

Mathematics Year 1, Calculus and Applications I

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Problem Sheet 3

Problems 7 and 15 and 16 are good candidates for starred questions

1. (a) Let $S_2 := \sum_{i=1}^n i^2$. Compute

$$\begin{aligned}\sum_{i=1}^n [(i+1)^3 - i^3] &= \sum_{i=1}^n [i^3 + 3i^2 + 3i + 1 - i^3] = 3 \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\ &= 3S_2 + \frac{3}{2}n(n+1) + n.\end{aligned}$$

The first sum in the equation above is a telescoping series and is equal to $2^3 - 1^3 + 3^3 - 2^3 + \dots + (n+1)^3 - n^3 = (n+1)^3 - 1$. Hence, after a little algebra,

$$S_2 = \frac{1}{6}n(n+1)(2n+1).$$

- (b) Partition $[0, 1]$ into $x_i = ih$ where $h = 1/n$ (note, $x_0 = 0$ and $x_n = 1$). The upper Riemann sum is

$$\sum_{i=1}^n h(ih)^2 = h^3 \sum_{i=1}^n i^3 = \frac{1}{n^3} \cdot \frac{1}{6}n(n+1)(2n+1) \rightarrow \frac{1}{3} \quad \text{as } n \rightarrow \infty.$$

2. Need to show $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} e^{i/n} = e - 1$. Let $r = e^{1/n}$. Then

$$\sum_{i=1}^n \frac{1}{n} e^{i/n} = \frac{1}{n} \sum_{i=1}^n r^i = \frac{1}{n} r \frac{1 - r^n}{1 - r} = \frac{(1/n)e^{1/n}}{(e^{1/n} - 1)}(e - 1) \rightarrow (e - 1) \quad \text{as } n \rightarrow \infty.$$

[E.g. use L'Hôpital's rule on $\lim_{x \rightarrow 0} \frac{x}{e^x - 1}$.]

3. Take any partition of $[0, 1]$ and calculate the lower Riemann sum L , and upper Riemann sum U . Since any interval of real numbers, however small, contains an infinite number of rationals and irrationals, it follows that

$$L = 0, \quad U = 1.$$

This is the case for *any* partition, hence $L \neq U$ for any partition and so the function is not Riemann integrable.

4. First part straightforward. Using this result we see that $\frac{d}{dx} \log(\sec x + \tan x) = \sec x$, hence we have the antiderivative of $\sec x$ as required.

The integral result can be applied on intervals that satisfy $\cos x \neq 0$ and $\sec x + \tan x > 0$, i.e. $\frac{1 + \sin x}{\cos x} > 0$ which is only possible if $\cos x > 0$. Hence the appropriate interval is $(-\pi/2, \pi/2)$ since $\cos(\pm\pi/2) = 0$.

5. (i) Substitute $x = \tan \theta$, i.e. $dx = \sec^2 \theta d\theta$ and $x^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta$:

$$\begin{aligned}\int \frac{1}{(x^2 + 1)^3} dx &= \int \frac{\sec^2 \theta}{\sec^6 \theta} d\theta = \int \cos^4 \theta d\theta = \int \frac{(1 + \cos 2\theta)^2}{4} d\theta \\ &= \int \left(\frac{1 + 2\cos 2\theta}{4} + \frac{1 + \cos 4\theta}{8} \right) d\theta = \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} + K.\end{aligned}$$

- (ii) Use partial fractions $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$. Solving we find $A = 1/3$, $B = -1/3$, $C = -2/3$, and the integral is

$$\frac{1}{3} \int \left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} \right) dx = \frac{1}{3} \int \left(\frac{1}{x-1} - \frac{1}{2} \frac{(2x+1)}{(x+1/2)^2 + (3/4)} - \frac{(3/2)}{(x+1/2)^2 + (3/4)} \right) dx,$$

where I have completed the square in the last term. Now we can integrate to find

$$\frac{1}{3} \log(x-1) - \frac{1}{2} \log(x^2+x+1) - (3/2) \frac{1}{\sqrt{3/2}} \tan^{-1} \frac{x+1/2}{\sqrt{3/2}} + K.$$

- (iii) Write $\frac{x^3+1}{x^3-1} = 1 + \frac{2}{x^3-1}$, and the integral of the second piece has just been done.

