Math40002 Analysis 1 — Term 2 Unseen 1

<u>Note:</u> You may need to use the results of Problem Sheet 1 (of Term 2) to solve some of the questions in this unseen sheet

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that if there is some $x_0 \in \mathbb{R}$ be such that f is continuous on x_0, then f is continuous on all $\mathbb{R}$.

   Let $x_1 \in \mathbb{R}$ and let $\epsilon > 0$. By continuity on x_0, there is some δ_0 such that for all $x \in \mathbb{R}$, if $x_0 - \delta_0 < x < x_0 + \delta_0$ then $|f(x) - f(x_0)| < \epsilon$. If $x_1 - \delta_0 < x < x_1 + \delta_0$, then $x_0 - \delta_0 < x - x_1 + x_0 < x_0 + \delta_0$. Therefore,

   $$|f(x) - f(x_1)| = |f(x - x_1 + x_1) - f(x_1)| = |f(x - x_1) + f(x_1) - f(x_1)|$$

   $$= |f(x - x_1) + f(x_0) - f(x_0)| = |f(x - x_1 + x_0) - f(x_0)| < \epsilon.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be such that f is continuous on x_0. Prove that there is some $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

   By Question 1, we may assume f is continuous on all $\mathbb{R}$.

   By Question 2 in Problem Sheet 1, it suffices to find some $a \in \mathbb{R}$ such that $f(q) = aq$ for all $q \in \mathbb{Q}$.

   Let $a := f(1)$.

   - $f(0) = f(0+0) = f(0) + f(0)$. **So** $f(0) = 0$.
   - **Let $n \in \mathbb{N}, x \in \mathbb{R}$. Then $f(nx) = f(\sum_{i=1}^{n} x) = \sum_{i=1}^{n} f(x) = n \cdot f(x)$.**
   - **Let $m \in \mathbb{N}, x \in \mathbb{R}$. Then $f(x) = f\left(m \cdot \frac{x}{m}\right) = m \cdot f\left(\frac{x}{m}\right)$. So $f\left(\frac{x}{m}\right) = \frac{f(x)}{m}$.**
   - **Let $x \in \mathbb{R}$. $f(x) + f(-x) = f(x - x) = f(0) = 0$. So $f(-x) = -f(x)$.**
   - **In conclusion, $f\left(\pm\frac{n}{m}\right) = \pm\frac{n}{m} \cdot f(1) = a \cdot \pm\frac{n}{m}$.**

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and let $I \subseteq \mathbb{R}$ be an interval such that f is monotone on I. Prove that there is some $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

   In Problem Sheet 1 of Term 2, you proved that the set of discontinuity points of a monotone function is at most countably infinite. Since I is uncountable, this implies there is some $x_0 \in I$ such that f is continuous on x_0. By Question 2, we're done.

   <u>Note:</u> The equation $f(x+y) = f(x) + f(y)$ is called Cauchy's functional equation. The existence of non-linear functions satisfying this equation relies on the Axiom of Choice – Recall that in Linear Algebra and Groups Unseen 6 from Term 1,

you proved using Zorn's Lemma that every vector space has a basis (in particular $\mathbb{R}$ over $\mathbb{Q}$). In fact, this is equivalent to the Axiom of Choice.

If you'd like, you can try proving by yourself (assuming Zorn's Lemma) that there are non-linear functions satisfying Cauchy's functional equation. However, please don't do so during this problem session, as this is more of a bonus question in Linear Algebra.

4. (a) Show that every non-constant periodic continuous function has a minimal period. Explicitly, show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous such that there is some $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, and f is not a constant function, then the set

$$\mathcal{T} := \{\, T \in (0, +\infty) | \forall x \in \mathbb{R} : f(x+T) = f(x) \,\}$$

has a minimum.

(b) Is the statement in Item 4a still true without the assumption of continuity? Prove your answer.

(a) By definition, $\mathcal{T}$ is bounded below, hence has an infimum, call it L. So it suffices to show that $L \in \mathcal{T}$. Indeed, let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. We'll show that $|f(x_0 + L) - f(x_0)| < \epsilon$. By continuity of f, there is some $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|(x_0 + L) - x| < \delta$ then $|f(x_0 - L) - f(x)| < \epsilon$. Since $L = \inf \mathcal{T}$, there is some $T \in \mathcal{T}$ such that $0 < T - L < \delta$. Then $|(x_0+L) - (x_0+T)| < \delta$, so $|f(x_0+L) - f(x_0+T)| < \epsilon$ By the triangle inequality,

$$|f(x_0 + L) - f(x_0)| \le |f(x_0 + L) - f(x_0 + T)| + |f(x_0 + T) - f(x_0)| < \epsilon.$$

It remains to show that $L \neq 0$, but if $L = 0$, then for every $\epsilon > 0$ there is some $T \in \mathcal{T}$ such that $0 \le T < \epsilon$. By definition of $\mathcal{T}$, it must be that $0 < T$.

Now, it is left as an exercise to construct, given $x_1, x_2 \in \mathbb{R}$, a sequence $(a_n)_{n=1}^{\infty}$ such that $a_1 = x_1$ and $f(a_{n+1}) = f(a_n)$ such that $a_n \to x_2$. It follows from continuity that $f(x_1) = f(a_n) \to f(x_2)$. So $f(x_1) = f(x_2)$.

(b) False. E.g.,

$$f(x) = \begin{cases} 1 & \textbf{if } x \in \mathbb{Q} \\ 0 & \textbf{if } x \notin \mathbb{Q}. \end{cases}$$

Clearly f is periodic, as every rational number is a period for f.