2D continuum limit of the applications framework

1. Consider the voltage distribution in a circular conductor of unit radius centred at the origin given by

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

$$h(z) = -\frac{m}{2\pi} \log z.$$

(a) Verify that $\nabla^2 \phi = 0$ everywhere inside the conductor except at $(0, 0)$.

(b) Show that $\phi = 0$ on the boundary $|z| = 1$ of the conductor.

(c) Assuming unit conductivity, find an expression for the complex current density $j_x - \mathrm{i}j_y$.

(d) Find the net current leaving the conductor through its boundary $|z| = 1$.

(e) The singularity of the voltage potential at $(0, 0)$ is known as a *current source singularity*. Can you see why it might have this name?

2. This question gives you another way to understand the *current source singularity* considered in question 1. In example 2 presented in lectures the voltage distribution $\phi(x, y)$ in an annular conductor $\rho < |z| < 1$, of uniform conductivity $\hat{c} = 1$, with unit voltage $\phi = 1$ imposed on $|z| = \rho$ and with $|z| = 1$ grounded, was shown to be

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

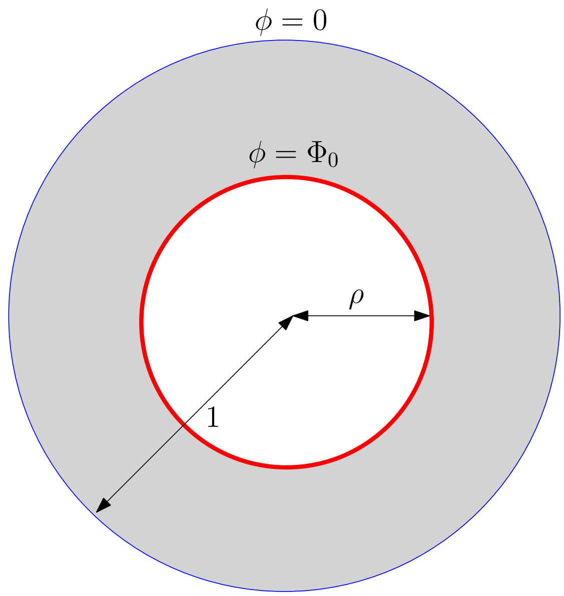
$$h(z) = \frac{\log z}{\log \rho}.$$

(a) Verify that if, as indicated in the figure, the inner boundary circle $|z| = \rho$ is set at voltage $\phi = \Phi_0$, where $\Phi_0 > 0$ is some real constant, then the solution is now given by

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where, now,

$$h(z) = \Phi_0 \frac{\log z}{\log \rho}.$$



(b) Show that, if Φ_0 remains fixed but $\rho \to 0$, so that the inner hole in the conductor disappears, then no current flows in the conductor.

(c) Suppose, on the other hand, that $\rho \to 0$ (so that the inner hole in the conductor disappears) but, *also*, $\Phi_0 \to \infty$ in such a way that it is always true that

$$\frac{\Phi_0}{\log \rho} = -\frac{m}{2\pi}, \tag{1}$$

where $m > 0$ is a positive constant, find the corresponding $h(z)$ and make a connection with the result in question 1.

(d) Find the total current into the conductor as a function of Φ_0 and ρ. What is the value of this total current in the limit considered in part (b)? What about in the limit considered in part (c)?

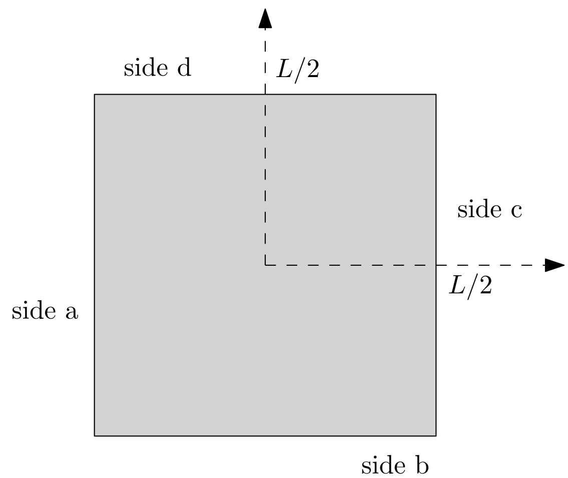
3. Consider the voltage distribution in a square conductor $-L/2 < x < L/2, -L/2 < y < L/2$ centred at the origin and with sides of length $L > 0$ given by

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

$$h(z) = -\frac{z}{L} - \frac{m}{2\pi} \log z. \tag{2}$$

Let the 4 sides of the conductor be denoted a, b, c and d as shown in the Figure.



(a) Verify that $\nabla^2\phi = 0$ everywhere inside the conductor except at $(0,0)$.

(b) Assuming unit conductivity, find an expression for the current density $j_x - \mathrm{i}j_y$.

(c) Use your answer in part (b) to calculate the net current leaving the conductor through each of its four sides a, b, c and d.

(d) Notice that the voltage potential (2) for this square conductor is made up of two contributions: one is given by

$$-\frac{z}{L}$$

which resembles the potential for a uniform current across the conductor from left to right running parallel to the x axis as considered in example 1 of the lecture notes; it also has a contribution

$$-\frac{m}{2\pi}\log z.$$

which we identified, in question 1, as a *current source singularity*. Can you see another way to find the answer to part (c) based on these observations?

4. In example 3 of the lecture notes it was determined that the current density j_y on the top boundary $y = \pi/2$ of an infinite strip conductor $-\infty < x < \infty, -\pi/2 < y < \pi/2$ is

$$j_y = \frac{m}{2\pi}\mathrm{sech}x.$$

(a) Use this result to find the total current leaving the strip through this top boundary of the strip.

(b) Could you have anticipated your result based on the observations in question 1 and 2?

5.*[1] Consider the voltage distribution in a circular conductor of unit radius centred at the origin given by

$$\phi(x,y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

$$h(z) = -\frac{m}{2\pi}\log\left(\frac{z^2 - a^2}{z^2a^2 - 1}\right), \qquad 0 < a < 1,$$

where a is a real parameter. The outer boundary of the conductor is the circle $|z| = 1$.

(a) Verify that $\nabla^2\phi = 0$ everywhere inside the conductor except at the two points $(a, 0)$ and $(-a, 0)$.

(b) Show that $\phi = 0$ on the boundary $|z| = 1$ of the conductor.

(c) Assuming unit conductivity, show that the complex current density in the direction normal to the conductor boundary is

$$\frac{m(a^4 - 1)}{\pi}\frac{1}{2a^2\cos 2\theta - (1 + a^4)},$$

where the angle θ is used to parametrize a point $(\cos\theta, \sin\theta)$ on the conductor boundary $|z| = 1$.

(d) Use your answer to part (c) to calculate the net current leaving the conductor through its boundary $|z| = 1$. (*Hint:* you may find the "t-substitution" from calculus helpful to carry out the integration).

(e) Could you have guessed the answer to part (d) using the observation on "current source singularities" from question 1?

[1]This question is a good exercise, so try it to make sure you understand the ideas, but the calculations are more difficult than I would give you in an examination.