M1M2 : Unseen 3 : Difference Equations

3).

(a). $2U(n+2) - 7U(n+1) + 3U(n) = 5^n$ (1)

Auxiliary equation: $2\lambda^2 - 7\lambda + 3 = 0$

$\iff (2\lambda - 1)(\lambda - 3) = 0$

$\iff \underline{\lambda = \frac{1}{2}}$ or $\underline{\lambda = 3}$

$\Rightarrow$ Complementary function: $U_{CF}(n) = A(3)^n + B(2)^{-n}$

Try $U_{PI}(n) = C5^n$. Substituting into (1):

$2C(5)^{n+2} - 7C(5)^{n+1} + 3C(5)^n = 5^n$

$\iff [50 - 35 + 3]\,C(5)^n = 5^n$

$\iff 18C = 1$

$\iff \underline{C = \frac{1}{18}}$

$\Rightarrow$ General Solution: $\underline{\underline{U(n) = A(3)^n + B(2)^{-n} + \frac{1}{18}(5)^n}}$

(b). Fibonacci : $U(n+2) - U(n+1) - U(n) = 0$

Auxiliary equation: $\lambda^2 - \lambda - 1 = 0$

$$\Leftrightarrow \lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\Rightarrow$ General solution: $U(n) = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Looking at the first terms of the sequence, we know:

$$U(1) = 1 \Rightarrow 1 = A\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right) + B\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$$

$$\Leftrightarrow (A+B) + \sqrt{5}(A-B) = 2 \quad (2)$$

$$U(2) = 1 \Rightarrow 1 = \frac{1}{4}A(6+2\sqrt{5}) + \frac{1}{4}B(6-2\sqrt{5})$$

$$\Leftrightarrow 3(A+B) + \sqrt{5}(A-B) = 2 \quad (3)$$

$(3)-(2)$: $2(A+B) = 0 \Leftrightarrow A = -B$

Sub. into (2): $2\sqrt{5}A = 2 \Leftrightarrow A = \frac{1}{\sqrt{5}}$ so $B = \frac{-1}{\sqrt{5}}$

So the general solution is indeed:

$$\boxed{U(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}$$

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(c). $S(n) = 1^2 + 2^2 + \dots + n^2$

$S(n+1) = 1^2 + 2^2 + \dots + n^2 + (n+1)^2$

$\Rightarrow \underline{S(n+1) - S(n) = (n+1)^2 = n^2 + 2n + 1}$

Auxiliary equation: $\lambda - 1 = 0 \Leftrightarrow \underline{\lambda = 1}$

$\Rightarrow$ Complementary function: $\underline{S_{CF}(n) = A(1)^n = A}$, A constant.

Try: $\underline{S_{PI}(n) = an^3 + bn^2 + cn}$ (one degree higher than RHS since CF is part of RHS)

Substituting in:

$$a(n^3 + 3n^2 + 3n + 1) + b(n^2 + 2n + 1) + c(n+1)$$
$$- an^3 - bn^2 - cn = n^2 + 2n + 1$$

$$\Leftrightarrow (a-a)n^3 + (3a + b - b)n^2 + (3a + 2b + c - c)n$$
$$+ (a + b + c) = n^2 + 2n + 1$$

$$\Leftrightarrow 3an^2 + (3a + 2b)n + (a + b + c) = n^2 + 2n + 1$$

Comparing coefficients: n^2: $3a = 1 \Leftrightarrow \underline{a = \frac{1}{3}}$

n: $3a + 2b = 2 \Rightarrow 1 + 2b = 2 \Leftrightarrow \underline{b = \frac{1}{2}}$

1: $a + b + c = 1 \Rightarrow \frac{5}{6} + c = 1 \Leftrightarrow \underline{c = \frac{1}{6}}$

$$\Rightarrow S(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + A$$

But $S(1) = 1 \Rightarrow 1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} + A \Leftrightarrow \underline{A = 0}$

$$\Rightarrow \boxed{S(n) = \frac{1}{6}n(2n^2 + 3n + 1) = \underline{\underline{\frac{1}{6}n(2n+1)(n+1)}}}$$

