

1. Find all solutions of the following systems of linear equations.

$$\begin{array}{ll}
 \text{(a)} & \begin{array}{rcl} x_1 - 2x_2 + x_3 - x_4 & = & 8 \\ 3x_1 - 6x_2 + 2x_3 & = & 18 \\ x_3 - 2x_4 & = & 5 \\ 2x_1 - 2x_2 + 3x_4 & = & 4 \end{array} & \text{(b)} & \begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 2 \\ 3x_1 - 8x_2 + 2x_3 & = & 5 \\ 2x_1 - 5x_2 + x_3 & = & 1 \end{array} \\
 \text{(c)} & \begin{array}{rcl} x_1 - 2x_3 + x_4 & = & 0 \\ 2x_1 - x_2 + x_3 - 3x_4 & = & 0 \\ 4x_1 - 3x_2 - x_3 - 7x_4 & = & 4 \end{array} & \text{(d)} & \begin{array}{rcl} -x_2 + x_3 - 3x_4 & = & 0 \\ x_1 + 3x_2 + x_3 - x_4 & = & 0 \\ 2x_1 + 5x_2 + 3x_3 - 5x_4 & = & 0 \end{array}
 \end{array}$$

In each case write down all solutions with $x_2 = 5$.

2. Which of these matrices A is invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Calculate A^{-1} when it exists.

3. * How can you use Gaussian elimination to solve $\begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$?

Find *all* solutions (x_1, x_2, x_3) .

4. Let A be an $n \times m$ matrix, and $b \in \mathbb{R}^n$. Suppose $Ax = b$ has at least one solution $x_0 \in \mathbb{R}^m$. Show all solutions are of the form $x = x_0 + h$, where h solves $Ah = 0$.
5. Let A and B be square $n \times n$ matrices with real entries. For each of the following statements, either give a **proof**, or find a **counterexample with** $n = 2$.
- (i) If $AB = 0$ then A and B cannot both be invertible.
 - (ii) If A and B are invertible then $A + B$ is invertible.
 - (iii) If A and B are invertible then AB is invertible.
 - (iv) If A and B are invertible and $(AB)^2 = A^2B^2$, then $AB = BA$.
 - (v) If $ABA = 0$ and B is invertible then $A^2 = 0$.
 - (vi) If $ABA = I$ then A is invertible and $B = (A^{-1})^2$.
 - (vii) If A has a left inverse B and a right inverse C then $B = C$.

6. Let $n \geq 2$ and let $A_n = (a_{ij})$ be the $n \times n$ matrix such that

$$\begin{aligned} a_{i-1,i} &= 1 \text{ for } i = 2, \dots, n, \\ a_{i+1,i} &= 1 \text{ for } i = 1, \dots, n-1, \end{aligned}$$

and $a_{ij} = 0$ for all other i, j . Write down A_2, A_3 and A_4 . Prove that A_n is invertible for all even values of n , and is not invertible for all odd values of n . Find A_2^{-1} and A_4^{-1} .