MECHANICS Explained in Six Pages

Excerpt from the new book *MATH and PHYSICS Prerequisites* by Ivan Savov

***Abstract*—Mechanics is the precise study of the motion of objects, the forces acting on them and more abstract concepts such as momentum and energy. You probably have an intuitive understanding of these concepts already, but in the next six pages you will learn how to use precise mathematical equations to support your intuition.**

I. INTRODUCTION

To solve a physics problem is to obtain the *equation of motion* $x(t)$, which describes the position of the object as a function of time. Once you know $x(t)$, you can answer any question pertaining to the motion of the object. To find the initial position x_i of the object, you simply plug $t = 0$ into the equation of motion $x_i = x(0)$. To find the time(s) when the object reaches a distance of 20[m] from the origin, we simply solve for t in $x(t) = 20$[m]. Many of the problems on the final exam will be of this form so if know how to find $x(t)$, you will be in good shape for the exam.

A. Dynamics is the study of forces

The first step towards finding $x(t)$ is to calculate all the *forces* that act on the object. Forces are the *cause* of motion, so if you want to understand motion you need to understand forces. Newton's second law $F = ma$ states that **a force acting on an object produces an acceleration** inversely proportional to the mass of the object. Once you have the acceleration, you can compute $x(t)$ using calculus. We will discuss the calculus procedure for getting from $a(t)$ to $x(t)$ shortly. For now, let us focus on the causes of motion: the forces acting on the object. There are many kinds of forces: the weight of an object $\vec{W}$ is a type force, the force of friction $\vec{F}_f$ is another type of force, the tension in a rope $\vec{T}$ is yet another type of force and there are many others. Note the little arrow on top of each force, which is there to remind you that forces are *vector* quantities. Unlike regular numbers, forces act in a particular direction, so it is possible that the effects of one force are counteracting another force. For example the force of the weight of a flower pot is exactly counter-acted by the tension in the rope on which it is suspended, thus, while there are two forces that may be acting on the pot, there is no *net force* acting on it. Since there is no net force to cause motion and since the pot wasn't moving to begin with, it will just sit there motionless despite the fact that there are forces acting on it! The first step when analyzing a physics problem is to calculate the *net force* acting on the object, which is the sum of all the forces acting on the object $\vec{F}_{net} \equiv \sum \vec{F}$. Knowing the net force, we can use $\vec{a}(t) = \frac{\vec{F}_{net}}{m}$ to find the acceleration.

B. Kinematics is the study of motion

If you know the acceleration of an object $a(t)$ and its initial velocity v_i, you can find its velocity $v(t)$ function at all later times. This is because the acceleration function $a(t)$ describes the change in the velocity of the object. If you know that the object started with an initial velocity of $v_i \equiv v(0)$, the velocity at a later time $t = \tau$ is equal to v_i plus the "total velocity change" between $t = 0$ and $t = \tau$. The mathematical way of saying this is $v(\tau) = v_i + \int_0^\tau a(t)\,dt$. The symbol $\int \cdot\, dt$ is called an *integral* and is fancy way of finding the total of some quantity over a given time period. In the above formula we were calculating the total of $a(t)$ between $t = 0$ and $t = \tau$.

To understand what is going on, it may be useful to draw an analogy with a scenario you are more familiar with. Consider the function $\text{ba}(t)$ which represents your account balance at time t, and the function $\text{tr}(t)$ which corresponds to the transactions (deposits and withdraws) on your account. The function $\text{tr}(t)$ describes the change in the function $\text{ba}(t)$, the same way the function $a(t)$ describes the change in $v(t)$. Knowing the initial balance of your account at the beginning of the month, you can calculate the balance at the end of the month as follows $\text{ba}(30) = \text{ba}(0) + \int_0^{30} \text{tr}(t)\,dt$.

If you know the initial position x_i and the velocity function $v(t)$ you can find the position function $x(t)$ by using integration again. We find the position at time $t = \tau$ by adding up all the velocity (changes in the position). The formula is $x(\tau) = x_i + \int_0^\tau v(t)\,dt$.

The entire procedure for predicting the motion of objects can be summarized as follows:

$$\frac{1}{m}\underbrace{\left(\sum \vec{F} = \vec{F}_{net}\right)}_{\text{dynamics}} = \underbrace{a(t) \overset{v_i + \int dt}{\longrightarrow} v(t) \overset{x_i + \int dt}{\longrightarrow} x(t)}_{\text{kinematics}}. \tag{1}$$

If you understand the above equation, then you understand mechanics. My goal for the next couple of pages is to introduce you to all the concepts that appear in that equation and the relationships between them.

C. Other concepts

Apart from equation (1), there is a number of other topics which are part of a standard Mechanics class.

Newton's second law can also be applied to the study of circular motion. Circular motion is described by the angle of rotation $\theta(t)$, the angular velocity $\omega(t)$ and the angular acceleration $\alpha(t)$. The causes of angular acceleration are angular forces, which we call *torques* $\mathcal{T}$. Apart from this change to angular quantities, the principles behind the circular motion are exactly the same as those for linear motion.

During a collision between two objects there will be a sudden spike in the contact force between them, which can be difficult to measure and quantify. It is therefore not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. In order to predict the motion of the objects after the collision we must use a *momentum* calculation. An object of mass m moving with velocity $\vec{v}$ has momentum $\vec{p} \equiv m\vec{v}$. The principle of conservation of momentum states that **the total amount of momentum before and after the collision is conserved**. Thus, if two objects with initial momenta $\vec{p}_{i1}$ and $\vec{p}_{i2}$ collide, the total momentum before the collision must be equal to the total momentum after the collision:

$$\sum \vec{p}_i = \sum \vec{p}_f \qquad \Rightarrow \qquad \vec{p}_{i1} + \vec{p}_{i2} = \vec{p}_{f1} + \vec{p}_{f2}.$$

Using this equation, it is possible to calculate the final momenta $\vec{p}_{f1}$, $\vec{p}_{f2}$ of the objects after the collision.

Another way of solving physics problems is to use the concept of energy. Instead of trying to describe the entire motion of the object, we can focus only on the initial parameters and the final parameters. The law of conservation of energy states that **the total energy of the system is conserved**. Knowing the total initial energy of a system allows us to find final energy, and from this calculate the final motion parameters.

D. Reality check

Of course, you must realize that reading a six page tutorial on Mechanics will not make an expert out of you. Mechanics expertise can only come from doing exercises on your own: "Il faut souffrir pour être bollé." What we *can* do in six pages is to go over all the important concepts and state the important formulas which connect the concepts. There are two ways of looking at Mechanics: either as an opportunity to play LEGO with scientific building blocks, or as a horrible chore inflicted upon requiring complicated mathematical prerequisites. The choice is up to you.

Speaking of prerequisites, I want to reassure you that you have nothing to worry about on that front. The hardest math you will have to do is solving a quadratic equation. We will cover everything you need to know about vectors and integrals in the next section.

II. PRELIMINARIES

In order to understand the equations of physics you need to be familiar with vector calculations and how to calculate some basic integrals. We introduce these concepts in the next two subsections.

A. Vectors

Forces, velocities, and accelerations are vector quantities. A vector quantity $\vec{v}$ can be expressed in terms of its *components* or in terms of its length and direction.

- x axis: The x axis is the horizontal the coordinate system.
- y axis: The y axis is an axis *perpendicular* to the x axis.
- v_x: the *component* of $\vec{v}$ along the x axis.
- v_y: the *component* of $\vec{v}$ along the y axis.
- $\hat{\imath} \equiv (1,0), \hat{\jmath} \equiv (0,1)$: Unit vectors in the x and y directions.
- $\|\vec{v}\|$: The length of the vector $\vec{v}$.
- θ: The angle that $\vec{v}$ makes with the x axis.

Given the xy coordinate system, we can denote a vector in three equivalent ways: $\vec{v} \equiv (v_x, v_y) \equiv v_x\hat{\imath} + v_y\hat{\jmath} \equiv \|\vec{v}\|\angle\theta$.

Given a vector expressed as a length and direction $\|\vec{v}\|\angle\theta$, we calculate its components using:

$$v_x = \|\vec{v}\|\cos\theta, \qquad \text{and} \qquad v_y = \|\vec{v}\|\sin\theta.$$

Alternately, a vector expressed in component notation $\vec{v} = (v_x, v_y)$ can be converted to the length-and-direction form as follows:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}, \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right).$$

It is important that you know how to convert between these two forms; the component notation is useful for calculations, whereas the length-and-direction form describes the geometry of vectors.

The *dot product* between two vectors $\vec{v}$ and $\vec{w}$ can be computed in two different ways:

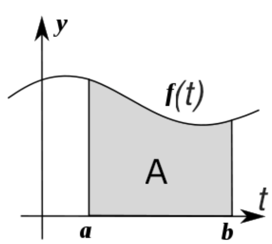
$$\vec{v}\cdot\vec{w} = v_x w_y + v_y w_y = \|\vec{v}\|\|\vec{w}\|\cos\phi,$$

where ϕ is the angle between the vectors $\vec{v}$ and $\vec{w}$. The dot product calculates how similar the two vectors are. For example, we have $\hat{\imath}\cdot\hat{\jmath} = 0$ since the vectors $\hat{\imath}$ and $\hat{\jmath}$ are orthogonal – they point in completely different directions.

B. Integrals

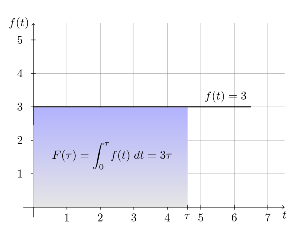
An integral corresponds to the computation of an *area* under a curve $f(t)$ between two points:

$$A(a,b) \equiv \int_{t=a}^{t=b} f(t)\ dt.$$



The symbol $\int$ is a mnemonic for *sum*, since the area under the curve corresponds in some sense to the sum of the values of the function $f(t)$ between $t = a$ and $t = b$. The integral is the total amount of f between a and b.

