Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 4 with solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1.* **Recap** — The following second order differential equation describes the time evolution of the linear elongation $x(t)$ of a damped harmonic oscillator

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0,$$

where k and ω are positive constants representing the damping of the medium and the intrinsic frequency of the system, respectively. Rewrite this equation as a system of two coupled linear first order ODEs and find the solution in terms of the eigenvalues and eigenvectors of the system.

This problem is done as a an example in the lectures. By defining $u = \frac{dx}{dt}$, we can convert this second order linear ODE into a sysem of 2 first oder linear ODEs:

$$\frac{d}{dt}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2k \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix}$$

The eigen values are

$$\lambda_{1,2} = -k \pm \sqrt{k^2 - \omega^2}$$

And the eignvectores are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}; \quad \vec{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ u \end{pmatrix} = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

2. Consider systems of two linear ODEs with constant coefficients given by:

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and } A \text{ is a } 2\times 2 \text{ matrix.}$$

Find the general solution of the following systems:

(a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = -1 \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 2 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(b) $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = -1 \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = \frac{1}{2} \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 2 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\frac{1}{2}t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(d) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = 1 + 2i \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$$

$$\lambda_2 = 1 - 2i \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}.$$

Defining $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$ as new real constants of integration, we can write the general solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 e^{t} \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + A_2 e^{t} \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}.$$

(e) $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = -1 + 2i \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$\lambda_2 = -1 - 2i \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 2 \\ i \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(-1+2i)t} \begin{pmatrix} 2 \\ -i \end{pmatrix} + c_2 e^{(-1-2i)t} \begin{pmatrix} 2 \\ i \end{pmatrix}.$$

Defining $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$ as new real constants of integration, we can write the general solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} + A_2 e^{-t} \begin{pmatrix} 2\sin 2t \\ -\cos 2t \end{pmatrix}.$$

(f) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = i \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 5 \\ 2 - i \end{pmatrix}$$

$$\lambda_2 = -i \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 5 \\ 2 + i \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{it} \begin{pmatrix} 5 \\ 2 - i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 5 \\ 2 + i \end{pmatrix}.$$

Defining $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$ as new real constants of integration, we can write the general solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + A_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

3. Find the solution for the following inhomogeneous system of ODEs with $x(0) = y(0) = 0$:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix}.$$

First we find the general solution to the corresponding homogenous ODE. Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = 1 \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 4 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then looking for a PI of the form $\begin{pmatrix} a \\ b \end{pmatrix}$, we have:

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \implies \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Given $x(0) = y(0) = 0$ we have $c_1 + c_2 = -2$ and $-2c_1 + c_2 = 1$, which results in $c_1 = c_2 = -1$.

4. * Find the general solution for the following system of ODEs in terms of its eigenvalues and eigenvectors:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Characterise the asymptotic behavior of the system.

Obtaining eigenvalues using the charactersitic equation:

$$-\lambda(\lambda^2 - 1) + (\lambda + 1) + (\lambda + 1) = 0 \quad \implies \quad \lambda_{1,2} = -1 \text{ (repeated) and } \lambda_3 = 2$$

$$\lambda_3 = 2 \quad \implies \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_{1,2} = -1 \quad \implies \quad v_{1x} + v_{1y} + v_{1z} = 0 \quad \implies \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So even though we have a repeated eigenvalue, there are 3 linearly independent eigenvalues and the matrix A is diagonalisable (no need to use Jordan normal form). So we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$