1. (a) Prove Jensen's inequality: Let $f : I \to \mathbb{R}$ be a convex function on an interval I. Let $x_1, \dots, x_k \in I$ and let $a_1, \dots, a_k > 0$. Then

$$f\left(\frac{\sum_{i=1}^k a_i x_i}{\sum_{i=1}^k a_i}\right) \le \frac{\sum_{i=1}^k a_i f(x_i)}{\sum_{i=1}^k a_i}$$

When is this an equality? **By defining $a'_j := a_j / (\sum_{i=1}^k a_i)$, we may assume $\sum_{i=1}^k a_i = 1$. So we need to prove $f\left(\sum_{i=1}^k a_i x_i\right) \le \sum_{i=1}^k a_i f(x_i)$. We prove the claim holds by induction on k. The case of $k = 1$ is trivial. For $k+1$, assuming the claim holds for k:**

$$f\left(\sum_{i=1}^{k+1} a_i x_i\right) = f\left(a_{k+1}x_{k+1} + \sum_{i=1}^{k} a_i x_i\right) =$$
$$f\left(a_{k+1}x_{k+1} + (1-a_{k+1})\sum_{i=1}^{k} \frac{a_i}{1-a_{k+1}} x_i\right) \le$$
$$a_{k+1} f\left(x_{k+1}\right) + (1-a_{k+1}) f\left(\sum_{i=1}^{k} \frac{a_i}{1-a_{k+1}} x_i\right) \le$$
$$a_{k+1} f\left(x_{k+1}\right) + \frac{1-a_{k+1}}{1-a_{k+1}} \sum_{i=1}^{k} a_i f\left(x_i\right) = \sum_{i=1}^{k+1} a_i f\left(x_i\right).$$

(b) Prove the inequality of arithmetic and geometric means (AM-GM inequality): Let $x_1, \dots, x_n \ge 0$, then

$$\sqrt[n]{x_1 \cdot \dots \cdot x_n} \le \frac{x_1 + \dots x_n}{n}.$$

$-\ln(x)$ is convex. (e.g., it is a negative of an inverse of a convex function e^x.) Therefore, Jensen's inequality holds for $a_1 = a_2 = \dots = a_n = 1$:

$$-\ln\left(\frac{\sum_{i=1}^n x_i}{n}\right) \le \frac{-\sum_{i=1}^n \ln(x_i)}{n} \iff e^{-\ln\left(\frac{\sum_{i=1}^n x_i}{n}\right)} \le e^{\frac{-\sum_{i=1}^n \ln(x_i)}{n}} \iff$$
$$\frac{1}{\frac{\sum_{i=1}^n x_i}{n}} \le \frac{1}{\sqrt[n]{x_1 \cdot \dots \cdot x_n}} \iff \sqrt[n]{x_1 \cdot \dots \cdot x_n} \le \frac{x_1 + \dots x_n}{n}$$

2. Let $I \subseteq$ be some interval. $f : I \to \mathbb{R}$ is *halving convex* if for all $x_1, x_2 \in I$:

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \le \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(a) Prove that if $f : I \to \mathbb{R}$ is halving convex, then for every $k, n \in \mathbb{N}$ such that $t = \frac{k}{2^n} \in [0,1]$:

$$\forall x_1, x_2 \in I : f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

By induction on n. The case $n = 1, k = 1$ is satisfied by definition and the case $n = 1, k \in 0, 2$ is trivial.

Assume the consequence holds for n. Let $k \in \mathbb{N}$ such that $\frac{k}{2^{n+1}} \in [0,1]$.

If $k = 2m$ is even, then $k/2^{n+1} = m/2^n$, and we are done, by the induction by the induction hypothesis.

If $k = 2m+1$ is odd, then $k = \frac{4m+2}{2}$. Since $k/2^{n+1} \le 1$ and k does not divide 2^{n+1}, it follows that $k/2^{n+1} < 1$. Therefore $\frac{2m+2}{2^{n+2}} = \frac{k+1}{2^{n+1}} \le 1$. So

