Recall a notation for a union of a set of sets called "unary union": Given a set $\mathcal{A}$ of sets, denote its union by $\cup\mathcal{A} := \bigcup_{X\in\mathcal{A}} X$.

1. Prove that a finite union of bounded sets is bounded.

   If $A_1, \ldots, A_n$ **are bounded, then** $\min\{\,\inf(A_i) | 1 \le i \le n\,\}$ **and** $\max\{\,\sup(A_i) | 1 \le i \le n\,\}$ **bound** $\bigcup_{i=1}^{n} A_i$ **below and above, respectively.**

2. Let (a_n) be a sequence and $a \in \mathbb{R}$. Prove that $a_n \to a$ if and only if for every open set $U \subseteq \mathbb{R}$ such that $x \in U$, there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

   $\Longrightarrow$ **Let** $U \subseteq \mathbb{R}$ **be open such that** $x \in \mathbb{R}$**, then there is some** $\delta > 0$ **such that** $(x - \delta, x + \delta) \subseteq U$**.**

   $\Longleftarrow$ $(x - \delta, x + \delta)$ **is open.**

3. (Second countability)

   (a) Prove that for every open set $U \subseteq \mathbb{R}$ and $x_0 \in U$, there is some open interval (a, b) with rational endpoints (namely, $a, b \in \mathbb{Q}$) such that $x_0 \in (a, b) \subseteq U$.

   Given an open set U **and** $x_0 \in U$**, there is some neighbourhood** $(x_0 - \delta, x_0 + \delta) \subseteq U$**. Let** $(a_n)_{n=1}^{\infty}$ **and** $(b_n)_{n=1}^{\infty}$ **be sequences of rational numbers such that** $a_n \nearrow x_0$ **and** $b_n \searrow x_0$**. (Why is this true?) By definition of convergence, there are some** $n_1, n_2 \in \mathbb{N}$ **such that** $x_0 - \delta < a_{n_1} < x_0 < b_{n_2} < x_0 + \delta$**. This proves that** $x_0 \in (a_{n_1}, b_{n_2}) \subseteq U$ **and** $a_{n_1}, b_{n_2} \in \mathbb{Q}$**.**

   (b) Prove that there is a countable set of open intervals $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ such that for every open set $U \subseteq \mathbb{R}$, there is some $\mathcal{C} \subseteq \mathcal{B}$ such that $U = \cup\mathcal{C} := \bigcup_{A\in\mathcal{C}} A$. I.e., U is a union of sets of $\mathcal{B}$.

   $\mathcal{B} := \{\,(a, b) | a, b \in \mathbb{Q}\,\}$. $\mathcal{B}$ **is countable since there is an embedding from** $\mathcal{B}$ **to** $\mathbb{Q} \times \mathbb{Q}$**. For every** $U \subseteq \mathbb{R}$ **open, and for every** $x \in U$**, let** $a_x, b_x \in \mathbb{Q}$ **such that** $x \in (a_x, b_x) \subseteq U$**. Let** $\mathcal{C} := \{\,(a_x, b_x) | x \in U\,\} \subseteq \mathcal{B}$**. It is left to convince oneself that** $\cup\mathcal{C} = U$ **(Prove by two-way inclusion.)**

<u>Side note:</u> *In general topology,* $\mathcal{B}$ *satisfying that every open set is a union of sets from* $\mathcal{B}$ *is called a* ***basis****, topological spaces which have a countable basis are called* ***second countable***

4. (Lindelöf's Lemma)

   (a) Prove that every open set $U \subseteq \mathbb{R}$ is a countable union of open intervals.

   Let $\mathcal{B}$ **be as in Question 1. By Question 1, there is some** $\mathcal{C} \subseteq \mathcal{B}$ **such that** $U = \cup\mathcal{C}$ **and** $\mathcal{C}$ **is countable, as a subset of a countable set.**

(b) Let $\mathcal{U}$ be a set of open sets. Prove that there is some countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 = \cup\mathcal{U}$.

Let $\mathcal{B}$ be as in Question 1. For every $U \in \mathcal{U}$, let $\mathcal{C}_U \subseteq \mathcal{B}$ such that $U = \cup\mathcal{C}_U$. Then

$$\cup\mathcal{U} = \bigcup_{U\in\mathcal{U}} U = \bigcup_{U\in\mathcal{U}} \cup\mathcal{C}_U = \bigcup_{U\in\mathcal{U}} \bigcup_{B\in\mathcal{C}_U} B.$$

Let $\mathcal{C} := \{ B \in \mathcal{B} | \exists U \in \mathcal{U} : B \in \mathcal{C}_U \}$. By the equation above, $\cup\mathcal{U} = \cup\mathcal{C}$.

