Mathematics Year 1, Calculus and Applications I

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Solutions: Problem Sheet 1

Problems 9 and 10 are good candidates for starred questions

1. (a) $h(x) = (x^2+1)^6 + 2$.

   (b) $f \circ g = (x^3-5)^{1/2}$, $g \circ f = x^{3/2} - 5$.

   (c) One way is $f \circ g = \sqrt{x^2+1}/[2+(1+x^2)^3]$ where $f(x) = \sqrt{x}/(2+x^3)$, $g(x) = 1+x^2$.

2. $\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$, and $\frac{d^2}{dx^2}(f(g(x)) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x)$.

3. Write the function with brackets as follows to make it clearer

$$\frac{d}{dx}\left\{1+\left[1+\left(1+x^2\right)^8\right]^8\right\}^8 = 8\{\ldots\}^7 \cdot 8[\ldots]^7 \cdot 8(1+x^2)^7 \cdot 2x,$$

and it is understood what goes in the curly and square brackets.

4. A point in the plane moves in such a way that it is always twice as far from $(0,0)$ as it is from $(0,1)$.

   (a) Let the coordinates of the point be (x,y). Distance from $(0,0)$ is $\sqrt{x^2+y^2}$, and distance from $(1,0)$ is $\sqrt{x^2+(y-1)^2}$. Hence the trajectory is

$$\sqrt{x^2+y^2} = 2\sqrt{x^2+(y-1)^2} \quad \Rightarrow \quad x^2+(y-4/3)^2 = \frac{4}{9}, \qquad (1)$$

   i.e. a circle of radius $2/3$ centered at $(0,4/3)$. (I got the equation by expanding and then completing the square.)

   (b) Two ways of doing this. First differentiate (1) with respect to t (since we are told t is the parameter) to find

$$2x\frac{dx}{dt} + 2\left(y - \frac{4}{3}\right)\frac{dy}{dt} = 0. \qquad (2)$$

   Putting $x = 0$ gives $dy/dt = 0$, since from (1) when $x = 0$ we obtain $y = 2, 2/3$, hence $y \neq 4/3$.
   Alternatively, we can parametrize the circle by

$$x = \frac{2}{3}\cos t, \quad y = \frac{4}{3} + \frac{2}{3}\sin t.$$

   [I picked the starting point $t = 0$ to be $x = 2/3$, $y = 4/3$ - this is immaterial.] Hence $x = 0$ implies $\cos t = 0$. But $dy/dt = (2/3)\cos t$, hence this is also 0.

   (c) Setting $\frac{dx}{dt} = \frac{dy}{dt}$ (assuming that dx/dt and dy/dt are not zero simultaneously) in (2) and cancelling, gives the straight line $y = -x + 4/3$.

5. (a) By the chain rule we have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$. Hence, the tangent is horizontal if $g'(t) = 0$, $f'(t) \neq 0$, and vertical when $f'(t) = 0$, $g'(t) \neq 0$.

   (b) For the curve $x = t^2$, $y = t^3 - t$, we have $f'(t) = 2t$ and $g'(t) = 3t^2 - 1$. Hence, the tangent is horizontal when $t = \pm 1/\sqrt{3}$ (I checked that $f'(\pm 1/\sqrt{3}) \neq 0$), and vertical when $t = 0$ (again $g'(0) \neq 0$).

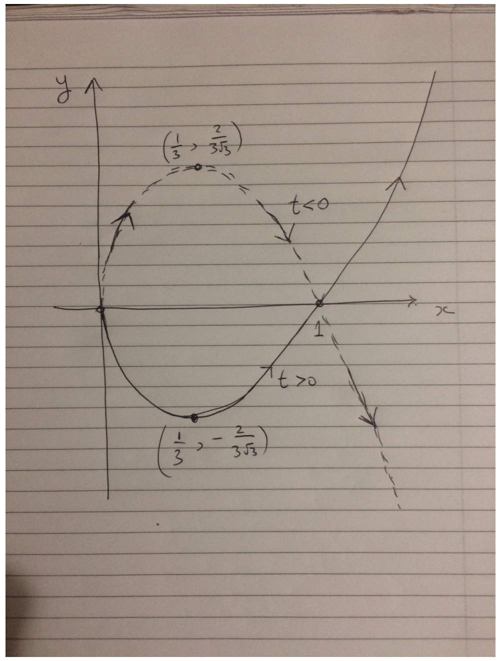

Figure 1: Sketch of the curve in problem 5.

