MATH40007 Introduction To Applied Mathematics

Professor Darren Crowdy

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1 Introduction to Graphs

1.1 The Matrix

Let's begin by considering the following matrix:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

This matrix has m rows and n columns, with $m = 5$ and $n = 4$. Note that A is *not* square.
Question: What is the rank of A? (*Recall: The rank of a matrix is the dimension of its row space and column space*)

Answer: Notice that the vector:

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

has the property of $Ax_0 = \mathbf{0}$, so we say that x_0 is in the *right null space* of A (we can also say null space or Kernel, but this way works best to distinguish it from other terms used later in this course). Thus we can think of the 4^{th} column of A as being *linearly dependent* on the other 3. Now we need to ask if the first 3 columns are linearly independent.

$$\text{Suppose}: \; c_1 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{0}$$

By looking at each of the final three elements of the LHS and RHS, we can immediately see that $c_1 = c_2 = c_3 = 0$, hence the first three columns are indeed linearly independent and we can say that the rank of $A = 3$

Question: How many vectors satisfy the equation $A^T\mathbf{w} = \mathbf{0}$?

Answer: Well let's look at some 'random' vectors and see if they work. (Don't worry, we will be more specific about how to find these vectors soon). Examining the vectors:

$$\mathbf{w_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{w_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{w_3} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

We get that all of these satisfy the required equation, but we can also observe that $\mathbf{w_3} = \mathbf{w_1} + \mathbf{w_2}$. So this raises the question of whether $\mathbf{w_1}$ and $\mathbf{w_2}$ are linearly independent. We can check this in a similar way to before.

$$\text{Suppose}: \ c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \mathbf{0}$$

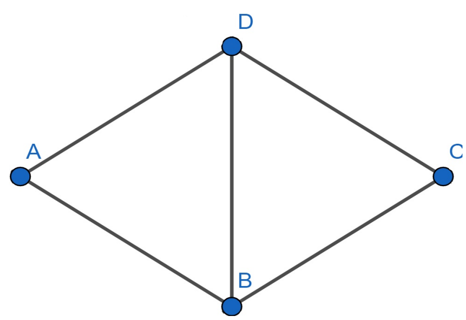
By looking at the first two elements of the LHS and RHS, we see that $c_1 = c_2 = 0$. So we have now found two linearly independent vectors which satisfy the original equation, but why can we not find any more? The answer lies in the *Rank-Nullity Theorem* (seen in Linear Algebra)
Note: $A^T\mathbf{w} = \mathbf{0} \iff \mathbf{w}^T A = \mathbf{0}$ by taking transposes of both sides, so we can say that $\mathbf{w}$ lies in the *Left Null Space* of A.
We can now use the statement from the rank-nullity theorem to say the following:

$$dim(Left\ Null\ Space) + Rank(A) = m = 5$$

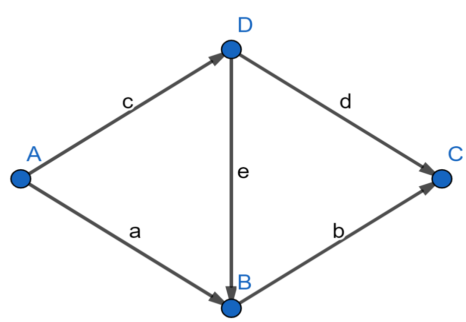
Hence $dim(Left\ Null\ Space) = 2$, and as we have found two linearly independent vectors in this set, they form a basis for this subspace.

1.2 Geometrical Representation of A

We can give a geometrical representation of the matrix A. Draw the following graph:



The graph has 4 'nodes' and 5 'edges' linking some of them.
Labelling: It is useful to label the nodes as 1,2,3,4 and the edges as a, b, c, d, e
Then create a direction on each edge (arbitrarily- any choice will do). But let's pick:



Now we will construct a matrix as follows:

1. Set the number of columns equal to the number of nodes
2. Set the number of rows equal to the number of edges
3. Set all entries equal to 0 initially
4. For each edge, alter a row with +1 in the column corresponding to node i and a -1 in the column corresponding to node j if the edge connects nodes i and j, and the direction of the edge goes from node j **to** node i.

Let's do it for the graph we just drew:

Step 1: writing out the original matrix

$$\begin{array}{c} \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e \end{array} \begin{array}{c} \begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step 2: filling in the row for edge a

$$\begin{array}{c} \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e \end{array} \begin{array}{c} \begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step 3: filling in the row for edge b

$$\begin{array}{c} \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e \end{array} \begin{array}{c} \begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step 4: filling in the row for edge c

$$\begin{array}{c} \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e \end{array} \begin{array}{c} \begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step 5: filling in the row for edge d

$$\begin{array}{c} \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e \end{array} \begin{array}{c} \begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step 6: filling in the row for edge e

$$\begin{array}{c} \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e \end{array} \begin{array}{c} \begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

And it doesn't come as much of a surprise that the matrix that we have constructed is *identical* to the matrix introduced at the start of this course!

We call this matrix, which has been formed from a (directed) graph, the *Incidence Matrix* and we can think of it as encoding the information from the graph.

1.3 Properties of A

Observation: Matrices can operate on vectors!

$$\text{Take } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$$

$$\text{Then } A\mathbf{x} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_3 - x_4 \\ x_2 - x_4 \end{pmatrix}$$

Hence, if we assign a value to each of the nodes, then $A\mathbf{x} \in \mathbb{R}^5$ can be seen as the difference in values across each of the 5 edges.

Definition: We will call a vector of values assigned to the set of nodes as a *Vector of Potentials*, denoted using $\mathbf{x}$.

The matrix A operating on $\mathbf{x}$ (the potentials) gives us the potential differences across the edges of the graph.

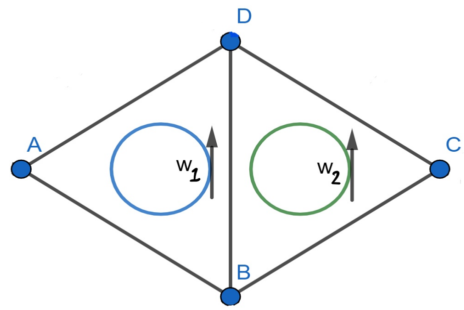
Summary: We have that $\mathbf{e} = A\mathbf{x}$ with A and $\mathbf{x}$ as above and $\mathbf{e}$ being the vector of potential differences. *Note*: If the potentials at all the nodes are the same, then all of the potential differences will be ZERO, exactly the first statement we came up with, so:

$$\mathbf{x_0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow A\mathbf{x_0} = \mathbf{0}$$

Provided the graph is *connected*- by which we mean that every node can be reached from any other node by some series of edges- then there will always be a right null vector of all ones.

Question: How do we interpret $\mathbf{w_1}$,$\mathbf{w_2}$ and $\mathbf{w_3}$ in a geometric sense?

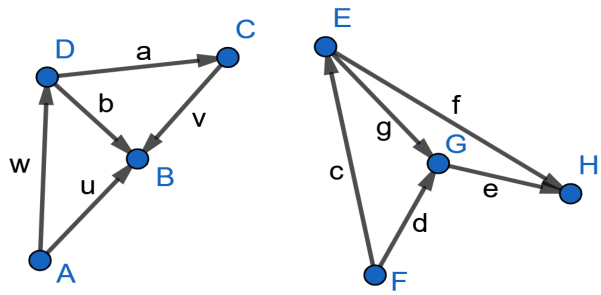
Answer: LOOPS!!!!!



Looking in more detail at $\mathbf{w_1}$, we see that edge a is traversed in the positive direction, and edges c and e are traversed in the negative direction. Edges b and d are not involved in this loop. Looking back at $\mathbf{w_1} = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \end{pmatrix}^T$, the geometric interpretation matches the algebraic deduction we made at the start.

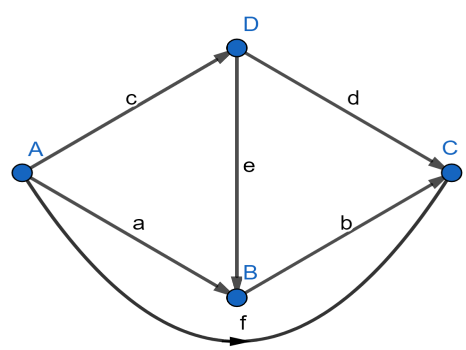
Exercise: Check $\mathbf{w_2}$ and $\mathbf{w_3}$ in a similar sense.

Question: What are the incidence matrices for the following graphs?



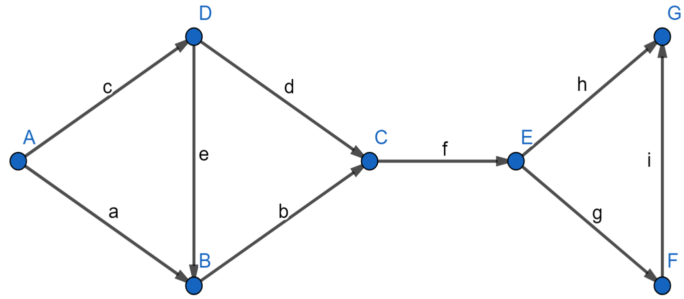
Answer: The same as the A we constructed. The point of this question, while it may seem obvious, is to recognise when two graphs are topologically identical, even if details of their geometry look different.

Let's take our original graph and add an edge, and observe what happens to the properties of the graph and the incidence matrix.



There will be a new edge added to A consisting of the entries $\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}$ which makes A now a 6-by-4 matrix. We now have that each node is now connected to every other node. We call this type of graph *complete*.

Let's look at another graph and its incidence matrix.



$$A' = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

Clearly $\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ is in the right null space of A'. Now the question is if there are any other elements of the right null space. The answer is no. The reason why is because the graph is connected. For connected graphs, it is easy to show and argue that the right null space only has one element, the vector consisting of all ones.

From this, we can deduce the idea that for graph, there will be at least one connected sub-graph, and the dimension of the right null space will of the incidence matrix be equal to the number of these connected sub-graphs in the overall graph.

<u>Hence</u> $Rank(A') = n - 1 = 7 - 1 = 6$ and for the number of loops, we have

$$dim(LNS) + Rank(A') = m$$

$$dim(LNS) = 9 - 6$$

$$dim(LNS) = 3$$

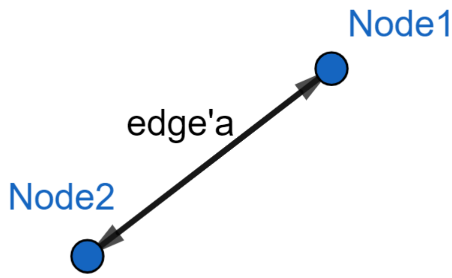
Which gives us that the number of loops in the graph is 3.

Exercise: Find the loops!

Now what happens if we remove edge F. We then have 2 disconnected sub-graphs, shown below. If A is the incidence matrix for this graph, then $Rank(A) = n - 2 = 7 - 2 = 5$ which gives us two linearly independent equipotential states.

Exercise: Find these equipotential states.

2 Node and Edge Variables



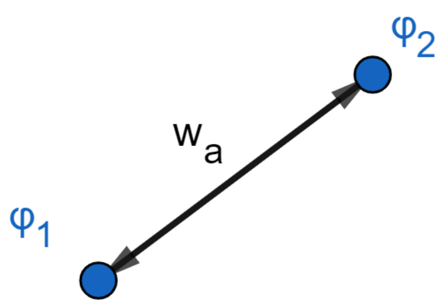
I'm sorry for the weird naming, Geogebra is very annoying

Given 2 nodes, 1 & 2, we can imagine that a VALUE is assigned to each node, ϕ_1 and $\phi_2 \in \mathbb{R}$, for example.

Definition: We will call these *Node Variables* or *Potentials*

We can also imagine that a value can be assigned to the *edge a* between the nodes, say $\mathbf{w}_a$

Definition: We will call these *Edge Variables* or *Flux* ('flux' because we thing of some quantity 'flowing' between the nodes)



FACT: Interesting applied mathematics problems involve relationships between node variables & edge variables (often dictated y physical principles/laws, but not always) as well as imposed constraints (again, arising from physical considerations or other factors).

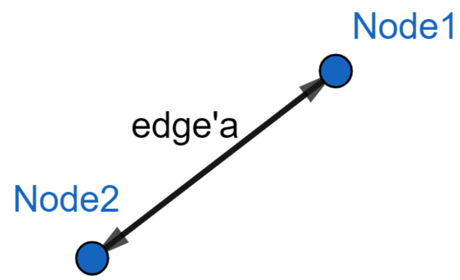
This course will survey several of these applications.

Mathematically, the framework is the SAME! Just the details of each problem is different.

3 Circuit Theory

The graphs we have been studying might be ELECTRIC CIRCUITS (one of many possibilities - we'll see others later)

If so, given any 2 nodes with an edge:



A CURRENT $\mathbf{w}_a$ may be flowing along the edge a joining nodes 1 and 2

Let the voltage at node 1 be ϕ_1 & the voltage at node 2 be ϕ_2. These are our 'potentials'.

Ohm's Law: This tells us that the strength of the current across an edge, or 'wire', or 'resistor', is related to the conductance c_a of that edge and the voltage drop (conductance is a material property).

In particular, Ohm's Law says:

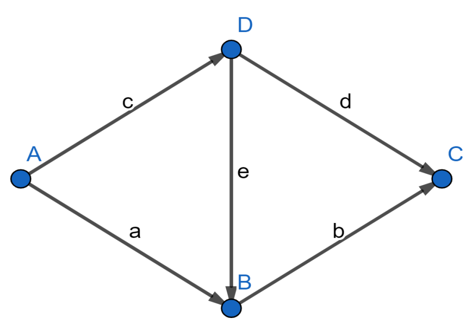
$$w_a = -c_a \cdot (\phi_2 - \phi_1)$$

Where $\phi_2 - \phi_1$ is the potential difference across edge a.

Note: Current flows from high voltage to low voltage.

We think of voltage as the 'potential' at the nodes (for this problem)

Example: Let's suppose the graph we considered earlier is a CIRCUIT of resistors:



Each edge here is a RESISTOR

Suppose for now that all resistors have conductance equal to UNITY, so:

$$c_a = c_b = c_c = c_d = c_e = 1$$

Let $\mathbf{x}$ be the vector of voltages at the nodes: $\mathbf{x} = \begin{pmatrix}\phi_1 & \phi_2 & \phi_3 & \phi_4\end{pmatrix}^T$

The <u>incidence matrix</u> was found to be:

$$
\begin{array}{c}
 \\ \text{edge } a \\ \text{edge } b \\ \text{edge } c \\ \text{edge } d \\ \text{edge } e
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 0 & -1
\end{pmatrix}
\end{array}
$$

Then:

$$
\mathbf{e} = A\mathbf{x} = \begin{pmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_2 \\ \phi_4 - \phi_1 \\ \phi_3 - \phi_4 \\ \phi_2 - \phi_4 \end{pmatrix}
$$

By Ohm's Law, with equal conductances equal to UNITY, the currents in each edge, $\mathbf{w} = \begin{pmatrix} w_a & w_b & w_c & w_d & w_e \end{pmatrix}^T$ are given by:

$$
\mathbf{w} = -A\mathbf{x} \tag{1}
$$

This is because $w_a = -c_a(\phi_2 - \phi_1) = -(\phi_2 - \phi_1)$ as $c_a = 1$

If $\mathbf{x} = \mathbf{x_0} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$, then all potentials are the same $\Rightarrow$ no potential differences $\Rightarrow$ <u>NO</u> currents, i.e. $\mathbf{w} = -A\mathbf{x} = \mathbf{0}$ which makes sense!!!

Question: What does $A^T\mathbf{w}$ correspond to in this context?

$$
A^T w = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} w_a \\ w_b \\ w_c \\ w_d \\ w_e \end{pmatrix} = \begin{pmatrix} -w_a - w_c \\ w_a - w_b + w_e \\ w_b + w_d \\ w_c - w_d - w_e \end{pmatrix}
$$

By inspection of the circuit, each component of $A^T\mathbf{w}$ is the net current <u>IN</u> to each of the corresponding nodes.

Equivalently, $-A^T\mathbf{w}$ is the vector whose elements are the net currents <u>OUT</u> of the nodes.

<u>Definition:</u> We will call $-A^T\mathbf{w}$ the DIVERGENCE of the currents at the nodes. We will use $\mathbf{f}$ to denote this. So $\mathbf{f} = -A^T\mathbf{w}$

But we know from (1) that $\mathbf{w} = -A\mathbf{x}$ because of Ohm's Law. So we now get that $\mathbf{f} = -A^T\mathbf{w} = -A^T(-A\mathbf{x}) = A^TA\mathbf{x}$

$$
\mathbf{f} = A^TA\mathbf{x}
$$

This is a *VERY IMPORTANT EQUATION* and we will see the importance of it very soon. For now, we have that $\mathbf{f}$, the vector of currents OUT of each node is connected to $\mathbf{x}$, the vector of the voltages at the nodes by this matrix A^TA, which we will call the *LAPLACIAN MATRIX*, and will be denoted by $K = A^TA$

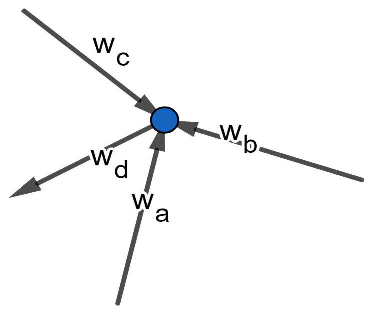
<u>Note:</u> It is remarkable that the Laplacian matrix of the graph appears in this physical law relating voltages to currents (with conductances hidden in there - more on this later)

4 Kirchoff's Current Law

The law governing the flow of current in an electrical circuit is Kirchoff;s Current Law (KCL for short): The net current out of a node in an electrical circuit must VANISH (unless it is connected to an external source/sink such as a battery)

In simpler terms: 'Flow of current in must equal the flow of current out of a node'

e.g.



Flux or current OUT of the node $= w_d - w_a - w_b - w_c = 0$ by KCL

5 Laplacian, degree & adjacency matrices

For the original graph that we began with, we had the following incidence matrix and its transpose:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \text{ and } A^T = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

Now, let's calculate the 'Laplacian Matrix' for this graph:

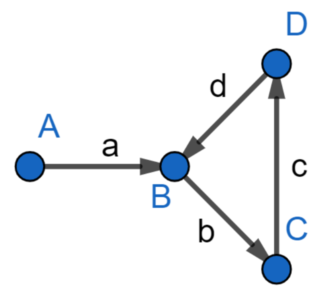
$$K = A^T A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Note: All of the off-diagonal entries are either -1 or 0

Question: Do you notice any significance to the diagonal entries?

Do another example:



Incidence matrix:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

Transpose:

$$A^T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Laplacian:

$$K = A^T A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

The same thing happens here in the off-diagonal. All the terms are either 0 or -1.

Now is it clearer what the significance of the diagonal entries are? (I wouldn't blame you if you don't get it yet, it's not the easiest to spot.)

The diagonal values of the Laplacian, K_{ii} is precisely the NUMBER OF EDGES connected to node i! (aka the 'degree' of node i)

Also notice that all K_{ij} for $i \neq j$ are either 0 or -1 (as mentioned several times earlier)

In particular:

$K_{ij} = 0$ if node i and node j are NOT connected by an edge.

$K_{ij} = -1$ if node i and node j ARE connected by an edge.

This is ALWAYS true [Exercise: Prove it]

We decompose the Laplacian K as follows:

$$K = D - W$$

Where K is the Laplacian matrix, D is the 'degree matrix', and W is the 'adjacency matrix'. W contains only '1' elements in off-diagonals corresponding to connected ('adjacent') nodes.

For the first graph, we had:

$$K = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, W = \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

For the second graph we calculated the Laplacian for:

$$K = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, W = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

Note: It is easy to construct the Laplacian matrix of a graph directly by using the representations of the degree and adjacency matrices. This means you don't need the incidence matrix A

Note again! The Laplacian matrix is independent of the assignment of the directions of the graph's edges.

6 Basic Circuit Theory

We know that the net current source at a set of nodes is given by $\mathbf{f} = -A^T\mathbf{w}$. Then suppose that Kirchoff's law holds at ALL nodes of our given graph. Then $\mathbf{f} = 0 \Rightarrow -A^T\mathbf{w} = 0$, or, on taking a transpose & dropping the minus sign, $\mathbf{w}A^T = 0$

.

Thus, $\mathbf{w}$ lies in the left null space of A! (We know non-trivial flows like this exist). But, by the rank-nullity theorems, this means that $\mathbf{w}$ is perpendicular to the column space of A. But the column space of A is the set of all vectors that can be written as $A\mathbf{x}$ for some set of coefficients of the columns collected in vector $\mathbf{x}$. Rank-nullity theorem states that $\mathbf{w}$ is perpendicular to this space.

Expressed differently, $\mathbf{w}$ cant NOT be expressed in the form $A\mathbf{x}$ for some non-zero $\mathbf{x}$.

From a physical perspective, there is no set of node voltages tha will sustain/produce this current in the circuit. (Mathematically, $\mathbf{w}$ must lie in the direct complement of the column space of A in $\mathbb{R}^m$)

Conclusion: The solutions $\mathbf{w_1}$ & $\mathbf{w_2}$ from our original graph in the first lecture correspond to 'currents' that cannot be derived from a set of voltage potentials ('unphysical' - just mathematical solutions)

Since under the assumption that KCL holds at all nodes, there is no admissible potentials that will produce a flow/current, then we are forced to relax this assumption that KCL holds at all nodes in a graph.

Hence we must have $\mathbf{f} \neq 0$ at a minimum of ONE node (i.e. $\mathbf{f}$ must be non-zero somewhere!). Indeed we can be more precise. Since $-A^T\mathbf{w} = \mathbf{f} \Rightarrow A^T\mathbf{w} = -\mathbf{f}$ which says that $-\mathbf{f}$ lies in the column space of A^T. But this is exactly the ROW SPACE of A.

By rank-nullity, the null-space of A is perpendicular to the row space of A, hence we must have $\mathbf{f}^T\mathbf{x_0} = 0 = \mathbf{x_0}^T\mathbf{f}$ where $\mathbf{x_0} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$, with n 1's, which is in the null space of any (connected graph's) incidence matrix.

7 Two-point Source/Sink Circuits

Given this constraint on the circuits we can consider, the simplest first case to consider is $\mathbf{f} = \begin{pmatrix}1 & -1 & 0 & \cdots & 0\end{pmatrix}$. This satisfies the condition stated at the end of the last section, $\mathbf{x_0}^T\mathbf{f} = 0$. Physically, this corresponds to a unit current entering the circuit at node 1 and a unit current leaving the current at node 2.

Note: This is EASY to effect in an actual current by attaching the +ve and -ve electrodes of a BATTERY to nodes 1 and 2 respectively.

Problem I ('Neumann Problem'): Specify current sources as $\mathbf{f} = \begin{pmatrix}1 & -1 & 0 & \cdots & 0\end{pmatrix}$ & find potentials at the nodes (i.e. voltages at the nodes)

Back to the graph! Suppose unit current enters node 1 & unit current leaves the circuit at node 2. Then $\mathbf{f} = \begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix}^T$. Now we need to find the potentials, or VOLTAGES at the nodes.

We know $K\mathbf{x} = \mathbf{f}$ or:

$$\begin{pmatrix}2 & -1 & 0 & -1\\ -1 & 3 & -1 & -1\\ 0 & -1 & 2 & -1\\ -1 & -1 & -1 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}+1\\ -1\\ 0\\ 0\end{pmatrix}$$

But isn't the solution simply $\mathbf{x} = K^{-1}\mathbf{f}$?

ABSOLUTELY NOT!!! Because K is singular, and so K^{-1} doesn't actually exist. K is singular because $K\mathbf{x_0} = (A^TA)\mathbf{x_0} = A^T(A\mathbf{x_0}) = 0$, if $\mathbf{x_0}$ is in the null space of A, it is also in the null space of the Laplacian K.

