M1M2: Unseen 6: Insect Outbreak

2.

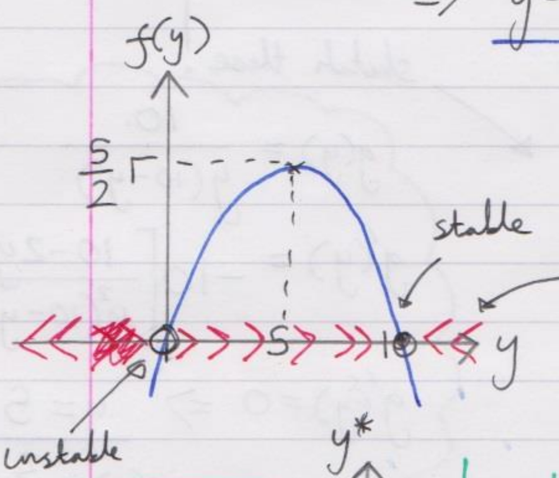
(a). $\dfrac{dy}{dt} = ry\left(1 - \dfrac{y}{10}\right)$ Logistic model.

(i). Fixed points : $f(y) = ry\left(1 - \frac{y}{10}\right) = 0$

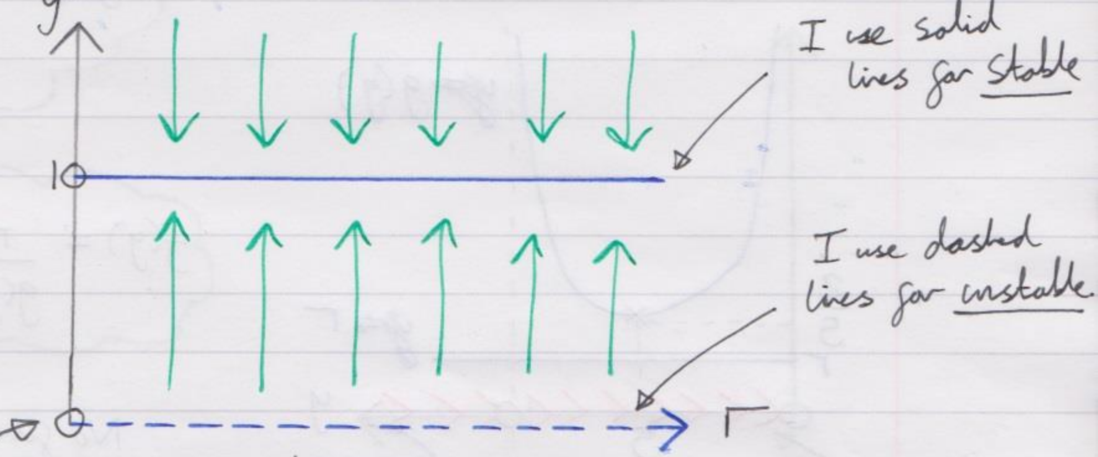
$\Rightarrow \underline{y = 0}$ or $\underline{y = 10}$

$f'(y) = r - \frac{r}{5} y$

$\Rightarrow$ Max at $\underline{y = 5,\ f(y) = \frac{5}{2} r}$



Bifurcation Diagram:



This is a $\underline{\text{transcritical}}$ bifurcation. (at $r=0$ so doesn't exist physically!)

(ii). For some $y_0 \in (0, 10)$, ~~the~~ regardless of the value of $r > 0$, the budworm population increases (away from the unstable fixed point $y^* = 0$) to $y^* = 10$ (equal to the carrying capacity $k = 10$) (arrives at the stable fixed point at $y^* = 10$).

Thus if we do nothing, the budworm will outbreak.

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(b) $\dfrac{dy}{dt} = ry\left(1-\dfrac{y}{10}\right) - 1$ (constant predation)

(i) $f(y) = ry\left(1-\dfrac{y}{10}\right) - 1$

Fixed points: $ry\left(1-\dfrac{y}{10}\right) - 1 = 0$ $\longrightarrow$ can solve this quadratic for y^* in terms of r

Maybe easier method!

or: write:

$$r = \frac{1}{y\left(1-\frac{y}{10}\right)}$$

$$\Leftrightarrow \quad r = \frac{10}{y(10-y)}$$

sketch these!

