MATH40007 - An Introduction to Applied Mathematics Coursework 2

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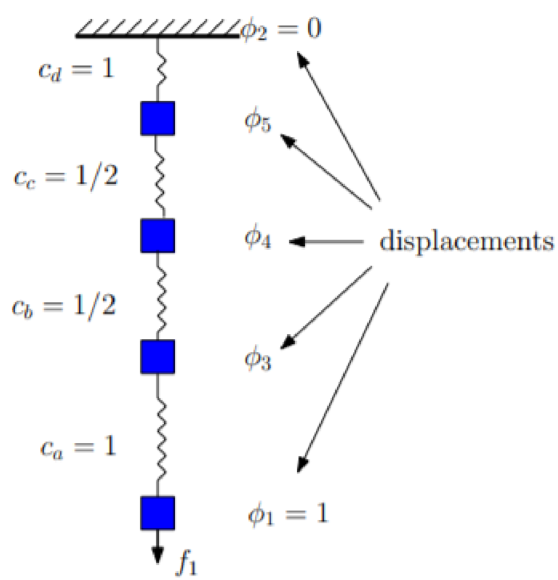
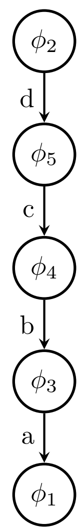
The figure shows 4 masses, all of unit mass, hanging from a ceiling with 4 linear springs labelled a, b, c, and d in between them. These springs have spring constants $c_a = c_d = 1$ and $c_b = c_c = 1/2$ as indicated in the figure. Displacements of the nodes are denoted by ϕ_j for $j = 1, ..., 5$. An external force f_1 holds the lowest mass at unit displacement from its equilibrium position so $\phi_1 = 1$ and the ceiling cannot be displaced implying $\phi_2 = 0$. There are no external forces on any of the other masses (masses $3, 4$ and 5).



(a)

Introduce the 5-vector of displacements $x = [\phi_1, \phi_2, \phi_3, \phi_4, \phi_5]^T$. By considering this system as a graph with 5 nodes (the masses and the ceiling labelled as shown being the nodes) and 4 edges (the 4 springs), write down the statement of equilibrium force balance in this system in terms of x and the weighted Laplacian matrix K for this graph (which you should find).

First, lets construct the graph representing our spring-mass system:



The incidence matrix is then

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Let f be the vector of external forces acting on each node. Then $f = [f_1, f_2, 0, 0, 0]^T$, where f_1 is the external force acting on node 1, f_2 is the wall reaction force on node 2, and the rest components are 0 since there is no external force acting on them.

Now let f_I be the vector of internal forces acting on the masses due to the springs. We have $f_I = -A^T T$, where $T = [T_a, T_b, T_c, T_d]^T$ represents the tension in the four strings. Further, by Hooke's law the tension can be written as

$$T = CAx, \text{ where } C = \begin{bmatrix} c_a & 0 & 0 & 0 \\ 0 & c_b & 0 & 0 \\ 0 & 0 & c_c & 0 \\ 0 & 0 & 0 & c_d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the matrix representing the spring constants across the system. Putting the pieces together, we obtain

$$f_I = -A^T T = -A^T C A x.$$

If the masses are in equilibrium, then the external and internal forces must add up to zero. Hence our equation becomes:

$$f = -f_I = \underbrace{A^T C A}_{K} x \implies$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & -1 & 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = Kx$$

where K is the weighted Laplacian matrix, obtained either by calculating $A^T CA$ or by writing the sum of the spring constants of the edges connected to node i on K_{ii} position and then putting $(-c)$ on the position K_{ij} if node j is connected to node i, where c is the spring constant of the edge between these nodes.

(b)

Find the values of the equilibrium displacements ϕ_3, ϕ_4 and ϕ_5 of masses $3, 4$ and 5.

$$Kx = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & -1 & 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 1-\phi_3 \\ -\phi_5 \\ \frac{3\phi_3-\phi_4-2}{2} \\ \frac{-\phi_3+2\phi_4-\phi_5}{2} \\ \frac{-\phi_4+3\phi_5}{2} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0. \end{bmatrix}$$

So we get the system

$$\begin{cases} \frac{3\phi_3-\phi_4-2}{2} = 0 \implies \phi_3 = \frac{\phi_4+2}{3} \\ \frac{-\phi_3+2\phi_4-\phi_5}{2} = 0 \\ \frac{-\phi_4+3\phi_5}{2} = 0 \implies \phi_5 = \frac{\phi_4}{3}. \end{cases}$$

Replacing ϕ_3 and ϕ_5 in the second equation we get $-\frac{\phi_4+2}{3} + 2\phi_4 - \frac{\phi_4}{3} = 0 \implies \phi_4 = \frac{1}{2}$, and so $\phi_3 = \frac{5}{6}, \phi_5 = \frac{1}{6}$.

