Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 4

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1.* **Recap** — The following second order differential equation describes the time evolution of the linear elongation $x(t)$ of a damped harmonic oscillator

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0,$$

where k and ω are positive constants representing the damping of the medium and the intrinsic frequency of the system, respectively. Rewrite this equation as a system of two coupled linear first order ODEs and find the solution in terms of the eigenvalues and eigenvectors of the system.

2. Consider systems of two linear ODEs with constant coefficients given by:

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and } A \text{ is a } 2 \times 2 \text{ matrix.}$$

Find the general solution of the following systems:

(a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$

(c) $A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$

(d) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

(e) $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

(f) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

3. Find the solution for the following inhomogeneous system of ODEs with $x(0) = y(0) = 0$:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix}.$$

4. * Find the general solution for the following system of ODEs in terms of its eigenvalues and eigenvectors:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Characterise the asymptotic behavior of the system.