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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2020

MATH40002 Analysis I

The following information must be completed:

Is the paper suitable for resitting students from previous years: No (new course)

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a)(i,iv) 2 marks, 1(c) 5 marks, 1(d)(i,ii,iii) 3 marks, 2(a)(b) 4 marks, 2(c)(i) 2 marks, 3(a)(i)(ii) 5 marks, 3(b)(i) 2 marks, 4a(i) 3 marks, 4b(i,ii) 5 marks, 5a(i) 2 marks, 5b(i) 3 marks, 5b(ii) 2marks, 6a(i,ii,iii) 3 marks, 6b(i,ii) 6 marks

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(a)(ii,iii) 2 marks, 1(b) 4 marks, 2(c)(ii) 7 marks, 3(a)(iii) 2 marks, 4a(ii) 2 marks, 4b(iii) 3 marks, 5a(ii) 3 marks, 5a(iii) 4 marks, 6b(iii) 3 marks

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(d) 7 marks, 3(b)(ii) 5 marks, 4c(ii) 3 marks, 5b(iii) 2 marks, 6a(iv) 1 mark

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(d) 4 marks, 3(b)(iii) 6 marks, 4c(i) 4 marks, 5(c) 4 marks, 6a(v) 2 marks, 6(c) 5 marks

Signatures are required for the final version:

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Analysis I

Date: ??

Time: ??

Time Allowed: 3 Hours

This paper has *6 Questions*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. In this question there is no need to give proofs, except where you are explicitly told to do so.

(a) Define what it means for a set S to be countably infinite. (1 mark)

Let S be the set of all sequences $(a_n)_{n\geq 1}$ with a_n in the set with two elements $\{0,1\}$.
Is S finite, countably infinite or uncountable? (1 mark)

Let $T := \Big\{(a_n)_{n\geq 1} \in S \;:\; \exists N \in \mathbb{N}$ such that $a_n = 0 \;\forall n \geq N\Big\}$ be the subset of sequences which are eventually zero. Is T finite, countably infinite or uncountable? (1 mark)

Fix $N \in \mathbb{N}$ and set $U := \Big\{(a_n)_{n\geq 1} \in S \;:\; a_n = 0 \;\forall n \geq N\Big\}$.
Is U finite, countably infinite or uncountable? (1 mark)

(b) Suppose $S \subset \mathbb{R}$ is nonempty and $A \in \mathbb{R}$ satisfies

$$\Big(\forall \epsilon > 0 \;\exists\, s_1 \in S \text{ such that } s_1 < A + \epsilon\Big) \;\; \textbf{and} \;\; \Big(\forall \epsilon > 0 \;\; \nexists\, s_2 \in S \text{ such that } s_2 < A - \epsilon\Big).$$

Either prove that $\sup S,\ \max S,\ \inf S$ or $\min S$ exists and equals A, or give an example where none of these is true. (4 marks)

(c) For each of the following statements about a real valued sequence $(a_n)_{n\geq 1}$,

(i) $\forall N \in \mathbb{N}\;\exists\, a \in \mathbb{R}$ such that $\forall \epsilon > 0,\ \forall n \geq N,\ |a_n - a| < \epsilon$

(ii) $\forall a \in \mathbb{R},\ \exists\, n \in \mathbb{N}$ such that $\forall \epsilon > 0,\ |a_n - a| < \epsilon$

(iii) $\exists\, a \in \mathbb{R}$ such that $\forall N \in \mathbb{N}\;\forall \epsilon > 0\;\exists\, n \geq N$ such that $|a_n - a| < \epsilon$

(iv) $\exists\, \epsilon > 0\;\exists\, a \in \mathbb{R}\;\exists\, N \in \mathbb{N}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$

(v) $\forall \epsilon > 0\;\exists\, N \in \mathbb{N}\;\exists\, a \in \mathbb{R}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$

state which of the following (A)-(E) it is equivalent to:

(A) Bounded (B) Has a convergent subsequence (C) Convergent
(D) Constant (E) Impossible (5 marks)

(d) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers.

