Math 260 Practice Solutions

1)

a) n^2 is odd iff $n+4$ is odd.

Pf: First show if n^2 is odd, then $n+4$ is odd. Use contrapositive. Suppose n+4 is even. Then $n+4=2k$ for some integer k. Since n=2k-4, $n^2=(2k-4)^2=2(2k^2-8k+8)$. Therefore n^2 is even.

Conversely, Suppose $n+4$ is odd. Then $n+4=2k+1$ for some integer. Solving for n, we get n=2k-3, $n^2=(2k-3)^2=2(2k^2-6k+4)+1$. Therefore n^2 is odd .

b) If $m \cdot n$ is even, then n is even or m is even.

Pf: Use contrapositive. That is, to prove $p \to q$, we show $\sim q \to \sim p$. That is , we prove that if n is odd and m is odd (change "or "to "and" in negation), then mn is odd. Suppose n and m are both odd. Then $m=2k+1$ and $n=2l+1$ some integers k and l. We have $m \cdot n=(2k+1)(2l+1)=2(kl+k+l)+1$ is odd. Therefore $m \cdot n$ is odd.

c) Use contradiction. Assume xy is rational. Then $xy=\frac{c}{d}$ for some $c,d \in \mathbb{Z}, d \neq 0$. It is given that x is rational.

Thus $x=\frac{a}{b}$ for some $a,b \in \mathbb{Z}, b \neq 0$. We have $xy=\frac{c}{d}, x=\frac{a}{b} \to \left(\frac{a}{b}\right)y=\frac{c}{d} \to y=\frac{bc}{ad}$,

$bc, ad \in \mathbb{Z}, ad \neq 0$. Therefore y is rational. This is a contradiction since y is given to be irrational. We conclude xy is irrational.

2)

a)

i) I am not majoring in math or I am not majoring in psychology.

j) Some students do not take calculus.

b) Write the converse and contrapositive of the following: If you can do math, then you can do physics.

Converse: If you can do physics, then you can do math.

Contrapositive. If you cannot do physics, then you cannot do math.

c) If x=3, then x^2=9 and if x^2=9, then x=3.

d) 1-1? Suppose $f(x_1)=f(x_2)$. Then $x_1-4=x_2-4$. Adding 4 to both sides, we get $x_1=x_2$. Therefore f is 1-1.

Onto? Let $b \in \mathbf{Z}$ in the codomain. We need to see if it is possible to find $a \in \mathbf{Z}$ in the domain such that $f(a)=b$. But $f(a)=b \leftrightarrow a-4=b \leftrightarrow a=b+4$. Since $b+4 \in \mathbf{Z}$ and $f(b+4)=b$, f is onto.

3)

a) $\begin{vmatrix} 3 & -2 & -2 \\ 1 & 1 & 2 \\ 5 & 5 & -4 \end{vmatrix} \to$ switch row 1 and row 2 and multiply by -1 $-\begin{vmatrix} 1 & 1 & 2 \\ 3 & -2 & -2 \\ 5 & 5 & -4 \end{vmatrix} \to$ b) Make the (2,1) entry and (3,1) entry 0 by Row1(-3)+Row2= new Row2, Row 1 (-5)+Row 3 = new Row 3: No change to the determinant.

$$-\begin{vmatrix} 1 & 1 & 2 \\ 0 & -5 & -8 \\ 0 & 0 & -14 \end{vmatrix} \rightarrow$$ c) next 'factor out' -5 from the row 2, -14 from the row 3

$$(-1)(-5)(-14)\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{8}{5} \\ 0 & 0 & 1 \end{vmatrix} \rightarrow \det A = (-1)(-5)(-14)(1) = -70$$

4)

a) The line of intersection is the set of solutions to the system of equations $\begin{matrix} x+2y+3z=8 \\ 2x+5y-z=10 \end{matrix}$.

In a row echelon form, the augmented matrix becomes $\begin{bmatrix} 1 & 2 & 3 & 8 \\ 2 & 5 & -1 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 8 \\ 0 & 1 & -7 & -6 \end{bmatrix}$. Z is a free variable, so let $z=t$. Do back substitution. Solving the second equation for y, $y-7z=-6 \rightarrow y=7z-6 \rightarrow y=7t-6$. Solving the first equation for x, $x+2y+3z=8 \rightarrow x=-2y-3z+8 \rightarrow x=-2(7t-6)-3t+8=-17t+20$. Thus the line of intersection can be given parametrically as $(x,y,z)=(-17t+20,7t-6,t)$

b) $$z=\frac{\begin{vmatrix} 2 & -1 & 0 \\ 3 & 0 & 10 \\ 5 & 0 & 14 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -5 \\ 3 & 0 & 1 \\ 5 & 0 & -1 \end{vmatrix}}=\frac{-(-1)(3(14)-5(10))}{-(-1)(-3-5)}=1$$

5) Find a row echelon form of $\begin{matrix} x-y-z=a \\ x+y+3z=b \\ 2y+4z=c \end{matrix}$

$$\begin{bmatrix} 1 & -1 & -1 & a \\ 1 & 1 & 3 & b \\ 0 & 2 & 4 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & a \\ 0 & 1 & 2 & \frac{b-a}{2} \\ 0 & 0 & 0 & a-b+c \end{bmatrix}$$. This system of equation has at least one solution iff the last row is entirely zero. Thus $a-b+c=0$.

6)

a) I) $T(\mathbf{v}+\mathbf{w})=A(\mathbf{v}+\mathbf{w})=A\mathbf{v}+A\mathbf{w}=T(\mathbf{v})+T(\mathbf{w})$ and ii) $T(c\mathbf{v})=A(c\mathbf{v})=cA(\mathbf{v})=cT(\mathbf{v})$. Thus T is a LT

b) A basis for the range of a linear transformation can be found by computing the column space of the matrix: The row echelon form of $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. The first column and the second column each contains a leading 1. Thus by going back to the original matrix, a basis for the range is {(2,1),(3,1)}

7)

a) $$\det(S^{-1}AS)=\det(S^{-1})\det(A)\det(S)=\frac{1}{\det(S)}\det(A)\det(S)=\det(A)$$

b) Since 2 can be factored out from each row, 2 becomes 8 outside the determinant function:

$$\det(2A^2) = 8\det(A^2) = 8\det(A)\det(A) = 128$$

8)

a) Its sign is -1 since there is one inversion.
b) rank=n-# of free variables=3-2=1.
c) False: they have the same dimension, but they are not equal.
d) True.
e) False: A matrix is invertible iff its determinant is not 0.
f) True. The number of solutions of a system of homogeneous equations is 1 or infinite.
g) True

9)

$T_1(a,b) = 3bx + (b-a) \to T((1,0)) = -1 = -\frac{1}{2}(2) = -\frac{1}{2}(2) + 0(5x)$ and

$T(0,1) = 3x + 1 = \frac{3}{5}(5x) + \frac{1}{2}(2) = \frac{1}{2}(2) + \frac{3}{5}(5x)$. Thus placing the coefficients into the columns,

$$T\Big|_B^C = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{5} \end{bmatrix}$$

Use the formula $\mathbf{w} = T\mathbf{v} \leftrightarrow [\mathbf{w}]_C = [T]_B^C[\mathbf{v}]_B$ to find T(2, 5): first express $(2,5)$ using B: But this easy since B is the standard basis. $\to (2,5) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}_B$

Then $T\Big|_B^C [\mathbf{v}]_B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix} = [\mathbf{w}]_C$

Finally using C, $\mathbf{w} = \frac{3}{2}(2) + 3(5x) = 3 + 15x$

10)

a) First show the zero vector is in the set: $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a diagonal matrix since all the off diagonal entries are 0.

