Mathematics Year 1, Calculus and Applications I

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Solutions Problem Sheet 6

1. The following functions are defined on the interval $[0, \pi]$. In each case (i) find the even and odd extensions of the given functions on $[-\pi, \pi]$ and extend them periodically with period 2π on the real line; (ii) sketch these over the interval $-4\pi < x < 4\pi$ making sure you include the assumed values of the function at any discontinuities; (iii) find the Fourier series for both even and odd extensions and state whether the convergence of the series is uniform or not. [You can state theorems without proof.]

$$f(x) = \cos x, \qquad f(x) = x^2, \qquad f(x) = e^x, \qquad f(x) = e^x - 1.$$

By inspecting your sketches, which of the Fourier series can be differentiated term-by-term to yield the Fourier series of new functions? Explain using theorems without proofs.

Solution

Parts (i) and (ii) are solved in figures 1-4. In each plot, I start off with the given function and plot it in $[0, \pi]$ - this is shown as a solid blue curve. The even or odd extension to $[-\pi, 0]$ is shown with a solid red curve, and the remaining $2\pi-$periodic extensions are in black. At points where the function is discontinuous, the Fourier series converges to the average $\frac{1}{2}(f_+ + f_-)$ as proved in class. I represent such points with open circles.

In all calculations that follow, even extensions have Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \tag{1}$$

while odd extensions have Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \tag{2}$$

(a) $f(x) = \cos x$.
The even extension gives $\cos x$ over the whole real line and so the Fourier series is simply $\cos x$, i.e. all Fourier coefficients are zero except $a_1 = 1$. Can do it as an even Fourier series in which case $b_n = 0$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi}\cos x \cos nx dx$, i.e. $a_n = 0$ except $a_1 = 1$. Convergence is uniform.
For the odd extension we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx dx$$

Note that $b_n \neq 0$ - the integral is over $[0, \pi]$. Use the formula $\sin nx \cos x = \frac{1}{2}[\sin(n+1)x + \sin(n-1)x]$, hence (need to do $n = 1$ separately)

$$b_n = \frac{2}{\pi}\frac{1}{2}\left[-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{1+\cos n\pi}{(n+1)} + \frac{1+\cos n\pi}{(n-1)}\right]$$
$$= \frac{2n}{\pi(n^2-1)}(1+\cos n\pi) = \frac{2n}{\pi(n^2-1)}(1+(-1)^n),$$

so only even n are non-zero. Also, if $n = 1$ we find $b_1 = 0$. Convergence is uniform everywhere except at $x = n\pi$ where the Fourier series is 0.

(b) $f(x) = x^2$.
The even extension follows from (1) with (use integration by parts)

$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\, dx = \frac{4\cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}.$$

The convergence of the Fourier series is uniform everywhere.
The odd extension follows from (2) with (again integrate by parts)

$$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin nx\, dx = \frac{2}{\pi}\left(-\frac{\pi^2\cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right).$$

Convergence is not uniform at $x = 2k\pi$ for any integer k.

(c) $f(x) = e^x$.
Even extension has series (1) with

$$a_n = \frac{2}{\pi}\int_0^\pi e^x \cos nx dx,$$

and the odd one

$$b_n = \frac{2}{\pi}\int_0^\pi e^x \sin nx dx.$$

Will do the integrals by considering real and imaginary parts of

$$\int_0^\pi e^x e^{inx} dx = \left[\frac{e^{(1+in)x}}{1+in}\right]_0^\pi = \frac{e^{(1+in)\pi} - 1}{1+in} = \frac{e^\pi \cos n\pi - 1}{1+n^2}(1-in)$$

Hence $a_n = \frac{2}{\pi}\frac{e^\pi \cos n\pi - 1}{1+n^2}$ and $b_n = \frac{2n}{\pi}\frac{e^\pi \cos n\pi - 1}{1+n^2}$.
Convergence is uniform for the even extension but not for the odd one at $x = k\pi$ for any integer k.

(d) $f(x) = e^x - 1$.
Similar to part (c). In fact the even extension has the *same* Fourier coefficients with the exception of $a_0 = \frac{2}{\pi}(e^\pi - 1 - \pi)$. The odd extension again same as in (c) but need to add the contribution $\frac{2}{\pi}\frac{(\cos n\pi - 1)}{n}$ that comes from the constant term -1. Convergence is not uniform at $x = \pm\pi, \pm 3\pi, \ldots$.

