EC504 ALGORITHMS AND DATA STRUCTURES FALL 2020 MONDAY & WEDNESDAY 2:30 PM - 4:15 PM

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New Data Structure: Disjoint Sets

- Problem of Interest: keep track of dynamic relations
  - e.g. who is connected to whom
  - Who belongs to same club
  - Which computers are in same network
  - Which web pages link on the Internet
- Abstraction
  - Have a set of objects
  - Have relations between objects:
    - symmetric, $a \sim b$, transitive $a \sim b$, $b \sim c$ —> $a \sim c$
  - Want to keep track of subsets of objects that are related as relations are added

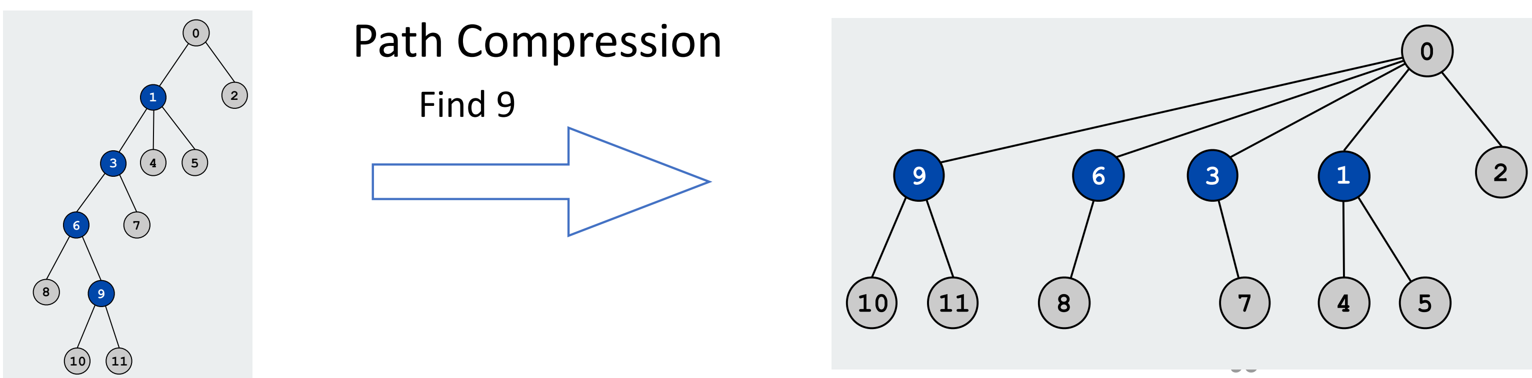
Disjoint Sets

- Operations
  - Find(a): Find the subset that contains element a
  - Union(a,b): Add a relation between two elements a, b which merges the subsets containing a, b
  - Often known as Union-Find problem

- Would like to do Unions and Find in O(1)
  - Seems really hard…
  - We'll come close with a disjoint set data structure

Disjoint Sets Algorithm

- Best Solution
  - Find: get root label of tree, moving up the tree
    - Restructure tree by making all ancestors have root as parent
  - Union: if new relation a ~ b, Find(a) and Find(b). If a, b are in different trees, merge trees
  - Merge: roots have ranks. Merge root so smaller rank is child of larger rank one; if same rank, increase new root rank by 1

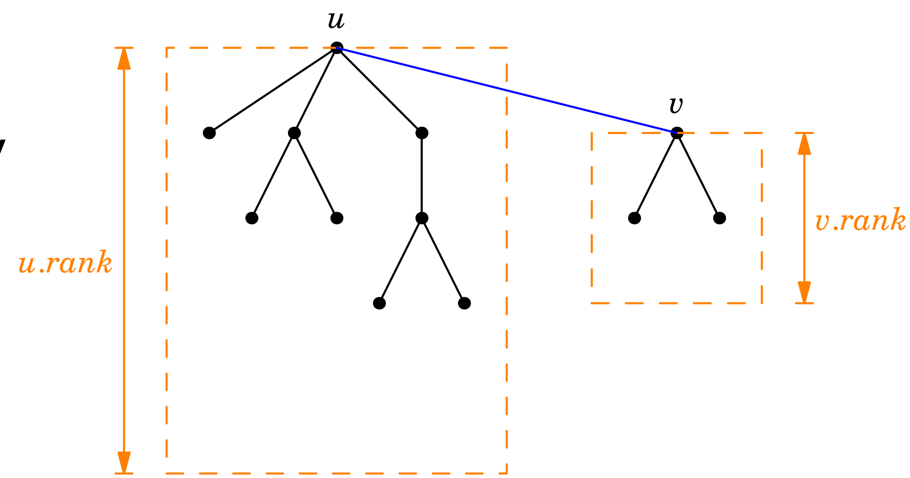


Disjoint Sets

- Theorem. Starting from an empty data structure, any sequence of M union and find operations on N objects takes $O(N + M\log^*(N))$ time
  - $\log^*(N)$: Ackerman function
    - $A(i) = 2^{A(i-1)}$; A(0) = 1
    - A(1) = 2, A(2) = 4, A(3) = 16, A(4) = 216 = 65536, A(5) = 265636,
    - A(6) = VERY VERY VERY BIG!
  - Inverse Ackerman: $i = \log^*(N)$
    - log*(N) = min number times you take $\log_2(\log_2(\cdots))$ to get smaller than or equal to 1
    - For all practical purposes, it is O(1) (Ch. 21, CLRS)

More on Rank and Path Compression

- Rank is an **upper bound** on height
  - Height is hard to keep track of exactly because of path compression
- How is rank assigned?
  - Rank of single nodes = 0
  - If 2 trees of equal ranks are merged, rank(new root) = rank(trees) + 1
  - Without path compression, this makes rank = height
  - After path compression, rank is upper bound to height

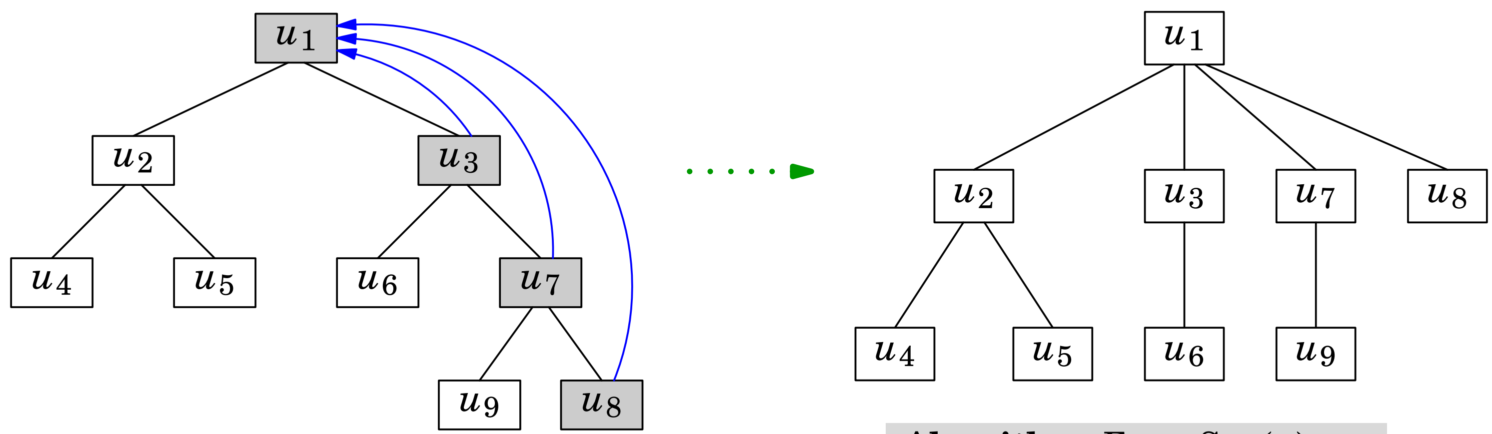


Algorithm: $\textsc{Union}(\widetilde{u}, \widetilde{v})$

$u \leftarrow \textsc{Find-Set}(\widetilde{u})$
$v \leftarrow \textsc{Find-Set}(\widetilde{v})$
if $u.rank = v.rank$ **then**
  $u.rank \leftarrow u.rank + 1$
  $v.parent \leftarrow u$
else if $u.rank > v.rank$ **then**
  $v.parent \leftarrow u$
else
  $u.parent \leftarrow v$

More on Rank and Path Compression

- Path compression illustration: Find u_8



- Rank does not change!
  - But height dropped by 1

Algorithm: FIND-SET(u)

$A \leftarrow \emptyset$
$n \leftarrow u$
while n is not the root **do**
    $A \leftarrow A \cup \{n\}$
    $n \leftarrow n.parent$
for each $x \in A$ **do**
    $x.parent \leftarrow n$
return n

