Mathematics Year 1, Calculus and Applications I

D.T. Papageorgiou

Problem Sheet 4 Solutions

1. Consider the function $f(x) = 1/x$ for $x \in [1, \infty)$. Calculate and compare the area under the curve, the surface area of the solid formed by revolving $f(x)$ about the $x-$axis, and the volume of the revolved solid. What do you conclude? [The revolved object is called *Gabriel's horn.*]

   Solution

   The area under $y = 1/x$ is $A = \lim_{M\to\infty} \int_1^M (1/x)dx = \lim_{M\to\infty} \log M = \infty$.

   The surface area of revolution is $S = \int_1^\infty \frac{2\pi}{x}\left(1+\frac{1}{x^4}\right)^{1/2} dx > \int_1^\infty \frac{2\pi}{x} dx = \infty$ by the result above. [Note, there is no need to find the exact value of the antiderivative.]

   The volume of revolution is $V = \int_1^\infty \frac{\pi}{x^2} = \pi$.

   Conclusion: Surface area of revolution is infinite but the volume is finite.

2. Find (i) $\int_0^1 \frac{dx}{8x^3+1}$ and (ii) $\int \frac{(1+x)^{3/2}}{x} dx$.

   Solution (i) Note first that $8x^3+1 = (x+\frac{1}{2})(8x^2-4x+2)$ and use partial fractions to write $\frac{1}{8x^3+1} = \frac{A}{x+\frac{1}{2}} + \frac{Bx+C}{8x^2-4x+2}$. Hence $A(8x^2-4x+2) + (x+\frac{1}{2})(Bx+C) = 1$, and equating coefficients of powers of x we find: $8A+B = 0$, $-4A+\frac{1}{2}B+C = 0$, and $2A+\frac{1}{2}C = 1$. Solve to find $A = \frac{1}{6}$, $B = -\frac{4}{3}$, $C = \frac{4}{3}$, hence

$$\int_0^1 \frac{dx}{8x^3+1} = \int_0^1 \left(\frac{1/6}{x+\frac{1}{2}} - \frac{4}{3}\frac{x-1}{(8x^2-4x+2)}\right) dx = \frac{1}{6}\log 3 - \frac{1}{6}\int_0^1 \frac{x-1}{(x-\frac{1}{4})^2+\frac{3}{16}} dx$$

$$= \frac{1}{6}\log 3 - \frac{1}{6}\int_0^1 \frac{(x-\frac{1}{4})-\frac{3}{4}}{(x-\frac{1}{4})^2+\frac{3}{16}} dx = \frac{1}{6}\log 3 - \frac{1}{6}\cdot\frac{1}{2}\log\left[(x-\frac{1}{4})^2+\frac{3}{16}\right]_0^1$$

$$+\frac{1}{8}\cdot\frac{4}{\sqrt{3}}\tan^{-1}\left[(x-\frac{1}{4})\frac{4}{\sqrt{3}}\right]_0^1 = \frac{1}{12}\log 3 + \frac{1}{2\sqrt{3}}\left(\tan^{-1}\sqrt{3} + \tan^{-1}(1/\sqrt{3})\right).$$

   (ii) Make the substitution $1+x = u^2$ (note: allowed to assume $1+x \geq 0$ - why?), to find $dx = 2udu$ and

$$\int \frac{(1+x)^{3/2}}{x} dx = \int \frac{2u^4}{u^2-1} du = 2\int \left(u^2 + \frac{u^2}{u^2-1}\right) du = 2\int \left(u^2+1+\frac{1}{u^2-1}\right) du$$

$$= 2\int \left(u^2+1-\frac{1}{2}\frac{1}{(u+1)}+\frac{1}{2}\frac{1}{(u-1)}\right) du = 2\left(\frac{u^3}{3}+u-\frac{1}{2}\log(u+1)+\frac{1}{2}\log|u-1|\right) + K$$

$$= \frac{2}{3}(1+x)^{3/2} + 2(1+x)^{1/2} + \log\frac{|\sqrt{1+x}-1|}{\sqrt{1+x}+1} + K.$$

3. After a glitch, a manufacturer only produced chains of variable density that start off with unit value but then become a linear function of distance from one end of the chain to the other. An order was delivered but the customer emailed back angrily saying that instead of the chains hanging evenly over their one unit tables, they rested

in such a way that one of the hanging pieces was twice as long as the other hanging piece. What is the density of the chain produced by the malfunctioning machine?

