Chapter 1

Mechanics

Mechanics is the precise study of moving objects, forces and energy. You already have an intuitive understanding of these concepts, but in this chapter I will teach you how to use precise mathematical models which will support your intuition.

Mechanics is the part of physics that is most well understood. Ever since Newton figured out the whole $F = ma$ thing and the law of gravitation, people have *used* mechanics in order to achieve great technological feats.

There will be math, yes, but nothing too complicated. In fact, the hardest type of equation you will have to solve is a quadratic equation, so don't worry too much about that. The upshot of understanding the math, is that you will be able to calculate and predict phenomena in the world around you simply by plugging numbers into the right equation.

In short, mechanics is powerful stuff, so let's get right into it.

1.1 Kinematics

Kinematics (from the Greek root kinema *motion*) is the study of trajectories of objects moving in space. It can be used to calculate how long a ball thrown upwards will stay in the air, or to calculate the acceleration needed to go from 0 to 100 km/h in 5 seconds.

To carry out these calculations we need to know the *equation of motion* and the *initial conditions* (like how fast you threw the ball up v_i) and then carry out some simple algebra steps to calculate anything we want. No seriously, this entire chapter boils down to three equations. It is all about the plug-number-into-equation skills.

1.1.1 Concepts

The basic concepts of kinematics are:

- t: time, measured in seconds [s].
- x: the *position* of an object with respect to a coordinate system, measured in meters [m].
- $x(t)$: the position as a function of time, also known as the equation of motion.
- $v(t)$: speed measured in meters per second [m/s].
- $a(t)$: acceleration measured in meters per second squared [m/s^2].
- $x_0 = x(0)$: the position at $t = 0$.
- $v_0 = v(0)$: the velocity at $t = 0$.

When solving some problem, where we calculate the motion of an object that starts form an *initial* point an goes to a *final* point we will use the following terminology:

- t_i: initial time (the beginning of the motion).
- t_f: final time (when the motion stops).
- x_i: initial position.
- x_f: final position.
- $\vec{d} = \Delta x = x_f - x_i$: delta x = change in position = displacement.
- $d = |\Delta x| = |x_f - x_i|$: distance traveled = absolute value of displacement.
- v_i: initial velocity of the object.
- v_f: final velocity of the object.

1.1.2 Formulas

There are basically three equations that you need to be aware of for this entire chapter. Together, these three equations fully describe all the aspects of motion with constant acceleration.

Uniform acceleration motion (UAM)

If the object undergoes a *constant* acceleration $a_{const} = a$, like your car if you floor the *accelerator* pedal, then the equations of motion are:

$$a(t) = a,$$
$$v(t) = at + v_0,$$
$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0.$$

There is also another very useful equation to remember:

$$v(t)^2 = v_0^2 + 2a[x(t) - x_0],$$

which is usually written

$$v_f^2 = v_i^2 + 2a\Delta x.$$

That is it. Memorize these equations, plug-in the right numbers, and you can solve any kinematics problem humanly imaginable. Chapter done.

Uniform velocity motion (UVM)

The special case where there is zero acceleration ($a = 0$), is called *uniform velocity motion* or UVM. The velocity is uniform (constant), because there is no acceleration. The following three equations describe the motion of the object under uniform velocity:

$$a(t) = 0,$$
$$v(t) = v_0,$$
$$x(t) = v_0 t + x_0.$$

As you can see, these are really the same equations as above, but because $a = 0$, some terms are missing.

1.1.3 Explanations

Calculus

The functions $x(t)$, $v(t)$ and $a(t)$ are connected. They all describe different aspects of the same motion. We have:

$$x(t) \xrightarrow{\frac{d}{dt}} v(t) \xrightarrow{\frac{d}{dt}} a(t),$$

which means that starting from the position function, $x(t)$, we can use the derivative operation to obtain the velocity and the acceleration. We can also use this relationship in the other direction:

$$x(t) \xleftarrow{\int dt} v(t) \xleftarrow{\int dt} a(t),$$

which means that we start from the acceleration $a(t)$ and use integration with respect to time to obtain the velocity $v(t)$. If we integrate the velocity we obtain the position function $x(t)$.

This is how the *equations of motion* are derived. Assuming that the acceleration is constant in time $a(t) = a$, we can calculate the velocity by *adding up* all the acceleration (integrating) to obtain the change in the velocity

$$v(t) = \int a(t)\ dt = \int a\ dt = at + v_i.$$

If you are moving at velocity 100[m/s] and you accelerate at a rate of 10[m/s^2] during 3 seconds, then your final velocity at the end of the three seconds will be

$$v(3) = \int_0^3 10\ dt + 100 = 10(3) + 100 = 130[m/s].$$

If you feel mystified by this integral sign, you are not alone. Just keep in mind that integration is *accumulating* some quantity to see what the total contribution is going to be.

The meaning of integration is actually a bit simpler to see with the relation:

$$x_f \equiv x(t_f) = \int_{t_i}^{t_f} v(t)\ dt + x_i,$$

which says that if you start at position $x_i = x(t_i)$ and move at a velocity $v(t)$ between t_i and t_f, then you final position will be x_f. You are probably most familiar with this kind of reasoning from the context of driving in your car. Indeed you know that if you start at Montreal $x_i = 0$[km], and drive at 110[km/h] for 2 hours, you can get to Quebec city which is at $x_f = 220$[km]. Most people have an intuitive feel about this kind of calculation, but mathematicians invented a fancy notation to describe it:

$$\int_{t_i=0}^{t_f=2} 110\ dt + 0 = 220.$$

We *accumulate* a total of 220[km] displacement. During the first hour we move 110[km], and during the second hour another 110[km].

So where does the equation of motion $x(t) = \frac{1}{2}at^2 + v_i t + x_i$ come from? Well, remember that we assumed that the acceleration is a constant $a(t) = a$, and we calculated that $v(t) = at + v_i$, so now we just do the next step:

$$x(t) = \int_{t_i}^{tf} v(t)\ dt = \int_{t_i}^{tf} at + v_i\ dt = \frac{1}{2}at^2 + v_i t + x_i.$$

So basically *every* time you have constant acceleration, you have $x(t) = \frac{1}{2}at^2 + v_i t + x_i$. Two steps of *calculus* say so.

1.2 Projectile motion

Ever since the invention of gun powder, generation after generation of men have found countless different ways of hurtling shrapnel and explosives at each other. Indeed, mankind has been stuck to the idea of two dimensional projectile motion like flies on shit. So long as there is money to be made in selling weapons, and TV stations to keep justifying the legitimacy of the use of these weapons, it is likely that the trend will continue.

As a first step towards reversing this trend, we will learn the physics of projectile motion, which is a type of two-dimensional kinematics problem. You need to know the techniques of the enemy (the Military-industrial complex) before you can fight them. We will see that, kinematics in two dimensions is nothing more than two parallel one-dimensional kinematics problems: UVM in the x direction and UAM in the y direction.