Outline of Proof of Complexity

- Properties of rank (without path compression): Let p(x) be parent(x)
  - x is not a root node, then rank[x] < rank[p[x]]
    - Rank k root created by linking two roots of rank k – 1
  - x is not a root node, then rank[x] never changes
  - p[x] changes, then rank[parent[x]] strictly increases
    - p[x] only changes for root
  - When p[x] changes, rank[x] < rank[p[x]]
    - p[x] only changes for roots, when one root goes under other

Algorithm: $\text{UNION}(\tilde{u}, \tilde{v})$

$u \leftarrow \text{FIND-SET}(\tilde{u})$
$v \leftarrow \text{FIND-SET}(\tilde{v})$
if $u.rank = v.rank$ **then**
    $u.rank \leftarrow u.rank + 1$
    $v.parent \leftarrow u$
else if $u.rank > v.rank$ **then**
    $v.parent \leftarrow u$
else
    $u.parent \leftarrow v$

Outline of Proof of Complexity - 2

- Properties of rank (without path compression): Let p(x) be parent(x)
  - Root of rank k has $\geq 2^k$ elements in its tree
    - Induction: true for k = 0
    - Assume true for k-1. Root gets rank k by merging two roots of rank k-1. Thus, no. elements $\geq 2^{k-1} + 2^{k-1} \geq 2^k$
  - Highest rank of node $\leq \lfloor \log_2(n) \rfloor$
  - For any integer k, there are $\leq n/2^k$ nodes with rank k
    - Non-root notes with rank k were roots with rank k at some point
    - Every rank k node has at least 2^k elements,

Algorithm: UNION($\tilde{u}$, $\tilde{v}$)

```
u ← FIND-SET(ũ)
v ← FIND-SET(ṽ)
if u.rank = v.rank then
    u.rank ← u.rank + 1
    v.parent ← u
else if u.rank > v.rank then
    v.parent ← u
else
    u.parent ← v
```

Outline of Proof of Complexity - 3

- Path compression: Let p(x) be parent(x)
  - Key property: roots, node ranks, tree elements don't change!

    All the properties of union by rank stay true

    parent pointers updated to root means rank(p(x)) > rank(x) still

- Iterated logarithm function

$$\log^*(n) = \begin{cases} 0 & n \leq 1 \\ 1 + \log^*(\log_2(n)) & \text{otherwise} \end{cases}$$

n	$\lg^* n$
1	0
2	1
$[3, 4]$	2
$[5, 16]$	3
$[17, 65536]$	4
$[65537, 2^{65536}]$	5

Outline of Proof of Complexity - 4

- Analysis: Buckets of elements
  - Divide elements by non-zero ranks into the following groups
    - {1}, {2, 3}, {4, …, 15}, …, $\{B, 2^B - 1\}$, $\{2^B, 2^{2^B} - 1\}$
    - Total number of buckets $\leq \log^*(n)$
    - Max elements in bucket {B, $2^B - 1$}: $\leq n/2^B$
  - Every node has rank in the first $\log^*(n)$ groups
- Creative accounting
  - Every node receives credits the moment it ceases to be a root:
    - If node rank is in $\{B, 2^B - 1\}$, get 2^B credits
    - Number of credits disbursed to all nodes $\leq n \log_2^*(n)$

Outline of Proof of Complexity - 5

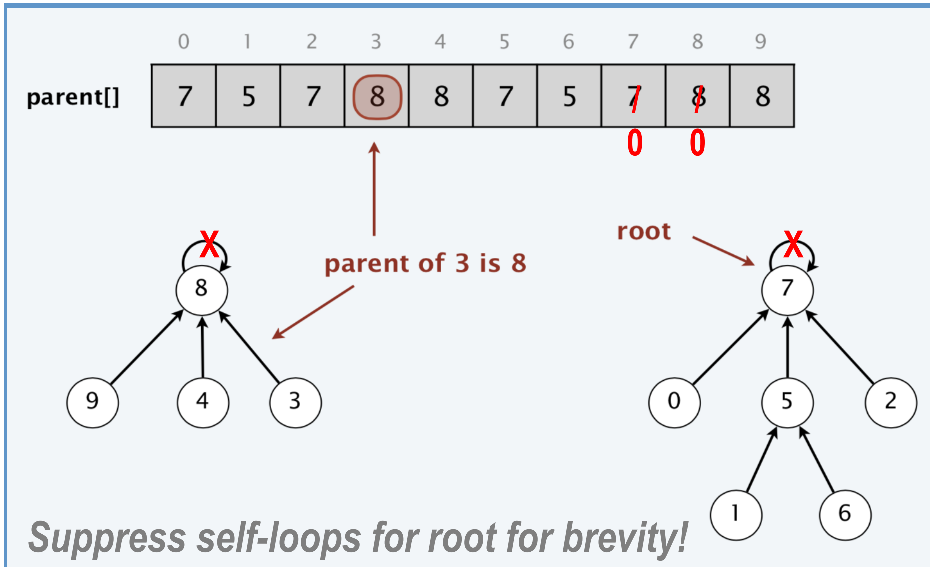
- Creative accounting
  - Every node receives credits the moment it ceases to be a root:
    - If node rank is in $\{B, 2^B - 1\}$, get 2^B credits
    - Number of credits disbursed to all nodes $\leq n \log^*(n)$
      - Max nodes in bucket {B, $2^B - 1$}: $\leq n/2^B$, so max of n credits in bucket
  - Run time of Find(x): bounded by number of parent pointers to root
    - Rank increases strictly as you search up the tree
    - 3 Cases:
      - Case 0: parent[x] is a root —> happens for 1 link per find
      - Case 1: rank[parent[x]] is in higher bucket than rank[x]
      - Case 2: rank[parent[x]] is in same bucket as rank[x]

Outline of Proof of Complexity - 6

- Case 1: rank[parent[x]] is in higher bucket than rank[x]
  - At most $\log^*(n)$ links have this property
- Case 2: charge 1 credit to follow parent pointer
  - If rank[x] is in bucket $\{B, 2^B - 1\}$, x will move to Case 1 before paying all its credits (and will still have some left over)
- So, since you started with $\leq n \log^*(n)$ credits across all nodes, and you spend less than that across all find operations, total amortized cost of m operations, for $m \geq n$, is $O(m \log^*(n))$

Representation of Disjoint Set Operations

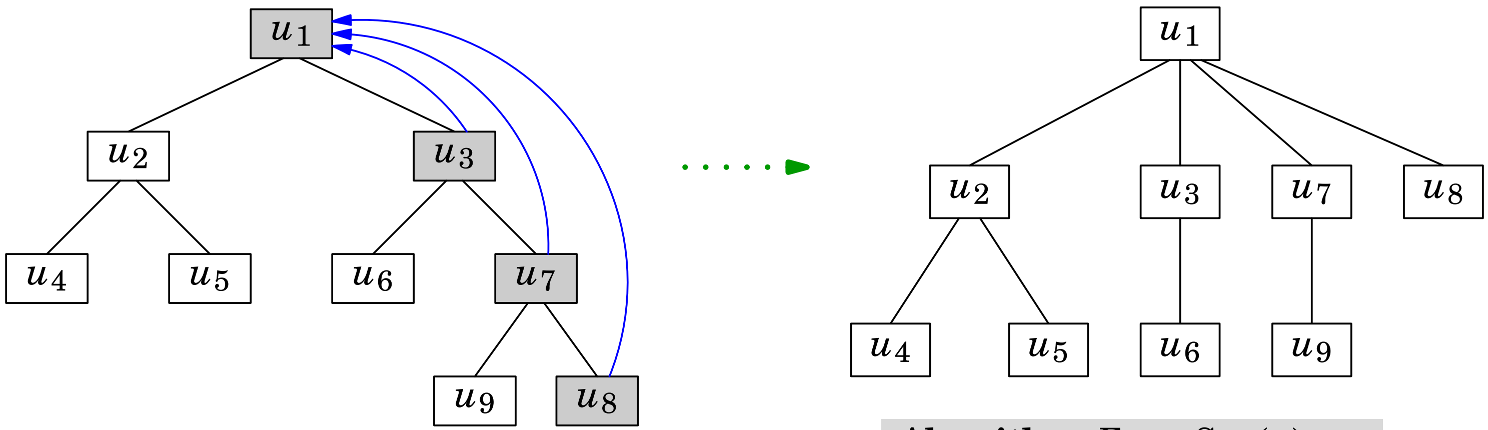
- Array: Objects indexed in the array, with value pointing to parent location
  - Roots point to self-location
  - Keep track of separate array for ranks



- How is rank assigned?
  - Rank of single nodes = 0
  - If 2 trees of equal ranks are merged, rank(new root) = rank(trees) + 1
  - Without path compression, this makes rank = height
  - After path compression, rank is upper bound to height

Bigger Example

- Path compression illustration: Find u_8



- Rank does not change!
  - But height dropped by 1

Algorithm: FIND-SET(u)

