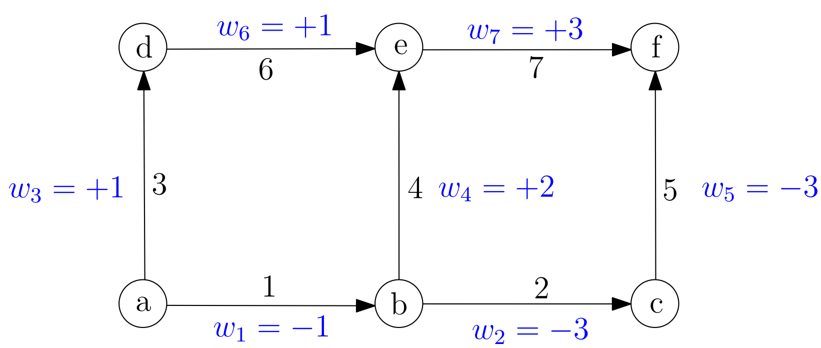
Electric circuit theory and resistor networks

Note: questions with an asterisk * require numerical computation and are optional.

1. Consider the following electrical circuit with edge currents given by

$$\mathbf{w} = (-1, -3, +1, +2, -3, +1, +3)^T.$$



(a) Show that this current $\mathbf{w}$ satisfies Kirchhoff's current law at all the nodes in the circuit.

(b) Find a basis for the left null-space of the incidence matrix for the graph associated with this circuit.

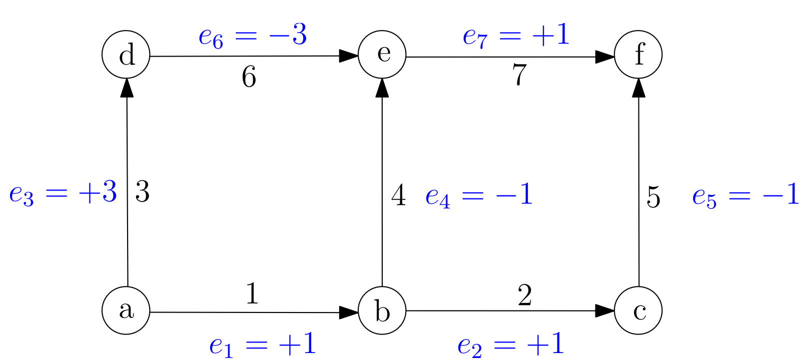
(c) What is the connection between your answers to parts (a) and (b)?

(d) Can you find a set of voltages at the 6 nodes of the circuit that, under Ohm's law and assuming all edges have unit conductance, will give rise to the current $\mathbf{w}$? If so, find this set of voltages.

(e) Suppose now that another current $\mathbf{e}$ flows in the circuit as shown in the figure below:

$$\mathbf{e} = (+1, +1, +3, -1, -1, -3, +1)^T.$$

Does this current satisfy Kirchhoff's current law at all the nodes? If not, find the net current out of each node.

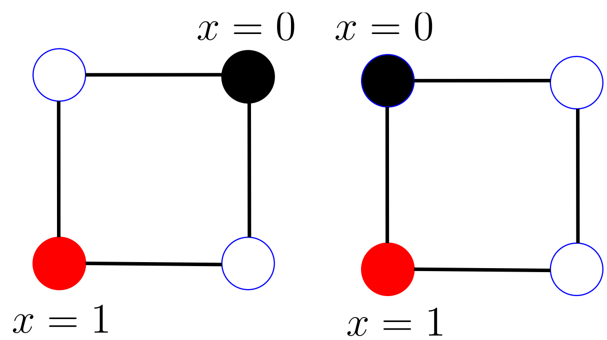


(f) Can you find a set of voltages at the 6 nodes of the circuit that, under Ohm's law and assuming unit conductance of all edges, will give rise to the current **e**? If so, find this set of voltages.

(g) Calculate $\mathbf{e}^T\mathbf{w}$ by direct calculation.

(h) Are you surprised by your answer to (g)? Could you have foreseen the answer, given you answers to parts (a)–(f), without calculating it directly as you did in part (g)?

