Problem Sheet 6

Math40002, Analysis 1

1. Define $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q}. \end{cases}$ Prove that f is not integrable, but that f^2 is.

2. Fix an integer $r \geq 0$ and define $f : [1, b] \to \mathbb{R}$ by $f(x) = x^r$.

   (a) Let $P_n = (1, b^{1/n}, b^{2/n}, \ldots, b^{(n-1)/n}, b)$ be a partition of $[1, b]$. Compute the lower Darboux sum $L(f, P_n)$, and show that $U(f, P_n) = b^{r/n} L(f, P_n)$.

   (b) Prove that f is integrable, and compute $\int_1^b x^r \, dx$.

3. Prove that any monotone increasing function $f : [a, b] \to \mathbb{R}$ is integrable, by considering its Darboux sums for partitions where every subinterval $[x_i, x_{i+1}]$ has the same length.

4. Define the *mesh* of a partition $P = (x_0, \ldots, x_k)$ to be the maximal length of any subinterval:
$$\mathrm{mesh}(P) = \max_{0 \leq i \leq k-1} \Delta x_i = \max_{0 \leq i \leq k-1} (x_{i+1} - x_i).$$
   Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\big(P_n\big)$ is any sequence of partitions of $[a, b]$ such that $\mathrm{mesh}(P_n) \to 0$ as $n \to \infty$, then
$$\int_a^b f(x) \, dx = \lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n).$$
   The proof should follow the argument we used in lecture to show that continuous functions are integrable.

5. (a) Prove for any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ that if $\sin(\frac{\theta}{2}) \neq 0$, then
$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$$
   using the formula $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.

   (b) Suppose for some $t > 0$ that $\sin(x)$ is monotone increasing on the interval $[0, t]$, and consider the partition $P_n = (0, \frac{t}{n}, \frac{2t}{n}, \ldots, \frac{(n-1)t}{n}, t)$ of $[0, t]$. Compute $U(\sin(x), P_n)$, and show that
$$\lim_{n \to \infty} U(\sin(x), P_n) = 2\sin^2(\tfrac{t}{2}).$$
   Remark: This limit is equal to $1 - \cos(t)$ by the double-angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$, so problem 4 tells us that $\int_0^t \sin(x) \, dx = \sin^2(\frac{t}{2}) = 1 - \cos(t)$.

6. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x)$ and the product $f(x)g(x)$ are both integrable, and $f(x) \geq 0$ for all $x \in [a, b]$. If $c \leq g(x) \leq d$ for all $x \in [a, b]$, prove that

$$c \int_a^b f(x)\, dx \leq \int_a^b f(x)g(x)\, dx \leq d \int_a^b f(x)\, dx.$$

7. (*) Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/|q|, & x = \frac{p}{q} \in \mathbb{Q}. \end{cases}$

   (We proved in problem sheet 1 that f is discontinuous at all rational numbers.)

   (a) Compute the lower Darboux integral $\underline{\int_0^1} f(x)\, dx$.

   (b) Consider the partition $P_n = (0, \frac{1}{n^3}, \frac{2}{n^3}, \ldots, \frac{n^3-1}{n^3}, 1)$ of $[0, 1]$. Show for n large that there are at most n^2 subintervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ on which $M_i = \sup\limits_{\frac{i}{n^3} \leq t \leq \frac{i+1}{n^3}} f(t)$ is at least $\frac{1}{n}$.

   (c) Prove that $U(f, P_n) \leq \frac{2}{n}$ for n large. (Hint: break the sum into terms where $M_i \geq \frac{1}{n}$ and terms where $M_i < \frac{1}{n}$.)

   (d) Conclude that f is integrable, and compute $\int_0^1 f(x)\, dx$.