Factorization systems

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November 20, 2024

A factorization system in a category $\mathcal{C}$ consists of two classes of morphisms (L, R), such that both L and R contain isomorphisms and are closed under composition, and every morphism $f : C \to D$ in $\mathcal{C}$ admits a factorization into a morphism $l \in L$ followed by a morphism $r \in R$, which is unique up to unique isomorphism among such factorizations.

$$\begin{tikzcd}
 & E \arrow[dd, "i"', "\cong"] \arrow[dr, "r"] & \\
C \arrow[ur, "l"] \arrow[dr, "l'"'] & & D \\
 & E' \arrow[ur, "r'"'] &
\end{tikzcd}$$

Plan of work:

- show that factorization systems descend to slices
- construct a few elementary examples (in the **Set**, **Grp**, **CRing** etc.)
- show that the left class in a f.s. has the right cancellation property, and, dually, the right class has the left cancellation property
- define the orthogonality relation and show various closure properties of orthogonal complements (under (co)limits, (co)base change, etc.)
- either of the classes in a f.s. determines the other via orthogonal complements
- recognition principle for factorization systems: (L, R) is a factorization system iff every morphism $f : C \to D$ in $\mathcal{C}$ admits an (apriori non-unique) (L, R)-factorization, both classes are replete, and $L \perp R$

If time permits it, we would also like to use the general theory of factorization systems to define (left-exact) modalities and study some of their properties. In particular, we would like to show that every (left-exact) modality determines a (left-exact) reflective subcategory and vice versa.