1(a). With the labelling of nodes given in the figure the Laplacian is

$$\mathbf{K} = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}.$$

The system to solve is

$$\mathbf{Kx} = \mathbf{f}, \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ X \\ Y \\ Z \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where X, Y, Z are the unknown potentials at nodes $4, 5$ and 6 and f_1 is the required effective conductance.

1(b). We can merge nodes 2 and 3 because they are both grounded, and by the symmetry of the graph and forcing we see that nodes 5 and 6 must be at the same potential/voltage so we can also merge those. Keeping all edges intact, the circuit becomes the "equivalent" circuit shown in the figure.

With the labelling of nodes given in the figure the Laplacian is

$$\hat{\mathbf{K}} = \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 4 & -2 \\ -2 & -2 & -2 & 6 \end{bmatrix}.$$

The system to solve is

$$\hat{\mathbf{K}}\mathbf{x} = \mathbf{f}, \tag{2}$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ \hat{X} \\ \hat{Y} \end{bmatrix}, \qquad \hat{\mathbf{f}} = \begin{bmatrix} \hat{f}_1 \\ -\hat{f}_1 \\ 0 \\ 0 \end{bmatrix},$$

and $\hat{X}, \hat{Y}$ are the unknown potentials at nodes 4 and 5/6 and $\hat{f}_1$ is the effective conductance.

1(c). Introduce the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is easily checked that

$$\mathbf{KS} = \mathbf{SK} \tag{3}$$

that is, matrices $\mathbf{S}$ and $\mathbf{K}$ commute. This means that if $\mathbf{x}$ satisfies

$$\mathbf{Kx} = \mathbf{f} \tag{4}$$

then

$$\mathbf{SKx} = \mathbf{Sf},$$

or since $\mathbf{K}$ and $\mathbf{S}$ commute,

$$\mathbf{KSx} = \mathbf{Sf}. \tag{5}$$

Note that, unlike the example considered in lectures, $\mathbf{Sf} \neq \mathbf{f}$. However, we know that

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{Sf} = \begin{bmatrix} f_1 \\ f_3 \\ f_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Let us introduce the following subblock decomposition of $\mathbf{K}$:

$$\mathbf{K} = \begin{bmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{bmatrix}, \tag{6}$$

where all subblock matrices are 3-by-3, and $\mathbf{R}$ is positive definite (and hence invertible). We will also write

$$\mathbf{x} = \begin{bmatrix} \mathbf{e}_1 \\ \hat{\mathbf{x}} \end{bmatrix}, \qquad \mathbf{Sx} = \begin{bmatrix} \mathbf{e}_1 \\ \hat{\mathbf{y}} \end{bmatrix},$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

It is then easy to check that (4) implies

$$\mathbf{Q}\mathbf{e}_1 + \mathbf{R}\hat{\mathbf{x}} = 0 \tag{8}$$

while (5) implies

$$\mathbf{Q}\mathbf{e}_1 + \mathbf{R}\hat{\mathbf{y}} = 0. \tag{9}$$

These equations imply that

$$\hat{\mathbf{x}} = -\mathbf{R}^{-1}\mathbf{Q}\mathbf{e}_1 = \hat{\mathbf{y}}.$$

Since

$$\hat{\mathbf{y}} = \begin{bmatrix} X \\ Z \\ Y \end{bmatrix}, \tag{10}$$

then we have proved that the potentials at nodes 5 and 6 are the same.

However (4) also implies

$$\mathbf{P}\mathbf{e_1} + \mathbf{Q}^T\hat{\mathbf{x}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

while (5) implies

$$\mathbf{P}\mathbf{e_1} + \mathbf{Q}^T\hat{\mathbf{y}} = \begin{bmatrix} f_1 \\ f_3 \\ f_2 \end{bmatrix}$$

However since $\hat{\mathbf{x}} = \hat{\mathbf{y}}$ we can also conclude that

$$f_2 = f_3.$$

This means that

$$\mathbf{S}\mathbf{f} = \mathbf{f}$$

and hence, on use of this in (4) and (5), we see that both $\mathbf{x}$ and $\mathbf{S}\mathbf{x}$ are solutions of the same circuit problem.