Thus it contains the zero vector.

b) Next show it is closed under addition:

Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ be diagonal matrices. Then $A + B = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$ is still a diagonal matrix with real entries. It is closed under addition.

c) Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, k be a real number. Then $kA = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}$ is still a diagonal matrix with real entries. It is closed under scalar multiplication.

Therefore, it is a subspace.

11)

a) Recall that $\|f\| = \sqrt{< f, f >}$. Thus

$$\|f\| = \left(\int_0^{2\pi} (\sin 2x)(\sin 2x)dx\right)^{\frac{1}{2}} = \left(\int_0^{2\pi} (\sin^2 2x)dx\right)^{\frac{1}{2}} = \left(\int_0^{2\pi} \frac{1-\cos 4x}{2}dx\right)^{\frac{1}{2}} = \sqrt{\pi}$$

b) Recall that two vectors are orthogonal iff their inner product is zero.

$< \sin 2x, \cos 2x >= \int_0^{2\pi} (\sin 2x)(\cos 2x)dx = \frac{\sin^2 2x}{4}\Big|_0^{2\pi} = 0$ (use u-sub with u=sin2x to integrate). Thus they are orthogonal.

12)

a) $(x_1, y_1) + (x_2, y_2) = (2x_1x_2, y_1 + y_2)$, $(x_2, y_2) + (x_1, y_1) = (2x_2x_1, y_2 + y_1) = (2x_1x_2, y_1 + y_2)$. It is commutative.

b) Looking for an element (a,b) such that $(x, y) + (a,b) = (x, y)$ for all x and y.

$(x, y) + (a,b) = (2xa, y + b) = (x, y) \rightarrow 2xa = x, y + b = y \rightarrow a = \frac{1}{2}, b = 0$. The zero vector is $(\frac{1}{2}, 0)$

c) Looking for an element (a,b) such that $(1,2) + (a,b) = (\frac{1}{2}, 0)$, the zero vector for all x and y.

$(1,2) + (a,b) = (\frac{1}{2}, 0) \rightarrow (2a, 2 + b) = (\frac{1}{2}, 0) \rightarrow a = \frac{1}{4}, b = -2$. Thus $-(1,2) = (\frac{1}{4}, -2)$

d) $1\mathbf{v} = 1(x, y) = (1x, y) = (x, y) = \mathbf{v}$. Yes, $1\mathbf{v} = \mathbf{v}$

13) A)

i) Pick two typical first degree polynomials $a + bx$ and $c + dx$. Then
$T((a + bx) + (c + dx)) = T((a + c)x + (b + d)x) =$
$3(a + c) + 2(b + d)x = (3a + 2bx) + (3c + 2dx) = T(a + bx) + T(c + dx)$

ii) Let k be a real number. Then $T(k(a + bx)) = T(ka + kbx) = 3ka + 2kbx$
$= k(3a + 2bx) = kT(a + bx)$

Thus it is a linear transformation.

c) The matrix with respect to $B = \{1 + x, -1 + 2x\}$ and $C = \{3, 5 + 2x\}$: Express the image of each vector in B using the vectors in C. Then their coefficients are the columns of the matrix.

$$T(1 + x) = 3 + 2x = c_1(3) + c_2(5 + 2x)$$

$$T(-1 + 2x) = -3 + 4x = d_1(3) + d_2(5 + 2x)$$

Skipping the usual 3 steps,

$$\begin{bmatrix} 3 & 5 & \vdots & 3 & -3 \\ 0 & 2 & \vdots & 2 & 4 \end{bmatrix} \rightarrow \text{RREF}= \begin{bmatrix} 1 & 0 & \vdots & -\frac{2}{3} & -\frac{13}{3} \\ 0 & 1 & \vdots & 1 & 2 \end{bmatrix}$$

Thus $[\mathbf{T}]_B^C = \begin{bmatrix} -\frac{2}{3} & -\frac{13}{3} \\ 1 & 2 \end{bmatrix}$

d) Use the formula $[T(\mathbf{v})]_C = [T]_B^C[\mathbf{v}]_B$ to compute $T(4x+1)$, first express 1+4x using the basis B:

$1+4x = c_1(1+x)+c_2(-1+2x) \rightarrow c_1 = 2, c_2 = 1$ (you may skip the usual 3 steps). Thus $4x+1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$

$[T(\mathbf{v})]_C = [T]_B^C[\mathbf{v}]_B = = \begin{bmatrix} -\frac{2}{3} & -\frac{13}{3} \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{17}{3} \\ 4 \end{bmatrix}$. Since $C = \{3, 2x+5\}$,

$\rightarrow \mathbf{w} = -\frac{17}{3}(3)+4(5+2x) = 3+8x$

14)

a) LI? Let $c_1v_1 + c_2v_2 = 0 \rightarrow c_1(1,0,2)+c_2(2,4,5) = (0,0,0) \rightarrow (c_1+2c_2, 4c_2, 2c_1+5c_2) = (0,0,0)$. Write the system of equations in a matrix form , we get

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since there are no free variables, $c_1 = c_2 = 0$. Thus the vectors are LI.

b) The subspace spanned by the vectors: Find (x,y,z) that $c_1(1,0,2)+c_2(2,4,5) = (x,y,z)$ has a solution: we use the row echelon form: $c_1(1,0,2)+c_2(2,4,5) = (x,y,z) \rightarrow (c_1+2c_2, 4c_2, 2c_1+5c_2) = (x,y,z)$. In

augmented matrix, $\begin{bmatrix} 1 & 2 & x \\ 0 & 4 & y \\ 2 & 5 & z \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & \frac{y}{4} \\ 0 & 0 & -2x-\frac{y}{4}+z \end{bmatrix}$

This system has a solution iff the last row is entirely 0 (then the rank A= rank $A^{\#}$=2). Thus the subspace spanned is given by the equation $-2x-\frac{y}{4}+z = 0$

15)

a) $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$ switch Row 1 and Row 3, multiply by -1. $-\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & -2 \end{vmatrix}$. Add Row1 (-3) to Row 3, Row1 (-1) to Row 2. No change in determinant. $\rightarrow -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix}$ 'Factor out" 2 from Row 2. $\rightarrow (-2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix}$. Add Row 2 (-1) to Row 3: $\rightarrow (-2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix}$. Thus the determinant is (-2) times the product of the diagonal elements. $\rightarrow \det(A) = 4$

b) The (2,3) entry of the adjoint matrix is the determinant of the 2x2 obtained by crossing off the third row and second column , times $\rightarrow (-1)^{i+j}$ (note that for (2,3) entry, use (3,2)). That is, $(-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2$

16) Recall that the matrix of a composition is the product of their matrices.. Since $T_1(1)=1+x,\ T_1(x)=-1+2x$ $[T_1]_B^C=\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}$, . Similarly $[T_2]_C^D=\begin{bmatrix}0 & 1\\ 3 & -1\end{bmatrix}$ Then the matrix of $[T_2\circ T_1]_B^D=\begin{bmatrix}0 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & -5\end{bmatrix}$. Thus

$$[T_2\circ T_1(1+3x)]_D=\begin{bmatrix}1 & 2\\ 2 & -5\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix}=\begin{bmatrix}7\\ -13\end{bmatrix}\to T_2\circ T_1(1+3x)=7-13x$$

CHECK(NOT REQURIED) :

Alternately, this problem can be solved without using matrices:

$$T_2T_1(1+3x)=T_2(1T_1(1)+3T_1(x))=T_2(1(1+x)+3(-1+2x))=T_2(-2+7x)$$
$$=-2T_2(1)+7T_2(x)=7(1-x)-2(3x)=7-13x$$

17) A linear transformation is invertible iff its matrix is invertible. Thus first write the LT in the matrix form. Then compute the determinant of the matrix to see if it is 0 or not

$$T(ax^2+bx+c)=(2a-c)x^2+bx\to T(1)=T(0x^2+0x+1)=-x^2\ ,T(x)=T(0x^2+1x+0)=x,T(x^2)=2x^2$$

Thus the matrix of T is $\left[T(e_1),T(e_2),T(e_3)\right]=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 2\end{bmatrix}$. The determinant of this matrix is not 0 since it has a row consisting of zeros. Therefore, the linear transformation is not invertible.

