Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 3

*You should prepare starred question, marked by * to discuss with your personal tutor.*

Reminder:

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad y''' = \frac{d^3y}{dx^3}, \ \ldots$$

1.* Consider a generic homogeneous second order linear differential equation:

$$\mathcal{L}_\alpha[y] = \alpha_2(x)\frac{d^2y}{dx^2} + \alpha_1(x)\frac{dy}{dx} + \alpha_0(x)y = 0.$$

The general solution of this ODE can be written as

$$y_{\text{GS}}(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants to be fixed by boundary conditions and $\{y_1(x), y_2(x)\}$ are two functions that form a basis of the two-dimensional vector space of solutions.

(a) Which of the following pairs of functions cannot be a basis of the vector space?

i. $\{e^x, e^{-x}\}$

ii. $\left\{1 - \sin^2(x), \left(1 + \tan^2(x)\right)^{-1}\right\}$

iii. $\{\ln x, \ln x^3\}$

iv. $\{e^{ax}, xe^{ax}\}$

v. $\left\{(x-1)^3, a(x^2 - 2x + 1)^{\frac{(x-1)}{4}}\right\}$

(b) Consider the functions $y_3 = \alpha y_1 + \beta y_2$ and $y_4 = \gamma y_1 + \delta y_2$. Find the condition that $\alpha, \beta, \gamma, \delta$ must fulfill so that the general solution can be expressed exclusively in terms of y_3 and y_4.

2. Find the general solution of the following homogeneous linear ODEs:

(a) $y'' + 13y' + 42y = 0$

(b) $y'' + 12y' + 36y = 0$

and the particular solution of

(c) $y'' + y' + y = 0$ with $y(0) = 0, y'(0) = 1$.

3. Find the general solution of the following inhomogeneous linear ODEs:

(a) $y'' - y' = xe^x$

(b) $y'' + 13y' + 42y = e^{-x}$

(c) $y'' + 13y' + 42y = e^{-6x}$

(d) $y'' + 12y' + 36y = x\left(1 + e^{-6x}\right)$

(e) $y'' - 2y' + 2y = \sin x$

(f) $y'' - 2y' + 2y = 4e^x \sin x$

(g) $y'' - 9y = \sinh 3x$

(h) $y'' + 4y' + 8y = e^{-2x}(1 + 3\cos x + 5\cos 2x)$

(i) $y'' + 5y' + 6y = e^{-3x}\left(1 + 4x + 3x^2\right)$

and the particular solution of

(j) $y'' - y' = xe^x$ with $y(0) = 0, y'(0) = 0$.

4. * The equation describing the elongation $x(t)$ of a harmonic oscillator of mass m under a force $F(t)$ is:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F(t)}{m},$$

where ω_0 is a positive constant.
Suppose we apply a constant force F_0 for a time T and we then stop the application of the force:

$$F(t) = \begin{cases} F_0, & 0 < t < T \\ 0, & t > T \end{cases}$$

(a) Solve the ODE for $x(t)$ given the initial conditions $x(0) = \frac{dx}{dt}(0) = 0$

(b) Find the amplitude of the oscillation for $t > T$

5. Solve the following third order linear ODEs with constant coefficients:

(a) $y''' - y = x$

(b) $y''' + 3y'' + 3y' + y = 0 \quad \text{with} \quad y(0) = y'(0) = y''(0) = 1$

(c) $y''' + 3y'' + 3y' + y = \cosh x$

6. Using the change of variables $x = e^z$, solve the following ODEs of the Euler type:

(a) $x^2y'' - 4xy' + 6y = x$

(b) $x^2y'' - 3xy' + 4y = x^2 \ln x$