Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Introduction to University Mathematics **MATH40001/MATH40009**

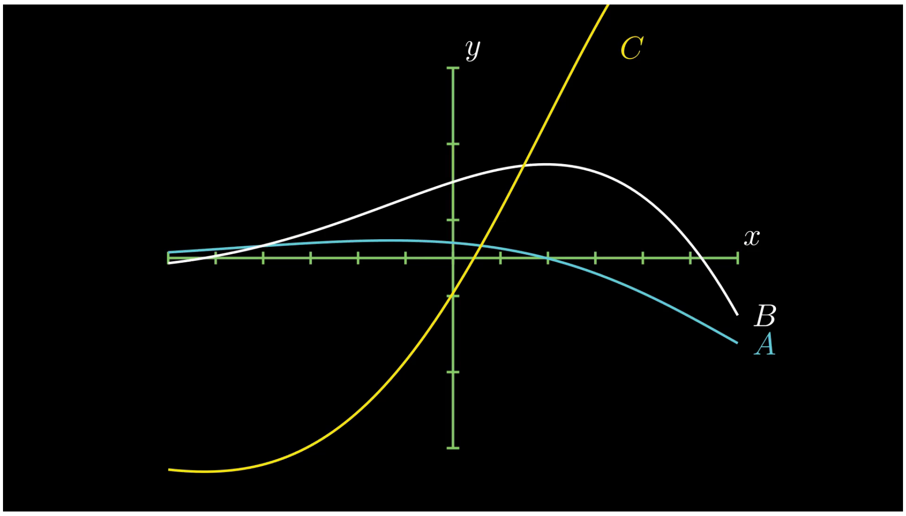
Problem Sheet 0

I – Language of Mathematics

1. Rewrite the following statements formally with quantifiers
   (a) If x and y are real numbers and y is strictly positive, $x + y$ is always bigger than x.
   (b) Every real number has a complex square root.
   (c) The average of two positive integers is positive.
   (d) The difference of two negative integers is not necessarily negative.
2. Let S be the set of all people living in London, and A a function that associates to every member of S their age.
   (a) Write the function formally.
   (b) Rewrite with quantifiers the following statement: in London, everybody is older than somebody.
3. Let n be an integer. Prove (carefully) that
   (a) if 2 divides n, then 2 divides n^2.
   (b) if 2 divides n^2, then 2 divides n.
4. Prove by induction that 3 divides $n^3 - n$ for all integers $n \geq 0$.

II – Real Functions

1. The following figure shows the graph of a function $f(x)$, its derivative $f'(x)$ and the definite integral $F(x) = \int_0^x f(t)dt$. Can you identify each graph? Explain your reasoning.



2. Try to sketch by hand the following functions:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 - x^2 \end{aligned}$$

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{-x^2} \end{aligned}$$

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto \sin\left(\frac{1}{x}\right) \end{aligned}$$

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto x\sin\left(\frac{1}{x}\right) \end{aligned}$$

3. Can you think of a **continuous function which is not differentiable**?
4. Can you think of a function which is **discontinuous everywhere**?
5. What about one which is **differentiable everywhere, but with a discontinuous derivative**?

III – Solving Equations

1. Find the family of solutions to $\frac{dy}{dx} = 2$, and draw them on the plane.
2. Find the solution to the initial value problem:

$$\frac{dy}{dx} = y, \; y(0) = 2$$

3. Check that $y(x) = a\sin(x + \lambda) + b\cos(x + \mu)$ is a solution to:

$$\frac{d^2y}{dx^2} = -y$$

4. Can an initial value problem have more than one solution?
5. Consider a box containing radioactive atoms, at any time t, we denote $x(t)$ the number of radioactive atoms remaining in the box. With time, these atoms decay; as each atom has the same chance to decay, the rate of change of atom number is proportional to the number of atoms remaining. Can you write an ordinary differential equation governing the number of atoms in the box? Find a solution to this problem, knowing that there was initially x_0 atoms in the box.

IV – Trigonometric and Hyperbolic Functions

1. By definition, $\tan\theta = \sin\theta/\cos\theta$, when $\theta \to \pi/2$, what value does $\cos\theta$ take? What happens to the tangent? Sketch the tangent function for $\theta \in [-\pi, \pi]$.
2. Sketch the polar graph $r = \cos\theta$, now consider what happens for $r = \cos n\theta$ as n varies over the integers.
3. Sketch the hyperbolic functions $\sinh$ and $\cosh$, you can use the definitions in terms of exponentials to help you.

4. How are the hyperbolic functions linked with hyperbolas?
5. Calculate the following integrals (it might be useful to work in polar coordinates):

$$\int \sqrt{4x^2 - 1} dx$$
$$\int x \cos(x^2) dx$$

6. Sketch the graph $r = 5 - \cos\theta - 4\sin\theta$, for $-\pi < \theta \leq \pi$. Then find the area bounded by the curve.

