M1M2 : Unseen 2 : The variation of parameters method

1)
(a). $$\frac{dy}{dx} + p(x)y = 0$$

$$\Leftrightarrow \quad \frac{dy}{dx} = -p(x)y$$

$$\Leftrightarrow \quad \int \frac{1}{y}\,dy = -\int p(x)\,dx \quad \text{(separating the variables)}$$

$$\Leftrightarrow \quad \log y = -\int p(x)\,dx + \text{constant}$$

$$\Leftrightarrow \quad \underline{\underline{y = Ce^{f(x)}}}, \quad \text{where } f(x) = -\int p(x)\,dx$$

(b). Substitute $y(x) = C(x)e^{f(x)}$ into $\frac{dy}{dx} + p(x)y = q(x)$

$$\Rightarrow \quad C'(x)e^{f(x)} + C(x)f'(x)e^{f(x)} + p(x)C(x)e^{f(x)} = q(x),$$

but recall $f(x) = -\int p(x)\,dx$, $\therefore f'(x) = -p(x)$

So therefore: $$C'(x)e^{f(x)} = q(x)$$

$$\Leftrightarrow \quad C'(x) = q(x)e^{-f(x)}$$

Integrating: $$\underline{\underline{C(x) = \int q(x)e^{-f(x)}\,dx + K}}, \quad K \text{ constant}$$

(c). The general solution is then $y(x) = C(x)e^{f(x)}$

recall: $f(x) = -\int p(x)\,dx$

$$\Rightarrow \quad y(x) = \Big(\underbrace{\int q(x)e^{\int p(x)\,dx}\,dx + K}_{=C(x)}\Big)e^{-\int p(x)\,dx}$$

$$\Leftrightarrow \quad y(x) = Ke^{-\int p(x)\,dx} + e^{-\int p(x)\,dx}\int q(x)e^{\int p(x)\,dx}\,dx$$

This is the same general solution found via the integrating factor method in lectures.

2).

(a). $$\frac{dq}{dt} + 2q = \sin t \quad (1), \qquad q(t=0) = 5$$

The homogeneous equation is: $\frac{dq}{dt} + 2q = 0$.

This has solution: $q_h(t) = Ce^{-2t}$ (separation of variables or use formula (3) from question sheet)

Now substitute the ansatz:

$$q(t) = C(t)e^{-2t} \text{ into } (1)$$

$$\Rightarrow C'(t)e^{-2t} - \cancel{2C(t)e^{-2t}} + \cancel{2C(t)e^{-2t}} = \sin t$$

$$\Leftrightarrow C'(t) = e^{2t}\sin t$$

Integrating by parts:

$u = e^{2t}$, $du = 2e^{2t}$
$dv = \sin t$, $v = -\cos t$

$$C(t) = -e^{2t}\cos t + 2\int e^{2t}\cos t\,dt + K, \quad K \text{ constant}$$

$u = 2e^{2t}$, $du = 4e^{2t}$
$dv = \cos t$, $v = \sin t$

And a second integration by parts:

$$C(t) = -e^{2t}\cos t + 2e^{2t}\sin t - 4\underbrace{\int e^{2t}\sin t \, dt}_{= C(t)} + K$$

So we have:

$$C(t) = -e^{2t}\cos t + 2e^{2t}\sin t - 4C(t) + K$$

$\Leftrightarrow$

$$C(t) = \frac{1}{5}e^{2t}\left(2\sin t - \cos t\right) + K$$

Therefore the general solution of 1 is: $q(t) = C(t)e^{-2t}$

$\Leftrightarrow$ $q(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + Ke^{-2t}$

when $t=0$, $q=5$, so: $5 = \frac{2}{5}(0) - \frac{1}{5}(1) + K$

$\Leftrightarrow$ $K = \frac{26}{5}$

$$\therefore \quad q(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + \frac{26}{5}e^{-2t}$$

when $t \to \infty$, $e^{-2t} \to 0$, so the charge oscillates according to

$$q \sim \frac{2}{5}\sin t - \frac{1}{5}\cos t.$$

3). (dropped dependence on x notation)

(a). Sub $y = c_1 y_1 + c_2 y_2$ into $y'' + p y' + q y = g$.

