Mathematics Year 1, Calculus and Applications I

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Solutions Problem Sheet 2

1. $\underline{y = x\exp(-x)}$: $y = 0$ at $x = 0$; $y < 0$ for $x < 0$; $y > 0$ for $x > 0$; $y \to 0$ as $x \to \infty$, and $y \to -\infty$ as $x \to -\infty$. In addition $y' = (1-x)\exp(-x)$, hence there is a local maximum at $x = 1$. This is the only critical point. See Figure 1.

   $\underline{y = x^2\exp(-x^2)}$: The function is symmetric about $x = 0$ and $y \geq 0$ for all x. $y = 0$ at $x = 0$ and $y \to 0$ as $|x| \to \infty$. $y' = 2x(1-x^2)\exp(-x^2)$, hence $x = 0$ is a local minimum and $x = \pm 1$ are local maxima. Sketch in Figure 2.

   $\underline{y = e^x/x}$: $y \to \pm\infty$ as $x \to 0\pm$. $y \to \infty$ as $x \to +\infty$, and $y \to 0$ as $x \to -\infty$. Also, $y' = e^x(1/x - 1/x^2)$, so $x = 1$ is the only critical point - it must be a local minimum. $y > 0$ for $x > 0$ and $y < 0$ for $x < 0$. Sketch in Figure 3.

2. For the function $f(x) = \exp(1/x)$, $x \neq 0$.

   (a) What are the limits

   $$\lim_{x\to 0+} f(x) = +\infty, \qquad \lim_{x\to 0-} f(x) = 0, \qquad \lim_{x\to +\infty} f(x) = \lim_{x\to -\infty} f(x) = 1.$$

   (b) Defining $f(0) = 0$, the function is differentiable everywhere except possibly at $x = 0$, Here we consider

   $$\lim_{h\to 0} \frac{\exp(1/h) - 0}{h},$$

   which clearly does not exist if $h > 0$.

   (c) Calculate derivatives:

   $$\frac{df}{dx} = -\frac{1}{x^2}\exp(1/x),$$

   $$\frac{d^2f}{dx^2} = \frac{1}{x^4}\exp(1/x) + \frac{2}{x^3}\exp(1/x),$$

   $$\cdots$$

   $$\frac{d^nf}{dx^n} = (-1)^n\frac{1}{x^{2n}}\exp(1/x) + g_n(x)\exp(1/x),$$

   where the function $g_n(x)$ contains terms of size x^{-2n+1} at most for small negative x. Now, $\lim_{x\to 0-} |\frac{\exp(1/x)}{x^{2n}}| = \lim_{t\to +\infty} t^{2n}\exp(-t) = 0$, and hence $\lim_{x\to 0-} g_n(x)\exp(1/x) = 0$ also by the comparison test (since it is x times something that already goes to 0).

   (d) From the result for d^2f/dx^2 we see that there is an inflection point at $x = -1/2$, $y = 1/e^2$. There are no critical points, and the asymptotes have been determined. The sketch is given in Figure 4.

3. The function $y = x\exp(1/x)$ is slightly different from that in problem 2. We have y behaving like x for large x and $\lim_{x\to 0-} x\exp(1/x) = 0$ but $\lim_{x\to 0+} x\exp(1/x) = +\infty$ as before. All derivatives are 0 at $x = 0-$ as before. Since $y' = (1-1/x)\exp(1/x)$ we must have a local minimum at $x = 1$, $y = e$. There are no other critical points. A sketch is given in Figure 5.

4. Need to show that the equation $\mathrm{e}^x = ax$ has at least one solution for any number a, except when $0 \le a < \mathrm{e}$.

   Lets do the easy cases first: (i) If $a = 0$ there is no root since $e^x > 0$. (ii) If $a < 0$ then $f(0) = 1$ and $\lim_{x\to-\infty}(e^x - ax) = -\infty$; by the intermediate value theorem there is at least one root (you can also see this graphically but that is not a proof).

