Topic: Probability and conditional probability

In today's problem class we will be reviewing the probability axioms and we will study problems involving conditional probabilities.

1. Given events $E, F, G \subseteq \Omega$, prove that

$$
\begin{aligned}
&(a) \quad \mathrm{P}(E^c \cap F) = \mathrm{P}(F) - \mathrm{P}(E \cap F) \\
&(b) \quad \mathrm{P}(E \cup F) \leq \mathrm{P}(E) + \mathrm{P}(F) \\
&(c) \quad E \subseteq F, F \subseteq G \Longrightarrow \mathrm{P}(E) \leq \mathrm{P}(G) \\
&(d) \quad \mathrm{P}(E \cap F) \geq \mathrm{P}(E) + \mathrm{P}(F) - 1
\end{aligned}
$$

[(d) is known as *Bonferroni's Inequality*.]

Solution: For general events E and F,

(a) $F \equiv (E \cap F) \cup (E^c \cap F)$, so by Axiom (iii)

$$\mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(E^c \cap F) \Longrightarrow \mathrm{P}(E^c \cap F) = \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

(b) $E \cup F \equiv E \cup (E^c \cap F)$, so by Axiom (iii)

$$\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(E^c \cap F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

but $\mathrm{P}(E \cap F) \geq 0$ so $\mathrm{P}(E \cup F) \leq \mathrm{P}(E) + \mathrm{P}(F)$.

(c) $E \subseteq F \subseteq G \Rightarrow E \cup G = G \Longrightarrow G = E \cup (E^c \cap G)$, so by Axiom (iii), as $\mathrm{P}(E^c \cap G) \geq 0$,

$$\mathrm{P}(G) = \mathrm{P}(E) + \mathrm{P}(E^c \cap G) \geq \mathrm{P}(E)$$

(d) *Bonferroni Inequality*: as $\mathrm{P}(E \cup F) \leq 1$,

$$\mathrm{P}(E \cap F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cup F) \geq \mathrm{P}(E) + \mathrm{P}(F) - 1$$

2. Suppose that E and F are events such that $\mathrm{P}(E) = x, \mathrm{P}(F) = y$ and $\mathrm{P}(E \cap F) = z$. Express the following terms in terms of x, y and z:
   (a) $\mathrm{P}(E^c \cup F^c)$
   (b) $\mathrm{P}(E^c \cap F)$
   (c) $\mathrm{P}(E^c \cup F)$
   (d) $\mathrm{P}(E^c \cap F^c)$

Solution:

(a) $E^c \cup F^c = (E \cap F)^c \Longrightarrow \mathrm{P}(E^c \cup F^c) = 1 - \mathrm{P}(E \cap F) = 1 - z$.

(b) $F = (E \cap F) \cup (E^c \cap F)$, so $\mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(E^c \cap F)$, so $\mathrm{P}(E^c \cap F) = y - z$.

(c) $E^c \cup F = E^c \cup (E \cap F) \Longrightarrow \mathrm{P}(E^c \cup F) = \mathrm{P}(E^c) + \mathrm{P}(E \cap F) = 1 - x + z$.

(d) $E^c \cap F^c = (E \cup F)^c \Longrightarrow \mathrm{P}(E^c \cap F^c) = 1 - \mathrm{P}(E \cup F) = 1 - x - y + z$.

3. A crime has been committed and a suspect is being held by police. He is either guilty, G, or not, G^c, and the probability of his being guilty on the basis of current evidence is $\mathrm{P}(G) = p$, say. Forensic evidence is now produced which shows that the criminal must have a property, A, which occurs in a proportion, π, of the general population. Suppose that if the suspect is innocent he can be treated as a member of the general population, so that $\mathrm{P}(A|G^c) = \pi$.

   The suspect is now interrogated and found to have property A. Show that the odds on his guilt have now risen from $\frac{\mathrm{P}(G)}{\mathrm{P}(G^c)} = p/(1-p)$ to $\frac{\mathrm{P}(G|A)}{\mathrm{P}(G^c|A)} = \frac{\mathrm{P}(G)}{\pi \mathrm{P}(G^c)}$.

