In this sheet, we define an alternative definition for a convergent series, and see its connection to the definitions we learned.

Definition 1. Let X be a set, and let $f : X \to \mathbb{R}$ be a function. (If $X = \mathbb{N}$ then this is a sequence.)

- For X finite, we define $\sum_{x\in X} f(x)$ simply to be the sum of the finite set $\{ f(x) \mid x \in X \}$.

- For X infinite, we say the sum $\sum_{x\in X} f(x)$ is convergent to a real number L, if for all $\epsilon > 0$, there is some finite subset $I_0 \subseteq X$, such that for every finite set I, if $I_0 \subseteq I \subseteq X$, then $\left|\left(\sum_{x\in I} f(x)\right) - L\right| < \epsilon$.

1. Prove the limit defined above is unique, in the following sense: If $\sum_{x\in X} f(x)$ is convergent to both L_1 and L_2, then $L_1 = L_2$.

   If $\sum_{x\in X} f(x)$ is convergent to both L_1 and L_2, let $\epsilon > 0$ be arbitrary. Then there are finite sets $I_1, I_2 \subseteq X$, such that for every finite I, if $I_1 \subseteq I \subseteq X$ then $\left|\left(\sum_{x\in I} f(x)\right) - L_1\right| < \epsilon/2$ and if $I_2 \subseteq I \subseteq X$ then $\left|\left(\sum_{x\in I} f(x)\right) - L_2\right| < \epsilon/2$. Choose $I_3 := I_1 \cup I_2$. It is finite, as a finite union of such. so $\left|\left(\sum_{x\in I_3} f(x)\right) - L_1\right| < \epsilon/2$ and $\left|\left(\sum_{x\in I_3} f(x)\right) - L_2\right| < \epsilon/2$. By the triangle inequality, we get:

$$|L_1 - L_2| \le \left|\left(\sum_{x\in I_3} f(x)\right) - L_1\right| + \left|\left(\sum_{x\in I_3} f(x)\right) - L_2\right| < \epsilon.$$

   So $|L_1 - L_2| < \epsilon$ for every $\epsilon > 0$. Therefore, $|L_1 - L_2| = 0$.

2. Prove that if $\sum_{n\in\mathbb{N}} a_n$ is convergent to L, then $\sum_{n=1}^{\infty} a_n$ is convergent to L.

   Assume $\sum_{n\in\mathbb{N}} a_n$ is convergent to L. Let $\epsilon > 0$, and let $I_0 \subseteq \mathbb{N}$ be finite, such that for every finite $I \subseteq \mathbb{N}$, if $I_0 \subseteq I \subseteq N$, then $\left|\left(\sum_{n\in I} a_n\right) - L\right| < \epsilon$. Now let $N \in \mathbb{N}$ be such that $n \le N$ for all $n \in I_0$. So for all $n > N$, the set $I_n := \{ 1, \dots, n \}$ is finite and $I_0 \subseteq I_n$. So

$$\left|\left(\sum_{n=1}^{\infty} a_n\right) - L\right| = \left|\left(\sum_{n\in I_n} a_n\right) - L\right| < \epsilon.$$

3. Let X_1, X_2 be sets such that $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = X$. Let $f : X \to \mathbb{R}$. Prove that if $\sum_{x\in X_1} f(x)$ is convergent to L_1, $\sum_{x\in X_2} f(x)$ is convergent to L_2, then $\sum_{x\in X} f(x)$ is convergent to $L_1 + L_2$.

For $i \in \{1,2\}$: Let $I_i \subseteq X_i$ be finite such that for every $I_i \subseteq I \subseteq X_i$ finite, $\left|\sum_{x\in I} f(x) - L_i\right| < \epsilon/2$. So let $I_0 := I_1 \cup I_2$. Let I be finite such that $I_0 \subseteq I \subseteq X$. Then $I \cap X_1 \supseteq I_1$ and $I \cap X_2 \supseteq I_2$, and $I = (I \cap X_1) \cup (I \cap X_2)$. So

$$\left|\left(\sum_{x\in I} f(x)\right) - (L_1 + L_2)\right| = \left|\left(\sum_{x\in I\cap X_1} f(x)\right) + \left(\sum_{x\in I\cap X_2} f(x)\right) - (L_1 + L_2)\right| \leq$$

$$\left|\left(\sum_{x\in I\cap X_1} f(x)\right) - L_1\right| + \left|\left(\sum_{x\in I\cap X_2} f(x)\right) - L_2\right| < \epsilon.$$

Definition 2. Let X be a set, and let $f : X \to \mathbb{R}$ be a function. We say the sum $\sum_{x\in X} f(x)$ is *Cauchy*, if for all $\epsilon > 0$, there is some finite subset $I_0 \subseteq X$, such that for every finite set $I \subseteq X$, if $I \subseteq (X \setminus I_0)$, then $\left|\left(\sum_{x\in I} f(x)\right)\right| < \epsilon$.

