Problem sheet 8

Questions 1 and 2

(a) $\frac{\partial u}{\partial x} = \frac{1}{\left[1+\left(\frac{y}{x}\right)^2\right]}\left(-\frac{y}{x^2}\right)$, $\frac{\partial u}{\partial y} = \frac{1}{\left[1+\left(\frac{y}{x}\right)^2\right]}\left(\frac{1}{x}\right)$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{\left[1+\left(\frac{y}{x}\right)^2\right]}\left[x\left(-\frac{y}{x^2}\right)+y\left(\frac{1}{x}\right)\right] = 0.$$

(b) $\frac{\partial u}{\partial x} = \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2} = \ln(x^2+y^2)+2$

$$\frac{\partial u}{\partial y} = \frac{2xy}{x^2+y^2} - 2\tan^{-1}\left(\frac{y}{x}\right) - \frac{2xy}{x^2+y^2} = -2\tan^{-1}\left(\frac{y}{x}\right)$$

so $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u+2x.$

Question 3

5) $V = (x^2+y^2+z^2)^{-1/2} \Rightarrow \frac{\partial V}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$

$\Rightarrow \frac{\partial^2 V}{\partial x^2} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$ ie. $\frac{\partial^2 V}{\partial x^2} = -\frac{1}{r^3}+\frac{3x^2}{r^5}$.

SIM for $\frac{\partial^2 V}{\partial y^2} = -\frac{1}{r^3}+\frac{3y^2}{r^5}$ and $\frac{\partial^2 V}{\partial z^2} = -\frac{1}{r^3}+\frac{3z^2}{r^5}$

and $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = -\frac{3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} = -\frac{3}{r^3}+\frac{3r^2}{r^5} = 0$

Question 4

$$s = \frac{x}{x^2+y^2}$$

$$t = \frac{y}{x^2+y^2}$$

$$ds = \frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial y}\,dy$$

$$dt = \frac{\partial t}{\partial x}\,dx + \frac{\partial t}{\partial y}\,dy$$

$$\frac{\partial s}{\partial x} = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = t^2 - s^2$$

$$\frac{\partial s}{\partial y} = \frac{-x.2y}{(x^2+y^2)^2} = -2st$$

$$\frac{\partial t}{\partial x} = \frac{y(-2x)}{(x^2+y^2)^2} = -2st$$

$$\frac{\partial t}{\partial y} = \frac{x^2+y^2 - 2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = s^2 - t^2$$

$$J = \begin{pmatrix} t^2-s^2 & -2st \\ -2st & s^2-t^2 \end{pmatrix} = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} y^2-x^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix}$$

1) Evidently
$$u_x = \bar{u}_s s_x + \bar{u}_t t_x$$
$$= \bar{u}_s\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) + \bar{u}_t\left(\frac{-2xy}{(x^2+y^2)^2}\right)$$
and
$$u_y = \bar{u}_s s_y + \bar{u}_t t_y$$
$$= \bar{u}_s\left(\frac{-2xy}{(x^2+y^2)^2}\right) + \bar{u}_t\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)$$
Square and add!
$$\left(u_x^2 + u_y^2\right) = \frac{\left(\bar{u}_s^2 + \bar{u}_t^2\right)}{(x^2+y^2)^2} \Rightarrow \text{result !}$$

For a general derivation see over

$$\begin{pmatrix} ds \\ dt \end{pmatrix} = J \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad J = \begin{pmatrix} t^2-s^2 & -2st \\ -2st & s^2-t^2 \end{pmatrix} = \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = J^{-1} \begin{pmatrix} ds \\ dt \end{pmatrix} \qquad \begin{cases} J^{-1} = \begin{pmatrix} -a & -b \\ -b & a \end{pmatrix} \dfrac{1}{-a^2-b^2} \\ J^{-1} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \dfrac{1}{a^2+b^2} \end{cases}$$

