MATH40004 - Calculus and Applications: Unseen Questions 6: 1D Bifurcations An insect outbreak?

1 The Spruce Budworm

An annoying insect called the spruce budworm has appeared in the forests of Eastern Canada and the United States. The budworm attacks the leaves of the coniferous fir trees and if allowed to reproduce unhindered and mass outbreak the budworms can defoliate and kill most trees in a small forest in a matter of a few years, causing huge damage to the ecosystem.

We need to provide some mathematical insight to help the entomologists choose how best to fight the problem and lessen the budworm population as much as possible.

1.1 Problem Setup

If we let y represent the budworm population (in billions) at time t (in months), it is believed from previous outbreaks that the budworm species follows a **logistic model** growth; that is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(1 - \frac{y}{k}\right), \tag{1}$$

where the **positive** real parameter r is called the **growth rate** of the budworm population y. The positive real parameter k is called the **carrying capacity** and in our investigation we will focus on a particular forest for which we estimate $k = 10$ is constant.

Our goal will be to reduce the threat the budworms pose by reducing their population as much as possible by affecting their logistic growth with a **predation term**. Taking all this into account the equation we will study is

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(1 - \frac{y}{10}\right) - p(y),} \tag{2}$$

where $p(y)$ is a function of y depending upon how we try to fight back the insect population's growth. Throughout the investigation to make physical sense we will take $r > 0$ and $y \geqslant 0$.

2 Exercises

(a). Suppose first we investigate what happens if we do nothing, i.e. we let $p(y) = 0$.

(i). Draw the bifurcation diagram for r for the equation (2) in this case. What type of bifurcation do we see?

(ii). For an initial budworm population in the forest of $y_0 \neq 0$, describe the possible resulting dynamics of the budworm population.

(b). Next investigate what happens if we try to do something about the problem and send a team of exterminators into the forest to attack the budworm by killing it at a **constant** rate, i.e. we let $p(y) = 1$.

(i). Draw the bifurcation diagram for r for the equation (2) in this case. What type of bifurcation do we see?

(ii). For an initial budworm population in the forest of $y_0 \neq 0$, describe the possible resulting dynamics of the budworm population.

(c). A mathematician theorises that applying widespread pesticides across the forest will generate a predation function of the form

$$p(y) = y^3 - \left(\frac{r}{10} + 9\right)y^2 + \left(\frac{12}{r} + r\right)y + 18\left(3 - \frac{2}{r}\right). \tag{3}$$

Finally, for this case:

(i). Draw the bifurcation diagram for r for the equation (2). What type of bifurcation do we see?

(ii). For an initial budworm population in the forest of $y_0 \neq 0$, describe the possible resulting dynamics of the budworm population.

(d). A period of data collection occurs, after which we are able to closely theorise the value of the growth rate r. Which method of predation (do nothing, constant extermination, widespread pesticide) should we recommend using to the entomologists in the cases when:

(i). We estimate $r = 3/10$ but the current population of budworms is unknown?

(ii). We estimate $r = 1$ and the current budworm population is close to $y = 5/2$?

(iii). We estimate $r = 10$ and the current budworm population has already managed to reach around $y = 4$?

3 Extension

A local birdwatching group inform you of another possible solution: they have noticed recently that the forest bird population have been feasting on the budworm rather than their alternative food sources. However this only happens if man leaves the forest alone - to send in a team of exterminators or use pesticides would scare off or poison the birds. One of the birdwatchers happens to be a very distinguished scientist and he conjectures that the birds eat the budworm with a predation function

$$p(y) = \frac{y^2}{1 + y^2}. \tag{4}$$

(a). In this case:

(i). Sketch the general shape of the bifurcation diagram for r for the equation (2) (i.e. draw the curve and mark which parts are stable and unstable, but do not worry about finding the precise values for r where bifurcations occur - unless you are very interested and want to use computational help!). What type(s) of bifurcation do we see?

(ii). For an initial budworm population in the forest of $y_0 \neq 0$, describe the possible resulting dynamics of the budworm population.