1) Find the kernel of the linear transformation $T: R^3 \to R^2$ defined by $T(x) = Ax$, where $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 3 \end{bmatrix}$

$$T(x) = 0$$

$$Ax = 0$$

$$x \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$c_1 = t$

$c_2 = -t$

$c_3 = t$

$\therefore \ker T = \{(1, -1, 1)\}$

2) Let $p(x) = 1 - 2x^2$, $q(x) = 4 - 2x + x^2$, and $r(x) = x + 2x^2$ be polynomials in P_2, $\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_1$, find each quantity

a) $\langle p, q \rangle$

b) $\langle q, r \rangle$

c) $\| q \|$

d) $d(p, q)$

a) $\langle p, q \rangle = \langle 1 - 2x^2, 4 - 2x + x^2 \rangle$

$= 1(4) + 0(-2) + (-2)(1) = 4 - 2 = 2$

$\therefore$ p and q are not orthogonal

b) $\langle q, r\rangle = \langle 4-2x+x^2, x+2x^2\rangle$

$= 4(0) + (-2)(1) + (1)(2) = -2+2 = 0$

$\therefore$ q and r are orthogonal

c) $\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{\langle 4-2x+x^2, 4-2x+x^2\rangle}$

$= \sqrt{4(4)+(-2)(-2)+(1)(1)}$

$= \sqrt{16+4+1}$

$= \sqrt{21}$

d) $d(p,q) = \|p-q\|$

$= \|1-2x^2-(4-2x+x^2)\|$

$= \|-3+2x-3x^2\|$

$= \sqrt{\langle -3+2x-3x^2, -3+2x-3x^2\rangle}$

$= \sqrt{(-3)(-3)+2(2)+(-3)(-3)}$

$= \sqrt{9+4+9} = \sqrt{22}$

3) The function $f(x) = x$ and $g(x) = x^2$ in $C[0,1]$

To find each quantity

a) $\|f\|$

b) $d(f,g)$

a) $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^1 x \cdot x \, dx} = \sqrt{\int_0^1 x^2 dx}$

$= \sqrt{x^3/3 \Big|_0^1} = \sqrt{1/3} = 1/\sqrt{3}$

b) $d(f,g) = \|f-g\| = \|x-x^2\| = \sqrt{\langle x-x^2, x-x^2\rangle}$

$= \sqrt{\int_0^1 (x-x^2)(x-x^2)dx}$

$= \sqrt{\int_0^1 (x^2-2x^3+x^4)dx}$

$= \sqrt{x^3/3 - 2x^4/4 + x^5/5 \Big|_0^1}$

$= \sqrt{1/3 - 1/2 + 1/5} = \sqrt{\frac{10}{30} - \frac{15}{30} + \frac{6}{30}} = \sqrt{\frac{1}{30}} = \frac{1}{\sqrt{30}}$

4) Let $T: P_2 \to P_3$ be a linear transformation defined by

$T(p(x)) = \int_0^x p(x)\,dx$

Find $[T]_B^C$ the matrix of T with respect to

$B = \{1, x, x^2\}$, $C = \{1, x, x^2, x^3\}$

Is T 1-1 and onto ?

$T(1,0,0) = \int_0^x 1\,dx = x\,\Big|_0^x = x$

$T(0,1,0) = \int_0^x x\,dx = \frac{x^2}{2}\Big|_0^x = \frac{1}{2}x^2$

$T(0,0,1) = \int_0^x x^2\,dx = \frac{x^3}{3}\Big|_0^x = \frac{1}{3}x^3$

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$[T]_B^C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

5.) a) Let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be LT, $T(1,1) = (4,8)$, $T(2,1) = (6,4)$

compute $T(7,5)$

b) find the matrix of T with respect to $B = \{(1,1),(2,1)\}$

and $C = \{(1,1),(1,-1)\}$

6) Let $V = C^0[-1,1]$ and define an inner product on V as

$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$

a) show $\langle f - 3g, h \rangle = \langle f, h \rangle - 3\langle g, h \rangle$

for $f, g, h \in V$

b) show $f(x) = \sin(\pi x)$ and $g(x) = \cos(\pi x)$

are orthogonal

c) find the projection of x^2 onto $2x+1$