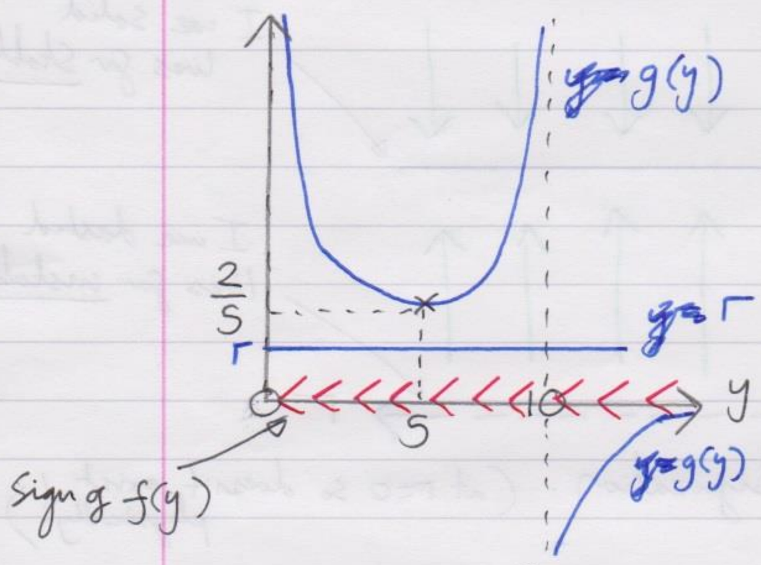
$$g(y) = \frac{10}{y(10-y)}$$

$$g'(y) = -10\left[\frac{10-2y}{y^2(10-y)^2}\right]$$

$$g'(y) = 0 \Rightarrow y = 5$$

$$g(y) = \frac{2}{5}$$

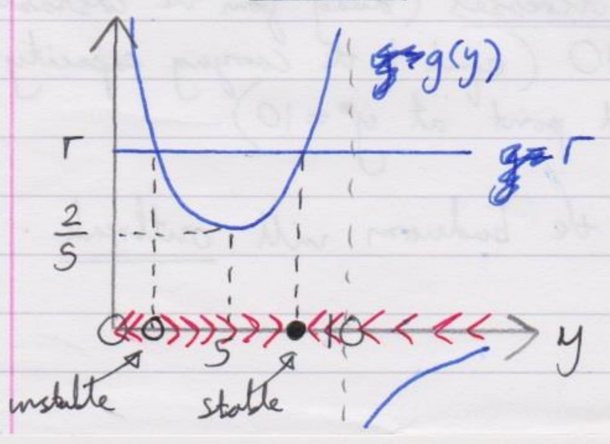
- Case (1): $0 < r < \dfrac{2}{5}$



$$f(y) = \frac{r - 1}{g(y)}$$

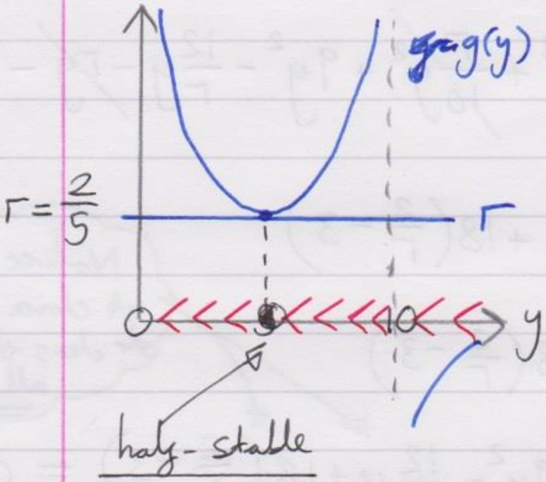
No fixed points

- Case (2): $r > \dfrac{2}{5}$



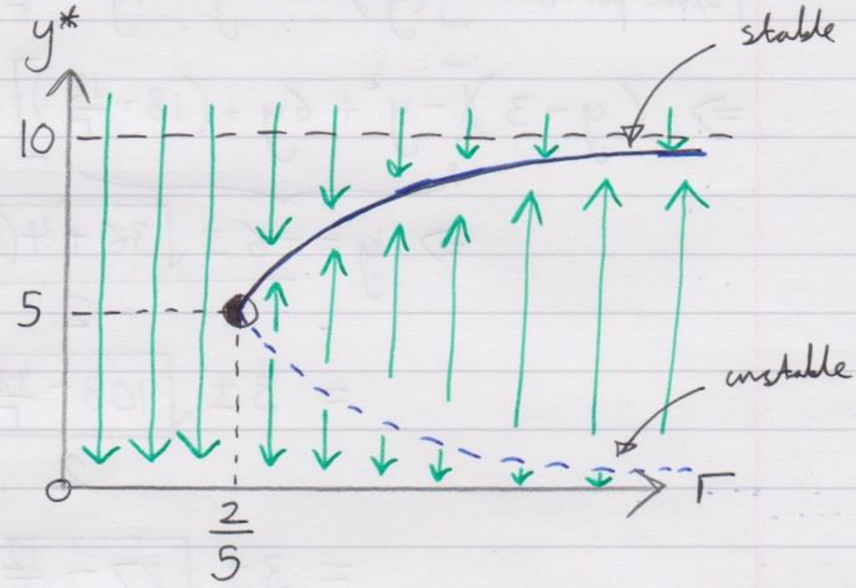
Two fixed points

- Case (3): $r = \frac{2}{5}$



One fixed point.

Bifurcation Diagram:



We have a saddle-node bifurcation.

(ii). For $0 < r < \frac{2}{5}$, and any y_0, the budworm population decays to $y^* = 0$. At $r = \frac{2}{5}$, for $y_0 \geq 5$, the budworm population decays to $y^* = 5$ and for $y_0 < 5$, decays to $y = 0$. For $r > \frac{2}{5}$, if the budworm population lies below a lower threshold (given by $y^* = 5 - \frac{1}{r}\sqrt{5r(5r-2)}$ (solve the quadratic $f(y)$ to see this)) then the population decays to $y = 0$. Otherwise the population grows/decreases to the upper branch $y^* = 5 + \frac{1}{r}\sqrt{5r(5r-2)}$.

As $r \to \infty$, the lower branch tends to $y = 0$ and the upper branch to $y = 10$.

