EC504 ALGORITHMS AND DATA STRUCTURES FALL 2020 MONDAY & WEDNESDAY 2:30 PM - 4:15 PM

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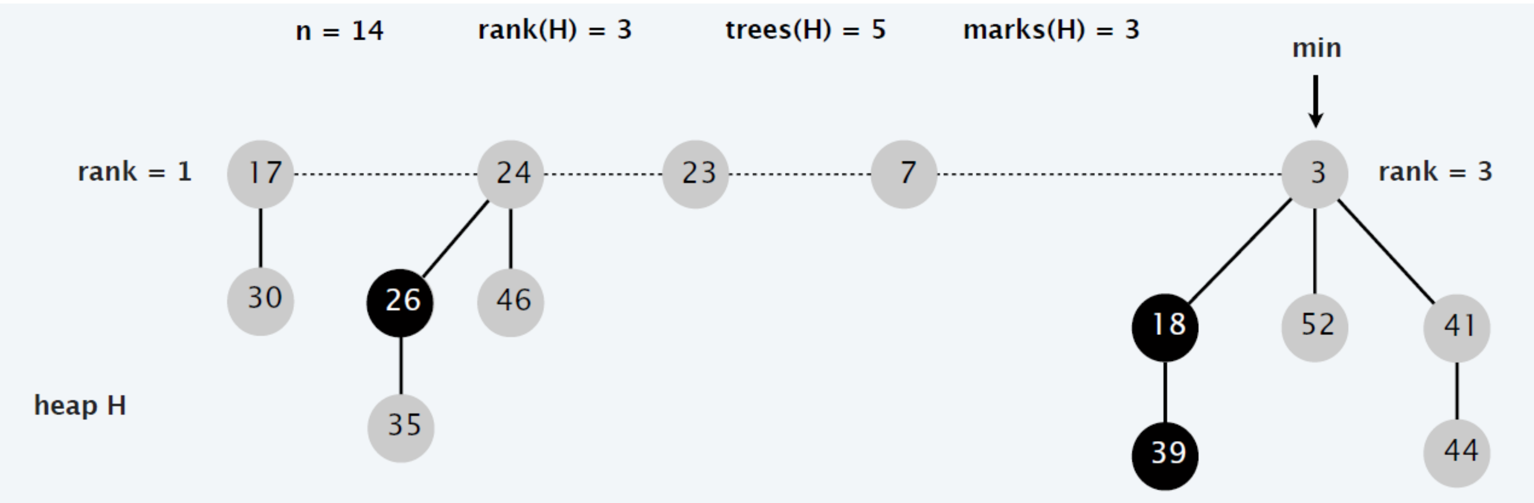
Fibonacci Heaps

Amortized Analysis of DecreaseKey

- Amortized analysis: Potential function $\Phi(\mathscr{H})$ = number of trees + 2 * number of marked nodes