1.2.1 Concepts

The basic concepts of kinematics in two dimensions are:

- $\hat{x}, \hat{y}$: a coordinate system.
- t: time, measured in seconds.
- x: the x position of an object as measured according to the coordinate system.
- $x(t)$: the x position as a function of time.
- $v_x(t)$: speed in the x direction, measured in meters per second.
- $a_x(t)$: acceleration in the x direction, measured in meters per second squared.
- $y(t)$: the y position as a function of time.
- $v_y(t)$: velocity in the y direction as a function of time.
- $a_y(t)$: the y-acceleration.
- $\vec{r}(t) = (x(t), y(t))$: position *vector*, the combined x and y positions.

When solving some problem, where we calculate the motion of an object that starts form an *initial* point an goes to a *final* point we will use the following terminology:

- t_i: initial time (the beginning of the motion).
- t_f: final time (when the motion stops).
- $\vec{v}_i = \vec{v}(t_i) = (v_{xi}, v_{yi})$: the initial velocity at $t = t_i$.
- $\vec{r}_i = \vec{r}(t_i) = (x_i, y_i)$: the initial position at $t = t_i$, where x_i is the initial x position and y_i the initial y position.
- $\vec{r}_f = \vec{r}(t_f) = (x_f, y_f)$: the final position at $t = t_f$, where x_f is the final x position, and y_f is the final y position.

1.2.2 Formulas

Motion in two dimensions

Sometimes you have to describe both the x and the y coordinate of the motion of a particle:

$$\vec{r}(t) = (x(t), y(t)).$$

We choose x to be the horizontal component of the projectile motion, and y to be its height.

The velocity of the projectile will be:

$$\vec{v}(t) = \frac{d}{dt}\left(\vec{r}(t)\right) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = (v_x(t), v_y(t)),$$

and the initial velocity is:

$$\vec{v}_i = \vec{v}(t_i) = |\vec{v}_i|\angle\theta = (v_x(t_i), v_y(t_i)) = (v_{ix}, v_{iy}) = ($$

The acceleration of the projectile will be

$$\vec{a}(t) = \frac{d}{dt}\left(\vec{v}(t)\right) = (a_x(t), a_y(t)) = (0, -9.81).$$

Note how we have zero acceleration in the x direction so we can use the UVM equations of motion for $x(t)$ and $v_x(t)$. In the y direction we have a uniform downward acceleration due to gravity.

Projectile motion

The equations of motion of a projectile are the following. First in the x direction we have uniform velocity motion (UVM):

$$\begin{aligned} x(t) &= v_{ix}t + x_i, \\ v_x(t) &= v_{ix}. \end{aligned}$$

In the y direction, you have the constant pull of gravity downwards which gives us a uniformly accelerated motion (UAM):

$$\begin{aligned} y(t) &= \frac{1}{2}(-9.81)t^2 + v_{iy}t + y_i, \\ v_y(t) &= -9.81t + v_{iy}, \\ v_{yf}^2 &= v_{yi}^2 + 2(-9.81)(\Delta y). \end{aligned}$$

1.3 Forces

Like a shepherd who brings in a stray sheep back, we need to rescue the word *force* and give it precise meaning. In physics force means something very specific. Not "the force" from Star Wars, not the "force of public opinion", and not the *force* in the battle of good versus evil.

Force in physics has a precise meaning as an amount of push or pull exerted on an object. Force is a vector. We measure force in Newtons [N], and we can use it in equations and solve for it just like any other unknown. In this section we will explore all the different kinds of forces.

1.3.1 Concepts

- $\vec{F}$: a force. This is something the object "feels" as a pull or a push. Force is a vector, so you must always keep in mind the direction in which the force $\vec{F}$ acts.

- $k, G, m, \mu_s, \mu_k, \ldots$: parameters on which the force F may depend. Ex: the heavier an object is (has large m parameter), the larger its gravitational pull will be: $\vec{W} = -9.81m\hat{z}$, where $\hat{z}$ points towards the sky.

1.3.2 Kinds of forces

We next list all the forces which you are supposed to know about for a standard physics class and define the relevant parameters for each kind of force. You need to practice exercises using each of these forces, until you start to *feel* how they act.

Gravitation

Manifestations of the gravitational pull of the planet Earth on massive objects:

- M: mass of the earth. $M = 5.9721986 \times 10^{24}$ [kg].

- m: mass of an object.

- $\vec{W} = \vec{F}_g$: The weight (the force on a object due to gravity).

- G: Gravitational constant $= 6.6710^{-11}$ $[\frac{Nm^2}{kg^2}]$.

- $\vec{F}_g = \frac{GMm}{r^2}$: Force of gravity between two objects of mass M and m respectively. Measured in Newtons [N].

- $\vec{F}_g = gm$ (downward): The force of gravity on the surface of the earth, where $g = \frac{GM}{r^2} \approx 9.81\ldots$ [N/kg]=[m/s^2].

The famous one-over-arr-squared law that describes the gravitational pull between two objects is:

$$F_g = \frac{GMm}{r^2}.$$

You will rarely use it, but it is extremely important as this is where all of mechanics began. This was Newton's big discovery. All the rest of mechanics is simple calculus, but this equation is *real* physics. It tells us something about how the Universe works.

At the surface of the earth:

$$\vec{F}_g = \frac{GMm}{r^2} = \underbrace{\left(\frac{GM}{r^2}\right)}_{g} m = \vec{g}m = \vec{W},$$

where the weight $\vec{W}$ of an object is a vector that points towards the centre of the earth, and $g = 9.81$[m/s^2].

Force of a spring

- $\vec{F}_s = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by x meters. The constant k [N/m] is a measure of the *strength* of the spring, or its stiffness.

Tension in a rope

- $\vec{T}$: Tension in a rope. Tension is always pulling *away* from an object: you can't push a dog on a leash.

Contact force

- $\vec{C}$: Contact force between two rigid objects. We generally brake-up contact forces into two components: perpendicular and parallel to the contact surface.

- $\vec{N} \equiv \vec{C}_\perp$: Normal force: the force between two surfaces. Normal is a mathematically precise way to say "perpendicular to a surface". Intuitively, you can think of $\vec{N}$ as the force that a surface exerts on an object to keep it where it is. The reason why my coffee mug does not fall to the floor, is that the table exerts a normal force on it keeping in place.

- $\vec{F}_f \equiv \vec{C}_{||}$: Force of friction between two surfaces. There are two kinds, both of which are proportional to the normal force

between the surfaces:
Kinetic:

$$F_{fk} = \mu_k |\vec{N}|.$$

Static:

$$F_{fs} = \mu_s |\vec{N}|.$$

Two kinds of friction forces

- $\vec{F}_{fs} = \mu_s|\vec{N}|$: Static force of friction, for objects that are not moving.

- μ_s: The static coefficient of friction. ex: 0.3. It describes the **maximum** amount of friction that can exist between two objects. If a horizontal force exists greater than $F_{fs} = \mu_s N$, then the object will start to slip.

- $\vec{F}_{fk} = \mu_k|\vec{N}|$: Kinetic force of friction acts when two objects are sliding relative to each other. It always acts in a direction opposing the motion.