First: $y' = c_1' y_1 + c_1 y_1' + c_2' y_2 + c_2 y_2'$

$$= c_1 y_1' + c_2 y_2' \quad \text{(since by (12) we let } c_1' y_1 + c_2' y_2 = 0\text{)}$$

$$\Rightarrow y'' = c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''$$

Now substituting in:

$$c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2'' + p\left[c_1 y_1' + c_2 y_2'\right] + q\left[c_1 y_1 + c_2 y_2\right] = g$$

$$\Leftrightarrow c_1 \underbrace{\left[y_1'' + p y_1' + q y\right]}_{=0} + c_2 \underbrace{\left[y_2'' + p y_2' + q y\right]}_{=0} + c_1' y_1' + c_2' y_2' = g$$

But y_1 and y_2 satisfy the homogeneous equation, so we are left with:

$$c_1' y_1' + c_2' y_2' = g(x),$$ as required.

(b). $c_1' y_1 + c_2' y_2 = 0$ (12)

$c_1' y_1' + c_2' y_2' = g(x)$ (13)

Consider: y_2'(12) $- y_2$(13): $c_1' y_1 y_2' + \cancel{c_2' y_2 y_2'} - c_1' y_1' y_2 - \cancel{c_2' y_2 y_2'} = -y_2 g(x)$

$$\Leftrightarrow c_1'\left(y_1 y_2' - y_1' y_2\right) = -y_2 g(x)$$

$$\Leftrightarrow c_1'(x) = \frac{-y_2(x) g(x)}{W(x)}, \quad \text{where } W(x) = y_1 y_2' - y_1' y_2.$$

Integrating: $$c_1(x) = -\int \frac{y_2(x) g(x)}{W(x)}\,dx + A, \quad A \text{ constant}$$

Similarly considering: y_1(13) $-y_1'$(12) gives the required expression for $c_2(x)$.

(c). The general solution is then: $y = c_1 y_1 + c_2 y_2$

$$\Rightarrow \quad y(x) = \left(-\int \frac{y_2 g}{W}\,dx + A\right) y_1 + \left(\int \frac{y_1 g}{W}\,dx + B\right) y_2$$

$$\Rightarrow \quad y(x) = Ay_1(x) + By_2(x) - y_1(x)\int \frac{y_2(x)g(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{W(x)}\,dx$$

4).

(a). $$y'' - 3y' + 2y = -\frac{e^{2x}}{e^x+1}$$

Auxiliary equation: $\lambda^2 - 3\lambda + 2 = 0 \iff (\lambda-1)(\lambda-2) = 0$

$\iff \underline{\lambda = 1 \text{ or } 2}$

Therefore $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are the independent solutions of the homogeneous equation.

Thus the Wronskian is:

$$W(x) = e^x(2e^{2x}) - e^x(e^{2x}) = 2e^{3x} - e^{3x} = e^{3x}$$

Therefore: $$c_1(x) = -\int \frac{e^{2x}\left(\frac{-e^{2x}}{e^x+1}\right)}{e^{3x}}\,dx = \int \left(\frac{e^x}{e^x+1}\right) dx$$

$$= \log(e^x+1) + A, \quad A \text{ constant}$$

And: $$c_2(x) = \int \frac{e^x\left(\frac{-e^{2x}}{e^x+1}\right)}{e^{3x}}\,dx = -\int \frac{1}{e^x+1}\,dx = -\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$= \log(e^{-x}+1) + B, \quad B \text{ constant}$$

Therefore the general solution is: $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

$\Leftrightarrow$

$$\boxed{y(x) = Ae^{x} + Be^{2x} + e^{x}\log(e^{x}+1) + e^{2x}\log(e^{-x}+1)}$$

(b). $y'' + 9y = 3\sec 3x$

Auxiliary equation: $\lambda^2 + 9 = 0 \Leftrightarrow (\lambda+3i)(\lambda-3i) = 0 \Leftrightarrow \underline{\lambda = \pm 3i}$

$\Rightarrow$ $y_1(x) = \cos 3x$, $y_2(x) = \sin 3x$ are the independent solutions of the homogeneous equation.

