Math40002 Analysis 1 — Term 2 Unseen 4

1. Find a function $f : [0, 1] \to \mathbb{R}$ such that the following hold:

   - Let $0 \leq a < b \leq 1$ and pick a value c between $f(a)$ and $f(b)$. Then there is some $x \in [a, b]$ such that $f(x) = c$.
   - f is *not* continuous.

2. Let $f : \mathbb{R} \to \mathbb{R}$ such that for every open interval $I \subseteq \mathbb{R}$: $f(I) := \{ f(x) | x \in I \} = \mathbb{R}$.

   (a) Prove that for every $a, b \in \mathbb{R}$ and c between $f(a)$ and $f(b)$. Then there is some $x \in [a, b]$ such that $f(x) = c$.

   (b) prove that f is *nowhere continuous*, i.e., f is not continuous for any $x \in \mathbb{R}$.

   Note: In the question above you proved that the converse of the Intermediate Value Theorem fails miserably for function satisfying that $f(I) = \mathbb{R}$ for every interval I. However, it is not so clear such functions exist!

3. Define $a \sim b$ iff $a - b \in \mathbb{Q}$.

   (a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$ and every equivalence class is countable.

   (b) For every $x \in \mathbb{R}$, let $cl(x)$ be the equivalence class of x with respect to $\sim$. Show that for every $x \in \mathbb{R}$: $cl(x)$ is dense in $\mathbb{R}$, namely, for every interval I, there is some $y \in cl(x) \cap I$.

   (c) Let $\mathcal{P} := \{ cl(x) | x \in \mathbb{R} \}$ be the set of all equivalence classes of $\sim$.
   Show that there is a bijection from $\mathcal{P}$ to $\mathbb{R}$.
   You may need to use the following fact from set theory:

Let $\mathcal{C}$ is an infinite set of countable non-empty sets such that $A \cap B = \emptyset$ for every $A, B \in \mathcal{C}$. Then there is a bijection between $\mathcal{C}$ and $\bigcup_{A \in \mathcal{C}} A$.

   (d) Let $g : \mathcal{P} \to \mathbb{R}$ be a bijection, as promised from Item 3c. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = g(cl(x))$. Prove that $f(I) = \mathbb{R}$ for every interval I. Deduce that f satisfies the intermediate value property, but is nowhere continuous.