

MECHANICS in FOUR PAGES

1 Introduction

Mechanics is the precise study of the motion of objects, the forces acting on them and more abstract concepts such as momentum and energy. You probably have an intuitive understanding of these concepts already, but in this tutorial you will learn how to use precise mathematical equations to support your intuition. Perhaps more importantly, I will also teach you where the equations come from.

The key concept which we want to obtain in a physics problem is the *equation of motion* $x(t)$ of an object which describes its position at all times. To find $x(t)$ we will start from Newton's second law $F = ma$, which says that forces *cause* objects to accelerate and that the amount of acceleration produced is inversely proportional to the mass of the object. Once we the acceleration of an object (the rate of change of its velocity), we can use *integration* to find the velocity of the object at all times $v(t)$ and from there we can also find its position as a function of time $x(t)$:

$$\frac{1}{m} \sum \vec{F} = a(t) \xrightarrow{\int dt} v(t) \xrightarrow{\int dt} x(t). \quad (1)$$

Note that the left hand side in the above equation involves the calculation of *net force* which is the sum of all the forces acting on the object: $\vec{F}_{\text{net}} \equiv \sum \vec{F}$.

$$U(h) = \int_0^h \vec{F}_g \cdot d\vec{y} = mgh. \quad (2)$$

You may have observed that both of the above equations contained vector quantities and involved integrals. Indeed, in order to properly understand the laws of physics, you need to know about vectors and a little bit of calculus. The background section will introduce these concepts.

2 Background

The basic quantities that will appear in the equations of physics are measured in meters [m], seconds [s], meters per second [m/s], meters per second squared [m/s²] and Newtons [N]. You should always keep in mind which units are being used (meters vs. kilometres) and *what type* the type quantity you are manipulating (length vs. velocity). You should also be aware whether the quantities are regular numbers or *vectors* with multiple components.

2.1 Vectors

Forces, velocities, and accelerations are vector quantities. When working with vectors you will have to work their *components*:

- $\vec{F} = (F_x, F_y)$: a vector quantity, a force.
- x -axis: The x -axis is the horizontal the co-ordinate system.
- y -axis: The y -axis is an axis *perpendicular* to the x -axis.
- F_x : the *component* of \vec{F} along the x axis.
- F_y : the *component* of \vec{F} along the y axis.
- \hat{i}, \hat{j} : Unit vectors in the x and y directions.

Given the xy coordinate system, we can denote a vector in three equivalent ways:

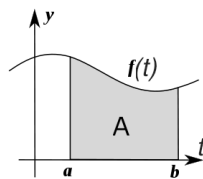
$$\vec{F} \equiv (F_x, F_y) \equiv F_x \hat{i} + F_y \hat{j} \equiv \|\vec{F}\| \angle \theta,$$

where $\|\vec{F}\|$ denotes the length of the vector \vec{F} and θ is the angle that it makes with the x axis.

2.2 Integrals

An integral corresponds to the computation of an area under a curve $f(t)$ between two points:

$$A(a, b) \equiv \int_{t=a}^{t=b} f(t) dt.$$

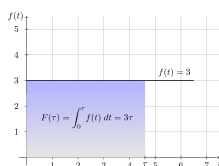


The symbol \int is a mnemonic for *sum*, since the area under the curve corresponds in some sense to the sum of the values of the function $f(t)$ between $t = a$ and $t = b$. The integral is the total amount of f between a and b .

Consider for example the constant function $f(t) = 3$, and let us find the expression $F(\tau) \equiv A(0, \tau)$ that corresponds to the area under $f(t)$ between $t = 0$ and the time $t = \tau$.

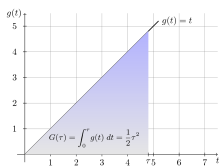
We can easily find this area because the region under the curve is rectangular:

$$F(\tau) = \int_0^\tau f(t) dt = 3\tau.$$



Indeed the area is equal to the height times the length of the base of the rectangle.

Another important calculation is the area under the function $g(t) = t$. We will compute $G(\tau) \equiv A(0, \tau)$, which corresponds to the area under $g(t)$ between 0 and τ . This area is also easily computed since the region under the curve is triangular:



$$G(\tau) = \int_0^\tau f(t) dt = \frac{\tau \times \tau}{2} = \frac{1}{2} \tau^2,$$

since the area of a triangle is the product of the length of the base times the height divided by two.