Define what it means for a_n to be convergent. (1 mark)

Define what it means for a_n to be divergent. (1 mark)

Give an example of a divergent sequence (a_n) that satisfies

$$\forall \epsilon > 0\;\exists\, N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_{n+1} - a_n| < \epsilon. \qquad (*)$$ (1 mark)

Suppose now that (a_n) satisfies $(*)$ and $a_{2n} \to a$. Prove carefully that $a_n \to a$. (4 marks)

(Total: 20 marks)

2. In this question you should work from first principles, carefully proving anything you use.

(a) For $n \geq 0$ show that $\sqrt{1 + \frac{1}{n}} \leq 1 + \frac{1}{2n}$. (2 marks)

(b) For $n \geq 0$ show that $\sqrt{n+1} - \sqrt{n} \leq \frac{1}{2\sqrt{n}}$. (2 marks)

(c) (i) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Define what it means for $\sum_{n=1}^{\infty} a_n$ to converge. (2 marks)

(ii) Prove carefully that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is not convergent. You should work from first principles, proving any results that you use. You may use part (b) if you wish. (7 marks)

(d) Suppose $a_n \geq 0 \ \forall n$ and converges to $a \in [0, 1)$. Prove $\sum_{n=1}^{\infty} a_n^n$ converges.
(You may assume that a sequence of real numbers converges if it is bounded above and monotonically increasing, but anything else you use should be proved from first principles.) (7 marks)

(Total: 20 marks)

3. In this question you may use any results from the course that you state correctly.
Let $(a_n)_{n\geq 1}$ be a sequence of real numbers.

(a) (i) Suppose $a_n \to a > 0$. Show $\exists\, N \in \mathbb{N}$ such that $a_n > 0 \ \forall n \geq N$. (2 marks)

(ii) Fix $k \in \mathbb{N}$ and define $b_n := \sqrt[k]{a_{n+1}a_{n+2}\dots a_{n+k}}$ for $n \geq N$. Prove $b_n \to a$. (3 marks)

(iii) Give without proof an example of a divergent sequence $a_n > 0$ for which b_n is convergent for some k. (2 marks)

(b) (i) State the alternating series test. (2 marks)

(ii) Let

$$a_n = \begin{cases} \frac{1}{n^2} + \frac{1}{n} & n \text{ even}, \\ -\frac{1}{n^2} & n \text{ odd}. \end{cases}$$

Is $\sum a_n$ convergent? Prove your answer carefully. (5 marks)

(iii) What is the radius of convergence of $\sum a_n z^n$? Prove your answer. (6 marks)

(Total: 20 marks)

4. (a) (i) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^3}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$. (3 marks)

(ii) Prove that f is continuous at all $x \neq 0$ as well. (2 marks)

(b) (i) State the intermediate value theorem. (2 marks)

(ii) Prove that there is some $x \in \mathbb{R}$ such that $4^x = \cos(x) + 2$. (3 marks)

(iii) Suppose that we have a continuous function $f : [0, 10] \to \mathbb{R}$, with $f(2k) = 1$ for integers $0 \leq k \leq 5$ and $f(2k+1) = -1$ for integers $0 \leq k \leq 4$. What is the least number of zeroes such an f must have? (3 marks)

(c) (i) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the following property:

For all $x \in [0, 1]$, there exists $y \in [0, 1]$ such that $|f(y)| \leq 0.99|f(x)|$.

Prove that there is some $t \in [0, 1]$ such that $f(t) = 0$. (4 marks)

(ii) Give an example (no proof necessary) of a discontinuous $f : [0, 1] \to \mathbb{R}$ with the same property as in part (i), such that $f(x) \neq 0$ for all $x \in [0, 1]$. (3 marks)

(Total: 20 marks)

5. In this question you may use any results from the course that you state correctly.

(a) (i) State the mean value theorem. (2 marks)

(ii) Deduce that $(1 + x)^r \geq 1 + rx$ for any real $x > 0$ and real $r \geq 1$. (3 marks)

(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $\lim_{x\to\infty} f(x)$ exists. Prove that if $\lim_{x\to\infty} f'(x)$ exists then it is zero. (4 marks)