- (iv) Use integration by parts. Write

$$\begin{aligned} \int x^3 \sqrt{x^2+1} dx &= \int x^2 \frac{d}{dx} \left(\frac{2}{3} (x^2+1)^{3/2} \right) dx = x^2 \frac{2}{3} (x^2+1)^{3/2} - \int 2x \frac{2}{3} (x^2+1)^{3/2} dx \\ &= \frac{2}{3} x^2 (x^2+1)^{3/2} - \frac{4}{15} (x^2+1)^{5/2} + K. \end{aligned}$$

- (v) Use the trigonometric substitution $u = \sin x$ so that $du = \cos x dx$, and the integral becomes

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x + \sin^3 x} dx &= \int \frac{du}{u(1+u)} = \int \left(\frac{1}{u} - \frac{1}{1+u} \right) du \\ &= \log \left(\frac{u}{1+u} \right) = \left[\log \left(\frac{\sin x}{1+\sin x} \right) \right]_{\pi/6}^{\pi/2} = \log(1/2) - \log(1/3) = \log(3/2). \end{aligned}$$

6. If $n = 1$, $I_1 = \tan^{-1} x$. Start with I_{n-1}

$$\begin{aligned} I_{n-1} &= \int \frac{dx}{(x^2+1)^{n-1}} = \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{x^2}{(x^2+1)^n} dx \\ &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{x^2+1-1}{(x^2+1)^n} dx \\ &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)I_{n-1} - 2(n-1)I_n, \end{aligned}$$

therefore, as required,

$$2(n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}.$$

7. We will need the double angle formulas

$$\begin{aligned} \sin A \cos B &= \frac{1}{2} [\sin(A+B) + \sin(A-B)], \\ \sin A \sin B &= \frac{1}{2} [\cos(A-B) - \cos(A+B)], \\ \cos A \cos B &= \frac{1}{2} [\cos(A-B) + \cos(A+B)]. \end{aligned}$$

(a) For any two integers m, n

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \cos nx dx &= \int_{-\pi}^{\pi} \frac{1}{2} [\sin(m+n)x + \sin(m-n)x] dx = 0. \\ \int_{-\pi}^{\pi} \sin mx \sin nx dx &= \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } n = m \end{cases}, \\ \int_{-\pi}^{\pi} \cos mx \cos nx dx &= \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } n = m \end{cases}.\end{aligned}$$

(b) We assume

$$f(x) = a_0 + \sum_{k=1}^N a_k \cos kx + b_k \sin kx, \quad (*)$$

and need to calculate the a 's and b 's. The constant a_0 can be found immediately by integrating over $[-\pi, \pi]$,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) dx &= 2\pi a_0 + \sum_{k=1}^N \int_{-\pi}^{\pi} (a_k \cos kx + b_k \sin kx) dx = 2\pi a_0 \Rightarrow \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,\end{aligned}$$

i.e. it is the average (or mean) value of the function over the domain.

