OBJECTIVES

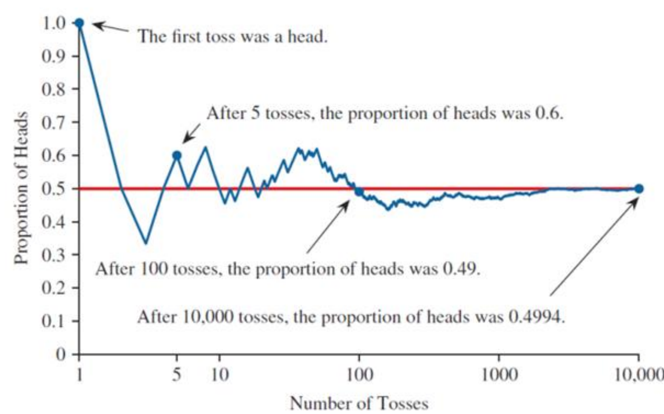
1. Construct sample spaces
2. Compute and interpret probabilities
3. Approximate probabilities using the Empirical Method
4. Approximate probabilities by using simulation

OBJECTIVE 1
CONSTRUCT SAMPLE SPACES

A _probability experiment_ is one in which we do not know what any individual outcome will be, but we do know how a long series of repetitions will come out. For example, if we toss a fair coin, we do not know what the outcome of a single toss will be, but we do know what the outcome of a long series of tosses will be – about half "heads" and half "tails". The _probability_ of an event is the proportion of times that the event occurs in the long run. So, for a "fair" coin, that is, one that is equally likely to come up heads as tails, the probability of heads is 1/2 and the probability of tails is 1/2.

LAW OF LARGE NUMBERS

The Law of Large Numbers says that as a probability experiment is repeated again and again, the proportion of times that a given event occurs will approach its probability.



SAMPLE SPACE

The collection of all the possible outcomes of a probability experiment is called a sample space.

EXAMPLE: Describe the sample space for each of the following experiments:
a) The toss of a coin
b) The roll of a die
c) Selecting a student at random from a list of 10,000 at a large university

SOLUTION:

a) The sample space is $S = \{H, T\}$

b) $S = \{1, 2, 3, 4, 5, 6\}$

c) sample space consists of 10000 students

PROBABILITY MODEL

We are often concerned with occurrences that consist of several outcomes. For example, when rolling a die, we might be interested in the probability of rolling an odd number. Rolling an odd number corresponds to the collection of outcomes {1, 3, 5} from the sample space {1, 2, 3, 4, 5, 6}. In general, a collection of outcomes of a sample space is called an event. (An event is a subset of the sample space)

Once we have a sample space, we need to specify a probability of each event. This is done with a probability model. We use the letter *"P"* to denote probabilities. For example, we denote the probability that a tossed coin lands heads by P(Heads).

In general, if *A* denotes an event, the probability of event *A* is denoted by ___P(A)___.

OBJECTIVE 2
COMPUTE AND INTERPRET PROBABILITIES

PROBABILITY RULES

The probability of an event is always between ___0___ and ___1___. In other words, for any event *A*,

$\underline{\ 0\ } \leq P(A) \leq \underline{\ 1\ }$.

- If *A* cannot occur, then ___$P(A) = 0$___.
- If *A* is certain to occur, then ___$P(A) = 1$___.

PROBABILITIES WITH EQUALLY LIKELY OUTCOMES

If a sample space has n equally likely outcomes and an event *A* has k outcomes, then

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in sample space}} = \frac{k}{n}$$

EXAMPLE: In the Georgia Cash-4 Lottery, a winning number between 0000 and 9999 is chosen at random, with all the possible numbers being equally likely. What is the probability that all four digits are the same?

SOLUTION: The event of all 4 digits are the same = {0000, 1111, 2222, ..., 9999} ← 10 outcomes

S = {0000, 0001, 0002, ... 9999} → 10000 outcomes

$P(\text{all 4 same}) = \frac{10}{10000} = 0.001$

EXAMPLE: A family has three children. Denoting a boy by B and a girl by G, we can denote the genders of these children from oldest to youngest. For example, GBG means the oldest child is a girl, the middle child is a boy, and the youngest child is a girl. There are eight possible outcomes: BBB, BBG, BGB, BGG, GBB, GBG, GGB, and GGG. Assume these outcomes are equally likely.

a) What is the probability that there are exactly two girls?
b) What is the probability that all three children are the same gender?