(d). $U(1) = -4,\ U(2) = -4$

$$U(n+2) - 2U(n+1) + 2U(n) = 0$$

(i). Auxiliary equation: $\lambda^2 - 2\lambda + 2 = 0 \iff \lambda = \dfrac{2 \pm \sqrt{4-8}}{2}$

$$\iff \underline{\lambda = 1 \pm i}$$

$$\Rightarrow \underline{U(n) = A(1+i)^n + B(1-i)^n}$$

$U(1) = -4 \Rightarrow \underline{-4 = (A+B) + i(A-B)}$ ④

$U(2) = -4 \Rightarrow -4 = A(2i) + B(-2i)$

$$\iff \underline{i(A-B) = -2} \quad ⑤$$

④ − ⑤: $\underline{A + B = -2}$

Sub into ⑤: $i(A - (-2-A)) = -2$

$$\iff i(A+1) = -1$$

$$\iff \underline{A = i - 1}, \text{ so: } \underline{B = -i-1}$$

$$\Rightarrow \boxed{U(n) = (-1+i)(1+i)^n - (1+i)(1-i)^n}$$

(ii). This can be written as:

$\underline{\underline{U(n) = 2\,\mathrm{Re}\{(-1+i)(1+i)^n\}}}$, using the fact that $\dfrac{z + \bar{z}}{2} = \mathrm{Re}\{z\}$ with $z = (-1+i)(1+i)^n$.

Hence clearly $U(n)$ is always a real number for all n.

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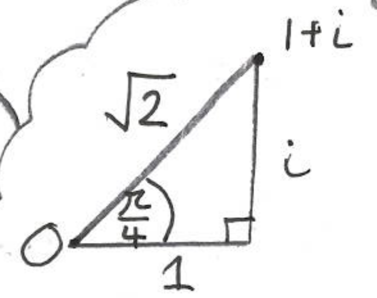
(iii) $U(n) = 2\,\mathrm{Re}\left\{(-1+i)(1+i)^n\right\}$



$(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$= 2\,\mathrm{Re}\left\{(-1+i)(\sqrt{2})^n\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^n\right\}$$

$$= 2\,\mathrm{Re}\left\{(-1+i)(\sqrt{2})^n\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right]\right\},$$

by using De Moivre's theorem.

$$\Rightarrow U(n) = 2(\sqrt{2})^n\,\mathrm{Re}\left\{-\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4} + i\cos\frac{n\pi}{4} - \sin\frac{n\pi}{4}\right\}$$

$$= \underline{-2(\sqrt{2})^n\left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]}$$

or can change to a cosine

If we let: $\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4} = R\sin\left(\frac{n\pi}{4} + \beta\right)$

$$= R\left(\sin\frac{n\pi}{4}\cos\beta + \cos\frac{n\pi}{4}\sin\beta\right)$$

$$\Rightarrow \left.\begin{aligned} R\cos\beta &= 1 \\ R\sin\beta &= 1 \end{aligned}\right\} \Rightarrow \underline{R = \sqrt{2}},\ \underline{\beta = \frac{\pi}{4}}$$

$$\Rightarrow U(n) = -2(\sqrt{2})^n\left[\sqrt{2}\sin\left(\frac{n\pi}{4} + \frac{\pi}{4}\right)\right]$$

