Problem Sheet 3

Math40002, Analysis 1

1. Give an example of a sequence of functions $f_1, f_2, f_3, \cdots : \mathbb{R} \to \mathbb{R}$ and constants $M_1, M_2, M_3, \cdots \in \mathbb{R}$ such that $|f_i(x)| \le M_i$ for all $x \in \mathbb{R}$ and the sum $\sum_{i=1}^{\infty} M_i$ converges, but $\sum_{i=1}^{\infty} f_i(x)$ is *not* continuous.

2. (a) Show that $f(x) = x^{1/2}$ is differentiable on $(0, \infty)$, and compute its derivative.
   (b) Do the same for $f(x) = x^{1/n}$, where n is any positive integer.
   (c) Now do the same for $f(x) = x^{m/n}$, where m and n are positive integers.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Hölder continuous* with exponent $\alpha > 0$ if there is a constant $C \ge 0$ such that
$$|f(x) - f(y)| \le C|x - y|^{\alpha}$$
for all $x, y \in \mathbb{R}$. Show that if $\alpha > 1$ then f is differentiable, and $f'(x) = 0$.
Remark: We will see in lecture soon that if $f' \equiv 0$ then f must be constant.

4. Find all $x \in \mathbb{R}$ where $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ x^2, & x \in \mathbb{Q} \end{cases}$ is differentiable and compute its derivative.

5. (a) Show, using $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, that
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 1.$$
   (b) Use the angle addition formulas to prove that $\sin(x)$ and $\cos(x)$ are differentiable and determine their derivatives. (Note: you may *not* just differentiate the power series term by term, because we have not proved that this gives the right answer.)

6. Recall that we defined $\log : (0, \infty) \to \mathbb{R}$ as the inverse function of e^x.
   (a) Using only this and formal properties of e^x, prove for $x > 0$ and $0 < |h| < x$ that
$$\frac{\log(x + h) - \log(x)}{h} = \frac{1}{x} \frac{\log\left(1 + \frac{h}{x}\right)}{h/x}.$$

(b) Prove by a substitution that $\lim_{y\to 0} \frac{\log(1+y)}{y} = \lim_{x\to 0} \frac{x}{e^x - 1}$, and that the latter limit is 1. (Hint: use the power series definition of e^x to evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$.)

(c) Show that $\log(x)$ is differentiable, and find its derivative.

7. (*) The goal of this problem is to construct a continuous function which is not differentiable anywhere. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$ for $-1 \le x \le 1$ and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. Then define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i f(4^i x).$$

(a) Draw a graph of $f(x)$, and convince yourself that it is continuous.

(b) Prove that g is continuous.

(c) Fix $x \in \mathbb{R}$ and an integer $n \in \mathbb{N}$. Let ϵ_n be $+\frac{1}{2}$ if there is no integer in the interval $(4^n x, 4^n x + \frac{1}{2})$, or $-\frac{1}{2}$ if there is no integer in $(4^n x - \frac{1}{2}, 4^n x)$. Check that one of these is always possible, and then define

$$d_i(x) = \frac{f(4^i(x + \frac{\epsilon_n}{4^n})) - f(4^i x)}{\epsilon_n / 4^n}.$$

Show that $|d_i(x)| = 4^i$ for all $i \le n$, and that $d_i(x) = 0$ for all $i > n$.

(d) Prove that $\left|\frac{g(x + \frac{\epsilon_n}{4^n}) - g(x)}{\epsilon_n / 4^n}\right| \ge 3^n - (3^{n-1} + 3^{n-2} + \cdots + 1) = \frac{3^n + 1}{2}$. Conclude that g is not differentiable at x.