Cover Sheet for Submission of Maths Examinations Summer 2020

We would advise preparing your coversheets with your CID, Module Name and Code and Date, before the exams are due to take place.

CID: 01738166

Module Name: An Introduction to Applied Maths

Module Code: MATH40007

Date: 18/05/2020

Questions Answered (in the file):

Please tick next to the question or questions you have answered in this file.

Q1	
Q2	
Q3	
Q4	✓
Q5	
Q6	

(Note: this is a coversheet for all students - not all students will have exams with 6 questions. Please tick the boxes which are appropriate for your exam and/or the file you are submitting).

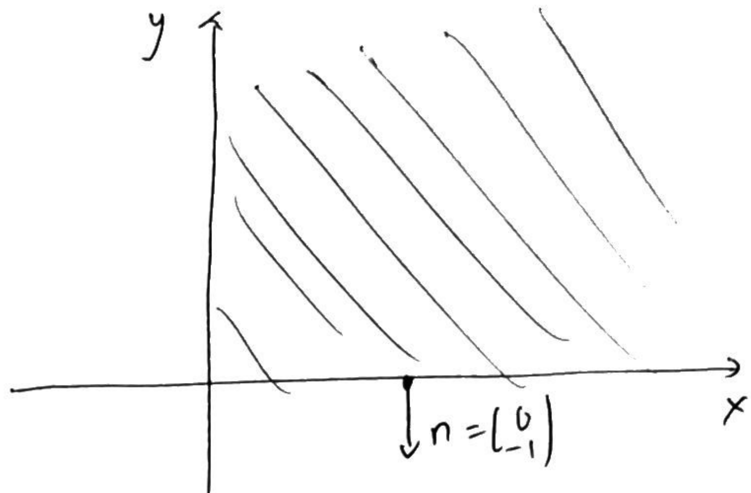
(Optional) Page Numbers for each question;

Page Number	Question Answered

If handwritten, please complete in CAPITAL Letters, in Blue or Black Ink, ensuring the cover sheet is legible.

$\varphi(x,y) = \mathrm{Re}[h(z)]$

$h(z) = \frac{-m}{2\pi} \log\left[\frac{z^2-i}{z^2+i}\right]$



(a) We have $\varphi = \frac{1}{2}\left[h(z) + \overline{h(z)}\right]$

$\Rightarrow$ To compute $\nabla^2 \varphi$, we compute $\nabla^2 h(z)$ and add its complex conjugate.

$h(z)$ is differentiable everywhere except at

$z^2 = i \iff x^2 - y^2 + (2xy-1)i = 0 \Rightarrow xy = \frac{1}{2}, \; x^2 = y^2$

$y = \frac{1}{2x} \Rightarrow x^2 = \frac{1}{4x^2} \Rightarrow 4x^4 = 1$

$x^4 = \frac{1}{4}$

$\Rightarrow x = y = \frac{1}{\sqrt{2}}$ (since $x, y > 0$)

$z^2 = -i \Rightarrow x^2 - y^2 + (2xy+1)i = 0 \Rightarrow xy = -\frac{1}{2}, \; x^2 = y^2$

$\rightarrow$ one should be negative

$\Rightarrow$ not on 1st quadrant.

$\Rightarrow h(z)$ is not differentiable at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

We have

$$\frac{\partial h(z)}{\partial x} = h'(z)\frac{\partial z}{\partial x} = h'(z) \; ; \quad \frac{\partial^2 h(z)}{\partial x^2} = h''(z)\frac{\partial z}{\partial x} = h''(z)$$

$$\frac{\partial h(z)}{\partial y} = h'(z)\frac{\partial z}{\partial y} = h'(z)\cdot i \; ; \quad \frac{\partial^2 h(z)}{\partial y^2} = i h''(z)\frac{\partial z}{\partial y} = -h''(z)$$

$\Rightarrow$

$$\frac{\partial^2 h(z)}{\partial x^2} + \frac{\partial^2 (h(z))}{\partial y^2} = 0 = \nabla^2 h(z) \Rightarrow$$

$$\nabla^2 \varphi = \nabla^2\left(h(z) + \overline{h(z)}\right) = 0.$$ (everywhere except at $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$)

