Imperial College London

Multidimensional Utility Maximisation in an Incomplete Market

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1.1 Problem Description

1.1.1 Primal Problem

We assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[t_0,T]}$ is the $\mathbb{P}$-augmentation of the natural filtration $\{F_t^W\}_{t\in[t_0,T]}$ generated by d-dimensional independent standard Brownian motions $\{(W_1(t),\dots,W_d(t))\}_{t\in[t_0,T]}$. Denote by $\mathcal{P}([t_0,T],\mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued progressively measurable processes on $[t_0,T]\times\Omega$, $\mathcal{H}([t_0,T],\mathbb{R}^n)$ the set of processes x in $\mathcal{P}([t_0,T],\mathbb{R}^n)$ such that $\mathbb{E}[\int_{t_0}^T |x(t)|^2 \mathrm{d}t] < \infty$ and $\mathcal{S}([t_0,T],\mathbb{R}^n)$ the set of processes x in $\mathcal{P}([t_0,T],\mathbb{R}^n)$ such that $\mathbb{E}[\sup_{t_0<t<T} |x(t)|^2 \mathrm{d}t] < \infty$.

In this section we define the set of admissible controls by

$$\mathcal{A} := \Big\{\pi(t) : \pi(t) \in \mathcal{H}([t_0,T],\mathbb{R}^m)\Big\}$$

Given any $\pi(t) \in \mathcal{A}$, consider an n–dimensional state process $X(t)$ satisfying the following SDE:

$$\begin{cases} \mathrm{d}X(t) &= \Big[(A(t)X(t) + B(t)\pi(t)\Big]\mathrm{d}t + \sum_{i=1}^d \Big[C_i(t)X(t) + D_i(t)\pi(t)\Big]\mathrm{d}W_i(t) \\ X(0) &= x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

where $A(t), C(t) \in \mathbb{R}^{n\times n}$, and $B_i(t), D_i(t) \in \mathbb{R}^{n\times m}$, for $i = \{1,\dots,d\}$ are $\mathbb{F}$–progressively measurable and uniformly bounded. The pair (X,π) is *admissible* if X is a solution to SDE (1) with control $\pi \in \mathcal{A}$. For simplicity, we denote

$$b(t, X(t), \pi(t)) := A(t)X(t) + B(t)\pi(t) \in \mathbb{R}^n$$

$$\sigma(t, X(t), \pi(t)) := \left[\begin{pmatrix} \\ C_1(t)X(t) + D_1(t)\pi(t) \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ C_d(t)X(t) + D_d(t)\pi(t) \\ \\ \end{pmatrix}\right] \in \mathbb{R}^{n\times d}$$

The system is then written as:

$$\begin{cases} \mathrm{d}X(t) &= b(t, X(t), \pi(t))\mathrm{d}t + \sigma(t, x(t), \pi(t))\mathrm{d}W(t) \\ X(0) &= x_0 \in \mathbb{R}^n \end{cases} \tag{2}$$

Consider the functional $J : \mathcal{A} \to \mathbb{R}$, defined by

$$J(\pi) := \mathbb{E}\left[\int_{t_0}^T f(t, X(t), \pi(t))\mathrm{d}t + g(X(T))\right], \tag{3}$$

where $f : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are defined by

$$f(t, X(t), \pi(t)) = \frac{1}{2}X^T(t)Q(t)X(t) + X^T(t)S^T(t)\pi(t) + \frac{1}{2}\pi^T(t)R(t)\pi(t) \tag{4}$$

$$g(X(T)) = \frac{1}{2}X^T(T)G(T)X(T) + X^T(T)L(T). \tag{5}$$

We assume that the processes $Q(t) \in \mathcal{P}([t_0,T],\mathbb{R}^{n\times n})$, $S(t) \in \mathcal{P}([t_0,T],\mathbb{R}^{m\times n})$, $R(t) \in \mathcal{P}([t_0,T],\mathbb{R}^{m\times m})$ are uniformly bounded, $G(T) \in \mathbb{R}^{n\times n}$, $L(T) \in \mathbb{R}^n$, Q, R and G are symmetric matrices and

$$\begin{pmatrix} Q(t) & S^T(t) \\ S(t) & R(t) \end{pmatrix} \geq 0.$$

Under these assumptions, we know that J is a convex functional of π.

We consider the following optimisation problem:

$$\text{Minimise } J(\pi) \text{ subject to } (X,\pi) \text{ admissible.} \tag{6}$$

An admissible control $\hat{\pi}$ is *optimal* if $J(\hat{\pi}) \leq J(\pi)$ for all $\pi \in \mathcal{A}$.

1.1.2 Dual Problem

We now derive the dual problem. The dual process $Y(t)$ satisfies the following SDE:

$$\begin{cases} dY(t) &= \Big[\alpha(t) - A(t)^T Y(t) - \sum_{i=1}^d C_i(t)^T \beta_i(t)\Big]dt + \sum_{i=1}^d \beta_i(t)dW_i(t) \\ Y(t_0) &= y, \end{cases} \tag{7}$$

where α, β_i and y are all to be determined. There is a unique solution to the SDE for given $(y,\alpha,\beta_1,\dots,\beta_d)$. We call $(\alpha,\beta_1,\dots,\beta_d)$ the *admissible dual control* and $(Y,\alpha,\beta_1,\dots,\beta_d)$ the *admissible dual pair*. Using Ito's lemma to $X(t)^T Y(t)$, we get

$$dX^TY = \Big[X^T\alpha + \pi^T\beta\Big]dt + \sum_{i=1}^d \Big(X^T\beta_i + Y^T(C_iX + D_i\pi)\Big)dW_i,$$

where

$$\beta = B^TY + \sum_{i=1}^d D_i^T\beta_i.$$

The process $X^T(t)Y(t) - \int_{t_0}^t [X^T(s)\alpha(s) + \pi^T(s)\beta(s)]ds$ is a local martingale and a supermartingale if it is bounded below by an integrable process, which gives

$$\mathbb{E}\left[X^T(T)Y(T) - \int_{t_0}^T \Big(X^T\alpha + \pi^T\beta\Big)ds\right] \leq x^Ty. \tag{8}$$

The optimisation problem (6) can be written equivalently as

$$\sup_\pi \mathbb{E}\left[-\int_{t_0}^T f(t,X(t),\pi(t))dt - g(X(T))\right].$$

Define the dual functions $\phi : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ by

$$\phi(t,\alpha,\beta) = \sup_{x,\pi}\Big\{x^T\alpha + \pi^T\beta - f(t,x,\pi)\Big\} \tag{9}$$

and $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(y) = \sup_x \Big\{ -x^T y - g(x) \Big\}. \tag{10}$$

Substituting f and g from (4) and (5), we can find the supremums by setting the derivatives to zero. We get

$$\phi(t,\alpha,\beta) = \sup_{x,\pi} \left\{ \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} \right\},$$

so setting the derivative to zero, we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0 \implies \begin{bmatrix} x^* \\ \pi^* \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Therefore

$$\begin{aligned} \pi^* &= \Big[S Q^{-1} S^T - R \Big]^{-1} (S Q^{-1} \alpha - \beta) \\ x^* &= Q^{-1}(\alpha - S^T \pi^*) \end{aligned}$$

Then ϕ is given by

$$\phi(t,\alpha,\beta) = \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Denoting

$$\begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1},$$

where

$$\tilde{Q} = Q^{-1} - Q^{-1} S^T (S Q^{-1} S^T - R)^{-1} S Q^{-1} \tag{11}$$

$$\tilde{R} = R^{-1} - R^{-1} S (S^T R^{-1} S - Q^{-1})^{-1} S^T R^{-1} \tag{12}$$

$$\tilde{S} = (S Q^{-1} S^T - R)^{-1} S Q^{-1} = R^{-1} S (S^T R^{-1} S - Q)^{-1} \tag{13}$$

we get

$$\phi(t,\alpha,\beta) = \frac{1}{2} \alpha^T \tilde{Q} \alpha + \alpha^T \tilde{S}^T \beta + \frac{1}{2} \beta^T \tilde{R} \beta \tag{14}$$

Similarly,

$$D_x \Big[-x^T y - \frac{1}{2} x^T G x - x^T L \Big] = -y - Gx - L \implies x^* = -G^{-1}(y + L).$$

Then $h(y)$ is given by

$$\begin{aligned} h(y) &= (y^T + L^T) G^{-1} y - \frac{1}{2} (y^T + L^T) G^{-1} (y + L) + (y^T + L^T) G^{-1} L \\ &= \frac{1}{2} \Big[y^T G^{-1} y + L^T G^{-1} y + y^T G^{-1} L + L^T G^{-1} L \Big] \\ &= \frac{1}{2} (y^T + L^T) G^{-1} (y + L) \end{aligned} \tag{15}$$

Combining (8), (9), (10), we get the following inequality:

$$\sup_{\pi} \mathbb{E}\left[-\int_{t_0}^{T} f(t,X,\pi)\mathrm{d}t - g(X(T))\right] \leq \inf_{y,\alpha,\beta_1,\ldots,\beta_d}\left[x^T y + \mathbb{E}\left[\int_{t_0}^{T} \phi(t,\alpha,\beta)\mathrm{d}t + h(Y(T))\right]\right]. \quad (16)$$

The dual control problem is defined by

$$\inf_{y,\alpha,\beta_1,\ldots,\beta_d}\left[x^T y + \mathbb{E}\left[\int_{t_0}^{T} \phi(t,\alpha,\beta)\mathrm{d}t + h(Y(T))\right]\right], \quad (17)$$

where Y satisfies (7). Problem (17) can be solved in two steps: first, for fixed y, solve a stochastic control problem

$$-\tilde{v}(t,y) = \inf_{\alpha,\beta_1,\ldots,\beta_d} \mathbb{E}\left[\int_{t_0}^{T} \phi(t,\alpha,\beta)\mathrm{d}t + h(Y(T))\right], \quad (18)$$

and second, solve a static optimisation problem

$$\inf_{y}(x^T y - \tilde{v}(t,y)).$$

1.2 Primal HJB Equation

1.2.1 Derivation of HJB equation

We transform the minimisation problem to maximisation by noting that

$$\inf_{\pi(t)\in\mathbb{R}^m} \mathbb{E}\left[\int_{t_0}^{T} f(t,X(t),\pi(t))\mathrm{d}t + g(X(T))\right] = -\sup_{\pi(t)\in\mathbb{R}^m} \mathbb{E}\left[\int_{t_0}^{T} -f(t,X(t),\pi(t))\mathrm{d}t - g(X(T))\right],$$

and denote the value function

$$v(t,X(t)) = \sup_{\pi(t)\in\mathbb{R}^m} \mathbb{E}\left[\int_{t_0}^{T} -f(t,X(t),\pi(t))\mathrm{d}t - g(X(T))\right] \tag{19}$$

Note that the optimal value of the optimisation problem is given by $-v(t,X(t))$. Consider the time interval $(t,t+h)$ and a constant control $\pi(t)=\pi$. According to the Dynamic programming principle,

$$v(t,X(t)) \geq \mathbb{E}\left[\int_{t}^{t+h} -f(s,X(s),\pi)\mathrm{d}s + v(t+h,X(t+h))\right], \tag{20}$$

where we denote $X(s)$ to be the solution of (2) given that we know the value of X at time t.

Applying Ito's formula between t and $t+h$ we get

$$\begin{aligned} v(t+h,X(t+h)) = v(t,X(t)) &+ \int_{t}^{t+h}\left(\frac{\partial v(s,X(s))}{\partial t} + \mathcal{L}^{\pi}[v(s,X(s)]\right)\mathrm{d}s \\ &+ \underbrace{\int_{t}^{t+h}\Big[D_x v(s,X(s))\Big]^T \sigma(s,X(s),\pi)\mathrm{d}W(s)}_{\text{(local) martingale}}, \end{aligned}$$

where $\mathcal{L}^{\pi}[v(t,x)]$ is the generator given by

$$\mathcal{L}^{\pi}[v(t,X(t))] = b^T(t,X(t),\pi)D_x v(t,X(t)) + \frac{1}{2}\mathrm{tr}\Big[\sigma(t,X(t),\pi)\sigma^T(t,X(t),\pi)D_x^2 v(t,X(t))\Big] \tag{21}$$

Substituting into equation (20), we get

$$0 \geq \mathbb{E}\left[\int_{t}^{t+h} \frac{\partial v}{\partial t}(s,X(s)) + \mathcal{L}^{\pi}[v(s,X(s))] - f(s,X(s),\pi)\mathrm{d}s\right]$$

Dividing by h and sending h to 0, this yields by the mean value theorem

$$0 \geq \frac{\partial v}{\partial t}(t,x) + \mathcal{L}^{\pi}[v(t,x)] - f(t,x,\pi).$$

Since this is true for any admissible π, we obtain the inequality

$$\frac{\partial v}{\partial t}(t,x) + \sup_{\pi\in\mathbb{R}^m}\left[\mathcal{L}^{\pi}v(t,x) - f(t,x,\pi)\right] \leq 0. \tag{22}$$

On the other hand, suppose that π^* is an optimal control. Then by the dynamic programming principle,

$$v(t,x) = \mathbb{E}\left[\int_t^{t+h} -f(s, X^*(s), \pi^*(s))\mathrm{d}s + v(t+h, X^*(t+h))\right], \tag{23}$$

where X^* is the solution to the initial SDE (2) with control π^* starting from x at time t. By similar reasoning, we get

$$\frac{\partial v}{\partial t}(t,x) + \mathcal{L}^{\pi^*}[v(t,x)] - f(t,x,\pi^*) = 0$$

which combined with (22) suggests that v should satisfy

$$\frac{\partial v}{\partial t}(t,x) + \sup_{\pi\in\mathbb{R}^m}[\mathcal{L}^{\pi(t)}[v(t,x)] - f(t,x,\pi)] = 0, \quad \forall (t,x) \in [t_0, T) \times \mathbb{R}^n. \tag{24}$$

with the terminal condition:

$$v(T,x) = -g(x) = -\frac{1}{2}x^T G(T)x - x^T L(T), \quad \forall x \in \mathbb{R}^n.$$

Equation (24) is called the Hamilton-Jacobi-Bellman equation.

1.2.2 Finding the Optimal Control

The supremum in the HJB equation (24) can be found by setting the derivative with respect to π to zero. The derivative of the generator $\mathcal{L}^\pi$, $D_\pi[\mathcal{L}^\pi] \in \mathbb{R}^m$, is given by:

$$D_\pi\Big[\mathcal{L}^\pi[v(t,x)]\Big] = D_\pi\Big[b(t,x,\pi)^T D_x[v(t,x)]\Big] + D_\pi\left[\frac{1}{2}\mathrm{tr}(\sigma(t,x,\pi)\sigma^T(t,x,\pi)D_x^2[v(t,x)])\right]. \tag{25}$$

We have that

$$\begin{aligned} D_\pi\Big[b^T(t,x,\pi)D_x[v(t,x)]\Big] &= D_\pi\Big[(x^T A^T(t) + \pi^T(t)B^T(t))D_x[v(t,x)]\Big] \\ &= B^T(t)D_x[v(t,x)] \end{aligned} \tag{26}$$

The latter derivative in (25) is given by:

$$\begin{aligned} D_\pi\left[\frac{1}{2}\mathrm{tr}[\sigma(t,x,\pi)\sigma^T(t,x,\pi)D_x^2[v]]\right] &= \frac{1}{2}D_\pi\left[\mathrm{tr}\left[\sum_{i=1}^d (C_i x + D_i\pi)(C_i x + D_i\pi)^T D_x^2[v]\right]\right] \\ &= \frac{1}{2}\sum_{i=1}^d D_\pi\Big[\mathrm{tr}[(C_i x + D_i\pi)(C_i x + D_i\pi)^T D_x^2[v]]\Big] \\ &= \frac{1}{2}\sum_{i=1}^d D_\pi\Big[(C_i x + D_i\pi)^T D_x^2[v](C_i x + D_i\pi)\Big] \\ &= \sum_{i=1}^d D_i^T D_x^2[v(t,x)](C_i x + D_i\pi) \end{aligned} \tag{27}$$

The derivative of $f(t,x,\pi)$ with respect to π is

$$D_\pi f(t,x,\pi) = Sx + R\pi \tag{28}$$

Combining the three equations,(26), (27), (28), we get that

$$D_\pi[\mathcal{L}^\pi(t)[v(t,x)] - f(t,x,\pi)] = B^T D_x[v(t,x)] + \sum_{i=1}^{d} D_i^T D_x^2[v(t,x)](C_i x + D_i\pi) - Sx - R\pi$$

Setting this to zero, we get

$$\pi^* = \left[\sum_{i=1}^{d} D_i^T D_x^2[v(t,x)]D_i - R\right]^{-1}\left[Sx - B^T D_x[v(t,x)] - \sum_{i=1}^{d} D_i^T D_x^2[v(t,x)]C_i x\right] \tag{29}$$

We now substitute (29) into (24) to get:

$$\frac{\partial v}{\partial t} + b(t,x,\pi^*)^T D_x[v] + \frac{1}{2}\mathrm{tr}\Big[\sigma(t,x,\pi^*)\sigma^T(t,x,\pi^*)D_x^2[v]\Big] - \frac{1}{2}x^T Qx - \frac{1}{2}x^T S^T\pi^* - \frac{1}{2}\pi^{*T}Sx - \frac{1}{2}\pi^{*T}R\pi^* = 0$$

