# Math40003 Linear Algebra and Groups Problem Sheet 3

1. * Let $\mathbb{R}[x]$ be the set of all polynomials with variable $x$ and real coefficients, with standard addition and scalar multiplication. Show that this is a vector space over $\mathbb{R}$.

   **A1** **Follows from associativity of $\mathbb{R}$ and definition of polynomial addition. i.e. let $f(x), g(x), h(x) \in \mathbb{R}[x]$ then we have:**

   $$\begin{aligned} f(x) &= \textstyle\sum_{i=1}^{m} a_i x^i \\ g(x) &= \textstyle\sum_{i=1}^{n} b_i x^i \\ h(x) &= \textstyle\sum_{i=1}^{s} c_i x^i \end{aligned}$$

   **Let $t = max\{m, n, s\}$ then define $a_i = 0$ for $m \leq i \leq t$, similarly define $b_i = 0$ for $n \leq i \leq t$, and define $c_i = 0$ for $s \leq i \leq t$. So we get:**

   $$\begin{aligned} f(x) &= \textstyle\sum_{i=1}^{t} a_i x^i \\ g(x) &= \textstyle\sum_{i=1}^{t} b_i x^i \\ h(x) &= \textstyle\sum_{i=1}^{t} c_i x^i \end{aligned}$$

   **Now**

   $$\begin{aligned} f(x) + (g(x) + h(x)) &= \textstyle\sum_{i=1}^{t} a_i x^i + (\sum_{i=1}^{t} b_i x^i + \sum_{i=1}^{t} c_i x^i) \\ &= \textstyle\sum_{i=1}^{t} (a_i + (b_i + c_i)) x^i \\ &= \textstyle\sum_{i=1}^{t} ((a_i + b_i) + c_i) x^i \\ &= (\textstyle\sum_{i=1}^{t} a_i x^i + \sum_{i=1}^{t} b_i x^i) + \sum_{i=1}^{t} c_i x^i \\ &= (f(x) + g(x)) + h(x) \end{aligned}$$

   **A2** **Follows from commutitivity of $\mathbb{R}$ and definition of polynomial addition.**

   **A3** **$0_V$ here is the polynomial $0$.**

   **A4** **The inverse of $f(x) = \sum_{i=1}^{m} a_i x^i$ is $-f(x) = \sum_{i=1}^{m} -a_i x^i$ clearly we get $f(x) + (-f(x)) = 0_V$.**

   **A5** **Follows from distributivity of $\times$ over $+$ in $\mathbb{R}$ and definition of polynomial addition.**

   **A6** **Follows from distributivity of $+$ over $\times$ in $\mathbb{R}$ and definition of multiplying a polynomial by a constant/scalar.**

   **A7** **Follows from commutativity of $\times$ in $\mathbb{R}$ and definition of multiplying a polynomial by a constant/scalar.**

   **A8** **Follows from definition of multiplying a polynomial by a constant/scalar.**

2. Decide whether the following sets together with the indicated operations of addition and scalar multiplication is a vector space:

(a) The set $\mathbb{R}^2$, with the usual addition but with scalar multiplication defined by

$$r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ry \\ rx \end{pmatrix}.$$

**Consider A7, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.**

$$\begin{aligned} r \odot (s \odot \mathbf{v}) &= r \odot \left( s \odot \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= r \odot \begin{pmatrix} sy \\ sx \end{pmatrix} \\ &= \begin{pmatrix} rsx \\ rsy \end{pmatrix} \end{aligned}$$

**and**

$$\begin{aligned} (rs) \odot \mathbf{v} &= (rs) \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rsy \\ rsx \end{pmatrix} \end{aligned}$$

**Because $r \odot (s \odot \mathbf{v}) \neq (rs) \odot \mathbf{v}$, A7 is not satisfied.**

(b) The set $\mathbb{R}^2$, with the usual scalar multiplication but with addition defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} y+s \\ x+r \end{pmatrix}.$$

**All of the axioms A5 - A8 are satisfied since the usual definition of scalar multiplication is used.**

**Consider the axiom A1, and let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, then**

$$\begin{aligned} (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} y_1 + y_2 \\ x_1 + x_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + y_3 \\ y_1 + y_2 + x_3 \end{pmatrix} \end{aligned}$$