Fibonacci Heaps: Demonstration

- Illustrate use in shortest path computations

- https://kbaile03.github.io/projects/fibo_dijk/fibo_dijk.html

Recent Ideas in Priority Queues

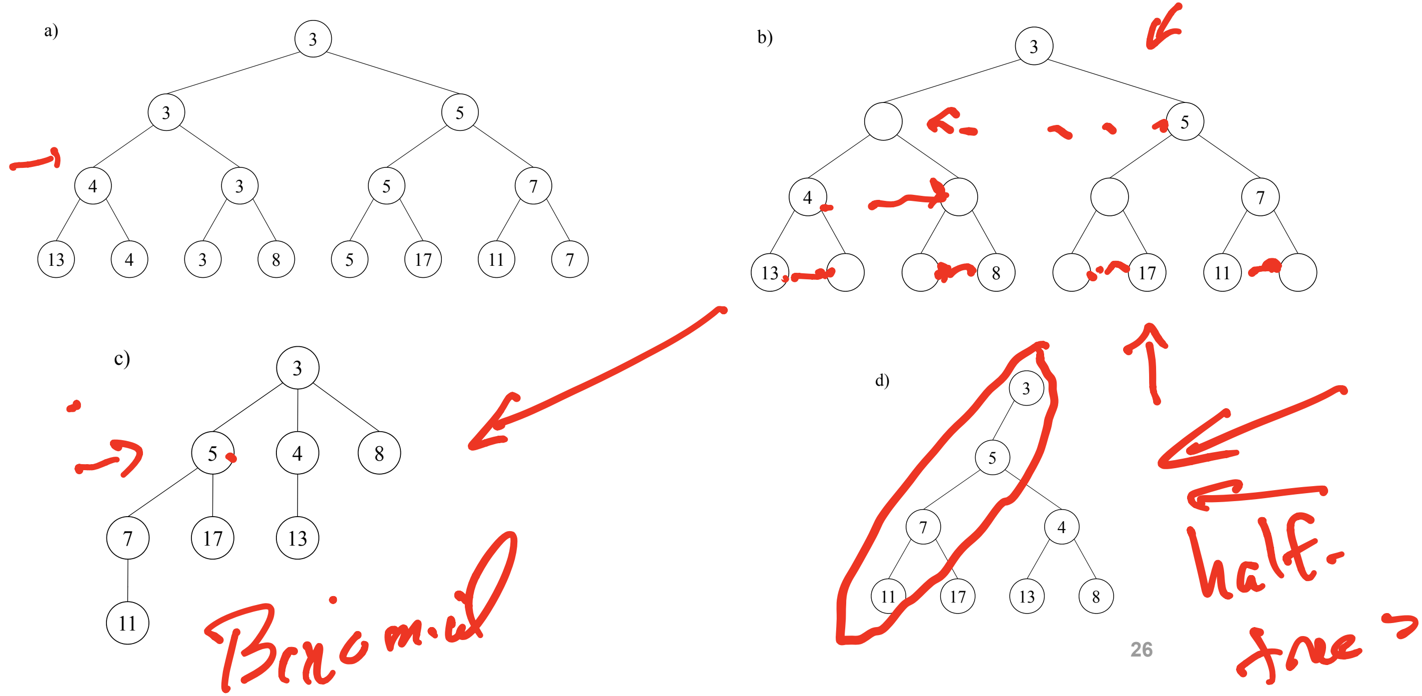
- Try to get complexity of Fibonacci heaps for DecreaseKey with simpler data structures 1985
- Novel concepts
  - Pairing Heaps
  - Quake heaps
  - Violation heaps
  - Rank-Pairing heaps

2008 -> ...

- Focus on rank-pairing heaps -> Tarjan ...

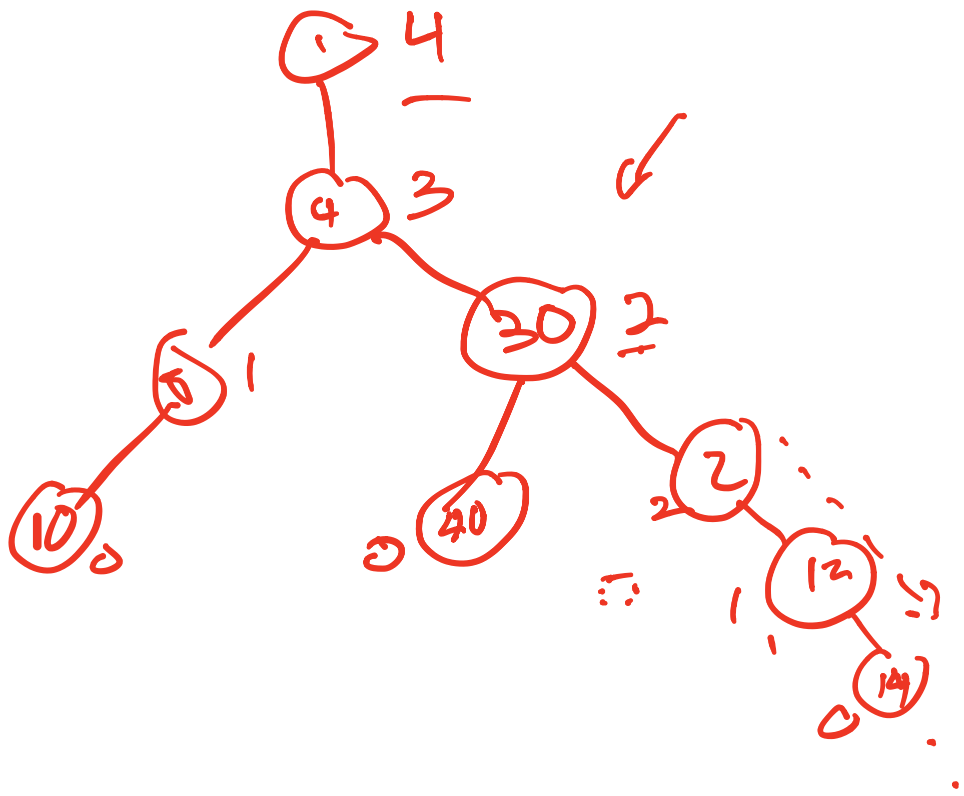
Rank-Pairing Heaps

- Background: Tournament trees



Binomial

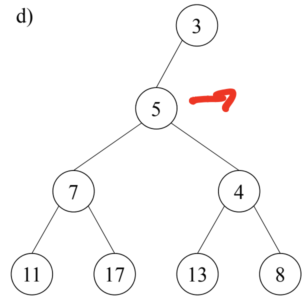
half-tree



Rank-Pairing Heaps

- Half-ordered binary trees (half-trees)
    - Root has single left child
    - Half-heap property: parent key less than or equal to left child