As $D_x^2[v]$ is a symmetric matrix, we can write

$$\begin{aligned}
\mathrm{tr}\Big[\sigma(t,x,\pi^*)\sigma^T(t,x,\pi^*)D_x^2[v]\Big] &= \sum_{i=1}^{d}\mathrm{tr}\Big[(C_i x + D_i\pi^*)(C_i x + D_i\pi^*)^T D_x^2[v]\Big] \\
&= \sum_{i=1}^{d}(C_i x + D_i\pi^*)^T D_x^2[v](C_i x + D_i\pi^*),
\end{aligned}$$

we get the HJB equation

$$\begin{aligned}
\frac{\partial v}{\partial t} + (Ax + B\pi^*)^T D_x[v] + \frac{1}{2}\sum_{i=1}^{d}(C_i x + D_i\pi^*)^T D_x^2[v](C_i x + D_i\pi^*) \\
- \frac{1}{2}x^T Qx - \frac{1}{2}x^T S^T\pi^* - \frac{1}{2}\pi^{*T}Sx - \frac{1}{2}\pi^{*T}R\pi^* = 0
\end{aligned} \tag{30}$$

where π^* is as in (29) and the terminal condition is given by

$$v(T,x) = -g(x) = -\frac{1}{2}x^T G(T)x - x^T L(T).$$

1.2.3 Solving the Primal HJB Equation

We assume that $v(t,x)$ is a quadratic function in x and we use the ansatz

$$v(t,x) = \frac{1}{2}x^T P(t)x + x^T M(t) + N(t), \tag{31}$$

with terminal conditions:

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0. \tag{32}$$

Then

$$\begin{aligned}
&\frac{\partial v}{\partial t}(t,x) = \frac{1}{2}x^T \dot{P}(t)x + x^T \dot{M}(t) + \dot{N}(t)\\
&D_x[v(t,x)] = P(t)x + M(t)\\
&D_x^2[v(t,x)] = P(t).
\end{aligned}$$

Substituting in (29) we get the optimal control

$$\pi^* = \left[\sum_{i=1}^d D_i^T P D_i - R\right]^{-1}\left[Sx - B^T Px - B^T M - \sum_{i=1}^d D_i^T P C_i x\right] \tag{33}$$

We can write this as

$$\pi^* = \vartheta_1 x + \kappa_1,$$

where

$$\vartheta_1 = \left(\sum_{i=1}^d D_i^T P D_i - R\right)^{-1}\left(S - B^T P - \sum_{i=1}^d D_i^T P C_i\right), \quad \kappa_1 = -\left(\sum_{i=1}^d D_i^T P D_i + R\right)^{-1} B^T M \tag{34}$$

Substituting this into (30), we get

$$\begin{aligned}
&\frac{\partial v}{\partial t} + (Ax + B(\vartheta_1 x + \kappa_1))^T D_x[v] + \frac{1}{2}\sum_{i=1}^d (C_i x + D_i(\vartheta_1 x + \kappa_1))^T D_x^2[v](C_i x + D_i(\vartheta_1 x + \kappa_1))\\
&-\frac{1}{2}x^T Qx - \frac{1}{2}x^T S^T(\vartheta_1 x + \kappa_1) - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T Sx - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T R(\vartheta_1 x + \kappa_1) = 0 \implies\\
&\frac{1}{2}x^T \dot{P}x + x^T \dot{M} + \dot{N} + (x^T A^T + x^T \vartheta_1^T B^T + \kappa_1^T B^T)(Px + M)\\
&+\frac{1}{2}\sum_{i=1}^d (x^T C_i^T + x^T \vartheta_1^T D_i^T + \kappa_1^T D_i^T)P(C_i x + D_i \vartheta_1 x + D_i \kappa_1)\\
&-\frac{1}{2}x^T Qx - \frac{1}{2}x^T S^T(\vartheta_1 x + \kappa_1) - \frac{1}{2}(x^T \vartheta_1^T + \kappa_1^T)Sx - \frac{1}{2}(x^T \vartheta_1^T + \kappa_1^T)R(\vartheta_1 x + \kappa_1) = 0
\end{aligned}$$

Rewriting this, we get

$$\begin{aligned}
&x^T\Bigg[\frac{1}{2}\dot{P} + \frac{1}{2}A^T P + \frac{1}{2}PA + \frac{1}{2}\vartheta_1^T B^T P + \frac{1}{2}PB\vartheta_1 + \frac{1}{2}\sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T)P(C_i + D_i\vartheta_1)\\
&-\frac{1}{2}Q - \frac{1}{2}\vartheta_1^T S - \frac{1}{2}S^T\vartheta_1 - \frac{1}{2}\vartheta_1^T R\vartheta_1\Bigg]x + x^T\Bigg[\dot{M} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M +\\
&\sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T)PD_i\kappa_1 - S^T\kappa_1 - \vartheta_1^T R\kappa_1\Bigg] + \dot{N} + \kappa_1^T B^T M + \frac{1}{2}\sum_{i=1}^d \kappa_1^T D_i^T P D_i \kappa_1 - \frac{1}{2}\kappa_1^T R\kappa_1 = 0
\end{aligned}$$

This equation must equal zero for all x, hence the coefficients in front of the quadratic term, as well as x and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\frac{1}{2}\dot{P}+\frac{1}{2}A^TP+\frac{1}{2}PA+\frac{1}{2}\vartheta_1^TB^TP+\frac{1}{2}PB\vartheta_1+\frac{1}{2}\sum_{i=1}^{d}(C_i^T+\vartheta_1^TD_i^T)P(C_i+D_i\vartheta_1)$$
$$-\frac{1}{2}Q-\frac{1}{2}\vartheta_1^TS-\frac{1}{2}S^T\vartheta_1-\frac{1}{2}\vartheta_1^TR\vartheta_1=0 \tag{35}$$

$$\dot{M}+A^TM+PB\kappa_1+\vartheta_1^TB^TM+\sum_{i=1}^{d}(C_i^T+\vartheta_1^TD_i^T)PD_i\kappa_1-S^T\kappa_1-\vartheta_1^TR\kappa_1=0 \tag{36}$$

$$\dot{N}+\kappa_1^TB^TM+\frac{1}{2}\sum_{i=1}^{d}\kappa_1^TD_i^TPD_i\kappa_1-\frac{1}{2}\kappa_1^TR\kappa_1=0, \tag{37}$$

where ϑ_1 and κ_1 are as in (34) and the terminal conditions are as in (32).

1.3 Primal BSDE

1.3.1 Solution via the Primal BSDE

We define the Hamiltonian $\mathcal{H} : \Omega \times [t_0, T] \times \mathbb{R}^n \times K \times R^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{H}(t,x,\pi,p,q) &= b^T p + \mathrm{tr}(\sigma^T q) - f(t,x,\pi)\\ &= x^T A^T p + \pi^T B^T p + \sum_{i=1}^{d}\left(x^T C_i^T q_i + \pi^T D_i^T q_i\right) - \frac{1}{2}x^T Q x - x^T S^T \pi - \frac{1}{2}\pi^T R\pi \end{aligned} \tag{38}$$

where we denote by $q_i \in \mathbb{R}^n$ the i^{th} column of the matrix $q \in \mathbb{R}^{n\times d}$. The adjoint process is given by

$$\begin{cases} \mathrm{d}p &= -D_x[\mathcal{H}(t,X(t),\pi(t),p(t),q(t))]\mathrm{d}t + \sum_{i=1}^{d} q_i(t)\mathrm{d}W_i(t) \\ p(T) &= -D_x[g(X(T))] = -G(T)X(T) - L(T) \end{cases} \tag{39}$$

According to the Stochastic Maximum Principle, the optimal control for the optimisation problem (6) satisfies the condition

$$D_\pi \mathcal{H}(t,X(t),\pi(t),p(t),q(t)) = 0.$$

We have

$$D_\pi \mathcal{H}(t,X(t),\pi(t),p(t),q(t)) = B^T p + \sum_{i=1}^{d} D_i^T q_i - SX - R\pi$$

so

$$B^T p + \sum_{i=1}^{d} D_i^T q_i - SX - R\pi = 0 \tag{40}$$

Solving this, we can find a linear solution for the control, which we denote by

$$\pi = \vartheta_2 X + \kappa_2. \tag{41}$$

Substituting the control in the Hamiltonian (38) we get

$$\begin{aligned}\mathcal{H} = X^T A^T p + (\vartheta_2 X + \kappa_2)^T B^T p + \sum_{i=1}^{d}\left(X^T C_i^T q_i + (\vartheta_2 X + \kappa_2)^T D_i^T q_i\right) - \frac{1}{2}X^T Q X \\ -\frac{1}{2}X^T S^T(\vartheta_2 X + \kappa_2) - \frac{1}{2}(\vartheta_2 X + \kappa_2)^T S X - \frac{1}{2}(\vartheta_2 X + \kappa_2)^T R(\vartheta_2 X + \kappa_2)\end{aligned} \tag{42}$$

The derivative of the Hamiltonian is then

$$D_x[\mathcal{H}] = A^T p + \vartheta_2^T B^T p + \sum_{i=1}^{d} C_i^T q_i + \sum_{i=1}^{d} \vartheta_2^T D_i^T q_i - QX - 2S^T\vartheta_2 X - S^T\kappa_2 - \vartheta_2^T R\vartheta_2 X - \vartheta_2^T R\kappa_2 \tag{43}$$

We try an ansatz for p of the form

$$p = \varphi(t)X(t) + \psi(t),$$

where $\varphi(t) \in \mathbb{R}^{n\times n}$ and $\psi(t) \in R^n$. Applying Ito's formula, we get

$$
\begin{aligned}
\mathrm{d}p &= \frac{\partial f}{\partial t}\mathrm{d}t + (D_x[f])^T \mathrm{d}X + \frac{1}{2}(\mathrm{d}X)^T D_x^2[f]\mathrm{d}X \\
&= (\dot{\varphi}X + \dot{\psi})\mathrm{d}t + \varphi \mathrm{d}X \\
&= (\dot{\varphi}X + \dot{\psi} + \varphi b(t,X,\pi))\mathrm{d}t + \varphi\sigma(t,X,\pi)\mathrm{d}W \\
&= \Big[\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B\pi\Big]\mathrm{d}t + \varphi\sum_{i=1}^{d}(C_iX + D_i\pi)\mathrm{d}W_i \\
&= \Bigg[\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B\vartheta_2 X + \varphi B\kappa_2\Bigg]\mathrm{d}t + \sum_{i=1}^{d}\varphi(C_iX + D_i\vartheta_2X + D_i\kappa_2)\mathrm{d}W_i
\end{aligned}
\tag{44}
$$

Equating the coefficients of (44) with (39), we get the system

$$
\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B\vartheta_2 X + \varphi B\kappa_2 = -D_x[\mathcal{H}(t,X(t),\pi(t),p(t),q(t))] \tag{45}
$$
$$
\varphi(C_iX + D_i\vartheta_2X + D_i\kappa_2) = q_i \quad i \in \{1,\dots,d\} \tag{46}
$$
$$
B^T\varphi X + B^T\psi + \sum_{i=1}^{d} D_i^T q_i - SX - R(\vartheta_1 X + \kappa_1) = 0, \tag{47}
$$

where the third equation is the Hamiltonian condition (40). We now substitute q_i from the second equation into (43) and (47), so that our system becomes

$$
\begin{aligned}
\dot{\varphi}(t)X(t) + \dot{\psi}(t) + \varphi(t)AX(t) + \varphi B\vartheta_2X + \varphi B\kappa_2 &= -A^T\varphi X - A^T\psi - \vartheta_2^T B^T\varphi X - \vartheta_2^T B^T\psi \\
&\quad - \sum_{i=1}^{d} C_i^T\varphi(C_iX + D_i\vartheta_2X + D_i\kappa_2) - \sum_{i=1}^{d}\vartheta_2^T D_i^T\varphi(C_iX + D_i\vartheta_2X + D_i\kappa_2) \\
&\quad + QX + 2S^T\vartheta_2X + S^T\kappa_2 + \vartheta_2^T R\vartheta_2X + \vartheta_2^T R\kappa_2
\end{aligned}
\tag{48}
$$
$$
B^T\varphi X + B^T\psi + \sum_{i=1}^{d} D_i^T\Big[\varphi(C_iX + D_i\vartheta_2X + D_i\kappa_2)\Big] - SX - R\vartheta_2x - R\kappa_2 = 0 \tag{49}
$$

From (49), we get

$$
\pi^* = \vartheta_2X + \kappa_2 = \Bigg[\sum_{i=1}^{d} D_i^T\varphi D_i - R\Bigg]^{-1}\Bigg[SX - B^T\varphi X - B^T\psi - \sum_{i=1}^{d} D_i^T\varphi C_iX\Bigg], \tag{50}
$$

i.e.,

$$
\vartheta_2 = \Bigg[\sum_{i=1}^{d} D_i^T\varphi D_i - 2R\Bigg]^{-1}\Bigg[S - B^T\varphi - \sum_{i=1}^{d} D_i^T\varphi C_i\Bigg], \quad \kappa_2 = -\Bigg[\sum_{i=1}^{d} D_i^T\varphi D_i - 2R\Bigg]^{-1} B^T\psi. \tag{51}
$$

We rewrite equation (48) as

$$
\Bigg[\dot{\varphi} + \varphi A + A^T\varphi + \varphi B\vartheta_2 + \vartheta_2^T B^T\varphi + \sum_{i=1}^{d}(C_i^T + \vartheta_2^T D_i^T)\varphi(C_i + D_i\vartheta_2) - Q - S^T\vartheta_2 - \vartheta_2^T S - \vartheta_2^T R\vartheta_2\Bigg]X
$$

$$+\Big[\dot{\psi}+\varphi B\kappa_2+A^T\psi+\vartheta_2^T B^T\psi+\sum_{i=1}^{d}C_i^T\varphi D_i\kappa_2+\sum_{i=1}^{d}\vartheta_2^T D_i^T\varphi D_i\kappa_2-S^T\kappa_2-\vartheta_2^T R\kappa_2\Big]=0$$

Since this must be true for all X, the coefficient in front of X must be equal to zero, so we get the system of ODEs

$$\dot{\varphi}+\varphi A+A^T\varphi+\varphi B\vartheta_2+\vartheta_2^T B^T\varphi+\sum_{i=1}^{d}(C_i^T+\vartheta_2^T D_i^T)\varphi(C_i+D_i\vartheta_2)-Q-S^T\vartheta_2-\vartheta_2^T S-\vartheta_2^T R\vartheta_2=0 \tag{52}$$

$$\dot{\psi}+\varphi B\kappa_2+A^T\psi+\vartheta_2^T B^T\psi+\sum_{i=1}^{d}C_i^T\varphi D_i\kappa_2+\sum_{i=1}^{d}\vartheta_2^T D_i^T\varphi D_i\kappa_2-S^T\kappa_2-\vartheta_2^T R\kappa_2=0, \tag{53}$$

where ϑ_2 and κ_2 are as in (51) and the terminal conditions are given by

$$\varphi(T)=-G(T),\quad \psi(T)=-L(T). \tag{54}$$

1.3.2 Equivalence of Primal HJB and Primal BSDE

From the Primal HJB, the optimal control is given by (33):

$$\pi^*=\left[\sum_{i=1}^{d}D_i^T P D_i-R\right]^{-1}\left[Sx-B^T Px-B^T M-\sum_{i=1}^{d}D_i^T PC_i x\right] \tag{55}$$

and from the Primal BSDE the optimal control is (50):

$$\pi^*=\left[\sum_{i=1}^{d}D_i^T\varphi D_i-R\right]^{-1}\left[SX-B^T\varphi X-B^T\psi-\sum_{i=1}^{d}D_i^T\varphi C_i X\right], \tag{56}$$

Comparing, we see that the equations are identical and we get the relation

$$\varphi=P,\quad \psi=M.$$

The ODE from the Primal BSDE for φ is (52). Substituting $\varphi=P$ and $\vartheta_2=\vartheta_1$ we get

$$\dot{P}+PA+A^T P+PB\vartheta_1+\vartheta_1^T B^T P+\sum_{i=1}^{d}(C_i^T+\vartheta_1^T D_i^T)P(C_i+D_i\vartheta_1)-Q-2S^T\vartheta_1-\vartheta_1^T R\vartheta_1=0,$$

which is equal to twice the ODE for P from the primal HJB (35), i.e.

$$\frac{1}{2}\left[\dot{P}+PA+A^T P+PB\vartheta_1+\vartheta_1^T B^T P+\sum_{i=1}^{d}(C_i^T+\vartheta_1^T D_i^T)P(C_i+D_i\vartheta_1)-Q-2S^T\vartheta_1-\vartheta_1^T R\vartheta_1\right]=0.$$

Similarly, substituting $\varphi=P,\psi=M$, $\vartheta_2=\vartheta_1$ and $\kappa_2=\kappa_1$ into (53) we get

$$\dot{M}+PB\kappa_1+A^T M+\vartheta_1^T B^T M+\sum_{i=1}^{d}C_i^T PD_i\kappa_1+\sum_{i=1}^{d}\vartheta_1^T D_i^T PD_i\kappa_1-S^T\kappa_1-\vartheta_1^T R\kappa_1=0$$

which is the same equation as (36):

$$\dot{M} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M + \sum_{i=1}^{d}(C_i^T + \vartheta_1^T D_i^T)PD_i\kappa_1 - S^T\kappa_1 - \vartheta_1^T R\kappa_1 = 0$$

The terminal conditions for the primal BSDE are given by (54):

$$\varphi(T) = -G(T), \quad \psi(T) = -L(T),$$

and from the primal HJB (32):

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0.$$

So the ODEs for solving P, M and φ, ψ are identical, hence we have equivalence between the two methods.