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for each $x \in A$ **do**
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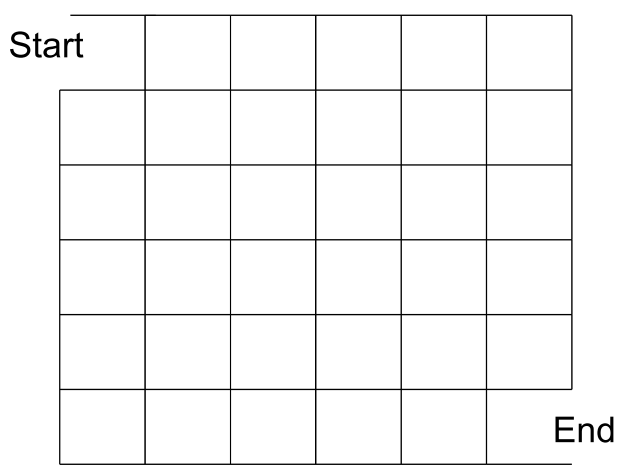
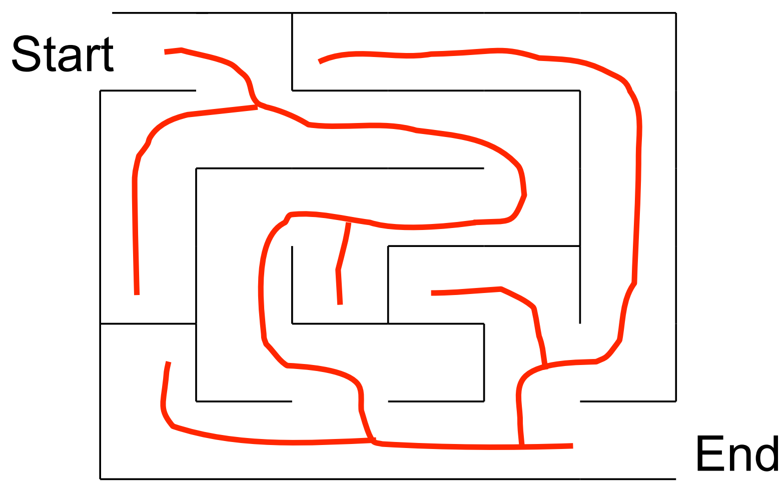
return n

Application

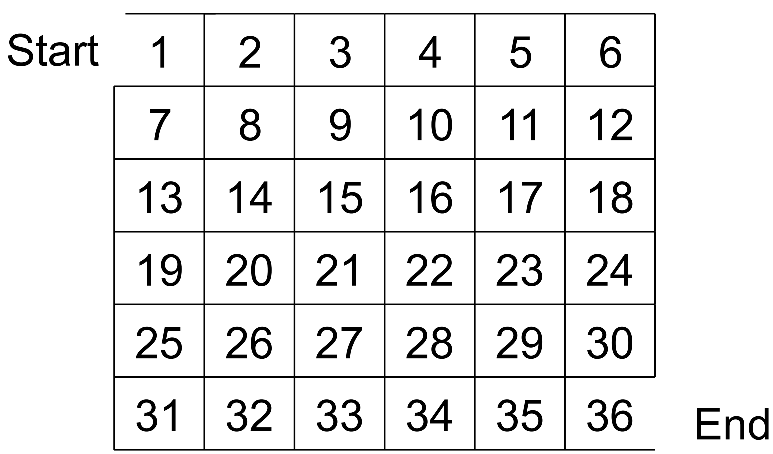
- Maze Building: take out walls to create a maze
  - Consider edges in random order (i.e. pick an edge)
  - Only delete an edge if it introduces no cycles (how? TBD)
  - When done, we will have a way to get from any place to any other place (including from start to end points)

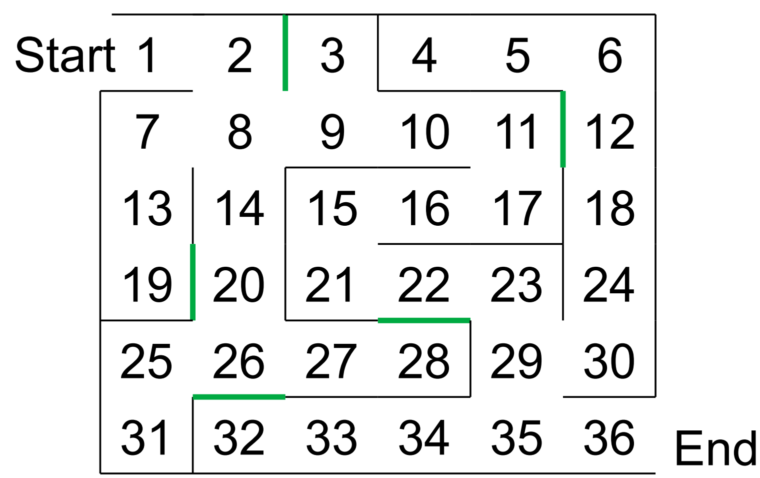




Application 2

- Maze Building: take out walls to create a maze
  - Number the cells; every edge allows a relation between cells
  - Keep track of disjoint sets: don’t remove edge if it is between two nodes in same subset





Graphs

in the "real world"

- Networks are graphs
  - Information: WWW, citation, …
  - Social: co-actor, dating, messenger, communities, …
  - Technological: Internet, power grids, airline routes, …
  - Biological: Neural networks, food web, blood vessels, …
- Object hierarchies are graphs
- Circuit layouts are graphs
- Computer programs are graphs

Undirected Graph G=(V,E) (or G=(N,A))

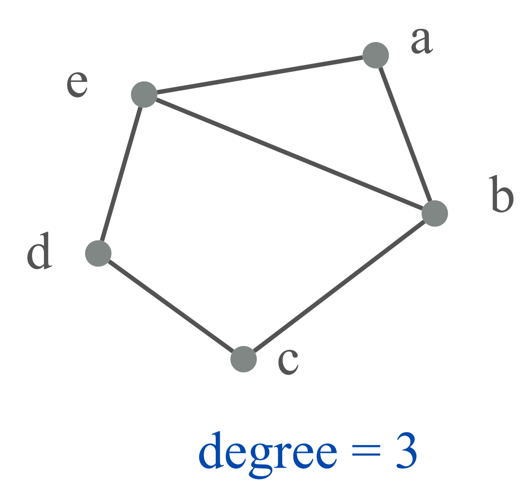
V: Set of vertices (nodes…)

Edge {a,b} is an unordered pair of nodes

$E \subset V \times V$: Set of vertices (arcs…)

Undirected graph: G=(V,E)

- Typically, don't allow {a,a} in E



Vertex Set: V = (a,b,c,d,e)

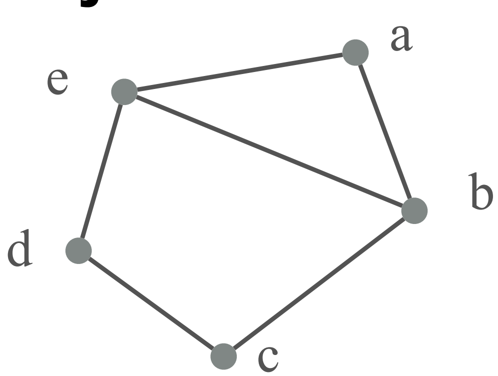
Edge Set: E = ({a,b}, {b,c}, {c,d},{d,e},{e,a},{e,b})

Undirected Graph G(V,E)

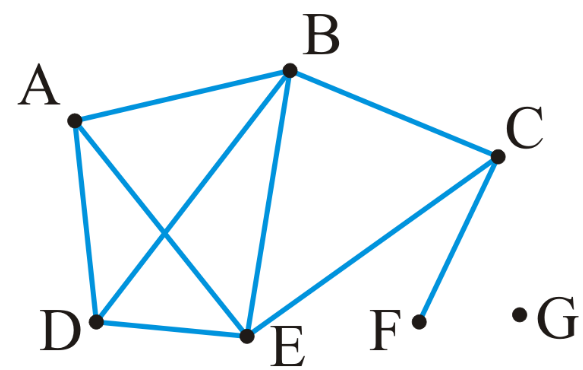
Maximum number of edges: $$|E| \le \binom{|V|}{2} = \frac{|V|(|V|-1)}{2} = O\left(|V|^2\right)$$