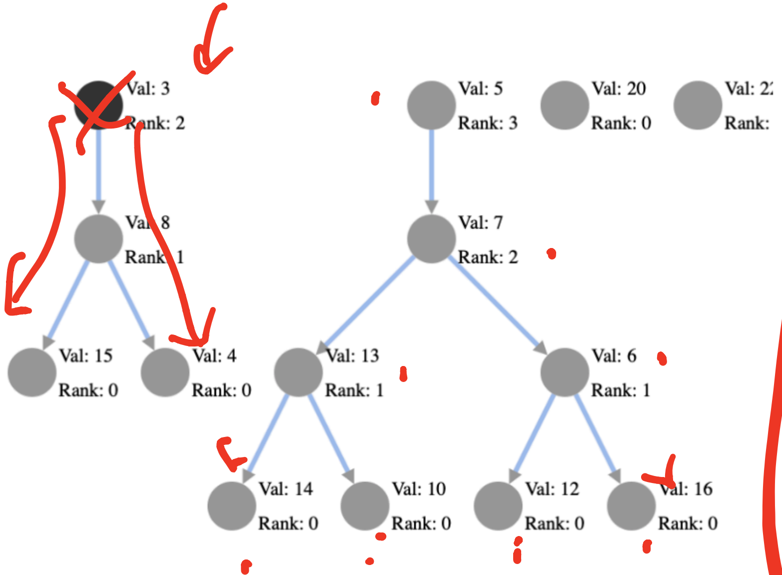
d)

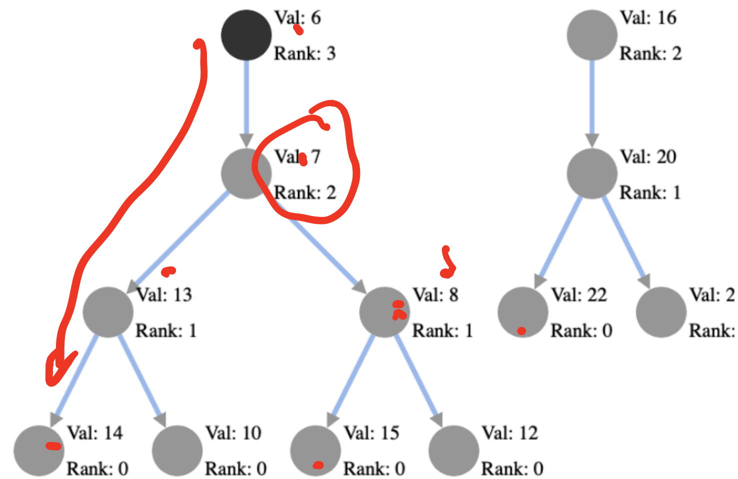


rank = 0

- Rank of a node in half trees: similar to height…
    - If node is leaf, rank = 0
    - If it is root, rank(parent) = 1+rank(left child)
    - If |rank(left child) - rank(right child)| <= 1: rank = 1 + max(rank(L),rank(R))
    - If |rank(L) - rank(R)| > 1: rank(parent)= max(rank(L), rank(R))

Half-Binary Trees

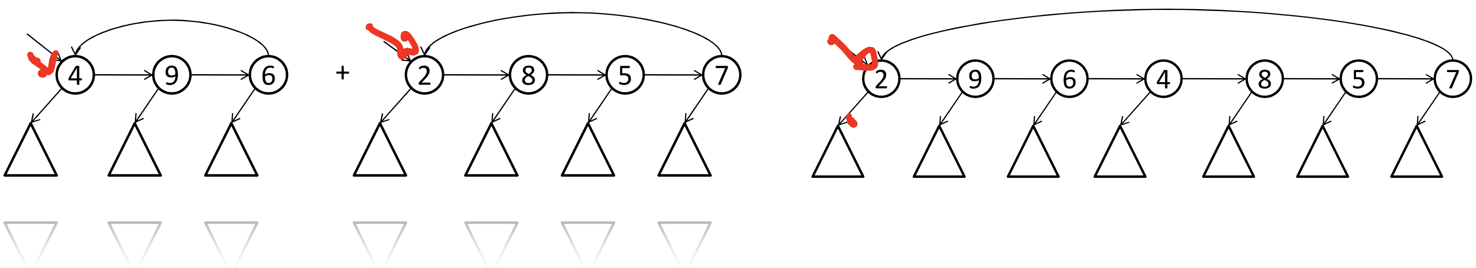




Rank-Pairing Heaps

- Rank-Pairing Heaps are a linked list of binary half-trees, with pointer to minimum root