SOLUTION:

a) $P(\text{exactly two girls}) = \frac{3}{8} = 0.375$

b) $P(\text{same gender}) = \frac{2}{8} = 0.25$

SAMPLING IS A PROBABILITY EXPERIMENT

Sampling an individual from a population is a probability experiment. The population is the sample space and members of the population are equally likely outcomes.

EXAMPLE: There 10,000 families in a certain town categorized as follows. A pollster samples a single family from this population.

Own a house	Own a condo	Rent a house	Rent an apartment
4753	1478	912	2857

→ total samples = 10000

a) What is the probability that the sampled family owns a house?
b) What is the probability that the sampled family rents?

SOLUTION:

a) $P(\text{own a house}) = \frac{4753}{10000}$

b) $P(\text{rents}) = \frac{912 + 2857}{10000} = \frac{3769}{10000}$

UNUSUAL EVENTS

An unusual event is one that is not likely to happen. In other words, an event whose probability is small. There are no hard-and-fast rules as to just how small a probability needs to be before an event is considered unusual, but we will use the following rule of thumb.

RULE OF THUMB ABOUT UNUSUAL EVENTS:
Any event whose probability is less than 0.05 is considered to be usual

EXAMPLE: In a college of 5000 students, 150 are math majors. A student is selected at random and turns out to be a math major. Is this unusual?

SOLUTION: Let's compute the probability of choosing a math major

$P(\text{math major}) = \frac{150}{5000} = 0.03$

since the probability is less than 0.05, the event would be considered an usual event

OBJECTIVE 3
APPROXIMATE PROBABILITIES USING THE EMPIRICAL METHOD

The Law of Large Numbers says that if we repeat a probability experiment a large number of times, then the proportion of times that a particular outcome occurs is likely to be close to the true probability of the outcome. The __Empirical method__ consists of repeating an experiment a large number of times, and using the proportion of times an outcome occurs to approximate the probability of the outcome.

EXAMPLE: In 2010, there were 2,046,935 boys and 1,952,451 girls born in the U.S. Approximate the probability that a newborn baby is a boy.

SOLUTION: The number of times the experiment has been repeated is

$2046935 + 1952451 = 3999386$ births.

Proportion of birth that are boys $= \frac{2046935}{3999386} = 0.5118$

Now, we approximate the probability:

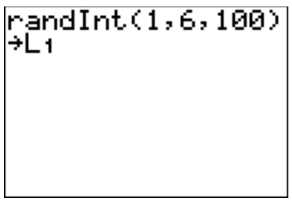
$P(\text{boy}) \approx 0.5118$

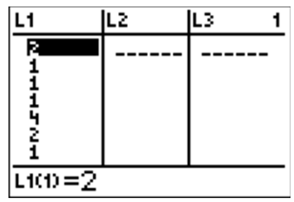
OBJECTIVE 4
APPROXIMATE PROBABILITIES BY USING SIMULATION

In practice, it can be difficult or impossible to repeat an experiment many times in order to approximate a probability with the Empirical Method. In some cases, we can use technology to repeat an equivalent virtual experiment many times. Conducting a virtual experiment in this way is called simulation.

SIMULATION ON THE TI-84 PLUS

The **randInt** command on the TI-84 PLUS calculator may be used to simulate the rolling of a single die. To access this command, press **MATH**, scroll to the **PRB** menu, and select **randInt**. The following screenshots illustrate how to simulate the rolling of a die 100 times. The outcomes are stored in list **L1**.





YOU SHOULD KNOW …

- The Law of Large Numbers
- The definition of:
  - Probability
  - Sample space
  - Event
- How to compute probabilities with equally likely outcomes
- The rule-of-thumb for determining when an event A is unusual (if $P(A) < 0.05$)
- How to approximate probabilities using the Empirical Method
- How to approximate probabilities using simulation

SECTION 5.2: THE ADDITION RULE AND THE RULE OF COMPLEMENTS

OBJECTIVES

1. Compute probabilities by using the General Addition Rule
2. Compute probabilities by using the Addition Rule for Mutually Exclusive Events
3. Compute probabilities by using the Rule of Complements

