Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 3 with solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

Reminder:

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad y''' = \frac{d^3y}{dx^3}, \ \dots$$

1.* Consider a generic homogeneous second order linear differential equation:

$$\mathcal{L}_\alpha[y] = \alpha_2(x)\frac{d^2y}{dx^2} + \alpha_1(x)\frac{dy}{dx} + \alpha_0(x)y = 0.$$

The general solution of this ODE can be written as

$$y_{\text{GS}}(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants to be fixed by boundary conditions and $\{y_1(x), y_2(x)\}$ are two functions that form a basis of the two-dimensional vector space of solutions.

(a) Which of the following pairs of functions cannot be a basis of the vector space?

i. $\{e^x, e^{-x}\}$

We use the Wronskain, which is the determinant of the Wronskian matrix to test for linear independence of these functions.

$$W(x) = \det \mathbb{W} = \det \begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} = -1 - 1 = -2 \neq 0.$$

So this pair of functions can be a basis of a vector space.

ii. $\left\{1 - \sin^2(x), \left(1 + \tan^2(x)\right)^{-1}\right\}$

These functions are proportional to each other as $y_1(x) = y_2(x)$. So they cannot be a basis of the vector space.

iii. $\{\ln x, \ln x^3\}$

These functions are proportional to each other as $3y_1(x) = y_2(x)$. So they cannot be a basis of the vector space.

iv. $\{e^{ax}, xe^{ax}\}$

We evaluate the Wronskain:

$$W(x) = \det \begin{bmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax} + axe^{ax} \end{bmatrix} = e^{2ax}(1 + ax - ax) = e^{2ax} \neq 0.$$

v. $\left\{(x-1)^3, a(x^2 - 2x + 1)\frac{(x-1)}{4}\right\}$

These functions are proportional to each other as $4y_1(x) = ay_2(x)$. So they cannot be a basis of the vector space.

(b) Consider the functions $y_3 = \alpha y_1 + \beta y_2$ and $y_4 = \gamma y_1 + \delta y_2$. Find the condition that $\alpha, \beta, \gamma, \delta$ must fulfill so that the general solution can be expressed exclusively in terms of y_3 and y_4.

$$W(x) = \det \mathbb{W} = \det \begin{bmatrix} \alpha y_1 + \beta y_2 & \gamma y_1 + \delta y_2 \\ \alpha y_1' + \beta y_2' & \gamma y_1' + \delta y_2' \end{bmatrix} = (y_1 y_2' - y_2 y_1')(\alpha\delta - \beta\gamma).$$

For the Wronskian to be non-zero $\alpha\delta - \beta\gamma \neq 0$, which is the determinant of the matrix of the coefficients. $y_1 y_2' - y_2 y_1' \neq 0$ as this is the Wronskian of y_1 and y_2 that are a basis for the solution vector space.

2. Find the general solution of the following homogeneous linear ODEs:

(a) $y'' + 13y' + 42y = 0$

Try $e^{\lambda x}$ we obtain the characteristic equation:

$$\lambda^2 + 13\lambda + 42 = 0 \quad \Rightarrow \quad \lambda = -6, -7$$

So the general solution is

$$y_{GS}(x) = c_1 e^{-6x} + c_2 e^{-7x}.$$

(b) $y'' + 12y' + 36y = 0$

Try $e^{\lambda x}$ we obtain the characteristic equation:

$$\lambda^2 + 12\lambda + 36 = 0 \quad \Rightarrow \quad \lambda = -6, -6$$

So the general solution is

$$y_{GS}(x) = c_1 e^{-6x} + c_2 x e^{-6x}.$$

and the particular solution of

(c) $y'' + y' + y = 0$ with $y(0) = 0, y'(0) = 1$.

Try $e^{\lambda x}$ we obtain the characteristic equation:

$$\lambda^2 + \lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

So the general solution (which is real) can be written as

$$y_{GS}(x) = e^{-x/2} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right].$$

We have $y(0) = c_1 = 0$ and $y'(0) = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 1$, which gives $c_2 = 2/\sqrt{3}$. So we have

$$y(x) = \frac{2}{\sqrt{3}} e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

3. Find the general solution of the following inhomogeneous linear ODEs:

(a) $y'' - y' = xe^x$

First we obtain the y_{CF}. Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^2 - \lambda = 0 \quad \Rightarrow \quad \lambda = 0, 1$$

So the general solution to the corresponding homogeneous ODE is

$$y_{CF}(x) = c_1 + c_2 e^x.$$

Next, we find a particular integral. Try the ansatz $y_{PI} = A(x)e^x$, we get

$$A'' + A' = x \qquad \text{try } A = Cx^2 + Dx \quad \Rightarrow \quad C = \frac{1}{2}, \quad D = -1.$$

