Problem Sheet 8

Math40002, Analysis 1

1. Evaluate $\displaystyle\int_0^x \frac{1}{1+e^t}\,dt$. Does $\displaystyle\int_0^\infty \frac{1}{1+e^t}\,dt$ exist, and if so, what is it?

   Solution. We substitute $t = \log(u)$ and decompose into partial fractions to get

$$\int_0^x \frac{1}{1+e^t}\,dt = \int_1^{e^x} \frac{1}{1+u}\cdot\frac{1}{u}\,du = \int_1^{e^x}\left(\frac{1}{u} - \frac{1}{1+u}\right)du.$$

   This is equal to

$$\log(u) - \log(1+u)\Big|_{u=1}^{u=e^x} = x - \log(1+e^x) + \log(2).$$

   We can rewrite this as

$$\int_0^x \frac{1}{1+e^t}\,dt = \log(2) - \log\left(\frac{1+e^x}{e^x}\right) = \log(2) - \log(1+e^{-x}),$$

   and this converges as $x \to \infty$ to $\displaystyle\int_0^\infty \frac{1}{1+e^t}\,dt = \log(2)$.

2. The prime number theorem says that the number $\pi(n)$ of primes between 1 and n is approximately $\int_2^n \frac{1}{\log(x)}\,dx$.

   (a) Prove that this integral equals $\frac{n}{\log(n)} + \int_2^n \frac{1}{(\log x)^2}\,dx$, up to a constant which does not depend on n.

   (b) Prove that there is a constant $C > 0$ such that $\int_2^n \frac{1}{(\log x)^2}\,dx < \frac{Cn}{(\log n)^2}$ for all sufficiently large n, by splitting the integral up into one with domain $[2, \sqrt{n}]$ and one with domain $[\sqrt{n}, n]$ and estimating each one separately.

   Solution. (a) We integrate by parts, using $\frac{d}{dx}(\log x)^{-1} = -\frac{1}{x(\log x)^2}$:

$$\begin{aligned}\int_2^n \frac{1}{\log(x)}\left(\frac{d}{dx}x\right)dx &= \frac{x}{\log(x)}\Big|_{x=2}^{x=n} + \int_2^n x\cdot\frac{1}{x(\log x)^2}\,dx\\ &= \frac{n}{\log(n)} - \frac{2}{\log(2)} + \int_2^n \frac{1}{(\log x)^2}\,dx.\end{aligned}$$

(b) We split the domain $[2,n]$ into $[2,\sqrt{n}] \cup [\sqrt{n}, n]$ and write

$$\int_2^n \frac{1}{(\log x)^2}\,dx = \int_2^{\sqrt{n}} \frac{1}{(\log x)^2}\,dx + \int_{\sqrt{n}}^n \frac{1}{(\log x)^2}\,dx.$$

On $[2,\sqrt{n}]$ we have $\frac{1}{(\log x)^2} \leq \frac{1}{(\log 2)^2}$, and on $[\sqrt{n}, n]$ we have

$$\frac{1}{(\log x)^2} \leq \frac{1}{(\log(\sqrt{n}))^2} = \frac{4}{(\log n)^2},$$

so we combine these bounds to get

$$\begin{aligned}\int_2^n \frac{1}{(\log x)^2}\,dx &\leq \int_2^{\sqrt{n}} \frac{1}{(\log 2)^2}\,dx + \int_{\sqrt{n}}^n \frac{4}{(\log n)^2}\,dx \\ &= \frac{\sqrt{n}}{(\log 2)^2} + \frac{4(n-\sqrt{n})}{(\log n)^2}.\end{aligned}$$

We now have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log x)^2} = \lim_{x\to\infty} \frac{(\log x)^2}{x^{1/2}} = \lim_{y\to\infty} \frac{y^2}{e^{y/2}} = 0$$

by the substitution $y = \log(x)$, since the power series for e^x shows that $e^{y/2} \geq \frac{(y/2)^3}{3!} = \frac{y^3}{48}$ for all $y > 0$. It follows that for all large enough n we have $\frac{\sqrt{n}}{(\log 2)^2} < \frac{n}{(\log n)^2}$, and so $\int_2^n \frac{1}{(\log x)^2}\,dx < \frac{5n}{(\log n)^2}$ for $n \gg 0$.

