Imperial College London
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Solutions to Question Sheet 1

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

All questions can be attempted before the problem class on Monday Week 3. Some of the questions are meant to be easier than others. If you can do the easy ones without much effort, skip to the harder ones. Question 2 or 7 could be suitable for tutorials in week 2 or early week 3. Solutions will be released on Wednesday of week 3.

Question 1 Compute the determinants of the following matrices, assuming that the entries are from the field $\mathbb{R}$. Which matrices are invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Solution: In order, the determinants are: $-2, 0, -1, 1-2a$. So the matrices are: invertible, non-invertible, invertible, and invertible iff $a \neq \frac{1}{2}$.

Question 2 (i) Factorise $\det\begin{pmatrix} x & y & x \\ y & x & x \\ x & x & y \end{pmatrix}$. (ii) Solve $\det\begin{pmatrix} t-1 & 3 & -3 \\ -3 & t+5 & -3 \\ -6 & 6 & t-4 \end{pmatrix} = 0$ for $t \in \mathbb{R}$.

Solution: (i) Using row operations $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - R_1$ shows the determinant is

$$\det\begin{pmatrix} x & y & x \\ y-x & x-y & 0 \\ 0 & x-y & y-x \end{pmatrix} = (x-y)^2 \det\begin{pmatrix} x & y & x \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = -(x-y)^2(2x+y)$$

by expanding the determinant about the first column.

(ii) Solutions $t = -2, 4$.

Question 3 Calculate the determinant of the matrix $\begin{pmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{pmatrix}$.

How would you check your answer?

Solution: -24: use whatever method you like. You can check by using a different method or using a computer-algebra package (Wolfram Alpha, Matlab, ...)

Question 4 For a real number α define

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & \alpha & 1 & -1 \\ -1 & \alpha & 1 & 1 \end{pmatrix} \in M_4(\mathbb{R}).$$

(a) Find the determinant of $A(\alpha)$.
(b) Find a value α_0 of α such that the system $A(\alpha_0)x = 0$ has a nonzero solution for $x \in \mathbb{R}^4$.
(c) Prove that when $\alpha < \alpha_0$, there is no real 4×4 matrix B such that $B^2 = A(\alpha)$.

Solution: (a) $|A(\alpha)| = \alpha - 1$.

(b) $\alpha_0 = 1$ (using result from lectures that system $Ax = 0$ has a nonzero solution for x iff $|A| = 0$).

(c) For $\alpha < 1$, $|A(\alpha)| < 0$. If $B^2 = A(\alpha)$ then by the multiplicativity of det, $|B|^2 = |A(\alpha)| < 0$, which is impossible if B is real.

Question 5 Let $A, B \in M_n(F)$ (where F is an arbitrary field). Prove that $|A| = 0$ or $|B| = 0$ if and only if $|AB| = 0$.
Note: you may NOT assume the result $|AB| = |A|\,|B|$ from lectures as this question is supposed to be part of the proof of that result. You MAY assume the result that says a matrix is invertible iff it has nonzero determinant.

Solution: $\Rightarrow$: Suppose $|A| = 0$. Then A is not invertible (result 5.1.10). It follows that AB is also not invertible (if it were, say the inverse was C, we'd have $ABC = I$, so BC would be the inverse of A, contradiction). Hence $|AB| = 0$, again by 5.1.10. Similarly, suppose $|B| = 0$. Then B is not invertible. It follows that AB is also not invertible (if it were, say the inverse was C, we'd have $CAB = I$, so CA would be the inverse of B, contradiction). Hence $|AB| = 0$.

$\Leftarrow$: If $|A| \neq 0$ and $|B| \neq 0$ then A, B are invertible. So AB is invertible (with inverse $B^{-1}A^{-1}$) and therefore $|AB| \neq 0$, again using 5.1.10.

Question 6 Let B_n be the $n \times n$ matrix

$$\begin{pmatrix} -2 & 4 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}$$

(a) Prove that if $n \geq 4$, then $|B_n| = 8|B_{n-3}|$.
(b) Prove that $|B_n| = 0$ if $n = 3k - 1$ (where k is a positive integer).
(c) Find $|B_n|$ if $n = 3k$ or $3k + 1$.

