Mathematics Year 1, Calculus and Applications I

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Solutions Problem Sheet 5

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1. Let $\{r_n\}$ denote the rational numbers in the interval $(0,1)$ arranged in the sequence whose first few terms are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$. Prove that the series $\sum_1^\infty r_n$ diverges.

   Solution $\sum_{n=1}^\infty r_n$ diverges because r_n does not thend to zero as $n \to \infty$. In fact $r_n \to 1$ as $n \to \infty$.

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2. Determine the convergence or divergence of the following infinite series:

$$(a)\ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \quad (b)\ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n \quad (c)\ \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \quad (d)\ \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} 4^n$$

$$(e)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \qquad (f)\ \sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1}-\sqrt{n}\right) \qquad (g)\ \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$$

$$(h)\ \sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}, \qquad (i)\ \sum_{1}^{\infty} \frac{2^{n^2}}{n!}, \qquad (j)\ \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}}\right)$$

Solution

(a) Ratio test $\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2(n+1))!}\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4}$ as $n \to \infty$. Converges.

(b) As above, only difference is the extra $\frac{5^{n+1}}{5^n}$ factor. Hence $\frac{a_{n+1}}{a_n} \to \frac{5}{4}$. Diverges.

(c) Can do this in different ways: (i) Write $a_n = \left[\left(\frac{n}{n+1}\right)^n\right]^n$; defining $x_n = \left(\frac{n}{n+1}\right)^n$, calculate

$$x_n = \left(1 - \frac{1}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^{n+1} \frac{1}{\left(1 - \frac{1}{n+1}\right)} \to e^{-1} \qquad \text{as} \qquad n \to \infty,$$

by definition of e. Hence $a_n \sim e^{-n}$ for large n and the series converges (geometric of ratio $e^{-1} < 1$).

(ii) Alternatively, note that

$$\log(a_n) = n^2 \log\left(\frac{n}{n+1}\right) = n^2 \log\left(1 - \frac{1}{n+1}\right) = n^2\left(-\frac{1}{n+1} + \ldots\right) \to -n,$$

as $n \to \infty$ hence $a_n \sim e^{-n}$ and the conclusion is the same as above. [I used Taylor's theorem to expand the log, only the first term is needed.]

(d) As above, and on including the extra factor 4^n we have $a_n \to e^{-n}\, 4^n = (4-e)^n$. Since $4 - e > 1$, the series diverges.

(e) Series converges by the alternating series test since $1/\sqrt{n} \to 0$ as $n \to \infty$.

(f) Here $a_n = \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)}$. Hence $a_n < \frac{1}{2n^{3/2}}$ for $n \geq 1$, and by comparison with $\int_1^\infty \frac{dx}{x^{3/2}}$ we have convergence.

(g) Use the comparison test with the integral $I := \int_2^\infty \frac{dx}{(\log x)^{\log x}}$. Calculate I first making the substitution $y = \log x$

$$I = \int_{\log 2}^\infty \frac{e^y dy}{y^y} = \int_{\log 2}^\infty \frac{e^y dy}{e^{y \log y}} = \int_{\log 2}^\infty e^{-y(\log y - 1)} dy = \left(\int_{\log 2}^{\log M} + \int_{\log M}^\infty \right) e^{-y(\log y-1)} dy$$

Now pick the constant M so that $\log M - 1 > 0$ (e.g. $M = 2e$ will do). The first integral is a finite number. The second integral can be estimated as follows

$$\int_{\log M}^\infty e^{-y(\log y-1)} dy < \int_{\log M}^\infty e^{-y} dy = \frac{1}{M},$$

hence the integral and the series converge.

(h) Ratio test $\frac{a_{n+1}}{a_n} = \frac{2}{(2n+3)(2n+2)} \to 0$ as $n \to \infty$. Converges.

(i) Ratio test $\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \frac{2^{n^2+2n+1}}{2^{n^2}} = \frac{2^{2n+1}}{n+1} \to \infty$ as $n \to \infty$.

(j) Compare with

$$\lim_{M\to\infty} \int_1^M \left(\frac{1}{x} - \frac{1}{\sqrt{x}} \right) dx = \lim_{M\to\infty} \left(\log M - 2\sqrt{M} + 2 \right) = \infty,$$

hence the series diverges.

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3. (a) Prove that the series

$$\sum_{n=1}^\infty \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2 \cdot 3} + \ldots = 1.$$

Use the result to prove that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, and obtain upper and lower bounds for this sum.

(b) Find the sum of the series $\sum_{n=1}^\infty \frac{n}{(n+1)!}$.

(c) Find the sum $\sum_{n=1}^\infty \frac{1+n}{2^n}$. [Hint: Differentiate a certain power series, justifying any operations.]

Solution

(a) Consider the partial sum $S_N = \sum_{n=1}^N \frac{1}{n(n+1)} = \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$ (telescoping series). Hence $\sum_{n=1}^\infty \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = 1$.
Now

$$\sum_{n=1}^\infty \frac{1}{n(n+1)} > \sum_{n=1}^\infty \frac{1}{(n+1)^2} = \sum_{k=2}^\infty \frac{1}{k^2} = \sum_{n=1}^\infty \frac{1}{n^2} - 1,$$

hence $\sum_{n=1}^\infty \frac{1}{n^2} < 1 + \sum_{n=1}^\infty \frac{1}{n(n+1)} = 2$, i.e. converges with upper bound 2.
To find a lower bound note that $\sum_{n=1}^\infty \frac{1}{n^2} > \sum_{n=1}^\infty \frac{1}{n(n+1)} = 1$.

