1(a) The function $h(z)$ is well-defined except when $z = 0$. Now, inside the conductor outside this point, we have

$$\frac{\partial h(z)}{\partial x} = h'(z)\frac{\partial z}{\partial x} = h'(z)$$

and

$$\frac{\partial^2 h(z)}{\partial x^2} = h''(z)\frac{\partial z}{\partial x} = h''(z). \tag{1}$$

Similarly,

$$\frac{\partial h(z)}{\partial y} = h'(z)\frac{\partial z}{\partial y} = \mathrm{i}h'(z)$$

and

$$\frac{\partial^2 h(z)}{\partial y^2} = \mathrm{i}h''(z)\frac{\partial z}{\partial y} = -h''(z). \tag{2}$$

It is clear on addition of (1) and (2) that

$$\nabla^2 h(z) = 0$$

and hence that

$$\nabla^2 h(z) + \nabla^2 \overline{h(z)} = \nabla^2 \left[h(z) + \overline{h(z)} \right] = 0.$$

Thus ϕ is harmonic at all points in D exterior to $(0,0)$.

1(b) Now

$$\phi = \mathrm{Re}[h(z)] = \mathrm{Re}\left[-\frac{m}{2\pi}\log z \right] = \mathrm{Re}\left[-\frac{m}{2\pi}\left(\log|z| + \mathrm{i}\arg[z]\right) \right] = -\frac{m}{2\pi}\log|z| = 0.$$

1(c) We know that

$$j_x - \mathrm{i}j_y = -\hat{c}h'(z) = -h'(z) = \frac{m}{2\pi z}.$$

1(d) We normal vector at a point on $|z| = 1$, i.e. at $z = e^{\mathrm{i}\theta}$ is

$$\mathbf{n} = (\cos\theta, \sin\theta)$$

In complex form this is $e^{\mathrm{i}\theta}$, which equals z. To work out

$$\mathbf{j.n}$$

we therefore need to compute

$$\mathrm{Re}[(j_x - \mathrm{i}j_y)z] = \frac{m}{2\pi}.$$

where we used part (c). To find the total current through the boundary we need to integrate this with respect to arclength around the boundary. But the arclength element is just $ds = rd\theta = d\theta$ since $|z| = r = 1$. Hence the total current through the boundary is

$$\int_0^{2\pi} \mathbf{j}.\mathbf{n} ds = \int_0^{2\pi} \frac{m}{2\pi} d\theta = m.$$

1(e) Since the only point inside the conductor where "Kirchhoff's current law" is not satisfied is the point $z = 0$, where $h(z)$ has a singularity, and since there is a net current m out of the conductor then this point singularity can be viewed as a "source" of current m. It is for this reason that it is called a *current source singularity*.

2(a) Concerning example 2 as considered in lectures, it is clear that all we need is *a rescaling of* $h(z)$ *by* $\Phi_0 > 0$ since

$$\begin{aligned}
\phi = \mathrm{Re}[h(z)] &= \mathrm{Re}\left[\frac{\Phi_0}{\log \rho} \log z\right] \\
&= \mathrm{Re}\left[\frac{\Phi_0}{\log \rho} (\log |z| + \mathrm{i}\arg[z])\right] \\
&= \frac{\Phi_0}{\log \rho} \log |z| \\
&= \begin{cases} 0, & \text{on } |z| = 1, \\ \Phi_0, & \text{on } |z| = \rho. \end{cases}
\end{aligned}$$

2(b) If $\rho \to 0$ then $\log \rho \to -\infty$ hence, if Φ_0 is fixed,

$$h(z) \to 0$$

hence there is no current in the conductor (all points of the conductor are at the same voltage, so there are no voltage differences present).

2(c) If $\rho \to 0$ then $\log \rho \to -\infty$ but if $\Phi_0 \to +\infty$ also, such that

$$\frac{\Phi_0}{\log \rho} \to -\frac{m}{2\pi} \tag{3}$$

where the right hand side is a (negative) constant then

$$h(z) \to -\frac{m}{2\pi} \log z.$$

This is exactly the $h(z)$ of question 1, which we called a *current source singularity*.

2(d) The total current into this conductor was computed in the lecture notes in the special case $\Phi_0 = 1$ so we just need to rescale that answer by a general Φ_0, i.e.,

$$-\frac{2\pi\Phi_0}{\log \rho}.$$

In the limit as $\rho \to 0$ (so that $\log \rho \to -\infty$) with Φ_0 fixed the limit of this total current into the circuit is **zero**.