Solution Start by modeling the facts given. Let the density of the chain be ρ. Then we are told that $\rho = 1 + \alpha x$ where α is a constant and x is the distance from one end, the left end say. The chain hangs over the table, hence can be taken to have length $L > 1$. The hanging parts of the chain have total length $L - 1$ (i.e. L minus the length of the table) and since they are in a $2 : 1$ proportion, the left hanging chain has length $\frac{2}{3}(L - 1)$ and the right hanging one has length $\frac{1}{3}(L - 1)$. (The right one is shorter because it is also heavier due to the density law and our choice of origin.) Now balance forces vertically, i.e. write an equation that states that the weight of the left hanging chain equals that of the right hanging one:

$$\int_0^{\frac{2}{3}(L-1)} (1 + \alpha x)dx = \int_{L-\frac{1}{3}(L-1)}^{L} (1 + \alpha x)dx$$

Now integrate and solve for α to find

$$\alpha = \frac{\frac{1}{3}(L - 1)}{\int_{L-\frac{1}{3}(L-1)}^{L} xdx - \int_0^{\frac{2}{3}(L-1)} xdx} = \frac{6(L - 1)}{L^2 + 4L - 5}.$$

Note that $\alpha > 0$ if $L > 1$, a consistency check.

4. Write an integral representing the area of the surface obtained by revolving the graph of $1/(1 + x^2)$ about the $x-$axis. Do not compute the integral but show that it is less than $2\sqrt{5}\pi^2$ no matter how long an interval is taken. Show also that an improved bound is $\sqrt{91}\pi^2/4$.

Solution The surface area is given by

$$S = \int_{-\infty}^{\infty} \frac{2\pi}{1 + x^2} \left[1 + \frac{4x^2}{(1 + x^2)^4}\right]^{1/2} dx$$

Clearly $\frac{x^2}{(1+x^2)^4} \leq 1$, hence

$$S < \int_{-\infty}^{\infty} 2\pi\sqrt{5}\frac{dx}{1 + x^2} = 2\pi \left[\tan^{-1} x\right]_{-\infty}^{\infty} = 2\pi^2\sqrt{5}.$$

For an improved estimate we can maximize the function $f(x) := \frac{x^2}{(1+x^2)^4}$ over $(-\infty, \infty)$. Local maxima/minima satisfy $f'(x) = 0$, i.e.

$$f' = \frac{2x(1 + x^2)^4 - 8x^3(1 + x^2)^3}{(1 + x^2)^8} = 0.$$

One root is $x = 0$ which gives a local minimum (in fact a global minimum), and the others are $x = \pm\frac{1}{\sqrt{3}}$. Due to symmetry, both of these are local maxima, hence $f(x) \leq f(1/\sqrt{3}) = \frac{27}{4^4}$. This implies

$$S < \int_{-\infty}^{\infty} \frac{2\pi}{1 + x^2} \left[1 + \frac{27}{64}\right]^{1/2} dx = \frac{\pi^2\sqrt{91}}{4}.$$

5. (a) Find the volume of the solid obtained by revolving the region under the graph of the function $y = \frac{1}{(1-x)(1-2x)}$ on the interval $[5,6]$ about the $y-$axis.

   (b) Find the centre of mass of the region under $1/(x^2+4)$ on $[1,3]$.

