1 Approximating derivatives

Given $y := f(x) = x^3$ we want to approximate the derivative at $x = 1$. We know that the exact value of $f'(x)$ is $\lim_{x \to 0} \frac{f(x+h)-f(x)}{h}$ and so $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = 3$.

Lets consider the following approximations:
(i)$f'(1) \approx \frac{f(1+h)-f(1)}{h} := D_+f, h > 0$
(ii)$f'(1) \approx \frac{f(1)-f(1-h)}{h} := D_-f, h > 0$
(iii)$f'(1) \approx \frac{f(1+h)-f(1-h)}{2h} := Df, h > 0$.

Our aim is to evaluate how accurate these formulas are compared to the exact value $f'(1) = 3$.

1.1

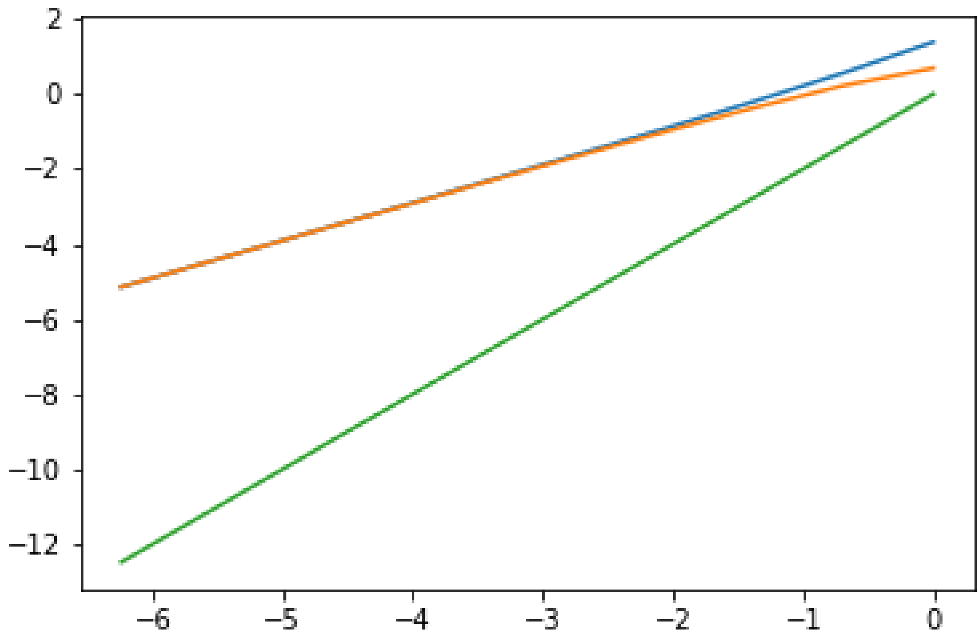
The table represents the values of $D_+f, D_-f,\ Df$ when we take $h = \frac{1}{2^n}$ for $n = 1, ..., 10$.

n	h	D_+f	D_-f	Df
1	$\frac{1}{2}^{1}$	4.75	1.75	3.25
2	$\frac{1}{2}^{2}$	3.8125	2.3125	3.0625
3	$\frac{1}{2}^{3}$	3.390625	2.640625	3.015625
4	$\frac{1}{2}^{4}$	3.19140625	2.81640625	3.00390625
5	$\frac{1}{2}^{5}$	3.0947265625	2.9072265625	3.0009765625
6	$\frac{1}{2}^{6}$	3.047119140625	2.953369140625	3.000244140625
7	$\frac{1}{2}^{7}$	3.02349853515625	2.97662353515625	3.00006103515625
8	$\frac{1}{2}^{8}$	3.0117340087890625	2.9882965087890625	3.0000152587890625
9	$\frac{1}{2}^{9}$	3.0058631896972656	2.9941444396972656	3.0000038146972656
10	$\frac{1}{2}^{10}$	3.0029306411743164	2.9970712661743164	3.00000009536743164