18) First find the row echelon form of $\begin{bmatrix}1 & 0 & 1 & 2\\ 0 & 1 & -1 & 2\\ 0 & -2 & 1 & 5\end{bmatrix}\to\begin{bmatrix}1 & 0 & 1 & 2\\ 0 & 1 & -1 & 2\\ 0 & 0 & 1 & -5\end{bmatrix}$

All the nonzero rows of the row echelon form form a basis for the row space: thus a basis for the row space is $\{(1,0,1,2),(0,1,01,2),(0,0,1,-5)\}$

For a basis of the column space, take the columns of the original matrix with leading 1 in the row echelon form: take the first 3 columns. Thus a basis for the column space is $\{(1,0,0),(0,1,-2),(1,-1,1)\}$

b)yes, it is true. They are both equal to the number of leading 1s.

19)

$$\begin{vmatrix}a_1+b_1 & a_1-b_1 & c_1\\ a_2+b_2 & a_2-b_2 & c_2\\ a_3+b_3 & a_3-b_3 & c_3\end{vmatrix}=\begin{vmatrix}a_1 & a_1-b_1 & c_1\\ a_2 & a_2-b_2 & c_2\\ a_3 & a_3-b_3 & c_3\end{vmatrix}+\begin{vmatrix}b_1 & a_1-b_1 & c_1\\ b_2 & a_2-b_2 & c_2\\ b_3 & a_3-b_3 & c_3\end{vmatrix}$$ (a column can be separated without affecting the determinant) $=\begin{vmatrix}a_1 & a_1 & c_1\\ a_2 & a_2 & c_2\\ a_3 & a_3 & c_3\end{vmatrix}+\begin{vmatrix}a_1 & -b_1 & c_1\\ a_2 & -b_2 & c_2\\ a_3 & -b_3 & c_3\end{vmatrix}+\begin{vmatrix}b_1 & a_1 & c_1\\ b_2 & a_2 & c_2\\ b_3 & a_3 & c_3\end{vmatrix}+\begin{vmatrix}b_1 & -b_1 & c_1\\ b_2 & -b_2 & c_2\\ b_3 & -b_3 & c_3\end{vmatrix}$ (I separated the columns

twice). But $\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$, since it has two identical columns. $\begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix} = (-1)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ since -1 factors out from the column 2. $\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ since two columns are switched, and

$\begin{vmatrix} b_1 & -b_1 & c_1 \\ b_2 & -b_2 & c_2 \\ b_3 & -b_3 & c_3 \end{vmatrix} = 0$ since two of the columns are identical after factoring out -1 from the second column. Therefore,

$$= \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & -b_1 & c_1 \\ b_2 & -b_2 & c_2 \\ b_3 & -b_3 & c_3 \end{vmatrix} = 0 - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0$$

$$= -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

20) A)

i) First show the zero vector is in the space: $(0,0,0)$ satisfies $0+0+2(0)=0$. Thus it contains the zero vector.

ii) Closed under addition: Let $(x_1,x_2,x_3)\in S$ and $(y_1,y_2,y_3)\in S$ (note that since $(x_1,x_2,x_3)\in S$ and $(y_1,y_2,y_3)\in S$, $x_1+x_2+2x_3=0$ and $y_1+y_2+2y_3=0$). Then $(x_1,x_2,x_3)+(y_1,y_2,y_3)=$ $(x_1+y_1,x_2+y_2,x_3+y_3)$ and $(x_1+y_1)+(x_2+y_2)+2(x_3+y_3)=(x_1+x_2+2x_3)+(y_1+y_2+2y_3)=0$. Thus it is closed under addition.

iii) Under scalar multiplication: let k be a real number, $(x_1,x_2,x_3)\in S$. Then $k(x_1,x_2,x_3)=(kx_1,kx_2,kx_3)$ and $kx_1+kx_2+2kx_3=k(x_1+x_2+2x_3)=0$. Thus it is closed under scalar multiplication.

Therefore it is a subspace.

B) To find a basis: $(x_1,x_2,x_3)\in S \to x_1+x_2+2x_3=0$. Thus there are two free variables since the row echelon form has rank 1. Let $x_2=s, x_3=t$. Then $(x_1,x_2,x_3)\in S \to x_1=-x_2-2x_3=-s-2t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$. Thus $(x_1,x_2,x_3)\in S \to$ it is a linear combination of $(-1,1,0)$ and $(-2,0,1)$.

They are clearly LI and spans the subspace. Therefore, $\{(-1,1,0),(-2,0,1)\}$ is a basis for S.

21)

Find the row echelon form of the augmented matrix $\begin{bmatrix} 1 & 2 & -1 & 4 & 0 \\ 3 & 5 & 1 & 2 & 0 \\ 2 & 4 & -2 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 4 & 0 \\ 0 & 1 & -4 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Solve this:

notice that x_3, x_4 are free variables: $x_3=s, x_4=t$. Use the second equation and solve for x_2:

$x_2 = 4x_3 - 10x_4 = 4s - 10t$. Take the first and solve for x_1 : $x_1 = -2x_2 + x_3 - 4x_4 = -7s + 16t$. Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$

$\begin{bmatrix} -7s+16t \\ 4s-10t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -7 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 16 \\ -10 \\ 0 \\ 1 \end{bmatrix}$. Thus $\{\begin{bmatrix} -7 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ -10 \\ 0 \\ 1 \end{bmatrix}\}$ spans the null space. Since they are clearly LI, it is a basis.

Thus the dimension of the null space is 2.

22)

a) First show it contains the zero vector: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in S since its second row entries are 0. Thus it contains the zero vector.

b) Next show it is closed under addition:

i) Let $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ be in S. Then $A + B = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$ is still in S since the second row is 0. It is closed under addition.

ii) Let $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, k be a real number. Then $kA = \begin{bmatrix} ka & kb \\ 0 & 0 \end{bmatrix}$ is still in S since the second row is 0. It is closed under scalar multiplication.

c)

i) Show LI: Suppose $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$A = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to c_1 = c_2 = 0$. Therefore they are LI.

ii) Spans: for any $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (a is like c_1, b is like c_2). Thus it spans S. Therefore, it is a subspace.

23) The subspace spanned by the vectors $\{(1,0,-2),(-2,1,4)\}$: Find all (x,y,z) that $c_1(1,0,-2) + c_2(-2,1,4) = (x, y, z)$ has a solution: we use the row echelon form: $c_1(1,0,-2) + c_2(-2,1,4) = (x, y, z) \to (c_1 - 2c_2, c_2, -2c_1 + 4c_2) = (x, y, z)$.

In augmented matrix, $\begin{bmatrix} 1 & -2 & x \\ 0 & 1 & y \\ -2 & 4 & z \end{bmatrix} \to \begin{bmatrix} 1 & -2 & x \\ 0 & 1 & y \\ 0 & 0 & 2x+z \end{bmatrix}$

This system has a solution iff the last row is entirely 0. That is, the subspace spanned is a plane given by the equation $2x + z = 0$ (notice that two LI vectors in R^3 spans a plane)

iii) (1,-4,2) is not in the span of {(1,0,-2),(-2,1,4)}. The vectors in the span of S must satisfy the equation $2x + z = 0$: here $2(1) + 2 \neq 0$.