On the other hand, in the limit of part (c) when $\rho \to 0$ with $\Phi_0 \to +\infty$ then we can use (3) to find that the limit of the total current into the conductor is

$$-2\pi \times \left(-\frac{m}{2\pi}\right) = m.$$

Note: This exercise is supposed to show you how the *current source singularity* inside a circular conductor considered in question 1 can be thought of as a limit of an annular conductor with an electrified inner boundary. As the radius of the inner boundary gets vanishingly small the imposed voltage on it much become infinitely large in order that it continues to inject a current m into the conductor.

3(a) By the same arguments as 1(a), it can be argued that ϕ satisfies $\nabla^2\phi = 0$ except at $z = 0$.

3(b) We know that

$$j_x - \mathrm{i}j_y = -\hat{c}h'(z) = -h'(z) = \frac{1}{L} + \frac{m}{2\pi z}.$$

3(c) On side a, where $z = -L/2 + \mathrm{i}y$, the unit normal *outward* from the conductor is $(-1, 0)$ so to compute the normal component of the current density we need

$$\begin{aligned} -j_x = -\mathrm{Re}\left[\frac{1}{L} + \frac{m}{2\pi z}\right]_{z=-L/2+\mathrm{i}y} &= -\mathrm{Re}\left[\frac{1}{L} + \frac{m\bar{z}}{2\pi z\bar{z}}\right]_{z=-L/2+\mathrm{i}y} \\ &= -\frac{1}{L} + \frac{mL}{4\pi(L^2/4 + y^2)}. \end{aligned}$$

To find the total current, we need to integrate this quantity with respect to the arclength along this boundary. On this boundary the arclength element $ds = dy$ hence the total current is

$$\int_{-L/2}^{L/2} \left[-\frac{1}{L} + \frac{mL}{4\pi(L^2/4 + y^2)}\right] dy = -1 + \int_{-L/2}^{L/2} \frac{mL}{4\pi(L^2/4 + y^2)} dy$$

If we change variables $y = Lu/2$ in the integral this becomes

$$-1 + \frac{m}{2\pi}\int_{-1}^{1} \frac{du}{1 + u^2} = -1 + \frac{m}{2\pi}\left[\tan^{-1} u\right]_{-1}^{1} = -1 + \frac{m}{2\pi}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = -1 + \frac{m}{4}.$$

On side b, where $z = x - \mathrm{i}L/2$, the unit normal *outward* from the conductor is $(0, -1)$ so to compute the normal component of the current density we need

$$\begin{aligned} -j_y = \mathrm{Im}\left[\frac{1}{L} + \frac{m}{2\pi z}\right]_{z=x-\mathrm{i}L/2} &= \mathrm{Im}\left[\frac{m\bar{z}}{2\pi z\bar{z}}\right]_{z=x-\mathrm{i}L/2} \\ &= \frac{mL}{4\pi(x^2 + L^2/4)}. \end{aligned}$$

To find the total current, we need to integrate this quantity with respect to the arclength along this boundary. On this boundary the arclength element $ds = dx$ hence the total current is

$$\int_{-L/2}^{L/2} \frac{mL}{4\pi} \frac{dx}{(x^2 + L^2/4)} = \frac{m}{4}$$

by the same change of variable as on side a.

On side c, where $z = L/2 + \mathrm{i}y$, the unit normal *outward* from the conductor is $(+1, 0)$ so to compute the normal component of the current density we need

$$\begin{aligned} j_x = \mathrm{Re}\left[\frac{1}{L} + \frac{m}{2\pi z}\right]_{z=L/2+\mathrm{i}y} &= \mathrm{Re}\left[\frac{1}{L} + \frac{m\bar{z}}{2\pi z\bar{z}}\right]_{z=L/2+\mathrm{i}y} \\ &= +\frac{1}{L} + \frac{mL}{4\pi(L^2/4 + y^2)}. \end{aligned}$$

On integration with respect to arclength $ds = dy$ on this side we get, in a similar way to the analysis on side a,

$$+1 + \frac{m}{4}.$$

On side d, where $z = x + \mathrm{i}L/2$, the unit normal *outward* from the conductor is $(0, +1)$ so to compute the normal component of the current density we need

$$\begin{aligned} j_y = -\mathrm{Im}\left[\frac{1}{L} + \frac{m}{2\pi z}\right]_{z=x+\mathrm{i}L/2} &= -\mathrm{Im}\left[\frac{m\bar{z}}{2\pi z\bar{z}}\right]_{z=x+\mathrm{i}L/2} \\ &= \frac{mL}{4\pi(x^2 + L^2/4)}. \end{aligned}$$

To find the total current, we need to integrate this quantity with respect to the arclength along this boundary. On this boundary the arclength element $ds = dx$ hence the total current is

$$\int_{-L/2}^{L/2} \frac{mL}{4\pi} \frac{dx}{(x^2 + L^2/4)} = \frac{m}{4}$$

by the same change of variable as on side b.