The second table shows the error between computed and exact values:

n	h	$\varepsilon_1 = \lvert D_+f - f'(1)\rvert$	$\varepsilon_2 = \lvert D_-f - f'(1)\rvert$	$\varepsilon_3 = \lvert Df - f'(1)\rvert$
1	$\frac{1}{2^1}$	1.75	1.25	0.25
2	$\frac{1}{2^2}$	0.8125	0.6875	0.0625
3	$\frac{1}{2^3}$	0.390625	0.359375	0.015625
4	$\frac{1}{2^4}$	0.19140625	0.18359375	0.00390625
5	$\frac{1}{2^5}$	0.0947265625	0.0927734375	0.0009765625
6	$\frac{1}{2^6}$	0.047119140625	0.046630859375	0.000244140625
7	$\frac{1}{2^7}$	0.02349853515625	0.02337646484375	0.00006103515625
8	$\frac{1}{2^8}$	0.0117340087890625	0.0117034912109375	0.0000152587890625
9	$\frac{1}{2^9}$	0.0058631896972656	0.005855560302734375	0.0000038146972656
10	$\frac{1}{2^{10}}$	0.0029306411743164	0.0029287338256835938	0.0000009536743164

1.2



1.3

According to the plot as $\log(h)$ gets smaller, $\log(\varepsilon)$ also gets smaller, so we can say that as h decreases (or as n increases) the errors $\varepsilon_1, \varepsilon_2$ and ε_3 get smaller and hence we get the better approximation of $f'(1)$. We can see that the smallest error is achieved by the green curve (representing ε_3). Further, we can see that the curve of ε_3 declines much faster than the other two (the slope appears much steeper), so method (iii) gets closer to the exact value of $f'(1)$ quicker.

2 Solving a differential equation numerically

First, lets confirm that the solution to

$$\frac{dy}{dx} = y, 0 < x \leq 1, y(0) = 1$$

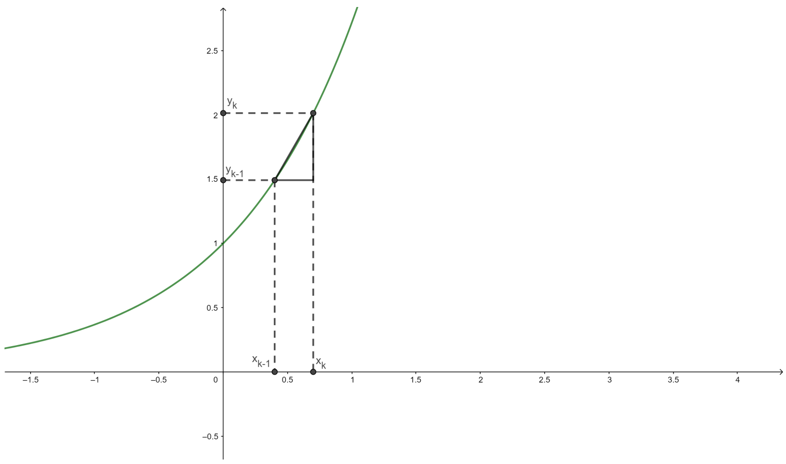
is $y = \exp(x)$.
If we integrate both sides of $\frac{1}{y}dy = dx$ we get $\ln(|y|) = x + C$ where C is some constant. Since we know that $y(0) = 1$, we have that $\ln(1) = 0 + C$ and hence $C = 0$. Therefore,

$$\ln(|y|) = x \implies \exp(\ln(|y|)) = \exp(x) \implies |y| = \exp(x) \implies y = \pm\exp(x),$$

but since $y = 1$ when $x = 0$ we get that the only solution is $y = \exp(x)$.

2.1

Using scheme (ii) we want to approximate $\frac{dy}{dx}(x_k)$.