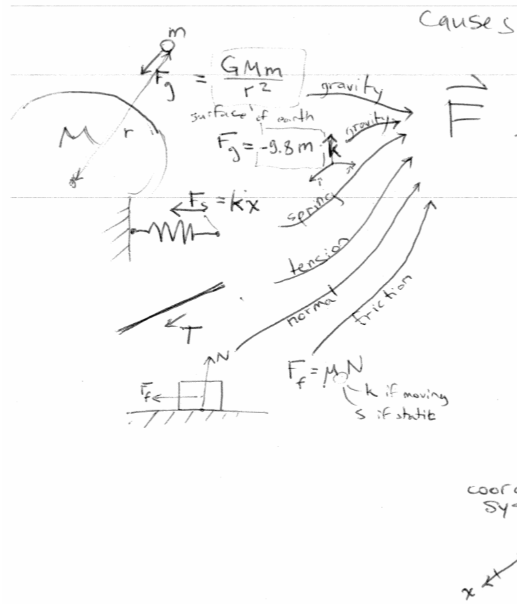
- μ_k: Kinetic coefficient of friction. ex: $\mu_k = 0.1$. Dimensionless. it is just the ratio that describes how much friction an object feels for a given amount of normal force.

1.3.3 Discussion

Ok so what is mechanics all about? You should know by now, since you have already learned about forces and about acceleration, velocity and position. Doesn't mechanics have something to do with $F = ma$?

You can think of the different forces as the "causes of motion", and the effect is the acceleration a. Once you have the acceleration, you can use the basic kinematics equations like $x(t) = \frac{1}{2}at^2 + v_i t + x_i$ which describe the motion at all times t.

The causes and effects of the motion of bodies are represented in this diagram:



To understand any physics problem you must first check which kinds of forces are relevant, make a force diagram and add up all the forces to obtain $\vec{F}_{net}$, the net force on the object. Then you use $\vec{a} = \frac{\vec{F}_{net}}{m}$ and you have your acceleration. Problem solved, because once you have a you just need to plug it into equations of motion:

$$\vec{a}(t) = \vec{a},$$

$$\vec{v}(t) = \vec{a}t + \vec{v}_i,$$

$$\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_i t + \vec{r}_i.$$

There you have it. All of undergraduate level mechanics boils down to your understanding and *usage* of the equation $F = ma$.

Disclaimer: Of course physics is more complicated than that, but what you are learning is an important and very useful subset. Namely, the above three equations make the assumption that the acceleration is constant, which in turn assumes that the balance of the forces acting on the object will always remain the same. We assume that during the entire time between t_i and t_f the force $\vec{F}_{net}$ stays constant. In a *lot* of situations this assumption is true.

In more complicated situations you could have $\vec{F}_{net}$ changing over time and thus $a(t)$ changing over time too. You would have to start from $a(t)$ and do the integration thing twice on $a(t)$ to get the $x(t)$ for the motion.

Sometimes the net force on the object $\vec{F}_{net}$ might be constant, but the mass of the object may change over time (like a multi-stage rocket losing its booster tanks) so you would still have $a(t) = \frac{F_{net}}{m(t)}$ changing with time because $m(t)$ would be changing. Again, for those situations you have to go back to calculus and integrate $a(t)$ twice to get the $x(t)$ function.

1.4 Force diagrams

Welcome to force-accounting 101. In this section we will learn how identify all the forces acting on an object and predict the resulting acceleration.

Newton's 2nd law says that the *net* force on an object causes an acceleration:

$$\sum \vec{F} = \vec{F}_{net} = m\vec{a},$$

so finding the net force must be a pretty important thing.

1.4.1 Concepts

Newton's second law is a relationship between these three concepts:

- m: the *mass* of an object.
- $\vec{F}$: vector used to denote any kind of *force*.
- $\vec{a}$: the *acceleration* of the object.

Both forces and accelerations are vectors. To work with vectors, we have work with their *components*:

- F_x: the *component* of $\vec{F}$ in the x direction.
- F_y: the *component* of $\vec{F}$ in the y direction.

Vectors are meaningless unless it is clear with respect to which *coordinate system* they are expressed.

- x-axis: Usually the x-axis is horizontal and to the right, however, for problems with inclines, it will be more convenient to put the x-axis parallel to the slope.
- y-axis: The y-axis is always *perpendicular* to the x-axis.
- $\hat{\imath}, \hat{\jmath}$: Unit vectors in the x and y directions. Any vector can be written as $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ or as $\vec{v} = (v_x, v_y)$.

Provided we have a coordinate system, we can write any force vector in three ways:

$$\vec{F} \equiv F_x\hat{\imath} + F_y\hat{\jmath} \equiv F_x\hat{x} + F_y\hat{y} \equiv (F_x, F_y),$$

note that there are two ways to write the unit vectors: $\hat{\imath} \equiv \hat{x}$ and $\hat{\jmath} \equiv \hat{y}$.

What types of forces are there in force diagrams?

- $\vec{W} \equiv \vec{F}_{gravity} = m\vec{g}$: The *weight*. This is the force on a object due to its gravity. The gravitational pull $\vec{g}$ always points downwards towards the center of the earth. $g = 9.81$ [N/kg].
- $\vec{T}$: Tension in a rope. Tension is always pulling *away* from the object.
- $\vec{N}$: Normal force the force between two surfaces.
- $\vec{F}_{fs} = \mu_s|\vec{N}|$: Static force of friction.
- $\vec{F}_{fk} = \mu_k|\vec{N}|$: Kinetic force of friction.
- $\vec{F}_{spring} = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by x meters.

1.4.2 Formulas

Newton's 2nd law

The sum of the forces acting on an object, divided by the mass gives you the acceleration of the object:

$$\sum F = \vec{F}_{net} = m\vec{a}.$$

Vector components

If a vector $\vec{v}$ makes an angle θ with the x-axis then:

$$v_x = |\vec{v}| \cos\theta, \qquad \text{and} \qquad v_y = |\vec{v}| \sin\theta.$$

The vector $v_x\hat{\imath}$ corresponds to the the part of $\vec{v}$ that points in the x-direction.

In what follows, you will be asked a countless number of times to

Find the component of $\vec{F}$ in the ? direction.

Which is another way of asking you to find the number $v_?$.

The answer is usually equal to the length $|\vec{F}|$ multiplied by either cos or sin and sometimes -1 all **depending on way the coordinate system is chosen**. So don't guess. Look at the coordinate system. If the vector points in the direction where x-increases, then v_x should be a positive number. If it points in the opposite direction, then v_x should be negative.

Vector components are important, because to add forces $\vec{F}_1$ and $\vec{F}_2$ you have to add their components:

$$(F_{1x}, F_{1y}) + (F_{1x}, F_{2y}) = \vec{F}_1 + \vec{F}_2 = (F_{1x} + F_{2x}, F_{1y} + I$$

Instead of dealing with vectors in the bracket notation as above, when solving force diagrams it is easier to simply write the x-equation on one line, and the y-equation on a separate line.

$$F_{netx} = F_{1x} + F_{2x},$$

$$F_{nety} = F_{1y} + F_{2y}.$$

It is a good idea to always write those two equations together as a block so it remains clear that you are talking about the same problem, but the first row represents the x-dimension and the second row represents the y-dimension.

Force check

It is important to account for *all* the forces acting on an object.

Any object with mass on the surface of the earth will feel a gravitational pull of magnitude $\vec{F}_{gravity} = \vec{W} = m\vec{g}$ downwards.

