

Topic: Discrete random variables and their distributions

In today's problem class we will be studying properties of discrete random variables.

1. Show that the function

$$p_X(x) = \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda} \right)^x$$

for parameter $\lambda > 0$ is a valid probability mass function for a discrete random variable X taking values on $\{0, 1, 2, \dots\}$. Also, find $P(X \leq x)$ for $x \in \mathbb{R}$.

Solution: Need to show p_X non-negative, and sums to one over the range of X . Clearly, p_X is nonnegative, and the sum of geometric progression gives result, that is,

$$\sum_{x=0}^{\infty} p_X(x) = \sum_{x=0}^{\infty} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda} \right)^x = \frac{1}{1+\lambda} \left(1 - \frac{\lambda}{1+\lambda} \right)^{-1} = 1$$

Let $x = 0, 1, \dots$, then

$$P(X \leq x) = \sum_{i=0}^x p_X(i) = \sum_{i=0}^x \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda} \right)^i = (1-\theta) \sum_{i=0}^x \theta^i = (1-\theta) \frac{1-\theta^{x+1}}{1-\theta} = 1 - \theta^{x+1},$$

where $\theta = \frac{\lambda}{1+\lambda}$. For general $x \in \mathbb{R}$, we have

$$P(X \leq x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \theta^{\lfloor x \rfloor + 1}, & \text{if } x \geq 0. \end{cases}$$

We note that $\lfloor x \rfloor$ denotes greatest integer less than or equal to x .

2. For what values of k is the following function a valid probability mass function?

$$p_X(x) = \begin{cases} \frac{k}{x(x+1)} & \text{if } x = n, n+1, n+2, \dots \\ 0 & \text{otherwise} \end{cases}$$

where n is a fixed positive integer.

Hint:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Solution:

$$\begin{aligned} \sum_{x=n}^{\infty} \frac{k}{x(x+1)} &= k \sum_{x=n}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) = k \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \dots \right) \\ &= \frac{k}{n}. \end{aligned}$$

Need $\sum p_X(x) = 1 \Rightarrow k = n$. Also, $p_X(x) \geq 0$ by construction (and also $p_X(x) \leq 1$).

3. If $X \sim \text{Poi}(\lambda)$ and we know that $P(X > 0) = 1 - e^{-0.5}$, determine $P(X \leq 1)$.

Solution: If $X \sim \text{Poi}(\lambda)$ so $p_X(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, 2, \dots$

$$P(X > 0) = 1 - P(X = 0) = 1 - e^{-0.5} \Rightarrow P(X = 0) = e^{-0.5}.$$

$$p_X(0) = P(X = 0) = e^{-\lambda} \Rightarrow \lambda = 0.5$$

$$P(X \leq 1) = p_X(0) + p_X(1) = e^{-0.5} + 0.5e^{-0.5} = 1.5e^{-0.5}.$$

4. If $X \sim \text{Poi}(\lambda)$ find the probability that X is odd.

Solution:

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{2x+1}}{(2x+1)!} = e^{-\lambda} \left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots \right) = e^{-\lambda} \sinh(\lambda).$$

5. If $X \sim \text{Bin}(n, \theta)$, find $g(x)$ such that

$$p_X(x+1) = g(x)p_X(x), \quad x = 0, 1, \dots, n-1.$$

Solution:

$$p_X(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x},$$

$$p_X(x+1) = \binom{n}{x+1} \theta^{x+1} (1-\theta)^{n-x-1}.$$

Note that

$$\binom{n}{x+1} = \frac{n!}{(x+1)!(n-x-1)!} = \frac{n!(n-x)}{(x+1)x!(n-x)!} = \frac{(n-x)}{(x+1)} \binom{n}{x}.$$

So we can write

$$p_X(x+1) = \frac{(n-x)\theta}{(x+1)(1-\theta)} \binom{n}{x} \theta^x (1-\theta)^{n-x}.$$

I.e. when we choose

$$g(x) = \frac{\theta(n-x)}{(1-\theta)(x+1)}, \quad x = 0, 1, 2, \dots, n-1,$$

then the result follows.

6. Let $X \sim \text{Bin}(n, p)$ and let $q = 1 - p$. Show that $Y = n - X \sim \text{Bin}(n, q)$.

Solution: For $y \in \{0, 1, \dots, n\}$, we have

$$\begin{aligned} p_Y(y) &= P(Y = y) = P(n - X = y) = P(X = n - y) = \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)} \\ &= \binom{n}{y} q^y (1-q)^{n-y}, \end{aligned}$$

since

$$\binom{n}{n-y} = \frac{n!}{(n-y)!(n-(n-y))!} = \frac{n!}{(n-y)!y!} = \binom{n}{y}.$$

So the p.m.f. of Y has the functional form of the p.m.f. of a $\text{Bin}(n, q)$ random variable.

7. A fair coin is tossed n times. Let X be the discrete random variable corresponding to the difference between the number of heads and the number of tails observed. Find the image/range and probability mass function of X .

Solution: Let U = “Number of Heads”, V = “Number of Tails”. Then $V = n - U$ and the difference between the number of heads and number of tails is given by $X = U - V = U - (n - U) = 2U - n$. Thus

$$\text{Im}X = \{-n, -n+2, -n+4, \dots, n-4, n-2, n\}$$

and

$$p_X(x) = P(X = x) = P(2U - n = x) = P(U = (x + n)/2) = p_U((x + n)/2)$$

where $U \sim \text{Bin}(n, 1/2)$.