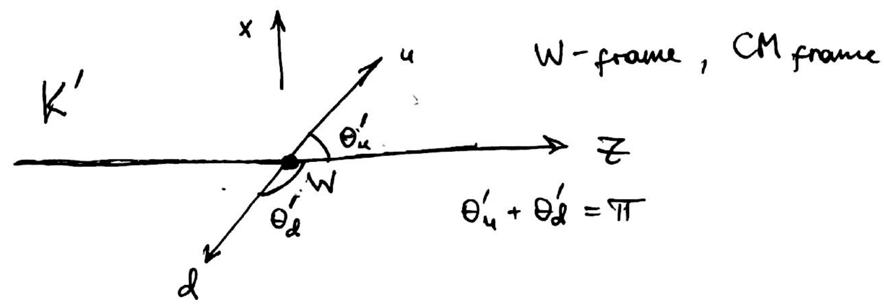
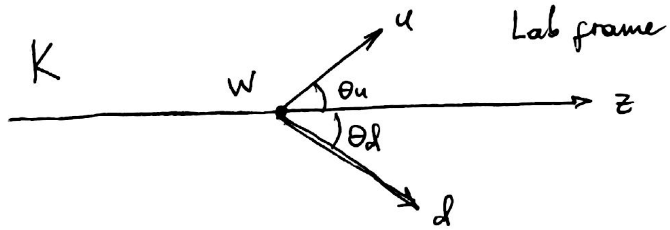



**In the lab frame, $K$:**

$$P_W = (E_W, 0, 0, p_W), \quad P_u = (E_u, p_u \sin\theta_u, 0, p_u \cos\theta_u), \quad P_d = (E_d, -p_d \sin\theta_d, 0, p_d \cos\theta_d) \tag{1}$$

Since $m_u \ll E_u$ and $m_d \ll E_d$,

$$P_u = (E_u, E_u \sin\theta_u, 0, E_u \cos\theta_u), \quad P_d = (E_d, -E_d \sin\theta_d, 0, E_d \cos\theta_d) \tag{2}$$

By conservation of momentum,

$$\begin{cases} E_W = E_u + E_d, \\ E_u \sin\theta_u = E_d \sin\theta_d, \\ p_W = E_u \cos\theta_u + E_d \cos\theta_d. \end{cases} \qquad \begin{array}{r} E_d^2 \cos^2\theta_d = p_W^2 + E_u^2 \cos^2\theta_u - 2 p_W E_u \cos\theta_u \\ + \quad E_u^2 \sin^2\theta_u = E_d^2 \sin^2\theta_d \\ \hline E_d^2 = p_W^2 + E_u^2 - 2 p_W E_u \cos\theta_u \end{array} \tag{3}$$

$$\therefore \cos\theta_u = \frac{p_W^2 + E_u^2 - E_d^2}{2 p_W E_u} \tag{4}$$

$$E^2 = p^2 + m^2 \quad \Rightarrow \quad p_W = \sqrt{E_W^2 - m_W^2} = E_W \sqrt{1 - \left(\frac{m_W}{E_W}\right)^2} \tag{5}$$

**In the CM frame, $K'$:**

$$P'_W = (m_W, 0, 0, 0), \quad P'_u = (E'_u, p'_u \sin\theta'_u, 0, p'_u \cos\theta'_u), \quad P'_d = (E'_d, -p'_d \sin\theta'_d, 0, p'_d \cos\theta'_d). \tag{6}$$

$$P'_u = (E'_u, E'_u \sin\theta'_u, 0, E'_u \cos\theta'_u), \quad P'_d = (E'_d, -E'_d \sin\theta'_d, 0, E'_d \cos\theta'_d). \tag{7}$$

There is no net momentum in the CM frame, so from Eq. (7) and 6,

$$E'_u = E'_d = \frac{m_W}{2}, \quad \text{and} \quad \theta'_u + \theta'_d = \pi \tag{8}$$

$$\sin\theta'_u = \sin\theta'_d, \quad \text{and} \quad \cos\theta'_u = -\cos\theta'_d \tag{9}$$

