M1M2: Unseen 5: Lotka–Volterra

2.
(a). Divide $\frac{dy}{dt} = cxy - dy$ by $\frac{dx}{dt} = ax - bxy$

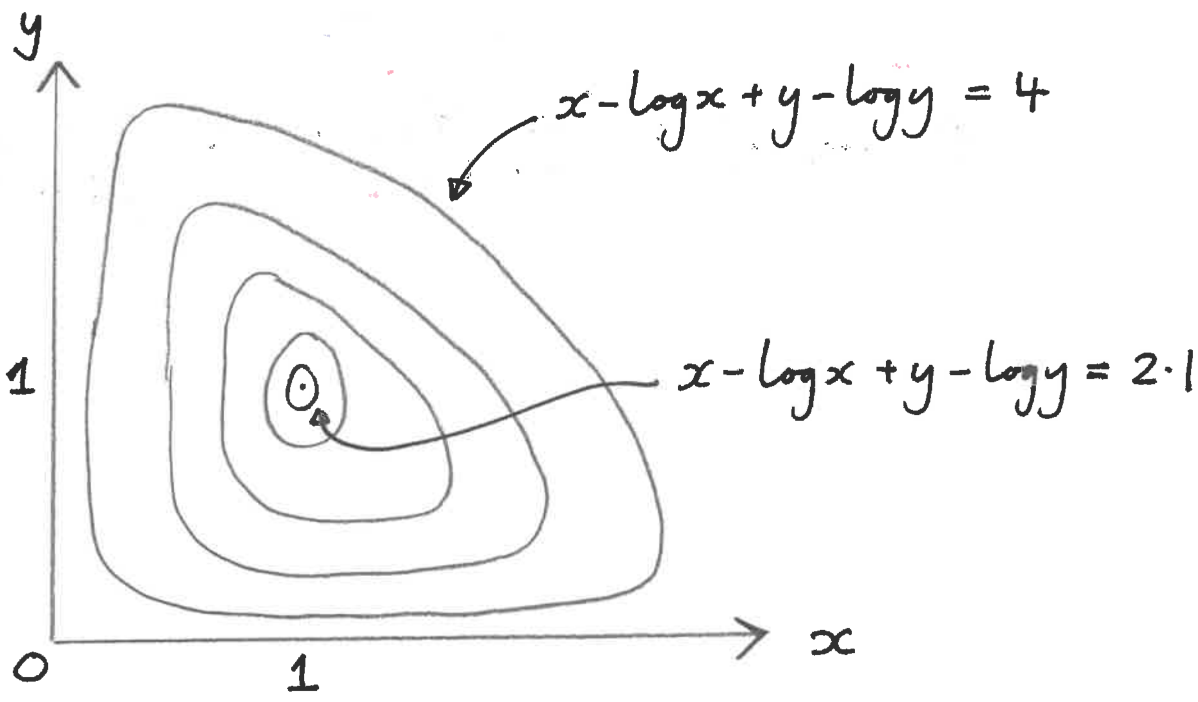
$$\Rightarrow \frac{dy}{dx} = \frac{y(cx-d)}{x(a-by)}$$

$$\Rightarrow \int \frac{(a-by)}{y}\,dy = \int \frac{(cx-d)}{x}\,dx$$ (separation of variables)

$$\Rightarrow a\log y - by = cx - d\log x - K$$, K constant of integration.

$$\Rightarrow \boxed{cx - d\log x + by - a\log y = K}$$

(b).



The curves look something like the sketch above.

$\boxed{2}$

3.

(a). We solve simultaneously:

$$ax - bxy = 0 \quad ①$$
$$cxy - dy = 0 \quad ②$$

We can write: $x(a - by) = 0$ in ①

$$\Rightarrow \underline{x = 0} \text{ or } \underline{y = \frac{a}{b}}$$

If $x = 0$; then in ②: $-dy = 0 \Rightarrow \underline{y = 0}$

$$\Rightarrow \boxed{(x_0, y_0) = (0,0)} \text{ is an equilibrium point.}$$

If $y = \frac{a}{b}$; then in ②: $\frac{ca}{b}x - \frac{da}{b} = 0$

$$\Rightarrow \underline{x = \frac{d}{c}}$$

$$\Rightarrow \boxed{(x_0, y_0) = \left(\frac{d}{c}, \frac{a}{b}\right)} \text{ is the other equilibrium point.}$$

$\boxed{3}$

4.1

(a). We put: $(x,y) = \underbrace{(x_0,y_0)}_{\text{constants}} + \varepsilon\underbrace{(x_1,y_1)}_{\text{variables}}$ into LV eqs:

$$\varepsilon \frac{dx_1}{dt} = a(x_0+\varepsilon x_1) - b(x_0+\varepsilon x_1)(y_0+\varepsilon y_1)$$

$$\Rightarrow \varepsilon \frac{dx_1}{dt} = \cancel{ax_0} + \varepsilon a x_1 - \cancel{bx_0y_0} - \varepsilon b x_0 y_1 - \varepsilon b x_1 y_0 - \cancel{\varepsilon^2 b x_1 y_1}$$

$$\Rightarrow O(\varepsilon): \quad \frac{dx_1}{dt} = (a-by_0)x_1 - bx_0y_1 \qquad (3)$$

