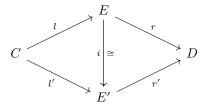
Factorization systems

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0.1 Definitions and basic properties

Definition 1. A factorization system in a category \mathcal{C} consists of two classes of morphisms (L,R), such that both L and R contain isomorphisms and are closed under composition, and every morphism $f:C\to D$ in \mathcal{C} admits a factorization into a morphism $l\in L$ followed by a morphism $r\in R$, which is unique up to unique isomorphism among such factorizations.



Definition 2. If W is a class of morphisms in a category \mathcal{C} and X is an object in \mathcal{C} , we define a class of morphisms W/X in \mathcal{C}/X , given by $f \in W/X$ iff $Uf \in W$, where $U : \mathcal{C}/X \to \mathcal{C}$ is the forgetful functor.

Lemma 3. If (L,R) is a factorization system in a category \mathcal{C} and X is an object in \mathcal{C} , then (L/X,R/X) is a factorization system in \mathcal{C}/X .

Lemma 4. If (L,R) is a factorization system in a category \mathcal{C} , then the intersection of L and R is precisely the class of isomorphisms in \mathcal{C} .

Lemma 5. If (L,R) is a factorization system in a category \mathcal{C} , then R has the left cancellation property and L has the right cancellation property.

Lemma 6. (Epi, Mono) is a factorization system in Set.

0.2 Orthogonality

Definition 7. Given two morphisms $l:A\to B$ and $r:X\to Y$ in \mathcal{C} , we say that l is left-orthogonal to g, or that g is right-orthogonal to l, if for every commutative square

$$\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow & \downarrow & \downarrow r \\
B & \xrightarrow{v} & Y,
\end{array}$$

there exists a unique diagonal filler d making both triangles commute. If L and R are two classes of maps in \mathcal{C} , we say that L is left-orthogonal to R if every morphism in L is left-orthogonal to every morphism in R.

Lemma 8. Given l and r as above, l is left-orthogonal to r iff the square

$$\begin{array}{ccc} \operatorname{Hom}(B,X) & \stackrel{l^*}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} \operatorname{Hom}(A,X) \\ & & \downarrow r_* \\ \operatorname{Hom}(B,Y) & \stackrel{l^*}{-\!\!\!\!-\!\!\!-\!\!\!-} \operatorname{Hom}(A,Y) \end{array}$$

is Cartesian in Set.

Definition 9. Let W be a class of morphisms in a category \mathcal{C} . The left orthogonal complement of W, denoted $^{\perp}W$, consists of those morphisms in \mathcal{C} which are left orthogonal to every morphism in W. The right orthogonal complement of W, denoted W^{\perp} , consists of those morphisms in \mathcal{C} which are right orthogonal to every morphism in W.

Lemma 10. For every class of morphisms W, W^{\perp} contains isomorphisms and is closed under limits, composition and base change, and has the left cancellation property. The left orthogonal complement enjoys dual properties.

Theorem 11. Given two classes of maps L, R in a category \mathcal{C} , there exists a (L, R)-factorization system on \mathcal{C} iff every morphism in \mathcal{C} has a (L, R)-factorization, L is left-orthogonal to R and both L and R are replete.