3. Let $f : [0,\infty) \to [0,\infty)$ be uniformly continuous, and suppose that $\int_0^\infty f(x)\,dx$ exists.

   (a) For each $\epsilon > 0$, prove that there is a $\delta > 0$ such that for all $y > 0$, if $f(y) \geq \epsilon$ then
   $$\int_y^{y+\delta} f(t)\,dt \geq \frac{\epsilon\delta}{2}.$$

   (b) Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

   (c) Describe a continuous function $g : [0,\infty) \to [0,\infty)$ such that $\displaystyle\int_0^\infty g(x)\,dx$ exists but $\lim\limits_{x\to\infty} g(x)$ does not. Can you make g differentiable as well?

Solution. (a) Given $\epsilon > 0$, uniform continuity says that there is a $\delta > 0$ such that

$$|y-t| < \delta \;\Rightarrow\; |f(y) - f(t)| < \frac{\epsilon}{2},$$

so if $f(y) \geq \epsilon$ then $f(t) > f(y) - \frac{\epsilon}{2} \geq \frac{\epsilon}{2}$ for all $t \in [y, y+\delta)$. Then

$$\int_y^{y+\delta} f(t)\,dt \geq \int_y^{y+\delta} \frac{\epsilon}{2}\,dt = \frac{\epsilon\delta}{2}.$$

(b) If $\lim_{x\to\infty} f(x)$ is not zero, then there is some $\epsilon > 0$ and a sequence $x_n \to \infty$ such that $f(x_n) \geq \epsilon$ for all n. Since f is nonnegative, there is a $\delta > 0$ such that for any such x_n we have

$$\int_{x_n}^{\infty} f(t)\,dt \geq \int_{x_n}^{x_n+\delta} f(t)\,dt \geq \frac{\epsilon\delta}{2}$$

by part (a).

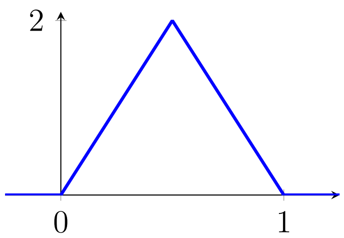
However, the convergence of $\int_0^\infty f(x)\,dx$ means that there is some $N > 0$ such that for all $x \geq N$, we have

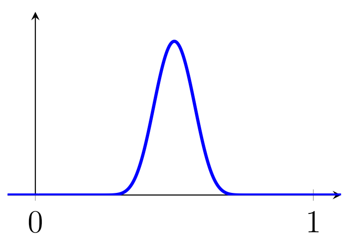
$$\left|\left(\lim_{b\to\infty}\int_0^b f(t)\,dt\right) - \int_0^x f(t)\,dt\right| < \frac{\epsilon\delta}{2},$$

hence $\int_x^\infty f(t)\,dt < \frac{\epsilon\delta}{2}$ for all $x \geq N$, and if we take x to be some $x_n \geq N$ then we have a contradiction.

(c) We take a continuous function $h : \mathbb{R} \to [0,\infty)$ satisfying $h(x) = 0$ for all $x \leq 0$ and all $x \geq 1$, $h(\frac{1}{2}) > 0$, and $\int_0^1 h(x)\,dx = 1$. Possible examples include

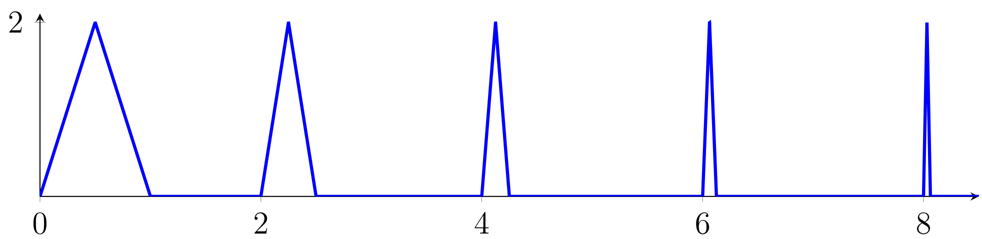
$$h_1(x) = \max(2 - |4x-2|, 0), \qquad h_2(x) = \begin{cases} ce^{-\frac{1}{x^2}-\frac{1}{(1-x)^2}}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$





