Math40003 Linear Algebra and Groups Problem Sheet 4

1. (a) Which of these sets of vectors are linearly independent? Which span $\mathbb{R}^3$?

   (i) $(5,3,0)$, $(2,1,1)$ (ii) $(1,0,1)$, $(-1,1,0)$, $(0,1,1)$

   (iii) $(1,3,1)$, $(2,1,1)$, $(-1,7,-5)$ (iv) $(1,-3,2)$, $(2,-1,1)$, $(2,-5,4)$, $(1,2,5)$

   (b) For which a,b,c are the vectors $(1,3,1)$, $(2,1,1)$, (a,b,c) linearly dependent?

   (a) (i) L.i., does not span

   (ii) L.d. since $v_1+v_2-v_3=0$; does not span

   (iii) L.i. and spans $\mathbb{R}^3$

   (iv) L.d. since $-17v_1-v_2+10v_3-v_4=0$, but spans $\mathbb{R}^3$

   (b) Those satisfying $2a+b-5c=0$.

2. *Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

   (a) If $\{v_1,\ldots,v_n\}$ is a basis, for V, and $\{x_1,\ldots,x_r\}$ is a linearly independent subset of V with $r<n$, and if $v_i \notin Span\{x_1,\ldots,x_r\}$ for all i, then $\{x_1,\ldots,x_r,v_{r+1},\ldots,v_n\}$ is a basis for V.

   (b) If U is a subspace of V, then $U+U=U$.

   (c) If U and W are subspaces of V, and $\dim U+\dim W=\dim V$, then $U\cap W=\{0_V\}$.

   (d) If $\dim V=n$ and $v_1\in V$, then there exist vectors $v_2,\ldots,v_n$ in V such that $\{v_1,\ldots,v_n\}$ spans V.

   (e) If W is a subspace of V , then $\dim W\le\dim V$ and $\dim W=\dim V$ if and only if $W=V$.

   (a) False: take $\{v_1,v_2,v_3\}$ to be the standard basis of $\mathbb{R}^3$, with $s=1$ and $x_1=(0,1,1)$.

   (b) True: since U is closed under addition, we have $U+U\subseteq U$; but since $0_V\in U$ we have $u=u+0_V\in U+U$ for all $u\in U$, and so $U\subseteq U+U$.

   (c) False: take $V=\mathbb{R}^2$, and $U=W=Span\{(1,0)\}$.

   (d) False: but only because v_1 might be 0_V. For any other v_1 it is true.

   (e) True: Let B be a basis for W. Now consider B as a subset of V. If $Span(B)=V$ then B is also a basis for V, so $\dim W=\dim V$. Otherwise, we have $v\in V\setminus Span(B)$, now $B'=\{v\}\cup B$ is LI. If this does not span V we continue to add vectors until we get a spanning set B^*. Now B^* will be basis for V and $B\subseteq B^*$, so $|B|\le|B^*|$.

   It remains to show that if $\dim W=\dim V$ then $V=W$. Suppose $v\in V\setminus W$ then $v\notin Span(B)$, so $\{v\}\cup B$ is an LI subset of V, so $dim(V)\ge dim(W)+1$ giving us a contradiction.

3. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly independent? Extend those which are linearly independent to a basis of $\mathbb{R}^4$.

   (i) $(1,2,3,0)$, $(-1,2,3,0)$ (ii) $(1,2,3,0)$, $(-1,2,3,0)$, $(0,1,2,3)$

   (iii) $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(-1,1,1,-1)$, $(0,1,2,-3)$

   (i) L.i. An example of a basis is $\{v_1, v_2, e_3, e_4\}$, where v_1, v_2 are the vectors given.

   (ii) L.i. Basis $\{v_1, v_2, v_3, e_1\}$.

   (iii) L.d.

