Math40002 Analysis 1 Problem Sheet 7

1. Let a_n, b_n be sequences of real numbers such that $b_n \neq 0$ and $a_n/b_n \to r \in \mathbb{R}$.

   - Prove that if $\sum b_n$ is absolutely convergent, then so is $\sum a_n$.
   - † Give examples (for any r) for which $\sum b_n$ is convergent but $\sum a_n$ diverges.

   Set $\epsilon = 1$, then $a_n/b_n \to r$ means $\exists N \in \mathbb{N}$ such that $n \geq N \ \Rightarrow\ |a_n/b_n - r| < 1 \ \Rightarrow\ |a_n| < (r+1)|b_n|$ so by the comparison test we see that $\sum |b_n|$ convergent $\Rightarrow\ \sum |a_n|$ convergent.

   The example $b_n = (-1)^n/\sqrt{n},\ a_n = rb_n + 1/n$ has $\sum b_n$ convergent (by alternating series test) but $\sum a_n$ divergent (because $\sum rb_n$ convergent and $\sum 1/n$ divergent).

2.† Give an example of sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ such that $a_n/b_n \to 1$ as $n \to \infty$ but $\sum_n a_n$ is convergent and $\sum_n b_n$ is divergent.

   $a_n = \frac{(-1)^n}{\sqrt{n}}$ and $b_n = \frac{1}{n} + \frac{(-1)^n}{\sqrt{n}}$. $\sum a_n$ convergent by alternating series test. Therefore $\sum b_n$ divergent because $\sum \frac{1}{n}$ is divergent.

3. Suppose that $a_n \in \mathbb{C} \setminus \{0\}\ \forall n$ and $a_{n+1}/a_n \to a \in \mathbb{C}$. What is the radius of convergence of $\sum_{n=1}^{\infty} a_n z^n$? Prove it!

   Compute the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(n!)^2 z^n}{(2n)!}$.

   Since $a_{n+1}z^{n+1}/a_n z^n \to az$ as $n \to \infty$, the ratio test tells us that the power series converges for $|az| < 1$ and diverges for $|az| > 1$.

   Thus it converges for $|z| < 1/|a|$ and diverges for $|z| > 1/|a|$, so $R = 1/|a|$.

   Taking $a_n = (n!)^2/(2n)!$ we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{(1+\frac{1}{n})^2}{4(1+\frac{1}{n})(1+\frac{1}{2n})} \longrightarrow \frac{1}{4}.$$

   Therefore $R = 4$.

4. Determine the radius of convergence of the following power series.

   (i) $\sum_{n=1}^{\infty} \frac{z^n}{3^n+5^n}$, (iii) $\sum_{n=1}^{\infty} \frac{n!}{1.3.5..(2n+1)} z^n$,

   (ii) $1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$, (iv) $\sum_{n=1}^{\infty} (n!)^{1/n} z^n$.

   (i) Ratio test gives $\frac{3^n+5^n}{3^{n+1}+5^{n+1}}|z| = \frac{(3/5)^n+1}{(3/5)^{n+1}+1}\frac{|z|}{5} \to \frac{|z|}{5}$ so $R = 5$.

   (ii) Write as $\sum (-1)^n \frac{z^{2n}}{(2n)!}$ and apply ratio test to this to give $-\frac{|z|^2}{(2n+2)(2n+1)} \to 0$ so always convergent: $R = \infty$.

   (iii) Ratio test gives $\frac{(n+1)|z|}{2n+3} \to \frac{|z|}{2}$ so $R = 2$.

   (iv) Ratio test: $\left|\frac{((n+1)!)^{\frac{1}{n+1}} z^{n+1}}{(n!)^{\frac{1}{n}} z^n}\right| = \frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n+1}+\frac{1}{n(n+1)}}}|z| = \frac{(n+1)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n(n+1)}}}|z|$. We showed on the last sheet that $n^{\frac{1}{n}} \to 1$. Similarly $1 \leq (n!)^{\frac{1}{n(n+1)}} \leq ((n^n))^{\frac{1}{n^2}} = n^{\frac{1}{n}} \to 1$ so by sandwich test the denominator $\to 1$ as well. Thus the ratio converges to $|z|$, so the series converges for $|z| < 1$ and diverges for $|z| > 1$. Therefore $R = 1$.

5.* What are the possible values of the radius of convergence of a series $\sum_{n=1}^{\infty} a_n z^n$ with $en^{-\pi} < |a_n| < \pi n^e \ \ \forall n$?