And into second LV equation:

$$\varepsilon \frac{dy_1}{dt} = c(x_0+\varepsilon x_1)(y_0+\varepsilon y_1) - d(y_0+\varepsilon y_1)$$

$$\Rightarrow \varepsilon \frac{dy_1}{dt} = \cancel{cx_0y_0} + \varepsilon c x_0 y_1 + \varepsilon c x_1 y_0 + \cancel{\varepsilon^2 c x_1 y_1} - \cancel{dy_0} - \varepsilon d y_1$$

$$\Rightarrow O(\varepsilon): \quad \frac{dy_1}{dt} = cy_0x_1 + (cx_0-d)y_1 \qquad (4)$$

So; Case 1: $(x_0,y_0)=(0,0)$, we have: (from (3),(4))

$$\frac{dx_1}{dt} = ax_1, \quad \frac{dy_1}{dt} = -dy_1$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{(linear system)}$$

e. values: $\lambda_1 = a$, $\lambda_2 = -d$

e. vectors:

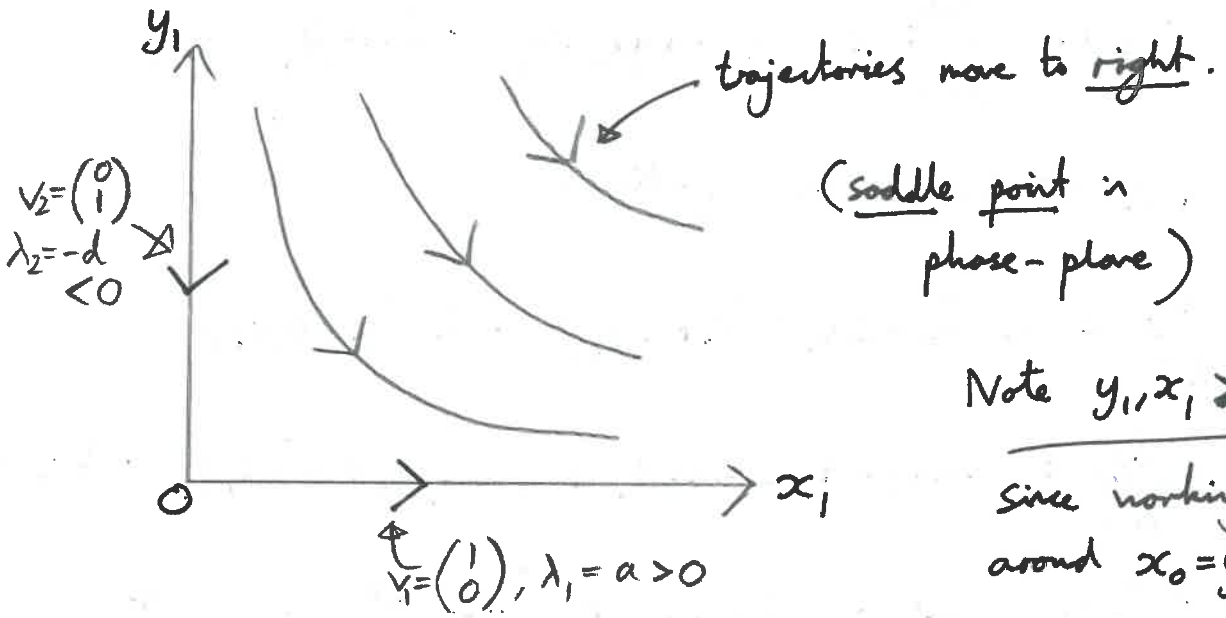
$$\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = a\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \Rightarrow \left.\begin{matrix} av_{1x} = av_{1x} \\ -dv_{1y} = av_{1y} \end{matrix}\right\} \Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and:

$$\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = -d\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} \Rightarrow \left.\begin{matrix} av_{2x} = -dv_{2x} \\ -dv_{2y} = -dv_{2y} \end{matrix}\right\} \Rightarrow \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + B\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-dt} \quad \text{(general solution)}$$

Phase-plane



(saddle point in phase-plane)

Note $y_1, x_1 \geq 0$. since working around $x_0 = y_0 = 0$.

Case 2: $(x_0, y_0) = \left(\frac{d}{c}, \frac{a}{b}\right)$, we have (from ③, ④):

$$\frac{dx_1}{dt} = -\frac{bd}{c} y_1, \quad \frac{dy_1}{dt} = \frac{ca}{b} x_1$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ca}{b} & 0 \end{bmatrix}}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{(linear system)}$$

e.values: $\begin{vmatrix} -\lambda & -\frac{bd}{c} \\ \frac{ca}{b} & -\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - \left(\frac{ac}{b}\right)\left(\frac{-bd}{c}\right) = 0$$

$$\Rightarrow \lambda^2 + ad = 0$$

$$\Rightarrow \lambda_1 = \sqrt{ad}\, i, \quad \lambda_2 = -\sqrt{ad}\, i$$

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e. vectors:

$$\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \sqrt{ad}\, i \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \Rightarrow \left.\begin{aligned} -\frac{bd}{c} v_{1y} &= \sqrt{ad}\, i\, v_{1x} \\ \frac{ac}{b} v_{1x} &= \sqrt{ad}\, i\, v_{1y} \end{aligned}\right\}$$

$$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ -\omega i \end{pmatrix}, \quad \text{where } \omega = \frac{c\sqrt{a}}{b\sqrt{d}}$$

