Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 2 with solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

In the following, we will sometimes use the notation:

$$y' = \frac{dy}{dx} \quad \text{and} \quad y'' = \frac{d^2y}{dx^2}.$$

1. Solve the following first order differential equations:

   (a) $y' = (1 + x)(1 + y)$

   The equation is separable, so the solution is

   $$\int \frac{dy}{1+y} = \int (1+x)\, dx \Longrightarrow \log|1+y| = x + \frac{1}{2}x^2 + c.$$

   (b) $y' = \frac{1+y}{2+x}$ with $y(0) = 1$

   The equation is separable, so the solution is

   $$\int \frac{dy}{1+y} = \int \frac{dx}{2+x}\, dx \Longrightarrow \log|1+y| = \log|2+x| + c.$$

   We can drop the absolute values as the sign can be absorbed in the new constant of integration A obtaining $y = A(2 + x) - 1$. Since $y(0) = 1$, we have $A = 1$, so we obtain

   $$y = x + 1.$$

   (c) $xyy' = x^2 + y^2$

   The equation can be written as

   $$y' = \frac{x}{y} + \frac{y}{x}$$

   which is homogenous. Using the substitution $u = y/x$ we obtain:

   $$u + x\frac{du}{dx} = u + \frac{1}{u} \Longrightarrow xu' = \frac{1}{u},$$

   which is separable so the solution is therefore

   $$\int u\, du = \int \frac{dx}{x} \Longrightarrow \frac{u^2}{2} = \frac{y^2}{2x^2} = \log|x| + c.$$

   (d) $xy' = y + xe^{y/x}$

   The equation can be written as

   $$y' = \frac{y}{x} + e^{y/x}$$

which is homogenous. Using the substitution $u = y/x$ we obtain:

$$u + x\frac{du}{dx} = u + e^u \Longrightarrow xu' = e^u,$$

Which is separable and the solution therefore is

$$-e^{-\frac{y}{x}} = \log|x| + c.$$

(e) $y' = -5y + x + e^{-2x}$ with $y(-1) = 0$

The equation is first-order and linear, so use the integrating factor $I = e^{5x}$, we have from the lectures

$$e^{5x}y = \int (xe^{5x} + e^{3x})\,dx = \frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + \frac{e^{3x}}{3} + c,$$

using integration by part. Using the condition $y(-1) = 0$ implies

$$c = \frac{6}{25e^5} - \frac{1}{3e^3}.$$

In summary

$$y(x) = \frac{x}{5} - \frac{1}{25} + \frac{e^{-2x}}{3} + \left(\frac{6}{25e^5} - \frac{1}{3e^3}\right)e^{-5x}.$$

(f) $y' = -y\cot x + \cos x$ with $y(0) = 0$

The ODE is linear and the integrating factor is

$$I = e^{\int \cot x\,dx} = \sin x.$$

So we have for the solution

$$y\sin x = \int \frac{\sin 2x}{2}\,dx = -\frac{\cos 2x}{4} + c.$$

Condition $y(0) = 0$ gives $c = 1/4$ so

$$y(x) = -\frac{\cos 2x}{4\sin x} + \frac{1}{4\sin x} = \frac{2\sin^2 x}{4\sin x} = \frac{1}{2}\sin x.$$

(g) $6y' = 2y + xy^4$ with $y(0) = -2$

The ODE is Bernoulli, so from the lectures we use the substitution $u = y^{-3}$. We obtain the following first order ODE:

$$u' + u = -\frac{1}{2}x$$

Using the integrating factor $I = e^x$, we obtain the solution

$$u(x) = y^{-3} = -\frac{1}{2}(x-1) + ce^{-x}.$$

Now appling the inital condition $y(0) = -2$, we obtain $c = -5/8$. So we have

$$y(x) = \frac{2}{(4 - 4x - 5e^{-x})^{\frac{1}{3}}}.$$

2. Solve the following second order differential equations:

(a) $y'' = \cos(2x)$ with $y(0) = 1$ and $y'(0) = 0$

y and y' missing from RHS so we can integrate twice to obtain:

$$y = -\frac{1}{4}\cos 2x + c_1 x + c_2.$$

Given the initial conditions we obtain $c_1 = 0$ and $c_2 = 5/4$.

