

# Equation quadratique

Ivan Savov

July 31, 2012

## Abstract

We review some basic math notions and the quadratic equation.

### 0.0.1 Example 1

Let's say your teacher doesn't like you and right away on the first day of classes he gives you a serious equation and wants you to find  $x$ :

$$\log_5 \left( 3 + \sqrt{6\sqrt{x} - 7} \right) = 34 + \sin(345) - \Psi(1).$$

Do you see now what I meant when I said that the teacher doesn't like you?

First note that it doesn't matter what  $\Psi$  is, since  $x$  is on the other side of the equation. We can just keep copying  $\Psi(1)$  from line to line and throw the ball back to the teacher: "My answer is in terms of your variables dude... you go ahead and figure out what the hell  $\Psi$  is, since you brought it up." The same goes with  $\sin(345)$ . If you don't have a calculator don't worry about it. We will just keep  $\sin(345)$  around and not try to evaluate it. Besides would you type in 345 radians or 345 degrees?

**Let's just find  $x$  and get it over with!**

On the right side we have the sum of a bunch of terms and no  $x$  in them so we can leave them alone. On the left hand side, the outer most function is a logarithm base 5. Cool... no problem look up in the table and find the row where we have:  $a^x \leftrightarrow \log_a(x)$ . We are going to apply the inverse on both sides:

$$5^{\log_5(3 + \sqrt{6\sqrt{x} - 7})} = 5^{34 + \sin(345) - \Psi(1)},$$

which simplifies to:

$$3 + \sqrt{6\sqrt{x} - 7} = 5^{34 + \sin(345) - \Psi(1)}.$$

From here on it's like if Bruce Lee walked into a place with lots of bad guys. Addition of 3 is undone by subtracting 3 on both sides:

$$\sqrt{6\sqrt{x} - 7} = 5^{34 + \sin(345) - \Psi(1)} - 3,$$

and to undo a square root you take the square

$$6\sqrt{x} - 7 = \left( 5^{34 + \sin(345) - \Psi(1)} - 3 \right)^2.$$

Add 7 to both sides

$$6\sqrt{x} = \left( 5^{34 + \sin(345) - \Psi(1)} - 3 \right)^2 + 7.$$

Divide by 6:

$$\sqrt{x} = \frac{1}{6} \left( \left( 5^{34 + \sin(345) - \Psi(1)} - 3 \right)^2 + 7 \right),$$

and then we square again plus some simplifications to get:

$$\begin{aligned} x &= \left[ \frac{1}{6} \left( \left( 5^{34 + \sin(345) - \Psi(1)} - 3 \right)^2 + 7 \right) \right]^2 \\ &= \frac{1}{36} \left[ \left( 5^{34 + \sin(345) - \Psi(1)} - 3 \right)^2 + 7 \right]^2. \end{aligned}$$

Do you see what I am doing? Next time a function stands in your way, hit it with its inverse!

### 0.0.2 Discussion

The recipe I have outlined above is not universal. Sometimes  $x$  isn't alone on one side. Sometimes  $x$  appears in several places in the same equation so just throwing complications from one side to the other won't help you. You need other techniques for solving equations like that.

The bad news is that there is no general formula for solving complicated formulas. The good news is that the above technique of digging towards  $x$  is sufficient for 80% of what you are going to be doing. You can get another 10% if you learn how to solve the quadratic equation:

$$ax^2 + bx + c = 0.$$

Solving third order equations  $ax^3 + bx^2 + cx + d = 0$  with pen and paper is still possible, but at this point you really might as well start using a computer.

There are all kinds of other equations which you can learn how to solve: equations with multiple unknowns, equations with logarithms, equations with exponentials, and equations with trigonometric functions. The essential principle of *digging* towards the unknown will get you pretty far though, so be sure to practice it.

## 1 Basic algebra rules

For  $a, b, c, d \in \mathbb{R}$  holds:

1. Distributive property:  $a(b + c) = ab + ac$ .
2. Associative property:  $a(bc) = b(ac) = c(ab)$ .
3. Commutative property:  $a + b = b + a$ ,  $ab = ba$ .