24) It is clearly LD since there are three vectors in a 2 dimensional space. We use the row echelon form to find dependency relation. Find c_1, c_2, c_3 satisfying $c_1(1,3)+c_2(3,-1)+c_3(0,4)=(0,0)$. Now

$c_1(1,3)+c_2(3,-1)+c_3(0,4)=(0,0) \rightarrow (c_1+3c_2, 3c_1-c_2+4c_3)=(0,0)$

In the augmented matrix form, $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & -1 & 4 & 0 \end{bmatrix}$. In row echelon form, $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & -1 & 4 & 0 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -2/5 & 0 \end{bmatrix}$. c_3 is a free variable: let $c_3 = s$. Solving the last equation for c_2, we have $c_2 = -2/5c_3 = -2/5s$. Solving the first equation for c_1, $c_1 = -3c_2 = -6/5s$. Thus $(c_1, c_2, c_3) = (-\frac{6}{5}s, -\frac{2}{5}s, s)$. Since we are looking for a relationship (one of many possible relationships), we can pick $s = 5$ (just a random number). This choice of s makes $(c_1, c_2, c_3) = (-6,2,5)$. Therefore,

$c_1(1,3)+c_2(3,-1)+c_3(0,4)=(0,0) \rightarrow -6(1,3)+2(3,-1)+5(0,4)=(0,0)$.

To find a LI subset of S that has the same span as S, drop any vector in the relationship, say (0,4). Then the resulting set is {(1,3),(3,-1)} and this set is LI since (1,3)=k(3,-1) is impossible.

25)

a) To show the product defined is an inner product, we must verity the three conditions:

i) (positive definiteness) $< f, f > = \int_0^1 f(x)f(x)dx = \int_0^1 f(x)^2 dx$. Since the integrand is never negative, $< f, f > \geq 0$ and $< f, f > = 0$ iff $f(x)$ is a zero polynomial.

ii) (Symmetry) $< f, g > = \int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx = < g, f >$

iii) (Linearity) $< af+bg, h > = \int_0^1 (af(x)+bg(x))h(x)dx = \int_0^1 (af(x)h(x)+bg(x)h(x))dx$

$= a\int_0^1 f(x)h(x)dx + b\int_0^1 g(x)h(x)dx = a < f, h > + b < g, h >$

Therefore $< f, g > = \int_0^1 f(x)g(x)dx$ is an inner product.

b) Next use Gram-Schdmit process to construct an orthogonal basis:

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{< u_2, v_1 >}{\|v_1\|^2}v_1$, $< v_2, u_1 > = < x, 1 > = \int_0^1 x(1)dx = \frac{1}{2}$, $\|u_1\|^2 = < u_1, u_1 > = < 1, 1 > = \int_0^1 1(1)dx = \int_0^1 1dx = 1$

Therefore, $v_2 = u_2 - \frac{< v_2, u_1 >}{\|u_1\|^2}u_1 = x - \frac{\frac{1}{2}}{1}(1) = x - \frac{1}{2}$. $\{1, x - \frac{1}{2}\}$ is an orthogonal basis.

26) Since $S = \{(1,3,2), (-4,2,4), (0,7,0)\}$ contains 3 vectors in R³ , it spans R³ if they are LI. We can compute their determinant to see if they are LI. $\left\|\begin{matrix} 1 & -4 & 0 \\ 3 & 2 & 7 \\ 2 & 4 & 0 \end{matrix}\right\| = (-1)^{2+3}7\begin{bmatrix} 1 & -4 \\ 2 & 4 \end{bmatrix} = -84$, which is not 0. Thus S spans R³

27) Since there are 3 vectors and dim(P2)=3, we need only to show LI.

Suppose $c_1(1+x)+c_2(2+x+x^2)+c_3(1-x)=0$

3 steps not shown.

$c_1(1+x)+c_2(2+x+x^2)+c_3(1-x)=0$

$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \text{RREF} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow c_1 = c_2 = c_3 = 0 \rightarrow \text{LI} \rightarrow$ it is a basis.

28) Subspace is omitted. Next show it is a basis.

Claim: $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ is a basis for all 2 x 2 upper triangular matrices.

Show LI: Suppose $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$A = \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow c_1 = c_2 = c_3 = 0$. Therefore they areLI.

Spans: for any $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ = (a is like c_1, b is like c_2). Thus it spans S. Therefore, it is a basis.

29)

To verify that it is a linear transformation, Show a)

$T((x_1, y_1)+(x_2, y_2)) = T(x_1, y_1)+T(x_2, y_2), \ T(k(x, y)) = kT(x, y)$

i) $T((x_1, y_1)+(x_2, y_2)) = T(x_1+x_2, y_1+y_2) = (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2) =$
$(x_1+y_1, x_1-y_1)+(x_2+y_2, x_2-y_2) = T(x_1, y_1)+T(x_2, y_2)$

ii) $T(k(x, y)) = T(kx, ky) = (kx+ky, kx-ky) = k(x+y, x-y) = kT(x, y)$.

Thus T is a linear transformation.

For the kernel, first find a matrix of T. Then compute the null space by finding its row echelon form.
$T(x,y)=(x+y,x-y)\to T(1,0)=(1,1),T(0,1)=(1,-1)$. Thus placing them into columns, the matrix of T is
$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. In row echelon form in augmented matrix, , $\begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}\to\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}$. You can see that the null space is trivial since there are no free variables. No basis for the null space. Using the rank-nullity theorem, the dimension of the range is 2-0=2.

30) Compute the determinant . Recall that a matrix is invertible iff $\det(A)\neq 0$. Use the cofactor expansion along the third column.

$$\begin{bmatrix}1 & 2 & 0\\ k & 0 & 2\\ 4k+1 & k & 0\end{bmatrix}\to\det(A)=0\to -2(k-2(4k+1))=0\to k=-\frac{2}{7}$$

Thus A is invertible iff $k\neq-\frac{2}{7}$

31)

a) $(a,b)+(c,d)=(2a+2c,2b+2d),\ (c,d)+(a,b)=(2c+2a,2d+2b)=(2a+2c,2b+2d)$. It is commutative.

b) Zero element is (c,d) such that $(a,b)+(c,d)=(a,b)$ for all (a,b). But $(a,b)+(c,d)=(2a+2c,2b+2d)=(a,b)\to 2a+2c=a,2b+2d=b\to c=-\frac{a}{2},d=-\frac{b}{2}$. But this choice of (c,d) depends on (a,b). Thus it is impossible to find (c,d) that works for all (a,b). Thus there is no zero element.

c)

32) Doing row echelon form is better than using the determinant: a dependency relation can be found if it is not LI. The determinant does not give you a relationship. Suppose $c_1(1,3,1)+c_2(-1,3,7)+c_3(-2,-3,2)=(0,0,0)$. Then $(c_1-c_2-2c_3,3c_1+3c_2-3c_3,c_1+7c_2+2c_3)=(0,0,0)$ (don't forget to show the 3 steps)

Using augmented matrix, $\begin{bmatrix}1 & -1 & -2 & 0\\ 3 & 3 & -3 & 0\\ 1 & 7 & 2 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & -\frac{3}{2} & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ Since there is a free variable (thus there is non-trivial solutions) , the set is dependent. To find a relation, find c_1,c_2,c_3. c_3 is a free variable, so let $c_3=t$. Then c_2 :
$c_2=-\frac{1}{2}c_3=-\frac{1}{2}t,c_1=\frac{3}{2}t$. . Thus $(c_1,c_2,c_3)=(\frac{3}{2}s,-\frac{1}{2}s,s)$. Since we are finding a relation (there are infinitely many of them), we can assign a number to t. So let $t=2$. Then $(c_1,c_2,c_3)=(3-1,2)$ Thus a relation is
$c_1(1,3,1)+c_2(-1,3,7)+c_3(-2,-3,2)=(0,0,0)\to 3(1,3,1)+(-1)(-1,3,7)+2(-2,-3,2)=(0,0,0)$. We can drop any vector in this relation without affecting the span. So drop (1,3,1) and the resulting set is LI that has the same span as S

