Math40003 Linear Algebra and Groups — Problem Sheet 6

1.* (a) Which of the following functions $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ are linear transformations?

   i. $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 + x_2)$

   ii. $T(x_1, x_2, x_3) = (0, \sqrt{2}x_3)$

   iii. $T(x_1, x_2, x_3) = (x_1x_2, x_3)$

(b) Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Which of the following functions $T : V \longrightarrow V$ are linear transformations?

   i. $T(A) = A^2$ for all $A \in V$

   ii. $T(A) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $A \in V$

(c) i. Find a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which sends $(1, 0)$ to $(1, 1, 0)$ and $(1, 1)$ to $(1, 0, -1)$.

   ii. Find two different linear transformations $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which send $(1, 1, 0)$ to $(1, 1)$ and $(0, 1, 1)$ to $(0, 1)$.

(d) Let V be the vector space (over $\mathbb{R}$) of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following are linear transformations (thinking of $\mathbb{R}$ as $\mathbb{R}^1$ in parts (i) and (iii)) ?

   i. $T_1 : V \to \mathbb{R}$ where $T_1(f) = f(1)$ (for $f \in V$).

   ii. $T_2 : V \to V$ where $T_2(f) = f \circ f$ (for $f \in V$).

   iii. $T_3 : \mathbb{R} \to V$ where $T_3(\mu)$ is the function $f_\mu \in V$ given by $f_\mu(x) = \mu x$ (for $\mu, x \in \mathbb{R}$).

2. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for exactly one vector $v \in \mathbb{R}^2$.

(b) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for no vector $v \in \mathbb{R}^2$.

(c) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for infinitely many vectors $v \in \mathbb{R}^2$.

(d) Show that there is no linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for exactly two vectors $v \in \mathbb{R}^2$.

3. (Harder) (i) Suppose V, W are vector spaces (over a field F) and $S, T : V \to W$ are linear transformations. Prove that $S + T : V \to W$ defined by $(S + T)(v) = S(v) + T(v)$ (for $v \in V$) is a linear transformation. If $\lambda \in F$, show that $\lambda S : V \to W$ defined by $(\lambda S)(v) = \lambda S(v)$ (for $v \in V$) is a linear transformation. Explain why the set U of all linear transformations from V to W is a vector space with these operations.

(ii) In the case where $V = F^2$ and $W = F^3$, what is the dimension of the vector space U? What is the dimension of U for arbitrary finite dimensional vector spaces V and W?

The following need material from the last week of term:

4. (a) Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$. Find bases of Ker T and Im T. For which values of k is the vector $(1, 3, k)$ in Ker T or Im T?

   (b) Let V be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Define $T : V \longrightarrow V$ by

   $$T(ax^2 + bx + c) = (a + b + c)x^2 + (c - a)x + (a + 3b + 5c).$$

   Find bases of Ker T and Im T.

   (c) Let V be as in part (b), and define $S : V \longrightarrow V$ by

   $$S(p(x)) = p(1 + x) - p(x) \text{ for } p(x) \in V.$$

   (So for example, $S(x^2) = (x + 1)^2 - x^2 = 2x + 1$.) Show that S is a linear transformation, and find bases of Ker S and Im S.

5. (a) Let V be a finite-dimensional vector space, and $T : V \longrightarrow V$ a linear transformation.

   i. Prove that T is injective if and only if Ker $T = \{0\}$.

   ii. Prove that T is surjective if and only if Ker $T = \{0\}$.

   (b) Find an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that Ker $T =$ Im T.

   (c) Prove that there does not exist a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that Ker $T =$ Im T.