Exam 3 Practice

13.) Let $T: P_1 \to P_1$ be defined by $T(a+bx) = 3a + 2bx$

a) verify that this is a Linear Transformation

b) Find $[T]_B^C$ with respect to $B = \{1+x, -1+2x\}$

$C = \{3, 5+2x\}$

c) Use $[T]_B^C$ to compute $T(1+4x)$

a) Is T a LT ?

i) $a+bx,\ c+dx \ni P_1$

$T(a+bx) + T(c+dx)$

$= 3a + 2bx + 3c + 2dx$

$= 3(a+c) + 2(b+d)x$

$= T((a+cx)+(b+dx))$

ii) $K \in \mathbb{Z},\ a+bx \in P_1$

$T(K(a+bx))$

$= T(Ka + Kbx)$

$= 3Ka + 2Kbx$

$= K(3a + 2bx)$

$= K\,T(a+bx)$

$\therefore T$ is a LT

b) find $[T]_B^C$

$T(1+x) = 3+2x = c_1(3+0x) + c_2(5+2x)$

$T(-1+2x) = -3+4x = c_1(3+0x) + c_2(5+2x)$

$(3c_1 + 0x) + (5c_2 + 2c_2x) = 3+2x = (3c_1 + 5c_2,\ 0 + 2c_2x) = 3+2x$

$(3c_1 + 0x) + (5c_2 + 2c_2x) = -3+4x = (3c_1 + 5c_2,\ 0 + 2c_2x) = -3+4x$

$$\left[\begin{array}{cc:cc} 3 & 5 & 3 & -3 \\ 0 & 2 & 2 & 4 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{cc:cc} 1 & 0 & -\frac{2}{3} & -\frac{13}{3} \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\therefore [T]_B^C = \begin{bmatrix} -\frac{2}{3} & -\frac{13}{3} \\ 1 & 2 \end{bmatrix}$$

c) Compute $T(1+4x)$

$1+4x = c_1(1+x) + c_2(-1+2x)$

$\begin{bmatrix} 1 & -1 & \vdots & 1 \\ 1 & 2 & \vdots & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & \vdots & 2 \\ 0 & 1 & \vdots & 1 \end{bmatrix}$

$\therefore \quad 1+4x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$

$[T(v)]_C = [T]_B^C [v]_B = \begin{bmatrix} -\frac{2}{3} & -\frac{13}{3} \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{17}{3} \\ 4 \end{bmatrix}$

$\therefore T(1+4x) = -\frac{17}{3}(3+0) + 4(5+2x)$

$= -17 + 20 + 8x$

$= 3 + 8x$

25.) Define a product on P_2 as follow : $\langle f,g \rangle = \int_0^1 f(x)g(x)\,dx$

a) show $\langle f,g \rangle$ is an inner product over $\mathbb{R}$

b) Find an orthogonal basis for the span $\{1, x\}$

a) show $\langle f,g \rangle$ is an inner product

i) Positive difinteness

$\langle f,f \rangle = \int_0^1 f(x)f(x)\,dx = \int_0^1 f(x)^2\,dx \geq 0$

and $\int_0^1 f(x)^2\,dx = 0$ iff $f(x)$ is zero polynomial

ii) Symmetry

$\langle f,g \rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g,f \rangle$

iii) Linearity

$\langle af + bg, h \rangle = \int_0^1 (af(x) + bg(x))h(x)\,dx$

$= \int_0^1 (af(x)h(x) + bg(x)h(x))\,dx$

$= \int_0^1 af(x)h(x)\,dx + \int_0^1 bg(x)h(x)\,dx$

$= a\int_0^1 f(x)h(x)\,dx + b\int_0^1 g(x)h(x)\,dx$

$= a\langle f,h \rangle + b\langle g,h \rangle$

$\therefore \quad \langle f,g \rangle$ is an inner product

b) Find an orthogonal basis for the span $\{\overset{u_1}{1}, \overset{u_2}{x}\}$

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$v_2 = x - \frac{\frac{1}{2}}{1}(1)$

$v_2 = x - \frac{1}{2}$

$\langle u_2, v_1 \rangle = \langle x, 1 \rangle$
$= \int_0^1 x(1)\,dx$
$= \frac{x^2}{2}\Big|_0^1$
$= \frac{1^2}{2} = \frac{1}{2}$

$\|v_1\|^2 = \langle v_1, v_1 \rangle = \langle 1, 1 \rangle$
$= \int_0^1 (1)(1)\,dx$
$= x\Big|_0^1 = 1$

$\therefore \{1, x - \frac{1}{2}\}$ is an orthogonal basis

38) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x_1, x_2, x_3) = (x_1 - 2x_3, x_2)$

a) show T is a linear transformation

b) Use the definition to find a basis for ker T

c) Find the dimension of the range. Is T onto?

