Problem Sheet 5

Math40002, Analysis 1

1. In lecture, we needed the claim that $\lim_{x\to\infty} xs^{x-1} = 0$ for any $s \in (0,1)$ in order to prove that the term-by-term derivative of a power series converges inside that power series's radius of convergence.

   (a) Prove that for all $c > 0$, there exists $N > 0$ such that $\log(x) < cx$ for all $x \geq N$.

   (b) Prove that $\lim_{x\to\infty} xs^x = 0$, and show that this implies the above claim.

Solution. (a) It's enough to prove that $\lim_{x\to\infty} \frac{\log(x)}{x} = 0$, since then there's an $N > 0$ such that $0 < \frac{\log(x)}{x} < c$ for all $x \geq N$. This limit exists by l'Hôpital's rule, which says that it is equal to $\lim_{x\to\infty} \frac{1/x}{1} = 0$.

(b) For any $c > 0$, part (a) says that $0 < xs^x < e^{cx}s^x = (e^c s)^x$ for all large enough x. Since $0 < s < 1$, we can choose a positive $c < \log(1/s)$ so that $0 < e^c s < 1$, and then

$$\lim_{x\to\infty} (e^c s)^x = 0.$$

Thus the squeeze theorem says that $\lim_{x\to\infty} xs^x = 0$ as well. We conclude that

$$\lim_{x\to\infty} xs^{x-1} = \frac{1}{s}\left(\lim_{x\to\infty} xs^x\right) = 0.$$

2. (a) Compute the Taylor series $P(x)$ of $f(x) = \log(1+x)$ centered at $x = 0$, and prove that it converges absolutely on $(-1,1)$.

   (b) Prove using Taylor's theorem that $f(x) = P(x)$ on some open neighborhood of 0, by showing that the sequence of nth order Taylor polynomials $P_n(x)$ converges uniformly to $f(x)$. Show that the same is true at $x = 1$, and so $\log(2) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$.

Solution. (a) We have $f'(0) = \frac{1}{1+x}$, and we claim by induction that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

for all $n \geq 1$: if it's true for $n = k$ then we have

$$f^{(k+1)}(x) = (-1)^{k-1}(k-1)! \cdot (-k)(1+x)^{-k-1} = (-1)^k k!(1+x)^{-(k+1)}$$

as desired. Then $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ for $n \geq 1$, and $f(0) = \log(1) = 0$, so $f(x)$ has Taylor series

$$P(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!x^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n},$$

which has the form $\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$. Absolute convergence follows from the comparison test, since $|\frac{(-1)^{n-1}x^n}{n}| \leq |x^n|$ and $\sum_{n=1}^{\infty} x^n$ is a geometric series which converges absolutely on $(-1, 1)$.

(b) By Taylor's theorem, if $x > -1$ is nonzero then we have

$$f(x) = P_n(x) + \frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1}$$

for some t between 0 and x. The same computation as in part (a) says that $f^{(n+1)}(t) = (-1)^n n!(1+t)^{-(n+1)}$, so

$$|f(x) - P_n(x)| = \left| \frac{(-1)^n n!(1+t)^{-(n+1)}}{(n+1)!}x^{n+1} \right| = \frac{1}{n+1}\left| \frac{x}{1+t} \right|^{n+1}.$$

Now if $0 < x \leq 1$ then we have $0 < t < x$, so $1+t > 1 \geq x$ and hence $\left|\frac{x}{1+t}\right| < 1$. If instead $-\frac{1}{2} \leq x < 0$ then we have $1+t > 1+x \geq \frac{1}{2} > |x|$, so again $\left|\frac{x}{1+t}\right| < 1$. Thus for any nonzero $x \in [-\frac{1}{2}, 1]$ we have

$$|f(x) - P_n(x)| \leq \frac{1}{n+1},$$

and so P_n converges uniformly to $f(x) = \log(1+x)$ on this interval.

Remark: In fact $f(x) = P(x)$ on all of $(-1, 1)$, but we need better control over t to prove this on the interval $(-1, \frac{1}{2})$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has at least six continuous derivatives, and that $f^{(i)}(0) = 0$ for $i = 1, 2, 3, 4, 5$ but $f^{(6)}(0) = 1$. Prove that $f(x)$ has a local minimum at $x = 0$.

