Problem Sheet 5

Math40002, Analysis 1

1. In lecture, we needed the claim that $\lim_{x\to\infty} xs^{x-1} = 0$ for any $s \in (0,1)$ in order to prove that the term-by-term derivative of a power series converges inside that power series's radius of convergence.

   (a) Prove that for all $c > 0$, there exists $N > 0$ such that $\log(x) < cx$ for all $x \geq N$.

   (b) Prove that $\lim_{x\to\infty} xs^x = 0$, and show that this implies the above claim.

2. (a) Compute the Taylor series $P(x)$ of $f(x) = \log(1+x)$ centered at $x = 0$, and prove that it converges absolutely on $(-1,1)$.

   (b) Prove using Taylor's theorem that $f(x) = P(x)$ on some open neighborhood of 0, by showing that the sequence of nth order Taylor polynomials $P_n(x)$ converges uniformly to $f(x)$. Show that the same is true at $x = 1$, and so $\log(2) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has at least six continuous derivatives, and that $f^{(i)}(0) = 0$ for $i = 1,2,3,4,5$ but $f^{(6)}(0) = 1$. Prove that $f(x)$ has a local minimum at $x = 0$.

4. (a) Suppose that some function $f : (-R,R) \to \mathbb{R}$ is equal to the power series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, which converges absolutely on $(-R,R)$. Prove that the Taylor series of f centered at $a = 0$ is precisely $\displaystyle\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, and hence that this power series is unique.

   (b) Compute the Taylor series of $f(x) = \frac{1}{1-x^2}$ centered at $a = 0$. What is $f^{(100)}(0)$?

5. (a) Prove that $f(x) = e^x$ is convex on all of $\mathbb{R}$.

   (b) Let $a, b > 0$. Use the convexity of e^x to prove the *arithmetic mean–geometric mean inequality*
   $$\frac{a+b}{2} \geq \sqrt{ab}.$$
   (Hint: think about $\alpha = \log(a)$ and $\beta = \log(b)$.)

   (c) Prove for any $a, b > 0$ and $s \in [0,1]$ that $sa + (1-s)b \geq a^s b^{1-s}$.

   (d) Prove *Young's inequality*: for any $x, y \geq 0$ and p, q positive with $\frac{1}{p} + \frac{1}{q} = 1$, we have
   $$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$

6. (*) Let (a_n) denote the Fibonacci sequence, with $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$.

   (a) Prove by induction that $a_n < 2^n$ for all $n \geq 0$. What is the radius of convergence of the *exponential generating function*

   $$F(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0 + 1x + \frac{1x^2}{2} + \frac{2x^3}{6} + \frac{3x^4}{24} + \ldots ?$$

   (b) Prove that $F''(x) = F'(x) + F(x)$, and that $F(0) = 0$ and $F'(0) = 1$.

   (c) Solve this differential equation for $F(x)$.

   (d) Use the solution from part (c) to prove *Binet's formula*:

   $$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

7. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

   (a) Prove that for all integers $n \geq 0$, there is a polynomial $p_n(x)$ such that

   $$f^{(n)}(x) = \frac{p_n(x)}{x^{3n}} e^{-1/x^2} \text{ for all } x \neq 0.$$

   (b) Prove that $f^{(n)}(0) = 0$ for all n, and hence that $f(x)$ does not equal its Taylor series (centered at $a = 0$) at any nonzero x.

   (c) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0. \end{cases}$ Prove that $g^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $g^{(n)}(0) = 0$ for all n.

   (d) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x)g(1-x)$. Prove that h is infinitely differentiable, meaning that $h^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $h(x) \neq 0$ if and only if $0 < x < 1$.