Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 5 with solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. Consider the following systems of ODEs, which you already solved in Problem Sheet 4:

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and } A \text{ is a } 2 \times 2 \text{ matrix.}$$

(a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

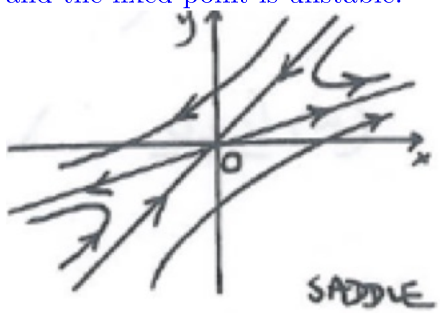
$$\lambda_1 = -1 \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 2 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This is a saddle-point and the fixed point is unstable.



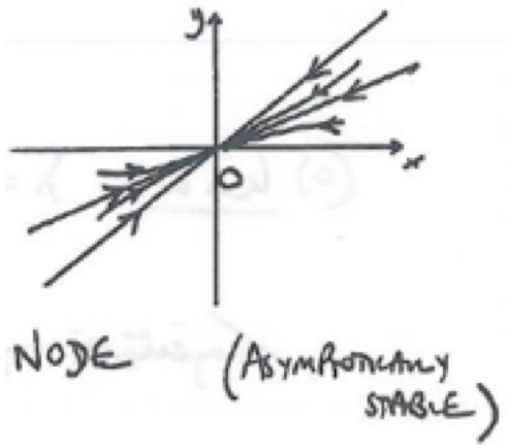
(b) $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$

$$\lambda_1 = -1 \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$



(c) $A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$

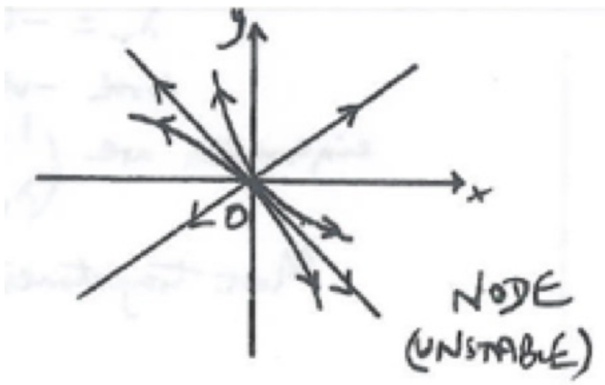
Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = \frac{1}{2} \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 2 \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\frac{1}{2}t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



(d) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = 1 + 2i \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$$

$$\lambda_2 = 1 - 2i \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$$

So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}.$$

Defining $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$ as new real constants of integration, we can write the general solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + A_2 e^t \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}.$$

This is an outward spiral and the fixed point is unstable. The direction of the trajectory can be obtained by evaluating the velocity vector field at some points.

(e) $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = -1 + 2i \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$\lambda_2 = -1 - 2i \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 2 \\ i \end{pmatrix}$$

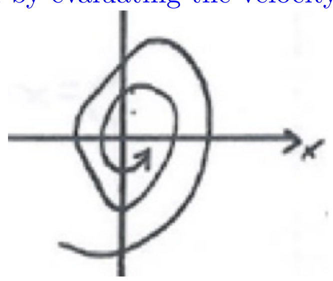
So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(-1+2i)t} \begin{pmatrix} 2 \\ -i \end{pmatrix} + c_2 e^{(-1-2i)t} \begin{pmatrix} 2 \\ i \end{pmatrix}.$$

Defining $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$ as new real constants of integration, we can write the general solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} + A_2 e^{-t} \begin{pmatrix} 2\sin 2t \\ -\cos 2t \end{pmatrix}.$$

This is an inward spiral and the fixed point is asymptotically stable. The direction of the trajectory can be obtained by evaluating the velocity vector field at some points.



(f) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

Obtaining eigenvalues and eigenvectors we have:

$$\lambda_1 = i \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 5 \\ 2 - i \end{pmatrix}$$

$$\lambda_2 = -i \quad \Longrightarrow \quad \vec{v}_2 = \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$$

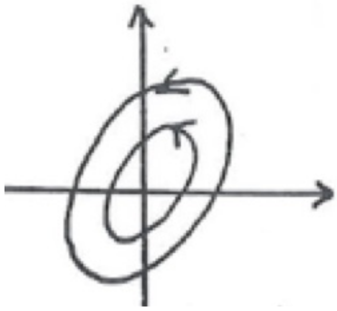
So we have for the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{it} \begin{pmatrix} 5 \\ 2-i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 5 \\ 2+i \end{pmatrix}.$$

Defining $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$ as new real constants of integration, we can write the general solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + A_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

The portrait is centres with eleptical colosed curves. The direction of the trajectory can be obtained by evaluating the velocity vector field at some points.



Sketch the phase portraits for all these systems in the (x, y) plane.

2. Consider the system of ODEs:

$$\frac{dx}{dt} = 3x - 2y$$
$$\frac{dy}{dt} = 2x - 2y$$

Solve $y(x)$ and relate this solution to the phase portrait in Question 1a.

$$\frac{dy}{dx} = \frac{2x-2y}{3x-2y} = \frac{2-2(y/x)}{3-2(y/x)}.$$

This ODE is homogenous and can be solved for $y(x)$, using the substitution $u = y/x$. The solution of the ODE is

$$(2x - y)(x - 2y)^2 = c$$

Where c is a constant. For $c = 0$, we obtain the eigenvectors. Other values of c will result in the different trajectories in the phase plane as seen in the phase portraint of 1a.

