Math40003 Linear Algebra and Groups Problem Sheet 5

1. (a) Let U and W be the following subspaces of $\mathbb{R}^4$:

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_4 = -x_1 + x_2 + x_3 = 0\},$$
$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 + x_3 - x_4 = -x_1 + 2x_2 + x_3 + x_4 = 0\}.$$

Find a basis for $U \cap W$. Find bases for U and for W, both of which contain your basis for $U \cap W$. Find a basis for $U + W$ containing your basis for $U \cap W$.

(b) Let X and Y be the following subspaces of $\mathbb{R}^4$:

$$X = \mathrm{Span}\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\},$$
$$Y = \mathrm{Span}\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}.$$

Find bases for $X \cap Y$ and $X + Y$.

(c) Let X and Y be as in part (ii). Find a subspace Z of $\mathbb{R}^4$ with the properties that $\mathbb{R}^4 = X + Z = Y + Z$, and $X \cap Z = Y \cap Z = \{0_V\}$.

2.* (a) Let U and W be 3-dimensional subspaces of $\mathbb{R}^5$, with $U \neq W$. Prove that $\dim U \cap W$ is either 1 or 2. Give examples to show that both possibilities can occur.

(b) Let U_1, U_2 and U_3 be 3-dimensional subspaces of $\mathbb{R}^4$. Give a proof that $\dim U_1 \cap U_2 \geq 2$. Deduce that $U_1 \cap U_2 \cap U_3 \neq \{0_V\}$.

(c) Now let V be the vector space of 2×3 matrices over $\mathbb{R}$. Find subspaces X and Y of V such that $\dim X = \dim Y = 4$, and $\dim X \cap Y = 2$.

(d) Let V be as in part (iii). Do there exist subspaces X and Y of V such that $\dim X = 3$, $\dim Y = 5$, and $\dim X \cap Y = 1$?

3. The *rank* of an $m \times n$ matrix A is defined to be the dimension of its row space $\mathrm{RSp}(A)$ and is denoted by $\mathrm{rank}\, A$. Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

(a) Let v be a row vector in $\mathbb{R}^n$. Prove that vB is a linear combination of the rows of B.

(b) Prove that the row space of AB is contained in the row space of B and $\mathrm{rank}\, AB \leq \mathrm{rank}\, B$.

(c) Prove that if $m = n$ and A is invertible, then $\mathrm{rank}\, AB = \mathrm{rank}\, B$.

(d) Prove that $\mathrm{rank}\, AB \leq \mathrm{rank}\, A$.

4. (a) Find the ranks of the matrices

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ 2 & 3 \end{pmatrix}$$

(b) Find an equation for a and b such that the following matrix has rank 2:

$$\begin{pmatrix} 3 & 2 & 5 \\ 1 & a & -1 \\ 1 & 3 & b \end{pmatrix}.$$

(c) Find an equation for b, c and d such that the matrices

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & -3 & 0 \\ 1 & 1 & 0 & b \\ 2 & -1 & 3 & c \\ 1 & 4 & -2 & d \end{pmatrix}$$

both have the same rank.