Chapter 1

Mechanics

In short, mechanics is powerful stuff, so let's get right into it.

1.1 Kinematics

Kinematics (from the Greek root kinema *motion*) is the study of trajectories of objects moving in space. It can be used to calculate how long a ball thrown upwards will stay in the air, or to calculate the acceleration needed to go from 0 to 100 km/h in 5 seconds.

1.1.1 Concepts

The basic concepts of kinematics are:

- t: time, measured in seconds [s].
- x: the *position* of an object with respect to a coordinate system, measured in meters [m].
- $x(t)$: the position as a function of time, also known as the equation of motion.
- $v(t)$: speed measured in meters per second [m/s].
- $a(t)$: acceleration measured in meters per second squared [m/s^2].
- $x_0 = x(0)$: the position at $t = 0$.
- $v_0 = v(0)$: the velocity at $t = 0$.

When solving some problem, where we calculate the motion of an object that starts form an *initial* point an goes to a *final* point we will use the following terminology:

- t_i: initial time (the beginning of the motion).
- t_f: final time (when the motion stops).
- x_i: initial position.
- x_f: final position.
- v_i: initial velocity of the object.
- v_f: final velocity of the object.

1.1.2 Formulas

There are basically three equations that you need to be aware of for this entire chapter. Together, these three equations fully describe all the aspects of motion with constant acceleration.

Uniform acceleration motion (UAM)

If the object undergoes a *constant* acceleration $a_{const} = a$, like your car if you floor the *accelerator* pedal, then the equations of motion are:

$$a(t) = a,$$
$$v(t) = at + v_0,$$
$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0.$$

There is also another very useful equation to remember:

$$v(t)^2 = v_0^2 + 2a[x(t) - x_0],$$

which is usually written

$$v_f^2 = v_i^2 + 2a\Delta x.$$

That is it. Memorize these equations, plug-in the right numbers, and you can solve any kinematics problem humanly imaginable. Chapter done.

Uniform velocity motion (UVM)

The special case where there is zero acceleration ($a = 0$), is called *uniform velocity motion* or UVM.

$$a(t) = 0,$$
$$v(t) = v_0,$$
$$x(t) = v_0 t + x_0.$$

As you can see, these are really the same equations as above, but because $a = 0$, some terms are missing.

1.1.3 Explanations

Calculus

The functions $x(t)$, $v(t)$ and $a(t)$ are connected. They all describe different aspects of the same motion. We have:

$$x(t) \xrightarrow{\frac{d}{dt}} v(t) \xrightarrow{\frac{d}{dt}} a(t),$$

which means that starting from the position function, $x(t)$, we can use the derivative operation to obtain the velocity and the acceleration. We can also use this relationship in the other direction:

$$x(t) \xleftarrow{\int dt} v(t) \xleftarrow{\int dt} a(t),$$

which means that we start from the acceleration $a(t)$ and use integration with respect to time to obtain the velocity $v(t)$. If we integrate the velocity we obtain the position function $x(t)$.

1.2 Projectile motion

1.2.1 Concepts

The basic concepts of kinematics in two dimensions are:

- $\hat{x}, \hat{y}$: a coordinate system.
- t: time, measured in seconds.
- x: the x position of an object as measured according to the coordinate system.
- $x(t)$: the x position as a function of time.
- $v_x(t)$: speed in the x direction, measured in meters per second.
- $a_x(t)$: acceleration in the x direction, measured in meters per second squared.
- $y(t)$: the y position as a function of time.
- $v_y(t)$: velocity in the y direction as a function of time.
- $a_y(t)$: the y-acceleration.
- $\vec{r}(t) = (x(t), y(t))$: position *vector*, the combined x and y positions.

