Math40002 Analysis 1 — Unseen 1

1. Prove that every root of a *monic* polynomial with integer coefficients (i.e, a polynomial of the form $p(x) = 1 \cdot x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, \ldots, a_{n-1} \in \mathbb{Z}$) is either irrational or an integer.

2. A subset $X \subseteq \mathbb{R}$ is *dense* if for all $a, b \in \mathbb{R}$ such that $a < b$, there is some $x \in X$ such that $a < x < b$.

   (a) Prove the irrational numbers are dense.

   (b) Prove the rational numbers are dense.

3. For the following sets, determine whether they are finite, countable, or uncountable.

   Prove your answer.

   (a) The set of all finite subsets of $\mathbb{R}$.

   (b) The set of all co-finite subsets of $\mathbb{R}$, that is, $\{ A \subseteq \mathbb{R} \mid \mathbb{R} \setminus A \text{ is finite } \}$.

   (c) The set of all finite subsets of $\mathbb{Q}$.

   (d) The set of all co-finite subsets of $\mathbb{Q}$, that is, $\{ A \subseteq \mathbb{Q} \mid \mathbb{Q} \setminus A \text{ is finite } \}$.

   (e) The set of all open intervals with endpoints in $\mathbb{R}$: $\{ (a, b) \mid a, b \in \mathbb{R} \}$.

   (f) The set of all open intervals with endpoints in $\mathbb{Q}$: $\{ (a, b) \mid a, b \in \mathbb{Q} \}$.

   (g) The set of all finite unions of open intervals with endpoints in $\mathbb{Q}$: The set of all sets of the form $\bigcup_{i=1}^{n} (a_i, b_i)$.

   (h) The set of all countable intersections of open intervals with endpoints in $\mathbb{Q}$: The set of all sets of the form $\bigcup_{i=1}^{\infty} (a_i, b_i)$.

   (i) The set of all finite intersections of open intervals with endpoints in $\mathbb{Q}$: The set of all sets of the form $\bigcap_{i=1}^{n} (a_i, b_i)$.

   (j) The set of all countable intersections of open intervals with endpoints in $\mathbb{Q}$: The set of all sets of the form $\bigcap_{i=1}^{\infty} (a_i, b_i)$.