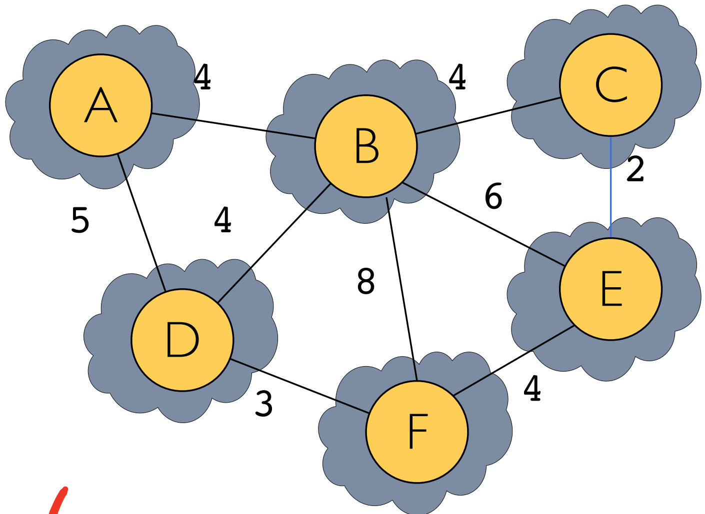
✓

```
edges = [(C,E),(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

2 3 4 4 4 4 5 6 8

A B C D E F

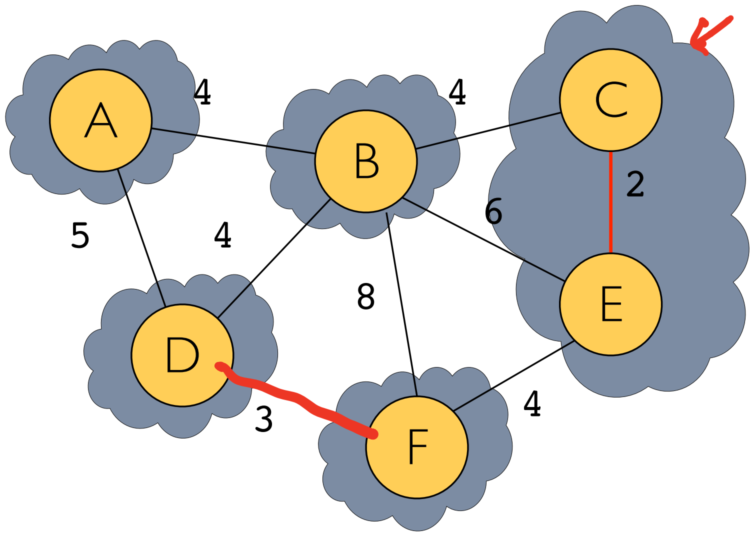
Example



✓

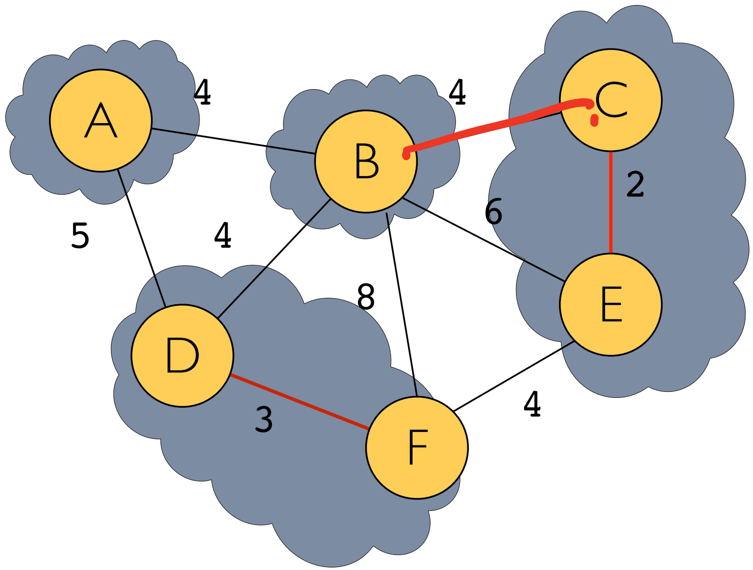
```
edges = [(C,E),(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



