Cover Sheet for Submission of Maths Examinations Summer 2020

We would advise preparing your coversheets with your CID, Module Name and Code and Date, before the exams are due to take place.

CID: 01738166

Module Name: Linear Algebra and Groups

Module Code: MATH40003

Date: 07/05/2020

Questions Answered (in the file):

Please tick next to the question or questions you have answered in this file.

Q1	
Q2	✓
Q3	
Q4	
Q5	
Q6	

(Note: this is a coversheet for all students - not all students will have exams with 6 questions. Please tick the boxes which are appropriate for your exam and/or the file you are submitting).

(Optional) Page Numbers for each question;

Page Number	Question Answered

If handwritten, please complete in CAPITAL Letters, in Blue or Black Ink, ensuring the cover sheet is legible.

(a)

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 0 & -3 & 4 & 1 \\ 1 & 4 & 1 & a & b \\ 0 & -2 & 0 & -1 & 0 \end{array}\right) = (A|c)$$

→ augmented matrix

(b)

$$(A|c) \xrightarrow[R_3-R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -4 & -1 & -2 & 1 \\ 0 & 2 & 2 & a-3 & b \\ 0 & -2 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \cdot \frac{-1}{4}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & +1 & \frac{1}{4} & \frac{1}{2} & -1/4 \\ 0 & 2 & 2 & (a-3) & b \\ 0 & -2 & 0 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[R_4+2R_2]{R_3-2R_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & 1/4 & 1/2 & -1/4 \\ 0 & 0 & 3/2 & a-4 & b+1/2 \\ 0 & 0 & 1/2 & 0 & -1/2 \end{array}\right) \xrightarrow[R_4 \cdot 2]{R_1-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & -3/2 & 2 & 1/2 \\ 0 & 1 & 1/4 & 1/2 & -1/4 \\ 0 & 0 & 3/2 & a-4 & b+1/2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 0 & -3/2 & 2 & 1/2 \\ 0 & 1 & 1/4 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3/2 & (a-4) & b+1/2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & a-4 & b+2 \end{array}\right)$$

If $a \neq 4$ $\exists$ sol

$$w = \frac{b+2}{a-4}$$

$$z = -1$$

$$y = -\frac{1}{2} - \frac{1}{2}\left(\frac{b+2}{a-4}\right)$$

$$x = -1 - 2\left(\frac{b+2}{a-4}\right)$$

If $a=4, b=-2$, sol. is

$\left(-1-2w, -\frac{1}{2}-2w, -1, w\right)$ – infinitely many sol.
$w \in \mathbb{R}$

If $a=4$, $b \neq -2$ we get no solutions.

(c)

(i)

- The row space of B is the span of the rows of A
- The row rank is $\dim(RSp[B])$.

(ii)

$a=4,\ b=2$:

$$A=\begin{pmatrix}1&2&-1&3\\2&0&-3&4\\1&4&1&4\\0&-2&0&-1\end{pmatrix}\xrightarrow[R_3=R_3-R_1]{R_2=R_2-2R_1}\begin{pmatrix}1&2&-1&3\\0&-4&-1&-2\\0&2&2&1\\0&-2&0&-1\end{pmatrix}\xrightarrow[R_4=R_4-\frac{1}{2}R_2]{R_3=R_3-\frac{1}{2}R_2}\begin{pmatrix}1&2&-1&3\\0&-4&-1&-2\\0&0&\frac{3}{2}&0\\0&0&\frac{1}{2}&0\end{pmatrix}$$

$$\xrightarrow{R_4=\frac{-1}{3}R_4-\frac{1}{3}R_3}\begin{pmatrix}1&2&-1&3\\0&-4&-1&-2\\0&0&\frac{3}{2}&0\\0&0&0&0\end{pmatrix}$$

$\Rightarrow$ Row rank of A is $\boxed{3}$

Row space is $\left\{\begin{pmatrix}1\\2\\-1\\3\end{pmatrix},\begin{pmatrix}0\\-4\\-1\\-2\end{pmatrix},\begin{pmatrix}0\\0\\3/2\\0\end{pmatrix}\right\}$

$$(A|c)=\left(\begin{array}{cccc|c}1&2&-1&3&0\\2&0&-3&4&1\\1&4&1&4&2\\0&-2&0&-1&0\end{array}\right)\xrightarrow[\text{ops}]{\text{Row}}\left(\begin{array}{cccc|c}1&2&-1&3&0\\0&-4&-1&-2&1\\0&0&\frac{3}{2}&0&5/2\\0&0&0&0&-4/3\end{array}\right)\Rightarrow$$

Row rank of A is $\boxed{4}$; Row space $\left\{\begin{pmatrix}1\\2\\-1\\3\\0\end{pmatrix},\begin{pmatrix}0\\-4\\-1\\-2\\1\end{pmatrix},\begin{pmatrix}0\\0\\3/2\\0\\5/2\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\-4/3\end{pmatrix}\right\}$

(d) Suppose row rank (A) = row rank $(A|c)$ $\Rightarrow$

Column rank (A) = row rank (A) = row rank $(A|c)$ = col. rank $(A|c)$ $\Rightarrow$

Col. rank (A) = col. rank $(A|c)$ $\Rightarrow$ The last column (c) can be represented as a linear combination of the first columns.

Using row operations (column operations) we can reduce the last column to $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, so we get the homogenous equation

$$A' \bar{x} = 0_v .$$

The solution space S is the kernel of the reduced matrix A', so it is a coset of K, where K is the kernel of A.

By rank nullity thm, we get that

$$\underline{\dim(\mathrm{Ker}(A))} = \dim(\mathbb{R}^n) - \underbrace{\dim(\mathrm{Im}(A))}_{= \mathrm{rank}(A)} = \underline{n - \mathrm{rank}(A)}$$