HW 10 6.2

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Find the nullity of T and give a geometric description of the kernel and range of T

33.) $\text{rank}(T) = 2$

$\dim \mathbb{R}^3 = 3$

$\text{Nullity} = 3 - 2 = 1$

$\text{Nullity} = 1 \quad \therefore \ker T$ is a line

$\text{rank}(T) = 2 \quad \therefore \text{Rng } T$ is a plane

35.) $\text{rank}(T) = 0$

$\dim \mathbb{R}^3 = 3$

$\text{Nullity} = 3 - 0 = 3$

$\text{Nullity} = 3 \quad \therefore \ker T$ is $\mathbb{R}^3$

$\text{rank}(T) = 0 \quad \therefore \text{Rng } T = \{(0,0,0)\}$

Find the Nullity of T

41.) $T: \mathbb{R}^4 \to \mathbb{R}^2$, $\text{rank}(T) = 2$

$\dim \mathbb{R}^4 = 4$

$\text{Nullity} = 4 - 2 = 2$

43.) $T: P_5 \to P_2$, $\text{rank}(T) = 3$

$\dim P_5 = 6$

$\text{Nullity} = 6 - 3 = 3$

56.) which vector spaces are isomorphic to $\mathbb{R}^6$?

(a) $M_{2,3}$

$\dim(M_{2,3}) = 2 \times 3 = 6 = \dim(\mathbb{R}^6)$

$\therefore$ Yes

b) P_6

$\dim(P_6) = 7 \neq \dim(\mathbb{R}^6) = 6$

$\therefore$ No

c) $C[0,6]$

$C[0,6]$ is a continuous function

$\dim(C[0,6]) = \infty \neq \dim(\mathbb{R}^6) = 6$

$\therefore$ It has infinite vector spaces

$\therefore$ No

d) $M_{6,1}$

$\dim(M_{6,1}) = 6 \times 1 = 6$

$\therefore$ Yes

e) P_5

$\dim(P_5) = 6$

$\therefore$ Yes

f) $C'[-3,3]$

$C'[-3,3]$ is a continuous function

$\dim(C'[-3,3]) = \infty \neq \dim(\mathbb{R}^6) = 6$

$\therefore$ It has infinite vector spaces

$\therefore$ No

g) $\{(x_1, x_2, x_3, 0, x_5, x_6, x_7) : x_i \in \mathbb{R}\}$

$\dim = 7$

$\therefore$ No

58) Proof: Let $T: V \to W$ be a linear transformation, Prove that T is one to one iff the rank of T equals the dimension of V

If T is 1-1 then $\text{Ran } T = \dim V$

L.t $\dim \text{Domain} = a$, $a \in \mathbb{Z}$

Suppose T is 1-1

So $\ker T = 0 = \text{Nullity}(T)$

then $\text{Rank } T = \dim \text{Domain} - \text{Nullity}(T)$

$\text{Rank } T = a - 0$

$\text{Rank } T = a = \dim \text{Domain}$

$\therefore \text{Rank } T = \dim V$

If $\text{Rank } T = \dim V$ then T is 1-1

Let $\dim \text{Domain} = a$, $a \in \mathbb{Z}$

Suppose $\dim V = \text{Rank } T$

so $\text{Rank } T = a$

then $\text{Rank } T = \dim \text{Domain} - \text{Nullity}(T)$

$a = a - \text{Nullity}(T)$

$\text{Nullity}(T) = 0$

Since $\text{Nullity} = 0$

$\ker T = 0$

$\therefore T$ is 1-1

70) Proof: Let $T: V \to W$ be a linear transformation, and let U be a subspace of W. Prove that the set $T^{-1}(U) = \{v \in V : T(v) \in U\}$ is a subspace of V. What is $T^{-1}(U)$ when $U = \{0\}$?