It follows that we can now add columns 5 and 6 in the linear system meaning that

$$\mathbf{K}\mathbf{x} = \mathbf{f}$$

becomes

$$\begin{bmatrix} 2 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 4 & -2 \\ -1 & -1 & 0 & -1 & 3 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can also remove columns 2 and 3 since they do not contribute:

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & -2 \\ -1 & -1 & 3 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can now eliminate the last row since it is a repeated equation; also, with $f_2 = f_3$, we can drop the third row since it is also a repeated equation:

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & 4 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix}$$

On multiplying rows 2 and 4 by 2 this system becomes

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ 2f_2 \\ 0 \\ 0 \end{bmatrix}$$

We also know that since

$$f_1 + f_2 + f_3 = 0, \qquad f_2 = f_3$$

then

$$2f_2 = -f_1.$$

However this is the **same** as system (2) once the second column of that system is removed (because it does not contribute), i.e., system (2) becomes

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ \hat{X} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ -\hat{f}_1 \\ 0 \\ 0 \end{bmatrix}.$$

We have therefore systematically shown that the two linear systems are equivalent.

2(a). We notice that

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{K}_3 & 0 \\ 0 & 0 & \mathbf{K}_2 \end{bmatrix}, \tag{11}$$

where a zero denotes a zero block matrix of appropriate size and $\mathbf{K}_n$ denotes an n-by-n circulant matrix. Since we know that $\mathbf{K_2}^2 = \mathbf{I}_2, \mathbf{K_3}^3 = \mathbf{I}_3$ it is clear that $\mathbf{S}^6 = \mathbf{I}_6$ (here $\mathbf{I}_n$ denotes the n-by-n identity matrix).

2(b). Suppose

$$\mathbf{S}\mathbf{x} = \lambda \mathbf{x} \tag{12}$$

then

$$\mathbf{S}^6\mathbf{x} = \lambda^6 \mathbf{x} = \mathbf{x} \tag{13}$$

which implies that

$$\lambda^6 = 1 = e^{2\pi i m}, \tag{14}$$

where m is any integer. It follows that any eigenvalues of $\mathbf{S}$ are among the 6th roots of unity (note however that it does *not* mean that every 6th root of unity is an eigenvalue). Indeed it is clear from the block decomposition above that the eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ \omega_3 \\ \omega_3^2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 1 \\ \omega_3^2 \\ \omega_3^4 \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

$$\mathbf{x}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \tag{16}$$

where $\omega_3 = e^{2\pi i/3}$. The corresponding eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = \omega_3, \qquad \lambda_4 = \omega_3^2, \qquad \lambda_5 = 1, \qquad \lambda_6 = -1. \tag{17}$$

3. (a) Write the unnormalized $\mathbf{\Phi}_k$ as

$$\mathbf{\Phi}_k = \text{Im} \begin{pmatrix} \omega_k \\ \omega_k^2 \\ \cdot \\ \cdot \\ \omega_k^n \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} \omega_k - \omega_k^{-1} \\ \omega_k^2 - \omega_k^{-2} \\ \cdot \\ \cdot \\ \omega_k^n - \omega_k^{-n} \end{pmatrix}, \qquad k = 1, 2, \cdots, n,$$

where

$$\omega_k = e^{k\pi i/(n+1)}. \tag{18}$$

Here Im[.] means take the imaginary part. It follows that

$$\overline{\mathbf{\Phi}_k}^T \mathbf{\Phi}_k = \frac{1}{4}\sum_{j=1}^{n}\left(\omega_k^j - \omega_k^{-j}\right)\left(\omega_k^{-j} - \omega_k^j\right)$$

since we must sum the squared modulus of each of the elements. However this is

$$\begin{aligned}\overline{\mathbf{\Phi}_k}^T \mathbf{\Phi}_k &= \frac{1}{4}\sum_{j=1}^{n}\left(2 - \omega_k^{2j} - \omega_k^{-2j}\right)\\ &= \frac{1}{4}\left(2n - \sum_{j=1}^{n}\omega_k^{2j} - \sum_{j=1}^{n}\omega_k^{-2j}\right)\end{aligned} \tag{19}$$

