1. (a) Labelling the top wall as node 1 and going vertically down to nodes 2, 3 and 4 (node 4 being the lower wall) the weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & C+1 & -C \\ 0 & 0 & -C & C \end{bmatrix}.$$

1. (b) The system to solve for the equilibrium displacements is

$$\mathbf{Kx} = \mathbf{f},$$

where

$$\mathbf{x} = \begin{bmatrix} 0 \\ \phi_1 \\ \phi_2 \\ 0 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} r_1 \\ m_1 g \\ m_2 g \\ r_2 \end{bmatrix}.$$

The middle two equations are easily solved by hand:

$$\phi_2 = \frac{(C+2)mg}{2C+1}$$

and

$$\phi_3 = \frac{3mg}{(2C+1)}$$

1. (c) As $C \to 0$,

$$\phi_2 = 2mg, \qquad \phi_3 = 3mg$$

As $C \to \infty$,

$$\phi_2 = mg/2, \qquad \phi_3 = 0.$$

1. (d) To compute the internal spring forces we need the incidence matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and spring constant matrix $\mathbf{C}$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & C \end{bmatrix}.$$

leading to the internal spring forces being

$$\mathbf{CAx} = mg \begin{bmatrix} \dfrac{C+2}{2C+1} \\[2ex] \dfrac{1-C}{2C+1} \\[2ex] -\dfrac{3C}{2C+1} \end{bmatrix}.$$

As $C \to 0$, we find

$$\mathbf{CAx} \to mg \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

As $C \to \infty$, we find

$$\mathbf{CAx} \to mg \begin{bmatrix} 1/2 \\ -1/2 \\ -3/2 \end{bmatrix}.$$

2.(a) The weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1+c_2+c_3 & -c_3 & -c_2 \\ 0 & -c_3 & c_3 & 0 \\ 0 & -c_2 & 0 & c_2 \end{bmatrix}.$$

2.(b) The system to solve for the equilibrium displacements is

$$\mathbf{Kx} = \mathbf{f},$$

where

$$\mathbf{x} = \begin{bmatrix} 0 \\ \phi_1 \\ \phi_2 \\ 0 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} r_1 \\ m_1 g \\ m_2 g \\ r_2 \end{bmatrix}.$$

The middle two equations are easily solved by hand:

$$\phi_2 = \frac{(m_1+m_2)g}{c_1+c_2}$$

and

$$\phi_3 = \frac{m_2 g}{c_3} - \frac{(m_1+m_2)g}{c_1+c_2}.$$

2.(c) The first and third equations give the reaction forces at the walls:

$$r_1 = -c_1\phi_2 = -c_1\frac{(m_1+m_2)g}{c_1+c_2}$$

and

$$r_2 = -c_2\phi_2 = -c_2\frac{(m_1+m_2)g}{c_1+c_2}$$

Note that

$$r_1 + r_2 = -(m_1+m_2)g.$$

2.(d) Assuming the walls at top and bottom are fixed and only the two masses can move, the free oscillations – that is, the oscillations when there are no external forces on the masses, only the internal spring forces – are the solutions of the governing equations which reduce to

$$-\hat{\mathbf{K}}\mathbf{x} = \mathbf{M}\frac{d^2\mathbf{x}}{dt^2},$$

where $\mathbf{x} = [\phi_1(t)\ \phi_2(t)]^T$ are the displacements of the two masses and

$$\hat{\mathbf{K}} = \begin{bmatrix} C & -c_3 \\ -c_3 & c_3 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

where, for convenience, we introduce the notation $C = c_1 + c_2 + c_3$. To find the natural modes of oscillation of the system we let

$$\mathbf{x} = \mathbf{\Phi}e^{\mathrm{i}\omega t}$$

for some real ω (this is the frequency of free, or natural, oscillation). Then, on substitution into the governing equation above we find

$$-\hat{\mathbf{K}}\mathbf{\Phi} = -\omega^2\mathbf{M}\mathbf{\Phi}.$$

If we let $\lambda = \omega^2$ then we need to find the eigenvalues λ satisfying

$$\begin{bmatrix} C/m_1 & -c_3/m_1 \\ -c_3/m_2 & c_3/m_2 \end{bmatrix}\mathbf{\Phi} = \lambda\mathbf{\Phi}.$$

These are the solutions of the characteristic equation

$$\det\begin{bmatrix} C/m_1 - \lambda & -c_3/m_1 \\ -c_3/m_2 & c_3/m_2 - \lambda \end{bmatrix} = 0.$$