(c) Here we need $\frac{3t^2-1}{2t} = 1$, hence $(3t+1)(t-1) = 0$, i.e. $t = -1/3, 1$.

(d) To sketch the curve we can use what we just found. At $t = 0$ we have $x = 0, y = 0$ and also

$$\lim_{t\to 0+} \frac{dy/dt}{dx/dt} = \lim_{t\to 0+} \frac{3t^2-1}{2t} = -\infty.$$

Also, if $0 < t < 1$ then $y = t(t^2-1) < 0$, and at $t = 1$ we cross the $x-$axis since $x(1) = 1$ and $y(1) = 0$. For $t > 1$ we have $y > 0$ (note that x is always non-negative).

From (b) the tangent is horizontal when $t = \pm 1/\sqrt{3}$, i.e. at the points $(\frac{1}{3}, -\frac{2}{3\sqrt{3}})$ and $(\frac{1}{3}, \frac{2}{3\sqrt{3}})$. The former point is a local minimum and the latter a local maximum.

From (c), at $t = 1$ we are at the point $(1,0)$ and the slope of the curve is 1. Also, at $t = -1/3$ we are at $(1/9, 8/27)$ and the tangent to the curve here is also unity.

As $t \to +\infty$, $x \to \infty$ and $y = x^{3/2} - x^{1/2} \to +\infty$.

One final observation and we can sketch the graph. Suppose I sketched the curve for $t => 0$. When $t < 0$ the curve is given by $x = (-t)^2 = t^2$ and $y = (-t)^3 - (-t) = -(t^3 - t)$. Hence we have symmetry about the $x - axis$. For $t > 0$ we get one branch (the lower one) and for $t < 0$ the upper branch. See Figure 1.

6. We have the function

$$f(x) = \begin{cases} x^n \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where n is a positive integer.

(a) With $n = 2$ the function is differentiable at 0 since

$$\lim_{h\to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h\to 0} h \sin(1/h) = 0,$$

by the squeezing theorem for limits.

Differentiating when $x \neq 0$ we find $f'(x) = 2x\sin(1/x) - \cos(1/x)$. Clearly this is continuous everywhere except at $x = 0$ since the limit of $\cos(1/x)$ is undefined as $x \to 0$. [Note that even if I fixed the derivative arbitrarily to be 1 at $x = 0$, the function is still not continuous. Why?]

(b) To find the smallest n that ensures that $\frac{d^2 f}{dx^2}$ exists and is continuous at $x = 0$, we don't need to differentiate fully (you should do this, however!), since the worst terms come from differentiation of the trig function. Hence the most singular term at the origin is proportional to $x^{n-4}\sin(1/x)$. To ensure existence and continuity of the limit (with value 0) we need $n \geq 5$.

7. Differentiating we find $f' = 1 + \cos x$, hence $f'(x) \geq 0$ (it is only zero at one point) and f is increasing for all x. The inverse exists for all x.

By the theorem regarding derivatives of inverse functions, we have

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{1 + \cos x}.$$

To find $g'(0)$ we need to find the x where $y = 0$, i.e. solve $0 = x + \sin x$. Clearly $x = 0$, hence $g'(0) = \frac{1}{1+\cos 0} = \frac{1}{2}$.

Similarly, solving $2\pi = x + \sin x$ gives $x = 2\pi$, hence $g'(2\pi) = \frac{1}{1+\cos 2\pi} = \frac{1}{2}$.

Finally, we need to solve $1 + \frac{\pi}{2} = x + \sin x$, hence $x = \pi/2$ by inspection, giving $g'(1 + \pi/2) = 1$.

8. Write $f(x) = x^{\frac{1}{\sin(x-1)}} = \exp\left(\frac{\log x}{\sin(x-1)}\right)$, and consider the limit $x \to 1$. The exponent is of the form $0/0$ so can use L'Hôpital's rule

$$\lim_{x\to 1} \frac{\log x}{\sin(x-1)} = \lim_{x\to 1} \frac{1/x}{\cos(x-1)} = 1.$$

By the continuity of the exponential we then have $\lim_{x\to 1} f(x) = \exp(1)$, hence we need to define $f(1) = e$.