But this gives us a clue as to why $\mathbf{x} = K^{-1}\mathbf{f}$ is not the solution. This is because if $\mathbf{x}$ is a solution, then another solution is $\mathbf{x} + c\mathbf{x_0}$ with $c \in \mathbb{R}$. To see this: $K(\mathbf{x} + c\mathbf{x_0}) = K\mathbf{x} + cK\mathbf{x_0} = \mathbf{f}$ as $cK\mathbf{x_0} = 0$, so $\mathbf{x} + c\mathbf{x_0}$ is also a solution to $K\mathbf{x} = \mathbf{f}$

Expressed differently, the set of potentials is ONLY defined up to an additive constant (added to the voltages at all nodes). This means we have the freedom to set one of the voltages as we like!

'GROUNDING': We therefore decide to 'ground' one of the nodes ,that is, set its voltage to ZERO. Any node will do. Let's choose to ground node 2 in our example (I really hope you're using colour at this point, because I use red to denote any deleted entries):

$$K\mathbf{x} = \begin{pmatrix}2 & -1 & 0 & -1\\ -1 & 3 & -1 & -1\\ 0 & -1 & 2 & -1\\ -1 & -1 & -1 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\ -1\\ 0\\ 0\end{pmatrix}$$

Grounding node 2 corresponds to DELETING row 2 and column 2 from the Laplacian, which leaves us with a new linear system:

$$\begin{pmatrix}2 & 0 & -1\\ 0 & 2 & -1\\ -1 & -1 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Now we can solve this by hand:

$$2x_1 - x_4 = 1 \tag{1}$$
$$2x_3 - x_4 = 0 \Rightarrow x_4 = 2x_3 \tag{2}$$
$$-x_1 - x_3 + 3x_4 = 0 \tag{3}$$

$$2 \ \& \ 3 \Rightarrow -x_1 - x_3 + 6x_3 = 0 \Rightarrow 5x_3 = x_1$$
$$2 \ \& \ 1 \Rightarrow 2x_1 - x_4 = 10x_3 - 2x_3 = 1$$

Which gives us that $\underline{x_3 = 1/8}$, $\underline{x_1 = 5/8}$, $\underline{x_4 = 1/4}$. Let's now check the consistency of this solution with the equation we deleted.

$$-x_1 + 3x_2 - x_3 - x_4 = -\frac{5}{8} - \frac{1}{8} - \frac{2}{8} = -\frac{8}{8} = -1$$

This is what we expected, as a unit current is leaving the circuit at node 2.

Sometimes the matrix we got from deleting one of the rows and columns is called the 'reduced Laplacian matrix', and we use $\widehat{K}$ (there is a hat there, trust me). So in this case, we had:

$$\widehat{K} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

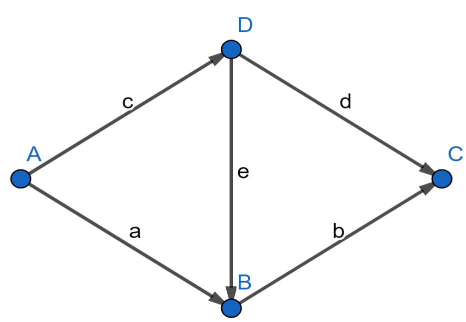
& the solution can be written as:

$$\widehat{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \widehat{K}^{-1}\widehat{\mathbf{f}}$$

Where $\widehat{\mathbf{f}} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ This is the REDUCED LINEAR SYSTEM $\widehat{K}\widehat{\mathbf{x}} = \widehat{\mathbf{f}}$.

<u>Problem II</u>:('Dirichlet Problem') Specify Potentials at 2 (or more) nodes and find the other potentials and possibly also the edge currents, and net current into the circuit.

<u>Back to the graph!</u>:



<u>Example</u>: Suppose Node 1 is at UNIT voltage, so $x_1 = 1$ & node 2 is grounded, so $x_2 = 0$. Now we <u>solve</u> for the potentials at the other nodes.

Question: Can you see how to do this using the previous solution?

We have $K\mathbf{x} = \mathbf{f}$ or:

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ -f_1 \\ 0 \\ 0 \end{pmatrix}$$

Where we do not yet know f_1, which is the current out of node 1. Notice that we set $f_2 = -f_1$, since we need this to hold.

Note that we have a mixture of unknowns on the <u>left</u> & <u>right</u> hand sides of this linear system. i.e. We need to determine x_3, x_4 and f_1. To solve such systems it is natural to introduce the notion of the *Schur Complement*.

$$\text{Let} K = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

Where in this example,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}, C = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

Let $\widehat{\mathbf{x}} = \begin{pmatrix} x_3 & x_4 \end{pmatrix}^T, \mathbf{e}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. We can now write the linear system as the following, with $\mathbf{f}_1 = \begin{pmatrix} f_1 & -f_1 \end{pmatrix}^T$:

$$K\mathbf{x} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \widehat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{0} \end{pmatrix}$$

Or:

$$A\mathbf{e}_1 + B^T\widehat{\mathbf{x}} = \mathbf{f}_1 \tag{4}$$
$$B\mathbf{e}_1 + C\widehat{\mathbf{x}} = \mathbf{0} \tag{5}$$

$$5 \Rightarrow C\widehat{\mathbf{x}} = -B\mathbf{e}_1 \Rightarrow \widehat{\mathbf{x}} = -C^{-1}B\mathbf{e}_1 = \begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix}$$

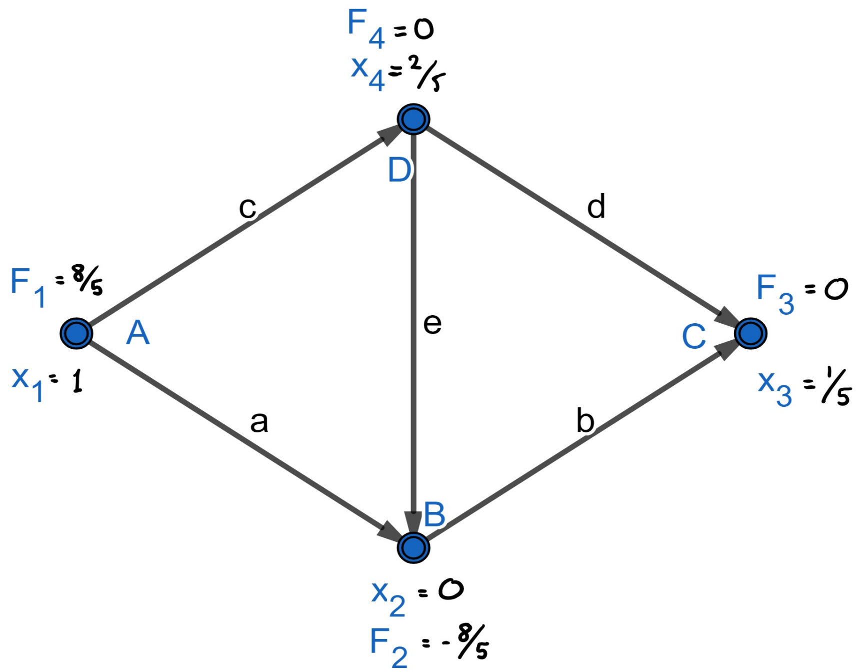
$$\begin{aligned} \text{Then } 4 \Rightarrow \mathbf{f}_1 &= A\mathbf{e}_1 - B^T C^{-1} B\mathbf{e}_1 \\ &= [A - B^T C^{-1} B]\mathbf{e}_1 \end{aligned}$$

This is known as the Schur Complement of the submatrix C in K.

Notice that $f_1 = \mathbf{e}_1^T [A - B^T C^{-1} B]\mathbf{e}_1$ where f_1 is the current into node 1 (& out of node 2).

Compute this quantity and show that <u>$f_1 = 8/5$</u>

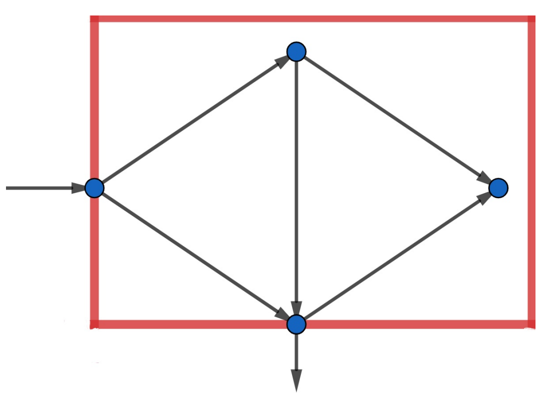
The final solution will look like this:



Definition: The effective conductance of a given circuit with UNIT VOLTAGE DROP specified between 2 given nodes is the CURRENT into the circuit at the input node (without loss of generality, the node held at UNIT VOLTAGE. KCL holds at all of the other nodes. We call this C_{eff}

Example: The effective conductance in the previous example is precisely the quantity f_1 just calculated! $C_{\text{eff}} = f_1$

The effective conductance is a property of the circuit topology and the individual conductances of the edges.



Imagine 'replacing' the entire circuit with a 'black box' (I know this one is red, it looks better) resistor with effective conductance f_1.

Note: Ohm's law still holds here as $f_1 = -C_{\text{eff}}(\phi_2 - \phi_1)$

8 Case of Non-equal Conductances: The matrix C

If the conductances of the edges are not all equal, then it is necessary to introduce a new diagonal matrix (which we will call C). Recall, for our example graph we had:

$$\mathbf{e} = A\mathbf{x} = \begin{pmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_2 \\ \phi_4 - \phi_1 \\ \phi_3 - \phi_4 \\ \phi_2 - \phi_4 \end{pmatrix}$$

From Ohm's law, we know that $w_a = -c_a(\phi_2 - \phi_1)$ with similar expressions for edges b,c,d and e. Thus, in this case, currents are given by $\mathbf{w} = -CA\mathbf{x}$ where:

$$C = \begin{pmatrix} c_a & & & & \\ & c_b & & 0 & \\ & & c_c & & \\ & 0 & & c_d & \\ & & & & c_e \end{pmatrix}$$

C is an $m - by - m$ diagonal matrix with the conductances of the edges in order along the diagonal.

$$\text{Then}\mathbf{f} = -A^T\mathbf{w} = -A^T(-CA\mathbf{x}) = A^TCA\mathbf{x}$$

If C is not the identity matrix, then we call K the 'stiffness matrix' or the 'weighted Laplacian' (if $C = I$ it is just the regular Laplacian matrix)

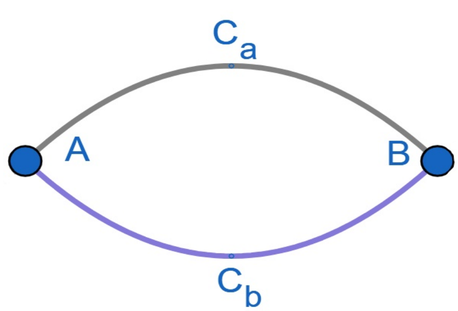
SUMMARY: The general equation relating current sources to potentials is the equation

$$\mathbf{f} = A^TCA\mathbf{x}$$

9 Combining Different Arrangements Of Resistors

9.1 Resistors in Parallel

Suppose 2 resistors connect two nodes 'in parallel' so that the corresponding graph is:



Conductance of edge $a = c_a$ and the conductance of edge $b = c_b$.

Question: What is the effective conductance of this circuit?

For this setup:

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, C = \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix}, A^T = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{f} = A^TCA\mathbf{x} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

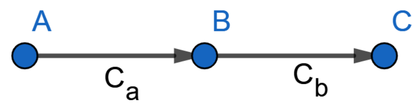
$$= \begin{pmatrix} c_a + c_b & -(c_a + c_b) \\ -(c_a + c_b) & c_a + c_b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_a + c_b \\ -(c_a + c_b) \end{pmatrix}$$

So we get that the effective conductance of two resistors in parallel is the sum of the individual conductances.

Rule: For resistors in parallel, ADD!

9.2 Resistors in series

Suppose the 2 resistors are 'in series' so that the corresponding graph is:



The required matrices are as follows:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},\ C = \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix}$$

$$CA = \begin{pmatrix} -c_a & c_a & 0 \\ 0 & -c_b & c_b \end{pmatrix}$$

$$A^T CA = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -c_a & c_a & 0 \\ 0 & -c_b & c_b \end{pmatrix}$$

$$= \begin{pmatrix} c_a & -c_a & 0 \\ -c_a & c_a + c_b & -c_b \\ 0 & -c_b & c_b \end{pmatrix}$$

Now set $\mathbf{x} = \begin{pmatrix} 1 & \phi_2 & 0 \end{pmatrix}^T$

$$\mathbf{f} = A^T CA\mathbf{x} = \begin{pmatrix} c_a & -c_a & 0 \\ -c_a & c_a + c_b & -c_b \\ 0 & -c_b & c_b \end{pmatrix} \begin{pmatrix} 1 \\ \phi_2 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ -f_1 \end{pmatrix}$$

$$\therefore\ (c_a + c_b)\phi_2 - c_a = 0\ \Rightarrow\ \phi_2 = \frac{c_a}{c_a + c_b}$$

$$-c_b\phi_2 = -f_1\ \Rightarrow\ f_1 = c_b\phi_2\ \Rightarrow\ \underline{f_1 = \frac{c_a c_b}{c_a + c_b}}$$

Some authors like to use the resistance of a resistor R_a instead of the conductance c_a. They are related by $\underline{R_a = (c_a)^{-1}}$

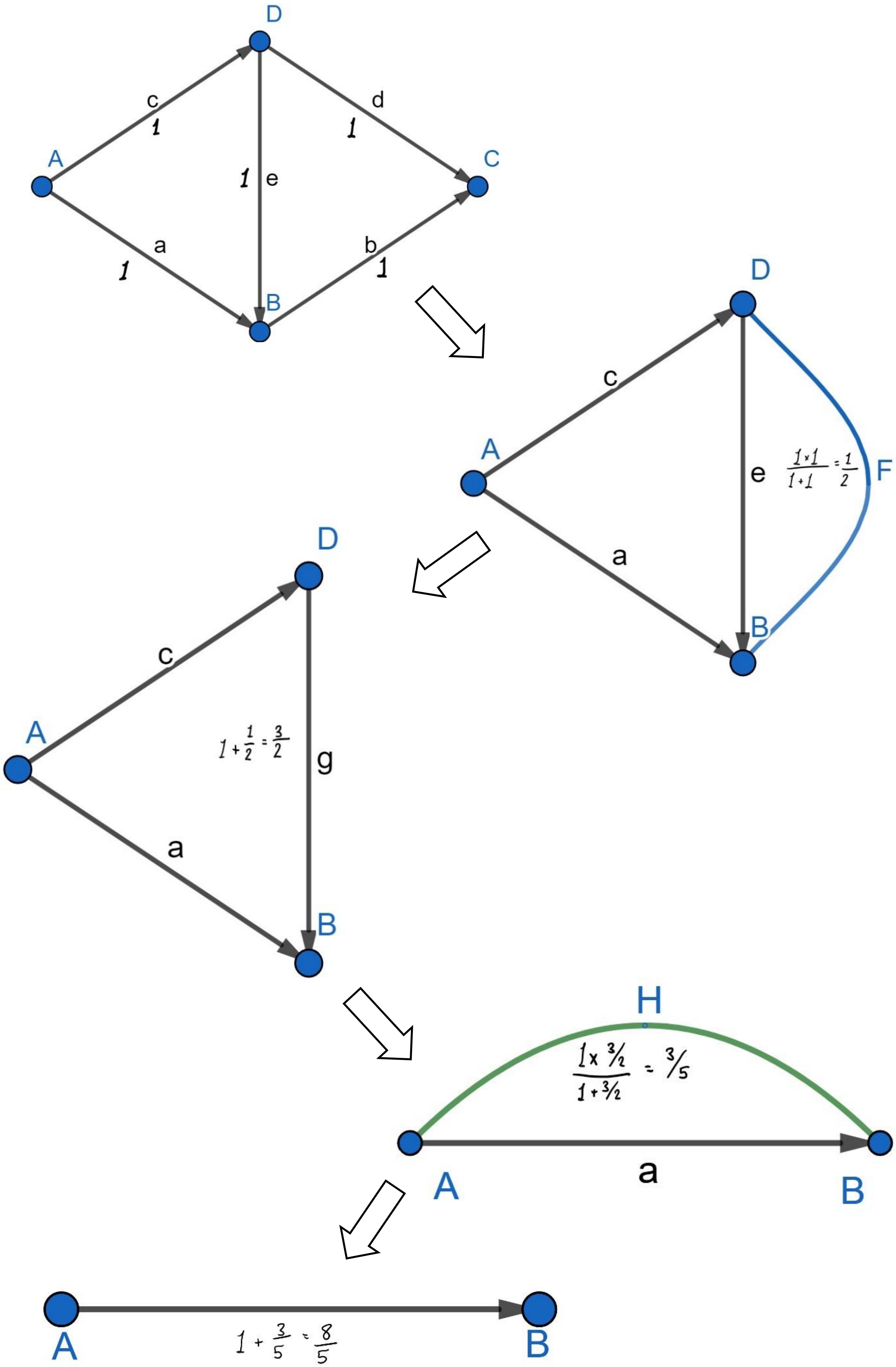
Hence the effective resistance of 2 resistors in series is:

$$R_{\text{eff}} = \frac{1}{C_{\text{eff}}} = \frac{c_a + c_b}{c_a c_b} = \frac{1}{c_a} + \frac{1}{c_b}$$

$$R_{\text{eff}} = R_a + R_b$$

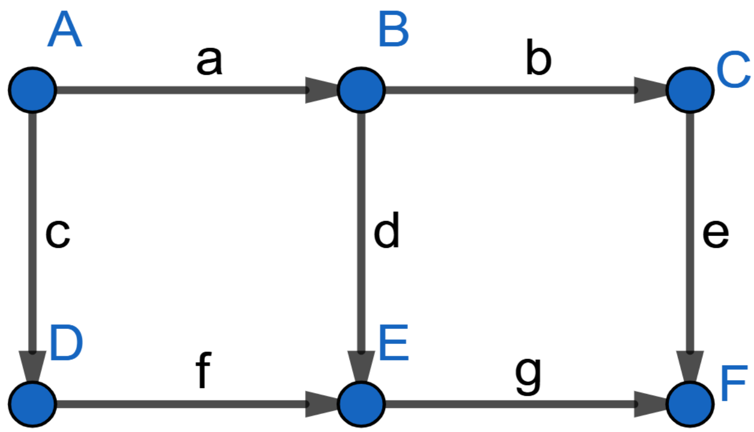
Note: It is useful to use the notion of effective conductance, especially the two rules for resistors in series and in parallel ,to compute effective conductance for more complicated graphs, or at least to reduce the size of the matrix calculation.

Example: Back to the usual!!



D
c
1
d
1
A
C
1
e
a
1
b
1
B
D
c
A
e
$\frac{1 \times 1}{1+1} = \frac{1}{2}$
F
a
B
D
c
A
$1 + \frac{1}{2} = \frac{3}{2}$
g
a
B
H
$\frac{1 \times 3/2}{1 + 3/2} = 3/5$
A
a
B
A
$1 + \frac{3}{5} = \frac{8}{5}$
B

Second Example: Consider the following graph:



Assume each edge is a resistor with unit conductance.

Question: Find the effective conductance between nodes A and F (i.e. find the current into node A if it is set to unit voltage while node F is grounded) *Solution:* There are several approaches to this problem, all of which are equivalent.

We can easily construct the Laplacian matrix and we ground node F in order to reduce the system:

$$K = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\widehat{K}\mathbf{x} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \\ \\ \widehat{\mathbf{x}} \\ \\ \end{pmatrix} = \begin{pmatrix} f_1 \\ \\ \\ \mathbf{0} \\ \\ \end{pmatrix}$$

Now we can solve this by the Schur Complement method (unless you enjoy computing 5 by 5 matrix inverses, in which, knock yourself out):

$$\widehat{K}\mathbf{x} = \begin{pmatrix} 2 & \mathbf{e}_1^T \\ \mathbf{e}_1 & C \end{pmatrix} \begin{pmatrix} 1 \\ \widehat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

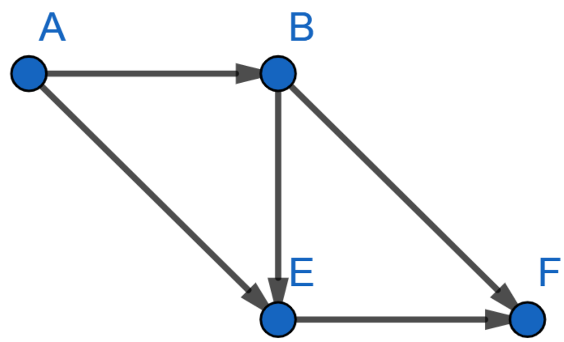
$$\text{Where } \mathbf{e}_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \ C = \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix}$$

$$\mathbf{e}_1 + C\widehat{\mathbf{x}} = 0 \Rightarrow \widehat{\mathbf{x}} = -C^{-1}\mathbf{e}_1$$

Hence $f_1 = 2 + \mathbf{e}_1^T\widehat{\mathbf{x}} = 2 - \mathbf{e}_1^T C^{-1}\mathbf{e}_1$

Exercise: Compute this!

Alternative: Consider edges c and f as well as edges b and e. We can see these pairs as resistors in series in order to reduce the circuit to a simpler circuit which is still equivalent, as shown:



Notice: This graph now has the same topology as the original graph!

Notice (again): The conductances are no longer equal

Notice (again(again)): Size of the linear system has been reduced

Stiffness matrix in this case, after recognising the change in conductances is:

$$K = \begin{array}{c} \\ A \\ B \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccc} \quad A \quad & \quad B \quad & \quad E \quad & \quad F \quad \end{array} \\ \begin{pmatrix} 1+\frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ -1 & 1+1+\frac{1}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & 1+1+\frac{1}{2} & -1 \\ 0 & -\frac{1}{2} & -1 & 1+\frac{1}{2} \end{pmatrix} \end{array}$$

Node F is grounded $\Rightarrow$ reduced linear system is

$$\begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -1 & \frac{5}{2} & -1 \\ -\frac{1}{2} & -1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \phi_2 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix}$$

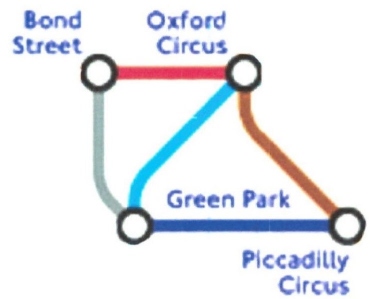
This is now small enough to be solved by hand!

$$\begin{pmatrix} \frac{5}{2} & -1 \\ -1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \phi_2 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \xrightarrow{\text{Solve}} \phi_2 = \frac{4}{7},\ \phi_5 = \frac{3}{7}$$

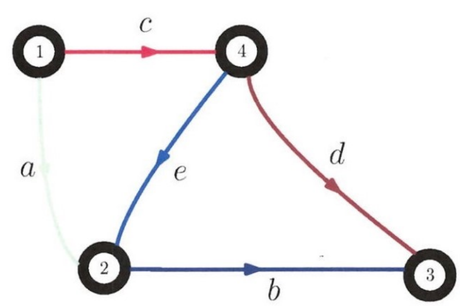
Therefore $f_1 = \frac{3}{2} - \phi_2 - \frac{\phi_5}{2} = \frac{5}{7}$

10 Random Walks On The Tube

Consider a 'fictional' London with the following Tube network:



We can think of this as a graph!!



Suppose a tourist, when she reaches any tube stop, decides at random which train to take next, with all possible routes from that station having equal probability (obviously by flipping a multi-sided coin, which is completely feasible)

Question 1: What is the probability, if she starts at Oxford Circus, of reaching Bond Street before she reaches Green Park?

Question 2: What is this probability if she starts, instead, at Piccadilly Circus?

Question 3: What is the probability if she starts at Bond Street and leaves, that she reaches Green Park BEFORE returning to Bond Street?