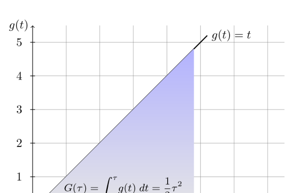
Consider for example the constant function $f(t) = 3$. Let us find the expression $F(\tau) \equiv A(0,\tau)$ that corresponds to the area under $f(t)$ between $t = 0$ and the time $t = \tau$. We can easily find this area because the region under the curve is rectangular:



$$F(\tau) \equiv A(0,\tau) = \int_0^\tau f(t)\ dt = 3\tau$$

since the area of a rectangle is equal to base times the height.

Anther important calculation is the area under the function $g(t) = t$. We will compute $G(\tau) \equiv A(0,\tau)$, which corresponds to the area under $g(t)$ between 0 and τ. This area is also easily computed since the region under the curve is triangular:



$$G(\tau) \equiv A(0,\tau) = \int_0^\tau g(t)\ dt = \frac{\tau\times\tau}{2} = \frac{1}{2}\tau^2,$$

since the area of a triangle is the product of the length of the base times the height divided by two.

For the purpose of understanding mechanics, what you need to know is that the integral of a function is the total amount of the function accumulated during some time period. You should try to remember the formulas:

$$\int_c^\tau a\ dt = a\tau + C, \qquad \int_c^\tau at\ dt = \frac{1}{2}a\tau^2 + C,$$

which correspond to the integral of a constant function and the integral of a line with slope a. Note that each time you give a general integral formula the answer will contain an additive constant term $+C$, which depends on the starting point of the area calculation. In the above examples we used $c = 0$ as the initial point so we had $C = 0$.

The integral of the sum of two functions is the sum of the integrals Using this fact and the two formulas above, we can also compute the integral for the function $f(t) = mt + b$ as follows:

$$\int_c^\tau (mt + b)\ dt = \frac{1}{2}m\tau^2 + b\tau + C. \tag{2}$$

Now that you know about vectors and integrals, we can start our discussion on the laws of physics.

III. Kinematics

Kinematics (from the Greek word *kinema* for *motion*) is the study of trajectories of moving objects. The equations of kinematics can be used to calculate how long a ball thrown upwards will stay in the air, or to calculate the acceleration needed to go from 0 to 100 km/h in 5 seconds.

A. Concepts

The key notions used to describe the motion of an objects are:

- t: the time, measured in seconds [s].
- $x(t)$: the position of an object as a function of time – also known as the equation of motion.
- $v(t)$: the velocity of the object as a function of time.
- $a(t)$: the acceleration of the object as a function of time.
- $x_i = x(0), v_i = v(0)$: initial position and velocity (initial conditions).

The position, velocity and acceleration functions ($x(t)$, $v(t)$ and $a(t)$) are connected. They all describe different aspects of the same motion. The function $x(t)$ is the main function since it describes the position of the object at all times. The velocity function describes the change in the position over time, hence it is measured in [m/s]. The acceleration function describes how the velocity changes over time.

Assume now that we know the acceleration of the object $a(t)$ and that we want to find $v(t)$. The acceleration is the change in the velocity of the object, thus if you know that the object stated with an initial velocity of $v_i \equiv v(0)$, and you want to find the velocity at later time $t = \tau$, you have to add up all the acceleration that the object felt during this time $v(\tau) = v_i + \int_0^\tau a(t)\ dt$. The velocity as a function of time is given by the initial velocity v_i plus the integral of the acceleration.

If we further integrate the velocity function, we will obtain the position function $x(t)$. Thus, the procedure for finding $x(t)$ starting from $a(t)$ can be summarized as follows:

$$a(t) \quad \xrightarrow{v_i+\int dt} \quad v(t) \quad \xrightarrow{x_i+\int dt} \quad x(t).$$

We will now illustrate how to apply this procedure for the important special case of motion with constant acceleration.

B. Uniform acceleration motion

Suppose the object starts from an initial position x_i with initial velocity v_i and undergoes a constant acceleration $a(t) = a$ from time $t = 0$ until $t = \tau$. What will be its velocity $v(\tau)$ and position $x(\tau)$ at time $t = \tau$?

We can find the velocity of the object by integrating the acceleration from $t = 0$ until $t = \tau$:

$$v(\tau) = v_i + \int_0^\tau a(t)\ dt = v_i + \int_0^\tau a\ dt = v_i + a\tau,$$

where we used the formula for the integral of a constant function. To obtain $x(t)$ we integrate $v(t)$ and obtain:

$$x(\tau) = x_i + \int_0^\tau v(t)\,dt = x_i + \int_0^\tau (at + v_i)\,dt = x_i + \frac{1}{2}a\tau^2 + v_i\tau,$$

where we used the integral formula from equation (2). Note that both of the above integrals calculations required the knowledge of the initial conditions x_i and v_i. This is because the integral calculations tell us about the *change* in the quantities relative to their initial value.

We can summarize our findings regarding uniform acceleration motion (UAM) in the following three equations:

$$a(t) = a, \qquad \text{(by the definition of UAM)}$$

$$v(t) = at + v_i, \tag{3}$$

$$x(t) = \tfrac{1}{2}at^2 + v_it + x_i. \tag{4}$$

These equations fully describe all aspects of the motion of an object undergoing a constant acceleration $a(t) = a$ starting from $x(0) = x_i$ with initial velocity $v(0) = v_i$. There is also another very useful formula to remember:

$$v_f^2 = v_i^2 + 2a(x_f - x_i), \tag{5}$$

which is obtained by combining equation (3) and (4) in a particular way.

A special case of the above equations is the case with zero acceleration $a(t) = 0$. If there is no acceleration (change in velocity) then the velocity of the motion will be constant so we call this *uniform velocity motion* (UVM). The equations of motion for UVM are: $v(t) = v_i$ and $x(t) = v_it + x_i$. If you understand the difference between UVM and UAM, and three formulas above, then you are ready to solve *any* kinematics problem.

C. Free fall

We say that an object is in *free fall* if the only force acting on it is the force of gravity. On the surface of the earth, the force of gravity will produce a constant acceleration of $a = -9.81$[m/s^2]. The negative sign is there because the gravitational acceleration is directed downwards, and we assume that the we measure distance from the ground up.

You can test your knowledge by trying the following practice problems.

0 to 100 in 5 seconds. You want to go from 0 to 100[km/h] in 5 seconds with your car. How much acceleration does your engine need to produce? Assume the acceleration is constant. Sol: Use (3). Ans: $a = 5.56$[m/s^2].

Moroccan example. Suppose your friend wants to send you a ball wrapped in aluminum foil from his balcony, which is located at a height of $x_i = 44.145$[m]. At $t = 0$[s] he *throws* the ball straight down with an initial velocity of $v_i = -10$[m/s]. How long does it take for the ball to hit the ground? Sol: Solve for t in (4) using $a = -9.81$. Ans: $t = 2.53$[s].

IV. Projectile motion

We will now analyze an important kinematics problem in *two* dimensions. The motion of a projectile is described by:

- $\vec{r}(t) \equiv (x(t), y(t))$: the position of the object at time t.
- $\vec{v}(t) \equiv (v_x(t), v_y(t))$: the velocity of the object as a function of time.
- $\vec{a}(t) \equiv (a_x(t), a_y(t))$: the acceleration as a function of time.

The motion of an object starts form an *initial* position an goes to a *final* position for which we will use the following terminology:

- $t_i = 0$: initial time (the beginning of the motion).
- t_f: final time (when the motion stops).
- $\vec{v}_i = \vec{v}(0) = (v_x(0), v_y(0)) = (v_{xi}, v_{yi})$: the initial velocity at $t = 0$.
- $\vec{r}_i = \vec{r}(0) = (x(0), y(0)) = (x_i, y_i)$: the initial position at $t = 0$.
- $\vec{r}_f = \vec{r}(t_f) = (x(t_f), y(t_f)) = (x_f, y_f)$: the final position at $t = t_f$.

Projectile motion is nothing more than two parallel one-dimensional kinematics problems: UVM in the x direction and UAM in the y direction.

A. Formulas

The acceleration felt by a flying projectile is:

$$\vec{a}(t) = (a_x(t), a_y(t)) = (0, -9.81) \quad [\text{m/s}^2].$$

There is no acceleration in the x direction (ignoring air friction) and we have a uniform downward acceleration due to gravity in the y direction. Therefore, the equations of motion of the projectile are the following:

$$\begin{aligned} x(t) &= v_{ix}t + x_i, & y(t) &= \tfrac{1}{2}(-9.81)t^2 + v_{iy}t + y_i, \\ v_x(t) &= v_{ix}, & v_y(t) &= -9.81t + v_{iy}, \\ & & v_{yf}^2 &= v_{yi}^2 + 2(-9.81)(y_f - y_i). \end{aligned}$$

In the x direction we have the equations of uniform velocity motion (UVM), while in the y direction, we have equations of uniformly accelerated motion (UAM). Indeed, projectile motion problems can be decomposed into two separate sets of equations coupled through the time variable t.

Example. Let us now consider the example illustrated in Figure 1 which shows an object being thrown with an initial velocity 8.96[m/s] at an angle of $51.3°$ from an initial height of 1[m]. Calculate the maximum height h that the object will reach, and the distance d where the object will hit the ground.

Your first step when reading any physics problem should be to extract the information from the problem statement. The initial position is $\vec{r}(0) = (x_i, y_i) = (0, 1)$[m]. The initial velocity is $\vec{v}_i = 8.96\angle 51.3°$[m/s], which is $\vec{v}_i = (8.96\cos 51.3°, 8.96\sin 51.3°) = (5.6, 7)$[m/s] in component form.

