Math40002 Analysis 1 **Unseen 2**

1. Say $S \subseteq \mathbb{R}$ is a set of real numbers, with the property that $\forall s \in S, \exists t \in S, s < t$. Can S be bounded above?

2. Say $S \subseteq \mathbb{R}$ is a set of real numbers, with the property that $\forall n \in \mathbb{N}, \exists t \in S, n < t$. Can S be bounded above?

3. (a) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded above. Assume:
$$\forall a \in A, \exists b \in B \text{ such that } a \leq b.$$
Prove $\sup A \leq \sup B$.
   (b) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded above.
Prove that if $A \subseteq B \subset \mathbb{R}$ then $\sup A \leq \sup B$.
   (c) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded below. Assume:
$$\forall a \in A, \exists b \in B \text{ such that } a \geq b.$$
Prove $\inf A \geq \inf B$.
   (d) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded below.
Prove that if $A \subseteq B \subset \mathbb{R}$ then $\inf A \geq \inf B$.

4. Say we have a sequence of real numbers $a_1, a_2, a_3, \ldots$. Assume:
$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N} : a_n < M$$
$$\exists m \in \mathbb{R} \text{ such that } \forall N \in \mathbb{N}, \exists n > N : m < a_n$$
Now let's define some sets $S_1, S_2, S_3, \ldots$ by
$$S_n = \{ a_{n+1}, a_{n+2}, a_{n+3} \ldots \}.$$
   (a) Prove that for all $n \geq 1$, there exists some $b_n \in \mathbb{R}$ such that $b_n = \sup(S_n)$.
   (b) Prove there exists some $l \in \mathbb{R}$ such that $l = \inf \{ b_1, b_2, b_3, \ldots \}$.
Such l is called the limpsup of the sequence $(a_1, a_2, a_3, \ldots)$
   (c) Find the limsup of the following sequence and prove your findings.
      i. $1, 1, 1, \ldots$
      ii. $0, 1, 0, 1, 0, 1, \ldots$
      iii. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
      iv. $-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots$
   (d) Prove:
$$\inf \{ a_1, a_2, a_3, \ldots \} \leq \limsup(a_1, a_2, a_3, \ldots) \leq \sup \{ a_1, a_2, a_3, \ldots \}.$$

(e) Let $X = \mathbb{Q} \cap (0,1)$ and let $f : \mathbb{N} \to X$ be a bijection. For every $i \in \mathbb{N}$, let $a_i = f(i)$. Prove $\limsup(a_1, a_2, a_3, \dots) = 1$.[1]

(f) If you like, then guess the definition of *liminf* and compute it for the examples above.

Which of these sequences converges? (we will see a rigorous definition of this notion next week, but perhaps you know what it means).

Can you tell just from looking at the limsup and liminf?

[1] Hint for (d): To prove $b_n = 1$, show that for every $c < 1$, $X \cap (c, 1)$ is infinite, therefore $S_n \cap (c, 1)$ is infinite.