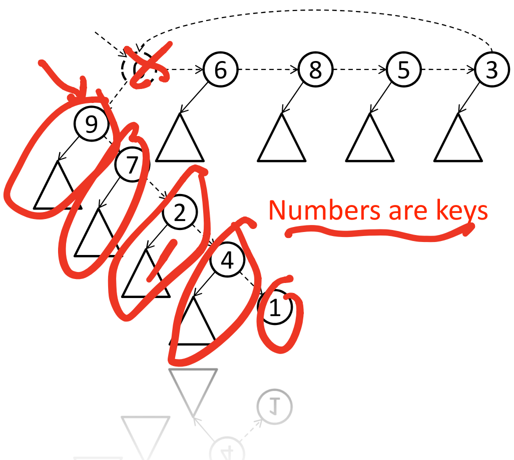
- Operations:
  - Insert: add a new single-node half-tree to list: O(1)
  - Merge: just link two lists, update min pointer: O(1)

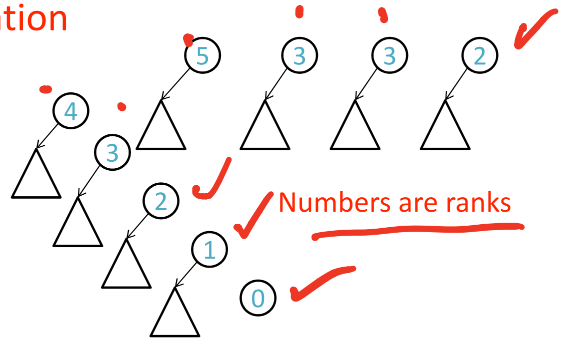


Rank-Pairing Heaps

- Operations (cont)
  - DeleteMin: Delete min-root.
    - Cut edges along right path down from new root to give new half-trees
    - Compress: merge roots of equal rank until no two roots have equal rank