2. Consider the two 4-node electrical circuits shown in the figure:

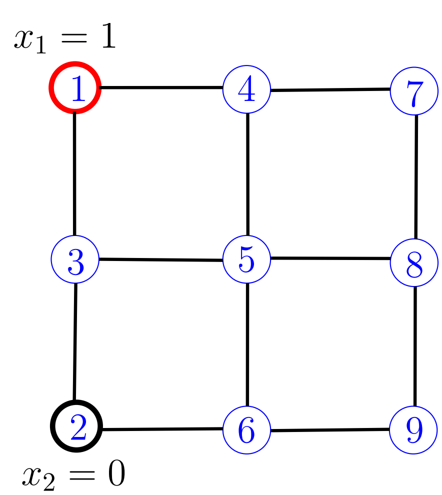


The lower left hand node in each circuit is held at unit voltage $x = 1$. The two circuits differ in which node is grounded. All edges have unit conductance.

(a) By reducing to simpler equivalent circuits, find the effective conductance of each circuit.

(b) Verify your answers in part (a) by solving a linear system for the unknown voltages in each (full) circuit and finding the net current coming out of the red node.

3. Consider the 3-by-3 square electrical circuit given by the graph shown in the figure:



Node 1 is held at unit voltage $x_1 = 1$ and node 2 is grounded $x_2 = 0$. Each edge is a resistor with unit conductance.

(a) Find the Laplacian matrix $\mathbf{K}$ for this graph.

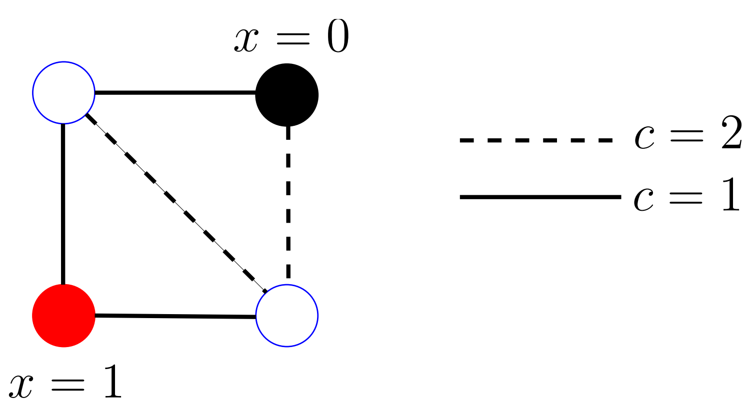
(b) By decomposing $\mathbf{K}$ into the block form

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

for suitable matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ (which you should write down), find an analytical expression for the effective conductance of this circuit in terms of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ and any other vector quantities that you might need to introduce.

(c)* Evaluate the analytical expression you found in part (b) to find a numerical value for the effective conductance.

4. Consider the electrical circuit given by



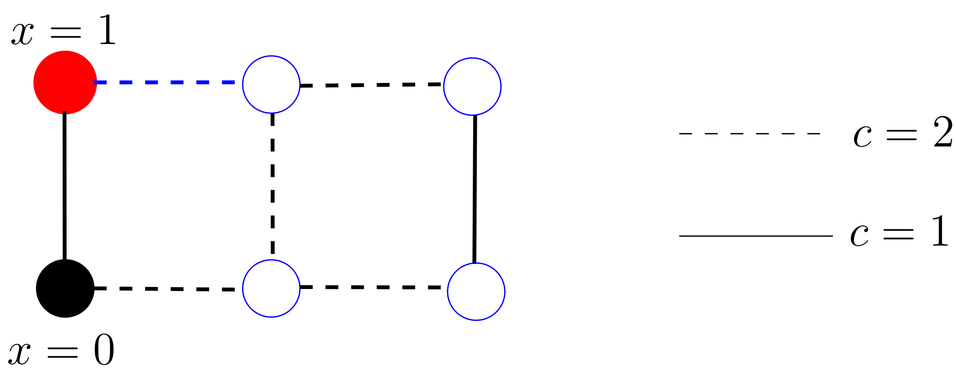
The wires, or edges, in this graph are resistors of two types having conductance $c = 1$ (shown as solid lines) and conductance $c = 2$ (shown as dashed lines). The red node is set to unit voltage $x = 1$ and the black node is grounded $x = 0$. Kirchhoff's current law holds at all other nodes. Find the effective conductance of this circuit.