Then you have to think about which of the other forces you know might be present: $\vec{T}$, $\vec{N}$, $\vec{F}_f$, $\vec{F}_{spring}$.

Anytime you see a rope tugging on the object you know you there must be some tension $\vec{T}$, which is a force vector pulling on the block.

Anytime you have an object sitting on a surface, the surface will push back with a *normal* force $\vec{N}$.

If the object is sliding on the surface there will be a force of friction acting acting against the direction of the motion:

$$F_{fk} = \mu_k |\vec{N}|.$$

If the object is not moving, then you have to use μ_s in the friction force equation, to get the maximum static friction force that the contact between the object and the ground can support before the object starts to slip:

$$\max\{F_{fs}\} = \mu_s |\vec{N}|.$$

If you see a spring that is either stretched or compressed by the object, then you must account for the spring force. The force of a spring is *restorative*: it always acts against the deformation you are making to the spring. If you stretch it by x[cm], then it will try to pull itself back to its normal length with a force of:

$$\vec{F}_s = -kx\hat{\imath}.$$

The constant of proportionality k is called the *spring constant* and in the above example would be measured in [N/cm].

1.4.3 Recipe for solving force diagrams

1. Draw a diagram centred on the object and draw all the forces acting on it.

2. Choose a coordinate system, and indicate clearly what you will call the x-direction, and what you will call the y-direction. All **equations are expressed *with respect to* this coordinate system**.

3. Write down this "template":

$$\sum F_x = \qquad\qquad = ma_x$$

$$\sum F_y = \qquad\qquad = ma_y$$

4. Fill the first line by finding the x-components of each force acting on the object.

5. Fill the second line by finding the y-components of each force acting on the object.

6. Consistency checks:

   (a) Check signs. If the force in the diagram is acting in the x-direction then its component must be positive. If the force is acting in the opposite direction to $\hat{x}$, then its component should be negative.

   (b) Verify that whenever $F_x \propto \cos\theta$, then $F_y \propto \sin\theta$. If instead we use an angle ϕ defined with respect to the y-axis we would have $F_x \propto \sin\phi$, and $F_y \propto \cos\phi$.

7. Solve the two equations finding the one or two unknowns. If there are two unknowns, you may need to solve two equations simultaneously by isolation and substitution.

Force diagrams are best explained through examples.

1.5 Momentum

Say you have a 1g piece of paper and a 1000kg car moving at the same speed 100km/h. Which would you rather get hit by? In physics the "amount moving stuff" has a precise name: *momentum*, and it is denoted with $\vec{p}$. You see, your gut feeling about the paper and the car is correct: in collisions it is the momentum that counts. In this section, we will add some mathematical precision to that gut feeling.

The momentum is equal to the velocity of the moving object multiplied by the object's mass ($\vec{p} = m\vec{v}$). Therefore, since the car weights $1000 \times 1000 = 10^6$ times more than the piece of paper, it has 10^6 times more momentum when moving at the same speed. A collision with it will "hurt" that much more.

Note that momentum is a vector, so we will have to do a lot of that length-and-direction-to-components transformation stuff:

$$(p_x, p_y) = \vec{p} = (|\vec{p}|\cos\theta, |\vec{p}|\sin\theta) = |\vec{p}|\angle\theta,$$

and also converting backwards from component notation to magnitude-direction:

$$|\vec{p}| = \sqrt{p_x^2 + p_y^2}, \qquad \theta = \tan^{-1}\left(\frac{p_y}{p_x}\right).$$

1.5.1 Concepts

- m: the *mass* of the moving object.
- $\vec{v}$: the *velocity* of the moving object.
- $\vec{p} = m\vec{v}$: the *momentum* of the moving object.

1.5.2 Formulas

Definition

The momentum of an object is the mass of the object times its velocity:

$$\vec{p} = m\vec{v}.$$

If you speed is $\vec{v} = (20, 0, 0)$[m/s], which is equivalent to saying "20[m/s] in the x-axis direction", and your mass is 100kg then your momentum is $\vec{p} = (2000, 0, 0)$[kg*m/s].

Newton's first law

Newton's first law states that whenever there is no acceleration ($\vec{a} = 0$), an object will maintain a constant velocity. This is kind of obvious if you know Calculus, since $\vec{a}$ is the derivative of $\vec{v}$. For example, if an object is stationary and there are no forces on it to cause it to accelerate, then it will stay stationary. If an object was moving with velocity $\vec{v}$ and there is no acceleration, then it will keep moving with velocity $\vec{v}$ forever. In the absence of acceleration, objects will conserve their velocity:

$$\vec{v}_{in} = \vec{v}_{out}.$$

This is equivalent to saying that objects conserve their momentum (just multiply the velocity by the mass if the mass stays constant and the velocity stays constant, then the momentum must stay constant).

Conservation of momentum

More generally, if you have a situation with multiple moving objects, you can say that the "overall momentum", i.e., the sum of the momenta of all the particles stays constant:

$$\sum \vec{p}_{in} = \sum \vec{p}_{out}.$$

This is amazingly powerful stuff, and one of the furthest reaching laws of physics. Whatever momentum comes into a collision must come out. This is a kind of Le Chatelier's principle "rien ne se cre, et rien ne disparait tout se transforme". Motion cannot simply appear or disappear, it can only be exchanged between systems.

1.6 Energy

Instead of thinking about velocities $v(t)$ and motion trajectories $x(t)$, we can solve physics problems using *energy* calculations. The key idea in this section is the principle *total energy conservation*, which tells us that the sum of the initial energies is equal to the sum of the final energies. We will define precisely the different kinds of energy, and then learn the rules of converting one energy into another.

1.6.1 Concepts

The concepts of energy come up in several different contexts.

Moving objects:

- m: the mass of an object.
- v: the velocity.
- $E_K = K$: kinetic energy $= \frac{1}{2}mv^2$.

Moving objects by force:

- $\vec{F}$: the force needed to move the object.
- $\vec{d}$: the displacement of the object. How far it moved.
- W: work done to move the object $= \vec{F} \cdot \vec{d}$.

Gravity:

- g: gravitational acceleration on the surface of earth. 9.81 [m/s^2].
- h: height of an object.
- U_g: Gravitational potential energy $= mgh$.

Springs:

- k: spring constant. Measured in [N/m].
- x: spring displacement from the relaxed position. If the spring is stretched then $x > 0$, and if it is compressed then $x < 0$.
- U_s: Spring potential energy $= \frac{1}{2}kx^2$.

There are all kinds of other forms of energy: electric energy, sound energy, thermal energy, etc. In this section we will focus on the types of *mechanical* energy.

1.6.2 Formulas

Kinetic energy:

$$K = \frac{1}{2}mv^2$$

Work:

$$W = \int \vec{F}(x) \cdot dx$$

for constant force:

$$W = \vec{F} \cdot \vec{d} = |F||d| \cos\theta.$$

Gravitational potential energy:

$$U_g = mgh,$$

which is the energy you have because of your height.

Spring energy:

$$U_s = \frac{1}{2}kx^2.$$

Conservation of energy:

$$\sum E_{in} + W_{in} = \sum E_{out} + W_{out}.$$

1.6.3 Explanations

Energy is measured in Joules $[J]$. The dimension of energy [J] is related to other dimension you are familiar with:

$$[\mathrm{J}] = [\mathrm{N}][\mathrm{m}] = [\mathrm{kg}][\mathrm{m}^2][\mathrm{s}^{-2}].$$

You can check the second equality from $F = ma$, which has dimensions [N] = [kg m/s^2].

