Math40002 Analysis 1 Problem Sheet 5

1. Fix $x > 0$. Prove $(1+x)^n \geq 1 + nx$ for any $n \in \mathbb{N}$. Deduce that $(1+x)^{-n} \to 0$. Deduce that if $r \in (0,1)$ then $r^n \to 0$, and if $r \in (1,\infty)$ then $r^n \to \infty$.

   By the binomial theorem, $(1+x)^n = 1+nx+\ldots \geq 1+nx$ because $\ldots$ is all > 0 (or empty for $n = 0, 1$). Hence $|(1+x)^{-n} - 0| \leq 1/(1+nx)$. Now

   $$1/(1+nx) < \epsilon \iff n > (\epsilon^{-1} - 1)/x.$$

   So given any $\epsilon > 0$ we pick $N > (\epsilon^{-1} - 1)/x$ so that

   $$n \geq N \ \Rightarrow\ |(1+x)^{-n} - 0| \leq 1/(1+nx) < \epsilon.$$

   We can write $r = (1+x)^{-1}$ by setting $x := r^{-1} - 1 > 0$, then apply previous result.

   If $r \in (1,\infty)$ then fix any $R > 0$. Now use the first part of the question to see that $r^n \geq 1+n(r-1) \geq R$ for all $n \geq \frac{R-1}{r-1}$. That is, $r^n \to \infty$.

2. Suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. In lectures we proved that if $L < 1$ then $a_n \to 0$.

   (a) Prove that if $L > 1$ then $|a_n| \to \infty$.

   (b) Give an example with $|a_{n+1}/a_n| < 1\ \forall n$ but $a_n \not\to 0$.

   Give (without proof) examples where $L = 1$ and

(i) $a_n \to 0$,	(iii) a_n divergent and bounded,
(ii) $a_n \to a \neq 0$,	(iv) $a_n \to \infty$.

   (a) If $L > 1$ then set $\epsilon = (L-1)/2 > 0$. Then $\exists N$ such that $\forall n \geq N$ we have $|a_{n+1}/a_n - L| < (L-1)/2$ and in particular $|a_{n+1}|/|a_n| > L - (L-1)/2 = (L+1)/2 > 1$.

   Let $\alpha := (L+1)/2 > 1$. Therefore we find inductively that $|a_{N+m}| > \alpha^m |a_N|$. But $\alpha^m \to \infty$ as $m \to \infty$ by previous question. In particular if we fix any $R > 0$ then $\exists M$ such that $\forall m \geq M$ we have $\alpha^m > R/|a_N|$.

   Putting it altogether we find that $\forall n \geq N + M$ we have $|a_n| > (R/|a_N|)|a_N| = R$. Thus $|a_n| \to \infty$ as $n \to \infty$.

   (b) Example: $a_n = 1 + 1/n$.

   (i) $a_n = 1/n$

   (ii) $a_n \equiv a$

   (iii) $a_n = (-1)^n$

   (iv) $a_n = n$

3. Let $(a_n)_{n\geq 1}$ be a sequence of *strictly positive* real numbers.
   Give an example such that $(1/a_n)_{n\geq 1}$ is unbounded.
   Suppose that $a_n \to a \neq 0$. Prove *from first principles* that $(1/a_n)_{n\geq 1}$ is bounded.

   Any example like $a_n = 1/n$ will do.

   Let $\epsilon = a/2 > 0$. Then $\exists N \in \mathbb{N}$ such that

   $$n \geq N \ \Rightarrow\ |a_n - a| < \epsilon \ \Rightarrow\ a_n > a - \epsilon = a/2 \ \Rightarrow\ 1/a_n < 2/a.$$

   Therefore $0 < 1/a_n \leq \max\left(a_1^{-1}, a_2^{-1}, \ldots, a_{N-1}^{-1}, 2/a\right)\ \forall n$ and so is bounded.

4.† Fix $r \in (0, 1/8)$. Define $(a_n)_{n\geq 1}$ by $a_1 := 1$ and $a_{n+1} = ra_n^2 + 1$.

