Imperial College London

MATH40002

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Analysis 1

Date: 4th May 2020

Time: 09.00am – 12.00 noon (BST)

Time Allowed: 3 Hours

Upload Time Allowed: 30 Minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. In this question there is no need to give proofs, except where you are explicitly told to do so.

(a) Define what it means for a set S to be countably infinite. (1 mark)

Let S be the set of all sequences $(a_n)_{n\geq 1}$ with a_n in the set with two elements $\{0,1\}$.
Is S finite, countably infinite or uncountable? (1 mark)

Let $T := \Big\{(a_n)_{n\geq 1} \in S \;:\; \exists N \in \mathbb{N}$ such that $a_n = 0 \;\forall n \geq N\Big\}$ be the subset of sequences which are eventually zero. Is T finite, countably infinite or uncountable? (1 mark)

Fix $N \in \mathbb{N}$ and set $U := \Big\{(a_n)_{n\geq 1} \in S \;:\; a_n = 0 \;\forall n \geq N\Big\}$.
Is U finite, countably infinite or uncountable? (1 mark)

(b) Suppose $S \subset \mathbb{R}$ is nonempty and $A \in \mathbb{R}$ satisfies

$$\Big(\forall \epsilon > 0 \;\exists\, s_1 \in S \text{ such that } s_1 < A + \epsilon\Big) \;\; \textbf{and} \;\; \Big(\forall \epsilon > 0 \;\; \nexists\, s_2 \in S \text{ such that } s_2 < A - \epsilon\Big).$$

Either prove that $\sup S,\ \max S,\ \inf S$ or $\min S$ exists and equals A, or give an example where none of these is true. (4 marks)

(c) For each of the following statements about a real valued sequence $(a_n)_{n\geq 1}$,

(i) $\forall N \in \mathbb{N}\;\exists\, a \in \mathbb{R}$ such that $\forall \epsilon > 0,\ \forall n \geq N,\ |a_n - a| < \epsilon$

(ii) $\forall a \in \mathbb{R},\ \exists\, n \in \mathbb{N}$ such that $\forall \epsilon > 0,\ |a_n - a| < \epsilon$

(iii) $\exists\, a \in \mathbb{R}$ such that $\forall N \in \mathbb{N}\;\forall \epsilon > 0\;\exists\, n \geq N$ such that $|a_n - a| < \epsilon$

(iv) $\exists\, \epsilon > 0\;\exists\, a \in \mathbb{R}\;\exists\, N \in \mathbb{N}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$

(v) $\forall \epsilon > 0\;\exists\, N \in \mathbb{N}\;\exists\, a \in \mathbb{R}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$

state which of the following (A)-(E) it is equivalent to:

(A) Bounded (B) Has a convergent subsequence (C) Convergent
(D) Constant (E) Impossible (5 marks)

(d) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers.

Define what it means for a_n to be convergent. (1 mark)

Define what it means for a_n to be divergent. (1 mark)

Give an example of a divergent sequence (a_n) that satisfies

$$\forall \epsilon > 0\;\exists\, N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_{n+1} - a_n| < \epsilon. \qquad (*)$$ (1 mark)

Suppose now that (a_n) satisfies $(*)$ and $a_{2n} \to a$. Prove carefully that $a_n \to a$. (4 marks)

(Total: 20 marks)

2. In this question you should work from first principles, carefully proving anything you use.

(a) For $n \geq 0$ show that $\sqrt{1+\frac{1}{n}} \;\leq\; 1+\frac{1}{2n}$. (2 marks)

(b) For $n \geq 0$ show that $\sqrt{n+1}-\sqrt{n} \;\leq\; \frac{1}{2\sqrt{n}}$. (2 marks)

(c) (i) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Define what it means for $\sum_{n=1}^{\infty} a_n$ to converge. (2 marks)

(ii) Prove carefully that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is not convergent. You should work from first principles, proving any results that you use. You may use part (b) if you wish. (7 marks)

(d) Suppose $a_n \geq 0$ $\forall n$ and converges to $a \in [0,1)$. Prove $\sum_{n=1}^{\infty} a_n^n$ converges.
(You may assume that a sequence of real numbers converges if it is bounded above and monotonically increasing, but anything else you use should be proved from first principles.) (7 marks)

(Total: 20 marks)

3. In this question you may use any results from the course that you state correctly.
Let $(a_n)_{n\geq 1}$ be a sequence of real numbers.