Inspection of the curves shows that the only series that can be differentiated term by term to yield other Fourier series that converge uniformly almost everywhere, are the even extensions since only they are piecewise continuous.

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2. Obtain the Fourier series of the function $f(x) = \pi x$ on the interval $0 \le x \le 1$ as a sine series and a cosine series (extend the function appropriately and note that the interval is $2-$periodic not $2\pi-$periodic).

Solution

The function is 2−periodic and so the sine/cosine series coefficients are

$$b_n = 2\int_0^1 \pi x \sin n\pi x \, dx, \qquad a_n = 2\int_0^1 \pi x \cos n\pi x \, dx.$$

Integrate by parts to find

$$b_n = \frac{2(-1)^{n+1}}{n}, \qquad a_n = \frac{2(\cos n\pi - 1)}{n^2 \pi}$$

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3. (a) Sketch the function $f(x) = |\sin x|$ defined on $-\pi \leq x \leq \pi$, and show that its Fourier series is given by

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

(b) What value does the Fourier series converge to at $x = 0, \pi, -\pi$?

(c) Use the series result to show that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

(d) Use your results to also show that

$$\sum_{n=1}^{\infty} \frac{1}{4(2n-1)^2 - 1} = \frac{1}{4 \cdot 1} + \frac{1}{4 \cdot 3^2 - 1} + \frac{1}{4 \cdot 5^2 - 1} + \ldots = \frac{\pi}{8}$$

Solution

(a) Sketch easy - see figure 5. The function is even so it has a cosine series given by (1) with

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] \, dx \\ &= \frac{1}{\pi}\left[\frac{1 - \cos(n+1)\pi}{n+1} - \frac{1 - \cos(n-1)\pi}{n-1}\right] = \frac{1}{\pi}\left[\frac{1 + \cos n\pi}{n+1} - \frac{1 + \cos n\pi}{n-1}\right] \\ &= -\frac{2(1 + \cos n\pi)}{\pi(n^2 - 1)} = -\frac{2(1 + (-1)^n)}{\pi(n^2 - 1)}. \end{aligned}$$

Only even terms are non-zero, $a_0 = 4/\pi$, and $a_{2k} = -\frac{4}{\pi(4k^2-1)}$, hence

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

(b) The function is continuous at $x = 0, \pi, -\pi$ and equal to 0 there, hence the Fourier series converges to 0 (uniformly) at all three points.

(c) Put $x = 0$ in the series, we obtain

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

(d) Here we pick $x = \pi/2$ in the series to find

$$\frac{\pi}{8} = \frac{1}{4} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos n\pi}{4n^2-1} = \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{4n^2-1} - \sum_{n=1}^{\infty}\frac{\cos n\pi}{4n^2-1}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{4n^2-1} \tag{3}$$

Here I used the result $\frac{1}{4} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ from part (c). Now $1-\cos n\pi = 1-(-1)^n$ which is 0 if n is even and 2 if n is odd. Hence only odd terms survive in (3) starting with $n = 1$. Hence

$$\frac{\pi}{8} = \sum_{n=1}^{\infty}\frac{1}{4(2n-1)^2-1}$$

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4. (a) Consider the function $f(x) = x\cos x$ on $-\pi < x < \pi$. Sketch the function. Is it even or odd?

(b) Find the Fourier series of $f(x)$ extended periodically over the whole of the real line. What values does the series converge to at $x = -\pi, +\pi$?

(c) Now introduce the function $\phi(x) = x$ on $-\pi < x < \pi$. Write down the Fourier series for $\phi(x)$ (extended periodically on the real line) and hence show that the Fourier series of $\chi(x) := x(1+\cos x)$ (extended periodically on the real line) is given by

$$\chi(x) = \frac{3}{2}\sin x + 2\left(\frac{\sin 2x}{1\cdot 2\cdot 3} - \frac{\sin 3x}{2\cdot 3\cdot 4} + \frac{\sin 4x}{3\cdot 4\cdot 5} + \dots\right) \tag{4}$$

(d) What values do you expect the Fourier series of $\chi(x)$ to converge to at the end points $x = -\pi$ and $x = \pi$? Is the periodic extension of χ continuous at the end points? Is the convergence uniform or not?

