Quantum interference channels

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quantum mechanics

popular view

In films and books by self-proclaimed life coaches:

- quantum mechanics is magic !
- stuff happens because you think about it ...

science view

- quantum mechanics is linear algebra

science view

- quantum mechanics is linear algebra
- replace bits^{01} with $\vec{\text{q}}$ubits
- qubits live in a two dimensional complex vector space :

$$|x\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha, \beta \in \mathbb{C} \tag{1}$$

- don't be fooled by the $|.\rangle$ notation...
  $|\psi\rangle \equiv \vec{\psi}$ it is just a neat notational trick for dot products...
- describe quantum systems with finite degrees of freedom like electron spin, and light polarization

quantum source

- A quantum source is probabilistic mixture of qubits $\{p(x), |x\rangle\}$
- Can be represented as a density matrix

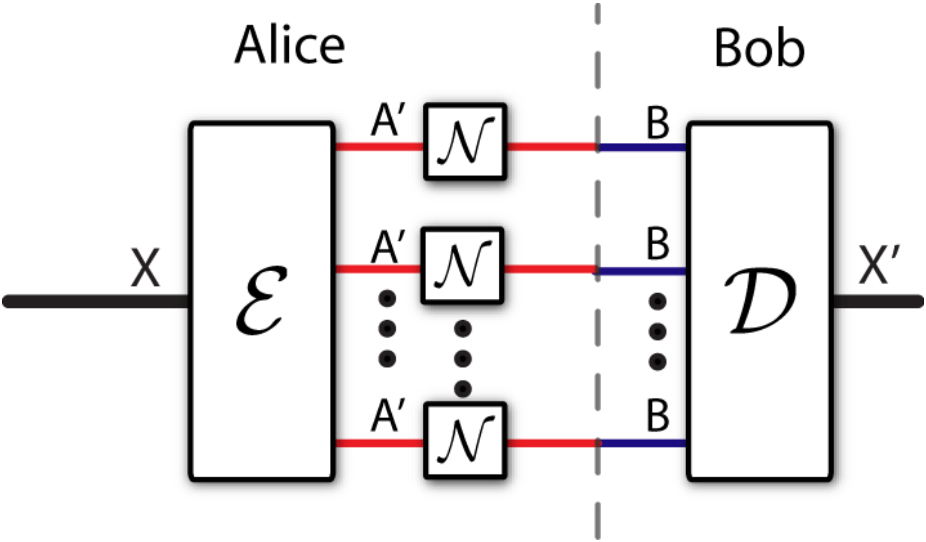
$$\rho = \sum p(x)|x\rangle\langle x| \tag{2}$$

- This way $H(\rho)$ models both classical uncertainty $p(x)$ (which quantum state did the source produce?)
- and quantum uncertainty (given that the sate produced was $|x\rangle$ the probability of finding $|0\rangle$ if measure is $|\alpha|^2$)

quantum channel

- A quantum channel takes states to states: $\mathcal{N}(\rho) = \rho'$
- Completely positive trace preserving (CPTP) maps

quantum channel



encoding

We can use a quantum channel to send either classical or quantum data.

For classical data, the encoding operation will be $\mathcal{E}\colon \mathcal{X} \to \mathbb{C}^2$,

$$\mathcal{E}(x) = |\sigma_x\rangle, \tag{3}$$

where $|\sigma\rangle_x$ is some predetermined set of “signal states” that will be send over the channel.

To encode some quantum data ρ we can do any quantum operation $\mathcal{E}\colon \mathbb{C}^2 \to \mathbb{C}^2$,

$$\mathcal{E}(\rho) = \sigma. \tag{4}$$

decoding

If we are sending classical data over the channel we will want to extract the classical message from the received quantum state.

Measurements act on density matrices to produce a classical output as well as a possibly modified quantum state. Measurements are sets of projection operators $\{M_i\}$.

The probability of outcome m occurring when the input system is ρ is

$$p(m) = \mathsf{Tr}\left(M_m^\dagger \rho M_m\right).$$

Our example from earlier

$$\Pr\{\text{measure } x = 0\} = \mathsf{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = |\alpha|^2$$

quantum decoding critetion

In the error criterion for classical information theory protocols is of the form $P_e = Pr\{M_1 \neq \hat{M}_1\}$. We need an analogous criterion for quantum protocols.

