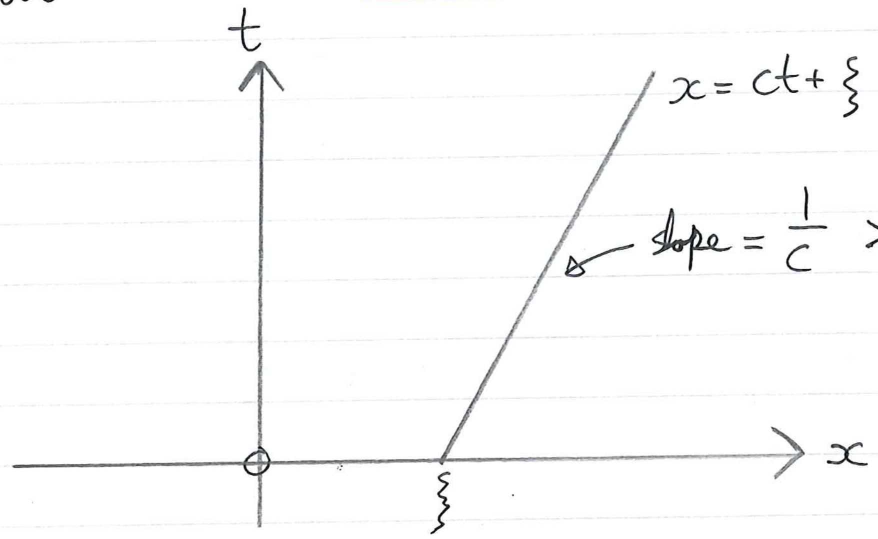
M1M2: Unseen 8: Assorted topics

1). $$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = F(x).$$

(a). Chain rule: $$\frac{d}{dt}(u(x,t)) = \frac{\partial u}{\partial t}\underbrace{\frac{dt}{dt}}_{=1} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$
$$= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}.$$

(b). $$\frac{dx}{dt} = c \Rightarrow x(t) = ct + \xi.$$



(c). Hence: $u(x,t) = u(\xi,0) = F(\xi)$, from initial condition

Re-arranging the characteristic curve: $\xi = x - ct$

$$\Rightarrow u(x,t) = F(\xi) = F(x-ct).$$

The solution corresponds to transporting the initial profile $F(x)$ unaltered (preserves the shape of u) along the characteristics with speed $\frac{dx}{dt} = c$.

Thus the solution after a later time t_1 is just a copy of the initial data $F(x)$, but displaced to the right a distance ct_1.

2).

(a). $f(x,y) = 9x^4 + 12x^2y^2 + 4y^4$

$$\frac{\partial f}{\partial x} = 36x^3 + 24xy^2 = 12x(3x^2 + 2y^2)$$

$$\frac{\partial f}{\partial y} = 24x^2y + 16y^3 = 8y(3x^2 + 2y^2)$$

Now $(3x^2 + 2y^2) \geqslant 0 \ \forall x, y$ unless $\underline{x = y = 0}$.

Indeed setting $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow$ This is only stationary point.
$\underline{(x,y) = (0,0)}$

$$\frac{\partial^2 f}{\partial x^2} = 108x^2 + 24y^2 = 0 \text{ at } (0,0).$$

$$\frac{\partial^2 f}{\partial y^2} = 24x^2 + 48y^2 = 0 \text{ at } (0,0).$$

$$\frac{\partial^2 f}{\partial x \partial y} = 48xy = 0 \text{ at } (0,0).$$

$\Rightarrow$ Eigenvalues of Hessian are zero

$\Rightarrow$ Test inconclusive!

Observe that: $f(x,y) = 9x^4 + 12x^2y^2 + 4y^4$

$$= (3x^2 + 2y^2)^2 \geqslant 0 \ \forall x, y.$$

Hence the point at $(0,0)$ must be a $\underline{\text{minimum}}$.

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(b). $f(x,y) = 2x^4 - 3x^2y + y^2$

$$\frac{\partial f}{\partial x} = 8x^3 - 6xy = 2x(4x^2 - 3y)$$

$$\frac{\partial f}{\partial y} = -3x^2 + 2y$$

Setting these equal to zero yields: $y = \frac{3}{2}x^2 \Rightarrow x = 0 \Rightarrow y = 0.$
or $y = \frac{4}{3}x^2$
or $x = 0$

So the only stationary point is at $(x,y) = (0,0)$

$$\frac{\partial^2 f}{\partial x^2} = 24x^2 - 6y = 0 \text{ at } (0,0)$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -6x = 0 \text{ at } (0,0)$$

