Duality geometrical intuition

Consider the vector space V and the basis $(v_1, v_2, \ldots, v_n)$ for V.

The concept of basis $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ for the the dual space V' is defined in 3.F in terms of the 1-0 properties: for $\varphi_i(v_j) = \delta_{ij}$, where δ_{ij} is the Dirac delta function (1 when $i = j$, and 0 otherwise).

Axler leaves the φs only defined abstractly, as functionals $\mathcal{L}(V, \mathbf{F})$, where V is any abstract vector space. If we restrict out attention to the case $V = \mathbb{R}^n$ (or $\mathbf{F}^n$ if you prefer), then we can gain some valuable intuition about the nature of the φs.

As Isak pointed out, we can think of the φs as function of the form:

$$\varphi(v) = \vec{a} \cdot v,$$

for some vector $\vec{a}$. In other words, each φ_i corresponds to some vector $\vec{a}_i$ and applying φ_i to vector v is equivalent to computing the dot product $\vec{a} \cdot v$.

Geometrically speaking, the functions φ_i compute the distance from a plane passing through the origin that has normal vector $\vec{a}_i$, up to a constant factor related to $\|\vec{a}_i\|$.

Example 1: dual to the standard basis

Consider $V = \mathbb{R}^2$ and the standard basis for V that consists of

$$e_1 = (1, 0) \quad \text{and} \quad e_2 = (0, 1).$$

The dual basis consists of the functionals

$$\varphi_1(v) = (1, 0) \cdot v \quad \text{and} \quad \varphi_2(v) = (0, 1) \cdot v,$$

which manifestly satisfy the 1-0 proprty $\varphi_i(e_j) = \delta_{ij}$.

Example 2: dual to a non-orthogonal basis

Consider again $V = \mathbb{R}^2$ and the basis for V that consisting of the vectors

$$v_1 = (2, 0) \quad \text{and} \quad v_2 = (1, 1).$$

The dual basis consists of the functionals

$$\varphi_1(v) = (\tfrac{1}{2}, -\tfrac{1}{2}) \cdot v \quad \text{and} \quad \varphi_2(v) = (0, 1) \cdot v,$$

which we can verify satisfies the 1-0 property $\varphi_i(v_j) = \delta_{ij}$.

Credit David for providing the algorithm for finding the vectors $\vec{a}_i$ needed to create the dual basis functionals φ_i. The algorithm is:

1. vertically stack vectors v_i into a matrix M ($M_{i;\cdot} = v_i$)
2. find the inverse of M
3. read-off a_is as the columns of M^{-1}

You can see SymPy code that implements the above steps for the vectors v_1 and v_2 here.

Understanding dual representations geometrically

Recall we said that the elements of the dual basis φ_i can be interpreted geometrically as functions that compute the distance to the plane passing through the origin with normal vector $\vec{a}_i$, up a scaling constant.

Let's revisit the two examples to see how this geometric intuition applies to them.

Example 1

Every vector $v \in V$ can be represented as a linear combination of the basis vectors e_1 and e_2:

$$v = \alpha_1 e_1 + \alpha_2 e_2,$$

where the coefficients α_i correspond to the "how much of e_i is in $\vec{v}$" calculations.

Equivalently, the same vector v can be represented in terms of two coefficients β_1 and β_2 obtained by asking the questions "how far is $\vec{v}$ from the planes with normal vector $\vec{a}_1 = (1, 0)$ and $\vec{a}_2 = (0, 1)$."

$$\beta_1 = \varphi_1(v) = \vec{a}_1 \cdot v \quad \text{and} \quad \beta_2 = \varphi_2(v) = \vec{a}_2 \cdot v.$$

In this case, we're dealing with the an orthonormal basis e_i, which leads to self-dual vectors $\vec{a}_i$, and the coefficients α_i and β_i are the same.

Example 2

Every vector $v \in V$ can be represented as a linear combination of the basis vectors v_1 and v_2:

$$v = \alpha_1 v_1 + \alpha_2 v_2,$$

and we know that α_1 and α_2 are unique coefficients that make up the linear combination (since v_1 and v_2 are linearly independent). We can still interpret α_1 and α_2 as "how much of v_i is needed to get $\vec{v}$".

Equivalently, the same vector v can be represented in terms of two coefficients β_1 and β_2 obtained by asking the questions "how far is $\vec{v}$ from the planes with normal vector $\vec{a}_1 = (\frac{1}{2}, -\frac{1}{2})$ and $\vec{a}_2 = (0, 1)$."

$$\beta_1 = \varphi_1(v) = \vec{a}_1 \cdot v \quad \text{and} \quad \beta_2 = \varphi_2(v) = \vec{a}_2 \cdot v.$$

Caveat: the coefficients β_i are not actually equal to the distances from the plane φ_i as claimed above, but related by a constant. If we wanted a to have a representation in terms of actual, geometrical distances to the planes, we'd have to use the coefficients $\hat{\beta}_i$ defined as

$$\hat{\beta}_1 = \hat{a}_1 \cdot v \quad \text{and} \quad \hat{\beta}_2 = \hat{a}_2 \cdot v,$$

where $\hat{a}_i = \frac{\vec{a}_i}{\|\vec{a}_i\|}$ are unit-length vectors in the same direction as $\vec{a}_i$.

Since the real geometrical distances $\hat{\beta}_i$ and the outputs of the functionals φ_i are related by the constant $\|\vec{a}_i\|$ we allow ourselves to use "loose" language and refer to the coefficients β_i as distances, but keep in mind actual distances are $\hat{\beta}_i = \beta_i / \|\vec{a}_i\|$.

Graphs of functionals

Another way to think about the functionals $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ is in terms of their "graph" in $\mathbb{R}^3$.

The graph of the multivariable function $f : \mathbb{R}^2 \to \mathbb{R}$ consists of all points $(x, y, f(x, y))$ in $\mathbb{R}^3$, for all possible $(x, y) \in \mathbb{R}^2$. In the case of the functions φ_1 and φ_2, the graph are planes.

Isak pointed out the general notion that three points in $\mathbb{R}^3$ uniquely define a plane. He also observed that we can use the 1-0 property $\varphi_i(v_j) = \delta_{ij}$ to obtain the coordinates of three points contained in each φ_i, and hence we can find the plane associted with φ_i this way (in particular this shows the dual basis φ_1 and φ_2 is unique).

Specifically, we know that:

- $\varphi_1(v_1) = 1$, $\varphi_1(v_2) = 0$, and $\varphi_1(0) = 0$
- $\varphi_2(v_1) = 0$, $\varphi_2(v_2) = 1$, and $\varphi_2(0) = 0$

The zero values tell us the graph of each φ_i intersects xy-plane on some line. The plane that corresponds to φ_1 some "twist" along the line 0 to v_i. The value $\varphi_i(v_i) = 1$ tells us how much to twist.