MATH40004 - Calculus and Applications
Coursework Term 2

Ivan Kirev
CID: 01738166

Problem 1

Find the general solution for the following system of differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

Lets denote

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}.$$

First, we find the eigenvalues of A.

$$\det(\lambda I - A) = 0 \iff \det\left(\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}\right) = 0 \iff$$

$$\det(\begin{pmatrix} \lambda - 2 & -2 & -3 \\ -1 & \lambda - 3 & -3 \\ 1 & 2 & \lambda + 2 \end{pmatrix} = 0.$$

So we get $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$. Therefore A has eigenvalue $\lambda_1 = 1$ with multiplicity of 3. We then find the eigenvectors as follows:

$$(A - \lambda_1 I) = (A - I) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}.$$

So solving $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we get that two eigenvectors are

$$v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}.$$

Since we only have 2 linearly independent eigenvectors, the matrix A is not diagonalizable. We look for a similar transformation to a Jordan normal form

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = W^{-1}AW, \text{ where } W = \begin{pmatrix} -2 & 3 & \alpha \\ 1 & 3 & \beta \\ 0 & -3 & \gamma \end{pmatrix}.$$

To derive the Jordan normal form we consider that:

(i) the geometric multiplicity of λ_1 is the dimension of $\mathrm{Ker}(A - \lambda_1 I)$, and it is the number of Jordan blocks corresponding to λ_1. As we have two eigenvectors, we know that there are two Jordan blocks.

(ii) the sum of the sizes of all Jordan blocks corresponding to the eigenvalue λ_1 is its algebraic multiplicity $= 3$. So now we can conclude that the sizes of the two blocks are 1×1 and 2×2 and therefore the Jordan normal form is

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now solving $WJ = AW$, we get

$$\begin{pmatrix} -2 & 3 & \alpha+3 \\ 1 & 3 & \beta+3 \\ 0 & -3 & \gamma-3 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 2\alpha+2\beta+3\gamma \\ 1 & 3 & \alpha+3\beta+3\gamma \\ 0 & -3 & -\alpha-2\beta-2\gamma \end{pmatrix} \implies$$

$$\alpha + 2\beta + 3\gamma = 3.$$

Let's pick $(\alpha, \beta, \gamma) = (0, 0, 1)$. Then $W = \begin{pmatrix} -2 & 3 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{pmatrix}$. Denote $\vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.From $\frac{d}{dt}\vec{p} = A\vec{p}$, we have

$$W^{-1}\frac{d\vec{p}}{dt} = \underbrace{[W^{-1}AW]}_{J} W^{-1}\vec{p}.$$

Now denote $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = W^{-1}\vec{p}$.

Then $\frac{d\vec{z}}{dt} = J\vec{z} \implies$

$$\begin{pmatrix} \frac{dz_1}{dt} \\ \frac{dz_2}{dt} \\ \frac{dz_3}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

We now have the system

$$\begin{cases} \frac{dz_1}{dt} = z_1 \\ \frac{dz_2}{dt} = z_2 + z_3 \\ \frac{dz_3}{dt} = z_3 \end{cases} \implies \begin{cases} z_1 = c_1 e^t \\ \frac{dz_2}{dt} - z_2 = c_2 e^t \\ z_3 = c_2 e^t. \end{cases}$$

Solving the differential equation in the second row:
It is of the form $\frac{dz_2}{dt} + p\,(t)\,z_2 = q\,(t) \implies$ integration factor $= \mu\,(t) = e^{-t}$, and multiplying our equation by $\mu(t)$, we get

$$\frac{d(\mu(t) z_2)}{dt} = \mu\,(t)\,q\,(t) \implies \frac{d(e^{-t} z_2)}{dt} = c_2.$$

Solving the last separable equation, we acquire $z_2 = e^t(c_2 t + c_3) = c_2 t e^t + c_3 e^t$. Now

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ c_2 t e^t + c_3 e^t \\ c_2 e^t \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = W\vec{z} = \begin{pmatrix} -2 & 3 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 t e^t + c_3 e^t \\ c_2 e^t \end{pmatrix} \implies$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} c_1 e^t + \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} (c_2 t e^t + c_3 e^t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} c_2 e^t.}$$

Problem 2

Consider the Euler ODE below ($x > 0$):

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 4y = 0.$$

In the lectures, we discussed how to solve these type ODE with a change of variables. Alternatively, here use the Ansatz $y = x^r$ to obtain two linearly independent solutions and hence find the general real solution for the ODE.