1.4 Dual HJB Equation

1.4.1 The Dual HJB

Recall that the dual process Y satisfies (7):

$$\begin{cases} \mathrm{d}Y(t) & = \Big[\alpha(t) - A(t)^T Y(t) - \sum_{i=1}^d C_i(t)^T \beta_i(t)\Big]\mathrm{d}t + \sum_{i=1}^d \beta_i(t)\mathrm{d}W_i(t) \\ Y(t_0) & = y \end{cases}$$

and also recall from (18) that the dual value function is given by

$$\tilde{v}(t, Y(t)) = \sup_{\alpha,\beta_1,\dots,\beta_d} \mathbb{E}\Bigg[-\int_{t_0}^T \phi(t,\alpha,\beta)\mathrm{d}t - h(Y(T))\Bigg],$$

where $\phi(t,\alpha,\beta)$ and $h(Y(T))$ are given in (14) and (15). Then, following the same steps as for the primal problem, the dual HJB equation is given by

$$\frac{\partial \tilde{v}}{\partial t}(t,y) + \sup_{\alpha,\beta_1,\dots,\beta_d}\Big[\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] - \phi(t,\alpha,\beta)\Big] = 0,$$

where the generator is given by

$$\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] = \left(\alpha^T - y^T A - \sum_{i=1}^d \beta_i^T C_i\right) D_y[\tilde{v}] + \frac{1}{2}\sum_{i=1}^d \beta_i^T D_y^2[\tilde{v}]\beta_i,$$

and the terminal condition is

$$\tilde{v}(T,y) = -h(y) = -\frac{1}{2}(y^T + L^T)G^{-1}(y+L).$$

1.4.2 Finding the Optimal Controls

To find the supremum, we set the derivatives with respect to $\alpha, \beta_1, \dots, \beta_d$ to zero. We have

$$D_\alpha\Big[\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] - \phi\Big] = D_y[\tilde{v}] - \tilde{Q}\alpha - \tilde{S}^T\left(B^T y + \sum_{i=1}^d D_i^T \beta_i\right) = 0 \tag{57}$$

$$D_{\beta_i}\Big[\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] - \phi\Big] = -C_i D_y[\tilde{v}] + D_y^2[\tilde{v}]\beta_i - D_i\left(\tilde{S}\alpha + \tilde{R}\left(B^T y + \sum_{i=1}^d D_i^T \beta_i\right)\right) = 0 \tag{58}$$

This is a system of $d+1$ equations in $d+1$ unknowns, so it can be solved and the optimal controls are linear functions of y, which we denote by α^* and β_i^*. The HJB equation then becomes

$$\frac{\partial \tilde{v}}{\partial t} + \left(\alpha^{*T} - Y^T A - \sum_{i=1}^d \beta_i^{*T} C_i\right) D_y[\tilde{v}] + \frac{1}{2}\sum_{i=1}^d \beta_i^{*T} D_y^2[\tilde{v}]\beta_i^* - \phi\left(t, \alpha^*, B^T y + \sum_{i=1}^d D_i^T \beta_i^*\right) = 0 \tag{59}$$

1.4.3 Solving the Dual HJB

Suppose that $\tilde{v}$ is a quadratic function in y and use the ansatz

$$\tilde{v}(t,y) = \frac{1}{2}y^T\tilde{P}(t)y + y^T\tilde{M}(t) + \tilde{N}(t), \tag{60}$$

with terminal conditions

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = \frac{1}{2}L^T(T)G^{-1}(T)L(T). \tag{61}$$

Then

$$\begin{aligned}
&\frac{\partial\tilde{v}}{\partial t}(t,y) = \frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t)y + y^T\dot{\tilde{M}}(t) + \dot{\tilde{N}}(t)\\
&D_y[v(\tilde{t},y)] = \tilde{P}(t)y + \tilde{M}(t)\\
&D_y^2[v(\tilde{t},y)] = \tilde{P}(t)
\end{aligned}$$

The system of equations (57) and (58) from which we derive the optimal controls α^* and β_i^* is now given by

$$\begin{cases}
\tilde{P}y + \tilde{M} - \tilde{Q}\alpha - \tilde{S}^T\left(B^TY + \sum_{i=1}^d D_i^T\beta_i\right) = 0\\
C_i(\tilde{P}y + \tilde{M}) - \tilde{P}\beta_i + D_i\left(\tilde{S}\alpha + \tilde{R}\left(B^TY + \sum_{i=1}^d D_i^T\beta_i\right)\right) = 0
\end{cases} \tag{62}$$

We do not solve this system explicitly, however, the solutions for α^* and β_i^* are linear in y, hence we simply denote by $\tilde{\vartheta}$ and $\tilde{\kappa}$ the coefficients before y and the free coefficient in α^* and similarly for β_i, i.e.

$$\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}, \quad \beta_i = \tilde{\vartheta}_iy + \tilde{\kappa}_i. \tag{63}$$

Substituting this into the HJB equation (59) we get

$$\begin{aligned}
&\frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}y + y^T\frac{\tilde{M}}{\mathrm{d}t} + \frac{\tilde{N}}{\mathrm{d}t} + \left(y^T\tilde{\vartheta}^T + \tilde{\kappa}^T - y^TA - \sum_{i=1}^d(y^T\tilde{\vartheta}_i^T + \tilde{\kappa}_i^T)C_i\right)(\tilde{P}y + \tilde{M})\\
&+\frac{1}{2}\sum_{i=1}^d(y^T\tilde{\vartheta}_i^T + \tilde{\kappa}_i^T)\tilde{P}(\tilde{\vartheta}_iy + \tilde{\kappa}_i) - \phi\left(t, \tilde{\vartheta}y + \tilde{\kappa}, B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_iy + \tilde{\kappa}_i)\right) = 0
\end{aligned}$$

Expanding $\phi\left(t, \tilde{\vartheta}y + \tilde{\kappa}, B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_iy + \tilde{\kappa}_i)\right)$ we get

$$\begin{aligned}
\phi(t,\alpha^*,\beta^*) = &\frac{1}{2}(\tilde{\vartheta}y + \tilde{\kappa})^T\tilde{Q}(\tilde{\vartheta}y + \tilde{\kappa}) + (\tilde{\vartheta}y + \tilde{\kappa})^T\tilde{S}^T\left(B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_iy + \tilde{\kappa}_i)\right)\\
&+\frac{1}{2}\left(B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_iy + \tilde{\kappa}_i)\right)^T\tilde{R}(B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_iy + \tilde{\kappa}_i))
\end{aligned}$$

Rearranging, we get

$$
\begin{aligned}
&\frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}y + y^T\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \frac{\mathrm{d}\tilde{N}}{\mathrm{d}t} + y^T\tilde{\vartheta}^T\tilde{P}y + y^T\tilde{P}\tilde{\kappa} - y^TA\tilde{P}y - y^T\sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{P}y - y^T\tilde{P}\sum_{i=1}^dC_i^T\tilde{\kappa}_i\\
&\quad + y^T\tilde{\vartheta}^T\tilde{M} + \tilde{\kappa}^T\tilde{M} - y^TA\tilde{M} - \sum_{i=1}^d y^T\tilde{\vartheta}_i^TC_i\tilde{M} - \sum_{i=1}^d\tilde{\kappa}_i^TC_i\tilde{M} + \frac{1}{2}y^T\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_iy\\
&\quad + y^T\sum_{i=1}^d\tilde{\vartheta}_i\tilde{P}\tilde{\kappa}_i + \frac{1}{2}\sum_{i=1}^d\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i - \frac{1}{2}y^T\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}y - y^T\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa} - y^T\tilde{\vartheta}^T\tilde{S}^T(B^T + \sum_{i=1}^dD_i^T\tilde{\vartheta}_i)y\\
&\quad - y^T\tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i - y^T(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i)\tilde{S}\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i\\
&\quad - \frac{1}{2}y^T\Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^dD_i^T\tilde{\vartheta}_i\Big)y - y^T\Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\sum_{i=1}^dD_i^T\tilde{\kappa}_i\\
&\quad - \frac{1}{2}\Big(\sum_{i=1}^d\tilde{\kappa}_i^TD_i\Big)\tilde{R}\Big(\sum_{i=1}^dD_i^T\tilde{\kappa}_i\Big) = 0
\end{aligned}
$$

Grouping together the coefficients in front of y we get:

$$
\begin{aligned}
&y^T\Bigg[\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{\vartheta}^T\tilde{P} - A\tilde{P} - \sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{P} + \frac{1}{2}\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} - \tilde{\vartheta}^T\tilde{S}^T\Big(B^T + \sum_{i=1}^dD_i^T\tilde{\vartheta}_i\Big)\\
&\quad - \frac{1}{2}\Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^dD_i^T\tilde{\vartheta}_i\Big)\Bigg]y\\
&+ y^T\Bigg[\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \tilde{P}\tilde{\kappa} - \tilde{P}\sum_{i=1}^dC_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{M} - A\tilde{M} - \sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{M} + \sum_{i=1}^d\tilde{\vartheta}_i\tilde{P}\tilde{\kappa}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i\\
&\quad - \Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{S}\tilde{\kappa} - \Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\sum_{i=1}^dD_i^T\tilde{\kappa}_i\Bigg]\\
&+ \frac{\mathrm{d}\tilde{N}}{\mathrm{d}t} + \tilde{\kappa}^T\tilde{M} - \sum_{i=1}^d\tilde{\kappa}_i^TC_i\tilde{M} + \frac{1}{2}\sum_{i=1}^d\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i - \frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i\\
&\quad - \frac{1}{2}\Big(\sum_{i=1}^d\tilde{\kappa}_i^TD_i\Big)\tilde{R}\Big(\sum_{i=1}^dD_i^T\tilde{\kappa}_i\Big) = 0
\end{aligned}
$$

This equation must equal zero for all y, hence the coefficients in front of the quadratic term, as well as y^T and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$
\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{\vartheta}^T\tilde{P} - A\tilde{P} - \sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{P} + \frac{1}{2}\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} - \tilde{\vartheta}^T\tilde{S}^T\Big(B^T + \sum_{i=1}^dD_i^T\tilde{\vartheta}_i\Big)
$$

$$-\frac{1}{2}\left(B+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\right)\tilde{R}\left(B^T+\sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\right)=0 \quad (64)$$

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}+\tilde{P}\tilde{\kappa}-\tilde{P}\sum_{i=1}^{d}C_i^T\tilde{\kappa}_i+\tilde{\vartheta}^T\tilde{M}-A\tilde{M}-\sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{M}+\sum_{i=1}^{d}\tilde{\vartheta}_i a\tilde{\kappa}_i-\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i$$
$$-\left(B+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\right)\tilde{S}\tilde{\kappa}-\left(B+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\right)\tilde{R}\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i=0 \quad (65)$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}+\tilde{\kappa}^T\tilde{M}-\sum_{i=1}^{d}\tilde{\kappa}_i^T C_i\tilde{M}+\frac{1}{2}\sum_{i=1}^{d}\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa}-\tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i$$
$$-\frac{1}{2}\left(\sum_{i=1}^{d}\tilde{\kappa}_i^T D_i\right)\tilde{R}\left(\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i\right)=0$$

where $\tilde{\vartheta}, \tilde{\kappa}, \tilde{\vartheta}_i$ and $\tilde{\kappa}_i$ satisfy the system (62):

$$\begin{cases} \tilde{P}y+\tilde{M}-\tilde{Q}\alpha-\tilde{S}^T\left(B^T Y+\sum_{i=1}^{d}D_i^T\beta_i\right)=0 \\ C_i(\tilde{P}y+\tilde{M})-\tilde{P}\beta_i+D_i\left(\tilde{S}\alpha+\tilde{R}\left(B^T Y+\sum_{i=1}^{d}D_i^T\beta_i\right)\right)=0 \end{cases}$$

and the terminal conditions are given by:

$$\tilde{P}(T)=-G^{-1}(T),\quad \tilde{M}(T)=-G^{-1}(T)L(T),\quad \tilde{N}(T)=\frac{1}{2}L^T(T)G^{-1}(T)L(T). \quad (66)$$

1.5 Dual BSDE

1.5.1 Solution via the Dual BSDE

The Hamiltonian $\tilde{\mathcal{H}} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^{nd} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ for the dual problem is defined as

$$\begin{aligned}\tilde{\mathcal{H}}(t, Y, \alpha, \beta_1, \ldots, \beta_d, p_2, q_2) &= -\phi\left(t, \alpha, B^T Y + \sum_{i=1}^{d} D_i^T \beta_i\right) + p_2^T(\alpha - A^T Y - \sum_{i=1}^{d} C_i^T \beta_i) + \sum_{i=1}^{d} \beta_i^T q_{2,i} \\ &= p_2^T \alpha - p_2^T A^T Y - p_2^T \sum_{i=1}^{d} C_i^T \beta_i + \sum_{i=1}^{d} \beta_i^T q_{2,i} - \phi\left(t, \alpha, B^T Y + \sum_{i=1}^{d} D_i^T \beta_i\right)\end{aligned} \tag{67}$$

The adjoin equation is given by the system

$$\begin{cases} \mathrm{d}p_2(t) & = -D_y[\tilde{\mathcal{H}}]\mathrm{d}t + \sum_{i=1}^{d} q_{2,i} \mathrm{d}W_i \\ p_2(T) & = -D_y[h(Y(T))] = -G^{-1} Y(T) - G^{-1} L \end{cases} \tag{68}$$

Due to the Stochastic Maximum Principle, the optimal control can be found by setting $D_\alpha[\tilde{\mathcal{H}}] = 0$ and $D_{\beta_i}[\tilde{\mathcal{H}}] = 0$ for all $i = 1, \ldots, d$, so we get the system

$$D_\alpha[\tilde{\mathcal{H}}] = p_2 - \tilde{Q}\alpha - \tilde{S}^T\left(B^T Y + \sum_{i=1} D_i^T \beta_i\right) = 0 \tag{69}$$

$$D_{\beta_i}[\tilde{\mathcal{H}}] = q_{2,i} - C_i p_2 - D_i \tilde{S}\alpha - D_i \tilde{R}\left(B^T Y + \sum_{i=1} D_i^T \beta_i\right) = 0 \tag{70}$$

There are $d+1$ equations in $d+1$ unknowns, so the system can be solved and the optimal controls are linear functions of Y, which we denote by

$$\alpha = \tilde{\vartheta} Y + \tilde{\kappa}, \quad \beta_i = \tilde{\vartheta}_i Y + \tilde{\kappa}_i, \quad i \in \{1, \ldots, d\}$$

Substituting into the Hamiltonian (67) we get

$$\begin{aligned}\tilde{\mathcal{H}} = p_2^T(\tilde{\vartheta} Y + \tilde{\kappa}) - p_2^T A^T Y - p_2^T \sum_{i=1}^{d} C_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i) + \sum_{i=1}^{d} (Y^T \tilde{\vartheta}_i^T + \tilde{\kappa}_i^T) q_{2,i} \\ -\phi\left(t, \tilde{\vartheta} Y + \tilde{\kappa}, B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\right)\end{aligned}$$

The derivative of the dual Hamiltonian is then

$$\begin{aligned}D_y[\tilde{\mathcal{H}}] = \tilde{\vartheta}^T p_2 - A p_2 - \sum_{i=1}^{d} \tilde{\vartheta}_i^T C_i p_2 + \sum_{i=1}^{d} \tilde{\vartheta}_i^T q_{2,i} - \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} Y - \tilde{\vartheta}^T \tilde{Q} \tilde{\kappa} \\ -2\tilde{\vartheta}^T \tilde{S}\left(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i\right) Y - \tilde{\vartheta}^T \tilde{S}^T \sum_{i=1}^{d} D_i^T \tilde{\kappa}_i - \left(B + \sum_{i=1}^{d} \tilde{\vartheta}_i^T D_i\right) \tilde{S} \tilde{\kappa}\end{aligned}$$

$$-\Big(B+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\Big)\tilde{R}\Big(B^T+\sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)Y-\Big(B^T+\sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)\tilde{R}\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i \tag{71}$$

We try the ansatz

$$p_2=\tilde{\varphi}(t)Y+\tilde{\psi}(t),$$

where $\varphi(t)\in\mathbb{R}^{n\times n}$ and $\psi(t)\in R^n$. Applying Ito's formula to p_2, we get

$$\begin{aligned}
\mathrm{d}p_2&=\Big(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y+\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\Big)\mathrm{d}t+\tilde{\varphi}\mathrm{d}Y\\
&=\Big(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y+\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\Big)\mathrm{d}t+\tilde{\varphi}\Big[\alpha-A^TY-\sum_{i=1}^{d}C_i^T\beta_i\Big]\mathrm{d}t+\tilde{\varphi}\sum_{i=1}^{d}\beta_i\mathrm{d}W_i\\
&=\Big[\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y+\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}+\tilde{\varphi}\alpha-\tilde{\varphi}A^TY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\beta_i\Big]\mathrm{d}t+\tilde{\varphi}\sum_{i=1}^{d}\beta_i\mathrm{d}W_i\\
&=\Big[\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y+\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}+\tilde{\varphi}\tilde{\vartheta}Y+\tilde{\varphi}\tilde{\kappa}-\tilde{\varphi}A^TY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\tilde{\vartheta}_iY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\tilde{\kappa}_i\Big]\mathrm{d}t+\tilde{\varphi}\sum_{i=1}^{d}(\tilde{\vartheta}_iY+\tilde{\kappa}_i)\mathrm{d}W_i
\end{aligned} \tag{72}$$