Degree of vertex: number of adjacent vertices

Vertex a **adjacent** to vertex b $\Longleftrightarrow$ $\{a, b\} \in E$



degree(b) = 3
degree(c) = 2

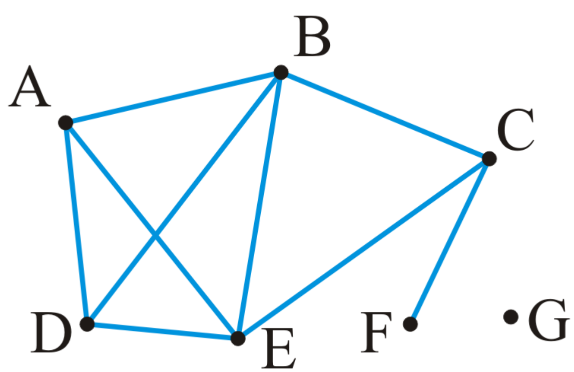


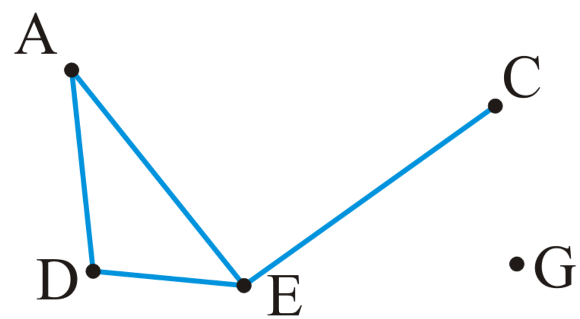
degree(E) = 4
degree(G) = 0

Subgraphs (V′,E′) of (V,E)

(V′,E′) is a subgraph of(V,E) if and only if

- $V' \subset V$, and $E' \subset E$
- If $\{a,b\} \in E' \Rightarrow a,b \in V'$

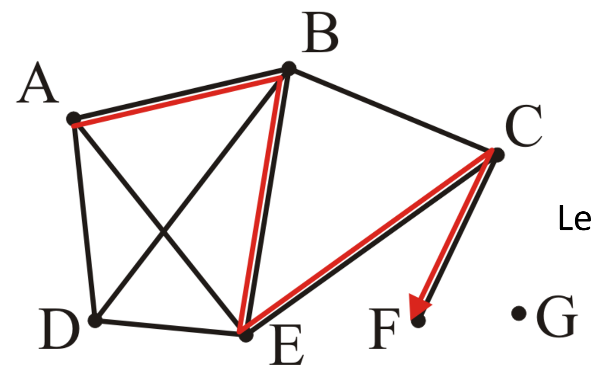


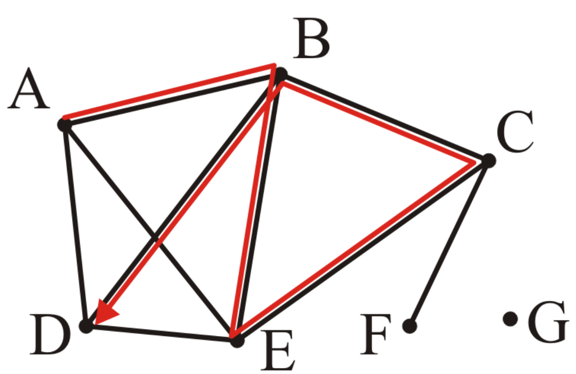


Paths in(V,E)

A path in an undirected graph is an ordered sequence of vertices $(v_0, v_1, v_2, \ldots, v_k)$

- $\{v_{j-1}, v_j\}$ is an edge in E for j = 1, ..., k
- Termed a path from v_0 to v_k
- The length of this path is k



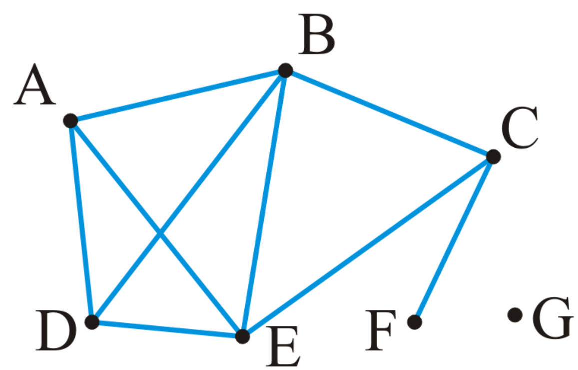


Simple paths in(V,E)

Simple path: no repeated nodes other than perhaps, the first and last vertices

Simple cycle: simple path with first and last nodes the same

- Example: (A,B,C,E) simple path
- (A,B,E,C,B,D) Not simple path
- (A,B,D,A) simple cycle



Connectedness

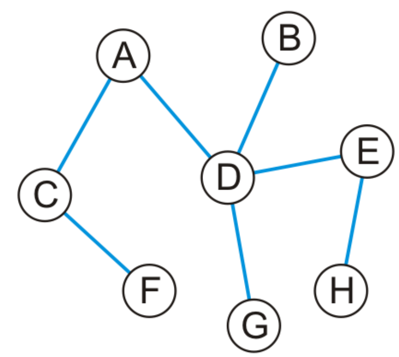
Vertices a, b in V are connected if there exists a path from a to b

Graph(V,E) is connected if, for every a, b in V, a and b are connected

Trees

Graph(V,E) is a tree if it is connected, and

- there is only one path between any two vertices a, b
- Equivalently, there are no cycles in the graph



Trees: |E| = |V| - 1

- Adding one more edge to a tree creates a cycle
- Removing an edge from a tree disconnects the graph
- Can covert to rooted tree picking any vertex as root

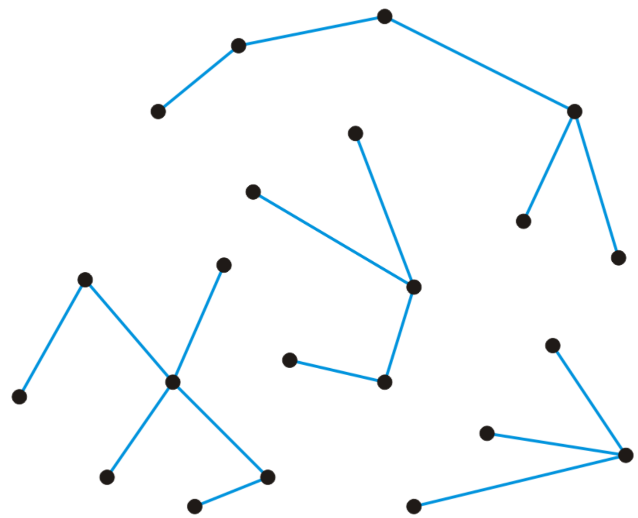
Forests

Forest is a graph (V,E) with no cycles

- Forest can be multiple disjoint trees

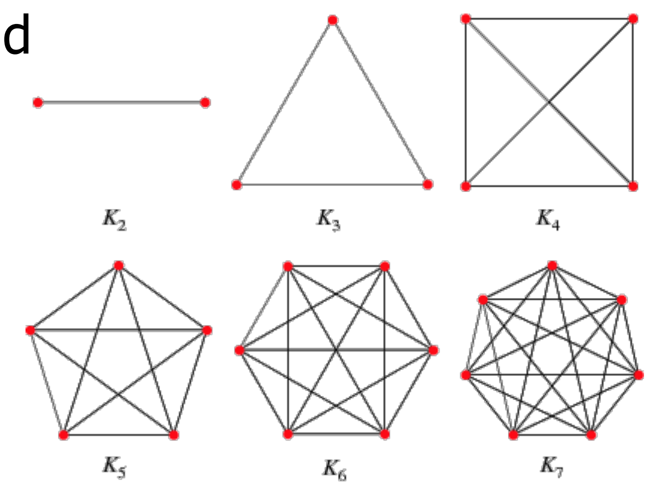
Properties:

- $|E| < |V|$
- Number of trees = $|V| - |E|$
- Removing edge: adds one more tree



Some Specific Graphs

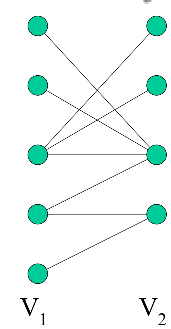
Complete graph K_n: undirected graph with n vertices, and an edge between every pair of vertices

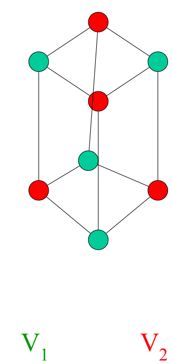


Bipartite graph $K_{m,n}$: undirected graph where V can be partitioned into vertex set V can be partitioned into disjoint, nonempty sets V_1 and V_2 such that

every edge connects a vertex in V_1 to a vertex in V_2

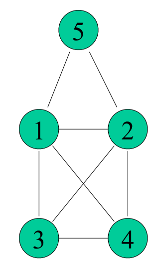
Equivalent: every cycle has even length!





