M1M2: Unseen 4: Series Solutions to differentiatial equations

2.

(a). We substitute the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (*)$$

into $\dfrac{d^2y}{dx^2} - x^4 y = 0$. This gives:

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x^4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^{n+4} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=4}^{\infty} a_{n-4} x^n = 0$$

$$\Rightarrow \left[2a_2 + 6a_3 x + 12a_4 x^2 + 20 a_5 x^3\right] + \sum_{n=4}^{\infty}\left[(n+1)(n+2)a_{n+2} - a_{n-4}\right]x^n = 0 \qquad 1$$

Now we have: $y(0) = 1 \Rightarrow \underline{a_0 = 1}$ (sub $x=0$ into $(*)$ to see this)

and: $y'(0) = 1 \Rightarrow \underline{a_1 = 1}$ (again sub $x=0$ into derivative of $(*)$)

$\boxed{2}$

Comparing coefficients of x, equation ① gives:

$$2a_2 = 0 \Rightarrow \underline{a_2 = 0}$$

$$6a_3 = 0 \Rightarrow \underline{a_3 = 0}$$

$$12a_4 = 0 \Rightarrow \underline{a_4 = 0}$$

$$20a_5 = 0 \Rightarrow \underline{a_5 = 0}$$

$$(n+1)(n+2)a_{n+2} - a_{n-4} = 0 \,, \quad n \geq 4 \qquad ②$$

Relationship ② (for $n=4$) gives:

$$30a_6 - a_0 = 0$$

$$\Rightarrow \underline{a_6 = \frac{1}{30}}$$

And (for $n=5$) ② gives:

$$42a_7 - a_1 = 0$$

$$\Rightarrow \underline{a_7 = \frac{1}{42}}$$

$$\Rightarrow \boxed{y(x) = 1 + x + \frac{1}{30}x^6 + \frac{1}{42}x^7 + \cdots}$$

$\boxed{3}$

(b). There are two ways to do this. One can first substitute $t = x - 1$ and transform the equation into one involving y and t and then substitute a power series solution in t about $t=0$ or, what I'll do here, substitute:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

into:

$$\frac{d^2y}{dx^2} - \Big[\underbrace{\underbrace{(x-1)^2}_{x^2-2x+1} + \underbrace{2(x-1)}_{+2x-2} + \underbrace{1}_{+1}}_{=x^2}\Big]\frac{dy}{dx} + y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - \left[(x-1)^2 + 2(x-1) + 1\right]\sum_{n=1}^{\infty} a_n n (x-1)^{n-1}$$
$$+ \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=1}^{\infty} a_n n (x-1)^{n+1}$$
$$-2\sum_{n=1}^{\infty} a_n n (x-1)^n - \sum_{n=1}^{\infty} a_n n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=2}^{\infty} a_{n-1}(n-1)(x-1)^n$$

$$-2\sum_{n=1}^{\infty} a_n n(x-1)^n - \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\Rightarrow \left[2a_2 - a_1 + a_0\right] + \left[6a_3 - 2a_1 - 2a_2 + a_1\right](x-1)$$

$$+ \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n-1)a_{n-1} - 2na_n - (n+1)a_{n+1} + a_n\right](x-1)^n$$

$$= 0$$

Comparing coefficients of $(x-1)$:

Again notice:
$a_1 = 1 = y'(1)$
$a_0 = 1 = y(1)$

$$2a_2 - 1 + 1 = 0$$

$$\Rightarrow \underline{a_2 = 0}$$

$$6a_3 - 1 - 2a_2 = 0 \quad \Rightarrow \quad \underline{a_3 = \frac{1}{6}}$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (1-2n)a_n - (n-1)a_{n-1} = 0, \quad n \geq 2 \qquad (3)$$

Relationship (3) for $n=2$:

$$12a_4 - 3a_3 - 3a_2 - 1 = 0$$

$$\Rightarrow 12a_4 = 1 + \frac{1}{2} \Rightarrow \underline{a_4 = \frac{3}{24} = \frac{1}{8}}$$

Put $n=3$ in (3): $\quad 20a_5 - 4a_4 - 5a_3 - 2a_2 = 0$

$$\Rightarrow 20a_5 = \frac{5}{6} + \frac{1}{2} \quad \Rightarrow \quad \underline{a_5 = \frac{1}{15}}$$

$$\Rightarrow \boxed{y(x) = 1 + (x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{8}(x-1)^4 + \frac{1}{15}(x-1)^5 + \cdots}$$

$\boxed{5}$

(c). Sub $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into: $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

Again: $y(0) = 1 \Rightarrow \underline{a_0 = 1}$
$y'(0) = 1 \Rightarrow \underline{a_1 = 1}$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} a_n n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_n n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow [2a_2 + 4a_0] + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} + (4-n)a_n\right]x^n = 0$$

Comparing coefficients of x^n: $2a_2 + 4a_0 = 0 \Rightarrow \underline{a_2 = -2}$

$$\underline{(n+1)(n+2)a_{n+2} + (4-n)a_n = 0, \quad n \geq 1} \quad (4)$$

(4) gives: $a_{n+2} = \frac{n-4}{(n+1)(n+2)} a_n$.

