Calculus and Applications: Unseen Questions 1
The uncertainty principle

1 Fourier Transform of the Gaussian distribution

Consider the Gaussian distribution $g(x)$ defined as

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}},$$

where $g(x)$ is normalised ($\int_{-\infty}^{\infty} g(x)dx = 1$). Prove that the Fourier transform of $g(x)$ is given by:

$$\hat{g}(\omega) = e^{-\frac{\sigma^2\omega^2}{2}}.$$

Hint: Consider the Fourier transform of $\frac{dg}{dx}$ and the properties of Fourier transforms discussed in the lecture.

2 The uncertainty principle

Generally speaking, the more concentrated $f(x)$ is, the more spread out its Fourier transform $\hat{f}(\omega)$ must be. In particular, the scaling property of the Fourier transform may be seen as saying: if we squeeze a function in x, its Fourier transform stretches out in ω domain. It is not possible to arbitrarily concentrate both a function and its Fourier transform. The trade-off between the compaction of a function and its Fourier transform can be formalized in the form of an uncertainty principle. This concept is closely related to Heisenberg's uncertainty principle that asserts a fundamental limit to the precision with which the values for certain pairs of conjugate variables such as position and momentum.

The statement for the uncertainty principle for the Fourier transforms uses the second moment of the squared modulus of the function $|f(x)|^2$ defined as

$$(\Delta x)^2 = \frac{\int_{-\infty}^{\infty} x^2 f(x) f^*(x)\, dx}{\int_{-\infty}^{\infty} f(x) f^*(x)\, dx};$$

Similarly, we define in the frequency domain

$$(\Delta \omega)^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \hat{f}(\omega) \hat{f}^*(\omega)\, d\omega}{\int_{-\infty}^{\infty} \hat{f}(\omega) \hat{f}^*(\omega)\, d\omega}.$$

Using these definitions the uncertainty principle for the Fourier transforms states that

$$\Delta x \Delta \omega \geqslant \frac{1}{2}.$$

To prove this statement we need the following two results. Firstly

$$\int_{-\infty}^{\infty} f'(x) f'^*(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 \hat{f}(\omega) \hat{f}^*(\omega)\, d\omega$$

Show this is true using the energy theorem.

Secondly, we need the *Schwarz's inequality* for complex functions that states if F and G are complex functions of x, we have

$$\left[\int (F^* G + F G^*)\, dx\right]^2 \leqslant 4 \int F F^*\, dx \int G G^*\, dx.$$

Prove this using the fact that for a real constant ϵ we have:

$$\int (F + \epsilon G)(F + \epsilon G)^*\, dx \geqslant 0$$

.

Using the above results prove the uncertainty principle.

Finally, show that for the Gaussian distribution the eqaulity in the uncertaintly principle is achieved.