Calculus and Applications: Unseen Questions 3: Difference Equations

1 What is a difference equation?

Suppose we have a function $U(n)$, where the independent variable n is a positive integer, and an implicit relationship between previous values of the function for generating the next value, for example:

$$U(n+1) = 5U(n), \quad U(n+2) = 2U(n+1) - 7U(n) + 5^n, \text{ or } U(n+3) = U(n+2) - n^2U(n). \quad (1)$$

These are examples of **difference equations**, and with sufficient information about the first values of the function, these allow us to generate the values of a sequence:

$$U(1) \to U(2) \to \cdots \to U(n) \to U(n+1) \to U(n+2) \to \cdots \quad (2)$$

1.1 The Fibonacci Sequence

The Fibonacci sequence is a famous sequence of numbers that describes many things in the natural world. The sequence goes as follows:

n	1	2	3	4	5	6	7	$\cdots$
$U(n)$	1	1	2	3	5	8	13	$\cdots$

Studying the sequence for a moment you might realise how it is constructed **implicitly**: i.e. the sum of the previous two terms gives the value of the next term! Or written mathematically

$$U(n+2) = U(n+1) + U(n), \qquad n \geqslant 1. \quad (3)$$

This might be useful for building up the first values of the sequence, but what if I asked you what the 100th value is? Or the 1000th? This implicit description would not be as useful as if you had an **explicit** description for the sequence, i.e a formula for $U(n)$ in terms of n. So given a difference equation, how do we extract the explicit formula for $U(n)$?

1.2 Characterising difference equations

Before we answer this question, let's first characterise difference equations in a very similar way to how we would with differential equations. Consider a general **linear** difference equation:

$$U(n+m) + a_1(n)U(n+m-1) + \cdots + a_{m-1}(n)U(n+1) + a_m(n)U(n) = f(n), \quad (4)$$

where m is a positive integer.

- Equation (4) is called **linear** since all the $U(n+i)$ quantities appear linearly (they are not squared or multiplied together etc.). Note the analogy to a linear differential equation.
- If the $a_i(n)$ are functions of n then the difference equation has **non-constant** coefficients, otherwise we say the equation has **constant** coefficients.

- The **order** of equation (4) is m. The order of a difference equation is defined to be the number of preceeding values needed to calculate the next value (even if not all of them appear in the difference equation), i.e. even if some of a_1, a_2, ..., $a_{m-1} = 0$ in (4) the order is still order m. You can think of it as how much more is the number of the largest term (here $n+m$) from the number of the smallest one (here n). Again notice the analogy to the order of a differential equation.
- If $f(n) = 0$ the difference equation (4) is said to be **homogeneous**. If $f(n) \neq 0$ we call equation (4) **inhomogeneous**. Note the analogy with differential equations again.

2 Linear difference equations with constant coefficients

A linear difference equation with constant coefficients can be written as in (4) but with the $a_i(n)$ as constants rather than functions of n. Namely

$$U(n+m) + a_1U(n+m-1) + \cdots + a_{m-1}U(n+1) + a_mU(n) = f(n). \tag{5}$$

The trick to solving (finding an explicit formula for $U(n)$) equations of this form is to tackle them just like you would with differential equations! In other words, the **general solution** can always be written as

$$U_{GS}(n) = U_{CF}(n) + U_{PI}(n), \tag{6}$$

where $U_{CF}(n)$ satisfies the homogeneous equation and $U_{PI}(n)$ satisfies the inhomogeneous equation. In a similar way to differential equations, $U_{CF}(n)$ contains m arbitrary constants for a difference equation of order m.

Example 1: Let $U(n+2) + 7U(n+1) - 18U(n) = 0$. Find an explicit formula for $U(n)$.

Solution: We sub $U(n) = \lambda^n$ into the equation. This gives

$$\lambda^{n+2} + 7\lambda^{n+1} - 18\lambda^n = 0 \tag{7}$$

$$\Rightarrow \quad \lambda^n(\lambda^2 + 7\lambda - 18) = 0. \tag{8}$$

So disregarding the trivial zero solution we need to solve the corresponding **auxiliary equation** just as we would if this was a differential equation. We find

$$\begin{aligned} & \lambda^2 + 7\lambda - 18 = 0 \\ \Leftrightarrow \quad & (\lambda+9)(\lambda-2) = 0 \\ \Leftrightarrow \quad & \lambda = -9 \quad \text{or} \quad 2. \end{aligned} \tag{9}$$

Thus the general solution can be written

$$U(n) = A(-9)^n + B(2)^n, \tag{10}$$

a linear combination of the two independent solutions. To determine the arbitrary constants A and B we would need extra information about the sequence, for example the values of the first two terms $U(1)$ and $U(2)$.

