Matt 260 Practice Problems

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1) Prove each of the following: Assume n and m represent integers.

a) n^2 is odd iff $n+4$ is odd.
b) If $m \cdot n$ is even, then n is even or m is even.
c) Let $x, y \in \mathbb{R}$. Prove that if x is rational and y is irrational, the xy is irrational.

(recall that 1) $t \in \mathbb{R}$ is rational if $t = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. 2) $h \in \mathbb{R}$ is irrational if h is not rational.)

2)

a) Write the negation for each the following: i) I am majoring in math and psychology. Ii) All students take calculus.
b) Write the converse and contrapositive of the following: If you can do math, then you can do physics.
c) Write x=3 iff x^2=9 as two conditional statements.
d) Determine if $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(x) = x - 4$ is a bijection.

3) Compute $\begin{vmatrix} 3 & -2 & -2 \\ 1 & 1 & 2 \\ 5 & 5 & -4 \end{vmatrix}$ by first transforming the matrix to a row-echelon form.

4)

a) Find an equation of the line the formed by intersecting the planes $x + 2y + 3z = 8$ and $2x + 5y - z = 10$.

b) Use Cramer's Rule to find z (do not find x, y) for the following system of equations:

$$2x - y - 5z = 0$$
$$3x + z = 10$$
$$5x - z = 14$$

5) Find an equation that a, b, c must satisfy for the following system of equations to have at least one solution:

$$x - y - z = a$$
$$x + y + 3z = b$$
$$2y + 4z = c$$

6) Define T(x)=Ax, where A=$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

a) Show T is a LT.
b) Find a basis for the range of T.

7)

a) Show that $\det(S^{-1}AS) = \det(A)$
b) If det (A)=4, and A is 3 x 3, find the det(2A^2)

8)

a) Find the sign of the permutation (1 3 2)
b) Find the rank of a 2 x 3 linear system of equations if its row echelon form has 2 free variables.
c) (true/false) The row space and column space of a matrix are equal.

d) (true/false) The determinant of a matrix does not change when a multiple of one row is added to another row.
e) (true/false) A matrix is invertible if and only if its determinant is 0.
f) (true/false) A system of homogeneous equations has at least one solution.
g) (true/false) $\det(A) = \det(A^T)$

9)Let $T: \mathbf{R}^2 \to P_1$ be defined by $T(a,b) = (b-a) + 3bx$. Find the matrix of T with respect to the bases $B = \{(1,0),(0,1)\}, C = \{2, 5x\}$. Then use the matrix to compute $T(2,5)$

10) Is the set of 2 X 2 diagonal matrices with real entries a subspace of the vector space of 2 x 2 matrices over R? Justify your answer.

11) (5 points each) Define an inner product on a vector space of all real-valued functions n, as $< f, g >= \int_0^{2\pi} f(x)g(x)\, dx$. A) Compute $\|f\|$ for $f(x) = \sin 2x$. B) Are $f(x) = \sin 2x$ and $g(x) = \cos 2x$ orthogonal? Justify your answer.

12) On $\mathbf{R}^2$ (all real ordered pairs), define the operation and multiplication by a real number as follows:

$$(x_1, y_1) + (x_2, y_2) = (2x_1x_2, y_1 + y_2)$$
$$\alpha(x, y) = (\alpha x, y)$$

a) Is + commutative?
b) Is there a 0 vector?
c) Use the definition of additive inverse to find $-(1,2)$
d) Does $1\mathbf{v} = \mathbf{v}$ hold for all $\mathbf{v} \in V$?

13) Let $\mathbf{T}: \mathbf{P}_1 \to \mathbf{P}_1$ be defined by $T(a+bx) = 3a + 2bx$
a) Verify that this is a linear transformation
b) Find $[\mathbf{T}]_B^C$ with respect to $B = \{1+x, -1+2x\}$ and $C = \{3, 5+2x\}$.
c) Use the matrix to compute $T(1+4x)$

14) Let $\mathbf{V} = \mathbf{R}^3$, $v_1 = (1,0,2)$, $v_2 = (2,4,5)$ Determine whether v_1, v_2 are LI or not and find the subspace spanned by { v_1, v_2 } and describe it geometrically.