   (a) Show that $a_{n+1} - a_n = r(a_n + a_{n-1})(a_n - a_{n-1})$.

   This is just $a_{n+1} - a_n = ra_n^2 - ra_{n-1}^2 = r(a_n + a_{n-1})(a_n - a_{n-1})$.

(b) Show that if $$0 < a_j < 2 \qquad \forall j \le n, \tag{1}$$
then $$|a_{n+1} - a_n| \; < \; (4r)^n/4. \tag{2}$$

Use $|a_{n+1} - a_n| < r(2+2)|a_n - a_{n-1}| = 4r|a_n - a_{n-1}| \le (4r)^2|a_{n-1} - a_{n-2}| \le \ldots \le (4r)^{n-1}|a_2 - a_1|$. But this equals $(4r)^{n-1}(r+1-1) = (4r)^n/4$, as required.

(c) Deduce that if (1) holds, then $a_{n+1} < r/(1-4r) + 1$.

By the triangle inequality, $a_{n+1} \le |a_{n+1} - a_n| + |a_n - a_{n-1}| + \ldots + |a_2 - a_1| + |a_1|$, which is $< \frac{1}{4}\Big((4r)^n + (4r)^{n-1} + ... + 4r\Big) + 1 \le r/(1-4r) + 1$ because $4r < 1$.

(d) Conclude that (1) holds for $j = n+1$ too, and so $\forall j$ by induction.

Since $r < 1/8$ we have $r/(1-4r) + 1 < 2$. (It is clear from the definition that $a_n > 0$ $\forall n$.)

(e) Using (2) deduce $|a_m - a_n| < (4r)^n/4(1-4r)$ for $m \ge n$.

By the same triangle inequality argument, $|a_m - a_n| < (4r)^{m-1}/4 + \ldots + (4r)^n/4$ which again is $\le (4r)^n/4(1-4r) \le (4r)^n/2$.

From Q1 $(4r)^n \to 0$ as $n \to \infty$ since $0 < 4r < 1$.

So $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that $n \ge N \;\Rightarrow\; (4r)^n < \epsilon \;\Rightarrow\; |a_m - a_n| < \epsilon/2$ for $m \ge n \ge N$. Thus a_m is Cauchy and so convergent.

(f) Deduce a_n is Cauchy. What does it converge to?

Let a be $\lim_{n\to\infty} a_n$. Taking limits in $a_{n+1} = ra_n^2 + 1$ gives $a = ra^2 + 1$ so that $a = \frac{1 \pm \sqrt{1-4r}}{2r}$. Then $\pm$ cannot be $+$ because we know from (1) that $a \in [0,2]$. So $a = \frac{1-\sqrt{1-4r}}{2r}$.

5.* Show that *any* sequence of real numbers $(a_n)_{n\ge0}$ has a subsequence which either converges, or tends to ∞, or tends to $-\infty$.

If (a_n) is bounded, it has a convergent subsequence by Bolzano-Weierstrass. Suppose instead it is unbounded above; we will show it has a subsequence tending to ∞ (unbounded below and $-\infty$ is similar).

We define a_{n_i} recursively such that $a_{n_i} > i$. Since 1 is not an upper bound, there is an $n_1 \in \mathbb{N}$ such that $a_{n_1} > 1$, so the recursion begins.

Assuming we've defined $n_1 < \ldots < n_i$ such that $a_{n_i} > i$, we need to define n_{i+1}. But $i+1$ is not an upper bound for the set $\{a_n : n > n_i\}$ (if it were then (a_n) would be bounded above by $\max(i+1, a_1, a_2, \ldots, a_{n_i})$.) So we can pick $a_{n_{i+1}} > i+1$ in it this set, as required.

Now given any $R \in \mathbb{R}$ pick $N \in \mathbb{N}$ with $N > R$. Then $\forall i \ge N$ we have $a_{n_i} > i \ge N > R$, which is the definition of $a_{n_i} \to \infty$.