$$= \underline{-2(\sqrt{2})^{n+1}\sin\left(\frac{\pi}{4}(n+1)\right)}$$

This is equal to 0 whenever $\frac{\pi}{4}(n+1) = k\pi,\quad k \in \mathbb{Z}^+$

$$\Leftrightarrow n + 1 = 4k$$

$$\Leftrightarrow \underline{\underline{n = 4k - 1}}, \quad k \in \mathbb{Z}^+$$

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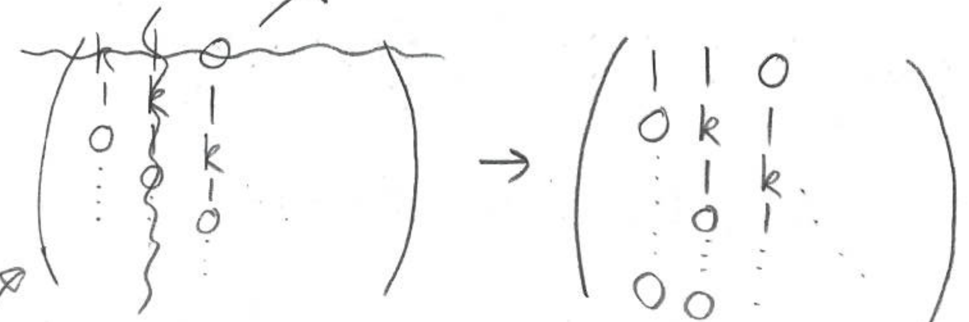
(e). Let $\det(M_n) = M(n)$. Then:

$$M(n) = kM(n-1) - \left[M(n-2) - 0\right]$$

First part of the determinant



For the second part remaining matrix to take det of looks like this!

So: $M(n) - kM(n-1) + M(n-2) = 0$

or: $\underline{M(n+2) - kM(n+1) + M(n) = 0}$, $k = 2\cosh\theta$.

Auxiliary equation: $\lambda^2 - (2\cosh\theta)\lambda + 1 = 0$

$$\Leftrightarrow \lambda = \frac{2\cosh\theta \pm \sqrt{4\cosh^2\theta - 4}}{2}$$

$$= \underline{\cosh\theta \pm \sinh\theta}$$

$$\Rightarrow M(n) = A(\cosh\theta + \sinh\theta)^n + B(\cosh\theta - \sinh\theta)^n$$

$$= \underline{Ae^{n\theta} + Be^{-n\theta}} \quad (6)$$

$\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$

$\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

$\Rightarrow \cosh\theta \pm \sinh\theta = e^{\pm\theta}$

Now:

$M(1) = \det(M_1) = |k| = k = 2\cosh\theta$

$\underline{= e^{\theta} + e^{-\theta}}$

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$$M(2) = \det(M_2) = \begin{vmatrix} k & 1 \\ 1 & k \end{vmatrix} = k^2 - 1 = 4\cosh^2\theta - 1$$

$$= (e^{\theta} + e^{-\theta})^2 - 1 = \underline{e^{2\theta} + e^{-2\theta} + 1}$$

sub $n=1$ in ⑥

Therefore:

$$M(1) = e^{\theta} + e^{-\theta} = Ae^{\theta} + Be^{-\theta} \Leftrightarrow \underline{e^{2\theta} + 1 = Ae^{2\theta} + B} \quad ⑦$$

$$M(2) = e^{2\theta} + e^{-2\theta} + 1 = Ae^{2\theta} + Be^{-2\theta} \Leftrightarrow \underline{e^{4\theta} + e^{2\theta} + 1 = Ae^{4\theta} + B} \quad ⑧$$

⑧ − ⑦: $Ae^{2\theta}(e^{2\theta} - 1) = e^{4\theta}$

$$\Leftrightarrow \underline{A = \frac{e^{2\theta}}{e^{2\theta} - 1}}, \quad \text{Sub into ⑦: } e^{2\theta} + 1 = \frac{e^{4\theta}}{e^{2\theta} - 1} + B$$

$$\Rightarrow B = \frac{(e^{2\theta}+1)(e^{2\theta}-1) - e^{4\theta}}{e^{2\theta} - 1} = \underline{\frac{-1}{e^{2\theta} - 1}}.$$

Therefore: $$M(n) = \frac{e^{(n+2)\theta} - e^{-n\theta}}{e^{2\theta} - 1} \times \frac{e^{-\theta}}{e^{-\theta}}$$

$$= \frac{e^{(n+1)\theta} - e^{-(n+1)\theta}}{e^{\theta} - e^{-\theta}} = \frac{\left(\frac{e^{(n+1)\theta} - e^{-(n+1)\theta}}{2}\right)}{\left(\frac{e^{\theta} - e^{-\theta}}{2}\right)}$$

$$= \boxed{\frac{\sinh(n+1)\theta}{\sinh\theta}}$$