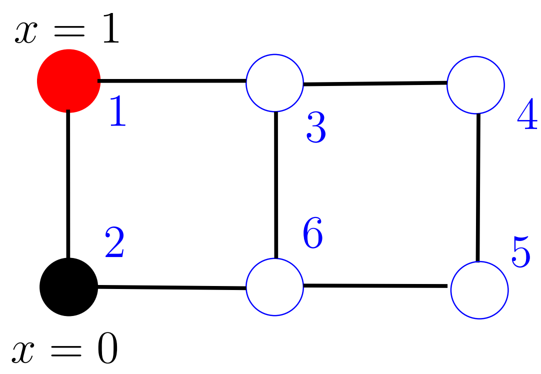
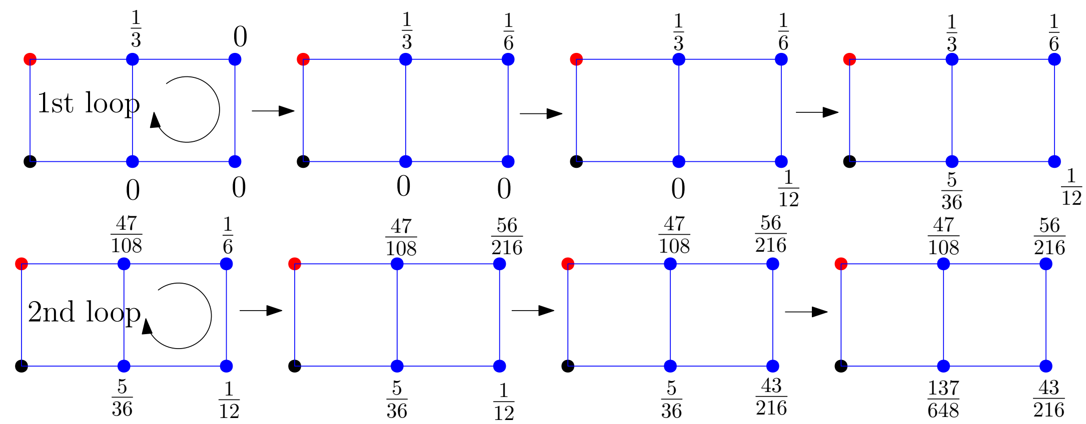
1 (a) Labelling nodes as follows:



and carrying out 2 loops of the method of relaxtion in the order 3,4,5 and then 6 we find:



At the end of the first loop we find

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{12} & \frac{5}{36} \end{bmatrix}^T \tag{1}$$

while at the end of the second loop we find

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & \frac{47}{108} & \frac{56}{216} & \frac{43}{216} & \frac{137}{648} \end{bmatrix}^T . \tag{2}$$

The energy dissipation has the 7 terms (a sum over the edges)

$$\begin{aligned} \mathcal{E} = & (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 \\ & + (x_5 - x_6)^2 + (x_3 - x_6)^2 + (x_2 - x_6)^2. \end{aligned} \tag{3}$$

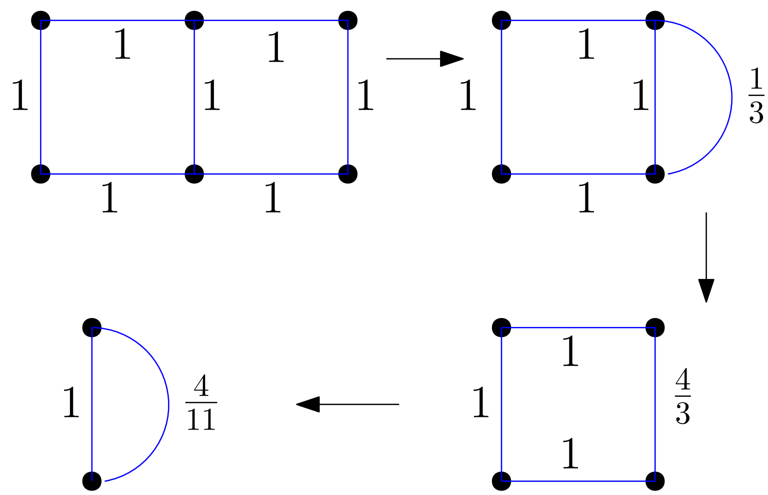
At the end of the first loop this has value

