# EC504 ALGORITHMS AND DATA STRUCTURES
# FALL 2020 MONDAY & WEDNESDAY
# 2:30 PM - 4:15 PM

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Prof: David Castañón, dac@bu.edu

GTF: Mert Toslali, toslali@bu.edu

Haoyang Wang: haoyangw@bu.edu

Christopher Liao: cliao25@bu.edu

# A Theory of Computation

- While we have introduced many problems with polynomial-time algorithms…
  - We haven't formally defined what this means
  - And not all problems enjoy fast computation
- Given a set I of problem instances, and a set S of problem solutions, an abstract problem is a binary relationship in I x S. That is, a set of pairs (i,s)
  - The problem shortest path associates each instance of a graph and an origin-destination with a sequence of vertices which connect the origin and destination
- D**ecision** problems have a yes or no solution. Abstract decision problem is a function which maps problem instances I into {yes, no}.
  - e.g. Is the shortest path in this graph between nodes 0 and nodes 30 have length > 10?

# P and NP

- A decision problem belongs to the class **P** if there is a algorithm solving the problem with a running time on  a deterministic machine that is polynomial in the input size.

- A decision problem belongs to the class **NP (**non-deterministic polynomial) if:
  - Any solution y leading to ‘yes' can be encoded in polynomial space with respect to the size of the input x.
  - Checking whether a given solution leads to `yes' can be done in polynomial time with respect to the size of x and y
  - problem is solvable in polynomial time in a non-deterministic machine
    - Can explore all possible solutions in parallel
    - Oracle selects best possible solution to check

<u>Slightly more formal:</u>

- Problem Q belongs to the class **NP**, if there exists a polynomial-time 2-argument algorithm A, such that:
  - For each instance i, i is a yes-instance to Q, if and only if, there is a polynomial-size certificate c for which A(i,c) = true.

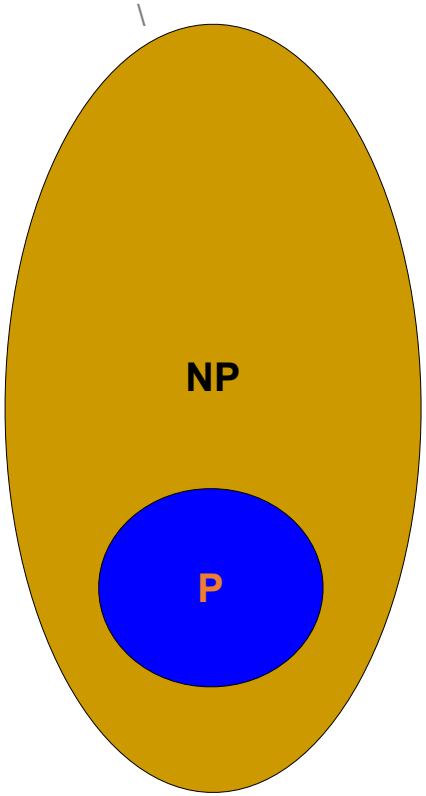
# Polynomially Reducible

- Definition: Problem A is polynomially reducible to problem B if there exists an algorithm for solving problem A in polynomial time if we could solve arbitrary instances of problem B at unit cost
  - Written as $A \leq_P B$
  - If $A \leq_P B$, and $B \leq_P A$, then we write $A =_P B$

- If $A \in \mathbf{P}$ and $B \leq_P A$, then $B \in \mathbf{P}$
  - e.g. Shortest path problem is in **P**. Assignment problem is polynomially reducible to shortest path problem. Then Assignment problem is in **P**.

# NP Complete

- We know class **P** is a subset of class **NP**
  - TSP is not known to be in class **P** but is in **NP**
  - It is a conjecture that **P** $\neq$ **NP**, but it has not been proven
  - We do know $\mathbf{P} \subset \mathbf{NP}$




- Definition: a problem A is NP-complete, if $A \in \mathbf{NP}$ and every problem $B \in \mathbf{NP}$ can be polynomially reduced to A. That is, $B \leq_P A$

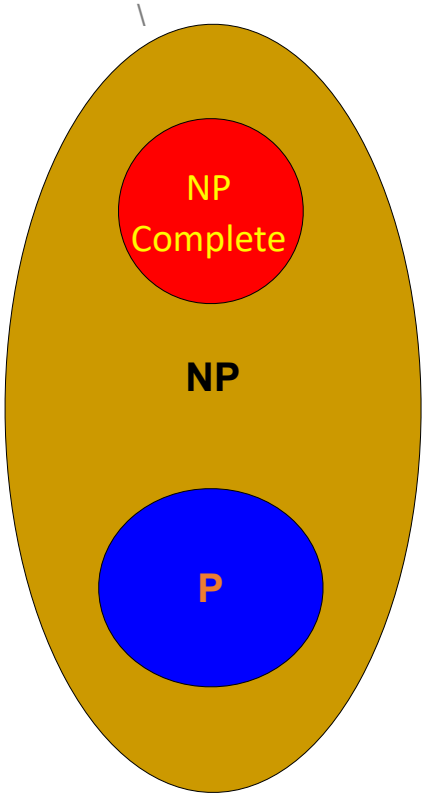
- Do such problems exist? If so, **and** one of them was in class **P**, then every problem in class **NP** could be solved in polynomial time

# Existence of NP-complete Problems

- Steven Cook and Leonid Levin (BU) proved in parallel that there exists an **NP** complete problem
  - Levin joined BU in 1980s
- Specifically: Boolean satisfiability (SAT)
  - Given a Boolean formula, is there an assignment of True and False to its variables so that the formula evaluates to True?
  - e.g. Can we find values of p, q, r, s so formula below is true?

$$(p \vee q \vee r) \wedge (p \vee \neg q \wedge s) \wedge (\neg p) \wedge (\neg q \wedge s) \wedge (\neg s)$$

- Cook-Levin Theorem
  - If a polynomial-time deterministic algorithm can solve this problem, then polynomial-time deterministic algorithms can solve all NP problems



# How Do We Find Other NP-Complete Problems

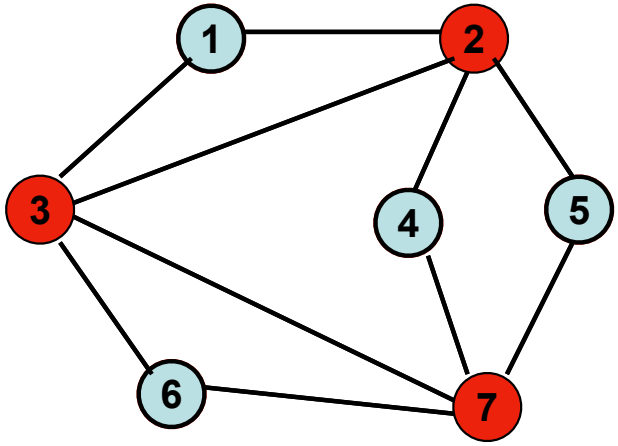
- Reduction!  We want to find if problem Y in **NP** is NP-complete
- If SAT $\leq_P$ Y, then Problem Y is NP-Complete!  SAT  is no easier than Y
- Note: to prove a problem A is in class **P**, we need to find a problem B in class **P** so that A $\leq_P$ B
- To prove a problem A is NP-complete class **P**, find an NP-complete problem so that B $\leq_P$ A
- Note the following: it is easy to show that SAT $=_P$ 3-SAT using standard logic transformations

# **NP**-complete problem 1: Clique

- Clique: Given a graph $G$ and a number $k$, decide whether $G$ has a clique of size $\geq k$.
  - Clique: a complete subgraph.
- Fact: Clique is in **NP**.
- Theorem: If one can solve Clique in polynomial time, then one can also solve $3$-SAT in polynomial time.
  - So Clique is at least as hard as $3$-SAT.
- Corollary: Clique is **NP**-complete.

# **NP**-complete problem 2: Vertex Cover

- Vertex Cover: Given a graph $G$ and a number $k$, decide whether $G$ has a vertex cover of size $\leq k$
  - $V'$ is a vertex cover if all edges in $G$ are "touched" by vertices from $V'$

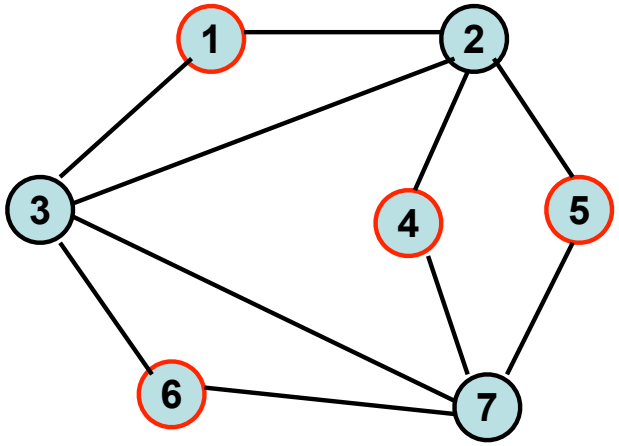



- Vertex Cover is in **NP**
  - Given a candidate subset $S \subseteq V$, it is easy to check whether "$|S| \leq k$ and $S$ touches all edges in $E$"

# Another NP-Complete Problem: Independent Set

- Independent Set: Decide whether a given graph has an independent set of size at least $k$

- The above argument shows that the Independent Set problem is also NP-Complete because Clique $\leq_P$ Independent set



