Problem Sheet 4

Math40002, Analysis 1

1. You are driving down a road whose speed limit is 60 miles per hour. A police officer sees your car at 12pm, and another officer 35 miles away sees your car at 12:30pm. Assuming they've attended their analysis lectures, how can they prove that you were speeding?

2. Prove using l'Hôpital's rule that $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r$. (Hint: take logs first.)

3. Use the mean value theorem to prove the following inequalities.

   (a) $|\sin(x) - \sin(y)| \le |x-y|$ for all $x, y \in \mathbb{R}$

   (b) $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$

4. Let H_n denote the harmonic sum $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

   (a) Using the mean value theorem, prove that $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

   (b) Prove that $H_n - 1 < \log(n) < H_{n-1}$ for all $n \ge 2$, where $H_k = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}$, and deduce that $\log(n+1) < H_n < \log(n) + 1$.

   (c) Prove that the sequence $\big(H_n - \log(n)\big)$ is decreasing, and that $\lim_{n\to\infty}(H_n - \log(n))$ exists. (This limit is called the *Euler–Mascheroni constant* $\gamma \approx 0.577\ldots$.)

5. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose there is a constant $C < 1$ such that $|f'(x)| \le C$ for all $x \in \mathbb{R}$. We will prove that f has exactly one fixed point, meaning there is a unique $y \in \mathbb{R}$ such that $f(y) = y$. Pick some $x_0 \in \mathbb{R}$ and let

$$x_{n+1} = f(x_n) \text{ for all } n \ge 0.$$

   (a) Prove that $|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$ for all n.

   (b) Prove that the sequence $\big(x_n\big)$ converges, and that if its limit is y then $f(y) = y$.

   (c) Prove that f cannot have two different fixed points.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that there is some $M > 0$ such that $|f'(x)| \le M$ for all $x \in \mathbb{R}$.

   (a) Prove that f is *Lipschitz*, meaning that there is some constant $C > 0$ such that $|f(x) - f(y)| \le C|x-y|$ for all $x, y \in \mathbb{R}$.

   (b) Prove that Lipschitz functions are uniformly continuous, and conclude that f is uniformly continuous.

7. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. We will prove that $f'(x)$ has the intermediate value property even though it may not be continuous. Throughout this problem, we will suppose that $f'(a) < f'(b)$ and fix some t such that $f'(a) < t < f'(b)$.

   (a) Let $g(x) = f(x) - tx$. Prove that there is some $c \in (a, b)$ such that $g(c) < g(a)$. (Hint: what is $g'(a)$?) Similarly, prove that there is some $d \in (a, b)$ such that $g(d) < g(b)$. In other words, $g(x)$ is not minimized at $x = a$ or at $x = b$.

   (b) Show that there is some $y \in (a, b)$ such that $g'(y) = 0$, and deduce that $f'(y) = t$.