15) For A= $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

a) Find the determinant by first writing A in upper triangular form.
b) Find the (2, 3) entry of the adjoint matrix.

16) Let $T_1 : P_1 \to P_1$ be defined by $T_1(1) = 1 + x, T_1(x) = -1 + 2x$ Let $T_2 : P_1 \to P_2$ be defined by $T_2(1) = 3x, T_2(x) = 1 - x$. Compute A) Find the matrix of $T_2 \circ T_1$ with respect to B=C=D={1,x}. b) Use the matrix found in a) to compute T(3x+1)

17) Let $T: P_2 \to P_3$ be defined by $T(ax^2 + bx + c) = (2a - c)x^2 + bx$. Is T invertible? Fully justify your answer.

18)

a) Find a basis for the column space and row space for $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & 1 \end{bmatrix}$

b) Is it true that dim(row space)=dim(column space)? Justify your answer.

19) Let k be the number that makes the following equation true. $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Find k without evaluating the determinants.

20) Let V= R^3 and $S = \{(x_1, x_2, x_3) \in R^3 : x_1 + x_2 + 2x_3 = 0\}$

a) Show S is a subspace of R^3.
b) Find a basis for S. (Hint: there are two free variables in this equation.)

21) Find vectors in R^4 that span the null(A), where A = $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 5 & 1 & 2 \\ 2 & 4 & -2 & 8 \end{bmatrix}$. What is the dimension of null(A)?

22) Let V = M_2(R) and $S = \{A \in V : A = \left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right]$

a) Show S is a subspace of V.
b) Prove that $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\}$ is a basis for S

23)

i. Describe the subspace of R^3 spanned by {(1,0,-2), (-2,1,4)}.
ii. Use i) to determine if (1,4,-2) in the span{(1,0,-2), (-2,1,4)}? Justify your answer.

24) Determine whether S={(1,3), (3,-1), (0,4)} is dependent or independent in R^2. If the set is dependent, find a dependent relationship and find a LI subset of S that has the same span as S. is, find a basis for the subspace for Span{S}.

25) Define a product on P_2 as follows: $< f, g >= \int_0^1 f(x)g(x)dx$

a) Show <f,g> is an inner product over R.
b) Find an orthogonal basis for the span($\{1, x\}$

26) Determine whether $S = \{(1,3,2), (-4,2,4), (0,7,0)\}$ spans R^3 or not. If $\mathrm{Span(S)} \neq \mathrm{R}^3$, then find a basis for Span(S).

27) Is $S = \{1 + x,\ 2 + x + x^2, 1 - x\}$ a basis for P_2? Justify your answer.

28) Let S be the subset of M_2(R) consisting of all upper triangular matrices. Show S is a subspace of M_2(R) and find a basis for S.

29) Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)$. Verify that T is a linear transformation and find a basis for Ker T. Also find the dim (Range T).

30) Let A= $\begin{bmatrix} 1 & 2 & 0 \\ k & 0 & 2 \\ 4k+1 & k & 0 \end{bmatrix}$, where k is a real number. Find the value(s) of k that makes A invertible.

31) Let V = R^2 and F= R. Define + and . on V as follows:

(a,b)+(c,d)=(2a+2c,2b+2d), k(a,b)=(2ka, 2kb).

a) Is the operation + commutative?
b) Is + associative?
c) Is there a 0 zero vector?
d) Does (c+d)v=cv+dv hold?

32) Determine whether $S=\{(1,3,1),(-1,3,7),(-2,-3,2)\}$ is LI or LD in R^3. If it is LD, find a dependency relationship and find a LI set of S that has the same span as S. That is, find a basis for Span S.