a) show T is a LT

i) show $T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$

$T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$
$= (x_1 + y_1 - 2(x_3 + y_3), x_2 + y_2)$
$= (x_1 + y_1 - 2x_3 - 2y_3, x_2 + y_2)$
$= (x_1 - 2x_3, x_2) + (y_1 - 2y_3, y_2)$
$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$

ii show $T(k(x_1, x_2, x_3)) = kT(x_1, x_2, x_3)$

$T(k(x_1, x_2, x_3)) = T(kx_1, kx_2, kx_3)$
$= (kx_1 - 2kx_3, kx_2)$
$= k(x_1 - 2x_3, x_2)$
$= kT(x_1, x_2, x_3)$

$\therefore$ T is a LT

b) Find the ker T

$T(x_1, x_2, x_3) = (x_1 - 2x_3, x_2) = (0,0)$

$x_1 + 0 - 2x_3 = 0$

$0 + x_2 + 0 = 0$

$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$c_1 = 2t$ Let $t = 1$ $c_1 = 2$

$c_2 = 0$ $c_2 = 0$

$c_3 = t$ $c_3 = 1$

$\ker T = \{2, 0, 1\}$ $\ker T = \{0, 0, 1\}$

c) $\dim \text{Domain} = \dim \ker T + \dim \text{Rng}\, T$

$3 = 1 + \dim \text{Rng}\, T$

Thus $\dim \text{Rng}\, T = 2$

$\therefore$ T is onto

since dim Codomain is also 2

45) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying $T(1,2) = (-1,3)$, $T(2,3) = (0,2)$. Find $T(3,4)$

$(3,4) = c_1(1,2) + c_2(2,3)$

$(c_1, 2c_1) + (2c_2, 3c_2) = (3,4)$

$(c_1 + 2c_2, 2c_1 + 3c_2) = (3,4)$

$c_1 + 2c_2 = 3$

$2c_1 + 3c_2 = 4$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$c_1 = -1$, $c_2 = 2$

$(3,4) = -1(1,2) + 2(2,3)$

$T(3,4) = T(-1(1,2) + 2(2,3))$

$= -1\,T(1,2) + 2\,T(2,3)$

$= -1(-1,3) + 2(0,2)$

$= (1,-3) + (0,4)$

$= (1,1)$

48.) Define $T: C^1[a,b] \to C^0[a,b]$ as $T(f(x)) = f'(x)$

a) show it is a LT

b) Find the kernel of T

c) Is T one to one ?

a) show T is a LT

i) show $T(f(x)+g(x)) = T(f(x)) + T(g(x))$

$$T(f(x)+g(x)) = (f(x)+g(x))'$$
$$= f'(x) + g'(x)$$
$$= T(f(x)) + T(g(x))$$

ii) show $T(kf(x)) = kT(f(x))$

$$T(kf(x)) = (kf(x))'$$
$$= kf'(x)$$
$$= kT(f(x))$$

$\therefore$ T is a LT

b)

c)

50.) b) Suppose the matrix of $T: \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis $B = C = \{(1,-1),(0,1)\}$ is given as

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, Find $T(2,3)$

$$T(1,-1) = 1(1,-1) + 1(0,1) = (1,0)$$
$$T(0,1) = 1(1,-1) + (-1)(0,1) = (1,-2)$$

$$(2,3) = c_1(1,-1) + c_2(0,1)$$
$$(c_1,-c_1) + (0,c_2) = (2,3)$$

$(c_1 + 0) = 2$

$(-c_1 + c_2) = 3$

$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 3 \end{array}\right] \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

$c_1 = 2, \quad c_2 = 5$

$(2,3) = 2(1,-1) + 5(0,1)$

$T(2,3) = T(2(1,-1) + 5(0,1))$

$= 2T(1,-1) + 5T(0,1)$

$= 2(1,0) + 5(1,-2)$

$= (2,0) + (5,-10)$

$= (7,-10)$

52) Define a function $T: P_1 \to P_2$ by

$T(ax+b) = (2a-3b) + (4a+5b)x + (16a+9b)x^2$

a) show T is a LT

b) Find a basis for the range of T (first pick bases and find the matrix)

a) Show T is a LT

i) show $T((a_1x+b_1)+(a_2x+b_2)) = T(a_1x+b_1) + T(a_2x+b_2)$

$T((a_1x+b_1)+(a_2x+b_2))$

$= T((a_1+a_2)x + (b_1+b_2))$

$= (2(a_1+a_2) - 3(b_1+b_2)) + (4(a_1+a_2) + 5(b_1+b_2))x + (16(a_1+a_2) + 9(b_1+b_2))x^2$

