Linear Algebra Hand-in Assignment

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Problem 1

We are given a transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. First, lets construct a matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$. Since $v \in \mathbb{R}^n$ we can write v as $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ for some scalars $v_1, ..., v_n$. The vector v can be written as follows:

$$\begin{aligned} v &= v_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + v_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= v_1 e_1 + v_2 e_2 + \cdots + v_n e_n \\ &= \sum_{i=1}^{n} v_i e_i, \end{aligned}$$

where e_i for $i \in (1, ..., n)$ form a basis for $\mathbb{R}^n$.

Now let's consider $T(v)$.

$$\begin{aligned} T(v) &= T(\sum_{i=1}^{n} v_i, e_i) \\ &= \sum_{i=1}^{n} T(v_i, e_i) \text{ (T preserves addition)} \\ &= \sum_{i=1}^{n} v_i T(e_i) \text{ (T preserves scalar multiplication).} \end{aligned}$$

We define the matrix $A_{m \times n}$ as

$$\begin{aligned} A &= \begin{pmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \end{aligned}$$

where each column of A is $T(e_i)$ for $i \in (1,...,n)$, a vector $\in \mathbb{R}^m$. Then by the definition of matrix and vector multiplication we get that

$$
\begin{aligned}
Av &= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= \begin{pmatrix} v_1 a_{11} & + \cdots + & v_n a_{1n} \\ & \vdots & \\ v_1 a_{m1} & + \cdots + & v_n a_{mn} \end{pmatrix} \\
&= v_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + v_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \\
&= v_1 T(e_1) + \cdots + v_n T(e_n) \\
&= \sum_{i=1}^{n} v_i T(e_i)
\end{aligned}
$$

Therefore, we got that

$$T(v) = \sum_{i=1}^{n} v_i T(e_i) = Av.$$

So we constructed matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$. Now all we need to show is that this matrix is unique.

Suppose B is a 4×4 matrix different from A, such that $Bv = T(v)$ for all $v \in \mathbb{R}^n$. Because $B \neq A$ there is at least one element in B, let's say $b_{jk} \neq a_{jk}$ for some $j,k \in (1,...,m),(1,...,n)$.

Now consider $T(e_k)$. We know $T(e_k) = Ae_k = A \begin{pmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{jk} \\ \vdots \\ a_{mk} \end{pmatrix} = Be_k$. But

$$Be_k = B \begin{pmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} b_{1k} \\ \vdots \\ b_{jk} \\ \vdots \\ b_{mk} \end{pmatrix} \neq \begin{pmatrix} a_{1k} \\ \vdots \\ a_{jk} \\ \vdots \\ a_{mk} \end{pmatrix}$$

since $a_{jk} \neq b_{jk}$, so we get a contradiction. Therefore A is unique.

Problem 2

First, let's find $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$ and $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$. We know that $\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$, so we get

$$\begin{aligned}
T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) &= T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right)\\
&= T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\right) + T\left(\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \text{(T preserves addition)}\\
&= \frac{1}{2}T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) + \frac{1}{2}T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) \text{(T preserves scalar multiplication)}\\
&= \frac{1}{2}\begin{pmatrix}-1\\8\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\0\\4\end{pmatrix}\\
&= \begin{pmatrix}-\frac{1}{2}\\4\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\0\\2\end{pmatrix}\\
&= \begin{pmatrix}-1\\4\\2\end{pmatrix}.
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) &= T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right)\\
&= T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\right) + T\left(-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \text{(T preserves addition)}\\
&= \frac{1}{2}T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) - \frac{1}{2}T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) \text{(T preserves scalar multiplication)}\\
&= \frac{1}{2}\begin{pmatrix}-1\\8\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\0\\4\end{pmatrix}\\
&= \begin{pmatrix}-\frac{1}{2}\\4\\0\end{pmatrix} - \begin{pmatrix}-\frac{1}{2}\\0\\2\end{pmatrix}\\
&= \begin{pmatrix}0\\4\\-2\end{pmatrix}.
\end{aligned}$$

Now we can set the transformation as follows:

$$\begin{aligned}
T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= T\left(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= xT\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + yT\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= x\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + y\begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} -1x & 0y \\ 4x & 4y \\ 2x & -2y \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

The matrix corresponding to T is

$$A = \begin{pmatrix} -1 & 0 \\ 4 & 4 \\ 2 & -2 \end{pmatrix}.$$

By problem 1 this matrix is unique.

Problem 3

By using that $Av_1 = v_3$ and $Av_3 = v_1$ we get that:

$$\begin{aligned}
Av_1 &= v_3 \\
A^2 v_1 &= Av_3 = v_1 \\
A^2 v_1 &= v_1.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
A^2 v_2 &= v_2 \\
A^2 v_3 &= v_3 \\
A^2 v_4 &= v_4.
\end{aligned}$$

First we will show that plugging in I_4 for A^2 is a solution. We know that $I_4 v_i = v_i$ for all $i \in (1,2,3,4)$ by properties of the identity matrix. Hence we get that $A^2 = I_4$ is a solution. Now we want to show that it is also unique.

First, by Proposition 4.1.7 in the Lecture Notes, we know that if we have two vector spaces V and W over F with $(v_1, v_2, ..., v_n)$ a basis for V and $(w_1, w_2, ..., w_n)$ any vectors in W, then there exists a unique linear transformation

$$T : V \longrightarrow W$$

$$T(v_i) = w_i \text{ for all } i \in (1, ..., n).$$

In our case, we have that $V = W = \mathbb{R}^4$, (v_1, v_2, v_3, v_4) is a basis for $\mathbb{R}^4$, and $v_1, v_2, v_3, v_4 \in \mathbb{R}^4$. Therefore, there exists a unique linear transformation

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$T(v_i) = v_i \text{ for all } i \in (1, 2, 3, 4).$$

Now using problem 1, we know that there exists a unique matrix $M_{4\times 4}$ such that $T(v) = Mv$ for all $v \in \mathbb{R}^4$. Because we know that $M = A^2 = I_4$ is one solution to our problem (we know that A^2 is a 4×4 matrix as A is a 4×4 matrix) and we now know that it is unique, we get

$$A^2 = I_4.$$