   Solution (a) The volume is given by

$$V = \int_5^6 \frac{2\pi x}{(1-x)(1-2x)} dx.$$

Use partial fractions, i.e. write $\frac{x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$, which gives $x = A(1-2x) + B(1-x)$, hence $A + B = 0$, and $-2A - B = 1$. Solving gives $A = -1$, $B = 1$. Hence

$$V = \int_5^6 2\pi \left(\frac{1}{x-1} - \frac{1}{2x-1} \right) dx = 2\pi \log(5/4) - \pi \log(11/9).$$

(b) The moment of the given area about the $y-$axis is

$$M_y = \int_1^3 xf(x)dx = \int_1^3 \frac{x}{x^2+4} dx = \frac{1}{2} \log(13/5).$$

The moment about the $x-$axis is

$$M_x = \int_1^3 \frac{1}{2}(f(x))^2 dx = \int_1^3 \frac{1}{2} \frac{1}{(x^2+4)^2} dx$$

Calculate using the substitution $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$ and hence

$$M_x = \int \frac{1}{2} \cos^2\theta d\theta = \frac{1}{8} \int (1+\cos 2\theta) d\theta = \frac{1}{8}(\theta + \frac{1}{2}\sin 2\theta) = \frac{1}{8} \left[\tan^{-1}\frac{x}{2} + \frac{2x}{4+x^2} \right]_1^3$$

If the centre os mass is $(\overline{x}, \overline{y})$ then

$$M_y = \overline{x} \int_1^3 \frac{dx}{x^2+4} = \overline{x}\frac{1}{2}\left(\tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{2}\right),$$

$$M_x = \overline{y}\frac{1}{2}\left(\tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{2}\right),$$

with M_y and M_x as found above.

6. As a circle of radius a and centre O rolls along a plane, the position of a point A on the circle's circumference is given parametrically by $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, where θ is the angle that AO makes with the vertical.

   (a) Find the distance travelled by A for $0 \le \theta \le 2\pi$. Is it bigger or smaller than the circle's circumference. Explain your finding.

   (b) Draw a diagram for one arch of the curve traced out by A (it is the cycloid encountered in tests!) and superimpose on it the circle when its centre is at $(\pi a, a)$, together with the line segment $0 \le x \le 2\pi a$ on the $x-$axis. Show that the three enclosed areas are equal.

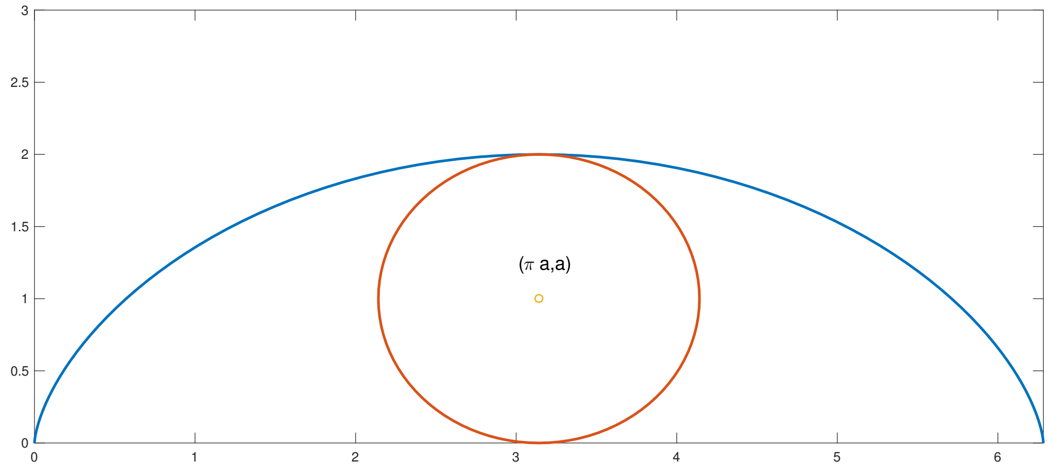

Figure 1: The cycloid and relevant regions in problem 6.