$= ((2a_1-3b_1)+(2a_2-3b_2)) + ((4a_1+5b_1)+(4a_2+5b_2))x + ((16a_1+9b_1)+(16a_2+9b_2))x^2$

$= (2a_1-3b_1) + (4a_1+5b_1)x + (16a_1+9b_1)x^2 + (2a_2-3b_2) + (4a_2+5b_2)x + (16a_2+9b_2)x^2$

$= T(a_1x+b_1) + T(a_2x+b_2)$

ii) show $T(k(ax+b)) = kT(ax+b)$

$T(k(ax+b)) = T(kax+kb)$

$= (2ka-3kb) + (4ka+5kb)x + (16ka+9kb)x^2$

$= k(2a-3b) + k(4a+5b)x + k(16a+9b)x^2$

$= k((2a-3b) + (4a+5b)x + (16a+9b)x^2)$

$= k\,T(ax+b)$

$\therefore$ T is a LT

b) $T(1) = T(0x+1) = -3+5x+9x^2$

$T(x) = T(x+0) = 2+4x+16x^2$

$$\Rightarrow \begin{bmatrix} -3 & 2 \\ 5 & 4 \\ 9 & 16 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rng T is $\{-3+5x+9x^2,\ 2+4x+16x^2\}$

6.1) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation

a.) Find $T(x_1, x_2)$ if $T(2,3) = (1,3)$ and $T(1,-3) = (-4,3)$

b) Find the inverse of the LT found in a)

(You are not required to show it is one to one and onto)

a) Find $T(x_1, x_2)$

$(x_1, x_2) = x_1(2,3) + x_2(1,-3)$

$$\begin{aligned} T(x_1,x_2) &= T(x_1(2,3) + x_2(1,-3)) \\ &= x_1 T(2,3) + x_2 T(1,-3) \\ &= x_1(1,3) + x_2(-4,3) \\ &= (x_1, 3x_1) + (-4x_2, 3x_2) \\ &= (x_1 - 4x_2,\ 3x_1 + 3x_2) \end{aligned}$$

62.) Let $T: P_2 \to P_2$ be a linear transformation be defined as

$T(a + bx + cx^2) = 5a + (-a+4b)x + (2c+b)x^2$

a) Find the matrix of T with $B = C = \{1, x, x^2\}$

b) Use the matrix found in a) to find $T(4x+3x^2)$

c) Is T invertible? If it is, find the inverse

a) Find $[T]_B^C$

$[T(1)]_C = 5 - x + 0$

$[T(x)]_C = 0 + 4x + x^2 \quad \Rightarrow \begin{bmatrix} 5 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$[T(x^2)]_C = 0 + 0 + 2x^2$

b) Find $T(4x+3x^2)$

$0 + 4x + 3x^2 = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$

$$[T(4x+3x^2)]_C = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \\ 10 \end{bmatrix}$$

$T(4x+3x^2) = 0 + 16x + 10x^2$

c) $\det([T]_B^C) = 40 \neq 0$

$\therefore [T]_B^C$ is invertible

$$[T^{-1}]_C^B = ([T]_B^C)^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 1/20 & 1/4 & 0 \\ -1/40 & -1/8 & 1/2 \end{bmatrix}$$

$T^{-1}(a + bx + cx^2) = \frac{1}{5}a + (\frac{1}{20}a + \frac{1}{4}b)x + (-\frac{1}{40}a - \frac{1}{8}b + \frac{1}{2}c)x^2$

67.) a) Find an orthonormal basis for the subspace of $C^0[0,1]$ spanned by $\{\overset{x_1}{1}, \overset{x_2}{x}, \overset{x_3}{x^2}\}$, where, $\langle f,g\rangle = \int_0^1 f(x)\,g(x)\,dx$

b) Find the Fourier expansion of x^2 using the orthonormal basis found in a)

a) $v_1 = x_1 = 1$

$$v_2 = x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$\langle x_2, v_1\rangle = \langle x, 1\rangle = \int_0^1 x(1)\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\langle v_1, v_1\rangle = \langle 1, 1\rangle = \int_0^1 1(1)\,dx = x\Big|_0^1 = 1$$

$$v_2 = x - \frac{\frac{1}{2}}{1}(1) = x - \frac{1}{2}$$

$$v_3 = x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$\langle x_3, v_1\rangle = \langle x^2, 1\rangle = \int_0^1 x^2(1)\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3$$

$$\langle v_1, v_1\rangle = 1$$

$$\langle x_3, v_2\rangle = \langle x^2, x - 1/2\rangle = \int_0^1 x^2(x - 1/2)\,dx = \int_0^1 (x^3 - \tfrac{1}{2}x^2)\,dx$$

