1D continuum limit of the applications framework

1. A conducting wire of uniform conductivity equal to unity $\hat{c}(x) = 1$ lies in the interval $x \in [0, 1]$. The voltage at $x = 0$ is set equal to unity and it is grounded at $x = 1$:

$$\phi(0) = 1, \qquad \phi(1) = 0.$$

There are no current sources for $x \in (0, 1)$.

(a) Find the solution for the voltage $\phi(x)$.

(b) This problem corresponds to the continuous version of a random walker problem on a discrete graph considered on Problem Sheet 3. Check that your solution in part (a) is consistent with the continuous limit of that random walker solution.

2. A conducting wire of uniform conductivity equal to unity $\hat{c}(x) = 1$ lies in the interval $x \in [0, 1]$. It is grounded at both ends, so the voltage there is zero

$$\phi(0) = \phi(1) = 0.$$

Current is fed into the wire with a current density

$$\hat{f}(x) = x$$

that increases linearly along the wire.

(a) Write down the boundary value problem for the voltage distribution $\phi(x)$.

(b) Find the solution of this boundary value problem.

(c) What is the current at each end of the wire?

(d) What is total current input into the wire?

3. A conducting wire of conductivity $\hat{c}(x) = 2 - x$ lies in the interval $x \in [0, 1]$. It is grounded at both ends. Current is fed into the wire with a current density

$$\hat{f}(x) = x(1 - x).$$

(a) Write down the boundary value problem for the voltage distribution $\phi(x)$.

(b) Find the solution of this boundary value problem.

(c) What is the current at each end of the wire?

(d) What is total current input into the wire?

4. Consider the continuum limit of the spring-mass network of n masses, all of unit mass, connected by springs, all with unit spring constant, between two fixed walls.

(a) Argue why, by direct extension of the arguments presented in lectures, the continuum limit of the discrete system

$$-K\mathbf{x} = \frac{d^2\mathbf{x}}{dt^2},$$

where $\mathbf{x}$ is the n-dimensional vector of displacements (from equilibrium) is

$$\frac{\partial^2\phi(x,t)}{\partial x^2} = \frac{\partial^2\phi(x,t)}{\partial t^2}, \tag{1}$$

where $\phi(x,t)$ is a continuous displacement function of x in the interval $x \in (0,1)$ and $\partial/\partial x$ is a "partial derivative" with respect to x but keeping t fixed, while $\partial/\partial t$ is a "partial derivative" with respect to t but keeping x fixed.[1]

(b) Suppose we seek solutions of the form

$$\phi(x,t) = \Phi(x)e^{\mathrm{i}\omega t},$$

where $\Phi(x)$ is purely a function of x, which is the analogue of

$$\mathbf{x} = \mathbf{x}_0 e^{\mathrm{i}\omega t}$$

which we used in the discrete case. Find the equation satisfied by $\Phi(x)$.

(c) Solve this equation for $\Phi(x)$ assuming that the displacements at the end-points are fixed, i.e.,

$$\Phi(0) = \Phi(1) = 0.$$

How many possible solutions do you find? How many solutions for $\mathbf{x}_0$ did we find in the case of finite n?

(d) Can you relate your answer to part (c) to the $n \to \infty$ analysis of the spring-mass system considered in lectures?

[1](1) is a partial differential equation known as the "1D wave equation"