(b) Write $\frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$, hence the partial sum

$$S_N := \sum_{n=1}^N \frac{n}{(n+1)!} = \sum_{n=1}^N \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) = 1 - \frac{1}{(N+1)!} \to 1 \quad \text{as} \quad N \to \infty.$$

(c) Consider the function $f(x) = \frac{1}{1-x}$ and in particular its Binomial expansion which converges absolutely for $|x| < 1$. The Binomial expansion is $f(x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$, and since the series converges absolutely we can differentiate to find $f'(x) = 1 + 2x + 3x^2 + \ldots$. In particular

$$f'(1/2) = 1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{n+1}{2^n} \tag{1}$$

But $f'(x) = \frac{1}{(1-x)^2}$ hence $f'(1/2) = 4$. Combining with (1) yields $\sum_{n=1}^{\infty} \frac{n+1}{n^2} = 3$.

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4. Suppose that $\{a_n\}$ is a decreasing sequence of positive terms such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $na_n \to 0$ as $n \to \infty$. [Hint - consider the sum $a_{n+1} + a_{n+2} + \ldots + a_{2n}$.]

Solution

Consider $a_{n+1} + a_{n+2} + \ldots + a_{2n} = \sum_{n=1}^{2n} a_n - \sum_{n=1}^{n} a_n$. Since the series has positive terms and is decreasing we have the bound

$$0 \leq a_{n+1} + a_{n+2} + \ldots + a_{2n} \leq na_{2n}.$$

But $\left(\sum_{n=1}^{2n} a_n - \sum_{n=1}^{n} a_n\right) \to 0$ as $n \to \infty$ by Cauchy's test for convergence. Hence, by the squeeze theorem $na_{2n} \to 0$, equivalently $(2n)a_{(2n)} \to 0$ and hence $na_n \to 0$ as $n \to \infty$.

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5. (a) For what values of α do the following series converge or diverge

$$(i) \quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}} \qquad (ii) \quad \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^{\alpha}}$$

(b) Show that the following series converges

$$\sum_{n=2}^{\infty} \frac{\log(n+1) - \log n}{(\log n)^2}.$$

Solution

(a) (i) Use the integral comparison test - we need to consider (for $\alpha \neq 1$ at the moment)

$$\int_2^M \frac{dx}{x(\log x)^{\alpha}} = \left.\frac{(\log x)^{1-\alpha}}{1-\alpha}\right|_2^M = \frac{(\log M)^{1-\alpha}}{1-\alpha} - \frac{\log 2}{1-\alpha},$$

and the improper integral converges as $M \to \infty$ only if $\alpha > 1$. Need to check $\alpha = 1$, i.e. the integral

$$\int_2^M \frac{dx}{x \log x} = \log(\log M) - \log(\log 2) \to \infty \quad \text{as} \quad M \to \infty,$$

hence the integral diverges.

(ii) Again use the integral test hence consider

$$\int_3^M \frac{dx}{(x\log x)(\log\log x)^\alpha} = \left.\frac{(\log\log x)^{1-\alpha}}{1-\alpha}\right|_3^M,$$

hence convergence as $M \to \infty$ only if $\alpha > 1$. For $\alpha = 1$

$$\int_3^M \frac{dx}{x\log x(\log\log x)} = \log\log\log M - \log\log\log 3 \to \infty \quad \text{as} \quad M \to \infty.$$

(b) Combine the logarithms in the numerator to write the series as $\sum_2^\infty \frac{\log\left(1+\frac{1}{n}\right)}{(\log n)^2}$. Next we show that $\log\left(1+\frac{1}{n}\right) < \frac{1}{n}$. Can do this many ways (e.g. $\left(1+\frac{1}{n}\right)^n < e$ from the definition of the exponential - see Chapter 1) but I will do it using Taylor's theorem that gives

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \frac{1}{3}\cdot\frac{1}{n^3} - \dots \tag{2}$$

The power series converges absolutely for $|1/n| < 1$ and since the series starts with $n = 2$ we are fine. The series (2) is alternating with decreasing terms, hence it is smaller than its first term, i.e. $\log\left(1+\frac{1}{n}\right) < \frac{1}{n}$ for $n \geq 2$.
Now we are in a position to use the comparison theorem for series. We have

$$\sum_2^\infty \frac{\log\left(1+\frac{1}{n}\right)}{(\log n)^2} < \sum_{n=2}^\infty \frac{(1/n)}{(\log n)^2},$$

and the latter sum is convergent by comparison with the integral

$$\int_2^M \frac{dx}{x(\log x)^2} = \frac{1}{\log 2} - \frac{1}{\log M} \to 0 \quad \text{as} \quad M \to \infty.$$

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6. For what values $p > 0$ does the series $\sum_{n=1}^\infty \left(1 - \frac{1}{n^p}\right)^n$ converge.

Solution

Recall and use the result $\lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k = e^{-1}$. Write $a_n = \left(1 - \frac{1}{n^p}\right)^n$. If $p < 0$ then a_n does not tend to zero and the series diverges. If $p = 0$ then the series converges and is equal to 0. Hence $p \geq 0$ is a necessary condition for convergence. Take $p > 0$ and re-write

$$a_n = \left(\left(1 - \frac{1}{n^p}\right)^{n^p}\right)^{n/n^p}.$$

For large n, therefore, we have

$$a_n \sim \left(\frac{1}{e}\right)^{n/n^p} \to \begin{cases} e^{-1} & p = 1 \\ 1 & p > 1 \\ \exp(-n^{1-p}) & p < 1 \end{cases}$$

Clearly if $p \geq 1$ the series does not converge (the terms do not tend to zero). So the only possibility is $0 \leq p < 1$. We need to prove this. One way to do is is to compare with the integral

$$\int_1^\infty \exp(-x^{1-p})dx = \int_1^\infty \frac{y^{\frac{p}{1-p}}}{1-p}\exp(-y)dy,$$

where the substitution $y = x^{1-p}$ has been used. Now for any $0 < p < 1$ we can find an integer $N > \frac{p}{1-p}$, and so

$$\int_1^\infty \frac{y^{\frac{p}{1-p}}}{1-p} \exp(-y)dy \le \int_1^\infty y^N \exp(-y)dy < \infty,$$

hence the series converges for $0 \le p < 1$.

