Imperial College London
Department of Mathematics

# Question Sheet 4

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Problem sheet released on Wednesday of week 5. All questions can be attempted before the problem class on Monday Week 7. Question 1 is suitable for tutorials. Solutions will be released on Wednesday of week 7.

**Question 1** The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Find a matrix $A \in M_2(\mathbb{R})$ with the property that

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}.$$

Compute the eigenvalues and eigenvectors of $A$ and express $(1,0)^T$ as a linear combination of eigenvectors. Hence, or otherwise, find a general expression for $F_n$ (in terms of $n$).

**Question 2** Suppose $A \in M_n(F)$ has characteristic polynomial

$$\chi_A(x) = a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + x^n.$$

The *Cayley - Hamilton Theorem* states that

$$\chi_A(A) = a_0I_n + a_1A + \ldots + a_{n-1}A^{n-1} + A^n = 0.$$

(A special case of this was given on Question 7 of sheet 3.) Here is the start of a proof of the result. Finish the proof:

Let $B = B(x)$ denote the adjugate matrix of $(xI_n - A)$. So each entry in $B$ is a cofactor of $xI_n - A$ and is therefore a polynomial of degree at most $n-1$ in $x$. Thus we can write $B(x) = B_{n-1}x^{n-1} + \ldots + B_1x + B_0$ for some matrices $B_{n-1}, \ldots, B_1, B_0 \in M_n(F)$ (so these do not involve the variable $x$). By 5.3.2 in the lectures:

$$(a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + x^n)I_n = \det(xI_n - A)I_n = B(x)(xI_n - A) = (B_{n-1}x^{n-1} + \ldots + B_1x + B_0)(xI_n - A). \quad (1)$$

Multiplying out the right-hand side and equating coefficients of the various powers of $x$ we obtain:

$$a_0I_n = -B_0A, \ldots$$

**Question 3** Suppose $S, T : V \to V$ are linear and $S \circ T = T \circ S$. For $\lambda \in F$ let $E_\lambda(S) = \{v \in V : Sv = \lambda v\}$. Show that if $v \in E_\lambda(S)$, then $T(v) \in E_\lambda(S)$.

**Question 4** (i) Suppose $v_1, \ldots, v_r \in \mathbb{R}^n$ is an orthogonal set of non-zero vectors. Show that $v_1, \ldots, v_r$ are linearly independent.

(ii) Suppose $A \in M_n(\mathbb{R})$. Prove that $A$ is an orthogonal matrix if and only if for all $u \in \mathbb{R}^n$ we have $\|Au\| = \|u\|$.

(iii) Suppose $A \in M_2(\mathbb{R})$ is an orthogonal matrix and $\det(A) = 1$. Show that there is $\theta \in \mathbb{R}$ such that $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (so $A$ is a rotation matrix).

(iv) Suppose $C \in M_3(\mathbb{R})$ is an orthogonal matrix and $\det(C) = 1$. Show that 1 is an eigenvalue of $C$. Is this true for $4 \times 4$ matrices?

**Question 5** Find an orthogonal matrix $A \in M_4(\mathbb{R})$ whose first column is $\frac{1}{2}(1,1,1,1)^T$.

**Question 6** Suppose $U$ is a subspace of $\mathbb{R}^n$ and $T : U \to U$ is linear. Suppose $B$ is an orthonormal basis of $U$. Prove that $[T]_B$ is symmetric if and only if for all $u, v \in U$ we have $T(u) \cdot v = u \cdot T(v)$.

**Question 7** Suppose $U$ is a subspace of $\mathbb{R}^n$. Let $U^\perp = \{v \in \mathbb{R}^n : v \cdot u = 0 \text{ for all } u \in U\}$.

i) Show that $U^\perp$ is a subspace of $\mathbb{R}^n$.

ii) Show that $U \cap U^\perp = \{0\}$.

iii) Show that $\dim(U^\perp) = n - \dim(U)$.

iv) In the case where $n = 4$ and $U$ is the subspace spanned by $u_1 = (1,1,0,1)^T$ and $u_2 = (1,1,1,0)^T$, compute $U^\perp$.