Kinetic energy

A moving object has energy, which we call *kinetic* energy from the Greek word for motion *kinema*. The kinetic energy is given by

$$K = \frac{1}{2}mv^2.$$

Note that velocity v and speed $|v|$ are not the same as energy. Suppose you have two objects of the same mass and one is moving twice faster than the other. The faster object will have twice the velocity, but four times more kinetic energy.

Work

When hiring someone to help you move, you have to pay them for the *work* they do. Work is the product of how much force is necessary to move your furniture and the distance of the move. The more force, the more work there will be for a fixed displacement. The more displacement (think moving to south shore versus moving next door) the more money the movers will ask for.

The amount of work a force $\vec{F}(x)$ (possibly changing) will produce when moved along some path p is given by:

$$W = \int_p \vec{F}(x) \cdot d\vec{x}.$$

If the force is constant, this expression simplifies to:

$$W = \int_0^d \vec{F}(x)\cdot d\vec{x} = \vec{F}\cdot \int_0^d d\vec{x} = \vec{F}\cdot\vec{d} = |F||d| \cos\theta,$$

where the factor of $\cos\theta$ comes from the dot product. Indeed, we only want to account for the part of $\vec{F}$ that is pushing in the direction of the displacement $\vec{d}$.

Potential energy is stored work

Some kinds of work are just a waste of your time, like working in a bank for example. You work and you get your paycheque, but nothing remains with you. Other kinds of work leave with you some *resource* at the end of the work day. Maybe you learn something, or you network with a lot of good people.

In physics, we make a similar distinction. Some types of work like work against friction are called *dissipative* since they just waste energy. Other kinds of work are called *conservative* since the work you do is not lost – it was just converted into a different form.

The gravitational and spring forces are conservative. Any work you do while lifting an object up into the air against the force of gravity is not lost it is *stored* in the height of the object. You can get *all* the work/energy back if you just let go of the object. The energy will come back in the form of kinetic energy as the object gets accelerated during the fall.

The negative of the work done against conservative forces is called *potential energy*. Being high in the air means you have a lot of potential to fall, and compressing a spring by a certain distance means it has the potential to spring back to its normal position. Let us look at the exact formulas now.

Gravitational potential energy

The force of gravity is given by:

$$F_g = W = mg.$$

The gravitational potential energy of lifting an object for a height h is:

$$U_g = -\int_0^h \vec{F}_g \cdot d\vec{y} = \int_0^h mgdy = mh\int_0^h 1dy = mhy\Big|_{y=0}^{y=h} = mgh.$$

Spring energy

The force of a spring when stretched a distance x is given by:

$$\vec{F}_s(\vec{x}) = -k\vec{x}.$$

The sprint potential energy is:

$$U_s = -\int_0^x \vec{F}_s(y)\cdot d\vec{y} = \int_0^x kydy = k\int_0^x ydy = k\frac{1}{2}y^2\Big|_{y=0}^{y=x} = \frac{1}{2}kx^2.$$

Conservation of energy

Energy cannot be created or destroyed. It just transforms from one form into another.

If there are no dissipative forces, then we have

$$\sum E_i = \sum E_f.$$

If there are dissipative forces like friction or external forces that do work, we must take their energy contributions into account as well:

$$\sum E_i \; + \; W_{in} = \sum E_f,$$

or

$$\sum E_i = \sum E_f \; + \; W_{out}.$$

This is one of the most important equations you will find in this book, because it will allow you to solve very complicated problems simply by accounting for all different kinds of energies.

1.7 Uniform circular motion

When a car makes a long left turn, the passenger riding shotgun will feel pushed towards the right and into the door. If we assume the car moves at a constant speed v in the turn, and that the radius of the curve is R, what is the force of contact between the passenger and the door? This question may sound complicated, but actually it boils down to a single formula. This entire section is dedicated to that formula and to objects moving around in a circle in general.

1.7.1 Concepts

- $\hat{x}, \hat{y}$: the a usual coordinate system.

In this section we will use a new type of coordinate system:

- $\hat{r}, \hat{t}$: the *radial* and *tangential* directions of a circular coordinate system. No matter where you are on the circle, the radial direction always points towards the centre, while the tangential direction is always perpendicular. On a bicycle wheel, the spokes are in the $\hat{r}$ direction, while the pavement is in the $\hat{t}$ direction.
- $\vec{v} = (v_x, v_y)_{\hat{x}\hat{y}} = (v_r, v_t)_{\hat{r}\hat{t}}$: Velocity of particle, can be expressed as in xy or rt coordinates
- $\vec{a} = (a_r, a_t)_{\hat{r}\hat{t}}$: the *acceleration* of the particle.

Every time you have uniform circular motion ($v_t = const$), these variables will be related:

- R: Radius of the circle of motion.
- v_t: Speed of the circular motion in $[m/s]$. Sometimes referred to as *tangential* speed.
- a_r: Radial acceleration. The relation is $a_r = \frac{v_t^2}{R}$.

There is some special terminology used to describe circular motion:

- C: the *circumference* of the circle of motion. For a circle of radius R, $C = 2\pi R$.
- T: the *period*, how long it takes for the object to complete one circle. Measured in seconds.
- f: the frequency of rotation. How many times per second does an object pass by the same point on the circle. $f = \frac{1}{T}$. Measured in Hertz [Hz]=[1/s].
- $\omega \equiv \frac{v_t}{R}$: angular velocity, how fast the angle of the object is rotating $\omega = 2\pi f$.
- RPM: the *revolutions per minute* which is the angular velocity expressed in units of revolutions (1[rev]=2π[rad]) and minutes (1[min]=60[s]).

The angular velocity ω is very useful because it describes the speed of circular motion of *any* radius. Indeed, different points on a rotating disk will have different tangential speeds v_t depending on how far from the radius they are, so it is much better to describe the angular velocity and multiply by the radius as needed.

1.7.2 Formulas

Three directions

Let's freeze time and zoom in on an object moving in a circle. We can draw a force diagram with three directions:

1. $\hat{r}$: towards the centre of the circle of rotation.
2. $\hat{t}$: in the instantaneous direction where the car is moving right now. This is called the *tangential* direction, from the greek to touch (imagine a straight line "touching" the circle).
3. $\hat{y}$: if necessary, we imagine a third *vertical* axis pointing up out of the plane of rotation.

Motion in a circle

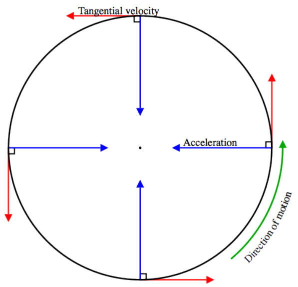
A circle of radius R has circumference:

$$C = 2\pi R.$$

An object moving along this circular path with tangential speed v_t. The period, T, is defined as how long it will take the object to complete one turn and is equal to:

$$T = \frac{2\pi R}{v_t} = \frac{2\pi}{\omega} = \frac{1}{f}.$$

Radial acceleration



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The one equation of this section is this:

$$a_r = \frac{v_t^2}{R},$$

and it relates the *radial* acceleration necessary to keep an object turning in a circle of radius R at constant speed v_t.

