Math40003 Linear Algebra and Groups   Problem Sheet 5

1. (a) Let U and W be the following subspaces of $\mathbb{R}^4$:

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_4 = -x_1 + x_2 + x_3 = 0\},$$
$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 + x_3 - x_4 = -x_1 + 2x_2 + x_3 + x_4 = 0\}.$$

Find a basis for $U \cap W$. Find bases for U and for W, both of which contain your basis for $U \cap W$. Find a basis for $U + W$ containing your basis for $U \cap W$.

(b) Let X and Y be the following subspaces of $\mathbb{R}^4$:

$$X = \mathrm{Span}\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\},$$
$$Y = \mathrm{Span}\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}.$$

Find bases for $X \cap Y$ and $X + Y$.

(c) Let X and Y be as in part (ii). Find a subspace Z of $\mathbb{R}^4$ with the properties that $\mathbb{R}^4 = X + Z = Y + Z$, and $X \cap Z = Y \cap Z = \{0_V\}$.

(a) An element (x_1, x_2, x_3, x_4) of $U \cap W$ satisfies the four equations

$$x_1 + x_2 + x_4 = 0, \qquad -x_1 + x_2 + x_3 = 0,$$

$$2x_1 + x_3 - x_4 = 0 \qquad -x_1 + 2x_2 + x_3 + x_4 = 0.$$

Expressing these equations in an augmented matrix and row reducing,

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ -1 & 2 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

we see that a general solution to these equations is $(0. -t, t, t)$, and so a basis for $U \cap W$ is $\{(0, -1, 1, 1)\}$.

There are many possible choices for the other bases. One possibility is $\{(0, -1, 1, 1), (1, -1, 2, 0)\}$ for U, $\{(0, -1, 1, 1), (2, -1, 0, 4)\}$ for W, and $\{(0, -1, 1, 1), (1, -1, 2, 0), (2, -1, 0, 4)\}$ for $U + W$.

(b) By the row-space method (or otherwise), we find that a basis for X is $\{(1, 1, 0, -1), (1, 2, 3, 0)\}$ (there are many other possibilities here). Call these vectors u_1, u_2. A basis for Y is $\{(1, 2, 2, -2), (2, 3, 2, -3)\}$; call these w_1, w_2. A vector v lies in $X \cap Y$ if $v = x_1u_1 + x_2u_2 = x_3w_1 + x_4w_2$ for some $x_i \in \mathbb{R}$. Solving this for $x_1, \ldots, x_4$, we get the general solution $(a, 0, -a, a)$ for $a \in \mathbb{R}$. So a general vector of $X \cap Y$ is au_1, and a basis is $\{u_1\} = \{(1, 1, 0, -1)\}$. Now $U + W$ is spanned by $\{u_1, u_2, w_1, w_2\}$, and using the row-space method we get the basis $\{u_1, u_2, w_1\}$ for $U + W$.

(c) One possibility is $Z = \mathrm{Span}\{e_1, e_2\} = \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{R}\}$**.**

2.* (a) Let U and W be 3-dimensional subspaces of $\mathbb{R}^5$, with $U \neq W$. Prove that $\dim U \cap W$ is either 1 or 2. Give examples to show that both possibilities can occur.

(b) Let U_1, U_2 and U_3 be 3-dimensional subspaces of $\mathbb{R}^4$. Give a proof that $\dim U_1 \cap U_2 \geq 2$. Deduce that $U_1 \cap U_2 \cap U_3 \neq \{0_V\}$.

(c) Now let V be the vector space of 2×3 matrices over $\mathbb{R}$. Find subspaces X and Y of V such that $\dim X = \dim Y = 4$, and $\dim X \cap Y = 2$.

(d) Let V be as in part (iii). Do there exist subspaces X and Y of V such that $\dim X = 3$, $\dim Y = 5$, and $\dim X \cap Y = 1$?