Next, to define $\mathcal{U}_0$, for every $B \in \mathcal{C}$, there is some $U \in \mathcal{U}$ such that $B \in \mathcal{C}_U$, which in turn implies $B \subseteq U$. So given $B \in \mathcal{C}$, choose such U and denote it U_B. Let $\mathcal{U}_0 := \{ U_B | B \in \mathcal{C} \}$. Since $\mathcal{C}$ is countable, so is $\mathcal{U}_0$.

$\mathcal{U}_0 \subseteq \mathcal{U}$, therefore $\cup\mathcal{U}_0 \subseteq \cup\mathcal{U}$. On the other hand, $B \subset U_B$ for every $B \in \mathcal{C}$, so $\cup\mathcal{U} = \cup\mathcal{C} = \bigcup_{B\in\mathcal{C}} B \subseteq \bigcup_{B\in\mathcal{C}} U_B = \cup\mathcal{U}_0$. In conclusion, $\cup\mathcal{U} = \cup\mathcal{U}_0$.

*Sidenote: In general, a topological space is called **Lindelöf** if for every set of open sets, there is a countable subset of it, such that their unions are equal. Lindelöf's Lemma proves that $\mathbb{R}$ is a Lindelöf space. In general, Lindelöf's Lemma can show that every second countable space is Lindelöf.*

5. Let $x_0 \in \mathbb{R}$. Find a set of open sets $\mathcal{U}$ such that $\cup\mathcal{U} = \mathbb{R} \setminus \{x_0\}$ and for every finite $\mathcal{U}_0 \subseteq \mathcal{U}$, there is some open set V such that $x \in V$ and $(\cup\mathcal{U}_0) \cap V = \emptyset$.

   $A_n := \mathbb{R} \setminus [x - 1/n, x + 1/n]$. $\mathcal{U} := \{ A_n | n \in \mathbb{N} \}$.

6. (Heine-Borel Theorem) Let $X \subseteq \mathbb{R}$. Prove that the following are equivalent:

   (i) X is compact.

   (ii) For every set $\mathcal{U}$ of open sets such that $\cup\mathcal{U} \supseteq X$, there is some finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 \supseteq X$.

   This is a complex question, and there are several things you need to prove. The best advice is not to give up! It may require time after the session and discussions with your mates, but once you succeed, you'll have the sense of triumph of proving a proper theroem all on your own. You may want to use the fact that S is compact if and only if every sequence in S has a subsequence that converges to a limit in S, as proved in Question sheet 2, Question 4.

(i)$\Longrightarrow$(ii) Assume (i) holds and let $\mathcal{U}$ be as in (ii). Assume towards a contradiction that there is no finite $\mathcal{U}_0$ as needed. By Question 4b, there is some countable $\mathcal{U}' \subseteq \mathcal{U}$ such that $X \subseteq \mathcal{U}'$. Let $\mathcal{U}' := \{ U_n | n \in \mathbb{N} \}$ be an enumeration of $\mathcal{U}'$. Let $A_n := \bigcup_{i=1}^{n} U_i$. By our assumption $X \nsubseteq A_n$ for all $n \in \mathbb{N}$. So for every $n \in \mathbb{N}$, there is some $x_n \in X \setminus A_n$. Now, since X is compact, (x_n) has a convergent sequence whose limit, call it y, is in X. Since $y \in X$, there is some $m \in \mathbb{N}$ such that $y \in U_m$. By Question 2, there is some $N \in \mathbb{N}$ such that $x_n \in U_m$ for all $n > N$. This contradicts the fact that $x_n \notin A_n$ and $U_m \subseteq A_n$ for $n > N, m$.

(i)$\Longleftarrow$(ii) Assume (ii) holds. Clearly $\mathcal{U} := \{ (x-1, x+1) | x \in X \}$ satisfies $\cup\mathcal{U} \supseteq X$. By (ii), there is some finite $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 \supseteq X$. By Question 1, $\cup\mathcal{U}_0$ is bounded, therefore so is X.

To show X is closed, let (x_n) be a sequence of elements of X such that $x_n \to x$, and assume towards a contradiction that $x \notin X$. Let $\mathcal{U}$ be a set of open sets as in 5 (for $x_0 = x$). $\cup\mathcal{U} \supseteq X$ since $x_0 \notin X$. So there is some $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 \supseteq X$. In particular $x_n \in \cup\mathcal{U}_0$ for all $n \in \mathbb{N}$. By 5, there is some open set $V \subseteq \mathbb{R}$ such that $x \in V$ and $(\cup\mathcal{U}_0 \cap V) = \emptyset$. By Question 2, there is some $n \in \mathbb{N}$ such that $x_n \in V$. So, in conclusion $x_n \in \emptyset$. Contradiction.