So,

$$P'_u = \frac{m_W}{2}(1, \sin\theta'_u, 0, \cos\theta'_u), \quad P'_d = \frac{m_W}{2}(1, -\sin\theta'_u, 0, -\cos\theta'_u). \tag{10}$$

**Special case, $\theta'_u = \pi/2$:**
By symmetry,

$$E_u = E_d = \frac{E_W}{2} \tag{11}$$

Substituting into Eq. (4),

$$\cos\theta_u = \frac{p_W^2 + E_u^2 - E_d^2}{2p_W E_u} = \frac{p_W}{2E_u} = \frac{p_W}{E_W} \tag{12}$$

$$\sin\theta_u = \sqrt{1-\cos^2\theta_u} = \sqrt{1-\frac{p_W^2}{E_W^2}} \tag{13}$$

Applying Eq. (5),

$$\sin\theta_u = \sqrt{1-\left(1-\left(\frac{m_W}{E_W}\right)^2\right)} = \frac{m_W}{E_W} \tag{14}$$

For high energy $W$, the angle between the transversely emitted quarks in the lab frame can be approximated as

$$\boxed{\theta = 2\theta_u \approx 2\sin\theta_u = \frac{2m_W}{E_W}} \tag{15}$$

This is in fact asymptotically minimum angle.

**General case:**

$$P_W = \Lambda P'_W \quad \Leftrightarrow \quad \begin{pmatrix} E_W \\ 0 \\ 0 \\ p_W \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} m_W \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} E_W = \gamma m_W \\ p_W = \gamma\beta m_W \end{cases} \tag{16}$$

$$\gamma = \frac{E_W}{m_W}, \quad \gamma\beta = \frac{p_W}{m_W}, \quad \beta = \frac{p_W}{E_W} \tag{17}$$

$$E_q \begin{pmatrix} 1 \\ \pm\sin\theta_q \\ 0 \\ \cos\theta_q \end{pmatrix} = P_q = \Lambda P'_q = \frac{m_W}{2} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ \pm\sin\theta'_u \\ 0 \\ \pm\cos\theta'_u \end{pmatrix} = \frac{m_W}{2} \begin{pmatrix} \gamma \pm \gamma\beta\cos\theta'_u \\ \pm\sin\theta'_u \\ 0 \\ \gamma\beta \pm \gamma\cos\theta'_u \end{pmatrix} \tag{18}$$

$$E_q = \frac{m_W}{2}(\gamma \pm \gamma\beta\cos\theta'_u) = \frac{E_W}{2}(1 \pm \beta\cos\theta'_u) \tag{19}$$

Substituting into Eq. (4),

$$\begin{aligned} \cos\theta_u &= \frac{p_W^2 + E_u^2 - E_d^2}{2p_W E_u} = \frac{\beta^2 E_W^2 + \frac{E_W^2}{4}\left[(1+\beta\cos\theta'_u)^2 - (1-\beta\cos\theta'_u)^2\right]}{\beta E_W^2 (1+\beta\cos\theta'_u)} \\ &= \frac{\beta^2 + \frac{1}{4}\left[(1+\beta\cos\theta'_u)^2 - (1-\beta\cos\theta'_u)^2\right]}{\beta(1+\beta\cos\theta'_u)} = \frac{\beta^2 + \beta\cos\theta'_u}{\beta(1+\beta\cos\theta'_u)} = \frac{\beta + \cos\theta'_u}{1+\beta\cos\theta'_u} \end{aligned} \tag{20}$$