So we have

$$y_{GS} = c_1 + c_2 e^x + (\frac{1}{2}x^2 - x)e^x.$$

(b) $y'' + 13y' + 42y = e^{-x}$

From 2(a)

$$y_{CF}(x) = c_1 e^{-6x} + c_2 e^{-7x}.$$

Next, we find a particular integral. Try the ansatz $y_{PI} = Ae^{-x}$, we get $30A = 1$ so

$$y_{GS} = c_1 e^{-6x} + c_2 e^{-7x} + \frac{1}{30} e^{-x}.$$

(c) $y'' + 13y' + 42y = e^{-6x}$

From 2(a)

$$y_{CF}(x) = c_1 e^{-6x} + c_2 e^{-7x}.$$

Next, we find a particular integral. Try the ansatz $y_{PI} = Axe^{-6x}$, we get $A = 1$ so

$$y_{GS} = c_1 e^{-6x} + c_2 e^{-7x} + xe^{-6x}.$$

(d) $y'' + 12y' + 36y = x\left(1 + e^{-6x}\right)$

From 2(b)

$$y_{CF}(x) = c_1 e^{-6x} + c_2 x e^{-6x}.$$

Next, we find a particular integral. Try the ansatz $y_{PI} = (a + bx) + (cx^2 + dx^3)e^{-6x}$, we obtain $a = -\frac{1}{108}$, $b = \frac{1}{36}$, $c = 0$ and $d = \frac{1}{6}$ so

$$y_{GS} = c_1 e^{-6x} + c_2 x e^{-6x} - \frac{1}{108} + \frac{1}{36}x + \frac{1}{6}x^3 e^{-6x}.$$

(e) $y'' - 2y' + 2y = \sin x$

First we obtain the y_{CF}. Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = 1 \pm i$$

So the general solution to the corresponding homogeneous ODE is

$$y_{CF}(x) = e^x (c_1 \cos x + c_2 \sin x).$$

Next, we find a particular integral. Try the ansatz $y_{PI} = A\cos x + B\sin x$, we get $A = \frac{2}{5}$ and $B = \frac{1}{5}$. So we have

$$y_{GS} = e^x (c_1 \cos x + c_2 \sin x) + \frac{2}{5}\cos x + \frac{1}{5}\sin x.$$

(f) $y'' - 2y' + 2y = 4e^x \sin x$

The y_{CF} is the same as part (e). Next, we find a particular integral. Try the ansatz $y_{PI} = Axe^{(1+i)x}$ aiming to take imaginary part later. We find $C = -2i$ so $y_{PI} = -2xe^x \cos x$. So we have

$$y_{GS} = e^x (c_1 \cos x + c_2 \sin x) - 2e^x \cos x.$$

(g) $y'' - 9y = \sinh 3x$

First we obtain the y_{CF}. Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda = \pm 3$$

So the general solution to the corresponding homogeneous ODE is

$$y_{CF}(x) = c_1 e^{3x} + c_2 e^{-3x}.$$

Next, we find a particular integral. Since $\sinh 3x = \frac{1}{2}(e^{3x} - e^{-3x})$ is all in the y_{CF}, Tty the ansatz $y_{PI} = Axe^{3x} + Bxe^{-3x}$, we get $A = \frac{1}{12}$ and $B = \frac{1}{12}$. So we have

$$y_{GS} = c_1 e^{3x} + c_2 e^{-3x} + \frac{1}{12} x e^{3x} + \frac{1}{12} x e^{-3x}.$$

(h) $y'' + 4y' + 8y = e^{-2x}(1 + 3\cos x + 5\cos 2x)$

First we obtain the y_{CF}. Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^2 + 4\lambda + 8 = 0 \quad \Rightarrow \quad \lambda = -2 \pm 2i$$

So the general solution to the corresponding homogeneous ODE is

$$y_{CF}(x) = e^{-2x} (c_1 \cos 2x + c_2 \sin 2x).$$

Next, we find a particular integral. Try the ansatz

$$y_{PI} = Ce^{-2x} + Re\left[Me^{(-2+i)x}\right] + Re\left[Mxe^{(-2+2i)x}\right],$$

we get $C = \frac{1}{4}$, $M = 1$ and $N = -\frac{5i}{4}$. So we have

$$y_{GS} = e^{-2x} (c_1 \cos 2x + c_2 \sin 2x) + e^{-2x}\left(\frac{1}{4} + \cos x + \frac{5}{4} x \sin 2x\right)$$

(i) $y'' + 5y' + 6y = e^{-3x}\left(1 + 4x + 3x^2\right)$

First we obtain the y_{CF}. Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = -2, -3$$

So the general solution to the corresponding homogeneous ODE is

$$y_{CF}(x) = c_1 e^{-2x} + c_2 e^{-3x}.$$

Next, we find a particular integral. Try the ansatz $y_{PI} = C(x)e^{-3x}$, we get

$$C'' - C' = 1 + 4x + 3x^2 \qquad \text{try} \quad C(x) = a + bx + cx^2 + dx^3 \quad \Rightarrow \quad a = 0, b = -11, c = -5, d = -1.$$

So we have

$$y_{GS} = c_1 e^{-2x} + c_2 e^{-3x} - e^{-3x}(11x + 5x^2 + x^3).$$

and the particular solution of

(j) $y'' - y' = xe^x$ with $y(0) = 0, y'(0) = 0$.