Notice: For question 1, we expect the answer to be at least 1/3 as there is a 1/3 chance of reaching Bond Street directly from Oxford Circus. For question 2, we expect the answer to be less than 1/2 as there is a 1/2 of reaching Green Park directly from Piccadilly Circus.For question 3, we expect this to be at least 1/2 as there is a 1/2 chance of succeeding on the first journey.

Now for some code (and poor quality images, with small amounts of writing on them):

```
%Random walk on Tube Map
rng('shuffle');
N=5000;          %number of trials
hit=0;           %initialize number of hits
for k=1:N;       %loop through N trials
    flag=1;      %set a flag: if flag=1 keep going, if flag=0 STOP trial
    state=4;     %choose to be either 3 or 4 (initial state)
    while flag == 1
        if state == 1
            flag=0;      %stop if state=1
            hit=hit+1;   %add a "hit"
        end;
        if state == 2
            flag=0;      %stop if state=2
        end;
        if state == 3
        toss=randi([1 2],1,1);       %toss a 2-sided coin
            if toss == 1;state=2;end;
            if toss == 2;state=4;end;
        end;
        if state == 4
        toss=randi([1 3],1,1);       %toss a 3-sided coin
            if toss == 1;state=3;end;
            if toss == 2;state=2;end;
            if toss == 3;state=1;end;
        end;
    end
end;
hit/N
```

To get an idea, we can carry out out a Direct Numerical Simulation (DNS)

The above MATLAB code starts a tourist off in ‘state’ 3 or 4 (depends on the user) and it simulates the journey $N = 5000$ times. It then counts the number of ‘hits’ on Bond Street (node 1) and it computes the proportion of the N journeys that hit Bond Street first.

The following are results of 3 of these simulations, starting at node 3:

```
Command Window

>> RandomWalk2

ans =

    0.2076

>> RandomWalk2

ans =

    0.1988

>> RandomWalk2

ans =

    0.1990
```

The answer from this appears to be approximately 0.2 or $\frac{1}{5}$.

The following are results of the same simulations but starting at node 4.

```
Command Window
  >> RandomWalk2

  ans =

      0.4002

  >> RandomWalk2

  ans =

      0.4056

  >> RandomWalk2

  ans =

      0.4042
```

The answer here appears to be approximately 0.4 or $\frac{2}{5}$

Summary: From these two simulations, and looking back at the graph that we had at the start, it appears as though these values mirror the potentials from the Dirichlet problem involving the electrical circuit!!!! So maybe the answer to question 3 will also have a link back to the circuit problem too.

```
1/26/20 2:48 PM  /Users/dgcrowdy/Dropbox.../RandomWalk3.m   1 of 1

%Random walk on Tube Map
clear all
rng('shuffle');
N=5000;          %number of trials
hit=0;           %initialize number of hits on node 2 ("escape node")
for k=1:N;       %loop through N trials
    flag=1;      %set a flag: if flag=1 keep going, if flag=0 STOP trial
    state=1;     %start at node 1 (initial state)
    toss=randi([1 2],1,1);       %toss a 2-sided coin
            if toss == 1;
                state=2;         %go to node 2
                hit=hit+1;       %record a "hit"
                flag=0;          %stop the simulation
            end;
            if toss == 2;state=4;end;
    while flag == 1
        if state == 1
            flag=0;      %stop if in state=1
        end;
        if state == 2
            hit=hit+1;   %record a "hit"
            flag=0;      %stop if in state=2
        end;
        if state == 3
        toss=randi([1 2],1,1);       %toss a 2-sided coin
            if toss == 1;state=2;end;
            if toss == 2;state=4;end;
        end;
        if state == 4
        toss=randi([1 3],1,1);       %toss a 3-sided coin
            if toss == 1;state=3;end;
            if toss == 2;state=2;end;
            if toss == 3;state=1;end;
        end;
    end
end;
hit/N
```

What about the probability of starting at Bond Street, leaving & reaching Green Park (node 2) beofre returning?

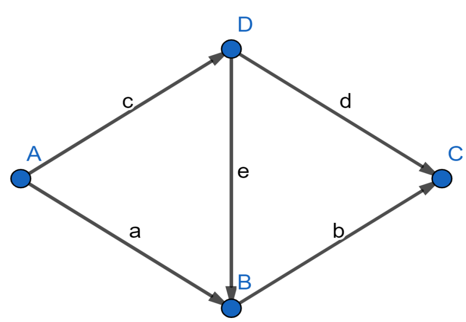
The code above should simulate this scenario.

```
Command Window
  >> RandomWalk3

  ans =

      0.8024

  >> RandomWalk3

  ans =

      0.7912

  >> RandomWalk3

  ans =

      0.8064
```

The answer appears to be approximately 0.8 or $\frac{4}{5}$.

Does this also relate back to the circuit problem? The only number we haven't considered yet is the 'effective conductance' of $\frac{8}{5}$, and although these numbers may seem connected, there is no concrete relationship.

Analytical Approach: Our graph is:



What is the probability, starting at any of nodes 1,2,3, or 4, of reaching node 1 before node 2 on a simple random walk?

Let p_i be this probability, with the walk starting at node i. It is clear that $p_1 = 1$ and $p_2 = 0$ by a mixture of inspection and common sense.

Basic Probability Law: Let E be any event and F & G be events such that only ONE of these can occur. The we have $P(E) = P(F) \cdot P(E|F) + P(G) \cdot P(E|G)$.

$$\text{Hence we have } p_3 = \frac{1}{2} \cdot p_2 + \frac{1}{2} \cdot p_4 \tag{1}$$

$$\underline{\text{and}}\ p_4 = \frac{1}{3} \cdot p_1 + \frac{1}{3} \cdot p_2 + \frac{1}{3} \cdot p_3 \tag{2}$$

So we get that $2p_3 = p_+ p_4$ and $3p_4 = p_1 + p_2 + p_3$, and substituting in $p_1 = 1$ and $p_2 = 0$, we get:

$$2p_3 = p_4 \Rightarrow p_3 = \frac{p_4}{2}$$

$$3p_4 - p_3 = 1 \Rightarrow 3p_4 - \frac{p_4}{2} = 1$$

$$\Rightarrow p_4 = \frac{2}{5}, \;\; p_3 = \frac{1}{5}$$

Remarkably we notice that these are the SAME values as the voltages $x_i, i = 1, 2, 3, 4$ found earlier! (Recall that these solutions were found using the Schur complement)

Question: Why are the solutions to 2 apparently different problems the same?

Key connection between Electrical circuits & Random walks - In solving for the potentials, $\mathbf{x}$ say, at any node a at which Kirchoff's Current Law holds, we look at the row of $A^T A\mathbf{x}$ corresponding to node a & set that component equal to ZERO (no net current out of node a)

However we know that the Laplacian matrix $K = A^T A = D - W$ where D is the degree matrix and W is the adjacency matrix. Hence the equation associated with KCL at node a is

$$deg(a) \cdot x_a - \sum_{\substack{\text{j connected} \\ \text{to a}}} x_j = 0$$

or more simply

$$x_a = \frac{\sum\limits_{\substack{\text{j connected} \\ \text{to a}}} x_j}{deg(a)} \tag{3}$$

But these are precisely the equations 1 and 2 which we had for the random walk! (where 'node a' is taken to be 3 or 4)

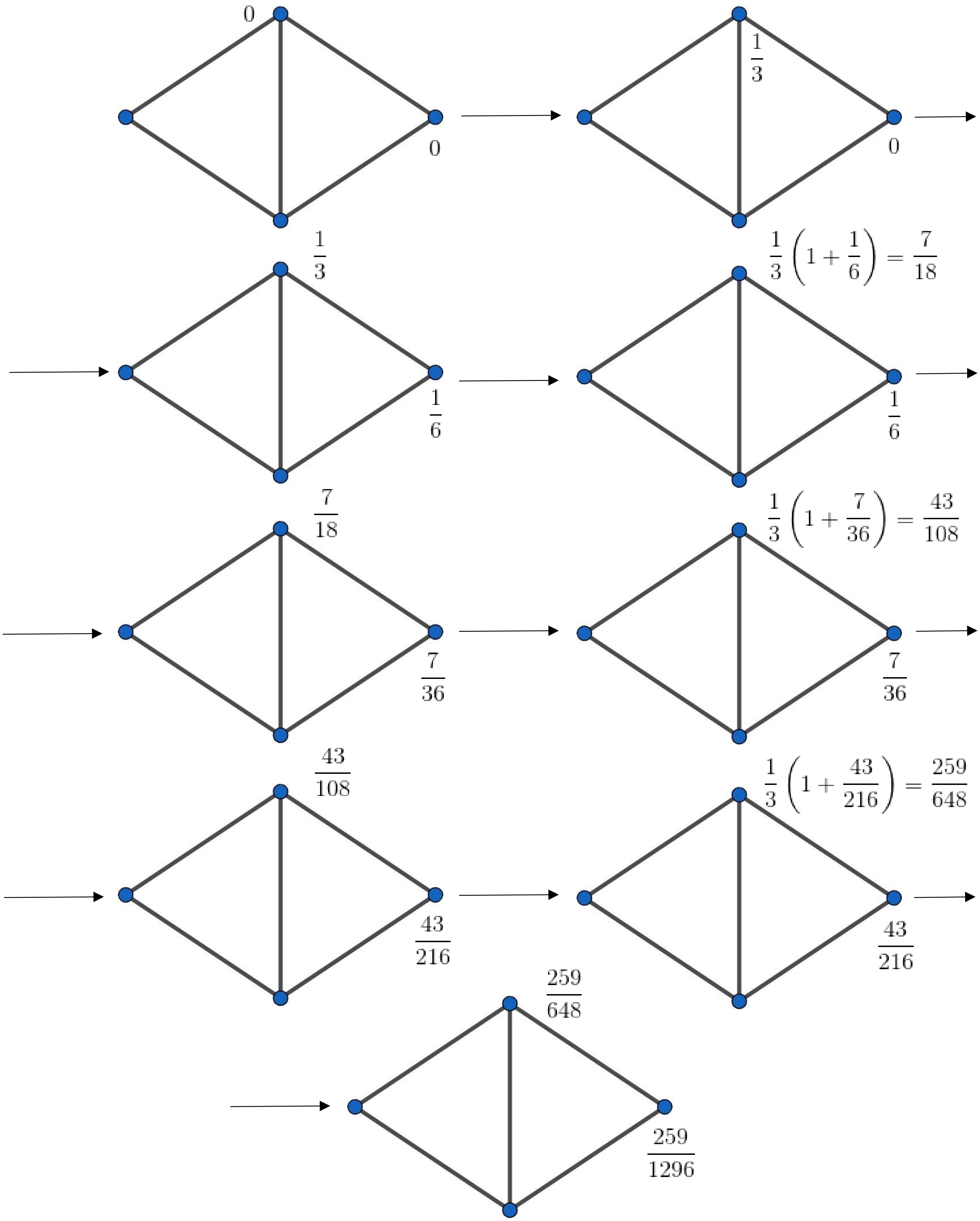
Observation: Equation 3 can also be interpreted as follows: the value of the potential x_a at any node at which KCL holds is the AVERAGE of the potentials at all neighbours (connected to it by an edge)

Definition: We call any potentials which have this property a *HARMONIC potential* (i.e. both $\mathbf{x}$ and $\mathbf{p}$ are harmonic potentials, both with $x_1 = p_1 = 1$ and $x_2 = p_2 = 0$)

Idea: This observation suggests a new construction of the solution as follows:

1. set potentials $x_1 = p_1 = 1$ and $x_2 = p_2 = 0$ & set all other potentials to zero as in initialisation.
2. cycle through each 'undetermined' node and calculate the average of its connected node potentials and reassign the potential at the node to be this average value.
3. Go to the next node until all nodes have been done
4. REPEAT! (iterate to convergence)

This is called the 'method of relaxation'. Let's see it in action for the usual example.


0
0
1/3
0
1/3
1/6
1/3 (1 + 1/6) = 7/18
1/6
7/18
7/36
1/3 (1 + 7/36) = 43/108
7/36
43/108
43/216
1/3 (1 + 43/216) = 259/648
43/216
259/648
259/1296

Now looking at these final values, we have $\frac{259}{648} \approx 0.39969$ and $\frac{259}{1296} \approx 0.19984$. Knowing the answer, we know we are close enough, so we can stop here, knowing we will arrive at the correct answer (after infinitely many iterations, but oh well)

Summary: We have connected the problem of finding voltages in an electrical circuit to simple random walks on graphs. We have also now got THREE different methods for computing the solution to EITHER problem.

1. Linear Algebra methods based on K (Schur Complements)
2. Perform a simulated random walk multiple times and take an average (definitely recommended for exams)
3. 'Method of relaxation' iterative scheme based on local 'averages'

However even after all of this, we need to prove equivalence for any graph. For this, we will introduce the new idea of the MAXIMUM PRINCIPLE

11 Maximum Principle & Minimum Principle for Harmonic Potentials

Given a (connected) graph, it is natural to separate nodes into two categories. These will be appropriately called 'Boundary Nodes', B, where the potential is specified, and 'Interior Nodes', where KCL holds & where, therefore, the potential is equal to the average of the potential of all of its neighbouring nodes. (e.g. In the random walk considered earlier, nodes 1 and 2 are the boundary nodes, and nodes 3 and 4 are the interior nodes)

FACT: The maximum possible value of the potential occurs at the boundary nodes (or one of them)

'Proof': Let M be the maximum value of the potential in a given graph. If $x_i = M$ for $x_i \in D$ (interior nodes), then the same must be true of all of its neighbours (since M is the average of these neighbouring values and none of them can exceed M). If all neighbours to x_i are still in D, then by the same argument, all of *their* neighbours must also have potential M. Continuing in this way, we eventually reach a boundary node and we find that the potential at this node is also M, so the maximum is attained on the boundary (because the graph is connected). Hence the maximum value is necessarily attained at the boundary node.

A similar argument can be used to prove the 'Minimum Principle', which works for the minimum value of the potentials in a graph.

Uniqueness Principle for harmonic potentials: How do we know from this that the solution of the circuit problem $\mathbf{x}$ and the random walk $\mathbf{p}$ are the same?

Let x_i and p_i be the harmonic potentials, such that they have the same boundary values, similar to what we had earlier. Define $d_i = x_i - p_i$. This has ZERO boundary values. By the maximum principle, the maximum & minimum values of d_i are ZERO $\longrightarrow$ $d_i = 0$ at all interior nodes!!

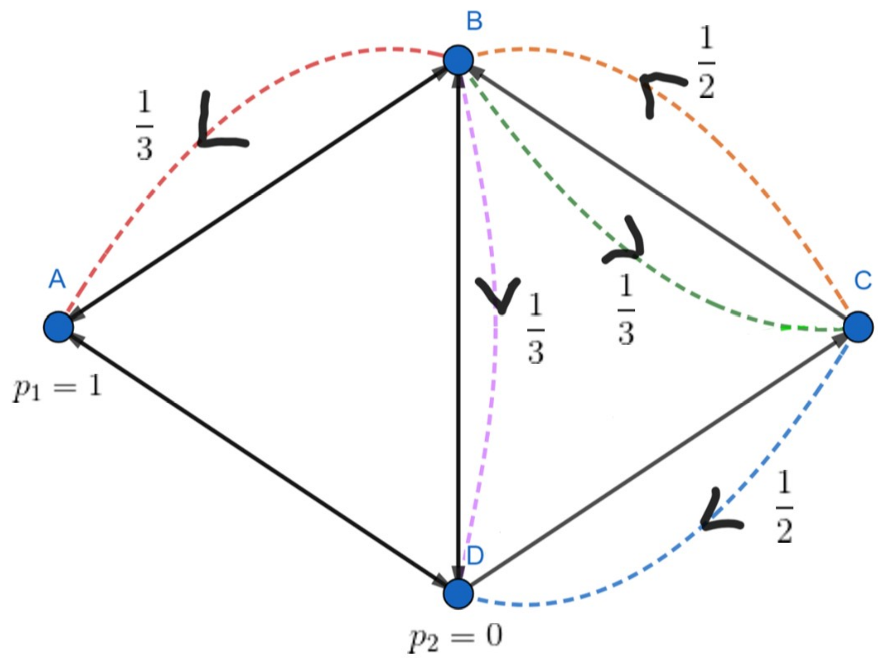
Analogy between random walks & resistor networks

Let the boundary sets be B_+ and B_- in a resistor network (or electrical circuit) and have them fixed at voltages 1 and 0 respectively with c_{ij} being the conductance of the resistor between node i and node j. Then the voltage x_i at any INTERIOR node i is the same as the probability p_i for a random walk starting at node i to reach B_+ before reaching B_- when the hopping probability from i

to j is $p_{ij} = c_{ij}/\sum_{\text{j adjacent}} c_{ij}$

Definition: We call p_i the 'hitting probability' for the boundary set B_+ starting at node (i)

Hopping Probabilities for our example



The hopping probabilities p_{ij} are shown as the dotted lines

Note: the hopping probabilities in each direction along the same edge need not be be equal (even the the 'conductances' are all unity)

Given this analogy between random walks & electrical circuits, it is natural to ask if there is a probabilistic interpretation of the effective conductance of the circuit? By ohm's Law, and the definition of the effective conductance

$$C_{\text{eff}}^{(i)} = \sum_{\substack{\text{j connected} \\ \text{to i}}} c_{ij}(x_i - x_j)$$

Where i is the input node and $C_{\text{eff}}^{(i)}$ is the current out of node i. Now introduce the notation $c_i = \sum_{\substack{\text{j connected} \\ \text{to i}}} c_{ij}$, then we have

$$\begin{aligned} C_{\text{eff}}^{(i)} &= x_i c_i - \sum_j c_{ij} x_j \frac{c_i}{c_i} \\ &= x_i c_i - c_i \sum_j x_j \frac{c_{ij}}{c_i} \end{aligned}$$

With $x_i = 1$ as it is the input node, $c_{ij}/c_i = p_{ij}$ which is the probability of hopping from node i to node j. x_j is the probability of returning to node i from node j before hitting B_-. So the sum in the final line is the total probability of starting at node i, leaving, and then returning before hitting B_-

Hence:

$$\sum_j \frac{c_{ij}}{c_i} x_j = \sum_j p_{ij} x_j$$

is the probability that you are at B_+, i.e. node i, are forced to leave it, but then return to B_+ before hitting B_-

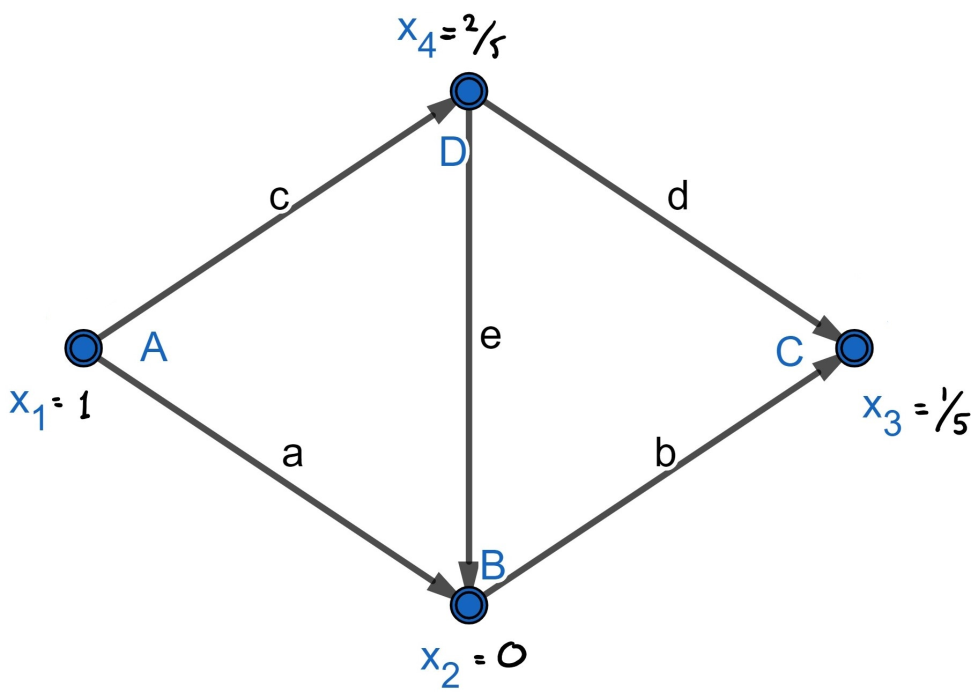
$$\therefore \; C_{\text{eff}}^{(i)} = c_i \left[1 - \sum_j p_{ij} x_j \right]$$

The bracket is the probability that you hit B_- BEFORE returning to B_+ (node i). We call this the escape probability, denoted by $p_{\text{esc}}^{(i)}$. This therefore means

$$C_{\text{eff}}^{(i)} = c_i \cdot p_{\text{esc}}^{(i)}$$

With $C_{\text{eff}}^{(i)}$ is the effective conductance with node i set to unit voltage 1, c_i is the sum of conductances of edges connected to node i and $p_{\text{esc}}^{(i)}$ is the probability of escaping from node i in a random walk that starts there.

For our example



$$\text{With } f_1 = C_{\text{eff}} = \frac{8}{5}$$

$$c_1 = 2 \Rightarrow p_{\text{esc}} = \frac{C_{\text{eff}}^{(1)}}{c_1} = \frac{4}{5}$$

This value is consistent with the DNS we performed earlier. This number therefore represents the probability of starting a random walk at node 1 and reaching 2 before returning.

12 Dirichlet's Principle

The energy dissipation associated any circuit with one node held at unit voltage with another grounded is defined to be:

$$\varepsilon(\mathbf{x}) = \mathbf{x}^T K \mathbf{x}$$

Where K is the circuit Laplacian, $\mathbf{x}$ is <u>ANY</u> potential defined at the nodes. It turns out that the solution given by the potentials $\mathbf{x}_*$ satisfying KCL at all internal nodes <u>minimises</u> $\varepsilon(\mathbf{x})$ i.e.

$$\varepsilon(\mathbf{x}_*) = \min_{\mathbf{x}} \varepsilon(\mathbf{x})$$

<u>Note:</u> It is a minimisation of the dissipation over the set of node <u>potentials</u> $\mathbf{x}$

<u>Preliminary exercises:</u>

Let $K = A^T C A$ be a weighted Laplacian where A is the incidence matrix of a connected graph & C is a diagonal matrix with strictly positive diagonal elements.

1. Prove that K is a positive semi-definite matrix, i.e. show that $\mathbf{x}^T K\mathbf{x} \geq 0$ for any $\mathbf{x}$
2. If we introduce the same sub-block decomposition as used earlier, $K = \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix}$, show that R is a positive definite matrix, so $\mathbf{x}^T R\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$

To prove Dirichlet's Principle, we consider first the governing linear equation fo the potential $\mathbf{x}$, $K\mathbf{x} = \mathbf{f}$.

Let x_1 correspond to the node at unit voltage and x_2 correspond to the grounded node. We can then introduce $\mathbf{x} = \begin{pmatrix} 1 & 0 & \widehat{\mathbf{x}} \end{pmatrix}^T$ and we then get that $\mathbf{f} = \begin{pmatrix} f & -f & \mathbf{0} \end{pmatrix}^T$. We can also introduce the sub-block decomposition $K = \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix}$ where P is 2-by-2, Q is $(n-2)$-by-2 and R is $(n-2)$-by-$(n-2)$.

