

4 Topics: Random variables and their distributions

4.1 Prerequisites: Lecture 10

Exercise 4- 1: (Suggested for personal/peer tutorial) Poisson approximation to the Binomial: If $X \sim \text{Bin}(n, p)$ and we have $n \rightarrow \infty$ and $p \rightarrow 0$ such that $\lambda = np$ remains constant, then the p.m.f. of X converges to the p.m.f. of a $\text{Poi}(\lambda)$ random variable. The same result holds, when for $n \rightarrow \infty$ and $p \rightarrow 0$, we have that np converges to a positive constant λ .

Hint: Use the result that for all $t \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^n = e^{-t}.$$

Solution: Consider the case when $\lambda = np$ is fixed when $n \rightarrow \infty$ and $p \rightarrow 0$. Let $0 \leq k \leq n$, then

$$\begin{aligned} P(X = k) &= \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= \frac{1}{k!} n(n-1) \cdots (n-k+1) \frac{n^k}{n^k} p^k (1-p)^{n-k} \\ &= \frac{\lambda^k}{k!} n(n-1) \cdots (n-k+1) \frac{1}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \frac{n(n-1) \cdots (n-k+1)}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \end{aligned}$$

Then, for fixed k :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n(n-1) \cdots (n-k+1)}{n^k} &= \lim_{n \rightarrow \infty} 1 \cdot (1 - 1/n) \cdots (1 - (k-1)/n) = 1, \\ \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n &= e^{-\lambda}, \\ \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} &= 1. \end{aligned}$$

Hence

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \rightarrow \frac{\lambda^k}{k!} e^{-\lambda}, \text{ as } n \rightarrow \infty,$$

where the right hand side is indeed the p.m.f. of a Poisson random variable with parameter λ .

Exercise 4- 2: A company wishes to make two of a group of six employees, comprising three female and three male employees, redundant, by selecting two employees at random. Let X and Y be the random variables corresponding to the number of female and male employees made redundant, respectively.

Find the probability mass functions of X and Y .

Solution: For both variables, the range is $\{0, 1, 2\}$, and distribution is given by Hypergeometric formula with $N = 6$, $K = 3$ and $n = 2$. Hence

$$p_X(x) = p_Y(x) = \frac{\binom{3}{x} \binom{3}{2-x}}{\binom{6}{2}} \quad x = 0, 1, 2$$

and zero otherwise.

Exercise 4- 3: Five balls numbered 1,2,3,4 and 5 are placed in a bag. Two balls are selected without replacement. Find the probability mass function of the following random variables:

- (a) X = the largest of the two selected numbers,
- (b) Y = the sum of the two selected numbers

Solution:

- (a) Range $\text{Im}X = \{2, 3, 4, 5\}$. Now $p_X(x) = P(X = x) = \text{card}(E)/\text{card}(\Omega)$, say, and $\text{card}(E)$ = “number of ways of choosing two from five with largest equal to x ” = $x - 1$, $\text{card}(\Omega)$ = “number of ways of choosing two from five” = $\binom{5}{2} = 10$. So $p_X(x) = P(X = x) = (x - 1)/10$.
- (b) Range $\text{Im}Y = \{3, 4, 5, 6, 7, 8, 9\}$. As above, define $p_Y(y) = P(Y = y) = \text{card}(E)/\text{card}(\Omega)$, say, and again $\text{card}(\Omega)$. Enumeration of $\text{card}(E)$ achieved by considering distinguishable partitions of y into the sum of two integers in the range $\{1, 2, 3, 4, 5\}$. Hence if $y = 3, 4, 8, 9$, $\text{card}(E) = 1$, but if $y = 5, 6, 7$, $\text{card}(E) = 2$, so

$$p_Y(y) = \begin{cases} 1/10 & y = 3, 4, 8, 9 \\ 2/10 & y = 5, 6, 7 \end{cases}$$

Exercise 4- 4: A surgical procedure is successful with probability θ . The surgery is carried out on five patients, with the success or failure of each operation independent of all other operations. Let X be the discrete random variable corresponding to the number of successful operations.

Find the probability mass function of X , and evaluate the probability that

- (a) all five operations are successful, if $\theta = 0.8$,
- (b) exactly four operations are successful, if $\theta = 0.6$,
- (c) fewer than two are successful, if $\theta = 0.3$.

Solution: $X \sim \text{Bin}(n, \theta)$, so

- (a) $\theta = 0.8, P(X = 5) = 0.3227$
- (b) $\theta = 0.6, P(X = 4) = 0.2592$
- (c) $\theta = 0.3, P(X < 2) = P(X = 0) + P(X = 1) = 0.5282$

Exercise 4- 5: If X has a Geometric distribution with parameter θ , so that

$$p_X(x) = (1 - \theta)^{x-1}\theta, \quad x = 1, 2, 3, \dots$$

and zero otherwise, show that, for $n, k \geq 1$,

$$P(X = n + k | X > n) = P(X = k).$$

This result is known as the *Lack of Memory* property (for a discrete random variable).

Solution: If $X \sim \text{Geo}(\theta)$, then

$$p_X(x) = (1 - \theta)^{x-1}\theta,$$

and

$$P(X \leq x) = 1 - (1 - \theta)^x, \text{ for } x \in \{1, 2, 3, \dots\}.$$

Thus $P(X > n) = (1 - \theta)^n$, and hence

$$\begin{aligned} P(X = n + k | X > n) &= \frac{P(X = n + k, X > n)}{P(X > n)} = \frac{P(X = n + k)}{P(X > n)} = \frac{(1 - \theta)^{n+k-1}\theta}{(1 - \theta)^n} \\ &= (1 - \theta)^{k-1}\theta = P(X = k). \end{aligned}$$

4.2 Prerequisites: Lecture 11

Exercise 4- 6: Suppose $X \sim \text{DUnif}(\{1, \dots, n\})$. Find the c.d.f. of X .

Solution: We have $P(X = x) = 1/n$ for $x \in \{1, \dots, n\}$ and zero otherwise. Hence

$$F_X(x) = P(X \leq x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{\lfloor x \rfloor}{n}, & \text{if } 0 \leq x < n, \\ 1, & \text{if } x \geq n, \end{cases}$$