Mathematics Year 1, Calculus and Applications I

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Problem Sheet 2

Problems 9 and 10 are good candidates for starred questions

1. Sketch the functions $y = x\exp(-x)$, $y = x^2\exp(-x^2)$, $y = \frac{\exp(x)}{x}$.

2. Consider the function $f(x) = \exp(1/x)$, $x \neq 0$.

   (a) What are the limits
   $$\lim_{x\to 0+} f(x), \qquad \lim_{x\to 0-} f(x), \qquad \lim_{x\to +\infty} f(x), \qquad \lim_{x\to -\infty} f(x).$$

   (b) Now define $f(0) = 0$. Is the function differentiable?

   (c) Calculate $\lim_{x\to 0-} \frac{d^n f}{dx^n}$ for any positive integer n.

   (d) Sketch $y = f(x)$.

3. Sketch the function $y = x\exp(1/x)$.

4. Show that the equation $\mathrm{e}^x = ax$ has at least one solution for any number a, except when $0 \le a < \mathrm{e}$.

5. Consider the function
   $$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) Show that $f(x)$ has a derivative at $x = 0$ and that $f'(0) = 0$.

   (b) Does f' have a derivative everywhere? If yes, what is it?

   (c) Do any further derivatives of $f(x)$ exist?

   (d) Sketch the function.

6. Find the derivative of the function $f(x) = x^x$, $x > 0$. Does the derivative at $x = 0+$ exist? Explain. Sketch the curve of $f(x)$.

7. Calculate $\frac{d}{dx}\left(x^{x^x}\right)$.

8. Is the logarithm to base 2 of an irrational number ever rational? If yes, give an example.

9. (a) Find $\lim_{a\to 0} \frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$.

   (b) Find $\lim_{a\to \infty} \frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$.

10. Find the following limits
    $$\lim_{x\to 1} x^{1/(1-x^2)}, \quad \lim_{x\to 0}(\tan x)^x, \quad \lim_{x\to\infty}[\log x - \log(x-1)], \quad \lim_{x\to 1}\frac{\log x}{e^x - 1}, \quad \lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4}.$$

11. Suppose that f is continuous at $x = x_0$, that $f'(x)$ exists for x in an interval about x_0, $x \neq x_0$, and that $\lim_{x\to x_0} f'(x) = m$. Prove that $f'(x_0)$ exists and equals m. [Hint. Use the mean value thorem.]