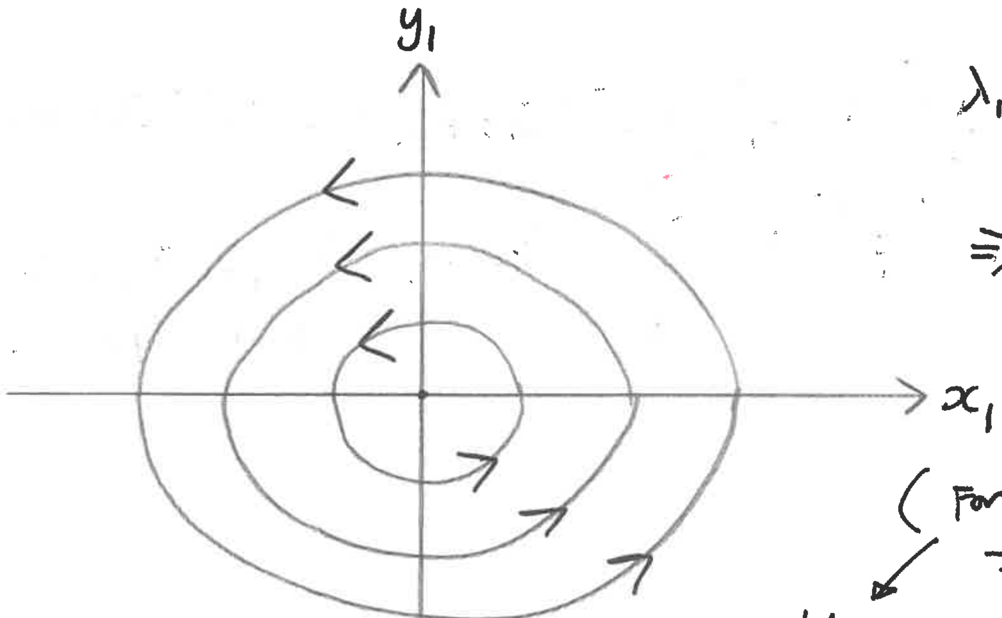
and:

$$\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = -\sqrt{ad}\, i \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} \Rightarrow \left.\begin{aligned} -\frac{bd}{c} v_{2y} &= -\sqrt{ad}\, i\, v_{2x} \\ \frac{ac}{b} v_{2x} &= -\sqrt{ad}\, i\, v_{2y} \end{aligned}\right\}$$

$$\Rightarrow \underline{v}_2 = \begin{pmatrix} -\frac{1}{\omega} i \\ 1 \end{pmatrix}, \quad \omega = \frac{c\sqrt{a}}{b\sqrt{d}}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A \begin{pmatrix} 1 \\ -\omega i \end{pmatrix} e^{\sqrt{ad}\, i t} + B \begin{pmatrix} -\frac{1}{\omega} i \\ 1 \end{pmatrix} e^{-\sqrt{ad}\, i t}, \quad \omega = \frac{c\sqrt{a}}{b\sqrt{d}}$$

Phase-plane



λ_1, λ_2 complex conjugate pair $\Rightarrow$ <u>centre</u>

(For $y_1 > 0$ $-\frac{bd}{c} y_1 < 0$.)

V. field at $\begin{pmatrix} 0 \\ y_1 \end{pmatrix}$: $\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = A \begin{pmatrix} 0 \\ y_1 \end{pmatrix} = \begin{pmatrix} -\frac{bd}{c} y_1 \\ 0 \end{pmatrix} \Rightarrow$ <u>Anti-clockwise direction.</u>

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(b). (i). The local solutions are:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A\begin{pmatrix} 1 \\ -\omega i \end{pmatrix} e^{\sqrt{ad}\, it} + B\begin{pmatrix} -\frac{1}{\omega} i \\ 1 \end{pmatrix} e^{-\sqrt{ad}\, it} \quad , \omega = \frac{c\sqrt{a}}{b\sqrt{d}}$$

This can be equivalently written as:

$$\underline{x_1(t) = C_1 \cos(\sqrt{ad}\, t + C_2)} \quad \text{S}$$

$$\underline{y_1(t) = C_1 \omega \sin(\sqrt{ad}\, t + C_2)} \quad , \quad C_1, C_2 \text{ arbitrary constants.}$$

To see this you could start from these expressions and use the formulae for $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$ to expand to retrieve the previous expressions.

Now it is clear to see the predator population $y_1(t)$ lags behind the prey population $x_1(t)$ by $\frac{\pi}{2}$ since sine lags behind cosine by $\frac{\pi}{2}$ (and they are sin/cosine of the <u>same</u> argument : $\sqrt{ad}\, t + C_2$).

(ii). With the local solutions as written in S, we see

$$x_1^2 + \left(\frac{y_1}{\omega}\right)^2 = C_1^2\left(\cos^2(\sqrt{ad}\, t + C_2) + \sin^2(\sqrt{ad}\, t + C_2)\right)$$

$$\Rightarrow \boxed{x_1^2 + \left(\frac{y_1}{\omega}\right)^2 = C_1^2} \quad \text{(i.e. } x_1 \text{ and } y_1 \text{ satisfy the equation of an ellipse).}$$

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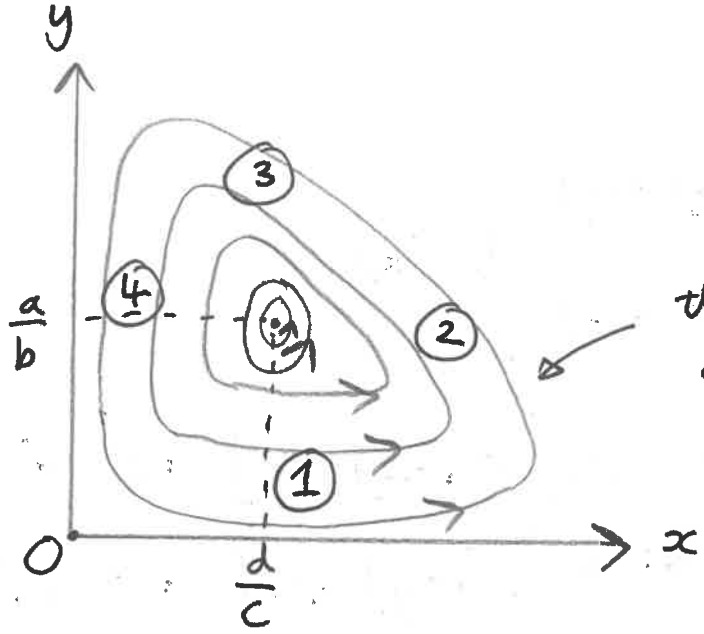
(c).



the trajectories are anticlockwise as this follows the direction found in the previous analysis.