(b) $y=0$

~~Let $R=\frac{z^2-i}{z^2+i}$~~

$$h(z) = -\frac{m}{2\pi}\log\left(\frac{(x+yi)^2-i}{(x+yi)^2+i}\right) \Rightarrow h(x,0) = -\frac{m}{2\pi}\log\left(\underbrace{\frac{x^2-i}{x^2+i}}_{R}\right)$$

$$\mathrm{Re}[h(z)] = \frac{-m}{2\pi}\cdot\log|R| = \phi(x,0)$$

$$R = \frac{x^2-i}{x^2+i} = \frac{(x^2-i)^2}{x^4+1} = \frac{x^4-2x^2i-1}{x^4+1} = \frac{x^4-1}{x^4+1} - \frac{2x^2}{x^4+1}i$$

$$\Rightarrow |R| = \sqrt{\frac{(x^4-1)^2+(2x^2)^2}{(x^4+1)^2}} = \sqrt{\frac{(x^4+1)^2}{(x^4+1)^2}} = 1 \Rightarrow \log|R| = 0$$

$$\Rightarrow \mathrm{Re}[h(z)] = 0 \Rightarrow \phi(x,0)=0.$$

(c) $x=0 \Rightarrow h(z) = -\frac{m}{2\pi}\log\left(\underbrace{\frac{-y^2-i}{-y^2+i}}_{R_1}\right) \Rightarrow \mathrm{Re}[h(z)] = \frac{-m}{2\pi}\log|R_1|$

$$R_1 = \frac{(y^2+i)^2}{(y^2-i)(y^2+i)} = \frac{y^4+2y^2i-1}{y^4+1} = \frac{y^4-1}{y^4+1} + \frac{2y^2}{y^4+1}i \Rightarrow |R_1| =$$

$$\frac{\sqrt{(y^4-1)^2+4y^4}}{y^4+1} = \frac{\sqrt{(y^4+1)^2}}{y^4+1} = \frac{y^4+1}{y^4+1} = 1 \Rightarrow \log|R_1| = 0 \Rightarrow \phi(0,y)=0.$$

(d) We know $j_x - ij_y = -\hat{c}\,h'(z)$, where $\hat{c}=1$.

$\Rightarrow j_x - ij_y = -h'(z)$.

When $y=0 \Rightarrow h(z) = \frac{-m}{2\pi}\log\left(\frac{x^2-i}{x^2+i}\right)$, $z=x \Rightarrow h'(z) =$

$$= \frac{-m}{2\pi}\cdot\frac{x^2+i}{x^2-i}\cdot\frac{d}{dx}\left[\frac{x^2-i}{x^2+i}\right] = \frac{-m}{2\pi}\cdot\frac{4x}{x^4+1}i$$

$\Rightarrow j_x - i j_y = \frac{m}{2\pi} \cdot \frac{4x}{x^4+1} i \Rightarrow j_x = 0$

$$\boxed{j_y = -\frac{2m}{\pi} \cdot \frac{x}{x^4+1}}$$

(e) We have $\underline{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2m}{\pi} \cdot \frac{x}{x^4+1} \end{pmatrix}$. $\underline{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \rightarrow$ normal vector out of the conductor

We want to integrate $\int_0^\infty \underline{j} \cdot \underline{n} \, dx = \int_0^\infty \frac{2m}{\pi} \cdot \frac{x}{x^4+1} dx$

$= \frac{2m}{\pi} \int_0^\infty \frac{x}{x^4+1} dx = I$. Substidute $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$I = \frac{1}{2} \int_0^\infty \frac{1}{u^2+1} du = \frac{1}{2} \left[\arctan(u) \right]_{x=0}^{x=\infty} = \frac{1}{2} \arctan(x^2) \Big]_0^\infty$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4} \Rightarrow \int_0^\infty j \cdot n \, dx = \frac{2m}{\pi} \cdot \frac{\pi}{4} = \boxed{\frac{m}{2}}$