Math40003 Linear Algebra and Groups **Problem Sheet 6**

1.* (a) Which of the following functions $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ are linear transformations?

i. $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 + x_2)$

ii. $T(x_1, x_2, x_3) = (0, \sqrt{2}x_3)$

iii. $T(x_1, x_2, x_3) = (x_1x_2, x_3)$

(b) Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Which of the following functions $T : V \longrightarrow V$ are linear transformations?

i. $T(A) = A^2$ for all $A \in V$

ii. $T(A) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $A \in V$

(c) i. Find a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which sends $(1, 0)$ to $(1, 1, 0)$ and $(1, 1)$ to $(1, 0, -1)$.

ii. Find two different linear transformations $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which send $(1, 1, 0)$ to $(1, 1)$ and $(0, 1, 1)$ to $(0, 1)$.

(d) Let V be the vector space (over $\mathbb{R}$) of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following are linear transformations (thinking of $\mathbb{R}$ as $\mathbb{R}^1$ in parts (i) and (iii)) ?

i. $T_1 : V \to \mathbb{R}$ where $T_1(f) = f(1)$ (for $f \in V$).

ii. $T_2 : V \to V$ where $T_2(f) = f \circ f$ (for $f \in V$).

iii. $T_3 : \mathbb{R} \to V$ where $T_3(\mu)$ is the function $f_\mu \in V$ given by $f_\mu(x) = \mu x$ (for $\mu, x \in \mathbb{R}$).

(a) i. Yes, since $T(x) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} x$ **(where** x **is written as a column vector).**

ii. Yes, since $T(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} x$**.**

iii. No, since e.g. $T(1, 0, 0) + T(0, 1, 0) \neq T(1, 1, 0)$**.**

(b) i. No, since $T(2I) = 4I \neq 2T(I)$**.**

ii. Yes; writing $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$**, we have**

$$T(A_1 + A_2) = M(A_1 + A_2) = MA_1 + MA_2 = T(A_1) + T(A_2), \text{ and}$$
$$T(\lambda A) = M(\lambda A) = \lambda MA = \lambda T(A).$$

(c) i. Since $(0, 1) = (1, 1) - (1, 0)$**, we must have**

$$T(0, 1) = T(1, 1) - T(1, 0) = (1, 0, -1) - (1, 1, 0) = (0, -1, -1).$$

Now we get

$$T(x_1, x_2) = x_1(1, 1, 0) + x_2(0, -1, -1) = (x_1, x_1 - x_2, -x_2).$$

ii. **Let $v_1 = (1,1,0)$, $v_2 = (0,1,1)$ and $v_3 = (0,1,0)$. Then $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$. (The vector v_3 here has been chosen arbitrarily from many possibilities.) Now let $w_1 = (1,1)$, $w_2 = (0,1)$, and let w_3 be any vector in $\mathbb{R}^2$. Then there is a unique linear transformation T such that $T(v_i) = w_i$ for $i = 1,2,3$. Then we have $T(1,0,0) = T(v_1 - v_3) = w_1 - w_3$, and $T(0,0,1) = T(v_2 - v_3) = w_2 - w_3$. So**

$$\begin{aligned} T(x_1, x_2, x_3) &= T(x_1(1,0,0) + x_2 v_3 + x_3(0,0,1)) \\ &= x_1(w_1 - w_3) + x_2 w_3 + x_3(w_2 - w_3) \\ &= (x_1, x_1 + x_3) + (-x_1 + x_2 - x_3) w_3. \end{aligned}$$

Taking $w_3 = (0,0)$ gives the transformation $T_1 : (x_1, x_2, x_3) \mapsto (x_1, x_1 + x_3)$. Taking $w_3 = (0,1)$ gives the transformation $T_1 : (x_1, x_2, x_3) \mapsto (x_1, x_2)$. So these are two transformations taking v_1 to w_1 and v_2 to w_2 as required. (There are infinitely many more, corresponding to different choices of w_3.)

