Uniform and Exponential Distribution

David Armstrong

UCI

Uniform Distribution

The *uniform distribution* is a distribution for a continuous random variable that can take on any value in an interval [a,b], with uniform density.

- Denoted $X \sim Uniform(a, b)$
- $\mathbb{S}_X = [a, b]$.
- $f(x) = \dfrac{1}{b-a}$.
  - We use this distribution for continuous variables where the values are all equally likely
- F$(X) = P(X \leq x) = \int_a^x f(x)dx = \dfrac{x-a}{b-a}$
- E(X)=$\dfrac{b+a}{2}$
- VAR$(X) = \dfrac{(b-a)^2}{12}$
- The parameters of the distribution are a (lower bound) and b (upper bound).

This is a valid probability density function.

- For all x in [a,b], $f(x) \geq 0$.
- $\int\limits_{\mathbb{S}_X} f(x)dx = \int\limits_a^b \dfrac{1}{b-a}dx = \dfrac{x}{b-a}\Big|_{x=a}^{x=b} = \dfrac{b}{b-a} - \dfrac{a}{b-a} = 1$

Uniform Distribution - R Code

To get the area to the left of a Uniform(0,1) variable:
$punif(u, min = 0, max = 1)$

To get the area to the right of a Uniform(0,1) variable:
$1 - punif(u, min = 0, max = 1)$

To get the area between two values say c and d (c < d):
$punif(d, min = 0, max = 1) - punif(c, min = 0, max = 1)$

To get the value of u related to the lower tail (α):
$qunif(\alpha, min = 0, max = 1)$

To get the value of u related to the upper tail:
$qunif(1 - \alpha, min = 0, max = 1)$

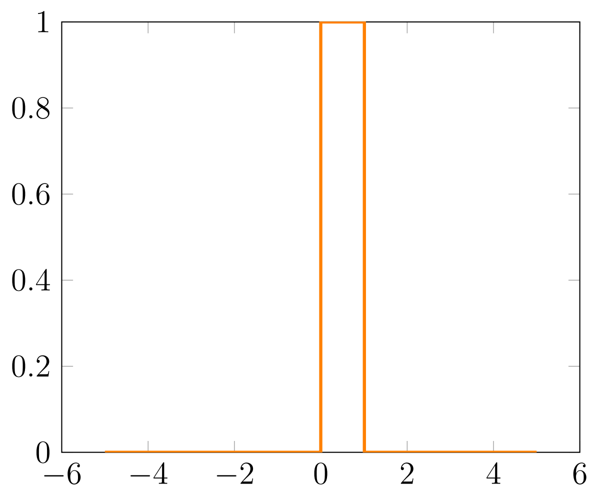
Uniform Distribution

Let X be a uniform continuous random variable on the interval $[a, b]$.

- $\mathbb{S}_X = [a, b]$.
- $f(x) = \dfrac{1}{b - a}$.

This is a valid probability density function.
Note: the CDF and pdf are only valid for x such that $a \leq x \leq b$

Ex: $X \sim Uniform(0, 1)$



Uniform Distribution - Expectation and Variance

Let X be a uniform continuous random variable on the interval (a, b).

- $E(X) = \int\limits_{\mathbb{S}_X} x\, f(x)dx = \int\limits_a^b x\, \frac{1}{b-a} dx$
  $= \left.\frac{x^2}{2(b-a)}\right|_{x=a}^{x=b} = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$

- The expected value of X is just the average of the two end points of the support.

- $E(X^2) = \int\limits_{\mathbb{S}_X} x^2\, f(x)dx = \int\limits_a^b x^2\, \frac{1}{b-a} dx$
  $= \left.\frac{x^3}{3(b-a)}\right|_{x=a}^{x=b} = \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} = \frac{(b^3-a^3)}{3(b-a)}$

Show that VAR(X)=$E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{12}$.

Uniform Distribution - CDF

Let X be a uniform continuous random variable on the interval (a, b).

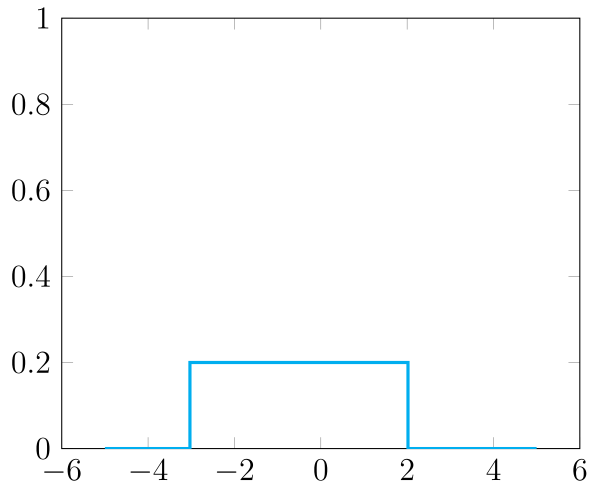
The cumulative distribution function is as follows:

- $F(x) = P(X < x) = \int\limits_{\tilde{x}<x} f(\tilde{x})d\tilde{x} = \int\limits_{a}^{x} \frac{1}{b-a}d\tilde{x}$
  $= \frac{\tilde{x}}{(b-a)}\Bigg|_{\tilde{x}=a}^{\tilde{x}=x} = \frac{x-a}{b-a}$

Thus for any x in (a,b), $F(x) = \frac{x-a}{b-a}$.