The linear system now becomes

$$K\mathbf{x} = \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \widehat{x} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{0} \end{pmatrix}$$

Where $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\mathbf{f} = \begin{pmatrix} f_1 & -f_1 \end{pmatrix}^T$

The equation for $\widehat{\mathbf{x}}$ is

$$Q\mathbf{e}_1 + R\widehat{\mathbf{x}} = 0 \Rightarrow \widehat{\mathbf{x}} = -R^{-1}Q\mathbf{e}_1$$

We will call this solution $\widehat{\mathbf{x}}_*$. We know that R is invertible because we know it is positive definite and is a square matrix. The dissipation can now be written as

$$\begin{aligned}
\varepsilon(\mathbf{x}) &= \begin{pmatrix} \mathbf{e}_1^T & \widehat{\mathbf{x}}^T \end{pmatrix} \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \widehat{\mathbf{x}} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_1^T & \widehat{\mathbf{x}}^T \end{pmatrix} \begin{pmatrix} P\mathbf{e}_1 + Q^T\widehat{\mathbf{x}} \\ Q\mathbf{e}_1 + R\widehat{\mathbf{x}} \end{pmatrix} \\
&= \mathbf{e}_1^T(P\mathbf{e}_1 + Q^T\widehat{\mathbf{x}}) + \widehat{\mathbf{x}}^T(Q\mathbf{e}_1 + R\widehat{\mathbf{x}}) \\
&= \mathbf{e}_1^T P\mathbf{e}_1 + \mathbf{e}_1^T Q^T\widehat{\mathbf{x}} + \widehat{\mathbf{x}}^T Q\mathbf{e}_1 + \widehat{\mathbf{x}}^T R\widehat{\mathbf{x}}
\end{aligned}$$

$$\text{Notice that } \; \mathbf{e}_1^T Q^T\widehat{\mathbf{x}} = (\mathbf{e}_1^T Q^T\widehat{\mathbf{x}})^T = \widehat{\mathbf{x}}^T Q\mathbf{e}_1$$

$$\text{Hence } \; \varepsilon(\mathbf{x}) = \mathbf{e}_1^T P\mathbf{e}_1 + +2\widehat{\mathbf{x}}^T Q\mathbf{e}_1 + \widehat{\mathbf{x}}^T R\widehat{\mathbf{x}}$$

Now for the magic ;)

Observation: Consider the quantity

$$\begin{aligned}(\widehat{\mathbf{x}} + R^{-1}Q\mathbf{e}_1)^T R(\widehat{\mathbf{x}} + R^{-1}Q\mathbf{e}_1) &= (\widehat{\mathbf{x}}^T + (R^{-1}Q\mathbf{e}_1)^T)(R\widehat{\mathbf{x}} + Q\mathbf{e}_1) \\ &= \widehat{\mathbf{x}}^T R\widehat{\mathbf{x}} + (R^{-1}Q\mathbf{e}_1)^T R\widehat{\mathbf{x}} + \widehat{\mathbf{x}}^T Q\mathbf{e}_1 + (R^{-1}Q\mathbf{e}_1)^T Q\mathbf{e}_1\end{aligned}$$

Now notice the following

$$\begin{aligned}(R^{-1}Q\mathbf{e}_1)^T R\widehat{\mathbf{x}} &= ((R^{-1}Q\mathbf{e}_1)^T R\widehat{\mathbf{x}})^T \\ &= \widehat{\mathbf{x}}^T R^T (R^{-1}Q\mathbf{e}_1) \\ &= \widehat{\mathbf{x}}^T R(R^{-1}Q\mathbf{e}_1) \\ &= \widehat{\mathbf{x}}^T Q\mathbf{e}_1\end{aligned}$$

So we can rewrite the quantity under observation as

$$\widehat{\mathbf{x}}^T R\widehat{\mathbf{x}} + 2\widehat{\mathbf{x}}^T Q\mathbf{e}_1 + (R^{-1}Q\mathbf{e}_1)^T Q\mathbf{e}_1$$

And we can see that the first two terms appear in the expression of $\varepsilon(\mathbf{x})$

$$\begin{aligned}\varepsilon(\mathbf{x}) &= \mathbf{e}_1^T P\mathbf{e}_1 + (\widehat{\mathbf{x}} + R^{-1}Q\mathbf{e}_1)^T R(\widehat{\mathbf{x}} + R^{-1}Q\mathbf{e}_1) - (R^{-1}Q\mathbf{e}_1)^T Q\mathbf{e}_1 \\ \underline{\text{OR}}\quad \varepsilon(\mathbf{x}) &= X^T R X + c \\ \text{Where}\quad X &= \widehat{\mathbf{x}} + R^{-1}Q\mathbf{e}_1 \\ c &= \mathbf{e}_1^T P\mathbf{e}_1 - (R^{-1}Q\mathbf{e}_1)^T Q\mathbf{e}_1\end{aligned}$$

Hence, since R is a positive definite matrix, $\varepsilon(\mathbf{x})$ is minimised when $X = 0$, or

$$\widehat{\mathbf{x}}_* = -R^{-1}Q\mathbf{e}_1$$

, which is exactly what we said earlier! Hence we have shown that the potential of voltages producing a current satisfying both Ohm's Law & KCL at the interior nodes of the circuit MINIMISES the energy dissipation $\varepsilon(\mathbf{x})$ among all possible potential functions defined over the interior nodes.

12.1 Thomson's Principle

The energy dissipation can also be written as

$$\begin{aligned}\varepsilon(\mathbf{x}) &= \mathbf{v}^T K\mathbf{x} = \mathbf{x}^T A^T C A\mathbf{x} \\ &= (-A\mathbf{x})^T(-CA\mathbf{x}) \\ &= \sum_{\text{edges}} w_k \left(\frac{w_k}{c_k}\right) = \tilde{\varepsilon}(\mathbf{w}) \\ \text{because}\qquad (-A\mathbf{x})^T_{1k} &= \left(\frac{w_k}{c_k}\right) \\ \text{and}\quad (-CA\mathbf{x})_{k1} &= w_k\end{aligned}$$

The final two lines are true because of Ohm's law and appropriate adjustments made. w_k represents the current in the k^{th} edge and c_k is the conductance of the k^{th} edge. This sum is now across the edges, and not over the nodes anymore, and that the potentials no longer appear at all in the expression for $\tilde{\varepsilon}(\mathbf{w})$

Thomson's Principle states that the current derived from a potential that satisfies KCL at all interior nodes minimises the dissipation $\tilde{\varepsilon}(\mathbf{w})$ over all possible currents $\mathbf{w}$ defines in the circuit (including ones which may not be derivable from a potential of voltages)

Note: This is a 'dual' statement to Dirichlet's Principle (where the minimisation is over the set of potentials at the nodes.

Proof: See Problem Sheet 4.

12.2 Tellegen's Theorem

Given a (connected) graph with incidence matrix A and any vector $\mathbf{w}$ satisfying KCL at all nodes, then if $\mathbf{e}$ is any vector of potential differences, then

$$\mathbf{e}^T\mathbf{w} = 0$$

Note: $\mathbf{w}$ does not need to necessarily come from a potential.

Note as well: $\mathbf{e}$ and $\mathbf{w}$ are completely unrelated flows in the edges.

Proof: (easy) Since $\mathbf{e}$ is a vector of potential differences, then $\mathbf{e} = -A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Hence $\mathbf{e}^T\mathbf{w} = -\mathbf{x}^T A^T\mathbf{w}$. But $A^T\mathbf{w} = 0$ since $\mathbf{w}$ satisfies KCL at all nodes. Hence $\mathbf{e}^T\mathbf{w} = 0$

Modified Version of Tellegen's Theorem

Suppose on some connected graph with incidence matrix A, $\mathbf{w}$ satisfies

$$-A^T\mathbf{w} = \mathbf{f}$$

Then if $\mathbf{e}$ is any vector of potential differences $\mathbf{e} = -A\mathbf{x}$, then

$$\mathbf{e}^T\mathbf{w} = -\mathbf{x}^T A^T\mathbf{w}$$

Both terms on the LHS are in $\mathbb{R}^m$ and both terms on the RHS are in $\mathbb{R}^n$

Note: This says that a dot product of two vectors in $\mathbb{R}^m$ equals a dot product of two vectors in $\mathbb{R}^n$!

Proof:

$$\begin{aligned}
\mathbf{e}^T\mathbf{w} &= (-A\mathbf{x})^T\mathbf{w} \\
&= -\mathbf{x}^T A^T\mathbf{w} \\
&= -\mathbf{x}^T(-\mathbf{f}) \\
&= \mathbf{x}^T\mathbf{f}
\end{aligned}$$

12.3 Effective conductance & Energy dissipation

Suppose 2 nodes in a graph, a and b, are set to voltages 1 and 0 respectively (when the graph is viewed as an electric circuit) Let $\mathbf{w}$ be the current in the circuit. Let $\mathbf{e}$ be the potential drops, so $\mathbf{e} = -A\mathbf{x}$

$$
\begin{aligned}
\text{Then} \quad \mathbf{e}^T\mathbf{w} &= \mathbf{x}^T\mathbf{f} && \text{(by Tellegen Theorem)} \\
&= x_a f_a - x_b f_a && \text{(since } f_a = -f_b\text{)} \\
&= f_a && \text{(since } x_a = 1, x_b = 0\text{)} \\
\underline{\text{But}} \quad \mathbf{e}^T\mathbf{w} &= (-A\mathbf{x})^T(-CA\mathbf{x}) \\
&= \mathbf{x}^T A^T CA\mathbf{x} \\
&= \mathbf{x}^T K\mathbf{x}
\end{aligned}
$$

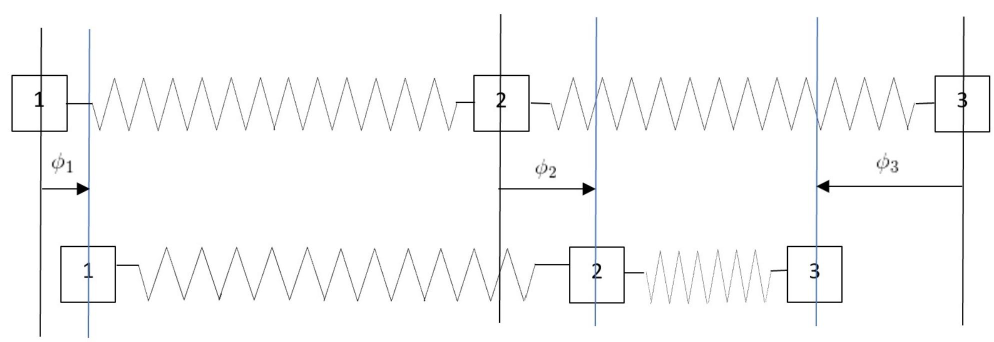
f_a is the effective conductance of the circuit and $\mathbf{x}^T K\mathbf{x}$ is the energy dissipation, and these are suddenly equal, so we get

$$f_a = C_{\text{eff}} = \varepsilon(\mathbf{x})$$

Thus, by Dirichlet's principle, the potential satisfying KCL and Ohm's law minimises the energy dissipation and the minimum value of this dissipation is the <u>effective conductance</u>

13 Spring-Mass Systems

Consider 3 masses connected by two spring as shown. Suppose the masses are then 'strained' or 'displaced':

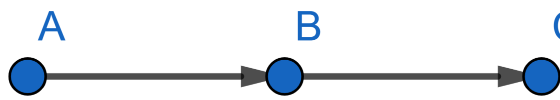


For each mass i, we consider x_i to be the 'equilibrium position', shown by the black line, and $\widehat{x}_i = x_i + \phi_i$ to be the 'strained position', shown by the blue line. The tension, T_1, in spring 1 is given, by <u>Hooke's Law</u> as

$$T_1 = c_1(\phi_2 - \phi_1)$$

where c_1 is the 'spring constant'. In a similar way, $T_2 = c_2(\phi_3 - \phi_2)$. Notice that $\phi_j = \widehat{x}_j - x_j$ is the displacement of masses $j = 1, 2, 3$

The spring-mass system can be modelled as a graph (just like the electrical circuit or random walk network):



The incidence matrix for this graph is:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ - & -1 & 1 \end{pmatrix}$$

$$\text{Let } \ \underline{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \ \text{ then } \ A\underline{\phi} = \begin{pmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_2 \end{pmatrix}$$

Therefore <u>Hooke's Law</u> can be written as

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = cA\underline{\phi}$$

$$\text{where } \ c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

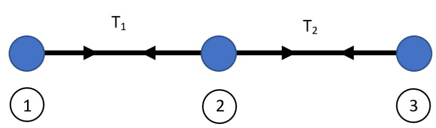
c is a diagonal matrix with the spring constants as the diagonal entries.

(<u>NB:</u> this is <u>entirely</u> analogous to 'Ohm's Law' in electric circuits but there is no minus sign here.)

The quantity $-A^T\mathbf{T}$ is the 'divergence of the tensions'. This can be interpreted as the <u>internal forces on the masses due to the springs</u>. In this example

$$-A^T\mathbf{T} = -\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 - T_1 \\ T_2 \end{pmatrix} = \mathbf{f}_I$$

Which can be seen as the TOTAL forces $\mathbf{f}_I$ on each mass due to the spring tensions as shown below



Putting the pieces together, we get

$$-A^T T = -A^T(cA\underline{\phi}) = -A^T cA\underline{\phi} = \mathbf{f}_I$$

What is the analogue of KCL at the nodes (masses in this case)?

It is the condition that, for the masses to be in equilibrium (not moving), all <u>forces</u> on the masses must add to ZERO! ('force balance')

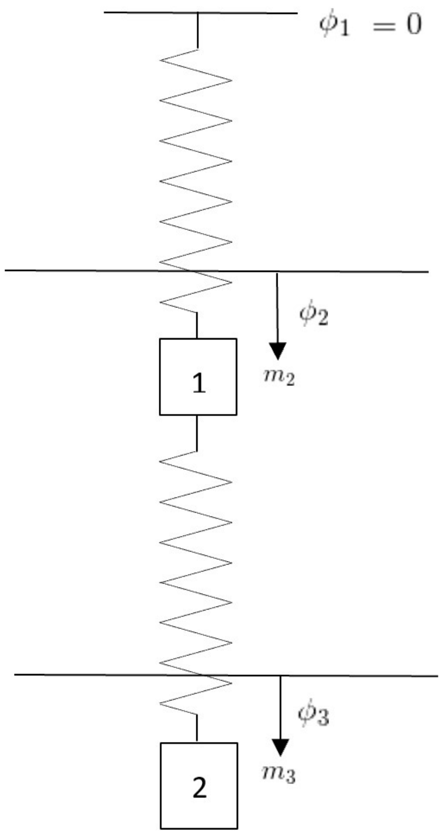
We will allow for an external force $\mathbf{f} = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}^T$ to act on each mass (e.g. due to gravity or wall reaction forces). Hence, force balance requires that

$$\mathbf{f} + \mathbf{f}_I = \mathbf{0}$$

$$\text{But } \ \mathbf{f}_I = -A^T cA\underline{\phi}$$

$$\underline{\text{Hence}} \ \ \mathbf{f} = A^T cA\underline{\phi}$$

Where $\mathbf{f}$ is the external forces at each node in equilibrium, A^TcA is the 'stiffness matrix' or weighted Laplacian and $\underline{\phi}$ is the displacements at equilibrium.



Example: (Equilibrium Problem)

Find the displacements of 2 masses, of mass m_2 and m_3, hanging under gravity from a fixed ceiling with 2 springs between them as shown in the figure (assuming the system is in equilibrium)

Stiffness matrix is (by usual construction)

$$K = A^TcA = \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1+c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{pmatrix}$$

Equilibrium Problem

$$\mathbf{f} = \begin{pmatrix} r_1 \\ m_2g \\ m_3g \end{pmatrix} = K \begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$$\begin{pmatrix} r_1 \\ m_2g \\ m_3g \end{pmatrix} = \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1+c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{pmatrix} \begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

With the quantity r being the 'reaction force' at the wall. It must also be found as a part of the solution.

Solve by hand (sufficiently simple):

$$(c_1+c_2)\phi_2 - c_2\phi_3 = m_2g \tag{1}$$
$$-c_2\phi_2 + c_2\phi_3 = m_3g \tag{2}$$
$$r_1 = -c_1\phi_2 \tag{3}$$

$$\begin{aligned} &1 + 2 \Rightarrow c_1\phi_2 = (m_2+m_3)g \Rightarrow \phi_2 = \left(\frac{m_2+m_3}{c_1}\right)g \\ &2 \Rightarrow \phi_3 = \frac{m_3g + c_2\phi_2}{c_2} = \frac{m_3}{c_2}g + \phi_2 \\ &\Rightarrow \phi_3 = \frac{m_3}{c_2} + \left(\frac{m_2+m_3}{c_1}\right)g \\ &3 \Rightarrow r_1 = -(m_2+m_3)g \end{aligned}$$

The reaction force is consistent with Newton's 3^{rd} Law, as the total force acting down, which is m_2g and m_3g is counteracted by the reaction at the wall.

13.1 Newton's Second Law of Mechanics

Suppose a mass m has position $\mathbf{x} \in \mathbb{R}^2$ (the restriction to $\mathbb{R}^2$ isn't necessary). Its velocity is $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, where we allow $\mathbf{x} = \mathbf{x}(t)$ to depend on time. Its acceleration is $\frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2} = \mathbf{a}$, which we will take to be the definiton of acceleration.

Newton's 2^{nd} Law tells us that

$$\mathbf{F} = m\mathbf{a}$$

Where $\mathbf{F}$ is the total force acting on the mass.

Important: The point here is that we want to consider a more general situation where we are no longer in equilibrium.

13.2 Spring-Mass systems OUT of equilibrium

We now allow our spring-mass system to be perturbed out of equilibrium. This means the displacements of the masses, $\underline{\phi}$, will become a function of time, $\underline{\phi} = \underline{\phi}(t)$, with the displacements being subject to Newton's Second Law.

The total force on the masses can be written in vector form as

$$\mathbf{f} + \mathbf{f}_I = \mathbf{f} - A^T cA\underline{\phi}$$

This is no longer equal to ZERO since we are no longer in equilibrium.

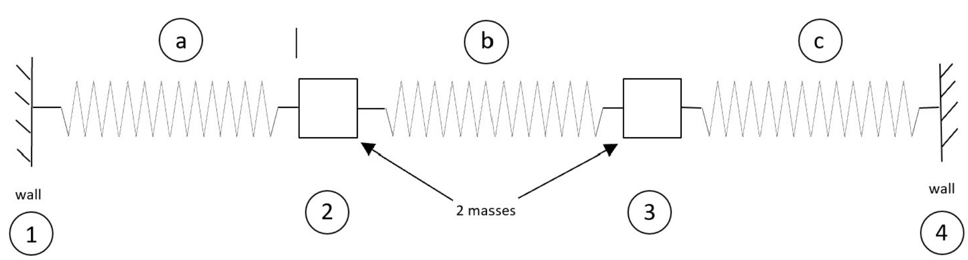
Instead, Newton's Second Law tells us that

$$\mathbf{f} + \mathbf{f}_I = \mathbf{f} - A^T cA\underline{\phi} = M\frac{d^2\underline{\phi}}{dt^2}$$

$$\text{Where} \quad M = \begin{pmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_n \end{pmatrix}$$

M is a diagonal matrix, and is positive definite since all masses must be positive. In our example, $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$, $n = 2$

13.3 Two masses, three spring between fixed walls



Let the displacements at the 4 nodes be x_i where $x_0 = x_3 = 0$ (walls are fixed)

Let $\underline{\phi} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the vector of displacements. Assume all masses are equal, so $m_1 = m_2 = 1$. Assume all spring constants equal to UNITY, so $c_a = c_b = c_c = 1$.

Laplacian Matrix:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

After $x_0 = x_3 = 0$, the reduced Laplacian $\widehat{K}$ is

$$\widehat{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

which is similar to 'grounding' two nodes. The reduced system now becomes

$$\widehat{\mathbf{f}} - \widehat{K}\widehat{\underline{\phi}} = M\frac{d^2}{dt^2}\widehat{\underline{\phi}}$$

where $\widehat{\underline{\phi}} = \begin{pmatrix} \mathbf{x_1} & \mathbf{x_2} \end{pmatrix}^T$ are the displacements of the masses and $\widehat{\mathbf{f}} = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^T$ are the external forces on the masses.

<u>Free oscillations:</u>

Assume there is <u>no</u> external force on either mass, so the system is 'free' of any external forces. Then $\widehat{\mathbf{f}} = 0$, and also $M = I$. Hence the system to be solved is

$$-\widehat{K}\widehat{\underline{\phi}} = \frac{d^2}{dt^2}\widehat{\underline{\phi}} \tag{4}$$

Notice that $\widehat{K}$ is <u>real</u> and symmetric.

We seek solutions of the form

$$\widehat{\underline{\phi}} = \Phi e^{i\omega t} \tag{5}$$

with $\omega \in \mathbb{R}$. The term $e^{i\omega t}$ is oscillatory in time, and $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ which is complex valued, which we will need to deal with, but more on this later.

We will let Φ be a complex valued vector. On substitution of 2 into 1, we get

$$-\widehat{K}\Phi e^{i\omega t} = -\omega^2\Phi e^{i\omega t}$$
$$\Rightarrow \widehat{K}\Phi = \omega^2\Phi$$

Recognise this as an <u>eigenvalue problem</u> for $\widehat{K}$! Φ is an <u>eigenvector</u> and ω^2 is an <u>eigenvalue</u>.

<u>NB:</u> $\widehat{K}$ is real symmetric $\Rightarrow$ its eigenvalues are real! ALL OF THEM :D

<u>Note 1:</u> Notice that since $\widehat{K}$ and ω^2 are <u>real</u>, then, on taking a complex conjugate

$$\overline{\widehat{K}}\overline{\Phi} = \overline{\omega^2}\overline{\Phi}$$
$$\Rightarrow \widehat{K}\overline{\Phi} = \omega^w\overline{\Phi}$$

Hence $\overline{\Phi}$ is also an eigenvector with the same eigenvalue ω^2

Note 2: Notice also that if $\Phi e^{i\omega t}$ is an eigensolution, then so too is $\Phi e^{-i\omega t}$

Note 3: Eigenvectors corresponding to distinct eigenvalues must be 'orthogonal' (i.e. $\Phi_i^T \Phi_j = 0$ if $\lambda_i \neq \lambda_j$

Real Valued Solutions:

All of these observations can be combined to show that if $\Phi e^{i\omega t}$ is an eigensolution, then so are

$$\begin{aligned} \text{Re}(\Phi e^{i\omega t}) &= \frac{1}{2}\Phi e^{i\omega t} + \frac{1}{2}\overline{\Phi} e^{-i\omega t} \\ \& \ \ \text{Im}(\Phi e^{i\omega t}) &= \frac{1}{2i}\Phi e^{i\omega t} - \frac{1}{2i}\overline{\Phi} e^{-i\omega t} \end{aligned}$$

Both of these are real-valued!

