# MECHANICS explained in four pages

Excerpt from the *MATH and PHYSICS Minireference* by Ivan Savov

***Abstract*—Mechanics is the precise study of the motion of objects, the forces acting on them and more abstract concepts such as momentum and energy. You probably have an intuitive understanding of these concepts already, but in the next four pages you will learn how to use precise mathematical equations to support your intuition. Perhaps more importantly, we will also learn where the equations come from.**

## I. Introduction

To solve a physics problem is to obtain the *equation of motion* $x(t)$, which describes position of the object as a function of time. Once you know $x(t)$, you can answer many of the question pertaining to the motion of the object. To find the initial position $x_i$ of the object, you simply plug $t = 0$ into the equation of motion $x_i = x(0)$. To find the time(s) when the object reaches a distance of 20[m] from the origin, we simply solve for $t$ in $x(t) = 20$[m]. Many of the problems on you final exam in physics will be of this form, so it is really important that you know how to find the equation of motion for any object.

The first step in finding $x(t)$ is to calculate all the *forces* that act on the object. Forces are the *cause* of motion, so if you want to understand motion you need to understand forces. Newton's second law $F = ma$ states that the amount force acting on an object produces an *acceleration* inversely proportional to the mass of the object. Once you have the acceleration, you can compute $x(t)$ using two simple calculus steps. For now, though, we want to focus on the causes of motion: the forces acting on the object. There are many kinds of forces: the weight of an object $\vec{W}$ is a type force, the force of friction $\vec{F}_f$ is another type another, the tension in a rope $\vec{T}$ yet another type of force and there are many others. Note the little arrow on top of each force, which is there to remind you that forces are *vector* quantities. Unlike regular numbers, forces act in a particular direction, so it is possible that the effects of one force are counteracting another force. For example the force of the weight of a flower pot is exactly counter-acted by the tension in the rope on which it is suspended, thus, while there are two forces that may be acting on the pot, there is no *net force* acting on it. Since there is no net force to cause motion and since the pot wasn't moving to begin with, it will just sit there motionless despite the fact that there are forces acting on it! The first step when analyzing a physics problem will be calculation of the *net force* acting on the object, which is the sum of all the forces acting on the object: $\vec{F}_{net} \equiv \sum \vec{F}$. Once we have found the net force we can use $\vec{F}_{net} = m\vec{a}$ to find the resulting acceleration.

It turns out that once you know the acceleration of an object $a(t)$, you can easily find its velocity $v(t)$ function and once you know the velocity function you can find the position function $x(t)$, which is what we wanted to find in the first place. The acceleration is the change in the velocity of the object, thus if you know that the object stated with an initial velocity of $v_i \equiv v(0)$, and you want to find the velocity at later time $t = \tau$, you have to add up all the acceleration that the object felt during this time $v(\tau) = v_i + \int_0^\tau a(t)\, dt$. The symbol $\int \cdot\, dt$ is called an *integral* and is fancy way of finding the total of some quantity over time. In order to find $x(t)$, we perform a second integration step in which we add up all the changes in the position (velocity) in order to find the final position $x(t)$. We can summarize the entire process which is used to find $x(t)$ in the following equation:

$$\frac{1}{m}\sum \vec{F} = a(t) \quad \xrightarrow{v_i + \int dt} \quad v(t) \quad \xrightarrow{x_i + \int dt} \quad x(t). \tag{1}$$

The right hand side of the equation is called a *dynamics* problem and involves the calculation of the *net force* $\vec{F}_{net} \equiv \sum \vec{F}$. The right hand side in the above equation is called *kinematics* and focusses on the use of integration in order to find $v(t)$ from $a(t)$ and $x(t)$ from $v(t)$. Don't worry about this integration business; it is quite simple and we will cover everything you need to know about it in the next section.

Newton's second law can also be applied to the study of circular motion. Circular motion is described by the angle of rotation $\theta(t)$, the angular velocity $\omega(t)$ and the angular acceleration $\alpha(t)$. The causes of angular acceleration are angular force, which we call *torques* $\mathcal{T}$. Apart from this change to angular quantities, the principles behind the circular motion are exactly the same as those for linear motion.