9. We are given the following functions with values of $x \geq 0$,

$$f_1(x) = x - \sin x \qquad f_2(x) = -1 + \tfrac{x^2}{2} + \cos x$$
$$f_3(x) = -x + \tfrac{x^3}{3\cdot 2} + \sin x \qquad f_4(x) = 1 - \tfrac{x^2}{2} + \tfrac{x^4}{4\cdot 3\cdot 2} - \cos x$$
$$f_5(x) = x - \tfrac{x^3}{3\cdot 2} + \tfrac{x^5}{5\cdot 4\cdot 3\cdot 2} - \sin x$$

(a) Calculating $f_1'(x) = 1 - \cos x$, hence $f_1'(x) \geq 0$ and the function is increasing. In addition, $f_1(0) = 0$. Since $f_1(x)$ is increasing we must have $f_1(x) \geq f_1(0) = 0$, hence $x \geq \sin x$. Alternatively we can use the Mean Value Theorem on the interval $(0, x)$, i.e. there is a $c_1 \in (0, x)$ so that

$$\frac{f_1(x) - 0}{x} = f_1'(c_1) \geq 0 \qquad \Rightarrow \qquad x \geq \sin x. \tag{3}$$

(b) Differentiating we find

$$\begin{aligned} f_2'(x) &= x - \sin x = f_1(x), & (4)\\ f_3'(x) &= -1 + \frac{x^2}{2} + \cos x = f_2(x), & (5)\\ f_4'(x) &= -x + \frac{x^3}{3\cdot 2} + \sin x = f_3(x), & (6)\\ f_5'(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 3\cdot 2} - \cos x. & (7) \end{aligned}$$

Have established that $f_1(x) \geq 0$, hence from (4) we have $f_2'(x)$ is increasing. Since $f_2(0) = 0$, we must have $f_2(x) \geq 0$, i.e.

$$\cos x \geq 1 - \frac{x^2}{2}. \tag{8}$$

The argument now repeats. $f_2(x) \geq 0$ hence $f_3'(x)$ is increasing, satisfies $f_3(0) = 0$, hence $f_3(x) \geq 0$ and

$$\sin x \geq x - \frac{x^3}{3 \cdot 2}. \tag{9}$$

Continuing like this gives $f_4(x) \geq 0$, and then $f_5(x) \geq 0$. From their definitions the following inequalities follow

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 3 \cdot 2}, \tag{10}$$

$$\sin x \leq x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2}. \tag{11}$$

Inequalities (8)-(11) are what you were asked to show.

(c) Can continue the procedure by defining functions $f_{2n}(x)$ and $f_{2n+1}(x)$ for $n = 1, 2, \ldots$, as follows

$$f_{2n}(x) = (-1)^n \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{2n}}{(2n)!} - \cos x\right], \tag{12}$$

$$f_{2n+1}(x) = (-1)^n \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{2n+1}}{(2n+1)!} - \sin x\right]. \tag{13}$$

(14)

Since for *any* integer k we have (by the procedure established) $f_k'(x) = f_{k-1}(x) \geq 0$, and since $f_k(0) = 0$ by construction, we must have

$$f_k(x) \geq 0 \qquad \text{for all } k. \tag{15}$$

To get the 4 inequalities we start with *any odd* integer n - it has to be an odd integer since the inequalities come from $f_2 \geq 0$, $f_3 \geq 0$ (i.e. $n = 1$) and $f_4 \geq 0$, $f_5 \geq 0$ (i.e. $n = 2$). Next we go to $n = 3, 4$ etc.. Evaluating (12) for n (odd) and $n + 1$ (even) and using condition (15) gives

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{2n}}{(2n)!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{2n+2}}{(2n+2)!}, \tag{16}$$

and identical substitutions into (13) again using condition (15) gives

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{2n+1}}{(2n+1)!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{2n+3}}{(2n+3)!}. \tag{17}$$

Putting $n = 1$ we arrive in what you did in part (b).

Note: These expansions will appear trivial once we do Taylor's Theorem and applications.

10. Need to prove

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad \text{for every} \qquad a > 0, \; b > 0.$$

(a) Since $(\sqrt{a}-\sqrt{b})^2 \geq 0$ we have $a - 2\sqrt{a}\sqrt{b} + b \geq 0$, and the result follows since $a, b > 0$.

(b) For the function $f(x) = (a+x)/\sqrt{ax}$ where $x > 0$, we can differentiate to find

$$f'(x) = \frac{\sqrt{ax} - (a+x)\frac{a}{2}(ax)^{-1/2}}{ax},$$

implying that there is a critical point at $x = a$. This must be a minimum because (i) $f(x) > 0$, (ii) $\lim_{x\to 0+} f(x) = +\infty$, and (iii) $\lim_{x\to+\infty} f(x) = +\infty$. The value of the minimum is $f(a) = 2$, hence

$$\frac{a+x}{\sqrt{ax}} \geq 2.$$

Now pick $x = b$ and the result follows.