Problem Sheet 8

Math40002, Analysis 1

1. Evaluate $\displaystyle\int_0^x \frac{1}{1+e^t}\,dt$. Does $\displaystyle\int_0^\infty \frac{1}{1+e^t}\,dt$ exist, and if so, what is it?

2. The prime number theorem says that the number $\pi(n)$ of primes between 1 and n is approximately $\int_2^n \frac{1}{\log(x)}\,dx$.

   (a) Prove that this integral equals $\frac{n}{\log(n)} + \int_2^n \frac{1}{(\log x)^2}\,dx$, up to a constant which does not depend on n.

   (b) Prove that there is a constant $C > 0$ such that $\int_2^n \frac{1}{(\log x)^2}\,dx < \frac{Cn}{(\log n)^2}$ for all sufficiently large n, by splitting the integral up into one with domain $[2, \sqrt{n}]$ and one with domain $[\sqrt{n}, n]$ and estimating each one separately.

3. Let $f : [0, \infty) \to [0, \infty)$ be uniformly continuous, and suppose that $\int_0^\infty f(x)\,dx$ exists.

   (a) For each $\epsilon > 0$, prove that there is a $\delta > 0$ such that for all $y > 0$, if $f(y) \geq \epsilon$ then
   $$\int_y^{y+\delta} f(t)\,dt \geq \frac{\epsilon\delta}{2}.$$

   (b) Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

   (c) Describe a continuous function $g : [0, \infty) \to [0, \infty)$ such that $\displaystyle\int_0^\infty g(x)\,dx$ exists but $\lim\limits_{x\to\infty} g(x)$ does not. Can you make g differentiable as well?

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly monotone increasing, with continuous first derivative on (a, b). Evaluate $\int_{f(a)}^{f(b)} f^{-1}(x)\,dx$ in terms of $\int_a^b f(x)\,dx$, and draw a picture to explain your answer.

5. Use problem 4 to evaluate $\displaystyle\int_1^x \frac{\sqrt{t^2-1}}{t}\,dt$ for $x \geq 1$.

6. Let $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$.

   (a) Prove that this improper integral converges for all $t > 0$. (In how many ways is it improper?)

   (b) Compute $\Gamma(1)$.

   (c) Prove that $\Gamma(n+1) = n\Gamma(n)$ for all integers $n \geq 1$, and deduce that $\Gamma(n+1) = n!$ for all $n \geq 0$.

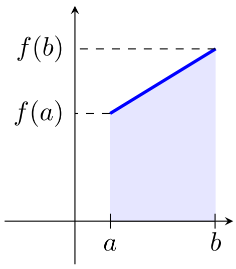
7. (*) Let $f : [a, b] \to \mathbb{R}$ be continuous, with $f''(x)$ continuous and bounded on (a, b).

(a) Use integration by parts twice to prove that

$$\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx = \int_a^b f(x)\,dx - (b-a)\left(\frac{f(a)+f(b)}{2}\right).$$

(b) If $|f''(x)| \le M$ for all $x \in (a, b)$, prove that

$$\left|\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx\right| \le \frac{M(b-a)^3}{12}.$$



In other words, $\int_a^b f(x)\,dx$ is the area of the trapezium shown at right, up to an error of at most $\frac{M(b-a)^3}{12}$. (Hint: check that $(x-a)(x-b) \le 0$ on $[a, b]$, and compute that $\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$.)

(c) Apply this to $f(x) = \log(x)$ to show that

$$\int_1^n \log(x)\,dx = \sum_{k=1}^{n-1}\left(\frac{\log(k)+\log(k+1)}{2} + e_k\right),$$

where $|e_k| \le \frac{1}{12k^2}$ for all k.

(d) Evaluate both the integral and the sum from part (c) to show that there is some constant $C > 0$ such that

$$\left|\log(n!) - \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right)\right| < C$$

for all n, or equivalently if $C_1 = e^{1-C}$ and $C_2 = e^{1+C}$ then

$$C_1\sqrt{n}\left(\frac{n}{e}\right)^n \le n! < C_2\sqrt{n}\left(\frac{n}{e}\right)^n.$$