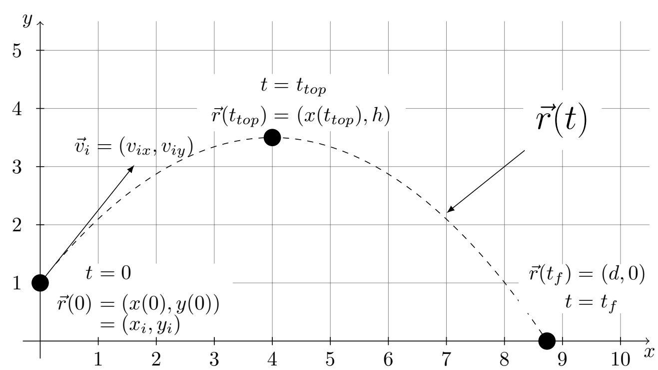


Figure 1. An object is thrown with $\vec{v}_i = 8.96\angle 51.3°$[m/s] from $\vec{r}_i = (0, 1)$[m]. What will be the maximum height reached h and distance travelled d by the object?

We can now plug the values of $\vec{r}_i$ and $\vec{v}_i$ into the equations of motion and find the desired quantities. When the object reaches its maximum height, it will have zero velocity in the y direction $v_y(t_{top}) = 0$. We can use this fact, and the $v_y(t)$ equation in order to find $t_{top} = 7/9.81 = 0.714$[s]. The maximum height is then obtained by evaluating the function $y(t)$ at $t = t_{top}$: $h = y(t_{top}) = 1 + 7(0.714) + \frac{1}{2}(-9.81)(0.714)^2 = 3.5$[m]. To find d, we must solve the quadratic equation $0 = y(t_f) = 1 + 7(t_f) + \frac{1}{2}(-9.81)(t_f)^2$ to find the time t_f when the object hits the ground. The solution is $t_f = 1.55$[s]. We then plug this value into the equation for $x(t)$ to obtain $d = x(t_f) = 0 + 5.6(1.55) = 8.68$[m]. You can verify that these answers match the trajectory in Figure 1.

V. Dynamics

Dynamics is the study of various forces that act on objects. Forces are vector quantities measured in Newtons [N]. In this section we will explore all the different kinds of forces.

A. Kinds of forces

We next list all the forces which you are supposed to know about.

1) Gravitational force: The force of gravity exists between any two massive objects. The magnitude of the gravitational force between two objects of mass M[kg] and m[kg] separated by a distance r[m] is given by the formula $\vec{F}_g = \frac{GMm}{r^2}$[N], where $G = 6.67 \times 10^{-11}[\frac{\text{Nm}^2}{\text{kg}^2}]$ is the *gravitational constant*.

On the surface of the earth, which has mass $M = 5.9721986 \times 10^{24}$[kg] and radius $r = 6.3675 \times 10^6$[m], the force of gravity on an object of mass m is given by

$$F_g = \frac{GMm}{r^2} = \underbrace{\frac{GM}{r^2}}_{g} m = 9.81m = W. \tag{6}$$

We call this force the *weight* of the object and to be precises we should write $\vec{W} = -mg\hat{\jmath}$ to indicate that the force acts *downwards* – in the negative y direction. Verify using your calculator that $\frac{GM}{r^2} = 9.81 \equiv g$.

2) Force of a spring: A spring is a piece of metal twisted into a coil that has a certain natural lenght. The spring will resist any attempts to stretch it or compress it. The force exerted by a spring is given by

$$\vec{F}_s = -k\vec{x}, \tag{7}$$

where x is the amount by which the spring is displaced from its natural length and the constant k[N/m] is a measure of the *strength* of the spring. Note the negative sign: if you try to stretch the spring (positive x) then the force of a spring will pull against you (in the negative x direction), if you try to compress the spring (negative x) it will push against you (in the positive x direction).

3) Normal force: The normal force is the force between two surfaces in contact. The word *normal* means "perpendicular to the surface of" in this case. The reason why my coffee mug does not fall to the floor right now, is that the table exerts a normal force $\vec{N}$ on it keeping in place.

4) Force of friction: In addition to the normal force between surfaces, there is also the force of friction $\vec{F}_f$ which acts to prevent or slow down any sliding motion between the surfaces. There are two kinds of force of friction and both kinds of are proportional to the amount of normal force between the surfaces:

$$\max\{\vec{F}_{fs}\} = \mu_s \|\vec{N}\| \quad \text{(static)}, \qquad \vec{F}_{fk} = \mu_k \|\vec{N}\| \quad \text{(kinetic)}, \tag{8}$$

where μ_s and μ_k are the static and dynamic *friction coefficients*. Note that it makes intuitive sense that the force of friction should be proportional to the magnitude of the normal force $\|\vec{N}\|$: the harder the surfaces push against each other the more difficult it should be to make them slide. The equations in (8) make this intuition precise.

The static force of friction acts on objects that are not moving. It describes the *maximum* amount of friction that can exist between two objects. If a horizontal force greater than $F_{fs} = \mu_s N$ is applied to the object, then it will start to slip. The kinetic force of friction acts when two objects are sliding relative to each other. It always acts in the direction opposite to the motion.

5) Tension: A force can also be exerted on an object remotely by attaching a rope to the object. The force exerted on the object will be equal to the *tension* in the rope $\vec{T}$. Note that tension always pulls *away* from an object: you can't push a dog on a leash.

B. Force diagrams

Newton's 2nd law says that the *net* force on an object causes an acceleration:

$$\sum \vec{F} = \vec{F}_{net} = m\vec{a}. \tag{9}$$

We will now learn how to calculate the net force acting on an object.

C. Recipe for solving force diagrams

1) Draw a diagram centred on the object. Draw all the vectors of all the forces acting on the object: $\vec{W}$, $\vec{N}$, $\vec{T}$, $\vec{F}_{fs}$, $\vec{F}_{fk}$, $\vec{F}_s$ as applicable.
2) Choose a coordinate system, and indicate clearly in the force diagram what you will call the positive x direction, and what you will call the positive y direction. All quantities in the subsequent equations will be expressed *with respect to* this coordinate system.
3) Write down this following "template":

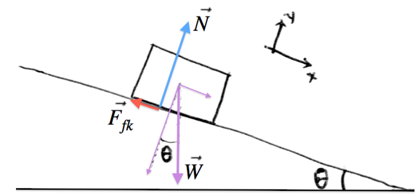
$$\sum F_x = \qquad\qquad = ma_x$$
$$\sum F_y = \qquad\qquad = ma_y$$

4) Fill in the template by calculating the x and y components of each force acting on the object.
5) Solve the equations for the unknown quantities.

Let us now illustrate the procedure by solving an example problem.

Example[1]**.** A block sliding down an incline with angle θ. The coefficient of friction between the block and the incline is μ_k. What is its acceleration?

Step 1: We draw a diagram which includes the weight and the contact forces between the object and the incline split into N and F_{fk}.



Step 2: We pick the coordinate system to be tilted along the incline. This is important because this way the motion is purely in the x direction, while the y direction will be static.

Step 3,4: We copy over the empty template, fill in the components

$$\sum F_x = |\vec{W}|\sin\theta - F_{fk} = ma_x,$$
$$\sum F_y = N - |\vec{W}|\cos\theta = 0,$$

and substitute known values to obtain:

$$\sum F_x = mg\sin\theta - \mu_k N = ma_x,$$
$$\sum F_y = N - mg\cos\theta = 0.$$

Step 5: We solve for a_x by first finding $N = mg\cos\theta$ and then substituting this value into the x equation:

$$a_x = \frac{1}{m}\left(mg\sin\theta - \mu_k mg\cos\theta\right) = g\sin\theta - \mu_k g\cos\theta.$$

VI. Momentum

During a collision between two objects there will be a sudden spike in the contact force between them, which can be difficult to measure and quantify. It is therefore not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. In order to predict the motion of the objects after the collision we must use a *momentum* calculation.

A. Definition

The momentum of a moving object is equal to the velocity of the moving object multiplied by the object's mass:

$$\vec{p} = m\vec{v} \qquad \text{[kg m/s]}.$$

Momentum is a vector quantity. If the velocity of the object is $\vec{v} = 20\hat{\imath} = (20, 0)$[m/s] and it has a mass of 100[kg] then its momentum is $\vec{p} = 2000\hat{\imath} = (2000, 0)$[kg m/s].

B. Conservation of momentum

The law of conservation of momentum states that the total amount of momentum before and after the collision is the same. In a collision involving two moving objects, if we know the initial momenta of the objects before the collision, we can calculate their momenta after the collision:

$$\sum \vec{p}_{in} = \sum \vec{p}_{out} \quad \Rightarrow \quad \vec{p}_{1,in} + \vec{p}_{2,in} = \vec{p}_{1,out} + \vec{p}_{2,out}. \tag{10}$$

This conservation law is one of the furthest reaching laws of physics you will learn in Mechanics. The quantity of motion (momentum) cannot be created or destroyed, it can only be exchanged between systems. This law applies very generally: for fluids, for fields, and even for collisions involving atomic particles described by quantum mechanics.

Example. You throw a piece of rolled up carton from your balcony on a rainy day. The mass of the object is 0.4[g] and it is thrown horizontally with a speed of 10[m/s]. Shortly after it leaves your hand, the carton collides with a rain drop of weight 2[g] falling straight down at a speed of 30[m/s]. What will be outgoing velocity of the objects if they stick together after the collision? Sol: The conservation of momentum equation says that: $\vec{p}_{in,1} + \vec{p}_{in,2} = \vec{p}_{out}$ so $0.4 \times (10, 0) \;+\; 2 \times (0, -30) \;=\; 2.4 \times \vec{v}_{out}$. Ans: $\vec{v}_{out} = (1.666, -25.0)$.

[1]See http://minireference.com/physics/force_diagrams for more examples.

VII. ENERGY

Instead of thinking about velocities $v(t)$ and motion trajectories $x(t)$, we can solve physics problems using *energy* calculations. The key idea in this section is the principle of *total energy conservation*, which tells us that, in any physical process, the sum of the initial energies is equal to the sum of the final energies.