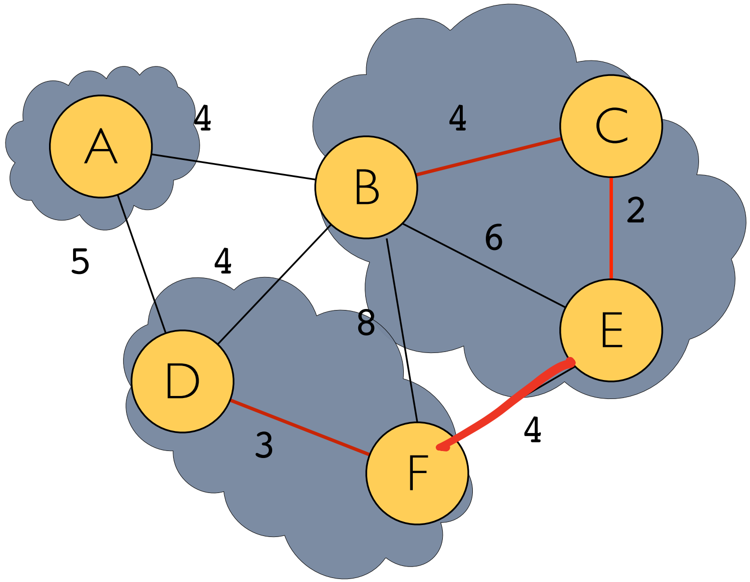

```
edges = [(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



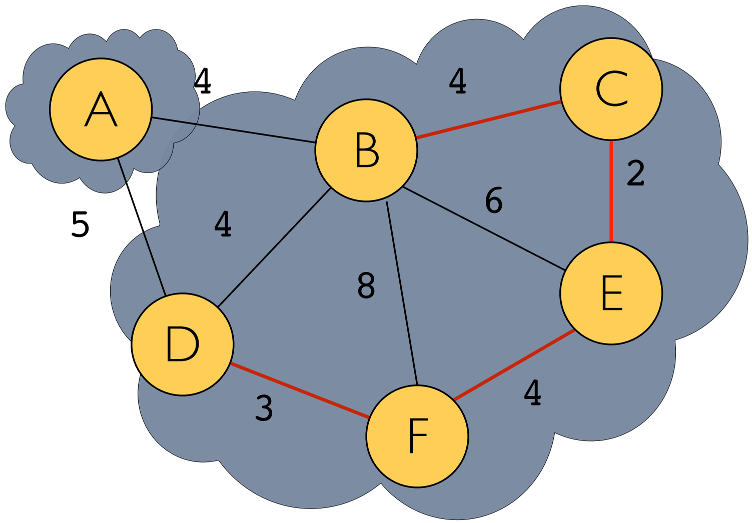