4. Let $V = \mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$). Show that the functions

   $$f_1(x) = 1, \quad f_2(x) = 1 + x + x^2, \quad f_3(x) = \sin x, \quad f_4(x) = \cos x$$

   are linearly independent. Which of the following functions lie in $\langle f_1, f_2, f_3, f_4 \rangle$?

   $$5 - 3x - 3x^2, \quad \tan x, \quad 10 - x - x^2 + \sin(x + \pi/3).$$

   (i) If $\exists \lambda_i \in \mathbb{R}$ such that $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 = \mathbf{0}$ (the zero function), then

   $$\lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) + \lambda_4 f_4(x) = 0 \text{ for all } x \in \mathbb{R}.$$

   Putting $x = 0, \pi, 2\pi$ gives the equations $\lambda_1 + \lambda_2 + \lambda_4 = 0$, $\lambda_1 + (1 + \pi + \pi^2)\lambda_2 - \lambda_4 = 0$, $\lambda_1 + (1 + 2\pi + 4\pi^2)\lambda_2 + \lambda_4 = 0$, from which it easily follows that $\lambda_1 = \lambda_2 = \lambda_4 = 0$. Now put $x = \pi/2$ to get $\lambda_3 = 0$ also. So f_1, f_2, f_3, f_4 are linearly independent.

   (ii) The function $\tan x$ is not in $\langle f_1, \ldots, f_4 \rangle$ — otherwise $\exists \lambda_i \in \mathbb{R}$ such that $\tan x = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) + \lambda_4 f_4(x)$ for all $x \in \mathbb{R}$, which we check to be impossible by putting $x = 0, \pi, 2\pi$ etc. as above.

   The other two functions are in $\langle f_1, \ldots, f_4 \rangle$.

5. (a) Write down an infinite number of different bases of $\mathbb{R}^2$ (in finite time).

   (b) Find a basis for $W = \langle x^2 - 1,\ x^2 + 1,\ 4,\ 2x - 1,\ 2x + 1 \rangle \leq \mathbb{R}[x]$.

   Recall: $\mathbb{R}[x]$ is the set of real polynomials in variable x

   (a) E.g. $\{(1,0), (0,t)\}$ are bases for any $t \in \mathbb{R} \backslash \{0\}$.

   (b) Call these polynomials $p_1, \ldots, p_5$. Since $p_1 \neq 0$ we keep it. Since p_2 is not a scalar multiple of p_1, we keep it. Next, $p_3 = 2(p_2 - p_1) \in \langle p_1, p_2 \rangle$, so we throw it away. Now p_4 has an x term, so cannot be in $\langle p_1, p_2 \rangle$, so keep it. Finally, $p_5 = p_4 + \frac{1}{2} p_3 \in \langle p_1, p_2, p_4 \rangle$ so chuck it.

   So a basis is $\{p_1, p_2, p_4\}$.

6. Let $M_{3,3}$ denote the vector space of all 3×3 matrices over $\mathbb{R}$.

   (i) Find a basis of $M_{3,3}$ consisting of invertible matrices.

(ii) Let $W = \{A \in M_{3,3} \colon A^t = A\}$. Show $W \leq V$ and compute $\dim W$.

(iii) Let $W \subset M_{3,3}$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W \leq V$ and find a basis for W.

(i) E.g. $I, I - 2E_{33}, I - 2E_{22}$ **together with** $I + E_{ij}$ **for all** $i \neq j$**, where** E_{ij} **has a 1 in the** ij**th position only.**

(ii) The general symmetric matrix

$$\begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$$

is $aE_{11} + bE_{22} + cE_{33} + x(E_{12} + E_{21}) + y(E_{13} + E_{31}) + z(E_{23} + E_{32})$**. So a basis for this subspace is** $E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}$**, and its dimension is 6.**

(iii) First show the middle entry must be 0 — e.g. add each diagonal and the middle row to get that $0 =$ **the sum of the 1st and third columns plus twice the middle entry.**

So the magic matrices have the form $\begin{pmatrix} a & b & -a-b \\ c & 0 & -c \\ -a-c & -b & -a \end{pmatrix}$**, with** $2a + b + c = 0$ **to get the last row to add to 0.**

Get a basis by taking first $a = 1, b = 0$ **then** $a = 0, b = 1$**. So dimension** $= 2$**.**