Imperial College London
Department of Mathematics

Question Sheet 5

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Problem sheet released on Wednesday of week 7. All questions can be attempted before the problem class on Monday Week 8. Question 3 is suitable for tutorials. Solutions will be released on Wednesday of week 8.

Question 1 Let S be the two-element set $\{a, b\}$. Show that there are precisely 16 distinct binary operations on S. How many of them make S a group?

Find a formula for the total number of binary operations on a set of n elements.

Question 2 Prove that multiplication of complex numbers is associative.

Question 3 Which of the following are groups?

(a) The set of all complex numbers z such that $|z| = 1$, with the usual complex multiplication.

(b) The set $\{x \in \mathbb{R} \mid x \geq 0\}$, with the operation $x * y = \max(x, y)$.

(c) The set $\mathbb{C} \setminus \{0\}$, with the operation $a * b = |a| \cdot b$.

(d) The set of all rational numbers with odd denominators, with the usual addition.

(e) The set $\{a, b\}$, where $a \neq b$, with the binary operation $*$ given by

$$a * a = a,\ b * b = b,\ a * b = b,\ b * a = b.$$

(f) The set $\{a, b\}$, with $a \neq b$, with the binary operation $*$ given by

$$a * a = a,\ b * b = a,\ a * b = b,\ b * a = b.$$

(g) The set $\mathbb{R}^3$, with the binary operation $v * w = v \times w$ (the vector product).

(h) The set $\mathbb{R}^3$, with the usual vector addition.

Question 4 Let S be the set of all real numbers except -1. For $a, b \in S$ define

$$a * b = ab + a + b.$$

Show that $(S, *)$ is a group. (Note: you need to check the closure axiom.)

Question 5 Let G be a group, and let $a, b, c \in G$. Prove the following facts.

(a) If $ab = ac$ then $b = c$.

(b) The equation $axb = c$ has a unique solution for $x \in G$.

(c) $(a^{-1})^{-1} = a$.

(d) $(ab)^{-1} = b^{-1}a^{-1}$.

Question 6 Let G be a group, and let e be the identity of G. Suppose that $x * x = e$ for all $x \in G$. Show that $y * z = z * y$ for all $y, z \in G$. Can you find infinitely many examples of groups G with the property that $x * x = e$ for all $x \in G$?

Question 7 (i) (Harder) Suppose X is a non-empty set and α, β are permutations of X with the property that any element of X moved by α is fixed by β and any element of X moved by β is fixed by α, i.e. for all $x \in X$:

$$(\alpha(x) \neq x \Rightarrow \beta(x) = x) \text{ and } (\beta(x) \neq x \Rightarrow \alpha(x) = x).$$

Prove that $\alpha \circ \beta = \beta \circ \alpha$. [Hint: consider $\alpha(\beta(x))$ in the cases where x is moved by β and where it is moved by α; note that if x is moved by β, then so is $\beta(x)$.]

(ii) Suppose $(G, \cdot)$ is a group and $g, h \in G$ are such that $gh = hg$. Show that for all $r \in \mathbb{N}$ we have $(gh)^r = g^r h^r$. Give an example of $g, h \in S_3$ (the symmetric group on $\{1, 2, 3\}$) where $(gh)^2 \neq g^2 h^2$.