Quantum interference channels

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Overview

1. Definitions
2. Best bounds on the classical Interference Channel
3. Intro to Quantum Information Theory
4. Quantum MAC
5. Quantum Interference Channel

Definitions

Definition (Interference network)

A two party interference network $(\mathcal{X}_1 \otimes \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1 \otimes \mathcal{Y}_2)$ is general model for communication networks with two inputs, two outputs and a probability transistion matrix $p(y_1, y_2|x_1, x_2)$.

Definition (Interference channel)

A two party interference channel is a particular use of an interference network $(\mathcal{X}_1 \otimes \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1 \otimes \mathcal{Y}_2)$ where messages M_1, M_2 are independently encoded into a codewords X_1, X_2 at rates (R_1, R_2) with Y_1, Y_2 being the intended receiver.

Best inner bound

Han Kobayashi / Cheng Motani Garg inner bound

$$G_{HK} = G_{CMG} = \cup_{p(x_1|u_1)p(x_2|u_2)p(u_1)p(u_2)}\{(R_1, R_2)|\text{ Eqn. (1) }\}$$

$$\begin{aligned}
R_1 &\leq I(X_1;Y_1|U_2Q) \\
R_2 &\leq I(X_2;Y_2|U_1Q) \\
R_1+R_2 &\leq I(X_1U_2;Y_1|Q)+I(X_2;Y_2|U_1U_2Q) \\
R_1+R_2 &\leq I(X_1;Y_1|U_1U_2Q)+I(X_2U_1;Y_2|Q) \\
R_1+R_2 &\leq I(X_1U_2;Y_1|U_1Q)+I(X_2U_1;Y_2|U_2Q) \\
2R_1+R_2 &\leq I(X_1U_2;Y_1|Q)+I(X_1;Y_1|U_1U_2Q)+I(X_2U_1;Y_2|U_2Q) \\
R_1+2R_2 &\leq I(X_2;Y_2|U_1U_2Q)+I(X_2U_1;Y_2|Q)+I(X_1U_2;Y_1|U_1Q)
\end{aligned} \tag{1}$$

First outer bound

$$G_{Naive} \triangleq \operatorname{conv} \cup_{p(x_1)p(x_2)} \{(R_1, R_2) | \text{ Eqn. (3) } \} \tag{2}$$

$$\begin{aligned} R_1 &\leq I(X_1; Y_2 | X_2) \\ R_2 &\leq I(X_2; Y_2 | X_1) \end{aligned} \tag{3}$$

Sato outer bound

$$G_{Sato} \triangleq \mathrm{conv} \cup_{p(x_1)p(x_2)} \{(R_1, R_2) | \text{ Eqn. (5) } \} \tag{4}$$

$$\begin{aligned} R_1 &\leq I(X_1; Y_2 | X_2) \\ R_2 &\leq I(X_2; Y_2 | X_1) \\ R_1 + R_2 &\leq I(X_1 X_2; Y_1 Y_2) \end{aligned} \tag{5}$$

Best outer bound

Consider the two random variables Z_1, Z_2 such that

$$\begin{array}{lll} Y_1 & \text{is a degraded version of} & Z_1 \\ Y_2 & \text{is a degraded version of} & Z_2 \\ Y_2 & \text{is a degraded version of} & (X_1, Z_1) \\ Y_1 & \text{is a degraded version of} & (X_2, Z_2) \end{array}$$

$$G_{Carl} \triangleq \operatorname{conv} \cup_{p(x_1)p(x_2)} \{ (R_1, R_2) | \text{ Eqn. (7) } \} \tag{6}$$

$$\begin{aligned} R_1 &\leq I(X_1; Y_1 | X_2) \\ R_2 &\leq I(X_2; Y_2 | X_1) \\ R_1 + R_2 &\leq \min\{ I(X_1 X_2; Z_1), I(X_1 X_2; Z_2) \} \end{aligned} \tag{7}$$

Quantum mechanics

Popular view

In films and books by self-proclaimed life coaches:

- quantum mechanics is magic !
- stuff happens because you think about it ...