(b) (i) Compute the second-order Taylor polynomial for $f(x) = -\log(\cos(x))$ centered at $x = 0$. (3 marks)

(ii) Prove that $f(x)$ is convex on the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. (2 marks)

(iii) Prove that $\cos(\frac{\pi}{3})\cos(\frac{\pi}{5}) \leq \Big(\cos(\frac{4\pi}{15})\Big)^2$. (2 marks)

(c) Let $f : (-1, 1) \to \mathbb{R}$ be an infinitely differentiable function satisfying $|f^{(n)}(x)| \leq (n-1)!$ for all $n \geq 1$ and all x. If $P_n(x)$ is the nth order Taylor polynomial for f, centered at $x = 0$, prove that $P_n \to f$ uniformly. (4 marks)

(Total: 20 marks)

6. (a) Which of the following are true? Give a one-sentence explanation or a counterexample for each.

All functions below have the form $[0,1] \to \mathbb{R}$ and are **bounded**.

(i) If f and g are both integrable and $f(x) \leq g(x)$ for all $x \in [0,1]$, then

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$$

if and only if $f(x) = g(x)$ for all $x \in [0,1]$. (1 mark)

(ii) If f and g are differentiable and $f(x)g'(x)$ is integrable, then so is $f'(x)g(x)$. (1 mark)

(iii) If e^f is integrable, then f is integrable. (1 mark)

(iv) If f_n is integrable for all $n \in \mathbb{N}$, and if $f(x) = \lim_{n\to\infty} f_n(x)$ exists and is continuous, then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

(1 mark)

(v) If f and g are integrable, then $\max(f(x), g(x))$ is integrable. (2 marks)

(b) (i) State a version of the fundamental theorem of calculus. (3 marks)

(ii) Define $f : [0,n] \to \mathbb{R}$ by $f(x) = \frac{1}{\sqrt{x+1}}$ for some $n \in \mathbb{N}$. Determine the lower and upper Darboux sums for f with respect to the partition $P = (0, 1, 2, \ldots, n)$; you can leave your answers in the form of a finite sum. (3 marks)

(iii) Give an integer estimate for $U(f,P)$ in the case $n = 10^6 - 1 = 999,999$. Your estimate need not be the closest integer to $U(f,P)$, but you should prove that it differs from the actual value by at most 1. (3 marks)

(c) Suppose that $f, g : [0,1] \to \mathbb{R}$ are integrable, and that $f(x) = g(x)$ for all rational $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

(5 marks)

(Total: 20 marks)

Solutions

1. (a) There exists a bijection $f\colon \mathbb{N} \to S$. (1 mark)

   S is uncountable. T is countably infinite. U is finite (with 2^{N-1} elements). (3 marks)

   (b) $A = \inf S$. (1 mark)

   Firstly A is a lower bound for S: if $S \ni s < A$ then set $\epsilon = \frac{1}{2}(A-s) > 0$ so $s = A-2\epsilon < A-\epsilon$, contradicting the second condition.

   Secondly if $B > A$ then set $\epsilon = B-A > 0$. By the first condition we find $S \ni s < A+\epsilon = B$ so B is not a lower bound. So A is the greatest lower bound. (3 marks)

   (c) (i) (D) constant (1 mark)

   (ii) (E) impossible (1 mark)

   (iii) (B) has a convergent subsequence (1 mark)

   (iv) (A) bounded (1 mark)

   (v) (C) convergent (1 mark)

   (d) $\exists a \in \mathbb{R}$ such that $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$. (1 mark)

   a_n is not convergent. (1 mark)

   $a_n = \sqrt{n}$ or $a_n = \sum_{i=1}^{n} \frac{1}{i}$ or ... (1 mark)

   Fix $\epsilon > 0$. We know $(*)$ and we know $\exists N' \in \mathbb{N}$ such that $\forall n \geq N',\ |a_{2n} - a| < \epsilon$. $(**)$

   Set $N'' := \max(N, 2N') + 1$ and take any $n \geq N''$.

   If $n = 2m$ is even then $m \geq \frac{1}{2}N'' > N'$ so by $(**)$ applied to m we see $|a_{2m}-a| = |a_n-a| < \epsilon$.

