Math40002 Analysis 1 — Problem Sheet 1

1. What is the biggest element of the set $\{x \in \mathbb{R} \colon x < 1\}$? Give a careful proof.

2. Prove that for every positive integer $n \neq 3$, the number $\sqrt{n} - \sqrt{3}$ is irrational.

3. * Show that any positive *eventually periodic* decimal expansion is rational, and in fact can be written as the fraction

$$p \Big/ 99\ldots9900\ldots00 \quad (m \text{ 9s and } n \text{ 0s})$$

for some integers $p, m, n \geq 0$.
Deduce that any integer divides some number of the form $99\ldots9900\ldots00$.

4. Irrational Kevin tries to show $\sqrt{12} - \sqrt{3}$ is rational, by the following argument.

$$\begin{aligned} & \sqrt{12} - \sqrt{3} = p/q, \quad p, q \in \mathbb{N}, \\ \Rightarrow \quad & 12 - 2\sqrt{12}\sqrt{3} + 3 = p^2/q^2, \\ \Rightarrow \quad & 15 - 2\sqrt{36} = p^2/q^2. \end{aligned}$$

Since $\sqrt{36} = 6$ is indeed rational, this looks good to him. Can you help him by pointing out three ways in which he's gone wrong? Be kind to him!

5. Suppose the sets $S_n,\ n = 1, 2, 3, \ldots$ are all disjoint and countable. Show that $S = \bigcup_{n=1}^{\infty} S_n$ is also countable. (Hint: recall the diagonal argument used in lectures.)

6. Suppose that S and T are countable. Show that $S \times T$ is countable. Hence show that $\bigcup_{n=1}^{\infty} S^n$ is countable, where $S^n := S \times \ldots \times S$ (n times).

7. † Show the set of polynomials $p(x)$ with integer coefficients is countable. (Hint: use Q6.)

   A real number is called *algebraic* if it is a root of a polynomial with integer coefficients. Show that rational numbers n/m and nth roots $\sqrt[n]{m}$ are algebraic. Show that the set of algebraic real numbers is countable.

   A real number is called *transcendental* if it is not algebraic. (Examples include π and e, but this is hard to prove.) Prove that transcendental numbers exist, and that in fact there are uncountably many of them.

8. Let $S^1 = \{s^1_1, s^1_2, s^1_3, \ldots\},\ S^2 = \{s^2_1, s^2_2, s^2_3, \ldots\},\ \ldots,\ S^n = \{s^n_1, s^n_2, s^n_3, \ldots\},\ \ldots$ be subsets of $\mathbb{N}$. Here the elements are ordered so that $s^n_i < s^n_{i+1}$ for all i and n.

   Define t_n recursively to be strictly larger than s^n_n and t_{n-1} (e.g. set $t_n = \max\{s^n_n, t_{n-1}\} + 1$), or $t_{n-1} + 1$ if $\nexists\, s^n_n$ (i.e. if S^n has $< n$ elements).

   Show that $T = \{t_1, t_2, \ldots\} \subseteq \mathbb{N}$ is not equal to any S^i. Conclude that the set of subsets of $\mathbb{N}$ is *not* countable.

*You should prepare starred questions * to discuss with your personal tutor.*
Questions marked † are slightly harder (more fun).