Question 1

The probability density function f for the χ^2_ν distribution (the chi-squared distribution with ν degrees of freedom) is

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2},$$

where the support is $x \in \mathbb{R}$ and $x > 0$, and the degrees of freedom $\nu \in \{1, 2, \ldots\}$.

(a) Let $Y \sim \chi^2_\nu$. Assuming that we know $\mathrm{E}(Y) = \nu$ and $\mathrm{E}\left(Y^2\right) = \nu(\nu+2)$, find $\mathrm{Var}(Y)$.

(b) Assume that $X_1, X_2, \ldots, X_n \sim \mathrm{N}\left(\mu, \sigma^2\right)$. Show that

$$\mathrm{Var}\left(S^2\right) = \frac{2\sigma^4}{n-1},$$

where S^2 is the sample variance, i.e. $S^2 = \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \overline{X}\right)^2$, as usual.

Question 2

Suppose you are on a gameshow and win a prize: there are two containers, labelled A and B, that are each filled with 1000 banknotes, where each banknote is either a £10 note, a £20 note or a £50 note. The prize is that you can choose to keep one of the containers. Before you choose a container, you are told the distributions of the banknotes in each of the two containers is different. Writing the sample space as $\Omega = \{10, 20, 50\}$, and considering the parameter θ as identifying the distribution, $\theta \in \{1, 2\}$, you are given the information that the two distributions are summarised by the following table, where f_θ is the probability mass function of the distribution when the parameter value is θ:

	$\omega = 10$	$\omega = 20$	$\omega = 50$
$f_1(\omega)$	0.3	0.4	0.3
$f_2(\omega)$	0.3	0.1	0.6

Furthermore, you are allowed to sample from one of the containers before making our choice: you can pull exactly one banknote out of exactly one of the containers, and then choose to keep either that container or the other container. We plan to use maximum likelihood estimation to aid us in choosing which container to keep (and you would like to keep the container with the most money).

(a) What is the maximum likelihood estimate (MLE) for θ if you sample $\omega = 50$?

(b) What is expected amount of money in each container?

(c) If you chose to sample from container A and pulled out a £50, would you prefer to keep container A or container B? (Supposing you want to keep the container with the most money.)

(d) Again, suppose you choose to sample from container A and pull out a £50. How many times more (or less) plausible/likely is it that container A is the container with the most money?

(e) What is the MLE for θ if you sample $\omega = 20$?

(f) Suppose you choose to sample from container A and pull out a £20. Would you choose to keep container A or container B?

(g) Again, suppose you choose to sample from container A and pull out a £20. How many times more (or less) plausible/likely is it that container A is the container with the most money?

(h) What is the MLE for θ if you sample $\omega = 10$?

(i) If you chose to sample from container A and pulled out a £10, would you choose to keep container A or container B?

(j) Is the MLE always unique?

Question 3

Suppose that the lifetime of a particular lightbulb is known to be distributed as $\operatorname{Exp}(\theta)$, i.e. an exponential distribution with parameter θ. Suppose that a group of n friends bought a multipack containing n of these lightbulbs and each person keeps one lightbulb to use at home. In a few years' time, they get together and share how long their lightbulbs lasted, and these measurements (lifetimes) are written down as $x_1, x_2, \ldots, x_n$.

(a) Write down the probability density function of the $\operatorname{Exp}(\theta)$ distribution.

(b) Given the sample of measurements $x_1, x_2, \ldots, x_n$, write down the likeihood function for θ based on these measurements.

(c) Write down the log-likelihood of θ given the measurements $x_1, x_2, \ldots, x_n$.

(d) Find the maximum likelihood estimate $\widehat{\theta}$.

(e) Are you sure that $\widehat{\theta}$ is a maximum, or could it be a minimum? If you have not already done so in (d), provide proof that $\widehat{\theta}$ is a maximum/minimum.

Question 4 (This question is an exercise in calculus)

Returning to, Question 1, the probability density function f for the χ^2_ν distribution is

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2},$$

where, as usual, $\Gamma(z)$ is the gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x.$$

(a) Show that the gamma function has the property $\Gamma(z+1) = z\Gamma(z)$ (Hint: use integration by parts).

(b) Show that if $Y \sim \chi^2_\nu$ then $\operatorname{E}(Y) = \nu$ (Hint: try to get the integral into a form that is a constant times 'something' that integrates to 1).

(c) Show that if $Y \sim \chi^2_\nu$ then $\operatorname{E}\left(Y^2\right) = \nu(\nu+2)$.