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7. This problem follows closely the derivation in class for the power series expansion for $\log(1+x)$.

   (a) Write down the sum of the geometric series $\sum_{k=0}^n r^k$.

   (b) Use (a) to show that

   $$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \ldots + (-1)^{n-1}t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}.$$

   (c) Use (b) to show that

   $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + R_n, \tag{3}$$

   where R_n is the remainder which you should express as an integral involving x.

   (d) Show that the power series for $\tan^{-1} x$ converges absolutely for x in the closed interval $[-1, 1]$.

   (e) Use the power series to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$. How many terms do we have to keep in this series in order to estimate π with accuracy to 10 decimal places, i.e. with error less than 10^{-10}?

Solution

(a) $\sum_{k=0}^n r^k = 1 + r + r^2 + \ldots + r^n = \frac{1-r^{n+1}}{1-r}$.

(b) Use the result above but for n terms not $n+1$, i.e.

$$1 + r + \ldots + r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r} \qquad \Rightarrow$$
$$\frac{1}{1-r} = 1 + r + \ldots + r^{n-1} + \frac{r^n}{1-r}$$

and putting $r = -t^2$ gives

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \ldots + (-1)^{n-1}t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}, \tag{4}$$

as desired.

(c) Now integrate (4) between 0 and x noting that $\int_0^x (1+t^2)^{-1}dt = \tan^{-1} x$ to find

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2}dt, \tag{5}$$

where the last integral is R_n.

(d) For absolute convergence we can use the ratio test of successive terms in the series for $\tan^{-1} x$ above. Find

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+1}}{2n+1} \cdot \frac{2n-1}{|x|^{2n-1}} \to |x|^2 \quad \text{as} \quad n \to \infty,$$

hence we have absolute convergence as long as $|x| < 1$. The boundary points $|x| = 1$ must be considered separately since the ratio test gives no information for them. This comes from the requirement that for absolute convergence the remainder $R_n \to 0$ as $n \to \infty$. To prove this, estimate the integral form of R_n as follows

$$|R_n| \leq \left| \int_0^x t^{2n} dt \right| \leq \frac{|x|^{2n+1}}{2n+1} \to 0 \quad \text{as} \quad n \to \infty \text{ if } |x| \leq 1.$$

Hence, we have absolute convergence of the power series for $\tan^{-1} x$ if $|x| \leq 1$.

(e) Putting $x = 1$ in the power series (5) and noting that $\tan^{-1} 1 = \pi/4$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

An upper bound for the error in estimating π using this series after truncating an n terms, is given by $\frac{4}{2n+1}$ (why? because the series is alternating the error is less than the absolute value of the first term dropped). For accuracy 10^{-10} we need $2n + 1 > 4 \times 10^{10}$, i.e. $n \gtrapprox 2 \times 10^{10}$, a lot of terms!

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8. Following up from the calculation of π above, here is a much more efficient way.

(a) Starting from the addition formula for the tangent

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \, \tan y},$$

introduce the inverse functions $x = \tan^{-1} u$ and $y = \tan^{-1} v$ to show that

$$\tan^{-1} u + \tan^{-1} v = \tan^{-1} \left(\frac{u + v}{1 - uv} \right). \tag{6}$$

(b) Show that choosing $(u+v)/(1-uv) = 1$ in expression (6), we have the following formula for π,

$$\frac{\pi}{4} = \tan^{-1} u + \tan^{-1} v, \tag{7}$$

and that restricting u and v to be in the interval $(0, 1)$ we can express them as the one-parameter family

$$u = \frac{1-p}{1+p}, \qquad v = p, \qquad 0 < p < 1, \tag{8}$$

or equivalently

$$u = \frac{n-m}{n+m}, \qquad v = \frac{m}{n}, \qquad 0 < m < n, \tag{9}$$

where we picked p to be the rational number $p = m/n$.

Use your earlier findings regarding the power series for $\tan^{-1} x$ (equation (3)) to explain why the choices (8)-(9) are useful.

(c) Hence show that (first derived and used by Euler)

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}. \tag{10}$$

Noting that $\frac{\frac{1}{3}+\frac{1}{7}}{1-\frac{1}{21}} = \frac{1}{2}$, show that $\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$, which when combined with (10) gives the formula (used by Jurij Vega, 1754-1802, a Slovenian mathematician who got 140 digits accuracy to π using this formula)

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}, \tag{11}$$

and on use of $\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}} = \frac{1}{3}$ and previous results we also have

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8}. \tag{12}$$

(d) If we use the expressions (10), (11) and (12), respectively, how many terms in the expansion (3) do we need to compute π to 10 decimals accuracy? Compare with your answer to question 8(e).