$$= \frac{x^4}{4} - \frac{1}{2}\frac{x^3}{3}\Big|_0^1 = 1/4 - 1/6 = 1/12$$

$$\langle v_2, v_2\rangle = \langle x - 1/2, x - 1/2\rangle = \int_0^1 (x - 1/2)(x - 1/2)\,dx$$

$$= \int_0^1 (x^2 - x + 1/4)\,dx = x^3/3 - \frac{x^2}{2} + 1/4\,x\Big|_0^1$$

$$= 1/3 - \frac{1}{2} + 1/4 = 1/12$$

$$v_3 = x^2 - \frac{1/3}{1} - \frac{1/12}{1/12}(x - 1/2) = x^2 - 1/3 - (x - 1/2)$$

$$= x^2 - 1/3 - x + \frac{1}{2} = x^2 - x + 1/6$$

$\therefore \{1, x - 1/2, x^2 - x + 1/6\}$ is an orthogonal basis

$$\|v_1\| = \langle 1, 1\rangle = \sqrt{\int_0^1 1(1)\,dx} = \sqrt{x\big|_0^1} = \sqrt{1} = 1$$

$$\|v_2\| = \langle x - 1/2, x - 1/2\rangle = \sqrt{1/12} = \frac{1}{2\sqrt{3}}$$

$$\|v_3\| = \langle x^2 - x + 1/6, x^2 - x + 1/6\rangle = \sqrt{\int_0^1 (x^2 - x + 1/6)^2\,dx}$$

$$= \sqrt{\int_0^1 (x^4 - x^3 + 1/6\,x^2 - x^3 + x^2 - 1/6\,x + 1/6\,x^2 - 1/6\,x + \tfrac{1}{36})\,dx}$$

$$= \sqrt{\int_0^1 (x^4 - 2x^3 + \tfrac{1}{3}x^2 + x^2 - \tfrac{1}{3}x + \tfrac{1}{36})\,dx}$$
$$= \sqrt{\int_0^1 (x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{1}{3}x + \tfrac{1}{36})\,dx}$$
$$= \sqrt{\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{4}{3}\frac{x^3}{3} - \frac{1}{3}\frac{x^2}{2} + \frac{1}{36}x \Big|_0^1}$$
$$= \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}}$$
$$= \sqrt{\frac{36}{180} - \frac{90}{180} + \frac{80}{180} - \frac{30}{180} + \frac{5}{180}} = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$$

$\therefore$ orthonormal basis $= \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

$= \left\{ \frac{1}{1}, 2\sqrt{3}\,(x - \tfrac{1}{2}), 6\sqrt{5}\,(x^2 - x + \tfrac{1}{6}) \right\}$

$= \left\{ 1, 2\sqrt{3}(x - \tfrac{1}{2}), 6\sqrt{5}(x^2 - x + \tfrac{1}{6}) \right\}$

b) Find the Fourier expansion of x^2 using

$f = \langle f, u_1 \rangle u_1 + \langle f, u_2 \rangle u_2 + \langle f, u_3 \rangle u_3$

$\langle f, u_1 \rangle = \int_0^1 x^2 (1)\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$\langle f, u_2 \rangle = \int_0^1 x^2 (2\sqrt{3}(x - \tfrac{1}{2}))\,dx = 2\sqrt{3}\int_0^1 (x^3 - \tfrac{1}{2}x^2)\,dx$

$= 2\sqrt{3}\left(\frac{x^4}{4} - \frac{1}{2}\frac{x^3}{3}\Big|_0^1\right) = 2\sqrt{3}\left(\frac{1}{4} - \frac{1}{6}\right) = 2\sqrt{3}\left(\frac{1}{12}\right) = \frac{\sqrt{3}}{6}$

$\langle f, u_3 \rangle = \int_0^1 x^2 (6\sqrt{5}(x^2 - x + \tfrac{1}{6}))\,dx = 6\sqrt{5}\int_0^1 (x^4 - x^3 + \tfrac{1}{6}x^2)\,dx$

$= 6\sqrt{5}\left(\frac{x^5}{5} - \frac{x^4}{4} + \frac{1}{6}\frac{x^3}{3}\Big|_0^1\right) = 6\sqrt{5}\left(\frac{1}{5} - \frac{1}{4} + \frac{1}{18}\right)$

$= 6\sqrt{5}\left(\frac{36}{180} - \frac{45}{180} + \frac{10}{180}\right) = 6\sqrt{5}\left(\frac{1}{180}\right) = \frac{\sqrt{5}}{30}$

