Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 8

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. Consider the function $u = \arctan(y/x)$. Show that:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

and that $u(x, y)$ is also a solution of the Laplace equation in two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2. Consider the function $u = x\ln(x^2 + y^2) - 2y\arctan(y/x)$. Show that:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + 2x$$

3. * Consider the function $u(x, y, z) = 1/r$ where $r = \sqrt{x^2 + y^2 + z^2}$. Show that this function is a solution of the Laplace equation in three dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

4. Consider the following change of variables:

$$s = \frac{x}{x^2 + y^2}$$
$$t = \frac{y}{x^2 + y^2}.$$

(a) Obtain the Jacobian matrix associated with this change of variables

(b) Consider a function $u(s, t)$ that obeys the partial differential equation:

$$\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = 0$$

Find the partial differential equation that this function obeys when it is expressed in terms of the variables x, y.

5. Solve the differential equations below. First, determine if they are exact or not. If exact, solve directly. If not, solve them by multiplying by a suitable integrating factor.

(a) $(3x - 2y)dy + (3x^2 + 3y)dx = 0$

(b) $e^y \sin x \dfrac{dy}{dx} + (1 + e^y) \cos x = 0$

(c) $(y^2 - x^2)\dfrac{dy}{dx} + 2xy = 0$

(d) $(y^4 + 2y^2 - x^3 + 5x^2y - 21xy^2)\dfrac{dy}{dx} + (x^3 - 3x^2y + 5xy^2 - 7y^3) = 0$

(e) $(x^3 - 2xy)\dfrac{dy}{dx} + (x + 2y^2) = 0$

(f) $(e^y + ye^x)dx + (e^x + xe^y + 1)dy = 0$

6. A few examples to practise sketching of functions in two variables, as well as finding and classifying their extrema:

(a) Find the stationary points (extrema) of

$$f(x, y) = x^3 + y^2 - 3x$$

and determine their character (i.e, whether they are maxima, minima or saddle points).

(b) Consider the function

$$f(x, y) = xy(x + y - 1)$$

Find the extrema and determine their character. Sketch the relevant contour lines of the function.

(c) * Consider the function

$$f(x, y) = x(y - 2)^2 + x^2 - x$$

Find the extrema and determine their character. Sketch the relevant contour lines of the function.