A. Concepts

Energy is measured in Joules [J] and it arises in several different contexts:

K **Moving objects:** An object of mass m moving at velocity $\vec{v}$ has a *kinetic energy* $K = \frac{1}{2}m\|\vec{v}\|^2$[J].

W **Moving objects by force:** If a constant force $\vec{F}$ acts on a object during a distance $\vec{d}$, then the *work* done by this force is $W = \vec{F}\cdot\vec{d}$[J]. Positive work corresponds to energy being added to the system while negative work corresponds to energy being removed from the system.

U_g **Gravitational potential energy:** The gravitational potential energy of an object raised to a height h above the ground is given by $U_g = mgh$[J], where m is the mass of the object and $g = 9.81$[m/s^2] is the gravitational acceleration on the Earth.

U_s **Spring potential energy:** The potential energy stored in a spring when it is displaced by $\vec{x}$[m] from its relaxed position is given by $U_s = \frac{1}{2}k\|\vec{x}\|^2$[J], where k[N/m] is the spring constant.

B. Conservation of energy

Consider a system which starts from an initial state (i), undergoes some motion and arrives at a final state (f). The law of conservation of energy states that **energy cannot be created or destroyed in any physical process**. This means that the initial energy of the system plus the work that was *in*put into the system must equal the final energy of the system plus any work that the was *out*put:

$$\sum E_i \quad + W_{in} \quad = \quad \sum E_f \quad + W_{out}. \tag{11}$$

The expression $\sum E_{(a)}$ corresponds to the sum of the different types of energy the system has in state (a). If we write down the equation in full we have:

$$K_i + U_{gi} + U_{si} \quad + W_{in} \quad = \quad K_f + U_{gf} + U_{sf} \quad + W_{out}.$$

Usually, some of the terms in the above expression can be dropped. For example, we do not need to consider the spring potential energy U_s in physics problems that do not involve springs.

C. Work

The amount of work done by a force $\vec{F}$ during a displacement $\vec{d}$ is:

$$W = \vec{F}\cdot\vec{d} = \|\vec{F}\|\|\vec{d}\|\cos\theta \quad = \int_0^d \vec{F}(x)\cdot d\vec{x}.$$

Note the use of the dot product since only the part of $\vec{F}$ that is pushing in the direction of the displacement $\vec{d}$ produces any work. The expression on the right must be used when the force is not constant.

D. Potential energy

Some kinds of work are just a waste of your time, like working in a bank for example. You work and you get your paycheque, but nothing remains with you. Other kinds of work leave you with some *resource* at the end of the work day. Maybe you learn something, or you network with a lot of good people. In physics, we make a similar distinction. Some types of work, like work against friction, are called *dissipative* since they just waste energy. Other kinds of work are called *conservative* since the work you do is not lost: it is converted into *potential energy*.

The gravitational and spring forces are conservative. Any work you do while lifting an object up into the air against the force of gravity is not lost, but *stored* in the potential energy of the object. The gravitational potential energy of lifting an object from a height of $y = 0$ to a height of $y = h$ is given by:

$$U_g(h) \equiv -W_{done} = -\vec{F}_g\cdot\vec{h} = -(-mg\hat{\jmath})\cdot h\hat{\jmath} = mgh.$$

You can get *all* the work/energy back if you let go of the object. The energy will come back in the form of kinetic energy as the object gets accelerated during the fall.

The spring potential energy stored in a spring as it is compressed from $y = 0$ to $y = x$[m] is given by:

$$U_s(x) = -W_{done} = -\int_0^x \vec{F}_s(y)\cdot d\vec{y} = k\int_0^x y\,dy = \frac{1}{2}kx^2.$$

Example. An investment banker is dropped (from rest) from a 100[m] tall building. What is his speed when he hits the ground? We will use $\sum E_i = \sum E_f$, where $_i$ is at the top of the building and $_f$ is at the bottom. We have $K_i + U_i = K_f + U_f$ and plugging-in the numbers we get: $0 + m\times 9.81\times 100 = \frac{1}{2}mv^2 + 0$. When we can cancel the mass m from both sides of the equation, we are left with $9.81\times 100 = \frac{1}{2}v_f^2$ which can be solved for v_f. We find $v_f = \sqrt{2\times 9.81\times 100} = 44.2945$[m/s] when he hits the ground. This is like 160[km/h]. That will definitely hurt.

VIII. UNIFORM CIRCULAR MOTION

TODO shorten ZZZZZZZZZZZ

Circular motion is different from linear motion and we will have to develop new techniques and concepts which are better suited for the description of circular motion.

A. A new coordinate system

Instead of the usual coordinate system $\hat{x}, \hat{y}, \hat{z}$ which is static, we can use a new coordinate system $\hat{t}, \hat{r}, \hat{z}$ that is "attached" to the rotating object.

Three important directions can be identified:

1) $\hat{t}$: The *tangential* direction in the instantaneous direction of motion of the object. The name comes from the Greek word for "touch" (imagine a straight line "touching" the circle).
2) $\hat{r}$: The *radial* direction always points towards the centre of the circle of rotation.
3) $\hat{z}$: The usual $\hat{z}$ direction, which is perpendicular to the plane of rotation.

We can use the new coordinate system to describe the position, velocity and acceleration of the object undergoing circular motion:

- $\vec{v} = (v_r, v_t)_{\hat{r}\hat{t}}$: The *velocity* of object expressed with respect to the $\hat{r}\hat{t}$ coordinates.
- $\vec{a} = (a_r, a_t)_{\hat{r}\hat{t}}$: The *acceleration* of the object in the $\hat{r}\hat{t}$ coordinates.

B. Radial acceleration

The defining feature of circular motion is the presence of an acceleration that acts perpendicularly to direction of motion. At each instant the object wants to continue moving along the tangential direction, but the radial acceleration causes the velocity to change direction. The result of this constant inward acceleration is that the object will follow a circular path.

The radial acceleration a_r of an object moving in a circle of radius R with a tangential velocity v_t is given by:

$$a_r = \frac{v_t^2}{R}.$$

This is an important equation which relates the three key parameters of circular motion.

We can calculate the magnitude of this radial force F_r as follows:

$$F_r = ma_r = m\frac{v_t^2}{R}.$$

Circular motion *requires* a radial force.

IX. ANGULAR MOTION

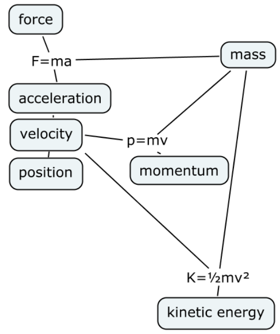
We will now study the physics of objects in rotation. A simple example of this kind of motion is a rotating disk. Other examples include rotating bicycle wheels, spinning footballs and spinning figure skaters.

As you will see shortly, the basic concepts used to describe angular motion are directly analogous to the concepts for linear motion: position, velocity, acceleration, force, momentum and energy.

A. Review of linear motion

It is instructive to begin our discussion with a brief review of the concepts and formulas used to describe the linear motion of objects.

The linear motion of an object is described by its position $x(t)$, velocity $v(t)$ and acceleration $a(t)$ as functions of time. The position function tells you where the object is, the velocity tells you how fast it is moving and the acceleration measures the change in the velocity of the object.



The motion of objects is governed by Newton's first and second laws. In the absence of external forces, objects will maintain a uniform velocity (UVM) which corresponds to the equations of motion: $x(t) = x_i + v_i t$, $v(t) = v_i$.

If there is a net force $\vec{F}$ acting on the object, the force will cause the object to accelerate and the magnitude of the acceleration is obtained using the formula $F = ma$. A constant force acting on an object will produce a constant acceleration (UAM), which corresponds to the equations of motion: $x(t) = x_i + v_i t + \frac{1}{2}at^2$, $v(t) = v_i + at$.

We also learned how to quantify the *momentum* $\vec{p} = m\vec{v}$ and the *kinetic energy* $K = \frac{1}{2}mv^2$ of moving objects. The momentum vector is the natural measure of the "quantity of motion", which plays a key role in collisions. The kinetic energy measures how much energy the object has by virtue of its motion.

The mass of the object m is an important factor in many of the equations of physics. In the equation $F = ma$, the mass m measures the objects *inertia*, i.e., how much resistance the object offers to being accelerated. The mass of the object also appears in the formulas for momentum and kinetic energy: the heavier the object is, the larger its momentum and kinetic energy will be.

B. Concepts

We now introduce the new concepts which are used to describe the angular motion of objects.

- The kinematics of rotating objects is described in terms of angular quantities:
  - $\theta(t)$[rad]: The angular position.
  - $\omega(t)$[rad/s]: The angular velocity.
  - $\alpha(t)$[rad/s^2]: The angular acceleration.
- I[kg m^2]: The *moment of inertia* of an object tells you how difficult it is to make it turn. The quantity I plays the same role in angular motion as the mass m plays in linear motion.
- $\mathcal{T}$[N m]: The *torque* is a measures the angular force.
- The angular equivalent of Newton's second law $\sum F = ma$ is given by the equation $\sum \mathcal{T} = I\alpha$. In words, this law states that applying an angular force (torque) $\mathcal{T}$ will produce an amount of angular acceleration α which is inversely proportional to the moment of inertia I of the object.
- $L = I\omega$[kg m^2/s]: The *angular momentum* of a rotating object describes the "quantity of spinning stuff".
- $K_r = \frac{1}{2}I\omega^2$[J]: The *angular* or *rotational* kinetic energy quantifies the amount of energy an object has by virtue of its rotational motion.

C. Formulas

1) Angular kinematics: Instead of talking about position x, velocity v and acceleration a, we will now talk about the angular position θ, angular velocity ω and angular acceleration α. Except for this change of ingredients, the *recipe* for fining the equations of motion remains the same:

$$\alpha(t) \quad \overset{\omega_i + \int dt}{\longrightarrow} \quad \omega(t) \quad \overset{\theta_i + \int dt}{\longrightarrow} \quad \theta(t).$$

Given the knowledge of the angular acceleration $\alpha(t)$, the initial velocity ω_i and the initial position θ_i, we can use integration in order to find the equation of motion $\theta(t)$ which describes the angular position of the rotating object at all times.