where $c > 0$ is a constant chosen so that $\int_0^1 h_2(x)\,dx = 1$. Now we define

$$g(x) = \sum_{n=0}^{\infty} h(2^n(x-2n)),$$



and we find for any even integer $2n \geq 0$ that

$$\int_{2n}^{2n+2} g(x)\,dx = \int_{2n}^{2n+2} h(2^n(x-2n)) = \int_0^{2^{n+1}} h(y)\cdot\frac{1}{2^n}\,dy = \frac{1}{2^n}$$

by the substitution $x = \frac{y}{2^n} + 2n$. Since $g(x)$ is nonnegative, the integral $\int_0^t g(t)\,dt$ is increasing, and it is bounded above by

$$\sum_{n=0}^{\infty} \int_{2n}^{2n+2} g(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2,$$

so g is integrable, with $\int_0^\infty g(x)\,dx = 2$. We also know that g is continuous or differentiable iff h is. But $\lim\limits_{x\to\infty} g(x)$ does not exist, because we have

$$g\left(2n + \frac{1}{2^{n+1}}\right) = h\left(\frac{1}{2}\right) > 0$$

for all $n \geq 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly monotone increasing, with continuous first derivative on (a, b). Evaluate $\int_{f(a)}^{f(b)} f^{-1}(x)\,dx$ in terms of $\int_a^b f(x)\,dx$, and draw a picture to explain your answer.

Solution. We make the substitution $x = f(y)$ and write

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,dx = \int_a^b f^{-1}(f(y))f'(y)\,dy = \int_a^b y f'(y)\,dy.$$

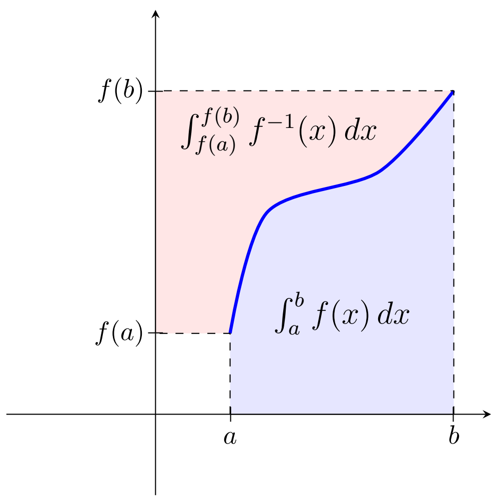
We can evaluate this last integral using integration by parts:

$$\int_a^b y f'(y)\,dy = y f(y)\big|_{y=a}^{y=b} - \int_a^b f(y)\,dy$$

and so

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx.$$

The integrals $\int_{f(a)}^{f(x)} f^{-1}(x)\,dx$ and $\int_a^b f(x)\,dx$ represent the red and blue shaded areas in the following diagram:



Their total area is that of the rectangle $0 \le x \le b$, $0 \le y \le f(b)$ minus that of the rectangle $0 \le x \le a$, $0 \le y \le f(a)$, so we should indeed expect that

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,dx + \int_a^b f(x)\,dx = bf(b) - af(a).$$

5. Use problem 4 to evaluate $\displaystyle\int_1^x \frac{\sqrt{t^2-1}}{t}\,dt$ for $x \ge 1$.

Solution. We note that $f(t) = \frac{\sqrt{t^2-1}}{t} = \sqrt{1-t^{-2}}$ is monotone increasing, and if $y = f(t)$ with $t \ge 1$ then

$$y^2 = 1 - \frac{1}{t^2} \;\Rightarrow\; f^{-1}(y) = t = \frac{1}{\sqrt{1-y^2}}.$$

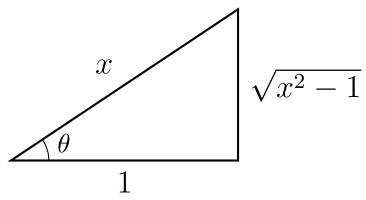
Thus we apply problem 4 with $a = 1$ (so $f(a) = 0$) and $b = x$ to get

$$\begin{aligned}\int_0^{f(x)} \frac{1}{\sqrt{1-t^2}}\,dt + \int_1^x f(t)\,dt &= xf(x) - 1f(1)\\ &= x\sqrt{1-\frac{1}{x^2}} = \sqrt{x^2-1}.\end{aligned}$$