There are 4 key regions:

1: The predator population, y, is low here, so the prey population <u>grows</u> as there are less predators to eat them.

2: The prey population, x, is high, hence there is more food for the predators, so the predator population <u>grows</u>.

3: The predator population, y, is high and eats the prey, so the prey population <u>decreases</u>.

4: The prey population is low, so there is less food for the predators, so the predator population starves and <u>decreases</u>.

⋮

The cycle then repeats itself.

5. Extension:

(a).
$$\left.\begin{aligned} \frac{dx}{dt} &= \alpha x(1-x) - xy \\ \frac{dy}{dt} &= \beta y(1-y) - xy \end{aligned}\right\} \quad (*)$$

Equilibrium points when: $x\left[\alpha(1-x) - y\right] = 0$ ⑥

and: $y\left[\beta(1-y) - x\right] = 0$ ⑦

⑥ gives: $x = 0$ or $y = \alpha(1-x)$

when $x = 0$; ⑦ gives: $\beta y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$

$\Rightarrow$ Two equilibria are: $(x_0, y_0) = (0,0)$

$(x_0, y_0) = (0,1)$

when $y = \alpha(1-x)$; ⑦ gives: $\alpha(1-x)\left[\beta\left(1 - \alpha(1-x)\right) - x\right] = 0$

$$\Rightarrow \alpha(1-x)\left[(\alpha\beta - 1)x + \beta(1-\alpha)\right] = 0$$

$$\Rightarrow x = 1 \quad \text{or} \quad x = \frac{\beta(\alpha-1)}{\alpha\beta - 1}$$

$y = \alpha(1-x)$

$\Rightarrow$ The remaining equilibria are: $(x_0, y_0) = (1, 0)$

$$(x_0, y_0) = \left(\frac{\beta(\alpha-1)}{\alpha\beta-1}, \frac{\alpha(\beta-1)}{\alpha\beta-1}\right)$$

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Substitute: $x = x_0 + \varepsilon x_1$, $y = y_0 + \varepsilon y_1$, $\varepsilon \ll 1$ into (*) equations.

$$\varepsilon \frac{dx_1}{dt} = \alpha(x_0+\varepsilon x_1)(1-(x_0+\varepsilon x_1)) - (x_0+\varepsilon x_1)(y_0+\varepsilon y_1)$$

$$\Rightarrow \varepsilon \frac{dx_1}{dt} = \cancel{\alpha x_0} + \varepsilon\alpha x_1 - \cancel{\alpha x_0^2} - \varepsilon\alpha x_0 x_1 - \varepsilon\alpha x_0 x_1 - \cancel{\varepsilon^2\alpha x_1^2} - \cancel{x_0 y_0}$$
$$- \varepsilon x_0 y_1 - \varepsilon x_1 y_0 - \cancel{\varepsilon^2 x_1 y_1}$$

$$\Rightarrow O(\varepsilon): \quad \underline{\frac{dx_1}{dt} = (\alpha - 2\alpha x_0 - y_0)x_1 - x_0 y_1}$$

Similarly from the other equation, one finds:

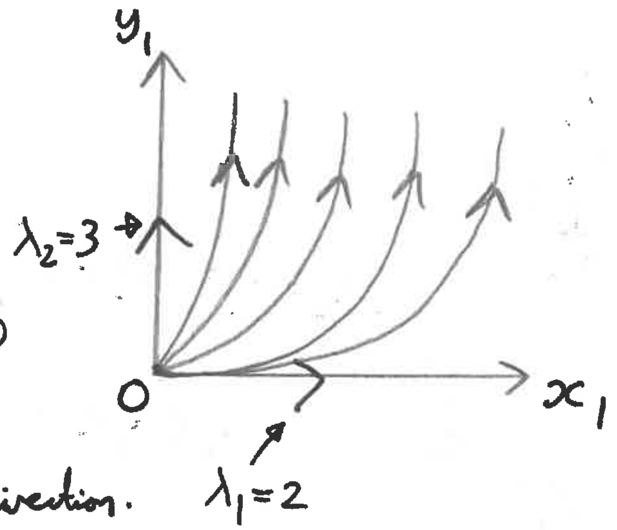
$$\underline{\frac{dy_1}{dt} = (\beta - 2\beta y_0 - x_0)y_1 - y_0 x_1}$$

(1) $(x_0, y_0) = (0,0)$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

e.values: $\underline{\lambda_1 = \alpha}$, $\underline{\lambda_2 = \beta}$. e.vecs: $\underline{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, $\underline{v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

