Question 1 (suggested for peer/personal tutorial)

Recall from Section 8.3.8 in Prof. Veraart's notes that the p.d.f. of the standard Cauchy distribution is

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \qquad \text{with support } x \in \mathbb{R}.$$

(a) Show that f_X is a probability density function (p.d.f.) and plot $y = f_X(x)$.

(b) Compute the first (raw) moment of X, $\mu = \mu_1' = E(X)$.

(c) Compute the kth central moment of X, $\mu_k = E\left((X-\mu)^k\right)$ for $k \in \{2, 3, \ldots\}$.

(d) Compute the second raw moment of X, $\mu_2' = E(X^2)$.

Question 2

Suppose that the random variable X is known to only take non-zero values in the bounded range $[a, b]$, i.e. the support of X is $[a, b]$.

(a) Prove that $\text{Var}(X) \leq \frac{(b-a)^2}{4}$.

(b) Conclude that if $X \sim \text{Bernoulli}(p)$, for some $p \in [0, 1]$, then $\text{Var}(X) \leq \frac{1}{4}$.

(c) Is the bound $\text{Var}(X) \leq \frac{(b-a)^2}{4}$ tight? In other words, is there a distribution F_X
   with support $[a, b]$ for which $\text{Var}(X) = \frac{(b-a)^2}{4}$?

Question 3

Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables all following a $N(0, 1)$ distribution.

(a) What is the distribution of the random variable $Z = X_1^2$?

(b) Find the distribution of $Y = X_1^2 + X_2^2 + \cdots + X_n^2$.

Question 4

Complete Exercise 1.2.5 in the lecture notes: Given a sample of observations $x_1, x_2, \ldots, x_n$, with the sample mean $\bar{x}$ and sample variance s^2 defined in Definition 1.2.9 of the lecture notes, prove that

$$\min_a \left[\sum_{i=1}^n (x_i - a)^2 \right] = \sum_{i=1}^n (x_i - \bar{x})^2 = (n-1)s^2.$$

Question 5 (using R)

(a) Use R to generate 10 observations from a normal distribution with mean 3 and variance 2. Save the values in a vector x.

(b) Use the built-in R commands to compute the sample mean, variance and standard deviation of x.

(c) Write your own R functions to compute the sample mean and sample variance of x.

Hints:

- Question 1(b): Check the definition of improper integrals from the Analysis module, or see Remark 1.1.18 in the lecture notes.
- Question 2(a): Derive the expression $(X-a)(X-b) = \left(X - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{4}$.
- Question 3: See Term 1, Problem Sheet 5, Exercise 8 and Term 1 Problem Sheet 7, Exercise 3.