   It remains to consider $a > 0$. There is probably another solution but I did it this way: Take the difference defined by $f(x) = e^x - ax$. Find the local minima for this (there is no local maximum since $f \to \infty$ as $x \to \infty$) by setting $f'(x_m) = 0$, i.e. $e^x_m - a = 0$, giving $x_m = \log a$. Hence $f(x_m) = a(1 - \log a)$ which immediately shows that $a = e$ gives a solution. If $a < e$ we have $a(1 - \log a) > 0$ hence $f(x) > 0$ and there cannot be a solution. If $a > e$ we have $f(x_m) < 0$, and since $f(0) = 1$, the intermediate value theorem guarantees a root.

5. We are given the function

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) Using the definition of the derivative we have

$$f'(0) = \lim_{h\to 0} \frac{\exp(-1/h^2) - 0}{h} = \lim_{t\to\infty} t\exp(-t^2) = 0,$$

   hence the derivative exists and $f'(0) = 0$.

   (b) Use the chain rule, $\frac{d}{dx}(e^{-1/x^2}) = \frac{2}{x^3}e^{-1/x^2}$ and combined with (a) above we have

$$f'(x) = \begin{cases} \frac{2}{x^3}\exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (c) We can see that $f^{(n)}(x)$ will contain a term proportional to $x^{-3n}e^{-1/x^2}$ along with smaller inverse powers of x (the x^{-3n} is the most singular as $x \to 0$). Since

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^{3n}} = 0, \qquad (why?)$$

   we also define $f^{(n)}(0) = 0$ and hence all higher derivatives exist.

   (d) To sketch the function we note that $f(x) \ge 0$, it is symmetric about $x = 0$, and $\lim_{|x|\to\infty} f(x) = e^0 = 1$. All derivatives are zero at $x = 0$ and there are inflection points at $x = \pm\sqrt{2/3}$, $y = e^{-3/2}$. A plot is provided in Figure 6.

6. Can write $f(x) = e^{x\log x}$, hence $f'(x) = x^x(1 + \log x)$. Considering $\lim_{x\to 0+} x^x(1 + \log x)$, we note that $\lim_{x\to 0+} x^x = 1$ (why?), and hence $\lim_{x\to 0+} f'(x)$ does not exist and in fact tends to $-\infty$. This means that the tangent at $x = 0$ is vertical. In addition, there is a local minimum at $x = e^{-1}$, $y = e^{-1/e}$, and clearly f is positive and becomes large for x large.

   A plot is given in Figure 7.

7. We can use the result $\frac{d}{dx}x^x = x^x(1 + \log x)$ in problem 7 also. Compute

$$\frac{d}{dx}\left(x^{x^x}\right) = \frac{d}{dx}e^{x^x \log x} = x^{x^x}\left(x^{x-1} + x^x(1 + \log x)\log x\right)$$

8. Yes. One example is $\log_2 \sqrt{2} = 1/2$.

9. (a) Need to find $\lim_{a\to 0}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$. The function $\frac{e^a-1}{a}$ has the form $'0/0'$ and so L'Hôpital's rule can be used to see that $\lim_{a\to 0}\frac{e^a-1}{a}=1$. Hence, $\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$ is of the form $'0/0'$ and what we have shown is that L'Hôpital's rule can be applied directly to find

$$\lim_{a\to 0}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)=\lim_{a\to 0}\frac{\frac{e^a}{e^a-1}-\frac{1}{a}}{1}=\lim_{a\to 0}\frac{ae^a-(e^a-1)}{a(e^a-1)}$$

$$=\lim_{a\to 0}\frac{ae^a}{ae^a+e^a-1}=\lim_{a\to 0}\frac{ae^a+e^a}{ae^a+2e^a}=\frac{1}{2}.$$

[We can do this much more easily using Taylor's Theorem that is coming a bit later.]

(b) For $\lim_{a\to\infty}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$ I can save myself all the differentiations by noting that $\log x$ is a strictly increasing function and hence $\log\left(\frac{e^a-1}{a}\right)<\log\left(\frac{e^a}{a}\right)=a-\log a$. Hence

$$\lim_{a\to\infty}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)<\lim_{a\to 0}\left(\frac{a-\log a}{a}\right)=\lim_{a\to 0}\left(1-\frac{\log a}{a}\right)=1,$$

since $\lim_{a\to\infty}\frac{\log a}{a}=0$, and by use of the squeezing theorem.