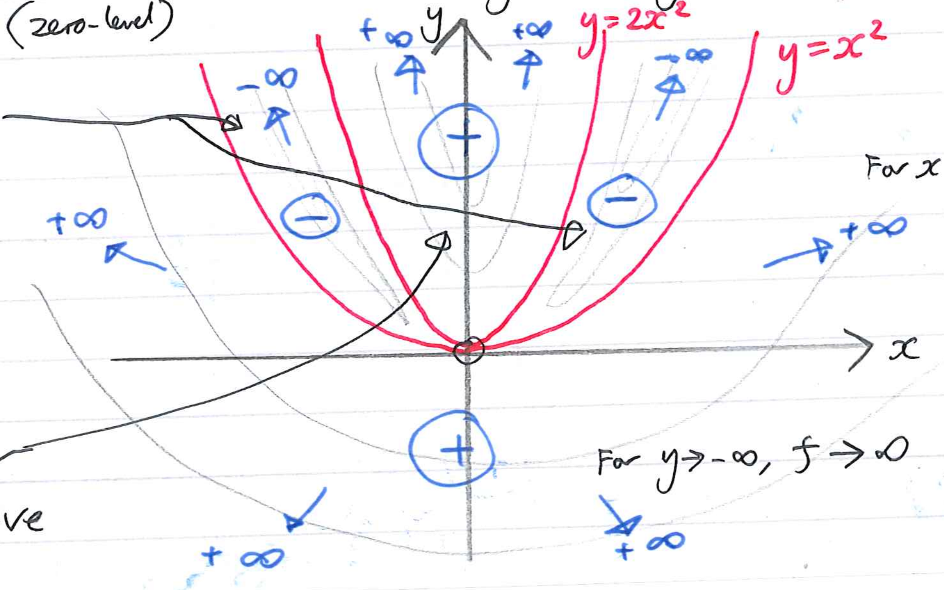
$\Rightarrow$ Eigenvalues of Hessian are zero $\Rightarrow$ Test inconclusive!

Contour sketch: $f(x,y) = 2x^4 - 3x^2y + y^2 = 0$

$$\Rightarrow (x^2 - y)(2x^2 - y) = 0$$

$\Rightarrow$ Level contours (zero-level) are: $y = x^2$, $y = 2x^2$

In these regions $f \to -\infty$ as one bracket +ve other −ve.



For $x \to \pm\infty$ $f \to \infty$

$\Rightarrow$ Saddle point

Here $f \to +\infty$ as both brackets +ve

For $y \to -\infty$, $f \to \infty$

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(c). $f(x,y) = x^3 - 3xy^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 = 3(x+y)(x-y)$$

$$\frac{\partial f}{\partial y} = -6xy$$

Setting these equal to zero gives: $\underline{\underline{(x,y) = (0,0)}}$ as only stationary point.

$$\frac{\partial^2 f}{\partial x^2} = 6x = 0 \text{ at } (0,0)$$

$$\frac{\partial^2 f}{\partial y^2} = -6x = 0 \text{ at } (0,0)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -6y = 0 \text{ at } (0,0)$$

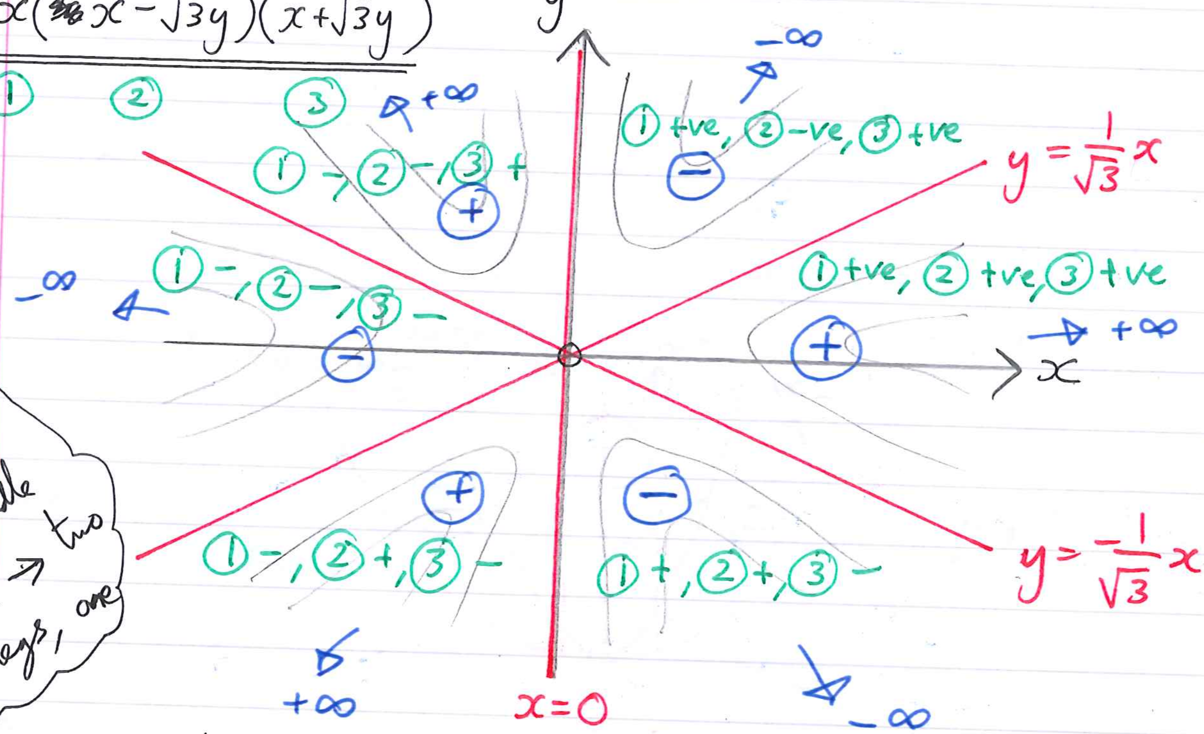
$\Rightarrow$ Eigenvalues of Hessian are zero $\Rightarrow$ Test inconclusive!