$$a^2+b^2 = t^4+s^4-2t^2s^2 + 4t^2s^2 = (t^2+s^2)^2$$

$$\frac{\partial u}{\partial s} = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \right)$$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \right)$$

$$\begin{pmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{pmatrix}}_{\equiv (J^{-1})^T} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

$$\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} \left[J^{-1} (J^{-1})^T \right] \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

$$J^{-1}(J^{-1})^T = \frac{1}{(a^2+b^2)^2} \cdot \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} =$$

$$= \frac{1}{(a^2+b^2)^2} \begin{pmatrix} a^2+b^2 & ab-ab \\ ab-ab & b^2+a^2 \end{pmatrix} = \frac{1}{a^2+b^2} I$$

$a^2+b^2 = (t^2+s^2)$ ✓

$$\Rightarrow \boxed{\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = \frac{1}{a^2+b^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right]}$$

Question 5

1) (a) $(3x^2+3y)dx + (3x-2y)dy = 0$ i.e. $P\,dx + Q\,dy = 0$.

$\partial P/\partial y = 3 = \dfrac{\partial Q}{\partial x}$ so EXACT

Find u such that $\dfrac{\partial u}{\partial x} = P = 3x^2+3y$ and $\dfrac{\partial u}{\partial y} = Q = 3x-2y$

$\Rightarrow$ Hence $u = x^3+3xy+k_1(y) = 3xy-y^2+k_2(x)$

So $u(x,y) = x^3+3xy-y^2+\text{constant}$

Hence solution $\underline{x^3+3xy-y^2 = A}$ (const)

(b) $(1+e^y)\cos x\,dx + e^y\sin x\,dy = 0$

$\dfrac{\partial P}{\partial y} = e^y\cos x = \dfrac{\partial Q}{\partial x}$ so EXACT

$\Rightarrow$ method as above $u(x,y) = (1+e^y)\sin x + \text{constant}$

$\Rightarrow \underline{(1+e^y)\sin x = A}$.

(c)(*) $2xy\,dx + (y^2-x^2)dy = 0 \Rightarrow \dfrac{\partial P}{\partial y} = 2x$, $\dfrac{\partial Q}{\partial x} = -2x$

so NOT EXACT

Mult by integrating factor I to make exact

Try $I(x)$ NO GOOD!

Try $I(y)$ $\Rightarrow \bar{P} = 2xy\,I(y)$, $\bar{Q} = (y^2-x^2)I(y)$

$\partial \bar{P}/\partial y = 2xI + 2xyI'$, $\dfrac{\partial \bar{Q}}{\partial x} = -2xI(y)$

so exact if $4xI + 2xyI' = 0 \Rightarrow \dfrac{dI}{I} = -\dfrac{2dy}{y} \Rightarrow I = \dfrac{1}{y^2}$

Then find $u(x,y)$ s.t. $\dfrac{\partial u}{\partial x} = \dfrac{2x}{y} \equiv \bar{P}$, $\dfrac{\partial u}{\partial y} = 1-\dfrac{x^2}{y^2} \equiv \bar{Q}$.

$\Rightarrow u = \dfrac{x^2}{y} + k_1(y) = y + \dfrac{x^2}{y} + k_2(x)$

$\Rightarrow u = \dfrac{x^2}{y} + y + \text{const}$. $\Rightarrow$ $\underline{\dfrac{x^2}{y} + y = A}$ i.e. $x^2+y^2 = Ax$.

(d) $(x^3-3x^2y+5xy^2-7y^3)\,dx+(y^4+2y^2-x^3+5x^2y-21xy^2)\,dy=0$

$$\frac{\partial P}{\partial y}=-3x^2+10xy-21y^2=\frac{\partial Q}{\partial x} \qquad \text{So EXACT}$$

$$\Rightarrow u=\frac{x^4}{4}-x^3y+\frac{5}{2}x^2y^2-7xy^3+k_1(y)=\frac{y^5}{5}+\frac{2}{3}y^3-x^3y+\frac{5}{2}x^2y^2-7xy^3+k_2(x)$$

$$\Rightarrow \qquad \underline{\frac{x^4}{4}-x^3y+\frac{5}{2}x^2y^2-7xy^3+\frac{y^5}{5}+\frac{2}{3}y^3=A} \text{ (constant)}$$