(e) Does the periodically extended function $\chi(x)$ have continuous derivatives of any order on the closed interval $[-\pi, \pi]$ (clearly the problematic points are the end points, so you may find it useful to carry out a local one-sided Taylor series expansion).

By considering the Fourier series (4) can you think of a series comparison test that would establish its absolute convergence for all $x \in [-\pi, \pi]$?

Solutions

(a) A sketch is given in Figure 6. The function is odd since $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$.

(b) The Fourier series is $x\cos x = \sum_{n=1}^{\infty} b_n \sin nx$ where

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x + \sin(n-1)x\right]dx \\
&= \frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{n+1}\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{\cos(n+1)x}{n+1}dx + \frac{1}{\pi}\left[-\frac{x\cos(n-1)x}{n-1}\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{\cos(n-1)x}{n-1}dx \\
&= \frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} = \frac{2n\cos n\pi}{n^2-1} = \frac{2n(-1)^n}{n^2-1} \qquad n \neq 1.
\end{aligned}$$

If $n = 1$ it follows that $b_1 = \frac{1}{\pi}\int_0^\pi x \sin 2x \, dx = -1/2$.
At $x = \pi, -\pi$ the series converges to 0.

(c) $\phi = x$ on $(-\pi, \pi)$ is also an odd function hence $x = \sum_{n=1}^\infty b_n \sin nx$ where

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx dx = -\frac{2\cos n\pi}{n}$$

Adding the two series together we get (add the $n = 1$ terms first)

$$\chi(x) = 2\sin x - \frac{1}{2}\sin x + \sum_{n=2}^\infty \left(-\frac{2\cos n\pi}{n} + \frac{2n\cos n\pi}{n^2-1}\right)\sin nx$$
$$= \frac{3}{2}\sin x + \sum_{n=2}^\infty \frac{2\cos n\pi}{n(n^2-1)}\sin nx = \frac{3}{2}\sin x + \sum_{n=2}^\infty \frac{2\cos n\pi}{n(n+1)(n-1)}\sin nx$$
$$= \frac{3}{2}\sin x + 2\left(\frac{\sin 2x}{1\cdot 2\cdot 3} - \frac{\sin 3x}{2\cdot 3\cdot 4} + \frac{\sin 4x}{3\cdot 4\cdot 5} + \dots\right)$$

(d) Adding x ensures that the function $\chi(x) = x(1 + \cos x)$ is continuous and equal to 0 at $x = \pm\pi$. The Fourier series of χ converges to 0 uniformly at these points.

(e) Clearly $\chi(\pm\pi) = 0$. Also $\chi' = 1 + \cos x - x\sin x$, hence $\chi'(\pm\pi) = 0$ also. A Taylor expansion near $x = \pi$ shows how many derivatives are continuous. Put $x = \pi - \epsilon$, $\epsilon > 0$. Then

$$\chi(\pi-\epsilon) = (\pi-\epsilon)[1+\cos(\pi-\epsilon)] = (\pi-\epsilon)[1-\cos\epsilon] = (\pi-\epsilon)\left(\frac{\epsilon^2}{2!} + \dots\right)$$
$$= \frac{\pi}{2}\epsilon^2 - \frac{\epsilon^3}{2} + \dots,$$

showing that the first derivative is continuous but not the second one (it is a constant that is different as the point is approached from above or below).

These observations allow us to conclude that the series is absolutely convergent on $[-\pi, \pi]$. Indeed we have

$$\left|\frac{\sin 2x}{1\cdot 2\cdot 3} - \frac{\sin 3x}{2\cdot 3\cdot 4} + \frac{\sin 4x}{3\cdot 4\cdot 5} + \dots\right| < \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots,$$

and the latter series converges by the integral test, for example.

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5. Consider the function $f(x) = \cos\alpha x$ for $-\pi < x < \pi$, where α is not an integer.

