Calculus and Applications: Unseen Questions 2: The variation of parameters method

1 First-order linear equations

Consider a first-order linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x). \tag{1}$$

You have already seen in lectures that one way to find solutions to equations of this form is to use the **integrating factor** method. Here we will explore another; the **method of variation of parameters**. You have seen a little of this technique in lectures but here we will take things further and develop the theory for finding solutions to second-order differential equations with non-constant coefficients.
First consider the homogeneous equation (i.e. where $q(x) = 0$), given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0. \tag{2}$$

(a). By separating the variables show that the solution to the homogeneous equation (2) can be given by

$$y_h(x) = Ce^{f(x)}, \qquad \text{where} \quad f(x) = -\int p(x)\mathrm{d}x \tag{3}$$

and C is an arbitrary constant (the subscript 'h' is used to denote that this is the homogeneous solution).

Now let's return to the inhomogeneous equation (1). Motivated by the solution to the homogeneous problem, we try a function of the form (3) where we allow the constant C to vary as a function of x; in other words we try the ansatz (solution guess):

$$y(x) = C(x)e^{f(x)}. \tag{4}$$

(b). Substitute the ansatz (4) into the equation (1) to determine that we must choose

$$C(x) = \int q(x)e^{-f(x)}\mathrm{d}x + K \tag{5}$$

where K is an arbitrary constant.

(c). Hence show that the general solution to equation (1) can be written as

$$y(x) = Ke^{-\int p(x)\mathrm{d}x} + e^{-\int p(x)\mathrm{d}x}\int q(x)e^{\int p(x)\mathrm{d}x}\mathrm{d}x. \tag{6}$$

Compare this general solution with the one obtained using the integrating factor method from lectures.

2 Exercise

(a). The charge, $q(t)$, on a capacitor in an electrical circuit as a function of time, t, is governed by the differential equation

$$\frac{\mathrm{d}q}{\mathrm{d}t} + 2q = \sin t. \tag{7}$$

Initially the charge on the capacitor is 5 colombs. Following through the variation of parameters method as above find the solution for the function $q(t)$. What happens to the charge on the capacitor in the long run ($t \to \infty$)?

3 Second-order linear equations

Now consider a second-order linear differential equation (whose coefficients are not necessarily constants)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = g(x). \tag{8}$$

The **method of undetermined coefficients** which you have seen in lectures allows us to find a particular solution of the equation (8) only in special cases; the coefficients must be constant (i.e. $p(x)$ and $q(x)$ are constants rather than functions of x) and $g(x)$ must belong to a certain class of functions. The **method of variation of parameters** is more general; it gives a formula for a particular solution provided only that the general solution of the corresponding homogeneous equation is known. There is no restriction on the need for constant coefficients or on the function $g(x)$.

Again, we start by considering the homogeneous equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0. \tag{9}$$

If we let $y_1(x)$ and $y_2(x)$ be two **independent** solutions to the homogeneous equation (9) then we can represent the general solution to equation (9) by

$$y_h(x) = c_1y_1(x) + c_2y_2(x), \tag{10}$$

where c_1 and c_2 are arbitrary constants.

Returning to the inhomogeneous equation (8), as before, motivated by the form of the solution (10) to the homogeneous equation, we allow the parameters c_1 and c_2 to vary with x; we make the ansatz

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{11}$$

(a). Substitute the ansatz (11) into the equation (8). There is a degree of flexibility in this method which we will use to set (you should see how this helps during the substitution process)

$$c_1'y_1 + c_2'y_2 = 0, \tag{12}$$

note the dependence on x is dropped here to save writing it out every time. Determine that

$$c_1'y_1' + c_2'y_2' = g(x). \tag{13}$$

(b). Using equations (12) and (13) determine that

$$c_1(x) = -\int \frac{y_2(x)g(x)}{W(x)}\mathrm{d}x + A \quad \text{and} \quad c_2(x) = \int \frac{y_1(x)g(x)}{W(x)}\mathrm{d}x + B, \tag{14}$$

where A and B are arbitrary constants and

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x). \tag{15}$$

(c). Deduce that the general solution to equation (8) can be written in terms of the solutions to the homogeneous equation (9) as

$$y(x) = Ay_1(x) + By_2(x) - y_1(x) \int \frac{y_2(x)g(x)}{W(x)} \mathrm{d}x + y_2(x) \int \frac{y_1(x)g(x)}{W(x)} \mathrm{d}x. \tag{16}$$

The method of undetermined coefficients seen in lectures is almost always easier to use when it can be applied, however the formula (16) derived here from the method of variation of parameters is useful for both theoretical purposes and for finding solutions to some differential equations for which you cannot apply the method of undetermined coefficients.

Remark: The function $W(x)$ defined above in (15) is called the **Wronskian**. In general, for a set of functions y_1, y_2, $\cdots$, y_n, it is defined to be the determinant of the matrix

$$\begin{bmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{bmatrix} \tag{17}$$

The Wronskian relates to the linear independence of the functions and you will see more of this in your second year differential equations class!

4 Exercises

(a). Find the general solution to the differential equation

$$y'' - 3y' + 2y = -\frac{e^{2x}}{e^x + 1}. \tag{18}$$

(b). Find the general solution to the differential equation

$$y'' + 9y = 3 \sec 3x. \tag{19}$$

5 Extension

The method of variation of parameters can be generalised to provide a general solution for nth-order linear differential equations in terms of the n linearly independent solutions for the corresponding nth-order homogeneous equation. Can you prove it?

Follow the same methodology as in the previous section. You will recall that we had the freedom to enforce a condition (12) above, now you will need to enforce $n-1$ extra conditions and decide what they should be!