**Math40002 Analysis 1** **Unseen 6**

1. Let $(a_n)$ be a sequence of real numbers. Prove/Disprove:

   (a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.
   **No, e.g.,** $a_n = (-1)^n/\sqrt{n}$**.**

   (b) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} |a_n^2|$ converges.
   **Yes. By comparison test. If** $\sum_{n=1}^{\infty} |a_n|$ **converges, then** $|a_n| \to 0$**. In particular, there is some** $N \in \mathbb{N}$ **such that** $|a_n| < 1$ **for all** $n > N$**. So** $|a_n^2| < |a_n|$ **for all** $n > N$**.**

   (c) If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
   **No, e.g.,** $a_n = 1/n$**.**

2. Assume $\sum_{n=1}^{\infty} \frac{1}{n^2} = S$. Find $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

   **So** $A = A/4 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ **and so** $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = 3A/4$**.**

3. (a) Assume $a_n$ is a monotonically decreasing sequence. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $na_n \to 0$.
   **First, since** $\sum_{n=1}^{\infty} a_n$ **converges,** $a_n \to 0$**. Since** $a_n$ **is monotonically decreasing,** $a_n \geq 0$ **for all** $n \in \mathbb{N}$**. Let** $\epsilon > 0$**, and let** $N \in \mathbb{N}$ **such that for all** $m > k > N$**:** $\sum_{n=1}^{m} a_n - \sum_{n=1}^{k} a_n < \epsilon/2$**. Then** $a_{k+1} + \cdots + a_m = \sum_{n=1}^{m} a_n - \sum_{n=1}^{k} a_n < \epsilon/2$**. Then for every** $n > 2N$**:**

$$\frac{n}{2} a_n < a_{\lceil n/2 \rceil} + \cdots + a_n < \epsilon/2.$$

   **So** $na_n < \epsilon$**.**

   (b) Is it true that in general, if $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges, then $na_n \to 0$?
   **No, e.g.,**

$$a_n = \begin{cases} \frac{1}{n} & \textbf{if } \sqrt{n} \in \mathbb{N} \\ 0 & \textbf{if otherwise.} \end{cases}$$

   **So** $\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \frac{1}{k^2}$**. But for** $\epsilon = 1/2$**, given** $N \in \mathbb{N}$**, let** $n > N$ **such that** $\sqrt{n} \in \mathbb{N}$**. Then** $na_n = n/n = 1 > \epsilon$**.**