During a collision between two objects there will be a sudden spike in the contact force between them, which can be difficult to measure and quantify. It is therefore not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. In order to predict the motion of the objects after the collision we must use a *momentum* calculation. An object of mass $m$ moving with velocity $\vec{v}$ has momentum $\vec{p} \equiv m\vec{v}$. The principle of conservation of momentum states that **the total amount of momentum before and after the collision is remains constant**. Thus, if two objects with initial momenta $\vec{p}_{i1}$ and $\vec{p}_{i2}$ collide, the total momentum before in the collision must be equal to the total momentum after the collision:

$$\sum \vec{p}_i = \sum \vec{p}_f \qquad \Rightarrow \qquad \vec{p}_{i1} + \vec{p}_{i2} = \vec{p}_{f1} + \vec{p}_{f2}.$$

Using this equation, it is possible to calculate the final momenta $\vec{p}_{f1}$, $\vec{p}_{f2}$ of the objects after the collision.

Another way of solving physics problems is to use the the concept of energy. Instead of trying to describe the entire motion of the object, we can focus only on the initial parameters and the final parameters. The law of conservation of energy states that **the total energy of the system is a constant**. Knowing the initial energy of a system, therefore, allows us to infer the final energy.

In the remainder of this document, we will learn more about each of the above concepts and ways of thinking. Before we begin with the physics material, we must introduce some mathematical background, which will allow us to better understand the concepts.

## II. Preliminaries

In order to understand the equations of physics you need to be familiar with vector calculations and how to calculate some basic integrals. We will introduce these concepts in the next two subsections.

### A. Vectors

Forces, velocities, and accelerations are vector quantities. A vector quantity $\vec{v}$ can be expressed in terms of its *components* or in terms of its length and direction.

- $x$-axis: The $x$-axis is the horizontal the coordinate system.
- $y$-axis: The $y$-axis is an axis *perpendicular* to the $x$-axis.
- $v_x$: the *component* of $\vec{v}$ along the $x$ axis.
- $v_y$: the *component* of $\vec{v}$ along the $y$ axis.
- $\hat{\imath} \equiv (1,0), \hat{\jmath} \equiv (0,1)$: Unit vectors in the $x$ and $y$ directions.
- $\|\vec{v}\|$: The length of the vector $\vec{v}$.
- $\theta$: The angle that $\vec{v}$ makes with the $x$ axis.

Given the $xy$ coordinate system, we can denote a vector in three equivalent ways:

$$\vec{v} \;\equiv\; (v_x, v_y) \;\equiv\; v_x\hat{\imath} + v_y\hat{\jmath} \;\equiv\; \|\vec{v}\|\angle\theta.$$

Given a vector expressed as a length and direction $\|\vec{v}\|\angle\theta$, we calculate its components as follows:

$$v_x = \|\vec{v}\|\cos\theta, \qquad \text{and} \qquad v_y = \|\vec{v}\|\sin\theta.$$

Alternately, a vector expressed in component notation $\vec{v} = (v_x, v_y)$ is converted to the length-and-direction form as follows:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}, \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right).$$

It is important to be able to be able to convert between these two forms; the component notation is useful for calculations whereas the length-and-direction form will provide you with a geometric intuition.

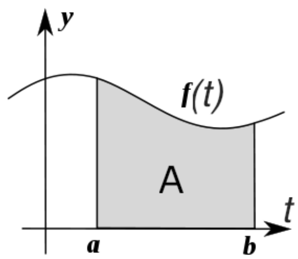
The *dot product* between two vectors $\vec{v}$ and $\vec{w}$ can be computed in two different ways:

$$\vec{v} \cdot \vec{w} = v_x w_y + v_y w_y = \|\vec{v}\|\|\vec{w}\| \cos\phi,$$

where $\phi$ is the angle between the vectors $\vec{v}$ and $\vec{w}$. The dot product calculates how similar the two vectors are. For example, we have $\hat{\imath} \cdot \hat{\jmath} = 0$, since the vectors $\hat{\imath}$ and $\hat{\jmath}$ are orthogonal – they point in completely different directions.

### B. Integrals

An integral corresponds to the computation of an *area* under a curve $f(t)$ between two points:

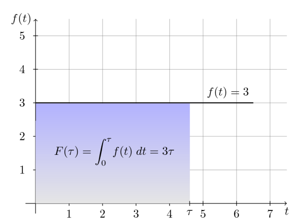



$$A(a, b) \equiv \int_{t=a}^{t=b} f(t)\; dt.$$

The symbol $\int$ is a mnemonic for *sum*, since the area under the curve corresponds in some sense to the sum of the values of the function $f(t)$ between $t = a$ and $t = b$. The integral is the total amount of $f$ between $a$ and $b$.

