Math40002 Analysis 1 **Unseen 3**

1. In this question you should work **from first principles**, proving any result you need.

   Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

   (a) Define what it means to say $a_n \to a \in \mathbb{R}$ as $n \to \infty$.

   (b) If $a_n = \frac{(n+1)(n+2)}{(2n-5)(n+3)}$, is $(a_n)_{n=1}^{\infty}$ convergent or not? Prove your answer carefully.

   (c) Instead of asking for the *difference* $a_n - a$ *to be close to 0*, we could ask for the *ratio* $(a_n + M)/(a + M)$ *to be close to 1* (for some $M \in \mathbb{R}$ included to avoid dividing by zero).

   So we make the following definition: $a_n \rightsquigarrow a$ if and only if there exists $M \neq -a$ such that

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \ \frac{a_n + M}{a + M} \in \big(1 - \epsilon, 1 + \epsilon\big).$$

   Prove that $a_n \rightsquigarrow a$ if and only if $a_n \to a$.

(a) $\forall \epsilon > 0$**,** $\exists N \in \mathbb{N}$**, such that** $\forall n > N$**,** $|a_n - a| < \epsilon$

(b) We will prove $a_n \to 1/2$**. Let** $\epsilon > 0$**, and let** $N \in \mathbb{N}$ **such that** $N > \frac{5}{2\epsilon}, 19$**. Then for all** $n > N$**:** $\epsilon > \frac{5}{2n}, n > 19, 2n - 5 > n$**. So**

$$|a_n - 1/2| = \left| \frac{(n+1)(n+2)}{(2n-5)(n+3)} - \frac{\frac{1}{2}(2n-5)(n+3)}{(2n-5)(n+3)} \right| =$$
$$\left| \frac{5n+19}{2(2n-5)(n+3)} \right| < \left| \frac{5n+n}{2n^2} \right| = \left| \frac{5}{2n} \right| = \frac{5}{2n} < \epsilon.$$

(c) $\Longrightarrow$ **Assume** $a_n \to a$**, and choose** $M \neq -a$ **(e.g.,** $M = 1 - a$**). Let** $\epsilon_0 > 0$**. Since** $a_n \to a$**, for every** $\epsilon > 0$**, there is some** $N \in \mathbb{N}$ **such that** $\forall n > N$**,** $|a_n - a| < \epsilon$**. In particular, this is true for** $\epsilon = |a + M|\,\epsilon_0$**. So let** N **be such that** $\forall n > N$**,** $|a_n - a| < |a + M|\,\epsilon_0$**. Then**

$$\left| \frac{a_n + M}{a - M} - 1 \right| = \left| \frac{a_n + M}{a + M} - \frac{a + M}{a + M} \right| = \left| \frac{a_n - a}{a + M} \right| = \frac{|a_n - a|}{|a + M|} < \epsilon_0$$

$\Longleftarrow$ **Assume** $a_n \rightsquigarrow a$**. Let** $\epsilon_0 > 0$**. Since** $a_n \rightsquigarrow a$**, there is some** $M \neq -a$ **such that for every** $\epsilon > 0$**, there is some** $N \in \mathbb{N}$ **such that** $\forall n > N$**,** $\left| \frac{a_n + M}{a - M} - 1 \right| < \epsilon$**. In particular, this is true for** $\epsilon = \frac{\epsilon_0}{|a+M|}$**. So let** N **be such that** $\forall n > N$**,** $|a_n - a| < |a + M|\,\epsilon_0$**. Then**

$$\frac{\epsilon_0}{|a + M|} > \left| \frac{a_n + M}{a - M} - 1 \right| = \left| \frac{a_n + M}{a + M} - \frac{a + M}{a + M} \right| = \left| \frac{a_n - a}{a + M} \right| = \frac{|a_n - a|}{|a + M|}.$$

Therefore, $\epsilon_0 > |a_n - a|$**.**

2. Show that if $a_n \to l$, and we define $b_n = \left(\sum_{k=1}^{n} a_n\right)/n$, then $b_n \to l$ too. Give an example to show that the converse does not hold.

Let $\epsilon > 0$ **and let** $N_1 \in \mathbb{N}$ **such that** $\forall n > N_1$**,** $|a_n - l| < \epsilon/2$**. Let** $M := \sum_{k=1}^{N_1} |a_n - l|$**. Let** $N_2 \in \mathbb{N}$ **be such that** $N_2 > 2M/\epsilon$ **(so** $M/N_2 < \epsilon/2$**). Let** $N := \max\{\, N_1, N_2 \,\}$**. We will show that for every** $n > N$**,** $|b_n - l| < \epsilon$**. Let** $n > N$**.**

$$|b_n - l| = \left|\frac{\sum_{k=1}^{n} a_k}{n} - l\right| = \left|\frac{\sum_{k=1}^{n} a_k}{n} - \frac{nl}{n}\right| = \left|\frac{\sum_{k=1}^{n}(a_k - l)}{n}\right| \leq \frac{\sum_{k=1}^{n} |a_k - l|}{n} =$$
$$\frac{\sum_{k=1}^{N_1} |a_k - l|}{n} + \frac{\sum_{k=N+1}^{n} |a_k - l|}{n} = \frac{M}{n} + \frac{\sum_{k=N+1}^{n} |a_k - l|}{n} <$$
$$\frac{M}{n} + \frac{(n - N_1)\epsilon}{2n} < \frac{M}{N_2} + \frac{\epsilon}{2} < \epsilon/2 + \epsilon/2 = \epsilon.$$

For $a_n = (-1)^n$**,** $b_n \to 0$**.**

3. Let $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. Compute $\lim_{n\to\infty} s_n$ for the following sequences:

Notice that $\frac{1}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$**. So**

$$s_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 - \frac{1}{n+1} \to 1.$$