$x^2 = \frac{1}{3}(1) + \frac{\sqrt{3}}{6}(2\sqrt{3}(x - \tfrac{1}{2})) + \frac{\sqrt{5}}{30}(6\sqrt{5}(x^2 - x + \tfrac{1}{6}))$

$= \frac{1}{3} + (x - \tfrac{1}{2}) + (x^2 - x + 16)$

Q3) Let $T: P_2 \to P_2$ be a LT. Let $B = \{1, x, x^2\}$, $C = \{3, 1+4x, 5+2x^2\}$

Suppose $[T]_B^C = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

a) Find $T(x)$

b) Find $T(a + bx + cx^2)$

a) Find $T(x)$

$[T]_B^C = [[T(1)]_C, [T(x)]_C, [T(x^2)]_C] = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

$$[T(x)]_C = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$T(x) = 3(3) + 0(1+4x) + 3(5+2x^2) = 9 + 15 + 2x^2 = 24 + 2x^2$

b) Find $T(a+bx+cx^2)$

$B = \{1, x, x^2\}$, $[a+bx+cx^2]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$[T(v)]_C = [T]_B^C [v]_B$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b+c \\ a+0+2c \\ a+3b+c \end{bmatrix}$$

$T(a+bx+cx^2) = 3(2a+3b+c) + (1+4x)(a+2c) + (5+2x^2)(a+3b+c)$

$= (6a+9b+3c) + (a+2c+4ax+8cx) + (5a+15b+5c+2ax^2+6bx^2+2cx^2)$

$= (12a+24b+10c) + (4a+8c)x + (2a+6b+2c)x^2$

77.) Let $T: \mathbb{R}^2 \to P_1$ defined by $T(a,b) = (a+b) + (a+2b)x$. Find the matrix of T with repect the bases $B = \{(2,0),(0,3)\}$ and $C = \{1+x, 2x\}$, Then use the matrix to compute $T(4,3)$

$T(2,0) = 2 + 2x = c_1(1+x) + c_2(0+2x)$

$(c_1 + c_1x) + (0 + 2c_2x) = 2 + 2x$

$(c_1+0) + (c_1+2c_2)x = 2 + 2x$

$c_1 + 0 = 2$

$c_1 + 2c_2 = 2$

$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

$c_1 = 2$, $c_2 = 0$

$T(0,3) = 3 + 6x = c_1(1+x) + c_2(0+2x)$

$(c_1 + c_1x) + (0 + 2c_2x) = 3 + 6x$

$(c_1+0) + (c_1x + 2c_2x) = 3 + 6x$

$c_1 + 0 = 3$

$c_1 + 2c_2 = 6$

$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3/2 \end{bmatrix}$

$c_1 = 3$, $c_2 = 3/2$

$$[T]_B^C = \begin{bmatrix} 2 & 3 \\ 0 & 3/2 \end{bmatrix}$$

$$[T(v)]_C = [T]_B^C [v]_B$$

$$(4,3) = c_1(2,0) + c_2(0,3)$$

$$\left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 3 & 3 \end{array}\right] \overset{RREF}{=} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$$

$$\begin{bmatrix} 2 & 3 \\ 0 & 3/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3/2 \end{bmatrix}$$

$$T(4,3) = 7(1+x) + 3/2\,(2x)$$
$$= 7 + 7x + 3x$$
$$= 7 + 10x$$

30.) a) Show $\{3x, -1+3x\}$ is a basis for P_1

b) Find the transition matrix from $B = \{3x, -1+3x\}$ to $C = \{x, 1+x\}$

c) Use b) to find a change of basis matrix from $C = \{x, 1+x\}$ to $B = \{3x, -1+3x\}$

d) Find $[-1+6x]_B$

e) use d) and the matrix found in b) to find $[-1+6x]_C$

a) $\{1, x\}$ is a standard basis for P_1

dim P_1 is 2

dim$(\{3x, -1+3x\})$ is also 2

$\{3x, -1+3x\}$ is LI

since one is not a scalar multiplication of the other

$\therefore \{3x, -1+3x\}$ is basis for P_1

b) Suppose $0 + 3x = c_1(0+x) + c_2(1+x)$

$-1 + 3x = c_1(0+x) + c_2(1+x)$

$(0 + c_1x) + (c_2 + c_2x) = 0 + 3x$

$(0 + c_1x) + (c_2 + c_2x) = -1 + 3x$

$(0+c_2)+(c_1+c_2)x = 0+3x$

$(0+c_2)+(c_1+c_2)x = -1+3x$

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & -1 \\ 1 & 1 & 3 & 3 \end{array}\right] \overset{RREF}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$\therefore$ transition matrix is $\begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}$

c) $P_C^B = (P_B^C)^{-1} = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 4/3 \\ 0 & -1 \end{bmatrix}$

d) $[-1+6x]_B = c_1(3x) + c_2(-1+3x)$

$c_1 = 1,\ c_2 = 1 \Rightarrow = 1(3x) + 1(-1+3x)$

$= 3x - 1 + 3x$

$= -1+6x$

$\therefore\ -1+6x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B$

e) $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_C = [P]_B^C \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

