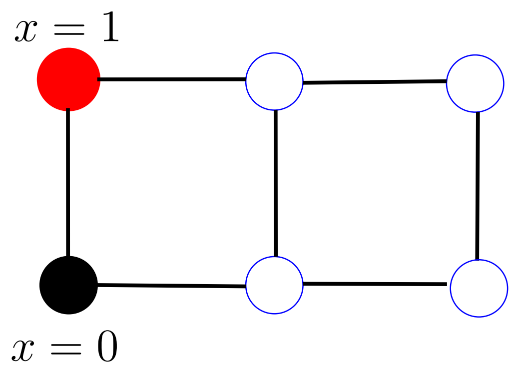
Energy and Minimum Principles

1. Consider the following electric circuit with one node held at unit voltage and another grounded. All edges are resistors with unit conductance. At the 4 interior nodes Kirchhoff's current law holds.



(a) By initializing all 4 interior nodes at zero voltage and looping around them twice use the method of relaxation to find an approximation to the voltages at the 4 interior nodes.

(b) Compute the dissipation $\mathcal{E}_j$, for $j = 1, 2$, associated with the flow calculated at the *end* of each loop around the circuit in the method of relaxation.

(c) Find the dissipation $\mathcal{E}$ associated with the actual current flowing in this circuit.

(d) Verify that $\mathcal{E} \leq \mathcal{E}_j$ for $j = 1, 2$ (that is, confirm Dirichlet's Principle for this example).

2. In establishing Dirichlet's Principle we assumed that two nodes, call them a and b, were set at voltages $x_a = 1$ and $x_b = 0$ and then showed that the voltages at the interior nodes minimized the energy dissipation

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^T K \mathbf{x},$$

where $\mathbf{x}$ is the vector of node potentials. Suppose we slightly change the problem now and, instead of insisting that $x_a = 1$, we instead insist that a unit current $f_a = 1$ drives the flow in the circuit, with $f_b = -1$ required for consistency (and KCL holding at all other nodes). Now x_a must also be found as part of the solution.

Show that the solution of this problem minimizes the *modified* energy dissipation function defined by

$$\mathcal{E}_0(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f},$$

where $\mathbf{f}$ is the (now known) vector of currents out of each node.

3. A *flow* of strength j_a from a to b in a given circuit (with n nodes, m edges, and with two boundary nodes a and b) is defined to be any set of edge fluxes, which can be collected in an m-dimensional vector $\mathbf{j}$, satisfying

$$-\mathbf{A}^T\mathbf{j} = \mathbf{f},$$

where

$$\mathbf{f} = [j_a \ \ -j_a \ \ \mathbf{0}]^T$$

and where nodes a and b correspond to the first two elements of $\mathbf{f}$, and $\mathbf{0}$ is the $(n-2)$-dimensional row vector of zeros. Note that a *flow*, as just defined, is distinct from a *current* since there is no requirement that a flow is derived from a set of voltages defined at the nodes (as must be true of an electrical current obeying Ohm's law).

(a) Let $\mathbf{x}$ be *any* potential assigned to the nodes of the circuit. Show that, for *any* flow $\mathbf{j}$ from a to b in the circuit,

$$-(\mathbf{A}\mathbf{x})^T\mathbf{j} = j_a(x_a - x_b).$$

(b) Show that the expression derived in part (a) can be alternatively written as

$$\sum_{\text{edges k}} (x_i - x_j)j_k = j_a(x_a - x_b),$$

where the sum is over all the edges in the graph and edge k is assumed to join node i to node j with the direction of the arrow (taken in constructing the incidence matrix) is from node i to node j.

(c) Let a flow $\mathbf{j}$ be a *unit flow* with unit strength so that $j_a = 1$. Define the energy dissipation to be

$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{j_k^2}{c_k},$$

where c_k is the conductance of edge k and j_k is the k-th element of $\mathbf{j}$. With the help of parts (a) and (b), prove that the particular unit flow, $\mathbf{w}$ say, that satisfies Ohm's Law on all edges and Kirchhoff's current law at all interior nodes (i.e., a unit "current" from a to b) minimizes this energy dissipation among all unit flows from a to b (this is *Thomson's Principle*).

4. In question 3(e) on Problem Sheet 3 you found the probability that a tourist starting a simple random walk at Leicester Square would reach Warren St before returning to Leicester Square. Suppose that, due to engineering works, the Bakerloo line between Oxford Circus and Piccadilly Circus is closed. Does this increase or decrease her probability of reaching Warren Street before returning to Leicester Square? (Hint: you should be able to use energy arguments to answer this question without carrying out any explicit calculations).