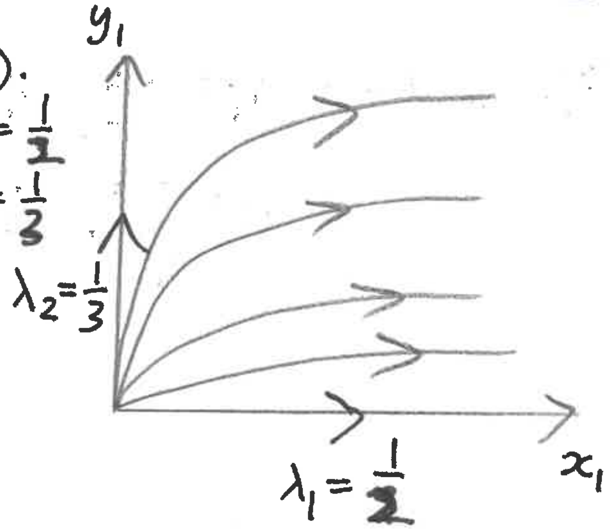
(i). $\alpha = 2$, $\beta = 3$



$\lambda_1, \lambda_2 > 0$

blow up in $\lambda_2 = 3$ direction.

(ii). $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$



(2) 10 $(x_0, y_0) = (1, 0)$

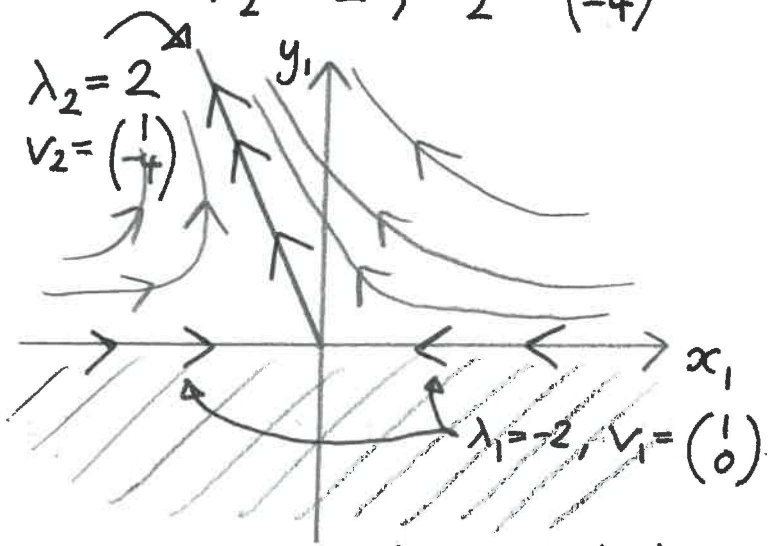
$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha & -1 \\ 0 & \beta-1 \end{bmatrix}}_{\text{e.values: } (-\alpha-\lambda)(\beta-1-\lambda) = 0} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\Rightarrow \underline{\lambda_1 = -\alpha}, \ \underline{\lambda_2 = \beta - 1}$$

e.vectors: $\underline{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, $\underline{v_2 = \begin{pmatrix} 1 \\ 1-\alpha-\beta \end{pmatrix}}$

(i). e.values: $\lambda_1 = -2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$, $v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

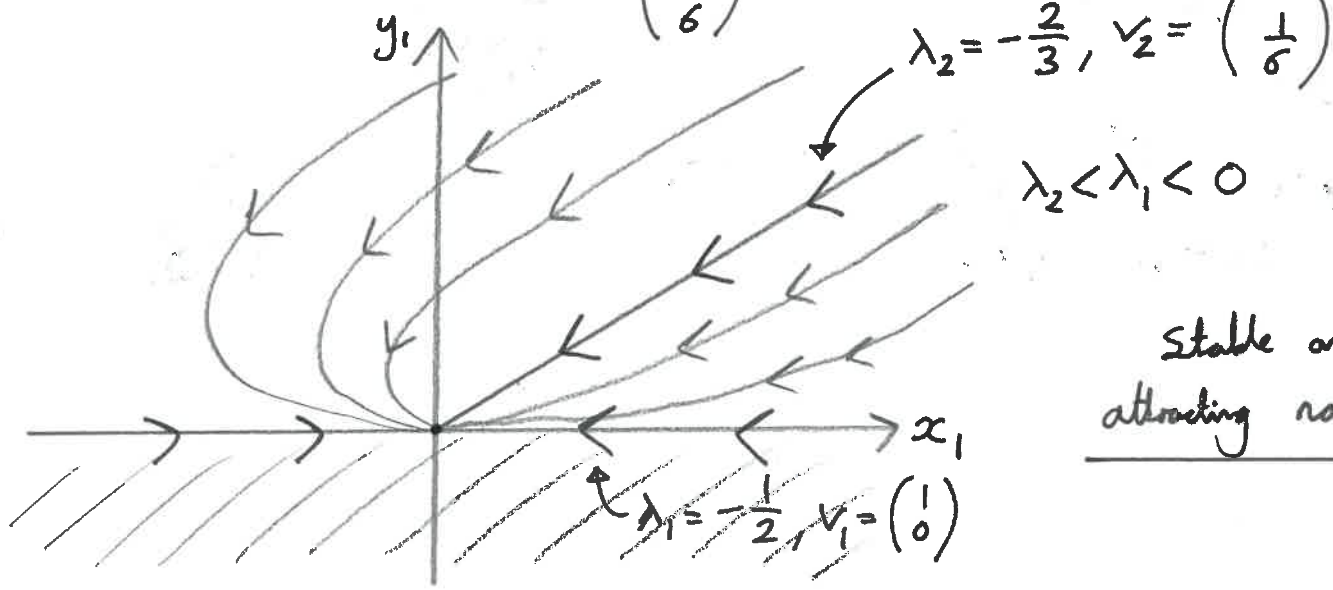


$y_1 > 0$ as $y_0 = 0$ here.
so lower half plane not needed

<u>Saddle point</u>

(ii). e.values: $\lambda_1 = -\frac{1}{2}$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -\frac{2}{3}$, $v_2 = \begin{pmatrix} 1 \\ \frac{1}{6} \end{pmatrix}$



