Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 7

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. * Find $\partial u/\partial x$ and $\partial u/\partial y$ for the following functions of two real variables:

   (a) $u = x^3 + 3xy + -y^2$
   (b) $u = e^{xy}\sin x$

   In each case:

   (i) write the expression for du
   (ii) verify that
   $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

2. The following are a few examples of the application of: the total differential, the chain rule, and the implicit function.

   (a) Using partial derivatives, find du/dt when
   $$u(x,y) = \frac{x-y}{x+y} \quad \text{with} \quad x = e^{ct},\ y = e^{-ct}.$$
   Check your answer otherwise.

   (b) Consider
   $$f(x,y) = x^2 + 3y^3 \quad \text{with} \quad x = s+t,\ y = 2s-t.$$
   Use the chain rule to obtain $\partial f/\partial t$ and $\partial f/\partial s$ and check your answer by direct substitution.

   (c) Consider
   $$u(x,y) = xy \quad \text{and} \quad \sin y + xy - x^3 = 0.$$
   Find du/dx.

   (d) The temperature in a region of space is given by the formula
   $$f(\mathbf{x}) = f(x,y,z) = kx^2(y-z),$$
   where k is a positive constant. An insect flies along a trajectory $\mathbf{x}(t) = (x(t), y(t), z(t)) = (t, t, 2t)$. Find the rate of change of the temperature along its path.

   (e) (The following is a classic result in Thermodynamics. Do not get flustered by the notation. Stick to the mathematical formulation to prove the result.)
   The equation of state of a gas is usually given by an implicit relation $f(p,V,T) = 0$ between the pressure p, the volume V, and the temperature T. Show that:
   $$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{\left(\frac{\partial f}{\partial V}\right)_{p,T}}{\left(\frac{\partial f}{\partial p}\right)_{V,T}},$$

and obtain similar expressions for $(\partial V/\partial T)_p$ and $(\partial T/\partial p)_V$. Hence derive the identity:

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = -1,$$

which is known as the reciprocity theorem.

3. 'Projectile man' needs to estimate bounds on the accuracy of his landing place for his next stunt. He knows that the horizontal range R of a projectile is given by:

$$R = \frac{U^2 \sin 2\alpha}{g},$$

where U is the projectile initial speed, α is the angle of elevation and g is the gravitational acceleration. If U and α are each known to $\pm 0.2\%$ (and g can be considered to be known exactly), find the % accuracy bounds for R when:

(a) $\alpha = 25°$, (b) $\alpha = 65°$, (c) $\alpha = 45°$.

4. * The cost P of a computer depends on the required CPU c and memory storage s according to the relation:

$$P = kc^2 s^3,$$

where k is some positive constant. Estimate the percentage change in cost if c and s are increased and decreased by 1%, respectively.

5. The following are a couple of examples to practise the Taylor expansion of functions of two variables:

   (a) Find the Taylor expansion up to quadratic terms for $f(x, y) = \ln(1 + x + 2y)$ about the point $(x_0, y_0) = (2, 1)$. Use your result to estimate the value of $\ln(5 + h + 2k)$ when $h = 0.2$ and $k = -0.05$ and compare your estimate to the 'true' value.

   (b) Find the Taylor expansion up to third-order terms for $f(x, y) = (x + 2y)\cos(2x + y)$ about the point $(x_0, y_0) = (0, 0)$ and compare the result in this case to an expansion based on the cosine function of one variable.