Math40002 Analysis 1 Unseen 5

1. Show that $n^{1/n} \to \infty$ as $n \to \infty$.[1]

   By Problem Sheet 5, Question 1, $1+\sqrt{n} = 1 + n\frac{1}{\sqrt{n}} \le \left(1+\frac{1}{\sqrt{n}}\right)^n$**. So**

   $$1 \le 1^{1/n} \le (\sqrt{n})^{1/n} \le (1+\sqrt{n})^{1/n} \le \left(\left(1+\frac{1}{\sqrt{n}}\right)^n\right)^{1/n} = 1 + \frac{1}{\sqrt{n}}$$

   Since $1 \to 1$ **and** $1 + \frac{1}{\sqrt{n}} \to 1$**, so does** $n^{1/n} \to \infty$**.**

2. Let $PL(a_n)$ be the set of all limits of convergent subsequences of (a_n), i.e.,

   $PL(a_n) = \{ L \in \mathbb{R} \mid \text{there is some subsequence}(a_{n_k}) \text{ such that } a_{n_k} \to L \text{ as } k \to \infty \}$.

   Elements of $PL(a_n)$ *are also called partial limits of* (a_n)*.*

   (a) For each one of the following items, give an example, without proof, of a sequence (a_n) such that $PL(a_n) = S$.

      i. $S = \{1, \ldots, m\}$.

      ii. $S = \mathbb{N}$.

   (b) Is there a sequence (a_n) such that $PL(a_n) = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$? You are not required to justify your answer, just come up with an answer – yes or no. You will prove the correct answer in a further question.

   (a) i. 1,. . . , m, 1,. . . , m, 1,. . . , m, ...

   ii. 1, 1,2, 1,2,3, 1,2,3,4, 1,2,3,4,5,. . .

   (b) No, because of Question 4.

3. Let (a_n) be a sequence, $L \in \mathbb{R}$. Prove that $L \in PL(a_n)$ if and only if for every $\epsilon > 0$, the set $\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \}$ is infinite.

   $\Longrightarrow$ **If** $L \in PL(a_n)$**, then there is a subsequence** (a_{n_k}) **such that** $a_{n_k} \to L$ **as** $k \to \infty$**. So for every** $\epsilon > 0$**, there is some** $N \in \mathbb{N}$ **such that** $\forall k > N \ |a_{n_k} - L| < \epsilon$**. So** $n_{N+1}, n_{N+2}, \ldots$ **is infinite and contained in** $\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \}$**.**

   $\Longleftarrow$ **If** $\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \}$ **is infinite for every** $\epsilon > 0$**, then we can choose** $n_1 \in \{ n \in \mathbb{N} \mid L - 1 < a_n < L + 1 \}$**. Assume we defined** $n_1 < \cdots < n_k$**. Rhere are infinitely many elements in** $\left\{ n \in \mathbb{N} \middle| L - \frac{1}{k+1} < a_n < L + \frac{1}{k+1} \right\}$**. In particular, there is some** $n_k < n_{k+1} \in \left\{ n \in \mathbb{N} \middle| L - \frac{1}{k+1} < a_n < L + \frac{1}{k+1} \right\}$**. Now, for every** $\epsilon > 0$**, let** $N > 1/\epsilon$**. So for all** $k > N$**,**

   $$|a_{n_k} - L| < 1/k < 1/N < \epsilon.$$

[1] Hint: Problem Sheet 5, Question 1

4. Prove that if (a_n) is a sequence and there is a sequence L_n of partial limits of $PL(a_n)$ such that $L_n \to L$, then L is also a partial limit of (a_n).

   Let (L_m) be a sequence of partial limits such that $L_m \to L$ as $m \to \infty$. By Question 1, it suffices to show that for every $\epsilon > 0$, the set $\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \}$ is infinite. Let $\epsilon > 0$. Since $L_m \to L$, there is some $m \in \mathbb{N}$ such that $|L_m - L| < \epsilon/2$. In particular,

$$\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \} \supseteq \{ n \in \mathbb{N} \mid L_m - \epsilon/2 < a_n < L_m + \epsilon/2 \} .$$

   So, as the right hand side is infinite, so is the left hand side.

5. In this question we give yet another definition of lim sup:
   Let (a_n) be a sequence. Show that

$$\lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right) = \sup(PL(a_n))$$

   in the sense that if one exists, so does the other and they are equal.

   - **Assume $L := \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)$ exists. Let $L' \in PL(a_n)$, so there is a subsequence (a_{n_k}) such that $a_{n_k} \to L'$. Since $n_k \geq k$, it follows that $\sup_{n\geq k} a_n \geq a_{n_k}$ for all $k \in \mathbb{N}$. So $L \geq L'$. As $L' \in PL(a_n)$ was arbitrary, $PL(a_n)$ is bounded above by L. In particular,**

$$\lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right) = L \geq \sup(PL(a_n)).$$

   - **Assume $S = \sup(PL(a_n))$ exists. Then by Question 4, $S \in PL(a_n)$. Let (a_{n_k}) be a subsequence such that $a_{n_k} \to S$. Then $\{ a_{n_k} | k \in \mathbb{N} \}$ is bounded. Since Since $n_k \geq k$, it follows that $\sup_{n\geq k} a_n \geq a_{n_k}$ for all $k \in \mathbb{N}$. So $\left\{ \sup_{n\geq m} a_n \middle| m \in \mathbb{N} \right\}$ is bounded below, and it is also descending, so it converges.**
   - **It is left to show that if both sides of the equation exist, then**

$$\lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right) \leq \sup(PL(a_n)).$$

     For that it suffices to find some subsequence a_{n_k} such that $a_{n_k} \to \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)$.
     - **Let $n_1 \geq 1$ be such that $|a_{n_1} - \sup_{n\geq 1} a_n| < 1$.**
     - **Let $n_2 \geq n_1 + 1$ be such that $|a_{n_2} - \sup_{n\geq(n_1+1)} a_n| < 1/2$**

       $\vdots$
     - **Assume $n_1 < n_2 < \cdots < n_k$.**
       Let $n_{k+1} \geq n_k + 1$ be such that $|a_{n_{k+1}} - \sup_{n\geq(n_k+1)} a_n| < 1/k$.

To prove $a_{n_k} \to \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)$, let $\epsilon > 0$ and let $N_1 \in \mathbb{N}$ be such that $\forall m > N$, $\left|\sup_{n\geq m} a_n - \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)\right| < \epsilon/2$. Let $N_2 := 2/\epsilon$. So for every $k > N_2$,

$$\left| a_{n_k} - \sup_{n\geq(n_k+1)} a_n \right| < 1/k < 1/N_1 = \epsilon/2.$$

Let $N := \max(N_1, N_2)$. So, if $k > N$, clearly also $n_k \geq k > N$ and, by the triangle inequality,

$$\left| a_{n_k} - \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)\right| \leq \left| a_{n_k} - \sup_{n\geq(n_k+1)} a_n \right| + \left| \sup_{n\geq(n_k+1)} a_n - \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)\right| < \epsilon/2 + \epsilon/2.$$