We were able to compute the above integrals thanks to the simple geometry of the areas under the curves. Later on in this book we will develop techniques for finding integrals of more complicated functions.

What you need to remember for now that the integral of a function is the total amount of the function accumulated during some time period. You should try to remember the formulas:

$$\int_c^\tau a dt = a\tau + C, \quad \int_c^\tau at dt = \frac{1}{2}a\tau^2 + C,$$

which correspond to the integral of a constant function $f(t) = a$ and the integral of $f(t) = at$, a line with slope a . Note that each time you give a general integral formula the answer will contain an additive constant term $+C$, which depends on the starting point $t = c$, which we use for the area calculation. In the above examples we used $c = 0$ as the initial point so we had $C = 0$.

Using the above formulas in combination, you can now compute the integral under of an arbitrary line $f(t) = mt + b$ as follows:

$$\int_c^\tau (mt + b) dt = \frac{1}{2}m\tau^2 + b\tau + C,$$

since the integral of the sum of two functions is the sum of the integrals.

Now that you know about some basic integrals, we can start learning about the laws of physics and derive the equation of motion for a particle in free fall.

3 Kinematics

Kinematics (from the Greek word *kinema* for *motion*) is the study of trajectories of moving objects. The equations of kinematics can be used to calculate how long a ball thrown upwards will stay in the air, or to calculate the acceleration needed to go from 0 to 100 km/h in 5 seconds.

3.1 Concepts

The key notions which we use to describe the motion of some object are:

- t : the time, measured in seconds [s].
- $x(t)$: the position of an object as a function of time, also known as the equation of motion.
- $v(t)$: the velocity of the object over time.
- $a(t)$: the acceleration of the object.
- $x_i = x(0), v_i = v(0)$: the initial conditions.

The position, velocity and acceleration functions ($x(t)$, $v(t)$ and $a(t)$) are connected. They all describe different aspects of the same motion. The function $x(t)$ is the main function since it describes the position of the object at all times. The velocity function describes the change in the position over time, hence it is measured in [m/s]. The acceleration function describes how the velocity changes over time. A constant positive acceleration means the velocity of the motion is steadily increasing, like when you press the gas pedal in your car. A constant negative acceleration means the velocity is steadily decreasing, like when pressing the brake pedal.

If you know the exact function $x(t)$, then you can compute its *derivative* and obtain the velocity function $v(t)$. You can obtain the acceleration function $a(t)$ by computing the derivative of the velocity $v(t)$:

$$a(t) \xleftarrow{\frac{d}{dt}} v(t) \xleftarrow{\frac{d}{dt}} x(t).$$

Alternately, **if you know the acceleration function $a(t)$,** you can use integration in order to obtain the velocity function $v(t)$ and then integrate the velocity function in order to obtain the position function $x(t)$:

$$a(t) \xrightarrow{\int dt} v(t) \xrightarrow{\int dt} x(t).$$

which means that we start from the acceleration $a(t)$ and use integration with respect to time to obtain the velocity $v(t)$. If we integrate the velocity we obtain the position function $x(t)$.

Recall that the integral is the calculation of the total

This is The main subject in this section is how to use

3.1.1 Uniform acceleration motion

(UAM)

This is how the *equations of motion* are derived. Assuming that the acceleration is constant in time $a(t) = a$, we can calculate the

velocity by *adding up* all the acceleration (integrating) to obtain the change in the velocity

$$v(t) = \int a(t) dt = \int a dt = at + v_i.$$

$$x(t) = \int v(t) dt = \int (at + v_i) dt = \frac{1}{2}at^2 + v_it + x_i.$$

3.1.2 Formula

If the object undergoes a *constant* acceleration $a(t) = a$, like your car if you floor the *accelerator* pedal, then the equations of motion are:

$$\begin{aligned} x(t) &= \frac{1}{2}at^2 + v_it + x_i, \\ v(t) &= at + v_i, \\ a(t) &= a. \end{aligned}$$

There is also another very useful equation to remember:

$$v(t)^2 = v_i^2 + 2a[x(t) - x_i],$$

which is usually written

$$v_f^2 = v_i^2 + 2a\Delta x.$$

That is it. Memorize these equations, plug-in the right numbers, and you can solve any kinematics problem humanly imaginable. Chapter done.