10. <u>$\lim_{x\to 1}x^{1/(1-x^2)}$</u> of form '$1^\infty$'.

$$x^{1/(1-x^2)}=\exp\left(\frac{1}{1-x^2}\log x\right);\quad \lim_{x\to 1}\frac{\log x}{1-x^2}=\lim_{x\to 1}\frac{1/x}{-2x}=-1/2$$

so $\lim_{x\to 1}x^{1/(1-x^2)}=e^{-1/2}$ since $\exp(x)$ is a continuous function.

<u>$\lim_{x\to 0}(\tan x)^x$</u>, $x>0$, is of form '0^0'.

$$(\tan x)^x=\exp\left(x\log(\tan x)\right);\quad x\log(\tan x)=\frac{\log(\tan x)}{(1/x)},$$

which is of the form '∞/∞' so can use L'H rule to find

$$\lim_{x\to 0}\frac{\log(\tan x)}{(1/x)}=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}}{-\frac{1}{x^2}}=-\lim x\to 0\frac{x^2}{\sin x}=0.$$

Hence $\lim_{x\to 0}(\tan x)^x=1$.

<u>$\lim_{x\to\infty}[\log x-\log(x-1)]$</u>$=\lim_{x\to\infty}\log\left(\frac{1}{1-1/x}\right)=0.$

<u>$\lim_{x\to 1}\frac{\log x}{e^x-1}$</u>$=\lim_{x\to 1}\left(\frac{1/x}{e^x}\right)=1.$

<u>$\lim_{x\to 0}\frac{\cos x-1+x^2/2}{x^4}$</u>$=\lim_{x\to 0}\frac{-\sin x+x}{4x^3}=\lim_{x\to 0}\frac{-\cos x+1}{12x^2}=\lim_{x\to 0}\frac{-\sin x}{24x}=-1/24.$

11. *Suppose that f is continuous at $x=x_0$, that $f'(x)$ exists for x in an interval about x_0, $x\neq x_0$, and that $\lim_{x\to x_0}f'(x)=m$. Prove that $f'(x_0)$ exists and equals m. [Hint. Use the mean value theorem.]*

We are given $\lim_{x\to x_0}f(x)=f(x_0)$. Also, $\lim_{x\to x_0}f'(x)=m$, hence I can write this as

$$\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}-m\right]=0,$$

and since f' exists near x_0 except possibly at x_0, we can use the MVT to find a c between x and x_0 such that the above limit has the form

$$\lim_{x \to x_0} \left[f'(c) - m \right] = 0.$$

Now as $x \to x_0$, the number c is squeezed between x and x_0, tends to x_0 in the limit, and the result follows.

