Math40002 Analysis 1 **Problem Sheet 2**

1. Fix $S \subset \mathbb{R}$ with an upper bound, and suppose that $S \neq \emptyset$ and $S \neq \mathbb{R}$. Give proofs or counterexamples to each of the following statements.

   (a) If $S \subset \mathbb{Q}$ then $\sup S \in \mathbb{Q}$.

   (b) If $S \subset \mathbb{R}\backslash\mathbb{Q}$ then $\sup S \in \mathbb{R}\backslash\mathbb{Q}$.

   (c) If $S \subset \mathbb{Z}$ then $\sup S \in \mathbb{Z}$.

   (d) $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : \ n, m \in \mathbb{N}, \ m \leq 10^{100}\right\}$ has a minimum if it is nonempty.

   (e) There exists a $\max S$ if and only if $\sup S \in S$.

   (f) $\sup S = \inf(\mathbb{R}\backslash S)$.

   (g) $\sup S = \inf(\mathbb{R}\backslash S) \iff S$ is an interval of the form $(-\infty, a)$ or $(-\infty, a]$.

(a) False, eg $S = \{x \in \mathbb{Q} \colon x < \sqrt{2}\}$ **with** $\sup S = \sqrt{2}$**.**

(b) False, eg $S = \{x \in \mathbb{R}\backslash\mathbb{Q} \colon x < 0\}$ **with** $\sup S = 0$**.**

(c) True. Pick $s_0 \in S$ **and an integer** N **larger than a given upper bound for** S**. Then** $[s_0, N] \cap \mathbb{Z}$ **is a finite set, so** $[s_0, N] \cap S$ **is also finite and nonempty set of integers so has a maximum** $m \in \mathbb{Z}$**. By (e) this is also** $\sup S$**.**

(d) True. Since it is $\neq \emptyset$ **pick an element** s_0**. Then** $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : \ n, m \in \mathbb{N}, \ m \leq 10^{100}\right\} \cap [0, s_0]$ **is nonempty and *finite*! (Because** m **runs through the finite set** $\{1, 2, \ldots, 10^{100}\}$ **and** n **runs through the finite set** $\{1, 2, \ldots, \lfloor s10^{100} \rfloor\}$**.) It therefore has a minimum, which is clearly the minimum of** $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : \ n, m \in \mathbb{N}, \ m \leq 10^{100}\right\}$ **because it is in the set and is** $\leq$ **all elements** s **of the set that satisfy** $s \leq s_0$**. It is therefore** $\leq$ **elements** s **of the set that satisfy** $s > s_0$ **too!**

(e) True. If $\sup S \in S$ **then it is the maximum element because it is an upper bound, so** $\sup S \geq s$ **for all** $s \in S$**.**

Conversely if there exists $m = \max S$ **then** m **is an upper bound, and given any other upper bound** M**,** $M \geq m$ **by definition of upper bound because** $m \in S$**. Therefore** m **is the least upper bound, i.e.** $\sup S = m \in S$**.**

(f) False. E.g. $S = \{0\}$ **has** $\sup S = 0$ **but** $\mathbb{R}\backslash S$ **is not even bounded below so has no infimum.**

(g) True. If $S = (-\infty, a)$ **then** $\sup S = a$ **while** $\mathbb{R}\backslash S = [a, \infty)$ **so** $\inf S^c = a$ **also.**

If $S = (-\infty, a]$ **then** $\sup S = a$ **while** $\mathbb{R}\backslash S = (a, \infty)$ **so** $\inf S^c = a$ **also.**

Conversely, if $\sup S = \inf S^c$ **then call this number** a**. Any** $x < a$ **must be in** S**: if not then** $x \in S^c$ **but** $x < \inf S^c$**, a contradiction. Similarly any** $x > a$ **must be in** S^c**: if not then** $x \in S$ **but** $x > \sup S$**, a contradiction.**

Therefore $(-\infty, a) \subseteq S$ **and** $(a, \infty) \subseteq S^c$**. Finally either** $a \in S$ **or** $a \in S^c$**, making** S **equal to** $(-\infty, a]$ **or** $(-\infty, a)$ **respectively.**

2. Fix nonempty sets $S_n \subset \mathbb{R}, \ n = 1, 2, 3, \ldots$. Prove that

$$\sup\{\sup S_1, \sup S_2, \sup S_3, \ldots\} \ = \ \sup\left(\bigcup_{n=1}^{\infty} S_n\right),$$

in the sense that if either exists then so does the other, and they are equal.