Though this recipe can be applied to any form of angular acceleration function, you are only *required* to know the equations of motion for two special cases: the case of constant angular acceleration $\alpha(t) = \alpha$ and the case of zero angular acceleration $\alpha(t) = 0$. These are the angular analogues of *uniform acceleration motion* and *uniform velocity motion* which we studied in the kinematics section.

The equations which describe *uniformly accelerated angular motion* are the following:

$$\begin{aligned}
\alpha(t) &= \alpha, \\
\omega(t) &= \alpha t + \omega_i, \\
\theta(t) &= \frac{1}{2}\alpha t^2 + \omega_i t + \theta_i, \\
\omega_f^2 &= \omega_i^2 + 2\alpha(\theta_f - \theta_i).
\end{aligned}$$

Note how the form of the equations is *identical* to the UAM equations. This should come as no surprise since the both sets of equations are obtained from the same integrals.

The equations of motion for *uniform velocity angular motion* are:

$$\begin{aligned}
\alpha(t) &= 0, \\
\omega(t) &= \omega_i, \\
\theta(t) &= \omega_i t + \theta_i.
\end{aligned}$$

2) Relation to linear quantities: The angular quantities θ, ω and α are the natural parameters for describing the motion of rotation objects. In certain situations, however, we may want to relate the angular quantities to linear quantities like distance, velocity and linear acceleration. This can be accomplished by multiplying the angular quantity by the radius of motion:

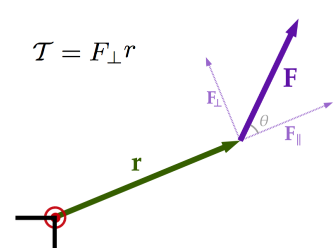
$$d = R\theta, \quad v = R\omega, \quad a = R\alpha.$$

For example, suppose you have a spool of network cable with radius 20[cm] and you need to measure out a length of 20[m] so as to connect your computer to your neighbours' router. How many turns from the spool will you need? To find out, we can solve for θ in the formula $d = R\theta$ and obtain $\theta = 20/0.2 = 200$[rad] which corresponds to 31.8 turns.

3) Torque: Torque is angular force. In order to make an object rotate, you must exert a torque on it. Torque is measured in Newton metres [N m].

The torque produced by a force depends on how far from the centre of rotation it is applied:

$$\mathcal{T} = F_\perp\, r = \|F\| \sin\theta\, r,$$



where r is called the leverage. Note that only the $F_\perp$ component of the force causes the rotation.

To understand the meaning of the torque equation, you should stop reading right now and go to experiment with a door. If you push the door close to the hinges, it will take a lot force to make it move than if you push far from the hinges. The more leverage r you have, the more torque you will produce. Also, if you pull on the door handle away from the hinges, your force will have only a $F_{||}$ component so no matter how hard you pull, you will not cause the door to move.

The standard convention is to call torques that produce counter-clockwise motion positive and torques that cause clockwise rotation negative.

The relationship between torque and force can also be used in the other direction. If an electric motor produces a torque of $\mathcal{T}$[N m] and is attached to a chain wheel of radius R then the pull in the chain will be:

$$F_\perp = \mathcal{T}/R \qquad \text{[N]}.$$

Using this equation, you could compute the maximum pulling force produced by your car. You will have to lookup the value of the maximum torque produced by your car's engine and then divide by the radius of your wheels.

4) Moment of inertia: The momentum of inertia of an object describes how difficult to make the object rotate:

$$I = \{ \text{ how difficult it is to make an object turn } \}.$$

The calculation of the moment of inertia takes into account the mass distribution of the object. An object which has most of its mass close to the centre will have a smaller moment of inertia, whereas objects which have their mass far from the centre will have a large moment of inertia.

The formula for calculating the moment of inertia is:

$$I = \sum m_i r_i^2 = \int_{obj} r^2 \, dm \qquad [\text{kg m}^2].$$

The above equation indicates that we need to weight each part of the object by the squared distance of that part from the centre, hence the units $[\text{kg}\,\text{m}^2]$.

We rarely calculate the moment of inertia of objects using the above formula. Most of the physics problems you will have to solve will involve geometrical shapes for which the moment of inertia is given by simple formulas:

$$I_{disk} = \frac{1}{2}mR^2, \quad I_{ring} = mR^2,$$
$$I_{sphere} = \frac{2}{5}mR^2, \quad I_{sph.shell} = \frac{2}{3}mR^2.$$

When you learn more about calculus, you will be able to derive on your own each of the above formulas. For now, however, just try to remember the formulas for the inertia of a disk and a ring as they are likely to come up in problems.

The quantity I plays the same role in the equations of angular motion as the mass m plays in the equations of linear motion.

5) Torques cause angular acceleration: Recall Newton's second law $F = ma$ which describes the amount of acceleration produced by a given force acting on an object. The angular analogue of Newton's second law is the following equation:

$$\mathcal{T} = I\alpha.$$

The above equation indicates that the angular acceleration produced by the a toque $\mathcal{T}$ is inversely proportional to the object's moment of inertia. Torque is the cause of angular acceleration.

6) Angular momentum: The angular momentum of a spinning object measures the "amount of rotational motion" that the object has. The formula for the angular momentum of a an object with moment of inertia I rotating at an angular velocity ω is:

$$L = I\omega \qquad [kg\,\text{m}^2/\text{s}].$$

The angular momentum of an object is a conserved quantity in the absence of torque:

$$L_{in} = L_{out}.$$

This is similar to the way momentum $\vec{p}$ is a conserved quantity in the absence of external forces.

7) Rotational kinetic energy: The kinetic energy of a rotating object is calculated as follows:

$$K_r = \frac{1}{2}I\omega^2 \qquad [\text{J}].$$

This is the rotational analogue to the linear kinetic energy $\frac{1}{2}mv^2$.

The amount of work produced by a torque $\mathcal{T}$ which is applied during an angular displacement of θ is given by:

$$W = \mathcal{T}\theta \qquad [\text{J}].$$

Using the above equations, we can now include the energy and work associated with rotational motion into conservation of energy calculations.

D. Examples

1) Rotational UVM: A disk is spinning at a constant angular velocity of 12[rad/s]. How many turns will the disk complete in one minute?

Since the angular velocity is constant, we can use the equation $\theta(t) = \omega t + \theta_i$ to find the total angular displacement after one minute. We obtain $\theta(60) = 12 \times 60 = 720$[rad]. To obtain the number of turns, we divide this number by 2π and obtain 114.6[turns].

2) Rotational UAM: A solid disk of mass 20[kg] and radius 30[cm] is initially spinning with an angular velocity of 20[rad/s]. A brake pad applied to the edge of the disk produces a friction force of 60[N]. How long before the disk stops?

To solve the kinematics problem, we need to find the angular acceleration produced by the brake. We can do this using the equation $\mathcal{T} = I\alpha$. We must find $\mathcal{T}$ and I_{disk} and solve for α. The torque produced by the brake is calculated using the force-times-leverage formula: $\mathcal{T} = F_{\perp}r = 60 \times 0.3 = 18$[N m]. The moment of inertia of a disk is given by $I_{disk} = \frac{1}{2}mR^2 = \frac{1}{2}(20)(0.3)^2 = 0.9$[kg m^2]. Thus we have $\alpha = 20$[rad/s^2]. We can now use the UAM formula for the angular velocity $\omega(t) = \alpha t + \omega_i$ and solve for the time when the motion will stop: $0 = \alpha t + \omega_i$. The disk will come to a stop after $t = \omega_i/\alpha = 1$[s].

3) Combined motion: A pulley of radius R and moment of inertia I has a rope wound around it and a mass m attached at the end of the rope. What will be the angular acceleration of the disk if we let the mass drop to the ground while unwinding the rope.

A force diagram on the mass tells us that $mg - T = ma_y$ (where $\hat{y}$ points downwards). The torque diagram on the disk tells us that $TR = I\alpha$. Adding R times the first equation to the second we get:

$$R(mg - T) + TR = Rma_y + I\alpha,$$

or after simplification we get:

$$Rmg = Rma_y + I\alpha.$$

But we know that the rope forms a solid connection between the disk and the mass block, so we must also have $R\alpha = a_y$ so if we substitute for a_y we get:

$$Rmg = RmR\alpha + I\alpha = (R^2m + I)\alpha.$$

Solving for α we obtain:

$$\alpha = \frac{Rmg}{R^2m + I}.$$

This answer makes sense intuitively. The numerator is the "cause" of the motion while the denominator is the effective moment of inertia of the mass-pulley system as a whole.

4) Conservation of angular momentum: A spinning figure skater starts from an initial angular velocity of $\omega_i = 12$[rad/s] with her arms far away from her body. The moment of inertia of her body in this configuration is $I_i = 3$[kg m^2]. She then brings her arms close to her body and in the process her moment of inertia becomes $I_f = 0.5$[kg m^2]. What will be her new angular velocity?

We will solve this problem using the law of conservation of angular momentum:

$$L_i = L_f \qquad \Rightarrow \qquad I_i\omega_i = I_f\omega_f,$$

which we can solve for the final angular velocity ω_f. The answer is $\omega_f = I_i\omega_i/I_f = 3 \times 12/0.5 = 72$[rad/s], which corresponds to 11.46 turns per second.

5) Conservation of energy: A 14[in] bicycle wheel with mass $m = 4$[kg] with all its mass concentrated near the rim is set in rolling motion at a velocity of 20[m/s] up an incline. How far up the incline will the wheel reach before it stops?

