Imperial College London

MATH40007

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

An Introduction to Applied Maths

Date: 18th May 2020

Time: 09.00am – 11.00am (BST)

Time Allowed: 2 Hours

Upload Time Allowed: 30 Minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

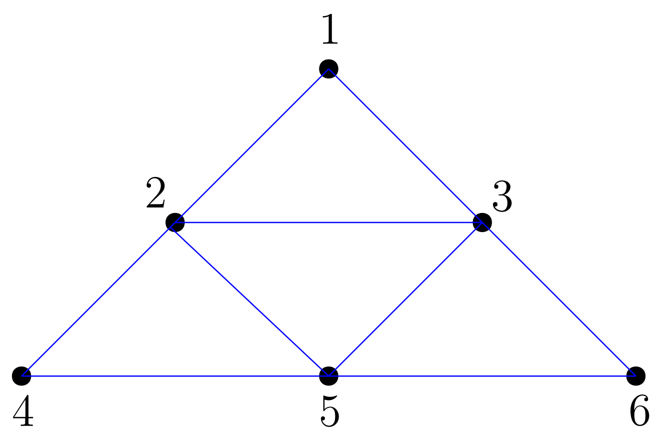
Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the electrical circuit shown in the Figure comprising 6 nodes connected by 9 resistors each of which has unit conductance.



(a) Using the ordering of nodes indicated in the Figure, find the Laplacian matrix of the graph associated with this circuit.

(4 marks)

(b) Node 1 is set at unit voltage, node 4 is grounded, and Kirchhoff's current law holds at all other nodes. Find the voltages at nodes 2, 3, 5 and 6.

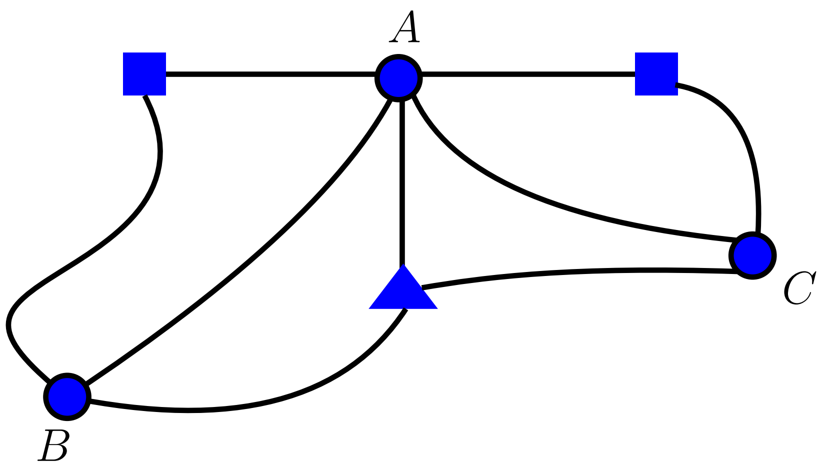
(8 marks)

(c) Suppose a unit current enters the circuit at node 1 and exits the circuit at node 4. Find the current into node 6 from node 3.

(8 marks)

(Total: 20 marks)

2. This question concerns a simple random journey around a train network with 6 stations connected with train lines as shown in this Figure:



At each station the random traveller picks at random a train line to take out of that station with all lines having equal probability, including possibly going back on the train line just taken.

(a) Suppose the random traveller can start at any of the 3 stations A, B or C. Also, he wants to maximize his probability of reaching one of the two square stations before he reaches the triangular station. Determine which of the 3 stations A, B or C he should start at, and find the associated probability of reaching a square station before the triangular station.

(6 marks)

(b) Suppose the random traveller starts at the triangular station. What is the probability that he reaches either of the two square stations before returning to the triangular station?