# **NP**-complete problem 3: Integer Programming (IP)

- Any 3-SAT formula can be expressed by integer programming.
- Consider a conjunctive clause, for example,
  - $\bar{x}_1 \vee x_2 \vee x_3, \quad x_1, x_2, x_3 \in \{0,1\}$
  - This is equivalent to $(1 - x_1) + x_2 + x_3 \geq 1, \quad x_1, x_2, x_3 \in \{0,1\}$

- Multiple conjunctive clauses:
  - Translate to multiple integer programming constraints which must be satisfied simultaneously
  - Finding a feasible solution to the set of integer programming inequalities would yield a feasible solution to 3-SAT

# Integer Programming example

- 3-SAT formula

  $(\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee x_4 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_5) \wedge (x_3 \vee \bar{x}_4 \vee x_5)$

- Integer program inequalities

- $(1 - x_1) + x_2 + x_3 \geq 1,$

  $(1 - x_2) + x_4 + (1 - x_5) \geq 1,$

  $x_1 + x_3 + x_5 \geq 1,$

  $x_3 + (1 - x_4) + x_5 \geq 1,$

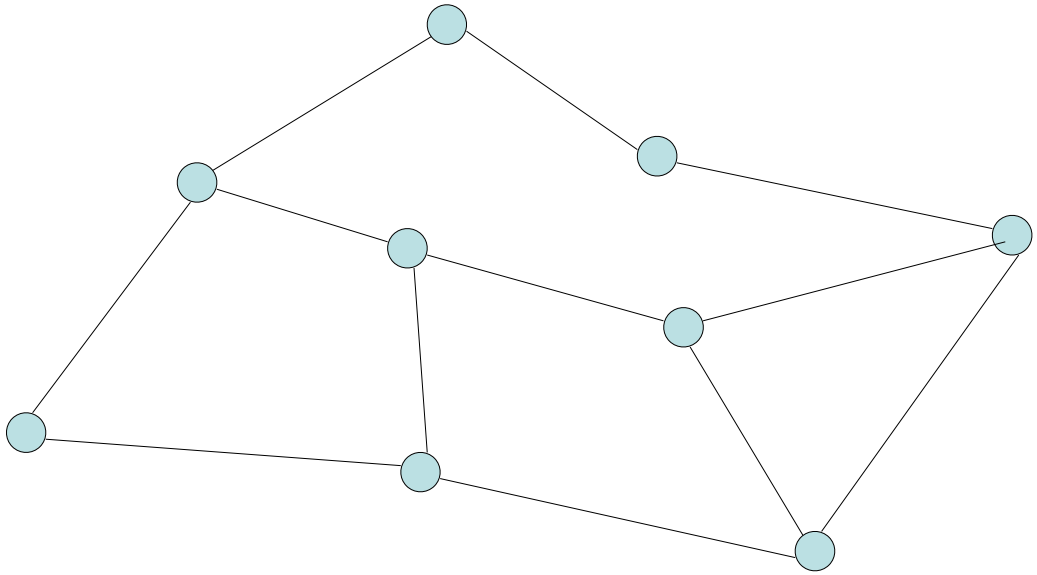
  $x_1, x_2, x_3, x_4, x_5 \in \{0,1\}$

- So if one can solve IP efficiently, then one can also solve 3-SAT efficiently

- 3-SAT $\leq_P$ Integer Programming

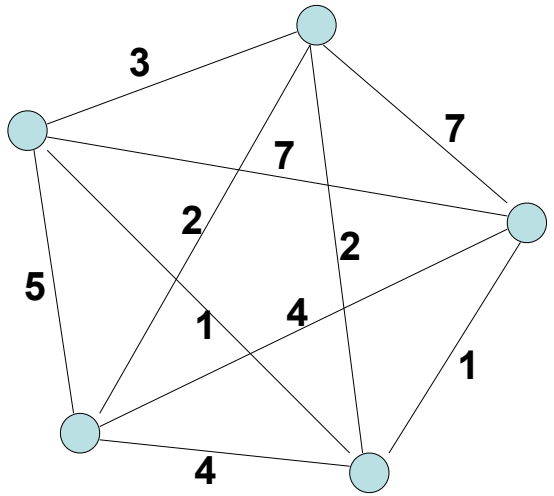
# Hamiltonian Circuit Problem

- Hamiltonian Circuit – does there exist a simple cycle including all the vertices of the graph?
  - Can show Vertex Cover $\leq_P$ Hamiltonian Circuit (very complicated construction)



# Traveling Salesman Problem

- Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)
  - Already proved Hamiltonian Circuit $\leq_P$ TSP




Find the minimum cost tour

# Subset Sums

The subset sum problem

- Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex. $\{ 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 \}$, $W = 1505$

Yes. $215 + 355 + 355 + 580 = 1505$.

- Subset sum is in NP
  - Easy to verify whether a subset sums up to W
- Important aspect: numbers are input in binary
  - Reduction must be polynomial in number of bits of input
  - Want to show 3-SAT $\leq_P$ Subset Sum

# Subset Sum via Dynamic Programming

- Break up into smaller problems, build up to full problem
- F(k,s) = True if there exists a subset of $w_1, \ldots, w_k$ that adds up to s, else false
- Initialization: F(k,0) = True, k = 0, …, n;  F(0, s) = False, s > 0
- Dynamic Programming recursion:

$$F(k,s) = \begin{cases} F(k-1,s) & \text{if } w_k > s \\ F(k-1,s) \vee F(k-1,s-w_k) & \text{if } w_k \leq s \end{cases}$$

- Can fill table to get solution: $O(n \times W)$
- But, this is not polynomial:  W is exponential in the size of the binary description of W in bits
  - Requires exponential space and time

# SAT Reduction to Subset Sum

- Given 3-SAT instance with n variables and k clauses, form 2n + 2k decimal integers, each having n + k digits as follows
  - For each variable $x_i$, construct numbers $t_i$ and $f_i$ of n + m digits: the i-th digit of $t_i$ and $f_i$ is equal to 1, other digits from 1 to n are 0
  - For n+1≤j≤n+m, the j-th digit of $t_i$ is equal to 1 if $x_i$ is in clause $c_{j-n}$

    For n+1≤j≤n+m,t he j-th digit of $f_i$ is equal to 1 if $\bar{x}_i$ is in clause $c_{j-n}$
  - All other digits of $t_i$ and $f_i$ are 0
  - Example: $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$

| | *i* | | | *j* | | | |
|---|---|---|---|---|---|---|---|
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 4 |
| $t_1$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $f_1$ | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| $t_2$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $f_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $t_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 |

# SAT Reduction to Subset Sum - 2

- Given 3-SAT instance with n variables and k clauses, form 2n + 2k decimal integers, each having n + k digits as follows
  - For each clause $c_j$, construct numbers $z_j$ and $y_j$ of n + m digits: The (n+j)-th digit of $z_j$ and $y_j$ is equal to 1, all other digits of $z_j$ and $y_j$ are 0
  - Example: $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$
  - Finally, construct a sum number s of n + m digits:
    For 1 ≤ j ≤ n, the j-th digit of s is equal to 1
    For n + 1 ≤ j ≤ n + m, the j-th digit of s is equal to 3
  - e.g. 1113333
  -

| | i | | | j | | | |
|---|---|---|---|---|---|---|---|
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 4 |
| $t_1$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $f_1$ | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| $t_2$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $f_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $t_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $z_1$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $y_1$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $z_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $y_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $z_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $y_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $z_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $y_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| sum | 1 | 1 | 1 | 3 | 3 | 3 | 3 |

# SAT Reduction to Subset Sum - 3

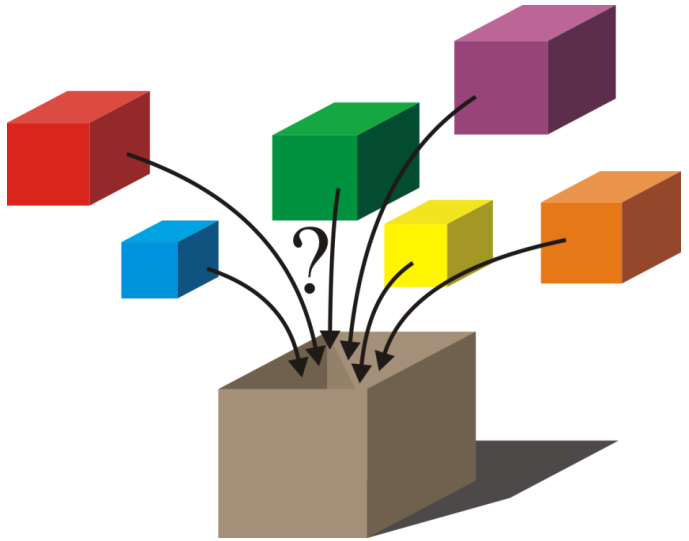
- If we find subset of rows that sum to the total sum, then we can convert the rows to Boolean variables $x_i, i = 1,\ldots,n$ such that every clause $c_j, j = 1,\ldots,m$ is true
  - $x_i$ true if $t_i$ is in subset; otherwise, $f_i$ must be in subset so $x_i$ is false
  - Since the j columns sum to 3, at least one of the literals in $c_j$ must be true because there are at most 2 in $z_j, y_j$
- If there exists a true assignment, then there is a set of rows that sums to sum
  - For clause $c_j$, add rows $y_j$ if two or less literals are true, add $z_j$ if only one literal is true
- Result: 3-SAT $\leq_P$ Subset Sum

| | i | | | j | | | |
|---|---|---|---|---|---|---|---|
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 4 |
| $t_1$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $f_1$ | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| $t_2$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $f_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $t_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $z_1$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $y_1$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $z_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $y_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $z_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $y_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $z_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $y_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| sum | 1 | 1 | 1 | 3 | 3 | 3 | 3 |