It is easily found that

$$\lambda = \frac{1}{2}\left[\frac{C}{m_1} + \frac{c_3}{m_2} \pm \left[\left(\frac{C}{m_1} + \frac{c_3}{m_2}\right)^2 - 4\left(\frac{Cc_3 - c_3^2}{m_1 m_2}\right)\right]^{1/2}\right].$$

It follows that the natural frequencies of oscillation are given by

$$\omega = \pm \left[\frac{1}{2} \left[\frac{C}{m_1} + \frac{c_3}{m_2} \pm \left[\left(\frac{C}{m_1} + \frac{c_3}{m_2} \right)^2 - 4 \left(\frac{Cc_3 - c_3^2}{m_1 m_2} \right) \right]^{1/2} \right] \right]^{1/2}.$$

Note: the next question helps us see why these frequencies are all real.

3(a). To find the natural modes of oscillation of the system we let

$$\mathbf{x} = \mathbf{\Phi} e^{\mathrm{i}\omega t}$$

for some real ω. Then, on substitution into the governing equation we find

$$-\hat{\mathbf{K}}\mathbf{\Phi} = -\omega^2 \mathbf{M}\mathbf{\Phi}$$

But $\mathbf{M}$ is clearly invertible with

$$\mathbf{M}^{-1} = \begin{bmatrix} 1/m_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/m_2 & 0 & \cdots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & \cdots & 0 & 1/m_N \end{bmatrix}.$$

Thus $\mathbf{\Phi}$ satisfies

$$\mathbf{M}^{-1}\hat{\mathbf{K}}\mathbf{\Phi} = \omega^2 \mathbf{\Phi}$$

from which we see that $\mathbf{\Phi}$ is an eigenvector of $\mathbf{M}^{-1}\hat{\mathbf{K}}$ and ω^2 is an eigenvalue.

(b) Even though $\hat{\mathbf{K}}$ is symmetric, once we multiply on the left by $\mathbf{M}^{-1}$ it is clear that the first row gets multiplied by $1/m_1$ while the second row gets multiplied by $1/m_2$. If $m_1 \neq m_2$ this clearly destroys the symmetry since the $(1,2)$ term in the matrix will now be different from the $(2,1)$ term. If $m_1 = m_2$ this argument will pertain at some later pair (i, j) if there exists some $m_i \neq m_j$, as has been assumed.

(c) Despite this lack of symmetry, we can still prove that $\mathbf{M}^{-1}\hat{\mathbf{K}}$ has N real eigenvalues and eigenvectors. To see this, notice that

$$\hat{\mathbf{K}}\mathbf{\Phi} = \omega^2 \mathbf{M}\mathbf{\Phi} = \omega^2 \mathbf{M}^{1/2}\mathbf{M}^{1/2}\mathbf{\Phi}$$

where $\mathbf{M}^{1/2}$ will also be diagonal with positive entries. It is clear that it is invertible with

$$\mathbf{M}^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 & 0 & \cdots & 0 \\ 0 & 1/\sqrt{m_2} & 0 & \cdots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & \cdots & 0 & 1/\sqrt{m_N} \end{bmatrix}.$$

Hence we can write

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{\Phi} = \omega^2 \mathbf{M}^{1/2}\mathbf{\Phi}$$

This can be rewritten as

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\underbrace{\mathbf{M}^{-1/2}\mathbf{M}^{+1/2}}_{\text{identity}}\mathbf{\Phi} = \omega^2 \mathbf{M}^{1/2}\mathbf{\Phi}$$

or as

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{M}^{-1/2}\mathbf{\Psi} = \omega^2\mathbf{\Psi}$$

where

$$\mathbf{\Psi} = \mathbf{M}^{+1/2}\mathbf{\Phi}.$$

Now the matrix

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{M}^{-1/2}$$

can be shown to be positive definite and symmetric, which means that is has N real eigenvalues and N real orthogonal eigenvectors, i.e., there are N solutions Ψ_j for $j = 1, \cdots, N$ satisfying

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{M}^{-1/2}\mathbf{\Psi}_j = \lambda_j \mathbf{\Psi}_j$$

where λ_j is real and positive. The real values $\{\lambda_j | j = 1, \cdots, N\}$ are the squares of the natural frequencies ω_j^2.