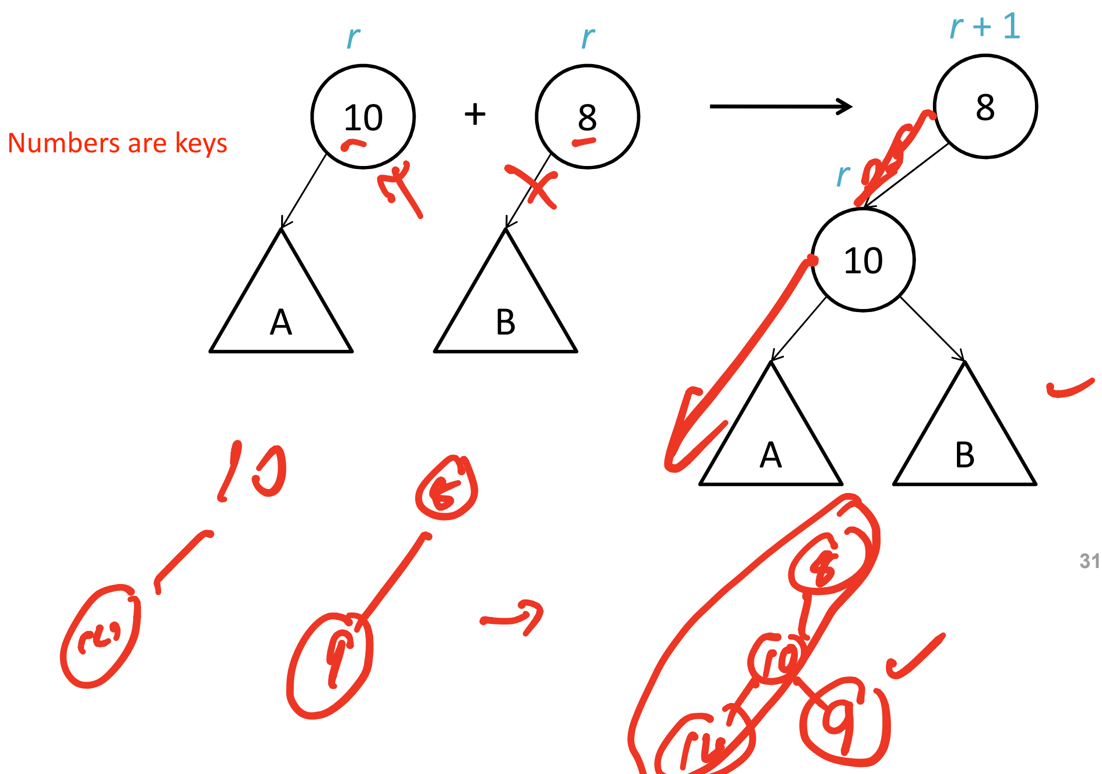
Cut Operation





Rank-Pairing Heaps

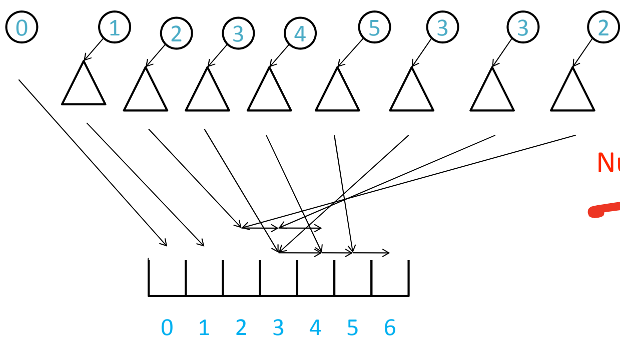
- Operations (cont)
  - Compress two trees of equal rank:



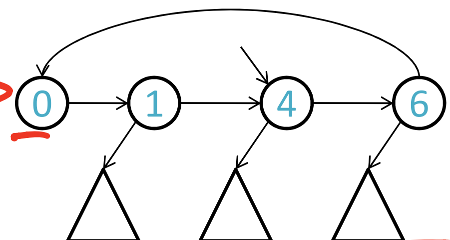
Rank-Pairing Heaps

- Operations (cont)
  - DeleteMin: Delete min-root.
    - Cut edges along right path down from new root to give new half-trees
    - Compress: merge roots of equal rank until no two roots have equal rank
    - Variation: Lazy compress: don't compress recursively

Merge

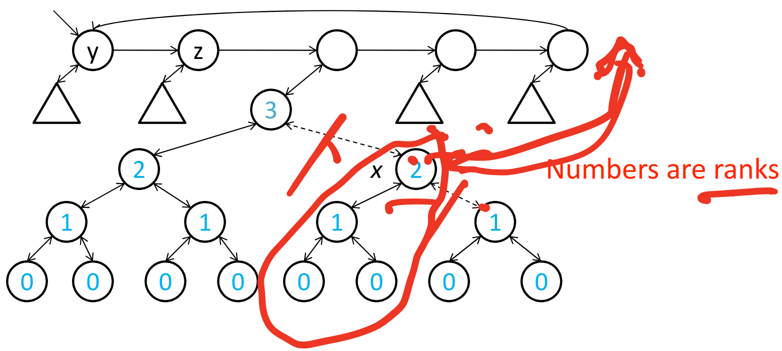


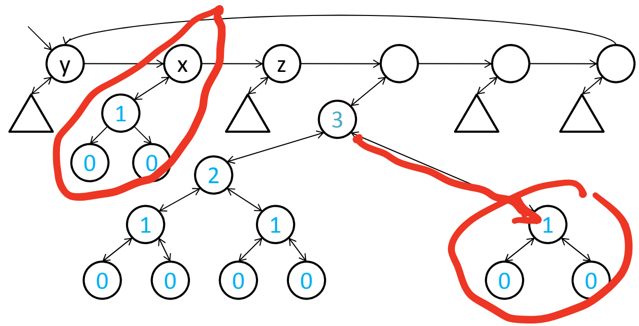
Numbers are ranks



Rank-Pairing Heaps

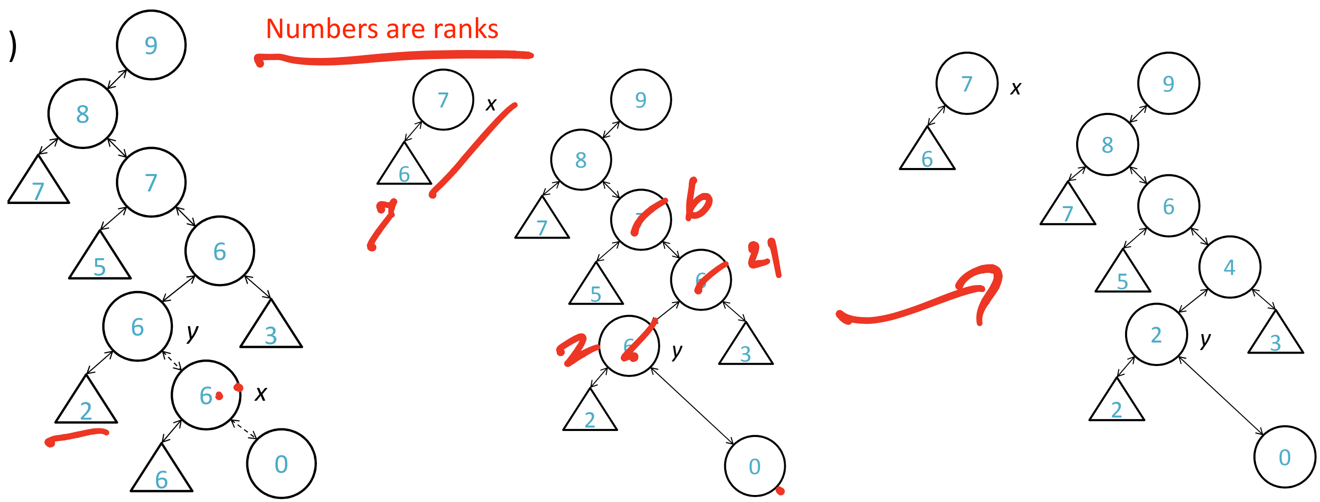
- Operations (cont)
  - DecreaseKey: Remove x and its left subtree to a new half-tree
    - Replace x by its right child. Change key of x to k. Add x to the list of half tree roots. Update the min-root.
    - Update the ranks