<u>Contour sketch:</u> $f(x,y) = x^3 - 3xy^2 = 0$

$$\Rightarrow x(x^2 - 3y^2) = 0$$

$$\Rightarrow \underline{\underline{x = 0}}, \quad \underline{\underline{y = \pm\frac{1}{\sqrt{3}}x}} \quad \text{(zero-level contours)}$$

$\underline{\underline{f(x,y) = x(\cancel{3}x - \sqrt{3}y)(x+\sqrt{3}y)}}$



Why called? Monkey saddle point → two depressions for legs, one for tail!

$\Rightarrow$ <u>Degenerate saddle point</u>: it has $\underline{\underline{3}}$ depressions and inclinations surrounding it, as can be seen in the contour sketch

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3.

(a) $\frac{dy}{dx} = -\frac{y}{2x\log x}$

$$\Rightarrow \underbrace{y}_{F(x,y)}\,dx + \underbrace{2x\log x}_{G(x,y)}\,dy = 0$$

$\frac{\partial F}{\partial y} = 1$, $\frac{\partial G}{\partial x} = 2\log x + 2$ <u>Not exact!</u>

We multiply through by the integrating factor $\lambda(x,y) = \frac{y}{x}$:

(why? Intuitively we need to "remove" the $\log x$ term from $\frac{\partial G}{\partial x}$ to have any chance of matching $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial x}$ together. So a factor of $\frac{1}{x}$ will do this! Then we fit the rest to make them equal!)

$$\Rightarrow \underbrace{\frac{y^2}{x}}_{H(x,y)}\,dx + \underbrace{2y\log x}_{L(x,y)}\,dy = 0$$

$\frac{\partial H}{\partial y} = 2\frac{y}{x}$, $\frac{\partial L}{\partial x} = 2\frac{y}{x}$ ✓ <u>exact</u>

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = H = \frac{y^2}{x} \Rightarrow u = y^2\log x + f(y) \\ \frac{\partial u}{\partial y} = L = 2y\log x \Rightarrow u = y^2\log x + g(x) \end{cases}$$

Clearly to match these we take $f(y), g(x)$ constant.

$$\Rightarrow u(x,y) = y^2\log x + \text{Const.}$$

$$\Rightarrow \boxed{y^2\log x = \text{constant.}}$$ is solution to ode.

$\boxed{6}$ [Method using separation of variables] — seeking an integrating factor $\lambda(x)$ yields similar calculation.

$$\frac{dy}{dx} = -\frac{y}{2x\log x}$$

$$\Rightarrow \int -\frac{1}{y}\,dy = \underbrace{\int \frac{1}{2x\log x}\,dx}$$

Trick: $\frac{d}{dx}\left[\log(\log x)\right] = \frac{1}{x} \times \frac{1}{\log x}$

$$\Rightarrow -\log y = \frac{1}{2}\log(\log x) + \text{const.}$$

$$\Rightarrow \frac{1}{y} = A(\log x)^{\frac{1}{2}}, \quad A \text{ constant.}$$

Care! This is $(\log x)^{\frac{1}{2}}$ NOT! $\log x^{\frac{1}{2}}$

$$\Rightarrow y(\log x)^{\frac{1}{2}} = \text{const.}$$

$$\Rightarrow \underline{\underline{y^2 \log x = \text{const.}}}$$

$\cancel{(\log x)^2 = 2\log x}$ $\quad$ $\cancel{(\log x)^{\frac{1}{2}} = \frac{1}{2}\log x}$ $\quad$ $\cancel{\log x^{\frac{1}{2}}}$ $\quad$ $\cancel{(\log x)^{\frac{1}{2}} = \frac{1}{2}\log x}$

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(b). $\lambda(t)$, $t(x,y)$.

