

6 Topics: Expectation, independence of random variables, probability generating functions

6.1 Prerequisites: Lecture 14

Exercise 6- 1: (Suggested for personal/peer tutorial) Indicator variables and their expectation: Recall that we defined the indicator of an event $A \in \mathcal{F}$ in Definition 7.3.2 as follows: For an event $A \in \mathcal{F}$, we denote by

$$\mathbb{I}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A, \end{cases}$$

the indicator variable of the event A .

- (a) For events $A, B \in \mathcal{F}$, show that
 - i. $(\mathbb{I}_A)^k = \mathbb{I}_A$ for any $k \in \mathbb{N}$,
 - ii. $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A$,
 - iii. $\mathbb{I}_{A \cap B} = \mathbb{I}_A \mathbb{I}_B$,
 - iv. $\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_A \mathbb{I}_B$.
- (b) Prove the fundamental bridge between probability and expectation, i.e. show that there is a one-to-one correspondence between events and indicator random variables and for any $A \in \mathcal{F}$ we have

$$P(A) = E(\mathbb{I}_A).$$

Exercise 6- 2: Prove Theorem 10.2.5: Consider a discrete/continuous random variable X with finite expectation.

- (a) If X is non-negative, then $E(X) \geq 0$.
- (b) If $a, b \in \mathbb{R}$, then $E(aX + b) = aE(X) + b$.

Exercise 6- 3: Prove Theorem 10.3.3: Let X be a discrete random variable with finite variance and consider deterministic constants $a, b \in \mathbb{R}$. Then

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

6.2 Prerequisites: Lecture 15

Exercise 6- 4: Consider a sequence of *Bernoulli* random variables X_1, \dots, X_n each with parameter θ resulting from independent binary trials, so that

$$P(X = 0) = 1 - \theta, \quad P(X = 1) = \theta.$$

Find the probability mass functions of the random variables

- (a) $Y = \text{Min} \{X_1, \dots, X_n\}$
- (b) $Z = \text{Max} \{X_1, \dots, X_n\}$

[Hint: find the ranges of Y and Z , and consider $P(Y = 1)$, $P(Z = 0)$.]

6.3 Prerequisites: Lecture 16

Exercise 6- 5: Suppose that $F_{X,Y}(x, y)$ is the joint distribution function of (X, Y) . Find an expression for $P(X \leq x, Y > y)$ in terms of $F_{X,Y}$.

Exercise 6- 6: Consider a probability space given by $\Omega = \{-1, 0, 1\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and P is defined by

$$P(-1) = P(0) = P(1) = \frac{1}{3}.$$

We define two discrete random variables by $X(\omega) = \omega$ and $Y(\omega) = |\omega|$.

- (a) Compute $P(X = 0, Y = 1)$, $P(X = 0)$ and $P(Y = 1)$. What can you conclude?
- (b) Compute $E(XY)$ and $E(X)E(Y)$. What can you conclude?

Exercise 6- 7: Show that discrete random variables X and Y on (Ω, \mathcal{F}, P) are independent if and only if

$$E(g(X)h(Y)) = E(g(X))E(h(Y)), \quad (6.1)$$

for all functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ for which the expectations on the right hand side exist.

Exercise 6- 8: Convolution theorem: Consider jointly discrete/continuous random variables X and Y and define their sum as $Z = X + Y$. Find the probability mass function/density function of Z (leaving your expression as a sum/integral). If you assume that X and Y are independent, can you simplify the p.m.f./p.d.f. of Z ?

Exercise 6- 9: Prove Theorem 11.6.2: For jointly discrete/continuous random variables X, Y with finite expectations, we have

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

Exercise 6- 10: Prove Theorem 11.6.6: Let X, Y denote two jointly discrete/continuous random variables with finite variances. Then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y).$$

6.4 Prerequisites: Lecture 17

Exercise 6- 11: Suppose that X_1, \dots, X_n are independent $\text{Ber}(p)$ random variables. Define $S_n = \sum_{i=1}^n X_i$. Use probability generating functions to show that $S_n \sim \text{Bin}(n, p)$.

Exercise 6- 12: Suppose that X_1, \dots, X_n are independent Poisson random variables – not necessarily with the same parameter, i.e. $X_i \sim \text{Poi}(\lambda_i)$. Define $S_n = \sum_{i=1}^n X_i$. Use probability generating functions to show that $S_n \sim \text{Poi}(\sum_{i=1}^n \lambda_i)$.