Imperial College London
Department of Mathematics

Question Sheet 6

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Problem sheet released on Wednesday of week 8. All questions can be attempted before the problem class on Monday Week 9. Questions 1 and 2 are suitable for tutorials. Solutions will be released on Wednesday of week 9.

Question 1 Suppose $(G, \cdot)$ is a group and H is a subgroup of G. Prove that each of the following is an equivalence relation on G (where g, h are elements of G):

(i) $g \sim_1 h$ if and only if there is $k \in G$ with $h = kgk^{-1}$;

(ii) $g \sim_2 h$ if and only if $h^{-1}g \in H$.

In the case where $(G, .)$ is the group $(\mathbb{R}^2, +)$ and H is the subgroup $\{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, describe geometrically the $\sim_2$-equivalence classes. What are the $\sim_1$-equivalence classes?

Question 2 Which of the following subsets H are subgroups of the given group G?

(a) $G = (\mathbb{Z}, +)$, $H = \{n \in \mathbb{Z} \mid n \equiv 0 \bmod 37\}$.

(b) $G = \mathrm{GL}(2, \mathbb{C})$, $H = \{A \in G \mid A^2 = I\}$.

(c) $G = \mathrm{GL}(2, \mathbb{R})$, $H = \{A \in G \mid \det(A) = 1\}$.

(d) $G = S_n$, $H = \{g \in G \mid g(1) = 1\}$ (for $n \in \mathbb{N}$).

(e) $G = S_n$, $H = \{g \in G \mid g(1) = 2\}$ (for $n \geq 2$).

(f) $G = S_n$, H is the set of all permutations $g \in G$ such that $g(i) - g(j) \equiv i - j \bmod n$ for all $i, j \in \{1, \dots, n\}$.

Question 3 Prove the following statements.

(a) Every cyclic group is abelian.

(b) The group S_n is *not* abelian, unless $n < 3$.

Question 4 Suppose $(G, \cdot)$ is a group and H, K are subgroups of G.

(i) Show that $H \cap K$ is a subgroup of G.

(ii) Show that if $H \cup K$ is a subgroup of G then either $H \subseteq K$ or $K \subseteq H$.

Question 5 Which of the following groups are cyclic?

(a) S_2.

(b) $\mathrm{GL}(2, \mathbb{R})$.

(c) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \{1, -1\} \right\}$ under matrix multiplication.

(d) $(\mathbb{Q}, +)$.

Question 6 Let G be a cyclic group of order n, and g a generator. Show that g^k is a generator for G if and only if $\gcd(k, n) = 1$.

Question 7 Let G and H be finite groups. Let $G \times H$ be the set $\{(g, h) \mid g \in G, h \in H\}$ with the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

(a) Show that $(G \times H, *)$ is a group.

(b) Show that if $g \in G$ and $h \in H$ have orders a, b respectively, then the order of (g, h) in $G \times H$ is the lowest common multiple of a and b.

(c) Show that if G and H are both cyclic, and $\gcd(|G|, |H|) = 1$, then $G \times H$ is cyclic. Is the converse true?

Question 8 Find an example of each of the following:

(a) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{C})$.

(b) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{R})$.

(c) an element of infinite order in the group $\mathrm{GL}(2, \mathbb{R})$.

(d) an element of order 12 in the group S_7.