The fidelity between two pure quantum states is the square of their inner product

$$F(|\varphi\rangle, |\psi\rangle) = |\langle\varphi|\psi\rangle|^2 . \tag{5}$$

Two states that are very similar have fidelity close to 1. Success criterion is

$$F(|M\rangle, |\hat{M}\rangle) \geq 1 - \epsilon. \tag{6}$$

new quantum resources

- Quantum Bit (q-bit)
- Shared maximally entangled pairs: [q q] (e-bits)
  ex: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- e-bits have no analog in classical i
- Noiseless Quantum Channel: $[q \to q]$
- Noisy channel: $\{q \to q\}$ usually denoted $\mathcal{N}$.

quantum channels

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quantum information theory

- Came from classical information theory
- Replace Shannon entropy with von Neumann entropy

Definition (von Neumann Entropy)

A quantum system described by the density matrix ρ has von Neumann entropy $S(\rho)$ given by the expression:

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \tag{7}$$

- `from classical import mutual_information, Fano inequality, random_coding, channel_capacity, compression, etc..`

classical capacity of quantum channel

Holevo, Schumacher and Westmoreland found a formula for the classical capacity of a quantum channel [H98,SW97].

Theorem

The classical capacity of a quantum channel $\mathcal{N}$ is given by

$$C(\mathcal{N}) = \max_{\rho} I(A;B) \tag{8}$$

where ρ is the output state

$$\rho = \sum_x p(x)|x\rangle\langle x|_A \otimes \mathcal{N}(\sigma_x)_B. \tag{9}$$

and σ_x are the input symbols.

quantum capacity

Lloyd, Shor and Devetak independently proved a formula for the quantum capacity of a channels.

Theorem (LSD Theorem)

Consider the input state $\rho^{AA'}$, half of which is sent through the channel $\mathcal{N}$ to obtain $w^{AB} = \mathcal{N}^{A' \to B}(\rho^{AA'})$. Define the quantity

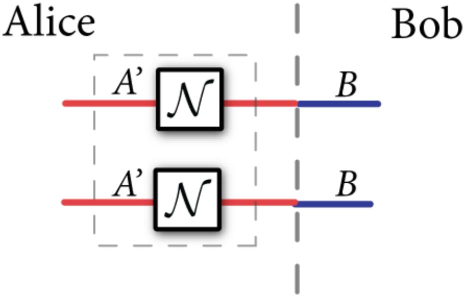
$$Q^{(1)}(\mathcal{N}) = \max_{\rho}(S(B)_w - S(AB)_w). \tag{10}$$

The quantum capacity Q of the channel $\mathcal{N}$ is given by the regularization of the $Q^{(1)}$ quantity

$$Q(\mathcal{N}) = \lim_{n \to \infty} \frac{Q^{(1)}(\mathcal{N}^{\otimes n})}{n}. \tag{11}$$

[L97, S00, D03]

limitations of these results



Not single letter ...

the end

thanks for your attention!

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physicist's view

- Very similar to what we learn in signals and systems

physicist's view

- Very similar to what we learn in signals and systems
- A quantum state (signal) is represented by a *wavefunction*

$$\psi\colon \mathbb{R}^3 \to \mathbb{C} \tag{12}$$

- The wave function of Hydrogen's ground state (n=1 ℓ=0)

$$\psi(r, \vartheta, \varphi) = \sqrt{\frac{1}{\pi a^3}} \exp\left(\frac{-r}{a}\right), \tag{13}$$

in the *position* basis. Apply Fourier transform to get the *momentum* basis

physicist's view (continued)

- The power of the signal is a probability density

$$\Pr\{\text{finding electron at } \vec{r}\,\} = |\psi(\vec{r})|^2 \tag{14}$$

- Verify that it is well normalized

$$\begin{aligned} P_{total} &= \iiint |\psi(\vec{r})|^2 \, d^3\vec{r} \\ &= \int_0^\infty \int_0^{2\pi} \int_0^\pi |\psi(r,\vartheta,\varphi)|^2 \, r^2 \, \sin\varphi d\varphi d\vartheta dr \\ &= \int_0^\infty \frac{4}{a^3} \exp\left(\frac{2r}{a}\right) r^2 \, dr = 1 \end{aligned} \tag{15}$$