Problem Sheet 7

Math40002, Analysis 1

1. Prove that if $f : [a, b] \to [0, \infty)$ is continuous and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int_a^b f(x)\, dx > 0$.

2. Suppose for some $f : [a, b] \to \mathbb{R}$ and integer $n \geq 1$ that the nth power f^n of f is integrable. Prove that if n is odd, then f is integrable. Why doesn't this work for n even, and can you find additional hypotheses on f that make it true in that case?

3. Let $C[a, b]$ denote the set of continuous functions $f : [a, b] \to \mathbb{R}$, and define a function $d : C[a, b] \times C[a, b] \to \mathbb{R}$ by

$$d(f, g) = \int_a^b |f(x) - g(x)|\; dx.$$

   (a) Prove that $d(f, g) = d(g, f)$ for all $f, g \in C[a, b]$.
   (b) Prove that $d(f, g) \geq 0$, with equality if and only if $f = g$.
   (c) Prove the triangle inequality $d(f, g) + d(g, h) \geq d(f, h)$.

   These properties say that d is a *metric*, which is a notion of distance on $C[a, b]$.

   (d) Prove that if $f_n \to f$ uniformly on $[a, b]$, then $\lim\limits_{n\to\infty} d(f_n, f) = 0$.

4. In problem sheet 5 we constructed a smooth (i.e., infinitely differentiable) function $f : \mathbb{R} \to [0, \infty)$ such that $f(x) > 0$ if and only if $x \in (0, 1)$.

   (a) Construct a smooth, monotone increasing function $g : \mathbb{R} \to [0, \infty)$ such that $g(x) = 0$ for all $x \leq 0$ and $g(x) = 1$ for all $x \geq 1$.
   (b) Given $a < b < c < d$, construct a smooth function $h : \mathbb{R} \to [0, \infty)$ satisfying

$$h(x) = 0 \text{ for all } x \notin [a, d], \qquad h(x) = 1 \text{ for all } x \in [b, c],$$

   and with h monotone increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

5. (a) Check that the derivative of $x \log(x) - x$ is $\log(x)$.
   (b) Use Darboux sums to prove for all integers $n \geq 1$ that

$$\log((n-1)!) \leq \int_1^n \log(x)\, dx \leq \log(n!).$$

(c) Evaluate the integral in (b) and deduce that

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1 + \frac{1}{n}\right) + \frac{\log(n+1)}{n}$$

for all $n \geq 1$.

(d) Conclude that $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Remark: this is a weak version of *Stirling's formula* $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

6. (*) Let $f : [N, \infty) \to [0, \infty)$ be a nonnegative, monotone decreasing function.

(a) Let $S_n = \sum\limits_{k=N}^{n} f(k)$ for all integers $n \geq N$. Use Darboux sums to prove that

$$S_n - f(N) \leq \int_N^n f(x)\,dx \leq S_{n-1}.$$

(b) Prove that the series $\sum\limits_{k=N}^{\infty} f(k)$ converges if and only if the limit

$$\int_N^\infty f(x)\,dx \overset{def}{=} \lim_{x\to\infty} \int_N^x f(t)\,dt$$

(called an *improper integral*) exists. This is the *integral test* for convergence.

(c) Prove that if the series $S = \sum\limits_{k=N}^{\infty} f(k)$ converges, so $I = \int_N^\infty f(x)\,dx$ exists, then $I \leq S \leq I + f(N)$.

7. Consider for any real s the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^s}$.

(a) Prove that this series is not convergent if $s \leq 0$.

(b) Use the integral test to prove that for $s > 0$, the series converges if and only if $s > 1$. If $s > 1$, show that $\frac{1}{s-1} < \zeta(s) < \frac{s}{s-1}$.

(c) Prove for any $a > 1$ that the series converges uniformly to a continuous function on $[a, \infty)$, and hence it defines a continuous function $\zeta : (1, \infty) \to \mathbb{R}$ called the *Riemann zeta function*. Can it be extended continuously to $[1, \infty)$?

(d) (Harder!) Prove that $\zeta(s)$ is continuously differentiable, and compute its derivative. It may help to first show that $\lim\limits_{x\to\infty} \frac{\log(x)}{x^\epsilon} = 0$ for any $\epsilon > 0$.