Math40002 Analysis 1 Unseen 4

1. Let $s_n = \sum_{k=1}^{n} \frac{1}{n+k}$. Prove s_n converges.

   $s_n = \sum_{k=1}^{n} \frac{1}{n+k} \leq \frac{n}{n+1} < 1$, so s_n is bounded above. We will show that s_n is increasing

$$s_{n+1} - s_n = \sum_{k=1}^{n+1} \frac{1}{n+k+1} - \sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=2}^{n+2} \frac{1}{n+k} - \sum_{k=1}^{n} \frac{1}{n+k} =$$
$$\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} =$$
$$\frac{(n+1)(2n+1) + (n+1)(2n+2) - (2n+1)(2n+2)}{(n+1)(2n+1)(2n+2)} =$$
$$\frac{(n+1)(2n+1) + (n+1)(2n+2) - 2(n+1)(2n+1)}{(n+1)(2n+1)(2n+2)} > 0.$$

   Therefore, by Theorem 3.13 in the lecture notes, s_n converges.

2. Define a sequence by $a_1 = 1$ and $a_{n+1} = (a_n + 1)^{1/2}$. Prove that $a_n \to (1+\sqrt{5})/2$.

   Let $\Phi = (1+\sqrt{5})/2$. Notice that $\Phi^2 - \Phi - 1 = 0$. So $(Phi+1)^{1/2} = \Phi$.

   First, to show that a_n converges to a non-negative real number, we will prove, by induction, that $\Phi > a_n > 0$ and $a_{n+1} > a_n$ for all $n \in \mathbb{N}$: For $n = 1$: $a_1 = 1, \Phi > 1 > 0$ and $a_{1+1} = (a_1+1)^{1/2} = (1+1)^{1/2} = \sqrt{2} > 1 = a_1$.

   Assume $\Phi > a_n > 0$ and $a_{n+1} > a_n$ for n. Sor for $n+1$:

   By the induction hypothesis, $a_n > 0$ and $a_{n+1} > a_n$. Therefore, $a_{n+1} = (a_n+1)^{1/2}$ and $\Phi = (\Phi+1)^{1/2} > (a_n+1)^{1/2} > 0$ and

$$a_{n+1+1} = (a_{n+1}+1)^{1/2} > (a_n+1)^{1/2} = a_{n+1}.$$

   So a_n is increasing and bounded above. Therefore, it converges to some limit $L \geq 0$. We will next prove $L = \Phi$.

   Indeed, Since $a_n \to L$, if b_n is defined to be a_{n+1}, then also $b_n \to L$. Notice that $a_n = (b_{n+1})^2 - 1$. Therefore, $a_n \to L^2 - 1$. By uniqueness of the limit, $L = L^2 - 1$, and the only non-negative solution to this is $L = \Phi$.

3. In Unseen 2, for a sequence $(a_n)_{n=1}^{\infty}$, we defined $\limsup(a_n)_{n=1}^{\infty}$ to be $\inf_{m\geq 1} \{ \sup_{n\geq m} \{ a_n \} \}$. Prove:

$$\limsup(a_n)_{n=1}^{\infty} = \lim_{m\to\infty} \left(\sup_{n\geq m} \{ a_n \} \right)$$

   in the sense that if one side of the equation exists, then so does the other and then they are equal.

In the same fashion, give two definitions for lim inf and show that they are equivalent in the same sense as above.