Combining the distributive and the associative property we can get this:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

**Example** This is useful when you are dealing with polynomials:

$$(x+3)(x+2) = x(x+2) + 3(x+2) = x^2 + x2 + 3x + 6.$$

Now for polynomials, we can use the commutative property on the second term  $x2 = 2x$ , and then combine the two  $x$  terms into one to obtain:

$$(x + 3)(x + 2) = x^2 + 5x + 6.$$

This kind of calculation happens so often that it is good idea to see it in more abstract form

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

My high school teacher used to repeat this to us all the time “a binomial times a binomial gives a trinomial”.

### 1.1 Factoring

Factoring means to take out some common part in a complicated expression to make it more compact. Given an expression, we can ask what the factors of each term are:

$$6x^2y + 15x = (3)(2)(x)(x)y + (5)(3)x.$$

We see that  $x$  and 3 appear in both terms. This means we can *factor them out* to the front like this:

$$6x^2y + 15x = 3x(2xy + 5).$$

Now is this simpler? On the left there are 8 characters and on the right there are 7 characters so we are saving space. More significant savings can occur in expressions like this:

$$2x^2y + 2x + 4x = 2x(xy + 1 + 2) = 2x(xy + 3).$$

**Example** You are asked to solve for  $t$  in

$$7(3 + 4t) = 11(6t - 4).$$

As you can see,  $t$  is on both sides. We don't like that. We have to bring all the  $t$  terms to one side and all the constant terms on the other. First let's expand the two brackets:

$$21 + 28t = 66t - 44.$$

Now we move things around to get:

$$21 + 44 = 66t - 28t.$$

If we factor out  $t$  on the right hand side we get:

$$21 + 44 = (66 - 28)t,$$

$$\text{so } t = \frac{21+44}{66-28} = \frac{65}{38}.$$

### 1.2 Quadratic factoring

When dealing with a quadratic functions, it is often useful to rewrite it as a product of two terms. If I give you the quadratic expression

$$x^2 + 5x + 6,$$

you can't really tell what its properties are. What are its roots? Where is it positive and where is it negative?

We can factor this expression, that is express it as a product of two things  $(x-?)(x-?)$  as follows:

$$x^2 + 5x + 6 = (x + 2)(x + 3),$$

and you see immediately that  $x = -2$  and  $x = -3$  are the solutions (roots). You can also see that for  $x > -2$  the expression is positive, since both terms will be positive. For  $x < -3$  both terms will be negative, but negative times negative gives positive, so the expression will be positive again. For values of  $x$  such that  $-3 < x < -2$ , the first term will be negative, and the second positive so the overall expression will be negative.

For some simple quadratics you can *guess* what the factors are. For more complicated ones, you need the quadratic formula.

Quadratic functions that never cross the  $x$ -axis cannot be factored.

### 1.3 Completing the square

Any quadratic formula can be written in the form  $A(x-h)^2+k$ . This is because all quadratic function with the same quadratic coefficient are essentially shifted version of each other. By *completing the square* we are making these shifts explicit. The value of  $h$  is how much the function is shifted to the *right* and the value  $k$  is the vertical shift.

Let's try to find the values  $A, k, h$  for our previous formula

$$x^2+5x+6 = A(x-h)^2+k = A(x^2-2hx+h^2)+k = Ax^2-2Ahx+Ah^2+k$$

By focussing on quadratic terms on both sides of the equation we see that  $A = 1$ , so we have

$$x^2 + 5x + 6 = x^2 - 2hx + h^2 + k.$$

Next we look at the terms multiplying  $x$ , and we see that  $h = -2.5$ , so we obtain

$$x^2 + 5x + 6 = x^2 + 2(2.5)x + (2.5)^2 + k.$$

Finally we pick value of  $k$  which would make the constant terms match

$$k = 6 - (2.5)^2 = 6 - \left(\frac{5}{2}\right)^2 = 6 \times \frac{4}{4} - \frac{25}{4} = \frac{24 - 25}{4}$$

This is how we complete the square, to obtain:

$$x^2 + 5x + 6 = (x + 2.5)^2 - \frac{1}{4},$$

which tells us that our function was just the basic  $x^2$ , shifted 2.5 units to the left, and 1/4 units downwards. This would be really useful information if you ever had to draw this function, since it is easy to plot the basic  $x^2$  graph, and then shift it appropriately.