We will solve this problem using the principle of conservation of energy $\sum E_i = \sum E_f$. We must take into account both the linear and rotational kinetic energies of the wheel:

$$\begin{aligned} K_i \quad + \quad K_{ri} \ + U_i &= K_f + K_{rf} + U_f \\ \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0 \ &= \ 0 \ + \ 0 \ + mgh. \end{aligned}$$

The first step is to calculate I_{wheel} using the formula $I_{wheel} = \frac{1}{2}mR^2 = 4 \times (0.355)^2 = 0.5$[kg m^2]. If the linear velocity of the wheel is 20[m/s], then its angular velocity is $\omega = 20/0.355 = 56.34$[rad/s]. We can now use these values in the energy equation:

$$\frac{1}{2}(4)(20)^2 + \frac{1}{2}(0.5)(56.34)^2 + 0 = 800.0 + 793.55 = (4)(9.81)h.$$

Therefore the maximum height reached will be $h = 40.61$[m] up the hill.

Note that roughly half of the kinetic energy of the wheel was stored in the rotational motion. This shows that it is important to take into account K_r when solving problems using energy principles.

E. Static equilibrium

We say that a system is in equilibrium when all the forces and torques acting on the system balance each other out. Since there is no net force on the system, it will just sit there motionless.

Conversely, if you see an object that is not moving, then the forces on it must be in equilibrium:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum \mathcal{T} = 0.$$

There must be zero net force in the x direction, zero net force in the y direction and zero torque on the object.

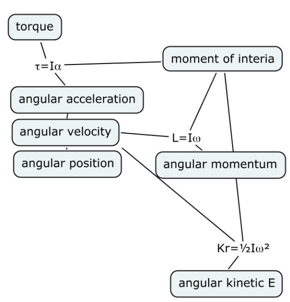
1) Example: Walking the plank: A heavy wooden plank is placed so that one third of its length protrudes from the side of a pirate shit. The plank has a length of 12[m] and total weight 120[kg]; 40[kg] of its weight is suspended above the ocean, while 80[kg] is lying on the ship's deck. How far out on the plank can a 80[kg] person walk before the plank tips over?

We will use the torque equilibrium equation $\sum \mathcal{T}_E = 0$ where we calculate the torques relative to the edge of the ship. The torque produced by person when he has walked a distance of x[m] from the edge of ship is $\mathcal{T}_1 = -80x$. The torque produced by the weight of the plank is given by $\mathcal{T}_2 = 120 \times 2 = 240$[N m] since the weight acts in the centre of gravity of the plank. The maximum distance that can be walked before the plank tips over is therefore $x = 240/80 = 3$[m].

F. Discussion

Our coverage of the ideas of rotational motion has been very brief. The reason for this, is that there was no new physics to be learned. In this section we used the techniques and ideas developed in the context of linear motion to describe the rotational motion of objects.

It is really important that you see the parallels between the new rotational concepts and their linear counterparts. To help you see the connections, you can compare the diagram shown on the right with the diagram from the beginning of this section.



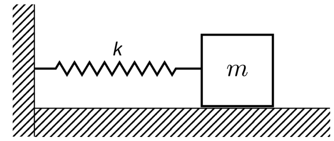
Let us summarize. If you know the torque acting on an object, then you can calculate its angular acceleration α. Knowing the angular acceleration $\alpha(t)$ and the initial conditions θ_i and ω_i, you can then find the other equations of motion $\omega(t)$ and $\theta(t)$ at all times.

Furthermore, the angular velocity ω is related to the *angular momentum* $L = I\omega$ and the *rotational kinetic energy* $K_r = \frac{1}{2}I\omega^2$ of the rotating object. The angular momentum measures the "quantity of rotational motion", while the rotational kinetic energy measures how much energy the object has by virtue of its rotational motion.

The moment of inertia I plays the role of the mass m in the rotational equations. In the equation $\mathcal{T} = I\alpha$, the moment of inertia I measures how difficult it is to make the object turn. The moment of inertia also appears in the formulas for the angular momentum and rotational kinetic energy.

X. Simple harmonic motion

Vibrations and oscillations are all around us. White light is made up of many oscillations of the electromagnetic field at different frequencies (colors). Sounds are made up of a combination of many air vibrations with different frequencies and strengths. In this section we will learn about *simple harmonic motion*, which describes the oscillation of a mechanical system at a fixed frequency and with a constant amplitude. By studying oscillations in their simplest form, you will pick up important intuition which you can apply to all other types of oscillations.



The canonical example of simple harmonic motion is the motion of a mass-spring system illustrated in the figure on the right. The block is free to slide along the horizontal frictionless surface. If the system is disturbed from its equilibrium position, it will start to oscillate back and forth at a certain *natural* frequency, which depends on the mass of the block and the spring constant.

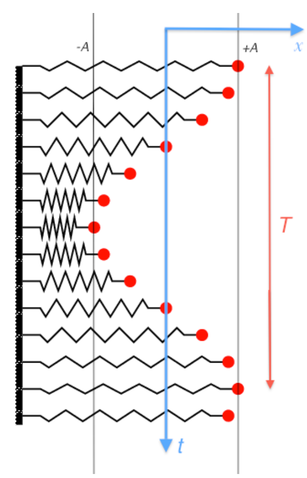
In this section we will focus our attention on two mechanical systems: the mass-spring system and the simple pendulum. We will follow the usual approach and describe the positions, velocities, accelerations and energies associated with these types of motion. The notion of *simple harmonic motion* (SHM) is far more important than just these two systems. The equations and intuition developed for the analysis of the oscillation of theseÂ simple mechanical systems can be applied much more generally to sound oscillations, electric current oscillations and even quantum oscillations. Pay attention, that is all I am saying.

A. Concepts

- A: The *amplitude* of the movement, how far the object goes back and forth relative to the centre position.
- $x(t)$[m], $v(t)$[m/s], $a(t)$[m/s^2]: The position, velocity and acceleration of the object as functions of time.
- T[s]: The *period* of the motion, i.e., how long it takes for the motion to repeat.
- f[Hz]: The *frequency* of the motion.
- ω[rad/s]: The *angular frequency* of the simple harmonic motion.
- ϕ[rad]: The phase constant. The Greek letter ϕ is pronounced "phee".

B. Simple harmonic motion

The figure on the right illustrates a spring-mass system undergoing simple harmonic motion. Observe that the position of the mass as a function of time behaves like the cosine function. From the diagram, we can also identify two important parameters of the motion: the amplitude A, which describes the maximum displacement of the mass from the centre position, and the period T, which describes how long it takes for the mass to come back to its initial position.



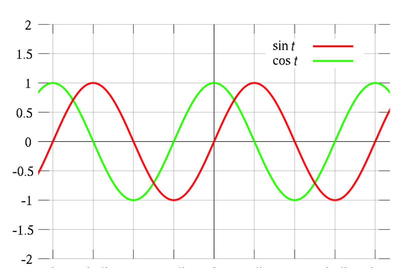
The equation which describes the position of the object as a function of time is the following:

$$x(t) = A\cos(\omega t + \phi).$$

The constant ω (omega) is called the *angular frequency* of the motion. It is related to the period T by the equation $\omega = \frac{2\pi}{T}$. The additive constant ϕ (phee) is called the *phase constant* or *phase shift* and its value depends on the initial condition for the motion $x_i \equiv x(0)$.

I don't want you to be scared by the formula for simple harmonic motion. I know there are a lot of Greek letters that appear in it, but it is actually pretty simple. In order to understand the purpose of the three parameters A, ω and ϕ, we will do a brief review of the properties of the cos function.

1) Review of sin and cos functions: The functions $f(t) = \sin(t)$ and $f(t) = \cos(t)$ are periodic functions which oscillate between -1 and 1 with a period of 2π. Previously we used the functions cos and sin in order to find



the horizontal and vertical components of vectors, and called the input variable θ (theta). However, in this section the input variable is the time t measured in seconds. Look carefully at the plot of $\cos(t)$ function. As t goes from $t = 0$ to $t = 2\pi$ the function cos completes one full cycle. The *period* of $\cos(t)$ is $T = 2\pi$, because this is how long it takes (in radians) for a point to go around the unit circle.

2) Time-scaling: To describe periodic motion with a different period, we can still use the cos function but we must add a multiplier in front of the variable t inside the cos function. This multiplier is called the *angular frequency* and is usually denoted ω (omega). The input-scaled cos function:

$$f(t) = \cos(\omega t),$$

has a period of $T = \frac{2\pi}{\omega}$.

If want to have a periodic function with period T, you should use the multiplier constant $\omega = \frac{2\pi}{T}$ inside the cos function. When you vary t from 0 to T, the function $\cos(\omega t)$ will go through one cycle because the quantity ωt goes from 0 to 2π. You shouldn't just take my word for this: try this for yourself by building a cos function with a period of 3 units.

The *frequency* of a periodic motion motion describes how many times per second the it repeats. The frequency is equal to the inverse of the period:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ [Hz]}.$$

The relation between f (frequency) and ω (angular frequency) is a factor of 2π. This multiplication is needed to convert the units: f measures "real world cycles", and we need a factor of 2π because the natural cycle length of the cos function is 2π radians.

3) Output-scaling: If we want to have oscillations that go between A and $-A$ instead of between -1 and 1, we can simply multiply the cos function by the appropriate *amplitude*:

$$f(t) = A\cos(\omega t).$$

The above function has period $T = \frac{2\pi}{\omega}$ and oscillates between $-A$ and A on the y axis.

4) Time-shifting: The function $A\cos(\omega t)$ starts from its maximum value at $t = 0$. In the case of the mass-spring system, this corresponds to the case when the motion begins with the spring maximally stretched $x_i \equiv x(0) = A$.

In order to describe other starting positions for the motion, it may be necessary to introduce a *phase shift* inside the cos function:

$$f(t) = A\cos(\omega t + \phi).$$

The constant ϕ must be chosen so that at $t = 0$, the function $f(t)$ correctly describes the initial position of the system.

For example, if the harmonic motion starts from the centre $x_i \equiv x(0) = 0$ and is initially going in the positive direction, then the equation of motion is described by the function $A\sin(\omega t)$. However, since $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$ we can equally well describe the motion in terms of a shifted cos function:

$$x(t) = A\cos\Big(\omega t - \frac{\pi}{2}\Big) = A\sin(\omega t).$$

Note that the function $x(t)$ correctly describes the initial position: $x(0) = 0$.

