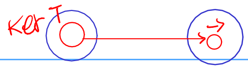
6.2

Recall Let V and W be vector spaces

$T: V \to W$ be a LT

① $\underbrace{\ker T = \{ v \in V : T(v) = \vec{0} \}}_{\text{vectors in } V \text{ that go to } \vec{0}}$



② $\text{Rng } T = \{ w \in W : w = T(v) \text{ for some } v \in V \}$

Rng T

Recall ① Ker T is a subspace of V

② Rng T is a subspace of W

Find basis for Ker T / Rng T

There are two ways

① use the definitions

or ② use $[T]_B^C$

eg. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a LT

defined by $T(x,y) = (x-y, 3x+y, x+4y)$

ⓐ find a basis for Ker T

ⓑ find a basis for Rng T

① Definition

ⓐ Ker T : set $T(x,y) = (0,0,0)$

$\Rightarrow (x-y, 3x+y, x+4y) = (0,0,0)$

$$\begin{aligned} x - y &= 0 \\ 3x + y &= 0 \\ x + 4y &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 0 \\ 1 & 4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x = 0$ $\qquad$ $\ker T = \{(0,0)\}$

$y = 0$ $\qquad$ no basis

(b) Range ① separate variables in $T(v)$ to find a generating set

② find a LI subset

$$T(x,y) = (x-y, 3x+y, x+4y)$$
$$= (x, 3x, x) + (-y, y, 4y)$$
$$= \underset{\text{like } c_1}{x}(1,3,1) + \underset{\text{like } c_2}{y}(-1,1,4)$$

$\Rightarrow$ Range of T = span$\{(1,3,1), (-1,1,4)\}$

since $\{(1,3,1), (-1,1,4)\}$ is LI,

not a scalar multiple

$\{(1,3,1), (-1,1,4)\}$ is a basis for Rng T

② Use $[T]_B^C$

Recall: $[T(v)]_C = [T]_B^C [v]_B$

where $[T]_B^C = [[T(v_1)]_C \, [T(v_2)]_C \, \dots \, [T(v_n)]_C]$

Theorem: Let $V \to W$ be a LT

If $\{v_1, v_2, \dots, v_n\}$ is a basis for V

then $\{T(v_1), T(v_2), \dots, T(v_n)\}$ spans Rng T

why? $T(v) = T(c_1v_1 + c_2v_2 + \dots + c_nv_n)$

$= c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n)$

$\Rightarrow$ all the vectors in Rng T are linear combinations of $\{T(v_1), T(v_2), \dots, T(v_n)\}$

To find basis for Ker T / Rng T

① Pick standard basis for B, C

② compute $[T]_B^C$

Recall: If A is a matrix,

null$(A) = \{v \in V : Av = 0\}$

③ Do REFF of $[T]_B^C$

Kernal of T = nullspace of $[T]_B^C$

$\Rightarrow$ go back to V using B

Range T $\Rightarrow$ column space of $[T]_B^C$

$\Rightarrow$ go back to W using C

Finding basis for $\ker T$ | $\operatorname{Rng} T$

$T(x,y) = (x-y, 3x+y, x+4y)$

Pick $B = \{(1,0),(0,1)\}$

$C = \{(1,0,0),(0,1,0),(0,0,1)\}$

$T(1,0) = \begin{matrix} x=1 \\ y=0 \end{matrix} = (1,3,1)$ 1^{st} column

$T(0,1) = \begin{matrix} x=0 \\ y=1 \end{matrix} = (-1,1,4)$ 2^{nd} column

$$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & 4 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

① nullspace

$$\left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \Rightarrow \ker T \{(0,0)\}$$

② columnspace $\Rightarrow$ take the columns in the original matrix with leading ones in RREF

$$\therefore \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\} \text{ is a basis for } col([T]_B^C)$$

$\Rightarrow \{(1,3,1),(1,-1,4)\}$ is a basis for $\operatorname{Rng} T$

eg. Let $T: P_2 \to P_2$ be a LT defined by

$$T(a+bx+cx^2) = (a-b+c) + (3a+4b+2c)x + (2a+5b+c)x^2$$

Find a basis for a) Ker T b) Rng T

① Definition

ⓐ Ker T $\Rightarrow$ $\boxed{\text{solve } T(v) = \vec{0}}$

suppose $T(a+bx+cx^2) = (a-b+c)+(3a+4b+2c)x+(2a+5b+c)x^2$

$$\begin{aligned} a-b+c &= 0 \\ 3a+4b+2c &= 0 \\ 2a+5b+c &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc:c} 1 & 0 & \frac{6}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $c = t$, $a = -\frac{6}{7}t$, $b = \frac{1}{7}t$