## 2 Solving quadratic equations

If you are asked to find  $x$  in the equation

$$x^2 = 45x + 23,$$

what would you do?

In math this happens a *whole* lot, so people came up with a formula for solving it.

To apply the formula you need to put the equation into the standard form

$$ax^2 + bx + c = 0,$$

where we have moved all the numbers and  $x$ 's to one side and left just 0 on the other side.

For our numerical example above this would mean that we subtract  $45x + 23$  on both sides of the equation to obtain

$$x^2 - 45x - 23 = 0.$$

What are the values of  $x$  that satisfy this formula?

### 2.0.1 Claim

The solutions to the equation

$$ax^2 + bx + c = 0,$$

are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Using the quadratic formula, solving the above complicated equation  $x^2 - 45x - 23 = 0$  becomes a mechanics task of plugging in the appropriate numbers into the formula:

$$x_1 = \frac{45 + \sqrt{45^2 - 4(1)(-23)}}{2} = 45.5054 \dots,$$

$$x_2 = \frac{45 - \sqrt{45^2 - 4(1)(-23)}}{2} = -0.5054 \dots$$

Answers obtained by using google.

### 2.0.2 Proof of claim

This is an important proof. You should know how to derive the quadratic formula in case your younger brother asks you one day to derive the formula from first principles. To understand the proof, all you need to know is how to "complete the square", which we just saw, so you can't say that you are not prepared.

We start with

$$ax^2 + bx + c = 0.$$

As a first step we move  $c$  to the other side

$$ax^2 + bx = -c,$$

and then we divide by  $a$  on both sides

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Now we *complete the square* on the left hand side, which is to say we ask the question: what are the values of  $h$  and  $k$  for this equation to hold?

$$(x + h)^2 + k = x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

The answer is  $h = \frac{b}{2a}$ . It can be a bit confusing to figure out what  $k$  is, so let's work it out step by step. Just remember that **to complete the square you have to use half of the linear term inside the bracket.**

So what do we have so far:

$$\left(x + \frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = x^2 + \frac{b}{2a}x + x\frac{b}{2a} + \frac{b^2}{4a^2}$$

which means that

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x.$$

Remember where we left off in the proof?

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

We replace the left hand side by the completed square expression:

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}.$$

From here on it is standard stuff from solving equations. We put all the constants on the same side

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2},$$

Since  $x^2$  maps both positive and negative numbers to the same value, there will be two solutions:

$$x + \frac{b}{2a} = \pm \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}}.$$

Let's take this moment to cleanup the mess on the right hand side.

$$\sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}} = \sqrt{-\frac{(4a)c}{(4a)a} + \frac{b^2}{4a^2}} = \sqrt{\frac{-4ac + b^2}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{exponent 2:}$$

Thus we have:

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

which is just one step away from the final answer

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which completes the proof.

### 2.0.3 Alternative proof of claim

We didn't need to necessarily show the derivation of the formula as we did. The claim was that  $x_1$  and  $x_2$  are solutions, so to prove the claim we could simply plug in  $x_1$  and  $x_2$  into the quadratic equation and verify that we get zero. **Verify on your own.**

## 2.1 Applications

### 2.1.1 The Golden Ratio

The golden ratio, usually denoted  $\phi = 1.6180339\dots$  is a very important proportion in geometry, art, aesthetics, biology and mysticism. It comes about from the solution to the equation

$$x^2 + x - 1 = 0.$$

Using the quadratic formula we get the two solutions which are

$$x_1 = \frac{-1 + \sqrt{5}}{2} = \frac{1}{\phi}, \quad x_2 = \frac{-1 - \sqrt{5}}{2} = -\phi.$$

You can read more about the golden ratio to learn about the various contexts in which it appears.