OBJECTIVE 1
COMPUTE PROBABILITIES BY USING THE GENERAL ADDITION RULE

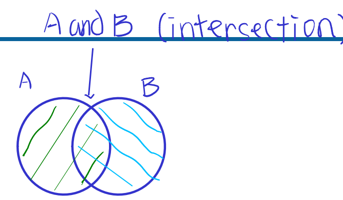
A OR *B* EVENTS AND THE GENERAL ADDITION RULE

A ______compound event______ is an event that is formed by combining two or more events. One type of compound event is of the form ___A or B___. The event *A* or *B* occurs whenever *A* occurs, *B* occurs, of *A* and *B* both occur. Probabilities of events in the form *A* or *B* are computed using the

___general addition rule___.

THE GENERAL ADDITION RULE: for any two event A and B

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$



EXAMPLE: 1000 adults were asked whether they favored a law that would provide support for higher education. In addition, each person was classified as likely to vote or not likely to vote based on whether they voted in the last election. What is the probability that a randomly selected adult is likely to vote or favors the law?

	Favor	Oppose	Undecided
Likely to vote	372	262	87
Not likely to vote	151	103	25

SOLUTION:

$P(\text{likely to vote or favors}) = P(\text{likely to vote}) + P(\text{favors}) - P(\text{likely to vote and favors})$

$$= \frac{372+262+87}{1000} + \frac{372+151}{1000} - \frac{372}{1000}$$

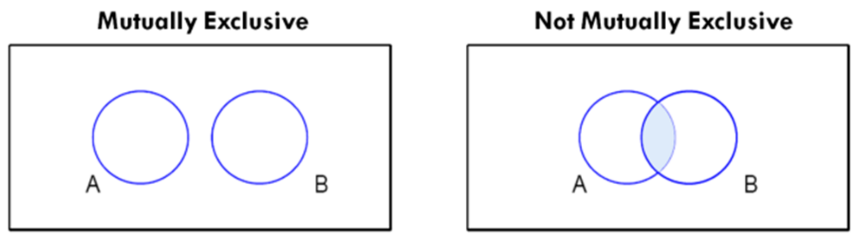
$$= 0.721 + 0.523 - 0.372$$

$$= 0.872$$

OBJECTIVE 2
COMPUTE PROBABILITIES BY USING THE ADDITION RULE FOR MUTUALLY EXCLUSIVE EVENTS

MUTUALLY EXCLUSIVE (disjoint events)

Two events are said to be **mutually exclusive** if it is __impossible__ for both events to occur.



EXAMPLE: A die is rolled. Event *A* is that the die comes up 3, and event *B* is that the die comes up an even number.

SOLUTION: These events are mutually exclusive since the dice cannot both come up 3 and an even number

EXAMPLE: A fair coin is tossed twice. Event *A* is that one of the tosses is heads, and Event *B* is that one of the tosses is tails.

SOLUTION: These events are not mutually exclusive

HT or TH

THE ADDITION RULE FOR MUTUALLY EXCLUSIVE EVENTS

If events *A* and *B* are mutually exclusive, then *P*(*A* and *B*) = 0. This leads to a simplification of the General Addition Rule.

ADDITION RULE FOR MUTUALLY EXCLUSIVE EVENTS:
If A and B are mutually exclusive events, then

$P(A \text{ or } B) = P(A) + P(B)$

EXAMPLE: In the 2012 Olympic Games, a total of 10,735 athletes participated. Of these, 530 represented the United States, 277 represented Canada, and 102 represented Mexico. What is the probability that an Olympic athlete chosen at random represents the U.S. or Canada?

SOLUTION: Since these events are mutually exclusive,

$P(\text{US or canada}) = P(\text{US}) + P(\text{canada}) = \frac{530}{10735} + \frac{277}{10735} = 0.0752$

OBJECTIVE 3
COMPUTE PROBABILITIES BY USING THE RULE OF COMPLEMENTS

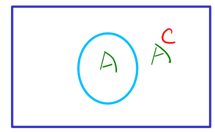
COMPLEMENTS

If there is a 60% chance of rain today, then there is a 40% chance that it will not rain. The events "Rain" and "No rain" are complements. The complement of an event says that the event does not occur.

If *A* is any event, the **complement** of *A* is the event that *A* does not occur. The complement of *A* is denoted A^c.