Solution

(a) From the definitions of u and v, the addition formula becomes

$$\begin{aligned} \tan(x+y) &= \frac{u+v}{1-uv} \qquad \Rightarrow \\ x+y &= \tan^{-1}\left(\frac{u+v}{1-uv}\right) \qquad \Rightarrow \\ \tan^{-1}u + \tan^{-1}v &= \tan^{-1}\left(\frac{u+v}{1-uv}\right), \end{aligned} \tag{13}$$

as required.

(b) Choosing $\frac{u+v}{1-uv} = 1$ in (13) immediately yields $\tan^{-1}u + \tan^{-1}v = \frac{\pi}{4}$. Continuing with $u, v \in (0,1)$, write $v = p$, $0 < p < 1$, so that $u = \frac{1-p}{1+p}$. Restricting to rationals, let $p = m/n$, $0 < m < n$, so that $v = \frac{m}{n}$ and $u = \frac{n-m}{n+m}$.
These choices for u, v ensure that we are within the radius of convergence of the power series for $\tan^{-1}u$ and $\tan^{-1}v$ found in problem 7.

(c) Pick $m = 1$, $n = 3$, so that $u = \frac{1}{2}$, $v = \frac{1}{3}$, and (13) gives

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \tag{14}$$

Picking $u = 1/3$, $v = 1/7$ makes $(u+v)/(1-uv) = 1/2$ as given, and hence (13) becomes

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{2} \tag{15}$$

Now eliminate $\tan^{-1}\frac{1}{2}$ between expressions (15) and (14) to find the required formula

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}. \tag{16}$$

Next picking $u = 1/5$, $v = 1/8$ gives $(u+v)/(1-uv) = 1/3$, hence

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{1}{3},$$

which when combined with (16) gives

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8}. \tag{17}$$

(d) To estimate how many terms we need for an error of 10^{-10} for each of formulas (14), (16) and (17), first note that it is sufficient to consider the power series for the largest argument of $\tan^{-1}$ appearing in each of these formulas, i.e. 1/2, 1/3 and 1/5, respectively. (Any smaller arguments present will contribute smaller errors automatically, so no need to bother with them further.) For (14) and on use of formula (5) we have

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)2^{2n-1}}.$$

Now, $2^{34} \approx 1.7\times 10^1 0$, so if $n = 17$ are used the error is guaranteed to be smaller than 10^{-10}.

For the Vega expression (16) the nth term has size $\frac{1}{(2n-1)3^{2n-1}}$. Since $3^2 1 \approx 1.05\times 10^{10}$, $n = 10$ will do.

Finally, for formula (17) the nth term has size $\frac{1}{(2n-1)5^{2n-1}}$ and since $13\times 5^{13} \approx 1.6\times 10^{10}$ we see that $2n-1 = 13$, i.e. $n = 6$ terms are enough.

The differences with the number of terms needed in question 6 are striking, i.e. 6 terms instead of 2×10^{10} terms! The moral of the exercise is - do analysis before you compute!

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9. (a) *Binomial Theorem.* Let $f(x) = (1+x)^s$ where s is a real number. Use induction arguments to show that $f^{(n)}(x) = s(s-1)\ldots(s-n+1)(1+x)^{s-n}$ and hence write down the Taylor series for $f(x)$ including the remainder term. Hence show that $(1+x)^s$ converges uniformly (i.e. it is analytic) for $|x| < 1$.

(b) Use the Binomial Theorem to compute $(126)^{1/3}$ and $\sqrt{96}$ to 4 decimals.

(c) Write out the Maclaurin series for $1/\sqrt{1+x^2}$ using the binomial series. What is $\frac{d^{20}}{dx^{20}}\left(\frac{1}{\sqrt{1+x^2}}\right)\Big|_{x=0}$?

(d) Find the Maclaurin series for $g(x) = \sqrt{1+x} + \sqrt{1-x}$, and hence calculate $g^{(20)}(0)$ and $g^{(2001)}(0)$.

Solution

(a) *Induction.* Start with $f(x) = (1+x)^s$ with s real. Differentiate once, $f'(x) = s(1+x)^{s-1}$, hence the formula is true for $n = 1$. Assume it is true for n, i.e. $f^{(n)} = s(s-1)\dots(s-(n-1))(1+x)^{s-n}$, differentiating once more we find $f^{(n+1)}(x) = s(s-1)\dots(s-(n-1))(s-n)(1+x)^{s-(n+1)}$, hence the formula holds for $n+1$ also and the induction proof is complete.

We can now write down the Taylor series of $f(x)$ about $x = 0$ (i.e. its Maclaurin series). This is, with remainder where ξ is a number between 0 and x,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \;\Rightarrow$$
$$= 1 + sx + s(s-1)\frac{x^2}{2!} + \dots + \frac{s(s-1)\dots(s-n+1)}{n!}x^n + \frac{s(s-1)\dots(s-n)}{(n+1)!}(1+\xi)^{s-n-1}.$$

Using the ratio test for convergence we find

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|s(s-1)\dots(s-n)|}{(n+1)!}\frac{|x|^{n+1}}{|x|^n}\frac{n!}{|s(s-1)\dots(s-n+1)|}$$
$$= \lim_{n\to\infty}\frac{|x|\,|s-n|}{n} = |x|,$$

hence we need $|x| < 1$ for absolute convergence.

