Coursework 2

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Problem 3.

(a) As suggested by the hint, let's break the problem into three parts:
Part 1
Let's show by induction that $n(n+1)$ is divisible by $2!$.
Basis step: $n_0 = 0$. Then $n_0(n_0+1) = 0$ which is divisible by $2!$.
Induction step: Assume that for an arbitrary n, $n(n+1)$ is divisible by $2!$.
We want to show that $(n+1)(n+2)$ is also divisible by $2!$.

$$(n+1)(n+2) = n(n+1) + 2(n+1).$$

According to the induction hypothesis, the first term is divisible by 2. The second term is also divisible by 2, therefore $(n+1)(n+2)$ is divisible by $2 = 2!$. Hence by induction $n(n+1)$ is divisible by $2!$ for all $n \in \mathbb{N}$.

Part 2
Let's show that $n(n+1)(n+2)$ is divisible by $3!$. Since we already know that $2! \mid n(n+1)$, it follows that $2! \mid n(n+1)(n+2)$. So all we need to show is that $n(n+1)(n+2)$ is divisible by 3 as well.
Basis step: $n_0 = 0$. Then $n_0(n_0+1)(n_0+2) = 0$ which is divisible by 3.
Induction step: Assume that for an arbitrary n, $n(n+1)(n+2)$ is divisible by 3.
We want to show that $(n+1)(n+2)(n+3)$ is also divisible by 3.

$$(n+1)(n+2)(n+3) = n(n+1)(n+2) + 3(n+1)(n+2).$$

According to the induction hypothesis, the first term is divisible by 3. The second term is also divisible by 3, hence $(n+1)(n+2)(n+3)$ is divisible by 3 and therefore by induction $3 \mid n(n+1)(n+2)$. So for all $n \in \mathbb{N}$, $3! \mid n(n+1)(n+2)$.

Part 3
Similarly to part 2, in order to show that $4! \mid n(n+1)(n+2)(n+3)$, we need

to prove that $n(n+1)(n+2)(n+3)$ is divisible by 4 (since we already know that it is divisible by 3! according to part 2).

Basis step: $n_0 = 0$. Then $n_0(n_0+1)(n_0+2)(n_0+3) = 0$ which is divisible by 4.

Induction step: Assume that for an arbitrary n, $n(n+1)(n+2)(n+3)$ is divisible by 4.

We want to show that $(n+1)(n+2)(n+3)(n+4)$ is also divisible by 4.

$$(n+1)(n+2)(n+3)(n+4) = n(n+1)(n+2)(n+3) + 4(n+1)(n+2)(n+3).$$

According to the induction hypothesis, the first term is divisible by 4. The second term is also divisible by 4, hence $(n+1)(n+2)(n+3)(n+4)$ is divisible by 4 and therefore by induction $4 \mid n(n+1)(n+2)(n+3)$ So for all $n \in \mathbb{N}$, $4! \mid n(n+1)(n+2)(n+3)$.

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(b) Let's denote $f(n,r) = n(n+1)(n+2)...(n+r-1)$ for $n, r \in \mathbb{N}, n \geq 1, r \geq 1$ (if $n = 0$, then $f(0,r) = 0$ so it is divisible by $r!$ for all r. We want to show that $r! \mid f(n,r)$. To do so we will use strong induction on the sum $n + r \leq k$ for some $k \in \mathbb{N} \setminus \{0,1\}$.

Basis step: $k_0 = 2$, then $n = 1, r = 1$, so $f(n,r) = f(1,1) = 1$ which is clearly divided by 1!.

Induction step: Suppose $r! \mid f(n,r)$ for all n, r such that $k_0 \leq n + r \leq k$. We want to show that $r'! \mid f(n', r')$ for all n', r' such that $n' + r' = k + 1$.

First we will show that $f(a,b) = (b)f(a,b-1) + f(a-1,b)$. We have that $f(a,b) = a(a+1)(a+2)...(a+b-1) = (a-1)a(a+1)(a+2)...(a+b-2) + b \cdot a(a+1)(a+2)...(a+b-2) = f(a-1,b) + rf(a,b-1)$.

Therefore,

$$f(n', r') = (r')f(n', r'-1) + f(n'-1, r').$$

According to the induction hypothesis, $f(n', r'-1)$ is divisible by $(r'-1)!$ (because $n' + r' - 1 = k$ which is $\leq k$), so the first term is divisible by $r'!$. The second terms is also divisible by $r'!$ because of the induction hypothesis, hence $r'! \mid f(n', r')$. By induction $r! \mid f(n,r)$ for all $n, r \geq 1$ such that $n + r \geq k_0$.

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