When solving some problem, where we calculate the motion of an object that starts form an *initial* point an goes to a *final* point we will use the following terminology:

- t_i: initial time (the beginning of the motion).
- t_f: final time (when the motion stops).
- $\vec{v}_i = \vec{v}(t_i) = (v_{xi}, v_{yi})$: the initial velocity at $t = t_i$.
- $\vec{r}_i = \vec{r}(t_i) = (x_i, y_i)$: the initial position at $t = t_i$, where x_i is the initial x position and y_i the initial y position.
- $\vec{r}_f = \vec{r}(t_f) = (x_f, y_f)$: the final position at $t = t_f$, where x_f is the final x position, and y_f is the final y position.

1.2.2 Formulas

Motion in two dimensions

Sometimes you have to describe both the x and the y coordinate of the motion of a particle:

$$\vec{r}(t) = (x(t), y(t)).$$

We choose x to be the horizontal component of the projectile motion, and y to be its height.

The velocity of the projectile will be:

$$\vec{v}(t) = \frac{d}{dt}\left(\vec{r}(t)\right) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = (v_x(t), v_y(t)),$$

and the initial velocity is:

$$\vec{v}_i = \vec{v}(t_i) = |\vec{v}_i|\angle\theta = (v_x(t_i), v_y(t_i)) = (v_{ix}, v_{iy}) = ($$

The acceleration of the projectile will be

$$\vec{a}(t) = \frac{d}{dt}\left(\vec{v}(t)\right) = (a_x(t), a_y(t)) = (0, -9.81).$$

Note how we have zero acceleration in the x direction so we can use the UVM equations of motion for $x(t)$ and $v_x(t)$. In the y direction we have a uniform downward acceleration due to gravity.

Projectile motion

The equations of motion of a projectile are the following. First in the x direction we have uniform velocity motion (UVM):

$$\begin{aligned} x(t) &= v_{ix}t + x_i, \\ v_x(t) &= v_{ix}. \end{aligned}$$

In the y direction, you have the constant pull of gravity downwards which gives us a uniformly accelerated motion (UAM):

$$\begin{aligned} y(t) &= \frac{1}{2}(-9.81)t^2 + v_{iy}t + y_i, \\ v_y(t) &= -9.81t + v_{iy}, \\ v_{yf}^2 &= v_{yi}^2 + 2(-9.81)(\Delta y). \end{aligned}$$

1.3 Forces

Like a shepherd who brings in a stray sheep back, we need to rescue the word *force* and give it precise meaning. In physics force means something very specific. Not "the force" from Star Wars, not the "force of public opinion", and not the *force* in the battle of good versus evil.

Force in physics has a precise meaning as an amount of push or pull exerted on an object. Force is a vector. We measure force in Newtons [N], and we can use it in equations and solve for it just like any other unknown. In this section we will explore all the different kinds of forces.

1.3.1 Concepts

- $\vec{F}$: a force. This is something the object "feels" as a pull or a push. Force is a vector, so you must always keep in mind the direction in which the force $\vec{F}$ acts.
- $k, G, m, \mu_s, \mu_k, \ldots$: parameters on which the force F may depend. Ex: the heavier an object is (has large m parameter), the larger its gravitational pull will be: $\vec{W} = -9.81 m\hat{z}$, where $\hat{z}$ points towards the sky.

1.3.2 Kinds of forces

We next list all the forces which you are supposed to know about for a standard physics class and define the relevant parameters for each kind of force. You need to practice exercises using each of these forces, until you start to *feel* how they act.

Gravitation

Manifestations of the gravitational pull of the planet Earth on massive objects:

- M: mass of the earth. $M = 5.9721986 \times 10^{24}$ [kg].
- m: mass of an object.
- $\vec{W} = \vec{F}_g$: The weight (the force on a object due to gravity).
- G: Gravitational constant $= 6.6710^{-11}$ $[\frac{Nm^2}{kg^2}]$.
- $\vec{F}_g = \frac{GMm}{r^2}$: Force of gravity between two objects of mass M and m respectively. Measured in Newtons [N].
- $\vec{F}_g = gm$ (downward): The force of gravity on the surface of the earth, where $g = \frac{GM}{r^2} \approx 9.81\ldots$ [N/kg]=[m/s^2].

