

Problem 1

By the Hillas cryterion,

$$E_{\max} \approx 2\beta cZeBr. \quad (1.1)$$

With $r = 4 \text{ km}$, $B = 10 \text{ T}$, $Z = 1$, and the particles traveling at nearly the speed of light ($\beta \rightarrow 1$),

$$E_{\max} \approx 2 \times 3 \times 10^8 \text{ m s}^{-1} \times 1 \times 1.602 \times 10^{-19} \text{ C} \times 10 \text{ T} \times 4 \text{ km} \approx \boxed{24 \text{ TeV}} \quad (1.2)$$

Discuss whether this places the LHC at about the right place on the Hillas diagram.

This places LHC below the theoretical cutoff line on the Hillas diagram, corresponding to $\sim 10^{20} \text{ eV}$. The LHC's magnetic field and size are about 7 orders of magnitude smaller than the CR sources at the theoretical cutoff.

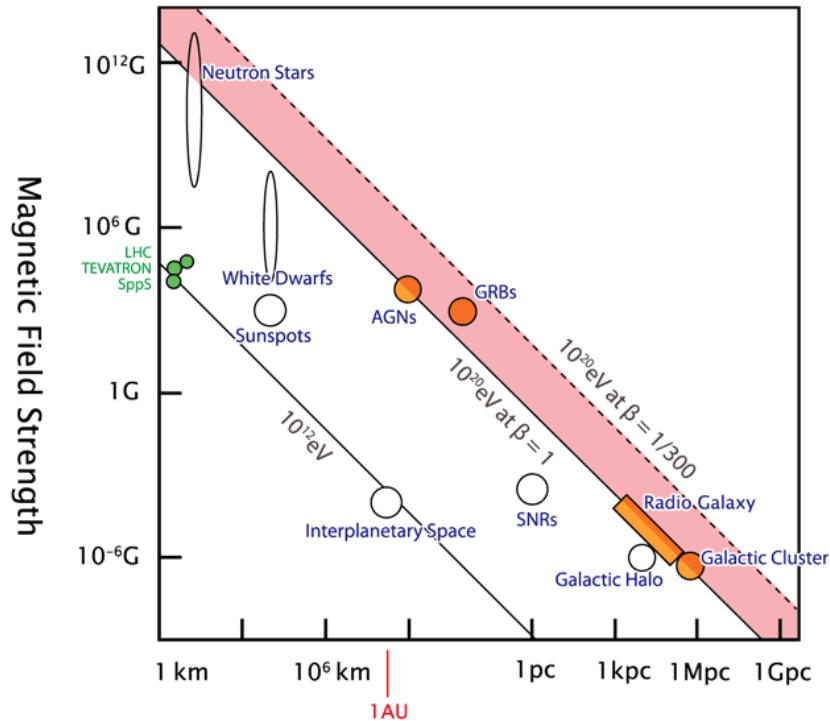


Figure from
Size <http://jemeuso.riken.jp/en/about2.html>

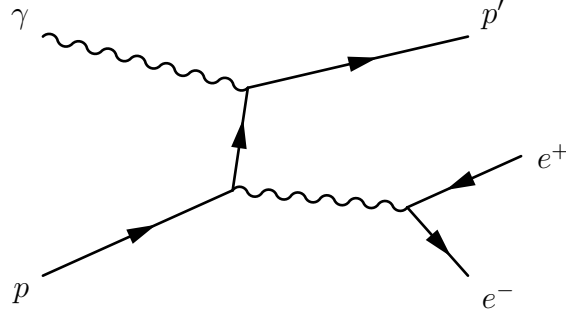
To accelerate particles to $E = 1 \text{ ZeV} = 1 \times 10^{21} \text{ eV}$, an accelerator of this size would need a magnetic field of

$$B_2 = \frac{E_2}{E_1} \times B_1 = \frac{1 \times 10^{21} \text{ eV}}{24 \times 10^{12} \text{ eV}} \times 10 \text{ T} = \boxed{4.16 \times 10^9 \text{ T}} \quad (1.3)$$