Note also that

$$\mathbf{\Psi}_i^T \mathbf{\Psi}_j = 0$$

if $i \neq j$. This means that

$$\mathbf{\Phi}_i^T \mathbf{M} \mathbf{\Phi}_j$$

if $i \neq j$. Thus while the vectors $\mathbf{\Phi}_j$ are not orthogonal they satisfy this generalized condition of "M-orthogonality".

4. On taking a derivative of the given quantity we find, using the product rule,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\hat{\mathbf{K}}\mathbf{x}\right] = \frac{1}{2}\left[\ddot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{M}\ddot{\mathbf{x}} + \dot{\mathbf{x}}^T\hat{\mathbf{K}}\mathbf{x} + \mathbf{x}^T\hat{\mathbf{K}}\dot{\mathbf{x}}\right]$$

Now we can use

$$M\ddot{\mathbf{x}} = -\hat{\mathbf{K}}\mathbf{x}, \qquad \ddot{\mathbf{x}}^T\mathbf{M} = -\mathbf{x}^T\hat{\mathbf{K}}$$

to eliminate the second derivatives and we find everything cancels. Hence the given quantity does not change in time, and is conserved by the dynamics.

5(a). The weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & C+1 & -C & 0 \\ 0 & -C & C+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

For the motion of the two masses the submatrix

$$\hat{\mathbf{K}} = \begin{bmatrix} C+1 & -C \\ -C & C+1 \end{bmatrix}$$

is relevant. According to Newton's second law the governing system of differential equations for the displacements ϕ_1 (left mass) and ϕ_2 (right mass) is

$$\hat{\mathbf{f}} - \hat{\mathbf{K}}\mathbf{x} = \frac{d^2\mathbf{x}}{dt^2},$$

where

$$\mathbf{x} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \qquad \hat{\mathbf{f}} = \begin{bmatrix} \cos\Omega t \\ 0 \end{bmatrix}.$$

(b) To find the particular solution, let

$$\mathbf{x}^{PS} = \mathbf{\Phi}\cos\Omega t$$

then on substitution into the equations derived in part (a) and cancellation of the common factor of $\cos\Omega t$,

$$\hat{\mathbf{f}}_0 - \hat{\mathbf{K}}\mathbf{\Phi} = -\Omega^2\mathbf{\Phi}.$$

where

$$\hat{\mathbf{f}}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{\mathbf{K}}\mathbf{\Phi} - \Omega^2\mathbf{\Phi}.$$

We proceed by finding the eigenvalues and eigenvectors of $\hat{\mathbf{K}}$. The determinant condition

$$\det\begin{bmatrix} C+1-\lambda & -C \\ -C & C+1-\lambda \end{bmatrix} = 0$$

yields

$$\lambda_1 = 1, \qquad \lambda_2 = 1+2C$$

with corresponding eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We now write

$$\mathbf{\Phi} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$$

and, on noticing that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$$

we can write the equation for $\mathbf{\Phi}$ as

$$\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 = \lambda_1 a_1 \mathbf{e}_1 + \lambda_2 a_2 \mathbf{e}_2 - \Omega^2 a_1 \mathbf{e}_1 - \Omega^2 a_2 \mathbf{e}_2$$

On equating coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$ we find

$$a_1 = \frac{1}{2(\lambda_1 - \Omega^2)}, \qquad a_2 = \frac{1}{2(\lambda_2 - \Omega^2)}.$$

Thus the particular solution is

$$\left[\frac{1}{2(\lambda_1 - \Omega^2)}\mathbf{e}_1 + \frac{1}{2(\lambda_2 - \Omega^2)}\mathbf{e}_2\right]\cos\Omega t$$

The general solution is obtained – using linearity – by adding a solution of the homogeneous system, i.e., a solution of

$$-\hat{\mathbf{K}}\mathbf{x} = \frac{d^2\mathbf{x}}{dt^2}.$$

On letting

$$\mathbf{x} = \mathbf{\Psi} e^{\mathrm{i}\omega t}$$

we must solve the eigenvalue problem

$$\hat{\mathbf{K}}\mathbf{\Psi} = \omega^2 \mathbf{\Psi}$$

which we already know has solutions

$$c_1\mathbf{e}_1 e^{\mathrm{i}t} + c_2\mathbf{e}_2 e^{\mathrm{i}\sqrt{1+2C}t}, \qquad c_1, c_2 \in \mathbb{C}.$$

Thus the general solution is given by

$$\begin{aligned}\mathbf{x} = &\left[\frac{1}{2(\lambda_1 - \Omega^2)}\mathbf{e}_1 + \frac{1}{2(\lambda_2 - \Omega^2)}\mathbf{e}_2\right]\cos\Omega t \\ &+ \mathbf{e}_1[A\cos t + B\sin t] + \mathbf{e}_2[D\cos(\sqrt{1+2C}t) + E\sin(\sqrt{1+2C}t)].\end{aligned}$$

(c) It is clear that the solution of part (b) is valid providing

$$\Omega^2 \neq \lambda_1, \lambda_2.$$

These are the "resonant" values where the forcing frequency equals one of the natural frequencies of the system.