$$f\left(\frac{k}{2^{n+1}}x_1 + \left(1 - \frac{k}{2^{n+1}}\right)x_2\right) =$$
$$f\left(\frac{4m+2}{2^{n+2}}x_1 + \left(1 - \frac{4m+2}{2^{n+2}}\right)x_2\right) =$$
$$f\left(\frac{1}{2}\cdot\frac{4m+2}{2^{n+1}}x_1 + \frac{1}{2}\left(2 - \frac{4m+2}{2^{n+1}}\right)x_2\right) =$$
$$f\left(\frac{1}{2}\left(\frac{2m}{2^{n+1}}x_1 + \left(1 - \frac{2m}{2^{n+1}}\right)x_2\right) + \frac{1}{2}\left(\frac{2m+2}{2^{n+1}}x_1 + \left(1 - \frac{2m+2}{2^{n+1}}\right)x_2\right)\right) \le$$
$$\frac{1}{2}f\left(\frac{2m}{2^{n+1}}x_1 + \left(1 - \frac{2m}{2^{n+1}}\right)x_2\right) + \frac{1}{2}f\left(\frac{2m+2}{2^{n+1}}x_1 + \left(1 - \frac{2m+2}{2^{n+1}}\right)x_2\right) \le \textbf{(by the even case)}$$
$$\frac{1}{2}\left(\frac{2m}{2^{n+1}}f(x_1) + \left(1 - \frac{2m}{2^{n+1}}\right)f(x_2) + \frac{2m+2}{2^{n+1}}f(x_1) + \left(1 - \frac{2m+2}{2^{n+1}}\right)f(x_2)\right) =$$
$$\frac{2m+1}{2^{n+1}}f(x_1) + \left(1 - \frac{2m+1}{2^{n+1}}\right)f(x_2).$$

Notice that the following proof is faulty, can you say why:

$$f\left(\frac{k}{2^{n+1}}x_1 + \left(1 - \frac{k}{2^{n+1}}\right)x_2\right) = f\left(\frac{k}{2^n}\cdot\frac{1}{2}x_1 + \left(\frac{1}{2} - \frac{k}{2^{n+1}}\right)x_2 + \frac{1}{2}x_2\right) =$$
$$f\left(\frac{1}{2}\left(\frac{k}{2^n}x_1 + \left(1 - \frac{k}{2^n}\right)x_2\right) + \frac{1}{2}x_2\right) \le$$
$$\frac{1}{2}f\left(\frac{k}{2^n}x_1 + \left(1 - \frac{k}{2^n}\right)x_2\right) + \frac{1}{2}f(x_2) \le \textbf{(induction hypothesis)}$$
$$\frac{1}{2}\left(\frac{k}{2^n}f(x_1) + \left(1 - \frac{k}{2^n}\right)f(x_2)\right) + \frac{1}{2}f(x_2) =$$
$$\frac{1}{2}\cdot\frac{k}{2^n}f(x_1) + \frac{1}{2}\cdot\left(1 - \frac{k}{2^n}\right)f(x_2) + \frac{1}{2}f(x_2) = \frac{k}{2^{n+1}}f(x_1) + \left(1 - \frac{k}{2^{n+1}}\right)f(x_2).$$

(b) Prove that if $f : I \to \mathbb{R}$ is halving convex and continuous, then it is convex.

There are several ways to see this, e.g., with epsilons and deltas. Perhaps one of the simplest is: Let $t \in [0,1]$, $x_1, x_2 \in I$. Let $a_n \to t$ such that $a_n = \frac{k_n}{2^n}$ for some $k_n \in \mathbb{N}$. (Why does there exist such a sequence?)

Then $a_n x_1 + (1-a_n)x_2 \to tx_1 + (1-t)x_2$. So, by continuity,

$$f(tx_1 + (1-t)x_2) = f(\lim_{a_n \to t} a_n x_1 + (1-a_n)x_2) = \lim_{a_n \to t} f(a_n x_1 + (1-a_n)x_2) \leq$$
$$\lim_{a_n \to t} a_n f(x_1) + (1-a_n) f(x_2) = tf(x_1) + (1-t)f(x_2).$$

3. Prove that a convex function on an open interval is continuous. Is it true for a closed interval?

   Let $f: I \to \mathbb{R}$ be convex on an open interval I. Let $a \in I$. So there are $b_1, b_2 \in I$ such that $a \in (b_1, b_2) \subseteq I$. Let g_i be the linear function such that $g_i(a) = f(a)$ and $g_i(b_i) = f(b_i)$. Then for every $x \in (b_1, a)$: $g_2(x) \leq f(x) \leq g_1(x)$ and for every $x \in (a, b_2)$: $g_1(x) \leq f(x) \leq g_2(x)$. (Why?) It follows that

$$f(a) = g_2(a) = \lim_{x \to a^-} g_2(x) \leq \lim_{x \to a^-} f(x) \leq \lim_{x \to a^-} g_1(x) = g_1(a) = f(a)$$

   and

$$f(a) = g_1(a) = \lim_{x \to a^+} g_1(x) \leq \lim_{x \to a^+} f(x) \leq \lim_{x \to a^+} g_2(x) = g_2(a) = f(a).$$

   So $\lim_{x \to a} f(x) = a$.

   For a closed interval, we can have a jump at an endpoint, e.g., $f : [0,1] \to \mathbb{R}$ defined as $f(x) = \begin{cases} 1 & \textbf{if } x = 1 \\ 0 & \textbf{if } x \neq 1 \end{cases}$.