Math40003 Linear Algebra and Groups **Problem Sheet 2**

1. *You should now be able to do questions 2 and 6 on PS1*

2. Describe the solution sets to the following sets of simultaneous equations in $\mathbb{R}^2$:

(a) $\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y &= 5 \end{aligned}$ (b) $x+2y=3$ (c) $\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y &= 5 \\ x+4y &= 6 \end{aligned}$

3. Describe the solution sets to the following sets of simultaneous equations in $\mathbb{R}^3$:

(a) $\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y-2z &= 5 \end{aligned}$ (b) $x+2y=3$ (c) $\begin{aligned} x+2y &= 3 \\ -4x+\tfrac{1}{2}y-2z &= 5 \\ x+4y+z &= 6 \end{aligned}$

4.* For which $a, b \in \mathbb{R}$ does the system of equations

$$\begin{aligned} x_1+x_2+x_3 &= -1 \\ 2x_1+x_2+ax_3 &= 1 \\ 3x_1+x_2+x_3 &= b \end{aligned}$$

have (i) no solutions, (ii) exactly one solution, (iii) infinitely many solutions?

What about the system

$$\begin{aligned} x_1+x_2+x_3+x_4 &= 0 \\ x_1-x_2+ax_3+x_4 &= 1 \\ 2x_1+ax_2+x_3+2x_4 &= b\ ? \end{aligned}$$

5. Which of the following are possible, find examples if possible:
   (a) Two simultaneous equations in two unknowns which defines a line in $\mathbb{R}^2$.
   (b) Two simultaneous equations in two unknowns which defines the empty set in $\mathbb{R}^2$.
   (c) One equation in no unknowns which defines the empty set.
   (d) Two simultaneous equations in three unknowns which defines a point in $\mathbb{R}^3$.

6. (a) Let M_θ be the reflection in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan\theta\}$. Using any school geometry or trigonometry you like, show that the matrix representing M_θ is
   $$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$
   (b) Let R_α be a rotation about the origin, and let M_β be the reflection in a line through the origin. Prove that $M_\beta R_\alpha$ is a reflection.
   (c) Let M_α and M_β be reflections in straight lines through the origin. Prove that $M_\alpha M_\beta$ is a rotation.