5 Topics: Continuous random variables, transformations of random variables, expectation

5.1 Prerequisites: Lecture 12

Exercice 5- 1: Let $X \sim \mathrm{Exp}(\lambda)$. Show that, if $x, y > 0$ then

$$\mathrm{P}(X > x + y | X > x) = \mathrm{P}(X > y).$$

This is called the *Lack of memory* property (for a continuous random variable).

Solution: Recall that, for $x, y > 0$, we have

$$f_X(x) = \lambda e^{-\lambda x},$$

and

$$\mathrm{P}(X > x) = \int_x^\infty \lambda e^{-\lambda u} du = e^{-\lambda x}.$$

Hence

$$\begin{aligned}\mathrm{P}(X > x + y | X > x) &= \frac{\mathrm{P}(X > x + y, X > x)}{\mathrm{P}(X > x)} = \frac{\mathrm{P}(X > x + y)}{\mathrm{P}(X > x)}\\ &= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = \mathrm{P}(X > y).\end{aligned}$$

Exercice 5- 2: (Suggested for personal/peer tutorial) The length of time (in hours) that a student takes to complete a one hour exam is a continuous random variable X with probability density function f_X defined by

$$f_X(x) = cx^2 + x, \quad 0 < x \leq 1,$$

for some constant c, and zero otherwise.

(a) Find the value of c.

(b) By integration, find the cumulative distribution function F_X of X.

(c) Find the probability that a student completes the exam in less than half an hour.

(d) **Given** that a student takes longer than fifteen minutes to complete the exam, find the probability that they require at least half an hour, that is, find the conditional probability

$$\mathrm{P}\left(X > \frac{1}{2} \middle| X > \frac{1}{4}\right)$$

(e) In a class of two hundred students, find the probability that at most three students complete the exam in fewer than ten minutes.

Assume that the exam completion times for the two hundred students are independent random variables having the distribution specified above.

Hint: Consider discrete random variables $Y_1, ...Y_{200}$ where

$$Y_i = \begin{cases} 1 & \textit{student } i \textit{ completes the exam in fewer than ten minutes} \\ 0 & \textit{otherwise} \end{cases}$$

You need to calculate $\mathrm{P}(Y \leq 3)$ *where* $Y = \sum_{i=1}^{200} Y_i$.

Solution:

(a) Density function must integrate to 1 over $\mathrm{Im}X = [0, 1]$, so

$$\int_0^1 f_X(x)\, dx = 1 \Longrightarrow \int_0^1 (cx^2 + x)\, dx = 1 \Longrightarrow \left[c\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1 \Longrightarrow c = \frac{3}{2}$$

(b) Distribution function F_X given for $0 \leq x \leq 1$ by

$$F_X(x) = \int_0^x f_X(t)\, dt = \frac{x^3 + x^2}{2}$$

and $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 1$.

(c) $\mathrm{P}(X < 1/2) = F_X(1/2) = \frac{3}{16}$.

(d) From the definition of the conditional probability, we have

$$\mathrm{P}(X > 1/2 | X > 1/4) = \frac{\mathrm{P}(X > 1/2, X > 1/4)}{\mathrm{P}(X > 1/4)} = \frac{\mathrm{P}(X > 1/2)}{\mathrm{P}(X > 1/4)} = \frac{1 - F_X(1/2)}{1 - F_X(1/4)} = \frac{104}{123}$$

(e) $Y \sim \mathrm{Bin}(200, \theta)$ where

$$\theta = \mathrm{P}(Y_i = 1) = \mathrm{P}(X < 1/6) = F_X(1/6) = \frac{7}{432}$$

Then

$$\mathrm{P}(Y \leq 3) = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) + \mathrm{P}(Y = 3) = 0.593$$

Exercice 5- 3: The probability density function of continuous random variable X taking values in the range $\mathrm{Im}X = (0, 2)$ is specified by

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and zero otherwise. Find the cumulative distribution function of X, F_X, and hence find $\mathrm{P}(0.8 < X \leq 1.2)$.

Solution: We need to consider the ranges of integration carefully;

$$F_X(x) = \begin{cases} \displaystyle\int_0^x t\, dt & = \frac{x^2}{2} & 0 \leq x \leq 1 \\ \displaystyle\int_0^1 t\, dt + \int_1^x (2 - t)\, dt & = 2x - \frac{x^2}{2} - 1 & 1 \leq x \leq 2 \end{cases}$$

and $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 2$. Hence $\mathrm{P}(0.8 < X < 1.2) = F_X(1.2) - F_X(0.8) = 0.36$.