Rank-Pairing Heaps

- Operations (cont)
  - DecreaseKey: Update the ranks

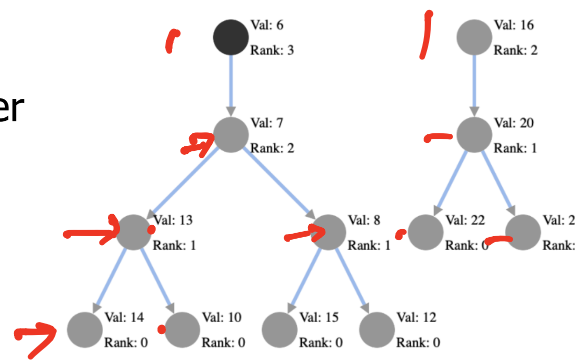


/ nodes in F.b. tree of rank h → $\boxed{\underline{k}^h \text{ nodes}}$

Rank-Pairing Heaps

$k^k \to \log(n)$

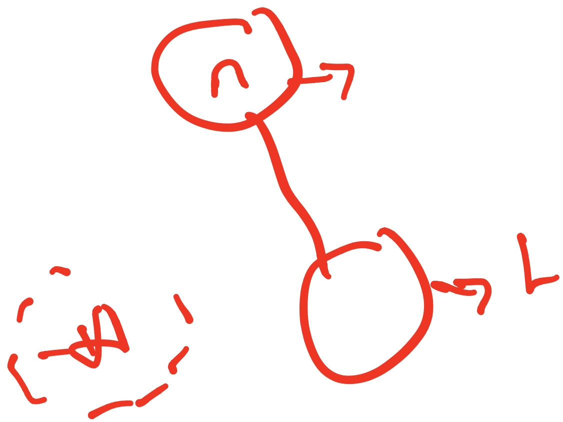
- Amortized analysis
  - Key result: number of nodes in a half tree with rank k is again bounded below by F_{k+2}, the Fibonacci number
  - Implies that the maximum height of half-trees in forest is O(log(n))
- Analysis based on potential function
  - Potential of a node: 1 if root, 0 if both children have same rank, 2 if one child's rank is 1 greater than other, j-1 if single child of rank j
  - Potential of heap: sum of potential of all node