(b) $y'' = 2y^3$ with $y(1) = y'(1) = 1$

x and y' missing from RHS so using $u = y'$ we have

$$\frac{1}{2}u^2 = \frac{1}{2}y^4 + c_1$$

We can already use the initial condition $y'(1) = 1$ and $y(1) = 1$ to set $c_1 = 0$ (proceeding with $c_1 \neq 0$ will be hard!). Then we have $u = y' = \pm y^2$ but we can reject the negative solution as $y'(1) = 1 > 0$. So we have $y' = y^2$, which is separable, so we have

$$\int \frac{dy}{y^2} = -\frac{1}{y} = x + c_2$$

Using $y(1) = 1$ we obtain $c_2 = -2$ so we have $y = 1/(2 - x)$.

(c) $y'' = (y')^2$

x and y missing from RHS so using $u = y'$ we have

$$u = -\frac{1}{x + c_1} = y'$$

[Note that $u = 0$ therefore $y = c_2$ is also a solution (which is the limit of $c_1 \to \infty$).] The ODE above is separable so we have $y = -\log|x + c_1| + c_2$.

(d) $yy'' + y' = (y')^2$

x is missing from RHS so using $u = y'$ we have $yu' + u = u^2$. We can write this as

$$y\left(u\frac{du}{dy}\right) + u = u^2,$$

which is separable so we obtain $\log|y| = \log|u - 1| + c_1'$ (or $u = 0$ and $y = c_2$). This can be rearranged using a new constant of integration as $u = 1 + c_1 y = y'$, which is separable. So the general solution is

$$\frac{1}{c_1}\log|1 + c_1 y| = x + c_2.$$

(e) $y'' = -x\,(y')^2$ with $y(0) = 0$ and $y'(1) = 1$

y missing from RHS so using $u = y'$, we have $u' = -xu^2$, which is separable. So

$$-\frac{1}{u} = -\frac{1}{2}x^2 + c_1$$

[Again, we also have the solution $u = 0$ and $y = c_1$ but does not satisfy the initial condition]. Using the initial condition $y'(1) = 1$, we obtain $c_1 = -1/2$. So

$$u = y' = \frac{2}{1 + x^2},$$

which is separable so we have $y = 2\tan^{-1} x + c_2$ and $c_2 = 0$ so that $y(0) = 0$.

(f) $y'' = y'y$

x missing from RHS so using $u = y'$ as above

$$u\frac{du}{dy} = uy \Rightarrow u = \frac{1}{2}y^2 + c_1$$

[or $u = 0$, $y = c_2$ as before.] The ODE above is separale. So we have

$$\frac{1}{2}x + c_2 = \int \frac{dy}{y^2 + 2c_1},$$

and the RHS depends on the sign of c_1:
$c_1 > 0 \Rightarrow y = (2c_1)^{1/2} \tan\left[(2c_1)^{1/2}(\frac{1}{2}x + c_2)\right]$.
$c_1 < 0 \Rightarrow y = -(2|c_1|)^{1/2} \tanh\left[(2|c_1|)^{1/2}(\frac{1}{2}x + c_2)\right]$.
$c_1 = 0 \Rightarrow y = -\frac{1}{\frac{1}{2}x + c_2}$.

(g) $y'' + \frac{1}{x}y' = 1$ with $y(1) = 0$ and $y(2) = 1$

y missing from RHS so using $u = y'$ we have $u' + \frac{1}{x}p = 1$, which is a linear first order ODE. Using integrating factor $I = x$ we have:

$$(xu)' = x \Rightarrow xu = \frac{1}{2}x^2 + c_1$$

So we have

$$u = y' = \frac{1}{2} + \frac{c_1}{x} \Rightarrow y = \frac{1}{4}x^2 + c_1 \log|x| + c_2$$

Given the initial conditions we obtain $c_1 = \frac{1}{4\log 2}$ and $c_2 = \frac{1}{4}$.

3.* When measuring the growth of a population $x(t)$, it was found that the Malthusian model of constant reproduction rate fails when the population gets large. An empirical model shows that the reproduction rate is not constant (as Malthus postulated) and depends on the population $x(t)$ as: $k\left(1 - (x/\beta)^\alpha\right)$.

