# Problem Sheet 4

Math40002, Analysis 1

1. You are driving down a road whose speed limit is 60 miles per hour. A police officer sees your car at 12pm, and another officer 35 miles away sees your car at 12:30pm. Assuming they've attended their analysis lectures, how can they prove that you were speeding?

   *Solution.* The mean value theorem guarantees that there is some time strictly between 12:00 and 12:30 when your velocity was
$$\frac{35 \text{ miles}}{\frac{1}{2} \text{ hour}} = 70 \text{ miles per hour.}$$

2. Prove using l'Hôpital's rule that $\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^r$. (Hint: take logs first.)

   *Solution.* We write the limiting term as
$$\left(1 + \frac{r}{x}\right)^x = e^{x \log\left(1+\frac{r}{x}\right)},$$
   so by the continuity of $f(x) = e^x$ it will suffice to compute
$$\lim_{x\to\infty} \frac{\log\left(1 + \frac{r}{x}\right)}{1/x} = \lim_{y\downarrow 0} \frac{\log(1+ry)}{y}.$$
   The derivative of $\log(1 + ry)$ is $\frac{r}{1+ry}$, so we apply l'Hôpital's rule to get
$$\lim_{y\downarrow 0} \frac{\log(1+ry)}{y} = \lim_{y\downarrow 0} \frac{r/(1+ry)}{1} = r,$$
   and so $\lim_{x\to\infty} x \log\left(1 + \frac{r}{x}\right) = r$ and it follows that $\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^r$.

3. Use the mean value theorem to prove the following inequalities.

   (a) $|\sin(x) - \sin(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$

   (b) $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$

   *Solution.* (a) When $x = y$ it is clearly satisfied. If $x \neq y$, then by the mean value theorem there is some $z$ between $x$ and $y$ such that
$$\frac{\sin(x) - \sin(y)}{x - y} = \cos(z) \;\Rightarrow\; |\sin(x) - \sin(y)| = |\cos(z)||x - y| \le |x - y|,$$
   where the last inequality uses the fact that $|\cos(z)| \le 1$.

(b) Since $f(x) = \sqrt{x}$ has derivative $f'(x) = \frac{1}{2\sqrt{x}}$, there is some $z \in (n, n+1)$ such that

$$\frac{\sqrt{n+1} - \sqrt{n}}{(n+1) - n} = \frac{1}{2\sqrt{z}} \in \left( \frac{1}{2\sqrt{n+1}}, \frac{1}{2\sqrt{n}} \right).$$

(Here we use the fact that $f'(x)$ has derivative $-\frac{1}{4}x^{-3/2} < 0$ to know that it is strictly decreasing on $[n, n+1]$, so it is maximized and minimized at the endpoints of this interval.) This simplifies to the desired

$$\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}.$$

Remark: this also admits an elementary proof, using the observation that

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \in \left( \frac{1}{2\sqrt{n+1}}, \frac{1}{2\sqrt{n}} \right).$$

4. Let $H_n$ denote the harmonic sum $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

(a) Using the mean value theorem, prove that $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

(b) Prove that $H_n - 1 < \log(n) < H_{n-1}$ for all $n \geq 2$, where $H_k = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}$, and deduce that $\log(n+1) < H_n < \log(n) + 1$.

(c) Prove that the sequence $\big(H_n - \log(n)\big)$ is decreasing, and that $\lim_{n\to\infty} (H_n - \log(n))$ exists. (This limit is called the *Euler–Mascheroni constant* $\gamma \approx 0.577\ldots$.)

*Solution.* (a) Since $\log(x)$ has derivative $\frac{1}{x}$, there is some $z \in (n, n+1)$ such that

$$\frac{\log(n+1) - \log(n)}{(n+1) - n} = \frac{1}{z} \in \left( \frac{1}{n+1}, \frac{1}{n} \right),$$

or equivalently $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$.

