Imperial College London
Department of Mathematics

Question Sheet 8

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

This is the final problem sheet for this module (released on Wednesday of week 10). Questions 2 and 5 are suitable for tutorials. Material for questions 7, 8, 9 will be covered on Wednesday and Friday of week 11. Solutions will be released on Friday of week 11.

Question 1 Suppose that $(G, .)$ is a group and H is a subgroup of G of index 2.

(a) Prove that the two left cosets of H in G are H and $G \setminus H$.

(b) Show that for every $g \in G$ we have $gH = Hg$.

Question 2 Suppose $(G, .)$ is a group. Invent a test which allows you to check whether a subset $X \subseteq G$ is a left coset (of some subgroup of G). Prove that your test works.

Question 3 Let X be any non-empty set and $G \leq \mathrm{Sym}(X)$. Let $a \in X$ and $H = \{g \in G : ga = a\}$ and $Y = \{g(a) : g \in G\}$.

(a) Prove that $H \leq G$ and for $g_1, g_2 \in G$ we have

$$g_1H = g_2H \Leftrightarrow g_1(a) = g_2(a).$$

Deduce that there is a bijection between the set of left cosets of H in G and the set Y. In particular, if G is finite, then $|G|/|H| = |Y|$.

(b) Use (a) to justify why the order of the group G of rotations of a cube (as in Question sheet 7) is 24.
[Hint: let X be the set of 6 faces of the cube, or the set of 8 vertices of the cube.]

Question 4 Let G be a finite group of order n, and H a subgroup of G of order m.

(a) For $x, y \in G$, show that $xH = yH \iff x^{-1}y \in H$.

(b) Suppose that $r = n/m$. Let $x \in G$. Show that there is an integer k in the range $1 \leq k \leq r$, such that $x^k \in H$.

Question 5 Prove that the following are homomorphisms:
(i) G is any group, $h \in G$ and $\phi : G \to G$ is given by $\phi(g) = hgh^{-1}$.
(ii) $G = \mathrm{GL}_n(\mathbb{R})$ and $\phi : G \to G$ is given by $\phi(g) = (g^{-1})^T$.
(Here $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible $n \times n$-matrices over $\mathbb{R}$ and the ^{T} denotes transpose.)
(iii) G is any abelian group and $\phi : G \to G$ is given by $\phi(g) = g^{-1}$.
(iv) $\phi : (\mathbb{R}, +) \to (\mathbb{C}^{\times}, \cdot)$ given by $\phi(x) = \cos(x) + \mathrm{i}\sin(x)$.

In each case say what is the kernel and the image of ϕ. In which cases is ϕ an isomorphism?

Question 6 (a) Use the inclusion - exclusion principle to give a formula for the number of permutations in S_n which have no fixed points. Prove that the proportion of such permutations in S_n tends to $1/e$ as $n \to \infty$.

(b) Give a formula for the number of permutation in S_n which have one fixed point.

(c) A standard deck of 52 cards is shuffled at random. What (approximately) is the probability that at least one card is still in the same place after the shuffle?

Question 7 (a) Write down all of the cycle shapes of the elements of S_5. For each cycle shape, calculate how many elements there are with that shape. (Check that your answers add up to $|S_5| = 120$.)

(b) How many elements of S_5 have order 2?

(c) How many subgroups of size 3 are there in the group S_5?

Question 8 What is the largest order of an element of S_8?

Question 9 Let G be a group, and let S be a subset of G. Recall that we say that S *generates* G if every element in G can be written as a product of elements of S and their inverses.

(i) Let $2 \leq k \leq n$. Show that a k-cycle $(a_1 \ldots, a_k)$ in S_n can be written as a product of $k-1$ distinct cycles of length 2. Deduce that the set of 2-cycles in S_n generates S_n.

(ii) (Harder) Let α be the n-cycle $(1234\ldots n)$ and β the 2-cycle (12). Prove that $\langle \alpha, \beta \rangle = S_n$.

[Hint: $\alpha\beta\alpha^{-1} = (23)$. Use tricks like this.]