Math40002 Analysis 1 — Problem Sheet 1

1. What is the biggest element of the set $\{x \in \mathbb{R} \colon x < 1\}$? Give a careful proof.

   It does not exist. Suppose it did, call it $m < 1$. Let $n = (m+1)/2$. Then $m = (m+m)/2 < (m+1)/2 < (1+1)/2 = 1$ shows that $m < n < 1$, so n is a larger element of the set: a contradiction.

2. Prove that for every positive integer $n \neq 3$, the number $\sqrt{n} - \sqrt{3}$ is irrational.

   Suppose $\sqrt{n} - \sqrt{3} = r$ is rational.
   Write as $\sqrt{n} = r + \sqrt{3}$ and square to give $n = r^2 + 2r\sqrt{3} + 3$.

   So either $r = 0$ (impossible; $n \neq 3$) or $\sqrt{3} = \frac{n-r^2-3}{2r}$. But this is rational, a contradiction.

3. * Show that any positive *eventually periodic* decimal expansion is rational, and in fact can be written as the fraction

$$p \Big/ 99\ldots9900\ldots00 \quad (m \text{ 9s and } n \text{ 0s})$$

   for some integers $p, m, n \geq 0$.
   Deduce that any integer divides some number of the form $99\ldots9900\ldots00$.

   Let the decimal expansion be $x = a_0.a_1a_2\ldots a_n(\overline{b_1\ldots b_m})$, where $\overline{\cdot}$ denotes recurring periodically. Then

$$\begin{aligned} x &= \frac{a_0a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n}(10^{-m} + 10^{-2m} + 10^{-3m} + \ldots) \\ &= \frac{a_0a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n}\frac{1}{10^m - 1} \\ &= \frac{(10^m-1)a_0a_1\ldots a_n + b_1\ldots b_m}{(10^m-1)10^n}, \end{aligned}$$

   which is of the form claimed, with $p = (10^m - 1)a_0a_1\ldots a_n + b_1\ldots b_m$.

   As proved in lectures, $x = 1/q$ has periodic decimal expansion since it is rational. Therefore we get $1/q = p/99\ldots9900\ldots00$ for some intreger p, and thus $99\ldots9900\ldots00/q = p$ as required.

4. Irrational Kevin tries to show $\sqrt{12} - \sqrt{3}$ is rational, by the following argument.

$$\begin{aligned} & \sqrt{12} - \sqrt{3} = p/q, \quad p, q \in \mathbb{N}, \\ \Rightarrow\ & 12 - 2\sqrt{12}\sqrt{3} + 3 = p^2/q^2, \\ \Rightarrow\ & 15 - 2\sqrt{36} = p^2/q^2. \end{aligned}$$

   Since $\sqrt{36} = 6$ is indeed rational, this looks good to him. Can you help him by pointing out three ways in which he's gone wrong? Be kind to him!

   1. Firstly, $\sqrt{12} - \sqrt{3} = \sqrt{3}$ is *not* rational.

   2. But if he is trying to show that $\sqrt{12} - \sqrt{3}$ is rational then he needs to end up with something implying $\sqrt{12} - \sqrt{3} = p/q$; it's no use to have $\sqrt{12} - \sqrt{3} = p/q$ implying something else. He's assumed the result he's trying to prove.

   3. However, his argument *is* the start of a good contradiction to be obtained from assuming that $\sqrt{12} - \sqrt{3} \in \mathbb{Q}$. He just needs to carry on from the last line to get $3q^2 = p^2$, therefore p is divisible by 3, therefore q is divisible by 3, etc...the usual contradiction one gets from assuming that $\sqrt{3} \in \mathbb{Q}$.

5. Suppose the sets S_n, $n = 1, 2, 3, \ldots$ are all disjoint and countable. Show that $S = \bigcup_{n=1}^{\infty} S_n$ is also countable. (Hint: recall the diagonal argument used in lectures.)

List the elements of the set $S_n = \{s_1^n, s_2^n, \ldots\}$. Then list the elements of $S = \bigcup_{n=1}^{\infty} S_n$ in an array with $s_1^1, s_2^1, \ldots$ on the first row, $s_1^2, s_2^2, \ldots$ on the second row, $s_1^n, s_2^n, \ldots$ on the nth row etc.

