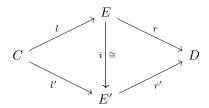# Factorization systems

Ivan Kobe

January 19, 2025

Definitions and basic properties

Definition 1. A factorization system in a category $\mathcal{C}$ consists of two classes of morphisms (L, R), such that both L and R contain isomorphisms and are closed under composition, and every morphism $f : C \to D$ in $\mathcal{C}$ admits a factorization into a morphism $l \in L$ followed by a morphism $r \in R$, which is unique up to unique isomorphism among such factorizations.

Definition 2. If W is a class of morphisms in a category $\mathcal{C}$ and X is an object in $\mathcal{C}$, we define a class of morphisms W/X in $\mathcal{C}/X$, given by $f \in W/X$ iff $Uf \in W$, where $U : \mathcal{C}/X \to \mathcal{C}$ is the forgetful functor.

Lemma 3. *If (L, R) is a factorization system in a category $\mathcal{C}$ and X is an object in $\mathcal{C}$, then $(L/X, R/X)$ is a factorization system in $\mathcal{C}/X$.*

Lemma 4. *If (L, R) is a factorization system in a category $\mathcal{C}$, then the intersection of L and R is precisely the class of isomorpihsms in $\mathcal{C}$.*

Lemma 5. *If (L, R) is a factorization system in a category $\mathcal{C}$, then R has the left cancellation property and L has the right cancellation property.*

Lemma 6. *($Epi, Mono$) is a factorization system in Set.*