The maximum energy of an accelerator circling the earth ($r = 6.4 \times 10^6 \text{ m}$) with a 10 T field would be

$$E_3 = \frac{r_3}{r_1} \times E_1 = \frac{6.4 \times 10^6 \text{ m}}{4 \times 10^3 \text{ m}} \times 24 \times 10^{12} \text{ eV} = \boxed{3.84 \times 10^4 \text{ TeV}} \quad (1.4)$$

Problem 2



$$P_\gamma + P_p = P_{p'} + P_{e^+} + P_{e^-} \quad (2.1)$$

Squaring both sides, and noting that in the center-of-mass frame the outgoing particles are produced at rest, we get

$$(P_\gamma + P_p)^2 = (m_p + 2m_e)^2. \quad (2.2)$$

$$(P_\gamma + P_p)^2 = P_\gamma^2 + P_p^2 + 2P_\gamma \cdot P_p = 0 + m_p^2 + 2P_\gamma \cdot P_p \quad (2.3)$$

Since the proton is ultra-relativistic,

$$P_\gamma = (E_\gamma \quad -E_\gamma \quad 0 \quad 0) \quad \text{and} \quad P_p \approx (E_p \quad E_p \quad 0 \quad 0). \quad (2.4)$$

$$\therefore P_\gamma \cdot P_p = 2E_p E_\gamma, \quad (2.5)$$

and

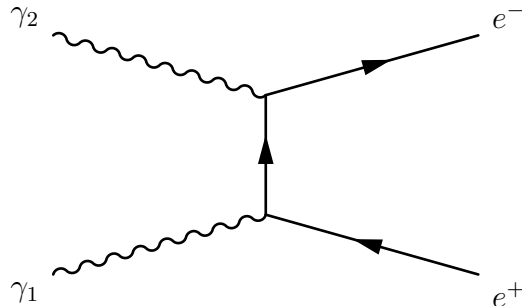
$$m_p^2 + 4E_p E_\gamma = m_p^2 + 4m_e m_p + 4m_e^2 \quad (2.6)$$

$$\boxed{E_p = \frac{m_e m_p + m_e^2}{E_\gamma}} \quad (2.7)$$

$m_e = 0.511$ MeV, $m_p = 938$ MeV. The temperature of the CMB is 2.726 K which corresponds to $E_\gamma = 0.235$ meV. Substituting, we get

$$E_p = \frac{0.511 \times 938 + 0.511^2}{0.235} \times \frac{10^{12}}{10^{-3}} \text{ eV} = \boxed{2.04 \times 10^{18} \text{ eV}} \quad (2.8)$$

Problem 3



$$P_1 + P_2 = P_+ + P_- \quad (3.1)$$

After squaring both sides, since $m_1 = m_2 = 0$,

$$2E_1E_2 = 4m_e^2 \quad \Rightarrow \quad E_1 = \frac{2m_e^2}{E_2}. \quad (3.2)$$

For $\lambda_2 = 1 \mu\text{m}$,

$$E_2 = \frac{hc}{\lambda} = 1.24 \text{ eV} \quad (3.3)$$

$$E_1 = \frac{2 \times (0.511 \times 10^6)^2}{1.24} \text{ eV} = 4.2 \times 10^{11} \text{ eV} = \boxed{0.42 \text{ TeV}} \quad (3.4)$$

which calculation is more precise

Problem 4

$$v = \frac{d}{t} \quad \Rightarrow \quad \frac{\delta v}{v} = \sqrt{\left(\frac{\delta d}{d}\right)^2 + \left(\frac{\delta t}{t}\right)^2} \quad (4.1)$$

We may as well ignore $\delta d/d$, because the fractional uncertainty of high measurement is usually much better than that of time. So,

$$\frac{\delta v}{v} \approx \frac{\delta t}{t} \quad (4.2)$$

For a particle traveling at nearly the speed of light,

$$\delta t \approx t \frac{\delta v}{v} \approx \frac{d}{c} \frac{\delta v}{v} \quad (4.3)$$

To measure the particle's velocity to 5%,

$$\delta t = \frac{1 \text{ m}}{3 \times 10^8 \text{ m s}^{-1}} \times 0.05 = 1.7 \times 10^{-10} \text{ s} = \boxed{0.17 \text{ ns}} \quad (4.4)$$

How accurate to tell an upmoving particle from a downmoving by 10 standard deviations?