Define $b_m := \sup_{n\geq m}\{a_n\}$ **for all** $m \in \mathbb{N}$**. Then, by definition of** b_m **and of** $\limsup$**, the left hand side of the equation is** $\inf\{b_m | m \in \mathbb{N}\}$ **and the right hand side is** $\lim_{m\to\infty} b_m$**. By 3b in unseen 2,** b_m **is decreasing. Therefore, if** $\inf\{b_m | m \in \mathbb{N}\}$ **exists, then** b_m **is bounded below, hence,** $b_m \to \inf\{b_m | m \in \mathbb{N}\}$**. On the other hand, if** $\lim_{m\to\infty} b_m$**, then** $(b_m)_{m=1}^{\infty}$ **converges and thus is bounded above and below, as seen in class. Therefore,** $\lim_{m\to\infty} b_m = \inf\{b_m | m \in \mathbb{N}\}$**.**

4. Let (a_n) be a sequence. Prove that $a_n \to a$ if and only if $a_{2n} \to a$ and $a_{2n+1} \to a$.

   Try to generalize.

   If $a_{2n} \to a$ **and** $a_{2n+1} \to a$**, then for every** ϵ**, there is some** N' **such that for all** $n > N$**:** $|a_{2n} - a|, |a_{2n+1} - a| < \epsilon$**. Choose** $N := 2N' + 1$**. Then for every** $n > N$**, either** $n = 2k$ **or** $n = 2k + 1$ **for some** $k > N'$ **(since** n **is either even or odd)**

   If $a_n \to a$**, then for every** ϵ**, there is some** $N \in \mathbb{N}$ **such that for all** $n > N$**:** $|a_n - a| < \epsilon$**. In particular,** $2n, 2n + 1 > N$**.**

5. The sequence b_n has b_1 and b_2 positive, and $b_{n+2} = b_n + b_{n+1}$ (note that then $b_n > 0$ for all n). Define $a_n = b_{n+1}/b_n$. Prove that (a_n) converges, and find the limit.

   First, clearly b_n **is increasing. Therefore, for every** $n \geq 2$**:** $a_{n+1} = b_{n+1}/b_n = \frac{b_n + b_{n-1}}{b_n}$ **and** $1 \leq \frac{b_n+b_{n-1}}{b_n} \leq 2b_n/b_n = 2$**. So** $1 \leq a_n \leq 2$**.**

   For all $n \in \mathbb{N}$**:**

$$a_{n+1} = \frac{b_{n+2}}{b_{n+1}} = \frac{b_{n+1} + b_n}{b_n} = 1 + 1/a_n.$$

   So

$$a_{n+2} = 1 + \frac{1}{1 + \frac{1}{a_n}}$$

   We will prove that $a_{2n} \to L$ **and** $a_{2n+1} \to L$ **for some** L**, so by Question 4,** $a_n \to L$**.**

   To prove a_{2n} **converges, we will prove it is monotone (either increasing or decreasing).**

   - **If** $a_4 \geq a_2$**: we will prove by induction, that** a_{2n} **is increasing: i.e, for all** $n \in \mathbb{N}$**:** $a_{2n+2} \geq a_{2n}$**. The base case** $n = 1$ **is the assumption of this item. Assume** $a_{2n+2} \geq a_{2n}$ **and we want to show** $a_{2n+4} \geq a_{2n+2}$**:**

$$a_{2n+4} = 1 + \frac{1}{1 + \frac{1}{a_{n+2}}} \geq 1 + \frac{1}{1 + \frac{1}{a_n}} = a_{2n}.$$

   - **If** $a_4 \leq a_2$**: The same proof shows that** a_{2n} **is decreasing, replacing** $\geq$ **with** $\leq$**.**

So a_{2n} is monotone and bounded, hence converges, to some limit $2 \geq L \geq 1$ (since $2 \geq a_{2n} \geq 1$ for all $n \in \mathbb{N}$.) Clearly $a_{2n+2} \to L$, but $a_{2n+2} = 1 + \frac{1}{1+\frac{1}{a_{2n}}}$. So if $a_{2n} \to L$, then $L = 1 + \frac{1}{1+\frac{1}{L}}$, which is equivalent to $L^2 - L - 1 = 0$ and the only positive solution is $L = (1+\sqrt{5})/2 = \Phi$.

A similar argument yields $a_{2n+1} \to \Phi$.