Cover Sheet for Submission of Maths Examinations Summer 2020

We would advise preparing your coversheets with your CID, Module Name and Code and Date, before the exams are due to take place.

CID: 01738166

Module Name: Probability and Statistics

Module Code: MATH40005

Date: 14/05/2020

Questions Answered (in the file):

Please tick next to the question or questions you have answered in this file.

Q1	
Q2	
Q3	
Q4	✓
Q5	
Q6	

(Note: this is a coversheet for all students - not all students will have exams with 6 questions. Please tick the boxes which are appropriate for your exam and/or the file you are submitting).

(Optional) Page Numbers for each question;

Page Number	Question Answered

If handwritten, please complete in CAPITAL Letters, in Blue or Black Ink, ensuring the cover sheet is legible.

(a)

(i) $\bar{X} = \sum_{i=1}^{n} \frac{1}{n} X_i \sim N\left(\sum_{i=1}^{n} \left(\frac{\mu}{n}\right), \sum_{i=1}^{n} \frac{\sigma^2}{n^2}\right) \Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$,

where we have used that if $X_i \sim N(\mu_i, \sigma_i^2)$ then for $Y = \sum_{i=1}^{n} X_i$, we have $Y \sim N(\mu, \sigma^2)$, where $\mu = \sum_{i=1}^{n} \mu_i$

$\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$

(ii) $E(S^2) = \sigma^2 \Rightarrow E\left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right) = \sigma^2$

$\Rightarrow \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right) = \sigma^2$ (Linearity)

$\Rightarrow E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right) = (n-1)\sigma^2$

$\underline{E(Z)} = E\left(\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2\right) = \frac{1}{n}\left(E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right)\right)$

$= \frac{1}{n} \cdot (n-1)\sigma^2 = \boxed{\left(\frac{n-1}{n}\right)\sigma^2}$

(iii) $Var(S^2) = \frac{2\sigma^2}{n-1} \Rightarrow Var\left(\frac{1}{n-1} \cdot P\right) = \frac{2\sigma^2}{n-1}$, $P := \sum_{i=1}^{n} (X_i - \bar{X})^2$

$\Rightarrow \frac{1}{(n-1)^2} Var(P) = \frac{2\sigma^2}{n-1}$ (Properties of Variance)

$\Rightarrow Var(P) = 2\sigma^2(n-1)$.

$Var(Z) = Var\left(\frac{1}{n} P\right) = \frac{1}{n^2} Var(P) = \frac{1}{n^2} 2\sigma^2(n-1)$

$= \boxed{\frac{2(n-1)}{n^2} \sigma^2}$

(iv) ~~$b_{\sigma^2}(Z) = E(Z)$~~

$$b_{\sigma^2}(Z) = E(Z) - \sigma^2 = \left(\frac{n-1}{n}\right)\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2 \neq 0$$

$\Rightarrow$ biased

(v) $$E\left[(Z-\sigma)^2\right] = \left[b_{\sigma^2}(Z)\right]^2 + Var(Z)$$

$$= \left(\frac{-1}{n}\sigma^2\right)^2 + \frac{2(n-1)}{n^2}\sigma^2 = \frac{1}{n^2}(\sigma^2)^2 + \frac{2n-2}{n^2}\sigma^2$$

(vi) We know $\frac{(n-1)}{\sigma^2}S^2 \sim \chi^2_{n-1}$ and $Z = \frac{n-1}{n}S^2$

$\Rightarrow nZ = (n-1)S^2 \Rightarrow \frac{n.Z}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow b = \frac{n}{\sigma^2}$, $n-1$ degrees of freedom.

(vii) $$Cov(\bar{X}, Z) = E\left[(\bar{X} - \mu)(Z - E(Z))\right] =$$

$$= E\left[(\bar{X} - \mu)\left(Z - \frac{n-1}{n}\sigma^2\right)\right] = E\left((\bar{X}-\mu)\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right)\right) = \text{(linearity)}$$

$$= \cancel{\frac{1}{n}E\left((\bar{X}-\mu)\right.} \frac{1}{n}\sum_{i=1}^{n} E\left[(\bar{X}-\mu)(X_i - \bar{X})^2\right] = \cancel{\frac{1}{n}\sum_{i=1}^{n}E}[$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\left[\bar{X}^3 - 2\bar{X}^2 X_i + \bar{X}X_i^2 - \bar{X}^2\mu + 2\mu\bar{X}X_i - \mu X_i^2\right]$$

(b) Fix $a>0$ and define

$$Y_a = \begin{cases} 0 & , \cancel{X}<a \\ a & , X \geq a \end{cases}$$

$\Rightarrow Y_a \leq X, \forall a, X \Rightarrow E(Y_a) \leq E(X)$.

$E(Y_a) = 0 \cdot P(X<a) + a\,P(X\geq a)$ (since Y_a is discrete)

$\Rightarrow a \cdot P(X \geq a) \leq E(\cancel{X}) \Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$ (since $a>0$)

(c) After Bonferroni correction $n=100$ we get $\alpha' = \frac{\alpha}{100} = \frac{0,01}{100}$

$= \underline{1 \times 10^{-4}}$.

We want p-value $< \alpha'$, so we only choose D, E, as

$2 \cdot 10^{-7} < 10^{-4}$ and $5 \times 10^{-6} < 10^{-4}$, but ~~b~~

(for C) $9 \cdot 10^{-3} > 10^{-4}$.