33)

$$A=\begin{bmatrix}0&2&2&1&0&0\\2&2&4&0&1&0\\0&3&1&0&0&1\end{bmatrix}(R1\leftrightarrow R2)\rightarrow\begin{bmatrix}2&2&4&0&1&0\\0&2&2&1&0&0\\0&3&1&0&0&1\end{bmatrix}(R1\div 2,R2\div 2)\rightarrow$$

$$\begin{bmatrix}1&1&2&0&1/2&0\\0&1&1&1/2&0&0\\0&3&1&0&0&1\end{bmatrix}(R2(-1)+R1\rightarrow R1,R2(-3)+R3)\rightarrow R3:\begin{bmatrix}1&0&1&-1/2&1/2&0\\0&1&1&1/2&0&0\\0&0&-2&-3/2&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&1&-1/2&1/2&0\\0&1&1&1/2&0&0\\0&0&1&3/4&0&-1/2\end{bmatrix}$$

$$R3(-1)+R2\rightarrow R2,R3(-1)+R1)\rightarrow R1:\rightarrow\begin{bmatrix}1&0&0&-5/4&1/2&1/2\\0&1&0&-1/4&0&1/2\\0&0&1&3/4&0&-1/2\end{bmatrix}$$

Thus $A^{-1}=\begin{bmatrix}-5/4&1/2&1/2\\-1/4&0&1/2\\3/4&0&-1/2\end{bmatrix}$

34)

a) (3,1) and (1,3) is a basis for R^2 since $\begin{vmatrix}3&1\\1&3\end{vmatrix}=8\neq 0$. Two LI vectors in R^2 is a basis.

b) First express (1,0) as a linear combination of (3,1) and (1,3). Then place the coefficients c_1,c_2 in the first column. Next express (0,1) as a linear combination of (3,1) and (1,3). Then place c_1,c_2 in the second column.

$$(1,0)=c_1(3,1)+c_2(1,3)\rightarrow(1,0)=(3c_1+c_2,c_1+3c_2)=(1,0)$$

$$\rightarrow 3c_1+c_2=1,c_1+3c_2=0$$

$$\rightarrow\left[\begin{array}{cc|c}3&1&1\\1&3&0\end{array}\right]\rightarrow\text{RREF}\left[\begin{array}{cc|c}1&0&\frac{3}{8}\\0&1&-\frac{1}{8}\end{array}\right]c_1=\frac{3}{8},c_2=-\frac{1}{8}\cdot$$

$$(0,1)=c_1(3,1)+c_2(1,3)\rightarrow(0,1)=(3c_1+c_2,c_1+3c_2)=(0,1)$$

$$\rightarrow c_1+3c_2=1,3c_1+c_2=0$$

$$\rightarrow\left[\begin{array}{cc|c}3&1&0\\1&3&1\end{array}\right]\rightarrow\text{RREF}\left[\begin{array}{cc|c}1&0&-\frac{1}{8}\\0&1&\frac{3}{8}\end{array}\right]c_1=-\frac{1}{8},c_2=\frac{3}{8}$$

Thus the matrix is $P_{C\leftarrow B}=[P]_B^C=\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\ -\frac{1}{8} & \frac{3}{8}\end{bmatrix}$

c) Using the formula $[\mathbf{v}]_C=P_{C\leftarrow B}[\mathbf{v}]_B$, $\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\ -\frac{1}{8} & \frac{3}{8}\end{bmatrix}[\begin{matrix}2\\3\end{matrix}]=[\begin{matrix}\frac{3}{8}\\ \frac{7}{8}\end{matrix}]$: i.e. $(2,3)=\frac{3}{8}(3,1)+\frac{7}{8}(1,3)$

d) $[P]_B^C=([P]_B^C)^{-1}=\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\ -\frac{1}{8} & \frac{3}{8}\end{bmatrix}^{-1}=8\begin{bmatrix}\frac{3}{8} & \frac{1}{8}\\ \frac{1}{8} & \frac{3}{8}\end{bmatrix}=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$

35) Write each of the element in $\{\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix},\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix},\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\}$ using the standard basis . Since $M_2(\mathbf{R})$ is a 4 dimensional vector space over R, the change of basis matrix is 4 x 4.

First , $\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix}=1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}-3\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+0\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$. Then write the coefficient 1, -3, 0, 1 in the first column.

Repeat this process for $\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix},\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ and we obtain $\begin{bmatrix}1 & 2 & 1 & 0\\ -3 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 1 & 2 & -1 & 1\end{bmatrix}$

36)

a) True. If AB is invertible then $\det(AB)\neq 0\rightarrow \det(A)\det(B)\neq 0\rightarrow \det(A)$ and $\det(B)$ are both not zero.
b) False. If $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is invertible but A,B are not invertible.
c) True. The rank is at most 2, and there are 3 variables. Thus at least one of the variables is free.
d) True. R satisfies all the vector space axioms.
e) False. A system of homogeneous equations always have the trivial solution.

37)

a) 0 is a unique element (a,b) such that $(x_1,y\ )+(a,b)=(x,y),\ \forall(x,y)$.
$(x,y)+(a,b)=(x+a,yb)=(x,y)\rightarrow x+a=x, yb=y\rightarrow a=0,b=1$. Thus $(0,1)$ is the zero vector.

b) $-(3,4)$ is an element (a,b) such that $(3,4)+(a,b)=(0,1)$,

$(3,4)+(a,b)=(3+a,4b)=(0,1) \rightarrow 3+a=0,\ 4b=1 \rightarrow a=-3, b=\frac{1}{4}$. Thus the additive inverse of $(3,4)$ is $(-3,\frac{1}{4})$

c) $(r+s)(x,y)=((r+s)x,1)=(rx+sx,1)$

On the other hand, $r(x,y)+s(x,y)=(rx,1)+(sx,1)=(rx+sx,1\cdot 1)=(rx+sx,1)$

38)

To verify that it is a linear transformation, Show a)

$T((x_1,x_2,x_3)+(y_1,y_2,y_3))=T(x_1,x_2,x_3)+T(y_1,y_2,y_3),\ T(k(x_1,x_2,x_3))=kT(x_1,x_2,x_3)$

a) $T((x_1,x_2,x_3)+(y_1,y_2,y_3))=T(x_1+y_1,x_2+y_2,x_3+y_3)=$
$(x_1+y_1-2(x_3+y_3),x_2+y_2)=(x_1-2x_3,x_2)+(y_1-2y_3,y_2)=$
$T(x_1,x_2,x_3)+T(y_1,y_2,y_3)$

b) $T(k(x_1,x_2,x_3))=T(kx_1,kx_2,kx_3)=(kx_1-2kx_3kx_2)=k(x_1-2x_3,x_2)=kT(x_1,x_2,x_3)$.

Thus it is a linear transformation.

For the kernel, first find the matrix of T. Then compute the null space by finding its row echelon form.

$T(x_1,x_2,x_3)=(x_1-2x_3,x_2) \rightarrow x_1-2x_3=0, x_2=0$. $\begin{bmatrix} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \end{bmatrix}$. This is already In RREF. You can see that x_3 is a free variable since there is no leading 1. Using the first and second equations, we get $x_1=2s, x_2=0$ the null space is $(2s,0,s)=s(2,0,1)$. Thus a basis for the null space is (0,0,1).

c) Dim(RngT)=3-1=2. Since the dimension of the range is the same as the dimension of the codomain, T is onto.