Consider for example the constant function $f(t) = 3$, and let us find the expression $F(\tau) \equiv A(0, \tau)$ that corresponds to the area under $f(t)$ between $t = 0$ and the time $t = \tau$.

We can easily find this area because the region under the curve is rectangular:




$$F(\tau) \equiv A(0, \tau) = \int_0^\tau f(t)\; dt = 3\tau.$$

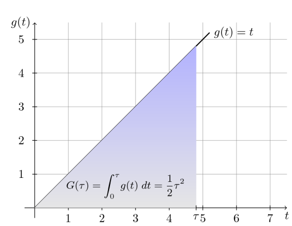
Indeed the area is equal to the height times the length of the base of the rectangle.

Anther important calculation is the area under the function $g(t) = t$. We will compute $G(\tau) \equiv A(0, \tau)$, which corresponds to the area under $g(t)$ between 0 and $\tau$. This area is also easily computed since the region under the curve is triangular:

$$G(\tau) \equiv A(0, \tau) = \int_0^\tau g(t)\; dt = \frac{\tau \times \tau}{2} = \frac{1}{2}\tau^2,$$

since the area of a triangle is the product of the length of the base times the height divided by two.

We were able to compute the above integrals thanks to the simple geometry of the areas under the curves. Later on in this book we will develop techniques for finding integrals of more complicated functions.




What you need to remember for now that the integral of a function is the total amount of the function accumulated during some time period. You should try to remember the formulas:

$$\int_c^\tau a\; dt = a\tau + C, \qquad \int_c^\tau at\; dt = \frac{1}{2}a\tau^2 + C,$$

which correspond to the integral of a constant function $f(t) = a$ and the integral of $f(t) = at$, a line with slope $a$. Note that each time you give a general integral formula the answer will contain an additive constant term $+C$, which depends on the starting point $t = c$, which we use for the area calculation. In the above examples we used $c = 0$ as the initial point so we had $C = 0$.

Using the above formulas in combination, you can now compute the integral under of an arbitrary line $f(t) = mt + b$ as follows:

$$\int_c^\tau (mt + b)\; dt = \frac{1}{2}m\tau^2 + b\tau + C,$$

since the integral of the sum of two functions is the sum of the integrals.

Now that you know about vectors and integrals, we can start our discussion on the laws of physics and derive the equation of motion for a particle in free fall.

## III. Kinematics

Kinematics (from the Greek word *kinema* for *motion*) is the study of trajectories of moving objects. The equations of kinematics can be used to calculate how long a ball thrown upwards will stay in the air, or to calculate the acceleration needed to go from 0 to 100 km/h in 5 seconds.

### A. Concepts

The key notions used to describe the motion of an objects are:

- $t$: the time, measured in seconds [s].
- $x(t)$: the position of an object as a function of time – also known as the equation of motion. Measured in meters [m].
- $v(t)$: the velocity of the object as a function of time. [m/s]
- $a(t)$: the acceleration of the object as a function of time. [m/s$^2$]
- $x_i = x(0), v_i = v(0)$: the starting position and velocity.

The position, velocity and acceleration functions ($x(t)$, $v(t)$ and $a(t)$) are connected. They all describe different aspects of the same motion. The function $x(t)$ is the main function since it describes the position of the object at all times. The velocity function describes the change in the position over time, hence it is measured in [m/s]. The acceleration function describes how the velocity changes over time. A constant positive acceleration means the velocity of the motion is steadily increasing, like when you press the gas pedal in your car. A constant negative acceleration means the velocity is steadily decreasing, like when pressing the brake pedal.

**If you know the exact function** $x(t)$, then you can compute its *derivative* and obtain the velocity function $v(t)$. You can obtain the acceleration function $a(t)$ by computing the derivative of the velocity $v(t)$:

$$a(t) \xleftarrow{\frac{d}{dt}} v(t) \xleftarrow{\frac{d}{dt}} x(t).$$

Alternately, **if you know the acceleration function** $a(t)$, you can use integration in order to obtain the velocity function $v(t)$ and then integrate the velocity function in order to obtain the position function $x(t)$:

$$a(t) \xrightarrow{\int dt} v(t) \xrightarrow{\int dt} x(t).$$

which means that we start from the acceleration $a(t)$ and use integration with respect to time to obtain the velocity $v(t)$. If we integrate the velocity we obtain the position function $x(t)$.