Note that if $x \leq a$ then F(x)=0 and if $x \geq b$ then F(x)=1.

Example: $X \sim Uniform(-3, 2)$



Example - Car Intersections

Say you are waiting for the first car to cross an intersection. The waiting time is assumed to be any number between 0 and 13 minutes. The waiting time X is a uniform random variable on the interval [0,13].

- What is the probability that you wait less than 5 minutes?

- How much time do you expect to wait?

- Find the variance for the waiting time.

Exponential Distribution

Now we come to what is known as the *exponential distribution*.
It is used to model the the time between events in a Poisson process.

Let X be a random variable that follows an exponential distribution.

- $X \sim Exponential(\lambda)$
- The probability density function of X (pdf) is $f(x) = \lambda e^{-\lambda x}$.
- The expectation is $E(X) = \dfrac{1}{\lambda}$.
- The variance is $VAR(X) = \dfrac{1}{\lambda^2}$
- The cumulative distribution function is $F(x) = 1 - e^{-\lambda x}$.
- $P(a < X < b) = F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$
- λ is the parameter where $\lambda > 0$.
- The support of X is $\mathbb{S}_X = [0, \infty)$

Exponential Distribution - R Code

To get the area to the left of a Exponential(1) variable:
$pexp(e, rate = 1)$

To get the area to the right of a Exponential(1) variable:
$1 - pexp(e, rate = 1)$

To get the area between two values say c and d (c < d):
$pexp(d, rate = 1) - pexp(c, rate = 1)$

To get the value of e related to the lower tail (α):
$qexp(\alpha, rate = 1)$

To get the value of e related to the upper tail:
$qexp(1 - \alpha, rate = 1)$

Exponential Distribution

Let X be a random variable that follows an exponential distribution with parameter $\lambda > 0$.

$$
\begin{aligned}
E(X) &= \int_0^\infty x f(x) dx \\
&= \int_0^\infty \lambda x e^{-\lambda x} dx \\
&= [-x e^{-\lambda x}]|_{x=0}^{x=\infty} + \int_0^\infty e^{-\lambda x} dx \\
&= (0 - 0) + \frac{1}{\lambda} x e^{-\lambda x}]|_{x=0}^{x=\infty} \\
&= 0 + \left(0 + \frac{1}{\lambda}\right) \\
&= \frac{1}{\lambda}
\end{aligned}
$$

The parameter λ can be viewed as the expected time until the next event is observed.

Similarly we can show that $E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = 2\left(\frac{1}{\lambda}\right)^2$

As a result, $VAR(X) = \frac{1}{\lambda^2}$.

Exponential Distribution

Let X be a random variable that follows an exponential distribution with parameter $\lambda > 0$.

The cumulative distribution function (cdf) F(x) is as follows.

$$\begin{aligned}
F(x) &= \int_0^x f(u)du \\
&= \int_0^x -\lambda e^{-\lambda u} du \\
&= -e^{-\lambda u}|_{u=0}^{u=x} \\
&= -e^{-\lambda x} - (-1) = 1 - e^{-\lambda x}
\end{aligned}$$

As a result: $P(a < X < b) = F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$.

Exponential Distribution

Let X follows an exponential distribution with a constant parameter $\lambda > 0$.

A property of this distribution is called its *memoryless* property.

- Notationally this is $P(X > x + t | X > x) = P(X > t)$ where $x, t > 0$.
- For example $P(X > 45 | X > 35) = P(X > 35 + 10 | X > 35) = P(X > 10)$

Given the wait time X for the next event is greater than x, the probability that the time X is greater than $x + t$ is just equal to the unconditional probability that X is greater than t.

Ex: The wait time (in minutes) to observe the next car that crosses an intersection follows an exponential distribution with $\lambda = \frac{1}{10}$.

a. What is the expected wait time for the next car to cross the intersection?

b. What is the variance of the wait time for the next car to cross the intersection?

c. Find the probability that the wait time is less than 5 minutes.

d. What is the probability that the wait time is between 4 and 6 minutes?

e. Now say we know (or that we are given, or condition on) that the wait time is more than 4 minutes. What is the probability that the wait time is more than 6 minutes?