$\text{Ker}T = \{ -\frac{6}{7}t + \frac{1}{7}tx + tx^2 : t \in \mathbb{R} \}$

Let $t = 7$

basis $\{ -6 + x + 7x^2 \}$

ⓑ Rng T

① separate varibles in $T(v)$ to find a spanning set

② find a LI subset

① $T(a+bx+cx^2) = (a-b+c)+(3a+4b+2c)x+(2a+5b+c)x^2$

a, b, c are the variables

$= (a+3ax+2ax^2) + (-b+4bx+5bx^2) + (c+2cx+cx^2)$

$= a(1+3x+2x^2) + b(-1+4x+5x^2) + c(1+2x+x^2)$

like c_1 like c_2 like c_3

$\Rightarrow \{ (1+3x+2x^2), (-1+4x+5x^2), (1+2x+x^2) \}$ spans Rng T

② To find a LI subset, find a dependency relation

suppose $c_1(1+3x+x^2) + c_2(-1+4x+5x^2) + c_3(1+2x+x^2) = 0 + 0x + 0x^2$

3 steps

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \left[\begin{array}{ccc:c} 1 & 0 & \frac{6}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 = t$, $c_2 = \frac{1}{7}t$, $c_1 = -\frac{6}{7}t$

$t = 7$ $c_1 = -6$, $c_2 = 1$, $c_3 = 7$

all non zero

drop the last one

$= \{ \underbrace{1+3x+2x^2, -1+4x+5x^2}_{\text{not scalar multiple} \Rightarrow \text{LI}} \}$

$\therefore \{ 1+3x+2x^2, -1+4x+5x^2 \}$ is a basis for Rng T

(2) Use $[T]_B^C$

Pick B and C

Let $B = \{1, x, x^2\}$ $C = \{1, x, x^2\}$

(a) standard basis for P_2

$T(1) = \begin{matrix} a=1 \\ b=0 \\ c=0 \end{matrix} = 1+3x+2x^2 \to [T(1)]_C = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$T(x) = \begin{matrix} a=0 \\ b=1 \\ c=0 \end{matrix} = -1+4x+5x^2 \to [T(x)]_C = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

$T(x^2) = \begin{matrix} a=0 \\ b=0 \\ c=1 \end{matrix} = 1+2x+x^2 \to [T(x^2)]_C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(b)

$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 4 & 2 \\ 2 & 5 & 1 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{6}{7} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix}$

(c) null space

$\left[\begin{array}{ccc|c} 1 & 0 & \frac{6}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (call the variables x_1, x_2, x_3)

$x_3 = t$ $x_2 = \frac{1}{7}t$ $x_3 = -\frac{6}{7}t$

$\Rightarrow$ nullspace : $\begin{bmatrix} -\frac{6}{7}t \\ \frac{1}{7}t \\ t \end{bmatrix} = \frac{1}{7}t \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}_B$ $\begin{matrix} \text{coeff of } 1 \\ \text{coeff of } x \\ \text{coeff of } x^2 \end{matrix}$

$\Rightarrow \{ -6+x+7x^2 \}$ is a basis for Ker T

(d) column space

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}_C, \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}_C \right\}$ is a basis for the column space

$\Rightarrow$ go back to P_2,

$\{1 + 3x + 2x^2, \; -1 + 4x + 5x^2\}$ is a basis for Rng T

Recall: $f: A \to B$ be a function

① f is $1-1$ if :

$f(a_1) = f(a_2)$ then $a_1 = a_2$

② f is onto if

$\forall\, b \in B$, $\exists\, a \in A$ with $f(a) = b$

eg.

$a \to d$ $\qquad\qquad$ $a \to d$

$b \to e$ $\qquad\qquad$ $b \to d$

$c \to f$ $\qquad\qquad$ $c \to e$

is $1-1$ $\qquad$ is not $1-1$ $\quad f(a) = f(b)$ but $a \neq b$

eg.

$a \to d$ $\qquad$ is onto $\qquad d = f(a)$

$b \to e$ $\qquad\qquad\qquad e = f(b)$

$c \to f$ $\qquad\qquad\qquad f = f(c)$

$a \to d$ $\qquad$ is not $\qquad f(a) \neq f$

$b \to d$ $\qquad$ onto $\qquad f(b) \neq f$

$c \to e$ $\qquad\qquad\qquad f(c) \neq f$

Theorem:

Let $T: V \to W$ be a LT

① T is one to one iff $\ker T = \{\vec{0}\}$

② T is onto iff $\dim \text{Rng}\, T = \dim W$ $\qquad$ Proof $\to$ HW

eg. Let $T: P_2 \to P_2$ be defined by

$T(a + bx + cx^2) = (a - b + c) + (3a + 4b + 2c)x + (2a + 5b + c)x^2$

① Is T one to one ?

② Is T onto ?

① $\ker T = \{ -6 + x + 7x^2 \} \neq \{ \vec{0} \} \Rightarrow$ not 1 to 1

② $\dim P_2 = 3$

$\dim \operatorname{Rng} T = 2 \quad \Rightarrow$ not onto

eg. $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x, y) = (x - y, 3x + y, x + 4y)$

① Is T 1-1 ?

② Is T onto ?

$\vec{0}$

① $T = \{ (0,0) \} \Rightarrow T$ is 1-1

② $\dim \mathbb{R}^3 = 3$

$\dim \operatorname{Rng} T = 2 \quad \Rightarrow$ not onto