5. In lectures we learned how to *directly* write down the Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{A}$ without the need to first find the incidence matrix $\mathbf{A}$. In the context of an electrical circuit this Laplacian matrix corresponds to the situation where all the conductances of the edges of the graph equal 1.

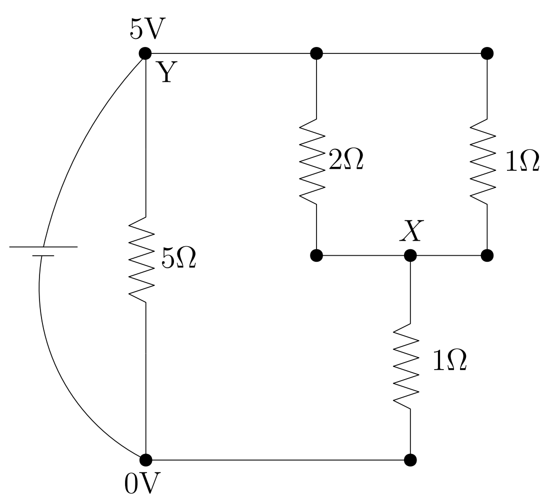
Can you identify how to *directly* write down the weighted Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ associated with a circuit where the diagonal matrix of conductances $\mathbf{C}$ is not the identity but is

$$\mathbf{C} = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ & & \cdots & & \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix}?$$

6. The figure shows another circuit where the red node is held at unit voltage and the black node is grounded. Kirchhoff's current law holds at all other nodes. The edges given by a solid line have unit conductance, those given by a dashed line have conductance 2. Find the effective conductance of the circuit by reducing the circuit to a system of "equivalent" circuits.



7. In electrical engineering it is common to draw circuit figures looking like this:



A zig-zag line indicates the presence of a resistor (resistance is measured in Ohms Ω). In this notation, if no resistor is shown along an edge – i.e., no zig-zag line – then the resistance of that edge is taken to be **zero**.

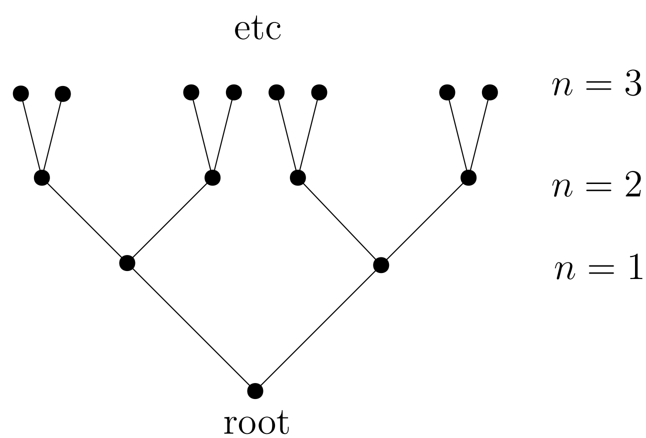
Note that the values of the *resistances* are given (not the conductances).

A battery (indicated by the symbol comprising two parallel lines of unequal length) ensures that the two sides of the 5 Ohm resistor have voltages 5 (at point Y) and 0 and this forces a current through the circuit.

(a) Find the voltage at point X.

(b) Find the current flowing into the circuit at node Y.

8. The figure shows a "tree" – that is, a graph without loops – which starts with a "root" having two branches; at the end of each branch, a copy (or next generation) of this original tree is copied; this process is then repeated. The figure shows $n = 3$ generations of such a tree.



(a) Suppose the tree with 3 generations as shown in the figure is an electrical circuit. Suppose that the root is set at unit voltage and all 8 nodes in the 3rd generation are grounded. What is the effective conductance, $C_{\text{eff}}^{(3)}$, of this circuit?

(b) Can you find the effective conductance, $C_{\text{eff}}^{(n)}$, if the root is set to unit voltage and all the nodes in the n-th generation are grounded?

(c) What is the limiting result $C_{\text{eff}}^{(\infty)}$?