**Question 1**

(a) Prove that for any random variable $X$ and any constant $a \in \mathbb{R}$,

$$\mathrm{Cov}\,(X, a) = 0.$$

(b) Prove that for any random variable $X$, $Y$ and $Z$, and constants $a, b \in \mathbb{R}$,

$$\mathrm{Cov}\,(aX + bY, Z) = a\mathrm{Cov}\,(X, Z) + b\mathrm{Cov}\,(Y, Z)\,.$$

(c) For any random variables $X$ and $Y$, and constants $a, b \in \mathbb{R}$, find an expression for

$$\mathrm{Cov}\,(aX + b, Y)$$

in terms of $\mathrm{Cov}\,(X, Y)$.

**Solution to Question 1**

**Part (a):**

Recall the definition of covariance:

$$\mathrm{Cov}\,(X, Y) = \mathrm{E}[(X - \mu_X)\,(Y - \mu_Y)],$$

where $\mu_X = \mathrm{E}[X]$ and $\mu_Y = \mathrm{E}[Y]$. Since $a$ is a constant, $\mathrm{E}[a] = a$. Therefore,

$$\mathrm{Cov}\,(X, a) = \mathrm{E}[(X - \mu_X)\,(a - a)] = \mathrm{E}[0] = 0.$$

**Part (b):**

Using the linearity of expectation,

$$\mu_{aX+bY} = \mathrm{E}[aX + bY] = a\mathrm{E}[X] + b\mathrm{E}[Y] = a\mu_X + b\mu_Y$$

Therefore, writing $\mu_Z = \mathrm{E}[Z]$, and using the linearity of expectation

$$\begin{aligned}
\mathrm{Cov}\,(aX + bY, Z) &= \mathrm{E}[(aX + bY - \mu_{aX+bY})\,(Z - \mu_Z)] \\
&= \mathrm{E}[(aX + bY - a\mu_X - b\mu_Y)\,(Z - \mu_Z)] \\
&= \mathrm{E}[(aX - a\mu_X + bY - b\mu_Y)\,(Z - \mu_Z)] \\
&= \mathrm{E}[(aX - a\mu_X)\,(Z - \mu_Z) + (bY - b\mu_Y)\,(Z - \mu_Z)] \\
&= \mathrm{E}[(aX - a\mu_X)\,(Z - \mu_Z)] + \mathrm{E}[(bY - b\mu_Y)\,(Z - \mu_Z)] \\
&= \mathrm{E}[a\,(X - \mu_X)\,(Z - \mu_Z)] + \mathrm{E}[b\,(Y - \mu_Y)\,(Z - \mu_Z)] \\
&= a\mathrm{E}[(X - \mu_X)\,(Z - \mu_Z)] + b\mathrm{E}[(Y - \mu_Y)\,(Z - \mu_Z)] \\
&= a\mathrm{Cov}\,(X, Z) + b\mathrm{Cov}\,(Y, Z)
\end{aligned}$$

**Part (c):**

Using Part (b),

$$\mathrm{Cov}\left(aX+b,Y\right)=a\mathrm{Cov}\left(X,Y\right)+\mathrm{Cov}\left(b,Y\right)$$

and using Part (a), $\mathrm{Cov}\left(b,Y\right)=0$, which implies

$$\mathrm{Cov}\left(aX+b,Y\right)=a\mathrm{Cov}\left(X,Y\right).$$

(To justify the first part, you could consider $b=bZ$, where $Z$ is the random variable which is identically equal to 1, and then proceed from there.)

**Question 2**

Do Exercise 7.2.7 in the notes: show that the conjugate prior for the exponential distribution is the gamma distribution.

**Hint:**

Start by assuming that a sample of independent random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is observed as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and that each $X_i$ follows an exponential distribution with the same unknown parameter $\theta$. Derive an expression the likelihood. Write down the probability density function for the gamma distribution - this is the prior. Then, compute the posterior.

**Solution to Question 2**

Suppose that a sample of random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is observed as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Suppose further that each $X_i$ follows an exponential distribution with the same unknown parameter $\theta$, i.e. each $X_i$ has the p.d.f. for $x_i > 0$,

$$f(x_i|\theta) = \theta \exp\left(-\theta x_i\right).$$

Therefore the likelihood is, for $x_i > 0$ for all $i = (1, 2, \ldots, n)$, and writing $n\overline{x} = \sum_{i=1}^{n} x_i$,

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{n} \left(\theta \exp\left(-\theta x_i\right)\right) = \theta^n \exp\left(-\theta n\overline{x}\right).$$

Suppose the prior for $\theta$ is a $\Gamma\left(\alpha, \beta\right)$. Then (using the shape-rate parametrisation):

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp\left(-\beta\theta\right)$$

Then the posterior p.d.f. is proportional to:

$$\begin{aligned} \pi(\theta|\mathbf{x}) \propto f(\mathbf{x}|\theta)\pi(\theta) &= \theta^n \exp\left(-\theta n\overline{x}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp\left(-\beta\theta\right) \\ &\propto \theta^{n+\alpha-1} \exp\left(-\theta\left(n\overline{x} + \beta\right)\right) \end{aligned}$$

which shows that the posterior is a $\Gamma\left(n+\alpha, n\overline{x}+\beta\right)$ distribution, i.e.

$$\pi(\theta|\mathbf{x}) = \frac{(n\overline{x}+\beta)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} \exp\left(-\theta\left(n\overline{x}+\beta\right)\right).$$

**Question 3**

Suppose we have the sample of observations

$$\mathbf{x} = (1.2, \quad 5.3, \quad 6.2, \quad 7.4, \quad 8.5)\,.$$

Which of the following samples are/are not bootstrap samples of $\mathbf{x}$? Justify your answer in each case.

(a) $(5.3, \quad 1.2, \quad 6.2, \quad 8.5, \quad 7.4)$

(b) $(6.2, \quad 7.4, \quad 7.4, \quad 1.2, \quad 5.3)$

(c) $(6.2, \quad 7.4, \quad 1.2, \quad 7.4)$

(d) $(7.4, \quad 5.3, \quad 7.4, \quad 9.9, \quad 8.5)$

(e) $(5.3, \quad 5.3, \quad 5.3, \quad 5.3, \quad 5.3)$

(f) $(7.4, \quad 5.3, \quad 7.4, \quad 8.5, \quad 1.2, \quad 6.2)$

**Solution to Question 3**

(a) This is a bootstrap sample, because it contains 5 values and all the values are in the set $\{1.2, \quad 5.3, \quad 6.2, \quad 7.4, \quad 8.5\}$.

(b) This is a bootstrap sample, because it contains 5 values and all the values are in the set $\{1.2, \quad 5.3, \quad 6.2, \quad 7.4, \quad 8.5\}$

(c) This is **not** a bootstrap sample because it does not contain 5 elements, and therefore does not contain the same number of elements as $\mathbf{x}$.

(d) This is **not** a bootstrap sample because 9.9 is not in the set $\{1.2, \quad 5.3, \quad 6.2, \quad 7.4, \quad 8.5\}$.

(e) This is a bootstrap sample because there are 5 elements and (even though they are all equal) all the elements are in the set $(5.3, \quad 5.3, \quad 5.3, \quad 5.3, \quad 5.3)$

(f) This is **not** a bootstrap sample because it contains 6 elements and not 5 elements, and therefore does not contain exactly the same number of elements as $\mathbf{x}$.