Exercice 5- 4: The *median* of a continuous random variable X is that value x such that $F_X(x) = 1/2$. Find the median of X when

(a) X has an *Exponential* distribution with parameter $\lambda > 0$, that is

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

and zero otherwise.

(b) $\log X$ has a normal distribution with parameters μ and σ^2.

Solution:

(a) $X \sim \text{Exp}(\lambda) \Rightarrow F_X(x) = 1 - e^{-\lambda x}$ for $x > 0$, so $F_X(x) = 1/2 \Longrightarrow x = \frac{\log 2}{\lambda}$.

(b) $\log X \sim \text{N}(\mu, \sigma^2)$, so

$$F_X(x) = 1/2 \Longrightarrow \text{P}[\, X \leq x \,] = 1/2 \Longrightarrow \text{P}[\, (\log X - \mu)/\sigma \leq (\log x - \mu)/\sigma] = 1/2$$

$$\Longrightarrow \Phi((\log x - \mu)/\sigma) = 1/2 \Longrightarrow (\log x - \mu)/\sigma = 0 \Longrightarrow \log x = \mu \Longrightarrow x = e^{\mu}.$$

5.2 Prerequisites: Lecture 13

Exercice 5- 5: Suppose that X is a continuous random variable with range $\text{Im}X = [0, 1]$, and probability density function f_X specified by

$$f_X(x) = 2(1 - x), \quad 0 \leq x \leq 1,$$

and zero otherwise. Find the probability distributions of random variables Y_1, Y_2 and Y_3 defined respectively by

(a) $Y_1 = 2X - 1$,

(b) $Y_2 = 1 - 2X$,

(c) $Y_3 = X^2$,

that is, in each case, find the range and the density function.

Solution: We can derive the densities from first principles:

(a) $Y_1 = 2X - 1 \Longrightarrow \text{Im}Y_1 = [-1, 1]$. Also,

$$F_{Y_1}(y) = \text{P}(Y_1 \leq y) = \text{P}(2X - 1 \leq y) = \text{P}(\, X \leq (1 + y)/2) = F_X((1 + y)/2).$$

$$\Longrightarrow f_{Y_1}(y) = \frac{1}{2} f_X((1 + y)/2) = (1 - y)/2, \quad -1 \leq y \leq 1.$$

(b) $Y_2 = 1 - 2X \Longrightarrow \text{Im}Y_2 = [-1, 1]$. Also,

$$F_{Y_2}(y) = \text{P}[\, Y_2 \leq y \,] = \text{P}(1 - 2X \leq y) = \text{P}((X \geq (1 - y)/2) = 1 - F_X((1 - y)/2)$$

$$\Longrightarrow f_{Y_2}(y) = \frac{1}{2} f_X((1 - y)/2) = (1 + y)/2, \quad -1 \leq y \leq 1.$$

(c) $Y_3 = X^2 \Longrightarrow \text{Im}Y_3 = [0, 1]$. Also,

$$F_{Y_3}(y) = \text{P}(Y_3 \leq y) = \text{P}(X^2 \leq y) = \text{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

$$\Longrightarrow f_{Y_3}(y) = \frac{1}{2\sqrt{y}} \left[\, f_X(\, \sqrt{y}\,) + f_X(\, -\sqrt{y}\,) \,\right] = \frac{1}{2\sqrt{y}}\, f_X(\, \sqrt{y}\,) = (1 - \sqrt{y})/\sqrt{y}, \quad 0 \leq y \leq 1.$$

We could also use the general transformation formula for (a) and (b), as the transformations are 1-1.

(a) $g(t) = 2t - 1 \Longleftrightarrow g^{-1}(t) = (1+t)/2$

(b) $g(t) = 1 - 2t \Longleftrightarrow g^{-1}(t) = (1-t)/2$

Then use

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dt} \left\{g^{-1}(t)\right\}_{t=y} \right|.$$

Exercice 5- 6: The continuous random variable X has a Uniform distribution on the interval $[-1, 1]$. Find the probability density function of random variables

(a) $Y = |X|$,

(b) $Z = X^2$.

