Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 1 with solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. Find the Fourier transforms of the following functions: (with $a > 0$)

   (i) $f(x) = \exp(-a\,|x|)$;

   $\mathcal{F}\{\exp(-a\,|x|)\} = \int_{-\infty}^{\infty} \exp(-a\,|x|) e^{-i\omega x}\,dx$

   $= \int_{-\infty}^{0} e^{(a-i\omega)x}\,dx + \int_{0}^{\infty} e^{-(a+i\omega)x}\,dx = \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \underline{\frac{2a}{a^2+\omega^2}}$

   (ii) $f(x) = \mathrm{sgn}(x)\exp(-a\,|x|)$; [$\mathrm{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$].

   $\mathcal{F}\{\mathrm{sgn}\,(x)\exp(-a\,|x|)\} = \int_{-\infty}^{0}(-1)e^{ax}e^{-i\omega x}\,dx + \int_{0}^{\infty} e^{-ax}e^{-i\omega x}\,dx = -\frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \underline{-\frac{2i\omega}{a^2+\omega^2}}.$

   (iii) $f(x) = 2a/(a^2 + x^2)$; (Hint: use the result of (i) and the symmetry formula from lectures)

   We know from (i) that if $f(x) = \exp(-a\,|x|)$, then $\widehat{f}(\omega) = 2a/(a^2 + \omega^2)$

   $\Rightarrow \widehat{f}(x) = 2a/(a^2 + x^2)$.

   By the symmetry formula $\mathcal{F}\{\widehat{f}(x)\} = 2\pi f(-\omega) = \underline{2\pi\exp(-a\,|\omega|)}$.

   (iv) $f(x) = 1 - x^2$ for $|x| \leq 1$ and zero otherwise; $f(x) = 1 - x^2$ for $|x| \leq 1 \Rightarrow \widehat{f}(\omega) = \int_{-1}^{1}(1 - x^2)e^{-i\omega x}dx$

   $= \int_{-1}^{1}(1 - x^2)\cos\omega x\,dx - i\int_{-1}^{1}(1 - x^2)\sin\omega x\,dx.$

   The second integral is zero since we are integrating an odd function.

   The first integral is an even function so can be written as twice the integral over $[0, 1]$.

   Thus $\widehat{f}(\omega) = 2\int_{0}^{1}(1 - x^2)\cos\omega x\,dx = \cdots$ (by parts twice) $\cdots = \underline{-(4/\omega^2)\cos\omega + (4/\omega^3)\sin\omega}$.

   (v) $f(x) = \sin(ax)/(\pi x)$; (Hint: use the transform of a rectangular pulse from the lectures and the symmetry formula).

   From your result in part (v), deduce that

   $$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

   From lectures, if $h(x) = 1$ for $|x| \leq a$ and zero otherwise, then $\widehat{h}(\omega) = (2/\omega)\sin(a\omega)$.

   Then by the symmetry formula, $\mathcal{F}\{(2/x)\sin(ax)\} = 2\pi h(-\omega) = 2\pi h(\omega)$ since h is even.

   Thus, $\mathcal{F}\{\sin(ax)/\pi x\} = \underline{h(\omega)}$. We therefore have that $\int_{-\infty}^{\infty}(\sin(ax)/\pi x)\,e^{-i\omega x}dx = h(\omega)$.

   Setting $\omega = 0$ and $a = 1 : \int_{-\infty}^{\infty}(\sin x)/x\,dx = \pi h(0) = \pi$.

   The integrand is even about $x = 0$, and so $\int_{0}^{\infty}(\sin x)/x\,dx = \pi/2$ as required.

2. If a function has Fourier transform $\widehat{f}(\omega)$, find the Fourier transform of $f(x)\sin(ax)$ in terms of $\widehat{f}$.

   $\mathcal{F}\{f(x)\sin ax\} = \int_{-\infty}^{\infty} f(x)\sin ax\, e^{-i\omega x}\,dx = \frac{1}{2i}\int_{-\infty}^{\infty} f(x)(e^{iax} - e^{-iax})e^{-i\omega x}\,dx$

   $= \frac{1}{2i}\int_{-\infty}^{\infty} f(x)e^{-i(\omega-a)x}\,dx - \frac{1}{2i}\int_{-\infty}^{\infty} f(x)\,e^{-i(\omega+a)x}\,dx = \underline{\frac{1}{2i}\widehat{f}(\omega - a) - \frac{1}{2i}\widehat{f}(\omega + a)}.$

3. By applying the inversion formula to the transforms obtained in 1(i) and 1(iv), establish the following results:

$$\text{(i)} \int_0^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a} \text{ if } a > 0;$$

From 1(i) $\mathcal{F}\{\exp(-a\,|x|)\} = 2a/(a^2+\omega^2)$.