(e)

B) $(x+2y^2)\,dx+(x^3-2xy)\,dy=0$

$$\frac{\partial P}{\partial y}=4y, \quad \frac{\partial Q}{\partial x}=3x^2-2y \qquad \text{SO NOT EXACT.}$$

TRY integrating factor x^k (where k is to be found).

$$\frac{\partial \bar{P}}{\partial y}=\frac{\partial}{\partial y}\left[x^k(x+2y^2)\right]=4x^ky.$$

$$\frac{\partial \bar{Q}}{\partial x}=\frac{\partial}{\partial x}\left[x^k(x^3-2xy)\right]=(k+3)x^{k+2}-2(k+1)x^ky.$$

For $\frac{\partial \bar{P}}{\partial y}\equiv\frac{\partial \bar{Q}}{\partial x}$ we <u>must</u> have $\underline{k=-3}$.

$$\Rightarrow \bar{P}=(x+2y^2)/x^3=\frac{\partial u}{\partial x}$$
$$\bar{Q}=(x^3-2xy)/x^3=\frac{\partial u}{\partial y}$$

$$\Rightarrow u=-\frac{1}{x}-\frac{y^2}{x^2}+k_1(y)=y-\frac{y^2}{x^2}+k_2(x)$$

$$\therefore \quad u(x,y)=-\frac{1}{x}-\frac{y^2}{x^2}+y+\text{constant}$$

$$\Rightarrow \underline{-\frac{1}{x}-\frac{y^2}{x^2}+y=A}.$$

(f)

$$(e^y + ye^x)dx + (e^x + xe^y + 1)dy$$

Now $\frac{\partial f}{\partial y} = e^y + e^x = \frac{\partial g}{\partial x}$

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so here we Do have an exact differential $\equiv dF(x,y)$ say

So $\frac{\partial F}{\partial x} = e^y + ye^x$ and $\frac{\partial F}{\partial y} = e^x + xe^y + 1.$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$F(x,y) = xe^y + ye^x + f(y)$ $\qquad$ $F(x,y) = ye^x + xe^y + y + g(x).$

$$F(x,y) = xe^y + ye^x + y + \left(\begin{matrix}\text{arb}\\ \text{constant}\end{matrix}\right).$$

Question 6

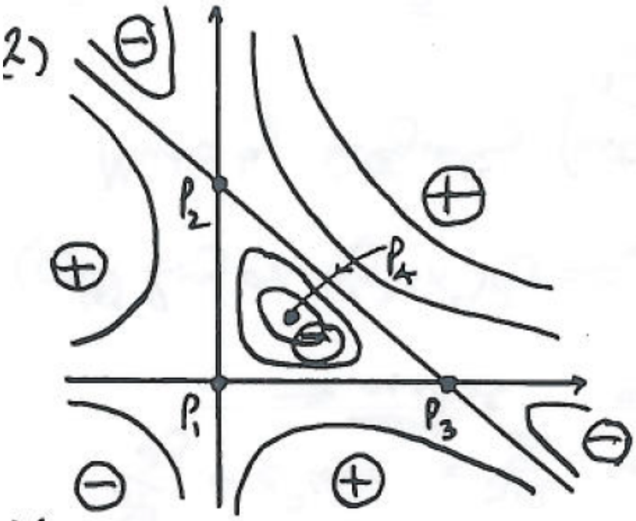
1) $f_x = 3x^2 - 3 = 0$ and $f_y = 2y = 0$ for stationary points $\Rightarrow (1,0)$ and $(-1,0)$

$f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 2$

At $(1,0)$: $A = 6$, $B = 0$, $C = 2 \Rightarrow AC - B^2 = 12 > 0$, $A > 0$
MINIMUM AT $(1,0)$