Solution. We apply Taylor's theorem to see that if $x \in (-\delta, \delta)$ is nonzero, then there is some t between 0 and x such that

$$f(x) = \sum_{i=0}^{5} \frac{f^{(i)}(0)x^i}{i!} + \frac{f^{(6)}(t)x^6}{6!} = f(0) + \frac{f^{(6)}(t)x^6}{6!}.$$

Since $f^{(6)}(x)$ is continuous, there is some $\delta > 0$ such that

$$|y - 0| < \delta \;\Rightarrow\; |f^{(6)}(y) - f^{(6)}(0)| < 1,$$

hence $f^{(6)}(y) > 0$ for all $y \in (-\delta, \delta)$. If we take $x \in (-\delta, \delta)$ above then $t \in (-\delta, \delta)$ as well, so $f^{(6)}(t) > 0$, and then since $\frac{x^6}{6!} \geq 0$ we conclude that $f(x) \geq f(0)$ for all $x \in (-\delta, \delta)$.

4. (a) Suppose that some function $f : (-R, R) \to \mathbb{R}$ is equal to the power series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, which converges absolutely on $(-R, R)$. Prove that the Taylor series of f centered at $a = 0$ is precisely $\displaystyle\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, and hence that this power series is unique.

   (b) Compute the Taylor series of $f(x) = \frac{1}{1-x^2}$ centered at $a = 0$. What is $f^{(100)}(0)$?

Solution. (a) Since we can differentiate power series term by term inside their radius of convergence, it follows by induction that $f^{(k)}(x)$ exists and that

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k},$$

absolutely convergent on the interval $(-R, R)$, for all k. This gives us $f^{(k)}(0) = k!a_k$, and so $f(x)$ has Taylor series

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{n!a_n \cdot x^n}{n!} = \sum_{n=0}^{\infty} a_n x^n.$$

In other words, if f is equal to some power series on $(-R, R)$ then that power series must be the Taylor series centered at $x = 0$, and so that power series is unique.

(b) Computing the derivatives of $f(x)$ gets messy very quickly, so instead we note that $f(x)$ is the sum of a geometric series

$$f(x) = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n}$$

on the interval $(-1, 1)$, and this is a power series, so it must be the Taylor series for $f(x)$. The coefficient of x^{100} is 1, and it's also supposed to be equal to $\frac{f^{(100)}(0)}{100!}$, so we must have $f^{(100)}(0) = 100!$.

5. (a) Prove that $f(x) = e^x$ is convex on all of $\mathbb{R}$.

   (b) Let $a, b > 0$. Use the convexity of e^x to prove the *arithmetic mean–geometric mean inequality*
   $$\frac{a+b}{2} \geq \sqrt{ab}.$$
   (Hint: think about $\alpha = \log(a)$ and $\beta = \log(b)$.)

   (c) Prove for any $a, b > 0$ and $s \in [0, 1]$ that $sa + (1-s)b \geq a^s b^{1-s}$.

   (d) Prove *Young's inequality*: for any $x, y \geq 0$ and p, q positive with $\frac{1}{p} + \frac{1}{q} = 1$, we have
   $$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$

Solution. (a) It suffices to check that that $f''(x) \geq 0$ for all x, and this is certainly true since $f''(x) = e^x$.

(b) Assuming $\alpha < \beta$ without loss of generality, the convexity of e^x implies for $\alpha < \frac{\alpha+\beta}{2} < \beta$ that

$$\frac{e^\alpha + e^\beta}{2} \geq e^{(\alpha+\beta)/2} = \sqrt{e^\alpha \cdot e^\beta}$$

which is equivalent to $\frac{a+b}{2} \geq \sqrt{ab}$.

(c) Since e^x is convex, we know that

$$se^\alpha + (1-s)e^\beta \geq e^{s\alpha+(1-s)\beta},$$

and the left side is $sa + (1-s)b$ while the right side is $(e^\alpha)^s(e^\beta)^{1-s} = a^s b^{1-s}$.

(d) We may assume that $x, y > 0$, since otherwise the inequality reduces to $\frac{x^p}{p} + \frac{y^q}{q} = 0$, which is true. We now use part (c), setting $s = \frac{1}{p}$ (so $1 - s = \frac{1}{q}$) and $(a, b) = (x^p, y^q)$, to get

$$\frac{x^p}{p} + \frac{y^q}{q} \geq (x^p)^{1/p}(y^q)^{1/q} = xy.$$

6. (*) Let (a_n) denote the Fibonacci sequence, with $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$.