$\lambda_2 < \lambda_1 < 0$

<u>Stable or attracting node</u>

(3) $(x_0, y_0) = (0, 1)$

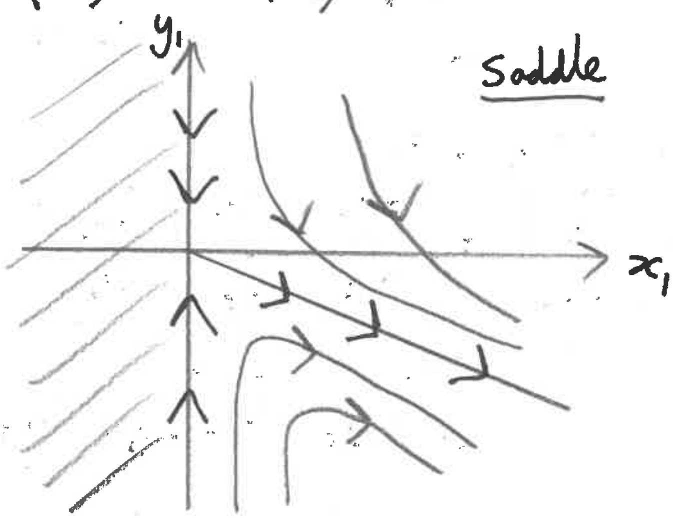
This will be qualitatively identical to $(1, 0)$.

$\lambda_1 = \alpha - 1$, $\lambda_2 = -\beta$,

$v_1 = \begin{pmatrix} 1-\alpha-\beta \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(i). $\lambda_1 = 1$, $\lambda_2 = -3$

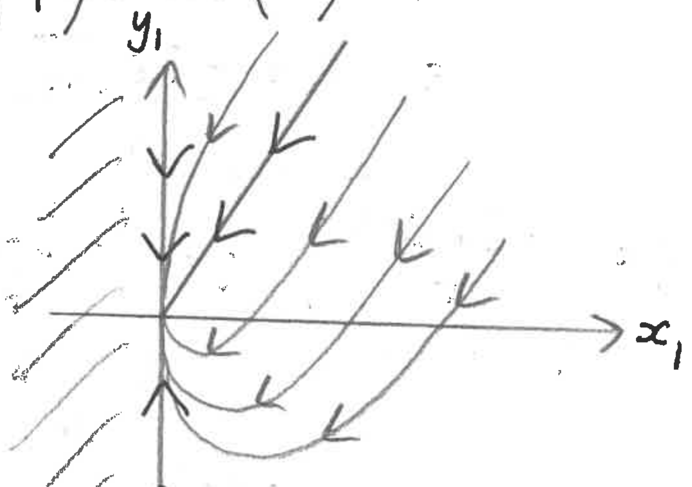
$v_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Saddle

(ii). $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -\frac{1}{3}$

$v_1 = \begin{pmatrix} \frac{1}{6} \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Stable node

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(4) $(x_0, y_0) = \left(\dfrac{\beta(\alpha-1)}{\alpha\beta-1}, \dfrac{\alpha(\beta-1)}{\alpha\beta-1} \right)$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha\beta(1-\alpha)}{\alpha\beta-1} & -\dfrac{\beta(\alpha-1)}{\alpha\beta-1} \\ \dfrac{-\alpha(\beta-1)}{\alpha\beta-1} & \dfrac{\alpha\beta(1-\beta)}{\alpha\beta-1} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

(i). $\alpha = 2, \beta = 3 \quad \rightarrow (x_0, y_0) = \left(\frac{3}{5}, \frac{4}{5}\right)$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{6}{5} & -\frac{3}{5} \\ -\frac{4}{5} & -\frac{12}{5} \end{bmatrix}} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

e.values: $\left(-\frac{6}{5} - \lambda\right)\left(\frac{-12}{5} - \lambda\right) - \left(\frac{-4}{5}\right)\left(\frac{-3}{5}\right) = 0$

