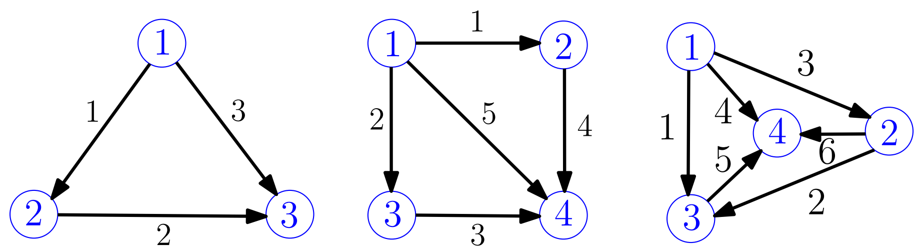
Linear algebra review and basic graph theory

1. Consider the following three graphs I, II and III:



Graph I Graph II Graph III

(a) Find the incidence matrix $\mathbf{A}$ for each graph (use the numbering of nodes and edges shown in the figures).

(b) For each graph, find all vectors in the nullspaces of $\mathbf{A}$ and $\mathbf{A}^T$.

(c) Find the degree matrix $\mathbf{D}$ for each graph.

(d) Find the adjacency matrix $\mathbf{W}$ for each graph.

(e) Find the Laplacian matrix $\mathbf{K}$ for each graph.

(f) Are any of the graphs complete?

2. Suppose that graphs I, II and III are the disconnected pieces of a **single** graph with 11 nodes and 14 edges.

(a) What is the rank of the incidence matrix $\mathcal{A}$ of this new single graph?

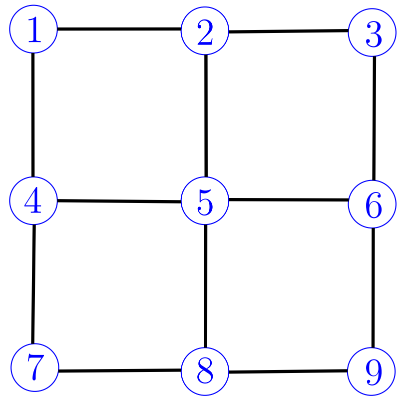
(b) Find all linearly independent solutions of

$$\mathcal{A}\mathbf{x} = 0.$$

(c) Find all linearly independent solutions of

$$\mathcal{A}^T\mathbf{w} = 0.$$

3. Consider a 3-by-3 square grid with $n = 9$ nodes and $m = 12$ edges as shown in the figure. Let the Laplacian matrix be $\mathbf{K} = \mathbf{A}^T\mathbf{A}$.

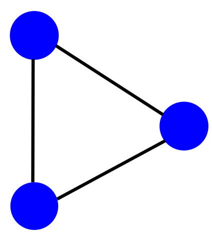


(a) How many of the 81 entries of $\mathbf{K}$ are zero?

(b) Write down the degree matrix $\mathbf{D}$.

4. In a graph with n nodes and n edges argue that there must be a loop.

5. Consider the 3-node graph



(a) Write down the incidence matrix $\mathbf{A}$ and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{A}$.

(b) Let $\omega = e^{2\pi \mathrm{i}/3}$ be a third root of unity so that

$$\omega^3 = 1.$$

Now introduce the vectors

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

Show that, for $n = 0, 1$ and 2,

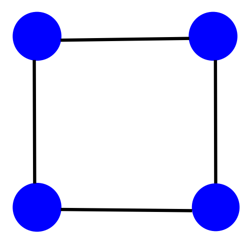
$$\mathbf{K}\mathbf{x}_n = \lambda_n \mathbf{x}_n$$

and find the values of the constants λ_0, λ_1 and λ_2.

(c) Why did we only consider the three possible values $n = 0, 1, 2$ in part (b)?

(d) The inner product (think "dot product") of two complex-valued vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be
$$\overline{\mathbf{u}}^T\mathbf{v},$$
where $\overline{\mathbf{u}}$ is the complex conjugate of the complex vector $\mathbf{u}$. Show that the vectors $\mathbf{x}_n$ for $n = 0, 1, 2$ are orthogonal with respect to this inner product (meaning that the inner product of any two vectors is zero).