For $\underline{n \text{ even}}$, this gives: $\underline{a_4 = \frac{-2}{12} a_2 = \frac{1}{3}}$ $(n=2)$

$\underline{a_6 = \frac{0}{30} a_4 = 0}$ $(n=4)$

$\underline{a_8 = \frac{2}{56} a_6 = 0}$ $(n=6)$

$\Rightarrow \underline{a_{10} = a_{12} = \dots = 0}$

(since $a_6, a_8 = 0$ all remaining terms must be 0 too).

For n odd, put $n = 2k-1$, then:

$$a_{2k+1} = \frac{2k-5}{(2k+1)(2k)} a_{2k-1} = \frac{(2k-5)(2k-7)}{(2k+1)(2k)\cdots(2k-2)} a_{2k-3}$$

$$= \cdots\cdots = \frac{(2k-5)(2k-7)\cdots(-3)}{(2k+1)!} \underbrace{a_1}_{=1}$$

$\Rightarrow$ Putting everything together:

$$\boxed{y(x) = 1 + x - 2x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \sum_{k=2}^{\infty} \frac{(2k-5)\cdots(-3)}{(2k+1)!} x^{2k+1}}$$

$\boxed{7}$

(d). Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into: $\frac{d^2y}{dx^2} - \sin(x)y = \cos(x)$.

$y(0) = 3 \Rightarrow a_0 = 3$

$y'(0) = 0 \Rightarrow a_1 = 0$

We represent the trigonometric functions $\sin x$ and $\cos x$ by their Maclaurin Series expansions:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots\right] \sum_{n=0}^{\infty} a_n x^n$$

$$= \left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^{n+1} + \frac{1}{6}\sum_{n=0}^{\infty} a_n x^{n+3}$$

$$-\frac{1}{120}\sum_{n=0}^{\infty} a_n x^{n+5} + \ldots = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n-1}x^n + \frac{1}{6}\sum_{n=3}^{\infty} a_{n-3}x^n$$

$$-\frac{1}{120}\sum_{n=5}^{\infty} a_{n-5}x^n + \ldots = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$$

8

$$\Rightarrow \left[2a_2\right] + \left[6a_3 - a_0\right]x + \left[12a_4 - a_1\right]x^2$$

$$+ \left[20a_5 - a_2 + \frac{1}{6}a_0\right]x^3 + \left[30a_6 - a_3 + \frac{1}{6}a_1\right]x^4$$

$$+ \ldots \quad = \quad 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$$

higher order terms not needed here as we only need up to $\underline{\underline{a_6}}$ (term in x^6)

Comparing coefficients of x^n: $\quad 2a_2 = 1 \Rightarrow \underline{a_2 = \frac{1}{2}}$

$$6a_3 - a_0 = 0 \Rightarrow \underline{a_3 = \frac{a_0}{6} = \frac{1}{2}}$$

$$12a_4 - a_1 = -\frac{1}{2} \Rightarrow \underline{a_4 = -\frac{1}{24}}$$

$$20a_5 - a_2 + \frac{1}{6}a_0 = 0 \Rightarrow \underline{a_5 = 0}$$

$$30a_6 - a_3 + \frac{1}{6}a_1 = \frac{1}{24} \Rightarrow \underline{a_6 = \frac{\frac{1}{24} + \frac{1}{2}}{30} = \frac{13}{720}}$$

$$\Rightarrow \boxed{y(x) = 3 + \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{24}x^4 + \frac{13}{720}x^6 + \ldots}$$

3.1: Investigation: Sub $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into: $x^2 \frac{d^2y}{dx^2} + y = 0$

$$x^2 \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow a_0 + a_1 x + \sum_{n=2}^{\infty} \left[n(n-1) + 1 \right] a_n x^n = 0$$

Compare coefficients of x^n:

$$a_0 = 0$$

$$a_1 = 0$$

$$\underbrace{\left[n(n-1) + 1 \right] a_n = 0}_{\text{but this gives } a_n = 0,\ \forall n \geq 2}, \quad n \geq 2$$

as $n(n-1) + 1 \neq 0$, $\forall n \geq 2$.