If $u, v \in W$, then $u+v$ is also $\in W$

W is close under addition

If $u \in W$, $c \in \mathbb{R}$

then cu is also $\in W$

W is close under scalar multiplication

If u and v are in the set $T^{-1}(U)$ then $u+v$ is the set $T^{-1}(U)$

$$T(u+v) = T(u) + T(v) \in U$$

If $u \in T^{-1}(U)$ and $c \in \mathbb{R}$

$$cu \in T^{-1}(U) \quad \text{as} \quad T(cu) = cT(u) \in U$$

6.3

Find the standard matrices A and A' for $T = T_2 \circ T_1$
and $T' = T_1 \circ T_2$

27.) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_1(x,y) = (x-2y, 2x+3y)$
$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2(x,y) = (y,0)$

$A = T = T_2 \circ T_1$
$T_1(x,y) = (x-2y, 2x+3y)$
$T_2 \circ T_1 = T_2(x-2y, 2x+3y) = (2x+3y, 0) = \begin{bmatrix} 2x+3y \\ 0+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$

$A' = T' = T_1 \circ T_2$
$T_2(x,y) = (y,0)$
$T_1 \circ T_2 = T_1(y,0) = (y-2(0), 2y+3(0)) = \begin{bmatrix} 0+y \\ 0+2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

29.) $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_1(x,y) = (-2x+3y, x+y, x-2y)$
$T_2 : \mathbb{R}^3 \to \mathbb{R}^2$, $T_2(x,y,z) = (x-2y, z+2x)$

$A = T = T_2 \circ T_1$
$T_1(x,y) = (-2x+3y, x+y, x-2y)$
$T_2 \circ T_1 = T_2(-2x+3y, x+y, x-2y)$
$= (-2x+3y-2(x+y), x-2y+2(-2x+3y))$
$= (-2x+3y-2x-2y, x-2y-4x+6y)$
$= (-4x+y, -3x+4y) = \begin{bmatrix} -4x+y \\ -3x+4y \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -3 & 4 \end{bmatrix}$

$A' = T' = T_1 \circ T_2$
$T_2(x,y,z) = (x-2y, z+2x)$
$T_1 \circ T_2 = T_1(x-2y, z+2x)$
$= (-2(x-2y)+3(z+2x), x-2y+z+2x, x-2y-2(z+2x))$
$= (-2x+4y+3z+6x, 3x-2y+z, x-2y-2z-4x)$

$= (4x+4y+3z,\ 3x-2y+z,\ -3x-2y-2z)$

$$= \begin{bmatrix} 4x+4y+3z \\ 3x-2y+z \\ -3x-2y-2z \end{bmatrix} = \begin{bmatrix} 4 & 4 & 3 \\ 3 & -2 & 1 \\ -3 & -2 & -2 \end{bmatrix}$$

Determine whether the linear transformation is invertible. If it is, find its inverse

31.) $T(x,y) = (-4x, 4y)$

① Show T is 1-1 and onto

ⓐ Show T is 1-1

suppose $T(x,y) = (0,0)$

$(-4x, 4y) = (0,0)$

$$\begin{aligned} -4x + 0y &= 0 \\ 0x + 4y &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} -4 & 0 & 0 \\ 0 & 4 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$c_1 = 0,\ c_2 = 0$

$\therefore \ker T = \{(0,0)\}$

$\therefore T$ is 1-1

ⓑ Show T is onto

$\dim \text{Domain} = \dim \ker T + \dim \text{Rng}\, T$

$2 = 0 + \dim \text{Rng}\, T$

$\therefore \dim \text{Rng}\, T = 2$

Since dim codomain is also

$\therefore T$ is onto

$\therefore T$ is invertible

② To find T^{-1}

Let $B = C = \{(1,0), (0,1)\}$

$[T]_B^C = [\,[T(1,0)]_C, [T(0,1)]_C\,]$

$[T(1,0)]_C = [(-4,0)]_C = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

$[T(0,1)]_C = [(0,4)]_C = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$$[T]_B^C = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$[T^{-1}]_B^C = ([T]_B^C)^{-1} = \begin{bmatrix} -1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ T^{-1}(1,0) & T^{-1}(0,1) \end{array}$$

