Problem Sheet 7

Math40002, Analysis 1

1. Prove that if $f : [a, b] \to [0, \infty)$ is continuous and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int_a^b f(x)\,dx > 0$.

   Solution. We can take $c \in (a, b)$ without loss of generality, since if $f(x) = 0$ for all $x \in (a, b)$ then $f(a) = f(b) = 0$ as well by continuity. Since $\frac{f(c)}{2} > 0$, we also have

   $$\exists \delta > 0 \text{ such that } |x - c| < \delta \;\Rightarrow\; |f(x) - f(c)| < \frac{f(c)}{2}$$

   by the continuity of f at c. Then if $|x - c| \leq \delta$ we have $f(x) \geq \frac{f(c)}{2}$. Taking a smaller δ if needed to ensure that $[c - \delta, c + \delta] \subset [a, b]$, we can write

   $$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx \\ &\geq \int_a^{c-\delta} 0 \quad dx + \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx + \int_{c+\delta}^b 0 \quad dx\end{aligned}$$

   since $f(x)$ is at least 0, $\frac{f(c)}{2}$, and 0 on the intervals $[a, c-\delta]$, $[c-\delta, c+\delta]$, and $[c+\delta, b]$ respectively. We evaluate these integrals one by one to get

   $$\int_a^b f(x)\,dx \geq 0 + \delta \cdot f(c) + 0 > 0.$$

2. Suppose for some $f : [a, b] \to \mathbb{R}$ and integer $n \geq 1$ that the nth power f^n of f is integrable. Prove that if n is odd, then f is integrable. Why doesn't this work for n even, and can you find additional hypotheses on f that make it true in that case?

   Solution. If n is odd then the nth root function $\sqrt[n]{\cdot} : \mathbb{R} \to \mathbb{R}$ is continuous, so if f^n is integrable then so is the composition $\sqrt[n]{f^n} = f$.

   When n is even, we still have a continuous $\sqrt[n]{\cdot} : [0, \infty) \to [0, \infty)$, and $f(x)^n \geq 0$ for all x, so we can again conclude that $\sqrt[n]{f^n}$ is integrable. But in this case $\sqrt[n]{f^n} = |f|$, so we have only proved that $|f|$ is integrable. If $f(x) \geq 0$ for all x then $f = |f|$, so this proves that a nonnegative function f is integrable if f^n is.

3. Let $C[a, b]$ denote the set of continuous functions $f : [a, b] \to \mathbb{R}$, and define a function $d : C[a, b] \times C[a, b] \to \mathbb{R}$ by

   $$d(f, g) = \int_a^b |f(x) - g(x)|\; dx.$$

(a) Prove that $d(f,g) = d(g,f)$ for all $f,g \in C[a,b]$.

(b) Prove that $d(f,g) \geq 0$, with equality if and only if $f = g$.

(c) Prove the triangle inequality $d(f,g) + d(g,h) \geq d(f,h)$.

These properties say that d is a *metric*, which is a notion of distance on $C[a,b]$.

(d) Prove that if $f_n \to f$ uniformly on $[a,b]$, then $\lim_{n\to\infty} d(f_n, f) = 0$.

Solution. (a) Since $|f-g| = |g-f|$ on $[a,b]$, their integrals are the same.

(b) We have $|f(x) - g(x)| \geq 0$ for all x by the definition of absolute value, so $\int_a^b |f(x) - g(x)|\,dx \geq \int_a^b 0\,dx = 0$. Since $|f(x) - g(x)|$ is continuous, problem 1 says that its integral is zero iff $|f(x) - g(x)| = 0$ for all x, or equivalently $f(x) = g(x)$ for all x.

(c) The usual triangle inequality says that for all x we have

$$|f(x) - g(x)| + |g(x) - h(x)| \geq |f(x) - h(x)|,$$

and each of the three terms above is continuous and hence integrable, so by basic properties of integration we have

$$\begin{aligned}\int_a^b |f(x) - g(x)|\,dx + &\int_a^b |g(x) - h(x)|\,dx \\ &= \int_a^b \left(|f(x) - g(x)| + |g(x) - h(x)|\right)\,dx \\ &\geq \int_a^b |f(x) - h(x)|\,dx\end{aligned}$$

which is equivalent to $d(f,g) + d(g,h) \geq d(f,h)$.