33) Use Gauss Jordan to find the inverse of $A=\begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 4 \\ 0 & 3 & 1 \end{bmatrix}$

34) Let $v_1=(1,3)$ and $v_2=(3,1)$.
a) Show v_1 and v_2 form a basis for R^2 and determine the components of each of $e_1=(1,0)$ and $e_2=(0,1)$ relative to this basis.
b) Find a change of basis matrix from $B=\{(1,0),(0,1)\}$ to $C=\{(1,3),(3,1)\}$ and use the matrix to find the component vector of (2,3)
c) Use b) (not from scratch) to find the change of basis matrix from $C=\{(1,3),(3,1)\}$ to $B=\{(1,0),(0,1)\}$

35) In $M_2(\mathbf{R})$, find the change of basis matrix from $\{\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ to $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

36)
a) (true/false) For square matrices A, B, if AB is invertible then A and B are both invertible.
b) (true/false) For matrices A, B, if A+B is invertible then A and B are both invertible.
c) (true/false) A system of equations whose augmented matrix is of dimensions 2x 4 has an infinite number of solutions.
d) (true/false) The set of real numbers R is a vector space over R under usual addition and multiplication.
e) (true/false) It is possible that a system of 3 x 3 homogeneous equations has no solution.

37) Define $+$ and $\bullet$ on R^2 over R as follows: $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1y_2),\ k(x,y)=(kx,1)$

a) Find the zero vector.
b) Find –(3,4)
c) Determine if (r+s)(c,y)=r(x,y)+s(x,y) holds.

38) Let $T_1: R^3 \to R^2$ be defined by $T(x_1,x_2,x_3)=(x_1-2x_3,x_2)$
a) Show T is a linear transformation
b) Use the definition to find a basis for Ker T.
c) Find the dimension of the range. Is T onto?

39)

Suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=5$. Compute $\begin{vmatrix} 3a+3b & 3b & 3c \\ -d-e & -e & -f \\ g+h & h & i \end{vmatrix}$

40) Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ using the adjoint of A.

41) Show that if V is a vector space ,{v_1, v_2,v_3} is LD and v_4 is another vector in V, the {v_1, v_2, v_3 ,v_4} is LD.

42) Solve
$$c_1 + 3c_2 = 0$$
$$c_1 + 4c_2 + c_3 + c_4 = 0$$

43) Let V= P_2 and $S = \{ax + bx^2 \in P_2 : a, b \in R\}$

a) Show that S is a subspace of P_2.
b) Find a basis for S.

44) Find a basis for the set of 2 x 2 skew symmetric matrices.

45) Let $T : R^2 \to R^2$ be a linear transformation satisfying $T(1,2) = (-1,3),\ T(2,3) = (0,2)$. Find $T(3,4)$.

46)

a) (true/false) A system of linear equations can have exactly two solutions. ____
b) (true/false) Let A and B be square matrices. If AB is nonsingular, then A and B are both nonsingular. ____
c) (true/false) $B = \dfrac{A + A^T}{2}$ is symmetric for any square matrix A. _______
d) (true/false) If A and B are $n \times n$ matrices, then $\det(A+B) = \det(A) + \det(B)$ ____
e) (true/false) A system of linear equations with two rows and three variables has at least one free variable.
f) Give an example of a $3\ x\ 3$ skew symmetric matrix.
g) (true/false) If a matrix A is invertible, then det(A)=0.

47) (5 points each)

a) If $\det(A) = 4,, \det(B) = 2, A, B$ are 4 x 4, compute $\det(2A^{-1}B^2)$

b) Suppose $\begin{vmatrix} a_1 + b_1 & 3b_1 & c_1 \\ a_2 + b_2 & 3b_2 & c_2 \\ a_3 + b_3 & 3b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Find k without evaluating the determinants

c) Solve $x + 2y + 3z = 0$

48) Define $\mathbf{T} : \mathbf{C}^1[a,b] \to \mathbf{C}^0[a,b]$ as $\mathbf{T}(f(x)) = f'(x)$

a) Show it is a LT
b) Find the kernel of T. (hint: use common sense)\
c) Is T one-to-one? Justify your answer.

