Math40002 Analysis 1 **Term 2 Unseen 7**

Throughout this sheet, we assume $f : [a, b] \to \mathbb{R}$ is a bounded function.

1. Prove that there exists $M \geq 0$ such that for all x, x', y, y' such that $a \leq x \leq x' \leq y' \leq y \leq b$:
$$\sup_{x \leq t \leq y} f(t) \leq \sup_{x' \leq t \leq y'} f(t) + M \text{ and } \inf_{x \leq t \leq y} f(t) \geq \inf_{x' \leq t \leq y'} f(t) - M.$$

2. Let $M \geq 0$ as promised from Question 1. Let $P' = \{ a = x_0, x_1, \ldots, x_n = b \}$ and $P = \{ a = y_0, y_1, \ldots, y_m = b \}$ be partitions of $[a, b]$.

   (a) Prove that if $y_k < x_i \leq x_j < y_{k+1}$ for some $0 \leq k < m$ and $0 < i \leq j < n$, then:

   i. $\sup_{y_k \leq t \leq y_{k+1}} f(t) \cdot (y_{k+1} - y_k) \leq$
$$\sup_{x_j \leq t \leq y_{k+1}} f(t) \cdot (y_{k+1} - x_j) + \sum_{l=i}^{j-1} \sup_{x_l \leq t \leq x_{l+1}} f(t) \cdot (x_{l+1} - x_l) + \sup_{y_k \leq t \leq x_i} f(t) \cdot (x_i - y_k) + M \cdot (y_{k+1} - y_k).$$

   ii. $\inf_{y_k \leq t \leq y_{k+1}} f(t) \cdot (y_{k+1} - y_k) \geq$
$$\inf_{x_j \leq t \leq y_{k+1}} f(t) \cdot (y_{k+1} - x_j) + \sum_{l=i}^{j-1} \inf_{x_l \leq t \leq x_{l+1}} f(t) \cdot (x_{l+1} - x_l) + \inf_{y_k \leq t \leq x_i} f(t) \cdot (x_i - y_k) - M \cdot (y_{k+1} - y_k).$$

   (a) Let $I := \{ 0 \leq k < m | \exists 0 < i < n : y_k < x_i < y_{k+1} \}$. Prove $|I| < n$.

   (b) Let $\mu := mesh(P) = \max \{ y_{k+1} - y_k | 0 \leq k < m \}$.
   Prove that $U(f, P) - U(f, P \cup P') < nM\mu$ and $L(f, P \cup P') - L(f, P) < nM\mu$.

3. Prove that for every $\epsilon > 0$ and for every partition $P' = \{ a = x_0, \ldots, x_n = b \}$, there is some $\delta > 0$ such that for every partition P of $[a, b]$:
if $mesh(P) < \delta$ then
$$U(f, P) < U(f, P \cup P') + \epsilon \text{ and } L(f, P) > L(f, P \cup P') - \epsilon.$$

4. Assume f is Darboux integrable. (the definition from the lectures.)
Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that for every partition $P = \{ a = y_0, y_1, \ldots, y_m = b \}$ of $[a, b]$ with $mesh(P) < \delta$ and for every choice of $t_0, \ldots, t_{m-1}$ such that $y_0 \leq t_0 \leq y_1 \leq t_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq t_{m-1} \leq y_m$:
$$\left| \sum_{i=0}^{m-1} f(t_i) \cdot (y_{i+1} - y_i) - \int_a^b f(x)\, dx \right| < \epsilon.$$

   Note: this is the definition of Riemann integrability, and the sum above is called the Riemann sum.

5. Find $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e^{i/n}$.

6. Find $\lim_{n\to\infty} \frac{\pi}{2n} \sum_{i=1}^{n} \cos\left(\frac{\pi}{2} + \frac{\pi i}{2n}\right)$.