4. Prove that if $\sum_{x\in X} f(x)$ is convergent, then it is Cauchy.

   Assume $\sum_{x\in X} f(x)$ is convergent to L, and let $\epsilon > 0$. Then there is some finite $I_0 \subseteq X$ such that for all finite I, if $I_0 \subseteq I \subseteq X$, then $\left|\left(\sum_{x\in I} f(x)\right) - L\right| < \epsilon/2$. Let $I \subseteq X$ be finite, such that $I \cap I_0 = \emptyset$. Then $\sum_{x\in I} f(x) = \sum_{x\in I\cup I_0} f(x) - \sum_{x\in I_0} f(x)$. By the triangle inequality, we get:

$$\left|\sum_{x\in I} f(x)\right| = \left|\sum_{x\in I\cup I_0} f(x) - \sum_{x\in I_0} f(x)\right| \leq \left|\sum_{x\in I\cup I_0} f(x) - L\right| + \left|\sum_{x\in I_0} f(x) - L\right| < 2\epsilon/2 = \epsilon.$$

5. Prove that if $\sum_{x\in X} f(x)$ is Cauchy, then it is convergent. (Hard!)

   Since $\sum_{x\in X} f(x)$ is Cauchy, for every $n \in \mathbb{N}$, we can choose $\epsilon_n := 1/n$, and then, there is some finite $I_n \subseteq X$, such that for every finite $I \subseteq X$, if $I \cap I_n = \emptyset$, then $|\sum_{x\in I} f(x)| < 1/n$. Let $J_n := I_1 \cup \cdots \cup I_n$, and let $b_n := \sum_{x\in J_n} f(x)$. We claim that (b_n) is Cauchy. Indeed: Let $\epsilon > 0$ and let $N > 1/\epsilon$. Then for $m > n > N$,

$$|b_m - b_n| = \left|\sum_{x\in J_m\setminus J_n} f(x)\right|.$$

   As $(J_m \setminus J_n) \cap I_n = \emptyset$, the RHS above is less than $1/n$, which in turn is less than $1/N < \epsilon$. So (b_n) is convergent to some L. We now finish by claiming that $\sum_{x\in X} f(x)$ is convergent to L. Let $\epsilon > 0$. Let $N_1 \in \mathbb{N}$ such that $N_1 > 2/\epsilon$ and let $N_2 \in N$ such that $|b_n - L| < \epsilon/2$ for all $n > N_2$. Let $N = \max N_1, N_2$. Finally, it suffices to show that for every finite I, if $J_N \subseteq I \subseteq X$, then

$$\left|\left(\sum_{x\in I} f(x)\right) - L\right| < \epsilon.$$

So here we go:

$$\left|\left(\sum_{x\in I} f(x)\right) - L\right| = \left|\left(\sum_{x\in J_N} f(x)\right) - L\right| + \left|\left(\sum_{x\in I\setminus J_N} f(x)\right) - L\right| < \epsilon.$$

6. Deduce that if $\sum_{x\in X} f(x)$ is convergent and $X' \subseteq X$, then $\sum_{x\in X'} f(x)$ is convergent.

   By Question 5, $\sum_{x\in X} f(x)$ is Cauchy. So let $I_0 \subseteq X$ be finite, such that for every finite $I \subseteq X$, if $I \cap I_0 = \emptyset$, then $\left|\sum_{x\in X} f(x)\right| < \epsilon$. Let $I_0' := I_0 \cap X'$. Clearly, for every finite $I \subseteq X'$, if $I \cap I_0' = \emptyset$, then $I \cap I_0 = \emptyset$ and thus $\left|\sum_{x\in X} f(x)\right| < \epsilon$. So $\sum_{x\in X'} f(x)$ is Cauchy. By 4 we are done.

7. Prove that if $\sum_{n=1}^{\infty} |a_n|$ is convergent to L, then $\sum_{n\in\mathbb{N}} |a_n|$ is convergent to L.

   Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that $0 < L - \sum_{n=1}^{\infty} |a_n| < \epsilon$ for all $m > N$. So let $I_0 := \{ 1, \ldots, N+1 \}$ and let I be finite such that $I_0 \subseteq I \subseteq \mathbb{N}$. Since I is finite, there is some $N_2 > N$ such that $n < N$ for all $n \in I$. So

$$0 < L - \sum_{n=1}^{N_2} |a_n| < L - \sum_{n\in I} |a_n| < L - \sum_{n=1}^{M+1} |a_n < \epsilon.$$

8. Prove that $\sum_{n\in\mathbb{N}} a_n$ is convergent to L if and only if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent to L.

   Divide $\mathbb{N}$ into $\mathbb{N}_+ := \{ n \in \mathbb{N} | a_n \geq 0 \}$ and $\mathbb{N}_- := \{ n \in \mathbb{N} | a_n < 0 \}$. By Questions 3 and 3, $\sum_{n\in\mathbb{N}} a_n$ is convergent if and only if there are $L_+, L_- \in \mathbb{R}$ such that $L_+ + L_- = L$ and $\sum_{n\in\mathbb{N}_+} a_n$ and $\sum_{n\in\mathbb{N}_-} a_n$ are convergent to L_+ and L_-, respectively. Applying Question 3 and Theorem 4.33 we get that the latter is equivalent to $\sum_{n=1}^{\infty} a_n$ being absolutely convergent to L.