Moroccan example Suppose your friend wants to send you a ball wrapped in aluminum foil from his balcony, which is located on the 14th floor (height of 44.145[m]). At $t = 0$ [s] he *throws* the ball straight down with an initial velocity of 10[m/s]. How long does it take for the ball to hit the ground?

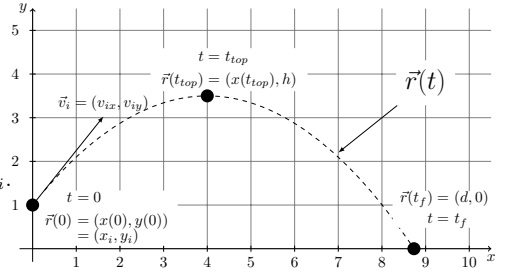
Assume that the y -axis measuring distance upwards starting from the ground floor. We know that the balcony is located at a height of $y_i = 44.145$ [m], and that at $t = 0$ [s] the ball starts with $v_i = -10$ [m/s]. The initial velocity is negative, because it points in the opposite direction to the y -axis. We know that there is an acceleration due to gravity of $a_y = -g = -9.81$ [m/s²].

We start by writing out the general UAM equation:

$$y(t) = \frac{1}{2}a_y t^2 + v_i t + y_i.$$

We want to find the time when the ball will hit the ground, so $y(t) = 0$. To find t , we plug in all the known values into the general equation:

$$y(t) = 0 = \frac{1}{2}(-9.81)t^2 - 10t + 44.145,$$



which is a quadratic equation in t . First rewrite the quadratic equation into the standard form:

$$0 = \underbrace{4.905}_{a} t^2 + \underbrace{5.0}_{b} t - \underbrace{44.145}_{c},$$

and then solve using the quadratic equation:

$$t_f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 + 866.12}}{9.81} = 2.53$$

We ignored the negative-time solution. Comparing with the first Moroccan example, we see that the answer makes sense: throwing a ball downwards will make it fall to the ground faster than just dropping it.

4 Projectile motion

The basic concepts of kinematics in two dimensions are:

- t : time, measured in seconds.
- $\vec{r}(t) \equiv (x(t), y(t))$: the position vector
- $\vec{v}(t) \equiv (v_x(t), v_y(t))$: the velocity as a function of time.
- $\vec{a}(t) \equiv (a_x(t), a_y(t))$: the acceleration as a function of time.

When solving some problem, where we calculate the motion of an object that starts from an *initial* point and goes to a *final* point we will use the following terminology:

- $t_i = 0$: initial time (the beginning of the motion).
- t_f : final time (when the motion stops).
- $\vec{v}_i = \vec{v}(t_i) = (v_x(0), v_y(0)) = (v_{xi}, v_{yi})$: the initial velocity at $t = t_i$.
- $\vec{r}_i = \vec{r}(0) = (x(0), y(0)) = (x_i, y_i)$: the initial position at $t = 0$.
- $\vec{r}_f = \vec{r}(t_f) = (x(t_f), y(t_f)) = (x_f, y_f)$: the final position at $t = t_f$.

4.1 Formulas

4.1.1 Motion in two dimensions

Sometimes you have to describe both the x and the y coordinate of the motion of a particle:

$$\vec{r}(t) = (x(t), y(t)).$$

We choose x to be the horizontal component of the projectile motion, and y to be its height.

The velocity of the projectile will be:

$$\vec{v}(t) = \frac{d}{dt}(\vec{r}(t)) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right) = (v_x(t), v_y(t)),$$

and the initial velocity is:

$$\vec{v}_i = \vec{v}(t_i) = |\vec{v}_i| \angle \theta = (v_x(t_i), v_y(t_i)) = (v_{ix}, v_{iy})$$

The acceleration of the projectile will be

$$\vec{a}(t) = \frac{d}{dt}(\vec{v}(t)) = (a_x(t), a_y(t)) = (0, -9.81).$$

Note how we have zero acceleration in the x direction so we can use the UVM equations of motion for $x(t)$ and $v_x(t)$. In the y direction we have a uniform downward acceleration due to gravity.

