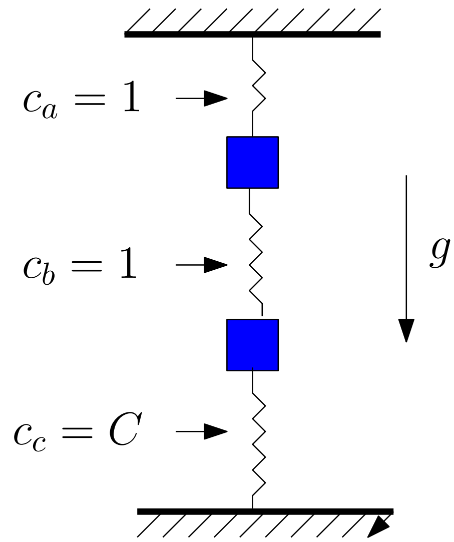
Spring-mass systems

1. Consider the arrangement of 2 masses and 3 springs between two fixed walls. The masses both have mass m and gravity acts on them. The spring constants are $c_a = 1, c_b = 1$ and $c_c = C$.



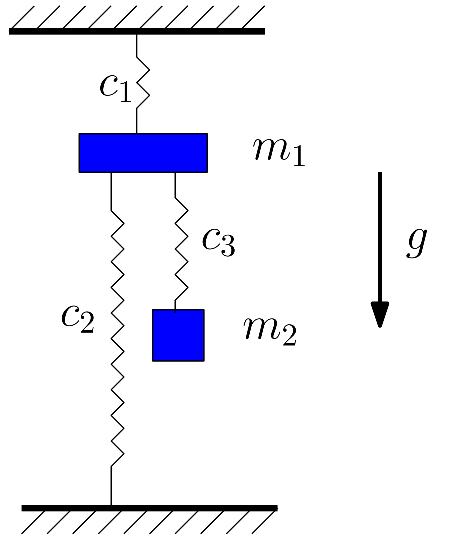
(a) By considering the system as having 4 nodes (the two masses, and the two walls) conneced by 3 edges, write down the weighted Laplacian for this spring-mass system.

(b) Assuming the system is in equilibrium, solve for the displacements of the two masses.

(c) What are the limiting values of these displacements as $C \to 0$ and as $C \to \infty$?

(d) Compute the internal forces due to the springs and find the limiting values as $C \to 0$ and ∞.

2. Another arrangement of 2 masses and 3 springs between two fixed walls is shown in the figure below. The masses are m_1 and m_2 and gravity acts on them. The spring constants are as shown in the figure.



(a) By considering the system as having 4 nodes (the two masses, and the two walls) conneced by 3 edges, write down the weighted Laplacian $\mathbf{K}$ for this spring-mass system.

(b) Assuming the system is in equilibrium, solve for the displacements of the two masses.

(c) Find the reaction forces at the two walls.

(d) Can you find the natural frequencies of free oscillation of this system?

3. According to Newton's second law, the equations for free oscillations of a line of N masses connected by springs has the form

$$-\hat{\mathbf{K}}\mathbf{x} = \mathbf{M}\frac{d^2\mathbf{x}}{dt^2},$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & \cdots & 0 & m_N \end{bmatrix}$$

where $m_j > 0$ for $j = 1, \cdots, N$ and where $\hat{\mathbf{K}}$ is the relevant N-by-N submatrix of the weighted Laplacian.

(a) Show that the natural frequencies of this system are related to eigenvalues of the matrix $\mathbf{M}^{-1}\hat{\mathbf{K}}$.

(b) Show that, unless all the system masses are equal, $\mathbf{M}^{-1}\hat{\mathbf{K}}$ is not a symmetric matrix.

(c) Given the lack of symmetry, can we be sure that there are N natural frequencies of the system as we know is true if all the masses are equal?

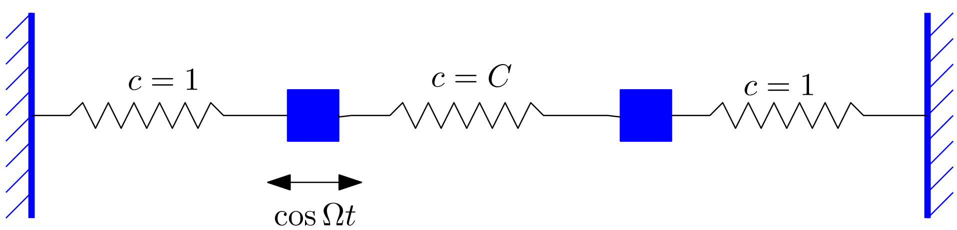
4. Show that for the dynamical system for free oscillations considered in question 3 the following quantity is conserved:

$$\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\hat{\mathbf{K}}\mathbf{x},$$

where we use the notation

$$\dot{\mathbf{x}} \equiv \frac{d}{dt}\mathbf{x}.$$

5. Consider the system of two masses of unit mass and three springs between two fixed walls. The values of the spring constants are indicated in the figure. The mass on the left is subject to an oscillatory external forcing of strength $\cos\Omega t$; there is no external force on the other mass.



(a) Find the matrix form of the system of second order ordinary differential equations satisfied by the 2-vector $\mathbf{x}$ of displacements of the two masses.

(b) Find the general solution of this system, for a *generic value* of the forcing frequency Ω, by finding a particular solution of the form

$$\mathbf{x}^{PS} = \mathbf{\Phi}\cos\Omega t$$

and adding this to a solution of the unforced system.

(c) Are there any values of the forcing frequency Ω for which your solution in part (b) breaks down?

6. The external forcing in the system in question 5 is now removed and free oscillation of the system is initiated by moving the mass to the right a distance A to the right with both masses initially at rest.

(a) If $\mathbf{x}$ is a vector $[\phi_1(t)\ \phi_2(t)]^T$ containing the displacments of the left mass and right mass respectively, show that the solutions for the subsequent displacements of the two masses can be written as

$$\mathbf{x} = \begin{bmatrix} A\sin\Omega t\sin\epsilon t \\ A\cos\Omega t\cos\epsilon t \end{bmatrix}$$

and find the values of Ω and ϵ.

(b) What condition on C is required in order that $\epsilon \ll \Omega$?

(c) If $\epsilon \ll \Omega$ draw graphs of the displacements $\phi_1(t)$ and $\phi_2(t)$ as functions of time and describe the nature of the oscillations.