   Hint: The odds on an event E are defined to be the ratio $\mathrm{P}(E)/\mathrm{P}(E^c)$, the odds-against E are $\mathrm{P}(E^c)/\mathrm{P}(E)$.

Solution: Given $\mathrm{P}(G) = p$, $\mathrm{P}(A|G) = 1$, $\mathrm{P}(A|G^c) = \pi$. Then

$$\mathrm{P}(G|A) = \frac{\mathrm{P}(A|G)\mathrm{P}(G)}{\mathrm{P}(A|G)\mathrm{P}(G) + \mathrm{P}(A|G^c)\mathrm{P}(G^c)} = \frac{1 \cdot p}{1 \cdot p + \pi \cdot (1-p)}.$$

This implies that

$$\mathrm{P}(G^c|A) = 1 - \mathrm{P}(G|A) = \frac{\pi(1-p)}{p + \pi \cdot (1-p)}$$

and hence

$$\frac{\mathrm{P}(G|A)}{\mathrm{P}(G^c|A)} = \frac{p}{\pi(1-p)} = \frac{\mathrm{P}(G)}{\pi \mathrm{P}(G^c)}.$$

4. A shop sells fuses produced by three manufacturers; each manufacturer supplies a deluxe and a standard type of fuse. A mixed batch of 500 fuses sold, and the number of faulty fuses of each type and for each manufacturer is recorded. By considering the following events; $M_i \equiv$ "fuse produced by manufacturer i" for $i = 1, 2, 3$, $D \equiv$ "Deluxe type of fuse" and $F \equiv$ "Fuse Faulty", a summary of the data can be presented as a 3-way table

	M_1		M_2		M_3	
	D	D^c	D	D^c	D	D^c
F	20	16	30	20	15	10
F^c	100	64	120	30	60	15

so that, for example, the number of deluxe fuses from manufacturer 1 that are faulty is 20, whereas the number of standard fuses from manufacturer 1 that are faulty is 16, etc.

(a) A fuse is selected with equal probability from the 500. What is the probability that
   i. it is faulty?
   ii. it was produced by manufacturer 1?

(b) Given that the selected fuse is faulty, what is the conditional probability that
   i. it is a deluxe fuse?
   ii. it is a fuse produced by manufacturer 1?
   iii. it is a deluxe fuse produced by manufacturer 1?

(c) Describe, evaluate, and comment on the following conditional probabilities:
   i. $\mathrm{P}(F|M_1)$, $\mathrm{P}(F|M_2)$, $\mathrm{P}(F|M_3)$
   ii. $\mathrm{P}(F|D)$, $\mathrm{P}(F|D^c)$
   iii. $\mathrm{P}(F|M_1 \cap D)$, $\mathrm{P}(F|M_2 \cap D)$, $\mathrm{P}(F|M_3 \cap D)$.
   iv. $\mathrm{P}(F|M_1 \cap D^c)$, $\mathrm{P}(F|M_2 \cap D^c)$, $\mathrm{P}(F|M_3 \cap D^c)$.

Solution: Recall that for a finite sample space Ω where all events are equally likely, we have

$$\mathrm{P}(A|B) = \frac{\mathrm{card}(A \cap B)}{\mathrm{card}(B)} = \frac{\mathrm{card}(A \cap B)/\mathrm{card}(\Omega)}{\mathrm{card}(B)/\mathrm{card}(\Omega)} = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

(a) i. $\frac{20+16+30+20+15+10}{500} = \frac{111}{500}$

   ii. $\frac{20+16+100+64}{500} = \frac{200}{500} = \frac{2}{5}$.