   If $n = 2m+1$ is odd then $2m+1 \geq N'' \geq 2N'+1$ so $m \geq N'$ so by $(**)$ applied to m we see $|a_{2m} - a| < \epsilon$ $(\dagger)$. Furthermore $n-1 \geq N''-1 \geq N$ so by $(*)$ applied to $n-1 = 2m$ we find $|a_n - a_{2m}| < \epsilon$ which by $(\dagger)$ and the triangle inequality gives $|a_n - a| < 2\epsilon$.

   So for both even and odd $n \geq N''$ we have shown $|a_n - a| < 2\epsilon$ so $a_n \to a$. (4 marks)

(Total: 20 marks)

2. **No marks in (a), (b) if get logical implications the wrong way round. “If" $\neq$ “only if"!**

(a) Suppose $\sqrt{1+\frac{1}{n}} > 1+\frac{1}{2n}$. Since both sides are positive we can square them to get the contradiction $1+\frac{1}{n} > 1+\frac{1}{n}+\frac{1}{4n^2}$. (2 marks)

(b) **Either** multiply (a) by $\sqrt{n}$ **or** use the trick of completing the square: $\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$. (2 marks)

(c) (i) It means that the sequence of partial sums $s_n := \sum_{i=1}^n a_i$ converges. (2 marks)

(ii) By (b) the partial sums are

$$s_n \;=\; \sum_{i=1}^n \frac{1}{\sqrt{i}} \;\geq\; 2\sum_{i=1}^n \left(\sqrt{i+1}-\sqrt{i}\right) \;=\; 2(\sqrt{n+1}-\sqrt{1}),$$

so they diverge. (4 marks)

Proof of this last claim from first principles: suppose $s_n \to s$. Then taking $\epsilon = 1$ we see $\exists N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|s_n - s| < 1 \implies s+1 > s_n \geq 2\sqrt{n+1}-2. \qquad (*)$$

Choose $n \geq \max(N, (s+1)^2)$. Then we get $s+1 > 2\sqrt{n+1}-2 \geq 2(s+2)-2 = 2(s+1)$ which implies $s+1 < 0$ which contradicts $(*)$. (3 marks)

(d) Pick $0 < \epsilon < \frac{1}{2}(1-a)$. Then $\exists N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|a_n - a| < \epsilon \implies a_n < a+\epsilon < A,$$

where $A := \frac{1}{2}(1+a) < 1$. Therefore, for $n \geq N$, the partial sums satisfy

$$s_n \;=\; \sum_{i=1}^n a_i^i \;\leq\; C + \sum_{i=N}^n A^i \;=\; C + \frac{A^N - A^{n+1}}{1-A} \;\leq\; C + \frac{A^N}{1-A}\,,$$

where $C = \sum_{i=1}^{N-1} a_i$. Therefore (s_n) is bounded above and monotonically increasing, so convergent. (7 marks)

(Total: 20 marks)

3. (a) (i) Set $\epsilon = a > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \epsilon$. In particular, $a_n > a - \epsilon = 0$. (2 marks)

(ii) Fix any $\epsilon \in (0, a)$ (to ensure $a - \epsilon > 0$). Then $\exists N' \in \mathbb{N}$ such that $\forall n \geq N'$, $|a_n - a| < \epsilon$. Thus for $n \geq \max(N, N')$ and $1 \leq i \leq k$ we have $a_n > 0$ and

$$a - \epsilon < a_{n+i} < a + \epsilon \implies \sqrt[k]{(a-\epsilon)^k} < b_n < \sqrt[k]{(a+\epsilon)^k} \implies a - \epsilon < b_n < a + \epsilon,$$

which is $|b_n - a| < \epsilon$. (3 marks)

(iii) E.g. $k = 2$ and $a_n = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$ is divergent but $b_n \equiv \sqrt{2} \to \sqrt{2}$. (2 marks)

(b) (i) If $a_n \downarrow 0$ (i.e. $a_n \geq 0$ is monotonically decreasing and tends to 0) then $\sum_{n+1}^{\infty} (-1)^n a_n$ is convergent. (2 marks)