Science view

- quantum mechanics is linear algebra and probability

Science view

- quantum mechanics is linear algebra and probability
- replace bits^{01} with $\vec{\text{q}}$ubits
- qubits live in a two dimensional complex vector space $\mathcal{H}$:

$$|x\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha, \beta \in \mathbb{C} \tag{8}$$

- don't be fooled by the $|.\rangle$ notation...
  $|\psi\rangle \equiv \vec{\psi}$ it is just a neat notational trick for dot products...
- Good for systems with finite degrees of freedom like electron spin, and light polarization
- Another *representation* exists for continuous systems via wave function

Quantum source

- A quantum source mixture of qubits $\{p(x), |\sigma_x\rangle\}$, where state $|\sigma_x\rangle$ is produced by the source with probability $p(x)$.
- We represent it as a density matrix

$$\rho = \sum_x p(x)|\sigma_x\rangle\langle\sigma_x| \tag{9}$$

- Models both classical uncertainty $p(x)$ (which quantum state did the source produce?)
- and quantum uncertainty (given that the sate produced was $|x\rangle$ the probability of finding $|0\rangle$ if measure is $|\alpha|^2$)

Examples

- Pure quantum state $|x\rangle = \alpha|0\rangle + \beta|1\rangle$ has a density matrix

$$\rho_x = |x\rangle\langle x| = \begin{bmatrix} |\alpha|^2 & \beta^*\alpha \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}. \tag{10}$$

- Specific example $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ becomes $\rho_+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$
- A mixed state $\rho = \{p(x), |\sigma_x\rangle\}$ with $p(0) = \frac{1}{4}, |\sigma_0\rangle = |0\rangle$ and $p(1) = \frac{3}{4}, |\sigma_1\rangle = |1\rangle$ gives a density matrix:

$$\rho = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1| = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}.$$

- Density matrices are positive and have unit trace

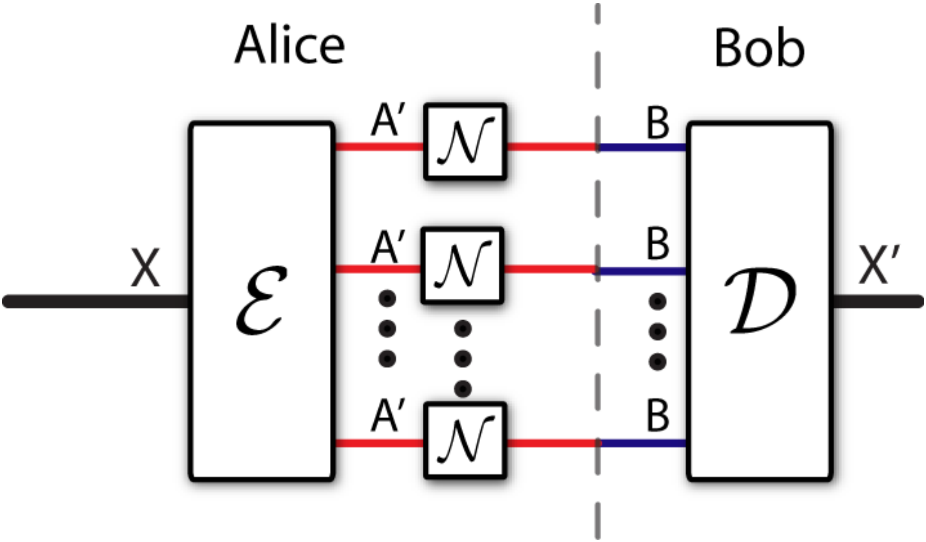
Quantum channel

- A quantum channel $\mathcal{N}: \mathcal{H}^A \to \mathcal{H}^B$ takes states to states

$$\mathcal{N}(\sigma^A) = \rho^B \tag{11}$$

- Completely positive, trace preserving (CPTP) map
- Analogue of cond. probability of a classical channel $p(y|x)$

Quantum channel



Encoding

We can use a quantum channel to send either classical or quantum data.