Equating the coefficients of (72) and (68) we get

$$\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y+\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}+\tilde{\varphi}\tilde{\vartheta}Y+\tilde{\varphi}\tilde{\kappa}-\tilde{\varphi}A^TY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\tilde{\vartheta}_iY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\tilde{\kappa}_i=-D_y[\tilde{\mathcal{H}}] \tag{73}$$

$$\tilde{\varphi}\tilde{\vartheta}_iY+\tilde{\varphi}\tilde{\kappa}_i=q_{2,i} \tag{74}$$

where the RHS of (73) is given by (71). We now substitute $q_{2,i}$ from equation (74) into the system (69) and (70) we get

$$\begin{cases}
\tilde{\varphi}Y+\tilde{\psi}-\tilde{Q}(\tilde{\vartheta}Y+\tilde{\kappa})-\tilde{S}^T\Big(B^TY+\sum_{i=1}^{d}D_i^T(\tilde{\vartheta}_iY+\tilde{\kappa}_i)\Big)=0\\
\tilde{\varphi}(\tilde{\vartheta}_iY+\tilde{\kappa}_i)-C_i(\tilde{\varphi}Y+\tilde{\psi})-D_i\tilde{S}(\tilde{\vartheta}Y+\tilde{\kappa})-D_i\tilde{R}\Big(B^TY+\sum_{i=1}^{d}D_i^T(\tilde{\vartheta}_iY+\tilde{\kappa}_i)\Big)=0
\end{cases} \tag{75}$$

which is the system we need to solve, to acquire the optimal controls $\alpha^*=\tilde{\vartheta}Y+\tilde{\kappa}$ and $\beta_i^*=\tilde{\vartheta}_i+\tilde{\kappa}_i$. Substituting $q_{2,i}$ into (73) we get

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y+\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}+\tilde{\varphi}\tilde{\vartheta}Y+\tilde{\varphi}\tilde{\kappa}-\tilde{\varphi}A^TY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\tilde{\vartheta}_iY-\tilde{\varphi}\sum_{i=1}^{d}C_i^T\tilde{\kappa}_i=-\tilde{\vartheta}^Tp_2+Ap_2+\sum_{i=1}^{d}\tilde{\vartheta}_i^TC_ip_2\\
&+\sum_{i=1}^{d}\tilde{\vartheta}_i^T(\tilde{\varphi}\tilde{\vartheta}_iY+\tilde{\varphi}\tilde{\kappa}_i)+\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}Y+\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa}+2\tilde{\vartheta}^T\tilde{S}\Big(B^T+\sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)Y+\tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i\\
&+\Big(B+\sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i\Big)\tilde{S}\tilde{\kappa}+\Big(B+\sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\Big(B^T+\sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)Y+\Big(B^T+\sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)\tilde{R}\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i
\end{aligned} \tag{76}$$

We rewrite equation (76) as

$$\Bigg[\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta} - \tilde{\varphi}A^T - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\vartheta}_i + \tilde{\vartheta}^T\tilde{\varphi} - A\tilde{\varphi} - \sum_{i=1}^{d}\tilde{\vartheta}_i C_i\tilde{\varphi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{\varphi}\tilde{\vartheta}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} \tag{77}$$

$$-2\tilde{\vartheta}^T\tilde{S}(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i) - (B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i)\tilde{R}(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i)\Bigg]Y \tag{78}$$

$$+\Bigg[\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{\psi} - A\tilde{\psi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{\psi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{\varphi}\tilde{\kappa}_i \tag{79}$$

$$-\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i - (B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i)(\tilde{S}\tilde{\kappa} + \tilde{R}\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i)\Bigg] = 0 \tag{80}$$

Since this must be true for all Y, the coefficient in front of Y must be equal to zero, so we get

$$\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta} - \tilde{\varphi}A^T - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\vartheta}_i + \tilde{\vartheta}^T\tilde{\varphi} - A\tilde{\varphi} - \sum_{i=1}^{d}\tilde{\vartheta}_i C_i\tilde{\varphi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{\varphi}\tilde{\vartheta}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}$$

$$-2\tilde{\vartheta}^T\tilde{S}(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i) - (B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i)\tilde{R}(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i) = 0 \tag{81}$$

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{\psi} - A\tilde{\psi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{\psi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{\varphi}\tilde{\kappa}_i$$

$$-\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i - (B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i)(\tilde{S}\tilde{\kappa} + \tilde{R}\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i) = 0, \tag{82}$$

where $\tilde{\vartheta}, \tilde{\kappa}, \tilde{\vartheta}_i$, and $\tilde{\kappa}_i$ can be found from the system (75):

$$\begin{cases} \tilde{\varphi}Y + \tilde{\psi} - \tilde{Q}(\tilde{\vartheta}Y + \tilde{\kappa}) - \tilde{S}^T\Big(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big) = 0 \\ \tilde{\varphi}(\tilde{\vartheta}_i Y + \tilde{\kappa}_i) - C_i(\tilde{\varphi}Y + \tilde{\psi}) - D_i\tilde{S}(\tilde{\vartheta}Y + \tilde{\kappa}) - D_i\tilde{R}\Big(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big) = 0 \end{cases}$$

and the terminal conditions are given by

$$\tilde{\varphi}(T) = -G^{-1}(T), \quad \tilde{\psi}(T) = -G^{-1}(T)L(T). \tag{83}$$

1.5.2 Equivalence between Dual HJB and Dual BSDE

From the dual HJB we get that the optimal controls $\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}$ and $\beta^* = \tilde{\vartheta}_i y + \tilde{\kappa}_i$ are solution to the system of equations (62):

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\vartheta}y + \tilde{\kappa}) - \tilde{S}^T(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\vartheta}_i y + \tilde{\kappa}_i) + D_i\tilde{S}(\tilde{\vartheta}y + \tilde{\kappa}) + D_i\tilde{R}\Big(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\Big) = 0 \end{cases}$$

Similarly, the optimal controls from the dual BSDE method are found by solving the system (75):

$$\begin{cases} \tilde{\varphi}Y+\tilde{\psi}-\tilde{Q}(\tilde{\vartheta}Y+\tilde{\kappa})-\tilde{S}^T\Big(B^TY+\sum_{i=1}^d D_i^T(\tilde{\vartheta}_iY+\tilde{\kappa}_i)\Big)=0 \\ \tilde{\varphi}(\tilde{\vartheta}_iY+\tilde{\kappa}_i)-C_i(\tilde{\varphi}Y+\tilde{\psi})-D_i\tilde{S}(\tilde{\vartheta}Y+\tilde{\kappa})-D_i\tilde{R}\Big(B^TY+\sum_{i=1}^d D_i^T(\tilde{\vartheta}_iY+\tilde{\kappa}_i)\Big)=0 \end{cases}$$

These systems are the same and therefore we get the relation

$$\tilde{\varphi}=\tilde{P},\quad \tilde{\psi}=\tilde{M}.$$

The first ODE from the dual BSDE is (81), so substituting $\tilde{P}=\tilde{\varphi}$ in it we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}+\tilde{P}\tilde{\vartheta}-\tilde{P}A^T-\tilde{P}\sum_{i=1}^d C_i^T\tilde{\vartheta}_i+\tilde{\vartheta}^T\tilde{P}-A\tilde{P}-\sum_{i=1}^d\tilde{\vartheta}_iC_i\tilde{P}-\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i-\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}$$
$$-2\tilde{\vartheta}^T\tilde{S}(B^T+\sum_{i=1}^d D_i^T\tilde{\vartheta}_i)-(B+\sum_{i=1}^d\tilde{\vartheta}_i^TD_i)\tilde{R}(B^T+\sum_{i=1}^d D_i^T\tilde{\vartheta}_i)=0$$

The first ODE from the dual HJB equation is given by (64):

$$\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}+\tilde{\vartheta}^T\tilde{P}-A\tilde{P}-\sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{P}+\frac{1}{2}\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i-\frac{1}{2}\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}-\tilde{\vartheta}^T\tilde{S}^T\Big(B^T+\sum_{i=1}^d D_i^T\tilde{\vartheta}_i\Big)$$
$$-\frac{1}{2}\Big(B+\sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\Big(B^T+\sum_{i=1}^d D_i^T\tilde{\vartheta}_i\Big)=0$$

The two equations are equivalent with the second being the first one divided by 2. Similarly, plugging in $\tilde{P}=\tilde{\varphi}$ and $\tilde{M}=\tilde{\psi}$ in (82) we get

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}+\tilde{P}\tilde{\kappa}-\tilde{P}\sum_{i=1}^d C_i^T\tilde{\kappa}_i+\tilde{\vartheta}^T\tilde{M}-A\tilde{M}-\sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{M}-\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\kappa}_i-\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^d D_i^T\tilde{\kappa}_i$$
$$-(B+\sum_{i=1}^d\tilde{\vartheta}_i^TD_i)(\tilde{S}\tilde{\kappa}+\tilde{R}\sum_{i=1}^d D_i^T\tilde{\kappa}_i)=0$$

The respective ODE from the dual HJB is (65):

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}+\tilde{P}\tilde{\kappa}-\tilde{P}\sum_{i=1}^d C_i^T\tilde{\kappa}_i+\tilde{\vartheta}^T\tilde{M}-A\tilde{M}-\sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{M}+\sum_{i=1}^d\tilde{\vartheta}_ia\tilde{\kappa}_i-\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^d D_i^T\tilde{\kappa}_i$$
$$-\Big(B+\sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{S}\tilde{\kappa}-\Big(B+\sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\sum_{i=1}^d D_i^T\tilde{\kappa}_i=0$$

The terminal conditions from the dual HJB method are given buy (66):

$$\tilde{P}(T)=-G^{-1}(T),\quad \tilde{M}(T)=-G^{-1}(T)L(T),\quad \tilde{N}(T)=\frac{1}{2}L^T(T)G^{-1}(T)L(T)$$

and from the dual BSDE, we have (83):

$$\tilde{\varphi}(T) = -G^{-1}(T), \quad \tilde{\psi}(T) = -G^{-1}(T)L(T).$$

As we can see the equations are identical, and their terminal conditions are also the same, so the two methods are equivalent.

1.6 Equivalence of Primal HJB and Dual HJB

Recall that we have the relation between the dual and primal value functions (16):

$$v(x) \leq \inf_y \{x^T y - \tilde{v}(y)\}$$

In this section, we impose no constraints on the control, and we will show that this leads to equality instead of inequality in the above equation. Substituting the respective ansatz, we have

$$\frac{1}{2}x^T Px + x^T M + N = \inf_y \left\{ x^T y - \frac{1}{2}y^T \tilde{P} y - y^T \tilde{M} - \tilde{N} \right\}$$

Setting the derivative of the RHS to zero, we get

$$y = \tilde{P}^{-1}(x - \tilde{M}),$$

so

$$\frac{1}{2}x^T Px + x^T M + N = -\frac{1}{2}(x^T - \tilde{M}^T)\tilde{P}^{-1}(x - \tilde{M}) - (x^T - \tilde{M}^T)\tilde{P}^{-1}\tilde{M} - \tilde{N} + x^T \tilde{P}^{-1}(x - \tilde{M})$$

Simplifying we get

$$\frac{1}{2}x^T Px + x^T M + N = \frac{1}{2}x^T \tilde{P}^{-1}x - x^T \tilde{P}^{-1}\tilde{M} + \frac{1}{2}\tilde{M}^T \tilde{P}^{-1}\tilde{M} - \tilde{N}.$$

Therefore, we get the relation

$$P = \tilde{P}^{-1}, \quad M = -\tilde{P}^{-1}\tilde{M}, \quad N = \frac{1}{2}\tilde{M}^T \tilde{P}^{-1}\tilde{M} - \tilde{N}.$$

To simplify computations, we consider a simpler case where $d = 1$ and $S = 0$. Then $\tilde{S} = 0, \tilde{Q} = Q^{-1}, \tilde{R} = R^{-1}$. The Riccati equation from the primal problem in this case is given by (35)

$$\dot{P} + 2PA + 2PB\vartheta_1 + (C_1^T + \vartheta_1^T D_1^T)P(C_1 + D_1\vartheta_1) - Q - \vartheta_1^T R\vartheta_1 = 0,$$

where

$$\vartheta_1 = (D_1^T PD_1 - R)^{-1}(-B^T P - D_1^T PC_1)$$

Therefore, our equation becomes

$$\begin{aligned} 0 &= \frac{\mathrm{d}P}{\mathrm{d}t} + 2PA - Q + C_1^T PC_1 + 2(PB + C_1^T PD_1)\vartheta_1 + \vartheta_1^T (D_1^T PD_1 - R)\vartheta_1 \\ &= \frac{\mathrm{d}P}{\mathrm{d}t} + 2PA - Q + C_1^T PC_1 - (PB + C_1^T PD_1)(D_1^T PD_1 - R)^{-1}(B^T P + D_1^T PC_1) \end{aligned}$$

Substituting $P = \tilde{P}^{-1}$, we get

$$\tilde{P}^{-1}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}\tilde{P}^{-1} - 2\tilde{P}^{-1}A + Q - C_1^T \tilde{P}^{-1}C_1 + (\tilde{P}^{-1}B + C_1^T \tilde{P}^{-1}D_1)(D_1^T \tilde{P}^{-1}D_1 - R)^{-1}(B^T \tilde{P}^{-1} + D_1^T \tilde{P}^{-1}C_1) = 0$$

Multiplying on the left and on the right by $\tilde{P}$, we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} - 2A\tilde{P} + \tilde{P}Q\tilde{P} - \tilde{P}C_1^T\tilde{P}^{-1}C_1\tilde{P} + \tilde{P}(\tilde{P}^{-1}B + C_1^T\tilde{P}^{-1}D_1)(D_1^T\tilde{P}^{-1}D_1 - R)^{-1}(B^T\tilde{P}^{-1} + D_1^T\tilde{P}^{-1}C_1)\tilde{P} = 0$$

Rewriting this, we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} - 2A\tilde{P} + \tilde{P}Q\tilde{P} + \tilde{P}C_1^T(\tilde{P}^{-1}D_1(D_1^T\tilde{P}D_1 - R)^{-1}D_1^T\tilde{P}^{-1} - \tilde{P}^{-1})C_1\tilde{P} + B(D_1^T\tilde{P}D_1 - R)B^T$$
$$+2B(D_1^T\tilde{P}D_1 - R)^{-1}D_1^t\tilde{P}^{-1}C_1\tilde{P} = 0 \tag{84}$$

On the other hand, for the dual problem we have:

$$\tilde{\vartheta} = Q\tilde{P}, \quad \tilde{\vartheta}_1 = (\tilde{P} - D_1R^{-1}D_1^T)^{-1}(C_1\tilde{P} + D_1R^{-1}B^T)$$

The dual Riccati equation is (64):

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + 2\tilde{\vartheta}^T\tilde{P} - 2A\tilde{P} - 2\tilde{\vartheta}_1^TC_1\tilde{P} + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - \tilde{\vartheta}^TQ^{-1}\tilde{\vartheta} - (B + \tilde{\vartheta}_1^TD_1)R^{-1}(B^T + D_1^T\tilde{\vartheta}_1) = 0$$

Substituting for $\tilde{\vartheta}$ we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - (B + \tilde{\vartheta}_1^TD_1)R^{-1}(B^T + D_1^T\tilde{\vartheta}_1) = 0$$

Substituting for $\tilde{\vartheta}_1$ we get

$$\begin{aligned}
0 &= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - (B + \tilde{\vartheta}_1^TD_1)R^{-1}(B^T + D_1^T\tilde{\vartheta}_1)\\
&= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - BR^{-1}B^T - 2BR^{-1}D_1^T\tilde{\vartheta}_1 - \tilde{\vartheta}_1^TD_1R^{-1}D_1^T\tilde{\vartheta}_1\\
&= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - BR^{-1}B^T - 2(BR^{-1}D_1^T + \tilde{P}C_1^T)\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T(\tilde{P} - D_1R^{-1}D_1^T)\tilde{\vartheta}_1\\
&= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - BR^{-1}B^T - (BR^{-1}D_1^T + \tilde{P}C_1^T)(\tilde{P} - D_1R^{-1}D_1^T)^{-1}(C_1\tilde{P} + D_1R^{-1}B^T)
\end{aligned}$$

This is then rewritten as

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} - 2A\tilde{P} + \tilde{P}Q\tilde{P} + \tilde{P}C_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}C_1\tilde{P} + 2BR^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}C_1\tilde{P}$$
$$+B(R^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}D_1R^{-1} - R^{-1})B^T = 0 \tag{85}$$

Now noting that

$$\begin{aligned}
(D_1R^{-1}D_1^T - \tilde{P})^{-1} &= \tilde{P}^{-1}D_1(D_1^T\tilde{P}D_1 - R)^{-1}D_1^T\tilde{P}^{-1} - \tilde{P}^{-1}\\
R^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}D_1R^{-1} - R^{-1} &= D_1^T\tilde{P}D_1 - R\\
R^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1} &= (D_1^T\tilde{P}D_1 - R)^{-1}D_1^T\tilde{P}^{-1}
\end{aligned}$$

we get exactly (84), so the dual and primal HJB methods are equivalent.