Next, take any integer $m \geq 1$, multiply (*) by $\cos mx$ and integrate between $-\pi$ and π . Using the *orthogonality* results from part (a) we see that only the term containing a_m in the sum will survive to give

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_m \int_{-\pi}^{\pi} \cos^2 mx dx = \pi a_m \Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx.$$

Now do the same calculation but multiply by $\sin mx$ and integrate to find

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx.$$

(c) For $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}.$$

we have by direct application of the formulas in part (b)

$$\begin{aligned}a_k &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx dx = \frac{1}{\pi} \left[\frac{\sin kx}{k} \right]_{-\pi/2}^{\pi/2} = \frac{2}{k\pi} \sin \frac{k\pi}{2}, \\ b_k &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin kx dx = \frac{1}{\pi} \left[-\frac{\cos kx}{k} \right]_{-\pi/2}^{\pi/2} = 0.\end{aligned}$$

The last result could have been anticipated because the given $f(x)$ is an even function of x on $[-\pi/2, \pi/2]$ and $\sin kx$ is odd there, hence the product $f(x) \sin kx$ is also odd, implying that the integral must be zero.

8. (i) Write $\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$. The first integral is a finite number. For the second integral we have

$$\int_1^\infty e^{-x^2} dx < \int_1^\infty x e^{-x^2} dx = \frac{1}{2e},$$

hence the integral is bounded.

- (ii) Again, given any $M > 0$ write $\int_0^\infty \frac{x^3}{(1+x^2)^2} dx = \int_0^M \frac{x^3}{(1+x^2)^2} dx + \int_M^\infty \frac{x^3}{(1+x^2)^2} dx$; the first integral is a finite number and for the second integral we have

$$\int_M^\infty \frac{x^3}{(1+x^2)^2} dx > \int_M^\infty \frac{x^3}{(x^2+x^2)^2} dx = \frac{1}{4} \int_M^\infty \frac{1}{x} dx.$$

The last integral diverges so by comparison our original integral also diverges.

- (iii) For $M > 0$ write $\int_0^\infty \frac{1}{\sqrt{x+x^3}} dx = \int_0^M \frac{1}{\sqrt{x+x^3}} dx + \int_M^\infty \frac{1}{\sqrt{x+x^3}} dx$, and consider separately $\lim_{a \rightarrow 0+} \int_a^M \frac{1}{\sqrt{x+x^3}} dx$ and $\lim_{b \rightarrow \infty} \int_M^b \frac{1}{\sqrt{x+x^3}} dx$. We have

$$\begin{aligned} \lim_{a \rightarrow 0+} \int_a^M \frac{1}{\sqrt{x+x^3}} dx &= \lim_{a \rightarrow 0+} \int_a^M \frac{1}{\sqrt{x}\sqrt{1+x^2}} dx \leq \lim_{a \rightarrow 0+} \int_a^M \frac{1}{\sqrt{x}} dx \\ &= \lim_{a \rightarrow 0+} 2(\sqrt{M} - \sqrt{a}) = 2\sqrt{M}. \end{aligned}$$

Similarly,

$$\int_M^\infty \frac{dx}{\sqrt{x+x^3}} < \int_M^\infty \frac{dx}{x^{3/2}} = \lim_{b \rightarrow \infty} \int_M^b \frac{dx}{x^{3/2}} = \lim_{b \rightarrow \infty} 2(M^{-1/2} - b^{-1/2}) = 2/\sqrt{M},$$

i.e. also bounded.

- (iv)

$$\int_0^1 \frac{\sin^2 x}{1+x^2} dx < \int_0^1 \frac{1}{1+x^2} dx = \pi/4.$$

- (v) Useful to compare with a function we know how to integrate. Let $f(x) = x - \log(1+x)$. We have $f(0) = 0$ and $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0$ for $x \geq 0$. Hence $f(x)$ is increasing and as a result

$$x > \log(1+x) \quad \text{for} \quad x > 0.$$

Hence, given $0 < \varepsilon < 1$ we have

$$\int_\varepsilon^1 \frac{dx}{\log(1+x)} > \int_\varepsilon^1 \frac{dx}{x} = \log(1/\varepsilon),$$

and sending $\varepsilon \rightarrow 0+$ proves that the integral is divergent.

- (vi) This is an improper integral whose integrand does not decay for large x . In fact infinitely more rapid oscillations take place and we can intuitively expect that there is cancellation to give a finite result. [In fact this and its cosine sister are known as Fresnel integrals and come up generically in wave propagation, optics etc.]