Solution (a) The length of the curve is

$$L = \int_0^{2\pi} \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \right]^{1/2} d\theta = \int_0^{2\pi} a \left[(1-\cos\theta)^2 + \sin^2\theta \right]^{1/2} d\theta$$

$$= \int_0^{2\pi} a\sqrt{2}\,[1-\cos\theta]^{1/2}\, d\theta = \int_0^{2\pi} 2a \sin\frac{\theta}{2}\, d\theta = 8a.$$

Note that $8a > 2\pi a$, the latter being the circumference of the circle. This makes sense since the particle moves along the $x-$axis as it goes around the circle, hence the trajectory is longer than the circumference.

(b) The area beneath the arc is

$$A = \int_0^{2\pi a} y dx = \int_0^{2\pi} y \frac{dx}{d\theta} d\theta = \int_0^{2\pi} a^2 (1-\cos\theta)^2 d\theta = a^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2}) d\theta = a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = 3\pi a^2.$$

The area of the circle is πa^2 and hence by symmetry, the areas on either side of the circle are $\frac{1}{2}(3\pi a^2 - \pi a^2) = \pi a^2$, so all three areas are equal.

7. Find the centre of mass of the triangular region (of uniform density per unit area) with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3). For convenience place the points so that $x_1 \leq x_2 \leq x_3$, $y_1 \leq y_3$ and $y_2 \leq y_3$.

   How does your answer compare with the problem of placing equal masses at the vertices of the triangle?

   Solution A diagram is given in Figure 2. We will need the point $a = x_2$ since the bounding regions change there. First write down the equations of the lines joining

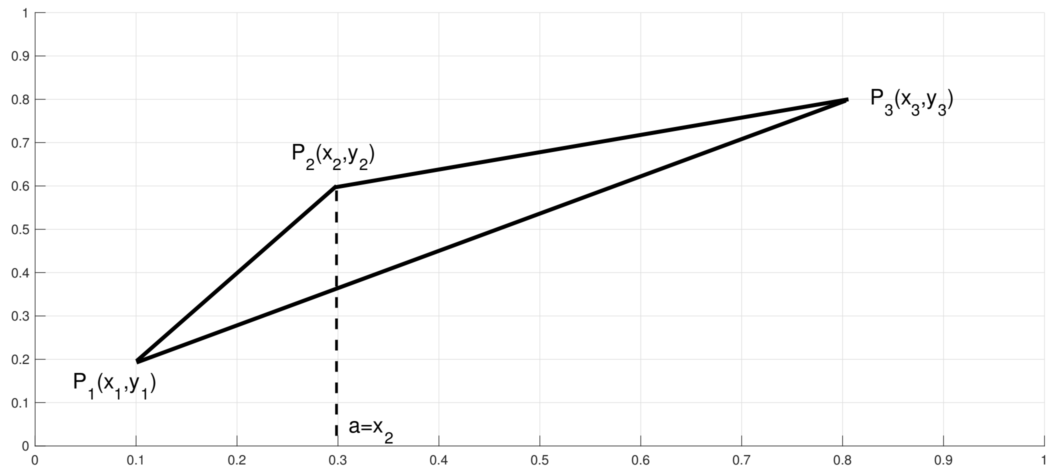

Figure 2: Diagram of the triangle in problem 7.

the three points:

$$\begin{array}{lll} P_1P_2 & \dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1} & y = (x-x_1)\dfrac{y_2-y_1}{x_2-x_1} + y_1 := \ell_{12}(x) \\ P_2P_3 & \dfrac{y-y_2}{x-x_2} = \dfrac{y_3-y_2}{x_3-x_2} & y = (x-x_2)\dfrac{y_3-y_2}{x_3-x_2} + y_2 := \ell_{23}(x) \\ P_1P_3 & \dfrac{y-y_1}{x-x_1} = \dfrac{y_3-y_1}{x_3-x_1} & y = (x-x_1)\dfrac{y_3-y_1}{x_3-x_1} + y_1 := \ell_{13}(x) \end{array}$$

