IMPERIAL COLLEGE LONDON
DEPARTMENT OF MATHEMATICS

Question Sheet 2

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Problem sheet released on Wednesday of week 3. All questions can be attempted before the problem class on Monday Week 4. Question 2 or 6 could be suitable for tutorials. Solutions will be released on Wednesday of week 4.

Question 1 Let $n \in \mathbb{N}$, $n \geq 2$. Suppose $D : M_n(\mathbb{R}) \to \mathbb{R}$ is a function on which elementary row operations have the same effect as they do for det (for example, if B is obtained from $A \in M_n(\mathbb{R})$ by interchanging two rows, then $D(B) = -D(A)$, etc.). Suppose also that $D(I_n) = 1$. Prove that $D(C) = \det(C)$ for all $C \in M_n(\mathbb{R})$.
Harder: What if we replace $\mathbb{R}$ by an arbitrary field F?

Question 2 For each of the following linear maps $T : V \to V$, choose a basis B for V and compute $[T]_B$. Hence, or otherwise, compute $\det(T)$.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) V is the vector space of all 2×2 matrices over $\mathbb{R}$, and $T(A) = MA$ for all $A \in V$, where $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(iii) V is the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

Question 3 Suppose $n \geq 2$ and $A \in M_n(F)$. The adjugate matrix $adj(A)$ is the transpose of the matrix of cofactors of A and we showed that $adj(A)A = \det(A)I_n$. Give an expression for $adj(adj(A))$ in the case where A is invertible.

Question 4 Suppose F is a field. Let $n \in \mathbb{N}$ and $a_0, ..., a_{n-1} \in F$, not all zero. Using the Vandermonde determinant, prove that the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

has at most $n-1$ distinct roots in F, i.e. there are at most $n-1$ distinct $\alpha \in F$ such that $f(\alpha) = 0$.

Question 5 Suppose U, V, W are vector spaces over a field F and $T : U \to V$ and $S : V \to W$ are linear transformations. Show that the composition $S \circ T : U \to W$ is a linear transformation. If U, V, W are finite dimensional with bases B, C, D, prove that

$${}_D[S \circ T]_B = {}_D[S]_C\, {}_C[T]_B.$$

Question 6 Let V be a vector space over a field F and $T : V \to V$ be a linear transformation. Suppose that $\lambda \in F$ is an eigenvalue of T. Let $m \geq 1$ be an integer and denote by T^m the composition $T \circ \ldots \circ T$ (m times). Note that this is a linear transformation $V \to V$.

i) Show that λ^m is an eigenvalue of T^m.

ii) If $a_0, \ldots, a_m \in F$ are such that $a_0 \operatorname{Id} + a_1 T + a_2 T^2 + \ldots + a_m T^m = 0$, show that λ is a root of the polynomial $p(x) = a_0 + a_1 x + \ldots + a_m x^m$.

Question 7 Suppose that $T : V \to V$ is a linear map with the property that $T(T(v)) = T(v)$ for all $v \in V$.

(i) Show that

$$V = \ker(T) + \operatorname{im}(T) \text{ and } \ker(T) \cap \operatorname{im}(T) = \{0\}.$$

Hint: Note that if $v \in V$ then $v = (v - T(v)) + T(v)$.

(ii) Deduce that if V is of dimension n, then there is a basis B of V such that

$$[T]_B = \begin{pmatrix} I_s & 0_{r \times n-s} \\ 0_{n-s \times s} & 0_{n-s \times n-s} \end{pmatrix},$$

where $s = \dim(\ker(T))$.