39)

$\begin{vmatrix} 3a+3b & 3b & 3c \\ -d-e & -e & -f \\ g+h & h & i \end{vmatrix} = (-1)(3)\begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix}$ ('factor out' 3 from the first row, (-1) from the second row) $=(-3)(\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix})$ (A column can be separated without changing the determinant).

But $\begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix} = 0$, since it has two identical columns. Therefore,

$$\begin{vmatrix} 3a+3b & 3b & 3c \\ -d-e & -e & -f \\ g+h & h & i \end{vmatrix} = (-1)(3)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-3)(5) = -15$$

40) To find the adjoint matrix of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$:

a) First compute the determinant: Since A is an upper triangular, it is the product of its diagonal elements: det(A)=1(2)(2)=4

b) Next find the cofactor matrix:

$$\begin{bmatrix} 4 & 0 & 0 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

c) Take the transpose of the cofactor matrix. $\begin{bmatrix} 4 & -2 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d) Divide each entry by the determinant: $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

41)

Since $\{v_1, v_2, v_3\}$, $\exists c_1, c_2, c_3$, not all 0 such that $c_1v_1 + c_2v_2 + c_3v_3 = 0$. To show $\{v_1, v_2, v_3, v_4\}$ is LD, observe that $c_1v_1 + c_2v_2 + c_3v_3 = 0 \rightarrow c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0$ and not all coefficients are 0. Thus $\{v_1, v_2, v_3, v_4\}$ is LD

42) First find the row echelon form of $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 1 & 4 & 1 & 1 & 0 \end{array}\right]$ is $\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$ by $R1(-1) + R2 \rightarrow new\ R2$ c_3 and c_4 are free variables: $c_3 = s, c_4 = t$. Solving the second equation for c_2, $c_2 = -s - t$. Using the first equation, $c_1 = -3c_2 = 3s + 3t$. Thus $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3s+3t \\ -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

43)

a) To show it is a subspace (observe that the requirement to be in S is the polynomial is degree 2 or less with 0 constant term),

i) Show 0 is in S: It is in s since $0 = 0x + 0x^2$

iv) To show it is closed under addition, $(ax + bx^2) + (cx + dx^2) = (a+c)x + (b+d)x^2 \in S$

v) To show it is closed under scalar multiplication, $k(ax + bx^2) = kax + kbx^2 \in S$

b) . Claim: $\{x, x^2\}$ is a basis. A) LI: Suppose $c_1x + c_2x^2 = \mathbf{0}$. This means $c_1x + c_2x^2 = 0x + 0x^2$ for all x. By equating coefficient, $c_1 = 0, c_2 = 0$

Spans: Let $ax + bx^2 \in S$. Then $ax + bx^2 = a(x) + b(x^2)$, a linear combination of x,x².

The dimension is 2.

44) Claim: $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ is a basis for the set of 2 x 2 skew symmetric matrices. \

a) Since there is only one vector, it is LI

b) Span: Let A be a skew symmetric matrix. Then $A=\begin{bmatrix}0 & -a\\ a & 0\end{bmatrix}=a\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

Thus it is a 1 dimensional subspace.

45) First express (3,4) in terms of $v_{1,}v_2 : c_1(1,2)+c_2(2,3)=(3,4)\rightarrow c_1=-1, c_2=2$. Thus $(3,4)=-(1,2)+2(2,3)$. T $T(3,4)=T(-(1,2)+2(2,3))=-T(1,2)+2T(2,3)=-(-1,3)+2(0,2)=(1,1)$

46)

a) False: the number of solutions is 0, 1 or infinite. This is same as in elementary algebra.

b) True. Since $\det(AB)=\det(A)\det(B)$, A and B must both have non zero determinant.

c) True: $B^T=(\frac{A+A^T}{2})^T=\frac{A^T+(A^T)^T}{2}=\frac{A^T+A}{2}=B$

d) False: If $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $\det A=\det B=0$, but $A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ so $\det(A+B)=1$

e) True: since there are two rows, its row echelon form has at most two leading 1s. This leaves at least two free variables (recall that the variables without the leading 1 become free variables)

f) $\begin{bmatrix}0 & 1 & -2\\ 1 & 0 & 0\\ 2 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ are some examples of 3 x 3 skew symmetric matrices.

g) False: if a matrix A is invertible then $\det(A)\neq 0$

47)

a) $\det(A)=4,,\det(B)=2,\ \det(2\mathrm{A}^{-1}\mathrm{B}^2)=2^4\frac{1}{\det(A)}\det(B)^2=16\frac{1}{4}(4)=16$

b) $\begin{vmatrix}a_1+b_1 & 3b_1 & c_1\\ a_2+b_2 & 3b_2 & c_2\\ a_3+b_3 & 3b_3 & c_3\end{vmatrix}=\begin{vmatrix}a_1 & 3b_1 & c_1\\ a_2 & 3b_2 & c_2\\ a_3 & 3b_3 & c_3\end{vmatrix}+\begin{vmatrix}b_1 & 3b_1 & c_1\\ b_2 & 3b_2 & c_2\\ b_3 & 3b_3 & c_3\end{vmatrix}$ (a column can be separated without affecting the determinant) $=3\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+3\begin{vmatrix}b_1 & b_1 & c_1\\ b_2 & b_2 & c_2\\ b_3 & b_3 & c_3\end{vmatrix}$ (3 can be factored out). But the second determinant is 0. K=3

c) This is already in a row echelon form. There are two free variables: y and z. $y=s, z=t\rightarrow x=-2s-3t$ The solutions are $(x,y,z)=(-2s-3t,s,t)=s(-2,1,0)+t(-3,0,1)$

48)

a) $T(f(x)+g(x))=(f(x)+g(x))'=f'(x)+g'(x)=T(f(x))+T(g(x))$
$T(kf(x))=(kf(x))'=k(f'(x))=kT(f(x))$
Thus it is a LT.

b) The kernel is all functions whose derivative is 0. Using common sense, it is all constant functions.

c) No, since the kernel is not trivial.

49) The operation is clearly commutative, thus you need only to check one direction.

i) $(x,y)+(0,1)=(x,y)$ for all x, y. Thus $\mathbf{0}=(0,1)$

ii) $(x,y)+(a,b)=(0,1)\to(x+a,by)=(0,1)\to a=-x, b=\dfrac{1}{y}$. Thus $-\mathbf{u}=(-x,\dfrac{1}{y})$ if $y\neq 0$. If $y=0$, then $-\mathbf{u}$ does not exist.

iii) $rs(x,y)=(rsx,(rs)^2y)=(rsx,r^2s^2y)$ and $r(s(x,y))=r(sx,s^2y)=(rsx,r^2s^2y)$. Thus it is true;

50)

a) First compute $T(1,1)$ and $T(0,1)$ (using standard basis vectors) and express them using the basis vectors of $C=\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&2\end{bmatrix}\}$.

$T(1,1)=\begin{bmatrix}1&0\\0&1\end{bmatrix}=c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}1&0\\0&2\end{bmatrix}\to c_1=1, c_2=0$ (use common sense). Place the values in the first column of the matrix.

$T(0,1)=\begin{bmatrix}0&0\\0&1\end{bmatrix}=c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}1&0\\0&2\end{bmatrix}\to c_1=-1, c_2=1$ (use common sense) Place the values in the second column of the matrix. Thus the matrix of T is $T(1,1)=\begin{bmatrix}1&1\\0&-1\end{bmatrix}$

b) You need to interpret the given matrix correctly.

$$\begin{array}{rc}
 & \begin{array}{cc} T(1,-1) & T(0,1) \end{array} \\
\begin{array}{r} \text{Coeff of } (1,-1) \\ \text{Coeff of } (0,1) \end{array} & \begin{bmatrix}1&1\\1&-1\end{bmatrix}
\end{array}$$

From the matrix,
$$T(1,-1)=1(1,-1)+1(0,1)=(1,0)$$
$$T(0,1)=1(1,-1)+(-1)(0,1)=(1,-2)$$

Next express (2,3) using the basis vectors (1,-1) and (0,1).