To find $T^{-1}(x,y)$

$$T^{-1}(x,y) = T^{-1}(x(1,0) + y(0,1))$$
$$= xT^{-1}(1,0) + yT^{-1}(0,1)$$
$$= x(-1/4, 0) + y(0, 1/4)$$
$$= (-1/4x, 0) + (0, 1/4y)$$
$$= (-1/4x, 1/4y)$$

33.) $T(x,y) = (x+y, 3x+3y)$

① show T is 1-1 and onto

ⓐ show T is 1-1

suppose $T(x,y) = (0,0)$

$(x+y, 3x+3y) = (0,0)$

$$\begin{matrix} x+y=0 \\ 3x+3y=0 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & 0 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 = -t$ Let $t=1$ $c_1 = -1$

$c_2 = t$ $c_2 = 1$

$\ker T = \{(-1,1)\} \neq \{(0,0)\}$

$\therefore$ T is not 1-1

$\therefore$ T is not invertible

35) $T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)$

① show T is 1-1 and onto

ⓐ show T is 1-1

suppose $T(x_1, x_2, x_3) = (0,0,0)$

$(x_1, x_1 + x_2, x_1 + x_2 + x_3) = (0,0,0)$

$$\begin{aligned} x_1 + 0 + 0 &= 0 \\ x_1 + x_2 + 0 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_1 = 0, \ C_2 = 0, \ C_3 = 0$

$\therefore \ker T = \{(0,0,0)\}$

$\therefore T$ is 1-1

ⓑ show T is onto

$\dim \text{Domain} = \dim \ker T + \dim \text{Rng}\, T$

$3 = 0 + \dim \text{Rng}\, T$

$\therefore \dim \text{Rng}\, T = 3$

Since dim Codomain is also 3

$\therefore T$ is onto

$\therefore T$ is invertible

To find T^{-1}

Let $B = C = \{(1,0,0), (0,1,0), (0,0,1)\}$

$[T]_B^C = [[T(1,0,0)]_C, [T(0,1,0)]_C, [T(0,0,1)]_C]$

$[T(1,0,0)]_C = [(1,1,1)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$[T(0,1,0)]_C = [(0,1,1)]_C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$[T(0,0,1)]_C = [(0,0,1)]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$[T^{-1}]_C^B = ([T]_B^C)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

To find $T^{-1}(x, y)$

$$\begin{aligned}
T^{-1}(x_1, x_2, x_3) &= T^{-1}(x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)) \\
&= x_1 T^{-1}(1,0,0) + x_2 T^{-1}(0,1,0) + x_3 T^{-1}(0,0,1) \\
&= x_1(1,-1,0) + x_2(0,1,-1) + x_3(0,0,1) \\
&= (x_1, -x_1, 0) + (0, x_2, -x_2) + (0, 0, x_3) \\
&= (x_1, -x_1 + x_2, -x_2 + x_3)
\end{aligned}$$

6.4

Find the matrix A' for T relative to the basic B'

5.) $T: R^2 \to R^2$, $T(x,y) = (-3x+y, 3x-y)$,

$B' = \{(1,-1), (-1,5)\}$

$T(1,-1) = (-4,4) = c_1(1,-1) + c_2(-1,5)$ $\quad$ $\begin{matrix} -4 & 4 \\ -3+(-1) & 3-(-1) \end{matrix}$

$T(-1,5) = (8,-8) = d_1(1,-1) + d_2(-1,5)$ $\quad$ $\begin{matrix} -3(-1)+5 & 3(-1)-5 \\ 8 & -8 \end{matrix}$

$(c_1,-c_1) + (-c_2, 5c_2) = (-4,4)$

$(d_1,-d_1) + (-d_2, 5d_2) = (8,-8)$

$(c_1-c_2, -c_1+5c_2) = (-4,4)$

$(d_1-d_2, -d_1+5d_2) = (8,-8)$

$\Rightarrow \left[\begin{array}{cc:cc} 1 & -1 & -4 & 8 \\ -1 & 5 & 4 & -8 \end{array}\right] \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & -4 & 8 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$A' = \begin{bmatrix} -4 & 8 \\ 0 & 0 \end{bmatrix}$