(ii) The example given is not convergent. (The alternating series test does not apply as $|a_n|$ is not monotonic.) (1 mark)

By the alternating series test, the partial sums s_n of $\sum \frac{(-1)^n}{n^2}$ converge. So if the partial sums σ_n of $\sum a_n$ also converge then by the algebra of limits $s_n - \sigma_n$ converges, i.e. the partials sums of $\sum_{n \text{ even}} \frac{1}{n} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m}$ converge, but we know they do not. (4 marks)

(iii) The radius of convergence is $R = 1$. (1 mark)

Beware applying the ratio test naively will **not** give the result! Instead, applying it to the sum over even n shows $\sum a_{2n} z^{2n}$ converges (absolutely) for $|z| < 1$ because

$$\left| \frac{a_{2n+2} z^{2n+2}}{a_{2n} z^{2n}} \right| = \frac{2 + \frac{3}{n}}{(1 + \frac{1}{n})^2 (2 + \frac{1}{n})} |z|^2 \longrightarrow |z|^2 < 1.$$

Similarly $\sum a_{2n+1} z^{2n+1}$ converges (absolutely) for $|z| < 1$ because

$$\left| \frac{a_{2n+1} z^{2n+1}}{a_{2n-1} z^{2n-1}} \right| = \frac{(2 - \frac{1}{n})^2}{(2 + \frac{1}{n})^2} |z|^2 \longrightarrow |z|^2 < 1.$$

Therefore, by the algebra of limits, $\sum a_n z^n$ converges for $|z| < 1$, so $R \geq 1$. (4 marks)

But we already noted that $\sum a_n z^n$ diverges for $z = 1$ so $R \leq 1$, so $R = 1$. (1 mark)

(Total: 20 marks)

4. (a) (i) By sequential continuity it's enough to prove that $\lim_{x\to 0} f(x) = f(0) = 0$, and since we need only consider nonzero x this is the same as $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^3}\right) = 0$. We have

$$0 \le \left| x^2 \sin\left(\frac{1}{x^3}\right)\right| \le x^2$$

for all x, since $|\sin(\frac{1}{x^3})| \le 1$. Then $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$, so the squeeze theorem says that $\left|x^2 \sin\left(\frac{1}{x^3}\right)\right| \to 0$ as $x \to 0$, and hence $x^2 \sin(\frac{1}{x^3}) \to 0$ as well. (3 marks)

(ii) We know that x^2 and $\sin(x)$ are continuous for all x, and $\frac{1}{x^3}$ is continuous for all $x \neq 0$. Then for $x \neq 0$ it follows that $\sin(\frac{1}{x^3})$ is a composition of two continuous functions, so it is continuous, and its product with the continuous x^2 is therefore continuous as well. (2 marks)

(b) (i) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for every y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$. (2 marks)

(ii) The function $f(x) = 4^x - \cos(x) - 2$ is continuous, and $f(0) = -2$ while $f(1) = 2 - \cos(1) > 0$, so the intermediate value theorem says that $f(c) = 0$ for some $c \in (0, 1)$. (3 marks)

(iii) The intermediate value theorem says that $f(c) = 0$ for some c in each interval $(k, k+1)$ where $x = 0, 1, \ldots, 9$, so it must have at least 10 zeroes. (The function $f(x) = \cos(\pi x)$ achieves exactly 10, though this is not a required part of the answer.) (3 marks)

(c) (i) We note that $g(x) = |f(x)|$ is also continuous, and since it is defined on the closed interval $[0, 1]$ its image must have the form $[g(a), g(b)]$ for some $a, b \in [0, 1]$. Then $g(a) = |f(a)| \ge 0$, so if $0 \notin [g(a), g(b)]$ then we must have $g(a) > 0$. But then there is $y \in [0, 1]$ such that $g(y) \le 0.99g(a) < g(a)$, contradicting the claim that $\inf_{x\in[0,1]} g(x) = g(a)$, so we must have $g(a) = 0$ after all, and then $f(a) = 0$. (4 marks)

(ii) Many examples will work, such as $f(x) = \begin{cases} x, & 0 < x \le 1 \\ 1, & x = 0. \end{cases}$ (3 marks)