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(c). $$\frac{dy}{dt} = ry\left(1-\frac{y}{10}\right) - \left[y^3 - \left(\frac{r}{10}+9\right)y^2 + \left(\frac{12}{r}+r\right)y + 18\left(3-\frac{2}{r}\right)\right]$$

$$\Rightarrow \frac{dy}{dt} = \cancel{ry} - \cancel{r\frac{y^2}{10}} - y^3 + \cancel{\frac{r}{10}y^2} + 9y^2 - \frac{12}{r}y - \cancel{ry} - 54 + \frac{36}{r}$$

$$\Rightarrow \frac{dy}{dt} = -y^3 + 9y^2 - \frac{12}{r}y + 18\left(\frac{2}{r}-3\right)$$

(i). $$f(y) = -y^3 + 9y^2 - \frac{12}{r}y + 18\left(\frac{2}{r}-3\right)$$

Notice only the choice $y=3$ clears this for $\underline{\text{all } r}$

Fixed points: $f(y) = -y^3 + 9y^2 - \frac{12}{r}y + 18\left(\frac{2}{r}-3\right) = 0$

So TRY $y=3$ as a factor!

$$\Rightarrow (y-3)\underbrace{\left[-y^2 + 6y + \left(18-\frac{12}{r}\right)\right]}_{} = 0$$

$$\Rightarrow y = \frac{-6 \pm \sqrt{36 + 4\left(18-\frac{12}{r}\right)}}{-2}$$

$$= \frac{3 \pm \sqrt{108 - \frac{48}{r}}}{2}$$

$$= 3 \pm \sqrt{27 - \frac{12}{r}}$$

$\Rightarrow$ Fixed points at: $\underline{y^* = 3 - \sqrt{3\left(9-\frac{4}{r}\right)}}$

$\underline{y^* = 3}$

$\underline{y^* = 3 + \sqrt{3\left(9-\frac{4}{r}\right)}}$

For $0 < r < \frac{4}{9}$, $9 - \frac{4}{r} < 0 \Rightarrow$ Only 1 Fixed point at $y^* = 3$