$\therefore\ -1+6x = 7(0+x) + (-1)(1+x)$

$= 7x - 1 - x$

$= -1+6x$

Q3) An inner product on $C^0[0,1]$ be defined by

$\langle f, g \rangle = \int_0^1 2f(x)g(x)\,dx$

a) Find an orthonormal basis for the subspace of $C^0[0,1]$

spanned by $\{\overset{x_1}{1}, \overset{x_2}{2x}\}$

b) Find the Fourier expanison of $4x+3$

a) Find an orthonormal basis

$v_1 = x_1 = 1$

$v_2 = x_2 - \dfrac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$\langle x_2, v_1 \rangle = \langle 2x, 1 \rangle = \int_0^1 2(2x)(1)\,dx = 4\left(\frac{x^2}{2}\right|_0^1 = 2$

$\langle v_1, v_1 \rangle = \int_0^1 2(1)(1)\,dx = 2x|_0^1 = 2$

$v_2 = 2x - \frac{2}{2}(1) = 2x-1$

$\langle v_2, v_2 \rangle = \int_0^1 2(2x-1)^2\,dx = 2\int_0^1 (4x^2-4x+1)\,dx$

$= 2\left(4\frac{x^3}{3} - 4\frac{x^2}{2} + x\right|_0^1 = 2\left(\frac{4}{3} - 2 + 1\right)$

$= \frac{8}{3} - 4 + 2 = \frac{2}{3}$

orthogonal basis $= \{1, 2x-1\}$

orthonoromal basis $= \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\} = \left\{ \sqrt{\frac{1}{2}}(1), \sqrt{\frac{3}{2}}(2x-1) \right\}$

$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}(2x-1) \right\}$

b) Find the Fourier expanison of $4x+3$

$f = \langle f, u_1 \rangle u_1 + \langle f, u_2 \rangle u_2$

$\langle f, u_1 \rangle = \int_0^1 2(4x+3)\left(\frac{1}{\sqrt{2}}\right)dx = \frac{2}{\sqrt{2}}\int_0^1 (4x+3)\,dx = \frac{2}{\sqrt{2}}\left(4\frac{x^2}{2} + 3x\right|_0^1$

$= \frac{2}{\sqrt{2}}(2+3) = \frac{10}{\sqrt{2}}$

$\langle f, u_2 \rangle = \int_0^1 2(4x+3)\left(\sqrt{\frac{3}{2}}(2x-1)\right)dx = \frac{2\sqrt{3}}{\sqrt{2}}\int_0^1 (4x+3)(2x-1)\,dx$

$\frac{2\sqrt{3}}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}}$ $= \sqrt{6}\int_0^1 (8x^2 - 4x + 6x - 3)\,dx = \sqrt{6}\int_0^1 (8x^2 + 2x - 3)\,dx$

$= \sqrt{6}\left(8\frac{x^3}{3} + 2\frac{x^2}{2} - 3x\right|_0^1 = \sqrt{6}\left(\frac{8}{3} + 1 - 3\right)$

$\frac{2\sqrt{6}}{2}$ $= \sqrt{6}\left(\frac{8}{3} - 2\right) = \sqrt{6}\left(\frac{2}{3}\right) = \frac{2\sqrt{6}}{3}$

$4x+3 = \frac{10}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{2\sqrt{6}}{3}\left(\frac{\sqrt{3}}{\sqrt{2}}\right)(2x-1)$

$\frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}}$ $= 5 + 2(2x-1) = 5 + (4x-2) = 4x+3$

$\frac{\sqrt{6}}{2}$

96.) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a LT given by $T(x,y) = (2x+y, x+y)$.

Let $B = \{(3,4),(4,5)\}$ and $C = \{(2,-1),(1,-1)\}$ be bases for $\mathbb{R}^2$

a) Find $[T]_B^B$

b) Find $[P]_C^B$ the transition matrix from C to B

c) Use similarity to find $[T]_C^C$

a) Find $[T]_B^B$

$T(3,4) = (10,7) = c_1(3,4) + c_2(4,5)$

$T(4,5) = (13,9) = d_1(3,4) + d_2(4,5)$

$(3c_1, 4c_1) + (4c_2, 5c_2) = (10,7)$

$(3d_1, 4d_1) + (4d_2, 5d_2) = (13,9)$

$(3c_1 + 4c_2, 4c_1 + 5c_2) = (10,7)$

$(3d_1 + 4d_2, 4d_1 + 5d_2) = (13,9)$

$$\left[\begin{array}{cc:cc} 3 & 4 & 10 & 13 \\ 4 & 5 & 7 & 9 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc:cc} 1 & 0 & -22 & -29 \\ 0 & 1 & 19 & 25 \end{array}\right]$$