```
edges = [(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



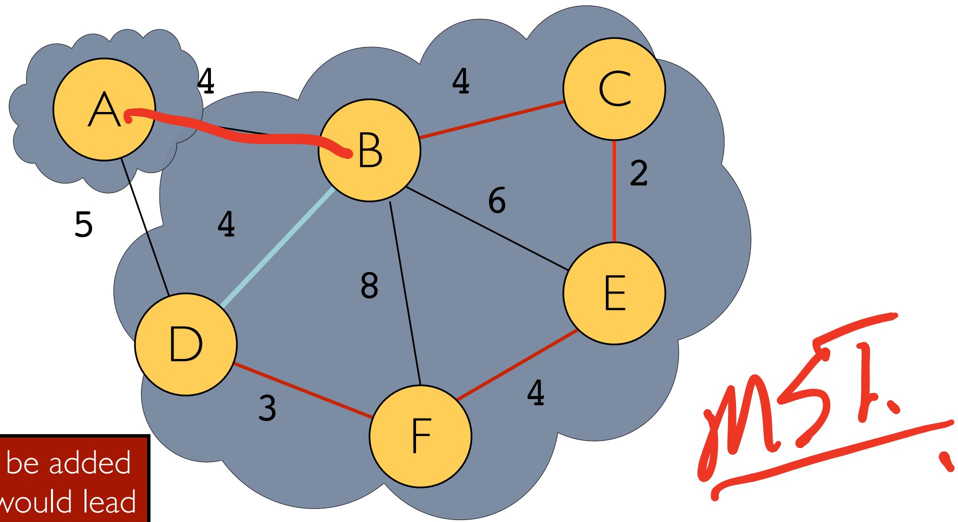

```
edges = [(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



edges = [~~(B,D)~~,(A,B),(A,D),(B,E),(B,F)]

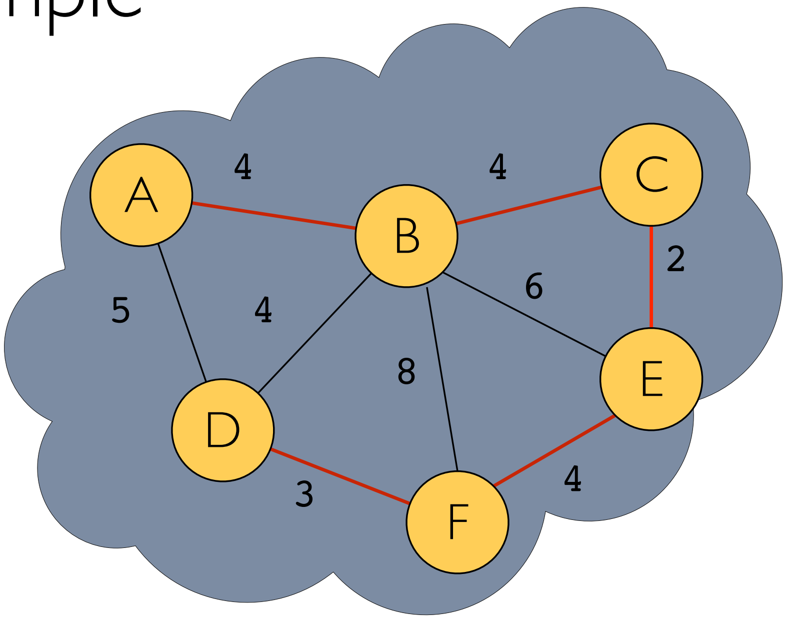
Example



BD cannot be added because it would lead to a cycle

```
edges = [(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(A,D),(B,E),(B,F)]
```

Merging Clouds Efficiently (Naive way)

- Keep track of clouds using disjoint sets (Union-Find)
  - Edge forms a cycle if both ends of the edge belong to the same disjoint set (have the same root!)
  - Checking that an edge forms a cycle has amortized complexity $O(\log^*(|V|)) \approx O(1)$

Kruskal Runtime

$T = \emptyset \rightarrow$ #V-1 ed..

- $O(|V|)$ for iterating through vertices
- $O(|E|\log|E|)$ for sorting edges
- $O(|E| \times 1)$ for iterating through edges and merging clouds with path compression
- $O(|V|+|E|\log|E|+|E|\times 1)$
  - $= O(|V|+|E|\log|E|)$
- $O(|V|+|E|\log|E|)$
  - Better than simple $O(|V|^3)$ without disjoint sets

Example

Disjoint components:

1 2 3 4 5 6 7 8 9

1) Add {3,6}

Disjoint components:

3 1 2 4 5 7 8 9

|

6

2) Add {4,6}

3 1 2 5 7 8 9

/ \

6 4

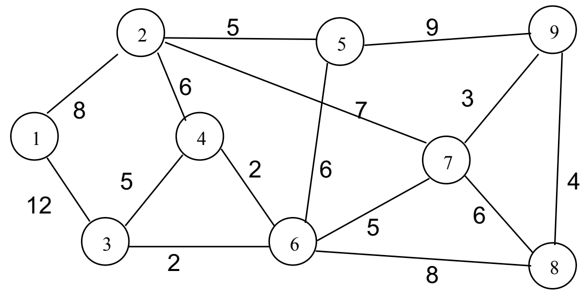
Minimum Spanning Tree

Empty

Minimum Spanning Tree

3--6

3--6--4



Sorted edges

{3,6}, {4,6},{7,9},{8,9},{3,4},{6,7}, {2,5},{2,4}{5,6},{7,8}, {2,7},{1,2}, {6,8},{5,9},{1,3}

Example

Disjoint components:

3) Add {7,9}

