MATH40004 - Calculus and Applications: Unseen Questions 4: Series solutions to ODEs

1 Power series solutions to differential equations

Consider a linear second order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = f(x), \tag{1}$$

where $p(x)$, $q(x)$ and $f(x)$ are functions of x. In many applications of differential equations we tend to be interested in the behaviour of the function $y(x)$ in specific regions or places; for example x might be limited to values in the range $[-1, 1]$ and near to $x = 0$ there might be a specific region of interest (maybe there is a transition region from laminar to turbulent air flow over the wing of an aircraft or a change in the temperature dynamics inside a combustion chamber).

Suppose we are interested in the solution to (1) near to (about) the point $x = x_0$. If we are unable to solve equation (1) nicely via any alternative methods one thing we can try to do is to find a **power series solution** to the equation about the point x_0. In other words, we assume the solution to (1) can be written in the form of a **Taylor series** about x_0:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \tag{2}$$

The goal is then to determine what the coefficients a_n are.

Example 1: Find the first five terms in a power series expansion about $x = 0$ for the general solution to the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0. \tag{3}$$

Solution: First we suppose that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{4}$$

Now we substitute our ansatz (4) into the differential equation (3), this gives

$$\underbrace{\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}}_{\text{This term is equivalent to} \frac{\mathrm{d}^2y}{\mathrm{d}x^2}} + 3x\underbrace{\sum_{n=1}^{\infty} a_n n x^{n-1}}_{=\frac{\mathrm{d}y}{\mathrm{d}x}} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{5}$$

Notice how we have just differentiated everything inside the sums the appropriate number of times to get this far - this is equivalent to differentiating the sums term by term so we can do this! However after differentiating inside the sums we need to be careful with the index we start the sums with: the power of x in the first sum is $n - 2$ after the two differentiations and we know that the resulting series should start with the term involving x^0 (i.e. there shouldn't be any terms involving x^{-1} or x^{-2} after the differentiation), so this is why this sum now starts from $n = 2$. Similarly for the second sum

starting from $n = 1$.
This can be written as

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 3na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0, \tag{6}$$

where we have relabelled the first sum (put $n \to n+2$) and multiplied the second sum by the x that was on the outside. The relabelling was to ensure that the power of x in each sum was equal, i.e. they are all x^n now. Continuing to simplify things:

$$a_2(0+2)(0+1)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 3na_n x^n - a_0 x^0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad (2a_2 - a_0) + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} + (3n-1)a_n]x^n = 0. \tag{7}$$

Comparing coefficients of x on both sides, we conclude

$$2a_2 - a_0 = 0, \tag{8}$$

$$(n+1)(n+2)a_{n+2} + (3n-1)a_n = 0, \quad n \geqslant 1. \tag{9}$$

If we let $a_0 = A$ and $a_1 = B$ (two arbitrary constants), (8)-(9) give

$$a_2 = \frac{a_0}{2} = \frac{A}{2}, \tag{10}$$

$$a_3 = \frac{-2}{3 \times 2} B = -\frac{1}{3} B, \tag{11}$$

$$a_4 = \frac{-5}{12} a_2 = -\frac{5}{24} A. \tag{12}$$

So putting everything together, we have

$$y(x) = A + Bx + \frac{A}{2}x^2 - \frac{B}{3}x^3 - \frac{5}{24}Ax^4 + \cdots \tag{13}$$

Remark 1: Notice that our solution contains **two** unknown arbitrary constants (A and B). This is to be expected since the general solution to a second order differential equation contains two arbitrary constants.

Remark 2: The relationship

$$a_{n+2} = \frac{1-3n}{(n+1)(n+2)} a_n, \tag{14}$$

found during the calculation gives us a recurrence relationship for generating all the coefficients in the series solution. Considering this relationship separately for n even and n odd we could find a general form for all of the coefficients in the series.

Remark 3: If we imposed two initial conditions, for example $y(0) = 2$ and $y'(0) = 0$, then the unknown values of A and B could be determined (these conditions would give $A = 2$ and $B = 0$) and hence all terms in the series solution could be found explicitly.