Results in Undirected Graphs

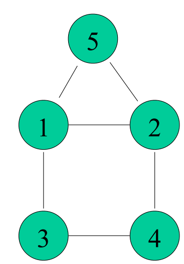
Let (V,E) be undirected. Then, $\sum_{k \in V} \text{degree}(k) = 2|E|$

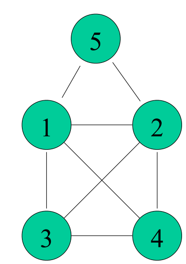


- Every edge connects 2 vertices…
  —> must have even number of odd-degree vertices

A path is Eulerian if it covers every edge in the graph exactly once

- Draw it without lifting pencil
- Theorem: a connected graph has an Eulerian cycle if and only if every node has even degree





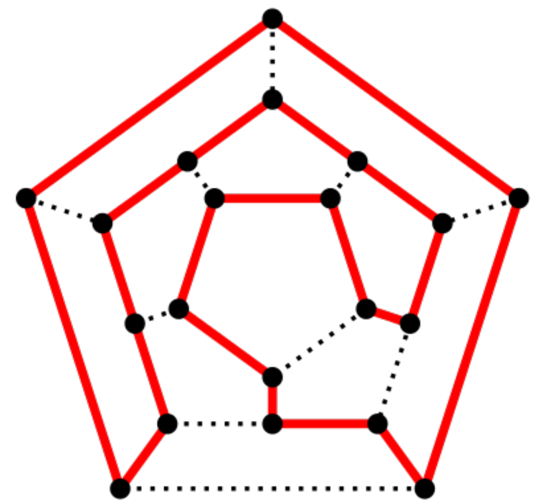
Results in Undirected Graphs

A path is Hamiltonian if it covers every vertex in the graph exactly once; it is a cycle if it covers every vertex but the start and end vertices exactly once

- Theorem: A graph with n vertices has Hamitonian cycle if degree of every vertex is at least n/2

- Theorem: For a graph G with $n \geq 3$ vertices, and degree(a) + degree(b) $\geq n$ whenever a, b are not adjacent, then G has a Hamiltonian cycle

  - Finding a Hamiltonian cycle is a hard problem!



Planar Graphs

A graph G is planar if it can drawn in the plane so that no edges cross

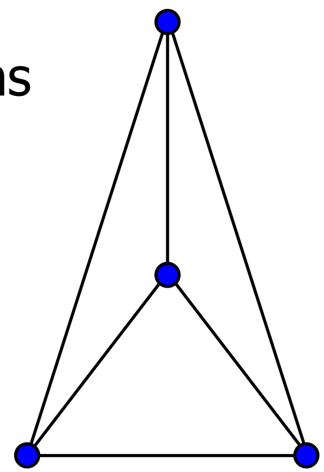
- Useful for VLSI, circuit design

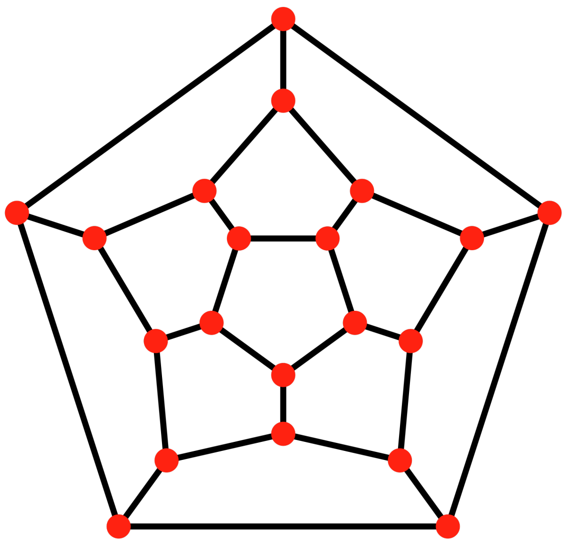
- Euler's Formula: a planar graph divides the plane into regions called faces. The number of regions r = m-n+2

- Sufficient conditions for planar graphs

  - $m \leq 3n - 6$

  - $r \leq 2n - 4$





Directed Graph G=(V,E)

V is the set of vertices

An edge e = (a,b) is an ordered pair from a to b, where a, b in V

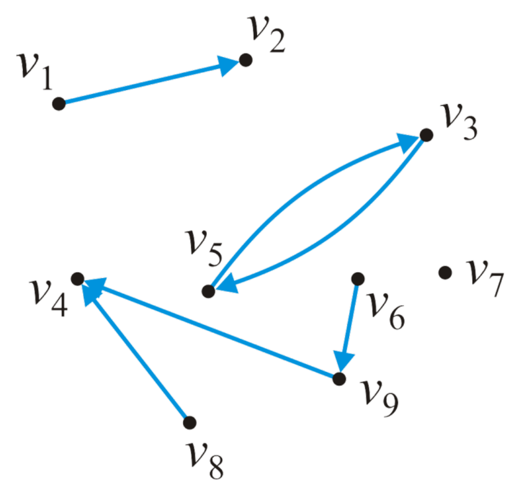
Note: edge (b, a) is different from edge (a, b)

Example: streets, …

Max. no. edges = $|V|^2$ (allow self loops)

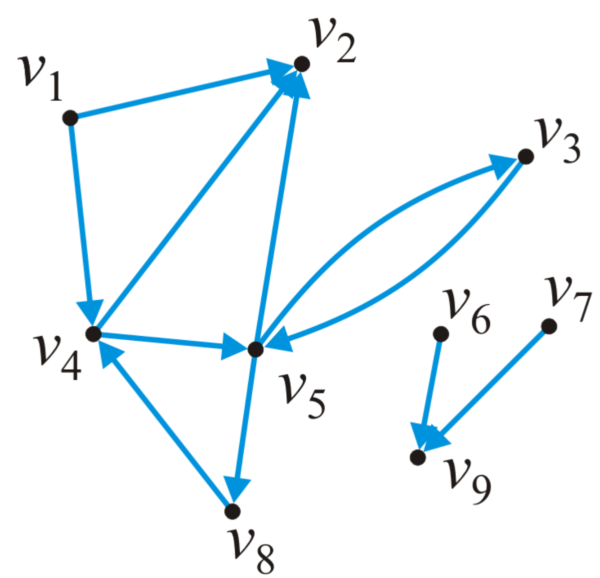
Out-degree of a vertex: no. edges leaving vertex

In-degree of a vertex: no. edges entering vertex



$E = \{(v_1, v_2), (v_3, v_5), (v_5, v_3), (v_6, v_9), (v_8, v_4), (v_9, v_4)\}$

Sources, Sinks, Paths



Vertex with in-degree 0: **Source** (V_1, V_6, V_7)

Vertex with out_degree 0: **Sink** (V_9)

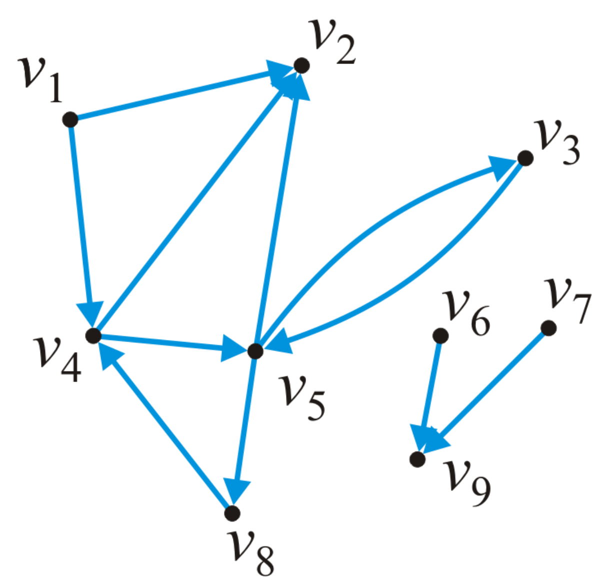
Path in directed graph (V,E) is an ordered sequence of vertices $(v_0, v_1, v_2, \ldots, v_k)$ such that $(v_{j-1}, v_j) \in E$ for j =1,…,k

Simple path: no repeated nodes in path

Cycle: path with $v_0 = v_k$. **Simple cycle**: no repeated nodes except start-end

Connectedness in Directed Graphs

Vertex a is connected to vertex b if there exists a path from a to b (directed path)



- Graph (V,E) is **strongly connected** if there exists directed path between any two vertices

- Graph (V,E) is **weakly connected** if there exists undirected path between any two vertices

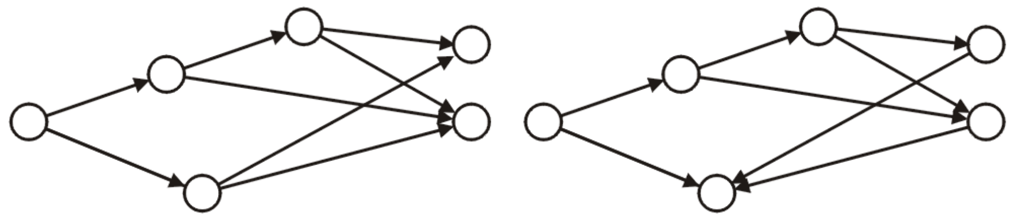
  – The sub-graph $\{v_3, v_4, v_5, v_8\}$ is strongly connected

