Topic: Counting

In today's problem class we will be reviewing concepts from combinatorics.

1. (a) 10 people say goodbye to each other and shake hands. Everybody goes home alone. How many handshakes are there in total?

   (b) 10 couples say goodbye to each other and shake hands. Every couple goes home alone. How many handshakes are there in total?

   (c) 10 couples say goodbye to each other: The men shake hands, the women kiss each other on each cheek and a man and a woman also kiss each other on each cheek. How many handshakes and how many kisses are there in total?

Solution:

(a) We need to find all possible pair, i.e. we are in the setting of sampling without replacement, where the order does not matter. There are $\binom{10}{2} = 45$ handshakes.

(b) You can argue that every time two couples say goodbye to each other, there are 4 handshakes in total. Since there are $\binom{10}{2}$ possible combinations of couples saying goodbye to each other we have $4 \cdot \binom{10}{2} = 180$ handshakes in total.
Alternatively, you can count the number of pairs in 20 people: $\binom{20}{2} = 190$ and you need to subtract the number of possibilities where the two people belonging to the same couple saying goodbye to each other (which should not happen!). Hence we have $190 - 10 = 180$ handshakes in total.

(c) There are 10 men in total, so $\binom{10}{2} = 45$ handshakes in total.

For the number of kisses, you can either argue that whenever two couples say goodbye to each other, there will be 12 kisses, hence $12 \cdot \binom{10}{2} = 540$ kisses in total. Or, you can count the number of pairs consisting of two women: $\binom{10}{2} = 45$ and the number of pairs consisting of one man and one woman: $\binom{10}{1}\binom{10}{1} = 100$. Out of the man-woman pairs, there are 10 pairs consisting of the 10 couples which will actually not kiss each other goodbye.

Each pair exchanges 4 kisses, so there are $(45 + 100 - 10) \cdot 4 = 540$ kisses in total.

2. Explain, without direct calculation that, for $n \in \mathbb{N}$,

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Use a proof where you only comment on sampling from sets of an appropriate cardinality.

Solution: The right hand side $\binom{2n}{n}$ gives us the number of possibilities of drawing n balls from an urn consisting of $2n$ balls which we shall label as $\{1, \ldots, n, n+1, \ldots, 2n\}$. This is the same, as drawing j balls from an urn consisting of the balls $\{1, \ldots, n\}$ and $n - j$ balls from an urn consisting of the balls $\{n+1, \ldots, 2n\}$ where we need to sum over all possibilities $j \in \{0, 1, \ldots, n\}$. Hence

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{n-j} = \sum_{j=0}^{n} \binom{n}{j}^2.$$

Remark 0.1 *In many situations, a probability calculation can be reduced to an exercise in counting equally likely sample outcomes using combinatorial techniques. If the sample space comprises*

$\mathrm{card}(\Omega)$ *equally likely outcomes, and event* E *represents a collection of* $\mathrm{card}(E)$ *of them, then we can legitimately define* $\mathrm{P}(E)$ *by*

$$\mathrm{P}(E) = \frac{\mathrm{card}(E)}{\mathrm{card}(\Omega)},$$

and so the probability calculation only requires enumeration of $\mathrm{card}(E)$ *and* $\mathrm{card}(\Omega)$.

3. Outlook: Hypergeometric distribution. Consider an urn filled with N balls, with $K \in \mathbb{N}$ being white balls and $N - K$ being black. Suppose we draw $n \in \mathbb{N}$ balls from the urn *without replacement* and we denote by x the number of observed white balls. We compute the probability of having x white balls when we draw n balls without replacement:

$$\mathrm{P}(X = x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, \text{ for } x \in \{0, 1, \ldots, K\} \text{ and } n - x \in \{0, 1, \ldots, N - K\},$$

and $\mathrm{P}(X = x) = 0$ otherwise. We justify the above formula as follows: For the denominator, we report the total number of possibilities of drawing n balls from an urn of N balls, so $\binom{N}{n}$ in total. For the numerator, we have $\binom{K}{x}$ possibilities of choosing x white balls from the total number of K white balls and $\binom{N-K}{n-x}$ possibilities of choosing $n - x$ black balls from the total number of $N - K$ black balls. We claim that

$$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} = \frac{\binom{n}{x}\binom{N-n}{K-x}}{\binom{N}{K}}.$$

(a) Prove the above identity by expanding the binomial coefficients/factorials.

(b) Describe in words what the left and right hand side represent.

(c) Suppose your sock drawer is a mess and contains 18 black socks and 10 blue socks that otherwise look alike. What is the probability that you randomly select two black socks if you select exactly 2 socks?

(d) If I deal you a hand of 13 cards at random from a well shuffled pack. What is the probability that your hand contains exactly 10 hearts?

(e) A 12-member jury for a criminal case will be selected from a pool of 14 men and 6 women. What is the probability that at least 3 of the jury will be women?