**and**

$$\begin{aligned}\mathbf{u} \oplus (\mathbf{v} + \mathbf{w}) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \left( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} y_2 + y_3 \\ x_2 + x_3 \end{pmatrix} \\ &= \begin{pmatrix} y_1 + x_2 + x_3 \\ x_1 + y_2 + y_3 \end{pmatrix}\end{aligned}$$

**Since $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ then A1 is not satisfied.**

(c) The set $\mathbb{R}^2$, with addition and scalar multiplication defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + a + 1 \\ y + b \end{pmatrix} \quad \text{and} \quad r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx + r - 1 \\ ry \end{pmatrix}.$$

**Consider A1, and let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, then**

$$\begin{aligned}(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + x_3 + 2 \\ y_1 + y_2 + y_3 \end{pmatrix}\end{aligned}$$

**and**

$$\begin{aligned}\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \left( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 + x_3 + 1 \\ y_2 + y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + x_3 + 2 \\ y_1 + y_2 + y_3 \end{pmatrix}\end{aligned}$$

**Because $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$, A1 is satisfied.**

**Consider A2 and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then**

$$\begin{aligned}\mathbf{v} \oplus \mathbf{w} &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix}\end{aligned}$$

**and**

$$\begin{aligned}\mathbf{w} \oplus \mathbf{v} &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix}\end{aligned}$$

**Because $\mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$, A2 is satisfied.**

**Consider A3, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then**

$$\begin{aligned}
\mathbf{e} \oplus \mathbf{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x - 1 + 1 \\ y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \mathbf{v}
\end{aligned}$$

**Therefore, A3 is satisfied.**

**Consider A4, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then**

$$\begin{aligned}
\mathbf{v} \oplus (-1 \odot \mathbf{v}) &= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \left(-1 \odot \begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} -x - 1 - 1 \\ -y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} -x - 2 \\ -y \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
&= \mathbf{e}
\end{aligned}$$

**Therefore, A4 is satisfied.**

**Consider A5 and let $\mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then**

$$\begin{aligned}
r \odot (\mathbf{v} \oplus \mathbf{w}) &= r \odot \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} r(x_1 + x_2 + 1) + r - 1 \\ r(y_1 + y_2) \end{pmatrix}
\end{aligned}$$

**and**

$$\begin{aligned}
(r \odot \mathbf{v}) \oplus (r \odot \mathbf{w}) &= r \odot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus r \odot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} rx_1 + r - 1 \\ ry_1 \end{pmatrix} \oplus \begin{pmatrix} rx_2 + r - 1 \\ ry_2 \end{pmatrix} \\
&= \begin{pmatrix} rx_1 + r - 1 + rx_2 + r - 1 + 1 \\ ry_1 + ry_2 \end{pmatrix} \\
&= \begin{pmatrix} r(x_1 + x_2 + 1) + r - 1 \\ r(y_1 + y_2) \end{pmatrix}
\end{aligned}$$

**Because $r \odot (\mathbf{v} \oplus \mathbf{w}) = (r \odot \mathbf{v}) \oplus (r \odot \mathbf{w})$, A5 is satisfied.**

**Consider A6, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then**

$$\begin{aligned}
(r+s) \odot \mathbf{v} &= (r+s) \odot \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} (r+s)x + (r+s) - 1 \\ (r+s)y \end{pmatrix}
\end{aligned}$$

**and**

$$\begin{aligned}
(r \odot \mathbf{v}) \oplus (s \odot \mathbf{v}) &= r \odot \begin{pmatrix} x \\ y \end{pmatrix} \oplus s \odot \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} rx + r - 1 \\ ry \end{pmatrix} \oplus \begin{pmatrix} sx + s - 1 \\ sy \end{pmatrix} \\
&= \begin{pmatrix} rx + r - 1 + sx + s - 1 + 1 \\ ry + sy \end{pmatrix} \\
&= \begin{pmatrix} (r+s)x + (r+s) - 1 \\ (r+s)y \end{pmatrix}
\end{aligned}$$