(i). Multiply (8) by $\lambda(t)$:

$$\lambda(t)F(x,y)\,dx + \lambda(t)G(x,y)\,dy = 0$$

For this to be exact, we require:

$$\frac{\partial}{\partial y}\left(\lambda(t)F(x,y)\right) = \frac{\partial}{\partial x}\left(\lambda(t)G(x,y)\right)$$

$$\Rightarrow \frac{\partial \lambda}{\partial y}F + \lambda\frac{\partial F}{\partial y} = \frac{\partial \lambda}{\partial x}G + \lambda\frac{\partial G}{\partial x}$$

$$\Rightarrow \frac{d\lambda}{dt}\frac{\partial t}{\partial y}F + \lambda\frac{\partial F}{\partial y} = \frac{d\lambda}{dt}\frac{\partial t}{\partial x}G + \lambda\frac{\partial G}{\partial x}$$

$$\Rightarrow \frac{d\lambda}{dt}\left[G\frac{\partial t}{\partial x} - F\frac{\partial t}{\partial y}\right] = \lambda\left(\frac{\partial F}{\partial y} - \frac{\partial G}{\partial x}\right)$$

$$\Rightarrow \frac{d\lambda}{dt} = \left(\frac{F_y - G_x}{Gt_x - Ft_y}\right)\lambda$$

$$\Rightarrow \int \frac{1}{\lambda}\,d\lambda = \int\left(\frac{F_y - G_x}{Gt_x - Ft_y}\right)dt$$

$$\Rightarrow \lambda = \exp\left\{\int\left(\frac{F_y - G_x}{Gt_x - Ft_y}\right)dt\right\}$$

$\boxed{8}$

(ii). $t = x + y \quad \Rightarrow \quad \dfrac{\partial t}{\partial x} = \dfrac{\partial t}{\partial y} = 1$

$$\frac{dy}{dx} = -\frac{(7x^3 + 3x^2y + 4y)}{(4x^3 + x + 5y)}$$

$$\Rightarrow \underbrace{(7x^3 + 3x^2y + 4y)}_{F(x,y)}\,dx + \underbrace{(4x^3 + x + 5y)}_{G(x,y)}\,dy = 0$$

$$\frac{\partial F}{\partial y} = 3x^2 + 4 \qquad \frac{\partial G}{\partial x} = 12x^2 + 1$$

Sub everything into formula for λ:

$$\lambda = \exp\left\{\int \frac{(3x^2+4) - (12x^2+1)}{(4x^3+x+5y) - (7x^3+3x^2y+4y)}\,dt\right\}$$

$$= \exp\left\{\int \frac{(-9x^2+3)}{-3x^3 - 3x^2y + x + y}\,dt\right\}$$

$$= (x+y)(1-3x^2)$$

$$= \exp\left\{\int \frac{3\cancel{(1-3x^2)}}{(x+y)\cancel{(1-3x^2)}}\,dt\right\}$$

$$= \exp\left\{\int \frac{3}{t}\,dt\right\}$$

$$= \exp\{3\log t\}$$

$$= t^3$$

$$= (x+y)^3$$

$\boxed{9}$

Multiply through by $\lambda(x,y)$:

$$\underbrace{(x+y)^3(7x^3+3x^2y+4y)}_{H(x,y)}\,dx + \underbrace{(x+y)^3(4x^3+x+5y)}_{L(x,y)}\,dy = 0$$

check that $\dfrac{\partial H}{\partial y} = \dfrac{\partial G}{\partial x}$ ✓ exact.

Then let:

$$H = \frac{\partial u}{\partial x} = (x+y)^3(7x^3+3x^2y+4y)$$

$$\Rightarrow u = x^7 + 4x^6y + x^4y + 6x^5y^2 + 4x^3y^2 + 4x^4y^3 + 6x^2y^3 + x^3y^4 + 4xy^4 + f(y)$$

$$L = \frac{\partial u}{\partial y} = (x+y)^3(4x^3+x+5y)$$

$$\Rightarrow u = 4x^6y + x^4y + 4x^3y^2 + 6x^5y^2 + 6x^2y^3 + 4x^4y^3 + 4xy^4 + x^3y^4 + y^5 + g(x)$$

Comparing these gives:

$$u = x^7 + 4x^6y + x^4y + 4x^3y^2 + 6x^5y^2 + 6x^2y^3 + 4x^4y^3 + 4xy^4 + x^3y^4 + y^5 + \text{Constant}$$

$$\Rightarrow \boxed{(x+y)^4(x^3+y) = \text{Constant.}}$$ is solution to the ode.