Solution:

(a) We have

$$\begin{aligned}
\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} &= \frac{K!}{x!(K-x)!}\frac{(N-K)!}{(n-x)!(N-K-(n-x))!}\frac{n!(N-n)!}{N!} \\
&= \frac{K!(N-K)!}{N!}\frac{n!}{x!(n-x)!}\frac{(N-n)!}{(K-x)!(N-n-(K-x))!} \\
&= \frac{\binom{n}{x}\binom{N-n}{K-x}}{\binom{N}{K}}
\end{aligned}$$

(b) On the left hand side, we consider balls IN sample, broken down by type (white or black). The denominator is the total number of ways of choosing n balls from N to be IN the sample. The numerator is the number of ways of choosing x white balls from K to be IN the sample, multiplied by the number of ways of choosing the $n - x$ black balls from $N - K$ to be IN the sample.

On the right hand side, we consider choosing balls in the urn to label "white" and "black", broken down by whether they are IN the sample or OUT of the sample. The denominator is the total number of ways of choosing K from N balls to label "white". The numerator is the number of ways of choosing x from n balls IN the sample to be white balls, multiplied by the number of ways of choosing the $K-x$ from $N-n$ balls OUT of the sample to be black balls.

(c)

$$\frac{\binom{18}{2}\binom{10}{0}}{\binom{28}{2}} = \frac{18\cdot 17}{2}\frac{2}{28\cdot 27} = \frac{2\cdot 17}{28\cdot 3} = \frac{17}{42}$$

(d)

$$\frac{\binom{13}{10}\binom{39}{3}}{\binom{52}{13}}$$

(e)

$$1 - \frac{\binom{6}{0}\binom{14}{12}}{\binom{20}{12}} - \frac{\binom{6}{1}\binom{14}{11}}{\binom{20}{12}} - \frac{\binom{6}{2}\binom{14}{10}}{\binom{20}{12}}$$

4. Use counting approaches in the solution of the following problems;
   (a) Each of n sticks is broken into two parts, long and short, and a new set of n sticks formed by pairing and joining the $2n$ parts at random. What is the probability that:
      i. each stick is paired and rejoined into its original form that is, there is a match between the rejoined long and short parts for all n sticks.
      ii. each of the n long parts are rejoined with a short part.
   (b) Six fair dice are rolled. What is the probability that a full set of scores $\{1, 2, 3, 4, 5, 6\}$ is obtained?
   (c) If the letters M,I,I,I,I,S,S,S,S,P,P are arranged at random, what is the probability that:
      i. the arrangement spells the word MISSISSIPPI?
      ii. the arrangement has no adjacent I's?
      iii. the arrangement has at least 2 consecutive S's?

Solution:

(a) Label sticks $l_1, l_2, \dots, l_n$ and $s_1, s_2, \dots s_n$ for long and short. $\text{card}(\Omega) = (2n)!$ (the number of ways of arranging the $2n$ pieces).

   i. e.g. $\{s_1, l_1, s_2, l_2, \dots, s_n, l_n\}$ would satisfy event E. There are $n!$ ways of arranging $\{1, 2, \dots, n\}$ but could have e.g. $\{s_1, l_1\}$ or $\{l_1, s_1\}$ for each pair, so $\text{card}(E) = 2^n n!$. Hence $\text{P}(E) = \frac{2^n n!}{(2n)!}$.

   ii. e.g. $\{s_1, l_5, s_3, l_1, \dots, s_6, l_7\}$, arrange $\{s_1, \dots, s_n\}$ in $n!$ ways and $\{l_1, \dots, l_n\}$ in $n!$ ways. Then can arrange each s, l pair in 2 ways. So $\text{card}(E) = 2^n n!n!$. Hence $\text{P}(E) = \frac{2^n n!n!}{(2n)!}$.

(b) $\text{card}(\Omega) = 6^6, \text{card}(E) = 6!$, hence $\text{P}(E) = \frac{6!}{6^6}$.

(c) The number of arrangements of M,I,I,I,I,S,S,S,S,P,P is

$$\text{card}(\Omega) = \frac{11!}{4!4!2!}.$$

i. Only 1 arrangement spells MISSISSIPPI so $\mathrm{card}(E) = 1$.

ii. Consider removing the I's: –M–S–S–S–S–P–P–
can arrange the remaining letters in $\frac{7!}{4!2!}$ ways. To ensure non-adjacent Is, must choose 4 of the –s as places to put the Is. Can do this in $\binom{8}{4}$ ways, so

$$\mathrm{card}(E) = \frac{7!}{4!2!}\frac{8!}{4!4!}.$$

iii. For at least 2 consecutive S's.
Find the number with no adjacent Ss as in ii. Then

$$\mathrm{card}(E) = \frac{11!}{4!4!2!} - \frac{7!}{4!2!}\binom{8}{4}.$$