11.) $T: R^3 \to R^3$, $T(x,y,z) = (x-y+2z, 2x+y-z, x+2y+z)$

$B' = \{(1,0,1), (0,2,2), (1,2,0)\}$

$T(1,0,1) = (3,1,2) = c_1(1,0,1) + c_2(0,2,2) + c_3(1,2,0)$

$T(0,2,2) = (2,0,6) = d_1(1,0,1) + d_2(0,2,2) + d_3(1,2,0)$

$T(1,2,0) = (-1,4,5) = e_1(1,0,1) + e_2(0,2,2) + e_2(0,2,2)$

$(c_1,0,c_1) + (0,2c_2,2c_2) + (c_3,2c_3,0) = (3,1,2)$ $\quad$ $(c_1+0+c_3) + (0+2c_2+2c_3) + (c_1+2c_2+0) = (3,1,2)$

$(d_1,0,d_1) + (0,2d_2,2d_2) + (d_3,2d_3,0) = (2,0,6)$ $\Rightarrow$ $(d_1+0+d_3) + (0+2d_2+2d_3) + (d_1+2d_2+0) = (2,0,6)$

$(e_1,0,e_1) + (0,2e_2,2e_2) + (e_3,2e_3,0) = (1,2,0)$ $\quad$ $(e_1+0+e_3) + (0+2e_2+2e_3) + (e_1+2e_2+0) = (1,2,0)$

$\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 3 & 2 & -1 \\ 0 & 2 & 2 & 1 & 0 & 4 \\ 1 & 2 & 0 & 2 & 6 & 3 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{7}{3} & \frac{10}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{4}{3} & \frac{8}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{4}{3} & -\frac{2}{3} \end{array}\right]$

$A' = \begin{bmatrix} \frac{7}{3} & \frac{10}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{4}{3} & \frac{8}{3} \\ \frac{2}{3} & -\frac{4}{3} & -\frac{2}{3} \end{bmatrix}$

Use the matrix P to show that the matrices A and A' are similar

19.) $P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 12 & 7 \\ -20 & -11 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

$$A' = P^{-1} A P$$
$$= \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 12 & 7 \\ -20 & -11 \end{bmatrix} P$$
$$= \begin{bmatrix} -4 & -3 \\ -8 & -4 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$$

$\therefore$ A and A' are similar

31.) Proof Let A be an $n \times n$ matrix such that $A^2 = 0$. Prove that if B is similar to A, then $B^2 = 0$

Suppose $B = P^{-1} A P$
$$PB = PP^{-1} A P$$
$$PBP^{-1} = APP^{-1}$$
$$PBP^{-1} = A$$
$$(PBP^{-1})^2 = A^2$$
$$(PBP^{-1})(PBP^{-1}) = A^2$$
$$(PB)(P^{-1}P)(BP^{-1}) = A^2$$
$$(PB)(BP^{-1}) = A^2$$
$$PB^2P^{-1} = A^2$$
$$P^{-1}PB^2P^{-1} = P^{-1}A^2$$
$$B^2P^{-1}P = P^{-1}A^2P$$

$$B^2 = P^{-1}A^2P$$
$$B^2 = P^{-1}(0)P$$
$$B^2 = 0$$

33.) Proof Prove Property 3 of Theorem 6.13: For square matrices A, B, and C of order n, if A is similar to B and B is similar to C, then A is similar to C.

suppose $B = P^{-1}AP$

$$PB = PP^{-1}AP$$

$$PBP^{-1} = APP^{-1}$$

$$PBP^{-1} = A$$

and B is similar to C

so $C = Q^{-1}BQ$

$$C = Q^{-1}(P^{-1}AP)Q$$

$$C = (QP)^{-1}A(PQ)$$

$\therefore$ A is similar to C