Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 5

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. Consider the following systems of ODEs, which you already solved in Problem Sheet 4:

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and } A \text{ is a } 2 \times 2 \text{ matrix.}$$

   (a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

   (e) $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

   (f) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

   Sketch the phase portraits for all these systems in the (x, y) plane.

2. Consider the system of ODEs:

$$\frac{dx}{dt} = 3x - 2y$$
$$\frac{dy}{dt} = 2x - 2y$$

   Solve $y(x)$ and relate this solution to the phase portrait in Question 1a.

3. * Consider the system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

   (a) Find the general solution of this system and explain carefully your calculations

   (b) Locating the system on the (τ, Δ) parameter plane or, equivalently, based on its eigenvalues, establish the expected asymptotic behaviour of the system

   (c) Sketch carefully the phase portrait, indicating in detail all the representative trajectories of the system

4. * Consider a combat model between two armies of strength $x(t)$ and $y(t)$. A simple model of their dynamical evolution under combat is given by:

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bx,$$

where a and b are positive constants.

(a) Explain the meaning of a and b

(b) Find the general solution $(x(t), y(t))$ in terms of the initial conditions $x(0) = x_0$ and $y(0) = y_0$ and sketch the phase portrait.

(c) Solve for $y(x)$ and relate this solution to the phase portrait in (b).

(d) Consider now the (relevant) solutions of the system in the positive quadrant $x(t), y(t) \geq 0$. Using the information above, characterise the qualitatively distinct outcomes of the system for any initial condition. (This is known as Lanchester's square law in Game Theory, and was influential during the Cold War.)

5. Consider a game-theoretical model for the collaboration between two political parties X and Y. The leaderships of both parties are keen on collaboration, but they need to evaluate the support for an alliance by their militants. The 'level of enthusiasm' for the alliance within each party is given by $x(t)$ and $y(t)$, respectively. A simple model of the time evolution of such system is given by:

$$\frac{dx}{dt} = 2x - 2y$$
$$\frac{dy}{dt} = 2x - 3y$$

(a) Find the general solution $(x(t), y(t))$ and sketch the phase portrait.

(b) Solve for $y(x)$ and relate this solution to the phase portrait in (a).

(c) Using the information above, consider the solutions in the positive quadrant $x(t), y(t) \geq 0$ and characterise the distinctive outcomes of the system for any initial condition (x_0, y_0).

(d) Based on the predictions of the model, the leadership of party X commissions a poll among the militants to gauge the initial level of enthusiasm for an alliance. What mathematical condition will they be looking for when examining the results of the poll?