$$\mathcal{E}_1 = 1.5394. \tag{4}$$

At the end of the second loop this has value

$$\mathcal{E}_2 = 1.4485. \tag{5}$$

1 (c) We can use the formula that says the dissipation with unit voltage at one node and another grounded is given by the effective conductance and then compute the effective conductance by reducing to a sequence of "equivalent circuit" as follows:



where we have used the rule for "3 resistors in series" twice. The conductance is therefore equilvalent to two resistors in parallel with conductances 1 and 4/11:

$$C_{\text{eff}} = \mathcal{E} = \frac{15}{11} \approx 1.3636. \tag{6}$$

This dissipation can also be found by first finding the exact potentials. Using the harmonicity of the potentials the equations to solve are

$$\begin{aligned} 3p_3 &= 1 + p_4 + p_6 \\ 2p_4 &= p_3 + p_5 \\ 2p_5 &= p_4 + p_6 \\ 3p_6 &= p_3 + p + 5 \end{aligned} \tag{7}$$

These equations are easily solved by hand to find

$$x_3 = \frac{7}{11}, \qquad x_4 = \frac{6}{11}, \qquad x_5 = \frac{6}{11}, \qquad x_6 = \frac{4}{11}. \tag{8}$$

These values can be substituted in (3) to find (6).

1 (d) We can see from these numerical values that

$$\mathcal{E}_1 \geq \mathcal{E}_2 \geq \mathcal{E}, \tag{9}$$

as expected in accordance with Dirichlet's Principle.

2 (a) The linear equation for the vector of potentials/voltages is

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \tag{10}$$

where

$$\mathbf{f} = [-1 \ \ +1 \ \ \mathbf{0}^T] = [-1 \ \hat{\mathbf{f}}^T], \qquad \hat{\mathbf{f}} = \begin{bmatrix} +1 \\ \mathbf{0} \end{bmatrix}, \tag{11}$$

and where we have grounded the node corresponding to the first entry of $\mathbf{x}$. Hence

$$\mathbf{x} = [0 \ \ \hat{\mathbf{x}}^T], \tag{12}$$

where $\hat{\mathbf{x}}$ is the set of voltages to be found and now includes the voltage at the node having unit current into it. Note that if we introduce the sub-block decomposition of $\mathbf{K}$:

$$\mathbf{K} = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & \mathbf{R} \end{bmatrix} \tag{13}$$

then we know that $\mathbf{R}$ is positive definite and invertible. It is also easy to verify that

$$\mathcal{E}_0(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} = \hat{\mathbf{x}}^T R \hat{\mathbf{x}} - 2\hat{\mathbf{x}}^T \hat{\mathbf{f}}.$$

The equation for $\hat{\mathbf{x}}$ is

$$\mathbf{R}\hat{\mathbf{x}} = \hat{\mathbf{f}} \tag{14}$$

so that

$$\hat{\mathbf{x}} = \mathbf{R}^{-1}\hat{\mathbf{f}}. \tag{15}$$

However we can write

$$\begin{aligned}\mathcal{E}_0(\mathbf{x}) &= \hat{\mathbf{x}}^T R \hat{\mathbf{x}} - 2\hat{\mathbf{x}}^T \hat{\mathbf{f}} = (\hat{\mathbf{x}} - \mathbf{R}^{-1}\hat{\mathbf{f}})^T \mathbf{R} (\hat{\mathbf{x}} - \mathbf{R}^{-1}\hat{\mathbf{f}}) - (\mathbf{R}^{-1}\hat{\mathbf{f}})^T \mathbf{R}\mathbf{R}^{-1}\hat{\mathbf{f}} \\ &= \mathbf{X}^T \mathbf{R} \mathbf{X} + \mathbf{c},\end{aligned}$$

where $\mathbf{c}$ is a vector that is independent of $\mathbf{x}$ (and hence $\hat{\mathbf{x}}$) and

$$\mathbf{X} \equiv \hat{\mathbf{x}} - \mathbf{R}^{-1}\hat{\mathbf{f}}. \tag{16}$$

It is clear, on use of the positive definiteness of $\mathbf{R}$, that $\mathcal{E}_0(\mathbf{x})$ is minimized when $\mathbf{X} = 0$, i.e., when

$$\hat{\mathbf{x}} = \mathbf{R}^{-1}\hat{\mathbf{f}} \tag{17}$$

which is the same as (15). Hence we have shown that the set of potentials determined by Ohm's law, satisfying KCL at the interior nodes and associated with unit input current, minimizes the dissipation function $\mathcal{E}_0(\mathbf{x})$ among all possible potentials defined at the ungrounded nodes.

