Mathematics Year 1, Calculus and Applications I

Problem Sheet 0

Problems 5, 6 and 7 are good candidates for starred questions

1. In the following examples you do not need to prove things but you should state which limit properties you are using.
   (a) Using the fact that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$, find $\lim_{x\to 0}\exp\left(\frac{3x}{\tan x}\right)$.
   (b) Using the fact that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$, find $\lim_{x\to 0}\cos\left(\frac{\pi\sin x}{4x}\right)$.

2. Find the following limits (do not use L'Hopital's rule, Binomial expansions etc.).
   (a) $\lim_{x\to 27}\frac{x^{1/3}-3}{x-27}$ (b) $\lim_{x\to 0}\frac{(3+x)^2-9}{x}$
   (c) $\lim_{x\to 1+}\frac{x(x+3)}{(x-1)(x-2)}$ (d) $\lim_{x\to 0+}\frac{(x^3-1)|x|}{x}$
   (e) $\lim_{x\to\frac{1}{2}-}\frac{2x-1}{\sqrt{(2x-1)^2}}$ (f) $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{ax+b}-\sqrt{ax+b/2}\right)$, $(a,b>0)$

3. (a) Establish the Comparison Test 2 given in the handout, using the $\varepsilon - A$ definition of the limit.
   (b) Use (a) above to find $\lim_{x\to\infty}\frac{1}{x}\sin\left(\frac{1}{x}\right)$.

4. (a) Use the $B-\delta$ definition of limits to show that if $\lim_{x\to x_0} f(x) = \infty$ and $g(x) \geq f(x)$ for x close to x_0, $x \neq x_0$, then $\lim_{x\to x_0} g(x) = \infty$.
   (b) Use (a) above to show that $\lim_{x\to 1}\frac{1+\cos^2 x}{1-x^2} = \infty$.

5. (a) Graph the function $y = f(x)$ where
$$f(x) = \begin{cases} |x|/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$
   Does $\lim_{x\to 0} f(x)$ exist? (Justify your answer.)
   (b) Graph the function $y = g(x)$ where
$$g(x) = \begin{cases} x+1 & x < 0 \\ 2x-1 & x \geq 0 \end{cases}$$
   Does $\lim_{x\to 0} g(x)$ exist? (Justify your answer.)
   (c) Graph $y = f(x) + g(x)$ with $f(x)$ as in (a) above and $g(x)$ as in (b). Does $\lim_{x\to 0}[f(x)+g(x)]$ exist? What can you conclude regarding the statement "*the limit of a sum can exist even though the limits of the summands do not*".

6. Suppose that a function f is defined on an open interval I containing the point x_0, and that there are numbers m and K such that we have the inequality
$$|f(x) - f(x_0) - m(x-x_0)| \leq K(x-x_0)^2 \qquad \forall\ x \in I.$$
   Prove that f is differentiable at x_0 with derivative $f'(x_0) = m$.

7. How close to 3 does x have to be to ensure that $|x^3 - 2x - 21| < \frac{1}{1000}$? Do not use your phone. The answer is not unique. Clearly 3 ± 10^{-n} with $n = 100$ works and it also does for $n = 10$. What you need to do is find an estimate for the largest interval around 3 that will give the desired smallness.