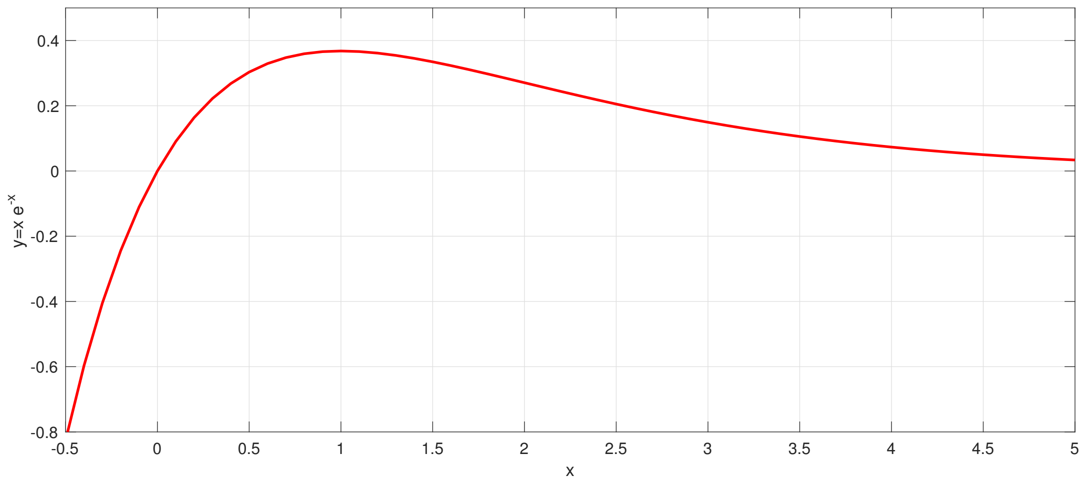

Figure 1: The function $y = x\exp(-x)$.

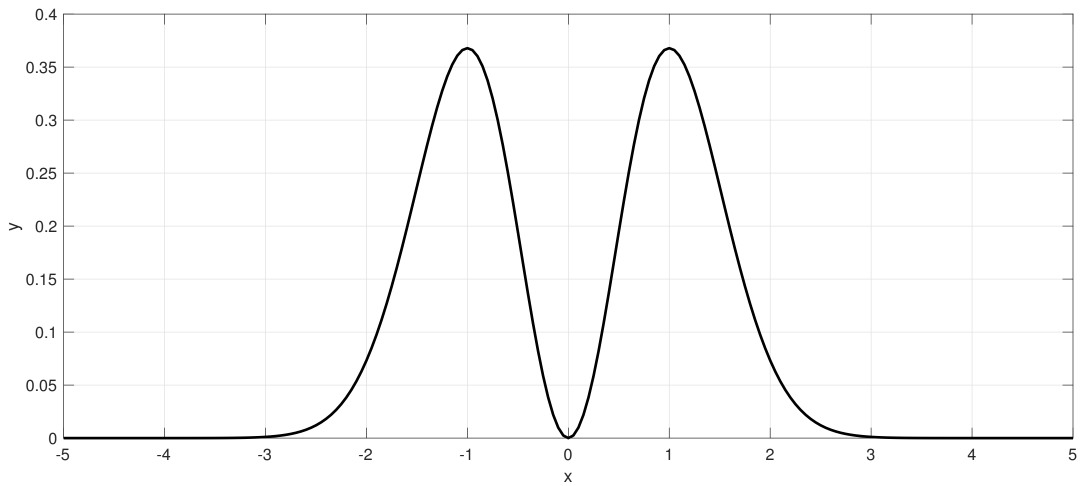

Figure 2: The function $y = x^2\exp(-x^2)$.

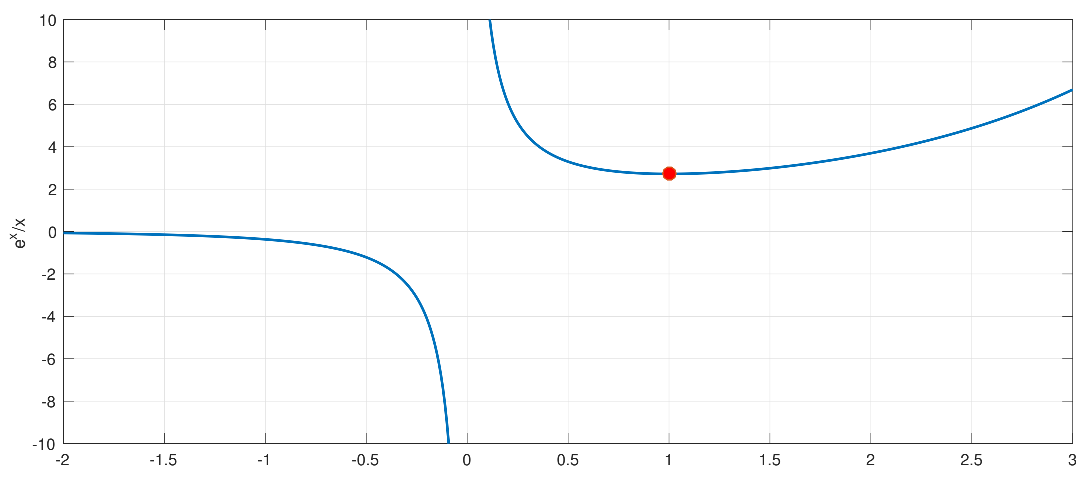

Figure 3: The function $y = \exp(x)/x$. The red dot denotes the point $(1, e)$ where the local minimum is attained.