(d) For any $\epsilon > 0$, uniform convergence means that we can find an $N > 0$ such that

$$n \geq N \;\Rightarrow\; |f_n(x) - f(x)| < \frac{\epsilon}{b-a} \;\forall x \in [a,b].$$

But then for all $n \geq N$ we have

$$d(f_n, f) = \int_a^b |f_n(x) - f(x)|\;dx < \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon.$$

(We have strict inequality because $\frac{\epsilon}{b-a} - |f_n(x) - f(x)|$ is continuous and strictly positive, hence its integral is also strictly positive.) Thus $0 \leq d(f_n, f) < \epsilon$ for all $n \geq N$, and we can find such an N for any $\epsilon > 0$, so this means that $d(f_n, f) \to 0$ as $n \to \infty$.

4. In problem sheet 5 we constructed a smooth (i.e., infinitely differentiable) function $f : \mathbb{R} \to [0,\infty)$ such that $f(x) > 0$ if and only if $x \in (0,1)$.

(a) Construct a smooth, monotone increasing function $g : \mathbb{R} \to [0, \infty)$ such that $g(x) = 0$ for all $x \leq 0$ and $g(x) = 1$ for all $x \geq 1$.

(b) Given $a < b < c < d$, construct a smooth function $h : \mathbb{R} \to [0, \infty)$ satisfying

$$h(x) = 0 \text{ for all } x \notin [a, d], \qquad h(x) = 1 \text{ for all } x \in [b, c],$$

and with h monotone increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

Solution. (a) We construct g by integrating f. This gives us a constant function for $x \geq 1$ since $f(x) = 0$ there, but its value won't be 1, so we rescale it and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{c}\int_0^x f(t)\,dt, & x \geq -1 \end{cases} \qquad \text{where } c = \int_0^1 f(x)\,dx.$$

We claim that g is differentiable, with $g'(x) = \frac{1}{c}f(x)$. Indeed, for all $x \leq 0$ we have $g(x) = 0$ – when $-1 \leq x \leq 0$ this follows from the fact that $f(x) = 0$ for $x \leq 0$ – so $g'(x) = 0 = \frac{1}{c}f(x)$ on $(-\infty, 0)$. And for $x > -1$ the fundamental theorem of calculus tells us that g is differentiable at x with $g'(x) = \frac{1}{c}f(x)$. Since $g'(x) = f(x) \geq 0$, we know that g is monotone increasing (and hence nonnegative, since it is zero for all $x \leq 0$), and it is smooth since its derivative is infinitely differentiable. It only remains now to check that for $x \geq 1$ we have

$$\begin{aligned} g(x) = \frac{1}{c}\int_0^x f(t)\,dt &= \frac{1}{c}\left(\int_0^1 f(t)\,dt + \int_1^x f(t)\,dt\right) \\ &= \frac{1}{c}\left(\int_0^1 f(t)\,dt + \int_1^x 0\,dt\right) = 1. \end{aligned}$$

(b) The functions $h_1(x) = g\left(\frac{x-a}{b-a}\right)$ and $h_2(x) = g\left(\frac{d-x}{d-c}\right)$ are smooth since they are compositions of two smooth functions (one linear in x, and the other one g), and they satisfy

$$h_1(x) = \begin{cases} 0, & x \leq a \\ 1, & x \geq b \end{cases} \qquad h_2(x) = \begin{cases} 0, & x \geq d \\ 1, & x \leq c \end{cases}$$

with h_1 and h_2 monotone increasing and constant, respectively, on $(-\infty, b]$ and constant and monotone decreasing, respectively, on $[c, \infty)$. It follows that $h(x) = h_1(x)h_2(x)$ has the desired properties: it is smooth, zero precisely outside (a, d) and one precisely on $[b, c]$, and increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

5. (a) Check that the derivative of $x\log(x) - x$ is $\log(x)$.

(b) Use Darboux sums to prove for all integers $n \geq 1$ that

$$\log((n-1)!) \leq \int_1^n \log(x)\,dx \leq \log(n!).$$

(c) Evaluate the integral in (b) and deduce that

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1+\frac{1}{n}\right) + \frac{\log(n+1)}{n}$$

for all $n \geq 1$.