(b) We sum each side of the inequality $\frac{1}{k+1} < \log(k+1) - \log(k) < \frac{1}{k}$ from $k = 1$ to $n - 1$ to get

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \log(n) - \log(1) < \sum_{k=1}^{n-1} \frac{1}{k},$$

noticing that lots of cancellation occurs in the middle. Since $\log(1) = 0$, this is equivalent to $H_n - 1 < \log(n) < H_{n-1}$. The left half of this gives us $H_n < \log(n) + 1$, and when we replace $n$ with $n + 1$ the right half gives us $\log(n+1) < H_n$, so we combine these to get $\log(n+1) < H_n < \log(n) + 1$.

(c) From part (b) we have

$$0 < \log(n+1) - \log(n) < H_n - \log(n) < 1,$$

so the sequence $a_n = H_n - \log(n)$ is bounded, and thus if it is monotone decreasing then it converges. We compute

$$a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log(n),$$

so if we let $f(x) = \frac{1}{x+1} - \log(x+1) + \log(x)$, then we want to show that $f(n) < 0$ for all integers $n \geq 1$. We have

$$\begin{aligned} f'(x) &= -\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x} \\ &= -\frac{1}{(x+1)^2} + \frac{1}{x(x+1)} = \frac{1}{x(x+1)^2} \end{aligned}$$

and so $f'(x) > 0$ for all $x > 0$, meaning that $f(x)$ is strictly increasing. But

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left( \frac{1}{x+1} - \log\left(1 + \frac{1}{x}\right) \right) = 0.$$

Since $f(x)$ is strictly increasing, this implies that $f(x) < 0$ for all $x > 0$. (Proof: assuming otherwise, let $\epsilon = f(y)$ be a positive value of $f$. Then there is no $N > 0$ such that if $x \geq N$ then $|f(x) - 0| < \epsilon$, because as soon as $x > \max(y, N)$ we have $f(x) > f(y) = \epsilon$. This contradicts $f(x) \to 0$.) So in particular, for all integers $n \geq 1$ we have $a_{n+1} - a_n = f(n) < 0$, hence $a_{n+1} < a_n$.

5. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose there is a constant $C < 1$ such that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. We will prove that $f$ has exactly one fixed point, meaning there is a unique $y \in \mathbb{R}$ such that $f(y) = y$. Pick some $x_0 \in \mathbb{R}$ and let

$$x_{n+1} = f(x_n) \text{ for all } n \geq 0.$$

(a) Prove that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all $n$.

(b) Prove that the sequence $(x_n)$ converges, and that if its limit is $y$ then $f(y) = y$.

(c) Prove that $f$ cannot have two different fixed points.

*Solution.* (a) If $x_{n+1} = x_n$ then $x_{n+2} = f(x_{n+1}) = f(x_n) = x_{n+1}$, and so both sides of the desired inequality are zero. Otherwise, the mean value theorem tells us that there is some $t$ between $x_n$ and $x_{n+1}$ such that

$$\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(t) \;\Rightarrow\; \left| \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \right| = |f'(t)| \leq C.$$

Thus $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ as desired.

(b) Write $d = |x_1 - x_0|$. Then $|x_{n+1} - x_n| \leq C^n d$ by induction and part (a). The triangle inequality says that for any integers $m \geq n$,

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n| \\ &\leq C^{m-1}d + \cdots + C^{n+1}d + C^n d \\ &< \sum_{i=n}^{\infty} C^i d = \frac{C^n d}{1 - C}. \end{aligned}$$

For any $\epsilon > 0$ we can find $N \geq 0$ such that $\frac{C^N d}{1-C} < \epsilon$, since the left side approaches 0 as $N \to \infty$. Then given $m, n \geq N$ we have shown that

$$|x_m - x_n| < \frac{C^N d}{1 - C} < \epsilon,$$

which proves that the sequence $(x_n)$ is Cauchy and hence convergent, say with limit $y$. Since $x_n \to y$ and $f$ is continuous, we have $f(x_n) \to f(y)$. But $f(x_n) = x_{n+1} \to y$, so it must be the case that $f(y) = y$.