Then draw the diagonals, where the nth diagonal is the list $s_n^1, s_{n-1}^2, \ldots, s_{n-i+1}^i, \ldots, s_1^n$ as in lectures.

Now put all these finite diagonals end to end in a long linear list, as in lectures, throwing out any repeated elements. The result is a list of all the elements of S, which is therefore countable.

6. Suppose that S and T are countable. Show that $S \times T$ is countable. Hence show that $\bigcup_{n=1}^{\infty} S^n$ is countable, where $S^n := S \times \ldots \times S$ (n times).

$S = \{s_1, s_2, \ldots\}$. $T = \{t_1, t_2, \ldots\}$. So write down the elements of $S \times T$ in an array with (s_i, t_j) in the ith row and jth column. Then apply the usual diagonal argument to list these elements.

So setting $T = S$, we see that S^2 is countable. Setting $T = S^2$ we see that S^3 is countable. Inductively then, S^n is countable. Therefore by Q5, $\bigcup_{n=1}^{\infty} S^n$ is countable.

7. † Show the set of polynomials $p(x)$ with integer coefficients is countable. (Hint: use Q6.)

A real number is called *algebraic* if it is a root of a polynomial with integer coefficients. Show that rational numbers n/m and nth roots $\sqrt[n]{m}$ are algebraic. Show that the set of algebraic real numbers is countable.

A real number is called *transcendental* if it is not algebraic. (Examples include π and e, but this is hard to prove.) Prove that transcendental numbers exist, and that in fact there are uncountably many of them.

The set of polynomials of degree n with integer coefficients is a subset of $\mathbb{Z}^{n+1}$ (each polynomial is equivalent to a list of $n+1$ integers, by taking the $n+1$ coefficients; we only get a subset because the first integer should be nonzero). Therefore the set of all polynomials with integer coefficients is a subset of $\bigcup_{n=1}^{\infty} \mathbb{Z}^n$, which is countable by Q6. And we showed in lectures that an infinite subset of a countable set is countable.

n/m is the root of $mx - n = 0$ while $\sqrt[n]{m}$ is the root of $x^n - m = 0$, so these are both algebraic.

Each polynomial has a finite number of roots. So in the list of polynomials, replace each polynomial by its finite list of roots to get a list of all algebraic numbers.

If the set of transcendental numbers were empty, finite or countable, then their union with the algebraic numbers would also be countable. Therefore $\mathbb{R}$ would be countable, but it is not. Therefore there are uncountably many transcendental numbers.

8. Let $S^1 = \{s_1^1, s_2^1, s_3^1, \ldots\}$, $S^2 = \{s_1^2, s_2^2, s_3^2, \ldots\}$, $\ldots$, $S^n = \{s_1^n, s_2^n, s_3^n, \ldots\}$, $\ldots$ be subsets of $\mathbb{N}$. Here the elements are ordered so that $s_i^n < s_{i+1}^n$ for all i and n.

Define t_n recursively to be strictly larger than s_n^n and t_{n-1} (e.g. set $t_n = \max\{s_n^n, t_{n-1}\} + 1$), or $t_{n-1} + 1$ if $\nexists\, s_n^n$ (i.e. if S^n has $< n$ elements).

Show that $T = \{t_1, t_2, \ldots\} \subseteq \mathbb{N}$ is not equal to any S^i. Conclude that the set of subsets of $\mathbb{N}$ is *not* countable.

Since we chose t_n to be larger than t_{n-1} then $T = \{t_1, t_2, t_3, \ldots\}$ is listed just like the S^is, with the elements in ascending order. That is, t_n is the nth smallest element of T.

By construction t_n is *not* equal to the nth smallest element s_n^n of S_n. (It was constructed to be larger.) Therefore $T \neq S_n$. But this is true for all n. Therefore T is not in the list $S_1, S_2, S_3, \ldots$.

Therefore given any list of subsets of $\mathbb{N}$, there is a subset $T \subseteq \mathbb{N}$ not in that list. Therefore the set of subsets of $\mathbb{N}$ is not countable.