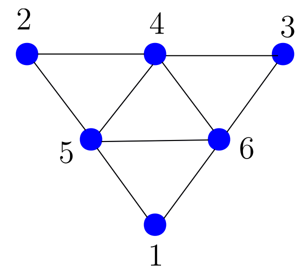
Symmetry, eigenvectors/eigenvalues

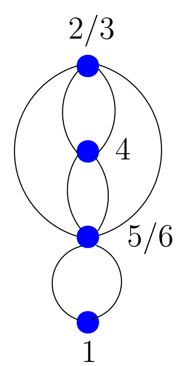
1. Consider the electric circuit with 6 nodes shown in the figure:



All edges have unit conductance. Suppose that node 1 is set to unit voltage and nodes 2 and 3 are grounded. It is required to find the effective conductance of this circuit.

(a) Write down a linear system involving the 6-by-6 Laplacian of the graph which can be solved to find the effective conductance.

(b) Using invoking symmetry, argue why we can alternatively find the effective conductance of the following "equivalent" circuit:



Write down a linear system involving the 4-by-4 Laplacian of *this* graph which can be solved to find the effective conductance.

(c) By introducing an appropriate symmetry matrix S prove that the effective conductances found using the linear systems in parts (a) and (b) are the same.

2. Consider the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Verify that $\mathbf{S}^6 = \mathbf{I}$ where $\mathbf{I}$ is the 6-by-6 identity.

(b) Can you use this fact to find the eigenvalues and eigenvectors of $\mathbf{S}$?

3. In lectures it was shown that eigenvectors of the n-by-n matrix

$$K_n \equiv \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

are

$$\mathbf{\Phi}_k = A_k \begin{pmatrix} \sin(k\pi/(n+1)) \\ \sin(2k\pi/(n+1)) \\ . \\ . \\ \sin(nk\pi/(n+1)) \end{pmatrix}, \qquad k = 1, 2, \cdots, n.$$

(a) Show that in order that each of these vectors satisfies $\overline{\mathbf{\Phi}_k}^T \mathbf{\Phi}_k = 1$ for $k = 1, \cdots, n$ then we must choose

$$A_k = \sqrt{\frac{2}{n+1}}, \qquad \text{for all } k = 1, \cdots, n.$$

(b) Verify by direct calculation that the set

$$\mathbf{\Phi}_k = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin(k\pi/(n+1)) \\ \sin(2k\pi/(n+1)) \\ . \\ . \\ \sin(nk\pi/(n+1)) \end{pmatrix}, \qquad k = 1, 2, \cdots, n$$

is then an *orthonormal* set of eigenvectors.

4. Let n be an even integer. Show that *half* of the eigenvectors of K_n can be found by considering the circulant matrix C_{n+1}.

5. Compute the Fourier sine series of

$$1 - \frac{x}{\pi}$$

and confirm that it agrees with the $n \to \infty$ result obtained in lectures of $n+1$ masses attached to a wall at one end and pulled with unit force at the other.