(a) Show that the Fourier series of $f(x) = \cos\alpha x$ is

$$\cos\alpha x = \frac{2\alpha\sin\alpha\pi}{\pi}\left(\frac{1}{2\alpha^2} - \frac{\cos x}{\alpha^2 - 1^2} + \frac{\cos 2x}{\alpha^2 - 2^2} + \dots\right) \tag{5}$$

(b) Confirm that the periodic extension of the function remains continuous at $x = \pm\pi$. Hence, select $x = \pi$ in (5) to show that the following expression holds

$$\cot\pi x = \frac{2x}{\pi}\left(\frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \dots\right). \tag{6}$$

This expression resolves $\cot\pi x$ into partial fractions!

(c) Re-write (6) in the form

$$\pi\left(\cot \pi x - \frac{1}{\pi x}\right) = -2x\left(\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \ldots\right), \tag{7}$$

and take x to lie in the interval $0 \le x \le \beta < 1$. Show that the series (7) converges uniformly in the given interval and can therefore be integrated term-by-term (consider the *nth* term and bound its absolute value by the term of a known convergent series).

(d) Integrate (7) from 0 to x and show that (careful with improper integrals at $x = 0$)

$$\log\left(\frac{\sin \pi x}{\pi x}\right) = \lim_{n\to\infty} \log \prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2}\right). \tag{8}$$

(e) Show that (8) is equivalent to (exponentiate both sides)

$$\sin \pi x = \pi x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\ldots$$

Show how your expression above can be used to produce the so-called *Wallis's* product

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots$$

Solution

(a) The function is even hence $b_n = 0$ and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \cos \alpha x \, \cos nx \, dx = \frac{1}{\pi}\int_0^\pi \left[\cos(\alpha+n)x + \cos(a-n)x\right] dx \\ &= \frac{1}{\pi}\left[\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right] = \frac{1}{\pi}\left[\frac{\sin \alpha\pi \, \cos n\pi}{\alpha+n} + \frac{\sin \alpha\pi \, \cos n\pi}{\alpha-n}\right] \\ &= \frac{2\alpha(-1)^n}{\pi(\alpha^2 - n^2)} \sin \alpha\pi, \end{aligned}$$

hence giving (5) as required.

(b) Since $\cos \alpha\pi = \cos(-\alpha\pi)$, the periodic extension remains continuous at the points $x = \pm\pi$. Hence, the Fourier series (5) converges uniformly to $\cos \alpha\pi$ at $x = \pi$, for instance. Putting $x = \pi$ into (5) yields

$$\cot \alpha\pi = \frac{2\alpha}{\pi}\left(\frac{1}{2\alpha^2} - \frac{\cos \alpha}{\alpha^2 - 1^2} + \frac{\cos 2\alpha}{\alpha^2 - 2^2} + \ldots\right)$$

which is a formula that holds for any α. Writing $\alpha = x$ (i.e. changing the independent variable) yields the required result.

(c) Manipulating (6) gives

$$\pi\left(\cot \pi x - \frac{1}{\pi x}\right) = -2x\left(\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \dots\right),$$

and so the nth term term can be bounded

$$\left|\frac{2x}{\pi(n^2 - x^2)}\right| < \frac{2}{\pi(n^2 - \beta^2)} < \frac{M}{n^2},$$

for some constant $M > 0$, hence the series converges absolutely by comparison with $\sum(1/n^2)$. Hence, the series can be integrated term by term.

(d) Integrate (7) between 0 and x. The left hand side is

$$\pi\int_0^x \left[\cot \pi t - \frac{1}{\pi t}\right] dt = \lim_{\epsilon\to 0}\left(\int_\epsilon^x \left[\pi \cot \pi t - \frac{1}{t}\right] dt\right)$$
$$= \log\frac{\sin \pi x}{\pi x} - \lim_{\epsilon\to 0}\log\frac{\sin \epsilon\pi}{\epsilon\pi} = \log\frac{\sin \pi x}{\pi x}.$$

Note that on integrating $-1/t$ above I expressed it as $-\log \pi t + \log \pi$; the constant term $\log \pi$ cancels when the limits are imposed. Next consider the integral of the right hand side of (7).

