Math40003 Linear Algebra and Groups Problem Sheet 4

1. (a) Which of these sets of vectors are linearly independent? Which span $\mathbb{R}^3$?

   (i) $(5,3,0)$, $(2,1,1)$ (ii) $(1,0,1)$, $(-1,1,0)$, $(0,1,1)$
   (iii) $(1,3,1)$, $(2,1,1)$, $(-1,7,-5)$ (iv) $(1,-3,2)$, $(2,-1,1)$, $(2,-5,4)$, $(1,2,5)$

   (b) For which a,b,c are the vectors $(1,3,1)$, $(2,1,1)$, (a,b,c) linearly dependent?

2. *Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

   (a) If $\{v_1,\dots,v_n\}$ is a basis, for V, and $\{x_1,\dots,x_r\}$ is a linearly independent subset of V with $r<n$, and if $v_i \notin Span\{x_1,\dots,x_r\}$ for all i, then $\{x_1,\dots,x_r,v_{r+1},\dots,v_n\}$ is a basis for V.

   (b) If U is a subspace of V, then $U+U=U$.

   (c) If U and W are subspaces of V, and $\dim U+\dim W=\dim V$, then $U\cap W=\{0_V\}$.

   (d) If $\dim V=n$ and $v_1\in V$, then there exist vectors $v_2,\dots,v_n$ in V such that $\{v_1,\dots,v_n\}$ spans V.

   (e) If W is a subspace of V , then $\dim W\le\dim V$ and $\dim W=\dim V$ if and only if $W=V$.

3. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly independent? Extend those which are linearly independent to a basis of $\mathbb{R}^4$.

   (i) $(1,2,3,0)$, $(-1,2,3,0)$ (ii) $(1,2,3,0)$, $(-1,2,3,0)$, $(0,1,2,3)$

   (iii) $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(-1,1,1,-1)$, $(0,1,2,-3)$

4. Let $V=\mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$). Show that the functions

   $$f_1(x)=1,\quad f_2(x)=1+x+x^2,\quad f_3(x)=\sin x,\quad f_4(x)=\cos x$$

   are linearly independent. Which of the following functions lie in $\langle f_1,f_2,f_3,f_4\rangle$?

   $$5-3x-3x^2,\quad \tan x,\quad 10-x-x^2+\sin(x+\pi/3).$$

5. (a) Write down an infinite number of different bases of $\mathbb{R}^2$ (in finite time).

   (b) Find a basis for $W=\langle x^2-1,\ x^2+1,\ 4,\ 2x-1,\ 2x+1\rangle\le\mathbb{R}[x]$.

   Recall: $\mathbb{R}[x]$ *is the set of real polynomials in variable* x

6. Let $M_{3,3}$ denote the vector space of all 3×3 matrices over $\mathbb{R}$.

   (i) Find a basis of $M_{3,3}$ consisting of invertible matrices.

   (ii) Let $W=\{A\in M_{3,3}\colon A^t=A\}$. Show $W\le V$ and compute $\dim W$.

   (iii) Let $W\subset M_{3,3}$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W\le V$ and find a basis for W.