- So the method fails to find a solution: all coefficients found are zero.
- This happens because at $x = 0$, the coefficient of the $\frac{d^2y}{dx^2}$ term vanishes "sufficiently fast" (it turns out if it was x rather than x^2 we would be okay!)

$\to 0$

4.

(a).

(i). Write into standard form:

$$\frac{d^2y}{dx^2} + \frac{(2x-1)}{2x}\frac{dy}{dx} + \frac{1}{2x^2}y = 0 \qquad (\dagger)$$

Then: $p(x) = 1 - \frac{1}{2x}$ so: $x\,p(x) = x - \frac{1}{2}$ is analytic at $x=0$ ✓

$q(x) = \frac{1}{2x^2}$ so: $x^2 q(x) = \frac{1}{2}$ is analytic at $x=0$ ✓

Since $x=0$ is a singular point that satisfies the above two conditions it is clearly a <u>regular</u> singular point.

(ii). Sub $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the equation $(\dagger)$:

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + \left[1 - \frac{1}{2x}\right] \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$+ \frac{1}{2x^2} \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

<u>not</u> $n=2$ $\because$ r not necessarily integer!

$\Rightarrow$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$- \frac{1}{2} \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-2} + \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n+r-2} = 0$$

we'll shift the index here so <u>all</u> are x^{n+r-2}

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=1}^{\infty} a_{n-1}(n+r-1)x^{n+r-2}$$

$$-\frac{1}{2}\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n x^{n+r-2} = 0$$

$$\Rightarrow \left[r(r-1) - \frac{r}{2} + \frac{1}{2}\right] a_0 x^{r-2}$$

$$+ \sum_{n=1}^{\infty}\left(\left[(n+r)(n+r-1) - \frac{1}{2}(n+r) + \frac{1}{2}\right]a_n + (n+r-1)a_{n-1}\right)x^{n+r-2} = 0$$

Comparing coefficients of x:

$$r(r-1) - \frac{r}{2} + \frac{1}{2} = 0 \quad (5)$$

(note: wlog $a_0 \neq 0$ otherwise we could redefine $a_1 \to a_0$ and r would shift by 1).

$$\left[(n+r)(n+r-1) - \frac{1}{2}(n+r) + \frac{1}{2}\right]a_n + (n+r-1)a_{n-1} = 0, \quad n \geq 1 \quad (6)$$

Equation (5) gives: $r_1 = \frac{1}{2}$ or $r_2 = 1$

For $r_1 = \frac{1}{2}$, (6) gives: $\left[n^2 - \frac{1}{4} - \frac{n}{2} - \frac{1}{4} + \frac{1}{2}\right]a_n + (n - \frac{1}{2})a_{n-1} = 0$

$$\Rightarrow n(n-\tfrac{1}{2})a_n + (n-\tfrac{1}{2})a_{n-1} = 0$$

$$\Rightarrow a_n = -\frac{1}{n}a_{n-1}$$

$$\Rightarrow a_n = \frac{(-1)^n}{n!}a_0$$

$$\Rightarrow y_1(x) = a_0|x|^{\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}x^n = a_0|x|^{\frac{1}{2}}e^{-x}$$

For $r_2 = 1$, ⑥ gives: $\left[n(n+1) - \frac{1}{2}(n+1) + \frac{1}{2}\right] a_n + n a_{n-1} = 0$

$$\Rightarrow n\left(n+\frac{1}{2}\right) a_n = -n a_{n-1}$$

$$\Rightarrow a_n = -\frac{2}{2n+1} a_{n-1}$$

$$\Rightarrow a_n = (-1)^n \frac{2^n}{(2n+1)(2n-1)\cdots 3} a_0$$

$$\Rightarrow y_2(x) = a_0 |x| \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n+1)(2n-1)\cdots 3} x^n$$

So putting a linear combination of both solutions together, we have:

$$\boxed{y(x) = A|x|^{\frac{1}{2}} e^{-x} + B|x| \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)(2n-1)\cdots 3} x^n}$$

5. Extension:

Sub $y(x) = (x-x_0)^r \sum_{n=0} a_n (x-x_0)^n$

into $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) = 0$.

The trick is to let:

$$\left.\begin{aligned} p(x)(x-x_0) &= \sum_{n=0}^{\infty} p_n (x-x_0)^n \\ q(x)(x-x_0)^2 &= \sum_{n=0}^{\infty} q_n (x-x_0)^n \end{aligned}\right\} \quad ⑦$$

and in the algebraic manipulations introduce these quantities by pulling out the $(x-x_0)$'s needed from the derivatives!

p_0 and q_0 are then as defined in ⑦