Simple harmonic motion is equally well described by the sin function or the cos function. The choice is up to you, but remember to add an appropriate phase shift ϕ (if necessary), so that the function you choose correctly describes the initial conditions.

By now the meaning of all the parameters in the simple harmonic motion equation should be clear to you. The constant in front of the cos tells us the amplitude A of the motion, the multiplicative constant ω inside the cos is related to the period/frequency of the motion $\omega = \frac{2\pi}{T} = 2\pi f$. Finally, the additive constant ϕ is chosen depending on the initial conditions.

C. Mass and spring

OK, enough math. It is time to learn about the first physical system which exhibits simple harmonic motion: the mass-spring system.

An object of mass m is attached to a spring with spring constant k. If disturbed from rest, this mass-spring system will undergo simple harmonic motion with angular frequency:

$$\omega = \sqrt{\frac{k}{m}}.$$

A stiff spring attached to a small mass will result in very rapid oscillations. A weak spring or a large mass will result in slow oscillations.

A typical exam question will tell you k and m and ask about the period T. If you remember the definition of T, you can easily calculate the answer:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

1) Equations of motion: The general equations of motion for the mass-spring system are as follows:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi), \\ v(t) &= -A\omega\sin(\omega t + \phi), \\ a(t) &= -A\omega^2\cos(\omega t + \phi). \end{aligned}$$

The general shape of the function $x(t)$ is cos-like. The *angular frequency* ω parameter is governed by the physical properties of the system. The parameters A and ϕ describe the specifics of the motion, namely, the *size* of the oscillation and where it starts from.

The function $v(t)$ is obtained, as usual, by taking the derivative of $x(t)$. The function $a(t)$ is obtained by taking the derivative of $v(t)$, which corresponds to the second derivative of $x(t)$.

2) Motion parameters: The velocity and the acceleration of the object are also periodic functions.

We can find the *maximum* values of the velocity and the acceleration by reading off the coefficient in front of the sin and cos in the functions $v(t)$ and $a(t)$.

1) The maximum velocity of the object:

$$v_{max} = A\omega.$$

2) The maximum acceleration:

$$a_{max} = A\omega^2.$$

The velocity is maximum as the object passes through the centre, while the acceleration is maximum when the spring is maximally stretched (compressed).

You will often be asked to solve for the quantities v_{max} and a_{max} in exercises and exams. This is an easy task if you remember the above formulas and you know the values of the amplitude A and the angular frequency ω.

3) Energy: The potential energy stored in a spring which is stretched (compressed) by a length x is given by the formula $U_s = \frac{1}{2}kx^2$. Since we know $x(t)$, we can obtain the potential energy of the mass-spring system as a function of time:

$$U_s(t) = \frac{1}{2}kx(t)^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi).$$

The potential energy reaches its maximum value $U_{s,max} = \frac{1}{2}kA^2$ when the spring is fully stretched or fully compressed.

The kinetic energy of the mass as a function of time is given by:

$$K(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi).$$

The kinetic energy is maximum when the mass passes through the center position. The maximum kinetic energy is given by $K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}mA^2\omega^2$.

4) Conservation of energy: The conservation of energy equation tells us that the total energy of the mass-spring system is conserved. The sum of the potential energy and the kinetic energy at any two instants t_1 and t_2 is the same:

$$U_{s1} + K_2 = U_{s2} + K_2.$$

It is also useful to calculate the total energy of the system:

$$E_T = U_s(t) + K(t) = \text{const}.$$

This means that even if $U_s(t)$ and $K(t)$ change over time, the total energy of the system always remains constant.

We can use the identity $\cos^2\theta + \sin^2\theta = 1$ to verify that the total energy is indeed a constant *and* that it is equal $U_{s,max}$ and K_{max}:

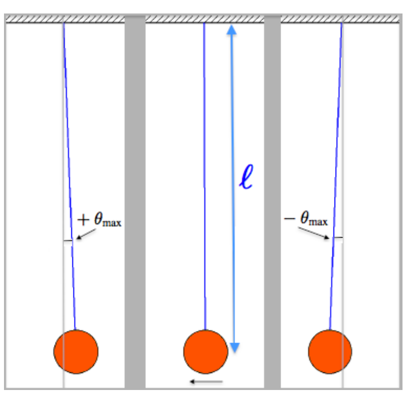
$$\begin{aligned}
E_T &= U_s(t) + K(t) \\
&= \frac{1}{2}kA^2\cos^2(\omega t) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t) \\
&= \frac{1}{2}m\omega^2A^2\cos^2(\omega t) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t) \quad (\text{since } k = m\omega^2) \\
&= \frac{1}{2}m\underbrace{\omega^2A^2}_{v_{max}^2}\underbrace{\left[\cos^2(\omega t) + \sin^2(\omega t)\right]}_{=1} = \frac{1}{2}mv_{max}^2 = K_{max} \\
&= \frac{1}{2}m(\omega A)^2 = \frac{1}{2}(m\omega^2)A^2 = \frac{1}{2}kA^2 = U_{s,max}.
\end{aligned}$$

The best way to understand SHM is to visualize how the energy of the system shifts between the potential energy of the spring and the kinetic energy of the moving mass. When the spring is maximally stretched $x = \pm A$, the mass will have zero velocity and hence zero kinetic energy $K = 0$. At this moment all the energy of the system is stored in the spring $E_T = U_{s,max}$. The other important moment is when the mass has zero displacement but maximal velocity $x = 0, U_s = 0, v = \pm A\omega, E_T = K_{max}$, which corresponds to all the energy being stored as kinetic energy.

D. Pendulum motion

We now turn our attention to another simple mechanical system whose motion is also described by the simple harmonic motion equations.

Consider an aa a **bb** object suspended at the end of a long string of length ℓ in a gravitational field of strength g. If we start the pendulum from a certain angle θ_{max} away from the vertical position and then release it, the pendulum will swing back and forth undergoing simple harmonic motion.



The period of oscillation is given by the following formula:

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

Note that the period does not depend on the amplitude of the oscillation (how far the pendulum swings) nor the mass of the pendulum. The only factor that plays a role is the length of the string ℓ. The angular frequency for a pendulum of length ℓ is going to be:

$$\omega \equiv \frac{2\pi}{T} = \sqrt{\frac{g}{\ell}}.$$

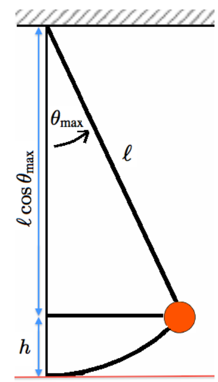
We describe the position of the pendulum in terms of the angle θ that it makes with the vertical line. The equations of motion are described in terms of the angular variables: the angular position θ, the angular velocity ω_θ and the angular acceleration α_θ:

$$\begin{aligned}
\theta(t) &= \theta_{max}\,\cos\left(\sqrt{\frac{g}{\ell}}t + \phi\right), \\
\omega_\theta(t) &= -\theta_{max}\sqrt{\frac{g}{\ell}}\,\sin\left(\sqrt{\frac{g}{\ell}}t + \phi\right), \\
\alpha_\theta(t) &= -\theta_{max}\frac{g}{\ell}\,\cos\left(\sqrt{\frac{g}{\ell}}t + \phi\right).
\end{aligned}$$

The angle θ_{max} describes the maximum angle that the pendulum swings to. Note how I had to invent a new name ω_θ for the angular velocity of the pendulum $\omega_\theta(t) = \frac{d}{dt}(\theta(t))$ so as not to confuse it with the constant $\omega = \frac{2\pi}{T}$ inside the cos function, which describes *angular frequency* of the periodic motion.

1) Energy: The motion of the pendulum is best understood by imagining how the energy of the system shifts between the gravitational potential energy of the mass and its kinetic energy.

The pendulum will have a maximum potential energy when it swings to the side by the angle θ_{max}. At that angle, the vertical position of the mass will be increased by a height h above the lowest point. We can calculate h as follows:



$$h = \ell - \ell\cos\theta_{max}.$$

Thus the maximum gravitational potential energy of the mass is:

$$U_{g,max} = mgh = mg\ell(1 - \cos\theta_{max}).$$

By the conservation of energy principle, the maximum kinetic energy of the pendulum must equal to the maximum of the gravitational potential energy:

$$mg\ell(1 - \cos\theta_{max}) = U_{g,max} = K_{max} = \frac{1}{2}mv_{max}^2,$$

where $v_{max} = \ell\omega_\theta$ is the linear velocity of the mass as it swings through the vertical position.

E. Explanations

It is worthwhile to understand how the equations of simple harmonic motion come about. In this subsection, we will discuss how the equations are derived from Newton's second law $F = ma$.

1) Trigonometric derivatives: The slope (derivative) of the function $\sin(t)$ varies between -1 and 1. The slope is largest when $\sin$ passes through the x axis and the slope is zero when it reaches its maximum and minimum values. A careful examination of the graphs of the bare functions sin and cos reveals that the derivative of the function $\sin(t)$ is described by the function $\cos(t)$ and vice versa:

$$\begin{aligned}
f(t) = \sin(t) \qquad &\Rightarrow \qquad f'(t) = \cos(t), \\
f(t) = \cos(t) \qquad &\Rightarrow \qquad f'(t) = -\sin(t).
\end{aligned}$$

When you learn more about calculus you will know how to find the derivative of any function you want, but for now just take my word that the above two formulas are true.