(b)

$$(126)^{1/3} = (125+1)^{1/3} = 5\left(1+\frac{1}{5}\right)^{1/3} = 5\left[1 + \frac{1}{3}\cdot\frac{1}{5} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\frac{1}{5^2}\right.$$
$$\left. + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\frac{1}{5^3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}\frac{1}{5^4} + \dots\right]$$

The last term has absolute value $\frac{5\times 80}{81\times 24\times 725} \approx 3\times 10^{-4}$, hence it needs to be kept. The next term can be neglected.

$$\sqrt{96} = \sqrt{100-4} = 10\left(1-\frac{1}{25}\right)^{1/2} = 10\left[1 - \frac{1}{2}\cdot\frac{1}{25} + \frac{(1/2)(-1/2)}{2!}\frac{1}{25^2}\right.$$
$$\left. + \frac{(1/2)(-1/2)(-3/2)}{3!}\frac{1}{25^3} + \dots\right]$$

(c) Use $x \to x^2$ and $s = -1/2$ to find

$$(1+x^2)^{-1/2} = 1 - \frac{1}{2}x^2 + \dots + \frac{(-1/2)(-3/2)\dots(-1/2-n+1)}{n!}x^{2n} + \dots$$

The only term that survives in $\frac{d^{20}}{dx^{20}}\left(\frac{1}{\sqrt{1+x^2}}\right)\Big|_{x=0}$ comes from the x^{20} term, i.e. $n = 10$. Hence

$$\frac{d^{20}}{dx^{20}}\left(\frac{1}{\sqrt{1+x^2}}\right)\Bigg|_{x=0} = \frac{(-1/2)(-3/2)\dots(-1/2-10+1)}{10!}20!$$

Note that the 20! factor comes from differentiating x^{20} twenty times.

(d) We know from part (a) that $\sqrt{1+x}+\sqrt{1-x}$ will give a series that only contains even powers of x. Hence we conclude immediately that $g^{(2001)}(0) = 0$ (in fact any odd derivative will be zero at $x = 0$).
For $g^{(20)}(0)$ the only non-zero term comes from the coefficient of x^{20}. Now

$$(1 \pm x)^{1/2} = 1 + \ldots + (1/2)(1/2-1)\ldots(1/2-20+1)\frac{x^{20}}{20!} + \ldots,$$

hence

$$g^{(20)}(0) = 2(1/2)(1/2-1)\ldots(1/2-20+1).$$

---

10. Find the radius of convergence of the following series:

$$(1)\ \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n \quad (2)\ \sum_{n=1}^{\infty}\frac{n^n}{(n!)}x^n \quad (3)\ \sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}x^n \quad (4)\ \sum_{n=1}^{\infty}\frac{n^{5n}}{(2n)!\,n^{3n}}x^n$$

$$(5)\ \sum_{n=1}^{\infty}\frac{(3n)!}{(n!)^2}x^n \quad (6)\ \sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{2^n}x^n \quad (7)\ \sum_{n=1}^{\infty}\frac{\log n}{2^n}x^n \quad (8)\ \sum_{n=1}^{\infty}\frac{1+\cos 2\pi n}{3^n}x^n$$

$$(9)\ \sum_{n=1}^{\infty}n\,x^n \quad (10)\ \sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n!}x^n \quad (11)\ \sum_{n=1}^{\infty}n^2x^n \quad (12)\ \sum_{n=1}^{\infty}\frac{\cos n^2}{n^n}x^n$$

$$(13)\ \sum_{n=1}^{\infty}\frac{n}{\log n}x^n \quad (14)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n!-1}x^n \quad (15)\ \sum_{n=1}^{\infty}\frac{n!}{n^n}x^n \quad (16)\ \sum_{n=1}^{\infty}\frac{(-1)^n+1}{n!}x^n$$

You may use Stirling's formula

$$n! = (2\pi n)^{1/2}\,n^n\ e^{-n}\,e^{\theta/12n}, \qquad 0 \le \theta \le 1,$$

in its appropriate form for large n.

[**Answers:** (1) $1/4$, (2) $1/e$, (3) 27, (4) $4/e^2$, (5) 0, (6) 2, (7) 2, (8) 3, (9) 1, (10) ∞, (11) 1, (12) ∞, (13) 1, (14) ∞, (15) e, (16) ∞.]

Solution

These problems are solved using the ratio test and the given Stirling formula if you wish. I didn't bother with it, but the way you could use it is to replace $n!$ in any of the sums by the formula - this is valid for large n and it would get rid of the n^n terms. I kept it as a ratio and that needs the familiar result $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Will denote by a_n the nth term in a series and by R the radius of convergence.

(1)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+2)!\,(n!)^2}{((n+1)!)^2\,(2n)!}|x| = \frac{(2n+2)(2n+1)}{(n+1)^2}|x| \to 4|x| \ \Rightarrow\ R = \frac{1}{4}$$

(2)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}\,n!}{(n+1)!\,n^n}|x| = \left(1+\frac{1}{n}\right)^n|x| \to e|x| \ \Rightarrow\ R = \frac{1}{e}$$

(3)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}|x| \to \frac{|x|}{27} \quad \Rightarrow R = 27$$

(4)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left(1+\frac{1}{n}\right)^{5n}}{\left(1+\frac{1}{n}\right)^{3n}} \frac{(n+1)^5}{(2n+2)(2n+1)(n+1)^3}|x| \to \frac{e^5}{e^3}\frac{|x|}{4} \quad \Rightarrow R = \frac{4}{e^2}$$

(5)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^2}|x| \sim 9n|x| \quad \Rightarrow R = 0$$

(6) Note first that $a_n = 0$ if n is even, so need to consider the new series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2^{2n-1}}$, so

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+1}2^{2n+1}}{|x|^{2n-1}2^{2n+1}} \to \frac{|x|^2}{4} \quad \Rightarrow R = 2$$

(7)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\log(n+1)\,2^n}{2^{n+1}\,\log(n)}|x| \to \frac{|x|}{2} \quad \Rightarrow R = 2$$

(8) Note that $\cos 2n\pi = 1$ for all $n \geq 1$, hence

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{3} \quad \Rightarrow R = 3$$

(9)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)|x|^{n+1}}{n|x|^n} \to |x| \quad \Rightarrow R = 1$$

(10) Since $\sin(2\pi n) = 0$ for all positive integers, the sum is 0 for all x. Hence $R = \infty$.