Solution: (a) Expanding by 1st col, get

$$|B_n| = -2|B_{n-1}| - 1 \det \begin{pmatrix} 4 & 0 & 0 & \dots \\ 1 & 2 & -4 & \dots \\ & & & \dots \end{pmatrix} = -2|B_{n-1}| - 4|B_{n-2}|.$$

Substitute for $|B_{n-1}|$ in this, using the same formula ($|B_{n-1}| = -2|B_{n-2}| - 4|B_{n-3}|$). This gives

$$|B_n| = 8|B_{n-3}|.$$

(b) When $n = 3k - 1$ this shows that

$$|B_n| = 8|B_{n-3}| = 8^2|B_{n-6}| = \cdots = 8^{k-1}|B_2| = 0.$$

(c) When $n = 3k$, we similarly get $|B_n| = 8^{k-1}|A_3| = 8^k$. And when $n = 3k+1$, we get $|B_n| = 8^k|A_1| = -2^{3k+1}$.

Question 7 The *trace* $\mathrm{tr}(A)$ of an $n \times n$ matrix $A = (a_{ij})$ over a field F is defined to be the sum of the diagonal entries of A: $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$. Prove that, for $n \times n$ matrices A, B (over F) we have: (a) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$; (b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Suppose V is an n-dimensional F-vector-space with a basis C and $T : V \to V$ is a linear transformation. We define the trace of T to be $\mathrm{tr}([T]_C)$. Prove that this does not depend on the choice of basis C.

Solution: (a) The ii-entry of $A + B$ is $(a_{ii} + b_{ii})$, so this is trivial.

(b) The trace of AB is $\sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}b_{ji})$. Reversing the order of summation and using commutativity of multiplication in the field, we see that this is equal to $\sum_{j=1}^{n}(\sum_{i=1}^{n} b_{ji}a_{ij})$. But this is $\mathrm{tr}(BA)$.

Last part: this does not depend on the choice of basis. Suppose D is another basis. By the change of basis formula, there is an invertible matrix P with $[T]_D = P^{-1}AP$, where $A = [A]_C$. By (b) we have $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(PP^{-1}A) = \mathrm{tr}(A)$.

Question 8 If A is an $n \times n$ matrix over a field F, the *characteristic polynomial* of A is $\det(xI_n - A)$. Prove that this is a polynomial of degree n over F and: (i) the coefficient of x^n is 1; (ii) the constant term is $(-1)^n\det(A)$; (iii) the coefficient of x^{n-1} is $-\mathrm{tr}(A)$.

Solution: Suppose C is an $n \times n$ matrix. By induction on n we can prove that:
$\det(C)$ is a sum of $n!$ terms each one of which is $\pm$ a product of n entries of C where there is exactly one entry from each row and column. Moreover the sign of the product of diagonal entries $c_{11} \ldots c_{nn}$ is $+$.
(See 5.5 for a more precise statement.)

Using this, $\det(xI_n - A)$ is equal to

$$(x - a_{11}) \ldots (x - a_{nn}) + \text{ a polynomial of degree } \leq n - 2.$$

So this is a polynomial of degree n where the coefficient of x^n is 1 and the coefficient of x^{n-1} is $-(a_{11} + \ldots + a_{nn}) = -\mathrm{tr}(A)$.

The constant term is obtained by putting $x = 0$ and this is $\det(-A) = (-1)^n\det(A)$.

Question 9 Let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

where the a_i are in the field F. Prove that the characteristic polynomial of A is
$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. (*Hint:* Try induction.)

Solution: By induction on n. The char poly is

$$p(x) = \det \begin{pmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{pmatrix}$$

Expand by the first row. By induction the det of the 11-minor is $x^{n-1}+a_{n-1}x^{n-2}+\cdots+a_1$. Also, the determinant of the $1n$-minor is $(-1)^{n-1}$ (as it is upper triangular). So we get

$$p(x) = x\,(x^{n-1}+a_{n-1}x^{n-2}+\cdots+a_1)+(-1)^{n+1}a_0.(-1)^{n-1} = x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.$$

Hence the result by induction.

Remark: The matrix A is called the *companion matrix* for the polynomial.