Cover Sheet for Submission of Maths Examinations Summer 2020

We would advise preparing your coversheets with your CID, Module Name and Code and Date, before the exams are due to take place.

CID: 01738166

Module Name: Analysis 1

Module Code: MATH40002

Date: 04/05/2020

Questions Answered (in the file):

Please tick next to the question or questions you have answered in this file.

Q1	
Q2	✓
Q3	
Q4	
Q5	
Q6	

(Note: this is a coversheet for all students - not all students will have exams with 6 questions. Please tick the boxes which are appropriate for your exam and/or the file you are submitting).

(Optional) Page Numbers for each question;

Page Number	Question Answered

If handwritten, please complete in CAPITAL Letters, in Blue or Black Ink, ensuring the cover sheet is legible.

(a) $n \geq 0$ $\sqrt{1+\frac{1}{n}} \leq 1 + \frac{1}{2n}$

For $a>0$ and $b>0$ we have $a^2 \geq b^2 \iff |a| \geq |b| \iff a \geq b$ (*)

$\sqrt{1+\frac{1}{n}} > 0$ and $1+\frac{1}{2n} > 0 \implies$

$$1+\frac{1}{2n} \geq \sqrt{1+\frac{1}{n}} \iff \left(1+\frac{1}{2n}\right)^2 \geq \left(\sqrt{1+\frac{1}{n}}\right)^2 = 1+\frac{1}{n}$$

We have

$$4n^3 + 4n^2 + n \geq 4n^3 + 4n^2 \quad (\text{as } n \geq 0) \iff$$

$$n(4n^2+4n+1) \geq 4n^3+4n^2 \quad \Big| \cdot \frac{1}{n} \iff$$

$$4n^2+4n+1 \geq \frac{4n^3+4n^2}{n} \iff$$

$$(2n+1)^2 \geq 4n^2\left(\frac{n+1}{n}\right) \quad \Big| \cdot \frac{1}{4n^2} \iff$$

$$\frac{(2n+1)^2}{4n^2} \geq \frac{n+1}{n} \iff \left(1+\frac{1}{2n}\right)^2 \geq 1+\frac{1}{n} \overset{(*)}{\implies} 1+\frac{1}{2n} \geq \sqrt{1+\frac{1}{n}}$$

(b) $\sqrt{n+1} - \sqrt{n} \leq \frac{1}{2\sqrt{n}} \iff$ multiply by $(\sqrt{n+1}+\sqrt{n}) > 0$ both sides:

$$(n+1-n) \leq \frac{\sqrt{n+1}+\sqrt{n}}{2\sqrt{n}} \iff \sqrt{n+1}+\sqrt{n} \geq 2\sqrt{n}$$

$$\iff \sqrt{n+1} \geq \sqrt{n}$$

from (*) we have $n+1 \geq n \implies \sqrt{n+1} \geq \sqrt{n}$, and since $n \geq 0$, $n+1 \geq n$

$$\implies \sqrt{n+1} \geq \sqrt{n} \implies \sqrt{n+1} - \sqrt{n} \leq \frac{1}{2\sqrt{n}}$$

(c)

(i) Denote $s_n := \sum_{i=1}^{n} a_i$

$\exists s : s_n \to s$.

(ii) ~~Suppose it is convergent : $s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$~~

~~Then $s_n \to s$ $\exists s$: $s_n \to s$. Also $s_{n+1} \to s \implies$~~

The partial sum $(a_n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 2\sum_{k=1}^{n} \frac{1}{2\sqrt{k}} \geq 2\sum_{k=1}^{n} \sqrt{k+1} - \sqrt{k}$ from (b)

$= 2\sqrt{n+1} - 2$ as the sum telescopes

We get $(a_n) \geq 2\sqrt{n+1} - 2$.

We have that $b_n = \sqrt{n}$ is unbounded (*) $\implies$ a_n is unbounded

Proof of (*):

Suppose $\sqrt{n}$ is bounded $\Rightarrow \exists B : B \geq \sqrt{n}, \forall n \in \mathbb{N}$

But $(B+1)^2 > B$ and $\sqrt{(B+1)^2} \in (a_n)$ for $\Rightarrow \overset{\circ}{\times}$

(d) Since $a_n \to a$, $\exists N_1 \in \mathbb{N}$ $|a_n - a| < \frac{1-a}{2}$ as $\frac{1-a}{2} > 0$ for $n \geq N_1$,

so $a_n < \frac{a+1}{2} < 1$.

~~So for $n > \max(N_1, N_2$~~

Similarly $\exists N_2 \in \mathbb{N}$: $|a_n - a| < \frac{a}{2}$ for $n \geq N_2$, so $a_n > \frac{a}{2} > 0$

So for $n > \max(N_1, N_2)$, $0 < a_n < \frac{a+1}{2} < 1$. Then

$$0 < a_n^n < \left(\frac{a+1}{2}\right)^n < 1$$

So $\sum_{n=1}^{\infty} a_n^n = \sum_{n=1}^{\max(N_1,N_2)} a_n^n + \sum_{n=\max(N_1,N_2)+1}^{\infty} a_n^n$, is bounded

above by

$\sum_{n=1}^{\max(N_1,N_2)} a_n^n + \sum_{n=\max(N_1,N_2)+1}^{\infty} \left(\frac{a+1}{2}\right)^n$, which obviously

converges as it is the sum of a finite number and a geometric series with $r<1$.

So the partial sums $\sum_{n=1}^{N} a_n^n$ are bounded above and increasing (as $a_n > 0$, $S_{n+1} - S_n = a_{n+1}^{n+1} > 0$, where $S_n = \sum_1^{\infty} a_n^n$), hence convergent.