Factorization systems

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January 19, 2025

0.1 Definitions and basic properties

Definition 1. A factorization system in a category $\mathcal{C}$ consists of two classes of morphisms (L, R), such that both L and R contain isomorphisms and are closed under composition, and every morphism $f : C \to D$ in $\mathcal{C}$ admits a factorization into a morphism $l \in L$ followed by a morphism $r \in R$, which is unique up to unique isomorphism among such factorizations.

$$\begin{tikzcd} & E \arrow[dd, "i"', "\cong"] \arrow[dr, "r"] & \\ C \arrow[ur, "l"] \arrow[dr, "l'"'] & & D \\ & E' \arrow[ur, "r'"'] & \end{tikzcd}$$

Definition 2. If W is a class of morphisms in a category $\mathcal{C}$ and X is an object in $\mathcal{C}$, we define a class of morphisms W/X in $\mathcal{C}/X$, given by $f \in W/X$ iff $Uf \in W$, where $U : \mathcal{C}/X \to \mathcal{C}$ is the forgetful functor.

Lemma 3. *If (L, R) is a factorization system in a category $\mathcal{C}$ and X is an object in $\mathcal{C}$, then $(L/X, R/X)$ is a factorization system in $\mathcal{C}/X$.*

Lemma 4. *If (L, R) is a factorization system in a category $\mathcal{C}$, then the intersection of L and R is precisely the class of isomorpihsms in $\mathcal{C}$.*

Lemma 5. *If (L, R) is a factorization system in a category $\mathcal{C}$, then R has the left cancellation property and L has the right cancellation property.*

Lemma 6. *$(Epi, Mono)$ is a factorization system in Set.*

0.2 Orthogonality

Definition 7. Given two morphisms $l : A \to B$ and $r : X \to Y$ in $\mathcal{C}$, we say that l is left-orthogonal to g, or that g is right-orthogonal to l, if for every commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "d"] & Y, \end{tikzcd}$$

there exists a unique diagonal filler d making both triangles commute. If L and R are two classes of maps in $\mathcal{C}$, we say that L is left-orthogonal to R if every morphism in L is left-orthogonal to every morphism in R.

Lemma 8. *Given l and r as above, l is left-orthogonal to r iff the square*

$$\begin{tikzcd} \mathrm{Hom}(B, X) \arrow[r, "l^*"] \arrow[d, "r_*"'] & \mathrm{Hom}(A, X) \arrow[d, "r_*"] \\ \mathrm{Hom}(B, Y) \arrow[r, "l^*"'] & \mathrm{Hom}(A, Y) \end{tikzcd}$$

is Cartesian in Set.

Definition 9. Let W be a class of morphisms in a category $\mathcal{C}$. The left orthogonal complement of W, denoted ${}^{\perp}W$, consists of those morphisms in $\mathcal{C}$ which are left orthogonal to every morphism in W. The right orthogonal complement of W, denoted $W^{\perp}$, consists of those morphisms in $\mathcal{C}$ which are right orthogonal to every morphism in W.

Lemma 10. *For every class of morphisms W, $W^{\perp}$ contains isomorphisms and is closed under limits, composition and base change, and has the left cancellation property. The left orthogonal complement enjoys dual properties.*

Theorem 11. *Given two classes of maps L, R in a category $\mathcal{C}$, there exists a $(L, R)-$factorization system on $\mathcal{C}$ iff every morphism in $\mathcal{C}$ has a $(L, R)-$factorization, L is left-orthogonal to R and both L and R are replete.*