Hence

$$M_y = \int_{x_1}^{x_2} x[\ell_{12}(x) - \ell_{13}(x)]dx + \int_{x_2}^{x_3} x[\ell_{23}(x) - \ell_{13}(x)]dx$$

$$M_x = \int_{x_1}^{x_2} \frac{1}{2}(\ell_{12}(x) + \ell_{13}(x))(\ell_{12}(x) - \ell_{13}(x))dx + \int_{x_2}^{x_2} \frac{1}{2}(\ell_{23}(x) + \ell_{13}(x))(\ell_{23}(x) - \ell_{13}(x))dx$$

$$= \int_{x_1}^{x_2} \frac{1}{2}[(\ell_{12}(x))^2 - (\ell_{13}(x))^2]dx + \int_{x_2}^{x_3} \frac{1}{2}[(\ell_{23}(x))^2 - (\ell_{13}(x))^2]dx$$

The area of the triangle is

$$\Delta = \int_{x_1}^{x_2} [\ell_{12}(x) - \ell_{13}(x)]dx + \int_{x_2}^{x_3} [\ell_{23}(x) - \ell_{13}(x)]dx$$

The centre of mass $(\overline{x}, \overline{y})$ is given by

$$\overline{x} = \frac{M_y}{\Delta} \qquad \overline{y} = \frac{M_x}{\Delta}$$

The centre of mass is the same as placing three equal masses at the vertices P_1, P_2, P_3.

8. Consider a sphere of radius r. Suppose the sphere is sliced into three pieces by two parallel planes that are a distance d apart, where $0 < d < r$. Show that the surface area of the middle piece is the same irrespective of where the cuts are made on the sphere.

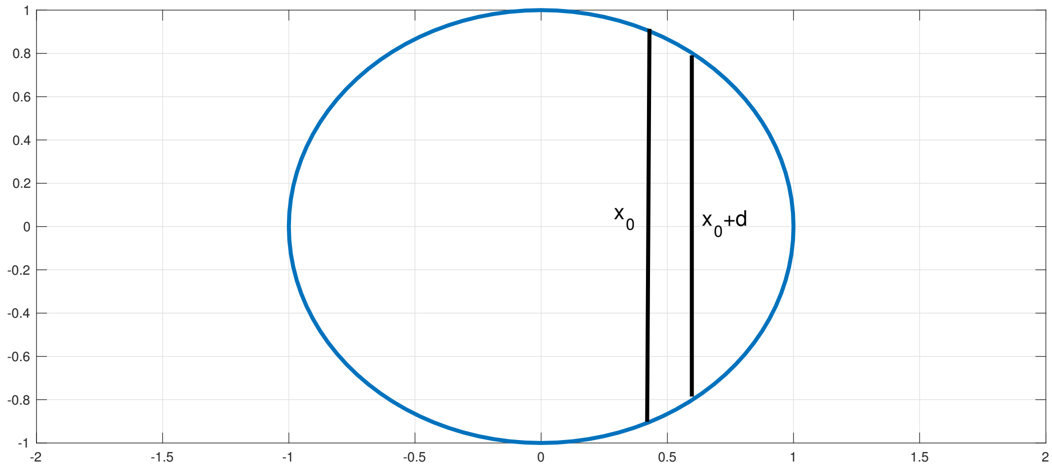

Figure 3: Diagram of the cross section of the sphere in problem 8.

Solution A diagram is given in Figure 3; this shows the cross section of the sphere and the two planes at $x = x_0$ and $x = x_0 + d$. Without loss of generality $x_0 > 0$. One way to find the area of the sphere between the two planes is to revolve the top part of the depicted circle in the interval $0 < x_0 \le x \le x_0 + d < r$ about the $x-$axis. The equation of the top half is $x^2 + y^2 = r^2$ therefore $y = \sqrt{r^2 - x^2}$. This area is

$$S = \int_{x_0}^{x_0+d} 2\pi y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx = 2\pi \int_{x_0}^{x_0+d} \sqrt{r^2 - x^2} \left[1 + \frac{x^2}{r^2 - x^2}\right]^{1/2} dx$$

$$= 2\pi \int_{x_0}^{x_0+d} \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} dx = 2\pi r d,$$

which is independent of x_0.