$\Rightarrow (5\lambda + 6)(5\lambda + 12) - 12 = 0$

$\Rightarrow 25\lambda^2 + 90\lambda + 60 = 0$

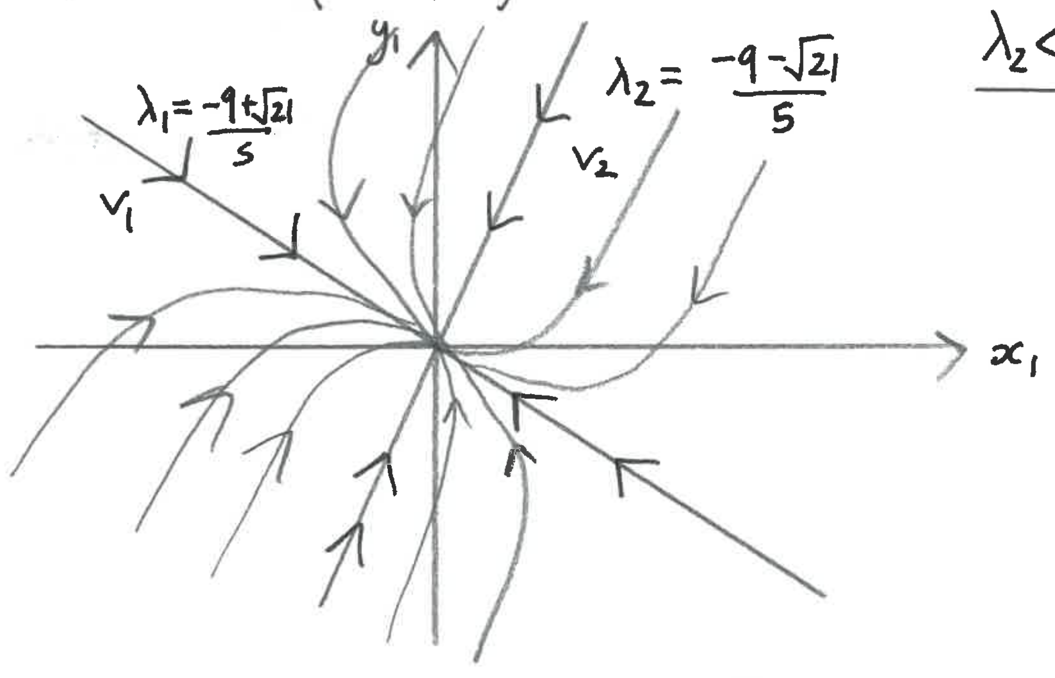
$\Rightarrow 5\lambda^2 + 18\lambda + 12 = 0$

$\Rightarrow \lambda = \dfrac{-9 \pm \sqrt{21}}{5} \quad \Rightarrow \lambda_1, \lambda_2 < 0$

$\Rightarrow$ Attracting or stable node.

$V_1 = \begin{pmatrix} -\frac{(3+\sqrt{21})}{4} \\ 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -\frac{(3-\sqrt{21})}{4} \\ 1 \end{pmatrix}$

$\lambda_2 < \lambda_1 < 0$



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(ii). $\alpha = \frac{1}{2}, \beta = \frac{1}{3} \longrightarrow (x_0, y_0) = \left(\frac{1}{5}, \frac{2}{5}\right)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{10} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{2}{15} \end{bmatrix}}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

e. values : $\left(-\frac{1}{10}-\lambda\right)\left(-\frac{2}{15}-\lambda\right) - \left(-\frac{2}{5}\right)\left(-\frac{1}{5}\right) = 0$

$\Rightarrow (10\lambda+1)(15\lambda+2) - 12 = 0$

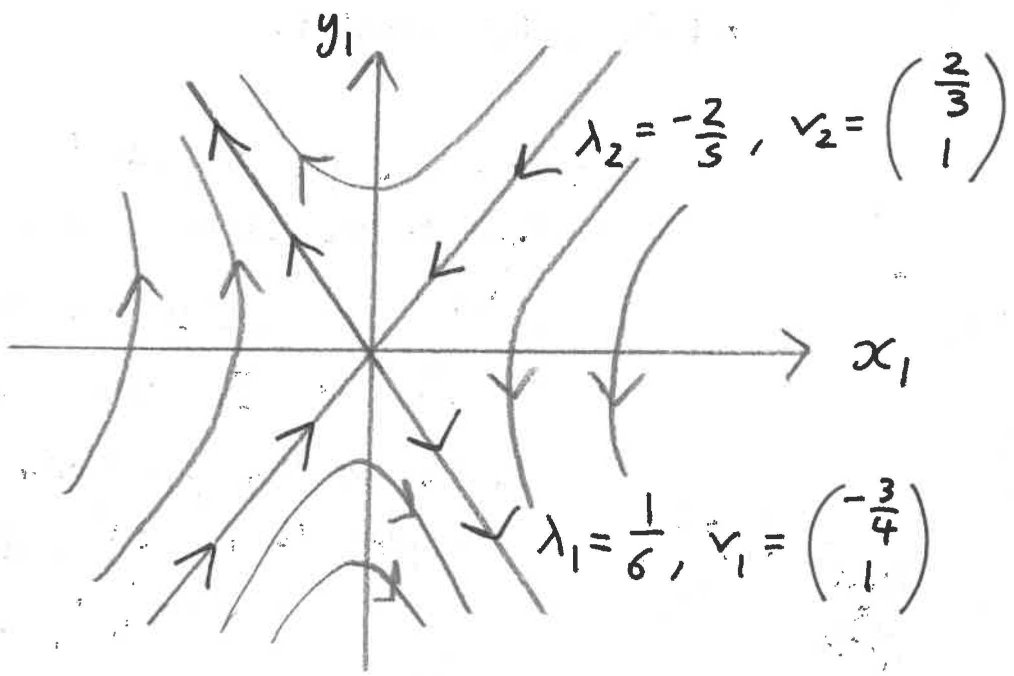
$\Rightarrow 150\lambda^2 + 35\lambda - 10 = 0$