Finding the eigensolutions:

Let $\lambda = \omega^2$ (for convenience). We need to solve $\widehat{K}\Phi = \lambda\Phi$ or

$$\begin{aligned} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Phi &= \lambda\Phi \\ \text{or} \quad \left(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \lambda I \right) \Phi &= \mathbf{0} \end{aligned}$$

We need $\det(\widehat{K} - \lambda I) = 0$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{aligned} \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} &= 0 \\ \iff (2-\lambda)^2 - 1 &= 0 \\ \Rightarrow (2-\lambda)^2 &= 1 \\ \Rightarrow (2-\lambda) &= \pm 1 \\ \Rightarrow \lambda &= 1, 3 \end{aligned}$$

$$\text{Hence } \underline{\omega = \pm 1, \pm\sqrt{3}}$$

which are the *frequencies of 'free' oscillation*

Eigenvectors: Finding the eigenvectors given the eigenvalues is straightforward:

$$\begin{aligned} \lambda = 1 : \widehat{K}\Phi = \Phi \Rightarrow (\widehat{K} - I)\Phi = 0 &\Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\ &\Rightarrow \Phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda = 3 : \widehat{K}\Phi = 3\Phi \Rightarrow (\widehat{K} - 3I)\Phi = 0 &\Rightarrow \begin{pmatrix} -1 & -1 \\ -1 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\ &\Rightarrow \Phi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

Upon combining this with the observations made above, the general solution to the problem of free oscillations becomes

$$\underline{\phi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (c_1 e^{it} + \overline{c_1} e^{-it}) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (c_2 e^{\sqrt{3}it} + \overline{c_2} e^{-\sqrt{3}it})$$

Since $c_1, c_2 \in \mathbb{C}$, then the general solution depends on 4 real constants. Physically, these are determined by the <u>initial</u> positions and velocities of each mass (4 quantities)

<u>Exercise:</u> Verify that if $c_1 = \frac{1}{2}(A - iB), c_2 = \frac{1}{2}(C - iD)$ with $A, B, C, D \in \mathbb{R}$, then

$$\underline{\phi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (A\cos(t) + B\sin(t)) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C\cos(\sqrt{3}T) + D\sin(\sqrt{3}t))$$

What are the reaction forces at the two walls? These are the <u>external</u> forces at nodes 1 and 4 required to keep the displacements there zero. The external force is $\mathbf{f} = \begin{pmatrix} r_1 & \widehat{\mathbf{f}} & r_4 \end{pmatrix}^T$ where r_1 is the reaction at node 1 and r_4 is the reaction at node 4

<u>Now</u>

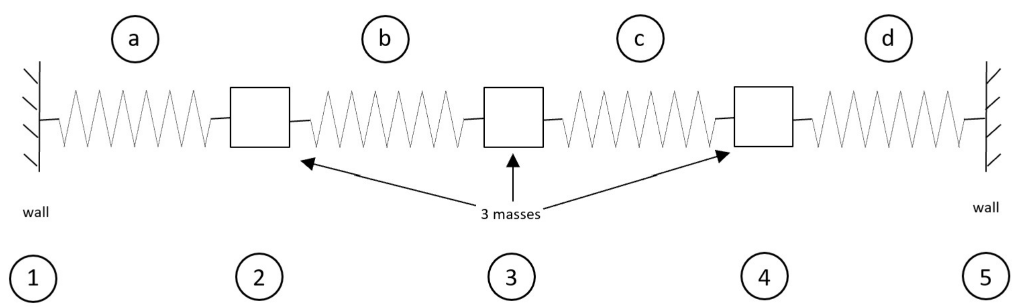
$$\mathbf{f} - K\underline{\phi} = M\frac{d^2}{dt^2}\underline{\phi} = -\omega^2\underline{\phi}$$

where $\underline{\phi} = \begin{pmatrix} 0 \\ x_1(t) \\ x_2(t) \\ 0 \end{pmatrix}$

$$\text{Hence,} \quad \begin{pmatrix} r_1 \\ 0 \\ 0 \\ r_4 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ x_1(t) \\ x_2(t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega^2 x_1(t) \\ -\omega^2 x_2(t) \\ 0 \end{pmatrix}$$

This gives the solutions for the reaction forces being $r_1 = -x_1(t)$ and $r_4 = -x_2(t)$

13.4 Three masses,4 springs between two fixed walls



Assume equal masses $m_1 = m_2 = m_3 = 1$
Assume equal spring constants $c_a = c_b = c_c = c_d = 1$
Look at <u>free</u> oscillations (same as the 2-mass case)

Laplacian for the nodes 1,2,3,4,5 is

$$K = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \underline{\phi} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Reduced system, after grounding nodes 1 and 5 by fixing them at 0 becomes

$$\widehat{K}\Phi = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \Phi = \omega^2 \Phi$$

With $\phi = \Phi e^{i\omega t}$ as before. We now set $\lambda = \omega^2$ and we solve the same equation as before $\widehat{K}\Phi = \lambda\Phi$ which is also an eigenvalue problem

Need $\det(\widehat{K} - \lambda I)$ were I is the 3-by-3 identity matrix.

$$\begin{aligned} \det \begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} &= 0 \\ \iff (2-\lambda)[(2-\lambda)^2 - 1] + 1[-1(2-\lambda)] &= 0 \\ \Rightarrow (2-\lambda)[(2-\lambda)^2 - 2] &= 0 \\ \Rightarrow \lambda = 2 \quad \text{or} \quad 2 - \lambda = \pm\sqrt{2} \Rightarrow \lambda = 2 \pm \sqrt{2} \end{aligned}$$

This now gives us 3 eigenvalues to work with. Now to find the eigenvectors:

$$\begin{aligned} \lambda = 2: &\quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \Phi_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \lambda = 2 + \sqrt{2}: &\quad \begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \Phi_2 = \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \\ \lambda = 2 - \sqrt{2}: &\quad \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \Phi_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \end{aligned}$$

From this, using the same ideas and methods as before, the general solution becomes

$$\begin{aligned} \underline{\phi} = &\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \left(c_1 e^{i\sqrt{2}t} + \overline{c_1} e^{-i\sqrt{2}t} \right) + \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \left(c_2 e^{i(2+\sqrt{2})^{\frac{1}{2}} t} + \overline{c_2} e^{-i(2+\sqrt{2})^{\frac{1}{2}} t} \right) \\ &+ \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \left(c_3 e^{i(2-\sqrt{2})^{\frac{1}{2}} t} + \overline{c_3} e^{-i(2-\sqrt{2})^{\frac{1}{2}} t} \right) \end{aligned}$$

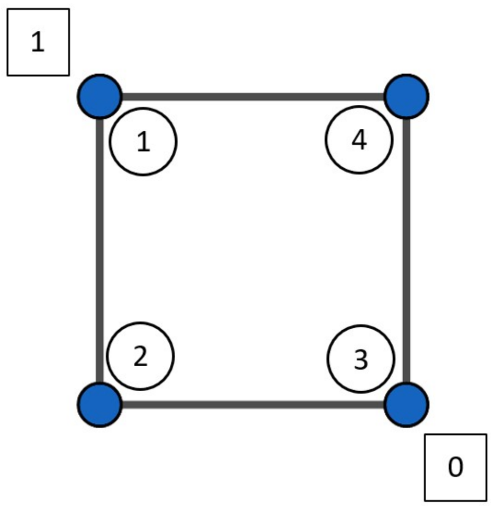
This solution depends on 3 complex constants, c_1, c_2, c_3, which is equivalent to 6 real constants.

14 Symmetry in Applied Mathematics

Symmetry is an important concept that can greatly aid the solution of a variety of applied mathematical problems.

Indeed, we have already seen how useful it can be in studying circuits and, in particular, in computing the effective conductance of a circuit by successively reducing the circuit to a set of 'equivalent circuits' having the same effective conductance.

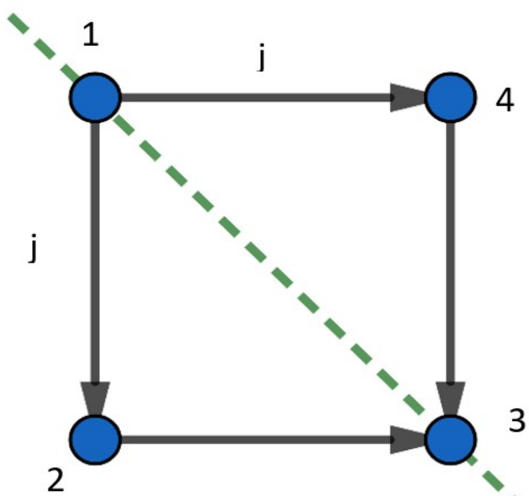
Let's study, in more detail, why these methods worked. Recall the example circuit:



- all conductances equal to 1
- node 1 set to unit voltage
- node 3 grounded

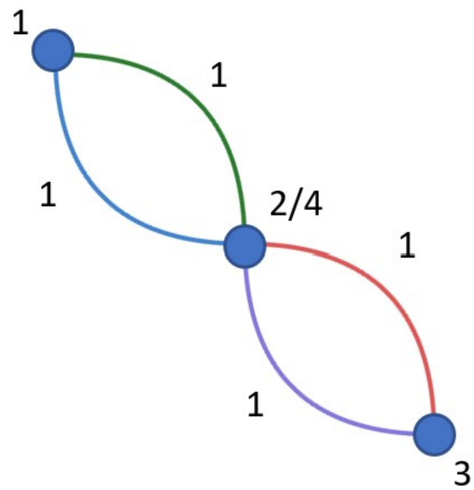
Now the task is to find the effective conductance of this circuit by symmetry.

Notice that 'by symmetry' of the circuit (i.e. the fact that all conductances are equal, and the configuration of the electrified/grounded nodes), we can see an axis of symmetry:

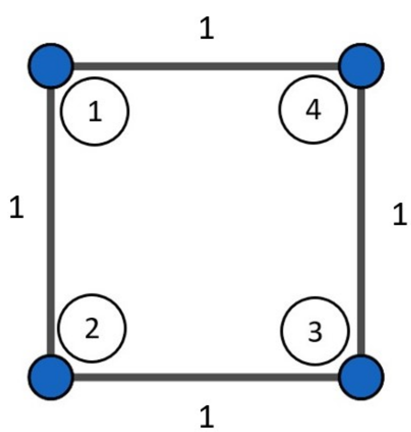


We expect the currents j along the edges shown to be equal, hence the voltages at nodes 2 and 4 to be equal also.

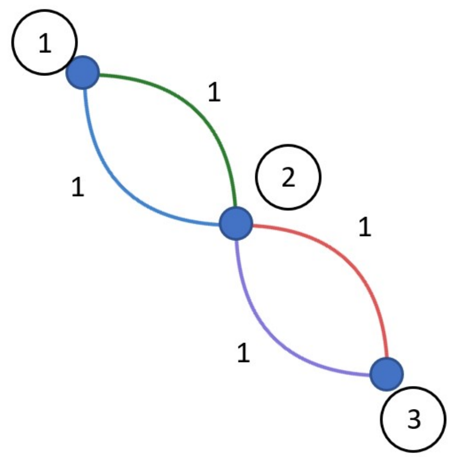
We therefore can consider merging these nodes and consider the 'equivalent circuit':



We therefore claim that the following circuits are equivalent (as far as computing the effective conductance is concerned): Mathematically, for System 1:



System 1



System 2

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ -f \\ 0 \end{pmatrix}$$

And for System 2:

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & -2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \widehat{f} \\ 0 \\ -\widehat{f} \end{pmatrix}$$

So how are these two systems 'equivalent'? To answer this, let's introduce a new matrix S, with

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Notice that

$$KS = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 \end{pmatrix}$$

$$SK = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 \end{pmatrix}$$

i.e. $\underline{KS = SK}$ So these two matrices, K and S commute!! This observation has important consequences.

Suppose we find a solution to

$$K\mathbf{x} = \mathbf{f} \tag{1}$$

Then we get the following equation

$$SK\mathbf{x} = S\mathbf{f} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ 0 \\ -f \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ -f \\ 0 \end{pmatrix} = \mathbf{f}$$

But since matrices K and S commute, we can rewrite this as

$$KS\mathbf{x} = \mathbf{f}$$

Therefore $S\mathbf{x}$ is also a solution of 1

$$\text{But} \qquad S\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \end{pmatrix}$$

Notice here how x_2 and x_4 have been swapped! However, by the uniqueness principle for harmonic potentials (two harmonic potentials with the same boundary data must be the same), we can conclude that

$$\mathbf{x} = S\mathbf{x}$$

And from this, we can say that $\underline{x_2 = x_4}$ Which is the same deduction we made earlier based on deduction!!

Since we know that $x_2 = x_4$, we can add columns 2 and 4 in System 1:

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ -f \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ -f \\ 0 \end{pmatrix}$$

Notice rows 2 and 4 are now identical, so we can either delete row 4 OR add rows 2 and 4 then just delete row 4 (OR just delete row 4 and multiply row 2 by 2)

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ -f \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & -2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ -f \end{pmatrix}$$

But this is exactly SYSTEM 2!!! Again, the uniqueness principle can be used to argue that the solutions are the SAME, and hence that $f = \widehat{f}$

14.1 Another Observation

S is a square matrix, hence we can look at its eigenvectors. Suppose $S\mathbf{x} = \lambda\mathbf{x}$ for some non-zero eigenvector $\mathbf{x}$ and non-zero eigenvalue λ where the dimension of the eigenspace is UNITY (i.e. there is only 1 eigenvalue of S that equals λ, all others are different)

$$\text{Then} \qquad KS\mathbf{x} = \lambda K\mathbf{x}$$

But K and S commute, hence

$$SK\mathbf{x} = \lambda K\mathbf{x}$$

From this, we conclude that $S(K\mathbf{x}) = \lambda(K\mathbf{x})$ or $K\mathbf{x}$ must be parallel to $\mathbf{x}$ since it is also an eigenvector with eigenvalue λ (and hence are both in the same 1-dimensional eigenspace)

But this means that

$$K\mathbf{x} = \widehat{\lambda}\mathbf{x}$$

The eigenvalue here may not be the same as before, hence the change in notation.

Let's check this.

What are the eigenvectors of S?

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

By inspection, we found that its 4 eigenvectors are:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

The first 3 have eigenvalue 1, but the 4^{th} vector has eigenvalue -1. Hence, from our previous observation, we expect $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ to be an eigenvector of K!

14.2 Eigenvectors of K

We know this! From Sheet 1, Question 6, we know the answer to this. The eigenvectors of K are

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix} \qquad n = 0, 1, 2, 3 \qquad \omega = e^{\frac{2i\pi}{4}} = i$$

With corresponding eigenvalue $\lambda_n = 2 - \omega^n - \omega^{-n}$, $\quad n = 0, 1, 2, 3$ ($\lambda_0 = 0, \lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 2$)

Hence

$$\mathbf{x_0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x_1} = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

Notice that

$$\mathbf{x_1} - \mathbf{x_3} = \begin{pmatrix} 0 \\ 2i \\ 0 \\ -2i \end{pmatrix} = 2i \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

This is the eigenvector of S with eigenvalue -1, thus we have checked that our observation holds in this example.

Note 1: The eigenvalue of $\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}^T$ as a vector of S is -1, but as a vector of K, it has an eigenvalue of 2.

Note 2: another shared eigenvector of K and S is $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$ by inspection.

How could we have established this using similar eigenvector considerations?

We could have started with the eigenvectors of K in our previous argument (instead of those of S) Some reasoning applies!

This means that any e-vector of K having multiplicity 1 is also and eigenvector of S. But the eigenvalues of K are (0,2,4,2). The eigenvector associated with the eigenvalue of 0 is $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$ and the eigenvector associated with the eigenvalue of 4 is $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$. This means we can immediately conclude that both of these vectors must also be eigenvectors of S as well. This can be easily checked.

15 Circulant Matrices

The matrix K that we just discussed, i.e. $K = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$, and considered on Problem Sheet 1 Q6, is an example of a Circulant Matrix.

How do we find its eigenvectors? (note: on Problem sheet 1, they were simply given to you, not worked out)

The answer lies in SYMMETRY

We can exploit symmetry to reduce the finding of the eigenvectors of K to a much simpler problem.

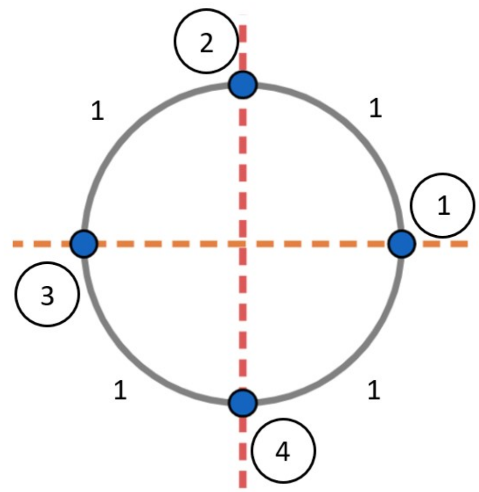
Why is it called a 'Circulant' matrix?

This is because column 2 of K is obtained from column 1 by shifting all elements by 1 position, with 'x_4' going to 'x_1'

$$\text{i.e.} \qquad x_1 \mapsto x_2, \quad x_2 \mapsto x_3, \quad x_3 \mapsto x_4, \quad x_4 \mapsto x_1 \tag{1}$$

Similarly, column 3 follows from column 2 and column 4 follows from column 3.

This mathematical observation can be understood as a consequence of the symmetry of the original graph from which we derived K as the Laplacian matrix:



It is clear from this graph that since the nodes are connected in this rotationally symmetric pattern, with all equal edge conductances, that nothing changes if we reassign vertex labels as in 1.

Now consider the matrix S given by

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This matrix encodes the reassignment embodied in 1:

$$S\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix}$$

What are the eigenvectors of S? These will be easier to find if we know its eigenvalues.

But suppose $S\mathbf{x} = \lambda\mathbf{x}$ for some λ and $\mathbf{x}$

Then it is clear that

$$\begin{aligned} & S^2\mathbf{x} = S(S\mathbf{x}) = S(\lambda\mathbf{x}) = \lambda S\mathbf{x} = \lambda^2\mathbf{x} \\ \& \quad & S^3\mathbf{x} = \cdots = \lambda^3\mathbf{x} \\ \& \quad & S^4\mathbf{x} = \cdots = \lambda^4\mathbf{x} \end{aligned}$$

But we know that if we carry out the reassignment 1 4 times in succession then we will <u>return</u> to the $\mathbf{x}$ that we started with. Therefore $S^4\mathbf{x} = \mathbf{x}$.

The conclusion is that $\underline{\lambda^4 = 1}$

Therefore, by the fundamental theorem of algebra, the eigenvalues of S are the 4^{th} roots of unity

$$\lambda_n = e^{\frac{2ni\pi}{4}} \qquad n = 0, 1, 2, 3$$

We chose $n = 0, 1, 2, 3$ because those are the values of n which give different values. We can introduce $\omega = e^{\frac{2i\pi}{4}}$ and then the eigenvalues of S become $1, \omega, \omega^2, \omega^3$. Having found the eigenvalues of S (the hard part usually), the eigenvectors can now be found easily.

<u>N.B.</u> Usually one has to resort to finding the zeros of a <u>determinant</u> to find eigenvalues! Symmetry has helped us avoid all that.

15.1 Eigenvectors of S

$$\lambda_0 = 1, \quad \text{need to solve} \quad S\mathbf{x_0} = S\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = \begin{pmatrix}\phi_2\\\phi_3\\\phi_4\\\phi_1\end{pmatrix} = \begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}$$

$$\therefore \quad \mathbf{x_0} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \text{setting } \phi_1 = 1 \text{ WLOG}$$

$$\lambda_1 = \omega, \quad \text{need to solve} \quad S\mathbf{x_1} = \begin{pmatrix}\phi_2\\\phi_3\\\phi_4\\\phi_1\end{pmatrix} = \omega\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = \omega\mathbf{x_1}$$

$$\therefore \quad \mathbf{x_1} = \begin{pmatrix}1\\\omega\\\omega^2\\\omega^3\end{pmatrix} \quad \text{setting } \phi_1 = 1 \text{ WLOG}$$

$$\lambda_2 = \omega^2, \quad \text{need to solve} \quad S\mathbf{x_2} = \begin{pmatrix}\phi_2\\\phi_3\\\phi_4\\\phi_1\end{pmatrix} = \omega^2\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = \omega^2\mathbf{x_2}$$

$$\therefore \quad \mathbf{x_2} = \begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{pmatrix} \qquad \text{setting } \phi_1 = 1 \text{ WLOG}$$

$$\lambda_3 = \omega^3, \quad \text{need to solve} \quad S\mathbf{x_3} = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_1 \end{pmatrix} = \omega^3 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \omega^3 \mathbf{x_3}$$

$$\therefore \quad \mathbf{x_3} = \begin{pmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega^9 \end{pmatrix} \qquad \text{setting } \phi_1 = 1 \text{ WLOG}$$

15.2 Summary

Eigenvalues of S are $\lambda_n = e^{\frac{2ni\pi}{4}}$ $\quad n = 0, 1, 2, 3$ with corresponding eigenvectors

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix} \qquad n = 0, 1, 2, 3$$

<u>Important Observation:</u>

$$KS = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 & -0 \\ 0 & -1 & 2 & -1 \\ -1 & -0 & -1 & 2 \\ 2 & -1 & 0 & -1 \end{pmatrix}$$

$$SK = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 & -0 \\ 0 & -1 & 2 & -1 \\ -1 & -0 & -1 & 2 \\ 2 & -1 & 0 & -1 \end{pmatrix}$$

Hence $SK = KS$ and we can see that K and S commute again!

By the same arguments introduced earlier, this means that any eigenvectors of S with eigenvalues of multiplicity 1 will <u>also</u> be eigenvectors of K!

However is it clear that <u>ALL</u> eigenvalues of S are distinct, hence are of multiplicity 1.

This means that we have now found <u>all</u> of the eigenvectors of K: they are the same as those of S:

$$\lambda_n = e^{\frac{2ni\pi}{4}} \quad n = 0, 1, 2, 3$$

This is consistent with Q6 on Problem Sheet 1

We still need to find the corresponding eigenvectors of K (these will generally be different to those of S):

$$K\mathbf{x}_n = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix} = (2 - \omega^n - \omega^{3n}) \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix}$$

Hence the eigenvalues are:

$$\begin{aligned}
2-\omega^n-\omega^{3n} &= 2-\omega^n-\omega^{(4-1)n} \\
&= 2-\omega^n-\left(e^{\frac{2i\pi}{4}}\right)^{(4-1)n} \\
&= 2-\omega^n-e^{2i\pi n}e^{-\frac{2i\pi n}{4}} \\
&= 2-e^{\frac{2i\pi n}{4}}-e^{-\frac{2i\pi n}{4}} \\
&= 2-2\cos\left(\frac{2\pi n}{4}\right)
\end{aligned}$$

$$\lambda_n^{[K]} = 2-2\cos\left(\frac{2\pi n}{4}\right) \qquad n=0,1,2,3$$

Or more simply {0,2,4,2}

15.3 Circulant Matrix of dimension N

It should be clear that this symmetry argument generalises, in a straightforward way, to a circulant matrix of any integer dimension $N \geq 1$.

Let

$$C_N = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \qquad C_N \text{ is an } N\text{-by-}N \text{ matrix}$$

Then the eigenvectors and associated eigenvalues of C_N are

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega_N^n \\ \omega_N^{2n} \\ \vdots \\ \omega_N^{(N-1)n} \end{pmatrix} \qquad n=0,1,\dots,N-1$$

$$\lambda_n = 2-2\cos\left(\frac{2\pi n}{N}\right) \qquad n=0,1,\dots,N-1$$

Important Observation: Pick $j = 0, 1 \ldots, N-1, N$

$$\text{Let} \quad n = j \qquad \underline{\text{AND}} \qquad n = N - j$$

$$\lambda_n = 2 - 2\cos\left(\tfrac{2\pi j}{N}\right) \qquad \lambda_n = 2 - 2\cos\left(\tfrac{2\pi(N-j)}{N}\right)$$

$$\lambda_n = 2 - 2\cos\left(2\pi - \tfrac{2\pi j}{N}\right)$$

$$\lambda_n = 2 - 2\cos\left(\tfrac{2\pi j}{N}\right)$$

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega_N^j \\ \omega_N^{2j} \\ \vdots \\ \omega_N^{(N-1)j} \end{pmatrix} \qquad \mathbf{x}_n = \begin{pmatrix} 1 \\ \omega_N^{N-j} \\ \omega_N^{2(N-j)} \\ \vdots \\ \omega_N^{(N-1)(N-j)} \end{pmatrix} = \begin{pmatrix} 1 \\ \omega_N^{-j} \\ \omega_N^{-2j} \\ \vdots \\ \omega_N^{-(N-1)j} \end{pmatrix}$$

The eigenvectors obtained at the end are complex conjugates of each other and the eigenvalues are identical! Hence since the eigenvalues are the same, we can form REAL eigenvectors by taking the real and imaginary parts of $\mathbf{x}_n$ for $n = j$ and $n = N - j$, which corresponds to taking linear combinations of the basis of the eigenspace.

It makes sense that we can form a set of real eigenvectors of C_N since it is a real symmetric matrix so we know that N real eigenvectors/eigenvalues exist (we also expect them to be mutually orthogonal).