  – The sub-graph $\{v_1, v_2, v_3, v_4, v_5, v_8\}$ is weakly connected

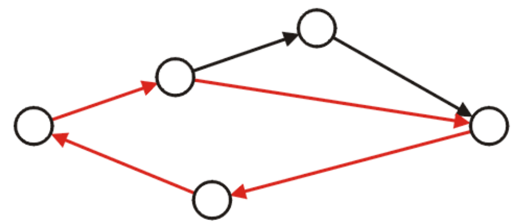
Directed Acyclic Graphs

Directed acyclic graph (DAG): Graph (V,E) with no directed cycles

- Represents partial order



- DAGs appear often
  - Parse tree constructed by a compiler



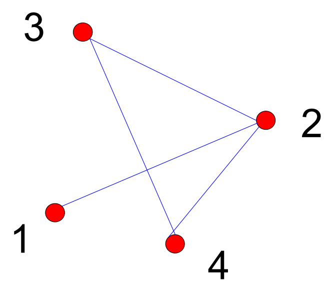
  - Dependency graphs such as those used in instruction scheduling and makefiles
  - Dependency formed by inheritance relationships in object-oriented programming languages

Representations of graphs

- Adjacency matrix:
  - $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E \end{cases}$$

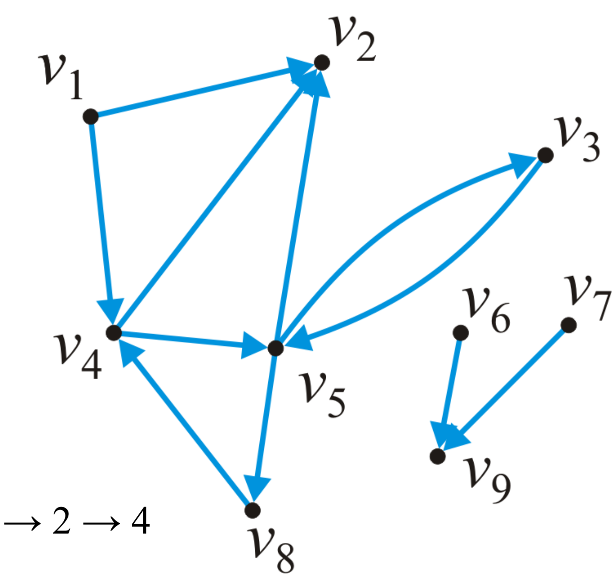
  - Symmetric for undirected graphs



$$A = \begin{array}{cc} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{l} \cdots 1 \\ \cdots 2 \\ \cdots 3 \\ \cdots 4 \end{array} \\ \begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 \end{array} & \end{array}$$

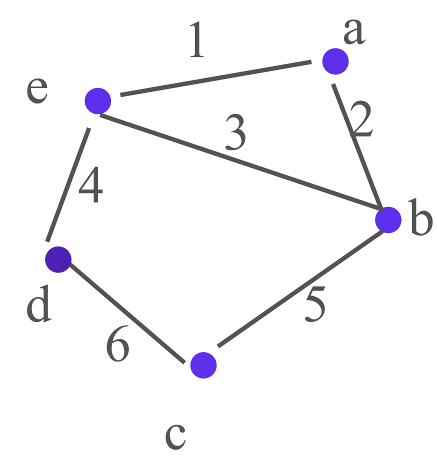
Adjacency List

- Each vertex has list of neighbor nodes
  - For directed graphs, out-list
  - Used in MATLAB for sparse matrices
  - Can be implemented as arrays
    - Forward-Star
  - or linked lists
    - |V|+|E| for directed graphs
  - Must store 2 edges for undirected



1 • → 2 → 4
2 •
3 • → 5
4 • → 2 → 5
5 • → 2 → 3 → 8
6 • → 9
7 • → 9
8 • → 4
9 •

V vertices
E edges

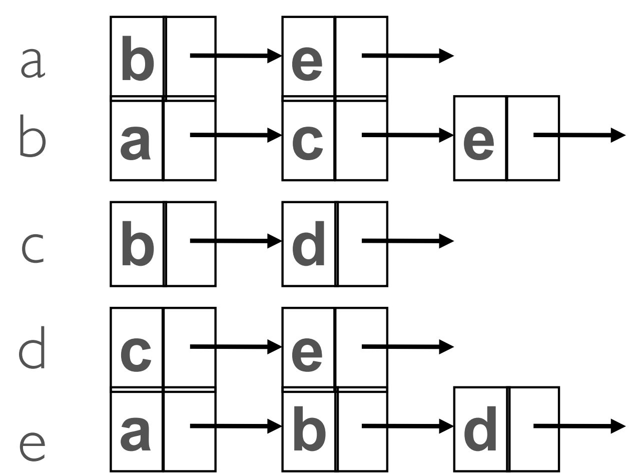


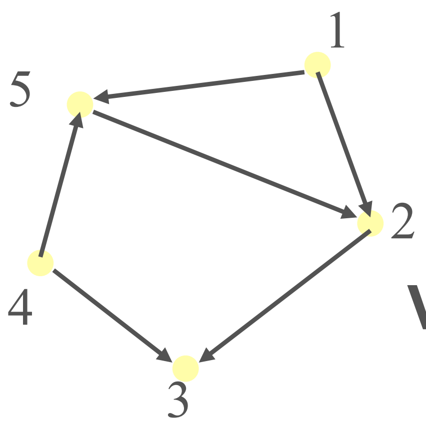
1. Edge list: {a,b}, {b,c}, {c,d},{d,e},{e,a}, {e,b}

2. Adjacency list

Vertex	Adjacencies
a	b, e
b	a, c, e
c	b, d
d	c, e
e	a, b, d

Linked List Form





Forward Star Representation

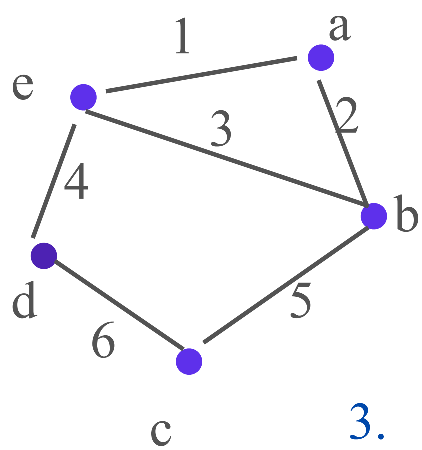
Vertex Array First[I]

i	First
i =1	1
i =2	3
i =3	4
i =4	4
i =5	6
i =6	7

Edges[j]

j	Edges
j =1	2
j =2	5
j =3	3
j=4	3
j=5	5
j =6	2
j =7	-1

Just the adjacency list put end to end in the arc array!



3. Adjacency matrix:

A=

	a	b	c	d	e
a	0	1	0	0	1
b	1	0	1	0	1
c	0	1	0	1	0
d	0	0	1	0	1
e	1	1	0	1	0

4. Incidence matrix:

Matrix of vertex rows, edges columns

M =

Directed: -1 in start of edge, 1 in end

Undirected: 1 in both

	1	2	3	4	5	6
a	1	1	0	0	0	0
b	0	1	1	0	1	0
c	0	0	0	0	1	1
d	0	0	0	1	0	1
e	1	0	1	1	0	0

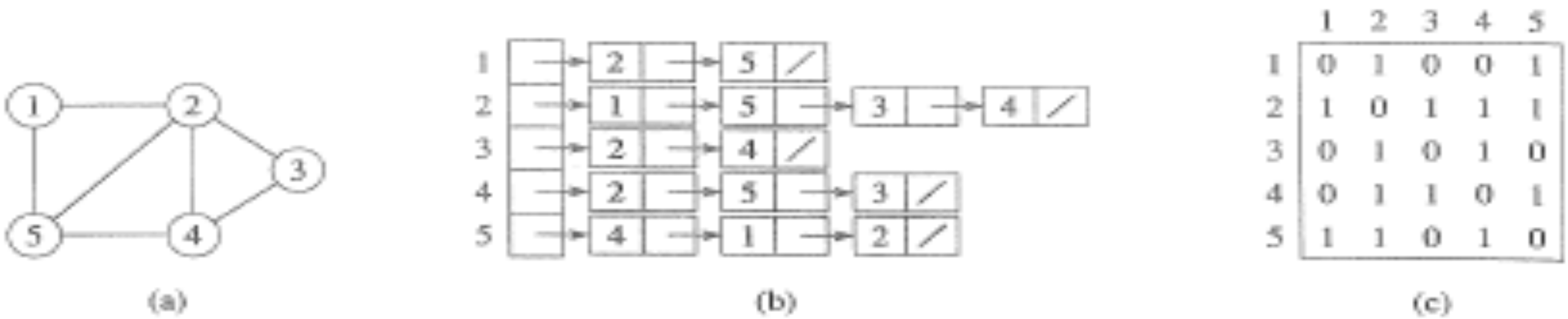


Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G having five vertices and seven edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

