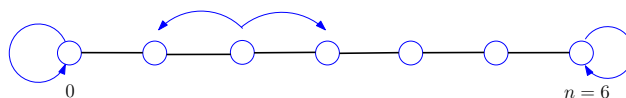


Random walks on graphs

1. Consider a simple random walk on a set of $n + 1$ nodes in a line as shown in the figure for the case $n = 6$. If the random walker is at either of the end nodes it stays there. If it is at any of the interior nodes it goes to either of its neighbours with equal probability. Nodes are labelled $0, 1, \dots, (n - 1), n$.



- (a) Find the probability that, starting at node j , the random walker reaches node n before it reaches node 0.
 - (b) Use the Maximum Principle for harmonic functions to prove that the random walker will eventually reach either node 0 or node n .
2. The figure shows the Circle Line in London. Assume that this is a train system with no connections to any other lines. To visit Imperial College one must alight at South Kensington; to visit UCL one must alight at Euston Square. During a train ride an undecided visitor to London gets off at every station, flips a (fair) coin, and goes clockwise if he gets “heads” and anticlockwise if he gets “tails”.



- (a) If the traveller starts his random journey at Tower Hill, is he more or less likely to reach Imperial College before UCL than if he starts at Paddington? Explain your answer.
- (b) The visitor is at Imperial College and restarts his random journey at South Kensington. What is the probability that he reaches UCL before getting back to Imperial?

3. The figure shows the Tube map in a fictional London.



An undecided tourist is visiting this fictional London. Her hotel is at Warren Street. Because she is undecided when she reaches any Tube station she stops and decides at random which train to take next (including going back the way she just came).

- Suppose the tourist starts her journey at Oxford Circus. Find the probability that she reaches her hotel before a Piccadilly Line station (dark blue).
- What if she starts at Tottenham Court Road instead?
- Find approximate answers to parts (a) and (b) by using the method of relaxation. Show your sequence of diagrams. Stop your iteration after 3 iterations of all the unknown potentials. What is the maximum relative error of your approximations?
- The next day, this undecided tourist has been sightseeing and is now at Tottenham Court Road station. She cannot decide whether to go back to her hotel, or to go to the theatre in Leicester Square. If her random journey takes her to Warren Street before Leicester Square she will go back to her hotel; if she reaches Leicester Square first, she will go to the theatre. Find a mathematical formula, in terms of suitably defined matrices and vectors, for the probability that she goes to the theatre.

- * (Optional) Calculate the numerical value of this probability?
- (e) Suppose the tourist ends up seeing a show at Leicester Square. She leaves Leicester square on another random journey. Find a formula, in terms of suitably defined matrices and vectors, for the probability that she gets back to her hotel before returning to Leicester Square.
- * (Optional) Calculate the numerical value of this probability?