EXAMPLE: Two hundred students were enrolled in a Statistics class. Find the complements of the following events:

a) Exactly 50 of them are business majors $x = 50$
b) More than 50 of them are business majors $x > 50$
c) At least 50 of them are business majors $x \geq 50$



SOLUTION: let x = # of bussiness majors

a) The complement is the number of bussiness majors $x \neq 50$ ($x < 50$ or $x > 50$)

b) The complement " " " " $x \leq 50$ (50 or fewer)

c) The complement " " " " $x < 50$ (fewer than 50)

THE RULE OF COMPLEMENTS

THE RULE OF COMPLEMENTS:

$P(A^c) = 1 - P(A)$ (Note: $P(A) + P(A^c) = 1$)

EXAMPLE: According to the *Wall Street Journal*, 40% of cars sold in July 2013 were small cars. What is the probability that a randomly chosen car sold in July 2013 is not a small car?

SOLUTION: Using the rule of complements,

$P(\text{not a small car}) = 1 - P(\text{small car})$
$= 1 - 0.40 = 0.60$

YOU SHOULD KNOW …

- How to use The General Addition Rule to compute probabilities of events in the form *A* or *B*
- How to determine whether events are mutually exclusive
- How to compute probabilities of mutually exclusive events
- How to determine the complement of an event
- How to use the Rule of Complements to compute probabilities

SECTION 5.3: CONDITIONAL PROBABILITY AND THE MULTIPLICATION RULE

OBJECTIVES

1. Compute conditional probabilities
2. Compute probabilities by using the General Multiplication Rule
3. Compute probabilities by using the Multiplication Rule for Independent Events
4. Compute the probability that an event occurs at least once

OBJECTIVE 1
COMPUTE CONDITIONAL PROBABILITIES

Approximately 15% of adult men in the U.S. are more than six feet tall. Therefore, if a man is selected at random, the probability that he is more than six feet tall is 0.15.

Now assume that you learn that the selected man is a professional basketball player. With this extra information, the probability that the man is more than six feet tall becomes much greater than 0.15.

A probability that is computed with the knowledge of additional information is called a

additional probability.

The conditional probability of an event *B* given an event *A* is denoted P(B|A). $P(B|A)$ is the probability that *B* occurs, under the assumption that *A* occurs.

The probability is computed as: $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

EXAMPLE: Consider the following table which presents the number of U.S. men and women (in millions) 25 years old and older who have attained various levels of education.

	Not a high school graduate	High school graduate	Some college, no degree	Associate's degree	Bachelor's degree	Advanced degree	total
Men	14.0	29.6	15.6	7.2	17.5	10.1	= 94.0
Women	13.7	31.9	17.5	9.6	19.2	9.1	= 101.0

total = 195.0

A person is selected at random.

a) What is the probability that the person is a man?
b) What is the probability that the person is a man with a Bachelor's degree?
c) What is the probability that the person has a Bachelor's degree, given that he is a man?

SOLUTION:

a) $P(\text{man}) = \frac{94.0}{195.0} = 0.4821$

b) $P(\text{man with bachelor's}) = P(\text{man and bachelor's}) = \frac{17.5}{195.0} = 0.0897$

c) $P(\text{bachelor's} \mid \text{man}) = \frac{P(\text{man and bachelor's})}{P(\text{man})} = \frac{17.5/195.0}{94.0/195.0} = \frac{17.5}{94.0} = 0.1862$

OBJECTIVE 2
COMPUTE PROBABILITIES BY USING THE GENERAL MULTIPLICATION RULE

THE GENERAL MULTIPLICATION RULE

The General Method for computing conditional probabilities provides a way to compute probabilities for events of the form "*A* and *B*." If we multiply both sides of the equation by $P(A)$ we obtain the **General Multiplication Rule**.

THE GENERAL MULTIPLICATION RULE: for any two events A and B,

$P(A \text{ and } B) = P(A) \cdot P(B|A)$

or $P(A \text{ and } B) = P(B) \cdot P(A|B)$

EXAMPLE: Among those who apply for a particular job, the probability of being granted an interview is 0.1. Among those interviewed, the probability of being offered a job is 0.25. Find the probability that an applicant is offered a job.