The famous one-over-arr-squared law that describes the gravitational pull between two objects is:

$$F_g = \frac{GMm}{r^2}.$$

You will rarely use it, but it is extremely important as this is where all of mechanics began. This was Newton's big discovery. All the rest of mechanics is simple calculus, but this equation is *real* physics. It tells us something about how the Universe works.

At the surface of the earth:

$$\vec{F}_g = \frac{GMm}{r^2} = \underbrace{\left(\frac{GM}{r^2}\right)}_{g} m = \vec{g}m = \vec{W},$$

where the weight $\vec{W}$ of an object is a vector that points towards the centre of the earth, and $g = 9.81$[m/s^2].

Force of a spring

- $\vec{F}_s = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by x meters. The constant k [N/m] is a measure of the *strength* of the spring, or its stiffness.

Tension in a rope

- $\vec{T}$: Tension in a rope. Tension is always pulling *away* from an object: you can't push a dog on a leash.

Contact force

- $\vec{C}$: Contact force between two rigid objects. We generally brake-up contact forces into two components: perpendicular and parallel to the contact surface.
- $\vec{N} \equiv \vec{C}_\perp$: Normal force: the force between two surfaces. Normal is a mathematically precise way to say "perpendicular to a surface". Intuitively, you can think of $\vec{N}$ as the force that a surface exerts on an object to keep it where it is. The reason why my coffee mug does not fall to the floor, is that the table exerts a normal force on it keeping in place.
- $\vec{F}_f \equiv \vec{C}_{||}$: Force of friction between two surfaces. There are two kinds, both of which are proportional to the normal force between the surfaces:
  Kinetic:
  $$F_{fk} = \mu_k|\vec{N}|.$$
  Static:
  $$F_{fs} = \mu_s|\vec{N}|.$$

Two kinds of friction forces

- $\vec{F}_{fs} = \mu_s|\vec{N}|$: Static force of friction, for objects that are not moving.
- μ_s: The static coefficient of friction. ex: 0.3. It describes the **maximum** amount of friction that can exist between two objects. If a horizontal force exists greater than $F_{fs} = \mu_s N$, then the object will start to slip.
- $\vec{F}_{fk} = \mu_k|\vec{N}|$: Kinetic force of friction acts when two objects are sliding relative to each other. It always acts in a direction opposing the motion.
- μ_k: Kinetic coefficient of friction. ex: $\mu_k = 0.1$. Dimensionless. it is just the ratio that describes how much friction an object feels for a given amount of normal force.

1.4 Force diagrams

Newton's 2nd law says that the *net* force on an object causes an acceleration:

$$\sum \vec{F} = \vec{F}_{net} = m\vec{a},$$

so finding the net force must be a pretty important thing.

1.4.1 Concepts

Newton's second law is a relationship between these three concepts:

- m: the *mass* of an object.
- $\vec{F}$: vector used to denote any kind of *force*.
- $\vec{a}$: the *acceleration* of the object.

Both forces and accelerations are vectors. To work with vectors, we have work with their *components*:

- F_x: the *component* of $\vec{F}$ in the x direction.
- F_y: the *component* of $\vec{F}$ in the y direction.

Vectors are meaningless unless it is clear with respect to which *coordinate system* they are expressed.

- x-axis: Usually the x-axis is horizontal and to the right, however, for problems with inclines, it will be more convenient to put the x-axis parallel to the slope.
- y-axis: The y-axis is always *perpendicular* to the x-axis.
- $\hat{\imath}, \hat{\jmath}$: Unit vectors in the x and y directions. Any vector can be written as $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ or as $\vec{v} = (v_x, v_y)$.