2 Unconstrained Markovian Optimisation

2.1 Problem Description

2.1.1 Primal Problem

We now adapt the problem from the last section by introducing a Markov chain, independent of the d–dimensional Brownian motion. Let $\eta(t)$ be a continuous-time finite state observable Markov chain. Let the Markov chain take values in the state space $I = \{1, 2, \ldots, k\}$ and start from an initial state $i_0 \in I$ with a $k \times k$ generator matrix $\mathcal{Q} = \{q_{ij}\}_{i,j=1}^{k}$. For each pair of distinct states (i, j) define the counting process $[\mathcal{Q}_{ij}] : \Omega \times [t_0, T] \to \mathbb{N}$ by

$$[\mathcal{Q}_{ij}](\omega, t) := \sum_{t_0 < s \leq t} \chi_{\{\eta(s-)=i\}}(\omega)\chi_{\{\eta(s)=j\}}(\omega), \quad \forall t \in [t_0, T],$$

and the compensator process $\langle \mathcal{Q}_{ij} \rangle : \Omega \times [t_0, T] \to [0, \infty)$ by

$$\langle \mathcal{Q}_{ij} \rangle(\omega, t) := q_{ij} \int_{t_0}^{t} \chi_{\{\eta(s-)=i\}}(\omega)\mathrm{d}s, \quad \forall t \in [t_0, T].$$

The process

$$\mathcal{Q}_{ij}(\omega, t) := [\mathcal{Q}_{ij}](\omega, t) - \langle \mathcal{Q}_{ij} \rangle(\omega, t)$$

is a purely discontinuous square-integrable martingale with an initial value zero.

Let $W(t)$ be a d-dimensional Brownian motion and consider an n–dimensional process $X(t)$ described by

$$\begin{cases} \mathrm{d}X(t) & = b(t, X(t), \pi(t), \eta(t-))\mathrm{d}t + \sigma(t, x(t), \pi(t), \eta(t-))\mathrm{d}W(t) \\ X(0) & = x_0 \in \mathbb{R}^n, \ \eta(0) = i_0 \in I \end{cases} \tag{86}$$

where $\pi(t) \in \mathbb{R}^m$ is the control and

$$b := A(t, \eta(t-))X(t) + B(t, \eta(t-))\pi(t) \in \mathbb{R}^n$$

$$\sigma := \left[\begin{pmatrix} \\ C_1(t, \eta(t-))X(t) + D_1(t, \eta(t-))\pi(t) \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ C_d(t, \eta(t-))X(t) + D_d(t, \eta(t-))\pi(t) \\ \\ \end{pmatrix} \right] \in \mathbb{R}^{n \times d}$$

where $A, C_i \in \mathbb{R}^{n \times n}$, and $B, D_i \in \mathbb{R}^{n \times m}, i \in \{1, \ldots, d\}$ are functions of both time and the Markov chain process.

The cost functional is given by

$$J(\pi) := \mathbb{E}\left[\int_{t_0}^{T} f(t, X(t), \pi(t), \eta(t))\mathrm{d}t + g(X(T), \eta(T)) \right], \tag{87}$$

where $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times I \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given by

$$f(t, X(t), \pi(t), \eta(t)) = \frac{1}{2} X^T(t) Q(t, \eta(t)) X(t) + X^T(t) S^T(t, \eta(t)) \pi(t) + \frac{1}{2} \pi^T(t) R(t, \eta(t)) \pi(t)$$
$$g(X(T), \eta(T)) = \frac{1}{2} X^T(T) G(T, \eta(T)) X(T) + X^T(T) L(T, \eta(T)).$$

The assumptions for this problem are the same to the ones in the previous section. We consider the following optimisation problem

$$\text{Minimise } J(\pi) \text{ subject to } (X, \pi) \text{ admissible.} \tag{88}$$

2.1.2 Dual Problem

We now derive the dual to the primal problem. First, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider the martingale $M : [t_0, T] \times \Omega \to \mathbb{R}^k$ generated by the Markov process $\eta(t)$:

$$M(t) := \eta(t) - \int_{t_0}^{t} Q^T \eta(s) \mathrm{d}s,$$

where Q is the generating matrix of the Markov chain. Assume that the dual process $Y(t)$ follows

$$\begin{cases} \mathrm{d}Y(t) &= \left[\alpha(t) - A(t, \eta(t-))^T Y(t) - \sum_{j=1}^{d} C_j^T(t, \eta(t-)) \beta_j(t))\right] \mathrm{d}t + \sum_{j=1}^{d} \beta_j(t) \mathrm{d}W_j(t) + \sum_{j=1}^{k} \gamma_j(t) \mathrm{d}M_j(t) \\ Y(t_0) &= y \end{cases} \tag{89}$$

where $\alpha(t) \in \mathcal{H}([t_0, T], \mathbb{R}^n)$, $\beta_j(t) \in \mathcal{H}([t_0, T], \mathbb{R}^n)$ and $\gamma_j(t) \in \mathcal{H}([t_0, T], \mathbb{R}^n)$ are the dual controls.Using Ito's lemma to $X(t)^T Y(t)$, we get

$$\mathrm{d}X^T Y = \left[X^T \alpha + \pi^T \beta\right] \mathrm{d}t + \sum_{j=1}^{d} \left(X^T \beta_j + Y^T (C_j X + D_j \pi)\right) \mathrm{d}W_j,$$

where

$$\beta = B^T Y + \sum_{j=1}^{d} D_j^T \beta_j.$$

The process $X^T(t)Y(t) - \int_{t_0}^{t} [X^T(s)\alpha(s) + \pi^T(s)\beta(s)] \mathrm{d}s$ is a local martingale and a supermartingale if it is bounded below by an integrable process, which gives

$$\mathbb{E}\left[X^T(T)Y(T) - \int_{t_0}^{T} \left(X^T \alpha + \pi^T \beta\right) \mathrm{d}s\right] \leq x^T y. \tag{90}$$

The optimisation problem (88) can be written equivalently as

$$\sup_{\pi} \mathbb{E}\left[-\int_{t_0}^{T} f(t, X(t), \pi(t), \eta(t)) \mathrm{d}t - g(X(T), \eta(T))\right].$$

Define the dual functions $\phi : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times I \to \mathbb{R}$ by

$$\phi(t,\alpha,\beta,\eta(t)) = \sup_{x,\pi} \Big\{ x^T \alpha + \pi^T \beta - f(t,x,\pi,\eta(t)) \Big\} \tag{91}$$

and $h : \mathbb{R}^n \times I \to \mathbb{R}$ by

$$h(y,\eta(T)) = \sup_{x} \Big\{ -x^T y - g(x,\eta(T)) \Big\}. \tag{92}$$

Substituting f and g, we can find the supremums by setting the derivatives to zero. We get

$$\phi(t,\alpha,\beta,\eta(t)) = \sup_{x,\pi} \left\{ \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} \right\},$$

so setting the derivative to zero, we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0 \implies \begin{bmatrix} x^* \\ \pi^* \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Therefore

$$\begin{aligned} \pi^* &= \Big[S Q^{-1} S^T - R \Big]^{-1} (S Q^{-1} \alpha - \beta) \\ x^* &= Q^{-1} (\alpha - S^T \pi^*) \end{aligned}$$

Then ϕ is given by

$$\phi(t,\alpha,\beta,\eta(t)) = \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Denoting

$$\begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1},$$

where

$$\tilde{Q} = Q^{-1} - Q^{-1} S^T (S Q^{-1} S^T - R)^{-1} S Q^{-1} \tag{93}$$

$$\tilde{R} = R^{-1} - R^{-1} S (S^T R^{-1} S - Q^{-1})^{-1} S^T R^{-1} \tag{94}$$

$$\tilde{S} = (S Q^{-1} S^T - R)^{-1} S Q^{-1} = R^{-1} S (S^T R^{-1} S - Q)^{-1} \tag{95}$$

we get

$$\phi(t,\alpha,\beta,\eta(t)) = \frac{1}{2} \alpha^T \tilde{Q}(t,\eta(t)) \alpha + \alpha^T \tilde{S}^T(t,\eta(t)) \beta + \frac{1}{2} \beta^T \tilde{R}(t,\eta(t)) \beta \tag{96}$$

Similarly,

$$D_x \Big[-x^T y - \frac{1}{2} x^T G x - x^T L \Big] = -y - Gx - L \implies x^* = -G^{-1}(y+L).$$

Then $h(y,\eta(T))$ is given by

$$h(y,\eta(T)) = (y^T + L^T) G^{-1} y - \frac{1}{2} (y^T + L^T) G^{-1} (y+L) + (y^T + L^T) G^{-1} L$$

$$
\begin{aligned}
&= \frac{1}{2}\Big[y^T G^{-1} y + L^T G^{-1} y + y^T G^{-1} L + L^T G^{-1} L\Big] \\
&= \frac{1}{2}(y^T + L^T) G^{-1} (y + L)
\end{aligned}
\tag{97}
$$

Combining (90), (91), (10), we get the following inequality:

$$
\sup_{\pi} \mathbb{E}\bigg[-\int_{t_0}^{T} f(t, X, \pi, \eta(t))\mathrm{d}t - g(X(T), \eta(T))\bigg] \leq \inf_{y,\alpha,\beta_j}\bigg[x^T y + \mathbb{E}\bigg[\int_{t_0}^{T} \phi(t, \alpha, \beta, \eta(t))\mathrm{d}t + h(Y(T), \eta(T))\bigg]\bigg]
\tag{98}
$$

The dual control problem is defined by

$$
\inf_{y,\alpha,\beta_1,\dots,\beta_d}\bigg[x^T y + \mathbb{E}\bigg[\int_{t_0}^{T} \phi(t, \alpha, \beta, \eta(t))\mathrm{d}t + h(Y(T), \eta(T))\bigg]\bigg],
\tag{99}
$$

This can be solved in two steps: first, for fixed y, solve a stochastic control problem

$$
-\tilde{v}(t, y, \eta(t)) = \inf_{\alpha,\beta_1,\dots,\beta_d} \mathbb{E}\bigg[\int_{t_0}^{T} \phi(t, \alpha, \beta, \eta(t))\mathrm{d}t + h(Y(T), \eta(T))\bigg],
\tag{100}
$$

and second, solve a static optimisation problem

$$
\inf_{y}\Big\{x^T y - \tilde{v}(t, y, \eta(t))\Big\}.
$$

2.2 Primal HJB

2.2.1 Deriving the Primal HJB

We transform the minimisation problem (88) to maximisation by noting that

$$\inf_{\pi}\mathbb{E}\left[\int_{t_0}^{T} f(t,X(t),\pi(t),\eta(t))\mathrm{d}t+g(X(T),\eta(T))\right]=-\sup_{\pi}\mathbb{E}\left[\int_{t_0}^{T} -f(t,X(t),\pi(t),\eta(t))\mathrm{d}t-g(X(T),\eta(T))\right]$$

and denote the value function

$$v(t,X(t),\eta(t))=\sup_{\pi}\mathbb{E}\left[\int_{t_0}^{T} -f(t,X(t),\pi(t),\eta(t))\mathrm{d}t-g(X(T),\eta(T))\right] \tag{101}$$

The HJB equations is given by

$$\frac{\partial v}{\partial t}(t,x,i)+\sup_{\pi}\Big[\mathcal{L}^{\pi}[v(t,x,i)-f(t,x,\pi,i)]\Big]+\sum_{j\neq i}^{k} q_{ij}(v(t,x,j)-v(t,x,i))=0 \tag{102}$$

with terminal conditions

$$v(T,x,i)=-g(x,i)=-\frac{1}{2}x^{T}G(T,i)x+x^{T}L(T,i).$$

2.2.2 Finding the Optimal Control

The supremum can be found by setting the derivative with respect to π to zero. The derivative of the generator $\mathcal{L}^{\pi}$, $D_{\pi}[\mathcal{L}^{\pi}]\in\mathbb{R}^{m}$, is given by:

$$D_{\pi}\Big[\mathcal{L}^{\pi}[v(t,x,i)]\Big]=D_{\pi}\Big[b(t,x,\pi,i)^{T}D_{x}[v(t,x,i)]\Big]+D_{\pi}\left[\frac{1}{2}\mathrm{tr}(\sigma(t,x,\pi,i)\sigma^{T}(t,x,\pi,i)D_{x}^{2}[v(t,x,i)])\right]. \tag{103}$$

We have that

$$\begin{aligned} D_{\pi}\Big[b^{T}(t,x,\pi,i)D_{x}[v(t,x,i)]\Big]&=D_{\pi}\Big[(x^{T}A^{T}(t,i)+\pi^{T}(t)B^{T}(t,i))D_{x}[v(t,x,i)]\Big]\\ &=B^{T}(t,i)D_{x}[v(t,x,i)] \end{aligned} \tag{104}$$

The latter derivative in (103) is given by:

$$\begin{aligned} D_{\pi}\left[\frac{1}{2}\mathrm{tr}[\sigma(t,x,\pi,i)\sigma^{T}(t,x,\pi,i)D_{x}^{2}[v(t,x,i)]]\right]&=\frac{1}{2}D_{\pi}\left[\mathrm{tr}\left[\sum_{j=1}^{d}(C_{j}x+D_{j}\pi)(C_{j}x+D_{j}\pi)^{T}D_{x}^{2}[v]\right]\right]\\ &=\frac{1}{2}\sum_{j=1}^{d}D_{\pi}\Big[\mathrm{tr}[(C_{j}x+D_{j}\pi)(C_{j}x+D_{j}\pi)^{T}D_{x}^{2}[v]]\Big]\\ &=\frac{1}{2}\sum_{j=1}^{d}D_{\pi}\Big[(C_{j}x+D_{j}\pi)^{T}D_{x}^{2}[v](C_{j}x+D_{j}\pi)\Big] \end{aligned}$$

$$= \sum_{j=1}^{d} D_j^T(t,i) D_x^2[v(t,x,i)](C_j(t,i)x + D_j(t,i)\pi) \tag{105}$$

The derivative of $f(t,x,\pi,i)$ with respect to π is

$$D_\pi f(t,x,\pi,i) = S(t,i)x + R(t,i)\pi \tag{106}$$

Combining the three equations,(104), (105), (106), we get that

$$\begin{aligned} D_\pi[\mathcal{L}^\pi(t)[v(t,x,i)] - f(t,x,\pi,i)] &= \sum_{i=j}^{d} D_j^T(t,i) D_x^2[v(t,x,i)](C_j(t,i)x + D_j(t,i)\pi) \\ &\quad + B^T(t,i) D_x[v(t,x,i)] - S(t,i)x - R(t,i)\pi \end{aligned}$$

The coefficients are all a function of time and the Markov chain process, however, for compactness, we write $S = S(t,i)$. Setting the derivative to zero, we get

$$\pi^* = \left[\sum_{j=1}^{d} D_j^T D_x^2[v(t,x,i)] D_j - R \right]^{-1} \left[Sx - B^T D_x[v(t,x,i)] - \sum_{j=1}^{d} D_j^T D_x^2[v(t,x,i)] C_j x \right] \tag{107}$$

We now substitute (107) into (102) to get:

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,x,i) + b(t,x,\pi^*,i)^T D_x[v(t,x,i)] + \frac{1}{2}\mathrm{tr}\Big[\sigma(t,x,\pi^*,i)\sigma^T(t,x,\pi^*,i) D_x^2[v(t,x,i)]\Big] \\ &\quad -\frac{1}{2}x^T Q x - \frac{1}{2}x^T S^T \pi^* - \frac{1}{2}\pi^{*T} S x - \frac{1}{2}\pi^{*T} R \pi^* + \sum_{j \neq i} q_{ij}(v(t,x,j) - v(t,x,i)) = 0 \end{aligned}$$

As $D_x^2[v(t,x,i)]$ is a symmetric matrix, we can write

$$\begin{aligned} \mathrm{tr}\Big[\sigma(t,x,\pi^*,i)\sigma^T(t,x,\pi^*,i) D_x^2[v(t,x,i)]\Big] &= \sum_{j=1}^{d} \mathrm{tr}\Big[(C_j x + D_j \pi^*)(C_j x + D_j \pi^*)^T D_x^2[v(t,x,i)]\Big] \\ &= \sum_{j=1}^{d} (C_j x + D_j \pi^*)^T D_x^2[v(t,x,i)](C_j x + D_j \pi^*), \end{aligned}$$

we get the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,x,i) + (Ax + B\pi^*)^T D_x[v(t,x,i)] + \frac{1}{2}\sum_{j=1}^{d} (C_j x + D_j \pi^*)^T D_x^2[v(t,x,i)](C_j x + D_j \pi^*) \\ &\quad -\frac{1}{2}x^T Q x - \frac{1}{2}x^T S^T \pi^* - \frac{1}{2}\pi^{*T} S x - \frac{1}{2}\pi^{*T} R \pi^* + \sum_{j \neq i}^{k} q_{ij}(v(t,x,j) - v(t,x,i)) = 0 \end{aligned} \tag{108}$$

where π^* is as in (107) and the terminal condition is given by

$$v(T,x,i) = -g(x,i) = -\frac{1}{2}x^T G(T,i)x - x^T L(T,i).$$

2.2.3 Solving the HJB equation

We solve (108) using the ansatz

$$v(t,x,i) = \frac{1}{2}x^T P(t,i)x + x^T M(t,i) + N(t,i) \tag{109}$$

with terminal conditions

$$P(T,i) = -G(T,i), \quad M(T,i) = -L(T,i), \quad N(T,i) = 0.$$

Then

$$\begin{aligned}
&\frac{\partial v}{\partial t}(t,x,i) = \frac{1}{2}x^T \frac{\mathrm{d}P(t,i)}{\mathrm{d}t}x + x^T \frac{\mathrm{d}M(t,i)}{\mathrm{d}t} + \frac{\mathrm{d}N(t,i)}{\mathrm{d}t} \\
&D_x[v(t,x,i)] = P(t,i)x + M(t,i) \\
&D_x^2[v(t,x,i)] = P(t,i)
\end{aligned}$$