Likewise,

$$\cos\theta_d = \frac{p_W^2 + E_d^2 - E_u^2}{2p_W E_d} = \frac{\beta^2 E_W^2 + \frac{E_W^2}{4}\left[(1-\beta\cos\theta'_u)^2 - (1+\beta\cos\theta'_u)^2\right]}{\beta E_W^2 (1-\beta\cos\theta'_u)} = \frac{\beta - \cos\theta'_u}{1-\beta\cos\theta'_u} \tag{21}$$

$$\sin\theta_q = \sqrt{1-\cos^2\theta_q} = \sqrt{1-\left(\frac{\beta\pm\cos\theta'_u}{1\pm\beta\cos\theta'_u}\right)^2} = \frac{\sin\theta'_u}{\gamma\pm\gamma\beta\cos\theta'_u} \tag{22}$$

$$\gamma = \frac{E_W}{m_W}, \quad \beta = \frac{p_W}{E_W} = \frac{\sqrt{E_W^2-m_W^2}}{E_W} = \sqrt{1-\left(\frac{m_W}{E_W}\right)^2} = \sqrt{1-\gamma^{-2}} \tag{23}$$

$$\gamma\beta = \frac{p_W}{m_W} = \frac{\sqrt{E_W^2-m_W^2}}{m_W} = \sqrt{\left(\frac{E_W}{m_W}\right)^2-1} = \sqrt{\gamma^2-1} \tag{24}$$

$$\boxed{\sin\theta_q = \frac{\sin\theta'_u}{\gamma\pm\sqrt{\gamma^2-1}\cos\theta'_u}} \tag{25}$$

This is the exact result. Only the approximation of massless quarks has been made, which is very accurate, given the mass of $W$ is $m^W = 80.4$ GeV and the masses of the quarks are $m^u = 2.3$ MeV and $m^d = 4.8$ MeV.

In the limit of large $\gamma$,

$$\sin\theta_q = \frac{\sin\theta'_u}{\gamma(1\pm\cos\theta'_u)} \approx \theta_q \tag{26}$$

The angle between emitted quarks is then

$$\theta \approx \frac{\sin\theta'_u}{\gamma(1+\cos\theta'_u)} + \frac{\sin\theta'_u}{\gamma(1-\cos\theta'_u)} = \frac{2}{\gamma}\frac{1}{\sin\theta'_u} \tag{27}$$

This gives Eq. (15), for $\theta'_u = \pi/2$.

Eq. (25) can be expressed as a tangent. Let $\sin\theta = A/B$, then

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{A/B}{\sqrt{1-A^2/B^2}} = \frac{A}{\sqrt{B^2-A^2}} \tag{28}$$

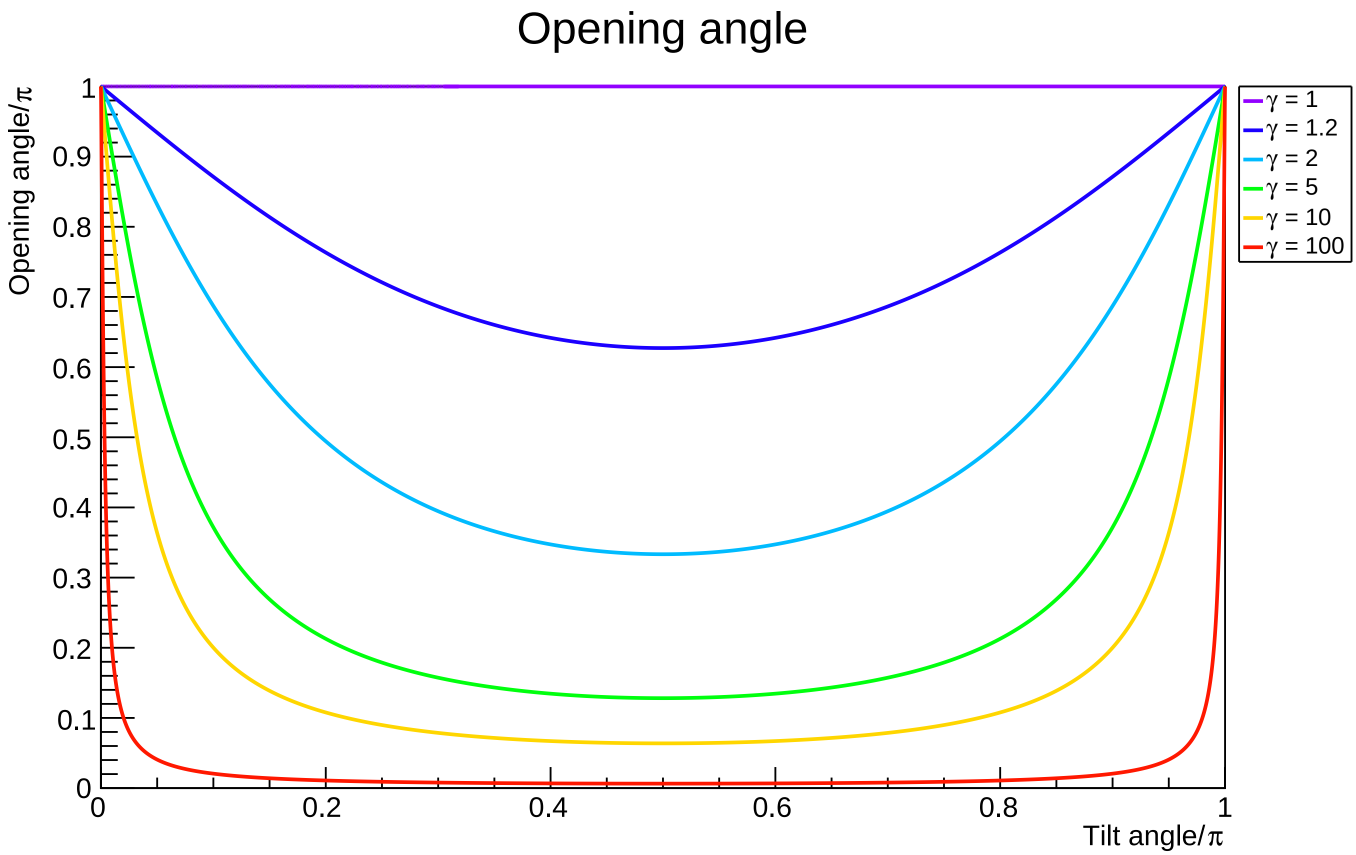
From Eq. (25),

$$\begin{aligned} B^2-A^2 &= \gamma^2 + (\gamma^2-1)\cos^2\theta'_u \pm 2\gamma\sqrt{\gamma^2-1}\cos\theta'_u - \sin^2\theta'_u \\ &= \gamma^2 + \gamma^2\cos^2\theta'_u \pm 2\gamma\sqrt{\gamma^2-1}\cos\theta'_u - 1 \\ &= (\gamma^2-1) \pm 2\gamma\sqrt{\gamma^2-1}\cos\theta'_u + \gamma^2\cos^2\theta'_u \\ &= \left(\gamma\cos\theta'_u \pm \sqrt{\gamma^2-1}\right)^2. \end{aligned} \tag{29}$$

Therefore,

$$\boxed{\tan\theta_q = \frac{\sin\theta'_u}{\gamma\cos\theta'_u\pm\sqrt{\gamma^2-1}}} \tag{30}$$

$$\boxed{\theta_{\text{opening}} = \tan^{-1}\left(\frac{\sin\theta_{\text{tilt}}}{\gamma\cos\theta_{\text{tilt}}+\sqrt{\gamma^2-1}}\right) + \tan^{-1}\left(\frac{\sin\theta_{\text{tilt}}}{\gamma\cos\theta_{\text{tilt}}-\sqrt{\gamma^2-1}}\right)} \tag{31}$$



Opening angle
Opening angle/π
Tilt angle/π
γ = 1
γ = 1.2
γ = 2
γ = 5
γ = 10
γ = 100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1