Therefore using the inversion formula:

$\exp(-a\,|x|) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(2a/(a^2+\omega^2)\right) e^{i\omega x}\,d\omega = (a/\pi)\left(\int_{-\infty}^{\infty}\frac{\cos(\omega x)}{a^2+\omega^2}\,d\omega + i\int_{-\infty}^{\infty}\frac{\sin(\omega x)}{a^2+\omega^2}\,d\omega\right)$

The second integral is zero since the integrand is odd in ω,

while the first integral has an even integrand and so doubles up over $[0,\infty]$.

Thus $\exp(-a\,|x|) = (2a/\pi)\int_0^\infty \cos(x\omega)/(a^2+\omega^2)\,d\omega$.

This expression is true for any x. Setting $x = 1$:

$$\underline{\frac{\pi e^{-a}}{2a} = \int_0^\infty \frac{\cos\omega}{a^2+\omega^2}\,d\omega}$$

as required.

$$\text{(ii)} \int_{-\infty}^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{2}.$$

From 1(iv) if we define $g(x) = 1-x^2$ for $|x| \le 1$ and zero otherwise, then by inversion:

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left(-\frac{4}{\omega^2}\cos\omega + \frac{4}{\omega^3}\sin\omega\right) e^{i\omega x}\,d\omega.$$

Set $x = 0$ and rearrange to obtain desired result.

4.* Sketch the function given by

$$f(x) = \begin{cases} 2d - |x| & \text{for } |x| \le 2d, \\ 0 & \text{otherwise.} \end{cases},$$

and show that $\widehat{f}(\omega) = (2/\omega)^2\sin^2(\omega d)$.

Use the energy theorem to demonstrate that

$$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{2\pi}{3}.$$

The function $f(x)$ is sketched in Figure 1.

$\widehat{f}(\omega) = \int_{-2d}^{2d}(2d-|x|)e^{-i\omega x}dx = \int_{-2d}^{2d}(2d-|x|)\cos\omega x\,dx - i\int_{-2d}^{2d}(2d-|x|)\sin\omega x\,dx$.

The second integral is zero since the integrand is odd in x.

The first integral has an even integrand and so doubles up over $[0, 2d]$.

Thus $\widehat{f}(\omega) = 2\int_0^{2d}(2d-x)\cos\omega x\,dx = \cdots$ (by parts)

$\cdots = (2/\omega^2)(1-\cos(2\omega d)) = (4/\omega^2)\sin^2(\omega d)$

Therefore $\left|\widehat{f}(\omega)\right|^2 = \underline{(16/\omega^4)\sin^4(\omega d)}$.

Now $\int_{-\infty}^\infty (f(u))^2\,du = \int_{-2d}^{2d}(2d-|u|)^2du = 2\int_0^{2d}(2d-u)^2du$

$= \cdots = \underline{(16/3)d^3}$.

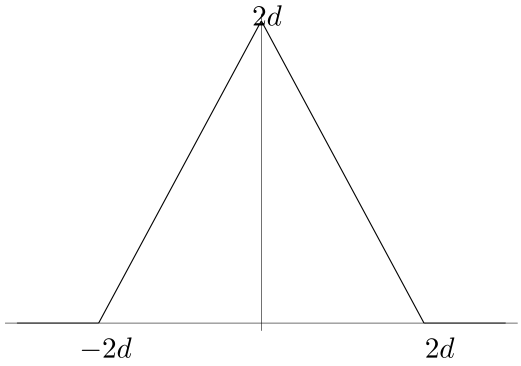

Figure 1: The function $f(x)$ in Question 4

By the energy theorem we have $32\pi d^3/3 = 16\int_{-\infty}^{\infty} \sin^4(\omega d)/\omega^4 d\omega$.

Setting $d = 1$ we get

$$\underline{\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\, dx = \frac{2\pi}{3}},$$

as required.

5. Show that the Fourier transform of $\exp(-cx)H(x)$,where H is the Heaviside function and c is a positive constant, is given by $1/(c + i\omega)$. Hence use the convolution theorem to find the inverse Fourier transform of

$$\frac{1}{(a+i\omega)(b+i\omega)},$$

where $a > b > 0$.

If $f(x) = \exp(-cx)H(x)$ then $\widehat{f}(\omega) = \int_{-\infty}^{\infty} (e^{-cx}H(x))\, e^{-i\omega x}\, dx = \int_0^{\infty} e^{-(c+i\omega)x} dx = \underline{1/(c + i\omega)}$.

Convolution $\Rightarrow (\mathcal{F})^{-1}(\widehat{g}(\omega)\widehat{h}(\omega)) = g(x) * f(x)$.

