Problem Sheet 2

Math40002, Analysis 1

1. Give an example of a compact set $S \subset R$ and a continuous function $f : S \to \mathbb{R}$ which does *not* satisfy the intermediate value theorem: in other words, there are points $a < b$ in S and some x between $f(a)$ and $f(b)$ such that $f(c) \neq x$ for all $c \in S$.

   Solution. Let $S = [0, 1] \cup [3, 4]$. This is closed (as a union of two closed intervals) and bounded, so it is compact. The function $f : S \to \mathbb{R}$ given by $f(x) = x$ is continuous, and it satisfies $f(1) = 1$ and $f(3) = 3$, but there is no $c \in S$ such that $f(c) = 2$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(c) = \{x \in \mathbb{R} \mid f(x) = c\}$ is closed.

   Solution. Let $\big(x_n\big) \subset f^{-1}(c)$ be a sequence which converges to a limit $x \in \mathbb{R}$. By sequential continuity we have $f(x_n) \to f(x)$, but $f(x_n) = c$ for all n, so $f(x) = c$ as well and thus $x \in f^{-1}(c)$. It follows that the limit of any convergent sequence in $f^{-1}(c)$ also lies in $f^{-1}(c)$, so $f^{-1}(c)$ is closed.

3. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the preimage
$$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$$
is open.

   Solution. $\Longrightarrow$: Suppose that f is continuous, and fix an open set $U \subset \mathbb{R}$. Let x be a point of $f^{-1}(U)$; then $f(x) \in U$ by definition, and since U is open, there is some $\epsilon > 0$ such that the whole open interval $(f(x) - \epsilon, f(x) + \epsilon)$ is a subset of U. Since f is continuous at x, there is $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$, hence
$$f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset U.$$
But then $y \in f^{-1}(U)$ for all such y, so $(x - \delta, x + \delta) \subset U$. Since we can find such a neighborhood for any $x \in f^{-1}(U)$, it follows that $f^{-1}(U)$ is open.

   $\Longleftarrow$: We will show that f is continuous at any $x \in \mathbb{R}$. Fix $\epsilon > 0$ and let $U = (f(x) - \epsilon, f(x) + \epsilon)$. Then $f^{-1}(U)$ contains x by definition, and since U is open, so is $f^{-1}(U)$. This means that $f^{-1}(U)$ contains an open neighborhood $(x - \delta, x + \delta)$ of x for some $\delta > 0$. Now if $|y - x| < \delta$ then
$$y \in f^{-1}(U) \implies f(y) \in U = (f(x) - \epsilon, f(x) + \epsilon) \implies |f(y) - f(x)| < \epsilon,$$
and we can do this for any $\epsilon > 0$, so f is continuous at x.

4. Prove that a set $S \subset \mathbb{R}$ is compact if and only if every sequence $(x_n) \subset S$ has a convergent subsequence whose limit is in S.

   Solution. $\Longrightarrow$: Let (x_n) be a sequence in S. Then (x_n) is bounded since it lies in the bounded set S, so by Bolzano–Weierstrass it has a convergent subsequence (x_{n_i}) whose limit is some real number. Since S is also closed, this limit must actually lie in S.

   $\Longleftarrow$: Let (x_n) be a convergent sequence of real numbers, with $x_n \in S$ for all n; write $x_n \to x$. By assumption there is a subsequence (x_{n_i}) which converges to some $y \in S$. But since x_n was already convergent, we must have $x = y$, so $\lim_{n\to\infty} x_n \in S$. This proves that S is closed. To see that S is bounded, we assume it is not and take a sequence $(x_n) \subset S$ with $|x_n| > n$ for all n; then (x_n) has no convergent subsequences at all, which is a contradiction.

5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $S \subset \mathbb{R}$ is compact, then the image $f(S)$ is also compact.

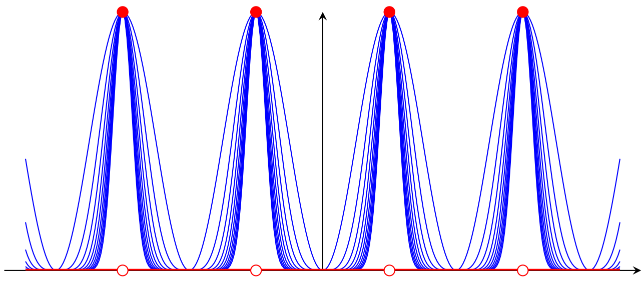
   Solution. Let $(y_n) \subset f(S)$ be an arbitrary sequence, and write $y_n = f(x_n)$ for $x_n \in S$. Since S is compact, there is a convergent subsequence (x_{n_i}), with $x_{n_i} \to x \in S$. But then by continuity we have $f(x_{n_i}) \to f(x)$, so the subsequence y_{n_i} converges to $f(x) \in f(S)$. Since every sequence in $f(S)$ has a convergent subsequence with limit in $f(S)$, we conclude by the previous problem that $f(S)$ is compact.