$\Rightarrow 30\lambda^2 + 7\lambda - 2 = 0$

$\Rightarrow \lambda_1 = \frac{1}{6}$ or $\lambda_2 = \frac{-2}{5}$

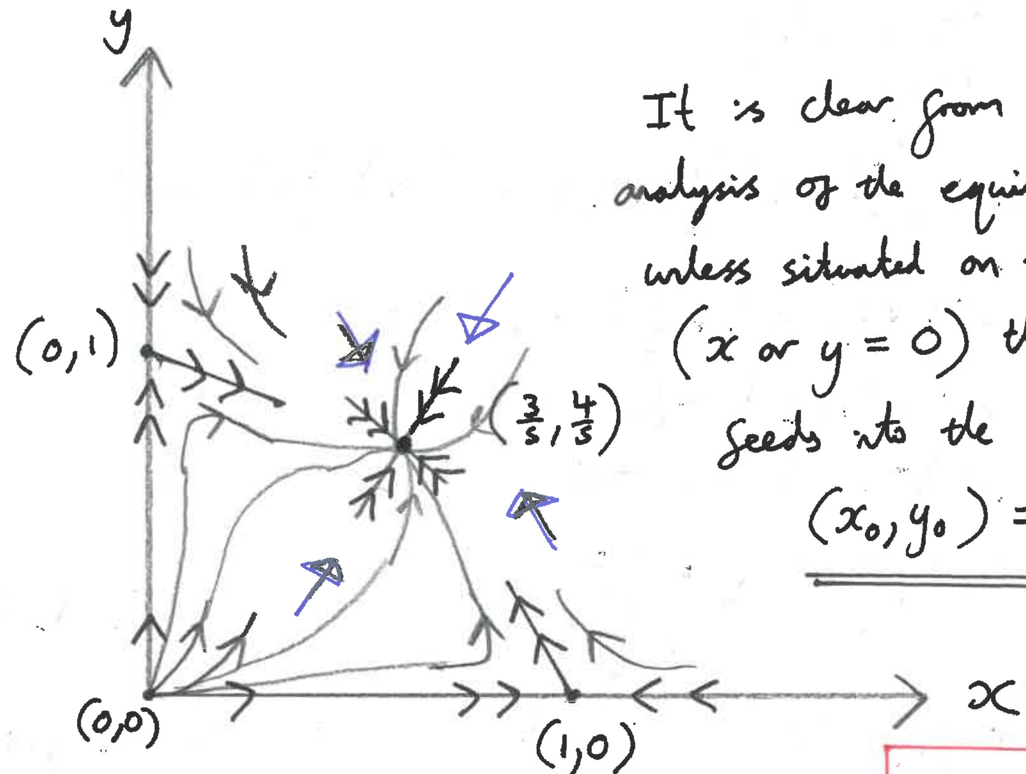
$\lambda_1 > 0, \lambda_2 < 0$

$\Rightarrow v_1 = \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$ $\Rightarrow$ Saddle point



So piecing everything together:

(i). $\alpha = 2, \beta = 3$

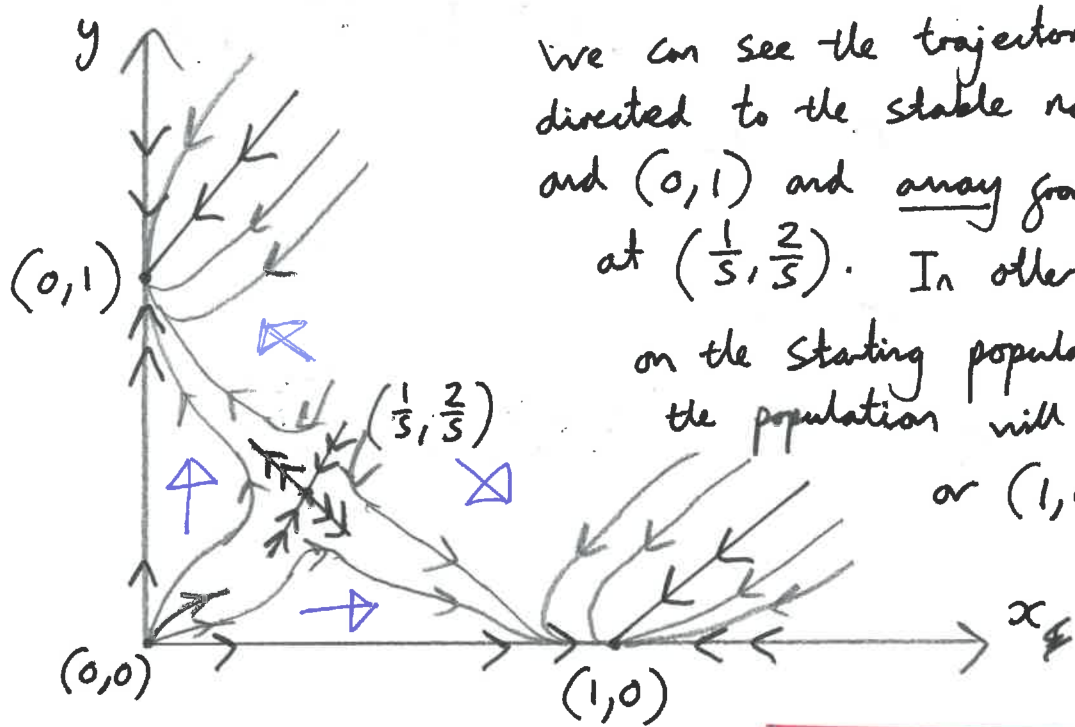


It is clear from our local analysis of the equilibria that unless situated on the boundary (x or $y = 0$) that everything feeds into the point $(x_0, y_0) = (\frac{3}{5}, \frac{4}{5})$.

$\Rightarrow$ Long-run population distribution tends to $\boxed{(x,y) = (\frac{3}{5}, \frac{4}{5})}$

- Both species coexist.

(ii). $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$



We can see the trajectories are directed to the stable nodes at (1,0) and (0,1) and <u>away</u> from the saddle at $(\frac{1}{5}, \frac{2}{5})$. In other words depending on the starting population distribution the population will tend to (1,0) or (1,0).

$\Rightarrow$ Long-run population distribution is $\boxed{(x,y) = (1,0) \text{ or } (0,1)}$

- One species extinct, other thrives.