(Total: 20 marks)

5. (a) (i) If $f : [a,b] \to \mathbb{R}$ is continuous, and it is differentiable on (a,b), then there is $c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. (2 marks)

(ii) Let $f(x) = x^r$. By the mean value theorem there is $c \in (1, 1+x)$ such that

$$\frac{f(1+x)-f(1)}{(1+x)-1} = f'(c) \;\Rightarrow\; \frac{(1+x)^r - 1}{x} = rc^{r-1} \geq r,$$

so upon rearranging we have $(1+x)^r \geq 1 + rx$.
Alternate solution: let $g(x) = (1+x)^r - rx$, and then $g(0) = 1$ and g is monotone increasing since $g'(x) = r(1+x)^{r-1} - r > r - r = 0$ for all $x > 0$. (3 marks)

(iii) Suppose that $L = \lim_{x\to\infty} f'(x)$ exists and is positive. Then there is an N such that $f'(x) > \frac{L}{2}$ for all $x \geq N$. By the mean value theorem, for all $x > N$ we have

$$\exists t \in (N,x) : \frac{f(x)-f(N)}{x-N} = f'(t) > \frac{L}{2} \;\Rightarrow\; f(x) > \frac{L}{2}(x-N) + f(N),$$

so $f(x) \to \infty$ as $x \to \infty$ and this contradicts the existence of $\lim_{x\to\infty} f(x)$. If $L < 0$ instead then we repeat the same argument with $-f(x)$ to get a contradiction, so if L exists then it must be 0. (4 marks)

(b) (i) We have $f(x) = -\log(\cos(x))$, and we compute that

$$f'(x) = \frac{\sin(x)}{\cos(x)}, \qquad f''(x) = \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)},$$

so $f(0) = f'(0) = 0$ and $f''(0) = 1$. The second-order Taylor polynomial is

$$P_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = \frac{x^2}{2}.$$

(3 marks)

(ii) The second derivative $f''(x) = \frac{1}{\cos^2(x)}$ is continuous and positive on this interval, so f is convex here. (2 marks)

(iii) By the convexity of f on $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have $\frac{1}{2}\left(f(\frac{\pi}{3}) + f(\frac{\pi}{5})\right) \geq f\left(\frac{\pi/3+\pi/5}{2}\right)$, or

$$-\log\left(\cos\left(\frac{\pi}{3}\right)\right) - \log\left(\cos\left(\frac{\pi}{5}\right)\right) \geq -2\log\left(\cos\left(\frac{4\pi}{15}\right)\right).$$

We rearrange both sides and use $\log(x) + \log(y) = \log(xy)$ to simplify this to

$$\log\left(\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{5}\right)\right) \leq \log\left(\cos^2\left(\frac{4\pi}{15}\right)\right),$$

and exponentiate both sides to get the desired result. (2 marks)

(c) By Taylor's theorem, for any nonzero $x \in (-1,1)$ there's a t between 0 and x such that

$$|f(x) - P_n(x)| = \left|\frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1}\right| \leq \frac{n!}{(n+1)!}|x^n| < \frac{1}{n+1}.$$

So for any $\epsilon > 0$, if $n \geq \frac{1}{\epsilon}$ then we have $|f(x) - P_n(x)| < \frac{1}{n+1} < \epsilon$ for all $x \in (-1,1)$, hence by definition the sequence $\left(P_n\right)$ converges uniformly to f. (4 marks)

(Total: 20 marks)

6. (a) (i) False: $f(x)=0$, $g(x)=\begin{cases}1, & x=0\\ 0, & x>0\end{cases}$ gives $\int_0^1 f(x)\,dx=\int_0^1 g(x)\,dx=0$. (1 mark)

