MATH40004 - Calculus and Applications - Term 2

Problem Sheet 6

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. Consider again the second order differential equation of the damped harmonic oscillator we saw in the previous problem sheet:
$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0,$$
where k and ω are positive constants representing the damping of the medium and the intrinsic frequency of the system, respectively.

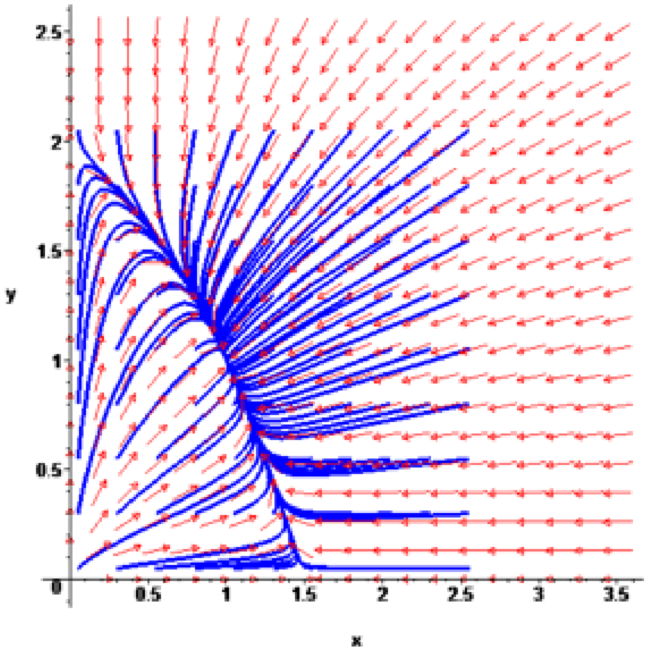
   We will now examine the system when the parameter k is tuned experimentally.

   (a) Sketch all the qualitatively different phase portraits that can appear as k is varied, and describe the associated asymptotic behaviours.

   (b) Establish all the values of k at which the system changes qualitative behaviour.

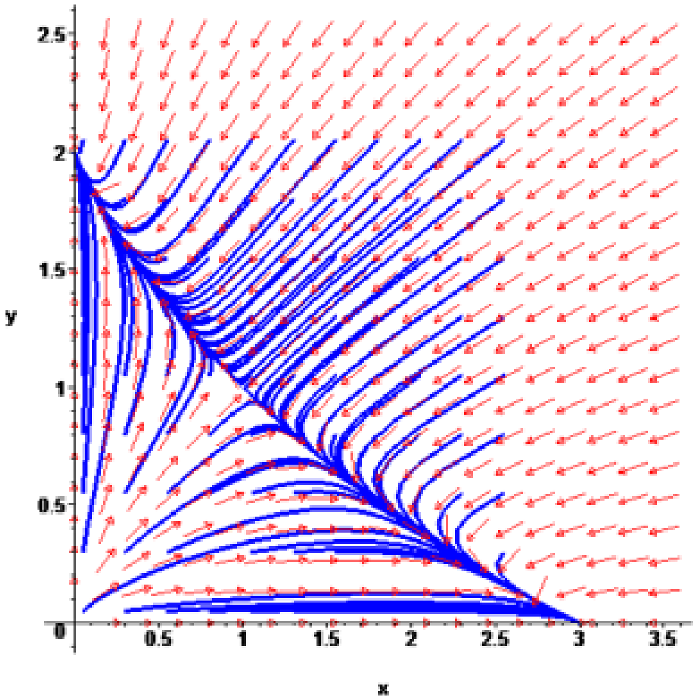
2. * You are given a system of two coupled nonlinear ODEs
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix},$$
which contain some parameters. The system describes two populations x and y which compete for resources.

   (a) If treated correctly, MATLAB or Maple can be your friends (really!). The following figure of the vector field and some sample trajectories can be obtained computationally. Note that x and y are populations, so the phase portrait is only of interest for $x, y > 0$.



Using the flows depicted in the figure, identify and classify all the fixed points of the system. What will be the long term behaviour of the system?

(b) As some of the parameters in the system change, we recompute the vector field and phase portrait and we obtain the following figure:



Again, use the flows depicted in the figure to identify and classify the fixed points of the system. What will be the long term behaviour of the system?

3. Consider the following three first order *nonlinear* ODEs:

$$\text{(a)}\ \frac{dx}{dt} = \frac{x}{\alpha + x} - x \qquad \text{(b)}\ \frac{dx}{dt} = \frac{1}{\alpha + x} - x \qquad \text{(c)}\ \frac{dx}{dt} = \frac{x}{\alpha + x^2} - x$$

where $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ is a parameter.

For each case:

(i) Perform the global stability analysis of the ODE, i.e., find the fixed (stationary) points x^* and their stability as a function of α.

(ii) Draw the bifurcation diagram x^* vs. α and classify any bifurcations observed.

4. * The following first order *nonlinear* ODE

$$\frac{dx}{dt} = x(1 - x) - h$$

describes a simple model of a fishery, where $x \in \mathbb{R}^+$ is the population of fish and $h \in \mathbb{R}^+$ is a parameter.

(a) Explain the meaning of each term in the equation.

(b) Plot the vector field (on the real line) for different values of h. Show that a bifurcation occurs at a certain value $h = h_c$ and classify this bifurcation.

(c) Discuss the long term behaviour of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.