Introduction to Applied Mathematics
Coursework 1

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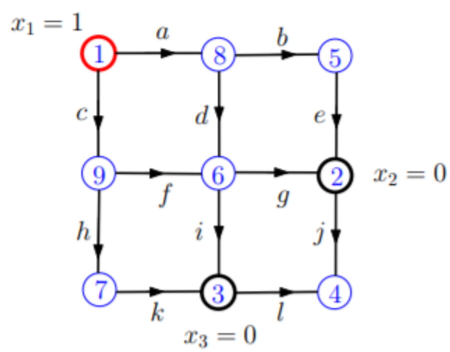
The figure shows an electrical circuit comprising a graph with 9 nodes linked by edges all having unit conductance. The nodes have been labelled 1–9. Node 1 is held at unit voltage: $x_1 = 1$. Nodes 2 and 3 are both grounded: $x_2 = x_3 = 0$.



(a)

Since we have that the graph is connected, the only non-zero solution to

$$Ax = 0$$

is the vector $x_0 = [1, 1, 1, 1, 1, 1, 1, 1, 1]^T$ (and the ones $\in$ the span of x_0). Therefore, the dimension of the right null-space of A is 1 and, by the Rank-Nullity Theorem, we get that

$$\text{Rank}(A) = 9 - 1 = 8 = \text{Rank}(A^T).$$

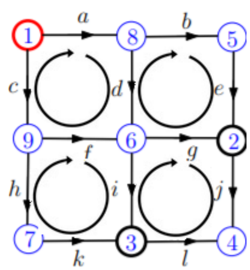
Now considering

$$A^T w = 0$$

and applying the Rank-Nullity Theorem once again, we get that

$$\dim(\text{null-space}(A^T)) = 12 - \text{Rank}(A^T) = 12 - 8 = 4.$$

Hence, there are 4 linearly independent vectors, call them w_1, w_2, w_3 and w_4, that are solution to $A^T w = 0$. We know that geometrically these vectors correspond to *closed loops* of the graph.



So considering the 4 small squares in the graph and drawing a loop in each of them with a clockwise direction, we get that four such linearly independent vectors are (putting 1 if the direction of the edge is the same as the direction of the loop, -1 if it is opposite, and 0 if the loop doesn't go through the edge):

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

It is easy to check that they are linearly independent (from the first two rows and last two). Further,we can also check that

$$A^T w_i = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} w_i = 0$$

for $i \in (1, ..., 4)$. So the *solutions* of $A^T w = 0$ are all linear combinations of the four vectors w_1, w_2, w_3, and w_4 (we know there aren't more since dim(null-space(A^T)) = 4).

(b)

The degree matrix D is a diagonal matrix with D_{ii} being the number of edges connected to node i (with all edges having unit conductance). Therefore, we have

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

The adjacency matrix W contains has 1 in elements w_{ij} if node j is connected to node i. Hence

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The Laplacian matrix K can be found using

$$K = D - W.$$

So we get:

$$K = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 4 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 3 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}.$$

(c)

We know that the vector of currents out of each node f is equal to Kx, where x is the vector of voltages at the nodes. Grounding nodes 2 and 3 corresponds to deleting rows 2 and 3 and columns 2 and 3 leaving us with the *reduced* linear system:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 & 3 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where x_i is the voltage at node i, f_1 is the effective conductance of the circuit, and the net currents out of nodes 4 to 9 are 0 since Kirchhoff's Current Law holds there. We set A, B and C as follows:

$$A = [2], B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & -1 & 0 & 3 & 0 \\ 0 & 0 & -1 & -1 & 0 & 3 \end{bmatrix},$$

so

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x} \end{bmatrix} = \begin{bmatrix} f_1 \\ \underline{0} \end{bmatrix},$$

where $\hat{x} = [x_4, x_5, x_6, x_7, x_8, x_9]^T$.Therefore,

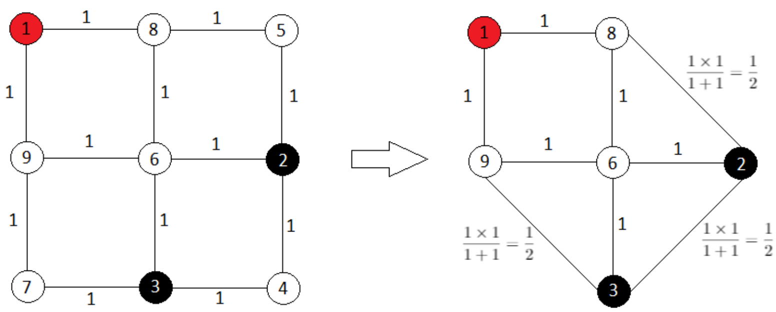
$$\begin{aligned} A + B^T \hat{x} &= f_1 \\ B + C\hat{x} &= 0. \end{aligned}$$