# Knapsack problem

The knapsack problem

- Suppose a container can hold a maximum weight $W$
- Given a number of items with specified weights $w_i$ and values $v_i$, is there some combination of items that has a total value greater than or equal to $V$ without exceeding the maximum weight $W$?
-



# Knapsack Decision Problem is NP-Complete

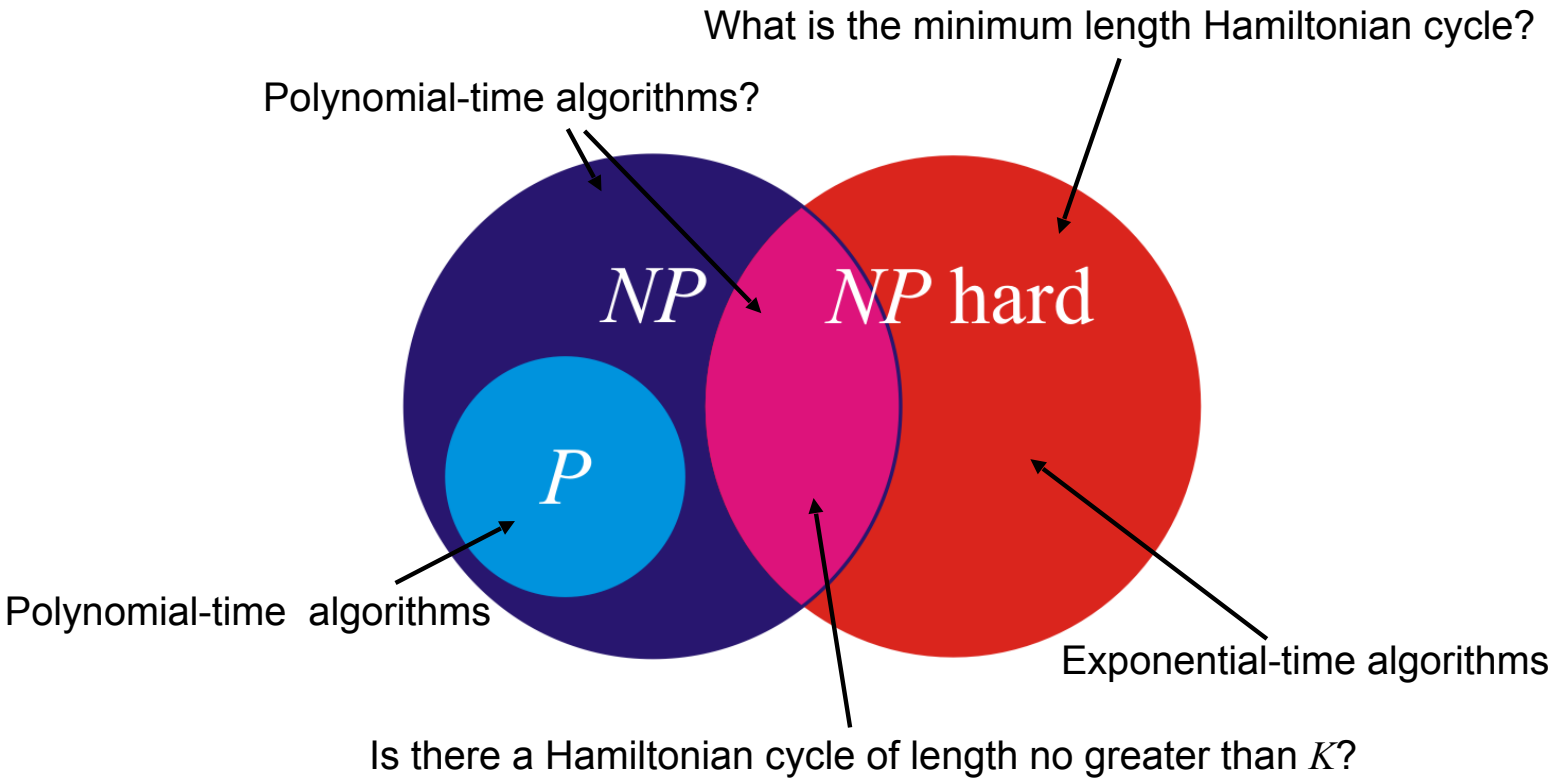
- Show that Subset Sum $\leq_P$ Knapsack
- Given subset sum problem instance with $w_1, \ldots, w_n$ and total weight W, form following Knapsack problem:
  - Let weight of object i be $w_i$
  - Let value $v_i$ of object i be $v_i = w_i$
- Solve the following knapsack decision problem: Is there a subset of items S such that the value is greater than or equal to W, while the total weight is less than or equal to W?
  - $\sum_{i \in S} v_i = \sum_{i \in S} w_i \geq W; \quad \sum_{i \in S} w_i \leq W;$
  - If answer is yes, then S is a subset with sum W. If no solution can be found for knapsack, then subset sum does not have a subset with sum W

# NP-Hard vs NP-Complete Problems

- The optimization version of NP-Complete problem is referred to as an **NP-Hard** problem
  - NP-complete problems are decision problems
  - E.g. finding the minimum length Hamiltonian circuit is NP-hard; finding whether there exists a Hamiltonian circuit with length less than L is NP-complete
  - Finding the maximum value that fits in a knapsack is NP-hard; finding whether we can fit value of at least V is NP-Complete
- Formally, a problem X is NP-hard, if there is an NP-complete problem Y, such that Y is reducible to X in polynomial time
  - But X does not have to be in NP, and does not have to be a decision problem

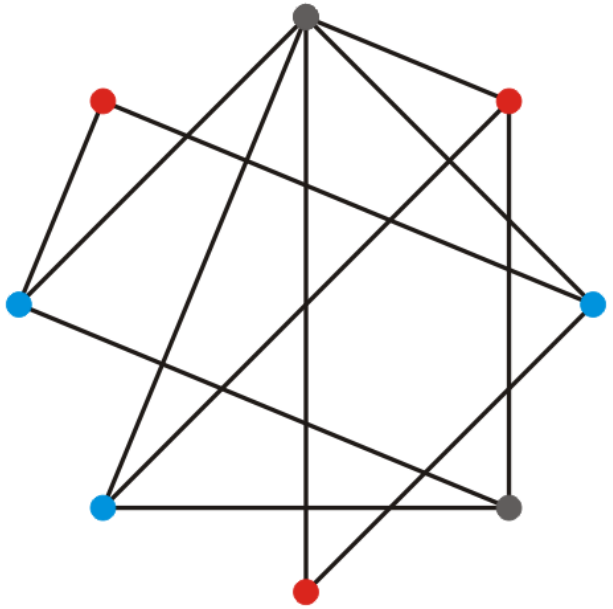
# $NP$ and $NP$ Hard

Consider this Venn diagram where $P \subseteq NP$



# NP-Hard: Graph colorings

- Given a graph and $n$ colors, is it possible to assign one of the colors to each of the vertices so that no adjacent vertices have the same color?



- Complete graphs requires $|V|$ colors
- Finding the smallest number of colors for a graph is $NP$ Hard

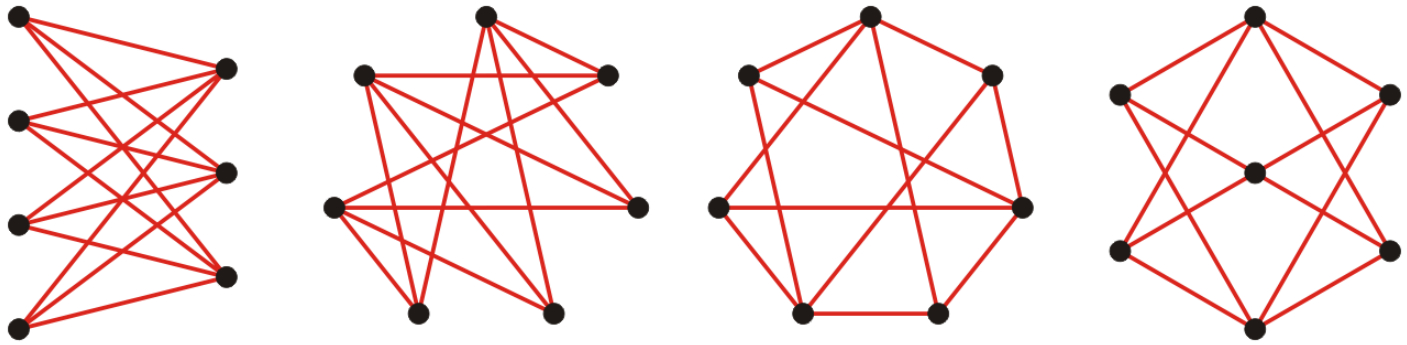
# NP-Complete: Graph isomorphisms

Recall the sub-graph isomorphism problem

- Given two graphs, is one graph isomorphic to a sub-graph of the other?

