Problem Sheet 1

Math40002, Analysis 1

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(\mathbb{R}) \subset \mathbb{Q}$. Prove that f is constant.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Is this still true if we only assume that $f(x) = g(x)$ for $x \in \mathbb{Z}$?

3. Consider the function $f : [1,2] \cap \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = \left|x - \sqrt{2}\right|$. Prove that f does *not* have a minimum value. Why doesn't the extreme value theorem apply?

4. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/n, & x = m/n \end{cases}.$$

   Here all rational numbers $x = \frac{m}{n}$ are written in lowest terms, with $n > 0$.

   (a) Prove that if x is rational, then f is not continuous at x.

   (b) Prove that if x is irrational, then f is continuous at x.

5. Let $f : [a,b] \to \mathbb{R}$ be continuous, and suppose that $f(a) \le y \le f(b)$.

   (a) Let $(a_0, b_0) = (a, b)$, and for all $n \ge 0$, define $m_n = \frac{a_n + b_n}{2}$ and

$$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, m_n), & f(m_n) \ge y \\ (m_n, b_n), & f(m_n) \le y. \end{cases}$$

   Prove that the sequences $\big(a_n\big)$ and $\big(b_n\big)$ converge to the same limit $L \in [a,b]$.

   (b) Prove that $f(L) = y$, concluding a new proof of the intermediate value theorem.

6. For any nonempty set $S \subset \mathbb{R}$, define $d_S : \mathbb{R} \to \mathbb{R}$ by $d_S(x) = \inf\limits_{s \in S} |x - s|$.

   (a) Describe or draw graphs of d_S when S is each of $\{0\}$, $\{-1, 3\}$, $\mathbb{Z}$, $\mathbb{Q}$.

   (b) Prove that $|d_S(y) - d_S(x)| \le |y - x|$ for all $x, y \in \mathbb{R}$, and conclude that d_S is continuous.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, not necessarily continuous. Define $S(x) = \sup\limits_{y<x} f(y)$ and $I(x) = \inf\limits_{y>x} f(y)$.

   (a) Prove for all $x \in \mathbb{R}$ that $S(x) \le f(x) \le I(x)$.

(b) Prove for all $x \in \mathbb{R}$ that $S(x) = I(x)$ if and only if f is continuous at x.

(c) Find an injective mapping

$$\{x \in \mathbb{R} \mid f \text{ is not continuous at } x\} \to \mathbb{Q}.$$

Conclude that the set of discontinuities of f is at most countably infinite.