1(a) Find the incidence matrix A for each graph.

Graph I

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \tag{1}$$

Graph II

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Graph III

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{3}$$

(b) For each graph, find all vectors in the nullspaces of A and $\mathbf{A}^\top$:

In this question, any vector in these spaces can be written as a *linear combination* of the basis vectors which are as follows:

Graph I ($\mathbf{A}$)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4}$$

Graph I ($\mathbf{A}^\top$) Note that the right null vector of $\mathbf{A}^\top$ corresponds to left null-vectors of $\mathbf{A}$ (just take a transpose) which correspond to loops in the graph.

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{5}$$

Graph II ($\mathbf{A}$)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{6}$$

Graph II ($\mathbf{A}^\top$)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{7}$$

Graph III ($\mathbf{A}$)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{8}$$

Graph III ($\mathbf{A}^\top$)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

(c) Find the degree matrix D for each graph.

Graph I

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{10}$$

Graph II

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{11}$$

Graph III

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{12}$$

(d) Find the adjacency matrix W for each graph:

Graph I

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{13}$$

Graph II

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{14}$$

Graph III

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{15}$$

(e) Find the Laplacian matrix K for each graph.

Note that graph Laplacian is given by $\mathbf{K} \equiv \mathbf{A}^\top \mathbf{A}$ or $\mathbf{K} = \mathbf{D} - \mathbf{W}$. It is easy to deduce these from the previous answers.

(f) Are any of the graphs complete?

Completeness of graphs can be checked by examining if all nodes are connected to each other.

Graph I: Complete

Graph II: Not complete

Graph III: Complete

2(a): What is the rank of the incidence matrix $\mathcal{A}$ of this new single graph? We define incident matrices of Graph I, II, III as $\mathbf{A}^{\mathrm{I}}$, $\mathbf{A}^{\mathrm{II}}$, and $\mathbf{A}^{\mathrm{III}}$, respectively. Because these graphs are disconnected, the incidence matrix $\mathcal{A}$ of this new single graph is written by

$$\mathcal{A} = \begin{pmatrix} \mathbf{A}^{\mathrm{I}} & & \mathcal{O} \\ & \mathbf{A}^{\mathrm{II}} & \\ \mathcal{O} & & \mathbf{A}^{\mathrm{III}} \end{pmatrix} \tag{16}$$

where $\mathcal{O}$ is short-hand for filling in the rest of the matrix with zero elements.

The ranks of $\mathbf{A}^{\mathrm{I}}$, $\mathbf{A}^{\mathrm{II}}$, and $\mathbf{A}^{\mathrm{III}}$ are 2, 3, and 3 respectively, so the rank of $\mathcal{A}$ is **8**.

(b) Find all linearly independent solutions of $\mathcal{A}\boldsymbol{x} = 0$.

From 1(b), we collect all vectors in the nullspaces of $\mathbf{A}$ of each graph I–III and pad them with an appropriate set of zeros (to make up a 11-dimensional vector). The (right) null-vectors of $\mathcal{A}$ are

$$\boldsymbol{x} = \begin{pmatrix} 1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\1\\1\\1 \end{pmatrix}. \tag{17}$$

(c) Find all linearly independent solutions of $\mathcal{A}^{\top}\boldsymbol{w} = 0$.

We define the i-th right null vector of graph j ($j = \mathrm{I}, \mathrm{II}, \mathrm{III}$) as $\boldsymbol{v}_i^j$ $(i = 1, 2, \cdots)$. For example,

$$\boldsymbol{v}_1^{\mathrm{I}} = \begin{pmatrix} 1\\1\\-1 \end{pmatrix}, \boldsymbol{v}_1^{\mathrm{II}} = \begin{pmatrix} 1\\0\\0\\1\\-1 \end{pmatrix}, \boldsymbol{v}_1^{\mathrm{III}} = \begin{pmatrix} 1\\0\\0\\-1\\1\\0 \end{pmatrix}. \tag{18}$$

Because graphs I, II, and III are disconnected, linearly independent solutions (the left null-vectors of $\mathcal{A}$) are

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{I}} \\ \mathbf{O}_5 \\ \mathbf{O}_6 \end{pmatrix}, \begin{pmatrix} \mathbf{O}_3 \\ \boldsymbol{v}_1^{\mathrm{II}} \\ \mathbf{O}_6 \end{pmatrix}, \begin{pmatrix} \mathbf{O}_3 \\ \boldsymbol{v}_2^{\mathrm{II}} \\ \mathbf{O}_6 \end{pmatrix}, \begin{pmatrix} \mathbf{O}_3 \\ \mathbf{O}_5 \\ \boldsymbol{v}_1^{\mathrm{III}} \end{pmatrix}, \begin{pmatrix} \mathbf{O}_3 \\ \mathbf{O}_5 \\ \boldsymbol{v}_2^{\mathrm{III}} \end{pmatrix}, \begin{pmatrix} \mathbf{O}_3 \\ \mathbf{O}_5 \\ \boldsymbol{v}_3^{\mathrm{III}} \end{pmatrix} \tag{19}$$

where $\mathbf{O}_n$ denotes a zero vector with n elements. There are 6 of these, in accordance with rank-nullity $(14 - 8 = 6)$.