6. Give a family of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ for all $n \in \mathbb{N}$ such that the f_n converge pointwise to a function $f : \mathbb{R} \to \mathbb{R}$ with infinitely many discontinuities.

   Solution. Let $f_n(x) = (\sin(x))^{2n}$. Then we define $f(x)$ by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\sin^2(x)\right)^n = \begin{cases} 1, & \sin^2(x) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

   The f_n are graphed below in blue for $1 \le n \le 10$, and the limit f is shown in red.



   This is discontinuous at every point of the form $x = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.

7. Recall that $\cos(x) = \mathrm{Re}(E(ix))$ and $\sin(x) = \mathrm{Im}(E(ix))$ have power series

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(a) Use the identity $E(ix)E(-ix) = E(0) = 1$ to prove that $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$.

(b) Prove that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$. (Hint: reduce to the case $0 \leq x \leq 1$.)

(c) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is uniformly continuous. (Hint: use the identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$.)

Solution. (a) We have $E(-ix) = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$, since $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ by inspecting their power series. So

$$\begin{aligned} 1 = E(ix)E(-ix) &= (\cos(x) + i\sin(x))(\cos(x) - i\sin(x)) \\ &= (\cos(x))^2 + (\sin(x))^2 . \end{aligned}$$

(b) By part (a) we have $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, so it suffices to prove that $|\sin(x)| \leq |x|$ for $|x| \leq 1$, since if $|x| > 1$ then $|\sin(x)| \leq 1 < |x|$ anyway. Moreover, since $|\sin(-x)| = |\sin(x)|$ and $|-x| = |x|$, we have $|\sin(-x)| \leq |-x|$ if and only if $|\sin(x)| \leq |x|$. So it suffices to consider $x \geq 0$, leaving only the case $0 \leq x \leq 1$ to be proved.

Restricting our attention to $[0, 1]$ now, we pair consecutive terms in the power series as follows:

$$\begin{aligned} \sin(x) &= x - \left(\frac{x^3}{3!} - \frac{x^5}{5!}\right) - \left(\frac{x^7}{7!} - \frac{x^9}{9!}\right) - \cdots - \left(\frac{x^{4n+3}}{(4n+3)!} - \frac{x^{4n+5}}{(4n+5)!}\right) - \ldots \\ &\leq x - 0 - 0 - \cdots - 0 - \cdots = x, \end{aligned}$$

where each term in parentheses is positive because $\frac{x^{4n+3}}{(4n+3)!} \geq \frac{x^{4n+5}}{(4n+5)!}$ on the interval $0 \leq x \leq 1$. So $\sin(x) \leq x$, and for a lower bound we group terms differently:

$$\begin{aligned} \sin(x) &= \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots + \left(\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n+3}}{(4n+3)!}\right) + \ldots \\ &\geq 0 + 0 + \cdots + 0 + \cdots = 0, \end{aligned}$$

because $\frac{x^{4n+1}}{(4n+1)!} \geq \frac{x^{4n+3}}{(4n+3)!}$ on the interval $0 \leq x \leq 1$ for each $n \geq 0$. Combining these inequalities, we have $0 \leq \sin(x) \leq x$, which implies that $|\sin(x)| \leq |x|$ on the interval $[0, 1]$, as claimed.

(c) The identity can be proved by writing

$$\begin{aligned} \sin(\alpha) - \sin(\beta) &= \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right) \\ &= \left(\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) \\ &\quad - \left(\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) \\ &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \end{aligned}$$

With it in hand, we have for any $x, y \in \mathbb{R}$ an inequality

$$\begin{aligned}|f(x) - f(y)| &= \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right|,\end{aligned}$$

since $|\cos(\theta)| \leq 1$ for all θ by part (a). Now we apply $|\sin(\theta)| \leq |\theta|$ from part (b) to get

$$|f(x) - f(y)| \leq 2\left|\frac{x-y}{2}\right| = |x - y|$$

for all $x, y \in \mathbb{R}$. Thus if we are given any $\epsilon > 0$, we can set $\delta = \epsilon > 0$, and we have

$$|x - y| < \delta \implies |f(x) - f(y)| \leq |x - y| < \delta = \epsilon$$

for all $x, y \in \mathbb{R}$, proving that f is indeed uniformly continuous.