$$\int_0^x \left\{-2t\left(\frac{1}{1^2 - t^2} + \frac{1}{2^2 - t^2} + \dots + \frac{1}{n^2 - t^2} + \dots\right)\right\} dt$$
$$= \left[\log(1^2 - t^2) + \log(2^2 - t^2) + \dots\right]_0^x = \log\left(1 - \frac{x^2}{1^2}\right) + \log\left(1 - \frac{x^2}{2^2}\right) + \dots$$
$$= \lim_{n\to\infty}\sum_{k=1}^n \log\left(1 - \frac{x^2}{k^2}\right)$$

Hence,

$$\log\frac{\sin \pi x}{\pi x} = \lim_{n\to\infty}\sum_{k=1}^n \log\left(1 - \frac{x^2}{k^2}\right) = \lim_{n\to\infty}\log\prod_{k=1}^n\left(1 - \frac{x^2}{k^2}\right) = \log\lim_{n\to\infty}\prod_{k=1}^n\left(1 - \frac{x^2}{k^2}\right)$$

(e) Now exponentiate both sides to get rid of the logs to find the desired result

$$\sin \pi x = \pi x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\dots \tag{9}$$

Put $x = 1/2$ in (9) to find

$$1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2\cdot 1^2}\right)\left(1 - \frac{1}{2^2\cdot 2^2}\right)\dots\left(1 - \frac{1}{2^2\cdot k^2}\right)\dots = \frac{\pi}{2}\prod_{k=1}^\infty \frac{(2^2k^2 - 1)}{2^2\cdot k^2}$$
$$= \frac{\pi}{2}\prod_{k=1}^\infty \frac{2k-1}{2k}\cdot\frac{2k+1}{2k} \qquad \Rightarrow \qquad \frac{\pi}{2} = \prod_{k=1}^\infty \frac{2k}{2k-1}\cdot\frac{2k}{2k+1}$$

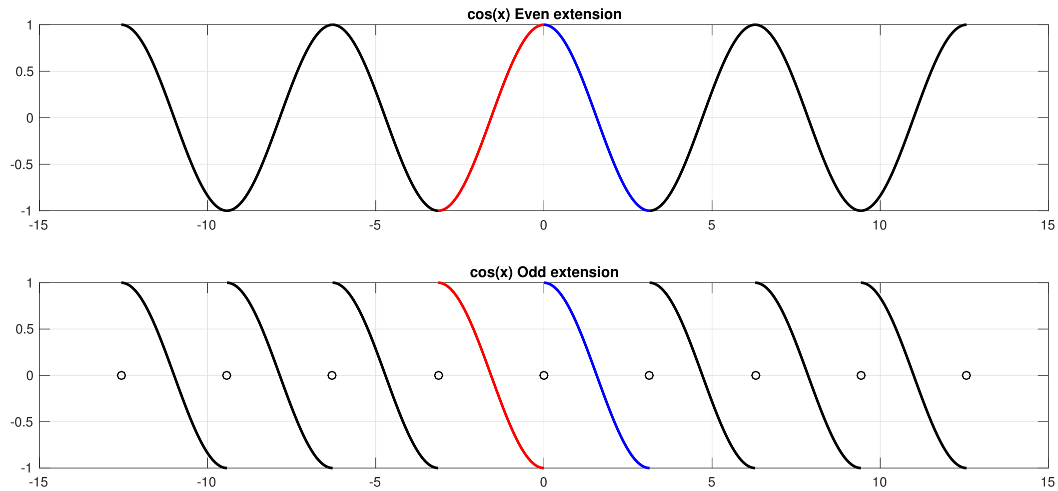

Figure 1: Even and odd extension of $f(x) = \cos x$ in problem 1.