(d) i. **If $f, g \in V$ and $\lambda \in \mathbb{R}$, then $T_1(f+g) = (f+g)(1) = f(1) + g(1) = T_1(f) + T_1(g)$ and $T_1(\lambda f) = \lambda f(1) = \lambda T_1(f)$. So T_1 is a linear transformation.**

ii. **Not a linear transformation. For example, consider $f \in V$ with $f(x) = x$. Then $T_2(2f) \neq 2T_2(f)$.**

iii. **This is a linear transformation. If $\mu_1, \mu_2, \lambda, x \in \mathbb{R}$, then $(T_3(\mu_1 + \mu_2))(x) = (\mu_1 + \mu_2)x = (T_3(\mu_1) + T_3(\mu_2))(x)$, so $T_3(\mu_1 + \mu_2) = T_3(\mu_1) + T_3(\mu_2)$. Similarly, $T_3(\lambda\mu_1)(x) = (\lambda\mu_1)x = \lambda(\mu_1 x) = \lambda T_3(\mu_1)(x)$. (The difficulty here is keeping track of the notation.)**

2. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for exactly one vector $v \in \mathbb{R}^2$.

   (b) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for no vector $v \in \mathbb{R}^2$.

   (c) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for infinitely many vectors $v \in \mathbb{R}^2$.

   (d) Show that there is no linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for exactly two vectors $v \in \mathbb{R}^2$.

(a) $T(x_1, x_2) = (x_1, x_2, 0)$ **is one example.**

(b) $T(x_1, x_2) = (0,0,0)$**.**

(c) $T(x_1, x_2) = (x_1, 0)$**.**

(d) Suppose v_1 and v_2 are distinct vectors in $\mathbb{R}^2$ with $T(v_1) = T(v_2) = (1,0,0)$. Then

$$T(v_2 - v_1) = (1,0,0) - (1,0,0) = (0,0,0).$$

So for any $\lambda \in \mathbb{R}$ we have

$$T(v_1 + \lambda(v_2 - v_1)) = (1,0,0) + \lambda(0,0,0) = (1,0,0).$$

So we have infinitely many vectors $v = (1-\lambda)v_1 + \lambda v_2$ such that $T(v) = (1,0,0)$.

3. (Harder) (i) Suppose V, W are vector spaces (over a field F) and $S, T : V \to W$ are linear transformations. Prove that $S + T : V \to W$ defined by $(S+T)(v) = S(v) + T(v)$ (for $v \in V$) is a linear transformation. If $\lambda \in F$, show that $\lambda S : V \to W$ defined by $(\lambda S)(v) = \lambda S(v)$ (for $v \in V$) is a linear transformation. Explain why the set U of all linear transformations from V to W is a vector space with these operations.

   (ii) In the case where $V = F^2$ and $W = F^3$, what is the dimension of the vector space U? What is the dimension of U for arbitrary finite dimensional vector spaces V and W?

(Harder)

(a) If $v_1, v_2 \in V$ then

$$(S+T)(v_1+v_2) = S(v_1+v_2)+T(v_1+v_2) = Sv_1+Sv_2+Tv_1+Tv_2 = (S+T)v_1+(S+T)v_2,$$

so $S+T$ preserves addition. And if $v \in V$ and $\lambda \in F$ then

$$(S+T)(\lambda v) = S(\lambda v) + T(\lambda v) = \lambda Sv + \lambda Tv = \lambda(Sv + Tv) = \lambda(S+T)v,$$

so λS preserves scalar multiplication.

If $v_1, v_2 \in V$ then

$$(\lambda S)(v_1 + v_2) = \lambda S(v_1 + v_2) = \lambda S v_1 + \lambda S v_2 = (\lambda S)v_1 + (\lambda S)v_2,$$

so $S+T$ preserves addition. And if $v \in V$ and $\mu \in F$ then

$$(\lambda S)(\mu v) = \lambda S(\mu v) = \lambda\mu S v = \mu\lambda S v = \mu(\lambda S)v,$$

so λS preserves scalar multiplication.

We have addition and scalar multiplication defined on U, so we just need to check that the vector space axioms are satisfied; this is routine. (The zero of U is the map which sends $v \mapsto 0_W$ for all $v \in V$. For $S \in U$, the negative $-S$ is the map $v \mapsto -(Sv)$.)

(b) From Question 5(i), we know that every element S of U corresponds to a 3×2 matrix A. And it is clear that every 3×2 matrix A corresponds to an element S of U, given by $S(v) = Av$. So U is "essentially" just the space of 3×2 matrices, and this has dimension 6.