(d) Conclude that $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Remark: this is a weak version of *Stirling's formula* $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Solution. (a) We have $\frac{d}{dx}(x\log x) = \log(x) + x\frac{1}{x} = \log(x)+1$ by the product rule, so $\frac{d}{dx}(x\log(x) - x) = (\log(x)+1) - 1 = \log(x)$.

(b) We know that $\log(x)$ is integrable on $[1,n]$ since it is continuous, so if we take the partition $P = (1,2,\dots,n)$ of $[1,n]$ then we have

$$L(\log(x),P) \leq \int_1^n \log(x)\,dx \leq U(\log(x),P).$$

Since $\log(x)$ is monotone increasing, we compute that

$$\begin{aligned} L(\log(x),P) &= \sum_{i=1}^{n-1} \log(i) \cdot 1 = \log(1\cdot 2\cdot \dots \cdot (n-1)) = \log((n-1)!) \\ U(\log(x),P) &= \sum_{i=1}^{n-1} \log(i+1)\cdot 1 = \log(2\cdot 3\cdot \dots \cdot n) = \log(n!), \end{aligned}$$

so we put these together to get $\log((n-1)!) \leq \int_1^n \log(x)\,dx \leq \log(n!)$.

(c) We use part (a) and the fundamental theorem of calculus to evaluate

$$\int_1^n \log(x)\,dx = x\log(x) - x\Big|_{x=1}^{x=n} = n\log(n) - (n-1).$$

Then part (b) says that $n\log(n) - (n-1) \leq \log(n!)$, or

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log(n) + 1 = \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right)$$

after some rearranging. Similarly, if we let $m = n-1$ then the leftmost inequality from (b) tells us that

$$\log(m!) \leq (m+1)\log(m+1) - m,$$

and we divide through by m and rearrange to get

$$\begin{aligned} \frac{\log(m!)}{m} - \log(m) + 1 &\leq \left(1+\frac{1}{m}\right)\log(m+1) - \log(m) \\ &= \log\left(\frac{m+1}{m}\right) + \frac{\log(m+1)}{m}. \end{aligned}$$

Relabeling the variable n gives $\frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1+\frac{1}{n}\right) + \frac{\log(n+1)}{n}$.

(d) Applying the squeeze theorem to part (c) shows us that

$$\lim_{n\to\infty}\left(\frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right)\right) = 0,$$

or equivalently

$$\lim_{n\to\infty}\log\left(\frac{\sqrt[n]{n!}}{n/e}\right) = 0 \;\Rightarrow\; \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n/e} = 1.$$

We apply the algebra of limits to conclude that $\displaystyle\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e$.

6. (*) Let $f : [N,\infty) \to [0,\infty)$ be a nonnegative, monotone decreasing function.

(a) Let $S_n = \displaystyle\sum_{k=N}^{n} f(k)$ for all integers $n \geq N$. Use Darboux sums to prove that

$$S_n - f(N) \leq \int_N^n f(x)\,dx \leq S_{n-1}.$$

(b) Prove that the series $\displaystyle\sum_{k=N}^{\infty} f(k)$ converges if and only if the limit

$$\int_N^\infty f(x)\,dx \overset{def}{=} \lim_{x\to\infty}\int_N^x f(t)\,dt$$

(called an *improper integral*) exists. This is the *integral test* for convergence.

(c) Prove that if the series $S = \displaystyle\sum_{k=N}^{\infty} f(k)$ converges, so $I = \int_N^\infty f(x)\,dx$ exists, then $I \leq S \leq I + f(N)$.

Solution. (a) Fix an integer $n \geq N$ and consider the partition

$$P_n = (N, N+1, N+2, \dots, n)$$

of $[N, n)$. Since f is monotone decreasing we can compute the Darboux sums of f with respect to P:

$$L(f, P_n) = \sum_{k=N}^{n-1} f(k+1), \qquad U(f, P_n) = \sum_{k=N}^{n-1} f(k).$$

We proved on the last problem sheet that f is integrable on $[N, n]$ since it is monotone, so we have

$$\sum_{k=N+1}^{n} f(k) = L(f, P_n) \leq \int_N^n f(x)\,dx \leq U(f, P_n) = \sum_{k=N}^{n-1} f(k).$$

The left and right sides are $S_n - f(N)$ and S_{n-1} respectively.

