# Math40002 Analysis 1 Term 2 Unseen 3

**Definition 1.** A function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ is *Lipschitz continuous* if there exists some $C \in \mathbb{R}$ such that $\forall x, y \in A : |f(x) - f(y)| \leq C|x - y|$.

For each of the following items, determine whether it is true or false and prove your answer. The items are totally independent from one another, so there is no need to solve them in any particular order.

1. If $f$ is Lipschitz continuous, then it is uniformly continuous.

   **True. It is clear from the definition of Lipschitz continuity that $C > 0$. Let $\epsilon > 0$ and let $\delta := \epsilon/C$. Let $x, y \in A$ such that $|x - y| < \delta$. Then**

   $$|f(x) - f(y)| \leq C|x - y| < C \cdot \epsilon/C = \epsilon.$$

2. If $f$ is uniformly continuous, then it is Lipschitz continuous.

   **False. Consider $f(x) = \sqrt{x}$ on $[0, 1]$. This is a continuous function on a compact set, therefore uniformly continuous. If $f$ would be Lipschitz continuous, then there would be some $C > 0$ such that $\forall x, y \in [0, 1]$ : $|f(x) - f(y)| \leq C|x - y|$. Let $y := \frac{min(1,(1/C)^2)}{2}$. Then $y \in [0, 1]$ and $yC < \sqrt{y}$. So**

   $$|f(y) - f(0)| = |\sqrt{y} - \sqrt{0}| = \sqrt{y} > yC = C|y - 0|.$$

3. If $f$ is continuous and bounded on the interval $(a, b)$ (meaning there exist $M, L \in \mathbb{R}$ such that $\forall x \in (a, b) : M \leq f(x) \leq L$), then $f$ is uniformly continuous on $(a, b)$.

   **False. Consider $f(x) = \sin\left(\frac{1}{x}\right)$. Let $\epsilon = 1/2$ and assume that there is some $\delta > 0$ such that for all $x, y \in (0, 1)$, if $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$. Let $k \in \mathbb{N}$ such that $k > \frac{1}{2\pi\delta}$. Then**

   $$\left|\frac{1}{2\pi k} - \frac{1}{2\pi k + \pi/2}\right| = \frac{1}{2\pi k} - \frac{1}{2\pi k + \pi/2} < \frac{1}{2\pi k} < \delta.$$

   **But**

   $$\left|f\left(\frac{1}{2\pi k}\right) - f\left(\frac{1}{2\pi k + \pi/2}\right)\right| = |\sin(2\pi k) - \sin(2\pi k + \pi/2)| = |0 - 1| = 1 > \epsilon.$$

4. If $f$ is bounded on $\mathbb{R}$ and uniformly continuous on every interval $[a, b]$ where $a, b \in \mathbb{R}$, then $f$ is uniformly continuous on $\mathbb{R}$.

**False. Consider $f(x) = \sin(x^2)$. Let $\epsilon = 1/2$ and assume that there is some $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Let $k \in \mathbb{N}$ such that $k > \frac{\pi}{16\delta^2}$. Let $x := \sqrt{2\pi k}, y := \sqrt{2\pi k + \pi/2}$. Then**

$$|x-y| = \frac{|(x-y)(x+y)|}{|x+y|} = \frac{|x^2-y^2|}{|x+y|} = \frac{\pi/2}{x+y} < \frac{\pi/2}{2\sqrt{2\pi k}}$$

$$\left|\frac{1}{2\pi k} - \frac{1}{2\pi k + \pi/2}\right| = \frac{1}{2\pi k} - \frac{1}{2\pi k + \pi/2} < \frac{1}{2\pi k} < \delta.$$

**But**

$$|f(x) - f(y)| = |\sin(2\pi k) - \sin(2\pi k + \pi/2)| = |0-1| = 1 > \epsilon.$$

5. If $f$ is bounded, continuous and monotonic on $(0,1)$, then $f$ is uniformly continuous on $(0,1)$.

   **True. Assume $f$ is monotonically increasing (for $f$ decreasing a symmetric argument works). Let $\hat{f} : [0,1] \to \mathbb{R}$ be defined as $\hat{f}(x) = f(x)$ for $x \in (0,1)$, $\hat{f}(0) = \inf\{\, f(x) | x \in (0,1) \,\}$, and $\hat{f}(1) = \sup\{\, f(x) | x \in (0,1) \,\}$. $\hat{f}$ is continuous on $(0,1)$ by definition and continuity of $f$. Since $\lim_{x\to 0} f(x) = \hat{f}(0)$ and $\lim_{x\to 1} f(x) = \hat{f}(1)$ (check that this is true), we have continuity at $0$ and $1$ as well. So $\hat{f}$ is continuous on $[0,1]$, therefore uniformly continuous on $[0,1]$, in particular, uniformly continuous on $(0,1)$. But $\hat{f} = f$ on $(0,1)$.**

6. If $f$ is uniformly continuous on $(0,1)$, then f is bounded on $(0,1)$.

   **True. By definition of uniform continuity, letting $\epsilon > 0$ there is some $\delta > 0$ such that for all $x, y \in (0,1)$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Let $N \in \mathbb{N}$ be the minimal natural number such that $(N+1) \cdot \delta \geq 1$. Then**

$$(0,1) = (0,\delta] \cup (\delta, 2\delta] \cup \cdots \cup ((N-1)\cdot\delta, N\cdot\delta] \cup (N\cdot\delta, 1).$$

   **For each interval in the union on the RHS, pick some $x$ in that interval, and let $S$ be the set of $x$'s picked. Since there are finitely many intervals, $S$ is finite. Now each $y \in (0,1)$ belongs to some interval in the RHS, therefore, there is some $x \in S$, such that $f(x) - \epsilon \leq f(y) \leq f(x) + \epsilon$. Let $L := \max(S) + \epsilon$ and $M := \min(S) - \epsilon$, we get that $\{\, f(y) | y \in (0,1) \,\}$ is bounded above and below by $L$ and $M$, respectively.**