(11)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2\,|x|^{n+1}}{n^2\,|x|^n} \to |x| \quad \Rightarrow R = 1$$

(12)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\cos(n+1)^2|\,n^n}{|\cos n^2|\,(n+1)^{n+1}}|x| = \frac{|\cos(n+1)^2|}{|\cos n^2|}\frac{1}{(n+1)}\frac{1}{\left(1+\frac{1}{n}\right)^n}|x|$$
$$\leq M\frac{|x|}{(n+1)} \to 0 \quad \Rightarrow R = \infty$$

The bound M follows since $|\cos n^2|$ with n an integer is always bounded away from 0.

(13)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)\,\log n}{n\,\log(n+1)}|x| \to 1 \quad \Rightarrow R = 1$$

(14)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n!-1)}{[(n+1)!-1]}|x| \to \frac{|x|}{n+1} \quad \Rightarrow R = \infty$$

(15) The coefficients of x^n in a_n are the reciprocals of those in problem (2). Hence $R = e$.

(16) Odd terms are zero, hence $\sum_{n=1}^{\infty} \frac{(-1)^n+1}{n!} x^n = \sum_{k=1}^{\infty} \frac{2}{(2k)!} x^{2k}$, hence

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|^2}{(2k+2)(2k+1)} \to 0 \quad \Rightarrow R = \infty$$

---

11. Find the Taylor series of the function $f(x) = \int_1^x \log t\, dt$ for x near 1. Do the same for the function $x \log x$ and compare the two. What do you conclude?

Solution

Generally, $f(x) = f(1 + x - 1) = f(1) + (x-1)f^{(1)}(1) + \frac{(x-1)^2}{2!} f^{(2)}(1) + \ldots$ is the Taylor expansion near $x = 1$. For $f(x) = \int_1^x \log t\, dt$ calculate

$$f'(x) = \log x \;\; f^{(2)}(x) = \frac{1}{x}, \;\; f^{(3)}(x) = -\frac{1}{x^2}, \;\; f^{(4)}(x) = \frac{2}{x^3}, \;\; \ldots$$

Hence

$$f(x) = \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} 2! - \frac{(x-1)^5}{5!} 3! + \ldots, \tag{18}$$

is the Taylor series for the given integral.

Now define $g(x) = x \log x$ and again calculate derivatives

$$g'(x) = 1 + \log x, \;\; g^{(2)}(x) = \frac{1}{x}, \;\; g^{(3)}(x) = -\frac{1}{x^2}, \ldots,$$

i.e. $f^{(k)}(x) = g^{(k)}(x)$ for all $k \geq 2$. Hence the Taylor expansion of $x \log x$ is

$$\begin{aligned} x \log x &= (x-1) + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} 2! - \frac{(x-1)^5}{5!} 3! + \ldots, \\ &= (x-1) + f(x), \end{aligned} \tag{19}$$

on use of (18). This of course is expected since

$$f(x) = \int_1^x \log t dt = [t \log t]_1^x - \int_1^x dt = x \log x - (x-1),$$

in complete agreement with (19).

---

12. Find the first four non-vanishing terms of the Maclaurin series for the following functions:

$$(a)\ x\cot x \qquad (b)\ e^{\sin x}, \qquad (c)\frac{\sqrt{\sin x}}{\sqrt{x}}$$
$$(d)\ e^{e^x}, \qquad (e)\ \sec x, \qquad (f)\ \log\sin x - \log x$$

Solution

(a) I avoided differentiating $x\cot x$ three times and used Maclaurin combined with Binomial - good to see it done this way also - can be much quicker.

$$x\cot x = x\frac{\cos x}{\sin x} = x\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right]\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right]^{-1}$$
$$=\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right]\left[1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\dots\right)\right]^{-1}$$
$$=\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right]\left[1+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\dots\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\dots\right)^2\right.$$
$$\left.+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\dots\right)^3+\dots\right]$$

and since we only need terms up to and including x^6 we only keep relevant terms in the second bracket, i.e.

$$=\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right]\left[1+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\dots\right)+\left(\frac{x^4}{(3!)^2}-2\frac{x^6}{(3!)(5!)}+\dots\right)\right.$$
$$\left.+\left(\frac{x^6}{(3!)^3}+\dots\right)\right]=\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right]\left[1+\frac{x^2}{3!}+\left(\frac{1}{(3!)^2}-\frac{1}{5!}\right)x^4\right.$$
$$\left.+\left(\frac{1}{(3!)^2}-\frac{2}{(3!)(5!)}\right)x^6+\dots\right]=1+\left(\frac{1}{3!}-\frac{1}{2!}\right)x^2+\left(\frac{1}{(3!)^2}-\frac{1}{5!}-\frac{1}{(2!)(3!)}+\frac{1}{4!}\right)x^4$$
$$+\left(\frac{1}{(3!)^2}-\frac{2}{(3!)(5!)}-\frac{1}{(2!)(3!)^2}+\frac{1}{(2!)(5!)}+\frac{1}{(3!)(4!)}-\frac{1}{6!}\right)x^6+\dots$$