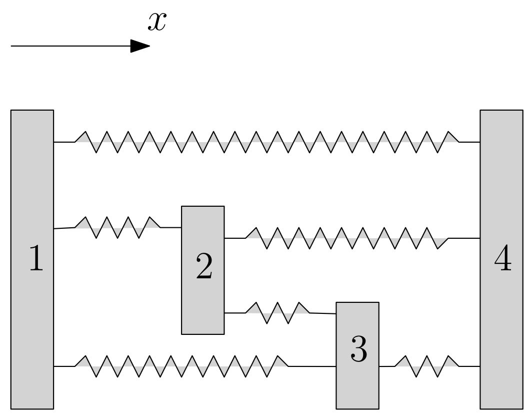
(6 marks)

(c) The square station on the right in the Figure is closed for engineering works so that the two train lines connecting it to stations A and C are out of service. Find the probability of a random traveller starting at station A reaching station B before returning to station A.

(8 marks)

(Total: 20 marks)

3. Consider the system of 4 masses linked by springs as shown in the Figure:



All springs have unit spring constant (even though, in the Figure, they have different lengths). All masses have unit mass (even though, in the Figure, they have different sizes). The masses can only experience displacements to the left and right in the x direction shown.

(a) By treating this spring-mass system as a graph with 4 nodes (the masses) and 6 edges (the springs) find the 4-by-4 generalized Laplacian matrix associated with the internal spring forces in this system. (4 marks)

(b) Suppose mass 1 is held fixed and let $\mathbf{X}(t) = (X_2(t), X_3(t), X_4(t))^T$ be the 3-vector of displacements under free oscillation (no external forces) of masses 2,3 and 4. On letting

$$\mathbf{X}(t) = e^{\mathrm{i}\omega_j t}\mathbf{e}_j, \qquad j = 1, 2, 3,$$

where $\mathbf{e}_j$ is a constant vector, find the three possible natural frequencies ω_j and the associated vectors $\mathbf{e}_j$ of free oscillation of this system. (6 marks)

(c) Consider now an equilibrium configuration in which mass 1 is held fixed in place so it cannot be displaced, while an external force $+1$ is imposed on mass 4. There are no external forces on masses 2 and 3. Let the associated equilibrium displacements of masses 2,3, and 4 be denoted by the 3-vector $\mathbf{\Phi}$ and write

$$\mathbf{\Phi} = \sum_{j=1}^{3} a_j \mathbf{e}_j,$$

where the vectors $\{\mathbf{e}_j | j = 1, 2, 3\}$ are those found in part (b). Find the coefficients a_1, a_2 and a_3.

(6 marks)

(d) Hence find the displacements of masses 2,3 and 4 for the equilibrium in part (c). (4 marks)

(Total: 20 marks)

4. Let D be the first quadrant $x > 0, y > 0$ in a two-dimensional (x, y) plane and suppose that it is an electrical conductor with unit conductivity. Suppose the voltage distribution $\phi(x, y)$ associated with a current in D is given by the function

$$\phi(x, y) = \mathrm{Re}[h(z)],$$

where $z = x + \mathrm{i}y$ and

$$h(z) = -\frac{m}{2\pi} \log\left[\frac{z^2 - \mathrm{i}}{z^2 + \mathrm{i}}\right].$$

The notation $\mathrm{Re}[\ .\ ]$ means take the real part of the complex-valued quantity in square brackets.

(a) Verify that $\nabla^2\phi(x, y) = 0$ everywhere in D except at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. (5 marks)

(b) Show that the boundary $y = 0$ is grounded, i.e. show that $\phi(x, 0) = 0$. (2 marks)

(c) Show that the boundary $x = 0$ is grounded, i.e. show that $\phi(0, y) = 0$. (2 marks)

(d) If the associated current density vector is written as $\mathbf{j} = (j_x, j_y)$ find the y-component of the current density distribution on the boundary $y = 0, x \geq 0$ as a function of x (that is, find the quantity $j_y(x, 0)$ as a function of x). (5 marks)

(e) Hence show that the total current passing across the boundary $y = 0, x \geq 0$ out of the conductor D is equal to $m/2$. (6 marks)

(Total: 20 marks)