Solution:

Rewriting the equation with $y = x^r$ we get:

$$\begin{aligned} 0 &= x^2 \frac{d^2 x^r}{dx^2} + 3x \frac{dx^r}{dx} + 4y \\ &= x^2 r(r-1) x^{r-2} + 3x r x^{r-1} + 4x^r \\ &= (r^2 - r) x^r + 3r x^r + 4x^r \\ &= r^2 + 2r + 4 = 0, \end{aligned}$$

where we have divided by x^r in the third line ($x > 0$). Solving for r, we get

$$r_{1,2} = -1 \pm \sqrt{3} i.$$

Hence we have two solutions for the differential equation:

$$y_1 = x^{-1+\sqrt{3}i}$$
$$y_2 = x^{-1-\sqrt{3}i}.$$

Now to check whether they are independent, we calculate the Wronskian:

$$\begin{aligned}
\det W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} &= \det \begin{bmatrix} x^{-1+\sqrt{3}i} & x^{-1-\sqrt{3}i} \\ (-1+\sqrt{3}i)x^{-2+\sqrt{3}i} & (-1-\sqrt{3}i)x^{-2-\sqrt{3}i} \end{bmatrix} \\
&= (-1-\sqrt{3}i)x^{-3} - (-1+\sqrt{3}i)x^{-3} \\
&= -\sqrt{3}ix^{-3} \neq 0.
\end{aligned}$$

Therefore, the y_1 and y_2 are linearly independent and $x^{-1+\sqrt{3}i}, x^{-1-\sqrt{3}i}$ form a basis for the solution space of our homogeneous differential equation. So the general solution y_{GS} can be expressed as follows:

$$\begin{aligned}
y_{GS} &= C_1 y_1 + C_2 y_2 \\
&= \boxed{C_1 x^{-1+\sqrt{3}i} + C_2 x^{-1-\sqrt{3}i}},
\end{aligned}$$

where C_1 and C_2 are some constants. Further, we can express y_{GS} in trigonometric form:

$$\begin{aligned}
y_{GS} &= C_1 x^{-1+\sqrt{3}i} + C_2 x^{-1-\sqrt{3}i} \\
&= x^{-1}(C_1 x^{\sqrt{3}i} + C_2 x^{-\sqrt{3}i}) \\
&= x^{-1}(C_1 e^{\ln(x^{\sqrt{3}i})} + C_2 e^{\ln(x^{-\sqrt{3}i})}) \\
&= x^{-1}(C_1 e^{\sqrt{3}i\ln(x)} + C_2 e^{-\sqrt{3}i\ln(x)}) \\
&= x^{-1}\Big(C_1\big[\cos(\sqrt{3}\ln x) + i\sin(\sqrt{3}\ln x)\big] + C_2\big[\cos(-\sqrt{3}\ln x) + i\sin(-\sqrt{3}\ln x)\big]\Big) \\
&= x^{-1}(\underbrace{(C_1+C_2)}_{C_1'}\cos(\sqrt{3}\ln x) + \underbrace{i(C_1-C_2)}_{C_2'}\sin(\sqrt{3}\ln x)).
\end{aligned}$$

But $y_{GS} \in \mathbb{R}$ and therefore we have that C_1' and C_2' are complex conjugates. So let $C_1' = A\cos\phi$, and $C_2' = A\sin\phi$. Lastly, we get the final result:

$$\boxed{y_{GS} = \frac{A\cos(\sqrt{3}\ln x - \phi)}{x}.}$$