Math40002 Analysis 1 — Term 2 Unseen 4

1. Find a function $f : [0,1] \to \mathbb{R}$ such that the following hold:

   - Let $0 \leq a < b \leq 1$ and pick a value c between $f(a)$ and $f(b)$. Then there is some $x \in [a,b]$ such that $f(x) = c$.
   - f is *not* continuous.

   $$f(x) = \begin{cases} \sin(1/x) & \textbf{if } x \neq 0 \\ 0 & \textbf{if } x = 0. \end{cases}$$

   The limit of f at 0 doesn't exist, so f is not continuous at 0.

   Another example $f'(x)$ for $f(x) = \begin{cases} x^2 \sin(1/x) & \textbf{if } x \neq 0 \\ 0 & \textbf{if } x = 0. \end{cases}$ **By Question 7 in Question Sheet 4, f' satisfies the intermediate value property, but is not continuous.**

2. Let $f : \mathbb{R} \to \mathbb{R}$ such that for every open interval $I \subseteq \mathbb{R}$: $f(I) := \{ f(x) | x \in I \} = \mathbb{R}$.

   (a) Prove that for every $a, b \in \mathbb{R}$ and c between $f(a)$ and $f(b)$. Then there is some $x \in [a,b]$ such that $f(x) = c$.

   (b) prove that f is *nowhere continuous*, i.e., f is not continuous for any $x \in \mathbb{R}$.

   (a) By definition, for every $c \in \mathbb{R}$, there is some $x \in (a,b)$ such that $f(x) = c$.

   (b) Let $x \in \mathbb{R}$. Let $\epsilon = 1 > 0$. For every $\delta > 0$: $f((x-\delta, x+\delta)) = \mathbb{R}$, therefore, there is some $x - \delta < x' < x + \delta$ such that $|f(x') - f(x)| > 1$.

   Note: In the question above you proved that the converse of the Intermediate Value Theorem fails miserably for function satisfying that $f(I) = \mathbb{R}$ for every interval I. However, it is not so clear such functions exist!

3. Define $a \sim b$ iff $a - b \in \mathbb{Q}$.

   (a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$ and every equivalence class is countable.

   - $a - a = 0 \in \mathbb{Q}$.
   - **If $a - b \in \mathbb{Q}$ then $b - a = -(a-b) \in \mathbb{Q}$.**
   - **If $a - b, b - c \in \mathbb{Q}$, then $a - c = (a-b) + (b-c) \in \mathbb{Q}$.**

   This shows that $\sim$ is an equivalence relation. For every $a \in \mathbb{R}$: $cl(a) = \{ b \in \mathbb{R} | b - a \in \mathbb{Q} \} = \{ a + (b-a) | (b-a) \in \mathbb{Q} \} = \{ a + q | q \in \mathbb{Q} \}$. Clearly the latter is countable.

(b) For every $x \in \mathbb{R}$, let $cl(x)$ be the equivalence class of x with respect to $\sim$. Show that for every $x \in \mathbb{R}$: $cl(x)$ is dense in $\mathbb{R}$, namely, for every interval I, there is some $y \in cl(x) \cap I$.

Denseness of $\mathbb{Q}$ was established in many different ways, e.g., by converging sequences and by decimal expansion. Now if (a, b) is an open interval, then by denseness of $\mathbb{Q}$, there is some $q \in \mathbb{Q} \cap (a - x, b - x)$. So $q \in \mathbb{Q}$ and $a < q + x < b$. Therefore $q + x \in cl(x) \cap (a, b)$.

(c) Let $\mathcal{P} := \{\, cl(x) | x \in \mathbb{R} \,\}$ be the set of all equivalence classes of $\sim$. Show that there is a bijection from $\mathcal{P}$ to $\mathbb{R}$.

You may need to use the following fact from set theory:

Let $\mathcal{C}$ is an infinite set of countable non-empty sets such that $A \cap B = \emptyset$ for every $A, B \in \mathcal{C}$. Then there is a bijection between $\mathcal{C}$ and $\bigcup_{A \in \mathcal{C}} A$.

Follows immediately from the fact that $\mathbb{R} = \bigcup_{A \in \mathcal{P}} A$ and two distinct equivalence classes are disjoint.

(d) Let $g : \mathcal{P} \to \mathbb{R}$ be a bijection, as promised from Item 3c. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = g(cl(x))$. Prove that $f(I) = \mathbb{R}$ for every interval I. Deduce that f satisfies the intermediate value property, but is nowhere continuous.

Let $I = (a, b)$ and let $y \in \mathbb{R}$. Then there is some $A \in \mathcal{P}$ such that $g(A) = y$. Let $x' \in \mathbb{R}$ such that $A = cl(x')$. By Item 3b, there is some $x \in cl(x') \cap I$. Therefore, $x \in I$ and $f(x) = g(cl(x)) = g(cl(x')) = g(A) = y$.