(ii) Intended solution: True: it's integration by parts, i.e., $f'g=(fg)'-fg'$ and $(fg)'$ is integrable. Unfortunately we didn't realize that this requires $(fg)'$ to be integrable, which isn't automatically true, but students who said this still got full credit.
Actual (unintended) solution: False: take $g(x)=1$ and $f(x)$ to be any differentiable function such that $f'(x)$ exists but isn't integrable, for example

$$f(x)=\begin{cases}x^2\sin\left(\frac{1}{x^2}\right), & x\neq 0\\ 0, & x=0;\end{cases}$$

then $fg'=0$ is integrable but $f'g=f'$ isn't, because

$$f'(x)=\begin{cases}2x\sin(\frac{1}{x^2})+\frac{2}{x}\cos(\frac{1}{x^2}), & x\neq 0\\ 0, & x=0\end{cases}$$

is unbounded as $x\downarrow 0$ and integrable functions are bounded by definition. (1 mark)

(iii) True: $\log(x)$ is continuous on $\mathrm{Image}(e^f)\subset(0,\infty)$, so $\log(e^f)=f$ is integrable as well. (1 mark)

(iv) False: if $f_n(x)=\begin{cases}n, & 0<x\leq\frac{1}{n}\\ 0 & \text{otherwise}\end{cases}$ then $\int_0^1 f_n(x)\,dx=1$ for all n, but $f(x)\equiv 0$ and so $\int_0^1 f(x)\,dx=0$. (1 mark)

(v) True: we have $\max(f,g)=\frac{1}{2}(f+g+|f-g|)$, and both linear combinations and absolute values of integrable functions are integrable. (2 marks)

(b) (i) Either of the following is acceptable:

1. if $f:[a,b]\to\mathbb{R}$ is continuous and $F(x)=\int_a^x f(t)\,dt$, then F is differentiable on (a,b) and $F'(x)=f(x)$ for all $x\in(a,b)$.
2. If $g:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and has a continuous derivative on (a,b), then $\int_a^b g'(x)\,dx=g(b)-g(a)$.

(3 marks)

(ii) Since $f(x)$ is monotone decreasing, we have

$$L(f,P)=\sum_{k=0}^{n-1}\left(\inf_{k\leq t\leq k+1} f(t)\right)\cdot 1=\sum_{k=0}^{n-1}\frac{1}{\sqrt{k+2}}$$
$$U(f,P)=\sum_{k=0}^{n-1}\left(\sup_{k\leq t\leq k+1} f(t)\right)\cdot 1=\sum_{k=0}^{n-1}\frac{1}{\sqrt{k+1}},$$

or $L(f,P)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n+1}}$ and $U(f,P)=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$. (3 marks)

(iii) We have the inequalities

$$L(f,P)\leq\underline{\int_0^n} f(x)\,dx=\int_0^n f(x)\,dx=\overline{\int_0^n} f(x)\,dx\leq U(f,P),$$

and $U(f,P)-L(f,P) = 1-\frac{1}{\sqrt{n+1}} < 1$, so the integral $\int_0^n f(x)\,dx$ is within 1 of $U(f,P)$. We use the fundamental theorem of calculus to compute

$$\int_0^n \frac{1}{\sqrt{x+1}}\,dx = 2\sqrt{x+1}\Big|_{x=0}^{x=n} = 2\sqrt{n+1}-2,$$

and for $n = 10^6-1$ this says that $\int_0^n \frac{1}{\sqrt{x+1}}\,dx = 1998$ is within 1 of $U(f,P)$. $(1999$ is also an acceptable answer.) (3 marks)

(c) The difference $h(x) = f(x)-g(x)$ is integrable, with $h(x) = 0$ for all rational $x \in [0,1]$, and by linearity it's enough to show that $\int_0^1 h(x)\,dx = 0$. Since the rationals are dense in $[0,1]$ we have $\inf h \leq 0$ on every interval, so

$$L(h,P) \leq 0 \text{ for every partition } P \;\Rightarrow\; \underline{\int_0^1} h(x)\,dx \leq 0.$$

Similarly we have $\sup h \geq 0$ on every interval, so

$$U(h,P) \geq 0 \text{ for every partition } P \;\Rightarrow\; \overline{\int_0^1} h(x)\,dx \geq 0.$$

But h is integrable, so the upper and lower Darboux integrals are equal, and this means that they must both be zero, so $\int_0^1 h(x)\,dx = 0$ as well. (5 marks)

(Total: 20 marks)