Solution: We know that $f_X(x) = 1/2, \ -1 \leq x \leq 1$ and zero otherwise. From first principles we get:

(a) $Y = |X| \Longrightarrow \mathrm{Im}Y = [0, 1]$ and

$$\begin{aligned} F_Y(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(|X| \leq y) = \mathrm{P}(-y \leq X \leq y) = F_X(y) - F_X(-y) \\ \Longrightarrow f_Y(y) = \ [\, f_X(y) + f_X(-y)\,] = 1, \quad 0 \leq y \leq 1. \end{aligned}$$

(b) $Z = X^2 \Longrightarrow \mathrm{Im}Z = [0, 1]$ and

$$\begin{aligned} F_Z(z) = \mathrm{P}(Z \leq z) = \mathrm{P}(X^2 \leq z) = \mathrm{P}(-\sqrt{z} \leq X \leq \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z}) \\ \Longrightarrow f_Z(z) = \frac{1}{2\sqrt{z}}\,[\, f_X(\,\sqrt{z}\,) + f_X(\,-\sqrt{z}\,)\,] = \frac{1}{2\sqrt{z}}, \quad 0 \leq z \leq 1. \end{aligned}$$

Exercice 5- 7: If X is **any** continuous random variable with distribution function F_X, show that

(a) Random variable $U = F_X(X)$ has a Uniform distribution on $[0, 1]$;

(b) Random variable $Y = -\log F_X(X)$ has an exponential distribution.

Solution: From first principles, if X has cdf F_X, then as X is continuous, F_X is 1-1 and monotone increasing.

(a) $U = F_X(X) \Longrightarrow \mathrm{Im}U = [0, 1]$ and

$$F_U(u) = \mathrm{P}(U \leq u) = \mathrm{P}(F_X(X) \leq u) = \mathrm{P}(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u, \quad 0 \leq u \leq 1,$$

which implies that $U \sim \mathrm{U}(0, 1)$.

(b) $Y = -\log F_X(X) = -\log U \Longrightarrow \mathrm{Im}Y = (0, \infty)$ and

$$F_Y(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(-\log U \leq y) = \mathrm{P}(U \geq e^{-y}) = 1 - F_U(e^{-y}) = 1 - e^{-y}, \quad y > 0,$$

which implies that $Y \sim \mathrm{Exp}(1)$.

Exercice 5- 8: Suppose that random variable X has a standard normal distribution.

(a) Find the cumulative distribution function (cdf) of $Y = X^2$ in terms of the standard normal c.d.f. Φ.

Hint: For the c.d.f. of Y, we have

$$\mathrm{P}(Y \leq y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(|X| \leq \sqrt{y}).$$

(b) Find the probability density function of Y, f_Y.

(c) Identify (by name) the probability distribution of Y.

Solution:

(a) $X \sim \mathrm{N}(0,1)$, and thus if Φ and ϕ are the standard normal c.d.f. and p.d.f. respectively. We have immediately that $Y = X^2 \Longrightarrow \mathrm{Im}Y = (0, \infty)$, and

$$F_Y(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}).$$

(b) By differentiation, we have

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[\phi(\sqrt{y}) + \phi(-\sqrt{y}) \right] = \left(\frac{1}{2\pi}\right)^{1/2} y^{-1/2} e^{-y/2}, \quad 0 \leq y \leq \infty.$$

(c) By inspection, we have that $Y \sim \mathrm{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \chi^2_1$.

5.3 Prerequisites: Lecture 14

Exercice 5- 9: The annual profit (in millions of pounds) of a manufacturing company is a function of product demand. If X is the continuous random variable corresponding to the demand in a given year, then the annual profit is also a continuous random variable, Y say, where

$$Y = 2(1 - e^{-2X})$$

If X has an Exponential distribution with parameter $\lambda = 6$, find the expected annual profit.

Solution: Using the general result for expectations of functions (LOTUS) with $g(X) = 2\left(1 - e^{-2X}\right)$

$$\begin{aligned}
\mathrm{E}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_0^{\infty} 2\left(1 - e^{-2x}\right) \lambda e^{-\lambda x}\, dx \\
&= 2\lambda \int_0^{\infty} \left(e^{-\lambda x} - e^{-(2+\lambda)x}\right) dx \\
&= 2\lambda \left[\frac{1}{\lambda} - \frac{1}{2+\lambda}\right] = \frac{4}{2+\lambda} = \frac{1}{2}.
\end{aligned}$$