We know from lecture that $\sin^{-1}(t)$ is an antiderivative of $\frac{1}{\sqrt{1-t^2}}$, so then

$$\begin{aligned}\int_1^x f(t)\,dt &= \sqrt{x^2-1} - \int_0^{f(x)} \frac{1}{\sqrt{1-t^2}}\,dt\\ &= \sqrt{x^2-1} - \sin^{-1}(f(x))\\ &= \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right).\end{aligned}$$

We can optionally simplify the last term by writing $\theta = \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$, where θ is the angle indicated in the right triangle below:



and observing that $\tan(\theta) = \sqrt{x^2-1}$; more precisely, we have

$$\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \frac{1}{x} \;\Rightarrow\; \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{x^2-1},$$

so then $\sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) = \tan^{-1}\left(\sqrt{x^2-1}\right)$. Thus $\int_1^x f(t)\,dt$ is equal to

$$\int_1^x \frac{\sqrt{t^2-1}}{t}\,dt = \sqrt{x^2-1} - \tan^{-1}(\sqrt{x^2-1}).$$

6. Let $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$.

   (a) Prove that this improper integral converges for all $t > 0$. (In how many ways is it improper?)

   (b) Compute $\Gamma(1)$.

   (c) Prove that $\Gamma(n+1) = n\Gamma(n)$ for all integers $n \geq 1$, and deduce that $\Gamma(n+1) = n!$ for all $n \geq 0$.

Solution. (a) This integral is improper in two ways: the upper limit is ∞, and when $t < 1$ the integrand $x^{t-1}e^{-x}$ is unbounded as $x \downarrow 0$. Thus we write

$$\Gamma(t) = \int_0^1 x^{t-1}e^{-x}\,dx + \int_1^\infty x^{t-1}e^{-x}\,dx$$

and ask for each of these integrals to exist individually. Since the integrand $x^{t-1}e^{-x}$ is positive, the two integrals

$$\int_a^1 x^{t-1}e^{-x}\,dx \qquad \text{and} \qquad \int_1^\infty x^{t-1}e^{-x}\,dx$$

both increase as $a \downarrow 0$ and $b \to \infty$ respectively, so it suffices to show that they are bounded above independently of $a \in (0,1]$ and $b \in [1,\infty)$.

For the first integral, we have $e^{-x} \leq 1$ for $0 \leq x \leq 1$, and so given $a > 0$ we have

$$\int_a^1 x^{t-1}e^{-x}\,dx \leq \int_a^1 x^{t-1}\,dx = \left.\frac{x^t}{t}\right|_{x=a}^{x=1} = \frac{1-a^t}{t}.$$

This is bounded above by $\frac{1}{t}$ since $t > 0$, so the improper integral exists.

For the second integral, we have $x^{t-1} \leq e^{x/2}$ for all sufficiently large x; in fact, it suffices to take $x \geq 2^t \cdot t!$, since then

$$e^{x/2} \geq \frac{(x/2)^t}{t!} = \frac{x^t}{2^t \cdot t!} \geq x^{t-1}.$$

So we write $N = 2^t \cdot t!$ for convenience and bound this integral by

$$\begin{aligned}
\int_1^b x^{t-1}e^{-x}\,dx &= \int_1^N x^{t-1}e^{-x}\,dx + \int_N^b x^{t-1}e^{-x}\,dx \\
&\leq \int_1^N x^{t-1}e^{-x}\,dx + \int_N^b e^{-x/2}\,dx \\
&= \int_1^N x^{t-1}e^{-x}\,dx + (-2e^{-x/2})\Big|_{x=N}^{x=b} \\
&= -2e^{-b/2} + \left(\int_1^N x^{t-1}e^{-x}\,dx + 2e^{-N/2}\right).
\end{aligned}$$

Thus $\int_1^b x^{t-1}e^{-x}\,dx$ is bounded above by the terms in parentheses, and so this improper integral exists as well.