Here we have divided the interval $[0, 1]$ into N parts. For some x_k, we want to approximate the slope of the tangent at $y(x_k)$ by taking x_{k-1} and approaching x_k from the left. We get:

$$\frac{dy}{dx}(x_k) \approx \frac{y(x_k) - y(x_{k-1})}{h} \approx \frac{dy}{dx}(x_{k-1}).$$

Given that $\frac{dy}{dx} = y$, we know that

$$y_{k-1} = \frac{dy}{dx}(x_{k-1}) \approx \frac{y(x_k) - y(x_{k-1})}{h}.$$

Hence, the approximation gives us that $hy_{k-1} = y_k - y_{k-1}$ and therefore

$$y_k = (h+1)y_{k-1}.$$

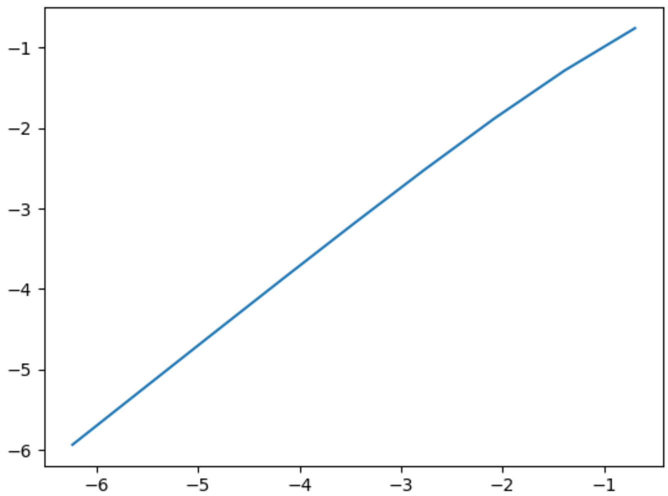
2.2

The following table shows the values of y_N and the error $\varepsilon = |y_N - y(1)|$, where $y(1) = \exp(1) = e$.

n	N	$y_N = (1 + \frac{1}{N})^N$	$\varepsilon = \|y_N - y(1)\|$
1	2^1	2.25	0.4682818284590451
2	2^2	2.44140625	0.2768755784590451
3	2^3	2.565784513950348	0.1524973145086972
4	2^4	2.637928497366600	0.08035333109244513
5	2^5	2.676990129378183	0.041291699080862276
6	2^6	2.697344952565099	0.020936875893946105
7	2^7	2.7077390196880207	0.010542808771024426
8	2^8	2.7129916242534344	0.00529020420561066
9	2^9	2.7156320001689913	0.0026498282900537795
10	2^{10}	2.7169557294664357	0.0013260989926093814

We can see that as N gets larger y_N gets closer to e (i.e. the error ε gets smaller).

This is a plot of $\log(\varepsilon)$ versus $\log(h)$.



From the graph we can see that as N gets larger (i.e. $\log(h)$ gets smaller) the error declines, so the approximation becomes more accurate as N gets larger.

2.3

In part 2.1 we showed that

$$y_k = (1+h)y_{k-1}.$$

Now we will show by induction that $y_k = (1+h)^k y_0$.
First, we have that $y_1 = (1+h)^1 y_0$. Suppose that $y_i = (1+h)^i y_0$ for some i. We want to show that $y_{i+1} = (1+h)^{i+1} y_0$. We have

$$y_{i+1} = (1+h)y_i = (1+h)(1+h)^i y_0 = (1+h)^{i+1} y_0.$$

Therefore by induction $y_k = (1+h)^k y_0$.

Now let's consider y_N. We have that

$$y_N = (1+h)^N y_0 = (1+\frac{1}{N})^N.$$

As $N \to \infty$, $h \to 0 \implies \log(h) \to -\infty$. From the graph we can see that as $\log(h) \to -\infty$, $\log(\varepsilon)$ approaches $-\infty$ as well and therefore $\varepsilon \to 0$. So as N goes to infinity, the error approaches 0. Hence we have:

$$\lim_{N\to\infty}(y_N - y(1)) = 0 \implies \lim_{N\to\infty} y_N = \lim_{N\to\infty} y(1) = y(1) = e$$

$$\implies \lim_{N\to\infty} y_N = e.$$

$\blacksquare$