Substituting in (107) we get

$$\begin{aligned}
\pi^* = \left[\sum_{j=1}^d D_j^T(t,i)P(t,i)D_j(t,i) - R(t,i)\right]^{-1} \Bigg[& S(t,i)x - B^T(t,i)P(t,i)x - B^T(t,i)M(t,i) \\
& - \sum_{j=1}^d D_j^T(t,i)P(t,i)C_j(t,i)x \Bigg]
\end{aligned} \tag{110}$$

We can write this as

$$\pi^* = \vartheta_1 x + \kappa_1,$$

where

$$\begin{aligned}
\vartheta_1 &= \left(\sum_{j=1}^d D_j^T(t,i)P(t,i)D_j(t,i) - R(t,i)\right)^{-1}\left(S(t,i) - B^T(t,i)P(t,i) - \sum_{j=1}^d D_j(t,i)P(t,i)C_j(t,i)\right) \\
\kappa_1 &= -\left(\sum_{j=1}^d D_j^T(t,i)P(t,i)D_j(t,i) - R(t,i)\right)^{-1} B^T(t,i)M(t,i)
\end{aligned}$$

Substituting into (108) we get

$$\begin{aligned}
\frac{1}{2}x^T \frac{\mathrm{d}P(t,i)}{\mathrm{d}t}x + x^T \frac{\mathrm{d}M(t,i)}{\mathrm{d}t} + \frac{\mathrm{d}N(t,i)}{\mathrm{d}t} + (A(t,i)x + B(t,i)(\vartheta_1 x + \kappa_1))^T (P(t,i)x + M(t,i)) \\
+\frac{1}{2}\sum_{j=1}^d (C_j(t,i)x + D_j(t,i)(\vartheta_1 x + \kappa_1))^T P(t,i)(C_j(t,i)x + D_j(t,i)(\vartheta_1 x + \kappa_1)) \\
-\frac{1}{2}x^T Q(t,i)x - x^T S^T(t,i)(\vartheta_1 x + \kappa_1) - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T R(t,i)(\vartheta_1 x + \kappa_1) \\
+\sum_{j\neq i}^k q_{ij}\left(\frac{1}{2}x^T(P(t,j) - P(t,i))x + x^T(M(t,j) - M(t,i)) + N(t,j) - N(t,i)\right) = 0
\end{aligned}$$

We rewrite this as

$$x^T\Bigg[\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t}+A^T(t,i)P(t,i)+\vartheta_1^TB^T(t,i)P(t,i)+\frac{1}{2}\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)$$
$$-\frac{1}{2}Q(t,i)-S^T(t,i)\vartheta_1-\frac{1}{2}\vartheta_1^TR(t,i)\vartheta_1+\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))\Bigg]x$$
$$+x^T\Bigg[\frac{\mathrm{d}M(t,i)}{\mathrm{d}t}+A^T(t,i)M(t,i)+\vartheta_1^TB^T(t,i)M(t,i)+P(t,i)B(t,i)\kappa_1$$
$$+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)D_j\kappa_1-S^T(t,i)\kappa_1-\vartheta_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(M(t,j)-M(t,i))\Bigg]$$
$$+\frac{\mathrm{d}N(t,i)}{\mathrm{d}t}+\kappa_1^TM(t,i)+\frac{1}{2}\sum_{j=1}^{d}\kappa_1^TP(t,i)\kappa_1-\frac{1}{2}\kappa_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(N(t,j)-N(t,i))=0$$

This equation must equal zero for all x, hence the coefficients in front of the quadratic term, as well as x and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t}+A^T(t,i)P(t,i)+\vartheta_1^TB^T(t,i)P(t,i)+\frac{1}{2}\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)$$
$$-\frac{1}{2}Q(t,i)-S^T(t,i)\vartheta_1-\frac{1}{2}\vartheta_1^TR(t,i)\vartheta_1+\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))=0 \tag{111}$$

$$\frac{\mathrm{d}M(t,i)}{\mathrm{d}t}+A^T(t,i)M(t,i)+\vartheta_1^TB^T(t,i)M(t,i)+P(t,i)B(t,i)\kappa_1$$
$$+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)D_j\kappa_1-S^T(t,i)\kappa_1-\vartheta_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(M(t,j)-M(t,i))=0 \tag{112}$$

$$+\frac{\mathrm{d}N(t,i)}{\mathrm{d}t}+\kappa_1^TM(t,i)+\frac{1}{2}\sum_{j=1}^{d}\kappa_1^TP(t,i)\kappa_1-\frac{1}{2}\kappa_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(N(t,j)-N(t,i))=0 \tag{113}$$

with terminal conditions

$$P(T,i)=-G(T,i),\quad M(T,i)=-L(T,i),\quad N(T,i)=0. \tag{114}$$

2.3 Primal BSDE

2.3.1 Solution via the Primal BSDE

Recall that the SDE (86) describing the state process is given by

$$\begin{cases} \mathrm{d}X(t) &= b(t, X(t), \pi(t), \eta(t-))\mathrm{d}t + \sigma(t, X(t), \pi(t), \eta(t-))\mathrm{d}W(t) \\ X(0) &= x_0 \in \mathbb{R}^n,\ \eta(0) = i_0 \in I \end{cases} \tag{115}$$

with cost functional (87)

$$J(\pi) := \mathbb{E}\left[\int_{t_0}^{T} f(t, X(t), \pi(t), \eta(t))\mathrm{d}t + g(X(T), \eta(T))\right]. \tag{116}$$

The Hamiltonian $\mathcal{H} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times I \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ is given by

$$\begin{aligned} \mathcal{H}(t, x, \pi, i, p, q) : &= -f(t, x, \pi, i) + b^T(t, x, \pi, i)p + \mathrm{tr}(\sigma^T(t, x, \pi, i)q) \\ &= -\frac{1}{2}x^T Q(t, i)x - x^T S^T(t, i)\pi - \frac{1}{2}\pi^T R(t, i)\pi + (x^T A^T(t, i) + \pi^T B^T(t, i))p \\ &+ \sum_{j=1}^{d}(x^T C_j^T(t, i) + \pi^T D_j^T(t, i))q_j, \end{aligned} \tag{117}$$

where $q_j \in \mathbb{R}^n$ is the j^{th} column of $q \in \mathbb{R}^{n\times d}$. Given an admissible pair (x, π), the adjoint equation in the unknown adapted processes $p(t), q(t)$ and $s(t) = (s^{(1)}(t), \ldots, s^{(n)}(t))$, where $s^{(l)}(t) \in \mathbb{R}^{k\times k}$ for $l \in \{1, \cdots, n\}$, is the following regime-switching BSDE:

$$\begin{cases} \mathrm{d}p(t) &= -D_x[\mathcal{H}(t, X(t), \pi(t), \eta(t-), p(t), q(t))]\mathrm{d}t + q(t)\mathrm{d}W(t) + s(t) \cdot \mathrm{d}\mathcal{Q}(t) \\ p(T) &= -D_x[g(X(T), \eta(T))] = -G(T, \eta(t))X(T) - L(T, \eta(t)) \end{cases} \tag{118}$$

where

$$s(t) \cdot \mathrm{d}\mathcal{Q}(t) = \left(\sum_{j\neq i} s_{ij}^{(1)}\mathrm{d}\mathcal{Q}_{ij}(t), \cdots, s_{ij}^{(n)}\mathrm{d}\mathcal{Q}_{ij}(t)\right)^T \tag{119}$$

We know that the optimal control maximises the Hamiltonian (117), that is, the derivative with respect to the control vanishes:

$$D_\pi[\mathcal{H}] = B^T(t, i)p + \sum_{i=1}^{d} D_i^T(t, i)q_i - S(t, i)X - R(t, i)\pi = 0 \tag{120}$$

From this, we know that the control π is linear in X, so it is of the form

$$\pi = \vartheta_2 X + \kappa_2, \tag{121}$$

where $\vartheta_2 \in \mathbb{R}^{m\times n}$ and $\kappa_2 \in \mathbb{R}^m$. Substituting the control in the Hamiltonian (117) we get

$$\mathcal{H} = X^T A^T(t, i)p + (\vartheta_2 X + \kappa_2)^T B^T(t, i)p + \sum_{j=1}^{d}\left(X^T C_j^T(t, i)q_j + (\vartheta_2 X + \kappa_2)^T D_j^T(t, i)q_j\right)$$

$$-\frac{1}{2}X^TQ(t,i)X - X^TS^T(t,i)(\vartheta_2 X+\kappa_2) - \frac{1}{2}(\vartheta_2 X+\kappa_2)^TR(t,i)(\vartheta_2 X+\kappa_2) \quad (122)$$

The derivative of the Hamiltonian is then

$$D_x[\mathcal{H}] = A^T(t,i)p + \vartheta_2^TB^T(t,i)p + \sum_{j=1}^d (C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))q_j - Q(t,i)X$$
$$-2S^T(t,i)\vartheta_2 X - S^T(t,i)\kappa_2 - \vartheta_2^TR(t,i)\vartheta_2 X - \vartheta_2^TR(t,i)\kappa_2 \quad (123)$$

We try an ansatz for p of the form:

$$p = \varphi(t,\eta(t))X(t) + \psi(t,\eta(t))$$

where $\varphi(t,\eta(t)) \in \mathbb{R}^{n\times n}$ and $\psi(t,\eta) \in \mathbb{R}^n$. Applying Ito's formula to $p = \varphi(t,\eta(t))X(t) + \psi(t,\eta(t))$, we have

$$\mathrm{d}p = \sum_{i=1}^k \chi_{\{\eta(t-)=i\}}\Big[\Big(\varphi(t,i)A(t,i)+\Delta\varphi(t,i)\Big)X + \Delta\psi(t,i) + \varphi(t,i)B(t,i)(\vartheta_2 X+\kappa_2)\Big]\mathrm{d}t$$
$$+\varphi(t,\eta(t-))\sigma(t,\eta(t-))\mathrm{d}W$$
$$+\sum_{i\neq j}\Big[\Big(\varphi(t,j)-\varphi(t,i)\Big)X + \psi(t,i)-\psi(t,i)\Big]\mathrm{d}Q_{ij}$$

where

$$\Delta\varphi(t,i) = \frac{\partial\varphi}{\partial t}(t,i) + \sum_{j=1}^k q_{ij}(\varphi(t,j)-\varphi(t,i))$$
$$\Delta\psi(t,i) = \frac{\partial\psi}{\partial t}(t,i) + \sum_{j=1}^k q_{ij}(\psi(t,j)-\psi(t,i))$$

Equating coefficients with (118) and setting $\eta(t-) = i$, we get

$$\Big(\varphi(t,i)A(t,i)+\Delta\varphi(t,i)\Big)X + \Delta\psi(t,i) + \varphi(t,i)B(t,i)(\vartheta_2 X+\kappa_2) = -D_x[\mathcal{H}] \quad (124)$$
$$\varphi(t,i)\sigma(t,X(t),\pi(t),i) = q(t) \quad (125)$$
$$\Big(\varphi(t,j)-\varphi(t,i)\Big)X + \psi(t,i)-\psi(t,i) = s_{ij}(t) \quad (126)$$
$$B^T(t,i)(\varphi X+\psi) + \sum_{j=1}^d D_j^T(t,i)q_j - S(t,i)X - R(t,i)(\vartheta_2 X+\kappa_2) = 0 \quad (127)$$

where the last equation is the Hamiltonian condition (120). We now substitute $q(t)$ from the second equation into the rest, and our system becomes

$$\Big(\varphi(t,i)A(t,i)+\Delta\varphi(t,i)\Big)X + \Delta\psi(t,i) + \varphi(t,i)B(t,i)(\vartheta_2 X+\kappa_2) = -A^T(t,i)(\varphi X+\psi)$$
$$-\vartheta_2^TB^T(t,i)(\varphi X+\psi) - \sum_{j=1}^d (C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))\varphi(C_j(t,i)X + D_j(t,i)(\vartheta_2 X+\kappa_2))$$

$$+Q(t,i)X + 2S^T(t,i)\vartheta_2 X + S^T(t,i)\kappa_2 + \vartheta_2^T R(t,i)\vartheta_2 X + \vartheta_2^T R(t,i)\kappa_2 \tag{128}$$

$$\Big(\varphi(t,j) - \varphi(t,i)\Big)X + \psi(t,i) - \psi(t,i) = s_{ij}(t) \tag{129}$$

$$B^T(t,i)(\varphi X + \psi) + \sum_{j=1}^{d} D_j^T(t,i)\varphi(C_j(t,i)X + D_j(t,i)(\vartheta_2 X + \kappa_2))$$
$$-S(t,i)X - R(t,i)(\vartheta_2 X + \kappa_2) = 0 \tag{130}$$

From (130) we find the optimal control $\pi^* = \vartheta_2 X + \kappa_2$:

$$\pi^* = \left[\sum_{j=1}^{d} D_j(t,i)\varphi D_j(t,i) - R(t,i)\right]^{-1}\left[S(t,i)X - \sum_{j=1}^{d} D_j^T(t,i)\varphi C_j(t,i)X - B^T(t,i)\varphi X - B^T(t,i)\psi\right] \tag{131}$$

i.e.,

$$\vartheta_2 = \left[\sum_{j=1}^{d} D_j^T(t,i)\varphi D_j(t,i) - R(t,i)\right]^{-1}\left(S(t,i) - B^T(t,i)\varphi - \sum_{j=1}^{d} D_j^T(t,i)\varphi C_j(t,i)\right)$$
$$\kappa_2 = -\left[\sum_{j=1}^{d} D_j^T(t,i)\varphi D_j(t,i) - R(t,i)\right]^{-1} B^T(t,i)\psi.$$

We rewrite equation (128) as

$$\Bigg[\frac{\partial\varphi}{\partial t}(t,i) + \sum_{j=1}^{k} q_{ij}(\varphi(t,j) - \varphi(t,i)) + \varphi(t,i)A(t,i) + A^T(t,i)\varphi + 2\vartheta_2^T B^T(t,i)\varphi$$
$$+\sum_{j=1}^{d}(C_j^T(t,i) + \vartheta_2^T D_j^T(t,i))\varphi(C_j(t,i) + D_j(t,i)\vartheta_2) - Q(t,i) - 2S^T(t,i)\vartheta_2 - \vartheta_2^T R(t,i)\vartheta_2\Bigg]X$$
$$+\Bigg[\frac{\partial\psi}{\partial t}(t,i) + \sum_{j=1}^{k} q_{ij}(\psi(t,j) - \psi(t,i)) + \varphi(t,i)B(t,i)\kappa_2 + A^T(t,i)\psi + \vartheta_2^T B^T(t,i)\psi$$
$$+\sum_{j=1}^{d}(C_j^T(t,i) + \vartheta_2^T D_j^T(t,i))\varphi D_j(t,i)\kappa_2 - S^T(t,i)\kappa_2 - \vartheta_2^T R(t,i)\kappa_2\Bigg] = 0$$

Since this must be true for all X, the coefficient in front of X must be equal to zero, so we get

$$\frac{\partial\varphi}{\partial t}(t,i) + 2A^T(t,i)\varphi(t,i) + 2\vartheta_2^T B^T(t,i)\varphi(t,i)$$
$$+\sum_{j=1}^{d}(C_j^T(t,i) + \vartheta_2^T D_j^T(t,i))\varphi(t,i)(C_j(t,i) + D_j(t,i)\vartheta_2)$$
$$-Q(t,i) - 2S^T(t,i)\vartheta_2 - \vartheta_2^T R(t,i)\vartheta_2 + \sum_{j=1}^{k} q_{ij}(\varphi(t,j) - \varphi(t,i)) = 0 \tag{132}$$

$$\begin{aligned}\frac{\partial\psi}{\partial t}(t,i)+\varphi(t,i)B(t,i)\kappa_2+A^T(t,i)\psi(t,i)+\vartheta_2^T B^T(t,i)\psi(t,i)\\+\sum_{j=1}^{d}(C_j^T(t,i)+\vartheta_2^T D_j^T(t,i))\varphi(t,i)D_j(t,i)\kappa_2\\-S^T(t,i)\kappa_2-\vartheta_2^T R(t,i)\kappa_2+\sum_{j=1}^{k}q_{ij}(\psi(t,j)-\psi(t,i))=0\end{aligned} \tag{133}$$

with terminal conditions

$$\varphi(T,i)=-G(T,i),\quad \psi(T,i)=-L(T,i). \tag{134}$$

2.3.2 Equivalence of Primal HJB and Primal BSDE

The optimal control from the primal HJB was given by (110):

$$\pi^*=\left[\sum_{j=1}^{d}D_j^T(t,i)PD_j(t,i)-R(t,i)\right]^{-1}\left[S(t,i)x-\sum_{j=1}^{d}D_j^T(t,i)PC_j(t,i)x-B^T(t,i)Px-B^T(t,i)M\right]$$

and from the primal BSDE (131)

$$\pi^*=\left[\sum_{j=1}^{d}D_j(t,i)\varphi D_j(t,i)-R(t,i)\right]^{-1}\left[S(t,i)X-\sum_{j=1}^{d}D_j^T(t,i)\varphi C_j(t,i)X-B^T(t,i)\varphi X-B^T(t,i)\psi\right]$$

Comparing, we see that the equations are the same and we get the relation

$$\varphi(t,i)=P(t,i),\quad \psi(t,i)=M(t,i),$$

so $\vartheta_1=\vartheta_2$ and $\kappa_1=\kappa_2$. The ODE from the primal HJB for $P(t,i)$ is given by (111):

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t}+A^T(t,i)P(t,i)+\vartheta_1^T B^T(t,i)P(t,i)+\frac{1}{2}\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^T P(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)\\-\frac{1}{2}Q(t,i)-S^T(t,i)\vartheta_1-\frac{1}{2}\vartheta_1^T R(t,i)\vartheta_1+\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))=0\end{aligned}$$

Letting $P(t,i)=\varphi(t,i)$ and $\vartheta_1=\vartheta_2,\kappa_1=\kappa_2$, and multiplying by 2 on both sides we get

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t,i)+2A^T(t,i)\varphi(t,i)+2\vartheta_1^T B^T(t,i)\varphi(t,i)+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^T\varphi(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)\\-Q(t,i)-2S^T(t,i)\vartheta_1-\vartheta_1^T R(t,i)\vartheta_1+\sum_{j\neq i}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i))=0\end{aligned}$$

which is the same ODE as the one from the primal BSDE (132).