Provided we have a coordinate system, we can write any force vector in three ways:

$$\vec{F} \equiv F_x\hat{\imath} + F_y\hat{\jmath} \equiv F_x\hat{x} + F_y\hat{y} \equiv (F_x, F_y),$$

note that there are two ways to write the unit vectors: $\hat{\imath} \equiv \hat{x}$ and $\hat{\jmath} \equiv \hat{y}$.

What types of forces are there in force diagrams?

- $\vec{W} \equiv \vec{F}_{gravity} = m\vec{g}$: The *weight*. This is the force on a object due to its gravity. The gravitational pull $\vec{g}$ always points downwards towards the center of the earth. $g = 9.81$ [N/kg].
- $\vec{T}$: Tension in a rope. Tension is always pulling *away* from the object.
- $\vec{N}$: Normal force the force between two surfaces.
- $\vec{F}_{fs} = \mu_s|\vec{N}|$: Static force of friction.
- $\vec{F}_{fk} = \mu_k|\vec{N}|$: Kinetic force of friction.
- $\vec{F}_{spring} = -kx$: The force (pull or push) of a spring that is displaced (stretched or compressed) by x meters.

1.4.2 Formulas

Newton's 2nd law

The sum of the forces acting on an object, divided by the mass gives you the acceleration of the object:

$$\sum F = \vec{F}_{net} = m\vec{a}.$$

Vector components

If a vector $\vec{v}$ makes an angle θ with the x-axis then:

$$v_x = |\vec{v}|\cos\theta, \qquad \text{and} \qquad v_y = |\vec{v}|\sin\theta.$$

The vector $v_x\hat{\imath}$ corresponds to the the part of $\vec{v}$ that points in the x-direction.

In what follows, you will be asked a countless number of times to

Find the component of $\vec{F}$ in the ? direction.

Which is another way of asking you to find the number $v_?$.

The answer is usually equal to the length $|\vec{F}|$ multiplied by either cos or sin and sometimes -1 all **depending on way the coordinate system is chosen**. So don't guess. Look at the coordinate system. If the vector points in the direction where x-increases, then v_x should be a positive number. If it points in the opposite direction, then v_x should be negative.

Vector components are important, because to add forces $\vec{F_1}$ and $\vec{F_2}$ you have to add their components:

$$(F_{1x}, F_{1y})+(F_{1x}, F_{2y}) = \vec{F_1}+\vec{F_2} = (F_{1x}+F_{2x}, F_{1y}+F_{2y}) = \vec{F}_{net}.$$

Instead of dealing with vectors in the bracket notation as above, when solving force diagrams it is easier to simply write the x-equation on one line, and the y-equation on a separate line.

$$F_{netx} = F_{1x} + F_{2x},$$

$$F_{nety} = F_{1y} + F_{2y}.$$

It is a good idea to always write those two equations together as a block so it remains clear that you are talking about the same problem, but the first row represents the x-dimension and the second row represents the y-dimension.

1.4.3 Recipe for solving force diagrams

1. Draw a diagram centred on the object and draw all the forces acting on it.

2. Choose a coordinate system, and indicate clearly what you will call the x-direction, and what you will call the y-direction. All **equations are expressed *with respect to* this coordinate system**.

3. Write down this "template":

$$\sum F_x = \qquad\qquad = ma_x$$

$$\sum F_y = \qquad\qquad = ma_y$$

4. Fill the first line by finding the x-components of each force acting on the object.

5. Fill the second line by finding the y-components of each force acting on the object.

6. Consistency checks:

   (a) Check signs. If the force in the diagram is acting in the x-direction then its component must be positive. If the force is acting in the opposite direction to $\hat{x}$, then its component should be negative.

   (b) Verify that whenever $F_x \propto \cos\theta$, then $F_y \propto \sin\theta$. If instead we use an angle ϕ defined with respect to the y-axis we would have $F_x \propto \sin\phi$, and $F_y \propto \cos\phi$.

7. Solve the two equations finding the one or two unknowns. If there are two unknowns, you may need to solve two equations simultaneously by isolation and substitution.

Force diagrams are best explained through examples.