3 (a) Since
$$-\mathbf{A}\mathbf{x} \tag{18}$$
is precisely the m-dimensional vector of potential drops across nodes it is clear that $-(\mathbf{A}\mathbf{x})^T\mathbf{j} = \mathbf{x}^T(-\mathbf{A}^T\mathbf{j}) = \mathbf{x}^T\mathbf{f}$.

3 (b) The previous expression can be written as
$$\sum_{\text{edges k}} (x_i - x_j)j_k = j_a(x_a - x_b), \tag{19}$$
where edge k is assumed to join node i to node j with the direction taken in forming the incidence matrix is from node i to node j.

3 (c) The energy dissipation is
$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges}} \frac{j_k^2}{c_k}. \tag{20}$$
Write
$$\mathbf{j} = \mathbf{w} + \mathbf{d} \tag{21}$$
where $\mathbf{w}$ is a current satisfying Ohm's law and KCL on all interior nodes; it is clear that
$$\mathbf{d} = \mathbf{j} - \mathbf{w} \tag{22}$$
is the difference between a general unit flow $\mathbf{j}$ and this unit current. Note that $\mathbf{d}$ is a flow of *zero* strength because
$$d_a = j_a - w_a = 1 - 1 = 0. \tag{23}$$
Writing d_k and w_k to be the values of $\mathbf{d}$ and $\mathbf{w}$ in edge k we can write
$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{j_k^2}{c_k} = \sum_{\text{edges k}} \frac{(w_k + d_k)^2}{c_k} = \sum_{\text{edges k}} \frac{w_k^2}{c_k} + \frac{d_k^2}{c_k} + 2\sum_{\text{edges k}} \frac{w_k d_k}{c_k} \tag{24}$$
But we have said that $\mathbf{w}$ is a current satisfying Ohm's law on each edge meaning that
$$w_k = c_k(x_i - x_j), \tag{25}$$
where edge k is assumed to join nodes i and j. Hence
$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{w_k^2}{c_k} + \frac{d_k^2}{c_k} + 2\sum_{\text{edges k}} (x_i - x_j)d_k. \tag{26}$$

But by part (a) with the relevant flow taken as **d** which has zero strength $d_a = 0$ we see that

$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{w_k^2}{c_k} + \frac{d_k^2}{c_k}, \tag{27}$$

where the first term on the right is the dissipation of the current **w** and the second on the right is another (positive) dissipation associated with **d**. It is therefore clear that

$$\tilde{\mathcal{E}}(\mathbf{w}) \leq \tilde{\mathcal{E}}(\mathbf{j}). \tag{28}$$

Thus the current **w** satisfying Ohm's law minimizes the dissipation.

4 We know the escape probability is given by the dissipation function which we know is

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^T \mathbf{K} \mathbf{x} = \frac{1}{2} \sum_i \sum_j c_{ij} (x_i - x_j)^2, \tag{29}$$

where we have written it as a double sum over the nodes of the graph and included a factor of $1/2$ since edges are counted twice in this double sum. If an edge of the graph is removed then the conductance of that edge is effectively set equal to zero. If $\overline{\mathcal{E}(\mathbf{x})}$ is the dissipation associated with this modified graph then

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^T \mathbf{K} \mathbf{x} = \frac{1}{2} \sum_i \sum_j c_{ij} (x_i - x_j)^2 \geq \overline{\mathcal{E}(\mathbf{x})} \tag{30}$$

since, in the modified graph, a positive contribution to the dissipation in the unperturbed dissipation is set equal to zero in computing $\overline{\mathcal{E}(\mathbf{x})}$. Thus the escape probability when an edge is removed is decreased.