Math40002 Analysis 1 **Term 2 Unseen 1**

<u>Note:</u> You may need to use the results of Problem Sheet 1 (of Term 2) to solve some of the questions in this unseen sheet

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that if there is some $x_0 \in \mathbb{R}$ be such that f is continuous on x_0, then f is continuous on all $\mathbb{R}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be such that f is continuous on x_0. Prove that there is some $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and let $I \subseteq \mathbb{R}$ be an interval such that f is monotone on I. Prove that there is some $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

   <u>Note:</u> The equation $f(x+y) = f(x)+f(y)$ is called Cauchy's functional equation. The existence of non-linear functions satisfying this equation relies on the Axiom of Choice – Recall that in Linear Algebra and Groups Unseen 6 from Term 1, you proved using Zorn's Lemma that every vector space has a basis (in particular $\mathbb{R}$ over $\mathbb{Q}$). In fact, this is equivalent to the Axiom of Choice.

   If you'd like, you can try proving by yourself (assuming Zorn's Lemma) that there are non-linear functions satisfying Cauchy's functional equation. However, please don't do so during this problem session, as this is more of a bonus question in Linear Algebra.

4. (a) Show that every non-constant periodic continuous function has a minimal period. Explicitly, show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous such that there is some $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, and f is not a constant function, then the set

   $$\mathcal{T} := \{\, T \in (0, +\infty) \,|\, \forall x \in \mathbb{R} : f(x + T) = f(x) \,\}$$

   has a minimum.

   (b) Is the statement in Item 4a still true without the assumption of continuity? Prove your answer.