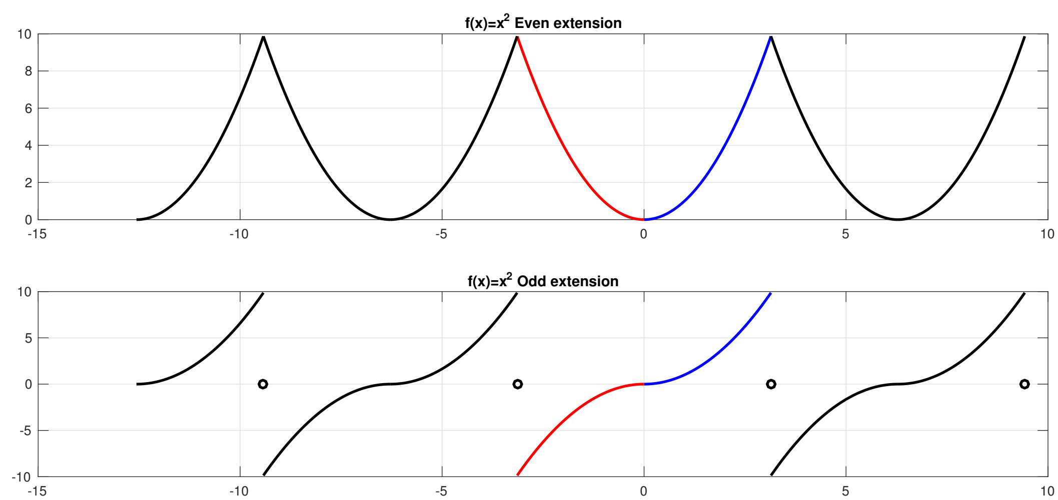

Figure 2: Even and odd extension of $f(x) = x^2$ in problem 1.

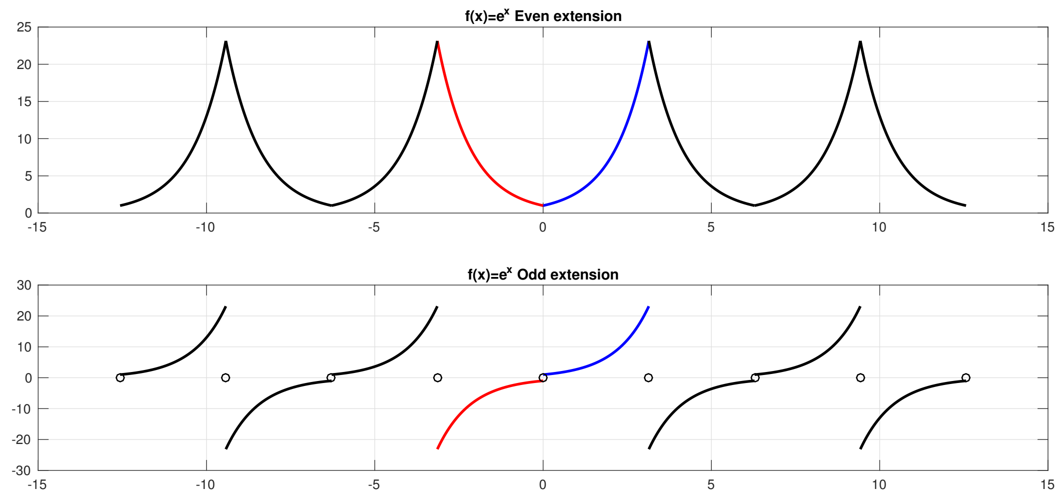

Figure 3: Even and odd extension of $f(x) = e^x$ in problem 1.

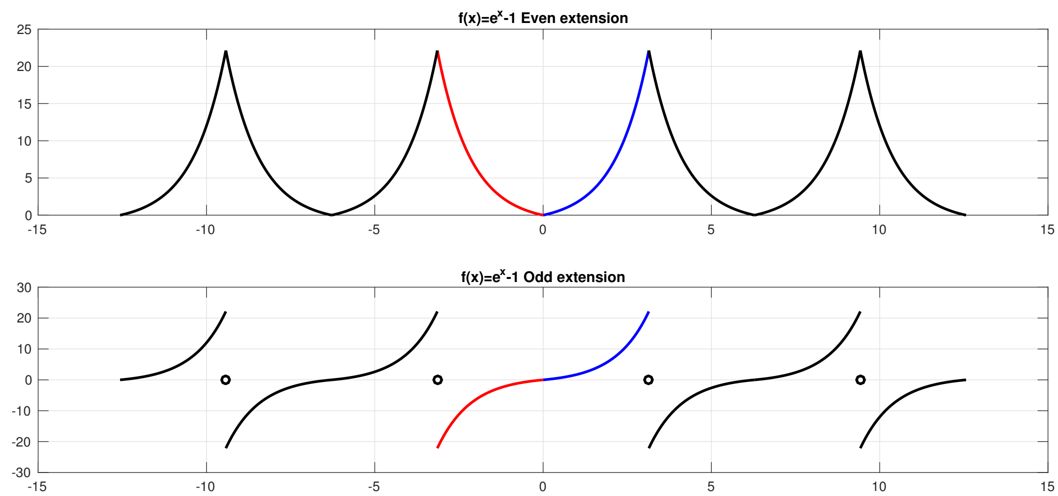

Figure 4: Even and odd extension of $f(x) = e^x - 1$ in problem 1.