3(d) Notice that the contribution -1 *out* of side a corresponds to a uniform current *entering* the conductor, and the contribution $+1$ *out* of side c corresponds to the same uniform current exiting the conductor. Meanwhile, from question 1, we recognize the source singularity at the centre of the square. Since, by symmetry and the uniformity of the conductivity, the current forced into the conductor at its centre must exit all sides equally, hence the remaining $m/4$ exiting each side.

4(a). The unit normal to the top boundary of the strip, and pointing *out* of the strip, is $(0,1)$ so, to find the total current out of the strip through this boundary, we need to integrate j_y with respect to arclength $ds = dx$ over the boundary:

$$\int_{-\infty}^{\infty} j_y dx = \frac{m}{2\pi} \int_{-\infty}^{\infty} \mathrm{sech} x dx.$$

But this is

$$\frac{m}{2\pi} \int_{-\infty}^{\infty} \frac{2dx}{e^x + e^{-x}} = \frac{m}{2\pi} \int_{-\infty}^{\infty} \frac{2e^x dx}{e^{2x}+1}.$$

Now introduce the change of variable $u = e^x$ and this becomes

$$\frac{m}{2\pi} \int_{0}^{\infty} \frac{2du}{u^2+1} = \frac{m}{\pi} \left[\tan^{-1} u\right]_0^{\infty} = \frac{m}{\pi} \times \frac{\pi}{2} = \frac{m}{2}.$$

4(b). This result might also have been anticipated on the grounds of "symmetry" since the current source is located on the centerline of the strip, and the strip has uniform conductivity, to there is no reason why more current would flow out of the top boundary than out of the lower boundary. Therefore if the singularity is producing a current m then the current leaving the strip along each boundary must be $m/2$.

5(a) The function $h(z)$ is singular when the argument of the logarithm vanishes, or equals infinity, which occurs when

$$z^2 = a^2, \qquad \text{and} \qquad z^2 = 1/a^2.$$

That is, at the four points

$$z = \pm a, \qquad z = \pm 1/a.$$

Since $0 < a < 1$, only the two points $z = \pm a$ lie inside the conductor. So ϕ satisfies $\nabla^2 \phi = 0$ everywhere in the conductor except for $(\pm a, 0)$, by similar arguments to those given in the solution to part 1(a).

5(b) Let

$$R = \frac{z^2 - a^2}{z^2 a^2 - 1}.$$

It is useful to take the complex conjugate of this quantity when $|z| = 1$:

$$\overline{R} = \frac{\overline{z}^2 - a^2}{\overline{z}^2 a^2 - 1} = \frac{1/z^2 - a^2}{a^2/z^2 - 1} = \frac{1 - z^2 a^2}{a^2 - z^2} = \frac{1}{R}.$$

Notice that we have used the fact that, for $|z| = 1$,

$$|z|^2 = 1, \qquad \text{or} \qquad \overline{z} z = 1, \qquad \text{or} \qquad \overline{z} = 1/z. \tag{4}$$

We conclude that on $|z| = 1$

$$|R| = 1.$$

Therefore, on $|z| = 1$,

$$\phi = \mathrm{Re}[h(z)] = \mathrm{Re}\left[-\frac{m}{2\pi}\log R\right] = \mathrm{Re}\left[-\frac{m}{2\pi}\left(\log|R| + \mathrm{i}\arg[R]\right)\right] = -\frac{m}{2\pi}\log|R| = 0.$$

5(c) We know that the current density, with unit conductivity, is given by

$$j_x - \mathrm{i}j_y = -h'(z) = \frac{m}{2\pi}\left[\frac{2z}{z^2 - a^2} - \frac{2za^2}{z^2a^2 - 1}\right].$$

The "complex form" of the unit normal vector $\mathbf{n}$ at any point z on the unit circle is z (since $\mathbf{n} = (\cos\theta, \sin\theta) \mapsto \cos\theta + \mathrm{i}\sin\theta = e^{\mathrm{i}\theta} = z$). To compute the normal component of the current density vector $\mathbf{j} = (j_x, j_y)$ we need to compute

$$\mathbf{j}.\mathbf{n}.$$

However this dot product is given, in complex notation, by the quantity

$$\mathrm{Re}\left[(j_x - \mathrm{i}j_y)z\right]$$

or

$$\mathrm{Re}\left[\frac{m}{2\pi}\left[\frac{2z^2}{z^2 - a^2} - \frac{2z^2a^2}{z^2a^2 - 1}\right]\right].$$