(a) (i) Suppose $a_n \to a > 0$. Show $\exists\, N \in \mathbb{N}$ such that $a_n > 0$ $\forall n \geq N$. (2 marks)

(ii) Fix $k \in \mathbb{N}$ and define $b_n := \sqrt[k]{a_{n+1}a_{n+2}\ldots a_{n+k}}$ for $n \geq N$. Prove $b_n \to a$. (3 marks)

(iii) Give without proof an example of a divergent sequence $a_n > 0$ for which b_n is convergent for some k. (2 marks)

(b) (i) State the alternating series test. (2 marks)

(ii) Let

$$a_n \;=\; \begin{cases} \frac{1}{n^2}+\frac{1}{n} & n \text{ even,} \\ -\frac{1}{n^2} & n \text{ odd.} \end{cases}$$

Is $\sum a_n$ convergent? Prove your answer carefully. (5 marks)

(iii) What is the radius of convergence of $\sum a_n z^n$? Prove your answer. (6 marks)

(Total: 20 marks)

4. (a) (i) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^3}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$. (3 marks)

(ii) Prove that f is continuous at all $x \neq 0$ as well. (2 marks)

(b) (i) State the intermediate value theorem. (2 marks)

(ii) Prove that there is some $x \in \mathbb{R}$ such that $4^x = \cos(x) + 2$. (3 marks)

(iii) Suppose that we have a continuous function $f : [0, 10] \to \mathbb{R}$, with $f(2k) = 1$ for integers $0 \leq k \leq 5$ and $f(2k + 1) = -1$ for integers $0 \leq k \leq 4$. What is the least number of zeroes such an f must have? (3 marks)

(c) (i) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the following property:

For all $x \in [0, 1]$, there exists $y \in [0, 1]$ such that $|f(y)| \leq 0.99|f(x)|$.

Prove that there is some $t \in [0, 1]$ such that $f(t) = 0$. (4 marks)

(ii) Give an example (no proof necessary) of a discontinuous $f : [0, 1] \to \mathbb{R}$ with the same property as in part (i), such that $f(x) \neq 0$ for all $x \in [0, 1]$. (3 marks)

(Total: 20 marks)

5. In this question you may use any results from the course that you state correctly.

(a) (i) State the mean value theorem. (2 marks)

(ii) Deduce that $(1 + x)^r \geq 1 + rx$ for any real $x > 0$ and real $r \geq 1$. (3 marks)

(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $\lim_{x\to\infty} f(x)$ exists. Prove that if $\lim_{x\to\infty} f'(x)$ exists then it is zero. (4 marks)

(b) (i) Compute the second-order Taylor polynomial for $f(x) = -\log(\cos(x))$ centered at $x = 0$. (3 marks)

(ii) Prove that $f(x)$ is convex on the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. (2 marks)

(iii) Prove that $\cos(\frac{\pi}{3})\cos(\frac{\pi}{5}) \leq \left(\cos(\frac{4\pi}{15})\right)^2$. (2 marks)

(c) Let $f : (-1, 1) \to \mathbb{R}$ be an infinitely differentiable function satisfying $|f^{(n)}(x)| \leq (n-1)!$ for all $n \geq 1$ and all x. If $P_n(x)$ is the nth order Taylor polynomial for f, centered at $x = 0$, prove that $P_n \to f$ uniformly. (4 marks)

(Total: 20 marks)

6. (a) Which of the following are true? Give a one-sentence explanation or a counterexample for each.

All functions below have the form $[0,1] \to \mathbb{R}$ and are **bounded**.

(i) If f and g are both integrable and $f(x) \le g(x)$ for all $x \in [0,1]$, then

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$$

if and only if $f(x) = g(x)$ for all $x \in [0,1]$. (1 mark)

(ii) If f and g are differentiable and $f(x)g'(x)$ is integrable, then so is $f'(x)g(x)$. (1 mark)

(iii) If e^f is integrable, then f is integrable. (1 mark)

(iv) If f_n is integrable for all $n \in \mathbb{N}$, and if $f(x) = \lim_{n\to\infty} f_n(x)$ exists and is continuous, then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

(1 mark)

(v) If f and g are integrable, then $\max(f(x), g(x))$ is integrable. (2 marks)

(b) (i) State a version of the fundamental theorem of calculus. (3 marks)

(ii) Define $f : [0,n] \to \mathbb{R}$ by $f(x) = \frac{1}{\sqrt{x+1}}$ for some $n \in \mathbb{N}$. Determine the lower and upper Darboux sums for f with respect to the partition $P = (0, 1, 2, \ldots, n)$; you can leave your answers in the form of a finite sum. (3 marks)

(iii) Give an integer estimate for $U(f,P)$ in the case $n = 10^6 - 1 = 999{,}999$. Your estimate need not be the closest integer to $U(f,P)$, but you should prove that it differs from the actual value by at most 1. (3 marks)

(c) Suppose that $f, g : [0,1] \to \mathbb{R}$ are integrable, and that $f(x) = g(x)$ for all rational $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

(5 marks)

(Total: 20 marks)