Upper bound is $M_n \in \mathbb{R}$ **such that** $M_n \geq s \ \forall s \in S_n$**.**

$\sup S_n \in \mathbb{R}$ **is an upper bound for** S_n **such that** $\sup S_n \leq M_n$ **for all upper bounds** M_n**. It exists if and only if** S_n **is nonempty and has an upper bound.**

Suppose the left hand side exists, call is M. Then $M \geq \sup S_i \geq s$ for each i and for each $s \in S_i$, so M is an upper bound for $\bigcup_{n=1}^{\infty} S_n$, so the right hand side N exists and $M \geq N$.

Similarly if the right hand side N exists then it is an upper bound for each S_i, so each $\sup S_i$ exists and is $\leq N$. Therefore the set on the left hand side has an upper bound N, therefore its supremum M exists and $M \leq N$.

Therefore if either exists so does the other. In this case we have shown that $M \geq N$ and $M \leq N$.

Therefore $M = N$.

3. Take bounded, nonempty $S,\, T \subset \mathbb{R}$. Define $S+T := \{s+t \colon s \in S,\ t \in T\}$. Prove

$$\sup(S+T) \;=\; \sup S + \sup T.$$

Any element $s+t$ of $S+T$ is $\leq \sup S + \sup T$, so $S+T$ is nonempty and has an upper bound $\sup S + \sup T$, so it has a supremum $\leq \sup S + \sup T$.

Given $\epsilon > 0$ we know $\sup S - \epsilon$ is not an upper bound for S because it is smaller than the *least* upper bound, so $\exists s \in S$ such that $s > \sup S - \epsilon$. Similarly $\exists t \in T$ such that $t > \sup T - \epsilon$.

Therefore $\exists\, s+t \in S+T$ such that $s+t > \sup S + \sup T - 2\epsilon$, so we have proved that

$$\sup S + \sup T - 2\epsilon \;<\; \sup(S+T) \;\leq\; \sup S + \sup T \quad \forall \epsilon > 0,$$

which implies $\sup(S+T) = \sup S + \sup T$.

4.∗ Fix $a \in (0,\infty)$ and $n \in \mathbb{N}$. We will prove $\exists\, x \in \mathbb{R}$ such that $x^n = a$. Set

$$S_a \;:=\; \{s \in [0,\infty) :\ s^n < a\}$$

and show S is nonempty and bounded above, so we may define $x := \sup S_a$.
For $\epsilon \in (0,1)$ show $(x+\epsilon)^n \leq x^n + \epsilon[(x+1)^n - x^n]$. (Hint: multiply out.)
Hence show that if $x^n < a$ then $\exists\, \epsilon \in (0,1)$ such that $(x+\epsilon)^n < a$. $(*)$
If $x^n > a$ deduce from $(*)$ that $\exists\, \epsilon \in (0,1)$ such that $(\frac{1}{x}+\epsilon)^n < \frac{1}{a}$. $(**)$
Deduce contradictions from $(*)$ and $(**)$ to show that $x^n = a$.

By the binomial theorem,

$$\begin{aligned} (x+\epsilon)^n &= x^n + n\epsilon x^{n-1} + \binom{n}{2}\epsilon^2 x^{n-2} + \ldots + \epsilon^n \\ &< x^n + n\epsilon x^{n-1} + \binom{n}{2}\epsilon x^{n-2} + \ldots + \epsilon \\ &= x^n + \epsilon((x+1)^n - x^n), \end{aligned}$$

for $\epsilon \in (0,1)$ (so that $\epsilon^k < \epsilon$) and $x > 0$.