9. (a) Consider a function $y = f(x)$ with $f(0) = 0$ and assume that its inverse $x = f^{-1}(y)$ exists. The function is rotated about the $y-$axis to produce a solid in the region $0 \le y \le y_0$. Use infinitesimals to show that the desired volume of revolution is

$$V = \pi \int_0^{y_0} [f^{-1}(y)]^2 dy.$$

(b) A bowl is created as described above by rotating $y = f(x)$ about the $y-$axis, and is filled with water to a height h_0. At its bottom ($x = y = 0$) a little hole is bored of radius r, that when open allows for the fluid to drain from the bowl. The speed of the exiting fluid is given by *Torricelli's* [1] *law* that states that at any given instant the speed equals $\sqrt{2gh}$ where h is the instantaneous height of the liquid remaining in the bowl. Formulate a conservation law that describes the physics of the problem, namely, *the rate of change of the volume at any given instant decreases by the rate at which fluid is exiting the small hole of radius* r to derive the equation

$$\frac{dV}{dt} = -\pi r^2 \sqrt{2gh},$$

[1] Evangelista Torricelli (1608-1647) was an Italian physicist and mathematician who is best known for the invention of the barometer

where V is the volume of fluid remaining in the bowl.

(c) Three different bowls are now created to be used as hourglasses. The functions describing the bowls are (using the notation of part (a)) (i) $y = \frac{1}{k}x^2$, where $k > 0$ has dimensions of length, (ii) $y = \alpha x$, and (iii) a hemispherical bowl of radius a centred at $(0, a)$. In all three cases the bowls are filled with liquid to an initial height h_0 (note that $0 < h_0 \leq a$), and have identical small holes of radius r at the bottoms. At $t = 0$ the hole is opened and the bowls are allowed to drain empty. Find α and a so that all three bowls empty at the same time.

Solution (a) Starting with $y = f(x)$ we know that $x = f^{-1}(y)$ exists by assumption. Take an infinitesimal slice of thickness dy between the planes y and $y + dy$. The radius is the rotated cylindrical shell is $f^{-1}(y)$, so its volume is $\Delta V = \pi[f^{-1}(y)]^2$, hence $V = \pi \int_0^{y_0} [f^{-1}(y)]^2 dy$ as required.

(b) Consider how much fluid exits the hole at the bottom of the bowl in time Δt. This is equal to the *fluid speed* $\times$ *area of the hole* $\times$ $\Delta t = \sqrt{2gh}\,\pi r^2\,\Delta t$. This is how much fluid has left, hence the change in volume of the fluid in the bowl in time Δt is $\Delta V = -\sqrt{2gh}\,\pi r^2\,\Delta t$. Note the minus sign, since the volume is decreasing. In the limit $\Delta t \to 0$, $\Delta V \to 0$ we have

$$\frac{dV}{dt} = -\pi r^2 \sqrt{2gh}, \tag{1}$$

as required.

(c) I will begin by finding how much time it takes for the parabolic bowl (i) to drain. The equation is $y = x^2/k$ (hence k has dimensions of length), and so $f^{-1}(y) = \sqrt{ky}$. Suppose that at time t the height of fluid in the bowl is $h(t)$. Using part (a) we can find the volume in the bowl

$$V = \pi \int_0^h ky dy = \pi \frac{h^2}{2} k.$$

This implies that equation (1) becomes

$$\pi k h \frac{dh}{dt} = -\pi r^2 \sqrt{2gh} \quad \Rightarrow \quad k \frac{dh}{dt} = -r^2 \sqrt{2g}\, h^{-1/2},$$