This can be simplied to

$$\mathrm{Re}\left[\frac{m(a^4 - 1)}{\pi}\left[\frac{1}{(1 - a^2/z^2)(z^2a^2 - 1)}\right]\right].$$

On setting $z = e^{\mathrm{i}\theta}$ this becomes

$$\mathrm{Re}\left[\frac{m(a^4 - 1)}{\pi}\left[\frac{1}{2a^2\cos 2\theta - (1 + a^4)}\right]\right] = \frac{m(a^4 - 1)}{\pi}\left[\frac{1}{2a^2\cos 2\theta - (1 + a^4)}\right],$$

as required.

5(d) We need to integrate this with respect to arclength around the boundary. But the arclength along a portion of the circular boundary (with unit radius) is just $d\theta$. Hence the total current through the boundary in the outward normal direction is

$$\int_{-\pi}^{\pi} \frac{m(a^4 - 1)}{\pi}\left[\frac{1}{2a^2\cos 2\theta - (1 + a^4)}\right] d\theta = \frac{m(a^4 - 1)}{2\pi a^2} I(a), \tag{5}$$

where

$$I(a) \equiv \int_{-\pi}^{\pi} \frac{d\theta}{\cos 2\theta - A}, \qquad A = \frac{1+a^4}{2a^2}.$$

The rest of this solution is just an exercise in calculus. Note that since $0 < a < 1$, then $A \geq \sqrt{a^2 \cdot \frac{1}{a^2}} = 1$. To compute $I(a)$ note that, because the integrand is even,

$$I(a) = 2\int_0^{\pi} \frac{d\theta}{\cos 2\theta - A}.$$

Splitting this up,

$$I(a) = 2\left[\int_0^{\pi/2} \frac{d\theta}{\cos 2\theta - A} + \int_{\pi/2}^{\pi} \frac{d\theta}{\cos 2\theta - A}\right]$$

Now change variable $\theta = \pi - \phi$ in the second of these integrals:

$$I(a) = 2\left[\int_0^{\pi/2} \frac{d\theta}{\cos 2\theta - A} - \int_{\pi/2}^{0} \frac{d\phi}{\cos 2\phi - A}\right] = 4\int_0^{\pi/2} \frac{d\theta}{\cos 2\theta - A}.$$

Now introduce the "t-substitution" from calculus:

$$t = \tan\theta, \qquad \frac{dt}{1+t^2} = d\theta, \qquad \cos\theta = \frac{1-t^2}{1+t^2}.$$

Then,

$$I(a) = 4\int_0^{\pi/2} \frac{d\theta}{\cos 2\theta - A} = -4\int_0^{\infty} \frac{dt}{A - 1 + (1+A)t^2}.$$

Another change of variable

$$\sqrt{A+1}t = \sqrt{A-1}u$$

yields

$$I(a) = -4\sqrt{\frac{A-1}{A+1}}\int_0^{\infty} \frac{1}{A-1}\frac{du}{1+u^2} = -\frac{2\pi}{\sqrt{A^2-1}}.$$

Now

$$A^2 - 1 = \frac{(1-a^4)^2}{4a^4}. \tag{6}$$

Hence the total current throught the boundary is, from (5) and (6),

$$-\frac{m(a^4-1)}{2\pi a^2}\frac{2\pi}{\sqrt{A^2-1}} = -\frac{m(a^4-1)}{2\pi a^2} \times \frac{2a^2}{1-a^4} = 2m. \tag{7}$$

5(e) This result could have been antipated if we write

$$h(z) = \underbrace{-\frac{m}{2\pi}\log(z-a) - \frac{m}{2\pi}\log(z+a)}_{\text{two current sources, each of strength m}} \\ + \frac{m}{2\pi}\log(z-1/a) + \frac{m}{2\pi}\log(z+1/a) + \frac{m}{2\pi}\log a^2$$

where we see two "current density source singularities", as encountered in question 1, at $\pm a$ inside the conductor (in question 1, the single singularity was at the origin). Thus, since there are two current source singularities of strength m inside the conductor, we expect a total current of $2m$ to be exiting the conductor through its boundary.

Note: Next year, in your complex analysis course, you will learn about much better ways to compute the integrals in this question (using the so-called "residue theorem", which follows from "Cauchy's theorem"). Watch out for these important results! It will save you having to do all the calculus I just showed you...