This can be done using common sense or use REF

$$\to\left[\begin{array}{cc|c}1&0&2\\-1&1&3\end{array}\right]\to REF\left[\begin{array}{cc|c}1&0&2\\0&1&5\end{array}\right] c_1=2, c_2=5$$

Then $T(2,3)=T(2(1,-1)+5(0,1))=2T(1,-1)+5T(0,1)=2(1,0)+5(1,-2)=(7,-10)$

51)

a) $5(\mathbf{0})+0\mathbf{v}_1+0\mathbf{v}_2=\mathbf{0}$. Since there is a nontrivial way to make the zero vector, it is LD.

(you can use any nonzero number for the coefficient of 0)

b) Suppose $c_1(w_1-w_2)+c_2(w_1+w_2)=\mathbf{0}$. (need to show $c_1=c_2=0$)
$c_1(w_1-w_2)+c_2(w_1+w_2)=\mathbf{0}\rightarrow(c_1+c_2)w_1+(c_1-c_2)w_2=\mathbf{0}$ Since $\{w_1,w_2\}$ is LI,
$c_1-c_2=0, c_1+c_2=0$. Solving the system with elimination, $c_1=c_2=0$. They are LI.

52)

a) $T((a_1x+b_1)+(a_2x+b_2))=T((a_1+a_2)x+(b_1+b_2))$
$=(2(a_1+a_2)-3(b_1+b_2))+(4(a_1+a_2)+5(b_1+b_2)x)+(16(a_1+a_2)+9(b_1+b_2))x^2$
$=(2a_1-3b_1)+(4a_1+5b_1)x+(16a_1+9b_1)x^2+(2a_2-3b_2)+(4a_2+5b_2)x+(16a_2+9b_2)x^2$
$=T(a_1x+b_1)+T(a_2x+b_2)$

And $T(c(ax+b))=T(cax+cb)=(2ca-3cb)+(4ca+5cb)x+(16ca+9cb)x^2$
$=c[(2a-3b)+(4a+5b)x+(16a+9b)x^2]=cT(ax+b)$
Thus T is a LT

b) Method1: Find a matrix representation M of T. Then the range of T is the column space of M. Here we use the ordered bases $B=\{1,x\}, C=\{1,x,x^2\}$

$T(1)=T(0x+1)=-3+5x+9x^2$. $T(x)=T(1x+0)=2+4x+16x^2$. Thus $\begin{bmatrix}3 & 2\\ 5 & 4\\ 9 & 6\end{bmatrix}$ is a matrix representation. Its REF is $\begin{bmatrix}1 & \frac{2}{3}\\ 0 & 1\\ 0- & 0\end{bmatrix}$, so the first column and the second column both have a leading 1. Going back to the original matrix, a basis for the range is $\{-3+5x+9x^2, 2+4x+16x^2\}$

Alternative solution:

$T(ax+b)=[(2a-3b)+(4a+5b)x+(16a+9b)x^2]=a(2+4x+16x^2)+b(-3+5x+9x^2)$. Thus the range is spanned by $\{-3+5x+9x^2, 2+4x+16x^2\}$. The polynomials are LI since one is not a multiple of the other. Thus $\{-3+5x+9x^2, 2+4x+16x^2\}$ is a basis for the range.

c) The rank-nullity theorem says dim(P_1) =dim(Kernel)+dim(Range). dim(P_1) =2 and dim(range)=2, so dim(kernel)=0. Thus T is 1-1 since its kernel is trivial.

53)

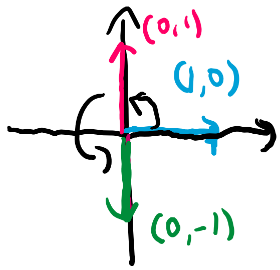
54)

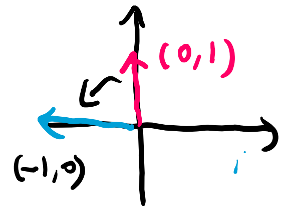
a) False: They span a line or a plane if the vectors are nonzero.
b) True. A scalar may be pulled out.
c) False: v^2 does not even make sense.
d) False: 1 is a real number, so it is always the usual 1.

55)

a) To find a matrix, find $T(1,0)$ and $T(0,1)$.

$T(1,0)=(0,-1)$ and $T(0,1)=(-1,0)$





Reflection about the x-axis does not change (-1,0).

Thus its matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

b) $\dim(\mathbf{R}^4) = \dim(\ker T) + \dim(RngT)$ and $\dim(\mathbf{R}^4) = 4,\ \dim(\ker T) = 1 \to \dim(RngT) = 3$. Since $\dim(\mathbf{R}^3) = \dim(RngT)$, T is onto.

56)

a) $\begin{bmatrix} 1 & 0 & x \\ 0 & 2 & y \\ -3 & -4 & z \end{bmatrix} \to$ REF $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & \frac{y}{2} \\ 0 & 0 & 3x+2y+z \end{bmatrix} \to 3x+2y+z=0$ when the system is consistent.

b) Make sure (2,1,z) is not in the plane $3x+2y+z=0$. So $3(2)+2(1)+z=0 \to z=-8$. Thus you must avoid $z=-8$.

57)

a) Recall that Span of a set is all the linear combinations of the vectors in the set.

Span $\{v_1, v_1+v_2\} = c_1v_1 + c_2(v_1+v_2)$ for all $c_1, c_2 \in \mathbf{R}$. But $c_1v_1 + c_2(v_1+v_2) = (c_1+c_2)v_1 + c_2v_2$ Since c_1+c_2 can also be any number, $c_1v_1 + c_2(v_1+v_2) = (c_1+c_2)v_1 + c_2v_2$ =span$\{\{v_1, v_2\}$

b) $T_2 \circ T_1(v_1+v_2) = T_2(T_1(v_1+v_2)) = T_2(T_1(v_1)+T_1(v_2)) = T_2(T_1(v_1)) + T_2(T_1(v_2)) = T_2 \circ T_1(v_1) + T_2 \circ T_1(v_2)$

And $T_2 \circ T_1(cv) = T_2(T_1(cv)) = T_2(cT_1(v)) = cT_2(T_1v) = cT_2 \circ T_1(v)$ Thus $T_2 \circ T_1$ is a LT.

c) Suppose $c_1T(v_1) + c_2T(v_2) = 0$. NTS $c_1 = c_2 = 0$. Since T is a LT, $c_1T(v_1) + c_2T(v_2) = 0 \to T(c_1v_1 + c_2v_2) = 0$. Since T is 1-1, its kernel is trivial. Thus $c_1v_1 + c_2v_2 = 0$. But this implies $c_1 = c_2 = 0$ since $\{v_1, v_2\}$ is LI. Therefore $\{T(v_1), T(v_2)\}$ is LI.

58) A) A set is LD if there is a relationship among its vectors. It is not necessary that all vectors in the set are used in a relationship. So if $\{v_1, v_2\}$ is LD, there is a relationship using $\{v_1, v_2\}$. This relationship still holds in $\{v_1, v_2, v_3\}$: simply do not use v_3

c) $\{(1,0), (0,1), (0,2)\}$ is LD in $\mathbf{R}^2$ but $\{(1,0), (0,1)\}$ is LI in $\mathbf{R}^2$,

59)

a)

$$((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1+x_2, 2y_1+2y_2) + (x_3, y_3) = (x_1+x_2+x_3, 4y_1+4y_2+2y_3)$$

but

$$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) = (x_1, y_1) + (x_2+x_3, 2y_2+2y_3) = (x_1+x_2+x_3, 2y_1+4y_2+4y_3)$$

Thus + is not associative.

b) Looking for (a,b) such that $(x,y)+(a,b)=(x,y)$ and $(a,b)+(x,y)=(x,y)$ for all x and y.