(b) ($\Longleftarrow$) If $I = \int_N^\infty f(x)\,dx$ exists then for any $\epsilon > 0$, there is an $M \geq 0$ such that

$$n \geq M \;\Rightarrow\; \left|\int_N^n f(x)\,dx - I\right| < \epsilon.$$

We use this together with part (a) to deduce that

$$n \geq M \;\Rightarrow\; S_n \leq \int_N^n f(x)\,dx + f(N) < I + f(N) + \epsilon.$$

The sequence $\big(S_n\big)$ of partial sums is therefore bounded above, and it is increasing since $f(n) \geq 0$ for all n, so it converges.

($\Longrightarrow$) If $\sum_{k=N}^{\infty} f(k)$ converges to some S, then given $\epsilon > 0$, there is an $M \geq 0$ such that

$$n \geq M \;\Rightarrow\; |S_n - S| = \left|\sum_{k=N}^{n} f(k) - S\right| < \epsilon.$$

Using part (a), we have

$$n \geq M + 1 \;\Rightarrow\; \int_N^n f(x)\,dx \leq S_{n-1} < S + \epsilon.$$

Now the function $F(x) = \int_N^x f(t)\,dt$ is increasing, since for any $x < y$ we have

$$\begin{aligned} F(y) = \int_N^y f(t)\,dt &= \int_N^x f(t)\,dt + \int_x^y f(t)\,dt \\ &\geq \int_N^x f(t)\,dt + \int_x^y \quad 0 \quad dt = F(x), \end{aligned}$$

and it is bounded above since for any $x \geq N$ we have $F(x) \leq F(m) < S + \epsilon$ for some integer $m \geq \max(x, M+1)$. Thus the limit $\lim_{x\to\infty} F(x) = \int_N^\infty f(t)\,dt$ exists, as desired.

Remark: we can't use the fundamental theorem of calculus to assert that $F'(x) = f(x) \geq 0$ and thus prove that $F(x)$ is increasing, because we do not know that f is continuous.

(c) In part (b) we proved for any $\epsilon > 0$ and all large enough x and n that $\int_N^x f(t)\,dt < S + \epsilon$ and $S_n < I + f(N) + \epsilon$ respectively, so

$$I = \lim_{x\to\infty} \int_N^x f(t)\,dt \leq S + \epsilon, \qquad S = \lim_{n\to\infty} S_n \leq I + f(n) + \epsilon.$$

These hold for any $\epsilon > 0$, so we must actually have $I \leq S$ and $S \leq I + f(N)$.

7. Consider for any real s the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$.

(a) Prove that this series is not convergent if $s \leq 0$.

(b) Use the integral test to prove that for $s > 0$, the series converges if and only if $s > 1$. If $s > 1$, show that $\frac{1}{s-1} < \zeta(s) < \frac{s}{s-1}$.

(c) Prove for any $a > 1$ that the series converges uniformly to a continuous function on $[a, \infty)$, and hence it defines a continuous function $\zeta : (1, \infty) \to \mathbb{R}$ called the *Riemann zeta function.* Can it be extended continuously to $[1, \infty)$?

(d) (Harder!) Prove that $\zeta(s)$ is continuously differentiable, and compute its derivative. It may help to first show that $\lim_{x\to\infty} \dfrac{\log(x)}{x^\epsilon} = 0$ for any $\epsilon > 0$.

Solution. (a) If $s = 0$ or $s < 0$ then $\lim_{n\to\infty} \dfrac{1}{n^s}$ is 1 or ∞, and since it is not 0 the series does not converge in either case.

(b) Since $f(x) = \frac{1}{x^s}$ is nonnegative and monotone decreasing on $[1, \infty)$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if the limit

$$\int_1^\infty \frac{1}{t^s}\, dt = \lim_{x\to\infty} \int_1^x \frac{1}{t^s}\, dt$$

exists. We apply the fundamental theorem of calculus: if $s \neq 1$ then

$$\int_1^x \frac{1}{t^s}\, dt = \left.\frac{t^{1-s}}{1-s}\right|_{t=1}^{t=x} = \frac{x^{1-s} - 1}{1-s}.$$