Example: $N = 4$

$$\text{Eigenvectors are} \quad \begin{pmatrix} 1 \\ \cos\left(\frac{2\pi n}{4}\right) \\ \cos\left(\frac{4\pi n}{4}\right) \\ \cos\left(\frac{6\pi n}{4}\right) \end{pmatrix}, \begin{pmatrix} 1 \\ \sin\left(\frac{2\pi n}{4}\right) \\ \sin\left(\frac{4\pi n}{4}\right) \\ \sin\left(\frac{6\pi n}{4}\right) \end{pmatrix} \qquad n = 0, 1, 2, 3$$

$$n = 0 \quad \text{gives} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \longleftarrow \lambda = 2 - 2\cos\left(\frac{2\pi n}{4}\right) = 0$$

$$n = 1 \quad \text{gives} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad \longleftarrow \lambda = 2 - 2\cos\left(\frac{2\pi n}{4}\right) = 2 - 2\cos\left(\frac{\pi}{2}\right) = 2$$

$$n = 2 \quad \text{gives} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \longleftarrow \lambda = 2 - 2\cos\left(\frac{2\pi n}{4}\right) = 2 - 2\cos(\pi) = 4$$

$n = 3 \quad \longrightarrow \quad$ nothing new

It is easy to check these are linearly independent and mutually orthogonal.

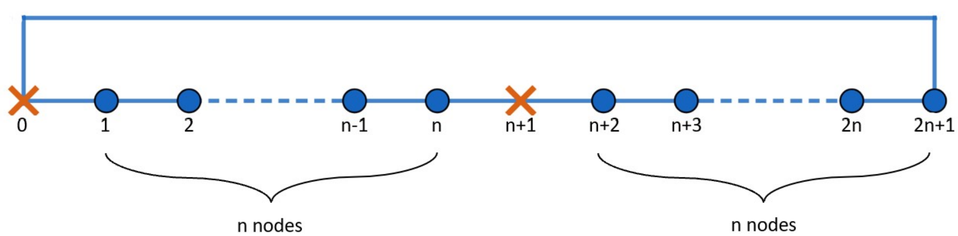
General Case: C_N has eigenvectors of the form:

$$\begin{pmatrix} 1 \\ \cos\left(\frac{2\pi n}{N}\right) \\ \cos\left(\frac{4\pi n}{N}\right) \\ \cos\left(\frac{6\pi n}{N}\right) \\ \vdots \\ \cos\left(\frac{2(N-1)\pi n}{N}\right) \end{pmatrix}, \begin{pmatrix} 1 \\ \sin\left(\frac{2\pi n}{N}\right) \\ \sin\left(\frac{4\pi n}{N}\right) \\ \sin\left(\frac{6\pi n}{N}\right) \\ \vdots \\ \sin\left(\frac{2(N-1)\pi n}{N}\right) \end{pmatrix}$$

Where $n = 0, 1, \ldots, \frac{N}{2}$ if N is even and $n = 0, 1, \ldots, \frac{N-1}{2}$ if N is odd. In the case of even N, $n = 0, \frac{N}{2}$ give 1 eigenvector each, and $n = 1, \ldots, \frac{N}{2} - 1$ gives 2 eigenvectors each. In the case of odd N, $n = 0$ gives 1 eigenvector, $n = 1, \ldots \frac{N-1}{2}$ each gives 2 eigenvctors.

16 On a Submatrix of a Circulant Matrix

Consider a circulant matrix with the corresponding graph of $2n + 2$ nodes



Let $\quad \mathbf{x} = \begin{pmatrix} x_0 \\ x_{n+1} \\ x_1 \\ \vdots \\ x_n \\ x_{n+2} \\ \vdots \\ x_{2n+1} \end{pmatrix} \quad \longleftarrow$ Note ordering of the node potentials

We mark out nodes 0 and $n + 1$ for special attention with the orange cross.

Between these nodes there are <u>two</u> sets of n nodes (joined by edges). So, what is the Laplacian of this graph? (Give me strength, this is probably the worst matrix I'll ever write)

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} 0 & n+1 & 1 & 2 & \dots & \dots & n-1 & n & n+2 & n+3 & \dots & \dots & 2n & 2n+1 \end{array} \\
\begin{array}{c} 0 \\ n+1 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ \vdots \\ n \\ n+2 \\ n+3 \\ \vdots \\ \vdots \\ \vdots \\ 2n+1 \end{array} &
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1 \\
0 & 2 & 0 & 0 & 0 & \dots & 0 & -1 & -1 & 0 & \dots & \dots & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
\vdots & \vdots & 0 & -1 & 2 & -1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 2 & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & 0 & \dots & \dots & -1 & 2 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & -1 & 0 & 0 & \dots & \dots & 0 & 0 & 2 & -1 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & -1 & 2 & -1 & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 2 & -1 \\
-1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2
\end{pmatrix}
\end{array}
$$

$$
= \begin{pmatrix} P & Q_1^T & Q_2^T \\ Q_1 & Q_1 & 0 \\ Q_2 & 0 & K_n \end{pmatrix}
$$

$$
\text{where} \quad K_n = \begin{pmatrix}
2 & -1 & 0 & 0 & \dots & 0 & 0 \\
-1 & 2 & -1 & 0 & \dots & 0 & 0 \\
0 & -1 & 2 & -1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 2 & -1 \\
0 & 0 & 0 & 0 & \dots & -1 & 2
\end{pmatrix}
$$

<u>Special Cases:</u>

$$
n = 2, \qquad K_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
$$

$$
n = 3, \qquad K_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 2 \end{pmatrix}
$$

<u>Note:</u> We saw <u>both</u> of these matrices earlier when considering 2 masses and 3 masses freely oscillating between two fixed walls.

We will now deduce some general properties concerning the eigenvectors/eigenvalues for any $n \geq 2$.

(Note: These are <u>NOT</u> themselves circulant matrices!)

We know the eigenvectors/eigenvalues of the $(2n+2)$-dimensional circulant matrix we started with.

In the natural ordering, they were the real and imaginary parts of

$$\mathbf{x}_m = \begin{pmatrix} 1 \\ \omega^m \\ \vdots \\ \omega^{2nm} \\ \omega^{(2n+1)m} \end{pmatrix} \qquad m = 0, 1, \ldots, 2n+1$$

Where $\omega = e^{\frac{2i\pi}{2(n+1)}} = e^{\frac{i\pi}{n+1}}$

The corresponding eigenvalue is

$$\begin{aligned} \lambda_m &= 2\left[1 - \cos\left(\frac{2\pi m}{2(n+1)}\right)\right] \\ &= 2\left[1 - \cos\left(\frac{\pi m}{n+1}\right)\right] \qquad m = 0, 1, \ldots, 2n+1 \end{aligned}$$

Let's rewrite these with our modified ordering:

$$\mathbf{x_m} = \begin{pmatrix} 1 \\ \omega^{(n+1)m} \\ \omega^m \\ \omega^{2m} \\ \vdots \\ \omega^{nm} \\ \omega^{(n+2)m} \\ \vdots \\ \omega^{(2n+1)m} \end{pmatrix} = \begin{pmatrix} 1 \\ (-1)^m \\ \omega^m \\ \omega^{2m} \\ \vdots \\ \omega^{nm} \\ \omega^{(n+2)m} \\ \vdots \\ \omega^{(2n+1)m} \end{pmatrix}$$

$$\text{Hence} \qquad \mathrm{Im}(\mathbf{x}_m) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{Im}(\omega^m) \\ \vdots \\ \mathrm{Im}(\omega^{nm}) \\ \mathrm{Im}(\omega^{(n+2)m}) \\ \vdots \\ \mathrm{Im}(\omega^{(2n+1)m}) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \widehat{\mathbf{x}}_m \\ \\ \widehat{\widehat{\mathbf{x}}}_m \\ \end{pmatrix} \equiv \underline{\phi}_m$$

$$\text{where} \qquad \widehat{\mathbf{x}}_m = \begin{pmatrix} \mathrm{Im}(\omega^m) \\ \vdots \\ \vdots \\ \mathrm{Im}(\omega^{nm}) \end{pmatrix}, \quad \widehat{\widehat{\mathbf{x}}}_m = \begin{pmatrix} \mathrm{Im}(\omega^{(n+2)m}) \\ \vdots \\ \vdots \\ \mathrm{Im}(\omega^{(2n+1)m}) \end{pmatrix}$$

Both of these vectors are n-dimensional. Now we introduce the idea of using the sub-block decomposition of 'God's Laplacian' to help us find the eigenvectors and eigenvalues of these two n-dimensional vectors.

$$\widetilde{C}_{2n+2}\underline{\phi}_m = \lambda_m \underline{\phi}_m \qquad \underline{\text{or}} \qquad \begin{pmatrix} P & Q_1^T & Q_2^T \\ Q_1 & Q_1 & 0 \\ Q_2 & 0 & K_n \end{pmatrix} \phi_{\mathbf{m}} = \begin{pmatrix} P & Q_1^T & Q_2^T \\ Q_1 & Q_1 & 0 \\ Q_2 & 0 & K_n \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{0}} \\ \widehat{\mathbf{x}}_m \\ \widehat{\widehat{\mathbf{x}}}_m \end{pmatrix} = \lambda_m \begin{pmatrix} \widehat{\mathbf{0}} \\ \widehat{\mathbf{x}}_m \\ \widehat{\widehat{\mathbf{x}}}_m \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} Q_1^T \widehat{\mathbf{x}}_m + Q_2^T \widehat{\widehat{\mathbf{x}}}_m &= 0 \\ K_n \widehat{\mathbf{x}}_m &= \lambda_m \widehat{\mathbf{x}}_m \\ K_n \widehat{\widehat{\mathbf{x}}}_m &= \lambda_m \widehat{\widehat{\mathbf{x}}}_m \end{aligned}$$

So both $\widehat{\mathbf{x}}_m$ and $\widehat{\widehat{\mathbf{x}}}_m$ are eigenvectors of K_n with eigenvalue λ_m!

However is it easy to show that $\widehat{\mathbf{x}}_m = (-1)^m \widehat{\widehat{\mathbf{x}}}_m$, so they are not linearly independent.

However, since $m = 0, 1, \ldots, 2n+1$, we appear to have generated $2n+2$ eigenvectors. But this is not the case! $\widehat{\mathbf{x}}_m = 0$ when $m = 0, n+1$ so we can discount these. Moreover, it is easy to check that

$$\widehat{\mathbf{x}}_{m+(n+1)} = -\widehat{\mathbf{x}}_m$$

which means that <u>only</u> the values $m = 1, 2, \ldots, n$ give independent eigenvectors.

<u>Summary:</u> K_n has eigenvectors

$$\underline{\phi}_m = \begin{pmatrix} \sin\left(\frac{m\pi}{n+1}\right) \\ \sin\left(\frac{2m\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nm\pi}{n+1}\right) \end{pmatrix}$$

$$\lambda_m = 2\left[1 - \cos\left(\frac{m\pi}{n+1}\right)\right] \qquad m = 1, 2, \ldots, n$$

<u>Exercise:</u> Check that these general formulae retrieve the values for $n = 2, 3$ we found earlier uskng direct calculation.

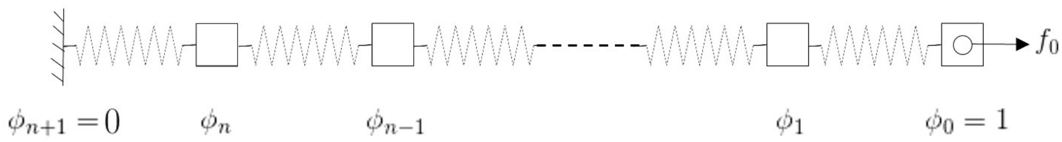
<u>Exercise:</u> Find the value of A_m in

$$\widehat{\underline{\phi}}_m = A_m \begin{pmatrix} \sin\left(\frac{m\pi}{n+1}\right) \\ \vdots \\ \vdots \\ \sin\left(\frac{nm\pi}{n+1}\right) \end{pmatrix}$$

If we require $\widehat{\underline{\phi}}_m^T \widehat{\underline{\phi}}_m = 1$ (See Problem Sheet 6 for the solution)

<u>Exercise:</u> Verify <u>directly</u> that these eigenvectors are mutually orthogonal (also see Problem Sheet 6 for the solution)

17 $n+1$ masses pulled from a fixed wall



Consider $n+1$ masses, each of unit mass, connected by springs, each with unit spring constant, with a fixed wall at one end and being pulled by a force f_0 at the other.

Describe this system as a graph with $n+2$ nodes with node 0 being the end mass with unit displacement i.e.

$$\phi_0 = 1$$

And node $n+1$ being the fixed wall, so

$$\phi_{n+1} = 0$$

The n masses in between are taken to be free of external forces.

$$\text{Let} \quad \mathbf{x} = \begin{pmatrix} \phi_0 \\ \phi_{n+1} \\ \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \qquad \longleftarrow \text{ note the } \underline{\text{ordering}} \text{ (putting 'forced' nodes at the start)}$$

The Laplacian for this graph is

$$K = \begin{array}{c} \\ 0 \\ n+1 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & n+1 & 1 & 2 & \dots & \dots & n-1 & n \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & \dots & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \end{array} \equiv \begin{pmatrix} P & Q^T \\ Q & K_n \end{pmatrix}$$

$$\text{where} \quad K_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \; P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Q is a n-by-2 matrix. The force balance condition is now

$$K\mathbf{x} = \mathbf{f} = \begin{pmatrix} f_0 \\ -f_0 \\ \\ \mathbf{0_n} \\ \\ \end{pmatrix}$$

$\mathbf{0_n}$ is an n-dimensional vector of zeroes. We can now rewrite this using the sub-block decomposition as

$$\begin{pmatrix} P & Q^T \\ Q & K_n \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \widehat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{0}_n \end{pmatrix}$$

$$\text{where} \quad , \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{f}_1 = \begin{pmatrix} f_0 \\ -f_0 \end{pmatrix}, \quad \widehat{\mathbf{x}} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

This is equivalent to

$$P\mathbf{e}_1 + Q^T\widehat{\mathbf{x}} = \mathbf{f}_1$$
$$Q\mathbf{e}_1 + K_n\widehat{\mathbf{x}} = \mathbf{0}_n$$

The second of this means that

$$K_n\widehat{\mathbf{x}} = -Q\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We must solve this equation for the unknown equilibrium displacements $\widehat{\mathbf{x}}$

Let us do this using the known eigenvectors of K_n:

$$K_n\underline{\phi}_j = \lambda_j\underline{\phi}_j, \qquad j = 1, 2, \ldots, n$$

$$\text{where} \quad \underline{\phi}_j = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}$$

$$\text{with} \quad \lambda_j = 2\left[1 - \cos\left(\frac{j\pi}{n+1}\right)\right]$$

This is an orthonormal set that can serve as a natural basis of $\mathbb{R}^n$

Therefore let us write the solution as

$$\widehat{\mathbf{x}} = \sum_{j=1}^{n} a_j \underline{\phi}_j$$

$$\underline{\text{Now}} \quad K_n \widehat{\mathbf{x}} = K_n \left(\sum_{j=1}^{n} a_j \underline{\phi}_j \right)$$

$$= \sum_{j=1}^{n} a_j K_n \underline{\phi}_j = \sum_{j=1}^{n} a_j \lambda_j \underline{\phi}_j$$

$$\text{But} \quad K_n \widehat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{Hence} \quad \sum_{j=1}^{n} a_j \lambda_j \underline{\phi}_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Use orthogonality to find $\{a_j\}$: Take product with $\underline{\phi}_j^T$

$$\sum_{j=1}^{n} a_j \lambda_j \underline{\phi}_m^T \underline{\phi}_j = \underline{\phi}_m^T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{m\pi}{n+1}\right)$$

$$\therefore \quad a_m \lambda_m 1 = \sqrt{\frac{2}{n+1}} \sin\left(\frac{m\pi}{n+1}\right)$$

$$\Rightarrow a_m = \sqrt{\frac{2}{n+1}} \; \frac{\sin\left(\frac{m\pi}{n+1}\right)}{2\left[1 - \cos\left(\frac{m\pi}{n+1}\right)\right]}$$

Hence the solution we seek is

$$\widehat{\mathbf{x}} = \sum_{j=1}^{n} \left(\sqrt{\frac{2}{n+1}} \; \frac{\sin\left(\frac{j\pi}{n+1}\right)}{2\left[1 - \cos\left(\frac{j\pi}{n+1}\right)\right]} \right) \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}$$

$$= \frac{1}{n+1} \sum_{j=1}^{n} \left(\frac{\sin\left(\frac{j\pi}{n+1}\right)}{2\left[1 - \cos\left(\frac{j\pi}{n+1}\right)\right]} \right) \begin{pmatrix} \sin\left(\frac{j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}$$

Or if we were to write this out component-wise,

$$\widehat{\mathbf{x}}_m = \frac{1}{n+1}\sum_{j=1}^{n}\left(\frac{\sin\left(\frac{j\pi}{n+1}\right)}{2\left[1-\cos\left(\frac{j\pi}{n+1}\right)\right]}\right)\sin\left(\frac{mj\pi}{n+1}\right)$$

$\underline{\underline{\underline{\text{BUT}}}}$ WE ALREADY <u>KNOW</u> THE ANSWER TO THIS PROBLEM!!!!!!!!!!!!

It is equivalent to the random walk on a line with $p_0 = 1, p_{n+1} = 0$ - see Problem Sheet 3, Q1

From this, the solution is $p_m = 1 - \frac{m}{n+1}$

We therefore conclude, from the uniqueness principle, that these solutions must be the <u>same</u> at the interior nodes:

$$p_m = \widehat{x}_m \qquad \text{for} \quad m = 1, \ldots, n$$

$$\underline{\text{or}} \quad 1 - \frac{m}{n+1} = \frac{1}{n+1}\sum_{j=1}^{n}\left(\frac{\sin\left(\frac{j\pi}{n+1}\right)}{2\left[1-\cos\left(\frac{j\pi}{n+1}\right)\right]}\right)\sin\left(\frac{mj\pi}{n+1}\right) \qquad \text{for} \quad m = 1, \ldots, n$$

17.1 Limit as $n \to \infty$

It is natural to ask what happens to this expression as $n \to \infty$.

Let us introduce $x = \frac{\pi m}{n+1}$. This replaces the integer index by a variable taking a discrete set of values $0 < x < \pi$ since $m = 1, 2, \ldots, n$

Our expression then becomes

$$\frac{1}{n+1}\sum_{j=1}^{\infty}\left(\frac{\sin\left(\frac{\pi j}{n+1}\right)}{1-\cos\left(\frac{\pi j}{n+1}\right)}\right)\sin(jx) = 1 - \frac{x}{\pi} \quad \text{for} \quad x = \frac{\pi m}{n+1}, \quad m = 1, \ldots, n$$

As $n \to \infty$, this becomes

$$\frac{1}{n+1}\sum_{j=1}^{\infty}\left(\frac{\frac{\pi j}{n+1} + \ldots}{1 - \left(1 - \frac{1}{2!}\left(\frac{\pi j}{n+1}\right)^2 + \ldots\right)}\right)\sin(jx) = \underline{\underline{\sum_{j=1}^{\infty}\frac{2}{\pi j}\sin(jx) = 1 - \frac{x}{\pi}}}$$

This looks like a 'Fourier Sine Series' like you encountered last term - well it is!

Let's compute the FOURIER SINE SERIES of $1 - \frac{x}{\pi}$ using ideas from last term! Write

$$1 - \frac{x}{\pi} = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Find the coefficients b_n by 'orthogonality': multiply this equation by $\sin(mx)$ and integrate between 0 and π

$$\int_0^{\pi}\left(1 - \frac{x}{\pi}\right)\sin(mx)dx = \int_0^{\pi}\sum_{n=1}^{\infty} b_n \sin(nx)\sin(mx)dx$$

Now, using integration by parts,

$$\begin{aligned}\int_0^\pi \left(1-\frac{x}{\pi}\right)\sin(mx)dx &= \left[-\left(1-\frac{x}{\pi}\right)\frac{\cos(mx)}{m}\right]_0^\pi - \frac{1}{\pi m}\int_0^\pi \cos(mx)dx \\ &= 0+\frac{1}{m}-\left[\frac{\sin(mx)}{\pi m^2}\right]_0^\pi \\ &= \frac{1}{m}\end{aligned}$$

Now for the other side:

$$\begin{aligned}\int_0^\pi \sin(nx)\sin(mx)dx &= \frac{1}{2}\int_0^\pi \left(\cos((m-n)x)-\cos((m+n)x)\right)dx \\ &= 0 \quad \text{unless} \quad m=n\end{aligned}$$

In the case that $m = n$

$$\begin{aligned}\int_0^\pi \sin^2(mx)dx &= \int_0^\pi \frac{1}{2}\left(1-\cos(2mx)\right)dx \\ &= \frac{\pi}{2} \\ \therefore \quad \frac{1}{m} &= \frac{\pi}{2}b_m \\ \Rightarrow \quad b_m &= \frac{2}{m\pi}\end{aligned}$$

$$\text{Hence} \qquad 1-\frac{x}{\pi} = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(nx)$$

Exactly as we wanted :)

In the previous analysis, we derived two different representations of the solution to the discrete system for $(n+1)$ masses.

We also introduced the idea of writing that solution as the values of a continuous function at a set of discrete points, i.e.

$$p_m = 1-\frac{m}{n+1} = p^{(n)}(x) \qquad \text{where} \quad x=\frac{\pi m}{n+1}, \quad m=1,\ldots,n$$

We then found that, for any n,

$$p^{(n)}(x) = \frac{1}{n+1}\sum_{j=1}^{n}\frac{\sin\left(\frac{\pi j}{n+1}\right)}{1-\cos\left(\frac{\pi j}{n+1}\right)}\sin(jx) = 1-\frac{x}{\pi}$$

$$\text{for} \quad x=\frac{\pi m}{n+1}, \quad m=1,\ldots,n$$

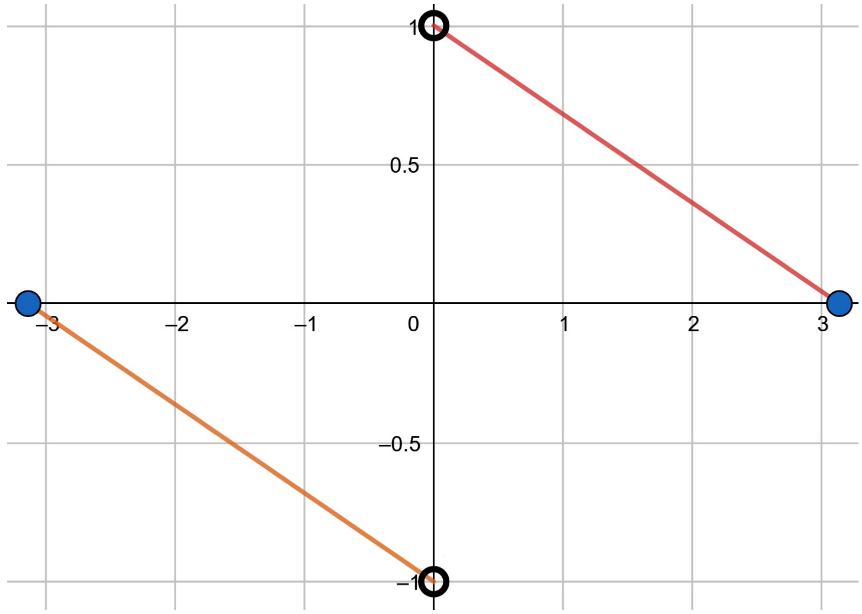
This gives us two different solutions for the same problem! As $n\to\infty$, $p^{(n)}(x)\to p^{(\infty)}(x)$ and we have

$$p^{(\infty)}(x) = \sum_{j=1}^{n}\frac{2}{j\pi}\sin(jx) = 1-\frac{x}{\pi} \qquad \text{for} \quad 0<x<\pi$$

Notice that these two expressions do NOT agree at the endpoint $x = 0$:

$$\begin{array}{ccc} \displaystyle\sum_{j=1}^{n}\frac{2}{j\pi}\sin(jx) & = & 1-\dfrac{x}{\pi} \\ 0 \quad \text{when} \quad x=0 & & 1 \quad \text{when} \quad x=0 \end{array}$$

This is a known feature of the left hand side, which is a FOURIER SINE SERIES of the continuous function on the right hand side. The left hand side is ODD and periodic:



The value it takes at the discontinuity at $x = 0$ is the average of $+1$ and -1, i.e. 0 (see Calculus last term)(indeed, Section 4.5.3 of notes 2019/20)

18 Continuum Equations

Instead of solving the discrete problem for finite n and taking a limit as $n \to \infty$, we can proceed differently.