$\therefore [T]_B^B = \begin{bmatrix} -22 & -29 \\ 19 & 25 \end{bmatrix}$

b) Find $[P]_C^B$ the transition matrix from C to B

$[P]_C^B = [[(2,-1)]_B, [(1,-1)]_B]$

$(2,-1) = c_1(3,4) + c_2(4,5)$

$(1,-1) = d_1(3,4) + d_2(4,5)$

$(3c_1, 4c_1) + (4c_2, 5c_2) = (2,-1)$

$(3d_1, 4d_1) + (4d_2, 5d_2) = (1,-1)$

$(3c_1 + 4c_2, 4c_1 + 5c_2) = (2,-1)$

$(3d_1 + 4d_2, 4d_1 + 5d_2) = (1,-1)$

$$\left[\begin{array}{cc:cc} 3 & 4 & 2 & 1 \\ 4 & 5 & -1 & -1 \end{array}\right] \xrightarrow{RREF} \begin{bmatrix} -14 & -9 \\ 11 & 7 \end{bmatrix}$$

c) use similarity to find $[T]_C^C$

$[T]_C^C = ([P]_C^B)^{-1} [T]_B^B [P]_C^B$

$= \begin{bmatrix} 4 & 1 \\ -5 & -1 \end{bmatrix}$

16.) Let $T_1 : P_1 \to P_1$ be defined by $T_1(1) = 1+x$, $T(x) = -1+2x$

Let $T_2 : P_1 \to P_2$ be defined by $T_2(1) = 3x$, $T_2(x) = 1-x$

a) Find the matrix of $T_2 \circ T_1$ with respect to $B = C = D = \{1, x\}$

b) Use the matrix found in a) to compute $T(3x+1)$

a) $[T_1]_B^C = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ $\quad [T_2]_C^D = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}$

$[T_2 \circ T_1]_B^D = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix}$

b) $[T_2 \circ T_1(1+3x)]_D = \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -13 \end{bmatrix}$

$T_2 \circ T_1(1+3x) = 7 - 13x$

55.) a) Find a matrix of the linear transformation that rotates (x,y) $90°$ counterclockwise followed by the reflection about the x-axis.

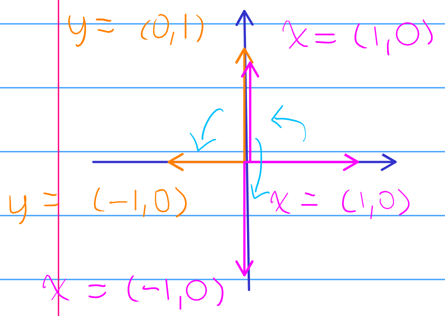
(hint : recall that a LT is determined by its values on basis vectors)

b) if $T : \mathbb{R}^4 \to \mathbb{R}^3$ is LT with $\dim(\ker T) = 1$,

show T is onto

a) $x = (1, 0)$

$y = (0, 1)$



$T(1,0) = (0,1)$

$T(0,1) = (-1,0)$

$\Rightarrow \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

b) $\dim \text{Domain} = \dim \text{Ker} T + \dim \text{Rng} T$

$4 = 1 + \dim \text{Rng} T$

$3 = \dim \text{Rng} T$

Since $\dim$ Codomain is also 3

$\therefore$ T is onto

57) b) Let V, W, L be vector spaces, $T_1 : V \to W$, $T_2 : W \to L$ be linear transformations. Prove that $T_2 \circ T_1 : V \to L$ is a LT.
(notes that you cannot use matrices in the proof since the vector spaces may be infinite dimensional).

c) Let V, W be vector spaces, $T : V \to W$ a LT. Prove that if T is 1-1, $\{v_1, v_2\}$ LI, then $\{T(v_1), T(v_2)\}$ is LI
(hint: first suppose $c_1 T(v_1) + c_2 T(v_2) = 0$)

