Mathematics Year 1, Calculus and Applications I

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Problem Sheet 4

Problems 3, 4, 8 and 9 are good candidates for starred questions

1. Consider the function $f(x) = 1/x$ for $x \in [1, \infty)$. Calculate and compare the area under the curve, the surface area of the solid formed by revolving $f(x)$ about the $x-$axis, and the volume of the revolved solid. What do you conclude? [The revolved object is called *Gabriel's horn.*]

2. Find (i) $\int_0^1 \frac{dx}{8x^3+1}$ and (ii) $\int \frac{(1+x)^{3/2}}{x} dx$.

3. After a glitch, a manufacturer only produce chains of variable density that starts off with unit value but then becomes a linear function of distance from one end of the chain to the other. An order was delivered but the customer emailed back angrily saying that instead of the chains hanging evenly over their one unit tables, they rested in such a way that one of the hanging pieces was twice as long as the other hanging piece. What is the density of the chain produced by the malfunctioning machine?

4. Write an integral representing the area of the surface obtained by revolving the graph of $1/(1+x^2)$ about the $x-$axis. Do not compute the integral but show that it is less than $2\sqrt{5}\pi^2$ no matter how long an interval is taken. Show also that an improved bound is $\sqrt{91}\pi^2/4$.

5. (a) Find the volume of the solid obtained by revolving the region under the graph of the function $y = \frac{1}{(1-x)(1-2x)}$ on the interval $[5, 6]$ about the $y-$axis.

   (b) Find the centre of mass of the region under $1/(x^2 + 4)$ on $[1, 3]$.

6. As a circle of radius a and centre O rolls along a plane, the position of a point A on the circle's circumference is given parametrically by $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, where θ is the angle that AO makes with the vertical.

   (a) Find the distance travelled by A for $0 \le \theta \le 2\pi$. Is it bigger or smaller than the circle's circumference. Explain your finding.

   (b) Draw a diagram for one arch of the curve traced out by A (it is the cycloid encountered in tests!) and superimpose on it the circle when its centre is at $(\pi a, a)$, together with the line segment $0 \le x \le 2\pi a$ on the $x-$axis. Show that the three enclosed areas are equal.

7. Find the centre of mass of the triangular region (of uniform density per unit area) with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3). For convenience place the points so that $x_1 \le x_2 \le x_3$, $y_1 \le y_3$ and $y_2 \le y_3$.

   How does your answer compare with the problem of placing equal masses at the vertices of the triangle?

8. Consider a sphere of radius r. Suppose the sphere is sliced into three pieces by two parallel planes that are a distance d apart, where $0 < d < r$. Show that the surface area of the middle piece is the same irrespective of where the cuts are made on the sphere.

9. (a) Consider a function $y = f(x)$ with $f(0) = 0$ and assume that its inverse $x = f^{-1}(y)$ exists. The function is rotated about the $y-$axis to produce a solid in the region $0 \leq y \leq y_0$. Use infinitesimals to show that the desired volume of revolution is

$$V = \pi \int_0^{y_0} [f^{-1}(y)]^2 dy.$$

(b) A bowl is created as described above by rotating $y = f(x)$ about the $y-$axis, and is filled with water to a height h_0. At its bottom ($x = y = 0$) a little hole is bored of radius r, that when open allows for the fluid to drain from the bowl. The speed of the exiting fluid is given by *Torricelli's* [1] *law* that states that at any given instant the speed equals $\sqrt{2gh}$ where h is the instantaneous height of the liquid remaining in the bowl. Formulate a conservation law that describes the physics of the problem, namely, *the rate of change of the volume at any given instant decreases by the rate at which fluid is exiting the small hole of radius* r to derive the equation

$$\frac{dV}{dt} = -\pi r^2 \sqrt{2gh},$$

where V is the volume of fluid remaining in the bowl.

(c) Three different bowls are now created to be used as hourglasses. The functions describing the bowls are (using the notation of part (a)) (i) $y = \frac{1}{k}x^2$, where $k > 0$ has dimensions of length, (ii) $y = \alpha x$, and (iii) a hemispherical bowl of radius a centred at $(0, a)$. In all three cases the bowls are filled with liquid to an initial height h_0 (note that $0 < h_0 \leq a$), and have identical small holes of radius r at the bottoms. At $t = 0$ the hole is opened and the bowls are allowed to drain empty. Find α and a so that all three bowls empty at the same time.

[1] Evangelista Torricelli (1608-1647) was an Italian physicist and mathematician who is best known for the invention of the barometer