(b) We have

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty} -e^{-x}\Big|_{x=0}^{x=b}$$
$$= \lim_{b\to\infty} (1 - e^{-b}) = 1.$$

(c) We integrate by parts: given $n \geq 1$ and $b > 0$, we have

$$\int_0^b x^n e^{-x}\,dx = \int_0^b x^n \frac{d}{dx}\left(-e^{-x}\right)\,dx$$
$$= -x^n e^{-x}\Big|_{x=0}^{x=b} - \int_0^b (nx^{n-1})(-e^{-x})\,dx$$
$$= -\frac{b^n}{e^b} + t\int_0^b x^{n-1}e^{-x}\,dx.$$

Taking limits as $b \to \infty$ and using the algebra of limits gives us $\Gamma(n+1) = n\Gamma(n)$, since $\lim_{b\to\infty} \frac{b^n}{e^b} = 0$.

Now the claim that $\Gamma(n+1) = n!$ follows by induction: we have already proved it when $n = 0$, meaning that $\Gamma(1) = 1 = 0!$, and if $\Gamma(k+1) = k!$ for some integer $k \geq 1$ then this identity proves that

$$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)\cdot k! = (k+1)!,$$

as desired.

Remark: In fact, we have $\Gamma(t+1) = t\Gamma(t)$ for all real $t > 0$ by the same argument, though if $t < 1$ then we have to be careful about taking limits on the right side because $x^{t-1}e^{-t}$ is unbounded as $x \downarrow 0$.

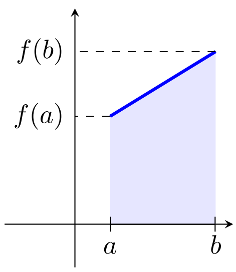
7. (*) Let $f : [a,b] \to \mathbb{R}$ be continuous, with $f''(x)$ continuous and bounded on (a,b).

(a) Use integration by parts twice to prove that

$$\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx = \int_a^b f(x)\,dx - (b-a)\left(\frac{f(a)+f(b)}{2}\right).$$

(b) If $|f''(x)| \leq M$ for all $x \in (a,b)$, prove that

$$\left|\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx\right| \leq \frac{M(b-a)^3}{12}.$$



In other words, $\int_a^b f(x)\,dx$ is the area of the trapezium shown at right, up to an error of at most $\frac{M(b-a)^3}{12}$. (Hint: check that $(x-a)(x-b) \leq 0$ on $[a,b]$, and compute that $\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$.)

(c) Apply this to $f(x) = \log(x)$ to show that

$$\int_1^n \log(x)\,dx = \sum_{k=1}^{n-1} \left(\frac{\log(k) + \log(k+1)}{2} + e_k \right),$$

where $|e_k| \le \frac{1}{12k^2}$ for all k.

(d) Evaluate both the integral and the sum from part (c) to show that there is some constant $C > 0$ such that

$$\left| \log(n!) - \log\left(\frac{n^{n+1/2}}{e^{n-1}} \right) \right| < C$$

for all n, or equivalently if $C_1 = e^{1-C}$ and $C_2 = e^{1+C}$ then

$$C_1\sqrt{n}\left(\frac{n}{e}\right)^n \le n! < C_2\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Solution. (a) We integrate by parts once to get

$$\begin{aligned}\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx &= \frac{(x-a)(x-b)}{2} f'(x)\bigg|_{x=a}^{x=b} - \int_a^b \left(x - \frac{a+b}{2} \right) f'(x)\,dx \\ &= \int_a^b \left(\frac{a+b}{2} - x \right) f'(x)\,dx,\end{aligned}$$

and then a second time to get

$$\begin{aligned}\int_a^b \left(\frac{a+b}{2} - x \right) f'(x)\,dx &= \left(\frac{a+b}{2} - x \right) f(x)\bigg|_{x=a}^{x=b} - \int_a^b (-1) f(x)\,dx \\ &= \frac{a-b}{2} f(b) - \frac{b-a}{2} f(a) + \int_a^b f(x)\,dx,\end{aligned}$$

which we rearrange slightly to get the desired answer.