4.1.2 Projectile motion

The equations of motion of a projectile are the following. First in the x direction we have uniform velocity motion (UVM):

$$\begin{aligned} x(t) &= v_{ix}t + x_i, \\ v_x(t) &= v_{ix}. \end{aligned}$$

In the y direction, you have the constant pull of gravity downwards which gives us a uniformly accelerated motion (UAM):

$$\begin{aligned} y(t) &= \frac{1}{2}(-9.81)t^2 + v_{iy}t + y_i, \\ v_y(t) &= -9.81t + v_{iy}, \\ v_{yf}^2 &= v_{yi}^2 + 2(-9.81)(\Delta y). \end{aligned}$$

5 Forces

Like a shepherd who brings in a stray sheep back, we need to rescue the word *force* and give it precise meaning. In physics force means something very specific. Not “the force” from Star Wars, not the “force of public opinion”, and not the *force* in the battle of good versus evil.

Force in physics has a precise meaning as an amount of push or pull exerted on an object. Force is a vector. We measure force in Newtons [N], and we can use it in equations and solve for it just like any other unknown. In this section we will explore all the different kinds of forces.

5.1 Concepts

- \vec{F} : a force. This is something the object “feels” as a pull or a push. Force is a vector, so you must always keep in mind the direction in which the force \vec{F} acts.
- $k, G, m, \mu_s, \mu_k, \dots$: parameters on which the force F may depend. Ex: the heavier an object is (has large m parameter), the larger its gravitational pull will be: $\vec{W} = -9.81m\hat{z}$, where \hat{z} points towards the sky.

5.2 Kinds of forces

We next list all the forces which you are supposed to know about for a standard physics class and define the relevant parameters for each kind of force. You need to practice exercises using each of these forces, until you start to *feel* how they act.

5.2.1 Gravitation

Manifestations of the gravitational pull of the planet Earth on massive objects:

- M : mass of the earth. $M = 5.9721986 \times 10^{24}$ [kg].
- m : mass of an object.
- $\vec{W} = \vec{F}_g$: The weight (the force on a object due to gravity).
- G : Gravitational constant = 6.67×10^{-11} [$\frac{Nm^2}{kg^2}$].
- $\vec{F}_g = \frac{GMm}{r^2}$: Force of gravity between two objects of mass M and m respectively. Measured in Newtons [N].
- $\vec{F}_g = gm$ (downward): The force of gravity on the surface of the earth, where $g = \frac{GM}{r^2} \approx 9.81 \dots$ [N/kg]=[m/s²].

The famous one-over-arr-squared law that describes the gravitational pull between two objects is:

$$F_g = \frac{GMm}{r^2}.$$

You will rarely use it, but it is extremely important as this is where all of mechanics began. This was Newton’s big discovery. All the rest of mechanics is simple calculus, but this equation is *real* physics. It tells us something about how the Universe works.

At the surface of the earth:

$$\vec{F}_g = \frac{GMm}{r^2} = \underbrace{\left(\frac{GM}{r^2} \right)}_g m = \vec{g}m = \vec{W},$$

where the weight \vec{W} of an object is a vector that points towards the centre of the earth, and $g = 9.81$ [m/s²].

5.2.2 Force of a spring

- $\vec{F}_s = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by x meters. The constant k [N/m] is a measure of the *strength* of the spring, or its stiffness.

5.2.3 Tension in a rope

- \vec{T} : Tension in a rope. Tension is always pulling away from an object: you can't push a dog on a leash.

5.2.4 Contact force

- \vec{C} : Contact force between two rigid objects. We generally break-up contact forces into two components: perpendicular and parallel to the contact surface.
- $\vec{N} \equiv \vec{C}_\perp$: Normal force: the force between two surfaces. Normal is a mathematically precise way to say "perpendicular to a surface". Intuitively, you can think of \vec{N} as the force that a surface exerts on an object to keep it where it is. The reason why my coffee mug does not fall to the floor, is that the table exerts a normal force on it keeping in place.
- $\vec{F}_f \equiv \vec{C}_\parallel$: Force of friction between two surfaces. There are two kinds, both of which are proportional to the normal force between the surfaces:
Kinetic:

$$F_{fk} = \mu_k |\vec{N}|.$$

Static:

$$F_{fs} = \mu_s |\vec{N}|.$$

5.2.5 Two kinds of friction forces

- $\vec{F}_{fs} = \mu_s |\vec{N}|$: Static force of friction, for objects that are not moving.
- μ_s : The static coefficient of friction. ex: 0.3. It describes the **maximum** amount of friction that can exist between two objects. If a horizontal force exists greater than $F_{fs} = \mu_s N$, then the object will start to slip.
- $\vec{F}_{fk} = \mu_k |\vec{N}|$: Kinetic force of friction acts when two objects are sliding relative to each other. It always acts in a direction opposing the motion.
- μ_k : Kinetic coefficient of friction. ex: $\mu_k = 0.1$. Dimensionless. it is just the ratio that describes how much friction an object feels for a given amount of normal force.