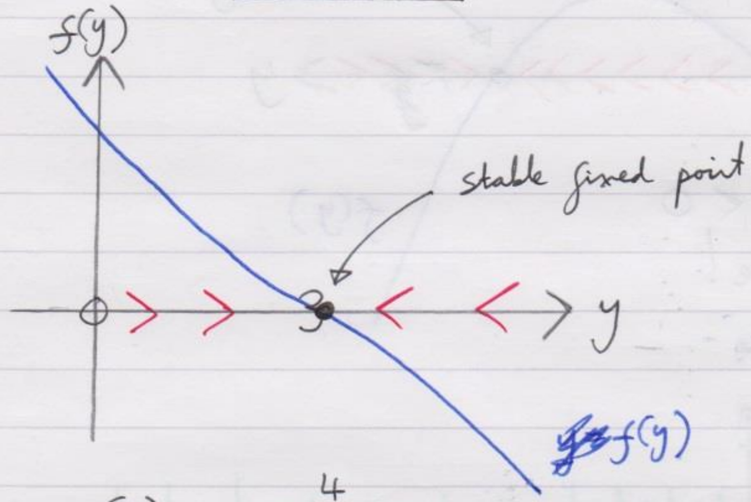
When $r = \frac{4}{9}$, $9 - \frac{4}{r} = 0 \Rightarrow$ Still 1 Fixed point at $y^* = 3$

For $\frac{4}{9} < r \leqslant \frac{2}{3}$, $3\left(9-\frac{4}{r}\right) \leqslant 9 \Rightarrow$ There are 3 fixed points!

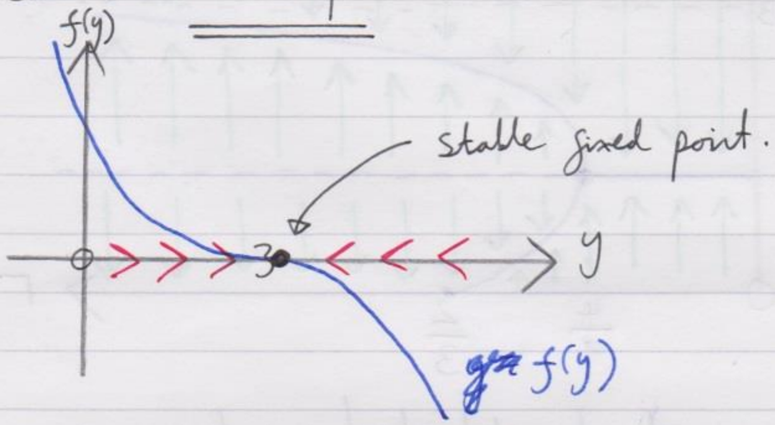
But for ~~r~~ $r > \frac{2}{3}$, the first fixed point $y^* = 3 - \sqrt{3\left(9-\frac{4}{r}\right)}$ becomes negative so even though it exists we note it is unphysical!

As $r \to \infty$, the fixed points end up at
$$y^* = 3 - 3\sqrt{3}$$
$$y^* = 3$$
$$y^* = 3 + 3\sqrt{3}$$