   Ratio test on a_n will not help here! Need to compare to $\sum_{n=1}^{\infty} \pi n^e z^n$ to see (by ratio test on $\pi n^e z^n$) that it converges absolutely for $|z| < 1$. Similarly by comparison with $en^{-\pi} z^n$ we see that

(by ratio test on $en^{-\pi}z^n$) that $|a_n z^n| \to \infty$ for $|z| > 1$. Thus $R = 1$.

Alternatively: $en^{-\pi/n}|z| < |a_n z^n|^{1/n} < \pi n^{e/n}|z|$ shows that $\lim_{n\to\infty} |a_n z^n|^{1/n}$ exists and equals $|z|$. Therefore, by the root test, $\sum a_n z^n$ is absolutely convergent for $|z| < 1$ and divergent for $|z| > 1$. So $R = 1$.

6. The great Professor Martin Lietype is not very good with complex numbers, but an ace with reals. He notices that the infinite series $1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ converges to the function

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{1}{1+x^2},$$

which is finite $\forall x \in \mathbb{R}$. He concludes the series converges $\forall x \in \mathbb{R}$. Is he right? If not, can you help him? Would it help if he was better with complex numbers?

The partial sum to n terms is $\frac{1-(-1)^n x^{2n}}{1+x^2}$ which tends to $1/(1+x^2)$ as required for $|x| < 1$. For $|x| \geq 1$ it clearly does not converge (and in fact the individual terms of the series $(-1)^n x^{2n} \not\to 0$).

If he was better with complex numbers he would see that $f(x)$ is ill-defined at $x = \pm i$ on the unit circle, which is why the radius of convergence is 1, not ∞.

7. Show the following sequence (a_n) is convergent:

$$a_n := \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}.$$

The first few terms seem to show that a_n is increasing, so we check:

$$\begin{aligned} a_{n+1} - a_n &= \left(\frac{1}{(n+1)+(n+1)} + \frac{1}{(n+1)+n}\right) - \left(\frac{1}{n+1}\right) \\ &= \frac{(2n^2+3n+1) + (2n^2+4n+2) - (4n^2+6n+2)}{(2n+2)(2n+1)(n+1)} \\ &= \frac{n+1}{(2n+2)(2n+1)(n+1)} \;=\; \frac{1}{(2n+2)(2n+1)} \;>\; 0. \end{aligned}$$

It is also bounded above by $n\frac{1}{n+1} < 1$, so convergent.

8. Suppose $a_n \geq 0\ \forall n$. Show that if $\sum a_n$ is convergent then $\sum \frac{a_n}{1+a_n}$ is convergent. Is the converse true?

Since $0 \leq \frac{a_n}{1+a_n} \leq a_n$, by comparison $\sum \frac{a_n}{1+a_n}$ is convergent.

Converse: if $\frac{a_n}{1+a_n}$ is convergent then $\frac{a_n}{1+a_n} \to 0$. In particular, $\exists\, N \in \mathbb{N}$ such that $\frac{a_n}{1+a_n} < \frac{1}{3}$ for all $n \geq N$, which implies $a_n < \frac{1}{2}$.

Therefore, for $n \geq N$, $0 \leq a_n = \frac{3}{2}\frac{a_n}{1+\frac{1}{2}} < \frac{3}{2}\frac{a_n}{1+a_n}$ so $\sum a_n$ convergent by comparison.

9. Let $s_n := \sum_{i=0}^{n} \frac{1}{i!}$. Show that $\frac{1}{(n+k)!} \leq \frac{1}{(n+1)^k n!}$ for all integers $n, k > 0$, and hence

$$s_N - s_n \;<\; \frac{1}{n \cdot n!} \qquad \forall N > n \geq 1 \tag{$*$}$$

Deduce (s_n) is bounded above and convergent to some $e := \sup\{s_n \colon n \in \mathbb{N}\} \in \mathbb{R}$ satisfying

$$0 \;<\; e - \sum_{i=0}^{n} \frac{1}{i!} \;\leq\; \frac{1}{n \cdot n!} \tag{$**$}$$

for all $n \geq 1$. If we could write $e = \frac{m}{n}$ with $m, n \in \mathbb{N}$ multiply (**) by $n!$ to get a contradiction. Conclude that e is irrational.