Now on summing a finite geometric progression,

$$\sum_{j=1}^{n}\omega_k^{2j} = \frac{\omega_k^{2n+2} - \omega_k^2}{\omega_k^2 - 1} = \omega_k^2\left[\frac{\omega_k^{2n} - 1}{\omega_k^2 - 1}\right].$$

But it follows from (18)

$$\omega_k^{2n} = \omega_k^{2(n+1)-2} = \frac{1}{\omega_k^2}.$$

Hence

$$\sum_{j=1}^{n}\omega_k^{2j} = \omega_k^2\left[\frac{\omega_k^{-2} - 1}{\omega_k^2 - 1}\right] = -1.$$

On taking a complex conjugate it follows immediately that

$$\sum_{j=1}^{n}\omega_k^{-2j} = -1.$$

On substitution of these results into (19) we find

$$\overline{\mathbf{\Phi}_k}^T \mathbf{\Phi}_k = \frac{1}{4}(2n+2) = \frac{n+1}{2} \tag{20}$$

Note that this result is independent of k (i.e. it is the same for every eigenvector). We therefore need to take

$$A = \sqrt{\frac{2}{n+1}} \tag{21}$$

in order to normalize the eigenvectors.

(b) Given

$$\mathbf{\Phi}_n = \begin{pmatrix} \sin(n\pi/(N+1)) \\ \sin(2n\pi/(N+1)) \\ . \\ . \\ \sin(nN\pi/(N+1)) \end{pmatrix}, \qquad n = 1, 2, \cdots, N$$

then

$$\boldsymbol{\Phi}_n^T \boldsymbol{\Phi}_m = \sum_{k=1}^{N} \sin\left(\frac{n\pi k}{N+1}\right) \sin\left(\frac{m\pi k}{N+1}\right) \tag{22}$$

Using a trigonometric identity we can write this as

$$\begin{aligned} \boldsymbol{\Phi}_n^T \boldsymbol{\Phi}_m &= \frac{1}{2} \sum_{k=1}^{N} \left[\cos\left(\frac{(n-m)\pi k}{N+1}\right) - \cos\left(\frac{(n+m)\pi k}{N+1}\right)\right] \\ &= \frac{1}{2} \mathrm{Re} \left\{ \sum_{k=1}^{N} \left[e^{\pi \mathrm{i}(n-m)k/(N+1)} - e^{\pi \mathrm{i}(n+m)k/(N+1)} \right] \right\}. \end{aligned} \tag{23}$$

Now using the sum of a finite geometric progression we know that

$$\sum_{k=1}^{N} x^k = \frac{x(1-x^N)}{1-x} \tag{24}$$

Hence

$$\begin{aligned} \sum_{k=1}^{N} e^{\pi \mathrm{i}(n-m)k/(N+1)} &= e^{\pi \mathrm{i}(n-m)/(N+1)} \frac{1 - e^{\pi \mathrm{i}(n-m)N/(N+1)}}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= e^{\pi \mathrm{i}(n-m)/(N+1)} \frac{1 - e^{\pi \mathrm{i}(n-m)(N+1-1)/(N+1)}}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= e^{\pi \mathrm{i}(n-m)/(N+1)} \frac{1 - (-1)^{(n-m)} e^{-\pi \mathrm{i}(n-m)/(N+1)}}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= \frac{e^{\pi \mathrm{i}(n-m)/(N+1)} - (-1)^{(n-m)}}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} \end{aligned} \tag{25}$$

A similar formula holds with $m - n \mapsto m + n$. Now $m - n$ and $m + n$ will either both be even or odd. If they are both even then

$$\begin{aligned} \frac{e^{\pi \mathrm{i}(n-m)/(N+1)} - (-1)^{(n-m)}}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} &= \frac{e^{\pi \mathrm{i}(n+m)/(N+1)} - (-1)^{(n+m)}}{1 - e^{\pi \mathrm{i}(n+m)/(N+1)}} \\ &= -1 \end{aligned} \tag{26}$$