1.5 Momentum

The momentum is equal to the velocity of the moving object multiplied by the object's mass ($\vec{p} = m\vec{v}$). Therefore, since the car weights $1000 \times 1000 = 10^6$ times more than the piece of paper, it has 10^6 times more momentum when moving at the same speed. A collision with it will "hurt" that much more.

Note that momentum is a vector, so we will have to do a lot of that length-and-direction-to-components transformation stuff:

$$(p_x, p_y) = \vec{p} = (|\vec{p}|\cos\theta, |\vec{p}|\sin\theta) = |\vec{p}|\angle\theta,$$

and also converting backwards from component notation to magnitude-direction:

$$|\vec{p}| = \sqrt{p_x^2 + p_y^2}, \qquad \theta = \tan^{-1}\left(\frac{p_y}{p_x}\right).$$

1.5.1 Concepts

- m: the *mass* of the moving object.
- $\vec{v}$: the *velocity* of the moving object.
- $\vec{p} = m\vec{v}$: the *momentum* of the moving object.

1.5.2 Formulas

Definition

The momentum of an object is the mass of the object times its velocity:

$$\vec{p} = m\vec{v}.$$

If you speed is $\vec{v} = (20, 0, 0)$[m/s], which is equivalent to saying "20[m/s] in the x-axis direction", and your mass is 100kg then your momentum is $\vec{p} = (2000, 0, 0)$[kg*m/s].

In the absence of acceleration, objects will conserve their velocity:

$$\vec{v}_{in} = \vec{v}_{out}.$$

This is equivalent to saying that objects conserve their momentum (just multiply the velocity by the mass if the mass stays constant and the velocity stays constant, then the momentum must stay constant).

Conservation of momentum

More generally, if you have a situation with multiple moving objects, you can say that the "overall momentum", i.e., the sum of the momenta of all the particles stays constant:

$$\sum \vec{p}_{in} = \sum \vec{p}_{out}.$$

This is amazingly powerful stuff, and one of the furthest reaching laws of physics. Whatever momentum comes into a collision must come out.

1.6 Energy

Instead of thinking about velocities $v(t)$ and motion trajectories $x(t)$, we can solve physics problems using *energy* calculations.

1.6.1 Concepts

The concepts of energy come up in several different contexts.

Moving objects:

- m: the mass of an object.
- v: the velocity.
- $E_K = K$: kinetic energy $= \frac{1}{2}mv^2$.

Moving objects by force:

- $\vec{F}$: the force needed to move the object.
- $\vec{d}$: the displacement of the object. How far it moved.
- W: work done to move the object $= \vec{F} \cdot \vec{d}$.

Gravity:

- g: gravitational acceleration on the surface of earth. 9.81 [m/s^2].
- h: height of an object.
- U_g: Gravitational potential energy $= mgh$.

Springs:

- k: spring constant. Measured in [N/m].
- x: spring displacement from the relaxed position. If the spring is stretched then $x > 0$, and if it is compressed then $x < 0$.
- U_s: Spring potential energy $= \frac{1}{2}kx^2$.

There are all kinds of other forms of energy: electric energy, sound energy, thermal energy, etc. In this section we will focus on the types of *mechanical* energy.

1.6.2 Formulas

Kinetic energy:

$$K = \frac{1}{2}mv^2$$

Work:

$$W = \int \vec{F}(x) \cdot dx$$

for constant force:

$$W = \vec{F} \cdot \vec{d} = |F||d| \cos\theta.$$

Gravitational potential energy:

$$U_g = mgh,$$

which is the energy you have because of your height.

Spring energy:

$$U_s = \frac{1}{2}kx^2.$$

Conservation of energy:

$$\sum E_{in} + W_{in} = \sum E_{out} + W_{out}.$$

1.7 Uniform circular motion

When a car makes a long left turn, the passenger riding shotgun will feel pushed towards the right and into the door. If we assume the car moves at a constant speed v in the turn, and that the radius of the curve is R, what is the force of contact between the passenger and the door? This question may sound complicated, but actually it boils down to a single formula. This entire section is dedicated to that formula and to objects moving around in a circle in general.