At $(-1,0)$: $A = -6$, $B = 0$, $C = 2 \Rightarrow AC - B^2 = -12 < 0$
SADDLE AT $(-1,0)$

2)



Evidently $f = 0 \Rightarrow x = 0$, $y = 0$, $x + y = 1$.
So the $f = 0$ contour is easy to draw.
x, y small +ve $\rightarrow f > 0$
Hence regions as shown and sketch contours

$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \Rightarrow \begin{array}{l} 2xy + y^2 - y = 0 \\ x^2 + 2xy - x = 0 \end{array} \Rightarrow P_1(0,0),\ P_2(0,1),\ P_3(1,0),\ P_4\left(\frac{1}{3},\frac{1}{3}\right)$

From the diagram we expect P_1 SADDLE, P_2 SADDLE, P_3 SADDLE, P_4 LOCAL MINIMUM

$\frac{\partial^2 f}{\partial x^2} = 2y$ (A), $\frac{\partial^2 f}{\partial y^2} = 2x$ (C), $\frac{\partial^2 f}{\partial x \partial y} = 2x + 2y - 1$ (B) $\Rightarrow$

	A	B	C	$AC - B^2$
P_1	0	-1	0	-1
P_2	2	1	0	-1
P_3	0	1	2	-1
P_4	$2/3$	$1/3$	$2/3$	$+\frac{1}{3}$

CONFIRMS P_1, P_2, P_3 SADDLES $AC - B^2 < 0$
and P_4 MINIMUM $AC - B^2 > 0$, $A > 0$.

(5) $f_x = (y-2)^2 + 2x - 1 = 0$, $f_y = 2x(y-2) = 0$

$\Rightarrow x = 0$ or $y = 2$

then $x = 0 \Rightarrow y - 2 = \pm 1 \Rightarrow y = 1$ or 3

and $y = 2 \Rightarrow x = \frac{1}{2}$ So stat pts $(0,1), (0,3), (\frac{1}{2}, 2)$.

$f_{xx} = 2$, $f_{xy} = 2(y-2)$, $f_{yy} = 2x$.

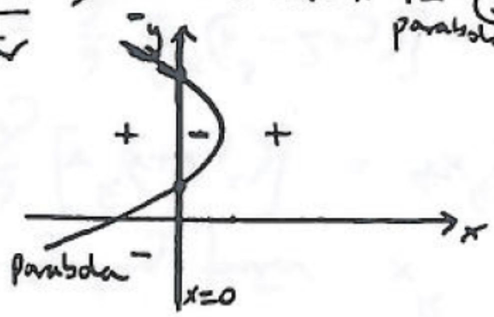
At $(0,1)$ $A = 2$, $B = -2$, $C = 0$. $AC - B^2 < 0$ SADDLE POINT

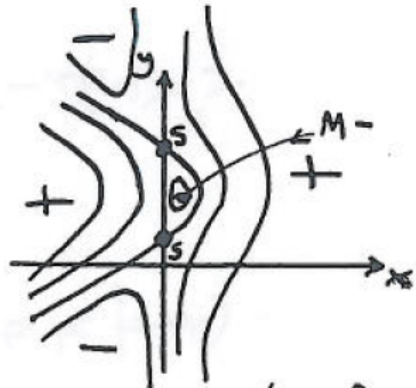
At $(0,3)$ $A = 2$, $B = 2$, $C = 0$. $AC - B^2 < 0$ SADDLE POINT

At $(\frac{1}{2}, 2)$ $A = 2$, $B = 0$, $C = 1$, $AC - B^2 > 0$, $A > 0$ MINIMUM .

$\underline{f = 0}$ $\Rightarrow$ $x = 0$ and $\underbrace{x - 1 = -(y-2)^2}_{\text{parabola}}$

contour





SADDLES $(0,1)$ $(0,3)$ where $f = 0$

MINIMUM $(\frac{1}{2}, 2)$ where $f = -\frac{1}{4}$