(b) The triangle inequality for integrals says that

$$\begin{aligned}\left| \int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx \right| &\le \int_a^b \left| \frac{(x-a)(x-b)}{2} f''(x) \right| dx \\ &\le \int_a^b \left| \frac{(x-a)(x-b)}{2} \right| \cdot M\,dx \\ &= M \int_a^b -\frac{1}{2}(x-a)(x-b)\,dx,\end{aligned}$$

since $\left|\frac{(x-a)(x-b)}{2}\right| = -\frac{(x-a)(x-b)}{2}$ for $x \in [a, b]$. We evaluate

$$\begin{aligned}
\int_a^b (x^2 - (a+b)x + ab)\,dx &= \frac{x^3}{3} - (a+b)\frac{x^2}{2} + abx\bigg|_{x=a}^{x=b} \\
&= \frac{b^3 - a^3}{3} - (a+b)\frac{b^2 - a^2}{2} + ab(b-a) \\
&= (b-a)\left(\frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{2} + ab\right) \\
&= (b-a)\left(-\frac{b^2}{6} + \frac{ab}{3} - \frac{a^2}{6}\right) \\
&= -\frac{b-a}{6}(b^2 - 2ab + a^2) = -\frac{(b-a)^3}{6},
\end{aligned}$$

and so the upper bound above becomes $-\frac{M}{2}\left(-\frac{(b-a)^3}{6}\right) = \frac{M(b-a)^3}{12}$.

(c) We have $f''(x) = -\frac{1}{x^2}$, so $|f''(x)| \leq \frac{1}{k^2}$ on the interval $[k, k+1]$. Thus

$$e_k = \int_k^{k+1} \log(x)\,dx - ((k+1) - k)\left(\frac{\log(k) + \log(k+1)}{2}\right)$$

satisfies $|e_k| \leq \frac{1}{k^2}\left(\frac{((k+1)-k)^3}{12}\right) = \frac{1}{12k^2}$. We sum over $1 \leq k \leq n-1$ to get

$$\int_1^n \log(x)\,dx = \sum_{k=1}^{n-1}\left(\frac{\log(k) + \log(k+1)}{2} + e_k\right)$$

as claimed.

(d) On the one hand, we can evaluate the integral explicitly as

$$\int_1^n \log(x)\,dx = x\log(x) - x|_{x=1}^n = n\log(n) - (n-1).$$

On the other hand, if we let $E = \sum_{k=1}^{n} e_k$ then the sum can be rearranged as

$$\begin{aligned}
\sum_{k=1}^{n-1}\left(\frac{\log(k) + \log(k+1)}{2} + e_k\right) &= \sum_{k=1}^{n-1}\frac{\log(k)}{2} + \sum_{k=1}^{n-1}\frac{\log(k+1)}{2} + \sum_{k=1}^{n-1} e_k \\
&= \left(\frac{\log(n!)}{2} - \frac{\log(n)}{2}\right) + \frac{\log(n!)}{2} + E \\
&= \log(n!) - \frac{\log(n)}{2} + E.
\end{aligned}$$

We equate the two sides to get

$$n\log(n) - (n-1) = \log(n!) - \frac{\log(n)}{2} + E,$$

or equivalently

$$\begin{aligned}\log(n!) &= \left(n+\frac{1}{2}\right)\log(n) - (n-1) - E \\ &= \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right) - E,\end{aligned}$$

and so $\left|\log(n!) - \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right)\right| \leq |E|$. Since $|E| \leq \displaystyle\sum_{k=1}^{n-1} \frac{1}{12k^2} < \frac{1}{12}\sum_{k=1}^{\infty}\frac{1}{k^2}$, or equivalently $|E| \leq \frac{\zeta(2)}{12}$, the left side is bounded above for all n.
Remark: on the last problem sheet we saw that $\zeta(s) < \frac{s}{s-1}$ for all $s > 1$, so $\zeta(2) < 2$ and hence $|E| < \frac{1}{6}$. Thus we have

$$0.8464 < e^{-E} \leq \frac{n!}{n^{n+1/2}/e^{n-1}} \leq e^{E} < 1.1814$$

for all n, and these bounds can be further improved if we know that $\zeta(2) = \frac{\pi^2}{6}$.