(a) Prove by induction that $a_n < 2^n$ for all $n \geq 0$. What is the radius of convergence of the *exponential generating function*

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0 + 1x + \frac{1x^2}{2} + \frac{2x^3}{6} + \frac{3x^4}{24} + \ldots?$$

(b) Prove that $F''(x) = F'(x) + F(x)$, and that $F(0) = 0$ and $F'(0) = 1$.

(c) Solve this differential equation for $F(x)$.

(d) Use the solution from part (c) to prove *Binet's formula*:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Solution. (a) We have $a_0 < 2^0$ and $a_1 < 2^1$, and if $a_n < 2^n$ and $a_{n+1} < 2^{n+1}$ then

$$a_{n+2} = a_{n+1} + a_n < 2^{n+1} + 2^n < 2 \cdot 2^{n+1} = 2^{n+2},$$

so it follows by induction that $a_k < 2^k$ for all $k \geq 0$.

We now have $\left|\frac{a_n x^n}{n!}\right| < \left|\frac{2^n x^n}{n!}\right| = \left|\frac{(2x)^n}{n!}\right|$, so the comparison test says that $F(x)$ converges absolutely whenever $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$ does. The latter is equal to e^{2x} for all $x \in \mathbb{R}$, so $F(x)$ has infinite radius of convergence.

(b) Since the power series for F has infinite radius of convergence, we can differentiate term by term to get

$$F'(x) = \sum_{n=0}^{\infty} \frac{na_n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{a_n x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{a_{m+1} x^m}{m!},$$

where in the last step we substitute $m = 1$, and this also has infinite radius of convergence. We repeat this argument to get

$$F''(x) = \sum_{n=0}^{\infty} \frac{na_{n+1} x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{a_{n+1} x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{a_{m+2} x^m}{m!}.$$

Since these power series all converge absolutely, we can rearrange them to get

$$\begin{aligned} F(x) + F'(x) &= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} + \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a_n + a_{n+1}) x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+2} x^n}{n!} = F''(x). \end{aligned}$$

We also have $F(0) = a_0 = 0$ and $F'(0) = a_1 = 1$ by inspection.

(c) The roots of $x^2 - x - 1 = 0$ are $r = \frac{1}{2}(1 + \sqrt{5})$ and $s = \frac{1}{2}(1 - \sqrt{5})$, so the general solution to $y'' - y' - y = 0$ is

$$y = c_1 e^{rx} + c_2 e^{sx}.$$

The initial conditions $y(0) = 0$ and $y'(0) = 1$ are equivalent to

$$\begin{aligned} c_1 + c_2 &= 0 \\ rc_1 + sc_2 &= 1, \end{aligned}$$

with solution $c_1 = \frac{1}{r-s} = \frac{1}{\sqrt{5}}$ and $c_2 = -c_1 = -\frac{1}{\sqrt{5}}$, so we have

$$F(x) = \frac{e^{rx} - e^{sx}}{\sqrt{5}}.$$

(d) From the above closed form for $F(x)$, we have

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} \frac{(rx)^n}{n!} - \sum_{n=0}^{\infty} \frac{(sx)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{r^n - s^n}{\sqrt{5}} \right) \frac{x^n}{n!}. \end{aligned}$$

Since this power series is equal to $\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, the coefficients of each x^n must be the same, so

$$a_n = \frac{r^n - s^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

7. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

(a) Prove that for all integers $n \geq 0$, there is a polynomial $p_n(x)$ such that

$$f^{(n)}(x) = \frac{p_n(x)}{x^{3n}} e^{-1/x^2} \text{ for all } x \neq 0.$$

(b) Prove that $f^{(n)}(0) = 0$ for all n, and hence that $f(x)$ does not equal its Taylor series (centered at $a = 0$) at any nonzero x.

(c) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0. \end{cases}$ Prove that $g^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $g^{(n)}(0) = 0$ for all n.

(d) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x)g(1-x)$. Prove that h is infinitely differentiable, meaning that $h^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $h(x) \neq 0$ if and only if $0 < x < 1$.

Solution. (a) When $n = 0$ we take $p_0(x) = 1$. If this holds for $n = k$, we compute

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} \left(\frac{p_k(x)}{x^{3k}} e^{-1/x^2} \right) \\ &= \frac{p_k'(x) x^{3k} - 3k x^{3k-1} p_k(x)}{x^{6k}} e^{-1/x^2} + \frac{p_k(x)}{x^{3k}} \left(\frac{2}{x^3} e^{-1/x^2} \right) \\ &= \left(\frac{p_k'(x) x^3 - 3k x^2 p_k(x) + 2 p_k(x)}{x^{3k+3}} \right) e^{-1/x^2}, \end{aligned}$$

so we can take $p_{k+1} = x^3 p_k' - (3kx^2 - 2) p_k$ and the proof follows by induction.