49)

Define an operation $+$ and a scalar multiplication $\cdot$ on $\mathbf{R}^2$ as follows:
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, x_2 y_2),\ \ c(x, y) = (cx, c^2 y)$$

i) (2 points) Show $\mathbf{0} = (0,1)$
ii) (4 points) Does $-\mathbf{u}$ exist for all $\mathbf{u}$? Justify your answer. (recall that $-\mathbf{u}$ is the vector such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$)
iii) (4 points) Does $(rs)\mathbf{v} = r(s\mathbf{v})$ hold? Justify your answer.

50) (5 points each)

a) Let $T : \mathbf{R}^2 \to \mathrm{diag}(\mathbf{R})$ be defined as $T(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Find the matrix of $[T]_B^C$ with respect to the bases $B = \{(1,1),(0,1)\}$ and $C = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\}$

b) Suppose the matrix of $T:\mathbf{R}^2 \to \mathbf{R}^2$ with respect to the basis $B = C = \{(1,-1),(0,1)\}$ is given as $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find $T(2,3)$

51) Let V be a vector space.

a) Show that the set $\{\mathbf{0}, \mathbf{v}_2, \mathbf{v}_3\}$ is LD for any vectors $\mathbf{v}_2, \mathbf{v}_3$ in V.

b) Show that if $\{\mathbf{w}_1, \mathbf{w}_2\}$ is LI, then $\{\mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_1 + \mathbf{w}_2\}$ is also LI

52) Define a function $\mathbf{T}:\mathbf{P}_1 \to \mathbf{P}_2$ by $\mathbf{T}(ax+b) = (2a-3b) + (4a+5b)x + (16a+9b)x^2$

a) (3 points) Show T is a LT.

b) (3 points) Find a basis for the range of T. (hint: First pick bases and find the matrix.)

53) (3 points)

54)

a) (true/false) If three vectors in $\mathbf{R}^3$ are LD, then they spans a plane isomorphic to $\mathbf{R}^2$.

b) (true/false) If $\mathbf{T}$ is a LT, then $\mathbf{T}(3\mathbf{v}) = 3\mathbf{T}(v)$

c) (true/false) If $\mathbf{T}$ is a LT, then $\mathbf{T}(\mathbf{v}^2) = \mathbf{T}(v)^2$

d) (true/false) If multiplication by $k \in \mathbf{R}$ on $\mathbf{R}^2$ is defined by $k(x_1, x_2) = (2kx_1 2kx_2)$, then $1 = \frac{1}{2}$

55) (3 points each)

a) Find a matrix of the linear transformation that rotates (x, y) 90° counterclockwise followed by the reflection about the x-axis. (hint: recall that a LT is determined by its values on basis vectors.

b) If $\mathbf{T}:\mathbf{R}^4 \to \mathbf{R}^3$ is LT with dim(ker T)=1, show T is onto.

56) (8 points) Consider the vector space $\mathbf{R}^3$.

a) (6 points) Describe geometrically the subspace of $\mathbf{R}^3$ spanned by the vectors $\{(1,0,-3),(0,2,-4)\}$

b) (2 points) Add a vector to $\{(1,0,-3),(0,2,-4)\}$ and extend the set to a basis for $\mathbf{R}^3$. You want to add a vector of the form $(2,1,z)$ to the set. Use part a) to find the number z you must avoid so that $\{(1,0,-3),(0,2,-4),(2,1,z)\}$ is a basis for $\mathbf{R}^3$.

57) (3 points each)

a) Let V be a vector space, $\{v_1, v_2\}$ be vectors in V. Show Span$\{v_1, v_2\}$=Span $\{v_1, v_1 + v_2\}$.

b) Let V, W, L be vector spaces, $T_1 : V \to W$, $T_2 : W \to L$ be linear transformations. Prove that $T_2 \circ T_1 : V \to L$ is a LT. (note that you cannot use matrices in the proof since the vector spaces may be infinite dimensional).

c) Let V, W be vector spaces, $T : V \to W$ a LT. Prove that if T is 1-1, $\{v_1, v_2\}$ LI, then $\{T(v_1), T(v_2)\}$ is LI. (hint: First suppose $c_1T(v_1) + c_2T(v_2) = \mathbf{0}$.)

