Cover Sheet for Submission of Maths Examinations Summer 2020

We would advise preparing your coversheets with your CID, Module Name and Code and Date, before the exams are due to take place.

CID: 01738166

Module Name: Linear Algebra and Groups

Module Code: MATH40003

Date: 07/05/2020

Questions Answered (in the file):

Please tick next to the question or questions you have answered in this file.

Q1	
Q2	
Q3	
Q4	
Q5	✓
Q6	

(Note: this is a coversheet for all students - not all students will have exams with 6 questions. Please tick the boxes which are appropriate for your exam and/or the file you are submitting).

(Optional) Page Numbers for each question;

Page Number	Question Answered

If handwritten, please complete in CAPITAL Letters, in Blue or Black Ink, ensuring the cover sheet is legible.

(a) Definition of det(A) by induction on n:

(i) $n=1$: $\det(A) = a_{11}$

(ii) $n=2$: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}\det(A_{11}) - a_{12}\det(A_{12})$

(iii) $\det(A_{n\times n}) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \dots + (-1)^{n+1} a_{1n} \det(A_{1n})$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j})$$

Let $1 \le l \le n$ → B is obtained by multiplying row l of A by α.

Case: $l=1$: The ij entry of B is αa_{1j} and $A_{1j} = B_{1j}$, so by definition

$$\det(B) = \sum_{j=1}^{n} (-1)^{j+1} \alpha a_{1j} \det(A_{1j}) = \alpha \det(A)$$

Case: $l>1$:

The $1j$-minor B_{1j} has $l-1$th row equal to α times the $l-1$th row of A_{1j}. So by induction,

$\det(B_{1j}) = \alpha \det(A_{1j})$ and as $b_{1j} = a_{1j}$ we obtain by definition $\det(B) = \alpha \det(A)$.

(b) $\det(X^2 Y^3 X^T Z) = \det(X) \cdot \det(X) \cdot \det(Y) \cdot \det(Y) \cdot \det(Y) \cdot \det(X^T) \cdot \det(Z)$

where we have used that $\det(AB) = \det(A)\cdot\det(B)$ and $\det(A^T) = \det(A)$

~~so de~~

$\det(X) = 1\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} + 7\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = \underline{2}$

$\det(Y) = 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = \underline{-1}$

$\det(Z) = \underbrace{0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=0} - 1\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = -1(-1) + 4(-2)$

$= 1 - 8 = \underline{-7}$

So $\det(X^2 Y^3 X^T Z) = 2^2 \cdot (-1)^3 \cdot 2 \cdot (-7) = 8 \cdot 7 = \boxed{56}$

(c) Have $H < (G, \cdot)$

(i) The left coset of H in G is $aH = \{ah \mid h \in H\}$ for fixed $a \in G$.

(ii) Want to show that the cosets of H partition G.

Suppose $c \in gH \cap g'H$ so $c \in gH$, so $c = ga$ for $a \in H$
and $c \in g'H$ so $c = g'b$ for $b \in H$

Then $g = g'ba^{-1}$, so $g \in g'H$
so for any $a \in H$, $ga \in g'H$, so $gH \subseteq g'H$

Similarly $g'H \subseteq gH$, so $gH = g'H$. Thus either $gH \cap g'H = \{\emptyset\}$ or $gH = g'H$.

(iii) Consider $f : X \to Y$
$a \mapsto g'g^{-1}a$, with $X = gH$ and $X = g'H$.

This is injective since $g'g^{-1}a = g'g^{-1}b \Rightarrow a = b$ after multiplying by gg'^{-1}

This is surjective since $gg'^{-1}a \mapsto a$ for any $a \in g'H$.

So f is a bijection.

(iv) Assume $X \cap Z \neq \emptyset$, so $\exists c \in X \cap Z$. Then $c = ah$ and $c = bk$ for $h \in H$, $k \in K$.

Now $aH = ch^{-1}H = c(h^{-1}H) = cH$ as $h^{-1} \in H$ and
$bK = ck^{-1}K = c(k^{-1}K) = cK$ as $k^{-1} \in K$.

So $aH \cap bK = cH \cap cK = c(H \cap K)$

Examples:

Consider $D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$

Then $\{e, r^2\} \cap \{e, s\} = \{e\}$ and $\{e, r^2\} \cap \{r, sr\} = \emptyset$