Math40002 Analysis 1 — Unseen 5

1. Show that $n^{1/n} \to \infty$ as $n \to \infty$.[1]

2. Let $PL(a_n)$ be the set of all limits of convergent subsequences of (a_n), i.e.,

   $PL(a_n) = \{ L \in \mathbb{R} \mid \text{there is some subsequence}(a_{n_k}) \text{ such that } a_{n_k} \to L \text{ as } k \to \infty \}$.

   Elements of $PL(a_n)$ are also called partial limits of (a_n).

   (a) For each one of the following items, give an example, without proof, of a sequence (a_n) such that $PL(a_n) = S$.

      i. $S = \{ 1, \ldots, m \}$.

      ii. $S = \mathbb{N}$.

   (b) Is there a sequence (a_n) such that $PL(a_n) = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$? You are not required to justify your answer, just come up with an answer – yes or no. You will prove the correct answer in a further question.

3. Let (a_n) be a sequence, $L \in \mathbb{R}$. Prove that $L \in PL(a_n)$ if and only if for every $\epsilon > 0$, the set $\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \}$ is infinite.

4. Prove that if (a_n) is a sequence and there is a sequence L_n of partial limits of $PL(a_n)$ such that $L_n \to L$, then L is also a partial limit of (a_n).

5. In this question we give yet another definition of lim sup:
   Let (a_n) be a sequence. Show that

   $$\lim_{m\to\infty} \left(\sup_{n \geq m} a_n \right) = \sup(PL(a_n))$$

   in the sense that if one exists, so does the other and they are equal.

[1] Hint: Problem Sheet 5, Question 1