Again we split the integral $\int_0^\infty \sin(x^2)dx = (\int_0^1 + \int_1^\infty) \sin(x^2)dx$. The first integral is perfectly fine and is equal to a finite number (we cannot find this in closed form). For the second integral we integrate by parts

$$\begin{aligned}\int_1^b \sin(x^2)dx &= \int_1^b \frac{x \sin(x^2)}{x} dx = \left[-\frac{1}{2x} \cos(x^2) \right]_1^b - \frac{1}{2} \int_1^b \frac{\cos(x^2)}{x^2} dx \\ &\leq \frac{1}{2} \cos 1 - \frac{1}{2b} \cos b^2 + \frac{1}{2} \int_1^b \frac{dx}{x^2} = \frac{1}{2} \cos 1 - \frac{1}{2b} \cos b^2 + \frac{1}{2} \left(1 - \frac{1}{b}\right),\end{aligned}$$

and as b becomes large we see that the integral is bounded above by 1.

Combining this with the integral over $[0, 1]$ proves that the Fresnel integral converges.

9. To prove that $\int_0^1 \frac{x^3}{2 - \sin^4 x} dx \leq \frac{1}{4} \log 2$ we use the inequality $\sin x \leq x$ in the interval $[0, 1]$ of interest. [In fact we have $|x| \leq |\sin x|$ for all $x \in \mathbb{R}$ - you have already proved this elsewhere by showing that the function $f(x) = x - \sin x$ is increasing.] This implies that $2 - \sin^4 x \geq 2 - x^4$, hence

$$\int_0^1 \frac{x^3}{2 - \sin^4 x} dx \leq \int_0^1 \frac{x^3}{2 - x^4} dx = \left[-\frac{1}{4} \log(2 - x^4) \right]_0^1 = \frac{1}{4} \log(2).$$

For the second integral, since we are on the interval $[0, \pi/2]$ we have $\cos x > 0$, hence

$$\left| \int_0^{\pi/2} \frac{x - \pi/2}{2 + \cos x} dx \right| \leq \int_0^{\pi/2} \frac{|x - \pi/2|}{2} dx = \frac{1}{2} \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) dx = \frac{\pi^2}{16}.$$

10. Proof of the *integral mean value theorem*: Let f and g be continuous on $[a, b]$ with $g(x) \geq 0$ for $x \in [a, b]$. Then there exists a c between a and b with

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Proof: Since f is continuous on $[a, b]$ it must have a maximum M and a minimum m on $[a, b]$, i.e. $m \leq f(x) \leq M$. Since $g(x) \geq 0$, we have for all $x \in [a, b]$

$$mg(x) \leq f(x)g(x) \leq Mg(x),$$

and by the properties of integrals we have in turn

$$m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx.$$

If $\int_a^b g(x)dx = 0$ then we also have $\int_a^b f(x)g(x)dx = 0$ and the result follows. If $\int_a^b g(x)dx \neq 0$, we have

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M,$$

and by the Intermediate Value Theorem there must be a c between a and b so that

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx},$$

and the result of the theorem follows.

An example that violates the conclusion of the theorem if we drop $g(x) \geq 0$ is, $f(x) = \sin x$, $g(x) = \sin x$, $a = -\pi/2$, $b = \pi/2$. Then $\int_{-\pi/2}^{\pi/2} fg dx \neq 0$ but $f(c) \int_{-\pi/2}^{\pi/2} \sin x dx = 0$ for all $c \in [-\pi/2, \pi/2]$.

11. These are fairly straightforward. I will do a couple of them.

x^2 on $[0, 1]$. Here we have $\mu = \int_0^1 x^2 dx = 1/3$ and $\sigma^2 = \int_0^1 (x^2 - 1/3)^2 dx = \frac{4}{45}$.