SOLUTION: let I = interviewed J = offered a job $P(I) = 0.1$ $P(J|I) = 0.25$

$P(J) = P(I \text{ and } J)$

$= P(I) \cdot P(J|I)$

$= 0.1 \cdot 2.5$

$= 0.025$

OBJECTIVE 3
COMPUTE PROBABILITIES BY USING THE MULTIPLICATION RULE FOR INDEPENDENT EVENTS

INDEPENDENCE

Two events are independent if the occurrence of one does not affect the probability that the other event occurs. If two events are not independent, we say they are dependent.

EXAMPLE: Determine whether the following events are independent or dependent.

A college student is chosen at random. The events are "being a freshman" and "being less than 20 years old."

SOLUTION: These events are not independent. If a student is freshman, the probability that the student is less than 20 year old is greater than a student who is not a freshman

EXAMPLE: Determine whether the following events are independent or dependent.

A college student is chosen at random. The events are "born on a Sunday" and "taking a statistics class."

SOLUTION: These events are independent. Given a student was born on Sunday, this has no effect on the probability that the student takes a statistics class

THE MULTIPLICATION RULE FOR INDEPENDENT EVENTS

When two events, A and B, are independent, then $P(B|A) = P(B)$, because knowing that A occurred does not affect the probability that B occurs. This leads to a simplified version of the multiplication rule.

THE MULTIPLICATION RULE FOR INDEPENDENT EVENTS:

If A and B are independent events, then

$P(A \text{ and } B) = P(A) \cdot P(B)$

EXAMPLE: According to recent figures from the U.S. Census Bureau, the percentage of people under the age of 18 was 23.5% in New York City, 25.8% in Chicago, and 26.0% in Los Angeles. If one person is selected from each city, what is the probability that all of them are under 18? Is this an unusual event?

SOLUTION: Since these events are independent,

$$\begin{aligned} P(\text{all of them under } 18) &= P(N \text{ and } C \text{ and } LA) \\ &= P(N) \cdot P(C) \cdot P(LA) \\ &= 0.235 \cdot 0.258 \cdot 0.260 \\ &= 0.0158 \end{aligned}$$

Since the probability is less than 0.05, this event is unusual

SAMPLING WITH AND WITHOUT REPLACEMENT

When we sample two items from a population, we can proceed in either of two ways. We can replace the first item drawn before sampling the second. This is known as **sampling with replacement**. The other option is to leave the first item out when sampling the second one. This is known as **sampling without replacement**.

- When sampling **with replacement**, the draws are independent.
- When sampling **without replacement**, the draws are not independent.

When sampling with replacement, each draw is made from the entire population, so the probability of drawing a particular item on the second draw does not depend on the first draw.
When a sample is very small compared to the population (less than 5%), a rule of thumb is that the items may be treated as independent.

OBJECTIVE 4
COMPUTE THE PROBABILITY THAT AN EVENT OCCURS AT LEAST ONCE

"AT LEAST ONE" TYPE PROBABILITIES

Sometimes we need to find the probability that an event occurs **at least once** in several independent trials. The easiest way to calculate these probabilities is by finding the probability of the complement. Remember that the complement of "At least one event occurs" is "No events occur".

To compute the probability that an event occurs at least once, find the probability that it does not occur at all, and subtract from 1.

EXAMPLE: Items are inspected for flaws by three inspectors. If a flaw is present, each inspector will detect it with probability 0.8. The inspectors work independently. If an item has a flaw, what is the probability that at least one inspector detects it?

SOLUTION: The complement of the event that at least one of the inspectors detects the flaw is that none of the inspectors detects the flaw.

Now, P(1 fail to detect) = 1 − P(1 detect flaw) = 1 − 0.8 = 0.2

Then, P(at least one detects flaw) = 1 − P(none detects flaw)
= 1 − P(all 3 inspectors fail to detect)
= 1 − P(1^{st} and 2^{nd} and 3^{rd} fail to detect)
= 1 − P(1^{st}) · P(2^{nd}) · P(3^{rd}) ← since the events are independent
= 1 − (0.2) · (0.2) · (0.2)
= 0.992

YOU SHOULD KNOW …

- How to compute conditional probabilities
- How to use the General Multiplication Rule to compute probabilities of events in the form "*A* and *B*"
- How to determine whether events are independent or dependent
- How to use the Multiplication Rule for Independent Events to compute probabilities
- The difference between sampling with replacement and sampling without replacement and the implications on the independence of the sampled items
- How to compute "at least one" type probabilities