Recall that the integral is the calculation of the total

This is The main subject in this section is how to use

### B. Uniform acceleration motion

(UAM)

This is how the *equations of motion* are derived. Assuming that the acceleration is constant in time $a(t) = a$, we can calculate the velocity by *adding up* all the acceleration (integrating) to obtain the change in the velocity

$$v(t) = \int a(t)\; dt = \int a\; dt = at + v_i.$$

$$x(t) = \int v(t)\; dt = \int (at + v_i)\; dt = \frac{1}{2}at^2 + v_i t + x_i.$$

*1) Formula:* If the object undergoes a *constant* acceleration $a(t) = a$, like your car if you floor the *accelerator* pedal, then the equations of motion are:

$$x(t) = \frac{1}{2}at^2 + v_i t + x_i, \tag{2}$$
$$v(t) = at + v_i, \tag{3}$$
$$a(t) = a. \tag{4}$$

There is also another very useful equation to remember:

$$v(t)^2 = v_i^2 + 2a[x(t) - x_i],$$

which is usually written

$$v_f^2 = v_i^2 + 2a\Delta x. \tag{5}$$

That is it. Memorize these equations, plug-in the right numbers, and you can solve any kinematics problem humanly imaginable. Chapter done.

*a) Moroccan example:* Suppose your friend wants to send you a ball wrapped in aluminum foil from his balcony, which is located on the 14th floor (height of $44.145$[m]). At $t = 0$[s] he *throws* the ball straight down with an initial velocity of 10[m/s]. How long does it take for the ball to hit the ground?

Assume that the $y$-axis measuring distance upwards starting from the ground floor. We know that the balcony is located at a height of $y_i = 44.145$[m], and that at $t = 0$[s] the ball starts with $v_i = -10$[m/s]. The initial velocity is negative, because it points in the opposite direction to the $y$-axis. We know that there is an acceleration due to gravity of $a_y = -g = -9.81$ [m/s$^2$].

We start by writing out the general UAM equation:

$$y(t) = \frac{1}{2}a_y t^2 + v_i t + y_i.$$

We want to find the time when the ball will hit the ground, so $y(t) = 0$. To find $t$, we plug in all the known values into the general equation:

$$y(t) = 0 = \frac{1}{2}(-9.81)t^2 - 10t + 44.145,$$

which is a quadratic equation in $t$. First rewrite the quadratic equation into the standard form:

$$0 = \underbrace{4.905}_{a}t^2 + \underbrace{5.0}_{b}\ t - \underbrace{44.145}_{c},$$

and then solve using the quadratic equation:

$$t_f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 + 866.12}}{9.81} = 2.53 \text{ [s]}.$$

We ignored the negative-time solution.

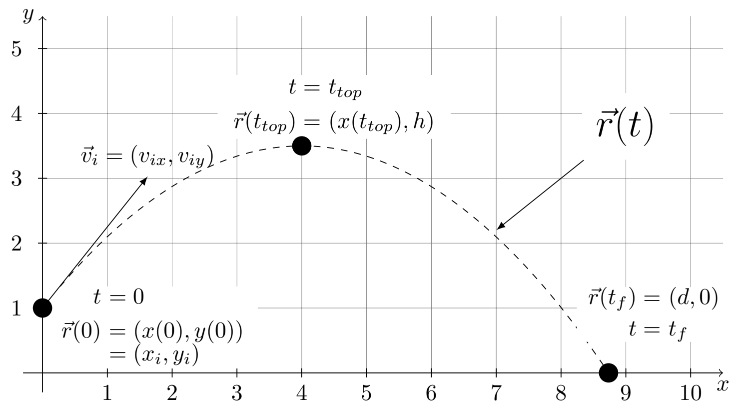
## IV. Projectile motion

The basic concepts of kinematics in two dimensions are:

- $t$: time, measured in seconds.
- $\vec{r}(t) \equiv (x(t), y(t))$: the position vector
- $\vec{v}(t) \equiv (v_x(t), v_y(t))$: the velocity as a function of time.
- $\vec{a}(t) \equiv (a_x(t), a_y(t))$: the acceleration as a function of time.

