Imperial College London

MATH40004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Calculus and Applications

Date: 12th May 2020

Time: 09.00am – 12.00 noon (BST)

Time Allowed: 3 Hours

Upload Time Allowed: 30 Minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Give definitions for the statements: (i) $f(x)$ is continuous at $x = x_0$, and (ii) $f(x)$ is differentiable at the point $x = x_0$ and has derivative m there. (4 marks)

(b) Consider the function

$$f(x) = \begin{cases} (\sin x^2)\,\sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is the function continuous at $x = 0$? Is it differentiable at $x = 0$? (4 marks)

(c) A particle trajectory is given by $x = t\cos t$, $y = t\sin t$ for $t \geq 0$.

(i) Find $\frac{dy}{dx}$ leaving your answer in terms of the parameter t. Show that $\frac{dy}{dx} = 0$ at $x = y = 0$. (2 marks)

(ii) Find all points where the trajectory crosses the $x-$axis, and all points where it crosses the $y-$axis. (2 marks)

(iii) Show that for $t > 0$ the trajectory has zero slope when t is such that $\tan t = -t$. Sketch the functions $g_1(t) = \tan t$ and $g_2(t) = -t$ on the same graph and use this to identify the points of zero slope. How many of them are there? Repeat to find points where the tangent is vertical, identify them graphically as above, and state how many there are. (5 marks)

(iv) Sketch the particle trajectory in the $x - y$ plane for $t \geq 0$. (3 marks)

(Total: 20 marks)

2. (a) Consider the improper integral $I = \int_0^\infty \frac{e^{-x}}{x^p} dx$ where $p > 0$. Determine for what values of p the integral converges. (You are not required to calculate the integral.) (6 marks)

(b) Use integration by parts to show that for $b > a > 0$

$$\int_a^b \cos(x^2) dx = \frac{1}{2} \left(\frac{\sin b^2}{b} - \frac{\sin a^2}{a} + \int_a^b \frac{\sin x^2}{x^2} dx \right).$$

Use this result to determine whether the integral $\int_0^\infty \cos(x^2) dx$ exists.

(6 marks)

(c) (i) Sketch the graph of the function $f(x) = 1 - x^2$ and shade the region A bounded by $f(x)$ and the axes $x \geq 0$, $y \geq 0$. (2 marks)

(ii) Calculate the area of A. (2 marks)

(iii) Find the centre of mass of A. (4 marks)

(Total: 20 marks)

3. (a) (i) Write down the $n-$term Taylor series with remainder about $x = 0$, for the function $f(x)$. (3 marks)

(ii) Find a rational approximation to $\sqrt{e}$ which has accuracy better than 1%. (3 marks)

(iii) Find

$$\lim_{x \to 0} \frac{e^{\sin x} - x - 1}{1 - \cos x}.$$

(4 marks)

(b) Consider the function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0 \\ -1 & -\pi < x < 0 \end{cases}$$

(i) Extend the function periodically for all x and sketch it in the interval $-3\pi \leq x \leq 3\pi$. (3 marks)

(ii) Find the Fourier series of the function in part (b)(i). (5 marks)

(iii) Use your answer to part (b)(ii) above to show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}.$$

(2 marks)

(Total: 20 marks)

4. (a) Solve the following ODE

$$y^2 \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3$$

with

$$y(0) = 1 \quad \text{and} \quad \frac{dy}{dx}(0) = -1.$$

(6 marks)

(b) Find the Fourier transform of $e^{-a|x-x_0|}$, where a and x_0 are real constants and $a > 0$.

(5 marks)

(c) Consider the following nonhomogenous linear ODE with a Dirac delta function on the right hand side and with $\omega_0 > 0$

$$-\frac{d^2y}{dx^2} + \omega_0^2\, y = \delta(x - x_0). \tag{1}$$

(i) Find the Complementary Function for this ODE.

(3 marks)

(ii) To find a particular integral (y_{PI}) for this problem, take Fourier transforms from both side of the equation (1) and use inverse Fourier transform and the result in part (b). Note that as this approach uses Fourier transforms, it obtains solutions that go to zero for large x ($y_{PI} \to 0$ as $|x| \to \infty$).

(6 marks)

(Total: 20 marks)

5. (a) Consider the following system of coupled linear differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

where the variable t corresponds to time.

(i) Find the general solution of this system in terms of its eigenvectors and eigenvalues.

(5 marks)

(ii) Evaluate the vector field on the $y = -x$ line. Sketch the phase portrait for this system in the (x, y) plane and describe the stability of the fixed point. Using the phase portrait or otherwise describe the asymptotic behaviour of the system as $t \to \infty$ starting from $(-3, 6)$.

(5 marks)

(iii) Now consider the non-homogenous system of coupled linear differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} e^{t} \\ e^{-3t} \end{pmatrix}.$$

Find a particular integral using an appropriate Ansatz for this equation.

(3 marks)

(b) Consider the following first order nonlinear ODE

$$\frac{dy}{dt} = ry - e^{y} + 1 \tag{2}$$

where $r \in \mathbb{R}$ is a parameter.

(i) By sketching the functions ry and $e^{y} - 1$ on the same diagram, or otherwise, identify the number of fixed points y^{*} and with arrows the trajectories along the y-axis for different values of r. Determine the stability of the fixed points.

(4 marks)

(ii) Sketch the bifurcation diagram (the possible values of y^{*} against r). Identify the bifurcation point(s) and classify them.

(3 marks)

(Total: 20 marks)

6. (a) Consider the equation

$$e^{2x+2z} - x^2 + \ln y^3 = 0$$

Evaluate the following partial derivatives and check the symmetry of the second order mixed partial derivatives:

$$\frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x}.$$

(8 marks)

(b) Consider the following ODE

$$\frac{d^3 y}{dx^3} + 6\frac{d^2 y}{dx^2} + 12\frac{dy}{dx} + 8y = xe^{-2x}.$$

(i) Find the complementary function to the corresponding homogenous ODE.

(4 marks)

(ii) Use the method of variation of parameter to obtain a particular integral and hence write the general solution.

(5 marks)

(iii) Rewrite this third order ODE as an equivalent system of coupled first oder ODEs.

(3 marks)

(Total: 20 marks)