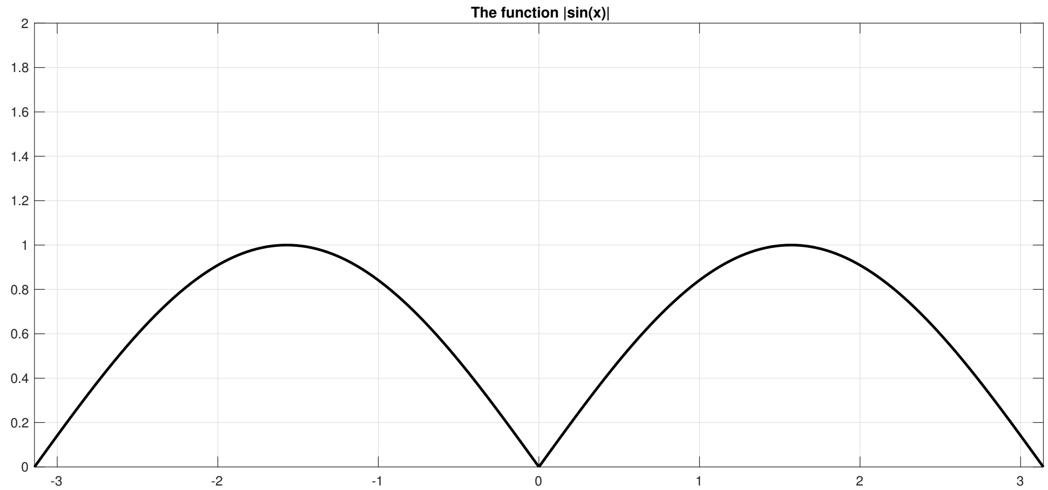

Figure 5: The function $|\sin x|$ of problem 3.

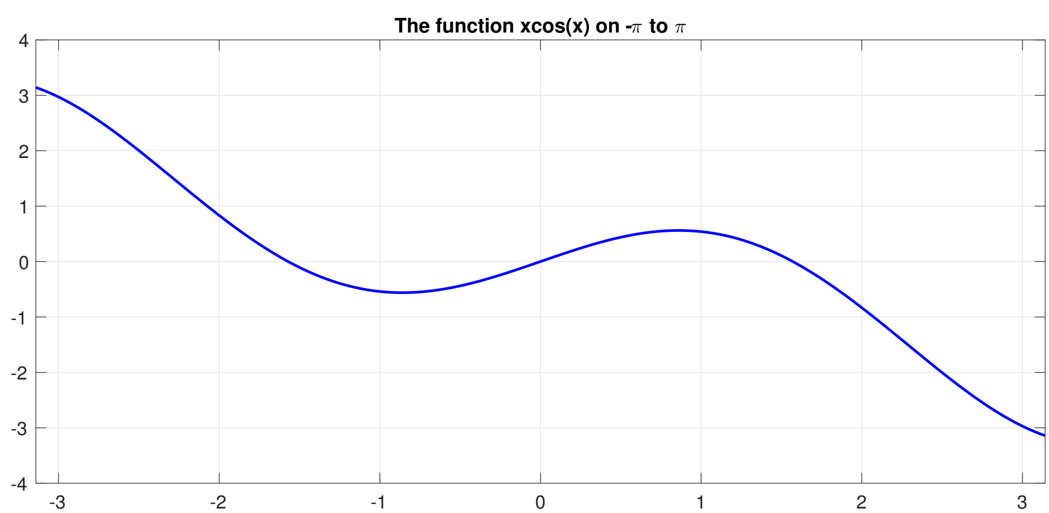

Figure 6: The function $x \cos x$ of problem 4.