(c)

Find the matrices P, Q and R, where P is a 2-by-2 matrix, in the following sub-block decomposition

$$K = \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}.$$

We already found K to be

$$K = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & -1 & 0 & -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix} \end{bmatrix}.$$

So denote

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, R = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix},$$

then

$$K = \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}.$$

(d)

In terms of these sub-block matrices write down the equation, coming from the condition of force balance, satisfied by the 3$-$vector $\hat{x} = [\phi_3, \phi_4, \phi_5]^T$.

We know that $x = [1, 0, \phi_3, \phi_4, \phi_5]^T$, so lets denote $e = [1, 0]^T$. Then we have $x = [e, \hat{x}]^T$. The equation satisfying equilibrium $Kx = f$ becomes

$$\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \hat{f} \\ \underline{0} \end{bmatrix},$$

where $\hat{f} = [f_1, f_2]^T$ and $\underline{0} = [0, 0, 0]^T$. Now we get the following system of equations:

$$\begin{cases} Pe + Q^T\hat{x} = \hat{f} \\ Qe + R\hat{x} = 0 \implies \hat{x} = -R^{-1}Qe \end{cases}$$

Substituting $\hat{x}$ in the first equation, we derive

$$\hat{f} = (P - Q^T R^{-1} Q)e.$$

(e)

Find the eigenvalues $\{\lambda_n | n = 1, 2, 3\}$ and eigenvectors $\{e_n | n = 1, 2, 3\}$ of R where all eigenvectors are normalized so that $|e_n| = 1$.

$$R = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix}$$

The eigenvalues of R are the roots of the characteristic equation $\det(R - \lambda I) = 0$.

$$\det\left(\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{-4\lambda^3 + 16\lambda^2 - 19\lambda + 6}{4}.$$

Hence we get $\lambda_1 = 1/2, \lambda_2 = 3/2, \lambda_3 = 2$.

Eigenvectors for λ_1 :

$$(R - \lambda_1 I)v_1 = 0 \iff$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$$

$$x = z, y = 2z = 2x.$$

So the eigenvector v_1 for λ_1 is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Normalizing it, we obtain

$$e_1 = \frac{v_1}{|v_1|} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

Eigenvectors for λ_2 :

$$(R - \lambda_2 I)v_2 = 0 \iff$$

$$\begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$$

$$x = -z, y = 0.$$

So the eigenvector v_2 for λ_2 is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalizing it, we obtain

$$e_2 = \frac{v_2}{|v_2|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Eigenvectors for λ_3 :

$$(R - \lambda_3 I)v_3 = 0 \iff$$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$$

$$x = z, y = -z.$$

So the eigenvector v_3 for λ_3 is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Normalizing it, we obtain

$$e_3 = \frac{v_3}{|v_3|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

(f)

Find the values of $e_i^T e_j$ for $i \neq j$.

R is symmetric, so $R = R^T$.Then for $i \neq j$ we have:

$$\begin{aligned} (Re_i) \cdot e_j &= (e_i^T R^T)e_j \\ &= e_i^T (R^T e_j) \\ &= e_i^T (Re_j) \\ &= e_i \cdot (Re_j) \end{aligned}$$

Replacing Re_i with $\lambda_i e_i$ and Re_j with $\lambda_j e_j$ (properties of eigenvectors and eigenvalues), we get:

$$(\lambda_i e_i) \cdot e_j = e_i \cdot (\lambda_j e_j) \implies$$
$$\lambda_i e_i \cdot e_j = \lambda_j e_i \cdot e_j$$

As $\lambda_i = \lambda_j$ for $i \neq j$, we get that $e_i \cdot e_j = 0$, or

$$e_i^T e_j = 0 \text{ for } i \neq j.$$

(g)

Write $\hat{x}$ as a sum over the eigenvectors of R, i.e as

$$\hat{x} = \sum_{n=1}^{3} a_n e_n$$

and find the coefficients $\{a_n\}$.

In part (b) we found ϕ_3, ϕ_4 and ϕ_5 to be $\frac{5}{6}, \frac{1}{2}, \frac{1}{6}$. So $\hat{x} = \begin{bmatrix} 5/6 \\ 1/2 \\ 1/6 \end{bmatrix}$. Expressing $\hat{x}$ in terms of e_i for $i \in \{1, 2, 3\}$, we get

$$\begin{bmatrix} 5/6 \\ 1/2 \\ 1/6 \end{bmatrix} = a_1 e_1 + a_2 e_2 + a_3 e_3 \implies$$
$$\begin{bmatrix} 5/6 \\ 1/2 \\ 1/6 \end{bmatrix} = \begin{bmatrix} a_1/\sqrt{6} - a_2/\sqrt{2} + a_3/\sqrt{3} \\ 2a_1/\sqrt{6} - a_3/\sqrt{3} \\ a_1/\sqrt{6} + a_2/\sqrt{2} + a_3/\sqrt{3} \end{bmatrix}$$

From the first and last row we get $a_2 = -\frac{\sqrt{2}}{3}$. Now we have the system:

$$\begin{cases} \frac{a_1}{\sqrt{6}} + \frac{a_3}{\sqrt{3}} = \frac{1}{2} \\ \frac{2a_1}{\sqrt{6}} - \frac{a_3}{\sqrt{3}} = \frac{1}{2} \end{cases}$$

Adding the two equations we get that $a_1 = \frac{\sqrt{6}}{3}$ and therefore $a_3 = \frac{\sqrt{3}}{6}$. So

$$\boxed{a_{1,2,3} = \{\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{3}}{6}\}}.$$

(h)

Find f_1.