(b) Need four derivatives of $f(x)=e^{\sin x}$, since $f^{(3)}(0)=0$ as we see below:

$$f'(x)=(\cos x)e^{\sin x}, \qquad f^{(2)}(x)=(\cos^2 x)e^{\sin x}-(\sin x)e^{\sin x},$$
$$f^{(3)}(x)=(\cos^3 x)e^{\sin x}-3(\cos x\sin x)e^{\sin x}-(\cos x)e^{\sin x}$$
$$f^{(4)}(x)=(\cos^4 x)e^{\sin x}-3(\cos^2 x\sin x)e^{\sin x}-3(\cos^2 x\sin x)e^{\sin x}$$
$$-3(\cos^2 x-\sin^2 x)e^{\sin x}-(\cos^2 x)e^{\sin x}+(\sin x)e^{\sin x}$$

Hence the first 4 non-zero terms are

$$e^{\sin x}=1+x+\frac{x^2}{2!}-3\frac{x^3}{3!}.$$

(c) Use the Binomial expansion after canceling x:

$$\frac{\sqrt{\sin x}}{\sqrt{x}} = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)^{1/2} = 1 - \frac{1}{2}\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\right)$$
$$+ \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\right)^2$$
$$+ \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\right)^3 + \dots$$
$$= 1 - \frac{x^2}{2\cdot 3!} + \left(\frac{1}{2\cdot 5!} - \frac{1}{4\cdot 2!\cdot 3!}\right)x^4 + \left(-\frac{1}{2\cdot 7!} + \frac{1}{2\cdot 2!\cdot 3!\cdot 5!} + \frac{1}{8(3!)^2}\right)x^6 + \dots$$

(d) Need 3 derivatives of $f(x) = e^{e^x}$:

$$f' = e^x\, e^{e^x}, \quad f^{(2)} = e^x\, e^{e^x} + (e^x)^2\, e^{e^x}, \quad f^{(3)} = e^x\, e^{e^x} + (e^x)^2\, e^{e^x} + 2(e^x)^2\, e^{e^x} + (e^x)^3\, e^{e^x},$$

hence

$$e^{e^x} = e + ex + 2e\frac{x^2}{2!} + 5e\frac{x^3}{3!} + \dots.$$

(e) Write

$$\sec x = (\cos x)^{-1} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)^{-1}$$

and use the Binomial expansion as in parts (a) and (c). The calculations are almost identical, so I will omit them.

If you try differentiating $f(x) = \sec x$ to find the Maclaurin series, you will need 6 derivatives since $f^{(1)}(0) = f^{(3)}(0) = f^{(5)}(0) = 0$ by virtue of the function being even. The terms increase substantially - try it that way and the Binomial way to appreciate the differences.

(f) Here we expand $\sin x$ and cancel the log:

$$f(x) := \log\sin x - \log x = \log x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) - \log x$$
$$= \cancel{\log x} + \log\left[1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \dots\right)\right] - \cancel{\log x} \tag{20}$$

If I define $y = \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \dots\right)$ then the Taylor series we need is

$$\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \dots,$$

and the first 4 non-zero terms follow by keeping terms up to and including x^8 in each y-term

$$f(x) = -\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \dots\right) - \frac{1}{2}\left(\frac{x^4}{(3!)^2} - \frac{2}{(3!)(5!)}x^6 + \frac{1}{(5!)^2}x^8 + \frac{2}{(3!)(7!)}x^8\right)$$
$$- \frac{1}{3}\left(\frac{x^6}{(3!)^2} - \frac{3x^8}{(3!)^2(5!)}\right) - \frac{1}{4}\frac{x^8}{(3!)^4} + \dots$$

13. Consider the function $h(x)$ defined on the interval $[-\pi, \pi]$ and given by

$$h(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(x/2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Use a Maclaurin expansion to show that $h(x)$ is continuous and has a continuous first derivative at $x = 0$.

Solution

Will expand $\sin(x/2)$ and then use a Binomial expansion:

$$\frac{1}{x} - \frac{1}{2\sin(x/2)} = \frac{1}{x} - \frac{1}{2\left(\frac{x}{2} - \frac{(x/2)^3}{3!} + \frac{(x/2)^5}{5!} + \ldots\right)}$$

$$= \frac{1}{x} - \frac{1}{x}\left(1 - \frac{(x/2)^2}{3!} + \frac{(x/2)^4}{5!} + \ldots\right)^{-1} = \frac{1}{x} - \frac{1}{x}\left(1 + \frac{(x/2)^2}{3!} - \frac{(x/2)^4}{5!} + \frac{(x/2)^4}{(3!)^2} + \ldots\right)$$

$$= -\frac{x}{24} + Kx^3 + \ldots,$$

where K is a known constant. It follows that the function is both continuous and has a continuous first derivative at $x = 0$. [In fact all derivatives exist at $x = 0$ and are continuous.]

---

14. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = f(x)/(1-x)$.

(a) By multiplying the power series of $f(x)$ and $1/(1-x)$, show that $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where $b_n = a_0 + \ldots + a_n$ is the nth partial sum of the series $\sum_{n=0}^{\infty} a_n$.

(b) Suppose that the radius of convergence of $f(x)$ is greater than 1 and that $f(1) \neq 0$. Show that $\lim_{n\to\infty} b_n$ exists and is not equal to zero. What does this tell you about the radius of convergence of $g(x)$?

(c) Let $\frac{e^x}{1-x} = \sum_{n=0}^{\infty} b_n x^n$. What is $\lim_{n\to\infty} b_n$?