For classical data, the encoding operation will be $\mathcal{E}\colon \mathcal{X} \to \mathcal{H}^A$,

$$\mathcal{E}(x) = |\sigma_x\rangle^A \tag{12}$$

where $|\sigma\rangle_x$ is some predetermined set of “signal states” that will be send over the channel.

To encode some quantum data ρ we can do any quantum operation $\mathcal{E}\colon \mathcal{H}^S \to \mathcal{H}^A$,

$$\mathcal{E}(\rho^S) = \sigma^A. \tag{13}$$

Decoding classical data

We want to extract the classical message from the received quantum state.

Measurements act on density matrices to produce a classical output as well as a possibly modified quantum state. Measurements are sets of projection operators $\{M_i\}$.

The probability of outcome m occurring when the input system is ρ is

$$p(m) = \mathrm{Tr}\left(M_m^\dagger \rho M_m\right).$$

Our example from earlier

$$\Pr\{\text{measure } x = 0\} = \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} |\alpha|^2 & \beta^*\alpha \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = |\alpha|^2$$

Decoding quantum data

In the error criterion for classical information theory protocols is of the form $P_e = Pr\{M_1 \neq \hat{M}_1\}$. We need an analogous criterion for quantum protocols.

The fidelity between two pure quantum states is the square of their inner product

$$F(|\varphi\rangle, |\psi\rangle) = |\langle\varphi|\psi\rangle|^2 . \tag{14}$$

Two states that are very similar have fidelity close to 1. Success criterion is

$$F(|M\rangle, |\hat{M}\rangle) \geq 1 - \epsilon. \tag{15}$$

Quantum information theory

Quantum information theory

- Came from classical information theory
- Replace Shannon entropy with von Neumann entropy

Definition (von Neumann Entropy)

A quantum system described by the density matrix ρ has von Neumann entropy $S(\rho)$ given by the expression:

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \tag{16}$$

- `from classical import mutual_information, Fano_inequality, random_coding, channel_capacity, compression, etc..`

Classical capacity of quantum channel

Holevo, Schumacher and Westmoreland found a multiletter formula for the classical capacity [H98,SW97].

Theorem

The classical capacity of a quantum channel $\mathcal{N}$ *is given by* $\mathcal{C}(\mathcal{N}) = \frac{1}{n} \cup_{n=1}^{\infty} C^{(1)}(\mathcal{N}^{\otimes n})$ *where*

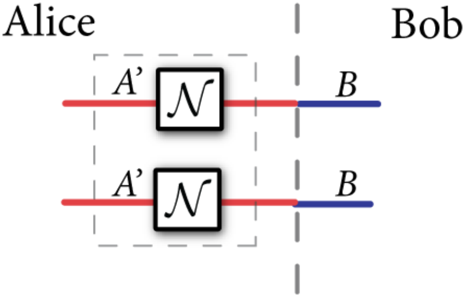
$$C^{(1)}(\mathcal{N}) = \max_{\rho} I(A;B) \tag{17}$$

where ρ *is the output state*

$$\rho = \sum_x p(x)|x\rangle\langle x|_A \otimes \mathcal{N}(\sigma_x)_B. \tag{18}$$

and $\{p(x), \sigma_x\}$ *is the input distribution.*

Limitations of these results



Not single letter ...

Quantum multiple access channel

Definition

The quantum multiple access channel (QMAC) is a map

$$\mathcal{M}^{A'B'\to C} \tag{19}$$

where systems A' and B' are the inputs and C is the output.