Wronskian, $W(x) = \cos 3x(3\cos 3x) - (-3\sin 3x)\sin 3x$

$$= 3(\cos^2 3x + \sin^2 3x)$$

$$= 3$$

$\sec 3x = \dfrac{1}{\cos 3x}$

$\Rightarrow$

$$c_1(x) = -\int \frac{\sin 3x \cdot 3\sec 3x}{3}\,dx = -\int \tan 3x\,dx$$

$$= -\frac{1}{3}\log(\sec 3x) + A = \frac{1}{3}\log(\cos 3x) + A, \quad A \text{ constant.}$$

$$c_2(x) = \int \frac{\cos 3x \cdot 3\sec 3x}{3}\,dx = \int dx = x + B, \quad B \text{ constant.}$$

$\therefore$ The general solution is: $y = c_1 y_1 + c_2 y_2$

$$\boxed{y(x) = A\cos 3x + B\sin 3x + \frac{1}{3}\log(\cos 3x)\cos 3x + x\sin 3x}$$

5). $$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = g(x) \quad (*)$$

Assume the n linearly independent solutions to the homogeneous equation:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0,$$

are <u>known</u>. Call these: $y_1, y_2, \cdots, y_n$.

Seek a solution to $(*)$ of form:

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + \cdots + c_n(x)y_n(x) \quad (\dagger)$$

Differentiating $(\dagger)$: $y' = (c_1y_1' + \cdots + c_ny_n') + (c_1'y_1 + \cdots + c_n'y_n)$

We <u>impose</u> the condition: $\underline{c_1'y_1 + \cdots + c_n'y_n = 0}$

Thus: $y' = c_1y_1' + \cdots + c_ny_n'$.

Differentiating this gives: $y'' = (c_1y_1'' + \cdots + c_ny_n'') + (c_1'y_1' + \cdots + c_n'y_n')$

This time we <u>impose</u>: $\underline{c_1'y_1' + \cdots + c_n'y_n' = 0}$

Continuing in this way, differentiating $n-1$ times, we impose the $n-1$ conditions:

$$\underline{\underline{c_1'y_1^{(k)} + \cdots + c_n'y_n^{(k)} = 0, \quad 0 \leq k \leq n-2}},$$

with the derivatives of y being:

$$\underline{\underline{y^{(k)} = c_1y_1^{(k)} + \cdots + c_ny_n^{(k)}, \quad 0 \leq k \leq n-1}} \quad (1)$$

A final n^{th} differentiation gives:

$$\underline{\underline{y^{(n)} = c_1y_1^{(n)} + \cdots + c_ny_n^{(n)} + c_1'y_1^{(n-1)} + \cdots + c_n'y_n^{(n-1)}}}. \quad (2)$$

To obtain a final condition on the C_i' functions we substitute the expressions 1 and 2 into the differential equation (*): giving:

$$\left[C_1 y_1^{(n)} + \cdots + C_n y_n^{(n)} + C_1' y_1^{(n-1)} + \cdots + C_n' y_n^{(n-1)}\right]$$

$$+ a_1(x)\left[C_1 y_1^{(n-1)} + \cdots + C_n y_n^{(n-1)}\right] + \cdots$$

$$+ a_n(x)\left[C_1 y_1 + \cdots + C_n y_n\right] = g(x)$$

Re-grouping terms:

$$\left[C_1' y_1^{(n-1)} + \cdots + C_n' y_n^{(n-1)}\right] + C_1\left[\cancelto{0}{y_1^{(n)} + \cdots + a_n y_1}\right]$$

$$+ \cancelto{0}{\cdots} + C_n\left[\cancelto{0}{y_n^{(n)} + \cdots + a_n y_n}\right] = g(x)$$

But $y_1, \cdots, y_n$ satisfy the homogeneous equation. So most terms vanish, leaving:

$$\underline{\underline{C_1' y_1^{(n-1)} + \cdots + C_n' y_n^{(n-1)} = g(x)}}$$

as a final condition.

We now have n conditions:

$$\left.\begin{aligned}
C_1' y_1 + \cdots + C_n' y_n &= 0 \\
C_1' y_1' + \cdots + C_n' y_n' &= 0 \\
&\vdots \\
C_1' y_1^{(n-2)} + \cdots + C_n' y_n^{(n-2)} &= 0 \\
C_1' y_1^{(n-1)} + \cdots + C_n' y_n^{(n-1)} &= g(x)
\end{aligned}\right\} \begin{array}{l} n \text{ equations} \\ n \text{ unknowns} \end{array}$$

The determinant of this system is the Wronskian of the functions $y_1, \cdots, y_n$. Since these functions were <u>linearly independent</u>, this Wronskian will be non-zero.

Hence the linear system has a unique solution for each C_i'.
Solving for these quantities then allows us to integrate to determine the C_i.

The general solution is then:

$$y(x) = C_1(x)y_1(x) + \dots + C_n(x)y_n(x).$$