6(a). We already computed the eigenvalues/eigenvectors so the general solution for free oscillation is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}[a\cos t + b\sin t] + \begin{bmatrix} 1 \\ -1 \end{bmatrix}[d\cos(\sqrt{1+2C}t) + e\sin(\sqrt{1+2C}t)].$$

Since the initial velocities are zero we immediately deduce

$$b = e = 0.$$

Hence

$$\mathbf{x} = \begin{bmatrix} a\cos t + d\cos(\sqrt{1+2C}t) \\ a\cos t - d\cos(\sqrt{1+2C}t) \end{bmatrix}$$

But at $t = 0$

$$\mathbf{x} = \begin{bmatrix} 0 \\ A \end{bmatrix}$$

implying that

$$a = -d, \qquad 2a = A.$$

Hence

$$\mathbf{x} = \frac{A}{2}\begin{bmatrix} \cos t - \cos(\sqrt{1+2C}t) \\ \cos t + \cos(\sqrt{1+2C}t) \end{bmatrix}.$$

On use of trigonometric identities we can write this as

$$\mathbf{x} = A\begin{bmatrix} \sin\Omega t\sin\epsilon t \\ \cos\Omega t\cos\epsilon t \end{bmatrix},$$

where

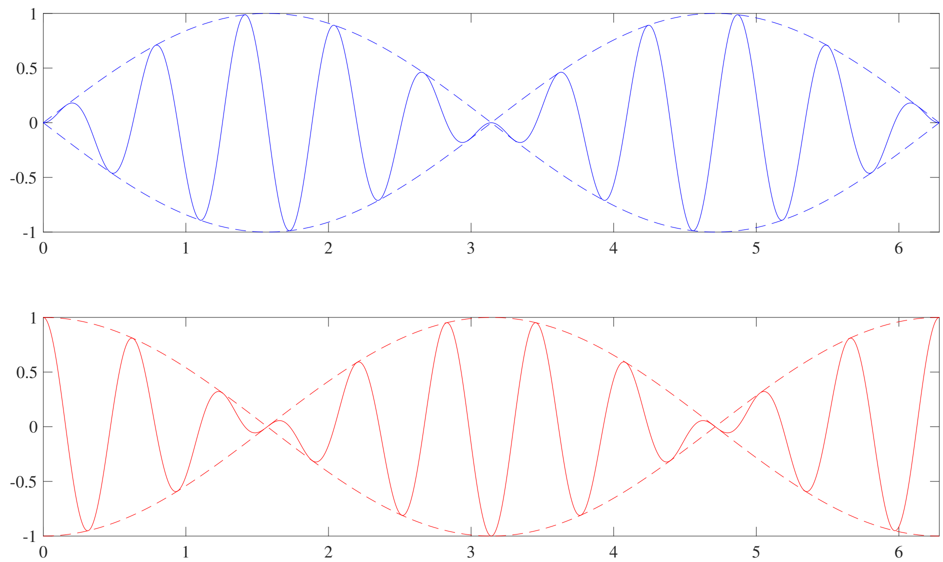
$$\begin{aligned} \Omega t - \epsilon t &= t, \\ \Omega t + \epsilon t &= \sqrt{1+2C}t. \end{aligned}$$

Hence

$$\Omega = \frac{\sqrt{1+2C}+1}{2}, \qquad \epsilon = \frac{\sqrt{1+2C}-1}{2}.$$

(b) It is clear that we need $C \ll 1$ if we require $\epsilon \ll \Omega$.

(c) If $\epsilon \ll \Omega$ then the displacements comprise a fast oscillation, with frequency Ω, with an amplitude that changes over a slow time scale with frequency $\epsilon \ll \Omega$. Also, for early times $t \ll 1$ the amplitude of displacement of the left mass is small, and the displacement of the right mass is large; however around $\epsilon t \approx \pi/2$ this situation reverses and it is the left mass where all the energy of the system of concentrated (with the right mass hardly moving). This exchange of energy continues.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