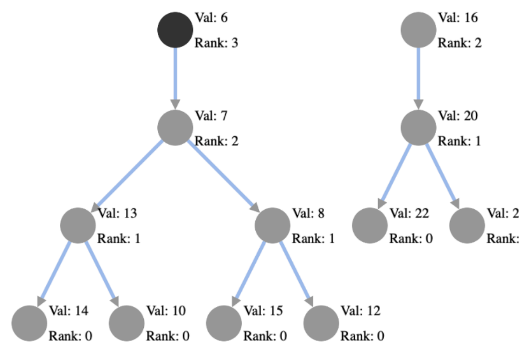


$\Phi(H) = 2$

Rank-Pairing Heaps: Demo

- https://skycocoo.github.io/Rank-Pairing-Heap/



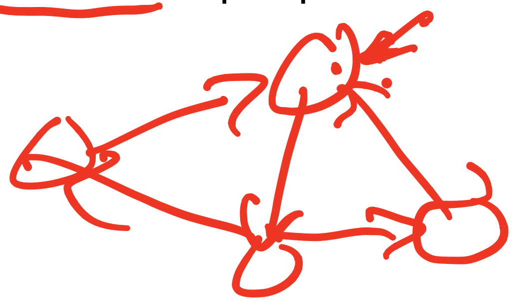


Rank-Pairing Heaps

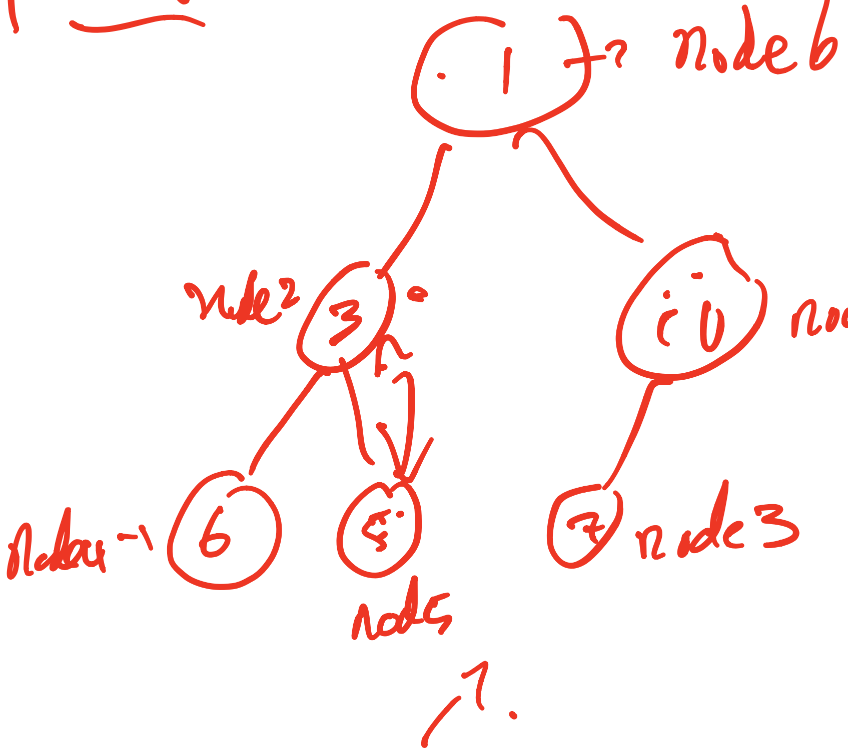
- Simpler implementation than Fibonacci heaps
- Same amortized complexity:
  - Insert: O(1)
  - DeleteMin: O(log(n))
  - DecreaseKey: O(1)
  - Merge: O(1)

Find = $O(n)$

- A subtle point in all heaps: need additional data structure to find nodes in O(1)
  - Not implemented in most standard libraries
  - Makes generic standard heap implementations have little utility …



Map: 10 → 2 ✓ . . .



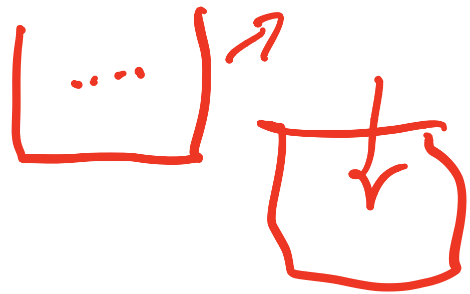
10 → 3 ✓

String Matching and Tries

- Consider the following problem:
  - Given a string T and k nonempty strings $P_1, P_2, \ldots, P_k$, find all occurrences of $P_1, P_2, \ldots, P_k$ in T.
  - T is called the text string and $P_1, P_2, \ldots, P_k$ are called pattern strings.
  - This problem was originally studied in the context of compiling indexes, but has found applications in computer security and computational genomics

- Simpler problem: k = 1
  - Knuth-Morris-Pratt (KMP) algorithm (CLRS)

- Harder problem: k > 1
  - Aho-Corasick tries



CLRS

Single Pattern Matching

- Given a string T[1:n-] of n characters in finite alphabet set S
- Nonempty string P = P[1:m] of characters in S

- Pattern P occurs with shift s in T[s:s+m-1] = P[1:m]

- Problem: Find all shifts s where Pattern occurs with shift s

- Naive algorithm:
  - For s in 1 to n-m+1:
    - If T[s:s+m-1] = P[1:m], add s to set of shifts where pattern occurs
  - Complexity: O(m(n-m+1)) $\approx O(m \cdot n)$

$\hookrightarrow O(n)$

Single Pattern Matching

- Example:
  - T[] = "AAAAAAAAAAAAAAAAAB"   (n=18)   s = 13
  - P[] = "AAAAB"   (m=5)

One shift matches: s=13

- Example   20
  - T[] = "ABABABCABABABCABABABC" (n=20) → 2
  - P[] = "ABABAC" (m=6)

No shifts match!

