Imperial College London

MATH40003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Linear Algebra and Groups

Date: 7th May 2020

Time: 09.00am – 12.00 noon (BST)

Time Allowed: 3 Hours

Upload Time Allowed: 30 Minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Let A be the following 3×3 matrix over $\mathbb{R}$:

$$A := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Compute A^{-1} using elementary row operations. (2 marks)

(b) Define what it means for an $n \times n$ matrix to be orthogonal. (1 mark)

(c) Show A is orthogonal. (1 mark)

(d) Let R_θ be the anti-clockwise rotation of $\mathbb{R}^2$ about the origin through θ radians. Show that R_θ can be represented by the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

You may assume R_θ is linear.

(4 marks)

(e) Recall that a reflection M_θ in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan\theta\}$ has matrix representation

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Let M be the reflection of the plane $\mathbb{R}^2$ in the x_1-axis, and M' the reflection in the line $x_1 = x_2$. Write down the matrices representing M, M' and show that MM' is a rotation.

(3 marks)

(f) Show that all real orthogonal 2×2 matrices are either rotations or reflections. (7 marks)

(g) Let $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an anticlockwise rotation by θ around the axis $\{l \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : l \in \mathbb{R}\}$. Suppose $B := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ is an orthonormal basis for $\mathbb{R}^3$.

Find the possibilities for $[T]_B$.

(2 marks)

(Total: 20 marks)

2. Consider the set of equations, where $a, b \in \mathbb{R}$:

$$\left.\begin{array}{rcl} x + 2y - z + 3w & = & 0 \\ 2x - 3z + 4w & = & 1 \\ x + 4y + z + aw & = & b \\ -2y - w & = & 0 \end{array}\right\} .$$

(a) Find the augmented matrix $(A|c)$ for this set of linear equations. (1 mark)

(b) For which values of $a, b \in \mathbb{R}$ does this set of equations have a solution? Find all solutions when possible. (6 marks)

(c) (i) Define the row space and row rank of a matrix $B \in M_{n\times m}(\mathbb{R})$. (1 mark)

(ii) Suppose $(A|c)$ is the augmented matrix obtained from the system of equations above. In the case where $a = 4$ and $b = 2$ find the row space and rank of A and $(A|c)$. (4 marks)

(d) Given a system of n real linear equations in n variables (not necessarily the one above) with augmented matrix $(A|c)$. Show that if the row rank of A equals the row rank of $(A|c)$ then the set of solutions S is a coset of a subspace $K \subseteq \mathbb{R}^n$ where the dimension of K is $n - rank(A)$.

In part (d) you may assume standard results about matrices in row reduced echelon form, and linear transformations (including the rank-nulity theorem). (8 marks)

(Total: 20 marks)

3. Let V and W be vector spaces over $\mathbb{R}$.

(a) Let U be a subset of V. Give conditions for U to be a subspace of V. (3 marks)

(b) Let $V = \mathbb{R}[x]$ the set of polynomials with real coefficients and variable x. For a fixed $n \in \mathbb{N}$ let

$$U_n = \{f(x) \in \mathbb{R}[x] : f(x) = \sum_{i=0}^{n} a_i x^i, a_{i+1} = 2a_i \text{ for } i \in \{0, 1, 2, ..., n-1\}\}.$$

Show that U_n is a subspace of V. Find a basis for U_n. (4 marks)

(c) Let $T : V \to W$ be a map from V to W.

- ∗ Give conditions for T to be a linear transformation.
- ∗ Define the kernel of T, $\ker(T)$.
- ∗ Define the image of T, $Im(T)$.
- ∗ Show that $\ker(T)$ is a subspace of V.

(8 marks)

(d) Let V and U_n be as in part (b). Let $T_n : U_n \to U_{n+1}$ be defined as follows:

$$T_n(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n+1} 2^i a_0 x^i.$$

- ∗ Show that T_n is a linear transformation.
- ∗ Find $Im(T)$.
- ∗ Find $\ker(T)$.
- ∗ Is U_n a subspace of U_{n+1}?

(5 marks)

(Total: 20 marks)

4. (a) Suppose $n \in \mathbb{N}$ and $A \in M_n(\mathbb{R})$ is such that $A^2 - 3A + 2I_n = 0_n$ (where 0_n denotes the $n \times n$ zero matrix).