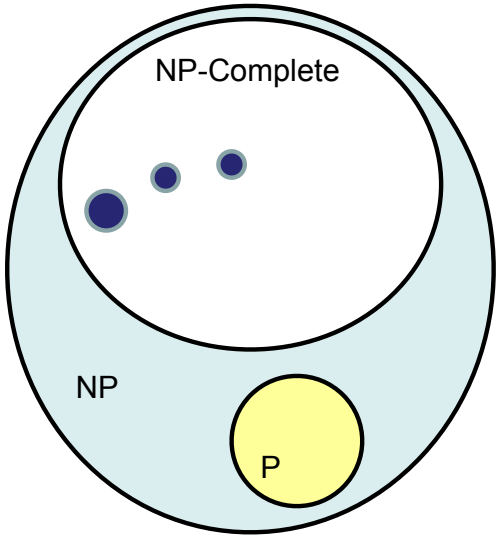
A possibly easier question is:

- Are two graphs isomorphic?



- So far, the best algorithm is $2^{O(\sqrt{n \ln(n)})}$

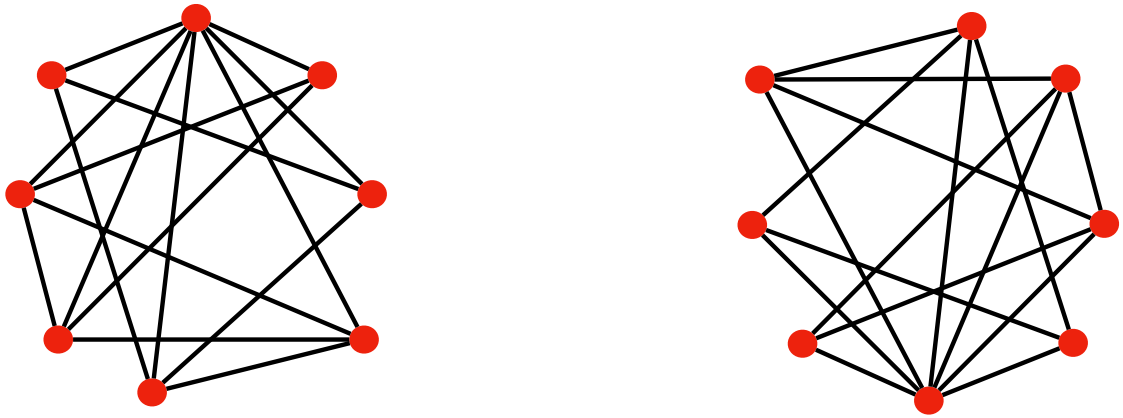
# Populating the NP-Completeness Universe




- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

# Problems in NP not in P or NP-Complete

- Graph Isomorphism
  - We don't have an algorithm to show problem is in **P**, and no reduction of an np-complete problem to it



# Problems in NP not in P or NP-Complete - 2

- These problems are often referred to as **NP-intermediate**
  - Circuit Minimization is the problem of obtaining the smallest logic circuit that represents a given Boolean function or truth table
  - Turnpike problem: Given $\frac{n(n-1)}{2}$ distances, find n points on a line
  - Cutting-stock problem: cutting standard-sized pieces of stock material, such as paper rolls or sheet metal, into pieces of specified sizes while minimizing material wasted
  - Prime factorization: decomposition of a composite number into a product of prime factors
  - Numbers in boxes: If you put the numbers 1,2,...,n into boxes, and If x,y,z are in a box then $x + y \neq z$. What is the smallest number of boxes you will need?

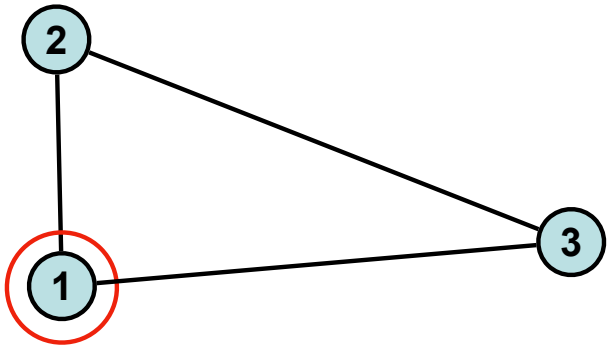
# Exercise

- *Dominating Set* problem: Given a graph $G = (V, E)$ and an integer $K$, decide whether $G$ contains a dominating set of size at most $K$.
  - Dominating set: $S \subseteq V$ such that $\forall v \in V$, either $v \in S$ or $v$ has a neighbor in $S$.
  - Namely, $S$ and $S$'s neighbors cover the entire $V$.
- Prove that Dominating Set is NP-complete.
- Hint: Reduction from Vertex Cover.

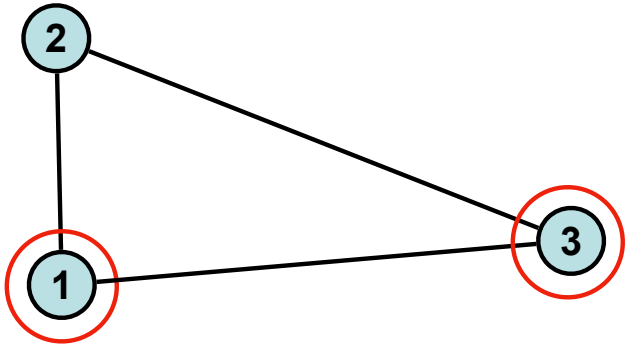
# Vertex Cover vs Dominating Set

- Every vertex cover is a dominating set (if graph is connected)
- But, size of minimal vertex cover is different from size of minimal dominating set! Minimal dominating set can be smaller…

Dominating Set

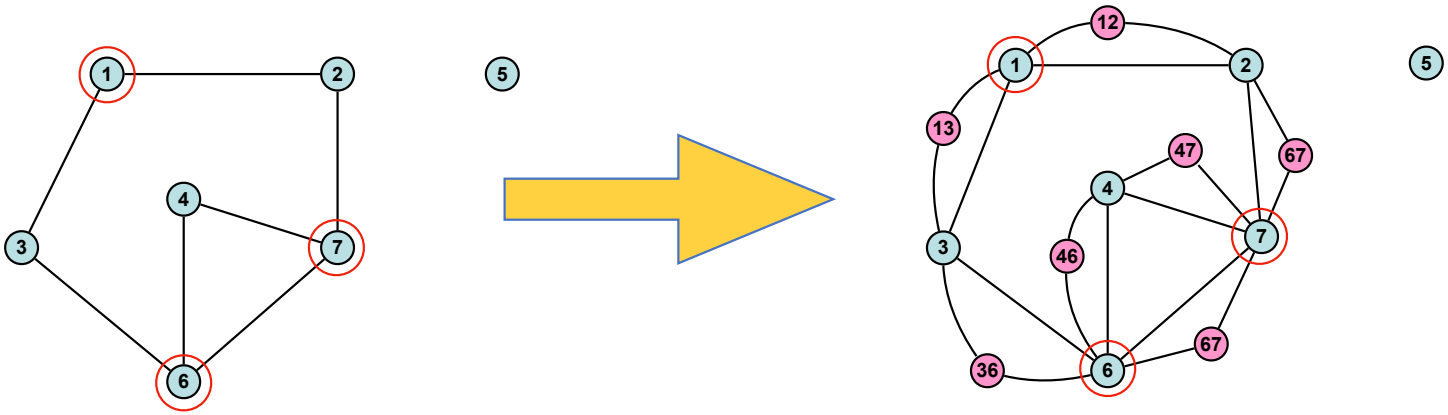



Vertex Cover.



# Reduction

- Add an extra vertex  for every edge
  - Minimum dominating set = k $\iff$ minimum vertex cover is $k - n_I$, where $n_I$ is number of isolated vertices



# Further Concepts in Complexity

- **Pseudopolynomial** complexity: If K is the size of the largest number, and n is the size of the input, then the worst case complexity is polynomial in n and K. (e.g. Integer knapsack)
- **Strongly Polynomial** complexity: worst case complexity is polynomial in input size, independent of largest value of number in input.
- **Strongly NP-complete problems**: If one restricts the size of the largest number in the problem to K, where K is a polynomial in the input size n, then the problem is still NP-complete.
  - e.g. Clique
- **PSPACE**: solved using an amount of storage which is polynomial in the size of the problem.
  - It is clear that P is in PSPACE; more importantly, NP is also in PSPACE
  - PSPACE-complete: Super Mario Brothers, Reversi, ….

# What to do if a problem is NP-complete?

- Solve on special cases where polynomial algorithms exists
  - e.g., TSP in one dimension, independent set on tree graphs, …
- Heuristics and approximations
  - Would like some guarantee that the performance is good
  - Algorithms that are fast on average
- Good exponential-time algorithms
  - Exact exponential-time algorithms is one of many options
  - Should be better than trivial algorithm: enumeration of all possibilities
    - e.g. TSP on dense graphs: (n-1)! possible tours, but can solve in much less time

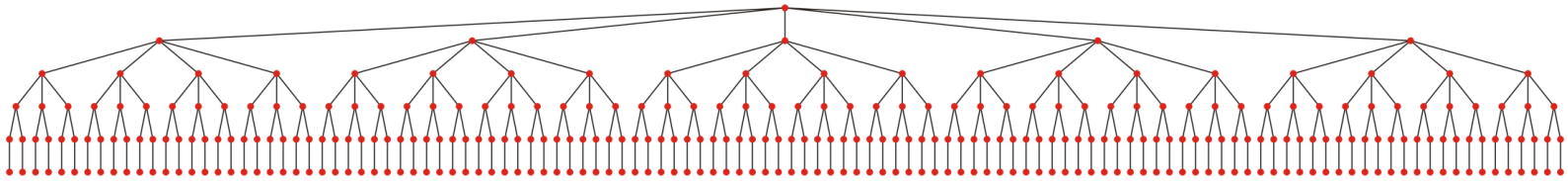
# Deterministic polynomial-time algorithms

Consider the traveling salesman problem:

In a complete weighed graph, what is the least weight Hamiltonian cycle?