## 3 Exponents

We often have to multiply number together the same number many times in math so we use the notation

$$x^n = \underbrace{xxx \cdots xx}_{n \text{ times}}$$

to denote some number  $x$  multiplied together  $n$  times.

### 3.1 Definitions

- $b^x$ :  $b$  exponent  $x$ :  
 –  $b$ : is called the *base*.  
 –  $x$ : is the *exponent*.
- $\exp(x) = e^x$ : the "natural" exponential function base  $e$  Euler's number.
- $2^x$ : exponential function base 2, very important in computer science.
- $b^0 = 1$ : by definitions zero times something gives 1.

### 3.2 Formulas

The number  $e = 2.7182818\dots$  is a special base that has lots of applications. We call that the *natural* base.

Another special base is 10 because we use the decimal system for our numbers. The  $\log_{10}$  of a number corresponds to how many digits there are in the number.

### Property 1

$$b^m b^n = b^{m+n},$$

which follows from first principles: Exponentiation means multiplying together the base many times. So  $b^m$  means you multiply  $b$  times itself  $m$  times. So if you count the total number of  $b$ s on the left side you will see that there is a total of  $m + n$  of them.

### Property 2

$$b^{-n} = \frac{1}{b^n},$$

### Property 3

$$\frac{b^m}{b^n} = b^{m-n},$$

### Property 4

$$(b^m)^n = b^{mn},$$

### Property 5

$$(ab)^n = a^n b^n,$$

### Property 5.2

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$

### Property 6

$$b^{1/n} = \sqrt[n]{b},$$

in particular  $b^{1/2} = \sqrt{b}$ .

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b},$$

$$\sqrt[n]{\left(\frac{a}{b}\right)} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},$$

If  $n$  is odd

$$\left(\sqrt[n]{b}\right)^n = \sqrt[n]{b^n} = b.$$

If  $n$  is even and  $b$  is positive

$$\left(\sqrt[n]{b}\right)^n = \sqrt[n]{b^n} = |b|,$$

but if  $b$  is negative then we can't evaluate  $\sqrt[n]{b}$  (there is no real number which multiplied times itself gives a negative number) so only the second part of the "cancellation" holds

$$\sqrt[n]{b^n} = |b|,$$

where  $|b|$  is the absolute value.

## 4 Logarithms

The word "logarithms" makes most people think about some mythical mathematical beast. Probably many headed and extremely difficult to understand.

None such thing! Logarithms are simple. It will take you at most a couple of pages to get used to manipulating them. Logarithms are used all over the place. The strength of your sound system is measured in decibels  $[dB]$ , which is because your ear is sensitive only to exponential differences in sound. Logarithms allow us to compare very large numbers and very small numbers on the same scale. If we were measuring sound in linear units instead of logarithmic then your sound system volume control would have to go from 1 to 1048576. Indeed, if the volume notches on your sound system go from 1 to 20, then each notch probably doubles the power instead of increasing it by a fixed amount.

Let's get started.

### 4.1 Definitions

You are probably familiar with these concepts already:

- $b^x$ : the exponential function base  $b$ .
- $\exp(x) = e^x$ : the "natural" exponential function base  $e$  Euler's number.
- $2^x$ : exponential function base 2, very important in computer science.
- $f(x)$ : some function.
- $f^{-1}(x)$ : the inverse of the function  $f(x)$ . It is defined in terms of  $f(x)$  such that the following holds  $f^{-1}(f(x)) = x$  i.e., if you apply  $f$  to some number and get the output  $y$ , and then you feed  $y$  into  $f^{-1}$  the output of  $f^{-1}$  will be  $x$ .

