Math40003 Linear Algebra and Groups Problem Sheet 2

1. *You should now be able to do questions 2 and 6 on PS1*

2. Describe the solution sets to the following sets of simultaneous equations in $\mathbb{R}^2$:

(a) $$\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y &= 5 \end{aligned}$$

The point$(-\mathbf{1},\mathbf{2})$

(b)$x+2y=3$

the line $\mathbf{y}=-\frac{\mathbf{x}}{\mathbf{2}}+\frac{\mathbf{3}}{\mathbf{2}}$

(c) $$\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y &= 5 \\ x+4y &= 6 \end{aligned}$$

The empty set

3. Describe the solution sets to the following sets of simultaneous equations in $\mathbb{R}^3$:

(a) $$\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y-2z &= 5 \end{aligned}$$

The line $\begin{pmatrix} 3 \\ 0 \\ -\frac{17}{2} \end{pmatrix}+\mathbf{t}\begin{pmatrix} -2 \\ 1 \\ \frac{17}{4} \end{pmatrix}$

(b)$x+2y=3$

the plane through $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

with normal vector $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

(c) $$\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y-2z &= 5 \\ x+4y+z &= 6 \end{aligned}$$

The point $\begin{pmatrix} -\frac{17}{25} \\ \frac{46}{25} \\ -\frac{17}{25} \end{pmatrix}$

4.* For which $a,b\in\mathbb{R}$ does the system of equations

$$\begin{aligned} x_1+x_2+x_3 &= -1 \\ 2x_1+x_2+ax_3 &= 1 \\ 3x_1+x_2+x_3 &= b \end{aligned}$$

have (i) no solutions, (ii) exactly one solution, (iii) infinitely many solutions?

Apply the row operations

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & a & 1 \\ 3 & 1 & 1 & b \end{array}\right) \xrightarrow[R_3\mapsto R_3-3R_1-2R_2']{R_2\mapsto R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & a-2 & 3 \\ 0 & 0 & 2-2a & b-3 \end{array}\right)$$

The final row gives $(2-2a)x_3=b-3$, so there are no solutions if $a=1$, $b\neq 3$. If $a=1$, $b=3$ then final row gives no information, but we have $x_2=(a-2)x_3-3$ and $x_1=-1-x_2-x_3$ from the second and first rows. Substituting $x_3=c$ for any $c\in\mathbb{R}$ we find infinitely many solutions.

When $a\neq 1$ then $x_3=\frac{b-3}{2-2a}$ is uniquely determined, as is x_2 from the second row and then x_1 from the first, so there is a unique solution.

What about the system

$$\begin{array}{rcl} x_1 + x_2 + x_3 + x_4 & = & 0 \\ x_1 - x_2 + a x_3 + x_4 & = & 1 \\ 2x_1 + a x_2 + x_3 + 2x_4 & = & b \ ? \end{array}$$

You have to be more careful with this one:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & a & 1 & 1 \\ 2 & a & 1 & 2 & b \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & a-1 & 0 & 1 \\ 0 & a-2 & -1 & 0 & b \end{array}\right)$$

(If $a = 2$ this is already in echelon form and the last row – and then back substitution – shows there are infinitely many solutions. But actually don't need to make this a special case, as it is also included in the general case below.)

$R_3 \mapsto 2R_3 + (a-2)R_2$ **gives**

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & a-1 & 0 & 1 \\ 0 & 0 & a^2 - 3a & 0 & 2b + a - 2 \end{array}\right)$$

If $a \notin \{0, 3\}$ then from the last row we see we get infinitely many solutions.

If $a = 0$ the last row shows we get 0 solutions if $b \neq 1$ and infinitely many otherwise.

If $a = 3$ the last row shows we get 0 solutions if $b = -\frac{1}{2}$ and infinitely many otherwise.

5. Which of the following are possible, find examples if possible:

   (a) Two simultaneous equations in two unknowns which defines a line in $\mathbb{R}^2$.

   (b) Two simultaneous equations in two unknowns which defines the empty set in $\mathbb{R}^2$.

   (c) One equation in no unknowns which defines the empty set.

   (d) Two simultaneous equations in three unknowns which defines a point in $\mathbb{R}^3$.

(a)

$$\begin{array}{rcl} x + y & = & 1 \\ 2x + 2y & = & 2 \end{array}$$

(b)

$$\begin{array}{rcl} x + y & = & 1 \\ x + y & = & 2 \end{array}$$

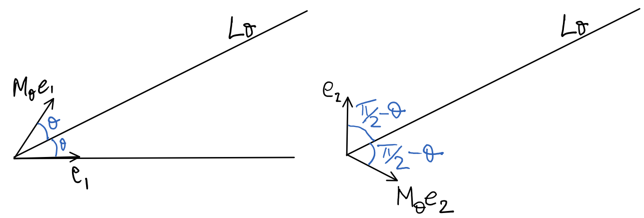
(c) $0 = 1$

(d) Not possible.

6. (a) Let M_θ be the reflection in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan\theta\}$. Using any school geometry or trigonometry you like, show that the matrix representing M_θ is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Drawing it, you see that e_1 gets reflected to the unit vector making an angle 2θ with the x_1-axis, i.e. $\binom{\cos 2\theta}{\sin 2\theta}$.



Similarly e_2 makes an angle $\pi/2 - \theta$ anticlockwise from L_θ, so gets reflected to a unit vector whose angle is $\pi/2 - \theta$ *clockwise* from L_θ. Thus it makes an angle $\theta - (\pi/2 - \theta) = 2\theta - \pi/2$ with the x_1-axis, so is the unit vector $\binom{\cos(2\theta - \pi/2)}{\sin(2\theta - \pi/2}) = \binom{\sin 2\theta}{-\cos 2\theta}$.

Thus the matrix is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ as claimed.

(b) Let R_α be a rotation about the origin, and let M_β be the reflection in a line through the origin. Prove that $M_\beta R_\alpha$ is a reflection.

We compute the product

$$\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos(2\beta - \alpha) & \sin(2\beta - \alpha) \\ \sin(2\beta - \alpha) & -\cos(2\beta - \alpha) \end{pmatrix}.$$

By (c) this is reflecton in $L_{\beta - \alpha/2}$

(c) Let M_α and M_β be reflections in straight lines through the origin. Prove that $M_\alpha M_\beta$ is a rotation.

We compute

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} = \begin{pmatrix} \cos 2(\alpha - \beta) & -\sin 2(\alpha - \beta) \\ \sin 2(\alpha - \beta) & \cos 2(\alpha - \beta) \end{pmatrix},$$

which is the rotation $R_{2(\alpha - \beta)}$.