1.1 Inhomogeneous equations and $p(x)$, $q(x)$ and $f(x)$ not polynomials

If there exists a function $f(x)$ on the right hand side of the differential equation then the same method still applies except when we compare coefficients of x on each side of the equation this function $f(x)$ will mean not every coefficient will be zero (as was the case in example 1). However what if the functions $p(x)$, $q(x)$ or $f(x)$ are not all polynomials in powers of $(x - x_0)^n$? Well we simply turn them into these! We Taylor expand these functions about $x = x_0$ so that they can be represented by power series in $(x - x_0)^n$ and then the same method applies.

2 Exercises

(a). Show that the first four non-zero terms in the power series solution about $x = 0$ to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x^4 y = 0, \tag{15}$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 1$ are

$$y(x) = 1 + x + \frac{1}{30}x^6 + \frac{1}{42}x^7 + \cdots \tag{16}$$

(b). Find the first five non-zero terms in the power series solution about $x = 1$ to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \tag{17}$$

subject to the conditions $y(1) = 1$ and $y'(1) = 1$.

(c). Find the general form of a power series solution about $x = 0$ to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0, \tag{18}$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 1$.

(d). Find a power series solution about $x = 0$ up to the term involving x^6 to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \sin(x)y = \cos(x), \tag{19}$$

subject to the conditions $y(0) = 3$ and $y'(0) = 0$.

3 Singular Points

The 'naive' approach to finding series solutions applied so far does not always work!

3.1 Investigation

In a similar way to before seek a solution of a power series form (2) about $x = 0$ to the equation

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0. \tag{20}$$

What happens? Any ideas why?

3.2 Definition

Consider the linear differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0. \tag{21}$$

- A point x_0 is called an **ordinary point** if both $p(x)$ and $q(x)$ are **analytic** (they have a Taylor series around $x = x_0$) at x_0 (or in other words $p(x_0)$ and $q(x_0)$ do not blow up to infinity!).

Otherwise we call x_0 a **singular point**. For a singular point x_0:

- If $p(x)(x - x_0)$ and $q(x)(x - x_0)^2$ are analytic at x_0 then the point x_0 is called a **regular singular point** and we can still find a solution via a generalised method called **the method of Frobenius**.

- Otherwise x_0 is called an **irregular singular point** and there is no good way to tackle these equations!

3.3 The Method of Frobenius

This is a modification of the previous power series method which is effective at dealing with regular singular points. The idea is to look for solutions of the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{22}$$

where now both r, which can take any value (not just positive integer), and the a_n are to be determined. It turns out that two values for r will be found and depending on what these values are the solutions will have very different forms - in fact if the two values for r are equal, or if they differ by an integer the solution will be more complex and contain a logarithm (google this if you're interested!). Otherwise the two linearly independent solutions for the equation (one for each value of r) can be represented in the form

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{and} \qquad y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{23}$$

where r_1 and r_2 are the two distinct values for r and I have abused notation by writing a_n twice but these coefficients might be different for each sum.

4 Exercise

(a). Consider the differential equation

$$2x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x(2x - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0. \tag{24}$$

(i). Show that $x = 0$ is a regular singular point of the equation.

(ii). Hence, using the method of Frobenius, show that the general solution to the equation can be written as

$$y(x) = A|x|^{1/2}e^{-x} + B|x| \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)(2n-1)\cdots 3} x^n. \tag{25}$$

5 Extension

I said earlier that it turns out there are always two values for r when using the method of Frobenius. If you have tried the last exercise you will have seen that a quadratic equation for r appears that gives rise to the two different values. This is not a coincidence and this quadratic equation in r is called the **indicial equation**. Starting with a general homogeneous linear second order differential equation as in (21), using the method of Frobenius, show that the indicial equation is

$$r(r-1) + p_0 r + q_0 = 0, \tag{26}$$

where you should explain what p_0 and q_0 are.

When problems involving regular singular points are encountered the values for r can then be found directly from this equation without the need for the long substitution process!