Math40002 Analysis 1 **Term 2 Unseen 2**

Recall a notation for a union of a set of sets called "unary union": Given a set $\mathcal{A}$ of sets, denote its union by $\cup\mathcal{A} := \bigcup_{X\in\mathcal{A}} X$.

1. Prove that a finite union of bounded sets is bounded.

2. Let (a_n) be a sequence and $a \in \mathbb{R}$. Prove that $a_n \to a$ if and only if for every open set $U \subseteq \mathbb{R}$ such that $a \in U$, there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

3. (Second countability)

   (a) Prove that for every open set $U \subseteq \mathbb{R}$ and $x_0 \in U$, there is some open interval (a, b) with rational endpoints (namely, $a, b \in \mathbb{Q}$) such that $x_0 \in (a, b) \subseteq U$.

   (b) Prove that there is a countable set of open intervals $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ such that for every open set $U \subseteq \mathbb{R}$, there is some $\mathcal{C} \subseteq \mathcal{B}$ such that $U = \cup\mathcal{C} := \bigcup_{A\in\mathcal{C}} A$. I.e., U is a union of sets of $\mathcal{B}$.

> <u>Side note:</u> *In general topology, $\mathcal{B}$ satisfying that every open set is a union of sets from $\mathcal{B}$ is called a* ***basis****, topological spaces which have a countable basis are called* ***second countable***

4. (Lindelöf's Lemma)

   (a) Prove that every open set $U \subseteq \mathbb{R}$ is a countable union of open intervals.

   (b) Let $\mathcal{U}$ be a set of open sets. Prove that there is some countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 = \cup\mathcal{U}$.

> <u>Sidenote:</u> *In general, a topological space is called* ***Lindelöf*** *if for every set of open sets, there is a countable subset of it, such that their unions are equal. Lindelöf's Lemma proves that $\mathbb{R}$ is a Lindelöf space. In general, Lindelöf's Lemma can show that every second countable space is Lindelöf.*

5. Let $x_0 \in \mathbb{R}$. Find a set of open sets $\mathcal{U}$ such that $\cup\mathcal{U} = \mathbb{R} \setminus \{x_0\}$ and for every finite $\mathcal{U}_0 \subseteq \mathcal{U}$, there is some open set V such that $x \in V$ and $(\cup\mathcal{U}_0) \cap V = \emptyset$.

6. (Heine-Borel Theorem) Let $X \subseteq \mathbb{R}$. Prove that the following are equivalent:

   (i) X is compact.

   (ii) For every set $\mathcal{U}$ of open sets such that $\cup\mathcal{U} \supseteq X$, there is some finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 \supseteq X$.

   This is a complex question, and there are several things you need to prove. The best advice is not to give up! It may require time after the session and discussions with your mates, but once you succeed, you'll have the sense of triumph of proving a proper theroem all on your own. You may want to use the fact that S is compact if and only if every sequence in S has a subsequence that converges to a limit in S, as proved in Question sheet 2, Question 4.