Math40002 Analysis 1 **Problem Sheet 5**

1. Fix $x > 0$. Prove $(1+x)^n \geq 1 + nx$ for any $n \in \mathbb{N}$. Deduce that $(1+x)^{-n} \to 0$. Deduce that if $r \in (0,1)$ then $r^n \to 0$, and if $r \in (1,\infty)$ then $r^n \to \infty$.

2. Suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. In lectures we proved that if $L < 1$ then $a_n \to 0$.

   (a) Prove that if $L > 1$ then $|a_n| \to \infty$.

   (b) Give an example with $|a_{n+1}/a_n| < 1 \ \forall n$ but $a_n \not\to 0$.

   Give (without proof) examples where $L = 1$ and
   (i) $a_n \to 0$, (iii) a_n divergent and bounded,
   (ii) $a_n \to a \neq 0$, (iv) $a_n \to \infty$.

3. Let $(a_n)_{n\geq 1}$ be a sequence of *strictly positive* real numbers.
   Give an example such that $(1/a_n)_{n\geq 1}$ is unbounded.
   Suppose that $a_n \to a \neq 0$. Prove *from first principles* that $(1/a_n)_{n\geq 1}$ is bounded.

4.† Fix $r \in (0, 1/8)$. Define $(a_n)_{n\geq 1}$ by $a_1 := 1$ and $a_{n+1} = ra_n^2 + 1$.

   (a) Show that $a_{n+1} - a_n = r(a_n + a_{n-1})(a_n - a_{n-1})$.

   (b) Show that if
   $$0 < a_j < 2 \qquad \forall\, j \leq n, \tag{1}$$
   then
   $$|a_{n+1} - a_n| \ < \ (4r)^n/4. \tag{2}$$

   (c) Deduce that if (1) holds, then $a_{n+1} < r/(1-4r) + 1$.

   (d) Conclude that (1) holds for $j = n+1$ too, and so $\forall j$ by induction.

   (e) Using (2) deduce $|a_m - a_n| < (4r)^n/4(1-4r)$ for $m \geq n$.

   (f) Deduce a_n is Cauchy. What does it converge to?

5.* Show that *any* sequence of real numbers $(a_n)_{n\geq 0}$ has a subsequence which either converges, or tends to ∞, or tends to $-\infty$.

6. At home Professor Papageorgiou has made a fully realistic mathematical model of a dart board. It is a copy of the unit interval $[0,1]$ in a frictionless vacuum. He throws a countably infinite number of darts at it, the nth landing at $a_n \in [0,1]$.

   He then makes a small dot $(x - \epsilon_x,\, x + \epsilon_x)$ around each point $x \in [0,1]$ with his pen. Prove that however small he makes each dot, at least one of them will contain an infinite number of darts $a_n \in [0,1]$.

   What if he only makes dots around each dart $a_n \in [0,1]$?

7. Let $(a_n)_{n\geq 1}$ be the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots$

   (i) Give (without proof) a subsequence of $(a_n)_{n\geq 1}$ which converges to $\ell = 0$, and one which converges to $\ell = 1$.

   (ii) Given any $\ell \in (0,1)$, give (with proof) a subsequence convergent to ℓ.

8. Professor Buzzard is teaching Lean about Cauchy sequences. Thomas has told him that it can be hard to find their limits, so he sets out to prove him wrong. He types some `mynat.definition` guff and then

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } n, m \geq N \;\Rightarrow\; |a_n - a_m| < \epsilon$$

$$\Rightarrow\; \forall n \geq N \quad |a_n - a_N| < \epsilon$$

$$\Rightarrow\; a_n \to a_N \;\text{ as }\; n \to \infty.$$

"Ha – who are you calling a computer scientist, Thomas?!" he exclaims, as he types `refl` with a flourish. Does Lean give him a "`Proof complete!`"?

*You should prepare starred questions * to discuss with your personal tutor.*
Questions marked † are slightly harder (closer to exam standard), but good for you.