(a) Write down and solve the ODE for $x(t)$

$$\frac{dx}{dt} = k\left(1 - \left(\frac{x}{\beta}\right)^\alpha\right)x \quad \Rightarrow \quad \frac{dx}{dt} - kx = -\frac{k}{\beta^\alpha}x^{\alpha+1}$$

This is a Bernoulli ODE [it is also separable so can be done by integration directly] using $u = x^{-\alpha}$ we obtain

$$\frac{du}{dt} = -\alpha k\left(u - \frac{1}{\beta^\alpha}\right) \quad \Rightarrow \quad u - \frac{1}{\beta^\alpha} = Ae^{-\alpha kt},$$

Where A is the constant of integration. Substituting $u = x^{-\alpha}$ we obtain

$$\frac{1}{x^\alpha} = \frac{1}{\beta^\alpha} + Ae^{-\alpha kt} \quad \Rightarrow \quad x^\alpha = \frac{\beta^\alpha}{1 + A\beta^\alpha e^{-\alpha kt}}.$$

(b) What is the value of the population as $t \to \infty$?

As $t \to \infty$ then $x \to \beta$ so the final population is finite. *beta* is referred to as carrying capacity.

4. By using a suitable substitution (or otherwise), find the solution of

(a) $y(xy+1)+x(1+xy+x^2y^2)y'=0$

On use of the substitution $u=yx$ we have

$$\frac{dy}{dx}=\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}$$

so that

$$y(u+1)+x(1+u+u^2)\left(\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}\right)=0$$

Multiplying by x and on rearrangement,

$$u^3=(1+u+u^2)x\frac{du}{dx}.$$

This is separable so we have

$$\log|x|=-\frac{1}{2x^2y^2}-\frac{1}{xy}+\log|xy|+c_1$$

(b) $y'=\frac{1-2y-x}{4y+2x}$

Let $u=x+2y$ then we have

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{du}{dx}-1\right)=\frac{1-u}{2u}\quad\Rightarrow\quad\frac{du}{dx}=\frac{1}{u}.$$

Thus we have

$$\frac{u^2}{2}=x+c_1\quad\Rightarrow\quad\frac{(x+2y)^2}{2}=x+c_1.$$

5. The vertical motion of an object is described by the equation of motion

$$z''=-g-\gamma z'$$

where the constant g is the acceleration due to gravity near surface of earth and the constant γ is a measure of friction or air-resistance. Find the general solution of the equation of motion. You don't need to know anything about mechanics to solve this problem!

Let $u=z'$ be the velocity of the object. We can write the ODE as

$$\frac{du}{dt}=-g-\gamma u,$$

which can be solved via an integrating factor or separation of variables and integrates to

$$\log(u+g/\gamma)=-\gamma t+c_1\quad\Rightarrow\quad u(t)=-\frac{g}{\gamma}+Ae^{-\gamma t}$$

where A is an arbitary constant of integration. Note that the velocity approaches so-called terminal velocity g/γ. A second integration yealds the general solution.

$$z(t)=-\frac{gt}{\gamma}-\frac{Ae^{-\gamma t}}{\gamma}+B.$$

A and B are constants of integration that can be determined for example by knowing the initial position and velocity of the object.

6.* Consider the motion of a rocket that is launched vertically with initial velocity v^*. Gravity further away from the surface of the earth is not constant and it decreases as the inverse of square distance from the centre of the earth r. Thus using second Newton law will result in the following ODE for the velocity v as a function of the distance to the center of the Earth r

$$r^2 v \frac{dv}{dr} = C$$

Solve the ODE above and find the critical initial velocity v_c^* above which the rocket is guaranteed to escape the Earth's pull.

The ODE is separable so we have

$$\int v\, dv = C \int \frac{dr}{r^2} \quad \Rightarrow \quad \frac{v^2}{2} = C\left(-\frac{1}{r} + c_1'\right) = -\frac{C}{r} + c_1$$

Initial condition is $v(R) = v*_c$ where R is the radiaus of earth. So we have:

$$c_1 = \frac{C}{R} + \frac{v_c^{*2}}{2} \quad \Rightarrow \quad \frac{v^2}{2} = -\frac{C}{r} + \frac{C}{R} + \frac{v_c^{*2}}{2}.$$

We note that C is negative as the gravity is a force towards the centre of earth. Given $C < 0$, then equation above suggests that the velocity of the rocket decreases as $r \to \infty$ which makes intuitive sense. For the rocket not to come back to earth we need $v^2 > 0$ for $\forall r$. This results then on the minimum critical initial velocity to be

$$v_c^* = \sqrt{2\frac{|C|}{R}}.$$