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Solutions to Question Sheet 4

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Problem sheet released on Wednesday of week 5. All questions can be attempted before the problem class on Monday Week 7. Question 1 is suitable for tutorials. Solutions will be released on Wednesday of week 7.

Question 1 The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Find a matrix $A \in M_2(\mathbb{R})$ with the property that

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}.$$

Compute the eigenvalues and eigenvectors of A and express $(1,0)^T$ as a linear combination of eigenvectors. Hence, or otherwise, find a general expression for F_n (in terms of n).

Solution: Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This has characteristic polynomial $x^2 - x - 1$ and eigenvalues $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$. We have corresponding eigenvectors $v_i = \begin{pmatrix} \lambda_i \\ 1 \end{pmatrix}$ (for $i = 1, 2$). (Note that these are orthogonal, which is what we should expect as A is symmetric.)

Now,

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

At this point, we can either produce a general expression for A^k, using the method in the lectures, or we can note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}(v_1 - v_2)$ and therefore

$$A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}(A^{n-1}v_1 - A^{n-1}v_2) = \frac{1}{\sqrt{5}}(\lambda_1^{n-1}v_1 - \lambda_2^{n-1}v_2).$$

Comparing the first coordinates gives $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$.

Question 2 Suppose $A \in M_n(F)$ has characteristic polynomial

$$\chi_A(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n.$$

The *Cayley - Hamilton Theorem* states that

$$\chi_A(A) = a_0 I_n + a_1 A + \ldots + a_{n-1}A^{n-1} + A^n = 0.$$

(A special case of this was given on Question 7 of sheet 3.) Here is the start of a proof of the result. Finish the proof:

Let $B = B(x)$ denote the adjugate matrix of $(xI_n - A)$. So each entry in B is a cofactor of $xI_n - A$ and is therefore a polynomial of degree at most $n-1$ in x. Thus we can write $B(x) = B_{n-1}x^{n-1} + \ldots + B_1x + B_0$ for some matrices $B_{n-1}, \ldots, B_1, B_0 \in M_n(F)$ (so these do not involve the variable x). By 5.3.2 in the lectures:

$$(a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + x^n)I_n = \det(xI_n - A)I_n = B(x)(xI_n - A) = \\ (B_{n-1}x^{n-1} + \ldots + B_1x + B_0)(xI_n - A). \quad (1)$$

Multiplying out the right-hand side and equating coefficients of the various powers of x we obtain:

$$a_0I_n = -B_0A, \ldots$$

Solution: We obtain

$$a_0I_n = -B_0A,\ a_1I_n = B_0 - B_1A,\ a_2I_n = B_1 - B_2A, \ldots, a_{n-1}I_n = B_{n-2} - B_{n-1}A,\ I_n = B_{n-1}.$$

Multiply (on the right) each of these equations in turn by $I_n, A, A^2, \ldots, A^{n-1}$ and add together. The left hand side is $\chi_A(A)$. The right-hand side is zero - everything cancels!

Question 3 Suppose $S, T : V \to V$ are linear and $S \circ T = T \circ S$. For $\lambda \in F$ let $E_\lambda(S) = \{v \in V : Sv = \lambda v\}$. Show that if $v \in E_\lambda(S)$, then $T(v) \in E_\lambda(S)$.

Solution: For $v \in E_\lambda(S)$, we have $S(T(v)) = T(S(v)) = T(\lambda v) = \lambda T(v)$, so $T(v) \in E_\lambda(S)$.

Question 4 (i) Suppose $v_1, \ldots, v_r \in \mathbb{R}^n$ is an orthogonal set of non-zero vectors. Show that $v_1, \ldots, v_r$ are linearly independent.

(ii) Suppose $A \in M_n(\mathbb{R})$. Prove that A is an orthogonal matrix if and only if for all $u \in \mathbb{R}^n$ we have $\|Au\| = \|u\|$.

(iii) Suppose $A \in M_2(\mathbb{R})$ is an orthogonal matrix and $\det(A) = 1$. Show that there is $\theta \in \mathbb{R}$ such that $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (so A is a rotation matrix).

(iv) Suppose $C \in M_3(\mathbb{R})$ is an orthogonal matrix and $\det(C) = 1$. Show that 1 is an eigenvalue of C. Is this true for 4×4 matrices?

Solution: (i) Consider $v = \sum_{i=1}^r a_iv_i$. Then $v_j \cdot v = \sum_i a_i(v_j \cdot v_i) = a_j\|v_j\|^2$ as the v_i are orthogonal. In particular, if $v = 0$, then as the v_i are non-zero, we deduce that $a_j = 0$ for all $j \leq r$.

(ii) Suppose $A^TA = I_n$. Then $\|Au\|^2 = (Au)^T(Au) = u^TA^TAu = u^TI_nu = \|u\|^2$, which gives one direction.

For the converse, consider the effect of A on the standard basis. The assumption implies that $\|Ae_i\| = \|e_i\| = 1$ and for $i \neq j$ we have $\|Ae_i - Ae_j\|^2 = \|A(e_i - e_j)\|^2 = \|e_i - e_j\|^2 = 2$. So

$$2 = \|Ae_i\|^2 + \|Ae_j\|^2 - 2(Ae_i) \cdot (Ae_j) = 2 - 2(Ae_i) \cdot (Ae_j).$$

Thus $Ae_i \cdot Ae_j = 0$. It follows that the columns Ae_i of A are unit vectors which are pairwise othogonal. So A is an orthogonal matrix.