In particular, knowing v_t and R we can fill-in the right hand side of the equation $F = ma$ in the r direction:

$$\sum \vec{F}_r = ma_r = m\frac{v_t^2}{R}.$$

This means that we can solve for F_r, which could be due to a rope pulling towards the centre, of a tire-road friction force F_f or some other force acting towards the centre.

1.7.3 Explanations

Radial coordinate system

To understand this whole $\hat{r}$ and $\hat{t}$ business, I want you to imagine for a second that you can freeze time. The $\hat{r}$ direction will be towards the centre of the circle and $\hat{t}$ is along a line touching the circle, i.e., though moving in a circle overall, instantaneously you are moving in the $\hat{t}$ direction. If you unfreeze time and one microsecond elapses the $\hat{r}$ and $\hat{t}$ directions will change, you will have moved a little along the circle and the $\hat{r}$ vector will have to rotate to keep pointing at the centre. Similarly, your instantaneous direction of motion, $\hat{t}$, will have shifted a bit.

Period

From kinematics we know that motion with constant velocity (UVM) is described by the formula $x = vt$, where the speed is *defined* as ratio of distance travelled over time: $v = \frac{\Delta x}{\Delta t}$. In this case the distance travelled, Δx, is not a straight line but a curve following the circumference of a circle:

$$C = 2\pi R.$$

The period is defined as the time it takes for the object to complete one full turn:

$$T = \frac{\text{distance}}{\text{speed}} = \frac{2\pi R}{v_t}.$$

Tangential speed v_t is measured in $[m/s]$, whereas R is measured in $[m]$ so if you divide the two you get seconds $[s]$.

Radial acceleration

Newton's first law says that if there is no acceleration acting on a moving object, then it will continue moving (1) in a straight line and (2) maintain the same speed. So far we have studied mostly problems involving *linear acceleration*, where some a_x is causing changes in speed. This section deals with problems where the speed $\|\vec{v}\|$ stays constant, but $\vec{v}$ changes direction.

Circular motion violates the requirement (1) in Newton's law (conservation of momentum). The logic is as follows. If you see any motion that is *not a straight line*, then there must be some acceleration involved which is producing the rotation. This is the radial acceleration.

Centrifugal force

People often get confused about the meaning of things when they talk about circular motion. There is the point of view of the passenger, and there is the point of view of the car door, which is holding the passenger into the circular trajectory.

- $\vec{F}_{dp}$: The amount of force the door is exerting on the passenger.
- $\vec{F}_{pd}$: The amount of force the passenger is exerting on the door.

This is the same force really, but one is relevant in the sum-of-the-forces calculation for the passenger $\sum F_{psngr}$, whereas the second should be added with all the other forces that act on the car (like the friction between the pavement and the car tires).

Viewed from the point of view of the passenger there is no confusion. The force F_{dp} acting towards the centre of rotation is causing a $ma_r = m\frac{v_t^2}{R}$, and thus the circular motion of the passenger results.

Viewed from the point of view of the door, it feels a certain push towards the outside of the circle which corresponds to the counter-force to the force that the passenger feels. This comes from Newton's third law, which says that for each contact force C_{12} exerted by object 1 on object 2, there exists a counter force exerted of equal magnitude and opposite direction, C_{21} which is the force of object 2 pushing back.

The confusion people get into, and the whole *false concept* of centrifugal force, comes from thinking of a force diagram on yourself and falsely including the outwards force on the door. You have to be clear. Which force diagram are you doing? If you are thinking of yourself as the object, the force that you feel is a *centripetal* force. The door pushes you each instant towards the centre of the circle. If it weren't for the door, you would fly straight on.

In some problems, it is also necessary to consider the y direction of the force diagram, where you will have the weight of the passenger $\vec{W} = -mg\hat{\jmath}$. You should be able to solve joint ry-dimension problems, with radial and vertical forces F_r, F_y and radial and vertical accelerations a_r, a_y.

1.8 Angular motion

We will now study the physics of rotating objects. Rotating disks, wheels, spinning footballs and ice skaters. Anything spinning really.

This is also a review chapter. You already know all the basic concepts for linear motion: position, velocity, acceleration, force, momentum and energy. In this chapter we define the *angular* equivalents for each of these, which will be appropriate whenever we have a rotating object.

1.8.1 Concepts

Angular force:

- $\mathcal{T}$: the *torque* is rotational force. Measured in newton-meters [Nm].

Angular mass:

- I: is the *moment of inertia* of an object, and tells you how difficult it is to make it turn. Measured in [kg m^2].

Angular F=ma:

- $\sum \mathcal{T} = I\alpha$: tells us that angular acceleration (α) is caused by angular forces (torques) and the constant of proportionality is the moment of inertia which takes into account the mass of an object, but also its shape.

Angular motion (kinematics):

- $\theta \equiv \frac{x_t}{R}$: angular displacement. Measured in radians [rad].
- $\omega \equiv \frac{v_t}{R}$: angular velocity. Measured in [rad/s] or [RPM].
- $\alpha \equiv \frac{a_t}{R}$: angular acceleration. Measured in [rad/s^2].

Angular momentum:

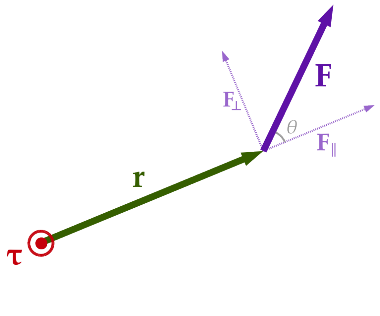
- $L = I\omega$: the *angular momentum* of a spinning object. Measured in [kg m^2/s].

Angular kinetic energy:

- $K_r = \frac{1}{2}I\omega^2$: The *angular kinetic energy* of a spinning object. Measured in joules [J]=[kg m^2/s^2].

1.8.2 Formulas

Torque



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Torque is defined as the rotational tendency of a force:

$$\mathcal{T} = F_{\perp} \times |r| = |F||r| \sin\theta.$$

Note that only the perpendicular part of the force $F_{\perp}$ is helping to cause a rotation.

To understand the meaning of the torque equation, you should stop reading and go experiment with a regular door. If you push close to the hinges, it will take a lot more force to produce the same torque since the r will be small. Also, if you grab the handles on both sides and pull away from the hinge, your entire force will be $F_{||}$, so no matter how hard you pull you will cause zero torque.

The predominant convention is to call torques that produce counter-clockwise motion positive and torques that cause clockwise rotation negative.

Torques cause angular acceleration

The same way we have forces that cause acceleration ($F = ma$), we have torques that cause angular acceleration:

$$\sum \mathcal{T} = \mathcal{T}_{net} = I\alpha.$$

Moment of inertia

$I = \{$ how difficult it is to make an object turn $\}$

$$I_{disk} = \frac{1}{2}mR^2, \quad I_{ring} = mR^2,$$

$$I_{sphere} = \frac{2}{5}mR^2, \quad I_{sph.shell} = \frac{2}{3}mR^2.$$

Angular kinematics

Instead of talking about position x, we will now talk about the angular orientation θ. Instead of talking about velocity v, we will talk about the angular velocity ω. Instead of acceleration, we have angular acceleration α.

