Mathematics Year 1, Calculus and Applications I

Challenge Problems 2

These are a little more physically inspired problems but with a rich mathematical history behind them as I will point out.

1. **The Catenary - *linea catenaria funicularis*** [1]

   Find the shape assumed by a flexible chain under the action of gravity, when it is hung from its two ends. A schematic of the problem is given in the figure - the chain is in blue and hangs from A and B.

   Hint: As shown in the figure, take an arbitrary piece PQ of the chain. The forces at P and Q are shown schematically (the force $\boldsymbol{T}(x)$ is tangent to the curve and the angle θ to the horizontal has been defined as shown). The force $\boldsymbol{T_c}$ represents the weight of the chain piece PQ.

   (a) Balance forces and come up with a differential equation that you need to solve.

   (b) Assume that the density of the chain, ρ say, is constant. Then you should be able to solve the equation and get the shape in closed form (define your preferred geometry - I picked everything to me 1 but that of course is a schematic - the only things you need to prescribe are (i) the chain length L, (ii) the distance AB; clearly $AB < L$).

   (c) What happens if ρ is not constant, e.g. take the density to be constant and ρ_0 in the middle third of the chain and another constant ρ_1 in the two outer thirds of the chain. If $\rho_1 = \rho_2$ you get the answer in part (b).

[1] Historical comments: Galileo wrote in 1638 that the curve is a parabola to within tiny errors (*ad unguem*). He is of course wrong, as was pointed out about 20 years later by Huygens. A solution was finally given independently by Leibniz and Johann Bernoulli in 1691. If your scientific Latin is reasonable, let me know and I can give you the references!

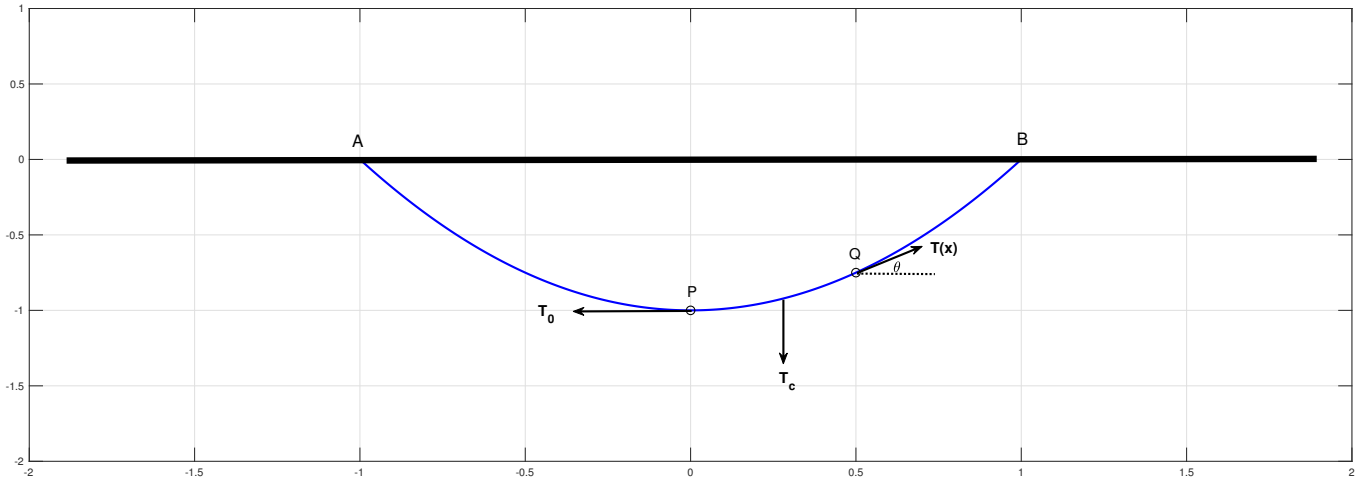

Figure 1: Schematic of the hanging chain.