(a) By a theorem from lectures, we have

$$\dim U \cap W = \dim U + \dim W - \dim(U + W).$$

Now since $\dim U + W \leq 5$**, and since** $\dim U + \dim W = 6$**, this gives** $\dim U \cap W \geq 1$**. Also since** $U \cap W \subset U$**, we have** $\dim U \cap W < \dim U = 3$**. So** $\dim U \cap W$ **is** 1 **or** 2**.**

Take $U = \mathrm{Span}\{e_1, e_2, e_3\}$**. For an example with** $\dim U \cap W = 1$**, we can take** $W = \mathrm{Span}\{e_3, e_4, e_5\}$**. For an example with** $\dim U \cap W = 2$**, take** $W = \mathrm{Span}\{e_2, e_3, e_4\}$**.**

(b) Once again, we have we have $\dim U_1 \cap U_2 = \dim U + \dim W - \dim U_1 + U_2$**. Since** $\dim U_1 + U_2 \leq 4$**, this gives** $\dim U_1 + U_2 \geq 2$**. Now if it were true that** $U_1 \cap U_2 \cap U_3 = \{0_V\}$**, then**

$$\dim(U_1 \cap U_2) + U_3 = \dim U_1 \cap U_2 + \dim U_3 - \dim U_1 \cap U_2 \cap U_3 \geq 2 + 3 - 0 = 5.$$

But this is impossible, since these are subspaces of $\mathbb{R}^4$**.**

(c) We write E_{ij} **for the matrix with** 1 **in the** ij**-entry and** 0 **elsewhere. Let**

$$X = \mathrm{Span}\{E_{11}, E_{12}, E_{13}, E_{21}\} \textbf{ and } Y = \mathrm{Span}\{E_{11}, E_{12}, E_{22}, E_{23}\}.$$

Then $\dim X = \dim Y = 4$ **and** $\dim X \cap Y = 2$**.**

(d) No such subspaces exist. Else we would have $\dim X + Y = 3 + 5 - 1 = 7$**, but** $X + Y$ **is a subspace of** V**, and** $\dim V = 6$**.**

3. The *rank* of an $m \times n$ matrix A is defined to be the dimension of its row space $\mathrm{RSp}(A)$ and is denoted by $\mathrm{rank}\, A$. Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

(a) Let v be a row vector in $\mathbb{R}^n$. Prove that vB is a linear combination of the rows of B.

(b) Prove that the row space of AB is contained in the row space of B and $\operatorname{rank} AB \leq \operatorname{rank} B$.

(c) Prove that if $m = n$ and A is invertible, then $\operatorname{rank} AB = \operatorname{rank} B$.

(d) Prove that $\operatorname{rank} AB \leq \operatorname{rank} A$.

(a) Let $v = (\lambda_1, \ldots, \lambda_n)$, and let $b_1, \ldots, b_n$ be the rows of B. Then

$$vB = \lambda_1 b_1 + \cdots + \lambda_n b_n,$$

which is a linear combination of the rows of B.

(b) Let $a_1, \ldots, a_m$ be the rows of A. Then the ith row of AB is $a_i B$, which is a linear combination of the rows of B (by part (i)). So every row of AB is in the row-space of B, and hence $\mathrm{R-Span}(AB) \subseteq \mathrm{RSp}(B)$. It follows immediately that $\operatorname{rank} AB \leq \operatorname{rank} B$.

(c) Given that A is invertible, we have $\operatorname{rank} A^{-1}AB \leq \operatorname{rank} AB$, by part (iii). So $\operatorname{rank} B \leq \operatorname{rank} AB$, and this gives $\operatorname{rank} AB = \operatorname{rank} B$ when combined with part (ii).

(d) To get $\operatorname{rank} AB \leq \operatorname{rank} A$ just apply the same argument to the columns: if v is a column vector in $\mathbb{R}^n$, then Av is a linear combination of the columns of A, and hence, for the column spaces, $\mathrm{CSp}(AB) \subseteq \mathrm{CSp}(A)$. Now use the result from the lectures which says that the dimensions of the row space and the column space of a matrix are equal.

4. (a) Find the ranks of the matrices

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ 2 & 3 \end{pmatrix}$$

(b) Find an equation for a and b such that the following matrix has rank 2:

$$\begin{pmatrix} 3 & 2 & 5 \\ 1 & a & -1 \\ 1 & 3 & b \end{pmatrix}.$$

(c) Find an equation for b, c and d such that the matrices

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & -3 & 0 \\ 1 & 1 & 0 & b \\ 2 & -1 & 3 & c \\ 1 & 4 & -2 & d \end{pmatrix}$$

both have the same rank.

(a) Both have rank 2.

(b) The matrix has rank 3 only if it is invertible. Otherwise it has rank 2 (since the rows are clearly not colinear for any a). So the condition for the rank to be 2 is that the determinant is 0. This gives the equation $3ab - 5a - 2b + 22 = 0$.

(c) Using the result that rank is the same as column rank, we require the fourth column to lie in the span of the first three. Solving this, we get the equation $23b - 7c - 6d = 0$.