1

Calculus and Applications:

Unseen 1 Solutions

1. Differentiating with respect to x.

we obtain:

$$\frac{dg(x)}{dx} = -\frac{x}{\sigma^2} g(x)$$

Taking Fourier transforms from both

sides and using property (vii) and (viii)

from the lectur notes we have:

$$\mathcal{F}\left\{\frac{dg(x)}{dx}\right\} = \frac{-1}{\sigma^2} \mathcal{F}\{x g(x)\}$$

$$i w \hat{g}(w) = -\frac{i}{\sigma^2} \frac{d\hat{g}(w)}{dw} \Rightarrow$$

$$\int_0^w \frac{d\hat{g}(w')}{\hat{g}(w')} = -\int_0^w w' \sigma^2 dw' \Rightarrow$$

$$\ln G(w) - \ln G(0) = -\frac{\sigma^2 w^2}{2}$$

Since the Gaussian distribution is normalized

$G(0) = 0$, thus we have

$$\ln G(w) = \frac{-\sigma^2 w^2}{2} \Rightarrow \boxed{G(w) = e^{-\frac{\sigma^2 w^2}{2}}}$$

as required.

2

2. To prove

$$\int_{-\infty}^{\infty} f'(x) f'^*(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 \hat{f}(\omega)\hat{f}^*(\omega)\, d\omega \qquad (*)$$

We start from the LHS and use energy theorem

$$\int_{-\infty}^{\infty} f'(x) f'^*(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{f'(x)\} F\{f'(x)\}^*\, d\omega$$

(using property vii from lectures)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -(i\omega)(i\omega)\hat{f}(\omega)\hat{f}^*(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 \hat{f}(\omega)\hat{f}^*(\omega)\, d\omega$$

as required.

To prove the Schwarz's inequality we use:

$$\int (F+\varepsilon G)(F+\varepsilon G)^* dx \geq 0$$

$$\int FF^* dx + \varepsilon\int (F^*G + FG^*)\, dx + \varepsilon^2 \int GG^* dx > 0$$

The LHS is a quadratic expression in ε, call it $c + b\varepsilon + a\varepsilon^2$.

The condition is satisfied if $b^2 - 4ac < 0$, which gives us the Schwarz's inequality for complex functions.

To prove the uncertainty principle, we use definitions of $(\Delta x)^2$ and $(\Delta \omega)^2$. We have:

(3)

$$(\Delta x)^2(\Delta w)^2 = \frac{\int x^2 f f^* \, dx \int w^2 \hat{f} \hat{f}^* \, dw}{\int f f^* \, dx \int \hat{f} \hat{f}^* \, dw}$$

using (*) and energy theorem

$$= \frac{\int x f . x f^* \, dx \int f' f'^* \, dx}{\left(\int f f^* \, dx\right)^2}$$

using Shwarz's inequality

$$\geq \frac{\left[\int (x f^* . f' + x f . f'^*) \, dx\right]^2}{4\left(\int f f^* \, dx\right)^2}$$

using integration by parts

$$= \frac{\left[\int x \frac{d}{dx}(f f^*) \, dx\right]^2}{4\left(\int f f^* \, dx\right)^2}$$

$$= \frac{\left[\cancel{\int f f^* \, dx}\right]^2}{4\left[\cancel{\int f f^* \, dx}\right]^2} = \frac{1}{4} \Rightarrow$$

$$\boxed{\Delta x \, \Delta S \geq \frac{1}{2}}$$

as required.

Using $g(x)$ and $\hat{g}(w)$ from part 1 for Gaussian and the following integrals : $\int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}}$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$

We obtain :

$$(\Delta x)^2(\Delta w)^2 = \frac{\sigma^2}{2} . \frac{1}{2\sigma^2} = \frac{1}{4} \Rightarrow \boxed{\Delta x \, \Delta w = \frac{1}{2}}$$