Similarly, the ODE from the primal HJB for $M(t,i)$ is given by (112):

$$\frac{\mathrm{d}M}{\mathrm{d}t}(t,i) + A^T M(t,i) + \vartheta_1^T B^T M(t,i) + P(t,i)B\kappa_1 + \sum_{j=1}^{d}(C_j + D_j\vartheta_1)^T P(t,i) D_j \kappa_1$$
$$-S^T\kappa_1 - \vartheta_1^T R\kappa_1 + \sum_{j\neq i}^{k} q_{ij}(M(t,j) - M(t,i)) = 0$$

Substituting $P(t,i) = \varphi(t,i), M(t,i) = \psi(t,i)$ we get

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t,i) + A^T(t,i)\psi(t,i) + \vartheta_1^T B^T(t,i)\psi(t,i) + \varphi(t,i)B(t,i)\kappa_1 + \sum_{j=1}^{d}(C_j(t,i) + D_j(t,i)\vartheta_1)^T \varphi(t,i) D_j \kappa_1$$
$$-S^T(t,i)\kappa_1 - \vartheta_1^T R(t,i)\kappa_1 + \sum_{j\neq i}^{k} q_{ij}(\psi(t,j) - \psi(t,i)) = 0$$

the same equations as the one from the primal BSDE (133). The terminal conditions from the primal HJB are (114)

$$P(T,i) = -G(T,i), \quad M(T,i) = -L(T,i), \quad N(T,i) = 0$$

and the terminal conditions from the primal BSDE are (134)

$$\varphi(T,i) = -G(T,i), \quad \psi(T,i) = -L(T,i). \tag{135}$$

As we can see, the terminal conditions are the same, so we can conclude that the two methods are equivalent.

2.4 Dual HJB

2.4.1 The dual HJB

The dual HJB is given by

$$0=\frac{\partial v}{\partial t}(t,y,i)+\sup_{\alpha,\beta,\gamma}\Bigg\{\Big(\alpha-A^Ty-\sum_{j=1}^{d}C_j^T\beta_j-\sum_{j=1}^{k}q_{ij}(\gamma_j-\gamma_i)^T\Big)D_y[v(t,y,i)]+\frac{1}{2}\sum_{j=1}^{d}\beta_j^TD_y^2[v(t,y,i)]\beta_j$$
$$-\phi\Big(t,\alpha,B^Ty+\sum_{j=1}^{d}D_j^T\beta_j\Big)+\sum_{j\neq i}^{k}q_{ij}v(t,y+\gamma_j-\gamma_i,j)\Bigg\}$$

with terminal condition

$$v(T,y,i)=-h(y,i)=-\frac{1}{2}(y^T+L^T(T,i))G^{-1}(T,i)(y+L(T,i)).$$

2.4.2 Finding the optimal control

We find the optimal controls $\alpha,\beta_1,\dots,\beta_d$ by setting the derivatives with respect to α and β_j to zero:

$$D_y[\tilde{v}]-\tilde{Q}\alpha-\tilde{S}^T(B^Ty+\sum_{i=1}^{d}D_i^T\beta_i)=0 \tag{136}$$

$$-C_jD_y[\tilde{v}]+D_y^2[\tilde{v}]\beta_j-D_j(\tilde{S}\alpha+\tilde{R}(B^Ty+\sum_{j=1}^{d}D_j^T\beta_j))=0 \tag{137}$$

This is a system of $d+1$ equations in $d+1$ unknowns, so it can be solved and the optimal controls are linear functions of y, which we denote by α^* and β_j^*. The HJB equation is then

$$0=\frac{\partial v}{\partial t}(t,y,i)+\sup_{\gamma}\Bigg\{\Big(\alpha^*-A^Ty-\sum_{j=1}^{d}C_j^T\beta_j^*-\sum_{j=1}^{k}q_{ij}(\gamma_j-\gamma_i)^T\Big)D_y[v(t,y,i)]$$
$$+\frac{1}{2}\sum_{j=1}^{d}\beta_j^{*T}D_y^2[v(t,y,i)]\beta_j^*-\phi\Big(t,\alpha^*,B^Ty+\sum_{j=1}^{d}D_j^T\beta_j^*\Big)+\sum_{j\neq i}^{k}q_{ij}v(t,y+\gamma_j-\gamma_i,j)\Bigg\} \tag{138}$$

2.4.3 Solving the dual HJB

Suppose that $\tilde{v}$ is a quadratic function in y and use the ansatz

$$\tilde{v}(t,y,i)=\frac{1}{2}y^T\tilde{P}(t,i)y+y^T\tilde{M}(t,i)+\tilde{N}(t,i), \tag{139}$$

where $\tilde{P}(t,i)\in\mathbb{R}^{n\times n}$, $\tilde{M}(t,i)\in\mathbb{R}^n$, $\tilde{N}(t,i)\in\mathbb{R}$, with terminal conditions

$$\tilde{P}(T,i)=-G^{-1}(T,i),\tilde{M}(T,i)=-G^{-1}(T,i)L(T,i),\tilde{N}(T,i)=\frac{1}{2}L^T(T,i)G^{-1}(T,i)L(T,i). \tag{140}$$

Then

$$\begin{aligned}
&\frac{\partial \tilde v}{\partial t}(t,y,i) = \frac{1}{2}y^T\frac{\mathrm{d}\tilde P}{\mathrm{d}t}(t,i)y + y^T\frac{\mathrm{d}\tilde M}{\mathrm{d}t}(t,i) + \frac{\mathrm{d}\tilde N}{\mathrm{d}t}(t,i)\\
&D_y[\tilde v(t,y,i)] = \tilde P(t,i)y + \tilde M(t,i)\\
&D_y^2[\tilde v(t,y,i)] = \tilde P(t,i)
\end{aligned}$$

The system of equations from which we derive the optimal controls α^* and β_i^* is now given by

$$\begin{cases} \tilde P y + \tilde M - \tilde Q\alpha - \tilde S^T(B^T Y + \sum_{i=1}^d D_i^T\beta_i) = 0\\ C_i(\tilde P y + \tilde M) - \tilde P\beta_i + D_i(\tilde S\alpha + \tilde R(B^T y + \sum_{j=1}^d D_j^T\beta_j)) = 0\end{cases} \tag{141}$$

We do not solve this system explicitly, however, the solutions for α^* and β^* are linear in y, hence we denote by $\tilde\vartheta$ and $\tilde\kappa$ the coefficients before y and the free coefficient in α^* and similarly for β_j, i.e.

$$\alpha^* = \tilde\vartheta y + \tilde\kappa, \quad \beta_j = \tilde\vartheta_j y + \tilde\kappa_j. \tag{142}$$

Substituting this into the HJB (138) equation we get

$$\begin{aligned}
&\frac{1}{2}y^T\frac{\mathrm{d}\tilde P}{\mathrm{d}t}(t,i)y + y^T\frac{\mathrm{d}\tilde M}{\mathrm{d}t}(t,i) + \frac{\mathrm{d}\tilde N}{\mathrm{d}t}(t,i)\\
&+\sup_\gamma\Bigg\{\Bigg(y^T\tilde\vartheta^T + \tilde\kappa^T - y^T A - \sum_{j=1}^d(\tilde\vartheta_j y + \tilde\kappa_j)^T C_j^T - \sum_{j\neq i}^k q_{ij}(\gamma_j-\gamma_i)^T\Bigg)(\tilde P(t,i)y + \tilde M(t,i))\\
&+\frac{1}{2}\sum_{j=1}^d(\tilde\vartheta_j y + \tilde\kappa_j)^T\tilde P(t,i)(\tilde\vartheta_j y + \tilde\kappa_j) - \phi\Big(t, \tilde\vartheta y + \tilde\kappa, B^T y + \sum_{j=1}^d D_j^T(\tilde\vartheta_j y + \tilde\kappa_j)\Big)\\
&+\sum_{j=1}^k q_{ij}\Big(\frac{1}{2}y^T\tilde P(t,j)y + y^T\tilde M(t,j) + \tilde N(t,j)\Big)\\
&+\sum_{j=1}^k q_{ij}\Big(\frac{1}{2}(\gamma_j-\gamma_i)^T\tilde P(t,j)(\gamma_j-\gamma_i) + (\gamma_j-\gamma_i)^T(\tilde P(t,j)y + \tilde M(t,j))\Big)\Bigg\} = 0
\end{aligned}$$

Setting the derivative w.r.t. γ_j, $j\neq i$ to zero we get

$$(\gamma_j-\gamma_i) = -\tilde P^{-1}(t,j)\Big[(\tilde P(t,j) - \tilde P(t,i))y + \tilde M(t,j) - \tilde M(t,i)\Big]$$

The HJB is then

$$\begin{aligned}
&\frac{1}{2}y^T\frac{\mathrm{d}\tilde P}{\mathrm{d}t}(t,i)y + y^T\frac{\mathrm{d}\tilde M}{\mathrm{d}t}(t,i) + \frac{\mathrm{d}\tilde N}{\mathrm{d}t}(t,i) + \Bigg(y^T\tilde\vartheta^T + \tilde\kappa^T - y^T A - \sum_{j=1}^d(\tilde\vartheta_j y + \tilde\kappa_j)^T C_j^T\Bigg)(\tilde P(t,i)y + \tilde M(t,i))\\
&+\frac{1}{2}\sum_{j=1}^d(\tilde\vartheta_j y + \tilde\kappa_j)^T\tilde P(t,i)(\tilde\vartheta_j y + \tilde\kappa_j) - \phi\Big(t, \tilde\vartheta y + \tilde\kappa, B^T y + \sum_{j=1}^d D_j^T(\tilde\vartheta_j y + \tilde\kappa_j), i\Big)
\end{aligned}$$

$$+\sum_{j\neq i}^{k} q_{ij}\Bigg\{\frac{1}{2}y^T\tilde{P}(t,j)y + y^T\tilde{M}(t,j) + \tilde{N}(t,j)$$

$$-\frac{1}{2}\Big[y^T(\tilde{P}(t,j)-\tilde{P}(t,i)) + \tilde{M}^T(t,j) - \tilde{M}^T(t,i)\Big]\tilde{P}^{-1}(t,j)\Big[(\tilde{P}(t,j)-\tilde{P}(t,i))y + \tilde{M}(t,j) - \tilde{M}(t,i)\Big]$$

Rearranging, we get

$$y^T\Bigg[\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i) + (\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i))\tilde{P}(t,i) + \frac{1}{2}\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\vartheta}_j - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$

$$-\tilde{\vartheta}^T\tilde{S}^T(t,i)\Big(B^T(t,i) + \sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\Big) - \frac{1}{2}\Bigg(B(t,i) + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i(t,i)\Bigg)\tilde{R}(t,i)\Bigg(B^T(t,i) + \sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\Bigg)$$

$$+\sum_{j\neq i}^{k} q_{ij}\Big[\tilde{P}(t,i) - \frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\Big]\Bigg]y$$

$$+y^T\Bigg[\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i) + \tilde{\vartheta}^T\tilde{M}(t,i) - A(t,i)\tilde{M}(t,i) + \tilde{P}(t,i)\tilde{\kappa} - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i)\tilde{M}(t,i) + \sum_{j=1}^{d}\tilde{P}(t,i)C_j(t,i)\tilde{\kappa}_j$$

$$\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\kappa}_j - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j - \Bigg(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Bigg)\tilde{S}(t,i)\tilde{\kappa}$$

$$-\Bigg(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Bigg)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j + \sum_{j\neq i}^{k}q_{ij}\Big[\tilde{M}(t,i) + \tilde{P}(t,i)\tilde{P}^{-1}(t,j)(\tilde{M}(t,j) - \tilde{M}(t,i))\Big]\Bigg]$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i) + \Bigg(\tilde{\kappa}^T - \sum_{j=1}^{d}\tilde{\kappa}_j^T C_j^T(t,i)\Bigg)\tilde{M}(t,i) + \frac{1}{2}\sum_{j=1}^{d}\tilde{\kappa}_j^T\tilde{P}(t,i)\tilde{\kappa}_j$$

$$-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}(t,i)\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^T\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j - \frac{1}{2}\Bigg(\sum_{j=1}^{d}\tilde{\kappa}_j^T D_j(t,i)\Bigg)\tilde{R}(t,i)\Bigg(\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\Bigg)$$

$$+\sum_{j\neq i}^{k}q_{ij}\Big[\tilde{N}(t,j) - \frac{1}{2}(\tilde{M}^T(t,j) - \tilde{M}^T(t,i))\tilde{P}^{-1}(t,j)(\tilde{M}(t,j) - \tilde{M}(t,i))\Big] = 0$$

This equation must equal zero for all y, hence the coefficients in front of the quadratic term, as well as y^T and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i) + (\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i))\tilde{P}(t,i) + \frac{1}{2}\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\vartheta}_j - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$

$$-\tilde{\vartheta}^T\tilde{S}^T(t,i)\Big(B^T(t,i) + \sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\Big) - \frac{1}{2}\Bigg(B(t,i) + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i(t,i)\Bigg)\tilde{R}(t,i)\Bigg(B^T(t,i) + \sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\Bigg)$$

$$+\sum_{j\neq i}^{k} q_{ij}\Big[\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\Big]=0 \tag{143}$$

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i)+\tilde{\vartheta}^T\tilde{M}(t,i)-A(t,i)\tilde{M}(t,i)+\tilde{P}(t,i)\tilde{\kappa}-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i)\tilde{M}(t,i)+\sum_{j=1}^{d}\tilde{P}(t,i)C_j(t,i)\tilde{\kappa}_j\\
&\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa}\\
&-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j+\sum_{j\neq i}^{k}q_{ij}\Big[\tilde{M}(t,i)+\tilde{P}(t,i)\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))\Big]=0
\end{aligned} \tag{144}$$

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i)+\Big(\tilde{\kappa}^T-\sum_{j=1}^{d}\tilde{\kappa}_j^T C_j^T(t,i)\Big)\tilde{M}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\kappa}_j^T\tilde{P}(t,i)\tilde{\kappa}_j\\
&-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\kappa}^T\tilde{S}^T\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\frac{1}{2}\Big(\sum_{j=1}^{d}\tilde{\kappa}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\Big)\\
&+\sum_{j\neq i}^{k}q_{ij}\Big[\tilde{N}(t,j)-\frac{1}{2}(\tilde{M}^T(t,j)-\tilde{M}^T(t,i))\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))\Big]=0
\end{aligned} \tag{145}$$

where $\tilde{\vartheta},\tilde{\kappa},\tilde{\vartheta}_j$ and $\tilde{\kappa}_j$ satisfy the system of equations (141):

$$\begin{cases}
\tilde{P}y+\tilde{M}-\tilde{Q}\alpha-\tilde{S}^T(B^TY+\sum_{i=1}^{d}D_i^T\beta_i)=0\\
C_i(\tilde{P}y+\tilde{M})-\tilde{P}\beta_i+D_i(\tilde{S}\alpha+\tilde{R}(B^Ty+\sum_{j=1}^{d}D_j^T\beta_j))=0
\end{cases}$$

and the terminal conditions are given by (140):

$$\tilde{P}(T,i)=-G^{-1}(T,i),\tilde{M}(T,i)=-G^{-1}(T,i)L(T,i),\tilde{N}(T,i)=\frac{1}{2}L^T(T,i)G^{-1}(T,i)L(T,i). \tag{146}$$

2.5 Dual BSDE

2.5.1 Solution via the Dual BSDE

Recall that the SDE (89) describing the state process is given by

$$\mathrm{d}Y = \left[\alpha(t) - A(t,\eta(t-))^T Y(t) - \sum_{j=1}^{d} C_j^T(t,\eta(t-))\beta_j(t)\right]\mathrm{d}t + \sum_{j=1}^{d}\beta_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^{k}\gamma_j(t)\mathrm{d}M_j(t)$$
$$Y(t_0) = y$$

and the cost functional is

$$\inf_{\alpha,\beta_1,\dots,\beta_d} \mathbb{E}\left[\int_{t_0}^{T} \phi\left(t,\alpha,B^T(t,i)Y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j,\eta(t)\right)\mathrm{d}t + h(Y(T),\eta(t)))\right]$$

where ϕ is given in (96):

$$\phi(t,\alpha,\beta,i) = \frac{1}{2}\alpha^T\tilde{Q}(t,i)\alpha + \alpha^T\tilde{S}^T(t,i)\beta + \frac{1}{2}\beta^T\tilde{R}(t,i)\beta$$

and h is given in (97). The Hamiltonian $\tilde{\mathcal{H}} : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^{n(d+1)}\times I\times\mathbb{R}^n\times\mathbb{R}^{nd}\times\mathbb{R}^{k\times nk}\to\mathbb{R}$ for the dual problem is defined as

$$\begin{aligned}\tilde{\mathcal{H}}(t,Y,\alpha,\beta_1,\dots,\beta_d,i,p,q,s) &= -\phi\left(t,\alpha,B^T(t,i)Y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j\right)\\ &+p^T\left(\alpha - A^T(t,i)Y - \sum_{j=1}^{d} C_j^T(t,i)\beta_j\right) + \sum_{j=1}^{d}\beta_j^T q_j + \sum_{j=1}^{k} s_{ij}\gamma_j \end{aligned} \tag{147}$$

where $p(t)\in\mathbb{R}^n, q(t)\in\mathbb{R}^{n\times d}$ and $s(t) = (s^{(1)}(t),\dots,s^{(n)}(t))$, where $s^{(l)}(t)\in\mathbb{R}^{k\times k}$ for $l = 1,\dots,n$. The adjoint equations are given by the system

$$\begin{cases}\mathrm{d}p &= -D_y[\tilde{\mathcal{H}}(t,Y,\alpha,\beta_1,\dots,\beta_d,\eta(t-),p,q,s]\mathrm{d}t + \sum_{j=1}^{d} q_j(t)\mathrm{d}W_j(t) + s(t)\cdot\mathrm{d}\mathcal{Q}(t)\\ p(T) &= -D_y[h(Y(T),\eta(T))] = -G^{-1}(T,\eta(T))Y(T) - G^{-1}(T,\eta(T))L(T,\eta(T))\end{cases} \tag{148}$$

where

$$s(t)\cdot\mathrm{d}\mathcal{Q}(t) = \left(\sum_{j\neq i} s_{ij}^{(1)}(t)\mathrm{d}\mathcal{Q}_{ij}(t),\dots,\sum_{j\neq i} s_{ij}^{(n)}(t)\mathrm{d}\mathcal{Q}_{ij}(t)\right)^T.$$