Problem 5

The Larmor radius is given by

$$r = \frac{mv}{qB} \quad (5.1)$$

Substituting $mv = 1 \text{ TeV}/c$, $q = e$, and $B = 5 \text{ kG}$,

$$r = \frac{1 \times 10^{12} \text{ eV}/3 \times 10^8 \text{ m s}^{-1}}{1.6 \times 10^{-19} \text{ C} \times 5 \times 10^3 \text{ G}} = \frac{1 \times 10^{12} \times 1.6 \times 10^{-15} \text{ m}}{1.6 \times 10^{-19} \times 5 \times 10^3 \times 3 \times 10^8} = 6671 \text{ m} \quad (5.2)$$

When the sagitta is small in comparison to the radius, as is the case here, it may be approximated by the formula

$$s \approx \frac{l^2}{2r}, \quad (5.3)$$

where s is the sagitta, r is the radius of curvature, and l is half the length of the chord.

$$\frac{\delta r}{r} = \sqrt{\left(2\frac{\delta l}{l}\right)^2 + \left(\frac{\delta s}{s}\right)^2} = \sqrt{4\frac{\delta l^2}{l^2} + 4\frac{\delta l^2}{l^4/r^2}} = 2\frac{\delta l}{l}\sqrt{1 + \frac{r^2}{l^2}} \approx 2\frac{\delta l}{l}\frac{r}{l} \quad (5.4)$$

$$\therefore \delta l = \frac{1}{2} \left(\frac{l}{r}\right)^2 \delta r \quad (5.5)$$

If the momentum is to be measured with 1% accuracy,

$$0.01 = \frac{\delta mv}{mv} = \frac{\delta r}{r} = \frac{2r}{l^2} \delta l. \quad (5.6)$$

For $l \approx 1$ m,

$$\delta l = 0.01 \times \frac{(1 \text{ m})^2}{2 \times 6671 \text{ m}} = 7.5 \times 10^{-7} \text{ m} = \boxed{750 \text{ nm}} \quad (5.7)$$

what technology

Problem 6

$$\frac{1 \times 3 \text{ GeV}}{\text{cm}^2 \text{ s}} \times \frac{\text{J}}{6.24 \times 10^9 \text{ GeV}} \times \left(\frac{100 \text{ cm}}{\text{m}}\right)^2 = 4.8 \times 10^{-6} \text{ W m}^{-2} \quad (6.1)$$

In comparison, the solar constant is 1.36 kW m^{-2} .

Problem 7

The Larmor radius is given by

$$r = \frac{mv}{qB} \approx \frac{E}{qBc} \Rightarrow E = qBcr \quad (7.1)$$

$$E = 1.6 \times 10^{-19} \text{ C} \times 10^{-5.3} \text{ G} \times 3 \times 10^8 \text{ m s}^{-1} \times 10^{2.5} \text{ pc} \times 3.086 \times 10^{16} \frac{\text{m}}{\text{pc}} = \boxed{1.465 \times 10^{18} \text{ eV}} \quad (7.2)$$

Problem 8

(a) The energy flux distribution of the CR is given by

$$\frac{d\Phi}{dE} = AE^{-2.7}. \quad (8.1)$$

There are

$$\int_{1 \text{ TeV}}^{5 \text{ TeV}} AE^{-2.7} dE \Big/ \int_{5 \text{ TeV}}^{\infty} AE^{-2.7} dE = \frac{1^{-1.7} - 5^{-1.7}}{5^{-1.7}} = \boxed{14.4} \quad (8.2)$$

times as many CR between 1 TeV and 5 TeV as there are above 5 TeV.

