Math40002 Analysis 1 — Unseen 2

1. Say $S \subseteq \mathbb{R}$ is a set of real numbers, with the property that $\forall s \in S, \exists t \in S, s < t$. Can S be bounded above?

   Yes, e.g, $\{ -1/n | n \in \mathbb{N} \}$.

2. Say $S \subseteq \mathbb{R}$ is a set of real numbers, with the property that $\forall n \in \mathbb{N}, \exists t \in S, n < t$. Can S be bounded above?

   No. If S is bounded above and M is an upper bound for S, let $N > M$ such that $N \in \mathbb{N}$. By the property, there is some $t \in S$ such that $M < N < t$, contradicting M being an upper bound.

3. (a) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded above. Assume:

   $$\forall a \in A, \exists b \in B \text{ such that } a \leq b.$$

   Prove $\sup A \leq \sup B$.

   Assume $\sup A > \sup B$. Then $\sup B$ is not an upper bound for A (otherwise it would contradict $\sup A$ being the *least* upper bound). So there is some $a \in A$ such that $a > \sup B$. By the assumption, there is some $b \in B$ such that $b \geq a > \sup B$, contradicting $\sup B$ being an upper bound

   (b) Prove that if $A \subseteq B \subset \mathbb{R}$ then $\sup A \leq \sup B$.

   If $A \subseteq B$, then for every $a \in A$, also $a \in B$ and $a \leq a$, so the assumption of (a) holds.

   (c) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded above. Assume:

   $$\forall a \in A, \exists b \in B \text{ such that } a \geq b.$$

   Prove $\inf A \geq \inf B$.

   (d) Prove that if $A \subseteq B \subset \mathbb{R}$ then $\inf A \geq \inf B$.

   Translate the proof of the first two items into lower bounds.

4. Say we have a sequence of real numbers $a_1, a_2, a_3, \ldots$. Assume:

   $$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N} : a_n < M$$

   $$\exists m \in \mathbb{R} \text{ such that } \forall N \in \mathbb{N}, \exists n > N : m < a_n$$

   Now let's define some sets $S_1, S_2, S_3, \ldots$ by

   $$S_n = \mathbb{R} \setminus \{ a_1, \ldots, a_n \} = \{ a_{n+1}, a_{n+2}, a_{n+3} \ldots \}.$$

   (a) Prove that for all $n \geq 1$, there exists some $b_n \in \mathbb{R}$ such that $b_n = \sup(S_n)$.

   Let M be an upper bound for $\{ a_1, \ldots, a_n \}$ (from the first assumption above). Then M is also an upper bound for S_n.

(b) Prove there exists some $l \in \mathbb{R}$ such that $l = \inf\{\, b_1, b_2, b_3, \dots \}$.

Let m be as promised from the second assumption. By the assumption, for every $N \in \mathbb{N}$, there is some $a_n \in S_N$ such that $m < a_n$. Therefore, $m < \sup S_N = b_N$. As N was arbitrary, the set $\{\, b_1, b_2, b_3, \dots \}$ is bounded below.

Such l is called the limpsup of the sequence $(a_1, a_2, a_3, \dots)$

(c) Find the limsup of the following sequence and prove your findings.

i. $1, 1, 1, \dots$

ii. $0, 1, 0, 1, 0, 1, \dots$

iii. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

iv. $-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots$

i. 1.

ii. 1. **For every $n \in \mathbb{N}$, $b_n = \sup S_n = 1$.**

iii. 0. **For every $n \in \mathbb{N}$, $b_n = \sup S_n = \frac{1}{n}$. So $\inf\{\, b_1, b_2, \dots \} = \inf\left\{\, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} = 0$.**

iv. 0. **For every $n \in \mathbb{N}$, $b_n = \sup S_n = 0$. So $\inf\{\, b_1, b_2, \dots \} = \inf\{\, 0 \,\} = 0$.**

(d) Prove:

$$\inf\{\, a_1, a_2, a_3, \dots \} \le \limsup(a_1, a_2, a_3, \dots) \le \sup\{\, a_1, a_2, a_3, \dots \}.$$

By 3b, $b_n \le \sup\{\, a_1, a_2, a_3, \dots \}$ for any $n \in \mathbb{N}$. Therefore

$$\limsup(a_1, a_2, a_3, \dots) = \inf\{\, b_1, b_2, b_3, \dots \} \le b_n \le \sup\{\, a_1, a_2, a_3, \dots \}.$$

On the other hand, since $b_n = \sup\{\, a_{n+1}, a_{n+2}, a_{n+3}, \dots \}$, $b_n \ge a_{n+1}$ for every $n \in \mathbb{N}$. Therefore, by 3c,

$$\limsup(a_1, a_2, a_3, \dots) = \inf\{\, b_1, b_2, b_3, \dots \} \ge \inf\{\, a_1, a_2, a_3, \dots \}.$$

(e) Let $X = \mathbb{Q} \cap (0, 1)$ and let $f : \mathbb{N} \to X$ be a bijection. For every $i \in \mathbb{N}$, let $a_i = f(i)$. Prove $\limsup(a_1, a_2, a_3, \dots) = 1$.[1]

Let $n_0 \in \mathbb{N}$. By definition of S_{n_0}, $S_{n_0} \subseteq X$, therefore it is bounded above by 1.

To show 1 is the least upper bound, assume towards a contradiction $c < 1$ is some other upper bound. Now, we will show that $X \cap (c, 1)$ is infinite. This can be deduced from the fact the rational numbers are dense, or by the assumption that X is countably infinite (as there is a bijection from $\mathbb{N}$) and showing that the map

[1] Hint for (d): To prove $b_n = 1$, show that for every $c < 1$, $X \cap (c, 1)$ is infinite, therefore $S_n \cap (c, 1)$ is infinite.

$g : X \to X \cap (c,1)$ **defined by** $g(x) = (1-c)x + c$ **is a bijection. Nonetheless, here is a straightforward proof: Let** $N \in \mathbb{N}$ **such that** $N > \frac{1}{1-c}$**. Then** $c < 1 - \frac{1}{N} < 1$**. In particular, for every** $n > N$**,** $1 - \frac{1}{N} \in X \cap (c,1)$ **so** $X \cap (c,1)$ **is infinite.**

Now $X \setminus S_{n_0}$ **is finite. In particular** $X \setminus S_{n_0} \cap (c,1)$ **is finite. As** $X = (X \setminus S_{n_0}) \cup S_{n_0}$**, it follows that**

$$X \cap (c,1) = ((X \setminus S_{n_0}) \cup S_{n_0}) \cap (c,1) = ((X \setminus S_{n_0}) \cap (c,1)) \cup (S_{n_0} \cap (c,1)).$$

Since $X \cap (c,1)$ **is infinite and** $(X \setminus S_{n_0}) \cap (c,1)$ **is finite,** $S_{n_0} \cap (c,1)$ **must be infinite. In particular, there is some** $a \in S_{n_0}$ **such that** $c < a$**, contradicting** c **is an upper bound for** S_{n_0}**.**

So $b_n = \sup S_n = 1$ **for all** $n \in \mathbb{N}$**, therefore** $\limsup(a_1, a_2, a_3, \dots) = \inf\{1, 1, 1, \dots\} = 1$**.**

(f) If you like, then guess the definition of *liminf* and compute it for the examples above.

Which of these sequences converges? (we will see a rigorous definition of this notion next week, but perhaps you know what it means).

Can you tell just from looking at the limsup and liminf?