As $x \to \infty$, this diverges if $p < 1$ and converges to $\frac{1}{s-1}$ if $s > 1$. If instead $s = 1$ then we have

$$\lim_{x\to\infty} \int_1^x \frac{1}{t}\, dt = \lim_{x\to\infty} \log(t)|_{t=1}^{t=x} = \lim_{x\to\infty} \log(x) = \infty.$$

So the improper integral $\displaystyle\int_1^\infty \frac{1}{t^s}\, dt$ exists if and only if $s > 1$, and if it exists then it is equal to $\frac{1}{s-1}$, so by part (c) we have

$$\frac{1}{s-1} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} \leq \frac{1}{s-1} + 1,$$

or $\frac{1}{s-1} \leq \zeta(s) \leq \frac{s}{s-1}$.

(c) We apply the Weierstrass M-test. Setting $b = \frac{1+a}{2}$, so that $1 < b < a$, we define $M_n = \frac{1}{n^b}$ for all s, and then we have

$$\left|\frac{1}{n^s}\right| \leq \frac{1}{n^b} = M_n \text{ for all } s \in [a, \infty).$$

The series $\displaystyle\sum_{n=1}^{\infty} M_n$ converges to $\zeta(b)$, so the M-test says that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on $[a, \infty)$ to a continuous function $\zeta(s)$.

Since $\zeta(s)$ is continuous on any interval $[a,\infty)$, if we are given $x > 1$ then we know that ζ is continuous on $[\frac{1+x}{2},\infty)$, and this interval contains x, so in particular $\zeta(s)$ is continuous at $s = x$. Thus it is continuous on all of $(1,\infty)$.

From part (b) we have $1 < (s-1)\zeta(s) < s$ for all $s > 1$, and so $\lim_{s\downarrow 1}(s-1)\zeta(s) = 1$ by the squeeze theorem. If it were possible to extend $\zeta(s)$ continuously to $s = 1$, then the algebra of limits would also tell us that $\lim_{s\downarrow 1}(s-1)\zeta(s) = 0\zeta(1) \neq 1$, and this is a contradiction.

(d) We write $f_n(x) = \sum_{k=1}^{n} \frac{1}{k^x}$ for all $n \geq 1$. Since $\frac{1}{n^x} = e^{-x\log(n)}$ has derivative $(-\log(n))e^{-x\log(n)} = \frac{-\log(n)}{n^x}$, we compute that

$$f_n'(x) = -\sum_{k=1}^{n} \frac{\log(x)}{k^x}.$$

Given $x > 1$, we set $a = \frac{1+x}{2}$ and $b = x+1$, so that $x \in (a,b)$ and $f_n \to f$ on $[a,b]$. Since we already know that $f_n \to \zeta$ uniformly on $[a,b]$, if we can show that f_n' converges uniformly on $[a,b]$ then it will follow that ζ is differentiable on $[a,b]$ (and in particular at $s = x$), with

$$\zeta'(s) = \lim_{n\to\infty} f_n'(s) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^s}.$$

The f_n' are continuous on $[a,b]$, so their uniform limit ζ' will be continuous on $[a,b]$ as well and in particular at $s = x$.

To prove that f_n' converges uniformly on $[a,b]$, we again apply the Weierstrass M-test. Let $\epsilon = \frac{a-1}{2} > 0$. We have $\log(n) < n^\epsilon$ for all sufficiently large n, because l'Hôpital's rule says that

$$\lim_{x\to\infty} \frac{\log(x)}{x^\epsilon} = \lim_{x\to\infty} \frac{1/x}{\epsilon x^{\epsilon-1}} = \lim_{x\to\infty} \frac{1}{\epsilon x^\epsilon} = 0.$$

Thus if we set $M_n = \frac{1}{n^{1+\epsilon}}$, then for all $s \geq a$ and large enough n we have

$$\left|-\frac{\log(n)}{n^s}\right| < \frac{n^\epsilon}{n^a} = \frac{1}{n^{a-\epsilon}} = \frac{1}{n^{1+\epsilon}} = M_n.$$

The series $\sum_{n=1}^{\infty} M_n$ converges to $\zeta(1+\epsilon)$, so the M-test says that $f_n'(s) = -\sum_{k=1}^{n} \frac{\log(k)}{k^s}$ converges uniformly on $[a,b]$, as desired.