General solution is as part (a). $y(0) = c_1 + c_2 = 0$ and $y'(0) = c_2 - 1 = 0$. So, we have $c_1 = -1$ and $c_2 = 1$. So

$$y_{GS} = -1 + e^x + (\frac{1}{2}x^2 - x)e^x.$$

4. * The equation describing the elongation $x(t)$ of a harmonic oscillator of mass m under a force $F(t)$ is:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F(t)}{m},$$

where ω_0 is a positive constant.

Suppose we apply a constant force F_0 for a time T and we then stop the application of the force:

$$F(t) = \begin{cases} F_0, & 0 < t < T \\ 0, & t > T \end{cases}$$

(a) Solve the ODE for $x(t)$ given the initial conditions $x(0) = \frac{dx}{dt}(0) = 0$

For $0 < t < T$ the ODE is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m},$$

and its general solution is

$$x(t) = x_{CF} + x_{PI} = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0}{m\omega_0^2}.$$

The initial conditions $x(0) = x'(0) = 0$ gives $c_1 = -\frac{F_0}{m\omega_0^2}$ and $c_2 = 0$. So that $x(t) = \frac{F_0}{m\omega_0^2}(1 - \cos\omega_0 t)$. For $t > T$ we have the following ODE

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

with the general solution

$$x(t) = x_{CF} = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

Evidently, $x(t)$ and $x'(t)$ at $t = T$ must agree between the two solutions (for the solution to be continues and physically valid). So we can obtain c_1 and c_2 by solving the following equations:

$$c_1 \cos \omega_0 T + c_2 \sin \omega_0 T = \frac{F_0}{m\omega_0^2}(1 - \cos \omega_0 T),$$
$$-\omega_0 c_1 \sin \omega_0 T + \omega_0 c_2 \cos \omega_0 T = \frac{F_0}{m\omega_0} \sin \omega_0 T.$$

(b) Find the amplitude of the oscillation for $t > T$

The solution for $t > T$ can also be written as

$$x(t) = A \cos (\omega_0 t + \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$. We can obtain A by squaring and adding the above equations (after dividing the second equation by ω_0) to be

$$A = \frac{F_0}{m\omega_0^2}\sqrt{2 - 2\cos \omega_0 T}.$$

5. Solve the following third order linear ODEs with constant coefficients:

(a) $y''' - y = x$

First we obtain the y_{CF}. Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^3 - 1 = 0 \quad \Rightarrow \quad \lambda = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So the general solution to the corresponding homogeneous ODE is

$$y_{CF}(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

Next, we find a particular integral. Try the ansatz $y_{PI} = ax^2 + bx + c$, we get $a = 0$, $b = -1$ and $c = 0$. So we have

$$y_{GS} = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) - x.$$

(b) $y''' + 3y'' + 3y' + y = 0 \quad$ with $\quad y(0) = y'(0) = y''(0) = 1$

Try $e^{\lambda x}$ to obtain the characteristic equation:

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1, -1, -1.$$

So the general solution is

$$y_{GS} = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

Given the inital conditions we have $y(0) = c_1 = 1$, $y'(0) = -c_1 + c_2 = 1$ and $y''(0) = c_1 - 2c_2 + 2c_3 = 1$ which gives $c_1 = 1$ and $c_2 = c_3 = 2$ so

$$y_{GS} = e^{-x} + 2xe^{-x} + 2x^2 e^{-x}.$$

(c) $y''' + 3y'' + 3y' + y = \cosh x$

The y_{CF} is the same as in part (b). We need to find a particular integral. As $\cosh x = \frac{1}{2}(e^x + e^{-x})$, we will try the ansatz $y_{PI} = Ae^x + Bx^3e^{-x}$, we get $A = \frac{1}{16}$ and $B = \frac{1}{12}$. So the general solution is

$$y_{GS} = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + \frac{1}{16}e^x + \frac{x^3}{12}e^{-x}.$$

6. Using the change of variables $x = e^z$, solve the following ODEs of the Euler type:

(a) $x^2y'' - 4xy' + 6y = x$

As this is a Euler ODE we put $x = e^t$ and we obtain the following linear ODE:

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = e^t$$

So using the standard method of finding complementary function and particular integral the solution we get is

$$y_{GS}(t) = c_1e^{3t} + c_2e^{2t} + \frac{1}{2}e^t$$

So in terms of x we have

$$y_{GS}(x) = c_1x^3 + c_2x^2 + \frac{1}{2}x.$$

(b) $x^2y'' - 3xy' + 4y = x^2 \ln x$

As this is a Euler ODE we put $x = e^t$ and we obtain the following linear ODE:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = te^{2t}$$

So using the standard method of finding complementary function and particular integral the solution we get is

$$y_{GS}(t) = c_1e^{2t} + c_2te^{2t} + \frac{1}{6}t^3e^{2t}$$

So in terms of x we have (for $x > 0$):

$$y_{GS}(x) = c_1x^2 + c_2x^2\ln x + \frac{1}{6}x^2(\ln x)^3.$$