Math40002 Analysis 1 **Term 2 Unseen 5**

1. (a) Prove Jensen's inequality: Let $f : I \to \mathbb{R}$ be a convex function on an interval I. Let $x_1, \ldots, x_k \in I$ and let $a_1, \ldots, a_k > 0$. Then

$$f\left(\frac{\sum_{i=1}^k a_i x_i}{\sum_{i=1}^k a_i}\right) \leq \frac{\sum_{i=1}^k a_i f(x_i)}{\sum_{i=1}^k a_i}$$

When is this an equality?

(b) Prove the inequality of arithmetic and geometric means (AM-GM inequality): Let $x_1, \ldots, x_n \geq 0$, then

$$\sqrt[n]{x_1 \cdot \cdots \cdot x_n} \leq \frac{x_1 + \ldots x_n}{n}.$$

2. Let $I \subseteq$ be some interval. $f : I \to \mathbb{R}$ is *halving convex* if for all $x_1, x_2 \in I$:

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(a) Prove that if $f : I \to \mathbb{R}$ is halving convex, then for every $k, n \in \mathbb{N}$ such that $t = \frac{k}{2^n} \in [0, 1]$:

$$\forall x_1, x_2 \in I : f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

(b) Prove that if $f : I \to \mathbb{R}$ is halving convex and continuous, then it is convex.

3. Prove that a convex function on an open interval is continuous. Is it true for a closed interval?