(i) Prove that A is invertible and give an expression for A^{-1}. (3 marks)

For $\lambda = 1, 2$, let $E_\lambda = \{v \in \mathbb{R}^n : Av = \lambda v\}$.

(ii) Prove that $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = \mathbb{R}^n$. (3 marks)

(iii) Prove that A is diagonalisable. (2 marks)

(iv) Suppose $v \in \mathbb{R}^n$ and $\{\|A^k v\| : k \in \mathbb{N}\}$ is bounded (where $\|\cdot\|$ is the usual norm on $\mathbb{R}^n$). Prove that $v \in E_1$. (2 marks)

(v) In the case $n = 2$ write down three matrices in $M_2(\mathbb{R})$ such that for any $A \in M_2(\mathbb{R})$ which satisfies $A^2 - 3A + 2I_2 = 0$, there is an invertible $P \in M_2(\mathbb{R})$ such that $P^{-1}AP$ is equal to one of these three matrices. Explain your answer briefly. (2 marks)

(b) Suppose $B \in M_n(\mathbb{R})$ is invertible and let $C = B^T B$.

(i) Prove that for every non-zero vector $v \in \mathbb{R}^n$ we have $v^T C v > 0$. (2 marks)

(ii) Show that there exist orthogonal matrices $P, Q \in M_n(\mathbb{R})$ and a diagonal matrix $D \in M_n(\mathbb{R})$ with $B = QDP$.
[You may use results from the lectures if they are quoted precisely.] (6 marks)

(Total: 20 marks)

5. (a) State the definition of the *determinant* $\det(A)$ of an $n \times n$ matrix A over a field F. Suppose $\alpha \in F$ and $A \in M_n(F)$. Let $i \leq n$ and let $B \in M_n(F)$ be the matrix which results from multiplying row i of A by α (and leaving all the other rows the same). Using your definition, prove that $\det(B) = \alpha \det(A)$.

(6 marks)

(b) Compute $\det(X^2 Y^3 X^T Z)$ where $X, Y, Z \in M_3(\mathbb{R})$ are given by:

$$X = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad Z = \begin{pmatrix} 0 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(3 marks)

(c) Suppose H is a subgroup of the group $(G, \cdot)$.

(i) Define what is meant by a *left coset* of H in G. (2 marks)

Suppose that X, Y are left cosets of H in G.

(ii) Prove that either $X \cap Y = \emptyset$ or $X = Y$. (3 marks)

(iii) Prove that there is a bijection $X \to Y$. (2 marks)

(iv) Suppose that K is also a subgroup of G and Z is a left coset of K in G. Prove that either $X \cap Z = \emptyset$ or $X \cap Z$ is a left coset of $H \cap K$ in G. Give examples to show that both of these possibilities can occur. (4 marks)

(Total: 20 marks)

6. (a) Suppose $(G, .)$ is a group and $g \in G$.

   (i) Define the *order* of g.

   Suppose G is a cyclic group of order 12.

   (ii) Give an example of such a group G.

   (iii) Determine the possible orders of elements of G and say how many elements of each order there are in G.

   (iv) Find subgroups A, B of G such that
   $A, B \neq G$, $A \cap B = \{e\}$ and $G = \{ab : a \in A, b \in B\}$. Justify your answer.

(9 marks)

(b) Suppose G, H are groups and $\phi : G \to H$ is a homomorphism.

   (i) Define the term *homomorphism* and say what is meant by the *kernel*, $\ker(\phi)$, of ϕ.

(2 marks)

   (ii) Prove that $N = \ker(\phi)$ is a subgroup of G and for all $g_1, g_2 \in G$ we have:

$$\phi(g_1) = \phi(g_2) \Leftrightarrow g_1 N = g_2 N.$$

(4 marks)

(c) For each of the following pairs of groups G, H, find a *non-trivial* homomorphism $\phi : G \to H$ (non-trivial here means that there is $g \in G$ with $\phi(g) \neq e_H$ and, if $G = H$, then ϕ is not the identity map.).

   (i) $G = H = GL_2(\mathbb{R})$;

   (ii) $G = (\mathbb{R}, +)$ and $H = (\mathbb{C}^\times, \cdot)$;

   (iii) $G = \mathrm{GL}_2(\mathbb{R})$ and $H = (\mathbb{R}^\times, \cdot)$;

   (iv) $G = (\mathbb{R}, +)$ and $H = \mathrm{GL}_2(\mathbb{R})$.

(5 marks)

(Total: 20 marks)