Except for these change of actors, the *play* remains the same. Just like in the linear case, there are three equations that fully describe uniform accelerated angular motion:

$$\alpha(t) = \alpha,$$

$$\omega(t) = \alpha t + \omega_0,$$

$$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0,$$

$$\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta.$$

A special case of these equations is when there is no net torque acting on the object. No torque means there will be no angular acceleration, so the equations become a little simpler:

$$\alpha(t) = 0,$$

$$\omega(t) = \omega_0,$$

$$\theta(t) = \omega_0 t + \theta_0.$$

We call that *uniform velocity angular motion*.

Angular momentum

The angular momentum of a spinning object is given by:

$$L = I\omega.$$

The angular momentum of an object is a conserved quantity in the absence of torque:

$$L_{in} = L_{out}.$$

This is similar to the way momentum $\vec{p}$ is a conserved quantity in the absence of external forces $\vec{F}_{net}$.

Rotational kinetic energy

$$K_r = \frac{1}{2}I\omega^2,$$

which is the rotational analogue to the linear kinetic energy $\frac{1}{2}mv^2$.

1.8.3 Static equilibrium II

Equilibrium is the situation where $\vec{F}_{net} = 0$. The forces on the objects balance each other, so there is no acceleration.

Conversely if you see an object that is not moving, then the forces on it must be in equilibrium. There must be zero net force on it. No net x-force:

$$\sum F_x = 0,$$

no net y-force on it

$$\sum F_y = 0,$$

and (this is the new part), if you see that it is not rotating, then it must also have no net toques on it:

$$\sum \mathcal{T}_{(A)} = 0,$$

where we can take the torques with respect to *any* centre of rotation (A).

Example: walking the plank

Plank is one third of the way out, and not attached to the ship. It weights 300 kg in total, so 100kg sticking out of the ship and 200kg on the ship. How far on the plank can you walk before it tips into the sea?

1.8.4 Explanations

Torque

People in the Mid West and other rural areas like to buy trucks because they have a lot of torque. Big engine, big torque. You can pull stuff. Wait isn't that force? But toque should be good too? What is the difference? I am confused.

Don't be confused. Torque is just force times distance, i.e. we have to take into account not only what force is exerted but also the leverage: how far from the center of rotation you are exerting the force:

$$\mathcal{T} = F_{rot} \times r.$$

The bigger the leverage r, the bigger torque you will create with a fixed amount of force.

When it comes down to the question how much you can pull, and you want to answer it by looking at the specs of your truck, you have to look at (1) the max torque $\mathcal{T}$ and (2) the radius of your wheels r, and divide the numbers to get the maximum pulling force F.

Rotational F=ma

The angular analogue of $F = ma$ is:

$$\mathcal{T} = I\alpha,$$

where

$$I = \sum m_i r_i^2 = \int_{obj} r^2 \; dm.$$

1.8.5 Examples

Rotational UVM

Something is spinning at a constant angular velocity during some time t. Just use $\theta(t) = \omega t + \theta_0$. Waaay too simple and boring.

Rotational UAM

A pulley of radius R and moment of inertia I has a rope wound around it and a mass m attached at the end of that rope. What is the angular acceleration caused by the mass as it unwinds the rope.

The force diagram on m tells us that $mg - T = ma_y$ (where $\hat{y}$ points downwards). The torque diagram on the disk tells us that $TR = I\alpha$. Adding R times the first equation to the second we get:

$$R(mg - T) + TR = Rma_y + I\alpha,$$

or after simplification we get:

$$Rmg = Rma_y + I\alpha.$$

But we know that the rope forms a solid connection between the disk and the mass block, so we must also have $R\alpha = a_y$, so if we substitute for a_y we get:

$$Rmg = RmR\alpha + I\alpha = (R^2m + I)\alpha,$$

or

$$\alpha = \frac{Rmg}{R^2m + I}.$$

This makes sense. The numerator is the "cause", and the denominator is the effective moment of inertia of the system as a whole.

1.8.6 Discussion

It is really important that you connect all of the above rotational concepts with their linear counterparts.

1.9 Simple harmonic motion

This law describes the motion of a mass attached to a spring, a pendulum and any system that goes back and forth in a cyclic fashion.

1.9.1 Concepts

- A: Amplitude of the movement, how far does the object go back and forth.
- t: time.
- $x(t)$: position of object at time t.
- ω: angular frequency.
- ϕ: phase constant.
- $(\omega t + \phi) = \theta$: phase, the argument of the function sin.
- T: the *period* is the time it takes for the movement to repeat. Measured in seconds [s].
- f: frequency [Hz]=[1/s].

1.9.2 Formulas

Frequency is defined as "how many cycles in one second" and is equal to the inverse of the period (how long one cycle takes):

$$f = 1/T = \frac{\omega}{2\pi} \text{ [Hz]}.$$

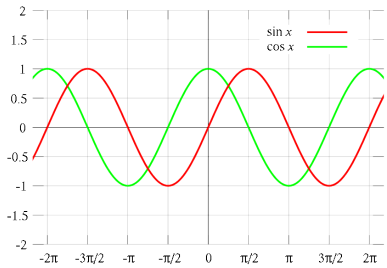
The relation between f (frequency) and ω (angular frequency) is a multiplication by 2π needed to match the units: 1 cycle needs to be multiplied by the number of radians it takes to make one full turn.

The equation of motion of an object undergoing simple harmonic motion is:

$$x(t) = A\cos(\omega t + \phi).$$

Now don't be scared. It is really simple. Let us just study the properties of $\sin(x)$ for a moment, so that you can be get familiar with the math of periodic functions.

The function sin(x)



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The functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ are both angular functions, that is, they take an angle θ as input. Sometimes we call the input x, but in the next paragraphs we will call the input θ so that we keep in mind that we are talking about angles.

The period T of $\sin(t)$ with no coefficients inside of it is 2π, because this is how long it takes (in radians) to do one full turn.

Time-scaling sin

If you want a sin function with a different period, you have to add a multiplier in front of x inside the sin. This multiplier is, by convention, called ω, so now we have a input-scaled sin function:

$$\sin(\omega t),$$

which will have period

$$T = \frac{2\pi}{\omega}.$$

No seriously, if you don't believe me check for yourself. If you let t go from 0 to T you should see the motion go though one cycle. Indeed, when multiplied with the scaling $\omega = \frac{2\pi}{T}$, the input to sin goes from 0 to 2π as t goes from 0 to T.

Sin or cos up to you

Simple harmonic motion is equally well described by the sin function or the cos function.

You should use cos when the spring starts from the stretched position: $x_i = x_{max} = A$, because naturally when $t = 0$ cos will be one, and the overall function will be:

$$x(t) = A\cos(\omega t).$$

If the harmonic motion starts from $x(t) = 0$ when $t = 0$, then sin is a more appropriate function to describe this motion:

$$x(t) = A\sin(\omega t),$$

which correctly predicts the initial position $x(0) = 0$.

