Imperial College London
Department of Mathematics

Solutions to Question Sheet 2

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Problem sheet released on Wednesday of week 3. All questions can be attempted before the problem class on Monday Week 4. Question 2 or 6 could be suitable for tutorials. Solutions will be released on Wednesday of week 4.

Question 1 Let $n \in \mathbb{N}$, $n \geq 2$. Suppose $D : M_n(\mathbb{R}) \to \mathbb{R}$ is a function on which elementary row operations have the same effect as they do for det (for example, if B is obtained from $A \in M_n(\mathbb{R})$ by interchanging two rows, then $D(B) = -D(A)$, etc.). Suppose also that $D(I_n) = 1$. Prove that $D(C) = \det(C)$ for all $C \in M_n(\mathbb{R})$.
Harder: What if we replace $\mathbb{R}$ by an arbitrary field F?

Solution: Suppose first that C is row equivalent to the identity matrix I_n. So there is a sequence of elementary row operations which turns I_n into C. Then, by induction on the number of operations used, $\det(C) = D(C)$ (the base step, where there are no operations used and $C = I_n$ is the assumption $D(I_n) = 1$).

If C is not row equivalent to I_n, then there is a sequence of elementary row operations which turns C into a matrix with a row of zeros. Adding some other row to this, we obtain a matrix E with two rows equal. Interchanging these equal rows we obtain $D(E) = -D(E)$ and so $D(E) = 0 = \det(E)$. Arguing as in the first part then gives $D(C) = 0 = D(E)$.

For an arbitrary field, the same argument will work unless $1 + 1 = 0$ in F: then we cannot deduce from $D(E) = -D(E)$ that $D(E) = 0$ (as $a + a = 0$ for any $a \in F$). So the argument will fail in $\mathbb{F}_2$, the field of integers modulo 2. The result also breaks down in this case: for example in $M_2(\mathbb{F}_2)$ we can let $D(C) = 1$ for all rank 1 matrices C and $D(C) = \det(C)$ in other cases, and the required properties hold.

Question 2 For each of the following linear maps $T : V \to V$, choose a basis B for V and compute $[T]_B$. Hence, or otherwise, compute $\det(T)$.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) V is the vector space of all 2×2 matrices over $\mathbb{R}$, and $T(A) = MA$ for all $A \in V$, where $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(iii) V is the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

Solution: (i) The matrix of T with respect to the standard basis is $\begin{pmatrix} -1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$.

So the determinant of T is 2.

(ii) Matrix of T w.r.t. basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$.

So the determinant is $\det(M)^2 = 36$.

(iii) T sends $1 \mapsto 0$, $x \mapsto 3x$, $x^2 \mapsto x + 6x^2$, so matrix of T wrt basis $1, x, x^2$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. Thus $\det(T) = 0$ (which we can also see from the fact that T is singular).

Question 3 Suppose $n \geq 2$ and $A \in M_n(F)$. The adjugate matrix $adj(A)$ is the transpose of the matrix of cofactors of A and we showed that $adj(A)A = \det(A)I_n$. Give an expression for $adj(adj(A))$ in the case where A is invertible.

Solution: Using the given equation, we have $adj(adj(A))adj(A) = \det(adj(A))I_n$ and (taking determinants) $\det(adj(A))\det(A) = \det(A)^n$. So if $\det(A) \neq 0$ we obtain that $adj(A)$ is invertible and

$$adj(adj(A)) = \det(A)^{n-1}(adj(A))^{-1} = \det(A)^{n-2}A.$$

Question 4 Suppose F is a field. Let $n \in \mathbb{N}$ and $a_0, ..., a_{n-1} \in F$, not all zero. Using the Vandermonde determinant, prove that the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

has at most $n-1$ distinct roots in F, i.e. there are at most $n-1$ distinct $\alpha \in F$ such that $f(\alpha) = 0$.

Solution: Suppose $x_1, ..., x_n \in F$ are roots, so $f(x_i) = a_0 + a_1 x_i + \cdots + a_{n-1}x_i^{n-1} = 0$, for $i = 1, ..., n$. Then

$$a_0 \begin{pmatrix} 1 \\ 1 \\ \ddots \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ \ddots \\ x_n \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \ddots \\ x_n^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ddots \\ 0 \end{pmatrix}.$$

Hence the columns of the Vandermonde determinant are linearly dependent so the determinant is 0. Hence $x_i = x_j$ for some $i \neq j$.