Three different capacities

- $\mathcal{C}$: classical info
- $\mathcal{Q}$: quantum info
- $\mathcal{C}_E$ or $\mathcal{Q}_E$: with additional resource *entanglement*

Classical capacity of QMAC

Theorem (Winter 01)

The capacity of the quantum MAC $\mathcal{M}$ to carry classical information is $\mathcal{C}_{QMAC}(\mathcal{M}) = \frac{1}{n} \cup_{n=1}^{\infty} \mathcal{C}_{QMAC}^{(1)}(\mathcal{M}^{\otimes n})$ where

$$\mathcal{C}_{QMAC}^{(1)} = \operatorname{conv} \cup_{p(x_1)p(x_2)\sigma_x} \{(R_1, R_2) | \text{ Eqn. (21) } \} \tag{20}$$

$$\begin{aligned} R_1 &\leq I(A;C|B)_\theta, \\ R_2 &\leq I(B;C|A)_\theta, \\ R_1 + R_2 &\leq I(AB;C)_\theta, \end{aligned} \tag{21}$$

where mutual informations are calculated with respect to

$$\theta = \sum_x p(x_1)p(x_2)|x_1\rangle\langle x_1|_A \otimes |x_2\rangle\langle x_2|_B \otimes \mathcal{M}(\sigma_{x_1} \otimes \sigma_{x_2})_C. \tag{22}$$

Quantum interference channel

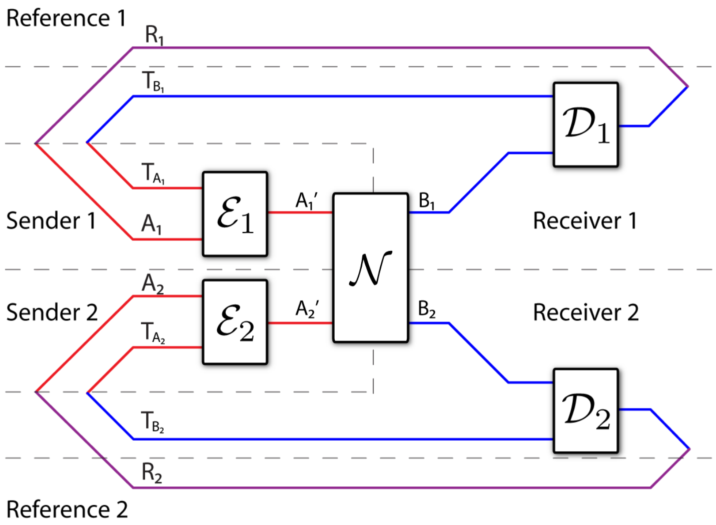
Definition

The quantum interference channel is a map

$$\mathcal{N}^{A_1' A_2' \to B_1 B_2} \tag{23}$$

Open problem.
can adapt some of the classical results...

Entanglement assisted



The end

thanks for your attention!

Physicist's view

- Very similar to what we learn in signals and systems

Physicist's view

- Very similar to what we learn in signals and systems
- A quantum state (signal) is represented by a *wavefunction*

$$\psi \colon \mathbb{R}^3 \to \mathbb{C} \tag{24}$$

- The wave function of Hydrogen's ground state (n=1 ℓ=0)

$$\psi(r, \vartheta, \varphi) = \sqrt{\frac{1}{\pi a^3}} \exp\left(\frac{-r}{a}\right), \tag{25}$$

  in the *position* basis. Apply Fourier transform to get the *momentum* basis

Physicist's view (continued)

- The power of the signal is a probability density

$$\Pr\{\text{finding electron at } \vec{r}\,\} = |\psi(\vec{r})|^2 \tag{26}$$

- Verify that it is well normalized

$$\begin{aligned} P_{total} &= \iiint |\psi(\vec{r})|^2 \, d^3\vec{r} \\ &= \int_0^\infty \int_0^{2\pi} \int_0^\pi |\psi(r,\vartheta,\varphi)|^2 \, r^2 \, \sin\varphi d\varphi d\vartheta dr \\ &= \int_0^\infty \frac{4}{a^3} \exp\left(\frac{2r}{a}\right) r^2 \, dr = 1 \end{aligned} \tag{27}$$