6. Consider the 4-node graph

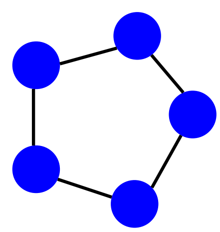


(a) Write down the incidence matrix $\mathbf{A}$ and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{A}$.

(b) Let $\omega = e^{2\pi i/4}$ be a fourth root of unity so that
$$\omega^4 = 1.$$
Now introduce the vectors
$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix}, \qquad n \in \mathbb{Z}.$$
Show that, for $n = 0, 1, 2$ and 3,
$$\mathbf{K}\mathbf{x}_n = \lambda_n \mathbf{x}_n$$
and find the values of the constants $\lambda_0, \lambda_1, \lambda_2$ and λ_3.

(c) Show that the vectors $\mathbf{x}_n$ for $n = 0, 1, 2$ and 3 are mutually orthogonal.

7. Consider the 5-node graph



(a) Write down the incidence matrix $\mathbf{A}$ and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{A}$.

(b) Let $\omega = e^{2\pi \mathrm{i}/5}$ be a fifth root of unity so that

$$\omega^5 = 1.$$

Now introduce the vectors

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \\ \omega^{4n} \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

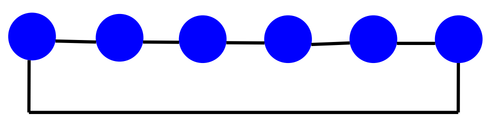
Show that, for $n = 0, 1, 2, 3$ and 4,

$$\mathbf{K}_0 \mathbf{x}_n = \lambda_n \mathbf{x}_n$$

and find the values of the constants $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ and λ_4.

(c) Show that the vectors $\mathbf{x}_n$ for $n = 0, 1, 2, 3$ and 4 are mutually orthogonal.

8. Consider the 6-node graph



(a) Write down the incidence matrix $\mathbf{A}$ and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{A}$.

(b) Let $\omega = e^{2\pi \mathrm{i}/6}$ be a sixth root of unity so that

$$\omega^6 = 1.$$

Now introduce the vectors

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \\ \omega^{4n} \\ \omega^{5n} \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

Show that, for $n = 0, 1, 2, 3, 4$ and 5,

$$\mathbf{K} \mathbf{x}_n = \lambda_n \mathbf{x}_n$$

and find the values of the constants $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5.

(c) Show that the vectors $\mathbf{x}_n$ for $n = 0, 1, 2, 3, 4$ and 5 are mutually orthogonal.

(d) In view of questions 5–7 you might have been expecting this question to put 6 vertices equally spaced around a circle. Instead we put them along a line and added an extra edge connecting the first and last node. Does this change make any difference to the linear algebra results?

9. Consider a **complete** graph with $n \geq 2$ nodes and edges between all pairs of nodes.

(a) Write down the general form of the Laplacian matrix $\mathbf{K} = \mathbf{A}^T\mathbf{A}$.

(b) Can you find n distinct non-zero vectors $\mathbf{x}$ satisfying the relation

$$\mathbf{K}\mathbf{x} = \lambda\mathbf{x}$$

for some value of λ? Find the corresponding values of λ.

(c) Suppose now that one of the nodes is grounded. Find the general form of the corresponding reduced Laplacian matrix $\mathbf{K}_0$.

(d) Can you find $n - 1$ distinct non-zero vectors $\mathbf{x}$ satisfying the relation

$$\mathbf{K}_0\mathbf{x} = \hat{\lambda}\mathbf{x}$$

for some value of $\hat{\lambda}$? Find the corresponding values of $\hat{\lambda}$.

(e) Unlike $\mathbf{K}$, the reduced Laplacian matrix $\mathbf{K}_0^{-1}$ is invertible. By directly computing $\mathbf{K}_0^{-1}$ for small values of $n = 2, 3, 4, \ldots$ and trying to spot a pattern (or indeed by any other method), can you propose a general formula for $\mathbf{K}_0^{-1}$ for general n?