Lets examine the possibility of taking the $n \to \infty$ limit at the level of the original governing equations and try to produce a set of (differential) equations governing the solution $p^{(\infty)}(x)$

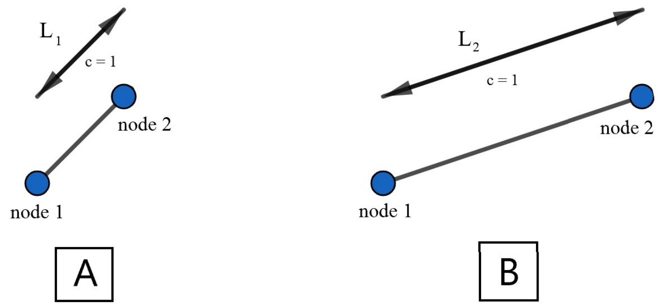
Then we might be able to solve that equation for $p^{(\infty)}(x)$ directly. (under the assumption that the two procedures 'commute')

Thus we are led to consider the 'continuum limit' of our governing equations.

18.1 Preliminary note on conductance

To take the 'continuum limit', it is useful to think about a definite application - Let's choose electric circuits (we know already our framework is very general)

Up to now, we have merely assigned a 'conductance' to an edge, without giving any details about why the conductance took that value. Thus, these two systems could be said to have the same edge conductance even though $L_1 < L_2$:



We want now, however to embed our graphs in $\mathbb{R}$ & $\mathbb{R}^2$ (i.e. in space) so the length will matter!

If the edges shown in A and B (I'm sorry i cant do squares yet) were actual wires, with lengths shown, and made of the same material (copper?), we would expect the conductance of system A to be LARGER than that of system B (more wire means greater resistance, so less conductance)

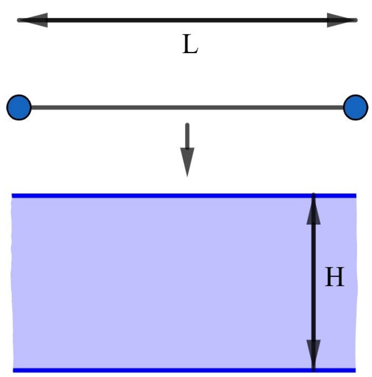
We therefore let

$$c = \frac{\widehat{c}}{L}$$

Where $\widehat{c}$ is called the 'conductivity' of the material. It is a redefinition that takes into account the geometry of the sample.

18.2 What about $2D$ effects

If the wire becomes

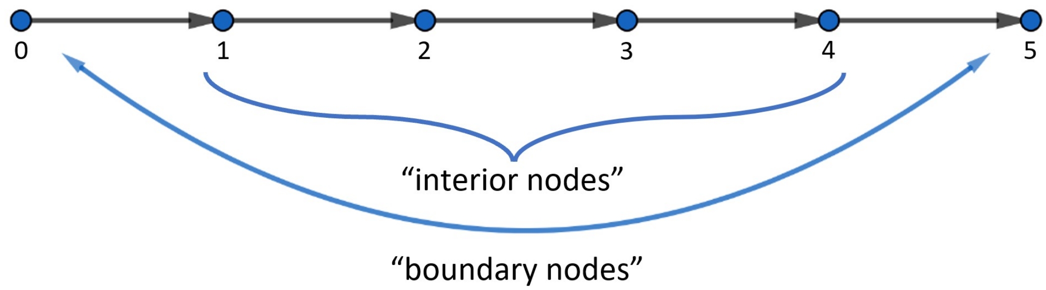


(i.e. it developed a width) then we would expect its conductance to increase as H increases. For a $2D$ situation, we therefore let

$$c = \frac{H}{L}\widehat{c}$$

18.3 1-dimensional continuum limit

We will do this via incidence matrices. Consider $N = 5$ and $N + 1$ nodes in a line:



The incidence matrix A is:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Let ϕ denote the vector of node variables

$$\underline{\phi} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} ; \quad \text{then} \quad A\underline{\phi} = \begin{pmatrix} \phi_1 - \phi_0 \\ \phi_2 - \phi_1 \\ \phi_3 - \phi_2 \\ \phi_4 - \phi_3 \\ \phi_5 - \phi 4 \end{pmatrix} = \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \\ \Delta\phi_5 \end{pmatrix} \qquad \text{where} \quad \Delta\phi_i = \phi_i - \phi_{i-1}$$

Embed in $\mathbb{R}$: we now embed this grpah in $\mathbb{R}$, indeed along the real interval [0.1], and use the location of the node in space as the node label

$$\text{i.e.} \quad \phi_i = \phi(x_i) \quad \text{where} \quad x_i = \frac{i}{N} = i\Delta x$$
$$\text{where} \quad \Delta x = \frac{1}{N}$$

Clearly $x_0 = 1$, $x_N = 1$. Also, we now thing of ϕ_i as the evaluation of a continuous function of a variable x at locations x_i

$$\text{Let edge variables be} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix}$$

$$-A^T\mathbf{w} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 - w_1 \\ w_3 - w_2 \\ w_4 - w_3 \\ w_5 - w_4 \\ -w_5 \end{pmatrix}$$

Use edge midpoints as a label once embedded in the interval $[0, 1]$

$$w_i = w\left(x_i - \frac{\Delta x}{2}\right)$$

Again, we thing of w_i as an evaluation of the continuous function $w(x)$ at $\left(x_i - \frac{\Delta x}{2}\right)$

$$\text{i.e.} \quad -A^T\mathbf{w} = \begin{pmatrix} w_1 \\ \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \\ -w_5 \end{pmatrix} \quad \text{where} \quad \Delta w_i = w_{i+1} - w_i \tag{1}$$

Now that we have embedded the graph in $\mathbb{R}$, it is useful to define

$$c_i = \frac{\widehat{c}_i}{\Delta x}$$

We do this because we want the conductance of a sample of a material used as an edge to get smaller as Δx gets bigger (more length measn more obstruction to the flow of current). Now we have related conductance and conductivity in our continuum model.

Also we want the point sources at any interior nodes to be 'smeared out' in space to give a source density $\widehat{f}_i$ defined by

$$f_i = \widehat{f}_i \Delta x$$

where f_i denotes the original current source at node i

Note: the hatted variables denote the relevant variables in the continuum limit.

18.4 Currents i the continuum limit $N \to \infty$

We want to examine what happens to $CA\underline{\phi}$ (currents) as $N \to \infty$. For $N = 5$, we have

$$\begin{aligned} -CA\underline{\phi} &= -\begin{pmatrix} \frac{\widehat{c}_1}{\Delta x} & & & & \\ & \frac{\widehat{c}_2}{\Delta x} & & 0 & \\ & & \frac{\widehat{c}_3}{\Delta x} & & \\ & 0 & & \frac{\widehat{c}_4}{\Delta x} & \\ & & & & \frac{\widehat{c}_5}{\Delta x} \end{pmatrix} \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \\ \Delta\phi_5 \end{pmatrix} \\ &= -\begin{pmatrix} \widehat{c}_1 \frac{\Delta\phi_1}{\Delta x} \\ \widehat{c}_2 \frac{\Delta\phi_2}{\Delta x} \\ \widehat{c}_3 \frac{\Delta\phi_3}{\Delta x} \\ \widehat{c}_4 \frac{\Delta\phi_4}{\Delta x} \\ \widehat{c}_5 \frac{\Delta\phi_5}{\Delta x} \end{pmatrix} \end{aligned}$$

As $N \to \infty$, this vector becomes an infinite dimensional evaluation of the function

$$w(x) = -\widehat{c}(x)\frac{d\phi}{dx}$$

At points in the interval $[0, 1]$. Notice that

$$\frac{\Delta\phi_i}{\Delta x} = \frac{\phi_i - \phi_{i-1}}{\Delta x} = \frac{\phi(x_i) - \phi(x_{i-1})}{\Delta x} \to \frac{d\phi}{dx} \quad \text{as} \quad N \to \infty \text{ or } \Delta x \to 0 \tag{2}$$

This is by the definition of the derivative

Similarly, at the <u>interior</u> points, $-A^T\mathbf{w} = \mathbf{f}$ is, for $N = 5$

$$\begin{pmatrix} \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \end{pmatrix} = \begin{pmatrix} \widehat{f}_1 \Delta x \\ \widehat{f}_2 \Delta x \\ \widehat{f}_3 \Delta x \\ \widehat{f}_4 \Delta x \end{pmatrix}$$

$$\underline{\text{or}} \qquad \begin{pmatrix} \frac{\Delta w_1}{\Delta x} \\ \frac{\Delta w_2}{\Delta x} \\ \frac{\Delta w_3}{\Delta x} \\ \frac{\Delta w_4}{\Delta x} \end{pmatrix} = \begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \\ \widehat{f}_3 \\ \widehat{f}_4 \end{pmatrix}$$

In the limit $N \to \infty$ or $\Delta x \to 0$, this becomes an ∞-dimensional vector of the evaluation of the continuous equation

$$\frac{dw}{dx} = \widehat{f}(x) \qquad \text{at points in } [0,1] \tag{3}$$

$\widehat{f}(x)$ is the continuous <u>source density function</u>

We can now COMBINE equations 2 and 3:

$$\frac{d}{dx}w = \frac{d}{dx}\left(-\widehat{c}(x)\frac{d\phi}{dx}(x)\right) = \widehat{h}(x) \tag{4}$$

<u>Note:</u> At the boundary nodes, the equation $-A^T\mathbf{w} = \mathbf{f}$ just gives $w_1 = f_1$ and $-w_5 = f_?$ (I can't see what number it is)

<u>Note 2:</u> Note that $-\mathbf{x}_0^T A^T \mathbf{w} = 0$ where $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, as expected.

On use of 1, this says (for $N = 5$)

$$w_1 + \sum_{i=1}^{4} (\Delta w_i) - w_5 = 0$$

$$\text{or} \qquad \sum_{i=1}^{4} \Delta w_i = w_5 - w_1 = w(1) - w(0)$$

So in the limit $N \to \infty$, this is

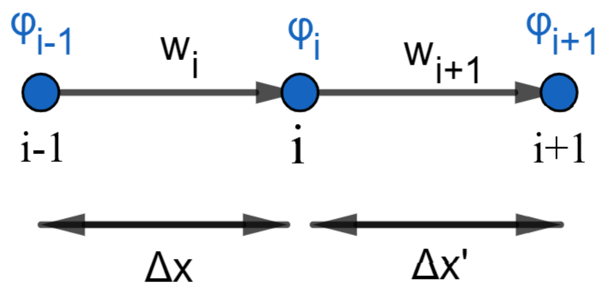
$$\lim_{N\to\infty} \left(\sum_{i=1}^{N} \frac{\Delta w_i}{\Delta x} \Delta x \right) = w(1) - w(0)$$

$$\text{or} \qquad \int_0^1 \frac{dw}{dx} dx = w(1) - w(0)$$

And this last statement is exactly the fundamental theorem of calculus, via the definition of the integral to get the last statement.

18.5 Alternative argument

This one doesn't use any incidence matrices. Another way to derive the same result is as follows. Assume UNIFORM conductance, so $c_j = c \quad \forall j$ and consider a typical INTERIOR node i:



Current:

$$\underline{-CA\underline{\phi} \quad \text{is}} \qquad w_{i+1} = -c(\phi_{i+1} - \phi_i) = -\frac{\widehat{c}}{\Delta x}(\phi_{i+1} - \phi_i) \tag{1}$$

$$w_i = -c(\phi_i - \phi_{i-1}) = -\frac{\widehat{c}}{\Delta x}(\phi_i - \phi_{i-1}) \tag{2}$$

$$\underline{\text{KCL is}} \qquad -A^T\mathbf{w} = \mathbf{f} \quad \Rightarrow \quad w_{i+1} - w_i = f_i = \widehat{f}_i \Delta x \tag{3}$$

Substituting 1 and 2 into 3:

$$-\frac{\widehat{c}}{\Delta x}\left((\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1})\right) = \widehat{f}_i \Delta x \quad \Rightarrow \quad -\frac{\widehat{c}}{(\Delta x)^2}\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right) = \widehat{f}_i \tag{4}$$

Now we change the equation 4 in the following way:

$$\phi_i = \phi(x), \quad \phi_{i+1} = \phi(x + \Delta x), \quad \phi_{i-1} = \phi(x - \Delta x), \quad \widehat{f}_i = \widehat{f}(x)$$

We also assume here that $\phi(x)$ is a differentiable function. Using this, 4 becomes:

$$-\frac{\widehat{c}}{(\Delta x)^2}\left(\phi(x + \Delta x) - 2\phi(x) + \phi(x - \Delta x)\right) = \widehat{f}(x) \tag{5}$$

Now we use a Taylor expansion on the first and last terms in the bracket,

$$\phi(x + \Delta x) = \phi(x) + \Delta x \frac{d\phi}{dx}(x) + \frac{(\Delta x)^2}{2!}\frac{d^2\phi}{dx^2} + \dots$$
$$\phi(x - \Delta x) = \phi(x) - \Delta x \frac{d\phi}{dx}(x) + \frac{(\Delta x)^2}{2!}\frac{d^2\phi}{dx^2} + \dots$$

We see tha 5 can be weitten as

$$-\frac{\widehat{c}}{(\Delta x)^2}\left(\frac{d^2\phi}{dx^2}(\Delta x)^2 + \dots\right) = \widehat{f}(x)$$

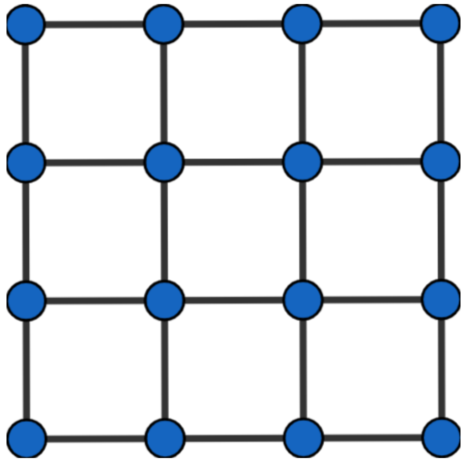
Hence as $\Delta x \to 0$, we find

$$-\widehat{c}\frac{d^2\phi}{dx^2} = \widehat{f}(x)$$

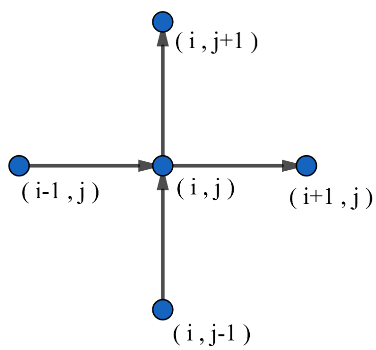
Which is precisely what we got from the initial method.

19 $2D$ Continuum Limit

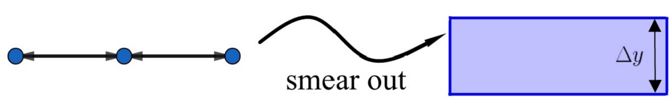
Consider the 'grid' graph, and as in the $1D$ case, we consider this embedded in the region $x \in [0,1]$ $y \in [0,1]$ of $\mathbb{R}^2$:



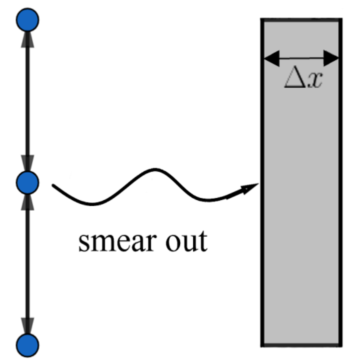
Let us emulate the simple derivation of the governing differential equation for the $1D$ case just given. Consider a TYPICAL INTERIOR NODE.



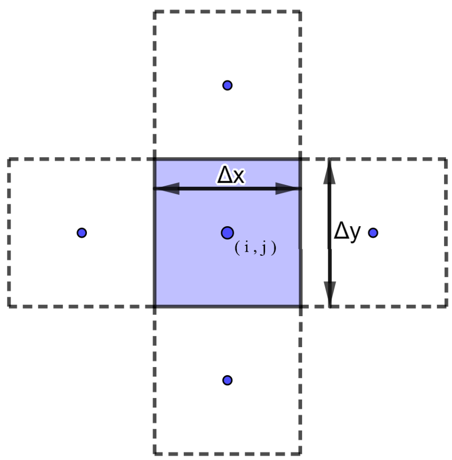
We now imagine that a typical set of edges along the x-direction gets 'smeared put' in the y-direction.



And similarly



We also suppose that any source of current from node (i, j) is smeared out over the local area



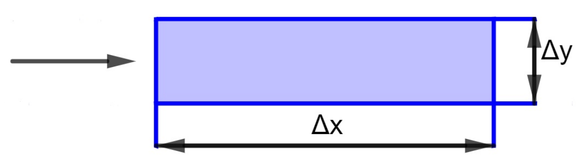
<u>Hence</u>

$$f_{ij} = \widehat{f}_{ij} \Delta x \Delta y$$

Where f_{ij} is the current source at node (i, j) and $\widehat{f}_{ij}$ is the 'source density' (source strength per unit area)

We now turn to the <u>conductance</u> which we also need to 'smear out' appropriately.

Consider a 'sample'



With current flowing in the direction of the arrow. We expect conductance to be of the form

$$c^{(x)} = \widehat{c} \frac{\Delta y}{\Delta x}$$

Since, as in the $1D$ case, we expect conductance to decrease if the sample gets <u>longer</u> (Δx increases), while we expect it to increase if width gets larger (Δy increases) as there is more 'room' for charge to flow.

Also, the current in the x-direction will be denoted by u_i and the current in the y-direction will be denoted by v_j

Hence '$-CA\underline{\phi}$ is

$$u_{i+1} = -c^{(x)}(\phi_{i+1,j} - \phi_{i,j}) = -\widehat{c}\frac{\Delta y}{\Delta x}(\phi_{i+1,j} - \phi_{i,j}) \tag{1}$$
$$u_i = -c^{(x)}(\phi_{i,j} - \phi_{i-1,j}) = -\widehat{c}\frac{\Delta y}{\Delta x}(\phi_{i,j} - \phi_{i-1,j}) \tag{2}$$

Where $\phi_{i,j}$ is the node value at node (i,j), and we have

$$v_{j+1} = -c^{(y)}(\phi_{i,j+1} - \phi_{i,j}) = -\widehat{c}\frac{\Delta x}{\Delta y}(\phi_{i,j+1} - \phi_{i,j}) \tag{3}$$
$$v_j = -c^{(y)}(\phi_{i,j} - \phi_{i,j-1}) = -\widehat{c}\frac{\Delta x}{\Delta y}(\phi_{i,j} - \phi_{i,j-1}) \tag{4}$$

Where we have used $c^{(y)} = \widehat{c}\frac{\Delta x}{\Delta y}$ for conductance in the y-direction (and have assumed that the constant of proportionality $\widehat{c}$ is the <u>SAME</u> as in the x-direction)

KCL at node (i,j) has <u>FOUR</u> contributions to the flux <u>out</u> of it:

$$u_{i+1} - u_i + v_{j+1} - v_j = f_{i,j} = \widehat{f}_{i,j}\Delta x\Delta y \tag{5}$$

$$\Rightarrow \quad -\widehat{c}\left(\frac{\Delta y}{\Delta x}(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}) + \frac{\Delta x}{\Delta y}(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1})\right) = \widehat{f}_{i,j}\Delta x\Delta y$$
$$\Rightarrow -\widehat{c}\left(\frac{1}{(\Delta x)^2}(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j})\right) - \widehat{c}\left(\frac{1}{(\Delta y)^2}(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1})\right) = \widehat{f}_{i,j} \qquad (*)$$

The object

$$\frac{1}{(\Delta x)^2}\left(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}\right)$$

looks exactly like its counterpart in the $1D$ case of nodes along the x-axis.

The object

$$\frac{1}{(\Delta y)^2}\left(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}\right)$$

Is exactly the same thing if, instead, the $1D$ lines of nodes was aligned along the y-axis. Since both expressions appear in the $2D$ case, we introduce the notation:

$$\frac{\partial^2\phi}{\partial x^2} = \lim_{\Delta x\to 0}\left[\frac{\phi(x+\Delta x, y) - 2\phi(x,y) + \phi(x-\Delta x, y)}{(\Delta x)^2}\right]$$
$$\frac{\partial^2\phi}{\partial y^2} = \lim_{\Delta y\to 0}\left[\frac{\phi(x, y+\Delta y) - 2\phi(x,y) + \phi(x, y-\Delta y)}{(\Delta y)^2}\right]$$

These are called <u>partial</u> derivatives (second partial derivatives to be precise). Using this notation, $(*)$ becomes

$$-\widehat{c}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] = \widehat{f}(x,y)$$

19.1 Currents are given by the first partial derivatives

$$\text{From } (2), \quad u_i = -\widehat{c}\Delta y \left[\frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x}\right]$$
$$= -\widehat{c}\Delta y \left[\frac{\phi(x,y) - \phi(x - \Delta x, y)}{\Delta x}\right]$$

Define current density in the x-direction, call it j_x, via

$$\lim_{\Delta x \to 0} [u_i] = \Delta y j_x^{(i)}$$

Hence,

$$j_x^{(i)} = -\widehat{c} \lim_{\Delta x \to 0} \left[\frac{\phi(x,y) - \phi(x - \Delta x, y)}{\Delta x}\right]$$
$$j_x(x) = -\widehat{c}\frac{\partial \phi}{\partial x}$$
$$\text{where} \quad \frac{\partial \phi}{\partial x} = \lim_{\Delta x \to 0} \left[\frac{\phi(x,y) - \phi(x - \Delta x, y)}{\Delta x}\right]$$

Take the x-derivative, keeping y fixed. Some people write $\left.\frac{\partial \phi}{\partial x}\right|_y$ for emphasis.

In a similar way, if we define j_y as the current density in the y-direction, then we can show, from (3) or (4), we can show

$$j_y = -\widehat{c}\frac{\partial \phi}{\partial y}$$
$$\text{where} \quad \frac{\partial \phi}{\partial y} = \lim_{\Delta y \to 0} \left[\frac{\phi(x, y + \Delta y) - \phi(x,y)}{\Delta y}\right]$$

19.2 Continuous Laplacian operator & Gradient operator

The notation ∇^2 is used to denote the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
$$\text{i.e.} \quad \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

The notation ∇ is used to denote the operation of taking a scalar ϕ and producing the vector with components $\begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix}$, so a sort of 'gradient' function.

$$\text{i.e.} \quad \nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix}$$

19.3 Divergence Operator

Returning to equation 5, where we imposed the 'divergence condition', i.e. $-A^T\mathbf{w} = \mathbf{f}$, at node (i, j), we note that we can write it as

$$(u_{i+1} - u_i) + (v_{j+1} - v_j) = f_{i,j} = \widehat{f}_{i,j}\Delta x \Delta y$$

$$\underline{\text{or}} \quad \left(\frac{u_{i+1} - u_i}{\Delta x}\right)\frac{1}{\Delta y} + \left(\frac{v_{j+1} - v_j}{\Delta y}\right)\frac{1}{\Delta x} = \widehat{f}_{i,j}$$

$$\underline{\text{or}} \quad \left(\frac{j_x^{(i+1)} - j_x^{(i)}}{\Delta x}\right) + \left(\frac{j_y^{(j+1)} - j_y^{(j)}}{\Delta y}\right) = \widehat{f}_{i,j}$$

Taking the limit $\Delta x, \Delta y \to 0$, we find

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y}$$

We call the operator $\frac{\partial}{\partial x} j_x + \frac{\partial}{\partial y} j_y$ acting on the vector $\begin{pmatrix} j_x \\ j_y \end{pmatrix}$ the <u>DIVERGENCE</u> of the vector, denoted by

$$\nabla \cdot \mathbf{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y}$$

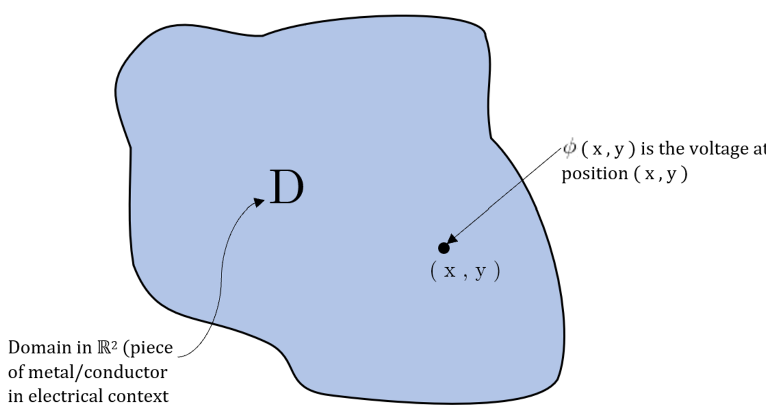
$$\text{where} \quad \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

20 Harmonic Functions

we need to solve the continuous version of '$K\underline{\phi} = 0$ at KCL nodes' which is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

At the interior points of some <u>domain D</u>



In analogy with the discrete case where we referred to a potential satisfying KCL at an interior node as a 'harmonic potential', we call the continuous potential function $\phi(x, y)$ <u>harmonic</u> if it satisfies $\nabla^2 \phi = 0$

20.1 A preview of next year

Next year, you will see a proper introduction to complex variable theory

You will learn that functions of the complex variable $z = x + iy$, i.e.

$$h = h(z)$$

Can be treated, when doing calculus (e.g. differentiating and integrating), in much the same way as real functions of x, $f(x)$, but with much more powerful (and useful!) consequences and properties.