58) (4 points each)

Let V be a vector space, $v_1, v_2, \cdots v_n$ be vectors in V.

a) Explain why if $\{v_1, v_2\}$ is LD, then $\{v_1, v_2, v_3\}$ is also LD. You may provide a formal proof, or give a brief explanation as to why the result holds.

b) Give an example of a nonzero vectors v_1, v_2, v_3 such that $\{v_1, v_2, v_3\}$ is LD but $\{v_1, v_2\}$ is LI

59) (4 points each)

Define an operation $+,\cdot$ on $\mathbb{R}^2$ over the real numbers as follows:

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 2y_1 + 2y_2)$, $a(x, y) = (a^2x, a^2x), a \in \mathbb{R}$

a) Is + associative? Justify your answer.

b) Is there a zero vector? Justify your answer.

c) Does $r(s\mathbf{v}) = (rs)\mathbf{v}$ hold for $r.s \in \mathbb{R}$ $\mathbf{v} \in \mathbb{R}^2$?

60)

a) Show $A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$ is an orthogonal matrix.

b) Find the inverse of A without performing row operations.

61) Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation.

a) Find $T(x_1, x_2)$ if $T(2,3) = (1,3)$ and $T(1,-3) = (-4,3)$

b) Find the inverse of the LT found in a) . You are not required to show it is 1-1 and onto.

62)

Let $T: \mathbf{P}_2 \to \mathbf{P}_2$ be a linear transformation be defined as $T(a + bx + cx^2) = 5a + (-a + 4b)x + (2c + b)x^2$.

a) Find the matrix of T with B=C= $\{1, x, x^2\}$

b) Use the matrix found in a) to find $T(4x + 3x^2)$

c) Is T invertible? If it is, find the inverse.

63)

a) Use DEFINITION to show $\{(1,3,1), (0,1,0)\}$ is a LI set in $\mathbf{R}^3$ (you cannot say that are LI since one is not a multiple of the other)

b) Find any vector that is not in the span of $\{(1,3,1), (0,1,0)\}$.

c) Extend the basis found in a) to a basis in $\mathbf{R}^3$

64) Construct an isomorphism between the following vector spaces. You are not required to show that the functions constructed are indeed isomorphisms.

a) Between $\mathbf{R}$ and the set of 2 x 2 skew symmetric matrices.

b) Between $\mathbf{R}^2$ and $\mathbf{P}_1$

65)

a) Prove that if $T: \mathbf{R}^2 \to \mathbf{R}^2$ is a one-to-one linear transformation, then T is invertible, that is, T^{-1} exists.

b) If $T: \mathbf{P}_1 \to \mathbf{P}_1$ is defined as $T(a + bx) = (a + b) + (a - b)x$, find $T^{-1}(a + bx)$ (you are not required to show T is a bijection)

66)

Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

a) Explain why it is possible to diagonalize A using an orthogonal matrix.

b) Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$. (hint: the characteristic polynomial of A is $(\lambda - 2)^2 (\lambda - 8)$

67) A) Find an orthonormal basis for the subspace of $C^0[0,1]$ spanned by $\{1, x, x^2\}$, where , $< f, g >= \int_0^1 f(x)g(x)\, dx$.

B) Find the Fourier expansion of x^2 using the orthonormal basis found in a)

68) Let $T: P_2 \to P_2$ be a LT. Let $B = \{1, x, x^2\}, C = \{3, 1 + 4x, 5 + 2x^2\}$

Suppose $[T]_B^C = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

a) Find $T(x)$

b) Find $T(a+bx+cx^2)$.

69) Let $T: V \to W$ be a linear transformation.

a) Show that if T is 1-1 and dim(W)>dim(V), then T is not onto.

b) Show that if T is onto and dimV=dimW, then T is one-to-one

70) Construct an isomorphism between R^3 and 2 x 2 symmetric matrices. You must verify that the function is a) LT b) 1-1 c) onto.

71) Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be defined as $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$.

a) (4 points)Find a basis for the range of T. Do not use the column space.

b) (2 points) Show $\{(1,0),(1,1)\}$ is a basis for $\mathbf{R}^2$

c) (4 points) Find the matrix of T with respect $B = \{(1,0),(1,1)\}, C = \{(1,-1),(0,1)\}$

d) Use the matrix found in c) to find T(2,3)

72)

Find an orthogonal matrix S and diagonal matrix D such that $S^T AS = D$ for $\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$. (The characteristic polynomial of A is $(\lambda - 3)^2(\lambda + 6)$)

73) Consider a function from $\mathbf{P}_2$ to $\mathbf{P}_2$ defined by $T(a+bx+cx^2) = (a-c)+(b-c)x+ax^2$

a) Show T is a LT

b) Find a basis for the range of T

74) Show if $T_1: \mathbf{R}^3 \to \mathbf{R}^3$ and $T_2: \mathbf{R}^3 \to \mathbf{R}^3$ are one-to-one linear transformations, then their composition $T_2 \circ T_1: \mathbf{R}^3 \to \mathbf{R}^3$ is also one-to-one. (recall that $T_2 \circ T_1(v) = T_2(T_1(v))$)

75) Find an orthogonal matrix S such that $S^T AS = D$ for $A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

76) Find an isomorphism between $\mathbf{R}^2$ and the set of 2x2 diagonal matrices. You are required to show that the function found is a LT, 1-1, onto. You may assume that the dimension of the set of 2x2 diagonal matrices is 2.

77) (8 points) Let $T: \mathbb{R}^2 \to P_1$ defined by $T(a,b) = (a+b)+(a+2b)x$. Find the matrix of T with respect the bases B= $B = \{(2,0),(0,3)\}$ and $C = \{1+x, 2x\}$. Then use the matrix to compute $T(4,3)$

78) Suppose a 3 x 3 matrix A can be diagonalized $S^{-1}AS = D$ for $S = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

a) (3 points) Find the eigenvalues of A and corresponding eigenvectors

b) (2 points) Show det(A)=0

79) Consider $C^0[-1,0]$. A product $< f,g >$ is defined as $< f,g >= \int_{-1}^{0} xf(x)g(x)dx$. Is the product an inner product? Justify your answer.

80) (4 points each)

a) Show $\{3x,-1+3x\}$ is a basis for P_1

b) Find the transition matrix from $B=\{3x,-1+3x\}$ to $C=\{x,1+x\}$

c) Use b) to find a change of basis matrix from $C=\{x,1+x\}$ to $B=\{3x,-1+3x\}$

d) Find $[-1+6x]_B$

e) Use d) and the matrix found in b) to find $[-1+6x]_C$

81) (4 points each) Define operations $+,\cdot$ on $\mathbf{R}^2$ as follows:

$$(x_1,y_1)+(x_2,y_2)=(x_1^{\ 2}x_2^{\ 2},y_1+y_2)$$
$$r(x,y)=(x+r,y+r)$$

a) Is $+$ associative?

b) Is $r(s\mathbf{v})=(rs)\mathbf{v}$ for $r.s\in\mathbf{R},v\in\mathbf{R}^2$?

c) Is there a zero vector?

82) (2 points each)

a) True/False Let V be a vector space. If $\{\mathbf{v}_1,\mathbf{v}_2\}$ is a basis for V and if $\mathbf{w}\neq\mathbf{v}_2$, then $\{\mathbf{w},\mathbf{v}_2\}$ is also a basis for V.

b) True/False Let V be a vector space. If $\{\mathbf{v}_1,\mathbf{v}_2\}$ spans V then it is impossible that $\{\mathbf{v}_1\}$ spans V.

c) True/False If T is a linear transformation, then T(1)=1

d) Find a matrix of a LT $T:\mathbf{R}^2\to\mathbf{R}^2$ that send (x,y) to (-x,y). (reflection about the y-axis)

83) (6 points) Let $T:P_2\to P_3$ be defined as $T(a+bx+cx^2)=(a-b)+(a+b+2c)x+(a+2c)x^2+(3a+b+4c)x^3$. Find a basis for Ker T and RngT

84) 3 points each) Let $T:P_1\to P_1$ be defined by $T(ax+b)=(a+b)x+(2a-b)$

a) Show T is a bijection. You may assume T is a LT

b) Find the inverse of T.

c) Find the matrix of T with respect to $B=\{2x,4\}$ and $C=\{x+1,x-1\}$

d) Use the matrix found in a) to find $T(2x+8)$ (you must first find the component vector of 2x+8 with respect to B)

85) (6 points) Let $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ be vectors in $\mathbf{R}^3$. Show if $\{\mathbf{v}_1,\mathbf{v}_2\}$ is LD, then $\{2\mathbf{v}_1,3\mathbf{v}_2\}$ is LD.