hence

$$\boldsymbol{\Phi}_n^T \boldsymbol{\Phi}_m = 0. \tag{27}$$

If $m - n$ and $m + n$ are both odd then

$$\begin{aligned} \frac{e^{\pi \mathrm{i}(n-m)/(N+1)} - (-1)^{(n-m)}}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} &= \frac{e^{\pi \mathrm{i}(n-m)/(N+1)} + 1}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= \frac{e^{\pi \mathrm{i}(n-m)/(N+1)} + 1}{1 - e^{\pi \mathrm{i}(n-m)/(N+1)}} \left[\frac{1 - e^{-\pi \mathrm{i}(n-m)/(N+1)}}{1 - e^{-\pi \mathrm{i}(n-m)/(N+1)}} \right] \end{aligned} \tag{28}$$

which is purely imaginary, as will the same quantity with $m-n \mapsto m+n$. Hence in this case too

$$\mathbf{\Phi}_n^T \mathbf{\Phi}_m = \frac{1}{2}\mathrm{Re}\left\{\sum_{k=1}^{N}\left[e^{\pi \mathrm{i}(n-m)k/(N+1)} - e^{\pi \mathrm{i}(n+m)k/(N+1)}\right]\right\} = 0. \tag{29}$$

4. A circulant matrix of dimension $n+1$ has the form

$$C_{n+1} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

This can be written in the sub-block decomposition

$$C_{n+1} = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & K_n \end{bmatrix}. \tag{30}$$

where

$$K_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & -1 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

is the matrix considered in lectures. We know eigenvectors $\mathbf{x}$ of C_{n+1} are

$$\mathbf{x}_m = \begin{bmatrix} 1 \\ \omega^m \\ \omega^{2m} \\ . \\ . \\ \omega^{nm} \end{bmatrix}, \qquad m = 0, 1, \cdots, n+1, \tag{31}$$

where $\omega = e^{2\pi \mathrm{i}/(n+1)}$ with eigenvalue

$$\lambda_m = 2\left[1 - \cos(2\pi m/(n+1))\right].$$

The imaginary part of this is also an eigenvector

$$\hat{\mathbf{x}}_m = \mathrm{Im}[\mathbf{x}_m] = \begin{bmatrix} 0 \\ \mathrm{Im}[\omega^m] \\ \mathrm{Im}[\omega^{2m}] \\ \cdot \\ \cdot \\ \mathrm{Im}[\omega^{nm}] \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{X}_m \end{bmatrix}, \qquad m = 0, 1, \cdots, n+1. \tag{32}$$

Note that the vectors corresponding to $m = 0$ and $m = n + 1$ are identically zero, while the vectors for each $m = j$ and $m = n + 1 - j$ are easily seen to be linearly dependent, meaning that there are only $n/2$ linearly independent vectors of this kind. But any such vector has the property

$$C_{n+1}\hat{\mathbf{x}}_m = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & K_n \end{bmatrix} \hat{\mathbf{x}}_m = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & K_n \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{X}_m \end{bmatrix} = \lambda_m \hat{\mathbf{x}}_m, \tag{33}$$

or equivalently,

$$\mathbf{q}^T\mathbf{X}_m = 0, \qquad K_n\mathbf{X}_m = \lambda_m\mathbf{X}_m. \tag{34}$$

This means that the n-dimension subvector $\mathbf{X}_m$ is an eigenvector of K_n, and there are $n/2$ of these. Thus, this construction produces *half* of the eigenvectors of K_n.

5. Writing the Fourier sine series as

$$1 - \frac{x}{\pi} = \sum_{n=1}^{\infty} b_n \sin nx \tag{35}$$

then we multiply by $\sin mx$ and integrate between 0 and π:

$$\int_0^{\pi} \sin mx \left[1 - \frac{x}{\pi}\right] dx = \int_0^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \sin mx dx. \tag{36}$$

Now on use of integration by parts we find

$$\int_0^{\pi} \sin mx \left[1 - \frac{x}{\pi}\right] dx = \frac{1}{m}.$$

Also, from a trigonometric identity,

$$\int_0^{\pi} \sin mx \sin nx dx = \frac{1}{2}\int_0^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$$

and the right hand side equals 0 unless $m = n$ when it equals $\pi/2$. On substitution of these results into (36) above we find

$$\frac{\pi}{2} b_m = \frac{1}{m}$$

so that, from (35),

$$1 - \frac{x}{\pi} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin nx$$

as found in lectures by a limiting process of a $n+1$ spring-mass system.