When solving some problem, where we calculate the motion of an object that starts form an *initial* point an goes to a *final* point we will use the following terminology:

- $t_i = 0$: initial time (the beginning of the motion).
- $t_f$: final time (when the motion stops).
- $\vec{v}_i = \vec{v}(t_i) = (v_x(0), v_y(0)) = (v_{xi}, v_{yi})$: the initial velocity at $t = t_i$.
- $\vec{r}_i = \vec{r}(0) = (x(0), y(0)) = (x_i, y_i)$: the initial position at $t = 0$.
- $\vec{r}_f = \vec{r}(t_f) = (x(t_f), y(t_f)) = (x_f, y_f)$: the final position at $t = t_f$.




### A. Formulas

*1) Motion in two dimensions:* Sometimes you have to describe both the $x$ and the $y$ coordinate of the motion of a particle:

$$\vec{r}(t) = (x(t), y(t)).$$

We choose $x$ to be the horizontal component of the projectile motion, and $y$ to be its height.

The velocity of the projectile will be:

$$\vec{v}(t) = \frac{d}{dt}\left(\vec{r}(t)\right) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = (v_x(t), v_y(t)),$$

and the initial velocity is:

$$\vec{v}_i = \vec{v}(t_i) = |\vec{v}_i|\angle\theta = (v_x(t_i), v_y(t_i)) = (v_{ix}, v_{iy}) = (|\vec{v}_i|\cos\theta, |\vec{v}_i|\sin\theta).$$

The acceleration of the projectile will be

$$\vec{a}(t) = \frac{d}{dt}\left(\vec{v}(t)\right) = (a_x(t), a_y(t)) = (0, -9.81).$$

Note how we have zero acceleration in the $x$ direction so we can use the UVM equations of motion for $x(t)$ and $v_x(t)$. In the $y$ direction we have a uniform downward acceleration due to gravity.

*2) Projectile motion:* The equations of motion of a projectile are the following. First in the $x$ direction we have uniform velocity motion (UVM):

$$x(t) = v_{ix}t + x_i,$$
$$v_x(t) = v_{ix}.$$

In the $y$ direction, you have the constant pull of gravity downwards which gives us a uniformly accelerated motion (UAM):

$$y(t) = \frac{1}{2}(-9.81)t^2 + v_{iy}t + y_i,$$
$$v_y(t) = -9.81t + v_{iy},$$
$$v_{yf}^2 = v_{yi}^2 + 2(-9.81)(\Delta y).$$

## V. Forces

Like a shepherd who brings in a stray sheep back, we need to rescue the word *force* and give it precise meaning. In physics force means something very specific. Not "the force" from Star Wars, not the "force of public opinion", and not the *force* in the battle of good versus evil.

Force in physics has a precise meaning as an amount of push or pull exerted on an object. Force is a vector. We measure force in Newtons [N], and we can use it in equations and solve for it just like any other unknown. In this section we will explore all the different kinds of forces.

### A. Concepts

- $\vec{F}$: a force. This is something the object "feels" as a pull or a push. Force is a vector, so you must always keep in mind the direction in which the force $\vec{F}$ acts.
- $k, G, m, \mu_s, \mu_k, \ldots$: parameters on which the force $F$ may depend. Ex: the heavier an object is (has large $m$ parameter), the larger its gravitational pull will be: $\vec{W} = -9.81m\hat{z}$, where $\hat{z}$ points towards the sky.

### B. Kinds of forces

We next list all the forces which you are supposed to know about for a standard physics class and define the relevant parameters for each kind of force. You need to practice exercises using each of these forces, until you start to *feel* how they act.

*1) Gravitation:* Manifestations of the gravitational pull of the planet Earth on massive objects:

- $M$: mass of the earth. $M = 5.9721986 \times 10^{24}$ [kg].
- $m$: mass of an object.
- $\vec{W} = \vec{F}_g$: The weight (the force on a object due to gravity).
- $G$: Gravitational constant = $6.67 \times 10^{-11}$ $[\frac{Nm^2}{kg^2}]$.
- $\vec{F}_g = \frac{GMm}{r^2}$: Force of gravity between two objects of mass $M$ and $m$ respectively. Measured in Newtons [N].
- $\vec{F}_g = gm$ (downward): The force of gravity on the surface of the earth, where $g = \frac{GM}{r^2} \approx 9.81\ldots$ [N/kg]=[m/s^2].