1.7.1 Concepts

- $\hat{x}, \hat{y}$: the a usual coordinate system.

In this section we will use a new type of coordinate system:

- $\hat{r}, \hat{t}$: the *radial* and *tangential* directions of a circular coordinate system. No matter where you are on the circle, the radial direction always points towards the centre, while the tangential direction is always perpendicular. On a bicycle wheel, the spokes are in the $\hat{r}$ direction, while the pavement is in the $\hat{t}$ direction.
- $\vec{v} = (v_x, v_y)_{\hat{x}\hat{y}} = (v_r, v_t)_{\hat{r}\hat{t}}$: Velocity of particle, can be expressed as in xy or rt coordinates

- $\vec{a} = (a_r, a_t)_{\hat{r}\hat{t}}$: the *acceleration* of the particle.

Every time you have uniform circular motion ($v_t = const$), these variables will be related:

- R: Radius of the circle of motion.
- v_t: Speed of the circular motion in $[m/s]$. Sometimes referred to as *tangential* speed.
- a_r: Radial acceleration. The relation is $a_r = \frac{v_t^2}{R}$.

There is some special terminology used to describe circular motion:

- C: the *circumference* of the circle of motion. For a circle of radius R, $C = 2\pi R$.
- T: the *period*, how long it takes for the object to complete one circle. Measured in seconds.
- f: the frequency of rotation. How many times per second does an object pass by the same point on the circle. $f = \frac{1}{T}$. Measured in Hertz [Hz]=[1/s].
- $\omega \equiv \frac{v_t}{R}$: angular velocity, how fast the angle of the object is rotating $\omega = 2\pi f$.
- RPM: the *revolutions per minute* which is the angular velocity expressed in units of revolutions (1[rev]=2π[rad]) and minutes (1[min]=60[s]).

The angular velocity ω is very useful because it describes the speed of circular motion of *any* radius. Indeed, different points on a rotating disk will have different tangential speeds v_t depending on how far from the radius they are, so it is much better to describe the angular velocity and multiply by the radius as needed.

1.7.2 Formulas

Three directions

Let's freeze time and zoom in on an object moving in a circle. We can draw a force diagram with three directions:

1. $\hat{r}$: towards the centre of the circle of rotation.
2. $\hat{t}$: in the instantaneous direction where the car is moving right now. This is called the *tangential* direction, from the greek to touch (imagine a straight line "touching" the circle).
3. $\hat{y}$: if necessary, we imagine a third *vertical* axis pointing up out of the plane of rotation.

Motion in a circle

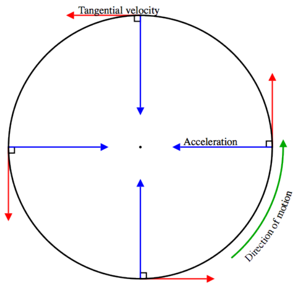
A circle of radius R has circumference:

$$C = 2\pi R.$$

An object moving along this circular path with tangential speed v_t. The period, T, is defined as how long it will take the object to complete one turn and is equal to:

$$T = \frac{2\pi R}{v_t} = \frac{2\pi}{\omega} = \frac{1}{f}.$$

Radial acceleration



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The one equation of this section is this:

$$a_r = \frac{v_t^2}{R},$$

and it relates the *radial* acceleration necessary to keep an object turning in a circle of radius R at constant speed v_t.

In particular, knowing v_t and R we can fill-in the right hand side of the equation $F = ma$ in the r direction:

$$\sum \vec{F}_r = ma_r = m\frac{v_t^2}{R}.$$

This means that we can solve for F_r, which could be due to a rope pulling towards the centre, of a tire-road friction force F_f or some other force acting towards the centre.