Solution

(a) The power series of $1/(1-x)$ follows from the identity $\frac{1-x^{n+1}}{1-x} = 1 + x + \ldots + x^n$ which we have seen often, by having $|x| < 1$ and sending $n \to \infty$. Thus

$$g(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$$
$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots + (a_0 + a_1 + \ldots + a_n)x^n + \ldots$$

and so $b_n = a_0 + a_1 + \ldots + a_n$ as required.

(b) If $f(x)$ has radius of convergence greater than 1 and $f(1) \neq 0$, we can evaluate the power series for f at $x = 1$ to obtain

$$f(1) = \sum_{n=0}^{\infty} a_n \neq 0.$$

Now $\lim_{n\to\infty} b_n = \sum_{n=0}^{\infty} a_n = f(1) \neq 0$. This means that the radius of convergence of $g(x)$ is 1 since on use of the ratio test for $\sum_{n=0}^{\infty} b_n x^n$ we find

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|}|x| = \frac{\cancel{|f(1)|}}{\cancel{|f(1)|}}|x|.$$

(c) You can do the expansion directly but that is not the way. Using (a) and (b) above, we identify $f(x) = e^x$. Clearly the radius of convergence is greater than 1 and $f(1) = e \neq 0$. Hence

$$\lim_{n\to\infty} b_n = e.$$

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15. (a) Write the Maclaurin series for the functions $1/\sqrt{1-x^2}$ and $\sin^{-1} x$. For what values of x do they converge?

(b) Find the terms up to and including x^3 in the series for $\sin^{-1}(\sin x)$ by substituting the series for $\sin x$ into the series for $\sin^{-1} x$.

(c) Use the substitution method from part (b) to obtain the first five terms of the series for $\sin^{-1} x$ by using the relation $\sin^{-1}(\sin x) = x$ and solving for a_0 to a_5.

(d) Find the terms up to and including x^5 of the Maclaurin series for the inverse function $g(s)$ of $f(x) = x^3 + x$. [Hint: Use the relation $g(f(x)) = x$ and solve for the coefficients in the series for g.]

Solution

(a) Use the Binomial expansion:

$$\left(1-x^2\right)^{-1/2} = 1 + (1/2)x^2 + (1/2)(3/2)\frac{x^4}{2!} + (1/2)(3/2)(5/2)\frac{x^6}{3!} + \ldots \qquad (21)$$

The radius of convergence is $|x| < 1$.
Since $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$, and as long as $|x| < 1$ we can integrate (21) to establish the power series for $\sin^{-1} x$ (the radius of convergence is 1 of course):

$$\sin^{-1} x = x + \frac{(1/2)}{3}x^3 + \frac{(1/2)(3/2)}{5\cdot 2!}x^5 + \frac{(1/2)(3/2)(5/2)}{7\cdot 3!}x^7 + \ldots \qquad (22)$$

(b) A direct substitution of $\sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$ into (22) gives, up to and including x^3,

$$\begin{aligned}
\sin^{-1}(\sin x) &= \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{(1/2)}{3}\left(x - \frac{x^3}{3!} + \ldots\right)^3 + \ldots \\
&= x - \cancel{\frac{x^3}{3!}} + \cancel{\frac{(1/2)}{3}x^3} + \ldots \\
&= x + \ldots
\end{aligned}$$

(c) Start by assuming a power series expansion for $\sin^{-1} y$ with the first 5 terms to be found, i.e. write

$$\sin^{-1} y = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \dots, \tag{23}$$

and now put $y = \sin x$ where $|x| < 1$. The left hand side of (23) becomes $\sin^{-1}(\sin x)$ which equals x. Hence we have

$$x = a_0 + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) + a_2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^2 + a_3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^3$$
$$+ a_4\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^4 + a_5\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^5 + \dots$$

The only way this can be satisfied is for the coefficients to balance, i.e.

$$\begin{array}{lll} x^0: & a_0 = 0 & \\ x^1: & a_1 = 1 & \\ x^2: & a_2 = 0 & \\ x^3: & -\dfrac{a_1}{3!} + a_3 = 0 & \Rightarrow \quad a_3 = \dfrac{1}{3!} \\ x^4: & a_4 = 0 & \\ x^5: & \dfrac{a_1}{5!} - \dfrac{3a_3}{3!} + a_5 = 0 & \Rightarrow \quad a_5 = \dfrac{1}{2 \cdot 3!} - \dfrac{1}{5!} = \dfrac{(1/2)(3/2)}{10} \end{array}$$

[Could have anticipated $a_0 = a_2 = a_4 = 0$ since the function is odd.]
These are exactly the first three non-zero coefficients found in (22).

(d) To find the expansion for the inverse function, assume

$$g(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + \dots \tag{24}$$

Since $f(x) = x + x^3$ is an odd function of x, then $g(s)$ is also an odd function of s, hence $a_0 = a_2 = a_4 = \dots = 0$. If you didn't use this it will come out of the calculations that I include for completeness (similar to part (c)). Using the identity $g(f(x)) = x$ in (24), we have

$$x = a_0 + a_1(x + x^3) + a_2(x + x^3)^2 + a_3(x + x^3)^3 + a_4(x + x^3)^4 + a_5(x + x^3)^5 + \dots$$

Equating powers of x gives the coefficients

$$\begin{array}{lll} x^0: & a_0 = 0 & \\ x^1: & a_1 = 1 & \\ x^2: & a_2 = 0 & \\ x^3: & a_1 + a_3 = 0 & \Rightarrow \quad a_3 = -1 \\ x^4: & a_4 = 0 & \\ x^5: & 3a_3 + a_5 = 0 & \Rightarrow \quad a_5 = 3 \end{array}$$

hence

$$g(s) = s - s^3 + 3s^5 + \dots$$