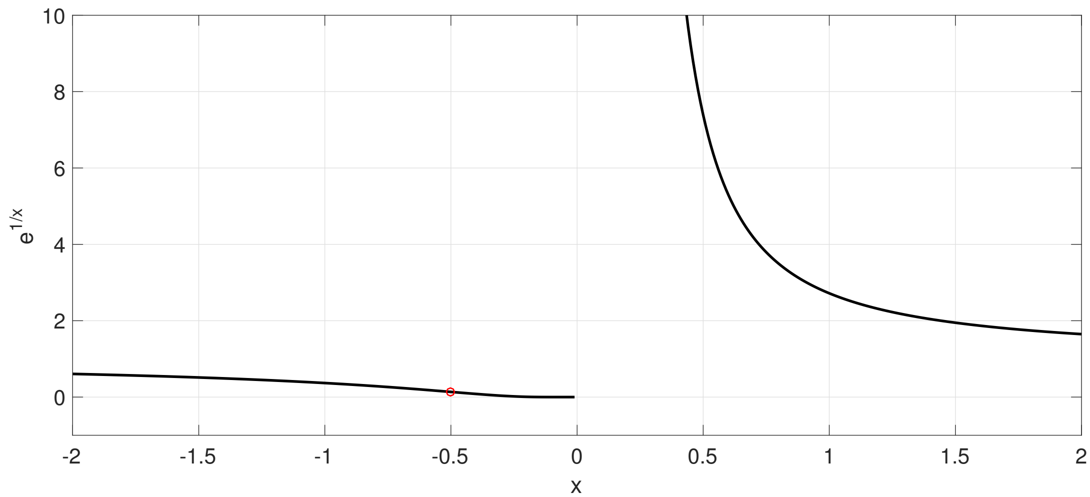

Figure 4: The function $y = \exp(1/x)$. The red dot denotes the point $(1/2, 1/e^2)$ where there is an inflection point

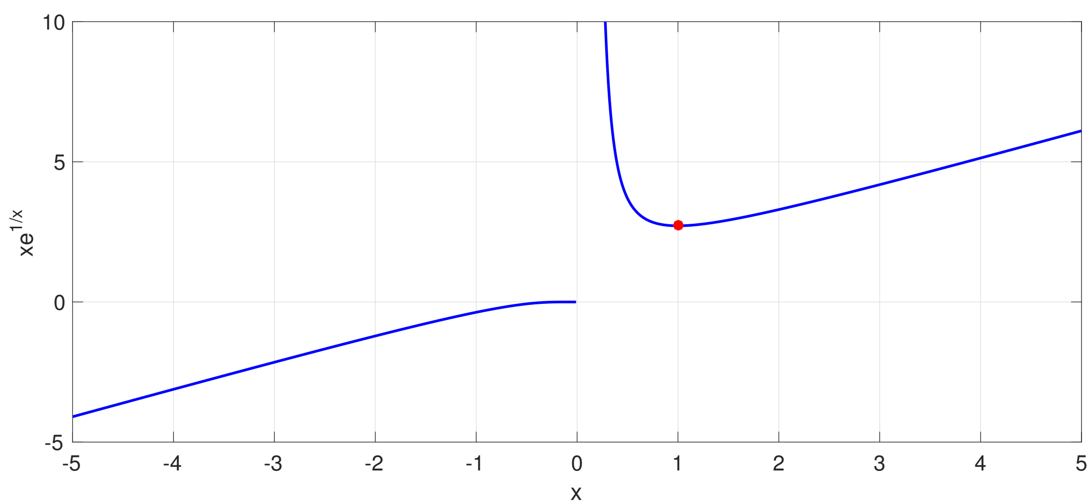

Figure 5: The function $y = x\exp(1/x)$. The red dot denotes the local minimum point $(1, e)$.