The famous one-over-arr-squared law that describes the gravitational pull between two objects is:

$$F_g = \frac{GMm}{r^2}.$$

You will rarely use it, but it is extremely important as this is where all of mechanics began. This was Newton's big discovery. All the rest of mechanics is simple calculus, but this equation is *real* physics. It tells us something about how the Universe works.

At the surface of the earth:

$$\vec{F}_g = \frac{GMm}{r^2} = \underbrace{\left(\frac{GM}{r^2}\right)}_{g} m = \vec{g}m = \vec{W},$$

where the weight $\vec{W}$ of an object is a vector that points towards the centre of the earth, and $g = 9.81$[m/s^2].

*2) Force of a spring:*

- $\vec{F}_s = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by $x$ meters. The constant $k$ [N/m] is a measure of the *strength* of the spring, or its stiffness.

*3) Tension in a rope:*

- $\vec{T}$: Tension in a rope. Tension is always pulling *away* from an object: you can't push a dog on a leash.

*4) Contact force:*

- $\vec{C}$: Contact force between two rigid objects. We generally brake-up contact forces into two components: perpendicular and parallel to the contact surface.
- $\vec{N} \equiv \vec{C}_\perp$: Normal force: the force between two surfaces. Normal is a mathematically precise way to say "perpendicular to a surface". Intuitively, you can think of $\vec{N}$ as the force that a surface exerts on an object to keep it where it is. The reason why my coffee mug does not fall to the floor, is that the table exerts a normal force on it keeping in place.
- $\vec{F}_f \equiv \vec{C}_{||}$: Force of friction between two surfaces. There are two kinds, both of which are proportional to the normal force between the surfaces:
  Kinetic:
  $$F_{fk} = \mu_k |\vec{N}|.$$
  Static:
  $$F_{fs} = \mu_s |\vec{N}|.$$

*5) Two kinds of friction forces:*

- $\vec{F}_{fs} = \mu_s |\vec{N}|$: Static force of friction, for objects that are not moving.
- $\mu_s$: The static coefficient of friction. ex: 0.3. It describes the **maximum** amount of friction that can exist between two objects. If a horizontal force exists greater than $F_{fs} = \mu_s N$, then the object will start to slip.
- $\vec{F}_{fk} = \mu_k |\vec{N}|$: Kinetic force of friction acts when two objects are sliding relative to each other. It always acts in a direction opposing the motion.
- $\mu_k$: Kinetic coefficient of friction. ex: $\mu_k = 0.1$. Dimensionless. it is just the ratio that describes how much friction an object feels for a given amount of normal force.

## VI. Force diagrams

Newton's 2nd law says that the *net* force on an object causes an acceleration:

$$\sum \vec{F} = \vec{F}_{net} = m\vec{a},$$

so finding the net force must be a pretty important thing.

### A. Concepts

Newton's second law is a relationship between these three concepts:

- $m$: the *mass* of an object.
- $\vec{F}$: vector used to denote any kind of *force*.
- $\vec{a}$: the *acceleration* of the object.

What types of forces are there in force diagrams?

- $\vec{W} \equiv \vec{F}_{gravity} = m\vec{g}$: The *weight*. This is the force on a object due to its gravity. The gravitational pull $\vec{g}$ always points downwards towards the center of the earth. $g = 9.81$ [N/kg].
- $\vec{T}$: Tension in a rope. Tension is always pulling *away* from the object.
- $\vec{N}$: Normal force the force between two surfaces.
- $\vec{F}_{fs} = \mu_s |\vec{N}|$: Static force of friction.
- $\vec{F}_{fk} = \mu_k |\vec{N}|$: Kinetic force of friction.
- $\vec{F}_{spring} = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by $x$ meters.

### B. Formulas

*1) Newton's 2nd law:* The sum of the forces acting on an object, divided by the mass gives you the acceleration of the object:

$$\sum F = \vec{F}_{net} = m\vec{a}.$$

It is a good idea to always write those two equations together as a block so it remains clear that you are talking about the same problem, but the first row represents the $x$-dimension and the second row represents the $y$-dimension.