**Because $(r+s) \odot \mathbf{v} = (r \odot \mathbf{v}) \oplus (s \odot \mathbf{v})$, A6 is satisfied.**

**Consider A7, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then**

$$\begin{aligned}
r \odot (s \odot \mathbf{v}) &= r \odot \left( s \odot \begin{pmatrix} x \\ y \end{pmatrix} \right) \\
&= r \odot \begin{pmatrix} sx + s - 1 \\ sy \end{pmatrix} \\
&= \begin{pmatrix} r(sx + s - 1) + r - 1 \\ rsy \end{pmatrix} \\
&= \begin{pmatrix} r(sx + s) - 1 \\ rsy \end{pmatrix}
\end{aligned}$$

**and**

$$\begin{aligned}
(rs) \odot \mathbf{v} &= rs \odot \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} rsx + rs - 1 \\ rsy \end{pmatrix}
\end{aligned}$$

**Because $r \odot (s \odot \mathbf{v}) = (rs) \odot \mathbf{v}$, then A7 is satisfied.**

**Consider A8, and let** $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$**, then**

$$\begin{aligned} 1 \odot \mathbf{v} &= 1 \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x + 1 - 1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \mathbf{v} \end{aligned}$$

**Therefore, A8 is satisfied.**

**Therefore, the set** $\mathbb{R}^2$ **with addition described by** $\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + a + 1 \\ y + b \end{pmatrix}$ **and scalar multiplication described by** $r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx + r - 1 \\ ry \end{pmatrix}$ **is a vector space.**

3. Show every $F$-vector space $V$ with additive identity $0_V$ has the following properties:

   (a) The vector $0_V$ is the unique vector satisfying the equation $0_V \oplus v = v$ for all vectors $v$ in $V$.

   (b) For 0 the additive identity in $F$, $0 \odot v = 0_V$ for all vectors $v$ in $V$.

**(a) Suppose that there are two vectors** $0_v$ **and** $0'_V$ **that satisfy**

$$0_V \oplus \mathbf{v} = \mathbf{v} \quad 0'_V \oplus \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in V.$$

**Let** $\mathbf{v} = 0'_V$ **in the first equation and** $\mathbf{v} = 0_V$ **in the second equation gives**

$$0_V \oplus 0'_V = 0'_V \quad 0'_V \oplus 0_V = 0_V$$

**By the commutative law, A2,** $0_V \oplus 0'_V = 0'_V \oplus 0_V$**, therefore,** $0_V = 0'_V$ **and hence the zero vector is unique.**

**(b) Using the distributive law, A6, for any** $\mathbf{v} \in V$**,**

$$0 \odot \mathbf{v} = (0 + 0) \odot \mathbf{v} = (0 \odot \mathbf{v}) \oplus (0 \odot \mathbf{v})$$

**By the additive identity axiom, A3, and the commutative law, A2,**

$$0 \odot \mathbf{v} = 0 \odot \mathbf{v} \oplus (0 \odot \mathbf{v}) = (0 \odot \mathbf{v}) \oplus 0_V$$

**Therefore,** $(0 \odot \mathbf{v}) \oplus (0 \odot \mathbf{v}) = (0 \odot \mathbf{v}) \oplus 0_V$**, and therefore,** $0 \odot \mathbf{v} = 0_V$**.**

4. Describe all subspaces of $\mathbb{R}^3$

   **(a) The set containing the zero vector,** $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$**, i.e. the zero subspace,**

   **(b) the set describing any straight line going through the origin (including the x, y, and z axes),**

   **(c) the set describing any plane that goes through the origin,**

   **(d) and** $\mathbb{R}^3$**.**

5. Let $U, W$ be subspaces of a vector space $V$ over $F$. Show that $U \cup W$ is a subspace of $V$ iff either $U \subseteq W$ or $W \subseteq U$. **(⇒) Suppose** $U \cup W$ **is a subspace. Suppose for contradiction we have** $w \in W \setminus U$ **and** $u \in U \setminus W$**. Then** $u + w \notin U \cup W$**:**

   **Suppose** $u + w \in U$ **then** $w = (u + w) + (-u) \in U$ **which contradicts** $w \in W \setminus U$**, so** $u + w \notin U$**. Similarly we get** $u + w \notin W$**.**

   **This contradicts SS2 for** $U \cup W$**, so either** $W \setminus U = \emptyset$ **(so** $W \subseteq U$**) or** $U \setminus W = \emptyset$ **(so** $U \subseteq W$**).**

   **(⇐) Clear as either** $U \cup W = U$ **or** $U \cup W = W$