The chain rule for derivatives tells us that the derivative of a composite function $f(g(x))$ is given by $f'(g(x)) \cdot g'(x)$, i.e., you must take the derivative of the outer function and then multiply by the derivative of the inner function. We can use the chain rule to the find derivative of the simple harmonic motion position function:

$$x(t) = A\cos(\omega t + \phi) \quad \Rightarrow \quad v(t) \equiv x'(t) = -A\sin(\omega t + \phi)\cdot(\omega) = -A\omega\sin(\omega t$$

where the outer function is $f(x) = A\cos(x)$ with derivative $f'(x) = -A\sin(x)$ and the inner function is $g(x) = \omega x + \phi$ with derivative $g'(x) = \omega$.

The same reasoning is used to obtain the second derivative:

$$a(t) \equiv \frac{d}{dt}\{v(t)\} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x(t).$$

Note that $a(t) = x''(t)$ has the same form as $x(t)$, but always acts in the opposite direction.

I hope this clarifies for you how we obtained the functions $v(t)$ and $a(t)$: we simply took the derivative of the function $x(t)$.

2) Derivation of mass-spring SHM equation: You may be wondering where the equation $x(t) = A\cos(\omega t + \phi)$ comes from. This formula looks very different from the kinematics equations for linear motion $x(t) = x_i + v_i t + \frac{1}{2}at^2$, which we obtained starting from Newton's second law $F = ma$ after two integration steps.

In this section, I suddenly pulled out the $x(t) = A\cos(\omega t + \phi)$ out of thin air, as if by revelation. Why did we suddenly start talking about cos functions and Greek letters with dubious names like phase. Are you phased by all of this? When I was first learning about simple harmonic motion, I was totally phased because I didn't see where the sin and cos came from.

The cos also comes from $F = ma$, but the story is a little more complicated this time. The force exerted by a spring is $F_s = -kx$. If you draw a force diagram on the mass, you will see that the force of the spring is the only force acting on it:

$$\sum F = F_s = ma \qquad \Rightarrow \qquad -kx = ma.$$

Recall that acceleration is the second derivative of the position:

$$a = \frac{dv}{dt} = \frac{d^2x(t)}{dt^2}.$$

We now rewrite the equation $-kx = ma$ entirely in terms of the function $x(t)$ and its second derivative:

$$\begin{aligned} -kx(t) &= m\frac{d^2x(t)}{dt^2} \\ 0 &= m\frac{d^2x(t)}{dt^2} + kx(t) \\ 0 &= \frac{d^2x(t)}{dt^2} + \frac{k}{m}x(t). \end{aligned}$$

This is called a *differential equation*. Instead of looking for an *unknown number* as in normal equations, in differential equations we are looking fro an *unknown function* $x(t)$. We do not know what $x(t)$ is, but do know one of its properties, namely, that the second derivative of $x(t)$ is equal to the negative of the function multiplied by some constant.

To solve a differential equation, you have to guess which function $x(t)$ satisfies these properties. There is an entire class called *Differential equations*, in which Engineers and Physicists learn how to do this guessing thing. Can you think of a function which is equal to its second derivative up to some constants?

OK, I thought of one:

$$x_1(t) = A_1 \cos\left(\sqrt{\frac{k}{m}}t\right),$$

and come to think of it I thought of a second one which also works:

$$x_2(t) = A_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

You should try this for yourself: verify that $0 = x_1''(t) + \frac{k}{m}x_1(t)$ and $0 = x_2''(t) + \frac{k}{m}x_2(t)$, which means that these functions are both *solutions* to the differential equation $0 = x''(t) + \omega^2 x(t)$.

Since both $x_1(t)$ and $x_2(t)$ are solutions, then any combination of them must also be a solution:

$$x(t) = A_1\cos(\omega t) + A_2\sin(\omega t).$$

This is *kind of* the answer we were looking for. I say *kind of* because the function $x(t)$ is specified in terms of two components describes by coefficients A_1 and A_2 instead of a single amplitude A and a phase ϕ.

Lo and behold, using the trigonometric identity $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ we can express the function $x(t)$ as a time-shifted trigonometric function:

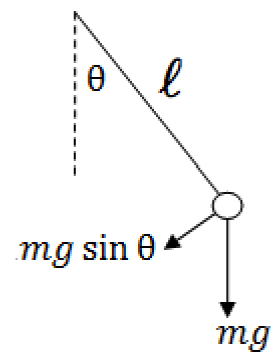
$$x(t) = A\cos(\omega t + \phi) = A_1\cos(\omega t) + A_2\sin(\omega t).$$

The expression on the left is the preferred way of describing SHM because the parameters A and ϕ can be measured more easily in the real world.

Let me go over what just happened here one more time. Our goal is to find the equation of motion which predicts the position of an object as a function of time $x(t)$. Let's draw an analogy with a situation which we have seen previously. In linear kinematics, uniform accelerated motion $a(t) = a$ is described by the equation $x(t) = x_i + v_i t + \frac{1}{2}at^2$ in terms of the parameters x_i and v_i. Depending on the initial velocity and the initial position of the object, we would obtain different trajectories. Simple harmonic motion with angular frequency ω is described by the equation $x(t) = A\cos(\omega t + \phi)$ in terms of the parameters A and ϕ, which are the *natural* parameters for describing SHM.

3) Derivation of pendulum SHM equation: To see how the SHM equation of motion arises in the case of the pendulum, we need to start from the torque equation $\mathcal{T} = I\alpha$.

The diagram on the right illustrates how we can calculate the torque on the pendulum which is caused by the force of gravity as a function of the displacement angle θ. Recall that the torque calculation only takes into account the $F_\perp$ component of any force, since it is the only part which causes a rotation:



$$\mathcal{T}_\theta = F_\perp \ell = mg\sin\theta\ell.$$

If we now substitute this into the equation $\mathcal{T} = I\alpha$, we obtain the following:

$$\begin{aligned} \mathcal{T} &= I\alpha \\ mg\sin\theta(t)\ell &= m\ell^2\frac{d^2\theta(t)}{dt^2} \\ g\sin\theta(t) &= \ell\frac{d^2\theta(t)}{dt^2} \end{aligned}$$

What follows is something which is not mathematically rigorous, but will allow us to continue and solve this problem. When θ is a small angle we can use the following approximation:

$$\sin(\theta) \;\approx\; \theta, \qquad\qquad \text{for } \theta \ll 1.$$

This type of equation is called a *small angle approximation*. You will see where it comes from later on when you learn about Taylor series approximations to functions. For now, you can convince yourself of the above formula by zooming many times on the graph of the function sin near the origin to see that $y = \sin(x)$ will look very much like $y = x$. Try this out.

Using the small angle approximation for $\sin\theta$ we can rewrite the equation involving $\theta(t)$ and its second derivative as follows:

$$\begin{aligned} g\sin\theta(t) &= \ell\frac{d^2\theta(t)}{dt^2} \\ g\theta(t) &\approx \ell\frac{d^2\theta(t)}{dt^2} \\ 0 &= \frac{d^2\theta(t)}{dt^2} + \frac{g}{\ell}\theta(t). \end{aligned}$$

At this point we can recognize that we are dealing with the same differential equation as in the case of the mass-spring system: $0 = \theta''(t) + \omega^2\theta(t)$, which has solution:

$$\theta(t) = \theta_{max}\cos(\omega t + \phi),$$

where the constant inside the cos function is $\omega = \sqrt{\frac{g}{\ell}}$.

F. Examples

When asked to solve word problems, you will usually be told the initial amplitude $x_i = A$ or the initial velocity $v_i = \omega A$ of the SHM and the question will ask you to calculate some other quantity. Answering these problems shouldn't be too difficult provided you write down the general equations for $x(t)$, $v(t)$ and $a(t)$, fill-in the knowns quantities and then solve for the unknowns.

1) Standard example: You are observing a mass-spring system build from a 1[kg] mass and a 250[N/m] spring. The amplitude of the oscillation is 10[cm]. Determine (a) the maximum speed of the mass, (b) the maximum acceleration, and (c) the total mechanical energy of the system.

First we must find the angular frequency for this system $\omega = \sqrt{k/m} = \sqrt{250/1} = 15.81$[rad/s]. To find (a) we use the equation $v_{max} = \omega A = 15.81 \times 0.1 = 1.58$[m/s]. Similarly, we can find the maximum acceleration using $a_{max} = \omega^2 A = 15.81^2 \times 0.1 = 25$[m^2/s]. There are two equivalent ways for solving (c). We can obtain the total energy of the system by considering the potential energy of the spring when it is maximally extended (compressed) $E_T = U_s(A) = \frac{1}{2}kA^2 = 1.25$[J], or we can obtain the total energy from the maximum kinetic energy $E_T = K = \frac{1}{2}mv_{max}^2 = 1.25$[J].

G. Discussion

In this section we learned about simple harmonic motion, which is described by the equation $x(t) = A\cos(\omega t + \phi)$. You may be wondering what *non-simple* harmonic motion is. A simple extension of what we learned would be to study oscillating systems where the energy is slowly dissipating. This is known as *damped harmonic motion* for which the equation of motion looks like $x(t) = Ae^{-\gamma t}\cos(\omega t + \phi)$, which describes an oscillation whose magnitude slowly decreases. The coefficient γ is known as the damping coefficient and indicates how fast the energy of the system is dissipated.

The concept of SHM comes up in many other areas of physics. When you learn about electric circuits, capacitors and inductors, you will run into equations of the form $0 = v''(t) + \omega^2 v(t)$, which indicates that the *voltage* in a circuit is undergoing simple harmonic motion. Guess what, the same equation used to describe the mechanical motion of the mass-spring system will be used to describe the voltage in an oscillating circuit!

H. Links

[Plot of the simple harmonic motion using a can of spray-paint]
http://www.youtube.com/watch?v=p9uhmjbZn-c

[15 pendulums with different lengths]
http://www.youtube.com/watch?v=yVkdfJ9PkRQ

XI. Summary

The numbered equations...

Appendix

I hope this short excerpt from the `MATH and PHYSICS Minireference` has given you some inspiration for compact teaching. No blah blah. Straight to the point.

If you liked this tutorial you can check out the other ones on http://minireference.com and order the printed book which has not only formulas but also compact explanations: http://minireference.com/order_book/.