1.8 Angular motion

We will now study the physics of rotating objects. Rotating disks, wheels, spinning footballs and ice skaters. Anything spinning really.

1.8.1 Concepts

Angular force:

- $\mathcal{T}$: the *torque* is rotational force. Measured in newton-meters [Nm].

Angular mass:

- I: is the *moment of inertia* of an object, and tells you how difficult it is to make it turn. Measured in [kg m^2].

Angular F=ma:

- $\sum \mathcal{T} = I\alpha$: tells us that angular acceleration (α) is caused by angular forces (torques) and the constant of proportionality is the moment of inertia which takes into account the mass of an object, but also its shape.

Angular motion (kinematics):

- $\theta \equiv \frac{x_t}{R}$: angular displacement. Measured in radians [rad].
- $\omega \equiv \frac{v_t}{R}$: angular velocity. Measured in [rad/s] or [RPM].
- $\alpha \equiv \frac{a_t}{R}$: angular acceleration. Measured in [rad/s^2].

Angular momentum:

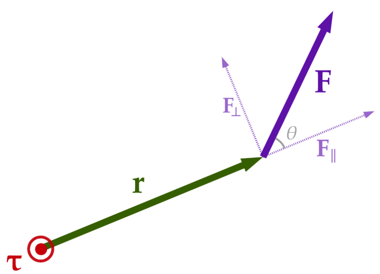
- $L = I\omega$: the *angular momentum* of a spinning object. Measured in [kg m^2/s].

Angular kinetic energy:

- $K_r = \frac{1}{2}I\omega^2$: The *angular kinetic energy* of a spinning object. Measured in joules [J]=[kg m^2/s^2].

1.8.2 Formulas

Torque



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Torque is defined as the rotational tendency of a force:

$$\mathcal{T} = F_{\perp} \times |r| = |F||r|\sin\theta.$$

Note that only the perpendicular part of the force $F_{\perp}$ is helping to cause a rotation.

To understand the meaning of the torque equation, you should stop reading and go experiment with a regular door. If you push close to the hinges, it will take a lot more force to produce the same torque since the r will be small. Also, if you grab the handles on both sides and pull away from the hinge, your entire force will be $F_{||}$, so no matter how hard you pull you will cause zero torque.

The predominant convention is to call torques that produce counter-clockwise motion positive and torques that cause clockwise rotation negative.

Torques cause angular acceleration

The same way we have forces that cause acceleration ($F = ma$), we have torques that cause angular acceleration:

$$\sum \mathcal{T} = \mathcal{T}_{net} = I\alpha.$$

Moment of inertia

$I = \{$ how difficult it is to make an object turn $\}$

$$I_{disk} = \frac{1}{2}mR^2, \quad I_{ring} = mR^2,$$

$$I_{sphere} = \frac{2}{5}mR^2, \quad I_{sph.shell} = \frac{2}{3}mR^2.$$

Angular kinematics

Instead of talking about position x, we will now talk about the angular orientation θ. Instead of talking about velocity v, we will talk about the angular velocity ω. Instead of acceleration, we have angular acceleration α.

Except for these change of actors, the *play* remains the same. Just like in the linear case, there are three equations that fully describe uniform accelerated angular motion:

$$\alpha(t) = \alpha,$$

$$\omega(t) = \alpha t + \omega_0,$$

$$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0,$$

$$\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta.$$

A special case of these equations is when there is no net torque acting on the object. No torque means there will be no angular acceleration, so the equations become a little simpler:

$$\alpha(t) = 0,$$

$$\omega(t) = \omega_0,$$

$$\theta(t) = \omega_0 t + \theta_0.$$

We call that *uniform velocity angular motion*.

Angular momentum

The angular momentum of a spinning object is given by:

$$L = I\omega.$$

The angular momentum of an object is a conserved quantity in the absence of torque:

$$L_{in} = L_{out}.$$

This is similar to the way momentum $\vec{p}$ is a conserved quantity in the absence of external forces $\vec{F}_{net}$.