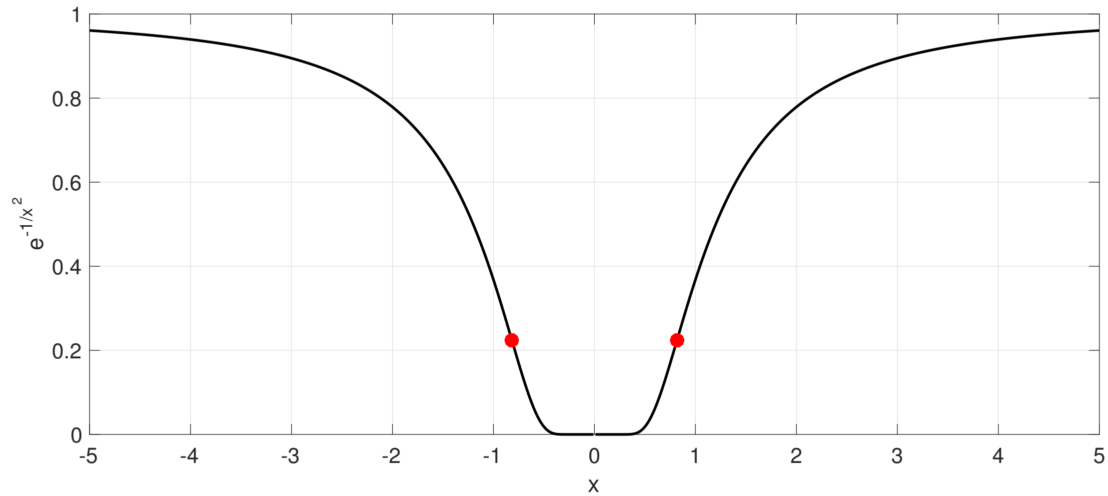

Figure 6: The function $y = \exp(-1/x^2)$. The red dots denote the inflection points $(\sqrt{2/3}, e^{-3/2})$.

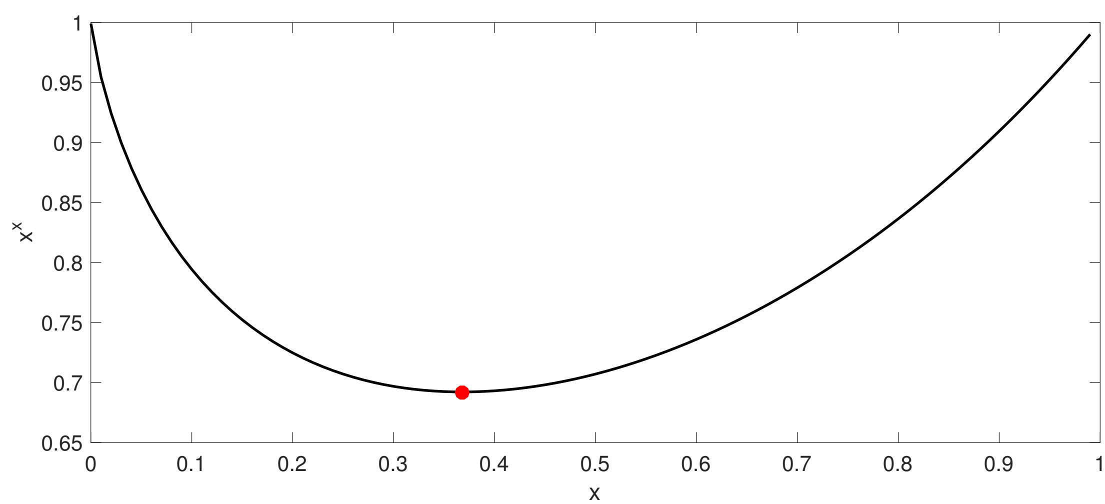

Figure 7: The function $y = x^x$. The red dot denotes the local minimum $(1/e, e^{-e})$.