Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 2

*You should prepare starred question, marked by * to discuss with your personal tutor.*

In the following, we will sometimes use the notation:

$$y' = \frac{dy}{dx} \quad \text{and} \quad y'' = \frac{d^2y}{dx^2}.$$

1. Solve the following first order differential equations:

   (a) $y' = (1+x)(1+y)$

   (b) $y' = \frac{1+y}{2+x}$ with $y(0) = 1$

   (c) $xyy' = x^2 + y^2$

   (d) $xy' = y + xe^{y/x}$

   (e) $y' = -5y + x + e^{-2x}$ with $y(-1) = 0$

   (f) $y' = -y\cot x + \cos x$ with $y(0) = 0$

   (g) $6y' = 2y + xy^4$ with $y(0) = -2$

2. Solve the following second order differential equations:

   (a) $y'' = \cos(2x)$ with $y(0) = 1$ and $y'(0) = 0$

   (b) $y'' = 2y^3$ with $y(1) = y'(1) = 1$

   (c) $y'' = (y')^2$

   (d) $yy'' + y' = (y')^2$

   (e) $y'' = -x\,(y')^2$ with $y(0) = 0$ and $y'(1) = 1$

   (f) $y'' = y'y$

   (g) $y'' + \frac{1}{x}y' = 1$ with $y(1) = 0$ and $y(2) = 1$

3.* When measuring the growth of a population $x(t)$, it was found that the Malthusian model of constant reproduction rate fails when the population gets large. An empirical model shows that the reproduction rate is not constant (as Malthus postulated) and depends on the population $x(t)$ as: $k\left(1 - (x/\beta)^\alpha\right)$.

   (a) Write down and solve the ODE for $x(t)$

   (b) What is the value of the population as $t \to \infty$?

4. By using a suitable substitution (or otherwise), find the solution of

   (a) $y(xy+1) + x(1 + xy + x^2y^2)y' = 0$

   (b) $y' = \frac{1-2y-x}{4y+2x}$

5. The vertical motion of an object is described by the equation of motion

$$z'' = -g - \gamma z'$$

   where the constant g is the acceleration due to gravity near surface of earth and the constant γ is a measure of friction or air-resistance. Find the general solution of the equation of motion. You don't need to know anything about mechanics to solve this problem!

6.* Consider the motion of a rocket that is launched vertically with initial velocity v^*. Gravity further away from the surface of the earth is not constant and it decreases as the inverse of square distance from the centre of the earth r. Thus using second Newton law will result in the following ODE for the velocity v as a function of the distance to the center of the Earth r

$$r^2 v \frac{dv}{dr} = C$$

Solve the ODE above and find the critical initial velocity v_c^* above which the rocket is guaranteed to escape the Earth's pull.