```
 3     7   1  2   5   8
/ \    |
6  4   9
```

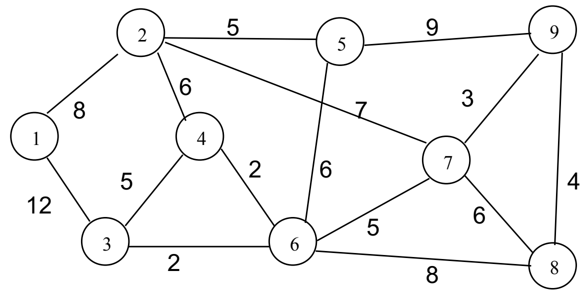
4) Add {8,9}

```
 3      7   1  2   5
/ \    /  \
6  4   9   8
```

Minimum Spanning Tree

3--6--4   7--9

3--6--4   7--9--8



Sorted edges

{3,6}, {4,6},{7,9},{8,9},{3,4},{6,7},
{2,5},{2,4}{5,6},{7,8}, {2,7},{1,2},
{6,8},{5,9},{1,3}

Example

Disjoint components:

5) Discard {3,4} (cycle); add {6,7}

```
   3       1  2   5
 /  | \
6  4   7
      /  \
    9    8
```

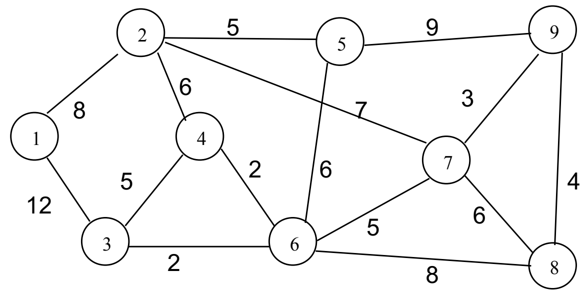
6) Add {2,5}

```
   3       1  2
 /  | \          \
6   4   7          5
      /  \
    9    8
```

Minimum Spanning Tree

```
3--6--4
      \
        7--9--8
```

```
3--6--4       2--5
     \
       7--9--8
```



Sorted edges

{3,6}, {4,6},{7,9},{8,9},{3,4},{6,7},
{2,5},{2,4}{5,6},{7,8}, {2,7},{1,2},
{6,8},{5,9},{1,3}

Example

Disjoint components:

7) Add {2,4}

```
  ( 3   )      1
 / | \   \
6  4  7  2
     / \  \
     9  8  5
```

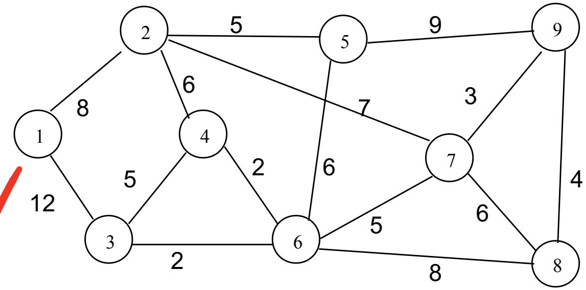
Minimum Spanning Tree

```
3--6--4--2--5
    \
     7--9--8
```

8) Discard {5,6},{7,8}, {2,7}. Select arc {1,2} . DONE

```
 (   3     )
  / | \   \  \
6  4  7  2  1
     / \  \
     9  8  5
```

```
3--6--4--2--5
    \
     \       1
      7--9--8
```



Sorted edges

{3,6}, {4,6},{7,9},{8,9},{3,4},{6,7},
{2,5},{2,4}{5,6},{7,8}, {2,7},{1,2},
{6,8},{5,9},{1,3}