Due to the Stochastic Maximum Principle, the optimal controls are found by setting $D_\alpha[\tilde{\mathcal{H}}]$, $D_{\beta_j}[\tilde{\mathcal{H}}]$ and $D_{\gamma_j}[\tilde{\mathcal{H}}]$ to zero, so we get the system

$$D_\alpha[\tilde{\mathcal{H}}] = p - \tilde{Q}(t,i)\alpha - \tilde{S}^T(t,i)\left(B^T(t,i)Y + \sum_{j=1} D_j^T(t,i)\beta_j\right) = 0 \tag{149}$$

$$D_{\beta_j}[\tilde{\mathcal{H}}] = q_j - C_j(t,i)p - D_j(t,i)\tilde{S}(t,i)\alpha - D_j(t,i)\tilde{R}(t,i)\left(B^T(t,i)Y + \sum_{k=1}^{d} D_k^T(t,i)\beta_k\right) = 0 \tag{150}$$

$$D_{\gamma_j}[\tilde{\mathcal{H}}] = s_{ij} = 0 \tag{151}$$

Without the last condition, there are $d+1$ equations in $d+1$ unknowns, so we know that we can find a linear solution for the controls, which we denote as

$$\alpha^* = \tilde{\vartheta} Y + \tilde{\kappa}, \quad \beta_j^* = \tilde{\vartheta}_j Y + \tilde{\kappa}_j, \quad j \in \{1, \ldots, d\}$$

Substituting into the Hamiltonian (147) we get

$$\begin{aligned}\tilde{\mathcal{H}} = p^T(\tilde{\vartheta} Y + \tilde{\kappa}) - p^T A^T(t,i) Y - p^T \sum_{j=1}^{d} C_j^T(t,i)(\tilde{\vartheta}_j Y + \tilde{\kappa}_j) + \sum_{j=1}^{d} (Y^T \tilde{\vartheta}_j^T + \tilde{\kappa}_j^T) q_j + \sum_{j=1}^{k} s_{ij}\gamma_j \\ -\phi\left(t, \tilde{\vartheta} Y + \tilde{\kappa}, B^T(t,i) Y + \sum_{j=1}^{d} D_j^T(t,i)(\tilde{\vartheta}_j Y + \tilde{\kappa}_j)\right)\end{aligned}$$

The derivative of the Hamiltonian w.r.t. Y is then

$$\begin{aligned}D_y[\tilde{\mathcal{H}}] = \tilde{\vartheta}^T p - A(t,i)p - \sum_{j=1}^{d} \tilde{\vartheta}_j^T C_j(t,i) p + \sum_{j=1}^{d} \tilde{\vartheta}_j^T q_j - \tilde{\vartheta}^T \tilde{Q}(t,i)\tilde{\vartheta} Y - \tilde{\vartheta}^T \tilde{Q}(t,i)\tilde{\kappa} \\ -2\tilde{\vartheta}^T \tilde{S}(t,i)\left(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\right) Y - \tilde{\vartheta}^T \tilde{S}^T(t,i) \sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j \\ -\left(B(t,i) + \sum_{j=1}^{d} \tilde{\vartheta}_j^T D_j(t,i)\right)\tilde{S}(t,i)\tilde{\kappa} - \left(B(t,i) + \sum_{j=1}^{d} \tilde{\vartheta}_j^T D_j(t,i)\right)\tilde{R}(t,i)\left(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\right) Y \\ -\left(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\right)\tilde{R}(t,i) \sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j\end{aligned} \tag{152}$$

We try an ansatz for p of the form

$$p = \varphi(t, \eta(t)) Y + \psi(t, \eta(t)).$$

where $\varphi(t,\eta(t)) \in \mathbb{R}^{n\times n}$ and $\psi(t,\eta(t)) \in \mathbb{R}^n$. Applying Ito's formula to $p = \varphi(t,\eta(t))Y(t) + \psi(t,\eta(t))$, we have

$$\begin{aligned}\mathrm{d}p = \sum_{i=1}^{k} \chi_{\{\eta(t-)=i\}}\Bigg[(\Delta\varphi(t,i) - \varphi(t,i)A(t,i)^T)Y + \Delta\psi(t,i) + \varphi(t,i)\alpha - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)\beta_j \\ -\varphi(t,i)\sum_{j=1}^{k} q_{ij}(\gamma_j - \gamma_i)\Bigg]\mathrm{d}t + \sum_{j=1}^{d} \varphi(t,i)\beta_j \mathrm{d}W_j \\ + \sum_{i\neq j}^{k} \Big[\big(\varphi(t,j) - \varphi(t,i)\big)Y + \psi(t,i) - \psi(t,i) + \varphi(t,j)(\gamma_j - \gamma_i)\Big]\mathrm{d}\mathcal{Q}_{ij}\end{aligned} \tag{153}$$

where

$$\Delta\varphi(t,i) = \frac{\partial\varphi}{\partial t}(t,i) + \sum_{j=1}^{k} q_{ij}(\varphi(t,j)-\varphi(t,i))$$
$$\Delta\psi(t,i) = \frac{\partial\psi}{\partial t}(t,i) + \sum_{j=1}^{k} q_{ij}(\psi(t,j)-\psi(t,i))$$

Equating the coefficients of (153) and (148) and setting $\eta(t-)=i$ we get

$$(\Delta\varphi(t,i)-\varphi(t,i)A(t,i)^T)Y + \Delta\psi(t,i) + \varphi(t,i)(\tilde{\vartheta}Y+\tilde{\kappa}) - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)(\tilde{\vartheta}_j Y + \tilde{\kappa}_j)$$
$$-\varphi(t,i)\sum_{j=1}^{k} q_{ij}(\gamma_j-\gamma_i) = -D_y[\tilde{\mathcal{H}}] \tag{154}$$
$$\varphi(t,i)\tilde{\vartheta}_j Y + \varphi(t,i)\tilde{\kappa}_j = q_j, \quad \forall j\in\{1,\dots,d\} \tag{155}$$
$$\Big(\varphi(t,j)-\varphi(t,i)\Big)Y + \psi(t,i)-\psi(t,i)+\varphi(t,j)(\gamma_j-\gamma_i) = s_{ij}(t) \tag{156}$$

where the RHS of (154) is given by (152). From (156) and (151) we get

$$\gamma_j-\gamma_i = -\varphi(t,j)^{-1}\Bigg(\Big(\varphi(t,j)-\varphi(t,i)\Big)Y + \psi(t,j)-\psi(t,i)\Bigg) \tag{157}$$

We now substitute q_j from equation (155) and $\gamma_j-\gamma_i$ from (157) into (154) and we get

$$\begin{aligned}
&(\Delta\varphi(t,i)-\varphi(t,i)A(t,i)^T)Y + \Delta\psi(t,i) + \varphi(t,i)(\tilde{\vartheta}Y+\tilde{\kappa}) - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)(\tilde{\vartheta}_j Y + \tilde{\kappa}_j)\\
&+\varphi(t,i)\sum_{j=1}^{k} q_{ij}\varphi(t,j)^{-1}\Bigg(\Big(\varphi(t,j)-\varphi(t,i)\Big)Y + \psi(t,j)-\psi(t,i)\Bigg) =\\
&(-\tilde{\vartheta}^T + A(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i))(\varphi(t,i)Y+\psi(t,i)) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)(\tilde{\vartheta}_j Y+\tilde{\kappa}_j)\\
&+\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}Y + \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa} + 2\tilde{\vartheta}^T\tilde{S}(t,i)\Bigg(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Bigg)Y + \tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\\
&+\Bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Bigg)\tilde{S}(t,i)\tilde{\kappa} + \Bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Bigg)\tilde{R}(t,i)\Bigg(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Bigg)Y\\
&+\Bigg(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Bigg)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j
\end{aligned} \tag{158}$$

We rewrite equation (158) as

$$\Bigg[\Delta\varphi(t,i)+\varphi(t,i)\Big(\tilde{\vartheta}-A^T(t,i)-\sum_{j=1}^{d}C_j^T(t,i)\tilde{\vartheta}_j\Big)+\Big(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\Big)\varphi(t,i)$$
$$+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\vartheta}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}-2\tilde{\vartheta}^T\tilde{S}(t,i)\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)$$
$$-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)+\sum_{j=1}^{k}q_{ij}(\varphi(t,i)-\varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i))\Bigg]Y$$
$$+\Bigg[\Delta\psi(t,i)+\varphi(t,i)\tilde{\kappa}-\varphi(t,i)\sum_{j=1}^{d}C_j^T(t,i)\tilde{\kappa}_j+\Big(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\Big)\psi(t,i)$$
$$+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa}$$
$$-\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j+\sum_{j=1}^{k}q_{ij}\varphi(t,i)\varphi^{-1}(t,j)(\psi(t,j)-\psi(t,i))\Bigg]=0$$

Since this must be true for all Y, the coefficient in front of Y must be equal to zero, so we get the ODEs

$$\frac{\partial\varphi}{\partial t}(t,i)+2\Big(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\Big)\varphi(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\vartheta}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$
$$-2\tilde{\vartheta}^T\tilde{S}(t,i)\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)$$
$$+\sum_{j=1}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i))=0 \tag{159}$$

$$\frac{\partial\psi}{\partial t}(t,i)+\varphi(t,i)\tilde{\kappa}-\varphi(t,i)\sum_{j=1}^{d}C_j^T(t,i)\tilde{\kappa}_j+\Big(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\Big)\psi(t,i)$$
$$+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j$$
$$-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa}-\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j$$
$$+\sum_{i=1}^{k}q_{ij}\Big[\psi(t,j)-\psi(t,i)+\varphi(t,i)\varphi^{-1}(t,j)(\psi(t,j)-\psi(t,i))\Big]=0, \tag{160}$$

with terminal conditions given by

$$\varphi(T,\eta(T)) = -G^{-1}(T,\eta(T)), \quad \psi(T,\eta(T)) = -G^{-1}(T,\eta(T))L(T,\eta(T)).$$

2.5.2 Equivalence of Dual HJB and Dual BSDE

From the dual HJB we get that the controls are given by $\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}$ and $\beta_j^* = \tilde{\vartheta}_j y + \tilde{\kappa}_j$, which can be computed from the system of equations (136) and (137):

$$\tilde{P}(t,i)y + \tilde{M}(t,i) - \tilde{Q}(t,i)\alpha - \tilde{S}^T(t,i)\left(B^T(t,i)y + \sum_{j=1}^d D_j^T(t,i)\beta_j\right) = 0$$

$$C_j(t,i)(\tilde{P}(t,i)y + \tilde{M}(t,i)) - \tilde{P}(t,i)\beta_j + D_j(t,i)\left(\tilde{S}(t,i)\alpha + \tilde{R}(t,i)\left(B^T(t,i)y + \sum_{j=1}^d D_j^T(t,i)\beta_j\right)\right) = 0$$

Similarly, the system of equations from the dual BSDE are given by (149) and (150)

$$\varphi(t,i)Y + \psi(t,i) - \tilde{Q}(t,i)\alpha - \tilde{S}^T(t,i)\left(B^T(t,i)Y + \sum_{j=1}^d D_j^T(t,i)\beta_j\right) = 0$$

$$C_j(t,i)(\varphi(t,i)Y + \psi(t,i)) - \varphi(t,i)\beta_j + D_j(t,i)\left(\tilde{S}(t,i)\alpha + \tilde{R}(t,i)\left(B^T(t,i)Y + \sum_{k=1}^d D_k^T(t,i)\beta_k\right)\right) = 0$$

These equations are the same, therefore we get the relation

$$\varphi(t,i) = \tilde{P}(t,i), \quad \psi(t,i) = \tilde{M}(t,i)$$

The first ODE from the dual HJB is (143):

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i) + (\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t,i))\tilde{P}(t,i) + \frac{1}{2}\sum_{j=1}^d \tilde{\vartheta}_j^T \tilde{P}(t,i)\tilde{\vartheta}_j - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta} \\ -\tilde{\vartheta}^T\tilde{S}^T(t,i)\left(B^T(t,i) + \sum_{i=1}^d D_i^T(t,i)\tilde{\vartheta}_i\right) - \frac{1}{2}\left(B(t,i) + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i(t,i)\right)\tilde{R}(t,i)\left(B^T(t,i) + \sum_{i=1}^d D_i^T(t,i)\tilde{\vartheta}_i\right) \\ + \sum_{j\neq i}^k q_{ij}\left[\tilde{P}(t,i) - \frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\right] = 0\end{aligned}$$

Letting $\tilde{P}(t,i) = \varphi(t,i)$ and multiplying by 2 we get

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t,i) + 2(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t,i))\varphi(t,i) + \sum_{j=1}^d \tilde{\vartheta}_j^T \varphi(t,i)\tilde{\vartheta}_j - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta} \\ -2\tilde{\vartheta}^T\tilde{S}^T(t,i)\left(B^T(t,i) + \sum_{i=1}^d D_i^T(t,i)\tilde{\vartheta}_i\right) - \left(B(t,i) + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i(t,i)\right)\tilde{R}(t,i)\left(B^T(t,i) + \sum_{i=1}^d D_i^T(t,i)\tilde{\vartheta}_i\right)\end{aligned}$$

$$+\sum_{j\neq i}^{k} q_{ij}\Big[2\varphi(t,i)-\varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i)\Big]=0$$

which is the same ODE as the first one from the dual BSDE (159):

$$\frac{\partial\varphi}{\partial t}(t,i)+2\left(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\right)\varphi(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\vartheta}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$
$$-2\tilde{\vartheta}^T\tilde{S}(t,i)\left(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\right)-\left(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\right)\tilde{R}(t,i)\left(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\right)$$
$$+\sum_{j=1}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i))=0$$

The first ODE from the dual BSDE is (159), so substituting $\tilde{P}(t,i) = \varphi(t,i)$ we get

$$\begin{aligned}\frac{\partial \tilde{P}}{\partial t}(t,i) + \tilde{P}(t,i)\Big(\tilde{\vartheta} - A^T(t,i) - \sum_{j=1}^{d} C_j^T(t,i)\tilde{\vartheta}_j\Big) + \Big(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d} \tilde{\vartheta}_j^T C_j(t,i)\Big)\tilde{P}(t,i) \\ + \sum_{j=1}^{d} \tilde{\vartheta}_j^T \tilde{P}(t,i)\tilde{\vartheta}_j - \tilde{\vartheta}^T \tilde{Q}(t,i)\tilde{\vartheta} - 2\tilde{\vartheta}^T \tilde{S}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big) \\ -\Big(B(t,i) + \sum_{j=1}^{d} \tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big) + \sum_{j\neq i} q_{ij}(\tilde{P}(t,j) - \tilde{P}(t,i)) \\ + \sum_{j=1}^{k} q_{ij}(\varphi(t,i) - \varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i)) = 0\end{aligned}$$

The first ODE from the dual HJB is (143):

$$\begin{aligned}\frac{1}{2}\frac{\partial \tilde{P}}{\partial t}(t,i) + (\tilde{\vartheta}^T - A - \sum_{j=1}^{d} \tilde{\vartheta}_j^T C_j^T)\tilde{P}(t,i) + \frac{1}{2}\sum_{j=1}^{d} \tilde{\vartheta}_j^T \tilde{P}(t,i)\tilde{\vartheta}_j - \frac{1}{2}\tilde{\vartheta}^T \tilde{Q}\tilde{\vartheta} - \tilde{\vartheta}^T \tilde{S}^T\Big(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i\Big) \\ -\frac{1}{2}\Big(B + \sum_{i=1}^{d} \tilde{\vartheta}_i^T D_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i\Big) + \sum_{j\neq i} \frac{1}{2} q_{ij}(\tilde{P}(t,j) - \tilde{P}(t,i)) \\ + \frac{1}{2}\sum_{j=1}^{k} q_{ij}(\varphi(t,i) - \varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i)) = 0\end{aligned}$$

which is the same equation as the one from the dual BSDE, divided by 2. So we conclude that the two methods are equivalent.

2.6 Equivalence of Primal HJB and Dual HJB

To do:

- Dual Markov Problem
- Derive primal HJB markov
- Derive dual HJB markov