Rotational kinetic energy

$$K_r = \frac{1}{2}I\omega^2,$$

which is the rotational analogue to the linear kinetic energy $\frac{1}{2}mv^2$.

1.8.3 Static equilibrium II

Equilibrium is the situation where $\vec{F}_{net} = 0$. The forces on the objects balance each other, so there is no acceleration.

Conversely if you see an object that is not moving, then the forces on it must be in equilibrium. There must be zero net force on it. No net x-force:

$$\sum F_x = 0,$$

no net y-force on it

$$\sum F_y = 0,$$

and (this is the new part), if you see that it is not rotating, then it must also have no net toques on it:

$$\sum \mathcal{T}_{(A)} = 0,$$

where we can take the torques with respect to *any* centre of rotation (A).

1.9 Simple harmonic motion

This law describes the motion of a mass attached to a spring, a pendulum and any system that goes back and forth in a cyclic fashion.

1.9.1 Concepts

- A: Amplitude of the movement, how far does the object go back and forth.
- t: time.
- $x(t)$: position of object at time t.
- ω: angular frequency.
- ϕ: phase constant.
- $(\omega t + \phi) = \theta$: phase, the argument of the function sin.
- T: the *period* is the time it takes for the movement to repeat. Measured in seconds [s].
- f: frequency [Hz]=[1/s].

1.9.2 Formulas

Frequency is defined as "how many cycles in one second" and is equal to the inverse of the period (how long one cycle takes):

$$f = 1/T = \frac{\omega}{2\pi} \text{ [Hz]}.$$

The relation between f (frequency) and ω (angular frequency) is a multiplication by 2π needed to match the units: 1 cycle needs to be multiplied by the number of radians it takes to make one full turn.

The equation of motion of an object undergoing simple harmonic motion is:

$$x(t) = A\cos(\omega t + \phi).$$

Mass and spring

Suppose you have a mass m attached to a spring with spring constant k. If disturbed from rest, this mass-spring system will undergo simple harmonic motion with angular frequency:

$$\omega = \sqrt{\frac{k}{m}}.$$

Pendulum

Consider an object suspended at the end of a long string of length ℓ in a gravitational field of strength g. If disturbed from equilibrium this system will undergo simple harmonic motion, by swinging from side to side.

The period of oscillation will be:

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

SHM equations of motion

These are the new equations of motion which you need to learn how to use:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi), \\ v(t) &= -A\omega\sin(\omega t + \phi), \\ a(t) &= -A\omega^2\cos(\omega t + \phi). \end{aligned}$$

Note that the velocity and the acceleration of the object are also periodic functions. Pay attention to the *amplitude* of the velocity:

$$v_{max} = \omega A,$$

and the *amplitude* of the acceleration:

$$a_{max} = \omega^2 A.$$

You will often be asked to solve for these quantities, which is going to be an easy task if you know A and ω.

Energy

The potential energy stored in the spring is:

$$U(t) = \frac{1}{2}kx(t)^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi).$$

The kinetic energy of the mass is:

$$K(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi).$$

The total energy is:

$$\begin{aligned}
E_{total} &= U(t) + K(t) \\
&= \frac{1}{2}kA^2\cos^2(\omega t) + \frac{1}{2}m\omega^2 A^2\sin^2(\omega t) \\
&= \frac{1}{2}kA^2 \underbrace{\left[\cos^2(\omega t) + \sin^2(\omega t)\right]}_{=1} = \frac{1}{2}kA^2 = U_{max} \\
&= \frac{1}{2}mv_{max}^2 = K_{max}.
\end{aligned}$$

1.10 Minireference

I hope this short excerpt from the `MATH and PHYSICS Minireference` has given you some inspiration for compact teaching. No blah blah. Straight to the point.

If you liked this tutorial you can check out the other ones on `http://minireference.com` and order the printed book which has not only formulas but also compact explanations: `http://minireference.com/order_book/`.