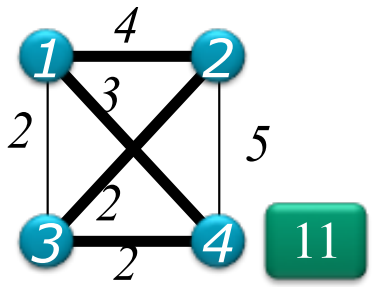
A deterministic algorithm to find a solution is to do tree traversal

- This tries every path
- The run time is $\Theta(|V|!)$
- The Held-Karp or Bellman's algorithm runs in $\Theta(|V|^2\ 2^{|V|})$ time
  - It uses dynamic programming



# Held-Karp Algorithm for TSP

- Take one starting vertex s arbitrarily.
  - For set of vertices $S \subset V - \{s\}$, and vertex $v \in S$, let $B(S, v)$ = minimum length of a path, that starts in s, visits all vertices in S (and no other vertices) and ends in v
- Dynamic Programming Recursion
  - B({v},v) = d(s,v)  (Initialization)
  - For j = 2 to |V|-1:
    - For all sets S, |S| = j, for all $v \in S$:
      - $B(S, v) = \min_{w \in S} B(S - \{v\}, w) + d(v, w)$
  - Return $\min_{v \in V, v \neq s} [B(V - \{s\}, v) + d(v, s)]$

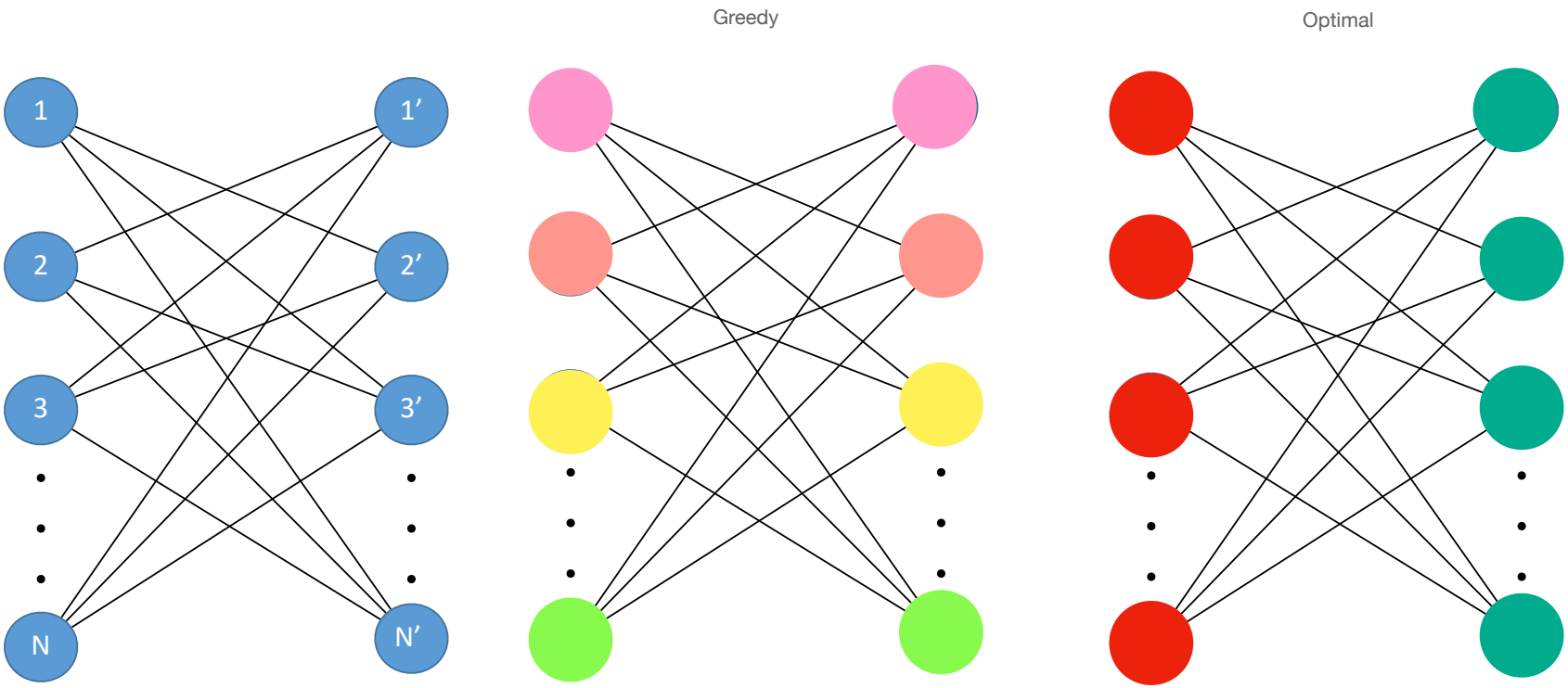



- Complexity: $O(n^2 2^n)$.  Must compute $B(S, v)$ for all subsets $S \subset V, v \in V$
  - $O(n2^n)$ values, each of which takes $O(n)$ to compute: Much better than n!

# Approximate Algorithms

- Objective: Find approximate algorithms for NP-hard problems with performance guarantees
  - Solution of approximate algorithm is within a factor of optimal solution
- Example: Graph coloring: Find the minimum number of colors for nodes so that no two neighbors have the same color
  - Greedy heuristic: Choose a color and an arbitrary starting node. Consider all other nodes: if a node can be painted with this color, do so; otherwise leave nodes unpainted.
  - Now, choose a second color. Loop over all unpainted nodes and paint them with this color if possible.
  - Repeat until done.
- Can show: There is an ordering of nodes for which this is the optimal thing to do: e.g. a line graph
- Can also show: There are graphs which can be colored with just k colors, for which the heuristic will use c colors, and c/k can be made as large as possible
  - No performance guarantee!
  - Variations of the heuristic can obtain performance guarantees close to O(n) (Not good!)

# Bad example for heuristic



# Guaranteed Performance: Integer Knapsack

- Greedy: Choose j in descending order of $V_j/w_j$ , until you exceed capacity.

  Let j be last item chosen that exceeds capacity. Return arg $\max[\sum_{k=1}^{j-1} V_k, V_j]$

- Theorem: Value of Greedy algorithm is no worse than 1/2 value of optimal algorithm
  - P = value of sequential greedy knapsack before overflow
  - $V_j$ = cost of item that overflows
  - P + $V_j$ is upper bound on optimal Knapsack. (by fractional argument).
- Bad example for Greedy:
  - 2 tasks. Task 1: Cap $\varepsilon$, value 2 $\varepsilon$; Task2: Cap C, value C

# Guaranteed Performance: Vertex Cover

- Greedy Algorithm: S $\leftarrow \varnothing$; E′ $\leftarrow$ E; $M \leftarrow \varnothing$
  - While (E′ $\neq \varnothing$)
    Let {u, v} $\in$ E′ be an arbitrary edge
    Add {u,v} to M: $M = M \cup \{\{u, v\}\}$. Note that M matches pairs of vertices in 1-1 manner
    Update vertex cover $S = S \cup \{u\} \cup \{v\}$
    Delete from E′ all edges incident to either u or v
  - Return S
- Let S* be a minimum vertex cover. Then, greedy algorithm computes a vertex cover S with $|S| \leq 2|S^*|$. It is a 2-approximation algorithm

  Pf.
  - S is a vertex cover
  - M is a matching, where no pair of edges share a vertex. Hence

    $|S^*| \geq |M|$, and $|S| = 2|M|$, so $|S| \leq 2|S^*|$

# Guaranteed Performance: Vertex Cover

- Theorem. [Dinur–Safra 2004] If P ≠ NP, then no ρ-approximation for VERTEX-COVER for any $\rho < 1.3606$.