(b) i.

$$\mathrm{P}(D|F) = \frac{\mathrm{card}(D \cap F)}{\mathrm{card}(F)} = \frac{20+30+15}{111} = \frac{65}{111}.$$

   ii.

$$\mathrm{P}(M_1|F) = \frac{\mathrm{card}(M_1 \cap F)}{\mathrm{card}(F)} = \frac{20+16}{111} = \frac{36}{111}.$$

iii.

$$\mathrm{P}(D \cap M_1 | F) = \frac{\mathrm{card}(D \cap M_1 \cap F)}{\mathrm{card}(F)} = \frac{20}{111}.$$

(c) i. First we compute the conditional probabilities, that given the fuse was produced by a particular manufacturer that it is faulty. We note that we have $\mathrm{P}(M_1) = 200/500 = 2/5$, $\mathrm{P}(M_2) = 200/500 = 2/5$ and $\mathrm{P}(M_3) = 100/500 = 1/5$. We have

$$\mathrm{P}(F|M_1) = \frac{\mathrm{card}(F \cap M_1)}{\mathrm{card}(M_1)} = \frac{20+16}{200} = \frac{36}{200} = \frac{9}{50} = 0.18,$$

$$\mathrm{P}(F|M_2) = \frac{\mathrm{card}(F \cap M_2)}{\mathrm{card}(M_2)} = \frac{(30+20)}{200} = \frac{50}{200} = \frac{1}{4} = 0.25,$$

$$\mathrm{P}(F|M_3) = \frac{\mathrm{card}(F \cap M_3)}{\mathrm{card}(M_3)} = \frac{(15+10)}{100} = \frac{25}{100} = \frac{1}{4} = 0.25.$$

We observe that the conditional probabilities are the same for manufactures 2 and 3 and it is lower for manufacturer 1.

ii. Next we compute the probability of a faulty fuse conditional on it being of deluxe type:

$$\mathrm{P}(F|D) = \frac{\mathrm{card}(F \cap D)}{\mathrm{card}(D)} = \frac{20+30+15}{(20+30+15+100+120+60)} = \frac{65}{345} = \frac{13}{69} \approx 0.188$$

and the probability of a faulty fuse conditional on it being of standard type:

$$\mathrm{P}(F|D^c) = \frac{\mathrm{card}(F \cap D^c)}{\mathrm{card}(D^c)} = \frac{16+20+10}{(16+20+10+64+30+15)} = \frac{46}{155} = \frac{13}{69} \approx 0.296.$$

iii. We have

$$\mathrm{P}(F|M_1 \cap D) = \frac{\mathrm{card}(F \cap M_1 \cap D)}{\mathrm{card}(M_1 \cap D)} = \frac{20}{120} = \frac{1}{6} \approx 0.166,$$
$$\mathrm{P}(F|M_2 \cap D) = \frac{\mathrm{card}(F \cap M_2 \cap D)}{\mathrm{card}(M_2 \cap D)} = \frac{30}{150} = \frac{1}{5} = 0.2,$$
$$\mathrm{P}(F|M_3 \cap D) = \frac{\mathrm{card}(F \cap M_3 \cap D)}{\mathrm{card}(M_3 \cap D)} = \frac{15}{75} = \frac{1}{5} = 0.2,$$

for the conditional probabilities of having a faulty fuse, given that it comes from a particular manufacturer and is of deluxe type.

iv. We have

$$\mathrm{P}(F|M_1 \cap D^c) = \frac{\mathrm{card}(F \cap M_1 \cap D^c)}{\mathrm{card}(M_1 \cap D^c)} = \frac{16}{80} = \frac{1}{5} = 0.2,$$
$$\mathrm{P}(F|M_2 \cap D^c) = \frac{\mathrm{card}(F \cap M_2 \cap D^c)}{\mathrm{card}(M_2 \cap D^c)} = \frac{20}{50} = \frac{2}{5} = 0.4,$$
$$\mathrm{P}(F|M_3 \cap D^c) = \frac{\mathrm{card}(F \cap M_3 \cap D^c)}{\mathrm{card}(M_3 \cap D^c)} = \frac{10}{25} = \frac{2}{5} = 0.4,$$

for the conditional probabilities of having a faulty fuse, given that it comes from a particular manufacturer and is of standard type.

These results confirm that events F, M_1, M_2, M_3 and D are not mutually independent.