In this section we will play with the following new concepts:

- $\log_b(x)$ : logarithm of  $x$  base  $b$ . This is the inverse function of  $b^x$ .
- $\ln(x)$ : the "natural" logarithm base  $e$ . This is the inverse of  $e^x$ .
- $\log_2(x)$ : logarithms base 2. The answer to the question, how many bits of computer memory does it take to store a large number  $N$  is  $\log_2(N)$  bits.

## 4.2 Formulas

The main thing to realize is that logs don't really exist on their own. They are defined as the inverses of the corresponding exponential function. The following statements are equivalent:

$$\log_b(x) = m \quad \Leftrightarrow \quad b^m = x.$$

For logarithms with base  $e$  one writes  $\ln(x)$  because  $e$  is a special base that has lots of applications. This is a subject for another lesson, but I wanted also to mention that  $e^x$  is actually related to  $\sin(x)$  and  $\cos(x)$ .

Another special base is 10 because we use the decimal system for our numbers.  $\log_{10}(x)$  tells you roughly the size of the number  $x$ : how many digits it has. When a system pawn with a high paying financial sector job boast about his or her "six-figure" salary, they are really talking about the log of how much money they make. What wankers!

Let me show you the two-three equations that you need to know for dealing with logs.

### Property 1

$$\log(x) + \log(y) = \log(xy),$$

from which we can derive two other useful ones:

$$\log(x^k) = k \log(x),$$

and

$$\log(x) - \log(y) = \log\left(\frac{x}{y}\right).$$

*Proof:* For all three equations above we have to show that the expression on the left is equal to the expression on the right.

We have only been acquainted with logarithms for a very short time. We don't know each other that well. In fact, the only thing we know about logs is the inverse relationship with the exponential function. So the only way to prove this property is to use this relationship.

The following statement is true for any base  $b$ :

$$b^m b^n = b^{m+n},$$

which follows from first principles. Exponentiation means multiplying together the base many times. So  $b^m$  means you multiply  $b$  times itself  $m$  times. So if you count the total number of  $b$ s on the left side you will see that there is a total of  $m+n$  of them.

If you define some new variables  $x$  and  $y$  such that  $b^m = x$  and  $b^n = y$  then the above equation will read

$$xy = b^{m+n},$$

if you take the logarithm of both sides you get

$$\log_b(xy) = \log_b(b^{m+n}) = m+n = \log_b(x) + \log_b(y).$$

In the last step we used the definition of log again which states that  $b^m = x \Leftrightarrow m = \log_b(x)$  and  $b^n = y \Leftrightarrow n = \log_b(y)$ .

Next we discuss another set some more rules about changing from one base to another. Is there some relation between  $\log_{10}(S)$  and  $\log_2(S)$ ?

### Property 2

$$\log_B(x) = \frac{\log_b(x)}{\log_b(B)}$$

This means that:

$$\log_{10}(S) = \frac{\log_{10}(S)}{1} = \frac{\log_{10}(S)}{\log_{10}(10)} = \frac{\log_2(S)}{\log_2(10)} = \frac{\ln(S)}{\ln(10)}$$

This property is very useful in case when you want to compute  $\log_{10}$ , but your calculator only gives you  $\ln$  then you simply compute the two numbers on the right hand side and divide them.

## 4.3 Discussion

**Applications of logarithms** In many sciences people often interchange multiplication and addition by using a logarithm transform

$$\log_3(f(x)) + \log_3(g(x)) = z,$$

which by the properties of logarithms (see definition) can be written as

$$\log_3(f(x)g(x)) = z,$$

which in turn means  $f(x)g(x) = 3^z$ .

This is useful in probability theory since if you have some probability distribution  $p(x, y)$  that is independent then  $p(x, y) = p_X(x)p_Y(y)$ . If you take the logarithm of this expression (the concept of log likelihood is the basis of *machine learning*) then you you get

$$\log(p_X(x)p_Y(y)) = \log(p_X(x)) + \log(p_Y(y)),$$

so very complicated product expressions can be evaluated as sums. Human and computers are much better at adding numbers than multiplying them, so we use this trick all the time.

## 5 Strudent loan interest rate

We need the exponential function here...