V – Complex Numbers

1. Show the following identity: $(r\cos\theta + ir\sin\theta)(s\cos\psi + is\sin\psi) = rs(\cos(\theta+\psi) + i\sin(\theta+\psi))$
2. Let $z = -3 + 4i$
   (a) Sketch z in the complex plane.
   (b) Calculate the modulus and argument of z.
   (c) Sketch $\bar{z}$ in the complex plane.
   (d) Find $z \cdot \bar{z}$.
3. Find all the complex roots of $x^3 - 1 = 0$. Do the same for $x^4 - 1 = 0$, try and sketch these roots on the complex plane. Can you guess where the roots of $x^n - 1 = 0$ will be located in the complex plane?
4. Prove the *conjugate root theorem*: for any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ with real coefficients $a_0 \ldots a_n$, if z is a root of P, then so is $\bar{z}$.
5. By considering the real and imaginary parts of $(e^{i\theta})^3$, derive the triple angle formulae for $\sin$ and $\cos$:

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$
$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

6. On the complex plane, find:
   (a) all the points z such that $|z + 1 - 3i| = 1$.
   (b) all the points z such that $|z - 2| = |z + i|$.
   (c) all the points z such that $\arg\left(\frac{z-1}{z}\right) = \frac{\pi}{3}$
7. Show that if $|z| = 1$, then

$$\operatorname{Im} \frac{z}{(z+1)^2} = 0.$$

   Is there a nice geometric interpretation of this equation? Find all the points on the complex plane such that $\operatorname{Im} \frac{z}{(z+1)^2} = 0$ — there are more of them than just the ones on the unit circle.
8. Find the real and imaginary parts of $\sinh z$ and $\cosh z$.

The geometric explanation of Euler's formula $e^{it} = \cos t + i \sin t$ in the video was shamelessly taken from the book *Visual Complex Analysis* by Tristan Needham. The first chapter of that book is well worth a look for a review of complex numbers with many nifty geometric arguments, and for the exercises at the end. The book is available from the Imperial College Library.

Around the same time as we were compiling the videos, *3Blue1Brown* on Youtube published a video on the same explanation. Check out their animation, with prettier colours and no crustaceans, here: https://youtu.be/v0YEaeIClKY.

VI – Sequences and Series

1. Can you see any similarities between sequences and functions? As you might have guessed, a_n can be seen as a function which takes as an input a natural number n and spits out a real number a_n. Then what are the domain and codomain of our function?

2. What are the next terms of the sequence $3, 5, 7, \ldots$?

3. Can you find the maximum number of pieces of pizza that can be made with a given number n of straight cuts?

4. Try to construct some squares whose sides follow the Fibonacci numbers. How do you form a spiral? What real life example comes first in your mind that might contain it in its shape?

5. What is the n^{th} term of an arithmetic sequence where the initial term is a and the common difference is d? What about the n^{th} term for the geometric sequence where the initial term is a and the common ratio is r?

6. For the sum of the first n terms in a geometric sequence, what happens if either the first term is 0 or the common ratio is 1?

7. For an arithmetic sequence, can you prove that

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}, \text{ for } n \geqslant 2?$$

What about a geometric sequence with terms a_n all positive, can you write a proof for the following relationship:

$$a_n = \sqrt{a_{n-1}a_{n+1}}, \text{ for } n \geqslant 2?$$

8. Can you think of a sequence that tends to a finite number?

9. Can you think of any sequences that don't converge and do not diverge to infinity?

10. Can you think of any similarities between the 'sequences-series' relationship and the 'derivative-integral' one? What happens if you integrate and then differentiate? What about if you go from a_n to S_n and then try to make a_n again?

11. Can you visualise why the infinite series determined by this

$$S_n = \sum_{i=1}^{n} \frac{1}{4^i}$$

converges? You can find a hint image in the video!

12. Can we take the sine or cosine of complex numbers? Can we use the series in the video to find $\sin(1+i)$ for example? Does that make sense giving that everything we know about sine function is $\frac{\text{opposite}}{\text{hypothenuse}}$?

VII – Linear Algebra

1. For $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$, calculate $M^2 - N^2$ and then $(M-N)(M+N)$. Are these equal?

2. Using matrices and vectors, solve this system of simultaneous equations:

$$\begin{cases} 3x + 2y = 5 \\ 2x - y = 8. \end{cases}$$

How many solutions does this have?

Now try to solve this system:

$$\begin{cases} 4x - 3y = 2 \\ 8x - 6y = 4. \end{cases}$$

3. (a) Find a matrix which describes an anticlockwise rotation of $\frac{3\pi}{4}$ radians about the origin.

   (b) What transformation does the matrix $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ represent?

   (c) What about this matrix: $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$?