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# Knapsack: Polynomial Time Approximation Scheme

- Knapsack: tasks $i = 1, \ldots, n$ with values $v_i$, weights $w_i$, total capacity W
- Pseudopolynomial complexity: O(nW)
  - Using Dynamic Programming with iteration in capacity and number of tasks consideredKnow pseudo-polynomial in C using dynamic programming
- Assume $w_i \leq W,\ i = 1, \ldots, n$
- A different dynamic programming algorithm (in Value)
  - OPT(i, v) = min weight of a knapsack for which we can obtain a solution of value ≥ v using a subset of items 1,..., I
  - Optimal value is the largest value v such that OPT(n, v) ≤ W
  - DP computes optimal value in $O(n^2 \max v_i)$ because max value is $O(n \max v_i)$

$$OPT(i, v) = \begin{cases} 0 & v \leq 0; \\ \infty & i = 0,\ v > 0; \\ \min(OPT(i-1, v), w_i + OPT(i-1, v - v_i)) & \text{otherwise} \end{cases}$$

# Knapsack: PTAS

- Knapsack Approximation: round up values
  - $\varepsilon \in (0,1]$ is precision parameter
  - Scaling parameter $\theta = \varepsilon v_{max}/(2n)$
  - Rounded values $\bar{v}_i = \lceil \frac{v_i}{\theta} \rceil \theta$
  - Compressed values $\hat{v}_i = \lceil \frac{v_i}{\theta} \rceil$
  - Optimal solutions are the same for rounded and compressed values
  - But compressed values have much smaller range!

| Item | Value | Comp. | Rounded | Weight |
|---|---|---|---|---|
| 1 | 934221 | 2 | 1000000 | 2 |
| 2 | 595634 | 2 | 1000000 | 3 |
| 3 | 1781001 | 4 | 2000000 | 1 |
| 4 | 2345741 | 5 | 2500000 | 3 |
| 5 | 5000000 | 10 | 5000000 | 4 |

$\varepsilon = 1,\ \theta = 500000$

# Knapsack: PTAS

- Theorem: If $S$ is solution to compressed problem, and $S^*$ is any other feasible solution, then $(1+\varepsilon)\sum_{i\in S} v_i \geq \sum_{i\in S^*} v_i$
  - Proof:
    $$\sum_{i\in S^*} v_i \leq \sum_{i\in S^*} \bar{v}_i \leq \sum_{i\in S} \bar{v}_i \leq \sum_{i\in S} (v_i+\theta) \leq \sum_{i\in S} v_i + n\theta$$
  - If $S^* = \{i_{v_{max}}\}$ then $n\theta = \varepsilon v_{max}/2$;
    $$\sum_{i\in S^*} v_i = v_{max} \leq \sum_{i\in S} v_i + \varepsilon v_max/2 \leq \sum_{i\in S} v_i + v_max/2 \Rightarrow v_{max} \leq 2\sum_{i\in S} v_i$$
  - Thus, $n\theta \leq \varepsilon \sum_{i\in S^*} v_i$ which shows $(1+\varepsilon)\sum_{i\in S} v_i \geq \sum_{i\in S^*} v_i$

# Knapsack: PTAS

- Theorem: for any $\varepsilon > 0$, the rounding algorithm computes a solution with value within a $(1+\varepsilon)$ of the optimal solution in $O(n^3/\varepsilon)$ time
  - Proof:
  - Running time is $O(n^2 \hat{v}_{max})$, and $\hat{v}_{max} = \lceil \frac{v_{max}}{\theta} \rceil = \lceil \frac{2n}{\varepsilon} \rceil$

# TSP Approximations

- Dynamic programming (Held-Karp, Bellman): $O(n^2 2^n)$ exact algorithm
- Restrict to Metric (or Euclidean) TSP: assume distances $d(i, j)$ satisfy the following
  - Symmetry: $d(i, j) = d(j, i)$
  - Triangle inequality: $d(i, j) \leq d(i, k) + d(k, j)$ for all i, j, k
- Approximate algorithm
  - Find a minimum spanning tree G = (V, T)
  - Add a duplicate edge to each edge in the minimum spanning tree T, with same distance, resulting in augmented edges T'
  - Find an Euler tour in G' = (V, T'). Note one exists because degree of each vertex is even.
  - List the vertices in order visited by Euler tour; skip vertices that appear more than once

# TSP Approximations

- Theorem: Approximate algorithm tour has distance no longer than twice the distance D* of the optimal TSP tour
  - Proof: Let D = weight of minimum spanning tree, D' the weight of the approximate tour

    Then, $D \leq D^*$, as the minimum spanning tree reaches every vertex, and has less constraints than the TSP tour

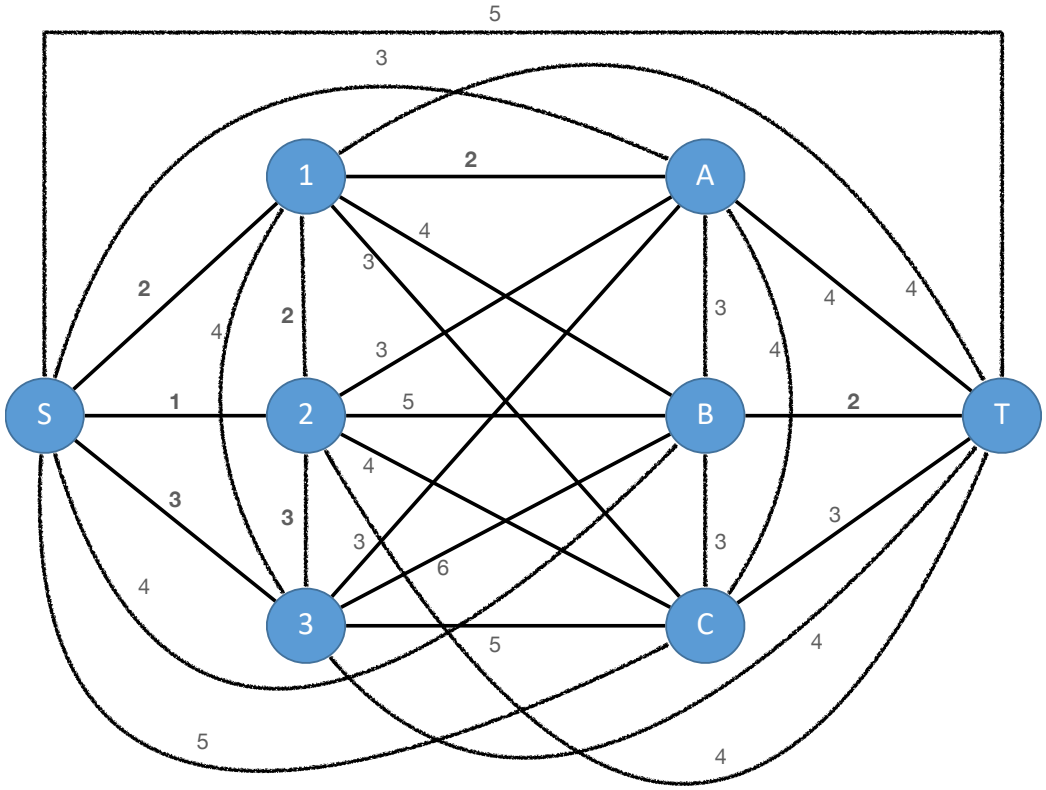
    Also, note the length of the Euler tour is 2D, as it travels every edge of the MST twice

    From the metric property, cutting the corners by deleting the duplicate vertices cannot increase the length,