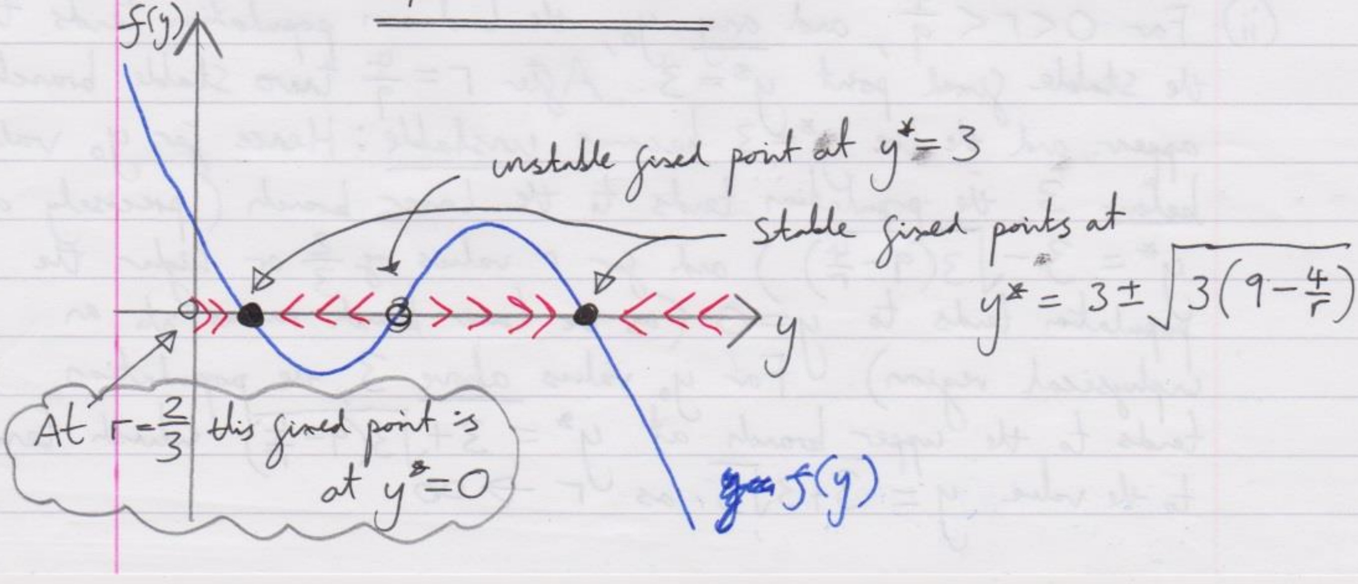
Case (1): $0 < r < \frac{4}{9}$



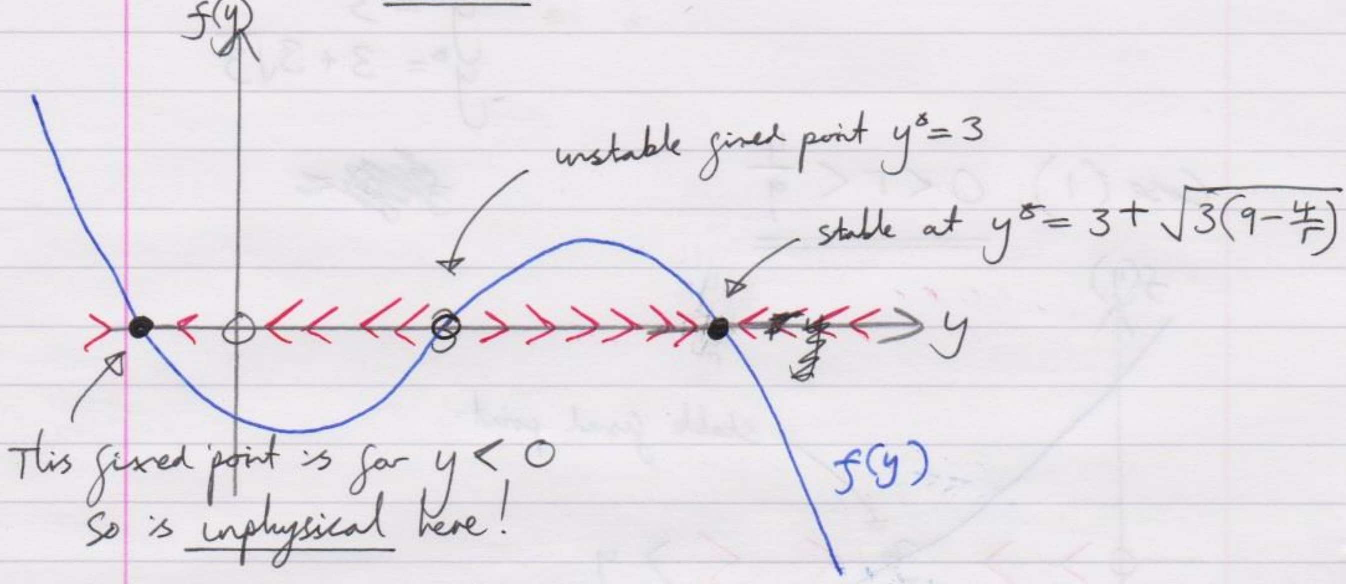
Case (2): $r = \frac{4}{9}$



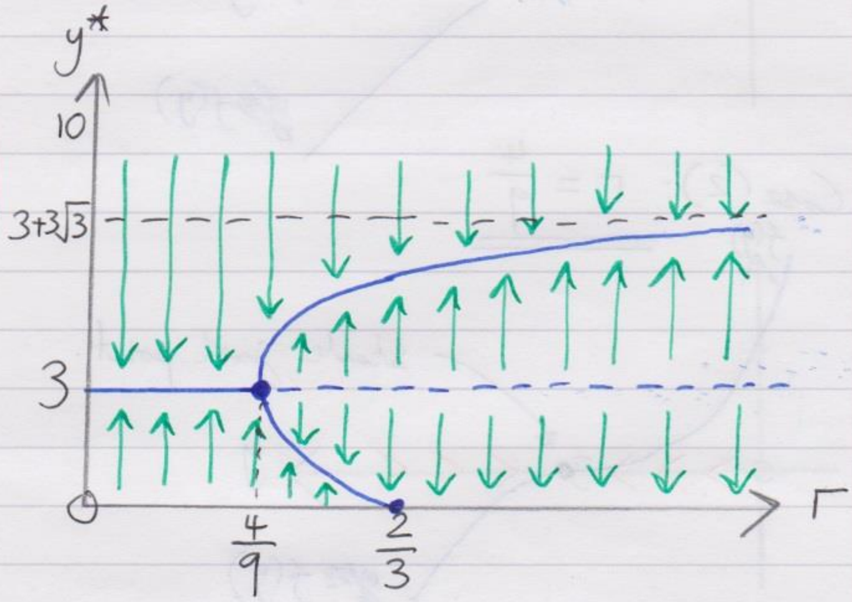
Case (3): $\frac{4}{9} < r \leqslant \frac{2}{3}$



Case (4): $r > \frac{2}{3}$



Bifurcation Diagram:



We have a Supercritical pitchfork bifurcation.

(ii) For $0 < r < \frac{4}{9}$, and any y_0, the budworm population tends to the stable fixed point $y^* = 3$. After $r = \frac{4}{9}$ two stable branches appear and the line $y^* = 3$ becomes unstable: Hence for y_0 values below 3, the population tends to the lower branch (precisely at $y^* = 3 - \sqrt{3(9 - \frac{4}{r})}$) and for r values of $\frac{2}{3}$ or higher the population tends to $y = 0$ (as the lower branch moves into an unphysical region). For y_0 values above 3, the population tends to the upper branch at $y^* = 3 + \sqrt{3(9 - \frac{4}{r})}$ which tends to the value $y = 3 + 3\sqrt{3}$ as $r \to \infty$.

