Coursework 3

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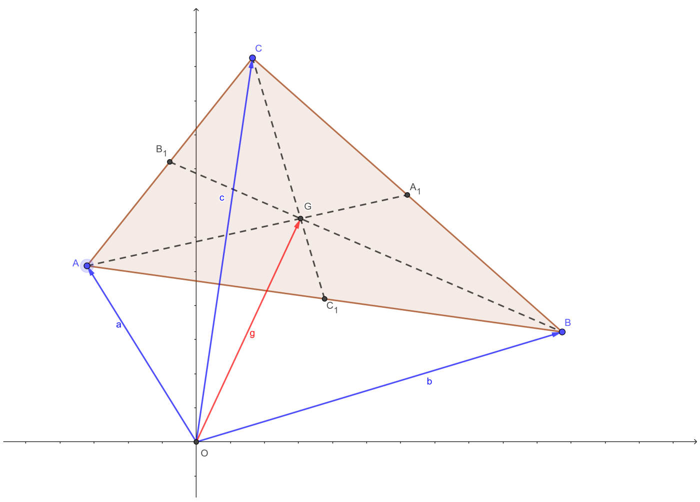
Problem 1. Let points A, B, and C be represented by the position vectors $\vec{a}, \vec{b}$, and $\vec{c}$, respectively. Further, let A_1, B_1, and C_1 be the midpoints of BC, AC, and AB, respectively.



(a)

By definition, the centroid of a polygon is $\overrightarrow{C} = (\vec{a_1} + \vec{a_2} + ... + \vec{a_n})/n$, where $\vec{a_i}$ are the vertices of the polygon. So if G is the centroid of $\triangle ABC$, then

$$\overrightarrow{G} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}.$$

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(b)

By considering the position vectors $\vec{a}, \vec{b}$, and $\vec{c}$, we can see that

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

$$\overrightarrow{BC} = \vec{c} - \vec{b}$$
$$\overrightarrow{CA} = \vec{a} - \vec{c}.$$

Since A_1, B_1, and C_1 are the midpoints of BC, AC, and AB, respectively, we find that

$$\overrightarrow{AC_1} = \frac{\vec{b} - \vec{a}}{2}$$
$$\overrightarrow{BA_1} = \frac{\vec{c} - \vec{b}}{2}$$
$$\overrightarrow{CB_1} = \frac{\vec{a} - \vec{c}}{2}.$$

Using vector addition we can see that

$$\overrightarrow{A_1} = \vec{b} + \frac{\vec{c} - \vec{b}}{2} = \frac{\vec{c} + \vec{b}}{2}$$
$$\overrightarrow{B_1} = \vec{c} + \frac{\vec{a} - \vec{c}}{2} = \frac{\vec{a} + \vec{c}}{2}$$
$$\overrightarrow{C_1} = \vec{a} + \frac{\vec{b} - \vec{a}}{2} = \frac{\vec{b} + \vec{a}}{2}.$$

Now to prove that the three medians are congruent at the centroid, we will show that point G with position vector $\vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$ lies on each of the three medians. To prove that G lies on AA_1, we must show that $\overrightarrow{AG} = \lambda\overrightarrow{AA_1}$ for some scalar λ.

$$\overrightarrow{AG} = \vec{g} - \vec{a} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} - \vec{a} = \frac{\vec{b} + \vec{c} - 2\vec{a}}{3}$$
$$\overrightarrow{AA_1} = \overrightarrow{A_1} - \overrightarrow{A} = \frac{\vec{c} + \vec{b}}{2} - \vec{a} = \frac{\vec{c} + \vec{b} - 2\vec{a}}{2}.$$

Therefore $\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AA_1}$, so point G lies on AA_1 and furthermore, it divides the segment in ratio $2:1$.
Similarly, we can show that G lies on BB_1 and CC_1 as well. We can now conclude that the three medians are congruent at G because it is a point on all three line segments, no two of which lie on a straight line.

$\blacksquare$

(c)
We have that $\vec{a} = (1,2), \vec{b} = (2,-1)$, and $\vec{c} = (0,3)$. Then

$$\vec{g} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} = \frac{1}{3}((1+2+0),(2+(-1)+3)) = (1, \frac{4}{3}).$$

Therefore the centroid G has coordinates $(1, \frac{4}{3})$.

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