Mathematics Year 1, Calculus and Applications I, 2019
Portfolio Marks Assessment 2

Ivan Kirev
CID: 01738166

1

The function $f(t)$ is defined and is differentiable in the neighborhood of a point $t = x$. For $h > 0$ we are given the following formulas:

$$F_1(x,h) := \frac{1}{h}\left[-\frac{1}{2}f(x-h) + \frac{1}{2}f(x+h)\right],$$
$$F_2(x,h) := \frac{1}{h}\left[\frac{1}{12}f(x-2h) - \frac{2}{3}f(x-h) + \frac{2}{3}f(x+h) - \frac{1}{12}f(x+2h)\right].$$

1.1

To investigate what $F_1(x,h)$ and $F_2(x,h)$ represent, we will use the formula

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^n(x) + R_n(x,h), \qquad (1)$$

where $R_n(x,h) = \frac{h^{n+1}}{(n+1)!}f^{n+1}(c)$ for some c between x and $x+h$.
So for $F_1(x,h)$ we get

$$\begin{aligned}
F_1(x,h) &= \frac{1}{2h}\big(f(x+h) - f(x-h)\big) \\
&= \frac{1}{2h}\Bigg(f(x) + hf'(x) + \frac{h^2}{2!}f^2(x) + \cdots + \frac{h^n}{n!}f^n(x) + R_n(x,h) \\
&\quad - f(x) + hf'(x) - \frac{h^2}{2!}f^2(x) + \cdots - \frac{(-h)^n}{n!}f^n(x) - R_n(x,-h)\Bigg) \\
&= \Bigg(f'(x) + \frac{h^2}{3!}f^3(x) + \frac{h^4}{5!}f^5(x) + \cdots + \frac{R_n(x,h) - R_n(x,-h)}{2h}\Bigg).
\end{aligned}$$

To approximate $F_1(x,h)$ lets consider $n=2$. Then we have

$$\begin{aligned}F_1(x,h) &= f'(x) + \frac{R_2(x,h) - R_2(x,-h)}{2h}\\ &= f'(x) + h^2\left(\frac{f^3(c_1) + f^3(c_2)}{12}\right),\end{aligned}$$

for some c_1 and c_2 in the intervals $[x, x+h]$ and $[x-h, x]$, respectively.
So we can say that $F_1(x,h)$ is an approximation for the derivative $f'(x)$ with error of order h^2:

$$F_1(x,h) = f'(x) + O(h^2).$$

For $F_2(x,h)$ we have

$$\begin{aligned}F_2(x,h) &= \frac{1}{h}\left(\frac{1}{12}f(x-2h) - \frac{2}{3}f(x-h) + \frac{2}{3}f(x+h) - \frac{1}{12}f(x+2h)\right)\\ &= \frac{1}{h}\left(\frac{1}{12}\big(f(x-2h) - f(x+2h)\big) + \frac{2}{3}\big(f(x+h) - f(x-h)\big)\right)\end{aligned}$$

Now if we use formula (1) with $n=4$ we get the following approximation for $F_2(x,h)$:

$$\begin{aligned}F_2(x,h) = {} & \frac{1}{12h}\Bigg(f(x) - \frac{2h}{1!}f'(x) + \frac{4h^2}{2!}f''(x) - \frac{8h^3}{3!}f^3(x) + \frac{16h^4}{4!}f^4(x) - \frac{32h^5}{5!}f^5(c_3)\\ & - f(x) - \frac{2h}{1!}f'(x) - \frac{4h^2}{2!}f''(x) - \frac{8h^3}{3!}f^3(x) - \frac{16h^4}{4!}f^4(x) - \frac{32h^5}{5!}f^5(c_4)\Bigg)\\ & + \frac{2}{3h}\Bigg(f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^3(x) + \frac{h^4}{4!}f^4(x) + \frac{h^5}{5!}f^5(c_5)\\ & - f(x) + \frac{h}{1!}f'(x) - \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^3(x) - \frac{h^4}{4!}f^4(x) + \frac{h^5}{5!}f^5(c_6)\Bigg)\\ = {} & \Bigg(-\frac{1}{3}f'(x) - \frac{2h^2}{9}f^3(x) - \frac{h^4}{45}\big(f^5(c_3) + f^5(c_4)\big) + \frac{4}{3}f'(x) + \frac{2h^2}{9}f^3(x) + \frac{h^4}{180}\big(f^5(c_5) + f^5(c_6)\big)\Bigg)\\ = {} & f'(x) + h^4\left(\frac{f^5(c_5) + f^5(c_6)}{180} - \frac{f^5(c_3) + f^5(c_4)}{45}\right),\end{aligned}$$