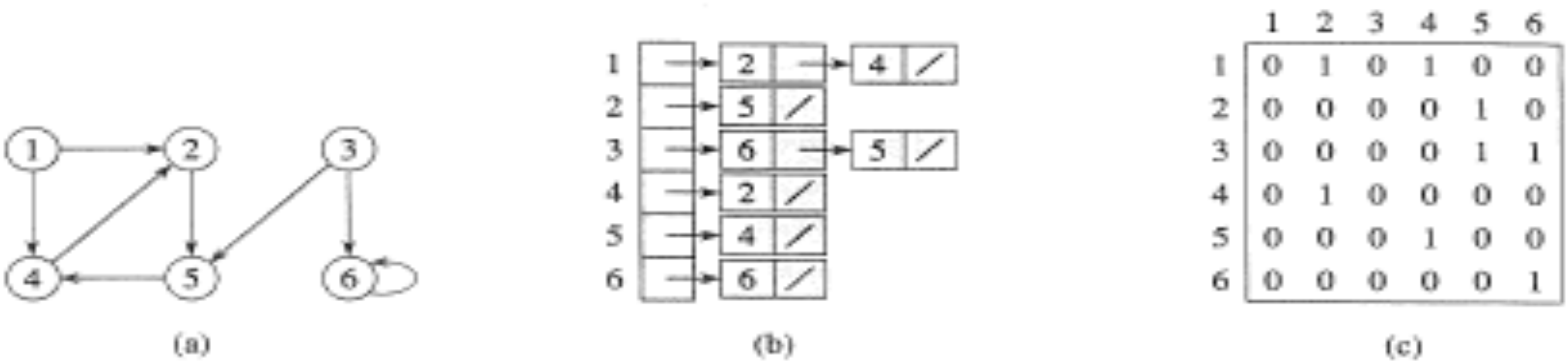


Figure 22.2 Two representations of a directed graph. **(a)** A directed graph G having six vertices and eight edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

TRAVERSALS DFS & BFS

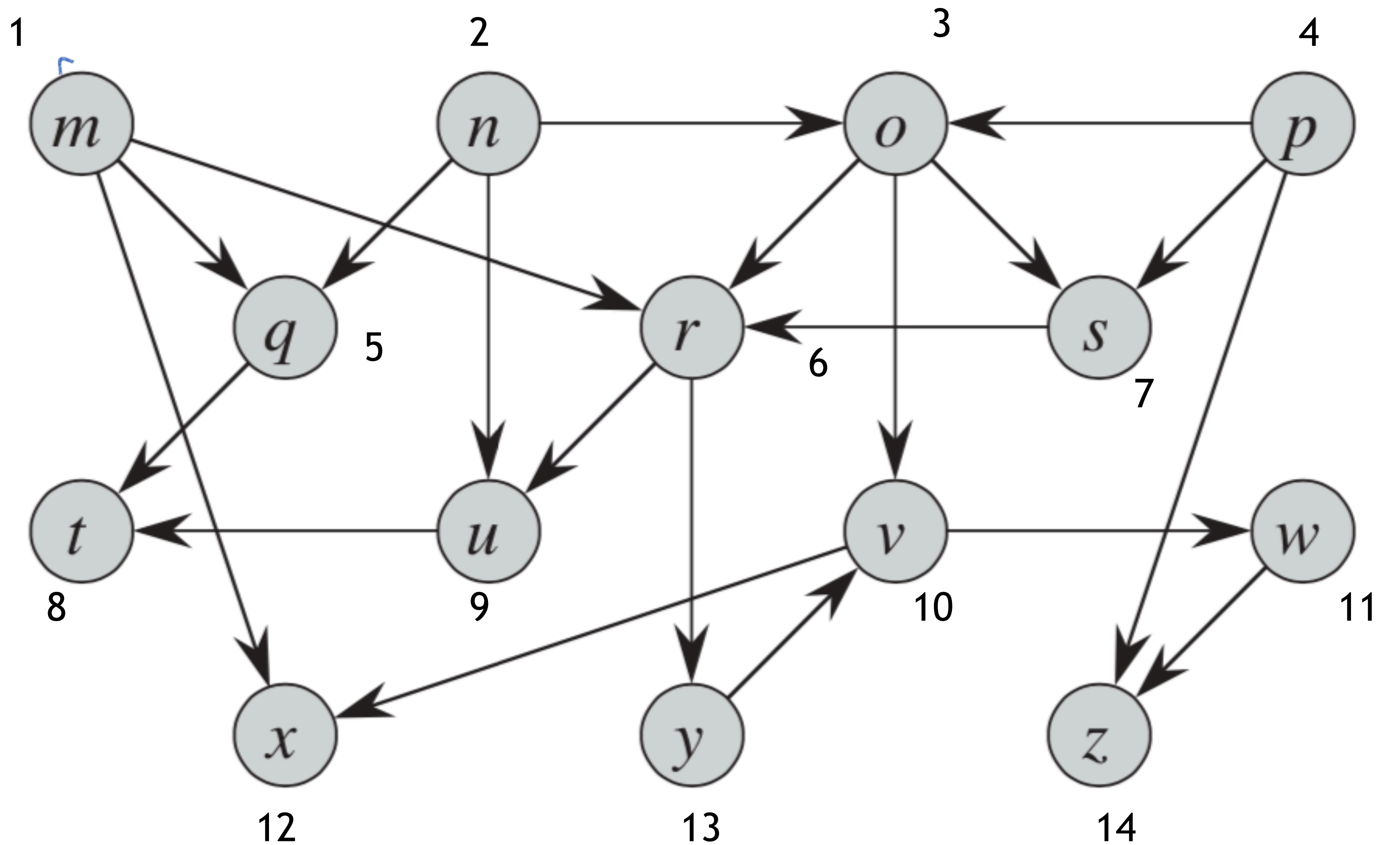
- *Depth First Search (DFS): Uses a Stack*
  - Select a start node: mark visited and push
  - (2) Pick unmarked neighbor: mark visited and push
  - Else pop and mark done
  - Return to 2
- *Breadth First Search (BFS): Uses a Queue*
  - *Select a start node set d = 0: mark visited and enqueue*
  - *(2) Dequeue mark done*
  - *All unmarked neighbors set d = d+1: mark all visited and enqueue*
  - *Return to 2*

(For a DAG this is a "reverse order topological sort")

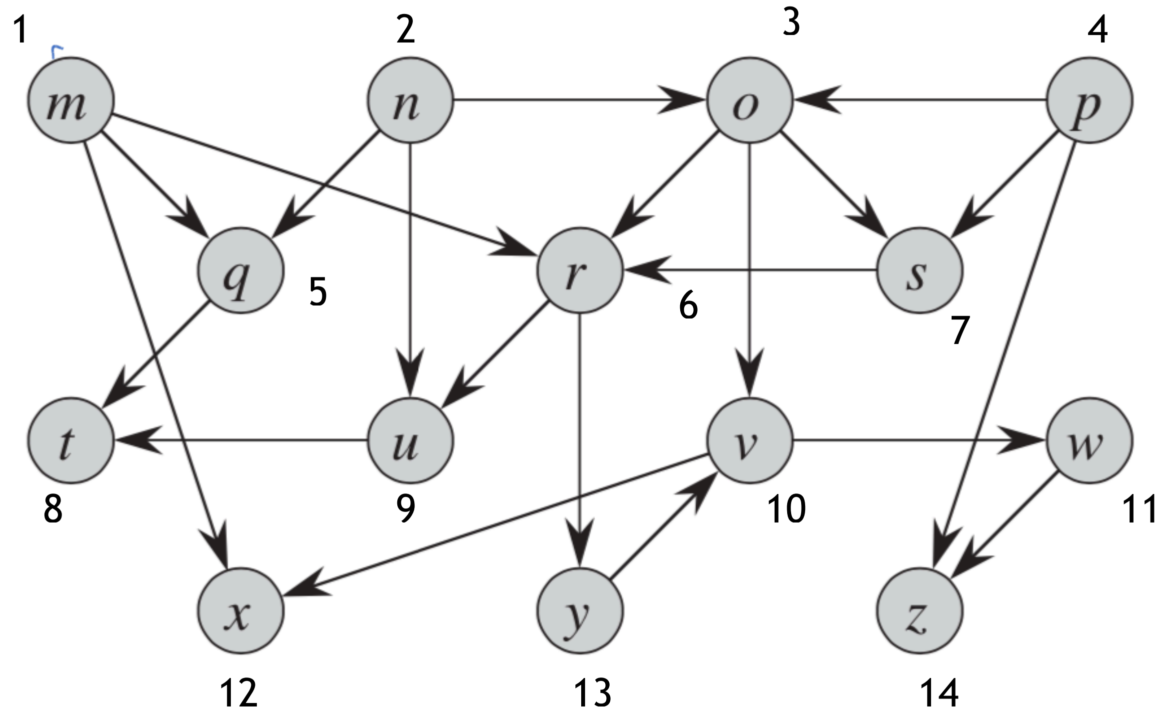
Set predecessor: Flag to reconstruct Tree.