with initial condition $h(0) = h_0$. Integrating we have

$$k \int_{h_0}^h h dh = -r^2 \sqrt{2g} \int_0^t dt \quad \Rightarrow \quad \frac{2}{3} k (h^{3/2} - h_0^{3/2}) = -r^2 t \sqrt{2g}$$

$$\Rightarrow \qquad k h^{3/2} = k h_0^{3/2} - \frac{3}{2} r^2 t \sqrt{2g}$$

The bowl empties when $h = 0$, i.e. after time t_E given by

$$t_E = \frac{2}{3} \frac{h_0^{3/2} k}{r^2 \sqrt{2g}}$$

[As an aside, we can check the dimension of this quantity and find that it is indeed time.]

For the conical bowl (ii) we have $y = \alpha x$ hence $x = f^{-1}(y) = y/\alpha$, where α is a dimensionless constant. Hence

$$V = \int_0^h \pi \frac{y^2}{\alpha^2} dy = \frac{\pi h^3}{3\alpha^2}, \tag{2}$$

and hence equation (1) becomes

$$h^{3/2}\frac{dh}{dt} = -\alpha^2 r^2 \sqrt{2g} \qquad \Rightarrow \qquad \frac{2}{5}h^{5/2} = -\alpha^2 r^2 t\sqrt{2g} + \frac{2}{5}h_0^{5/2},$$

where the initial condition $h(0) = h_0$ has been used. Hence the time to empty is

$$t_E^{(ii)} = \frac{2}{5}\frac{h_0^{5/2}}{\alpha^2 r^2 \sqrt{2g}}.$$

For the spherical bowl (c) the equation describing the curve to be revolved about the $y-$axis is $x^2 + (y-a)^2 = a^2$ (a circle centered at $(0,a)$ and of radius a). Hence $x = f^{-1}(y) = [a^2 - (y-a)^2]^{1/2}$, and so the volume of the bowl is

$$V = \int_0^h \pi[a^2 - (y-a)^2]dy = \pi[a^2 y - \frac{(y-a)^3}{3}]_0^h = \pi\left[a^2 h - \frac{1}{3}(h-a)^3 - \frac{a^3}{3}\right]$$
$$= \pi h^2\left(a - \frac{h}{3}\right).$$

Hence, equation (1) becomes

$$\pi(2ah - h^2)\frac{dh}{dt} = -\pi r^2 h^{1/2}\sqrt{2g} \quad \Rightarrow \quad (2ah^{1/2} - h^{3/2})\frac{dh}{dt} = -r^2\sqrt{2g}.$$

Integrate using $h(0) = h_0$ to find

$$\frac{4}{3}ah^{3/2} - \frac{2}{5}h^{5/2} - \left(\frac{4}{3}ah_0^{3/2} - \frac{2}{5}h_0^{5/2}\right) = -r^2\sqrt{2g}\,t,$$

hence the emptying time is

$$t_E^{(iii)} = \left(\frac{4}{3} - \frac{2}{5}h\right)\frac{h_0^{3/2}}{r^2\sqrt{2g}}.$$

[Note, as a check, that $4a/3 - 2h_0/5 > 0$, i.e. $t_E^{(iii)} > 0$ as expected.]

Now for all bowls to empty at the same time we need

$$\frac{2}{3}\frac{h_0^{3/2}k}{r^2\sqrt{2g}} = \frac{2h_0^{5/2}}{5\alpha^2 r^2\sqrt{2g}} \qquad \Rightarrow \qquad \alpha = \left(\frac{3h_0}{5k}\right)^{1/2},$$
$$\frac{2kh_0^{3/2}}{3r^2\sqrt{2g}} = \left(\frac{4a}{3} - \frac{2h_0}{5}\right)\frac{h_0^{3/2}}{r^2\sqrt{2g}} \qquad \Rightarrow \qquad a = \frac{k}{2} + \frac{3h_0}{10}.$$