For us, we willuse the fact that all the rules of calculus apply - next year you will see a fuller explanation of this, and learn about 'analytic functions', and the 'Cauchy-Riemann equations', etc.

$$\begin{aligned}
\underline{\text{Notice}} \qquad \text{Let} \quad z &= x + iy \\
\text{Let} \quad h(z) &= z^4 = (x+iy)^4 \\
\frac{\partial h}{\partial x} &= 4(x+iy)^3 \cdot \frac{\partial z}{\partial x} = 4(x+iy)^3 \\
\frac{\partial^2 h}{\partial x^2} &= 12(x+iy)^2 \cdot \frac{\partial (x+iy)}{\partial x} = 12(x+iy)^2 && (1) \\
\frac{\partial h}{\partial y} &= 4(x+iy)^3 \cdot \frac{\partial (x+iy)}{\partial x} = 4i(x+iy)^3 \\
\frac{\partial^2 h}{\partial y^2} &= 12i(x+iy)^2 \cdot \frac{\partial (x+iy)}{\partial x} = 12i^2(x+iy)^2 && (2) \\
1 + 2 : \quad & \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h = 0 && (3)
\end{aligned}$$

h is complex valued, but taking a complex conjugate of 3

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \overline{h} = 0 \qquad (4)$$

$$3 + 4 : \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (h + \overline{h}) = 0 \qquad (5)$$

20.2 A Useful Observation

Let us introduce the complex co-ordinate $z = x + iy$ where $i^2 = -1$ and let $h(z)$ be any function of this variable z, assumed to the differentiable. In general, it will be a complex valued function, but we can still ask about the value of

$$\nabla^2 h = \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h(z)$$

By the chain rule, and the definition of partial differentiation,

$$\left.\frac{\partial h}{\partial x}\right|_y = \frac{dh}{dz}\left.\frac{\partial z}{\partial x}\right|_y$$

$$= h'(z) \cdot \left.\frac{\partial}{\partial x}\right|_y (x + iy) = h'(z) \tag{1}$$

$$\frac{\partial^2 h}{\partial x^2} = h''(z) \qquad \text{by similar steps} \tag{2}$$

$$\underline{\text{Also}} \qquad \left.\frac{\partial h}{\partial y}\right|_x = \frac{dh}{dz}\left.\frac{\partial z}{\partial y}\right|_x$$

$$= h'(z) \cdot \left.\frac{\partial}{\partial y}\right|_x (x + iy) = ih'(z) \tag{3}$$

$$\frac{\partial^2 h}{\partial y^2} = (i)^2 h''(z) = -h''(z) \tag{4}$$

$$\text{Now} \qquad 3 + 4 \Rightarrow \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = h''(z) - h''(z) = 0$$

$$\text{Since} \qquad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (\text{for any } h(z)!) \tag{5}$$

We can take a complex conjugate of 5

$$\frac{\partial^2 \overline{h}}{\partial x^2} + \frac{\partial^2 \overline{h}}{\partial x^2} = 0 \tag{6}$$

Where we have used that fact that x and y are <u>real</u>. Then 5+6 implies

$$\frac{\partial^2}{\partial x^2}(h + \overline{h}) + \frac{\partial^2}{\partial y^2}(h + \overline{h}) = 0$$

$$\text{or} \qquad \phi(x, y) = \frac{h(z) + \overline{h(z)}}{2} = \text{Re}[h(z)]$$

$$\text{satisfies} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0$$

20.3 Summary

$\phi = \text{Re}[h(z)]$ is a harmonic (<u>real</u>) function. $h(z)$ is an arbitrary, differentiable function of z [<u>Note:</u> there are called <u>ANALYTIC FUNCTIONS</u> - There is an entire course on these in Year 2!]

20.4 Current Density in terms of $h(z)$

We know the current density, in the x and y direction, can be written as the vector

$$\mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = -\widehat{c} \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix}$$

Using this, we have

$$\left.\frac{\partial \phi}{\partial x}\right|_y = \frac{1}{2}\frac{\partial}{\partial x}\left[h(z) + \overline{h(z)}\right] = \frac{1}{2}\left[h'(z)\frac{\partial z}{\partial x} + \overline{h'(z)\frac{\partial z}{\partial x}}\right]$$
$$= \frac{1}{2}\left[h'(z) + \overline{h'(z)}\right] \tag{7}$$
$$\left.\frac{\partial \phi}{\partial y}\right|_x = \frac{1}{2}\frac{\partial}{\partial y}\left[h(z) + \overline{h(z)}\right] = \frac{1}{2}\left[h'(z)\frac{\partial z}{\partial y} + \overline{h'(z)\frac{\partial z}{\partial y}}\right]$$
$$= \frac{1}{2}\left[ih'(z) - i\overline{h'(z)}\right] \tag{8}$$

Using these two equations, 7 - i8 gives us:

$$\underline{\underline{\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}}} = \frac{1}{2}\left[h'(z) + \overline{h'(z)}\right] + \frac{1}{2}\left[h'(z) - \overline{h'(z)}\right] = \underline{\underline{h'(z)}}$$

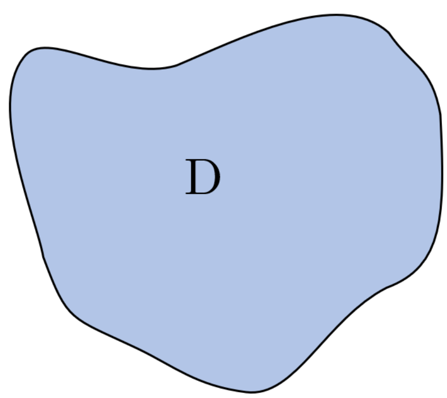
Hence

$$\underline{\underline{\underline{j_x - ij_y = -\widehat{c}h'(z)}}}$$

Using this, we can get the current density components from the real and imaginary parts of $-\widehat{c}h'(z)$

20.5 Boundary conditions on domain boundary

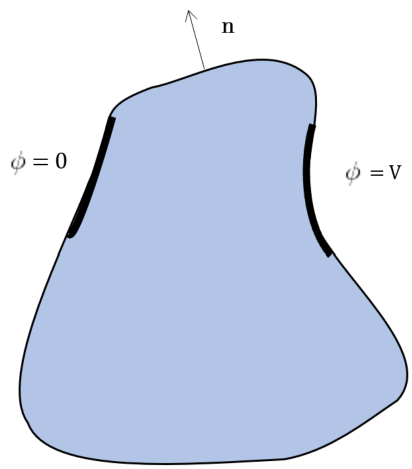
Any bounded domain, D, now assumed to be a conductor, say, will have a physical boundary.



The black line is the boundary. Just as in a discrete conductor graph, these boundary nodes can either be KCL nodes, or we can externally impose the value of the potential at those nodes. If we specify the voltage, this is called a boundary condition of DIRICHLET TYPE; if we specify that the node is of KCL type, then we call the boundary condition of NEUMANN TYPE.

The form of the KCL condition in the continuous case is that the current flow at a boudary point in the direction NORMAL to the boundary at that point must be ZERO (no current can exit the circuit at these points.)

Note: The boundary conditions on different parts of the physical boundary can be of different type! For example,

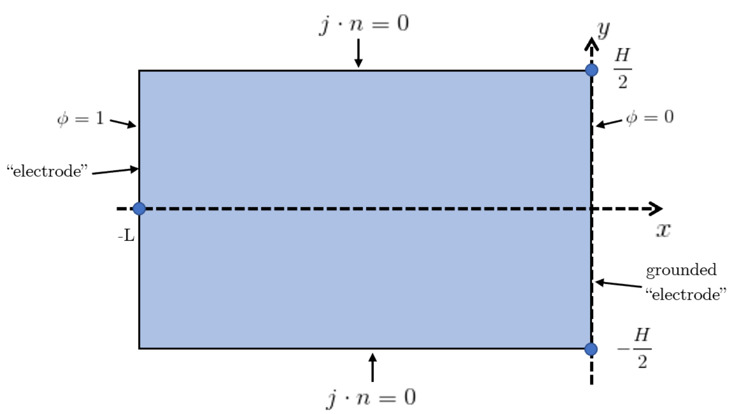


We can then ask the usual questions:

1. What is the potential function (i.e. voltage distribution) $\phi(x, y)$ inside the domain D?
2. What is the effective conductance of the conductor set-up (if this question makes sense)?
3. If a DIRICHLET type boundary condition holds at some boundary point, then what is the current there; or; if the NEUMANN type condition holds, what is the voltage there ('Dirichlet-to-Neumann' map)

21 Some Examples

21.1 *Example 1:* Rectangular conductor of uniform conductivity $\widehat{c} = 1$



Let the left hand side of the conductor be attached to an electrode held at unit voltage; and the right hand side is grounded. The top/bottom of the conductor are 'insulating' ($\mathbf{j} \cdot \mathbf{n} = 0$)

In physics, this is the 'two terminal conductance problem'

What is the corresponding $h(z)$?

Our physical problem, by earlier considerations, is reduced to finding a function $h(z)$ such that $\phi(x, y) = \text{Re}[h(z)]$ satisfies all the conditions on sides as shown in the figure.

$$\underline{\text{Try}} \qquad h(z) = -\frac{z}{L}$$

$$\text{Then} \qquad \phi(x, y) = \text{Re}[h(z)] = -\frac{x}{L} = \begin{cases} 1 & x = -L \\ 0 & x = 0 \end{cases}$$

So BOUNDARY CONDITIONS on the left and right sides are satisfies.

$$\underline{\text{Also}} \qquad j_x - ij_y = -h'(z) = \frac{1}{L} \Rightarrow \underline{j_x = \frac{1}{L},\ j_y = 0}$$

This last statement holds <u>everywhere</u>, and so certainly at the top and bottom of the conductor. Hence current is purely <u>tangential</u> at the top and bottom $\Rightarrow$ BOUNDARY CONDITIONS on top/bottom are also satisfied.

$$\underline{\text{Indeed}} \qquad h(z) = -\frac{z}{L} \quad \text{is the required function}$$

<u>Note:</u> There is a continuous version of the uniqueness theorem for this class of problems (see Year 2)

21.2 What is the effective conductance

As in the discrete case, it is defined to be the current entering the circuit via the electrode held at unit voltage (and leaving the circuit at the other electrode). But $j_x = \frac{1}{L}$ at each point on the left side (electrode).

Hence, on integration with respect to y (recall j_x is current density), the total current into the circuit from this electrode (i.e. C_{eff}) is

$$C_{\text{eff}} = \int_{-\frac{H}{2}}^{\frac{H}{2}} j_x dy = \frac{H}{L}$$

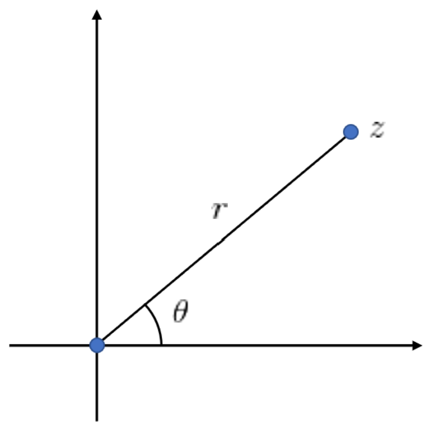
$$\underline{\underline{C_{\text{eff}} = \frac{H}{L}}}$$

Here we used the fact that $u = \Delta y j_x$ with u being the current, and j_x being the current density.

21.3 A note on the complex logarithm

The polar representation of a general complex number z is

$$z = re^{i\theta}$$



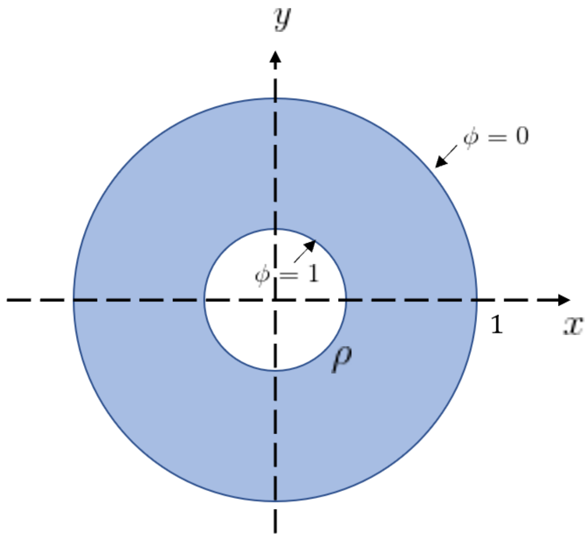
Where $r = |z|$, the modulus of z and $\theta = \arg z$, the argument of z

$$\text{Hence} \qquad z = |z|e^{i\theta}$$

'Taking a logarithm':

$$\log z = \log |z| + \log e^{i\theta} = \log |z| + i\theta$$
$$\text{i.e.} \qquad \mathrm{Re}(\log z) = \log |z|$$
$$\mathrm{Im}(\log z) = \theta = \arg z$$

21.4 *Example 2:* Annular conductor of uniform conductivity $\widehat{c} = 1$



Let $Z = X + iY$. These are different variable names now. Suppose the inner boundary circle $|Z| = \rho$ is help at unit voltage Iit is a circular electrode) while the outer boundary is grounded ($|Z| = 1$)

What is the required function $\widehat{h}(Z)$?

$$\text{Try} \qquad \widehat{h}(Z) = \frac{1}{\log \rho} \log Z$$

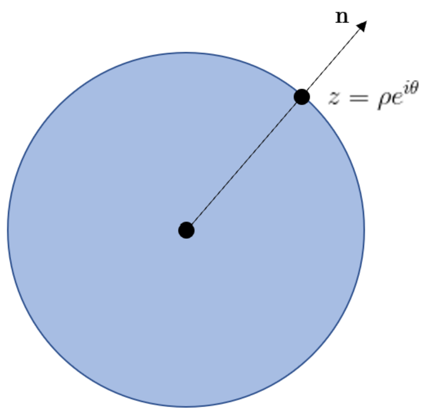
$$\text{Now} \qquad \widehat{\phi} = \mathrm{Re}[\widehat{h}(Z)] = \frac{\log |Z|}{\log \rho} = \begin{cases} 1 & |Z| = \rho \\ 0 & |Z| = 1 \end{cases}$$

Indeed this is the required function $\widehat{h}(Z)$! (this time, we only had one set of boundary conditions to check, which were the inner and outer rings of the conductor)

21.5 What is the effective conductance

$$\widehat{j}_x - i\widehat{j}_y = -\widehat{h}'(Z) = -\frac{1}{(\log \rho)Z}$$

For a circle of radius ρ, centred at the origin, the normal $\mathbf{n}$ to the boundary (assuming unit length) is $e^{i\theta}$ in complex form.



Hence the normal component of $\mathbf{j}$ at any point on this circle is

$$\mathrm{Re}\left[(\widehat{j}_x - i\widehat{j}_y)e^{i\theta}\right]$$

Exercise: Confirm that if $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, then $\mathbf{a}\cdot\mathbf{b} = \mathrm{Re}[\overline{a}b]$ where $a = a_x + ia_y$ and $b = b_x + ib_y$

But this quantity is

$$\mathrm{Re}\left[-\frac{1}{(\log \rho)\rho e^{i\theta}}e^{i\theta}\right] = -\frac{1}{\rho \log \rho}$$

Note that this is constant everywhere on the boundary circle.

This is the normal current density at any point on the boundary $|Z| = \rho$.

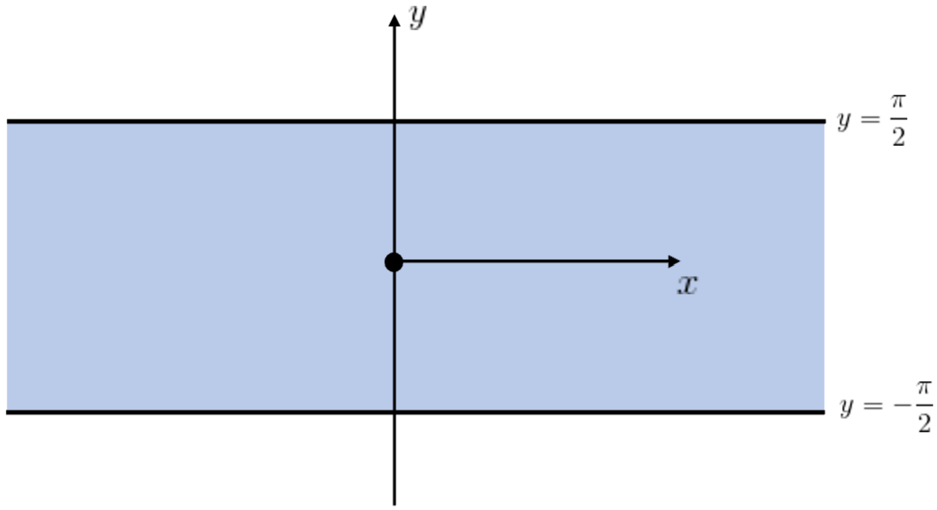
Integrating this around the boundary means multiplying by the perimeter $2\pi\rho$ (since normal current density is constant, and we could have used this in the first example too)[see note at end of this subsection]

$\int_0^{2\pi} \rho d\theta = 2\pi\rho$, which implies that the effective conductance is

$$\underline{\underline{\widehat{C}_{\text{eff}} = -\frac{2\pi}{\log \rho}}}$$

Note: What we are really doing here is computing the integral of the normal current density with respect to arc length ds around the boundary. (here it is $ds = \rho d\theta$)

21.6 *Example 3:* Consider an infinite conducting two dimensional plate of width π



Assume $\widehat{c} = 1$.

The potential in this plate is

$$\phi = \text{Re}[h(z)]$$

$$\text{where} \qquad h(z) = -\frac{m}{2\pi} \log\left[\frac{1-e^z}{1+e^z}\right] \qquad m \in \mathbb{R}$$

1. Show that $\nabla^2 \phi = 0$ in the plate except at $z = 0$ or $(0, 0)$.
2. Show that the edges of the plate are grounded i.e. $\phi = 0$
3. Find an expression for the current density components on the upper boundary of the plate as a function of x

Solution:

1. $h(z)$ is a well-defined defined function of z except when the argument of the logarithm vanishes, i.e. when

$$e^z = 1 = e^{2n\pi i} \qquad n \in \mathbb{Z}$$

$$\text{or} \qquad e^z = -1 = e^{i\pi + 2n\pi i} \qquad n \in \mathbb{Z}$$

Only the point $z = 0$ is inside the plate. Hence, provided $z \neq 0$, $h(z)$ is well-defined and differentiable. Thus, for $z \neq 0$,

$$\frac{\partial h}{\partial x} = h'(z)\frac{\partial z}{\partial x} = h'(z)$$

$$\frac{\partial^2 h}{\partial x^2} = h''(z)\frac{\partial z}{\partial x} = h''(z)$$

$$\frac{\partial h}{\partial y} = h'(z)\frac{\partial z}{\partial y} = ih'(z)$$

$$\frac{\partial^2 h}{\partial y^2} = ih''(z)\frac{\partial z}{\partial y} = i^2 h''(z) = -h''(z)$$

Using these and by taking a complex conjugate, we get the following statements which basically give us the result

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h = 0 \tag{1}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \overline{h} = 0 \tag{2}$$

$$\begin{aligned} 1 + 2 \Rightarrow \quad & \frac{\partial^2}{\partial x^2}(h + \overline{h}) + \frac{\partial^2}{\partial y^2}(h + \overline{h}) = 0 \\ \iff \quad & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \forall\, z \neq 0 \end{aligned}$$

2. On the top boundary of the plate:

$$\begin{aligned} y = \frac{z - \overline{z}}{2i} &= \frac{\pi}{2} \\ \Rightarrow \quad z - \overline{z} &= \pi i \\ \Leftrightarrow \quad \overline{z} &= z - \pi i \end{aligned}$$

(On the bottom edge, we get $\overline{z} = z + \pi i$ in a similar way)

$$\begin{aligned} \text{Let} \qquad R &\equiv \frac{1 - e^z}{1 + e^z} \\ \text{Then} \qquad \overline{R} &= \frac{1 - e^{\overline{z}}}{1 + e^{\overline{z}}} \\ &= \frac{1 - e^{z - \pi i}}{1 + e^{z - \pi i}} \\ &= \frac{1 + e^z}{1 - e^z} \\ &= \frac{1}{R} \end{aligned}$$

This was because $e^{-i\pi} = -1$

Hence, on this boundary, we have that

$$\begin{aligned} \overline{R} &= \frac{1}{R} \\ \Leftrightarrow R\overline{R} &= 1 \\ \Leftrightarrow |R| &= 1 \\ \Leftrightarrow \log |R| &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \phi = \mathrm{Re}\left[-\frac{m}{2\pi} \log R\right] &= -\frac{m}{2\pi} \mathrm{Re}\left[\log |R| + i \arg R\right] \\ &= -\frac{m}{2\pi} \log |R| = 0 \end{aligned}$$

A similar analysis shows $\phi = 0$ on the bottom edge too!!

3. We know that the current density is

$$
\begin{aligned}
j_x - ij_y &= -\widehat{c}h'(z) = -h'(z) \qquad \text{(since } \widehat{c} = 1\text{)} \\
&= \frac{m}{2\pi}\left[\frac{-e^z}{1-e^z} - \frac{e^z}{1+e^z}\right]
\end{aligned}
$$

On the top wall,

$$
\begin{aligned}
z &= x + i\frac{\pi}{2} \\
\therefore\ e^z &= e^{x+i\frac{\pi}{2}} = ie^x \\
\therefore\ j_x - ij_y &= -\frac{m}{2\pi}\left[\frac{ie^x}{1-ie^x} + \frac{ie^x}{1+ie^x}\right] \\
&= -\frac{im}{2\pi}e^x\left[\frac{1+\cancel{ie^x}+1-\cancel{ie^x}}{1+e^{2x}}\right] \\
&= -\frac{im}{2\pi}\left[\frac{2e^x}{1+e^{2x}}\right] \\
&= -\frac{im}{2\pi}\left[\frac{2}{e^x+e^{-x}}\right] \\
&= -\frac{im}{2\pi}\ \frac{1}{\cosh x}
\end{aligned}
$$

$$
\therefore\ \underline{\underline{j_x = 0,\ j_y = \frac{m}{2\pi}\operatorname{sech} x}}
$$