What if we have repeated roots of the auxiliary equation?

Example 2: Let $U(n+2) - 6U(n+1) + 9U(n) = 0$.

Solution: Auxiliary equation:

$$\begin{aligned} & \lambda^2 - 6\lambda + 9 = 0 \\ \Leftrightarrow \quad & (\lambda-3)^2 = 0 \\ \Leftrightarrow \quad & \lambda = 3. \end{aligned} \tag{11}$$

So certainly $A(3)^n$ is a solution. What about the other?
Just like with differential equations, we try $Bn(3)^n$. Substituting this in confirms it's the other solution we're after - I'll leave the details to you! So the general solution here is

$$U(n) = A(3)^n + Bn(3)^n. \tag{12}$$

Similarly for inhomogeneous equations pick the equivalent $U_{PI}(n)$ just as you would for differential equations and if part of the complementary function appears in the function $f(n)$ on the right-hand side try a polynomial times that part for the particular integral! Here is one last example of an inhomogeneous difference equation.

Example 3: Suppose $U(n+2) - U(n+1) - 6U(n) = (3)^{n+1}$. Find the general solution for $U(n)$.

Solution: Auxiliary equation:

$$\begin{aligned} & \lambda^2 - \lambda - 6 = 0 \\ \Leftrightarrow\quad & (\lambda+2)(\lambda-3) = 0 \\ \Leftrightarrow\quad & \lambda = -2 \quad \text{or} \quad 3. \end{aligned} \tag{13}$$

Therefore

$$U_{CF}(n) = A(-2)^n + B(3)^n. \tag{14}$$

Notice now the right hand side of the difference equation can be written as $3(3)^n$. So part of the complementary function appears on the right hand side. Hence we must try a polynomial times this function as our choice of particular integral. We try $U_{PI}(n) = Cn(3)^n$ where C is to be determined. Substituting in

$$\begin{aligned} & [C(n+2)(3)^{n+2}] - [C(n+1)(3)^{n+1}] - 6Cn(3)^n = 3(3)^n \\ \Rightarrow\quad & [9C(n+2) - 3C(n+1) - 6Cn](3)^n = 3(3)^n \\ \Rightarrow\quad & 15C = 3 \\ \Rightarrow\quad & C = \frac{1}{5}. \end{aligned} \tag{15}$$

Therefore the general solution can be written as

$$U(n) = A(-2)^n + B(3)^n + \frac{1}{5}n(3)^n. \tag{16}$$

3 Exercises

(a). Find the general solution to the difference equation

$$2U(n+2) - 7U(n+1) + 3U(n) = 5^n. \tag{17}$$

(b). For the Fibonacci sequence discussed in section 1.1, with difference equation as given in (3), show that

$$U(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{18}$$

(c). Let $S(n)$ be the sum of the first n square numbers, i.e

$$S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2. \tag{19}$$

Find a difference equation relating $S(n+1)$ to $S(n)$. Solve this difference equation to find an explicit formula for $S(n)$ in terms of n only.

(d). Suppose that $U(1) = -4$, $U(2) = -4$ and

$$U(n+2) - 2U(n+1) + 2U(n) = 0. \tag{20}$$

(i). Find $U(n)$ in terms of n.

(ii). From your formula for $U(n)$ from part (i) explain why $U(n)$ is a real number for all values of n.

(iii). For what values of n does $U(n) = 0$ hold?

(e). Let the $n \times n$ matrix M_n be defined by

$$M_n = \begin{bmatrix} k & 1 & 0 & 0 & \dots & 0 \\ 1 & k & 1 & 0 & \dots & \vdots \\ 0 & 1 & k & 1 & \ddots & \vdots \\ 0 & 0 & 1 & k & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 & k \end{bmatrix}, \tag{21}$$

where $k = 2\cosh\theta$. Prove that

$$det(M_n) = \frac{\sinh(n+1)\theta}{\sinh\theta}. \tag{22}$$