(c) Suppose that $y$ and $z$ are distinct fixed points of $f$. By the mean value theorem, there is some $t$ between $y$ and $z$ such that

$$\frac{f(y) - f(z)}{y - z} = f'(t) \;\Rightarrow\; \frac{y - z}{y - z} = f'(t) \;\Rightarrow\; f'(t) = 1.$$

But this contradicts the assumption that $|f'(x)| \leq C < 1$ for all $x \in \mathbb{R}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that there is some $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$.

   (a) Prove that $f$ is *Lipschitz*, meaning that there is some constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$.

   (b) Prove that Lipschitz functions are uniformly continuous, and conclude that $f$ is uniformly continuous.

*Solution.* (a) If $x = y$ then the Lipschitz condition is automatically satisfied, so we can assume that $x \neq y$. The mean value theorem tells us that there is some $z$ between $x$ and $y$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(z),$$

so taking absolute values gives

$$|f(x) - f(y)| = |f'(z)||x - y| \leq M|x - y|$$

which is precisely the Lipschitz condition with $C = M$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz; fix $\epsilon > 0$ and set $\delta = \frac{\epsilon}{C}$. Then for any $x, y \in \mathbb{R}$, we have

$$|x - y| < \delta \;\Rightarrow\; |g(x) - g(y)| \leq C|x - y| < C\delta = C \cdot \frac{\epsilon}{C} = \epsilon,$$

so $g$ satisfies the definition of uniform continuity. In particular, this applies to $f(x)$ by part (a).

7. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. We will prove that $f'(x)$ has the intermediate value property even though it may not be continuous. Throughout this problem, we will suppose that $f'(a) < f'(b)$ and fix some $t$ such that $f'(a) < t < f'(b)$.

   (a) Let $g(x) = f(x) - tx$. Prove that there is some $c \in (a, b)$ such that $g(c) < g(a)$. (Hint: what is $g'(a)$?) Similarly, prove that there is some $d \in (a, b)$ such that $g(d) < g(b)$. In other words, $g(x)$ is not minimized at $x = a$ or at $x = b$.

   (b) Show that there is some $y \in (a, b)$ such that $g'(y) = 0$, and deduce that $f'(y) = t$.

*Solution.* (a) We know that $g(x)$ is differentiable, with $g'(x) = f'(x) - t$, so in particular $g'(a) < 0$. Fixing $\epsilon = |g'(a)| > 0$, there is $\delta > 0$ such that

$$a < x < a + \delta \;\Rightarrow\; \left| \frac{g(x) - g(a)}{x - a} - g'(a) \right| < \epsilon = -g'(a),$$

and since $x - a$ is positive, this implies that

$$\frac{g(x) - g(a)}{x - a} - g'(a) < -g'(a) \;\Rightarrow\; g(x) - g(a) < 0.$$

Thus $g(x) < g(a)$ for all $x \in (a, a + \delta)$ and we can take $c = \min(a + \frac{\delta}{2}, \frac{a+b}{2})$. (The point of taking this minimum is just to make sure that $c \in (a, b)$.)
Similarly, we have $g'(b) > 0$, so for $\epsilon = g'(b)$ we can find $\delta > 0$ such that

$$b - \delta < x < b \;\Rightarrow\; \left| \frac{g(x) - g(b)}{x - b} - g'(b) \right| < \epsilon = g'(b).$$

We deduce from this and the fact that $x - b < 0$ that

$$\frac{g(x) - g(b)}{x - b} - g'(b) \geq -g'(b) \;\Rightarrow\; g(x) - g(b) \leq 0,$$

so $g(x) < g(b)$ for $x \in (b - \delta, b)$ and we can take $d = \max(b - \frac{\delta}{2}, \frac{a+b}{2})$.

(b) We know that $g$ is continuous since it is differentiable, so the extreme value theorem says that $g(x)$ achieves a minimum at some $y \in [a, b]$. By part (a) we know that $y \neq a$ and $y \neq b$, so $y \in (a, b)$, and since $y$ is a local minimum of $g$ it follows that $g'(y) = 0$. Since $g'(x) = f'(x) - t$, we must have $f'(y) = t$.