6. At home Professor Papageorgiou has made a fully realistic mathematical model of a dart board. It is a copy of the unit interval $[0,1]$ in a frictionless vacuum. He throws a countably infinite number of darts at it, the nth landing at $a_n \in [0,1]$.

He then makes a small dot $(x - \epsilon_x,\, x + \epsilon_x)$ around each point $x \in [0,1]$ with his pen. Prove that however small he makes each dot, at least one of them will contain an infinite number of darts $a_n \in [0,1]$.

What if he only makes dots around each dart $a_n \in [0,1]$?

By Bolzano-Weierstrass there exists a subsequence b_n of the a_n which is convergent to some $b \in [0,1]$. Therefore consider any neighbourhood $(b - \epsilon_b, b + \epsilon_b)$ of the limit. There exists $N \in \mathbb{N}$ such that $b_n \in (b - \epsilon_b, b + \epsilon_b)$ $\forall n \ge N$, so there are an *infinite* number of darts in this dot.

For some sequences (a_n) it is possible to find a neighbourhood of each dart with only finitely many darts in it. Eg if $a_n = 1/n$ then we can choose the neighbourhood $(1/(n+1), 1/(n-1))$ of a_n.

For some it is not; eg if $a_1 = 0$ and $a_n = 1/n$ for $n > 1$ – then any neighbourood of a_1 has infinitely many darts.

The general condition is that no point a_n of the sequence should be a limit of any subsequence.

7. Let $(a_n)_{n\geq 1}$ be the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots$

   (i) Give (without proof) a subsequence of $(a_n)_{n\geq 1}$ which converges to $\ell = 0$, and one which converges to $\ell = 1$.

   (ii) Given any $\ell \in (0, 1)$, give (with proof) a subsequence convergent to ℓ.

(i) The subsequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ converges to $\ell = 0$.

The subsequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ converges to $\ell = 1$.

(ii) Let $\ell_n/10^n$ ($\ell_n \in \mathbb{N}$) be the decimal expansion of ℓ truncated at the nth decimal place. Since $\ell \neq 0$, the decimal expansion is nonzero so there is a k such that $\ell_k \neq 0$. Now take the subsequence of $(a_n)_{n\geq 1}$ given by

$$\frac{\ell_k}{10^k}, \frac{\ell_{k+1}}{10^{k+1}}, \ldots.$$

Notice we *do not* cancel the fractions into lower terms – the denominators must keep increasing so the ith term a_{n_i} satisfies that $n_i < n_{i+1}$ – i.e. subsequences always "move to the right" in the original sequence. By its definition, $|\ell_n/10^n - l| \leq 10^{-n}$. So given any $\epsilon > 0$, choose $N > 1/\epsilon$ and

$$\left|\frac{\ell_n}{10^n} - \ell\right| \leq 10^{-n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

for all $n \geq N$. So the subsequence $\to \ell$, as required.

8. Professor Buzzard is teaching Lean about Cauchy sequences. Thomas has told him that it can be hard to find their limits, so he sets out to prove him wrong. He types some `mynat.definition` guff and then

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n, m \geq N \ \Rightarrow \ |a_n - a_m| < \epsilon$$

$$\Rightarrow \ \forall n \geq N \quad |a_n - a_N| < \epsilon$$

$$\Rightarrow \ a_n \to a_N \ \text{ as } \ n \to \infty.$$

   "Ha – who are you calling a computer scientist, Thomas?!" he exclaims, as he types `refl` with a flourish. Does Lean give him a "`Proof complete!`"?

The problem is that N can depend on ϵ; we only found N after fixing ϵ. So he only proves that $|a_n - a_N| < \epsilon$ for a fixed $\epsilon > 0$. To prove that $a_n \to a_N$ we need to prove $|a_n - a_N| < \epsilon$ for *any* $\epsilon > 0$, so we need to be able to change ϵ, but that may change N and so the "limit" a_N.