Any combination of the above scenarios is possible too. So now it becomes difficult to see what the motion is, with *two* amplitudes (one for stretched initial conditions, and one for centered initial conditions) and a sum of sin and cos:

$$x(t) = A_1\cos(\omega t) + A_2\sin(\omega t).$$

Lo and behold. There is actually a simpler way of writing the above equation. Sin and cos are identical except for the fact that one is a shifted version of the other. By using trigonometric identity Kung Fu, we can rewrite the above in terms of sin or cos with some combined amplitude and a *phase shift*:

$$x(t) = A\cos(\omega t + \phi).$$

The initial conditions x_i and v_i are used to find either the individual coefficients A_1 and A_2 of the sin and the cos or the common coefficient A and the phase shift ϕ.

Let me go over what just happened here one more time, because this is some crazy stuff. You are not supposed to learn about this stuff until the *differential equations* class, but I will tell you about it right now because it is like *woooow*. Let's draw an analogy with a situation which we have seen previously. We are trying to describe (and predict) the position of an object as a function of time $x(t)$. In kinematics there was *one* equation $x(t) = x_i + v_i t + \frac{1}{2}at^2$, and depending on the initial velocity and the initial position we got different trajectories, because of the different values of x_i and v_i. In the case of simple harmonic motion we have *two* functions and the initial position x_i and initial velocity v_i determine how much of each function we should include in the formula for $x(t)$.

The thing to remember is that you can pick cos or sin as the situation requires and, if the situation requires it, add a phase factor. I assure you that the fact that $a\sin(x) + b\cos(x)$ can be expressed as $A\cos(x+\phi)$ or $A\sin(x-\psi)$ is not magic, but simple application of trig_identities.

Mass and spring

Suppose you have a mass m attached to a spring with spring constant k. If disturbed from rest, this mass-spring system will undergo simple harmonic motion with angular frequency:

$$\omega = \sqrt{\frac{k}{m}}.$$

A typical exam question is to tell you k and m and ask about the period T, at which point you have to remember the definition of period to obtain the answer:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

Pendulum

Consider an object suspended at the end of a long string of length ℓ in a gravitational field of strength g. If disturbed from equilibrium this system will undergo simple harmonic motion, by swinging from side to side.

The period of oscillation will be:

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

Note in particular that this equation does not depend on the amplitude of the oscillation (how far the pendulum swings) nor the mass of the pendulum. Kind of cool no? For a long while, this was how people kept track of time: with pendulum clocks.

To describe the angular position of the pendulum we can use a the formula:

$$\theta(t) = \theta_{max}\cos(\omega t + \phi),$$

where θ_{max} is the maximum angle the pendulum swings to. The coefficient inside the cos is equal to:

$$\omega \equiv \frac{2\pi}{T} = \sqrt{\frac{g}{\ell}}.$$

SHM equations of motion

These are the new equations of motion which you need to learn how to use:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi), \\ v(t) &= -A\omega\sin(\omega t + \phi), \\ a(t) &= -A\omega^2\cos(\omega t + \phi). \end{aligned}$$

Note that the velocity and the acceleration of the object are also periodic functions. Pay attention to the *amplitude* of the velocity:

$$v_{max} = \omega A,$$

and the *amplitude* of the acceleration:

$$a_{max} = \omega^2 A.$$

You will often be asked to solve for these quantities, which is going to be an easy task if you know A and ω.

And equivalent set of equations of motion exists with $x(t) \propto \sin(\omega t + \psi)$, but in this section we will just talk about the cos kind of $x(t)$. Everything I say would be true if we had started with $x(t) \propto \sin(\omega t + \psi)$ instead.

Energy

The potential energy stored in the spring is:

$$U(t) = \frac{1}{2}kx(t)^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi).$$

The kinetic energy of the mass is:

$$K(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi).$$

The total energy is:

$$\begin{aligned} E_{total} &= U(t) + K(t) \\ &= \frac{1}{2}kA^2\cos^2(\omega t) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t) \\ &= \frac{1}{2}kA^2\underbrace{\left[\cos^2(\omega t) + \sin^2(\omega t)\right]}_{=1} = \frac{1}{2}kA^2 = U_{m} \\ &= \frac{1}{2}m\omega^2A^2\cos^2(\omega t) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t) \quad \text{(sin} \\ &= \frac{1}{2}m\underbrace{\omega^2A^2}_{v_{max}^2}\underbrace{\left[\cos^2(\omega t) + \sin^2(\omega t)\right]}_{=1} = \frac{1}{2}mv_{max}^2 \end{aligned}$$

Conservation of energy

In simple harmonic motion, the total energy is conserved:

$$E_T = U + K.$$

Energy shifts between the potential energy of the spring and the kinetic energy of the moving mass. For any two points t_i and t_f:

$$U_i + K_i = U_f + K_f.$$

Two particularly important moments of the SHM are the points where it is at its maximum position and zero velocity $x = \pm A$, $K = 0$, $E_T = U$, which corresponds to all the energy being stored in the spring, and the point where it has zero displacement but maximal velocity $x = 0, U = 0, v = \pm A\omega, E_T = K$, which corresponds to all the energy being kinetic.

1.9.3 Explanations

Derivation of SHM equation

OK, wait a minute, you are saying. First I told you about uniform velocity $x(t) = v_i t + x_i$ and then I added in acceleration $x(t) = x_i + v_i t + \frac{1}{2}at^2$, which is a little more complex but still manageable. Now I am talking about sin and cos and Greek letters with dubious names like phase. Are you phased by all of this? When I was learning this stuff, I was totally phased because I didn't understand where the sin and cos came from.

The sin comes from $F = ma$, and the fact that the force of a spring is $F_s = -kx$.

Recall the definition of acceleration:

$$a = \frac{dv}{dt} = \frac{d^2x(t)}{dt^2}.$$

We now substitute these facts into $F = ma$ to get:

$$\begin{aligned}
F &= ma \\
-kx &= ma \\
-kx(t) &= m\frac{d^2x(t)}{dt^2} \\
0 &= m\frac{d^2x}{dt^2} + kx(t) \\
0 &= \frac{d^2x}{dt^2} + \frac{k}{m}x(t).
\end{aligned}$$

This is as far as Newton's second law will be able to bring us. The formula $F = ma$ is not enough to tell us the exact function $x(t)$, like in the UVM and UAM kinematics motion in the earlier chapters. Instead of knowing the function $x(t)$, we know one of its properties, namely, that its second derivative is equal to the negative of itself multiplied by some numbers.

Can you think of a function whose second derivative is equal to itself times $\frac{k}{m}$? Ok I thought of one:

$$x_1(t) = A\cos(\sqrt{\frac{k}{m}}t + \phi),$$

and come to think of it I thought of a second one which also works:

$$x_2(t) = A\sin(\sqrt{\frac{k}{m}}t + \psi).$$

Hmm, now you see where the sin comes from and why we said that $\omega = \sqrt{\frac{k}{m}}$ for a mass-spring system. It is all from $F = ma$.

1.9.4 Examples

In various word problems you will either be told the initial amplitude $x_i = A$ or the initial velocity $v_i = \omega A$ and asked to solve for some unknown. Nothing should be too difficult provided you write down the equations $x(t)$, $v(t)$ and $a(t)$ and fill in the knowns.

1.9.5 Links