where c_3, c_4, c_5, c_6 are in the intervals $[x-2h, x]$, $[x, x+2h]$, $[x, x+h]$, and $[x-h, x]$, respectively.
So we can say that $F_2(x,h)$ is an approximation for the derivative $f'(x)$ with error of order h^4:

$$F_2(x,h) = f'(x) + O(h^4).$$

1.2

Since F_1 is an approximation with error of order h^2 and F_2 with error of order h^4 we expect that F_2 is a better approximation for the derivative $f'(x)$ when h is small.
When $f(x) = x^4$, then we have:

$$\begin{aligned}
F_1(1, 0.1) &= \frac{1}{0.1}\Big(-\frac{(1-0.1)^4}{2} + \frac{(1+0.1)^4}{2}\Big) \\
&= 4.04. \\
F_2(1, 0.1) &= \frac{1}{0.1}\Big(\frac{(1-0.2)^4}{12} - \frac{2}{3}(1-0.1)^4 + \frac{2}{3}(1+0.1)^4 - \frac{(1+0.2)^4}{12}\Big) \\
&= 4.
\end{aligned}$$

The derivative of $f(x)$ is $4x^3$ so $f'(1) = 4$. F_1 gives the approximation 4.04 with error 0.04 while F_2 gives the exact value. This is because the error of F_2 depends on the fifth derivative of $f(x)$, which in this case is 0. So the error becomes 0 as well and we get the exact value of the derivative.

1.3

We are given the formula

$$\begin{aligned}
F_3(x,h) := \frac{1}{h}\Big[&-\frac{1}{60}f(x-3h) + \frac{3}{20}f(x-2h) - \frac{3}{4}f(x-h) \\
&+ \frac{3}{4}f(x+h) - \frac{1}{20}f(x+2h) + \frac{1}{60}f(x+3h)\Big].
\end{aligned}$$

For $f_n(t) = t^n$ we have

$$\begin{aligned}
F_3^n(x,h) = \frac{1}{h}\Big[&-\frac{1}{60}(x-3h)^n + \frac{3}{20}(x-2h)^n - \frac{3}{4}(x-h)^n \\
&+ \frac{3}{4}(x+h)^n - \frac{1}{20}(x+2h)^n + \frac{1}{60}(x+3h)^n\Big].
\end{aligned}$$

The following table represents the values of $F_3^n(1,h)$ for $n = 1, 2, ..., 6$ and $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

h	$F_3^1(1,h)$	$F_3^2(1,h)$	$F_3^3(1,h)$	$F_3^4(1,h)$	$F_3^5(1,h)$	$F_3^6(1,h)$
1/2	1	2	3	4	5	6
1/4	1	2	3	4	5	6
1/8	1	2	3	4	5	6

We know that

$$\frac{df_n(t)}{dt} = \frac{dt^n}{dt} = nt^{n-1},$$

so

$$\frac{df_n(1)}{dt} = n.$$

For $n = 1, 2, ..., 6$ and $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ we see that $F_3^n(1, h)$ gives the exact result of the derivative of $f_n(1)$.

If we consider $n > 6$ the values of $F_3^n(1, h)$ don't give us the exact values of $\frac{df_n(1)}{dt} = n$, but instead an approximation with an error. The following is a table representing the values of $F_3^n(1, h)$ for $n = 7, 8, ..12$ and $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$:

h	$F_3^7(1,h)$	$F_3^8(1,h)$	$F_3^9(1,h)$	$F_3^{10}(1,h)$	$F_3^{11}(1,h)$	$F_3^{12}(1,h)$
1/2	7.5625	12.5	31.21875	97.1875	310.0742	952.6406
1/4	7.0088	8.0703	9.3241	11.1316	14.3284	20.7134
1/8	7.000137	8.0011	9.00497	10.01678	11.04698	12.11543

Here we get an approximation of the derivative at $x = 1$ which gets more accurate as h gets smaller.

Therefore, it appears that $F_3(x, h)$ is an approximation of the derivative of $f(x)$, similar to F_1 and F_2. The values of F_3^n equal the exact values of the derivative of $f_n(t) = t^n$ for $n = 1, 2, ..6$ because the error of the approximation depends on the seventh derivative of $f_n(t)$. (i.e we should get that $F_3(x, h)$ is an approximation of $f'(x)$ with error of order h^6 ($F_3(x, h) = f'(x) + O(h^6)$) if we expand $F_3(x, h)$ using formula (1) until we get a remainder that contains the seventh derivative of $f(c_i)$ for some constants c_i.)