## Cover Sheet for Submission of Maths Examinations Summer 2020

We would advise preparing your coversheets with your CID, Module Name and Code and Date, before the exams are due to take place.

**CID: 01738166**

**Module Name: Probability and Statistics**

**Module Code: MATH40005**

**Date: 14/05/2020**

### Questions Answered (in the file):

Please tick next to the question or questions you have answered in this file.

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|---|---|
| **Q1** | |
| **Q2** | |
| **Q3** | |
| **Q4** | |
| **Q5** | |
| **Q6** | ✓ |

(Note: this is a coversheet for all students - not all students will have exams with 6 questions. Please tick the boxes which are appropriate for your exam and/or the file you are submitting).

### (Optional) Page Numbers for each question;

| Page Number | Question Answered |
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If handwritten, please complete in CAPITAL Letters, in Blue or Black Ink, ensuring the cover sheet is legible.

(a)

(i) 90% condidence $\Rightarrow 1-\alpha = 0,9 \Rightarrow \alpha = 0,1 \Rightarrow \frac{\alpha}{2} = 0,05$

$\Rightarrow 1-\frac{\alpha}{2} = 0,95$

$\Rightarrow z_{1-\frac{\alpha}{2}} = z_{0,95} = \underline{1,645}$. By symmetry $z_{\alpha/2} = z_{0,05} = -1,645$

We have

$$P\left(\bar{y} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \bar{y} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha = 90\%$$

$\Rightarrow \bar{y} = 5, \ \sqrt{n} = 10, \ \sigma = \sqrt{9} = 3 \Rightarrow$

Our interval is $\boxed{\left(5 - 1,645 \cdot \frac{3}{\sqrt{10}}, \ 5 + 1,645 \cdot \frac{3}{\sqrt{10}}\right)}$

(ii) We have $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\bar{y} - y_i)^2 = 4 \Rightarrow s = \sqrt{4} = 2$

$T = \dfrac{\bar{y} - \mu}{s/\sqrt{n}} \sim$ Student's t-distribution with n-1 degr. of freedom.

degrees of dreedom $= n-1 = 9$

95% cond. interval $\Rightarrow \alpha = 0,05 \Rightarrow 1-\frac{\alpha}{2} = 0,975$.

$P(T < t_{0,975}) = 0,975$ :

$t_{0,975} = (9$ degr. of freedom$) = 2,262 \Rightarrow$

$$P\left(\bar{y} - t_{0,975} \frac{s}{\sqrt{n}} < \mu < \bar{y} + t_{0,975} \frac{s}{\sqrt{n}}\right) = 0,95$$

$\Rightarrow$ Interval is $\boxed{\left(5 - 2,265 \cdot \frac{2}{\sqrt{10}}, \ 5 + 2,265 \cdot \frac{2}{\sqrt{10}}\right)}$

(iii) Use Chebyshev's inequality:

$$P\left(\left|\bar{X} - \overset{E(\bar{X})}{\cancel{\mu}}\right| \geq k\sqrt{Var(\bar{X})}\right) \leq \frac{1}{k^2} \iff$$

$$P\left(\left|\bar{X} - \overset{E(\bar{X})}{\cancel{\mu}}\right| < k\sqrt{Var(\bar{X})}\right) \geq 1 - \frac{1}{k^2}$$

$E(\bar{x}) = \mu$

$Var(\bar{x}) = \frac{\sigma^2}{n}$ => we get:

$$P\left(|\bar{X} - \mu| < k \cdot \sqrt{\frac{\sigma^2}{n}}\right) \geq 1 - \frac{1}{k^2}$$

We want $1 - \frac{1}{k^2} = 0,99 \Rightarrow \frac{1}{k^2} = \frac{1}{100} \Rightarrow \underline{k = 10}$

$$\Rightarrow P\left(|\bar{X} - \mu| < 10 \cdot \frac{4}{\sqrt{10}}\right) \geq 99\%$$

$\Rightarrow P(|5 - \mu| < 4\sqrt{10}) \geq 99\%$ => confidence interval is

$$\boxed{(5 - 4\sqrt{10},\ 5 + 4\sqrt{10})}$$

Note: Used x instead of y.

(b)

(i) Normal with mean 0, variance $\sigma^2$.

(ii) Model 1 - doesn't fit, data seems to be distributed in U-shape => suggest wrong model.

Model 2 - fits, data seems to be randomly distributed.

(c) $\pi(\theta|\mathbf{x}) = \dfrac{f(\mathbf{x}|\theta)\cdot\pi(\theta)}{m(\mathbf{x})}$

Note: using x instead of z.

$f(\mathbf{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n}\left(\theta e^{-\theta x_i}\right) = \underline{\theta^n \cdot e^{-\theta y}}$, where $y = \sum_{i=1}^{n} x_i$

$\downarrow$ independent

$\Rightarrow \pi(\theta|\mathbf{x}) = \dfrac{\theta^n \cdot e^{-\theta y} \cdot \dfrac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\cdot e^{-\beta\theta}}{m(\mathbf{x})} = \dfrac{\theta^{n+\alpha-1}\cdot\beta^\alpha\cdot e^{-\theta(y+\beta)}}{\Gamma(\alpha)\cdot m(\mathbf{x})}$

Notice $\Gamma(\alpha+n, \beta+y) = \dfrac{\beta^{\alpha+n}\cdot\theta^{\alpha+n-1}\cdot e^{-\theta(\beta+y)}}{\Gamma(\alpha+n)}$, and

$\pi(\theta|\mathbf{x}) \propto \boxed{\Gamma(\alpha+n, \beta+y)} \Rightarrow \pi(\theta|\mathbf{x})$ follows gamma distribution with shape parameter $\alpha+n$, rate parameter $\beta+y$.

(d) $E(X) = 2$

$E(Y) = 3$ $\qquad E(XY) = 4$, $2 \le Y \le 5$

$E(X)\cdot E(Y) = 6 \ne 4 = E(XY) \Rightarrow X$ and $Y$ are not independent.

~~Since $Y\in[2,5] \Rightarrow E(Y) \ge 2$.~~

We know

$Cov(X,Y) \le \sigma_X\sigma_Y$ and $Cov(X,Y) = E(XY) - E(X)\cdot E(Y) = 4 - 2\cdot 3 = \underline{-2}$

Since $Y\in[2,5] \Rightarrow Var(Y) \le \dfrac{(5-2)^2}{4} = \dfrac{9}{4} \Rightarrow \sigma_Y^2 \le \dfrac{9}{4} \Rightarrow \underline{\sigma_Y \le \dfrac{3}{2}}$

$Cov(X,Y) \le \sigma_X\sigma_Y \Rightarrow -2 \le \sigma_X\cdot\dfrac{3}{2} \Rightarrow \sigma_X \ge \dfrac{-2\cdot 2}{3}$

$\Rightarrow \sigma_X \ge \dfrac{-4}{3}$ - trivial :c

$$\frac{\mathrm{Cov}(x,y)}{\sigma_x \sigma_y} \in [-1,1]$$

$$\Rightarrow \frac{-2}{\frac{3}{2}\cdot \sigma_x} \in [-1;1] \Rightarrow \frac{-2}{\frac{3}{2}\sigma_x} \geq -1 \Rightarrow \boxed{\sigma_x \geq \frac{4}{3}}$$

lower bound