    Hence $D \leq D^* \leq D' \leq 2D$

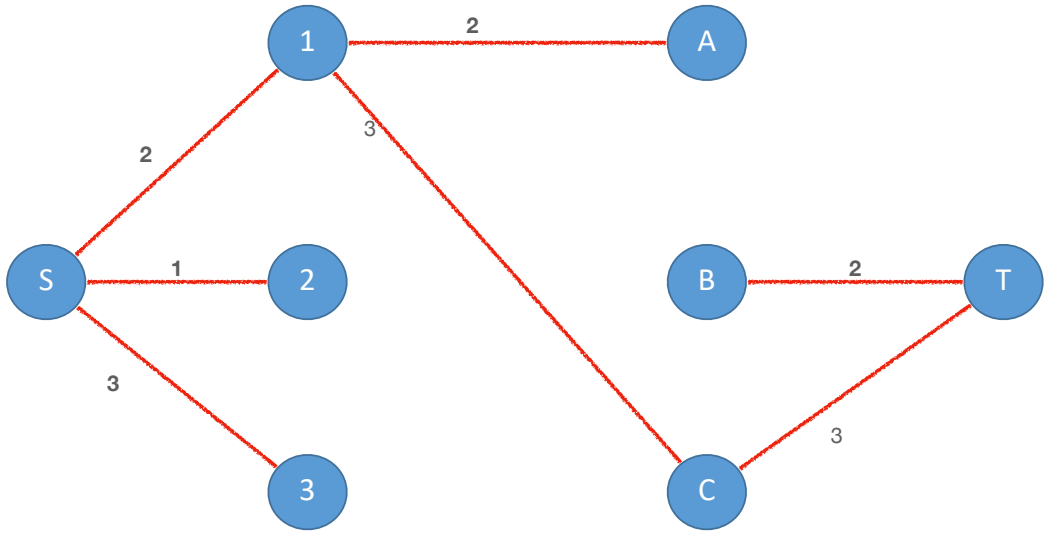
# Example

- Find MST: Weight 16



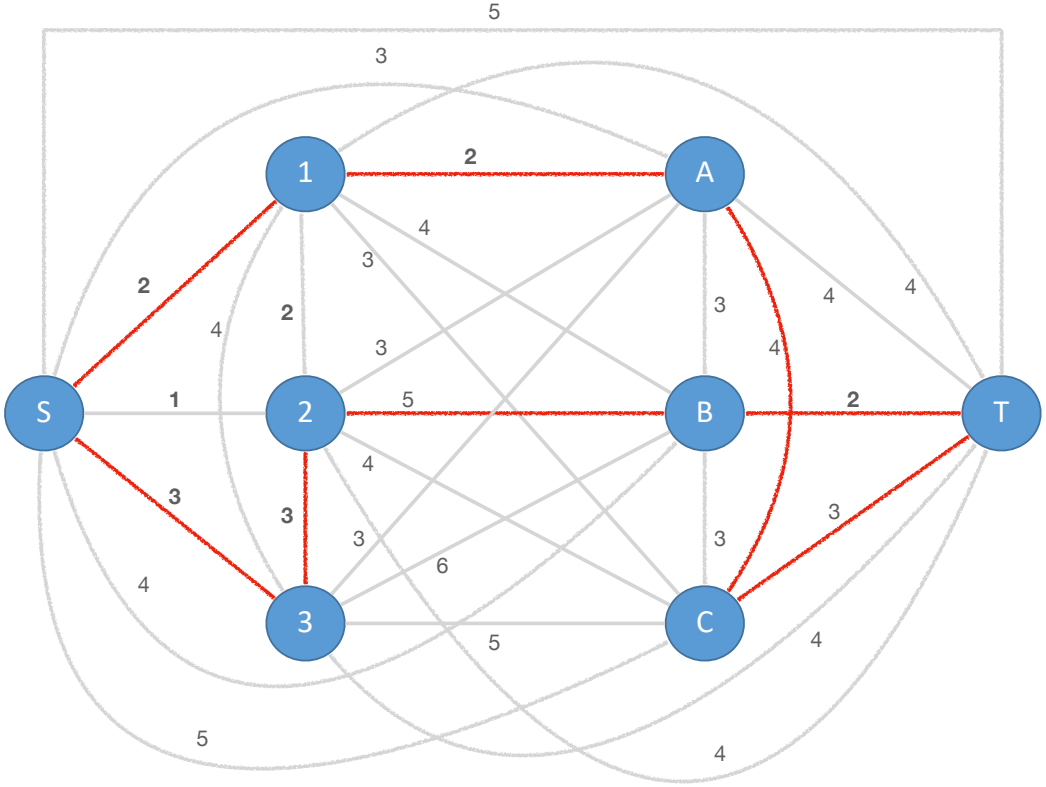
# Example - 2

- Double edges



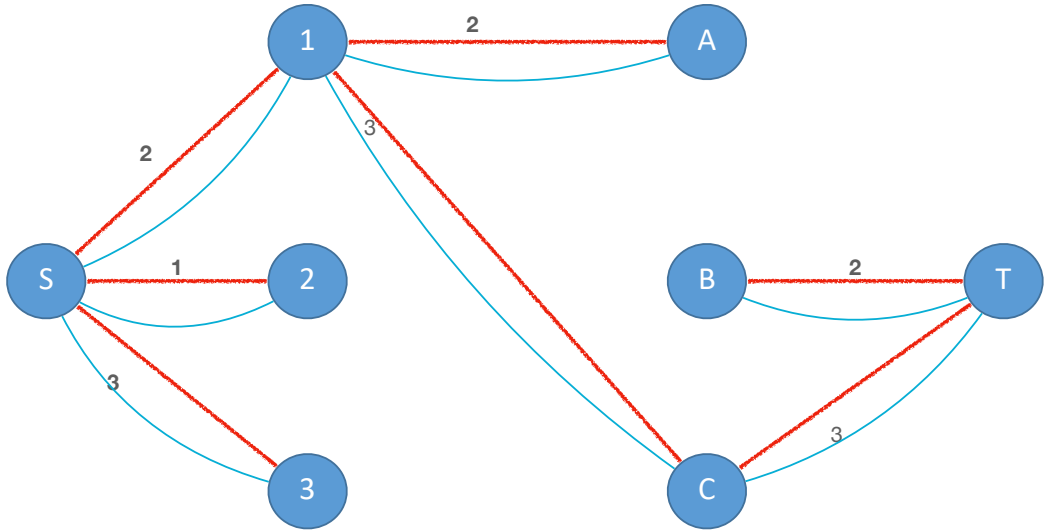
# Example - 3

- Euler tour length: 32
- Cut corners: S-1-A-C-T-B-2-3-S. Length 24



# Example - 3

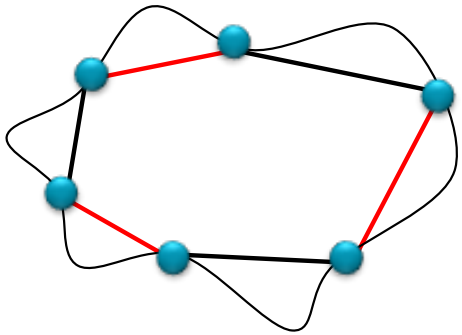
- Find Euler Tour: S-1-A-1-C-T-B-T-C-1-S-2-S-3-S. Length 32




- Cut corners: S-1-A-C-T-B-2-3-S. Length

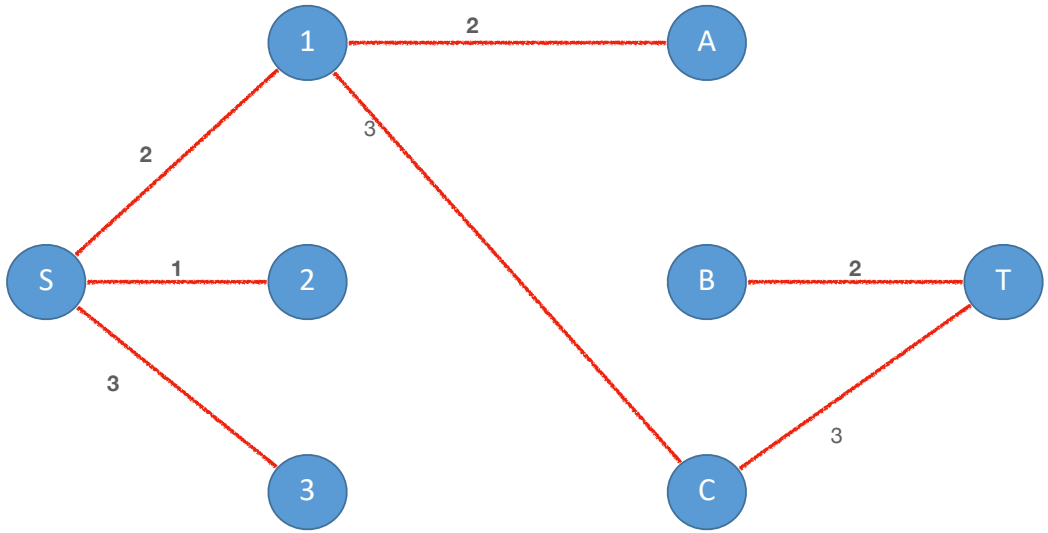
# TSP Approximations

- Better approximation: Christofides' algorithm
  - Suboptimality factor 1.5 for Euclidean TSP
  - Key idea: find a minimum weight matching M between all vertices of odd degree in minimum spanning tree T
    - Can be done in polynomial time by algorithm beyond our scope
  - Note: There must be an even number of odd degree vertices
    - Property of undirected graphs: sum of degrees is even
  - The graph $G(V, T \cup M)$ is Eulerian: every vertex has even degree
  - Can find Euler tour in graph
  - Using the same shortcut ideas, can get the 1.5 bound!



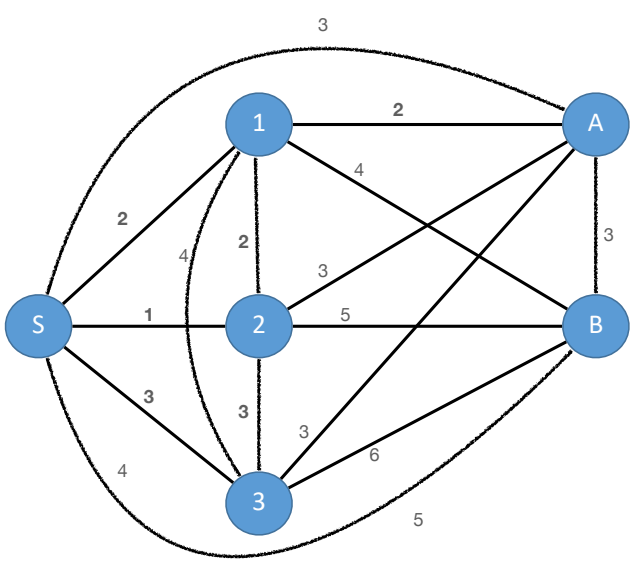
# Example - Christofides Approximation

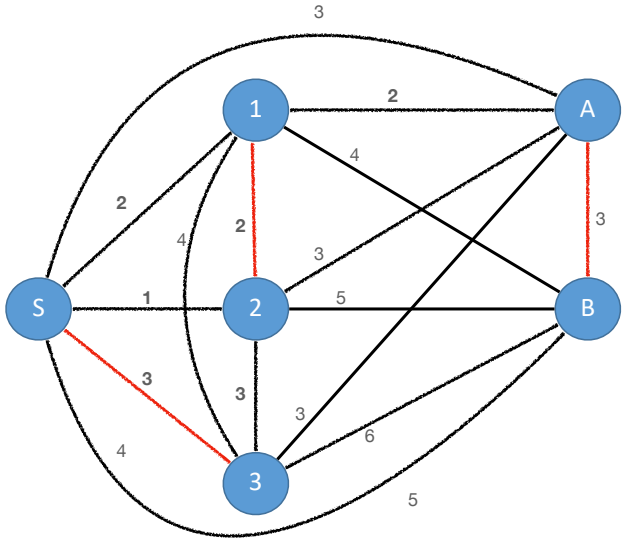
- Vertices S, 1, 2, 3, A, B have odd degree