Question 5 Suppose U, V, W are vector spaces over a field F and $T : U \to V$ and $S : V \to W$ are linear transformations. Show that the composition $S \circ T : U \to W$ is a linear transformation. If U, V, W are finite dimensional with bases B, C, D, prove that

$${}_D[S \circ T]_B = {}_D[S]_C\, {}_C[T]_B.$$

Solution: To check that $S \circ T$ is a linear transformation, just use the definition.

For the next part, take a deep breath and repeatedly use 4.3.4 from last term:
Let $v \in V$. Then:

$${}_D[S \circ T]_B[v]_B = [S(T(v))]_D = {}_D[S]_C[T(v)]_C = {}_D[S]_C\, {}_C[T]_B[v]_B.$$

As the vector $[v]_B$ is arbitrary, we obtain the required matrix equality.

Question 6 Let V be a vector space over a field F and $T : V \to V$ be a linear transformation. Suppose that $\lambda \in F$ is an eigenvalue of T. Let $m \geq 1$ be an integer and denote by T^m the composition $T \circ \ldots \circ T$ (m times). Note that this is a linear transformation $V \to V$.

i) Show that λ^m is an eigenvalue of T^m.

ii) If $a_0, \ldots, a_m \in F$ are such that $a_0 \operatorname{Id} + a_1 T + a_2 T^2 + \ldots + a_m T^m = 0$, show that λ is a root of the polynomial $p(x) = a_0 + a_1 x + \ldots + a_m x^m$.

Solution:

i) Suppose $0 \neq v \in V$ is an eigenvector with eigenvalue λ. Then $T(v) = \lambda(v)$. Furthermore (see by induction or otherwise) $T^m(v) = \lambda^m(v)$. So v is an eigenvector of T^m with eigenvalue λ^m.

ii) Again, let v be an eigenvector of T with eigenvalue λ. Then

$$0 = 0v = (a_0 \operatorname{Id} + a_1 T + a_2 T^2 + \ldots + a_m T^m)v = a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \ldots + a_m \lambda^m v = p(\lambda) v.$$

As $v \neq 0$ this implies that $p(\lambda) = 0$.

Question 7 Suppose that $T : V \to V$ is a linear map with the property that $T(T(v)) = T(v)$ for all $v \in V$.

(i) Show that

$$V = \ker(T) + \operatorname{im}(T) \text{ and } \ker(T) \cap \operatorname{im}(T) = \{0\}.$$

Hint: Note that if $v \in V$ then $v = (v - T(v)) + T(v)$.

(ii) Deduce that if V is of dimension n, then there is a basis B of V such that

$$[T]_B = \begin{pmatrix} I_s & 0_{r \times n-s} \\ 0_{n-s \times s} & 0_{n-s \times n-s} \end{pmatrix},$$

where $s = \dim(\operatorname{im}(T))$.

Solution: (i) We need to show that $\ker(T) \cap \operatorname{im}(T) = \{0\}$ and that $V = \ker(T) + \operatorname{im}(T)$. Suppose that $v \in \ker(T) \cap \operatorname{im}(T)$. Then there exists $w \in V$ such that $v = T(w)$. Then

$$0 = T(v) = (T(T(w))) = T(w) = v$$

so that $\ker(T) \cap \operatorname{im}(T) = \{0\}$. Now suppose $v \in V$, so that $v = (v - T(v)) + T(v)$. Note that $T(v) \in \operatorname{im}(T)$. Also $T(v - T(v)) = T(v) - T^2(v) = 0$ so $v - T(v) \in \ker(T)$. So $v \in \ker(T) + \operatorname{im}(T)$.

(ii) Note that all (non-zero) vectors in $\ker(T)$ are eigenvectors with eigenvalue 0. Moreover, if $v \in \operatorname{im}(T)$ there is $w \in V$ with $T(w) = v$, so $T(v) = T(T(w)) = T(w) = v$. So all non-zero vectors in $\operatorname{im}(T)$ are eigenvectors with eigenvalue 1. Let $v_1, \ldots, v_t$ be a basis for $ker(T)$ and $w_1, \ldots, w_s$ a basis for $\operatorname{im}(T)$. By rank + nullity $\dim(V) = t + s$ and by (i), $w_1, \ldots, w_s, v_1, \ldots, v_t$ span V. So these $s + t$ vectors form a basis B for V. The matrix $[T]_B$ is of the required form.
[Remark: Such a T is called a *projection*: can you see why?]