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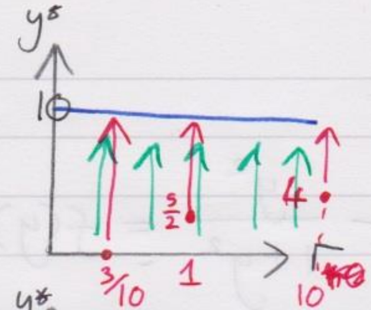
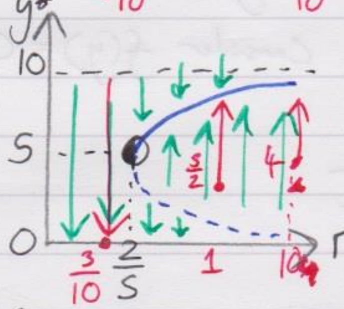
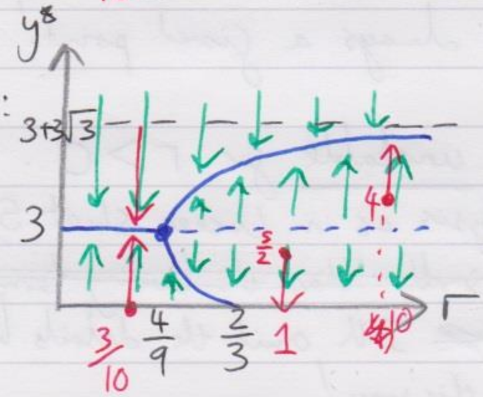
(d) Nothing:



Constant:



pesticide: (cubic)



(i). $r = \frac{3}{10}$. Marked in red.

We see that for Nothing, $y \to 10$. For pesticide $y \to 3$ and for constant $y \to 0$.

So we recomend:

Constant extermination

(ii). $r = 1$, $y_0 = \frac{5}{2}$.

For nothing, $y \to 10$.

For Constant we are above lower branch so $y \to 5 + \sqrt{15}$

For pesticide we are below $y^* = 3$, so we decay to 0.

$\Rightarrow$ recomend:

widespread pesticide

(iii). In this case we are too late to do anything (although $r = 10$ is very unphysical $\to$ budworm would be raining from the sky!) but for do nothing $y \to 10$. For constant: $y \to$ upper branch $= 5 + \frac{\sqrt{2400}}{10} < 10$

$9 <$

For pesticide:

$y \to$ upper branch $= 3 + \sqrt{\frac{129}{5}} < 9$.

So the budworm outbreak in all cases but widespread pesticide keeps it as low as possible.