There are

$$\frac{0.1^{-1.7} - 1^{-1.7}}{5^{-1.7}} = \boxed{758} \quad (8.3)$$

times as many CR between 0.1 TeV and 1 TeV as there are above 5 TeV.

(b) $x \equiv \log_{10}(E/\text{eV})$. The x flux distribution is

$$\frac{d\Phi}{dx} = \frac{d\Phi}{dE} \frac{dE}{dx}. \quad (8.4)$$

$$E = 10^x \Rightarrow \frac{dE}{dx} = \ln(10) 10^x \Rightarrow \frac{d\Phi}{dx} = A (10^x)^{-2.7} \ln(10) 10^x = A \ln(10) 10^{-1.7x} = A' e^{-3.914x}. \quad (8.5)$$

Let x be the measured value of energy, and $\langle x \rangle$ be the true value. Then, by Bayes' theorem,

$$P(\langle x \rangle | x) = \frac{P(x | \langle x \rangle) P(\langle x \rangle)}{P(x)}. \quad (8.6)$$

$$P(\langle x \rangle) = 9.83 \times 10^{20} \times e^{-3.914x}, \quad (8.7)$$

$$P(x | \langle x \rangle) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x - \langle x \rangle)^2}{2\sigma^2}}, \quad (8.8)$$

$$P(x) = \int_{1 \text{ TeV}}^{\infty} P(x | \langle x \rangle) P(\langle x \rangle) d\langle x \rangle \quad (8.9)$$

The flux of particles with measured energy greater than x that had true energy greater than $\langle x \rangle$ is the CDF of the joint distribution $P(\langle x \rangle | x)$.

$$P(\langle x \rangle > a | x > b) = \int_a^{\infty} \int_b^{\infty} P(\langle x \rangle | x) dx d\langle x \rangle. \quad (8.10)$$

The fraction of CRs measured as above 5 TeV, that are actually above 5 TeV, is then

$$P(\langle x \rangle > 5 \text{ TeV} | x > 5 \text{ TeV}) / P(\langle x \rangle > 1 \text{ TeV} | x > 5 \text{ TeV}) = \boxed{0.50}. \quad (8.11)$$

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In[1]:= c = Log[10-1.7]
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Out[1]= -3.91439
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```
In[2]:= TeV[x_] := N[Log10[x] + 12]
```

```
In[3]:= flux[x_] := Exp[c x]
```

```
In[4]:= G1[x_, μ_, σ_] := PDF[NormalDistribution[μ, σ], x]
```

```
G[x_, μ_] := G1[x, μ, 0.2]
```

```
G1[x, μ, σ]
```

```
G[x, μ]
```

```
Out[6]= 
$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

```

```
Out[7]= 1.99471 e-12.5 (x-μ)2
```

Distribution of particles with measured energy x

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In[8]:= Px[x_] := Integrate[G[x, μ] * flux[x], {μ, TeV[12], ∞}]
```

The join distribution

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In[9]:= joint[x_, μ_] := G[x, μ] * flux[x] / Px[x]
```

Flux of particles with measured energy above 5 TeV whose energy was really above 5 TeV

```
In[10]:= A = NIntegrate[joint[x, μ], {x, TeV[5], ∞}, {μ, TeV[5], ∞}]
```

```
Out[10]= 5.1445
```

The flux of all particles with measured energy above 5 TeV

```
In[11]:= B = NIntegrate[joint[x, μ], {x, TeV[5], ∞}, {μ, TeV[1], ∞}, PrecisionGoal → 12,  
MaxRecursion → 20, Method → {GlobalAdaptive, MaxErrorIncreases → 10 000}]
```

NIntegrate::slwcon :

Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small. >>

```
Out[11]= 10.2955
```

Fraction of particles with measured energy above 5 TeV whose energy was really above 5 TeV

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In[12]:= A / B
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Out[12]= 0.499686
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