Knuth-Morris-Pratt Algorithm

- Idea: exploit that whenever we find a mismatch, we have already looked at a subset of the pattern
  - T[] = "AAAA   A   AAAAAAAAAAAAB"   (n=18)
  - P[] = "AAAA B"   (m=5)
  - If we get a mismatch on T[5], we don't have to start searching from k=2 for the next match! We know no shift can match before s = 6
- Idea: patterns have a prefix function
  - A string w is a **proper prefix** of a string x if x = w + v for some string non-empty string v, where + is concatenation
  - A string w is a **proper suffix** of a string x if x = v + w for some non-empty string v

ABCD

D suffix,
CD
BCD

Pre A AB ABC

Knuth-Morris-Pratt Algorithm

- Algorithm: preprocess pattern P[0:m-1] to compute function π[j]
  - π[j] = longest proper prefix of P[0:j] that is also a suffix of P[0:j], for j = 0 to m-1

  - Examples: P = "AAAA" —> π[] = [0, 1, 2, 3]
  - P = "ABCDE" —> π[] = [0,0,0,0,0]
  - P = "AABAACAABAA" —> π[] = [0,1,0,1,2,0,1,2,3,4,5]
  - P = "AAACAAAAAC" —> π[] = [0,1,2,0,1,2,3,3,4]
  - P = "AAABAAA" —> π[] = [0,1,2,0,1,2,3]

Knuth-Morris-Pratt Algorithm

- Analysis
  - String P: "a b a b a b a b c a"; Prefix function: $\pi(i)$: [0,0,1,2,3,4,5,6,0,1]
- Match into T: "a b a b **d** a a b a b a b a b c a d c d"

```
a b a b d a a b a b a b a b c a d c d
a b a b a b a b c a     Shift by π(3)

a b a b d a a b a b a b a b c a d c d
    a b a b a b a b c a shift by π(1)

a b a b d a a b a b a b a b c a d c d
        a b a b a b a b c a
```

```
a b a b d a a b a b a b a b c a d c d
            a b a b a b a b c a

a b a b d a a b a b a b a b c a d c d
              a b a b a b a b c a
```

Knuth-Morris-Pratt Algorithm

- Computing $\pi[j]$ from P[0:m-1]:

```
k= 0; j = 1;
π[0]=0;
While j < m:
  If P[k]==P[j]:
    k++; π[j]=k; j++;
  else if k == 0:   # first character did not match
    π[j] = 0; j++;
  else:   #mismatch after first character
    k = π[k-1]
```

w	c	a	b	c	a

arr	0	0	0	1	2

P	A	A	B	A	A	C	A	A	B
Pi	0	1	0	1	2	0	1	2	3
K	0	1	0	1	2	0	1	2	3
J	1	2	3	4	5	6	7	8	9

Knuth-Morris-Pratt Algorithm

```
k = 0; j = 0;
while (j < n)
    if P[k] == T[j]
        k++; j ++;
    if  k == m:
        print("match at  ", j-i)
        k=pi[k-1]
    else if (j < n and P[k] != T[j]):
        if k != 0
            k= pi[k-1]
        else
            j++;
return
```

T	A	A	B	A	A	A	A	A	B
P	A	A							
pi	0	1							
s	1	0	0	1	1	1	1	0	0
k	0/1	1/0	0	0/1	1/0	1/0	1/0	1/0	0
j	0	1	2	3	4	5	6	7	8

T	A	A	B	A	A	A	A	A	B
P	A	A	A						
pi	0	1	2						
s	0	0	0	1	1	1	0	0	0
k	0/1	1/2	2/0	0/1	1/2	2/1	2/1	2/1	0
j	0	1	2	3	4	5	6	7	8

Knuth-Morris-Pratt Algorithm

- Complexity analysis
  - For each position j in the string T:
    - If it does not match, you slide the pattern to all possible suffixes and check for a match
    - Worst case pattern: "AAAAAAA": $\pi[k] = k$ so lots of prefixes
- Let q = number of digits matched when checking position j.
  - Potential $\Phi(j) = q$. Note $\Phi(0) = 0,\ \Phi(end) = 0$
  - Cost of checking position j: c(j) = number of iterations of k = $\pi[k-1]$ for this j
    - Cost is 1 + c(j)
  - Each iteration k = $\pi[k-1]$ decreases number matched by at least one
  - Amortized cost for position j: 1 + c(j) + $\Phi(j) - \Phi(j-1)$ = 1 + c(j) - c(j): $\Theta(1)$
- To do n iterations: $\Theta(n)$

$k \rightarrow \pi(k-1) \rightarrow$

Knuth-Morris-Pratt Algorithm

- Demo: http://jovilab.sinaapp.com/visualization/algorithms/strings/kmp