b) $T_2 \circ T_1 (V_1 + V_2) = T_2(T_1(V_1 + V_2)) = T_2(T_1(V_1) + T_1(V_2))$

$= T_2(T_1(V_1)) + T_2(T_1(V_2))$

$= T_2 \circ T_1(V_1) + T_2 \circ T_1(V_2)$

$T_2 \circ T_1(kV) = T_2(T_1(kV)) = T_2(kT_1(V)) = kT_2(T_1(V))$

$= k\, T_2 \circ T_1(V)$

$\therefore T_2 \circ T_1$ is a LT

c) Suppose $c_1 T(v_1) + c_2 T(v_2) = 0$

So $c_1 = c_2 = 0$

Since T is LT

$c_1 T(v_1) + c_2 T(v_2) = 0$

Since T is 1-1

$c_1 v_1 + c_2 v_2 = 0$

It implies $c_1 = c_2 = 0$

since v_1 and v_2 are LI

$\therefore \{T(v_1), T(v_2)\}$ is LI

64.) Construct an isomorphism between the following vector spaces. You are not required to show that the functions constructed are indeed isomorphisms

a) between $\mathbb{R}$ and the set of 2×2 skew symmetric matrices

b) between $\mathbb{R}^2$ and P_1

a) $T : \mathbb{R} \to SSM$ as $T(a) = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$

b) $T : \mathbb{R}^2 \to P_1$ as $T(a, b) = a + bx$

69.) Let $T : V \to W$ be a linear transformation

a) Show that if T is 1-1 and $\dim(W) > \dim(V)$, then T is not onto

b) Show that if T is onto and $\dim(V) = \dim(W)$, then T is 1-1

a) Suppose T is 1-1

thus $\dim \operatorname{Ker} T = 0$

$\dim \text{Domain} = \dim \ker T + \dim \operatorname{Rng} T$

$\dim \text{Domain} = 0 + \dim \operatorname{Rng} T$

$\dim \text{Domain} = \dim \operatorname{Rng} T$

But $\dim(W) > \dim(V)$

$\therefore$ T is not onto

b) Suppose T is onto

thus $\dim \text{Rng } T = \dim(W)$

$\dim \text{Domain} = \dim \ker T + \dim \text{Rng } T$

$\dim(V) = \dim \ker T + \dim(W)$

$\dim \ker T = \dim(V) - \dim(W)$

since $\dim(V) = \dim(W)$

$\dim \ker T = 0$

$\therefore$ T is 1-1

70.) Construct an isomorphism between $\mathbb{R}^3$ and 2×2 symmetric matrices.

You must verify that the function is

a) LT

b) 1-1

c) onto

$$T: \mathbb{R}^3 \to M_{2,2} \text{ as } T(a,b,c) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

a) Is T a LT ?

i) show $T(a_1+a_2, b_1+b_2, c_1+c_2) = T(a_1,b_1,c_1) + T(a_2,b_2,c_2)$

$$T(a_1+a_2, b_1+b_2, c_1+c_2) = \begin{bmatrix} a_1+a_2 & c_1+c_2 \\ c_1+c_2 & b_1+b_2 \end{bmatrix} = \begin{bmatrix} a_1 & c_1 \\ c_1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & c_2 \\ c_2 & b_2 \end{bmatrix}$$

$$= T(a_1,b_1,c_1) + T(a_2,b_2,c_2)$$

(ii) show $T(k(a,b,c)) = kT(a,b,c)$

$$T(k(a,b,c)) = T(ka,kb,kc) = \begin{bmatrix} ka & kc \\ kc & kb \end{bmatrix} = k\begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

$$= kT(a,b,c)$$

b) Is T 1-1 ?

suppose $\ker T = T(a,b,c) = \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore (a,b,c) = (0,0,0)$

$\therefore T$ is 1-1

c) Is T onto ?

$\dim \text{Domain} = \dim \ker T + \dim \text{Rng}\, T$

$3 = 0 + \dim \text{Rng}\, T$

$\therefore \dim \text{Rng}\, T = 3$

Since the $\dim \text{Rng}\, T = 3$, by the rank-nullity theorem

$\dim M_{2,2}$ is also 3

$\therefore T$ is onto

95.) Suppose A and B are similar matrices and B and C are similar matrices. Show A and C are similar matrices

(Suggestion: to show A and C are similar, you need to find a matrix E such that $C = E^{-1}AE$. We know since A and B are similar matrices, there is a matrix S_1 such that $S_1^{-1}AS = B$. In addition, since B and C are similar matrices, there is a matrix S_2 such that $S_2^{-1}BS_2 = C$. Now can you find E?)

Since $A \sim B$, there is a non-singular matrix S

such that $B = S^{-1}AS$

Since $B \sim C$, there is a non-singular matrix Q

such that $C = Q^{-1}BQ$

$C = Q^{-1}BQ$

$C = Q^{-1}(S^{-1}AS)Q$

$C = (Q^{-1}S^{-1})A(SQ)$

since S and Q are non-singular,

SQ is also non-singular

$C = (QS)^{-1}A(SQ)$

$C = (SQ)^{-1}A(SQ)$

$\therefore A \sim C$