But let's turn that "essentially" into something rigorous. Let $\mathrm{Mat}_{3,2}(F)$ be the vector space of 3×2 matrices with entries from F. Then the map $S \mapsto A$ gives a bijection $\Phi : U \to \mathrm{Mat}_{2,3}(F)$. It is easy to check that this map is a linear transformation. Now since $\ker \Phi = \{0\}$ and $\operatorname{im} \Phi = \mathrm{Mat}_{2,3}(F)$, Rank–Nullity tells us that $\dim U = \dim M_{2\times 3}(F)$.

In the general case, if $\dim V = m$ and $\dim W = n$, then $\dim U = mn$, by the same argument.

The following need material from the last week of term:

4. (a) Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$. Find bases of Ker T and Im T. For which values of k is the vector $(1, 3, k)$ in Ker T or Im T?

   (b) Let V be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Define $T : V \longrightarrow V$ by

   $$T(ax^2 + bx + c) = (a + b + c)x^2 + (c - a)x + (a + 3b + 5c).$$

   Find bases of Ker T and Im T.

   (c) Let V be as in part (b), and define $S : V \longrightarrow V$ by

   $$S(p(x)) = p(1 + x) - p(x) \text{ for } p(x) \in V.$$

   (So for example, $S(x^2) = (x + 1)^2 - x^2 = 2x + 1$.) Show that S is a linear transformation, and find bases of Ker S and Im S.

(a) A basis for $\mathrm{Ker}(T)$ is $\{(1,1,1)\}$. A basis for $\mathrm{Im}(T)$ is $\{(1,0,-1),(1,-1,0)\}$. (There are many other possibilities.) The vector $(1,3,k)$ is not in $\mathrm{Ker}(T)$ for any k, and it is in $\mathrm{Im}(T)$ if and only if $k = -4$.

(b) We can treat this as the matrix equation

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Now we get bases B_K for the kernel and B_I for the image of this matrix transformation:

$$B_K = \{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\}, \quad B_I = \{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\}.$$

We can translate these back into polynomials, getting $\{x^2 - 2x + 1\}$ as a basis for $\mathrm{Ker}(T)$, and $\{x^2 - x + 1, x^2 + 3\}$ as a basis for $\mathrm{Im}(T)$.

(c) We have $S(p(x)+q(x)) = p(1+x)+q(1+x)-p(x)-q(x) = S(p(x))+S(q(x))$**, and** $S(\lambda p(x)) = \lambda p(x+1) - \lambda p(x) = \lambda S(p(x))$**. So** S **is a linear transformation. A basis for** $\mathrm{Ker}(S)$ **is** $\{1\}$**. A basis for** $\mathrm{Im}(S)$ **is** $\{1, x\}$**.**

5. (a) Let V be a finite-dimensional vector space, and $T : V \longrightarrow V$ a linear transformation.

   i. Prove that T is injective if and only if Ker $T = \{0\}$.

   ii. Prove that T is surjective if and only if Ker $T = \{0\}$.

   (b) Find an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that Ker $T =$ Im T.

   (c) Prove that there does not exist a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that Ker $T =$ Im T.

(a) i. If T **is injective, then there can be at most one solution to** $T(v) = 0_V$**. So** $v = 0_V$ **is the only solution, and so** $\mathrm{Ker}(T) = \{0_V\}$**. Conversely, suppose that** $\mathrm{Ker}(T) = \{0_V\}$**. Suppose that** $T(v) = T(w)$**. Then** $T(v-w) = T(v) - T(w) = 0_V$**, and so** $v - w \in \mathrm{Ker}(T)$**. But then** $v - w = 0_V$**, and so** $v = w$**.**

ii. Since $\dim \mathrm{Im}(T) + \dim \mathrm{Ker}(T) = \dim V$**, we see that** $\dim \mathrm{Im}(T) = \dim V$ **if and only if** $\dim \mathrm{Ker}(T) = 0$**. But** $\dim \mathrm{Im}(T) = \dim(V)$ **if and only if** T **is surjective, and** $\dim \mathrm{Ker}(T) = 0$ **if and only if** $\mathrm{Ker}(T) = \{0_V\}$**.**

(b) $T(x_1, x_2) = (x_2, 0)$ **is one such transformation.**

(c) If $\mathrm{Im}(T) = \mathrm{Ker}(T)$ **then** $\dim \mathrm{Im}(T) = \dim \mathrm{Ker}(T)$**. But then since** $\dim V = \dim \mathrm{Im}(T) + \dim \mathrm{Ker}(T)$**, we have** $\dim V = 2 \dim \mathrm{Im}(T)$**, which is even.**