Question 1

Suppose you are tracking the value of two companies listed on the London Stock Exchange over the course of one week. Rather than record the actual values of the share prices, you record the increase or decrease in each share price value at the daily close to the nearest pound. You record the following table:

	Monday	Tuesday	Wednesday	Thursday	Friday
Company X	5	4	8	6	2
Company Y	3	2	7	4	-1

Table 1: Daily change in share price (£)

Without using a calculator:

(a) Compute the sample covariance between the two sequences to two decimal places.

(b) Compute the sample correlation between the two sequences. You may leave your answer as a fraction.

(c) Compute the sample correlation between the two sequences to two decimal places.

(d) Are the two sequences significantly correlated?

Question 2

Suppose that for every batch of lightbulbs produced in a factory an unknown proportion θ are defective. Suppose that a random sample of n lightbulbs is taken from a batch, and for $i = 1, 2, \ldots, n$ let the random variable $X_i = 1$ if the ith lightbulb is defective and let $X_i = 0$ otherwise. We assume that the random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are independent, and we observe $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

(a) What distribution can we use to model each X_i?

(b) Given your answer in (a), write down the probability mass/density function $f(x_i|\theta)$.
Hint: the p.m.f./p.d.f. should be a polynomial in θ.

(c) Given your answer in (b), compute down the likelihood function $L(\theta|\mathbf{x})$.

(d) Compute the maximum likelihood estimate for θ, given the observations $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

(e) Write down the maximum likelihood estimator for θ, given the random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$.

(Please turn over for Questions 3 and 4)

Question 3

Young's inequality states that if a and b are non-negative real numbers, and p and q are any positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab.$$

Use Young's inequality to prove Hölder's Inequality: Let X and Y be random variables and let p and q be two positive numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$|\mathrm{E}\,(XY)| \leq \left(\mathrm{E}\,(|X|^p)\right)^{1/p} \left(\mathrm{E}\,(|Y|^q)\right)^{1/q}.$$

(a) If Z is a non-negative random variable, i.e. $Z \geq 0$, prove that $\mathrm{E}\,(Z) \geq 0$.

(b) Prove that $\mathrm{E}\,(|XY|) \geq 0$.

(c) Prove that $|\mathrm{E}\,(XY)| \leq \mathrm{E}\,(|XY|)$.

(d) Use Young's inequality to prove $\mathrm{E}\,(|XY|) \leq \left(\mathrm{E}\,(|X|^p)\right)^{1/p} \left(\mathrm{E}\,(|Y|^q)\right)^{1/q}$.

(e) Conclude that Hölder's Inequality is true.

(f) Use Hölder's Inequality to prove the Cauchy-Schwarz Inequality: $|\mathrm{E}\,(XY)| \leq (\mathrm{E}(|X|^2))^{1/2}(\mathrm{E}(|Y|^2))^{1/2}$.

(g) Use the Cauchy-Schwarz inequality to prove Theorem 4.1.6: $|\mathrm{Cov}\,(X,Y)| \leq \sigma_X\sigma_Y$, where σ_X^2 and σ_Y^2 are the variances of X and Y, respectively.

Question 4 (Knowledge of partial derivatives required)

Suppose the random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are assumed to be independent and identically distributed as $\mathrm{N}\left(\mu, \sigma^2\right)$, where μ and σ^2 are unknown. Suppose further that $\mathbf{X}$ is observed as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

(a) Compute the likelihood $L(\mu, \sigma^2|\mathbf{x})$.

(b) Compute the log-likelihood $\log L(\mu, \sigma^2|\mathbf{x})$.

(c) Compute the partial derivative $\frac{\partial}{\partial\mu} \log L(\mu, \sigma^2|\mathbf{x})$.

(d) Set the partial derivative in (c) equal to 0 and solve for μ. Show that this value of μ maximises $L(\mu, \sigma^2|\mathbf{x})$ globally for fixed σ^2.

(e) Compute the partial derivative $\frac{\partial}{\partial\sigma^2} \log L(\mu, \sigma^2|\mathbf{x})$. **Hint**: it may help to set $z = \sigma^2$ and then compute the partial derivative with respect to z.

(f) Set the partial derivative in (e) equal to 0 and solve for σ^2. Show that this value of σ^2 is a (local) maximum for $L(\mu, \sigma^2|\mathbf{x})$, for fixed μ.

(g) Show that the value of σ^2 found in (f) maximises the likelihood $L(\mu, \sigma^2|\mathbf{x})$ globally for fixed μ.

(h) Write down the maximum likelihood estimators for μ and σ^2 given the random variables $\mathbf{X}$.