3(a) We know that the diagonal element of the Laplacian matrix K_{ii} is the number of edges of each node, and the off-diagonal element K_{ij} is -1 if node i is connected to node j, all other elements are zero.

The quickest way to compute the number of zero elements is to count the non-zero elements as follows. Notice that there are 9 nodes and 12 edges. All diagonal elements will be non-zero, and there are 9 of these; each edge produce 2 non-zero elements because a -1 will appear in K_{ij} (for $i \neq j$) as well as in K_{ji}. Hence the total number of non-zero elements is

$$9 + 12 \times 2 = 33. \tag{20}$$

The number of non-zero elements is therefore

$$81 - 33 = 48. \tag{21}$$

Another argument goes as follows. For each row of $\mathbf{K}$, corresponding to a given node, the number of zeros in that row is the number of nodes that are **not** connected to the given node by an edge. By the symmetry of this graph there are 3 types of nodes: 4 corner nodes, 4 nodes in the middle of each side, and one central node. Each corner node is **not** connected to 6 other nodes; each middle-side node is **not** connected to 5 other nodes; the central node is **not** connected to 4 nodes. The total number of zeros in the Laplacian is therefore

$$\underbrace{4 \times 6}_{\text{from corner nodes}} + \underbrace{4 \times 5}_{\text{from middle-side nodes}} + \underbrace{1 \times 4}_{\text{from middle node}} = 48. \tag{22}$$

(b) The degree matrix is the diagonal matrix containing the number of nodes connected to each node. Using the node numbering given in the figure:

$$D_0 = \mathrm{diag}(2, 3, 2, 3, 4, 3, 2, 3, 2). \tag{23}$$

4. Consider a graph which has n nodes and n edges. Then it must have at least one connected subgraph and hence the graph's n-by-n incidence matrix $\mathbf{A}$ will have a corresponding right null vector with all ones in components corresponding to the nodes in this connected subgraph (and zeros elsewhere) meaning that the rank is $n - 1$ or less. By rank-nullity there must therefore be at least one left null-vector, or equivalently, a vector in the nullspace of $\mathbf{A}^\top$. This corresponds to a loop.

5. Note that $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$ and $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$.

(a) Write down the incidence matrix A and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^\top \mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{24}$$

(b)

$$\mathbf{K}\boldsymbol{x_n} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix} = (2 - \omega^n - \omega^{2n}) \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix} \tag{25}$$

Therefore, $\lambda_n = 2 - \omega^n - \omega^{2n}$.

$$\lambda_0 = 2 - \omega^0 - \omega^0 = 0, \tag{26}$$
$$\lambda_1 = 2 - \omega - \omega^2 = 3, \tag{27}$$
$$\lambda_2 = 2 - \omega^2 - \omega^4 = 3. \tag{28}$$

(c) Why did we only consider the three possible values $n = 0, 1, 2$ in part (b)?

For any integer n, $n \geq 0$, we can classify $n = 3k$, $n = 3k+1$, $n = 3k+2$. It is easy to check $\lambda_{3k} = \lambda_0$, $\lambda_{3k+1} = \lambda_1$, and $\lambda_{3k+2} = \lambda_2$.

(d)

$$\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \ \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \tag{29}$$

It is easy to check that both $\overline{\boldsymbol{x_0}}^\top \boldsymbol{x}_1$ and $\overline{\boldsymbol{x_0}}^\top \boldsymbol{x}_2$ equal zero because $1+\omega+\omega^2 = 0$. Furthermore,

$$\overline{\boldsymbol{x}}_1^\top \boldsymbol{x}_2 = 1 + \bar{\omega}\omega^2 + \overline{\omega^2}\omega = 1 + \omega + \frac{1}{\omega} = \frac{1}{\omega}(1 + \omega + \omega^2) = 0, \tag{30}$$

where the relations of $\bar{\omega}\omega = \overline{\omega^2}\omega^2 = 1$ are used.

6–8: These problems are all based on the same idea as question 5 except that the number of nodes, N, changes from $N = 4$ to $N = 6$; the idea is for the students to see the pattern emerging (these facts will be used later in the course). For example, in question 6, where $N = 4$,

$$\lambda_n = 2 - \omega^n - \omega^{3n}, \qquad n = 0, 1, 2, 3 \tag{31}$$

where $\omega = e^{2\pi \mathrm{i}/4}$ and we can use $1+\omega+\omega^2+\omega^3 = 0$. The vectors $\mathbf{x}_n$ ("eigenvectors") will be

$$\boldsymbol{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix} \qquad n = 0,1,2,3. \tag{32}$$

For general N we find

$$\lambda_n = 2 - \omega^n - \omega^{(N-1)n} = 2 - \omega^n - \frac{1}{\omega^n}, \qquad n = 0,1,2,\ldots,N-1, \tag{33}$$

where $\omega = e^{2\pi \mathrm{i}/N}$.

