Topic: Continuous random variables and their distributions

In today's problem class we will be studying properties of continuous random variables.

1. For each of the function $f(x)$ given below determine whether $f(x)$ is a valid probability density function (p.d.f.). If $f(x)$ is not a valid p.d.f., determine if there exists a constant c such that $cf(x)$ is a valid p.d.f.. Note that in each case, $f(x) = 0$ for all x not in the interval specified.
   (a) $f(x) = 2x, \quad 0 < x < 1.$
   (b) $f(x) = |x|, \quad |x| < \frac{1}{2}.$
   (c) $f(x) = 1 - |x|, \quad |x| < 1.$
   (d) $f(x) = \log(x), \quad 0 < x < 1.$
   (e) $f(x) = \log(x), \quad 0 < x < 2.$
   (f) $f(x) = \frac{2}{3}(x-1), \quad 0 < x < 3.$
   (g) $f(x) = e^{-2x}, \quad x > 0.$
   (h) $f(x) = 4e^{-2x} - e^{-x}, \quad x > 0.$
   (i) $f(x) = e^{-|x|}, \quad |x| < 1.$

Solution: Need to satisfy $\int f(x)dx = 1$ and $f(x) \geq 0$.

(a) valid: $f(x) = 2x \geq 0$ for all $x \in (0,1)$ and

$$\int_0^1 2x dx = x^2|_0^1 = 1.$$

(b) not valid, but $c = 4$ works: $f(x) = |x| \geq 0$ and

$$\int_{-1/2}^{1/2} |x| dx = \int_{-1/2}^{0} (-x) dx + \int_0^{1/2} x dx = \left.\frac{-x^2}{2}\right|_{-1/2}^{0} + \left.\frac{x^2}{2}\right|_0^{1/2} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

(c) valid: $f(x) = 1 - |x| \geq 0$ for all $|x| < 1$ and

$$\int_{-1}^{1} (1 - |x|) dx = \int_{-1}^{0} (1+x) dx + \int_0^1 (1-x) dx = x - x^2/2\Big|_{-1}^{0} + x + x^2/2\Big|_0^1 = \frac{3}{2} - \frac{1}{2} = 1.$$

(d) not valid, but $c = -1$ works: $f(x) = \log(x) < 0$ for all $0 < x < 1$ and

$$\int_0^1 \log(x) dx = x\log(x) - x|_0^1 = -1.$$

(e) not valid, no c possible: Note that $f(x) = \log(x) < 0$ for all $0 < x < 1$ and $f(x) = \log(x) > 0$ for all $1 < x < 2$.

(f) not valid, no c possible: We have that $f(x) \geq 0$ for $x \geq 1$ and $f(x) \leq 0$ for $x \leq 1$.

(g) not valid, but $c = 2$ works: $f(x) = e^{-2x} \geq 0$ for all $x > 0$ and

$$\int_0^\infty e^{-2x} dx = \left.\frac{-1}{2} e^{-2x}\right|_0^\infty = \frac{1}{2}.$$

(h) not valid, no c possible: $f(x) = 4e^{-2x} - e^{-x} = e^{-x}(4e^{-x} - 1)$. We know that $e^{-x} > 0$ for all x, but the second factor switches sign in the range $x > 0$ and hence we cannot find a suitable c: $4e^{-x} - 1 \geq 0 \Leftrightarrow e^{-x} \geq \frac{1}{4} \Leftrightarrow -x \geq \log(1/4) \Leftrightarrow x \leq \log(4)$.

(i) not valid but $c = \frac{e}{2(e-1)}$ works: $f(x) = e^{-|x|} \geq 0$ for all $|x| < 1$. Also,

$$\int_{-1}^{0} e^x dx + \int_0^1 e^{-x} dx = e^x\big|_{-1}^{0} - e^{-x}\Big|_0^1 = 1 - e^{-1} - e^{-1} + 1 = 2(1 - e^{-1}) = \frac{2(e-1)}{e}.$$

2. Let $Z \sim \mathrm{N}(0,1)$. Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Find the c.d.f. and the p.d.f. of the random variable $X = \sigma Z + \mu$. Note that you can express the c.d.f. of X in terms of the c.d.f. Φ of Z.

Solution: We start by computing the c.d.f. of X: Let $x \in \mathbb{R}$, then

$$F_X(x) = \mathrm{P}(X \leq x) = \mathrm{P}(\sigma Z + \mu \leq x) = \mathrm{P}(Z \leq (x-\mu)/\sigma) = F_Z((x-\mu)/\sigma) = \Phi((x-\mu)/\sigma).$$

Differentiating F_X gives us the corresponding density for any $x \in \mathbb{R}$:

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\Phi((x-\mu)/\sigma) = \phi((x-\mu)/\sigma)\frac{1}{\sigma} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2\right).$$

3. You are bidding against a competitor for an item on eBay. The amount, X, in pounds, of the bid placed by your competitor has probability density function given by:

$$f_X(x) = \begin{cases} c(20-x), & 0 < x < 20; \\ 0, & \text{otherwise.} \end{cases}$$

You make a bid without knowing your competitor's bid.

(a) Determine the value of c.

(b) Find $F_X(x)$, the cumulative distribution function (cdf) of X.

(c) What is the probability that you lose the bid if you place a bid of £16?

(d) How much should you bid in order to have a 75% chance of winning?

Solution:

(a) $\int_0^{20} c(20-x)\,\mathrm{d}x = 1 \Rightarrow c = 1/200$

(b)

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0, \\ \int_0^x \frac{1}{200}(20-t)dt = \frac{40x - x^2}{400}, & \text{for } 0 \leq x < 20, \\ 1, & \text{for } x \geq 20. \end{cases}$$

(c) $\mathrm{P}(X > 16) = 1 - F_X(16) = 1/25$.

(d) Solve $F_X(x) = 0.75 \Rightarrow 0.75 \times 400 = 40x - x^2 \Leftrightarrow x_1 = 10, x_2 = 30$. Since $x_2 = 30 > 20$ (the range we are considering) we deduce that we need to bid $x = £10$.