### C. Recipe for solving force diagrams

1) Draw a diagram centred on the object and draw all the forces acting on it.
2) Choose a coordinate system, and indicate clearly what you will call the $x$-direction, and what you will call the $y$-direction. All **equations are expressed *with respect to* this coordinate system**.
3) Write down this "template":
   $$\sum F_x = \qquad\qquad = ma_x$$
   $$\sum F_y = \qquad\qquad = ma_y$$
4) Fill the first line by finding the $x$-components of each force acting on the object.
5) Fill the second line by finding the $y$-components of each force acting on the object.
6) Consistency checks:
   a) Check signs. If the force in the diagram is acting in the $x$-direction then its component must be positive. If the force is acting in the opposite direction to $\hat{x}$, then its component should be negative.
   b) Verify that whenever $F_x \propto \cos\theta$, then $F_y \propto \sin\theta$. If instead we use an angle $\phi$ defined with respect to the $y$-axis we would have $F_x \propto \sin\phi$, and $F_y \propto \cos\phi$.
7) Solve the two equations finding the one or two unknowns. If there are two unknowns, you may need to solve two equations simultaneously by isolation and substitution.

Force diagrams are best explained through examples.

## VII. Momentum

The momentum is equal to the velocity of the moving object multiplied by the object's mass ($\vec{p} = m\vec{v}$). Therefore, since the car weights $1000 \times 1000 = 10^6$ times more than the piece of paper, it has $10^6$ times more momentum when moving at the same speed. A collision with it will "hurt" that much more.

### A. Concepts

- $m$: the *mass* of the moving object.
- $\vec{v}$: the *velocity* of the moving object.
- $\vec{p} = m\vec{v}$: the *momentum* of the moving object.

### B. Formulas

*1) Definition:* The momentum of an object is the mass of the object times its velocity:

$$\vec{p} = m\vec{v}.$$

If you speed is $\vec{v} = (20, 0, 0)$[m/s], which is equivalent to saying "20[m/s] in the $x$-axis direction", and your mass is 100kg then your momentum is $\vec{p} = (2000, 0, 0)$[kg*m/s].

In the absence of acceleration, objects will conserve their velocity:

$$\vec{v}_{in} = \vec{v}_{out}.$$

This is equivalent to saying that objects conserve their momentum (just multiply the velocity by the mass if the mass stays constant and the velocity stays constant, then the momentum must stay constant).

*2) Conservation of momentum:* More generally, if you have a situation with multiple moving objects, you can say that the "overall momentum", i.e., the sum of the momenta of all the particles stays constant:

$$\sum \vec{p}_{in} = \sum \vec{p}_{out}.$$

This is amazingly powerful stuff, and one of the furthest reaching laws of physics. Whatever momentum comes into a collision must come out.

## VIII. Energy

Instead of thinking about velocities $v(t)$ and motion trajectories $x(t)$, we can solve physics problems using *energy* calculations.

### A. Concepts

The concepts of energy come up in several different contexts.

Moving objects:

- $m$: the mass of an object.
- $v$: the velocity.
- $E_K = K$: kinetic energy $= \frac{1}{2}mv^2$.

Moving objects by force:

- $\vec{F}$: the force needed to move the object.
- $\vec{d}$: the displacement of the object. How far it moved.
- $W$: work done to move the object $= \vec{F} \cdot \vec{d}$.

Gravity:

- $g$: gravitational acceleration on the surface of earth. 9.81 [m/s^2].
- $h$: height of an object.
- $U_g$: Gravitational potential energy $= mgh$.

Springs:

- $k$: spring constant. Measured in [N/m].
- $x$: spring displacement from the relaxed position. If the spring is stretched then $x > 0$, and if it is compressed then $x < 0$.
- $U_s$: Spring potential energy $= \frac{1}{2}kx^2$.

There are all kinds of other forms of energy: electric energy, sound energy, thermal energy, etc. In this section we will focus on the types of *mechanical* energy.

### B. Formulas

Kinetic energy:

$$K = \frac{1}{2}mv^2$$

Work:

$$W = \int \vec{F}(x) \cdot dx$$

for constant force:

$$W = \vec{F} \cdot \vec{d} = |F||d| \cos\theta.$$

Gravitational potential energy:

$$U_g = mgh,$$

which is the energy you have because of your height.

Spring energy:

$$U_s = \frac{1}{2}kx^2.$$

Conservation of energy:

$$\sum E_{in} + W_{in} = \sum E_{out} + W_{out}.$$

## IX. Conclusion

The numbered equations...

## X. Minireference

I hope this short excerpt from the `MATH and PHYSICS Minireference` has given you some inspiration for compact teaching. No blah blah. Straight to the point.

If you liked this tutorial you can check out the other ones on http://minireference.com and order the printed book which has not only formulas but also compact explanations: http://minireference.com/order_book/.