$(n+k)! = (n+k)(n+k-1)\cdots(n+1)n! \geq (n+1)(n+1)\cdots(n+1)n! = (n+1)^k n!$ so $\frac{1}{(n+k)!} \leq \frac{1}{(n+1)^k \cdot n!}$ for all $n, k > 0$. Therefore

$$\begin{aligned} s_N - s_n &= \sum_{k=1}^{N-n} \frac{1}{(n+k)!} \leq \sum_{k=1}^{N-n} \frac{1}{(n+1)^k n!} = \frac{1}{n!} \cdot \frac{1}{n+1} \cdot \frac{1-(n+1)^{-N-n}}{1-(n+1)^{-1}} \\ &< \frac{1}{n!} \cdot \frac{1}{n+1-1} = \frac{1}{n \cdot n!} \end{aligned}$$

for all $N > n \geq 1$, where the second equality comes from summing the finite geometric series.

Therefore s_N is bounded above by $s_n + \frac{1}{n \cdot n!}$ for all N. (Or put $n = 1$ to see that s_n is bounded above by $s_1 + 1 = 3$ for all n.) Since s_n is monotonically increasing it converges to $\sup\{s_n : n \in \mathbb{N}\} =: e$.

Since $s_N < s_n + \frac{1}{n \cdot n!}$, we have $\sup\{s_N\} \leq s_n + \frac{1}{n \cdot n!}$. This gives the second inequality of

$$0 < e - s_n \leq \frac{1}{n \cdot n!}.$$

The first inequality comes from $e = \sup S \geq s_n \in S$, and we cannot have equality (otherwise $s_{n+1} = s + \frac{1}{(n+1)!} > e$; a contradiction).

If $e = \frac{m}{n}$ then by (**), $n!\,e - \sum_{i=0}^{n} \frac{n!}{i!}$ is an integer in $\left(0, \frac{1}{n}\right]$ – a contradiction.

10.† Celebrity computer scientist Professor Buzzard has taught Thomas and Liebeck a game. They each flip a fair coin repeatedly until they get a tail. The winner is the one who got the most heads, and receives £4^n from the loser, where n is the loser's number of heads.[1]

Liebeck declares confidently "Ah ha Thomas, if you throw h heads, my expected winnings are 50p, whatever h is." Check he is right. He's pretty sure he's going to clean up.

He throws k heads (and then a tail) with probability $1/2^{k+1}$. If $k < h$ heads he loses £4^k; if $k > h$ heads he wins £4^h so his expected winnings are

$$\sum_{k=0}^{h-1} \frac{1}{2^{k+1}}(-4^k) + \sum_{k=h+1}^{\infty} \frac{1}{2^{k+1}} 4^h = -\frac{1}{2}(2^h - 1) + \frac{1}{2} 2^h = £\frac{1}{2} = 50\text{p}.$$

Thomas replies "Ah but Liebeck, if *you* throw h heads, your expected winnings are -50p, whatever h is." Check he is also right.

Thomas throws k heads (and then a tail) with probability $1/2^{k+1}$. If $k < h$ heads Liebeck wins £4^k; if $k > h$ heads he loses £4^h so his expected winnings are

$$\sum_{k=0}^{h-1} \frac{1}{2^{k+1}}(4^k) - \sum_{k=h+1}^{\infty} \frac{1}{2^{k+1}} 4^h = \frac{1}{2}(2^h - 1) - \frac{1}{2} 2^h = -£\frac{1}{2} = -50\text{p}.$$

"Lean says the game is symmetric between the pair of you, so don't you think your expected winnings should be zero?" says Buzzard. What is going on? (Hint: we're meant to be studying absolute convergence, not coin tossing.)

Liebeck's expected winnings are the sum over all $a \neq b \in \mathbb{N}$ of the probability that he throws a heads (then a tail) and Thomas throws b heads (then a tail) times by his winnings (4^b if $a > b$, or -4^a if $a < b$). I.e.

$$\sum_{a>b} \frac{1}{2^{a+1}} \frac{1}{2^{b+1}} 4^b - \sum_{a<b} \frac{1}{2^{a+1}} \frac{1}{2^{b+1}} 4^a = \frac{1}{4} \sum_{a>b} \frac{1}{2^{a-b}} - \frac{1}{4} \sum_{a<b} \frac{1}{2^{b-a}}.$$

[1] If they flip the same number of heads it is a draw and no money changes hands.

Here you can see the symmetry (if it converged it would equal 0) but also that the whole sum is *not absolutely convergent* because *neither* of the two sums on the right hand side is convergent (think of summing the first over all $a > b$ for *fixed* b, then sum over b to get ∞). So by rearranging you can make it converge to anything you like. In real life the expectation is not defined – if Liebeck and Thomas play forever the average winnings will go all over the place, never settling down close to a fixed value.