From the system we get that $\hat{x} = -C^{-1}B$ and so

$$f_1 = \boxed{A - B^T C^{-1} B}.$$

(d)

Using the rule that the effective conductance of two resistors with conductances c_a and c_b in *series* is $\frac{c_a c_b}{c_a + c_b}$ we can reduce our circuit at nodes $4, 5$ and 7 using that resistors b and e, j and l, and h and k are in series as follows:



We have now reduced the circuit to a smaller equivalent circuit.Because we now no longer have equal conductances across the edges, we need to compute the weighted Laplacian matrix:

$$K = A^TCA = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1/2 & -1 & -1/2 & 0 \\ 0 & -1/2 & 2 & -1 & 0 & -1/2 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1/2 & 0 & -1 & 5/2 & 0 \\ -1 & 0 & -1/2 & -1 & 0 & 5/2 \end{bmatrix},$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \text{ and } A = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

are the conductance matrix and reduced incidence matrix for our smaller equivalent circuit.

K can also be obtained by writing the sum of the conductances of the edges connected to node i on K_{ii} position and then putting $(-c)$ on the position K_{ij} if node j is connected to node i, where c is the conductance of the edge between these nodes.

Our linear system is then:

$$Kx = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1/2 & -1 & -1/2 & 0 \\ 0 & -1/2 & 2 & -1 & 0 & -1/2 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1/2 & 0 & -1 & 5/2 & 0 \\ -1 & 0 & -1/2 & -1 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ x_6 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Since nodes 2 and 3 are grounded,we can delete rows 2 and 3 and columns 2 and 3. Also, the net current out of nodes $6, 8$ and 9 is 0 since Kirchhoff's Current Law holds there. Our reduced linear system is then the following:

$$\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 4 & -1 & -1 \\ -1 & -1 & 5/2 & 0 \\ -1 & -1 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ x_6 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We get the system:

$$\begin{aligned} 2 - x_8 - x_9 &= f_1 \\ 4x_6 - x_8 - x_9 &= 0 \\ -1 - x_6 + \tfrac{5}{2}x_8 &= 0 \\ -1 - x_6 + \tfrac{5}{2}x_9 &= 0. \end{aligned}$$

Solving it, we get that

$$x_6 = \frac{1}{4}, x_8 = \frac{1}{2}, x_9 = \frac{1}{2}, f_1 = 1.$$

Hence the effective conductance of the circuit is $\boxed{1}$.

(e)

Plugging in the values we found for x_6, x_8 and x_9 we get:

$$Kx = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1/2 & -1 & -1/2 & 0 \\ 0 & -1/2 & 2 & -1 & 0 & -1/2 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1/2 & 0 & -1 & 5/2 & 0 \\ -1 & 0 & -1/2 & -1 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1/4 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the linear system we get that $f_2 = f_3 = -\frac{1}{2}$, which makes sense since $x_0^T f = [1, 1, 1, 1, 1, 1, 1, 1, 1][1, f_2, f_3, 0, 0, 0, 0, 0, 0]^T$ should equal 0. Therefore unit current enters node 1 and $\frac{1}{2}$ current leaves at nodes 2 and 3. So the net current out of node 2 is $\boxed{\frac{1}{2}}$.

(f)

If we ground node 6 as well ($x_6 = 0$), our linear system

$$Kx = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1/2 & -1 & -1/2 & 0 \\ 0 & -1/2 & 2 & -1 & 0 & -1/2 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1/2 & 0 & -1 & 5/2 & 0 \\ -1 & 0 & -1/2 & -1 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ x_6 = 0 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_6 \\ 0 \\ 0 \end{bmatrix}$$

can be reduced further by deleting row 4 and column 4 in addition to the second and third ones.

The reduced system then becomes:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 5/2 & 0 \\ -1 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix}.$$

Solving it we obtain

$$\begin{aligned} x_8 &= \frac{2}{5} \\ x_9 &= \frac{2}{5} \\ f_1 &= \frac{6}{5}. \end{aligned}$$

So the effective conductance is $\boxed{\frac{6}{5}}$.