MATH40004 - Calculus and Applications:
Unseen Questions 7:
Assorted topics in multivariable calculus

1 Partial differential equations: Introduction to the method of characteristics: The transport equation

Consider the partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{1}$$

with initial condition $u(x, t = 0) = u(x, 0) = F(x)$. This equation is known as the transport equation. Here x represents some one dimensional distance, t time and $u(x, t)$ the information or data we want to transport. $c > 0$ is a positive constant known as the advection or transport speed. The function $F(x)$ is the initial profile or data of u.

(a). Use the chain rule to show that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x}. \tag{2}$$

Comparing (2) with (1) we see that along the **special curves** in the (x, t) plane where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c, \tag{3}$$

we have that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0, \tag{4}$$

or in other words, u is **constant** along these special curves. These special curves are called **characteristics** and they tell us a lot of information about the partial differential equation.

(b). Solve equation (3) to find $x(t)$ when $x = \xi$ at $t = 0$ (ξ being some constant). Sketch this characteristic in the (x, t) plane.

Since u is constant along these characteristics then the value of u at a later time t must be the same value of u on that characteristic at time $t = 0$. That is

$$u(x, t) = u(\xi, 0). \tag{5}$$

(c). Conclude that the solution to the transport equation can be written in terms of the initial data as

$$u(x, t) = F(x - at). \tag{6}$$

Describe the physical meaning of this solution.

2 Exercise: Stationary points of surfaces

As you have seen in lectures, when we try to classify the stationary points of functions of two variables $f(x, y)$ we observe the sign of the quantity

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \tag{7}$$

along with the sign of $\frac{\partial^2 f}{\partial x^2}$ (or alternatively the signs of the eigenvalues of the Hessian matrix). However this method is inconclusive if these quantities (or the eigenvalues of the Hessian matrix) are 0 and more investigation needs to be done.

Find and classify the stationary points of the following surfaces. One of the stationary points is new to you, it is a degenerate stationary point called the **monkey saddle point**; you should describe what distinguishes it from the stationary points you already know about.

(a). $f(x, y) = 9x^4 + 12x^2y^2 + 4y^4$

(b). $f(x, y) = 2x^4 - 3x^2y + y^2$

(c). $f(x, y) = x^3 - 3xy^2$

3 Exercise: Exact ordinary differential equations

Consider the ordinary differential equation

$$F(x, y)\mathrm{d}x + G(x, y)\mathrm{d}y = 0. \tag{8}$$

If this equation is **exact** (or in other words $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$) then it is easy to solve. As you have seen in lectures, sometimes we have to multiply through by an integrating factor in order to change a differential equation to an **exact** differential equation. In lectures the only integrating factors you have seen are functions of x or y only ($\lambda(x)$ or $\lambda(y)$), but in general the integrating factor can be both a function of x and y ($\lambda(x, y)$). There is no method to know what to pick, it's a matter of intuition and trial and improvement!

(a). Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{2x\log x}. \tag{9}$$

By multiplying through by an integrating factor $\lambda(x, y)$ which you need to choose show that the equation can be made exact. Hence solve the equation (note: it is possible to solve this equation using an integrating factor in just one variable and also by separation of variables, but these methods require excellent integration skills).

(b). Suppose we seek an integrating factor of the form $\lambda(t)$, where $t = t(x, y)$ is some function of x and y to multiply equation (8) by to make exact.

(i). Show using the compatibility condition for exactness that a general formula for the integrating factor is then

$$\lambda(t(x, y)) = \exp\left\{\int\left(\frac{F_y - G_x}{Gt_x - Ft_y}\right)\mathrm{d}t\right\}, \tag{10}$$

where the subscript indicates partial differentiation with respect to that variable.

(ii). Taking $t(x, y) = x + y$ in the above formula, find an integrating factor for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{7x^3 + 3x^2y + 4y}{4x^3 + x + 5y}. \tag{11}$$

Hence solve the equation.