Let $\widehat{g}(\omega) = 1/(a + i\omega) \Rightarrow g(x) = \exp(-ax)H(x)$.

Let $\widehat{h}(\omega) = 1/(b + i\omega) \Rightarrow h(x) = \exp(-bx)H(x)$.

$\Rightarrow (\mathcal{F})^{-1}((a + i\omega)^{-1}(b + i\omega)^{-1}) = (\exp(-ax)H(x)) * (\exp(-bx)H(x))$.

RHS $= \int_{-\infty}^{\infty} \exp(-a(x-u))H(x-u) \exp(-bu)H(u)\, du$

$= \int_0^{\infty} \exp(-a(x-u))H(x-u) \exp(-bu)\, du$

The function $H(x - u)$ is non-zero (and equal to 1) only if $0 < u < x$.

Therefore RHS $= \int_0^x \exp(-ax)\exp((a-b)u)\, du = \cdots = \underline{(\exp(-bx) - \exp(-ax))/(a - b)\ \ (x > 0)}$.

$\underline{\text{RHS} = 0 \text{ if } x < 0.}$

6. Use the symmetry rule to show that

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}(\widehat{f}(\omega) * \widehat{g}(\omega)).$$

Convolution $\Rightarrow \mathcal{F}\{\widehat{f}(x) * \widehat{g}(x)\} = \mathcal{F}\{\widehat{f}(x)\}\mathcal{F}\{\widehat{g}(x)\}$

$=$ (symmetry formula) $= 4\pi^2 f(-\omega)g(-\omega)$.

Take RHS, change ω to x and take Fourier transform again from both sides:

$\mathcal{F}\{4\pi^2 f(-x)g(-x)\} = 2\pi(\widehat{f}(-\omega) * \widehat{g}(-\omega))$ using the symmetry rule again.

Thus: $\underline{\mathcal{F}\{f(x)g(x)\} = (\widehat{f}(\omega) * \widehat{g}(\omega))/(2\pi)}$, as required.

7. Suppose that $f(x)$ is a function such that $\widehat{f}(\omega) = 0$ for all ω with $|\omega| > M$, where M is a positive constant. Let $g(x) = \sin(ax)/(\pi x)$. Show that if the constant $a > M$:

$$f(x) * g(x) = f(x).$$

Hint: Use the transform of $g(x)$ from Q1(v).

From 1(v) we have that $\widehat{g}(\omega) = 1$ if $|\omega| \leq a$ and zero otherwise.

By convolution: $f(x) * g(x) = (\mathcal{F})^{-1}(\widehat{f}(\omega)\widehat{g}(\omega))$.

Inversion formula $\Rightarrow$ RHS $= (1/2\pi)\int_{-\infty}^{\infty} \widehat{f}(\omega)\widehat{g}(\omega)e^{i\omega x}d\omega = (1/2\pi)\int_{-a}^{a} \widehat{f}(\omega)e^{i\omega x}\,d\omega$

$= (1/2\pi)\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}d\omega$ (since $a > M$).

Thus: $f(x) * g(x) = (\mathcal{F})^{-1}(\widehat{f}(\omega))\underline{= f(x)}$, as required.

8.* By considering suitable integration formulae, establish the following results involving the Dirac delta function:

(i) $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$; (ii) $x\delta'(x) = -\delta(x)$; (iii) $\delta(-x) = \delta(x)$.

Here $f(x)$ is continuous. [In each case multiply by an arbitrary continuous test function $\phi(x)$ and integrate from $-\infty$ to ∞].

(i) $\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\phi(x)\,dx = f(x_0)\phi(x_0) = f(x_0)\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,dx$

$= \int_{-\infty}^{\infty} f(x_0)\delta(x - x_0)\phi(x)\,dx$

$\Rightarrow \int_{-\infty}^{\infty} [f(x)\delta(x - x_0) - f(x_0)\delta(x - x_0)]\phi(x)\,dx = 0$ for arbitrary ϕ.

$\Rightarrow \underline{f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)}$.

(ii) $\int_{-\infty}^{\infty} x\phi(x)\delta'(x)\,dx =$ (by parts) $= [\delta(x)x\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)\,(\phi(x) + x\phi'(x))\,dx$.

Term in square brackets is zero.

Integral reduces to $-\phi(0)$ which can also be written as $-\int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx$.

Thus $\underline{x\delta'(x) = -\delta(x)}$.

(iii) $\int_{-\infty}^{\infty} \delta(-x)\phi(x)\,dx =$ (subst $x = -s$) $= \int_{-\infty}^{\infty} \delta(s)\phi(-s)\,ds = \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx$

$\Rightarrow \underline{\delta(-x) = \delta(x)}$.