Math40002 Analysis 1 **Problem Sheet 3**

1. Without looking at your notes, say out loud (ideally to a friend) the definition of $a_n \to a$ *in English* (not maths!). Pass back and forwards between maths and English (e.g. $\forall\,\epsilon > 0 \quad \longleftrightarrow \quad$ "However close I want to get", etc.).
   Write down your definition. Now check your notes. Are there any subtle differences (things in a different order, $\forall$ replaced by $\exists$, etc.?) If so they're VERY important. Is your definition still correct ? There are many correct – and incorrect – ways of writing the same definition. If it's only nearly correct, it's very wrong – can you find a counterexample to your definition?

2. * Which of the following sequences are convergent and which are not? What is the limit of the convergent ones? Give proofs for each.

   (a) $\frac{n+7}{n}$
   (b) $\frac{n}{n+7}$
   (c) $\frac{n^2+5n+6}{n^3-2}$
   (d) $\frac{n^3-2}{n^2+5n+6}$
   (e) $\frac{1-n(-1)^n}{n}$

3. We've defined what it means for (a_n) to converge to a real number $a \in \mathbb{R}$ as $n \to \infty$. Professor Lee Beck thinks infinity is cool, so he comes up with some definitions of $a_n \to +\infty$ as $n \to \infty$. Which are right and which are wrong? For any wrong ones, illustrate its wrongness with an example.

   (a) $\forall a \in \mathbb{R},\ a_n \not\to a$.
   (b) $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $n \geq N \ \Rightarrow\ |a_n - \infty| < \epsilon$.
   (c) $\forall R > 0\ \exists N \in \mathbb{N}$ such that $n \geq N \ \Rightarrow\ a_n > R$.
   (d) $\forall a \in \mathbb{R}\ \exists \epsilon > 0$ such that $\forall N \in \mathbb{N}\ \exists n \geq N$ such that $|a_n - a| \geq \epsilon$.
   (e) $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ a_n > \frac{1}{\epsilon}$.
   (f) $\forall n \in \mathbb{N},\ a_{n+1} > a_n$.
   (g) $\forall R \in \mathbb{R},\ \exists n \in N$ such that $a_n > R$.
   (h) $1/\max(1, a_n) \to 0$.

4. Let $S \subset \mathbb{R}$ be nonempty and bounded above. Show that there exists a sequence of numbers $s_n \in S$, $n = 1, 2, 3, \ldots$, such that $s_n \to \sup S$.

5. Give *without proof* examples of sequences (a_n), (b_n) with the following properties.

   (i) Neither of a_n, b_n is convergent, but $a_n + b_n$, $a_n b_n$ and a_n/b_n all converge.
   (ii) a_n converges, b_n is *un*bounded, but $a_n b_n$ converges.
   (iii) a_n converges, b_n bounded, but $a_n b_n$ diverges.

*You should prepare starred questions * to discuss with your personal tutor.*