3. * Consider the system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find the general solution of this system and explain carefully your calculations

(b) Locating the system on the (τ, Δ) parameter plane or, equivalently, based on its eigenvalues, establish the expected asymptotic behaviour of the system

(c) Sketch carefully the phase portrait, indicating in detail all the representative trajectories of the system

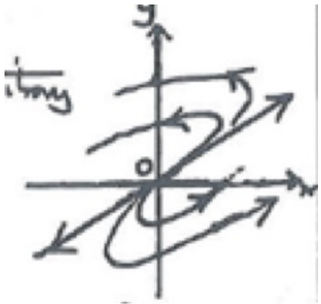
The system has trace $\tau = 2$ and determinant $\Delta = 1$. So it is on the line $\tau^2 - 4\Delta$, which is case 4 and there is repeated eigenvalues. It is clear that we are in the non-diagonalizable case, so this needs to be solved through transforming matrix A into a Jordan nomral form.

$$\lambda = 1, 1 \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

By solving for matrix W that could result in the Jordan normal form we obtain $w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so we have the General solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right]$$

. The fixed point is unstable and the solutions blow up in the direction of v_1. The phase portrait is give below. Check notes from class for full characterisation of the phase portrait.



4. * Consider a combat model between two armies of strength $x(t)$ and $y(t)$. A simple model of their dynamical evolution under combat is given by:

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bx,$$

where a and b are positive constants.

(a) Explain the meaning of a and b

(b) Find the general solution $(x(t), y(t))$ in terms of the initial conditions $x(0) = x_0$ and $y(0) = y_0$ and sketch the phase portrait.

(c) Solve for $y(x)$ and relate this solution to the phase portrait in (b).

(d) Consider now the (relevant) solutions of the system in the positive quadrant $x(t), y(t) \geq 0$. Using the information above, characterise the qualitatively distinct outcomes of the system for any initial condition. (This is known as Lanchester's square law in Game Theory, and was influential during the Cold War.)

a and b denote the rate each army destroys the other army. This system can be charectrised by the following linear system of ODEs

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We have

$$\lambda_1 = \sqrt{ab} \quad \Longrightarrow \quad \vec{v}_1 = \begin{pmatrix} -\sqrt{a} \\ \sqrt{b} \end{pmatrix}$$

$$\lambda_2 = -\sqrt{ab} \implies \vec{v}_2 = \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \end{pmatrix}$$

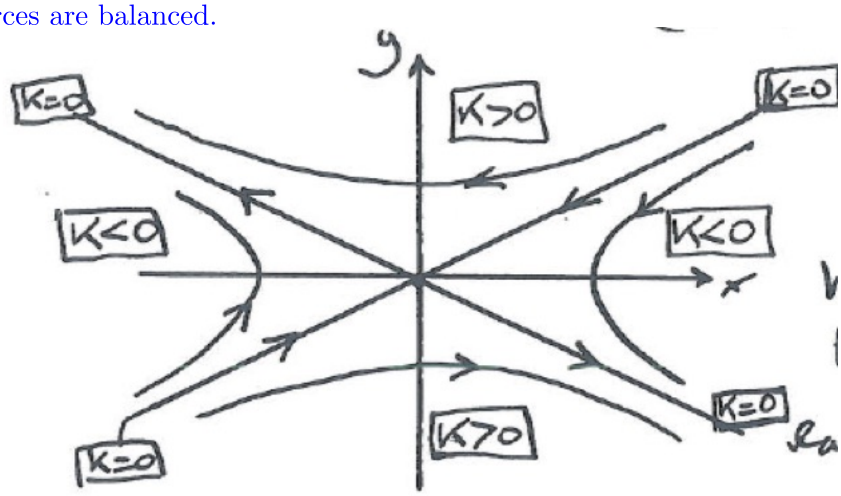
This is a saddle-point. Solving the following separable ODE

$$\frac{dy}{dx} = \frac{bx}{ay}$$

Gives the solution $ay^2 - bx^2 = K$, where K is a constant and can be determined from the inital values x_0 and y_0:

$$K = ay_0^2 - bx_0^2$$

The phase portrait below shows that, if we have $K > 0$ then $x \to 0$ and $y \to \sqrt{K/a}$ meaning that y wins and when $K < 0$ then $y \to 0$ and $x \to \sqrt{-K/b}$ meaning that x wins. If $K = 0$ we have the forces are balanced.



5. Consider a game-theoretical model for the collaboration between two political parties X and Y. The leaderships of both parties are keen on collaboration, but they need to evaluate the support for an alliance by their militants. The 'level of enthusiasm' for the alliance within each party is given by $x(t)$ and $y(t)$, respectively. A simple model of the time evolution of such system is given by:

$$\frac{dx}{dt} = 2x - 2y$$
$$\frac{dy}{dt} = 2x - 3y$$

   (a) Find the general solution $(x(t), y(t))$ and sketch the phase portrait.

   (b) Solve for $y(x)$ and relate this solution to the phase portrait in (a).

   (c) Using the information above, consider the solutions in the positive quadrant $x(t), y(t) \geq 0$ and characterise the distinctive outcomes of the system for any initial condition (x_0, y_0).

   (d) Based on the predictions of the model, the leadership of party X commissions a poll among the militants to gauge the initial level of enthusiasm for an alliance. What mathematical condition will they be looking for when examining the results of the poll?

We have

$$\lambda_1 = 1 \implies \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2 \implies \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We have $(2x-y)^2(x-2y) = c$ where c is a constant. Initial positive enthusiasms place $x(0)$ and $y(0)$ in the 1st quadrant. By inspection of the trajectories in this quadrant intersection with one the axis can occur only on the y axis (as $x \to 0$) and only when $y(0) > 2x(0)$ (for example * in the phase portrait below).