A vector boson, e.g. W^-, has spin $s^W = 1$, with 3 possible projections on the quantization axis, $s_z^W = \pm 1, 0$. Consider a situation when a W^- decays into 2 quarks, $\bar{u}$ and d. For concreteness, let the quantization axis z be along the momentum vector of W^- in the lab reference frame. With this choice of axes, the W^- spin projection is its helicity.

The produced quarks both have spin 1/2, and 2 possible spin projections on the z' axis (along the momentum of $\bar{u}$), $s_{z'}^q = \pm 1/2$. But since the weak interactions all have left-handed chirality, the helicities of the product quarks are not random. The d quark must be left-handed, $s_{z'}^d = 1/2$, and the $\bar{u}$ quark must be right-handed, $s_{z'}^u = 1/2$.

Initial state	Final state	
W	$\bar{u}$	d
$s^W = 1$	$s^u = 1/2$	$s^d = 1/2$
$s_z^W = \pm 1, 0$	$s_{z'}^u = 1/2$	$s_{z'}^d = 1/2$

By the triangle rule, the possible total final state spin values are

$$\left| s_{z'}^u - s_{z'}^d \right| \le S^f \le s_{z'}^u + s_{z'}^d \quad \Rightarrow \quad S^f = 0, 1. \tag{1}$$

But the total spin projection in the final state is

$$S_{z'}^f = s_{z'}^u + s_{z'}^d = 1, \tag{2}$$

which constrains the total spin to $S^f = 1$.

Thus, there are three possible initial states, each corresponding to a particular W^- helicity, but only one final state. The amplitudes of overlap between the states are:

$$\begin{aligned} A_+ &= \left\langle S^i = 1,\, S_z^i = 1 \middle| S^f = 1,\, S_{z'}^f = 1 \right\rangle = \left\langle \chi_+^i \middle| \chi'^f \right\rangle \\ A_0 &= \left\langle S^i = 1,\, S_z^i = 0 \middle| S^f = 1,\, S_{z'}^f = 1 \right\rangle = \left\langle \chi_0^i \middle| \chi'^f \right\rangle \\ A_- &= \left\langle S^i = 1,\, S_z^i = -1 \middle| S^f = 1,\, S_{z'}^f = 1 \right\rangle = \left\langle \chi_-^i \middle| \chi'^f \right\rangle \end{aligned} \tag{3}$$

To take the inner products, the initial and final state spinors need to be expressed in the same basis. The transformation is given by the rotation operator,

$$\hat{R}(\theta) : \chi'^f \mapsto \chi^f \tag{4}$$

The rotation operator can be expressed in terms of the projection operator as

$$\hat{R}(\theta) = e^{-i\theta \hat{\vec{S}} \cdot \vec{n}} = e^{-i\theta \hat{S}_y}. \tag{5}$$

Note, that we are interested in the passive rotation of the coordinates.

A definite projection state is an eigenstate of $\hat{S}_z$ with the projection as the eigenvalue. We can get the form of $\hat{S}_z$ directly from this statement, for an arbitrary spin:

$$(\hat{S}_z)_{nm} = \left\langle n \middle| \hat{S}_z \middle| m \right\rangle = m \left\langle n \middle| m \right\rangle = m\delta_{nm}. \tag{6}$$

Similarly[1], to get $\hat{S}_x$ and $\hat{S}_y$, we use the raising and lowering operator eigenvalues

$$(\hat{S}_\pm)_{nm} = \left\langle n \middle| \hat{S}_\pm \middle| m \right\rangle = \sqrt{(s \mp m)(s \pm m + 1)}\,\delta_{nm}, \tag{7}$$

[1] Problem 4.53 in J.D. Griffiths Quantum Mechanics (2nd ed.)

and the definition of the raising and lowering operators,

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \quad \text{and} \quad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-). \tag{8}$$

Thus, the spin matrices for $S = 1$ are

$$\hat{S}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{1}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

Applying the Rodrigues' rotation formula

$$e^{-i\theta\hat{S}_n} = \hat{1} - i\sin\theta\,\hat{S}_n - (1-\cos\theta)(\hat{S}_n)^2 \tag{10}$$

to Eq. (5), we get

$$\hat{R}(-\theta) = e^{-i\theta\hat{S}_y} = \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1+\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \end{pmatrix}. \tag{11}$$

The overlap amplitudes defined in Eq. (3) can now be found by

$$\begin{aligned} A_+ &= \langle \chi^i_+ | \chi'^f \rangle = \langle 1 | \hat{R}(\theta) | 1 \rangle = \tfrac{1}{2}(1+\cos\theta) \\ A_0 &= \langle \chi^i_0 | \chi'^f \rangle = \langle 0 | \hat{R}(\theta) | 1 \rangle = \tfrac{1}{\sqrt{2}}\sin\theta \\ A_- &= \langle \chi^i_- | \chi'^f \rangle = \langle -1 | \hat{R}(\theta) | 1 \rangle = \tfrac{1}{2}(1-\cos\theta) \end{aligned} \tag{12}$$

We recognize these as the Wigner $d^J_{m,m'}$ functions for $J = 1$,

$$d^1_{1,1} = \frac{1+\cos\theta}{2}, \quad d^1_{1,0} = \frac{\sin\theta}{\sqrt{2}}, \quad d^1_{1,-1} = \frac{1-\cos\theta}{2}, \quad d^1_{0,0} = \cos\theta. \tag{13}$$

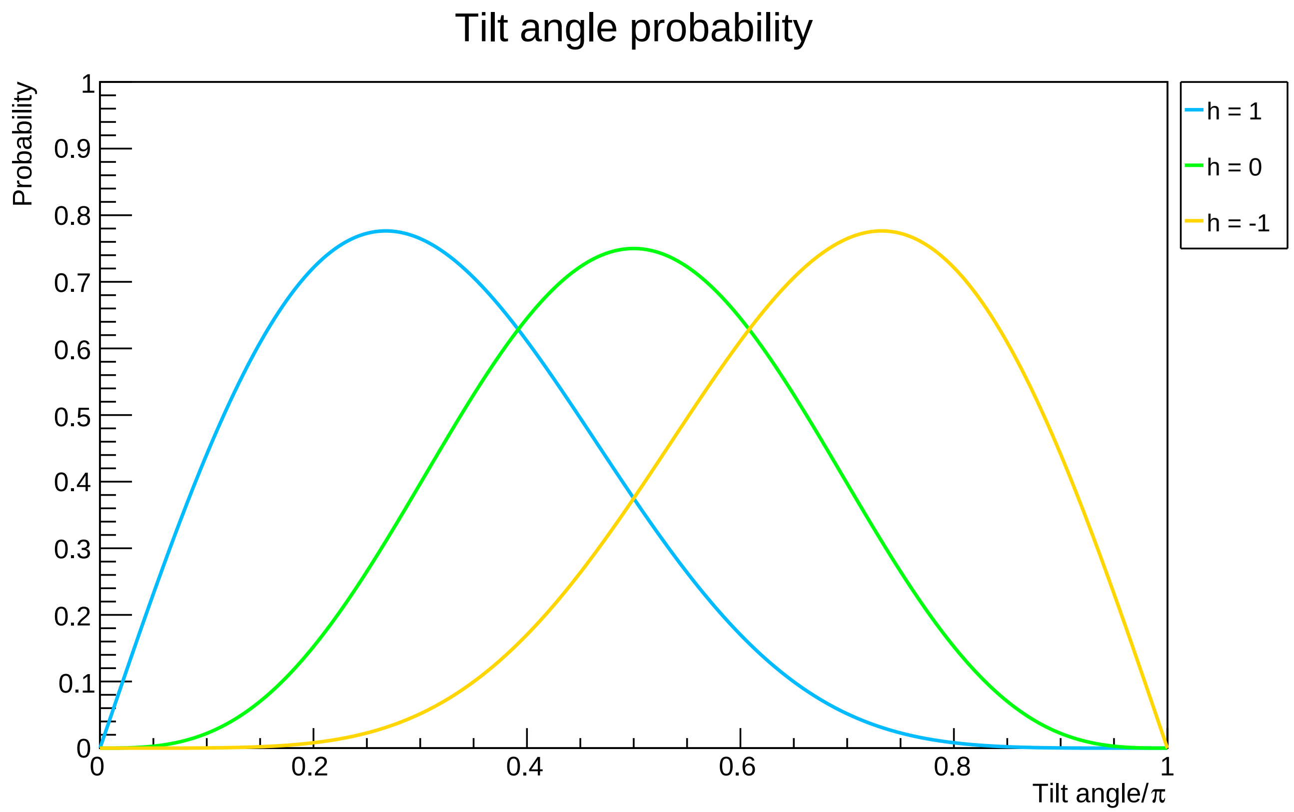
The probability of observing the corresponding decay product within a certain solid angle is given by

$$P = c\int d\Omega\ |A(\theta)|^2 = c\int d\phi \int d\theta\,\sin\theta\,|A(\theta)|^2, \tag{14}$$

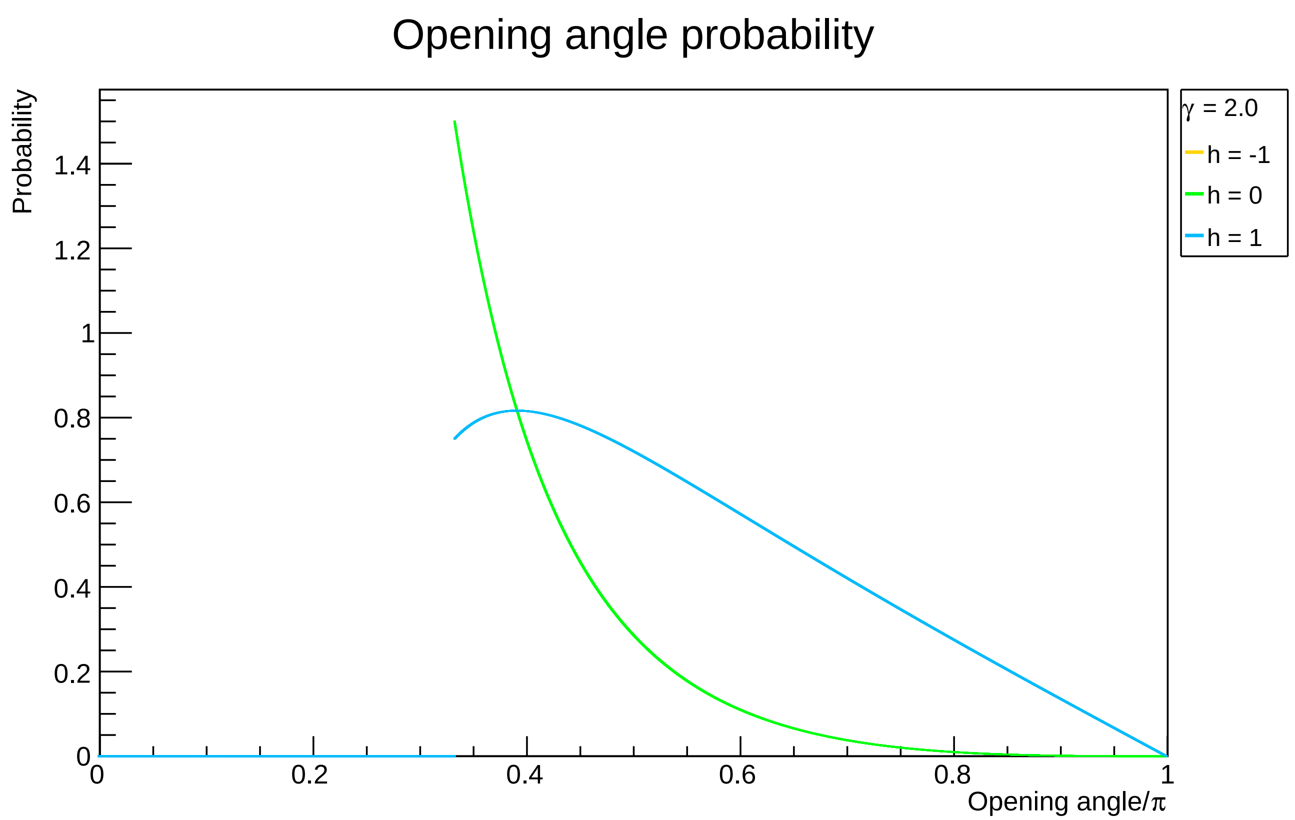
which, by normalization condition, must integrate to unity over the whole solid angle of 4π. The normalized probability density functions, $P_h(\theta)$, of decay at angle θ given a particular W^- helicity are then:

$$P_+(\theta) = \frac{3}{8}(1+\cos\theta)^2\sin\theta, \quad P_0(\theta) = \frac{3}{4}\sin^3\theta, \quad P_-(\theta) = \frac{3}{8}(1-\cos\theta)^2\sin\theta. \tag{15}$$

These funstions are plotted below.



Combining this result with the previously derived opening angle dependence on the tilt angle, we can find the opening angle probability density. This is plotted below for $\gamma = 2$. Note that the opening probability density functions are the same for helicity ± 1, so only transversely and longitudinally polarized vector bosons can be distinguished. There plot has a cut off, because there is minimum opening angle allowed for a given γ.



To find the probability density of the cosine of the titl angle, we rewrite Eq. (14) as

$$P = c \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \; |A(\theta)|^2 \,. \tag{16}$$

The respective probability density functions are then,

$$P_+(\cos\theta) = \frac{3}{8}(1+\cos\theta)^2, \quad P_0(\cos\theta) = \frac{3}{4}(1-\cos^2\theta), \quad P_-(\cos\theta) = \frac{3}{8}(1-\cos\theta)^2. \tag{17}$$