(The distance label is the minimum number of hops to each node from start)

DAG Topological Sort (Execution order)



DAG Topological Sort (Execution order)



in			14		in	in
			11	12		
			10	10		
8		9	13	13	7	
5		6	6	6	3	
1	1	1	1	1	2	4

4 2 3 7 1 6 13 10 12 11 14 9 5 8

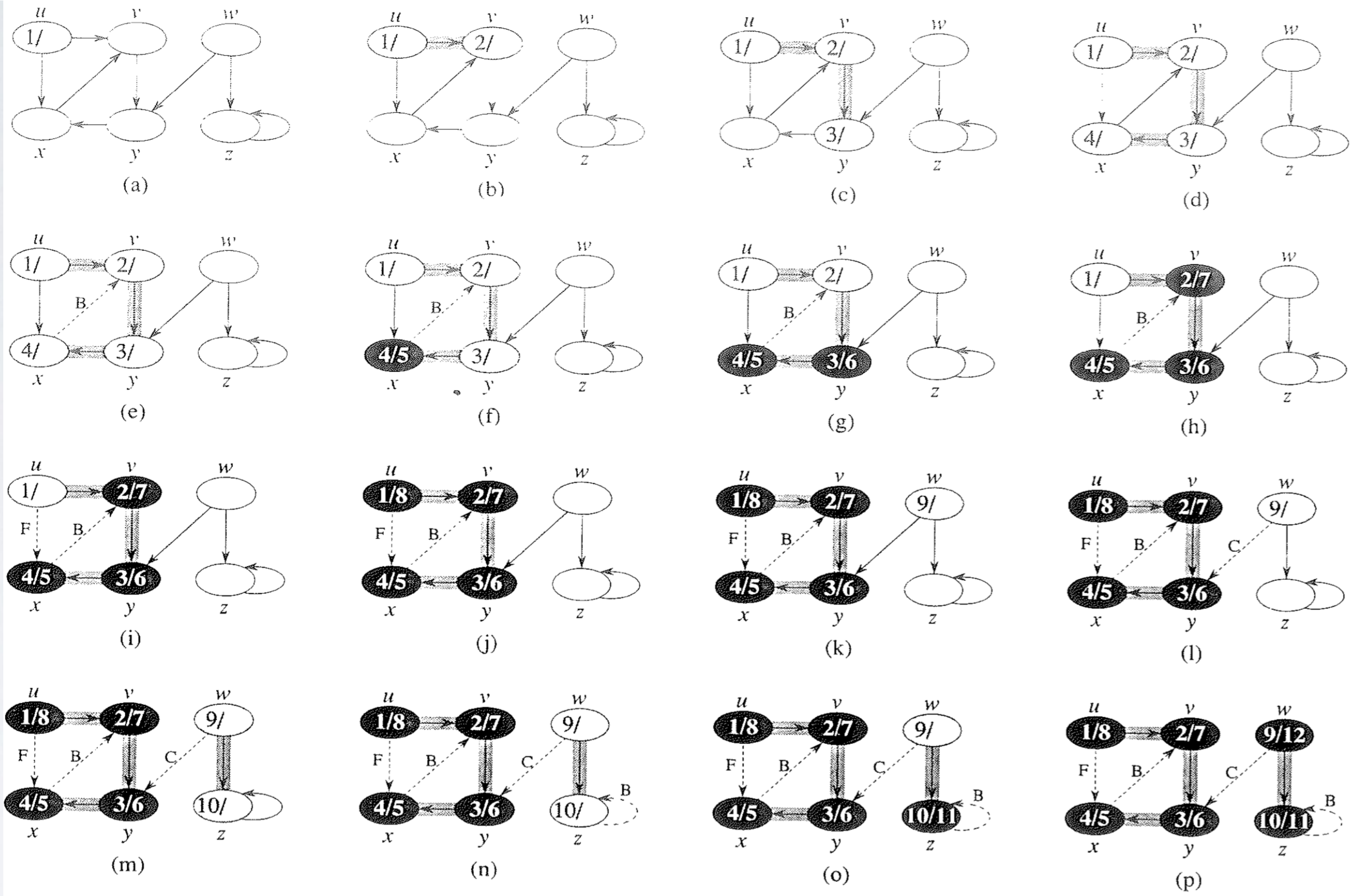


Figure 22.4 The progress of the depth-first-search algorithm DFS on a directed graph. As edges are explored by the algorithm, they are shown as either shaded (if they are tree edges) or dashed (otherwise). Nontree edges are labeled B, C, or F according to whether they are back, cross, or forward edges. Vertices are timestamped by discovery time/finishing time.

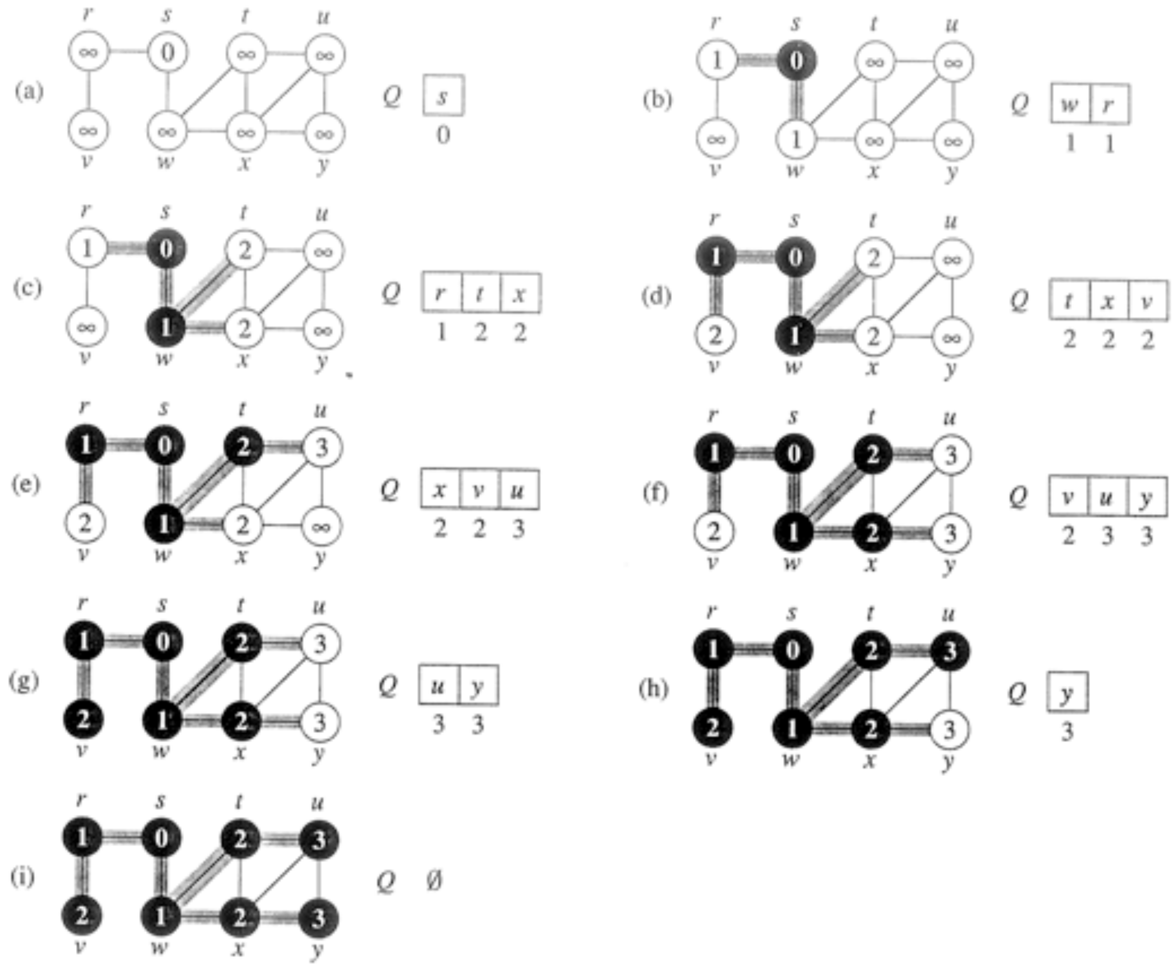


Figure 22.3 The operation of BFS on an undirected graph. Tree edges are shown shaded as they are produced by BFS. Within each vertex u is shown $d[u]$. The queue Q is shown at the beginning of each iteration of the **while** loop of lines 10–18. Vertex distances are shown next to vertices in the queue.