(b) When $n = 0$ it is true by definition. If we have proved it for all $n \leq k$, then for $n = k+1$ we have

$$f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x \to 0} \frac{f^{(k)}(x)}{x},$$

assuming the limit exists, and so using part (a) we wish to prove that

$$\lim_{x \to 0} \frac{p_k(x)}{x^{3k+1} e^{1/x^2}} = 0.$$

Since $p_k(x)$ is continuous, it will suffice to prove that $\lim_{x\to 0} |x^{3k+1}e^{1/x^2}| = \infty$. By the substitution $y = \frac{1}{x}$ we have

$$\lim_{x\to 0} |x^{3k+1}e^{1/x^2}| = \lim_{|y|\to\infty} \left| \frac{e^{y^2}}{y^{3k+1}} \right| = \lim_{y\to\infty} \frac{e^{y^2}}{y^{3k+1}}.$$

But since $y^2 \geq 0$, every term in the power series $e^{y^2} = \sum_{i=0}^{\infty} \frac{(y^2)^i}{i!}$ is nonnegative, and so if we single out the $i = 2k+1$ term then

$$e^{y^2} \geq \frac{(y^2)^{2k+1}}{(2k+1)!} \quad \Rightarrow \quad \frac{e^{y^2}}{y^{3k+1}} \geq \frac{y^{4k+2}/(2k+1)!}{y^{3k+1}} = \frac{y^{k+1}}{(2k+1)!}.$$

The right side certainly goes to ∞ as $y \to \infty$, hence $\lim_{x\to 0} |x^{3k+1}e^{1/x^2}| = \infty$ and this proves that $f^{(k+1)}(0) = 0$. The proof follows for all n by induction.

The Taylor series of f at $a = 0$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$, but clearly for all $x \neq 0$ this is not equal to $f(x) = e^{-1/x^2} > 0$.

(c) For all n, we have

$$g^{(n)}(x) = \begin{cases} 0, & x < 0 \\ f^{(n)}(x), & x > 0. \end{cases},$$

so we only need to check that $g^{(n)}(0)$ exists and is zero for all n. Again, we induct: it is true when $n = 0$, and if it is true for for $n = k$ then

$$\frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0} = \frac{g^{(k)}(x)}{x} = \begin{cases} 0, & x < 0 \\ f^{(k)}(x), & x > 0. \end{cases}$$

Thus $\lim_{x\uparrow 0} \frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0} = 0$ by inspection, and

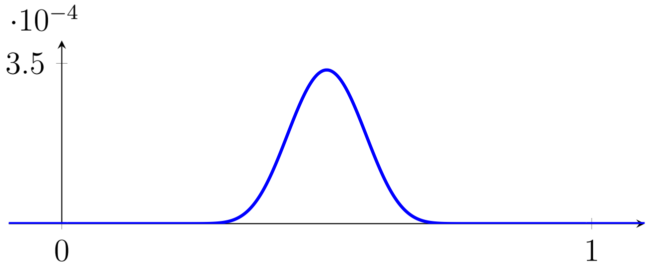
$$\lim_{x\downarrow 0} \frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0} = \lim_{x\downarrow 0} \frac{f^{(k)}(x)}{x} = f^{(k+1)}(0) = 0$$

by part (b), so $g^{(k+1)}(0) = \lim_{x\to 0} \frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0}$ exists and is zero as well.

(d) Since $g(x)$ and $g(1-x)$ are infinitely differentiable, repeated application of the product rule says that $h(x) = g(x)g(1-x)$ has n derivatives for all n as well. Moreover, we have $g(x) = 0$ for $x \leq 0$ and $g(1-x) = 0$ for $x \geq 1$, so $h(x) = 0$ for all $x \notin (0,1)$; and if $0 < x < 1$ then

$$h(x) = g(x)g(1-x) = e^{-1/x^2} \cdot e^{-1/(1-x)^2} > 0.$$

Here is a graph of $h(x)$:



Note that $h(x)$ is very small on the interval $(0,1)$ – the maximum value is $h(\frac{1}{2}) = e^{-8} \approx 0.000335\ldots$ – and it decays to zero so quickly that it's hard to see from the graph that $h(x) > 0$ for most x on this interval, but $h(x)$ is indeed positive there.