6 Force diagrams

Newton's 2nd law says that the *net* force on an object causes an acceleration:

$$\sum \vec{F} = \vec{F}_{net} = m\vec{a},$$

so finding the net force must be a pretty important thing.

6.1 Concepts

Newton's second law is a relationship between these three concepts:

- m : the *mass* of an object.
- \vec{F} : vector used to denote any kind of *force*.
- \vec{a} : the *acceleration* of the object.

What types of forces are there in force diagrams?

- $\vec{W} \equiv \vec{F}_{gravity} = m\vec{g}$: The *weight*. This is the force on a object due to its gravity. The gravitational pull \vec{g} always points downwards towards the center of the earth. $g = 9.81$ [N/kg].
- \vec{T} : Tension in a rope. Tension is always pulling away from the object.
- \vec{N} : Normal force the force between two surfaces.
- $\vec{F}_{fs} = \mu_s |\vec{N}|$: Static force of friction.
- $\vec{F}_{fk} = \mu_k |\vec{N}|$: Kinetic force of friction.
- $\vec{F}_{spring} = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by x meters.

6.2 Formulas

6.2.1 Newton's 2nd law

The sum of the forces acting on an object, divided by the mass gives you the acceleration of the object:

$$\sum F = \vec{F}_{net} = m\vec{a}.$$

6.2.2 Vector components

If a vector \vec{v} makes an angle θ with the x -axis then:

$$v_x = |\vec{v}| \cos \theta, \quad \text{and} \quad v_y = |\vec{v}| \sin \theta.$$

The vector $v_x \hat{i}$ corresponds to the the part of \vec{v} that points in the x -direction.

In what follows, you will be asked a countless number of times to

Find the component of \vec{F} in the ? direction.

Which is another way of asking you to find the number $v_?$.

The answer is usually equal to the length $|\vec{F}|$ multiplied by either \cos or \sin and sometimes -1 all **depending on way the coordinate system is chosen**. So don't guess. Look at the coordinate system. If the vector points in the direction where x -increases, then v_x should be a positive number. If it points in the opposite direction, then v_x should be negative.

Vector components are important, because to add forces \vec{F}_1 and \vec{F}_2 you have to add their components:

$$(F_{1x}, F_{1y}) + (F_{1x}, F_{2y}) = \vec{F}_1 + \vec{F}_2 = (F_{1x} + F_{2x}, F_{1y} + F_{2y})$$

Instead of dealing with vectors in the bracket notation as above, when solving force diagrams it is easier to simply write the x -equation on one line, and the y -equation on a separate line.

$$F_{netx} = F_{1x} + F_{2x},$$

$$F_{nety} = F_{1y} + F_{2y}.$$

It is a good idea to always write those two equations together as a block so it remains clear that you are talking about the same problem, but the first row represents the x -dimension and the second row represents the y -dimension.

6.3 Recipe for solving force diagrams

1. Draw a diagram centred on the object and draw all the forces acting on it.
2. Choose a coordinate system, and indicate clearly what you will call the x -direction, and what you will call the y -direction. All **equations are expressed with respect to this coordinate system**.
3. Write down this "template":

$$\sum F_x = \quad = ma_x$$

$$\sum F_y = \quad = ma_y$$

4. Fill the first line by finding the x -components of each force acting on the object.
5. Fill the second line by finding the y -components of each force acting on the object.
6. Consistency checks:

- (a) Check signs. If the force in the diagram is acting in the x -direction then its component must be positive. If the force is acting in the opposite direction to \hat{x} , then its component should be negative.
- (b) Verify that whenever $F_x \propto \cos \theta$, then $F_y \propto \sin \theta$. If instead we use an angle ϕ defined with respect to the y -axis we would have $F_x \propto \sin \phi$, and $F_y \propto \cos \phi$.