Therefore, if $x^n < a$ we can set $\epsilon := \min\left(\frac{1}{2}, \frac{a-x^n}{(x+1)^n - x^n}\right) \in (0,1)$ so that

$$(x+\epsilon)^n \;<\; x^n + \epsilon((x+1)^n - x^n) \;\leq\; x^n + (a - x^n) \;=\; a,$$

as required. So $x+\epsilon$ is both in S_a and $> \sup S_a$ – a contradiction.

If $x^n > a$ then $\left(\frac{1}{x}\right)^n < \frac{1}{a}$ so by $(*)$ applied to $\frac{1}{x}$ and $\frac{1}{a}$ we find $\exists\, \epsilon \in (0,1)$ such that $\left(\frac{1}{x}+\epsilon\right)^n < \frac{1}{a}$.

Thus $\left(\frac{x}{1+\epsilon x}\right)^n > a \;\Rightarrow\; y^n > a$ for all $y \geq \frac{x}{1+\epsilon x}$. That is, $\frac{x}{1+\epsilon x}$ is an upper bound for S_a, but is $< x = \sup S_a$ – a contradiction.

5. Suppose $0 < q \in \mathbb{Q}$ and $a \in (0, \infty)$. Write $q = \frac{m}{n}$ with $m, n \in \mathbb{N}$ and define

$$a^q \;:=\; x^m,$$

where $x =: a^{1/n}$ is defined in the last question. Show this is well defined, and make a definition of a^{-q}.

Show that $(ab)^q = a^q b^q$ and $(a^{q_1})^{q_2} = a^{q_1 q_2}$ for any $a, b \in (0, \infty)$ and $q, q_1, q_2 \in \mathbb{Q}$.

To show this is well defined we have to show that if we replace $\frac{m}{n}$ by $\frac{pm}{pn}$ for some $p \in \mathbb{N}$ then we get the same answer. (Why is this enough?) That is, we must show

$$\left(a^{\frac{1}{n}}\right)^m \;=\; \left(a^{\frac{1}{pn}}\right)^{pm}.$$

It would be sufficient to prove

$$a^{\frac{1}{n}} \;=\; \left(a^{\frac{1}{pn}}\right)^p.$$

The LHS is the unique (why?) $x \in (0, \infty)$ such that $x^n = a$. So it is enough to show the RHS has this property, i.e. that

$$\left(\left(a^{\frac{1}{pn}}\right)^p\right)^n \;=\; a.$$

But the LHS is

$$\left(a^{\frac{1}{pn}}\right)^{np},$$

which is indeed a by the definition of $a^{\frac{1}{pn}}$.

Now define $a^{-q} := 1/a^q$ and play similar games as above to prove the identities.

6. For real numbers x, y, z, consider the following inequalities.

(a) $|x + y| \;\le\; |x| + |y|$

(b) $|x + y| \;\ge\; |x| - |y|$

(c) $|x + y| \;\ge\; |y| - |x|$

(d) $|x - y| \;\ge\; \big||x| - |y|\big|$

(e) $|x| \;\le\; |y| + |x - y|$

(f) $|x| \;\ge\; |y| - |x - y|$

(g) $|x - y| \;\le\; |x - z| + |y - z|$

Prove (a) from first principles. Why is it called the "triangle inequality"?

Deduce (b,c,d,e,f,g) from (a).

First prove that $x \le |x|$ by considering the two cases $x \le 0$, $x > 0$.

Then it follows that $x + y \le |x| + |y|$ and $-(x + y) \le |-x| + |-y| = |x| + |y|$. Combining these two gives (a).

Called triangle rule because it says that if you make a triangle with vertices at $0, x, x+y$ then the length of the side $[0, x + y]$ is $\le$ the sum of the lengths of the other two sides $[0, x]$ and $[x, x + y]$. It's a bit more convincing when $0, x, x + y$ are all vectors in higher dimensions, eg. R^2, where the same result holds.

For (b) replace x, y in (a) by $x + y, -y$ and rearrange. Similarly for all the others. For (g) write $x - y = (x - z) - (y - z)$.

*You should prepare starred questions * to discuss with your personal tutor.*