# Example - Christofides 2

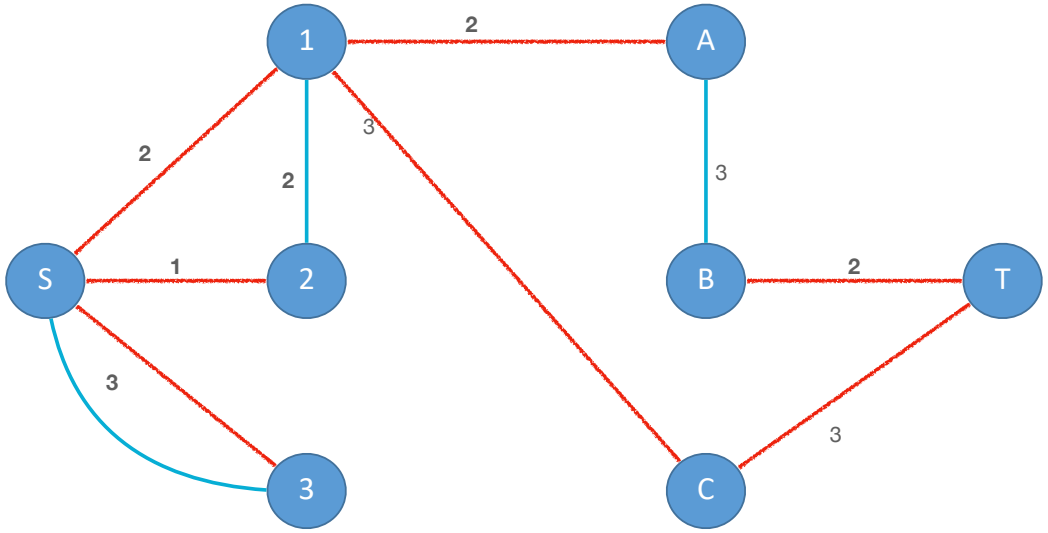
- Graph for matching



# Example - Christofides 3

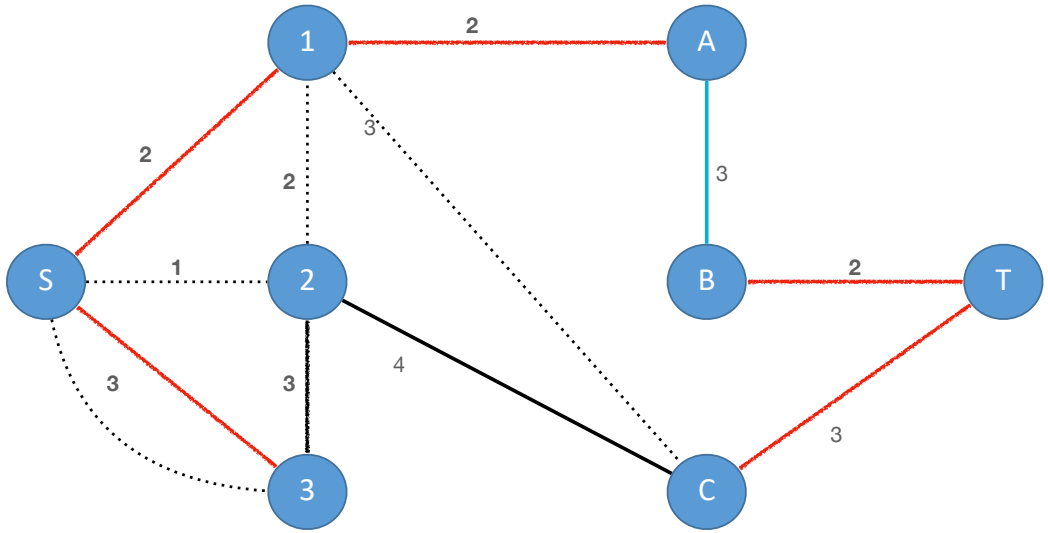
- Eulerian Network




- Euler Tour: S-1-A-B-T-C-1-2-S-3-S:  length 24

# Example - Christofides 3

- Euler Tour: S-1-A-B-T-C-1-2-S-3-S:  Length 24
- Cut corners: S-1-A-B-T-C-2-3-S:  Length 22



# Further Improvements

- Approximate algorithms for NP-Hard problems are fast, but often have significant gaps in performance relative to optimal algorithms
- Can often improve on approximate algorithms
- Local Search
  - Starts with an initial solution given by a fast algorithm
  - Searches a “neighborhood” of the initial solution for improved solutions
  - Fast, but may not find optimal
- Branch and Bound
  - Similar in spirit to A* search
  - Intelligent enumeration of search space, using bounds to eliminate regions to search
  - Can be slow

# Local Search

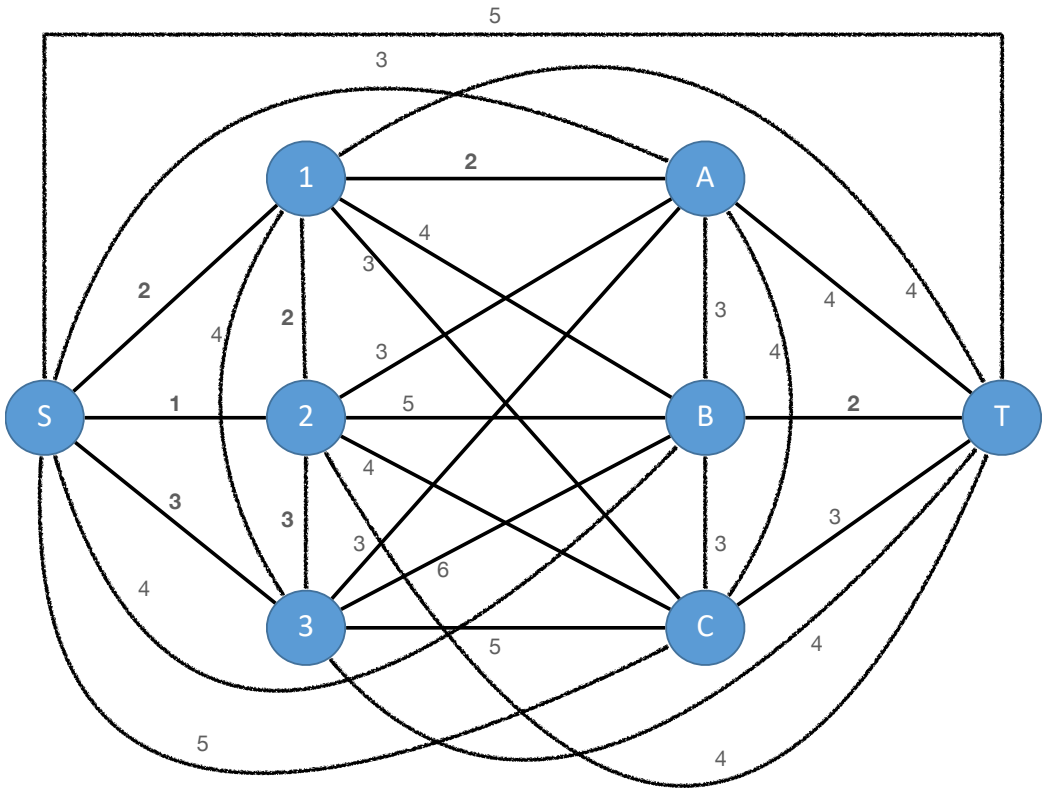
- Assume we have a candidate solution x to an optimization problem
  - Value function V(x) which we seek to maximize
- Define a neighborhood of x: N(x) to be solutions easily obtained from x
  - Easy = in linear or polynomial time
  - Ideally, N(x) is not very large
- Local search algorithm:
  - While there exists $y \in N(x)$ such that $V(y) > V(x)$,
    - Find $y \in N(x)$ such that $V(y) > V(x)$
    - Set x = y
- Algorithm terminates with a solution at least as good as initial solution x with limited extra computation

# Example: Local Search for TSP

- Assume we have an initial tour x = $v_1, v_2, \ldots, v_n$
- Neighborhood N(x): any tour where the position of one vertex is changed and inserted in a different position:
  - e.g. y = $v_1, v_n, v_2 \ldots, v_{n-1}$ where $v_n$ was inserted into a different order
- Algorithm:
  - Compute an initial tour x, and set it to current best tour x.
  - Search neighborhood N(x) for shorter tour y
    - If y is found, switch current best tour x = y
    - Repeat search
  - If no shorter tour is found in N(x), return x
- Many variations possible by changing neighborhood (e.g. swap pairs, …)
- May converge to local minimum, not global

# Example

- Initial tour: Greedy  S-2-1-A-B-T-C-3-S, Length
- Initial tour: Christofides, S-1-A-B-T-C-2-3-S, length 22



# Example

- Initial tour x: MST tour S-1-A-C-T-B-2-3-S,  Length 24
  - y = S-2-1-A-C-T-B-3-S  Length 23, swap
  - y = S-2-1-A-B-C-T-3-S  Length 21, swap
  - No further improvements possible

- Initial tour x: Christofides tour
  - x = S-1-A-B-T-C-2-3-S , Length 22
  - y = S-2-1-A-B-T-C-3-S , Length 21: swap
  - No further improvements possible