7. Solve the two equations finding the one or two unknowns. If there are two unknowns, you may need to solve two equations simultaneously by isolation and substitution.

Force diagrams are best explained through examples.

7 Momentum

The momentum is equal to the velocity of the moving object multiplied by the object's mass ($\vec{p} = m\vec{v}$). Therefore, since the car weighs $1000 \times 1000 = 10^6$ times more than the piece of paper, it has 10^6 times more momentum when moving at the same speed. A collision with it will "hurt" that much more.

Note that momentum is a vector, so we will have to do a lot of that length-and-direction-to-components transformation stuff:

$$(p_x, p_y) = \vec{p} = (|\vec{p}| \cos \theta, |\vec{p}| \sin \theta) = |\vec{p}| \angle \theta,$$

and also converting backwards from component notation to magnitude-direction:

$$|\vec{p}| = \sqrt{p_x^2 + p_y^2}, \quad \theta = \tan^{-1} \left(\frac{p_y}{p_x} \right).$$

7.1 Concepts

- m : the mass of the moving object.
- \vec{v} : the velocity of the moving object.
- $\vec{p} = m\vec{v}$: the momentum of the moving object.

7.2 Formulas

7.2.1 Definition

The momentum of an object is the mass of the object times its velocity:

$$\vec{p} = m\vec{v}.$$

If you speed is $\vec{v} = (20, 0, 0)[\text{m/s}]$, which is equivalent to saying "20[m/s] in the x -axis direction", and your mass is 100kg then your momentum is $\vec{p} = (2000, 0, 0)[\text{kg} \cdot \text{m/s}]$.

In the absence of acceleration, objects will conserve their velocity:

$$\vec{v}_{in} = \vec{v}_{out}.$$

This is equivalent to saying that objects conserve their momentum (just multiply the velocity by the mass if the mass stays constant and the velocity stays constant, then the momentum must stay constant).

7.2.2 Conservation of momentum

More generally, if you have a situation with multiple moving objects, you can say that the “overall momentum”, i.e., the sum of the momenta of all the particles stays constant:

$$\sum \vec{p}_{in} = \sum \vec{p}_{out}.$$

This is amazingly powerful stuff, and one of the furthest reaching laws of physics. Whatever momentum comes into a collision must come out.

8 Energy

Instead of thinking about velocities $v(t)$ and motion trajectories $x(t)$, we can solve physics problems using *energy* calculations.

8.1 Concepts

The concepts of energy come up in several different contexts.

Moving objects:

- m : the mass of an object.
- v : the velocity.
- $E_K = K$: kinetic energy = $\frac{1}{2}mv^2$.

Moving objects by force:

- \vec{F} : the force needed to move the object.
- \vec{d} : the displacement of the object. How far it moved.
- W : work done to move the object = $\vec{F} \cdot \vec{d}$.

Gravity:

- g : gravitational acceleration on the surface of earth. $9.81 \text{ [m/s}^2\text{]}$.
- h : height of an object.
- U_g : Gravitational potential energy = mgh .

Springs:

- k : spring constant. Measured in $[\text{N/m}]$.
- x : spring displacement from the relaxed position. If the spring is stretched then $x > 0$, and if it is compressed then $x < 0$.

- U_s : Spring potential energy = $\frac{1}{2}kx^2$.

There are all kinds of other forms of energy: electric energy, sound energy, thermal energy, etc. In this section we will focus on the types of *mechanical* energy.

8.2 Formulas

Kinetic energy:

$$K = \frac{1}{2}mv^2$$

Work:

$$W = \int \vec{F}(x) \cdot dx$$

for constant force:

$$W = \vec{F} \cdot \vec{d} = |F||d| \cos \theta.$$

Gravitational potential energy:

$$U_g = mgh,$$

which is the energy you have because of your height.

Spring energy:

$$U_s = \frac{1}{2}kx^2.$$

Conservation of energy:

$$\sum E_{in} + W_{in} = \sum E_{out} + W_{out}.$$

9 Minireference

I hope this short excerpt from the **MATH and PHYSICS Minireference** has given you some inspiration for compact teaching. No blah blah. Straight to the point.

If you liked this tutorial you can check out the other ones on <http://minireference.com> and order the printed book which has not only formulas but also compact explanations: http://minireference.com/order_book/.