Math40002 Analysis 1 — Term 2 Unseen 5

In this sheet we give an equivalent definition of integrability to that which you learned in the lectures. In the lectures you learned the definition of *Darboux integrals and integrability.*

Definition 1.

(a) A *tagged partition* (P, τ) of an interval $[a, b]$ is a partition P together with a finite sequence of numbers $\tau = (t_0, \ldots, t_{n-1})$ subject to the conditions that for each i: $t_i \in [x_i, x_{i+1}]$. In other words, it is a partition together with a distinguished point of every sub-interval.

(b) Given a function $f : [a, b] \to \mathbb{R}$ and a tagged partition (P, τ) as above, we define the Riemann sum of f with respect to (P, τ) to be $R(f, P, \tau) := \sum_{i=0}^{n-1} f(t_i) \cdot \Delta x_i$.

Definition 2.

(a) A function $f : [a, b] \to \mathbb{R}$ is *Riemann* integrable* with integral L if for every $\epsilon > 0$, there is some partition P such that for every tagged partition (S, σ) such that S is a refinement of P: $|R(f, S, \sigma) - L| < \epsilon$.

(b) A function $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* with integral L if for every $\epsilon > 0$, there is some δ such that for every tagged partition (P, τ) with $mesh(P) < \delta$: $|R(f, P, \tau) - L| < \epsilon$.

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that if f is Riemann integrable on $[a, b]$ with integral L, then it is Riemann* integrable on $[a, b]$ with integral L.

2. Let (P, τ) be a tagged partition of $[a, b]$. Prove that $L(f, P) \leq R(f, P, \tau) \leq U(f, P)$.

3. Let P be a partition of $[a, b]$ and let $\epsilon > 0$. Prove that there are tags τ_1, τ_2 of P such that $R(f, P, \tau_1) < L(f, P) + \epsilon$ and $U(f, P) - \epsilon < R(f, P, \tau_2)$.

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that the following are equivalent:

   (i) f is integrable on $[a, b]$ and $\int_a^b f(x)\, dx = L$.

   (ii) f is Riemann* integrable on $[a, b]$ with integral L.

Next week, we'll have a guided exercise proving that Darboux integrability implies Riemann integrability, thus concluding that Darboux, Riemann* and Riemann integrals are all equivalent and equal.