86) (3 points extra credit)
Let A, B be square matrices. Prove that if A is similar to B, then A^2 is similar to B^2.

87) (6 points) Let A be a square matrix. Suppose $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ are eigenvectors for the eigenvalue 4. Compute $A\cdot\begin{bmatrix}7\\6\end{bmatrix}$. (hint: first express $\begin{bmatrix}7\\6\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$)

88) 6 points each)

a) Solve $\begin{aligned} x_1+2x_2+3x_3+4x_4&=10\\ x_1+3x_2+2x_3+5x_4&=8\end{aligned}$ using Gauss-Jordan method.

b) Use the INVERSE of a matrix to solve $\begin{array}{l} x_1 + 2x_2 = 5 \\ x_1 + 3x_2 = -5 \end{array}$. You must solve the system of equations by multiplying a vector by the inverse of an appropriate matrix.

89) Consider a set $A = \{(1,2,3),(1,0,1)\}$ in $\mathbf{R}^3$

a) (2 points) Explain why the set A is LI without doing any row operations.

b) 1 points) How many vectors must be added to extend A to a basis for $\mathbf{R}^3$? Justify your answer.

c) (2 points) You would like to extend A to a basis for $\mathbf{R}^3$. Find three vectors not in A that must be avoided in order to extend A to a basis for $\mathbf{R}^3$ Justify your answer.

90)

a) Let A be a n x n matrix such that $A^3 = I_n$. Find the inverse of A. You must fully justify your steps.

b) Let A, B be nonsingular matrices. Show $(AB)^{-1} = B^{-1}A^{-1}$

91) Define a product on R^2 as follows: For v=(a,b), w=(c,d), $< v, w > = < (a,b),(c,d) > = ac + 3bd$. Determine if the product is an inner product. If it is, find $\|(2,4)\|$

92) Suppose λ is an eigenvalue of a matrix A. Show λ^2 is an eigenvalue of a matrix A^2.

93) (6 points, 2 points)

An inner product on $C^0[0,1]$ be defined by $< f, g > = \int_0^1 2f(x)g(x)dx$

a) Find an orthonormal basis for the subspace of $C^0[0,1]$ spanned by $\{1, 2x\}$

b) Find the Fourier expansion of $4x+3$.

94) Let A be a nondefective square matrix. Show that the determinant of A is the product of its eigenvalues. (note: this result is true even when A is defective. But it is easier to prove the result with the assumption)

95) Suppose A and B are similar matrices and B and C are similar matrices. Show A and C are similar matrices. (suggestion: To show A and C are similar, you need to find a matrix E such that $C=E^{-1}AE$. We know since A and B are similar matrices, there is a matrix S_1 such that $S_1^{-1}AS_1=B$. In addition, since B and C are similar matrices, there is a matrix S_2 such that $S_2^{-1}BS_2=C$. Now can you find E?)

96) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a LT given by $T(x,y) = (2x+y, x+y)$. Let $B = \{(3,4),(4,5)\}$ and $C = \{(2,-1),(1,-1)\}$ be bases for $\mathbb{R}^2$

a) Find $[T]_B^B$

b) Find $[P]_C^B$ the transition matrix from C to B.

c) Use similarity to find $[T]_C^C$.

97) (3 points each) Let S be the set of 2 x 2 symmetric matrices

a) Show S is a subspace of $M^2(R)$

b) Find a basis for S and prove that the set you found is indeed a basis.

c) Extend the basis found in b) to a basis for $M_2(R)$. You must carefully justify (not necessarily by computations) all the steps.