In part (a) we derived the equation

$$f = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & -1 & 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix}$$

and then found that $\phi_3 = 5/6, \phi_4 = 1/2, \phi_5 = 1/6$. We can now obtain

$$f_1 = 1 - \phi_3 = 1 - \frac{5}{6} = \frac{1}{6}$$
$$f_2 = -\phi_5 = -\frac{1}{6}.$$

So $\boxed{f_1 = \frac{1}{6}}$.

(i)

Suppose now that we fix mass 1 at unit displacement but we perturb masses 3, 4 and 5 a small amount from the equilibrium positions you have just worked out. Find general mathematical expressions for the time-evolving displacements of masses $3, 4$ and 5.

The total force (external and internal) is no longer zero, since we don't have equilibrium. The vector of displacement is $\phi = [0, 0, \phi_3(t), \phi_4(t), \phi_5(t)]^T$, since the displacements of masses 1 and 2 are fixed and at the other three nodes are a function of time. The vector $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 = 0 \\ f_4 = 0 \\ f_5 = 0 \end{bmatrix}$ represents the external forces acting on each of the masses, where we have denoted that there are no external forces on the middle masses (free oscillation). The vector $f_I = -K\phi$ represents the internal forces. The net force is then $f + f_I = f - K\phi$ and equals $M\frac{d^2\phi}{dt^2}$ by Newton's second law, where $M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is the matrix of the masses (the diagonal is all ones since all masses are unit). Hence we have the equation:

$$\begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & -1 & 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \frac{d^2}{dt^2} \begin{bmatrix} 0 \\ 0 \\ \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \end{bmatrix}$$

We can think of nodes 1 and 2 as being 'grounded' and since we only look for $\phi_3(t), \phi_4(t)$ and $\phi_5(t)$, we can delete the first two rows and columns of our equation to obtain the following reduced equation:

$$-\underbrace{\begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \end{bmatrix}}_{\hat{\phi}} = \frac{d^2}{dt^2} \underbrace{\begin{bmatrix} \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \end{bmatrix}}_{\hat{\phi}}$$

We seek solutions of the form

$$\hat{\phi} = \Phi e^{iwt}$$

so that $\frac{d^2}{dt^2}\hat{\phi} = -w^2\Phi e^{iwt}$. Our equation becomes an eigenvector problem for R:

$$R\Phi = w^2\Phi.$$

Here we notice that if Φe^{iwt} is a solution, then so is Φe^{-iwt}. So we can conclude that if Φe^{iwt} is a solution, then so are

$$\begin{aligned}
\mathrm{Re}[\Phi e^{iwt}] &= \frac{\Phi e^{iwt} - \overline{\Phi} e^{-iwt}}{2} \\
\mathrm{Im}[\Phi e^{iwt}] &= \frac{\Phi e^{iwt} - \overline{\Phi} e^{-iwt}}{2i}
\end{aligned}$$

In part (e) we found the eigenvalues of R to be $\lambda_{1,2,3} = \frac{1}{2}, \frac{3}{2}, 2 \implies$

$$w = \pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{3}{2}}, \pm\sqrt{2}.$$

Also we found three orthogonal eigenvectors $e_{1,2,3} = \Phi_{1,2,3} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

So the general solution for $\hat{\phi}$ is

$$\hat{\phi} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \left(c_1 e^{i\frac{1}{\sqrt{2}}t} + \overline{c_1} e^{-i\frac{1}{\sqrt{2}}t}\right) + \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \left(c_2 e^{i\sqrt{\frac{3}{2}}t} + \overline{c_2} e^{-i\sqrt{\frac{3}{2}}t}\right) + \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \left(c_3 e^{i\sqrt{2}t} + \overline{c_3} e^{-i\sqrt{2}t}\right),$$

where c_1, c_2 and c_3 are complex numbers with real and imaginary part. So if $c_1 = (A - Bi)/2, c_2 = (C - Di)/2, c_3 = (E - Fi)/2$, using the fact that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ we can express the general solution as

$$\begin{aligned}
\hat{\phi} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \left(A\cos(\frac{1}{\sqrt{2}}t) + B\sin(\frac{1}{\sqrt{2}}t)\right) &+ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \left(C\cos(\sqrt{\frac{3}{2}}t) + D\sin(\sqrt{\frac{3}{2}}t)\right) \\
&+ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \left(E\cos(\sqrt{2}t) + F\sin(\sqrt{2}t)\right).
\end{aligned}$$