Problem Sheet 2

Math40002, Analysis 1

1. Give an example of a compact set $S \subset R$ and a continuous function $f : S \to \mathbb{R}$ which does *not* satisfy the intermediate value theorem: in other words, there are points $a < b$ in S and some x between $f(a)$ and $f(b)$ such that $f(c) \neq x$ for all $c \in S$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(c) = \{x \in \mathbb{R} \mid f(x) = c\}$ is closed.

3. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the preimage
$$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$$
is open.

4. Prove that a set $S \subset \mathbb{R}$ is compact if and only if every sequence $\big(x_n\big) \subset S$ has a convergent subsequence whose limit is in S.

5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $S \subset \mathbb{R}$ is compact, then the image $f(S)$ is also compact.

6. Give a family of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ for all $n \in \mathbb{N}$ such that the f_n converge pointwise to a function $f : \mathbb{R} \to \mathbb{R}$ with infinitely many discontinuities.

7. Recall that $\cos(x) = \mathrm{Re}(E(ix))$ and $\sin(x) = \mathrm{Im}(E(ix))$ have power series
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$
   (a) Use the identity $E(ix)E(-ix) = E(0) = 1$ to prove that $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$.
   (b) Prove that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$. (Hint: reduce to the case $0 \leq x \leq 1$.)
   (c) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is uniformly continuous. (Hint: use the identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$.)