(iii) Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The conditions give $ad - bc = 1$, $ab + cd = 0$, $a^2 + c^2 = 1$, $b^2 + d^2 = 1$. Multiply the first by a and use $ab = -cd$: we obtain $a = (a^2 + c^2)d = d$.

Similarly $b = -c$. Finally, by the 3rd of equation, there is $\theta \in \mathbb{R}$ with $a = \cos\theta$ and $c = \sin\theta$.

(iv) It suffices to show that $\det(I_3 - A) = 0$. We know that $\det(A^T) = \det(A) = 1$. Also $A^T(I_3 - A) = A^T - I_3 = -(I_3 - A^T)$. So using the product formula $\det(I_3 - A) = \det(A^T)\det(I_3 - A) = \det(-(I_3 - A^T))$. As these are 3×3 matrices, this is $-\det(I_3 - A^T) = -\det((I_3 - A)^T) = -\det(I_3 - A)$ and the result follows.

The result does not hold for 4×4 matrices. For example the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ is orthogonal and has characteristic polynomial $(x^2+1)^2$.

Question 5 Find an orthogonal matrix $A \in M_4(\mathbb{R})$ whose first column is $\frac{1}{2}(1,1,1,1)^T$.

Solution: The systematic way is to extend the given vector v_1 to a basis (for example u_1, e_1, e_2, e_3) and apply Gram - Schmidt. Then take the orthonormal set of vectors as the columns of the required matrix. A quicker way is to spot some orthogonal vectors such as $v_1 = (1,1,1,1)^T, v_2 = (1,-1,0,0)^T, v_3 = (0,0,1,-1)^T$, extend to a basis (eg. include e_1) and then use Gram - Schmidt to obtain

$$v_4 = e_1 - (e_1 \cdot v_1)v_1/4 - (e_1 \cdot v_2)v_2/2 - (e_1 \cdot v_3)v_3/2 = \frac{1}{4}(1,1,-1,-1)^T,$$

then normalise these.

Question 6 Suppose U is a subspace of $\mathbb{R}^n$ and $T : U \to U$ is linear. Suppose B is an orthonormal basis of U. Prove that $[T]_B$ is symmetric if and only if for all $u, v \in U$ we have $T(u) \cdot v = u \cdot T(v)$.

Solution: Let $v_1, \ldots, v_n$ be the basis of U and $A = (a_{ij})$ be the matrix $[T]_B$. Thus $T(v_j) = \sum_{i=1}^n a_{ij}v_i$. So (as in question 4(i)) $v_k \cdot (T(v_j)) = a_{kj}$ for $1 \le k, j \le n$. Thus, if $T(u)\cdot v = u\cdot T(v)$ for all $u, v \in U$ we obtain (by putting $u = v_k$ and $v = v_j$) that $a_{kj} = a_{jk}$ for all j, k, so A is symmetric. Conversely if A is symmetric, then the calculation shows that for all k, j we have $v_k \cdot (T(v_j)) = v_j \cdot T(v_k) = T(v_k) \cdot v_j$. As $v_1, \ldots, v_n$ span U, we can then use linearity of T to deduce that $v_k \cdot T(v) = T(v_k) \cdot v$ for all k and all $v \in U$, and then finally that $u \cdot T(v) = T(u) \cdot v$ for all $u, v \in U$.

Question 7 Suppose U is a subspace of $\mathbb{R}^n$. Let $U^\perp = \{v \in \mathbb{R}^n : v\cdot u = 0 \text{ for all } u \in U\}$.

i) Show that $U^\perp$ is a subspace of $\mathbb{R}^n$.

ii) Show that $U \cap U^\perp = \{0\}$.

iii) Show that $\dim(U^\perp) = n - \dim(U)$.

iv) In the case where $n = 4$ and U is the subspace spanned by $u_1 = (1,1,0,1)^T$ and $u_2 = (1,1,1,0)^T$, compute $U^\perp$.

Solution: (i) Use the test for being a subspace.

(ii) If $u \in U \cap U^{\perp}$ then $u \cdot u = 0$, so $u = 0$.

(iii) By the modular law $n \geq \dim(U + U^{\perp}) = \dim(U) + \dim(U^{\perp}) - \dim(U \cap U^{\perp})$. So using (iii), $\dim(U^{\perp}) \leq n - \dim(U)$. On the other hand, if we take an orthonormal basis $u_1, \ldots, u_k$ of U, then by the Gram - Schmidt process, we can find an orthonormal basis $u_1, \ldots, u_k, u_{k+1}, \ldots, u_n$ of $\mathbb{R}^n$. Then $u_{k+1}, \ldots, u_n \in U^{\perp}$, so $\dim(U^{\perp}) \geq n - \dim(U)$ and we therefore have the equality.

(iv) Note that $U^{\perp} = \{x \in \mathbb{R}^n : u_1 \cdot x = 0 \text{ and } u_2 \cdot x = 0\}$. This is a system of two homogeneous linear equations and you know how to solve such things.