9(a) The general form of $\mathbf{K}$ is the n-by-n matrix

$$\mathbf{K} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & \cdots & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ -1 & -1 & \cdots & \cdots & n-1 & -1 \\ -1 & -1 & \cdots & \cdots & -1 & n-1 \end{bmatrix}. \tag{34}$$

(b) Notice that $\mathbf{K}$ can be written

$$\mathbf{K} = n\mathbf{I} - \mathbf{J}, \tag{35}$$

where $\mathbf{I}$ is the n-by-n identity matrix and $\mathbf{J}$ is the rank-one n-by-n matrix of all ones:

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}. \tag{36}$$

By rank-nullity, since $\mathbf{J}$ has rank 1, it has $n-1$ (right) null vectors which are easy to work out. They are the following n dimensional vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ . \\ . \\ 0 \end{pmatrix}, \quad \cdots, \mathbf{x}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ 1 \\ -1 \end{pmatrix}. \tag{37}$$

These satisfy

$$\mathbf{J}\mathbf{x}_j = 0, \qquad j = 1, \cdots, n-1. \tag{38}$$

Moreover the vector

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}, \tag{39}$$

clearly satisfies

$$\mathbf{J}\mathbf{x}_0 = n\mathbf{x}_0. \tag{40}$$

By (43) the vectors $\{\mathbf{x}_j | j = 0, 1, \cdots, n-1\}$ just found satisfy

$$\mathbf{K}\mathbf{x}_0 = (n\mathbf{I} - \mathbf{J})\mathbf{x}_0 = n\mathbf{x}_0 - n\mathbf{x}_0 = 0, \tag{41}$$

and

$$\mathbf{K}\mathbf{x}_j = (n\mathbf{I} - \mathbf{J})\mathbf{x}_j = n\mathbf{x}_j, \qquad j = 1, \cdots, n-1. \tag{42}$$

Hence we have found the required vectors and associated values of $\lambda = 0, \underbrace{n, n, \cdots, n}_{n-1 \text{ times}}$.

(c) The general form of $\mathbf{K}_0$ is the same as $\mathbf{K}$ except you can remove the last column and row making it now an $(n-1)$-by-$(n-1)$ matrix.

(d) The arguments here are exactly as in part (b). We write

$$\mathbf{K}_0 = n\hat{\mathbf{I}} - \hat{\mathbf{J}}, \tag{43}$$

where $\hat{\mathbf{I}}$ is the $(n-1)$-by-$(n-1)$ identity matrix and $\hat{\mathbf{J}}$ is the rank-one $(n-1)$-by-$(n-1)$ matrix of all ones. Now it can be shown that the modified $(n-1)$-dimensional vectors given by

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ . \\ . \\ 0 \end{pmatrix}, \quad \cdots, \hat{\mathbf{x}}_{n-2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ 1 \\ -1 \end{pmatrix}. \tag{44}$$

(there are $(n-2)$ of these) satisfy

$$\hat{\mathbf{J}}\hat{\mathbf{x}}_j = 0, \qquad j = 1, \cdots, n-2. \tag{45}$$

and the $(n-1)$-dimensional vector

$$\hat{\mathbf{x}}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}, \tag{46}$$

clearly satisfies

$$\hat{\mathbf{J}}\hat{\mathbf{x}}_0 = (n-1)\hat{\mathbf{x}}_0. \tag{47}$$

Hence the vectors $\{\hat{\mathbf{x}}_j | j = 0, 1, \cdots, n-1\}$ just found satisfy

$$\mathbf{K}_0\hat{\mathbf{x}}_0 = (n\hat{\mathbf{I}} - \hat{\mathbf{J}})\hat{\mathbf{x}}_0 = n\hat{\mathbf{x}}_0 - (n-1)\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0, \tag{48}$$

and

$$\mathbf{K}_0\hat{\mathbf{x}}_j = (n\hat{\mathbf{I}} - \hat{\mathbf{J}})\hat{\mathbf{x}}_j = n\hat{\mathbf{x}}_j, \qquad j = 1, \cdots, n-1. \tag{49}$$

Hence we have found the required vectors and associated values of $\lambda = 1, \underbrace{n, n, \cdots, n}_{n-2 \text{ times}}$.

(e) By explicitly computing the inverses of $\mathbf{K}_0$ for $n = 2, 3, 4$ we can guess that, for general n,

$$\mathbf{K}_0^{-1} = \frac{1}{n}\begin{bmatrix} 2 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 2 & \cdots & \cdots & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & \cdots & \cdots & 1 & 2 \end{bmatrix}. \tag{50}$$

and it is easily verified that this is the correct inverse: i.e., check that $\mathbf{K}_0\mathbf{K}_0^{-1} = \mathbf{I}$.