$f(x) = \begin{cases} 1 & \text{on } [0, 1] \\ 2 & \text{on } (1, 2] \end{cases}$. Here $\mu = \frac{1}{2} \left(\int_0^1 1 dx + \int_1^2 2 dx \right) = \frac{3}{2}$. Next, $(f(x) - \mu)^2$ is equal to $(1 - 3/2)^2 = 1/4$ on $[0, 1]$ and $(2 - 3/2)^2 = 1/4$ on $(1, 2]$. Hence $\sigma^2 = \frac{1}{2} \int_0^2 (1/4) dx = 1/4$.

12. (a)

$$\mu = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{b-a} \sum_{i=1}^n \int_{x_{i-1}}^{x_i} k_i dx = \frac{1}{b-a} \sum_{i=1}^n k_i (x_i - x_{i-1})$$

Hence

$$\sigma^2 = \frac{1}{b-a} \int_a^b (f(x) - \mu)^2 dx = \frac{1}{b-a} \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x) - \mu)^2 dx = \frac{1}{b-a} \sum_{i=1}^n (k_i - \mu)^2 (x_i - x_{i-1}).$$

- (b) If the partition consists of equally spaced points, we have $x_i - x_{i-1} = \frac{b-a}{n}$, hence the formulas are

$$\mu = \frac{1}{n} \sum_{i=1}^n k_i,$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (k_i - \mu)^2.$$

- (c) The standard deviation of a step function is a sum of non-negative terms (in both cases of uniform and non-uniform partitions). The only way this can be zero is if the function is a constant (and hence equal to its average).
- (d) For a list of numbers a_1, a_2, \dots, a_n the mean is $\mu = \frac{1}{n} \sum_{i=1}^n a_i$ and $\sigma^2 = \frac{1}{n} \sum_{i=1}^n (a_i - \mu)^2$, so completely analogous to the uniform partition result for functions.
- (e) If its standard deviation is zero, then the numbers are equal.

13. (a) Calculate using definition of $f(x)$:

$$\int_a^b g(x) \delta_n(x) dx = \int_{-1/2n}^{1/2n} g(x) n dx = \int_{-1/2}^{1/2} g(y/n) dy \rightarrow g(0) \quad \text{as } n \rightarrow \infty,$$

since g is a continuous function.

(b) Using the result above, we have

$$\lim_{n \rightarrow \infty} \int_a^t g(x) \delta_n(x) dx = \begin{cases} g(0) & t > 0 \\ 0 & t < 0 \end{cases}.$$

(c) Here take $g(x) = 1$ and use the results above to find

$$\lim_{n \rightarrow \infty} \int_{t_1}^{t_2} \delta_n(x) dx = \begin{cases} 0 & t_1 < t_2 < 0 \\ 0 & 0 < t_1 < t_2 \\ 1 & t_1 < 0 < t_2 \end{cases}.$$

(d) We have shown in (c) that the anti-derivative of $\delta(x)$ is the function $H(x)$ (called the Heaviside function) defined by

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

14. Begin by constructing $f(x)$: For $0 \leq x < 1$ we have $f(x) = 1$, and for $1 \leq x < 2$, $f(x) = 2$. Hence, for $0 \leq x \leq 1$ we have $F(x) = \int_0^x dx = x$, and for $1 \leq x \leq 2$, $F(x) = \int_0^1 dx + \int_1^x 2dx = 2x - 1$:

$$F(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2x - 1 & \text{for } 1 \leq x \leq 2 \end{cases}.$$

The function $F(x)$ is continuous but $F'(1)$ does not exist. This does not contradict the fundamental theorem of calculus since $f(x)$ is not continuous. Recall the statement of the FTC:

Fundamental Theorem of Calculus

If f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F(x)$ is continuous on $[a, b]$. If in addition f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ and $F' = f$.

15. Can evaluate the integral using integration by parts:

$$\int_1^n \log x dx = [x \log x]_1^n - \int_1^n x \cdot \frac{1}{x} dx = n \log n - (n - 1).$$

Now partition $[1, n]$ in unit intervals as instructed to find the upper Riemann sum U and lower Riemann sum L :

$$U = \log 2 + \log 3 + \dots + \log n = \log[n!],$$

$$L = \log(1) + \log(2) + \dots + \log(n - 1) = \log[(n - 1)!],$$

therefore we have the inequality

$$\log[(n - 1)!] \leq n \log n - (n - 1) \leq \log[n!],$$

and using $n \log n - (n-1) = \log(n^n) + \log(e^{-(n-1)}) = \log(n^n e^{-n} e)$, along with the fact that \log is monotonic increasing we obtain the desired result

$$(n-1)! \leq n^n e^{-n} e \leq n!$$

Using this we can bound $n!/n^n$ as follows

$$e^{-n} e \leq \frac{n!}{n^n} \leq n e^{-n} e \Rightarrow e^{-1} e^{1/n} \leq \left(\frac{n!}{n^n}\right)^{1/n} \leq n^{1/n} e^{-1} e^{1/n}.$$

As n gets large we know that $e^{-1/n} \rightarrow 1$ and $n^{1/n} \rightarrow 1$ (why?), hence by the squeezing theorem the result $\lim_{n \rightarrow \infty} \left(\frac{n!}{n^n}\right)^{1/n} = 1/e$ follows.

16. (a) This question involves integration by parts. Calculate

$$I_0 = \int_0^\infty e^{-x} dx = [-e^{-x}]_0^\infty = 1$$

$$J_0 = \int_0^\infty e^{-x} \cos x dx = [e^{-x} \sin x]_0^\infty + \int_0^\infty e^{-x} \sin x dx = [-e^{-x} \cos x]_0^\infty - J_0$$

Hence $J_0 = 1/2$.

[Alternatively use complex variables and write $e^{-x} \cos x = \mathcal{R}[e^{(i-1)x}]$ so that $J_0 = \mathcal{R} \left[\frac{e^{(i-1)x}}{(i-1)} \right]_0^\infty = \mathcal{R} \left(\frac{1+i}{2} \right) = 1/2.$]

$$I_n = [-e^{-x}(\sin x)^n]_0^\infty + \int_0^\infty n e^{-x}(\sin x)^{n-1} \cos x dx = n J_{n-1}.$$

For any non-negative integer n , let

$$I_n = \int_0^\infty e^{-x}(\sin x)^n dx, \quad J_n = \int_0^\infty e^{-x}(\sin x)^n \cos x dx.$$

To get the last expression we write $e^{-x} = d(-e^{-x})$ and integrate that first by parts, i.e.

$$\begin{aligned} J_n &= [-e^{-x}(\sin x)^n \cos x]_0^\infty - \int_0^\infty (-e^{-x}) [-(\sin x)^{n+1} + n(\sin x)^{n-1} \cos^2 x] dx \\ &= \int_0^\infty e^{-x} [n(\sin x)^{n-1} - (1+n)(\sin x)^{n+1}] dx = n I_{n-1} - (n+1) I_{n+1} \Rightarrow \\ J_n &= n I_{n-1} - (n+1) I_{n+1} \end{aligned} \quad (*)$$

as required.

Note: J_n can also be integrated by parts by noting that $(\sin x)^n \cos x = d\left(\frac{(\sin x)^{n+1}}{n+1}\right)$ and integrating this first, to immediately obtain the alternative expression

$$J_n = \left[\frac{e^{-x}(\sin x)^{n+1}}{n+1} \right]_0^\infty + \int_0^\infty e^{-x} \frac{(\sin x)^{n+1}}{n+1} dx \Rightarrow J_n = \frac{1}{n+1} I_{n+1}. \quad (**)$$

As we will see, this is of course consistent with the recursion formula for I_n we find next.

- (b) Use $n = 1$ in the formula $I_n = nJ_{n-1}$ to find $I_1 = J_0 = 1/2$.

From (*) $J_1 = I_0 - 2I_2 = I_0 - 2(2J_1)$, hence $5J_1 = 1$ as needed.

We have $I_n = nJ_{n-1}$. Evaluate (*) at $n - 1$ to find $J_{n-1} = (n - 1)I_{n-2} - nI_n$ and substitute into the expression for I_n to get

$$I_n = n(n - 1)I_{n-2} - n^2I_n \quad \Rightarrow \quad I_n = \frac{n(n - 1)}{(1 + n^2)}I_{n-2}. \quad (*_3)$$

Note: This result should of course be consistent with equating (*) = (*) which implies $I_{n+1} = \frac{n(n+1)}{1+(n+1)^2}I_{n-1}$ which is identical to (*) once we shift the index $n + 1 \rightarrow n$.

To find the recursion for J , we now eliminate I_{n-1} and I_{n+1} in (*) to find

$$J_n = n(n - 1)J_{n-2} - (n + 1)^2J_n \quad \Rightarrow \quad J_n = \frac{n(n - 1)}{1 + (n + 1)^2}J_{n-2}. \quad (*_4)$$

- (c) We need to calculate explicit expressions from (*) and (*) for $n \geq 2$. In both cases we have starting values $I_0 = 1$, $I_1 = 1/2$ and $J_0 = 1/2$, $J_1 = 1/5$, and all other values will be in terms of these. Inspection of (*) and (*) shows that all even indices will involve I_0 or J_0 and odd indices will involve I_1 and J_1 . Lets calculate a few terms and you will get the general formula.

$$\begin{aligned} I_2 &= \frac{2 \cdot 1}{(1+2^2)}I_0 & I_3 &= \frac{3 \cdot 2}{(1+3^2)}I_1 \\ I_4 &= \frac{4 \cdot 3}{(1+4^2)}I_2 = \frac{4!}{(1+2^2)(1+4^2)}I_0, & I_5 &= \frac{5 \cdot 4}{(1+5^2)}I_3 = \frac{5!}{(1+3^2)(1+5^2)}I_1 \\ &\dots & \dots & \\ I_{2k} &= (2k)! \prod_{p=1}^k \left(\frac{1}{1+(2p)^2} \right), & I_{2k+1} &= \frac{1}{2}(2k+1)! \prod_{p=1}^k \left(\frac{1}{1+(2p+1)^2} \right) \end{aligned}$$

Similarly for the J s we have

$$\begin{aligned} J_2 &= \frac{2 \cdot 1}{(1+3^2)}J_0 & J_3 &= \frac{3 \cdot 2}{(1+4^2)}J_1 \\ J_4 &= \frac{4!}{(1+3^2)(1+5^2)}J_0 & J_5 &= \frac{5!}{(1+4^2)(1+6^2)}J_1 \\ &\dots & \dots & \\ J_{2k} &= \frac{1}{2}(2k)! \prod_{p=1}^k \left(\frac{1}{(1+(2p+1)^2)} \right) & J_{2k+1} &= \frac{1}{5}(2k+1)! \prod_{p=1}^k \left(\frac{1}{(1+(2p+2)^2)} \right) \end{aligned}$$

We can compare directly I_{2k} with J_{2k} found above; the products in J_{2k} are smaller since $\frac{1}{(1+(2p+1)^2)} < \frac{1}{1+(2p)^2}$, in addition to the factor of $1/2$. Hence $I_{2k} > J_{2k}$. Analogous reasoning shows also that $I_{2k+1} > J_{2k+1}$ and so $I_n > J_n$ for all $n \geq 0$. This is reasonable since comparison of the integrands of J_n and I_n , respectively shows that

$$|e^{-x}(\sin x)^n \cos x| \leq |e^{-x}(\sin x)^n|.$$