$\Rightarrow$ recomend: widespread pesticide

[Of course in practice for such a little difference in outcomes ($y \to 10, \approx 10, \approx 8$) wouldn't be worth the negative effects of all that pesticide!]

3. Extension:

(a). $$\frac{dy}{dt} = ry\left(1-\frac{y}{10}\right) - \frac{y^2}{1+y^2} = f(y)$$

(i). Looking for fixed points we consider $f(y)=0$, so:

$$ry\left(1-\frac{y}{10}\right) = \frac{y^2}{1+y^2}$$

Clearly $\boxed{y^*=0}$ is always a fixed point!

In fact it is always unstable for $r>0$. To see this we can do a local analysis as in Unseen sheet 5, ~~or simply see the terms of $O(y^2)$ are smaller than the leading terms of $O(y)$ and $O(1)$ which are positive~~ I'll omit the details here!

actually not clear this way!

The remaining fixed points are the solutions to:

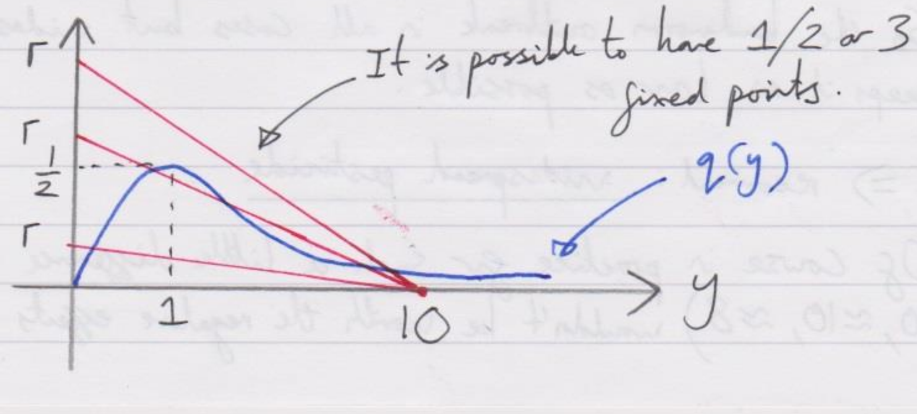
$$r\left(1-\frac{y}{10}\right) = \frac{y}{1+y^2}.$$

The LHS is a straight line passing through the axes at 10 and r with gradient $-\frac{r}{10}$.

For the RHS, let $q(y) = \frac{y}{1+y^2}$. Then $q'(y) = \frac{1-y^2}{(1+y^2)^2}$

$\Rightarrow$ Max. at $y=1$, $q(y)=\frac{1}{2}$.

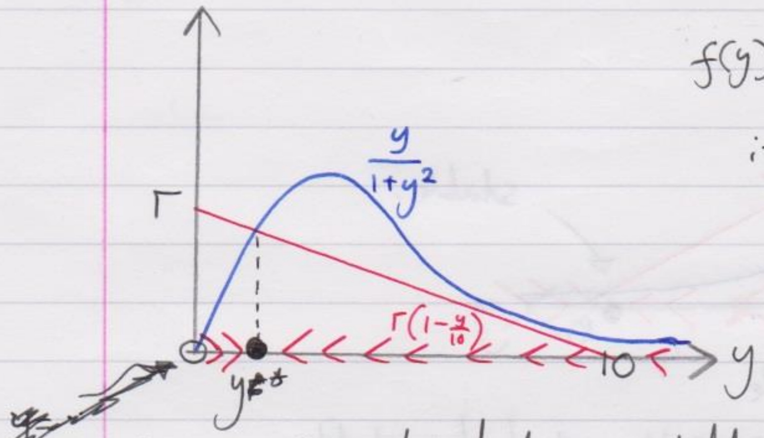
A quick sketch shows:



Case (1): $0 < \Gamma < \Gamma_1$

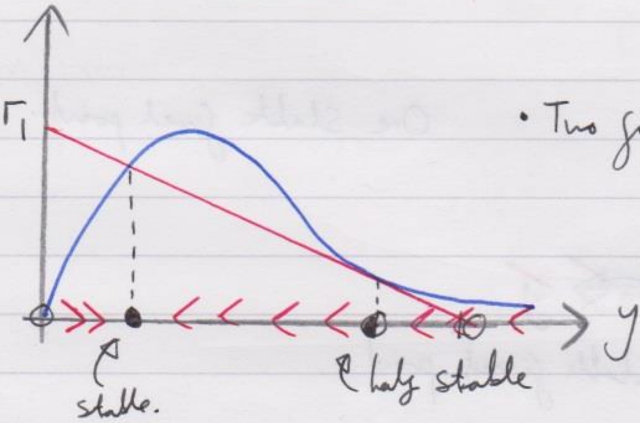
$$f(y) = y\left[\Gamma\left(1-\frac{y}{10}\right) - \frac{y}{1+y^2}\right]$$

i.e. $= \underbrace{y}_{>0}\left[\text{red} - \text{blue}\right]$



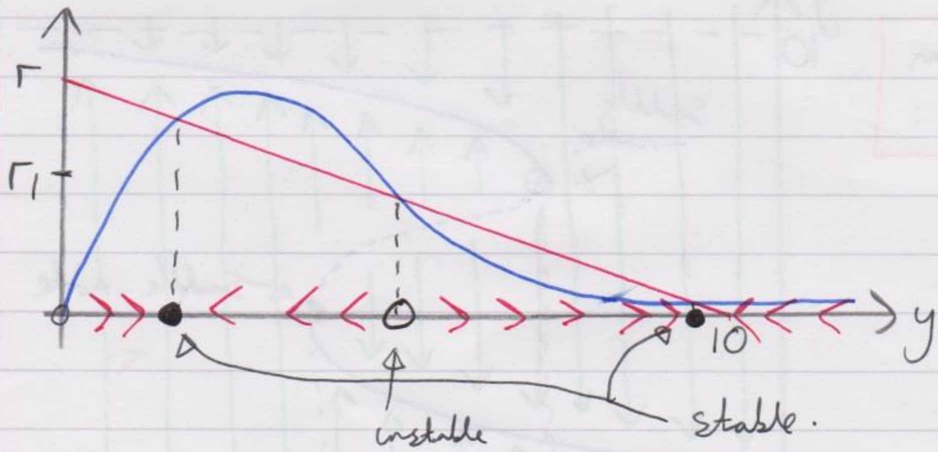
- One fixed point which is Stable.

Case (2): $\Gamma = \Gamma_1$



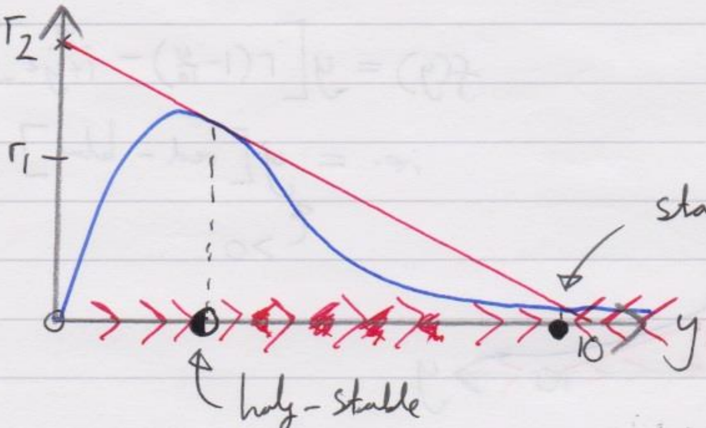
- Two fixed points: 1 stable, 1 half-stable

Case (3): $\Gamma_1 < \Gamma < \Gamma_2$



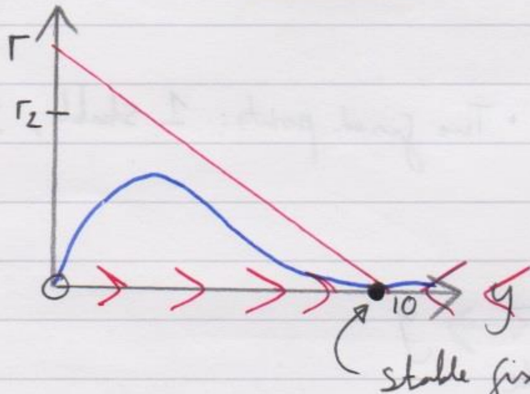
- 3 fixed points, 2 stable, 1 unstable.

Case (4): $\underline{\underline{r = r_2}}$:



- Two fixed points: 1 stable, 1 half-stable.

Case (5): $\underline{\underline{r > r_2}}$



One stable fixed point.

Putting all together:

Bifurcation diagram: