Problem 1

By the Hillas cryterion,

$$E_{\max} \approx 2\beta c Z e B r. \tag{1.1}$$

With $r = 4\,\mathrm{km}$, $B = 10\,\mathrm{T}$, $Z = 1$, and the particles traveling at nearly the speed of light ($\beta \to 1$),

$$E_{\max} \approx 2 \times 3 \times 10^8\,\mathrm{m\,s^{-1}} \times 1 \times 1.602 \times 10^{-19}\,\mathrm{C} \times 10\,\mathrm{T} \times 4\,\mathrm{km} \approx \boxed{24\,\mathrm{TeV}} \tag{1.2}$$

This places LHC below the theoretical cutoff line on the Hillas diagram, corresponding to $\sim 10^{20}$ eV. The LHC's magnetic field and size are about 7 orders of magnitude smaller than the CR sources at the theoretical cutoff.

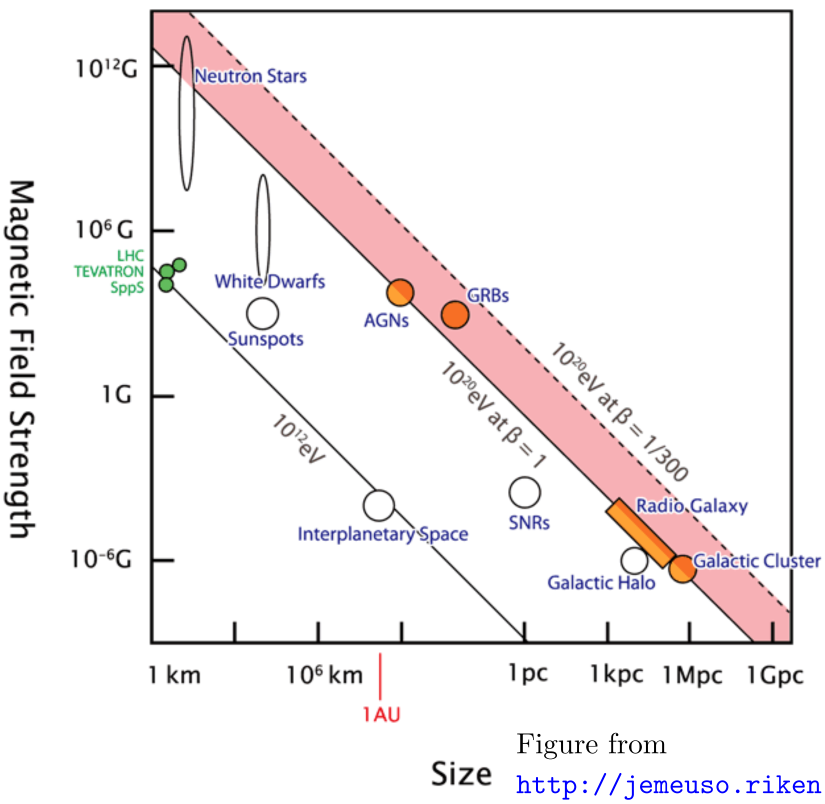


Figure from http://jemeuso.riken.jp/en/about2.html

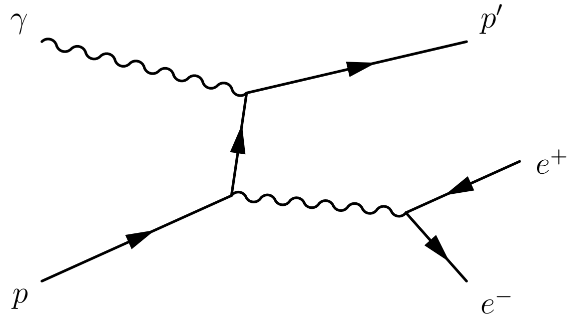
To accelerate particles to $E = 1\,\mathrm{ZeV} = 1 \times 10^{21}\,\mathrm{eV}$, an accelerator of this size would need a magnetic field of

$$B_2 = \frac{E_2}{E_1} \times B_1 = \frac{1 \times 10^{21}\,\mathrm{eV}}{24 \times 10^{12}\,\mathrm{eV}} \times 10\,\mathrm{T} = \boxed{4.16 \times 10^9\,\mathrm{T}} \tag{1.3}$$

The maximum energy of an accelerator circling the earth ($r = 6.4 \times 10^6\,\mathrm{m}$) with a 10 T field would be

$$E_3 = \frac{r_3}{r_1} \times E_1 = \frac{6.4 \times 10^6\,\mathrm{m}}{4 \times 10^3\,\mathrm{m}} \times 24 \times 10^{12}\,\mathrm{eV} = \boxed{3.84 \times 10^4\,\mathrm{TeV}} \tag{1.4}$$

Problem 2



$$P_\gamma + P_p = P_{p'} + P_{e^+} + P_{e^-} \tag{2.1}$$

Squaring both sides, and noting that in the center-of-mass frame the outgoing particles are produced at rest, we get

$$(P_\gamma + P_p)^2 = (m_p + 2m_e)^2. \tag{2.2}$$

$$(P_\gamma + P_p)^2 = P_\gamma^2 + P_p^2 + P_\gamma P_p = 0 + m_p^2 + 2P_\gamma \cdot P_p \tag{2.3}$$

Since the proton is ultra-relativistic,

$$P_\gamma = \begin{pmatrix} E_\gamma & -E_\gamma & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_p \approx \begin{pmatrix} E_p & E_p & 0 & 0 \end{pmatrix}. \tag{2.4}$$

$$\therefore P_\gamma \cdot P_p = 2E_p E_\gamma, \tag{2.5}$$

and

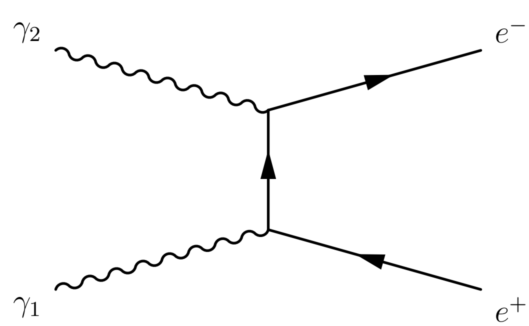
$$m_p^2 + 4E_p E_\gamma = m_p^2 + 4m_e m_p + 4m_e^2 \tag{2.6}$$

$$\boxed{E_p = \frac{m_e m_p + m_e^2}{E_\gamma}} \tag{2.7}$$

$m_e = 0.511$ MeV, $m_p = 938$ MeV. The temperature of the CMB is 2.726 K which corresponds to $E_\gamma = 0.235$ meV. Substituting, we get

$$E_p = \frac{0.511 \times 938 + 0.511^2}{0.235} \times \frac{10^{12}}{10^{-3}} \text{ eV} = \boxed{2.04 \times 10^{18} \text{ eV}} \tag{2.8}$$

Problem 3



$$P_1 + P_2 = P_+ + P_- \tag{3.1}$$

After squaring both sides, since $m_1 = m_2 = 0$,

$$2E_1E_2 = 4m_e^2 \quad \Rightarrow \quad E_1 = \frac{2m_e^2}{E_2}. \tag{3.2}$$

For $\lambda_2 = 1\,\mu\text{m}$,

$$E_2 = \frac{hc}{\lambda} = 1.24 \text{ eV} \tag{3.3}$$

$$E_1 = \frac{2 \times (0.511 \times 10^6)^2}{1.24} \text{ eV} = 4.2 \times 10^{11} \text{ eV} = \boxed{0.42 \text{ TeV}} \tag{3.4}$$

Problem 4

$$v = \frac{d}{t} \quad \Rightarrow \quad \frac{\delta v}{v} = \sqrt{\left(\frac{\delta d}{d}\right)^2 + \left(\frac{\delta t}{t}\right)^2} \tag{4.1}$$

We may as well ignore $\delta d/d$, because the fractional uncertainty of high measurement is usually much better then that of time. So,

$$\frac{\delta v}{v} \approx \frac{\delta t}{t} \tag{4.2}$$

For a particle traveling at nearly the speed of light,

$$\delta t \approx t\frac{\delta v}{v} \approx \frac{d}{c}\frac{\delta v}{v} \tag{4.3}$$

To measure the particle's velocity to 5%,

$$\delta t = \frac{1\,\text{m}}{3 \times 10^8\,\text{m}\,\text{s}^{-1}} \times 0.05 = 1.7 \times 10^{-10}\,\text{s} = \boxed{0.17\,\text{ns}} \tag{4.4}$$

Problem 5

The Larmor radius is given by

$$r = \frac{mv}{qB} \tag{5.1}$$

Substituting $mv = 1$ TeV$/c$, $q = e$, and $B = 5$ kG,

$$r = \frac{1 \times 10^{12}\,\text{eV}/3 \times 10^8\,\text{m}\,\text{s}^{-1}}{1.6 \times 10^{-19}\,\text{C} \times 5 \times 10^3\,\text{G}} = \frac{1 \times 10^{12} \times 1.6 \times 10^{-15}\,\text{m}}{1.6 \times 10^{-19} \times 5 \times 10^3 \times 3 \times 10^8} = 6671\,\text{m} \tag{5.2}$$

When the sagitta is small in comparison to the radius, as is the case here, it may be approximated by the formula

$$s \approx \frac{l^2}{2r}, \tag{5.3}$$

where s is the sagitta, r is the radius of curvature, and l is half the length of the chord.

$$\frac{\delta r}{r} = \sqrt{\left(2\frac{\delta l}{l}\right)^2 + \left(\frac{\delta s}{s}\right)^2} = \sqrt{4\frac{\delta l^2}{l^2} + 4\frac{\delta l^2}{l^4/r^2}} = 2\frac{\delta l}{l}\sqrt{1 + \frac{r^2}{l^2}} \approx 2\frac{\delta l}{l}\frac{r}{l} \tag{5.4}$$

$$\therefore \delta l = \frac{1}{2}\left(\frac{l}{r}\right)^2 \delta r \tag{5.5}$$

If the momentum is to be measured with 1% accuracy,

$$0.01 = \frac{\delta m v}{m v} = \frac{\delta r}{r} = \frac{2r}{l^2}\delta l. \tag{5.6}$$

For $l \approx 1\,\mathrm{m}$,

$$\delta l = 0.01 \times \frac{(1\,\mathrm{m})^2}{2 \times 6671\,\mathrm{m}} = 7.5 \times 10^{-7}\,\mathrm{m} = \boxed{750\,\mathrm{nm}} \tag{5.7}$$

This length is comporable to the longest wavelength of visible light. A scintilator based detector could be used for this perpose.

Problem 6

$$\frac{1 \times 3\,\mathrm{GeV}}{\mathrm{cm}^2\,\mathrm{s}} \times \frac{\mathrm{J}}{6.24 \times 10^9\,\mathrm{GeV}} \times \left(\frac{100\,\mathrm{cm}}{\mathrm{m}}\right)^2 = 4.8 \times 10^{-6}\,\mathrm{W\,m^{-2}} \tag{6.1}$$

In comparison, the solar constant is $1.36\,\mathrm{kW\,m^{-2}}$.

Problem 7

The Larmor radius is given by

$$r = \frac{mv}{qB} \approx \frac{E}{qBc} \quad \Rightarrow \quad E = qBcr \tag{7.1}$$

$$E = 1.6 \times 10^{-19}\,\mathrm{C} \times 10^{-5.3}\ \mathrm{G} \times 3 \times 10^8\,\mathrm{m\,s^{-1}} \times 10^{2.5}\ \mathrm{pc} \times 3.086 \times 10^{16}\frac{\mathrm{m}}{\mathrm{pc}} = \boxed{1.465 \times 10^{18}\,\mathrm{eV}} \tag{7.2}$$

Problem 8

(a) The energy flux distribution of the CR is given by

$$\frac{d\Phi}{dE} = AE^{-2.7}. \tag{8.1}$$

There are

$$\int_{1\ \mathrm{TeV}}^{5\ \mathrm{TeV}} AE^{-2.7}dE \bigg/ \int_{5\ \mathrm{TeV}}^{\infty} AE^{-2.7}dE = \frac{1^{-1.7} - 5^{-1.7}}{5^{-1.7}} = \boxed{14.4} \tag{8.2}$$

times as many CR between 1 TeV and 5 TeV as there are above 5 TeV.

There are

$$\frac{0.1^{-1.7} - 1^{-1.7}}{5^{-1.7}} = \boxed{758} \tag{8.3}$$

times as many CR between 0.1 TeV and 1 TeV as there are above 5 TeV.

(b) $x \equiv \log_{10}(E/\mathrm{eV})$. The x flux distribution is

$$\frac{d\Phi}{dx} = \frac{d\Phi}{dE}\frac{dE}{dx}. \tag{8.4}$$

$$E = 10^x \;\Rightarrow\; \frac{dE}{dx} = \ln(10)\,10^x \;\Rightarrow\; \frac{d\Phi}{dx} = A\left(10^x\right)^{-2.7}\ln(10)\,10^x = A\ln(10)\,10^{-1.7x} = A' e^{-3.914x}. \tag{8.5}$$

Let x be the measured value of energy, and $\langle x\rangle$ be the true value. Then, by Bayes' theorem,

$$P\left(\langle x\rangle | x\right) = \frac{P\left(x | \langle x\rangle\right) P\left(\langle x\rangle\right)}{P\left(x\right)}. \tag{8.6}$$

$$P\left(\langle x\rangle\right) = 9.83\times 10^{20}\times e^{-3.914x}, \tag{8.7}$$

$$P\left(x | \langle x\rangle\right) = \frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}}, \tag{8.8}$$

$$P\left(x\right) = \int_{1\ \mathrm{TeV}}^{\infty} P\left(x | \langle x\rangle\right) P\left(\langle x\rangle\right)\,d\langle x\rangle \tag{8.9}$$

The flux of particles with measured energy greater than x that had true energy greater than $\langle x\rangle$ is the CDF of the joint distribution $P\left(\langle x\rangle | x\right)$.

$$P\left(\langle x\rangle > a | x > b\right) = \int_a^{\infty}\int_b^{\infty} P\left(\langle x\rangle | x\right)\,dx\,d\langle x\rangle\,. \tag{8.10}$$

The fraction of CRs measured as above 5 TeV, that are actually above 5 TeV, is then

$$P\left(\langle x\rangle > 5\ \mathrm{TeV} | x > 5\ \mathrm{TeV}\right)/P\left(\langle x\rangle > 1\ \mathrm{TeV} | x > 5\ \mathrm{TeV}\right) = \boxed{0.417}. \tag{8.11}$$

In[1]:=
```
c = Log[10^-1.7]
```
Out[1]= -3.91439

In[2]:=
```
TeV[x_] := N[Log10[x] + 12]
```

In[3]:=
```
flux[x_] := Exp[c x]
```

In[4]:=
```
G1[x_, μ_, σ_] := PDF[NormalDistribution[μ, σ], x]
G[x_, μ_] := G1[x, μ, 0.2]
G1[x, μ, σ]
G[x, μ]
```

Out[6]= $\dfrac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$

Out[7]= $1.99471\, e^{-12.5\,(x-\mu)^2}$

Distribution of particles with measured energy *x*

In[8]:=
```
Px[x_] := Integrate[G[x, μ] * flux[x], {μ, TeV[1], ∞}]
```

The join distribution

In[9]:=
```
joint[x_, μ_] := G[x, μ] * flux[x] / Px[x]
```

Flux of particles with measured energy above 5 TeV whose energy was really above 5 TeV

In[10]:=
```
A = NIntegrate[joint[x, μ], {x, TeV[5], ∞}, {μ, TeV[5], ∞}]
```
Out[10]= 2.92022

The flux of all particles with measured energy above 5 TeV

In[11]:=
```
B = NIntegrate[joint[x, μ], {x, TeV[5], ∞}, {μ, TeV[1], ∞}, PrecisionGoal → 12,
  MaxRecursion → 20, Method → {GlobalAdaptive, MaxErrorIncreases → 10 000}]
```

NIntegrate::slwcon :
Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small. ≫

Out[11]= 7.

Fraction of particles with measured energy above 5 TeV whose energy was really above 5 TeV

In[12]:=
```
A / B
```
Out[12]= 0.417174