$(x,y)+(a,b)=(x,y)\to(x+a,2y+2b)=(x,y)\to x+a=x,2y+2b=y\to a=0,b=-\frac{y}{2}$

b depends on y. Thus there is no zero vector that works for all x and y.

c) $a(x,y)=(a^2x,a^2x)\to r(s(x,y))=r(s^2x,s^2y)=(r^2s^2x,r^2s^2y)$

and

$(rs)(x,y)=((rs)^2x,(rs)^2y)=(r^2s^2x,r^2s^2y)$

Yes, $r(s\mathbf{v})=(rs)\mathbf{v}$ hold for $r.s\in\mathbb{R}\ \ \mathbf{v}\in\mathbb{R}^2$?

60)

a) Recall that matrix A is orthogonal if and only if $A^TA=AA^T=I_n$

$$A=\begin{bmatrix}\frac{3}{7} & \frac{2}{7} & \frac{6}{7}\\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7}\end{bmatrix}\to A^T=\begin{bmatrix}\frac{3}{7} & -\frac{6}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7}\end{bmatrix}.$$ Now verify that $AA^T=I_3$

b) The inverse of an orthogonal matrix is its transpose. Thus $A^{-1}=\begin{bmatrix}\frac{3}{7} & -\frac{6}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7}\end{bmatrix}$

61)

a) $T(x_1,x_2)$ is determined by $T(1,0)$ and $T(0,1)$, since $T(x_1,x_2)=x_1T(1,0)+x_2T(0,1)$.
$T(2,3)=(1,3)\to 2T(1,0)+3T(0,1)=(1,3)$ and $T(1,-3)=(-4,3)\to T(1,0)-3T(0,1)=(-4,3)$. Now we have two equations and two unknowns. We can use the elimination method (Math 50) to solve for $T(1,0)$ and $T(0,1)$. We obtain $3T(1,0)=(-3,6)$ by adding these equations. Thus $T(1,0)=(-1,2)$. Using $T(1,0)-3T(0,1)=(-4,3)$, $3T(0,1)=T(1,0)-(-4,3)=(-1,2)-(-4,3)=(3,-1)\to T(0,1)=(1,-\frac{1}{3})$. Thus

$$T(x_1,x_2)=x_1T(1,0)+x_2T(0,1)=x_1(-1,2)+x_2(1,-\frac{1}{3})=(-x_1+x_2,2x_1-\frac{1}{3}x_2)$$

b) Compute $[T]_B^C=[T((1,0)),T((0,1))]$. $T(1,0)=(-1,2),T(0,1)=(1,-\frac{1}{3})\to[T]_B^C=\begin{bmatrix}-1 & 1\\ 2 & -\frac{1}{3}\end{bmatrix}$ Use the adjoint formula to find the inverse: $[T]_B^C=\begin{bmatrix}\frac{1}{5} & \frac{3}{5}\\ \frac{6}{5} & \frac{3}{5}\end{bmatrix}\to[T^{-1}]_C^B=([T]_B^C)^{-1}=\begin{bmatrix}\frac{1}{5} & \frac{3}{5}\\ \frac{6}{5} & \frac{3}{5}\end{bmatrix}$.

$$\rightarrow [T^{-1}(x_1,x_2)]_C^B = ([T]_B^C)^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{6}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{5}x_1+\frac{3}{5}x_2 \\ \frac{6}{5}x_1+\frac{3}{5}x_2 \end{bmatrix}$$

Therefore $T^{-1}(x_1,x_2)=(\frac{1}{5}x_1+\frac{3}{5}x_2, -\frac{6}{5}x_1+\frac{3}{5}x_2)$

62)

a) Let B=C= $\{1,x,x^2\}$ is an ordered basis for $\mathbf{P}_2$ (The basis can be ordered in any way, for example, it can be ordered as $\{x^2,x,1\}$, as long as you are consistent with the order). Compute $[T]_B^C=[[T(1)]_C,[T(x)]_C,[T(x^2)]_C]$.
$T(1)=5-x$ (use a=1,b=0,c=0). Thus the first column is 5, -1, 0. $T(x)=4x+x^2$ (use a=0 b=1 c=0), making the second column 0,4,1. $T(x^2)=2x^2$ (use a=0 b=0 c=1), making the third column 0,0,2., Therefore, the matrix with respect to the standard basis is $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

b) $[T(4x+3x^2)]_C \rightarrow \begin{bmatrix} 5 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \\ 10 \end{bmatrix} \rightarrow T(4x+3x^2)=16x+10x^2$

c) Yes, it is 1-1 since the matrix found in a) is invertible since the determinant is 40, a nonzero number. (the determinant is the product of its diagonal entries since it is lower triangular).

$[T^{-1}]_C^B = ([T]_B^C)^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{20} & \frac{1}{4} & 0 \\ -\frac{1}{40} & -\frac{1}{8} & \frac{1}{2} \end{bmatrix} \rightarrow [T^{-1}(\mathbf{v})]_B = [T^{-1}]_C^B[\mathbf{v}]$, so

$$[T^{-1}(a+bx+cx^2)]_B \rightarrow \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{20} & \frac{1}{4} & 0 \\ -\frac{1}{40} & -\frac{1}{8} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5}a \\ \frac{1}{20}a+\frac{1}{4}c \\ -\frac{1}{40}a-\frac{1}{8}b+\frac{1}{2}c \end{bmatrix}$$

$$\rightarrow T^{-1}(a+bx+cx^2)] = \frac{1}{5}a+(\frac{1}{20}a+\frac{1}{4}c)x+(-\frac{1}{40}a-\frac{1}{8}b+\frac{1}{2}c)x^2$$

63)

a) $c_1(1,3,1)+c_2(0,1,0)=(0,0,0)$ Show this has only the trivial solution. Write in the matrix form to obtain REF:

$\begin{bmatrix}1&0&0\\3&1&0\\1&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$. Since rank A =2, it has only the trivial solution.. Thus it is LI.

b) $(1,0,0)$ is not in the span since any nonzero coefficients c_1, c_2,

$c_1(1,3,1)+c_2(0,1,0)=(1,0,0)\rightarrow c_1=1,\ 3c_1+c_2=0,\ \ c_1=0$. The first and the third equations are inconsistent. (geometrically, $(1,0,0)$ does not lie in the plane formed by $\{(1,3,1),(0,1,0)\}$.

c) To extend a basis of the subspace to a basis for $\mathbf{R}^3$, add any vector that is not in the span of $\{(1,3,1),(0,1,0)\}$. In b) it was shown that $(1,0,0)$ is not in the span of $\{(1,3,1),(0,1,0)\}$. Thus $\{(1,3,1),(0,1,0),(1,0,0)\}$ is a basis for $\mathbf{R}^3$

64)

a) $T:\mathbf{R}\rightarrow SKSYM$ as $T(a)=\begin{bmatrix}0&-a\\a&0\end{bmatrix}$

b) $T:\mathbf{R}^2\rightarrow\mathbf{P}_1$ as $T(a,b)=a+bx$

65)

a) Since the dimension of the kernel is 0, the dimension of the range is 2 since $2=\dim(Ker)+\dim(range)$. Thus T is onto. Since it is given that T is 1-1, T is a bijection. Therefore, T^{-1} exists. (recall that T is invertible iff it is a bijection)

b) Pick B=C={1,x}. $T(ax+b)=(a+b)+(a-b)x\rightarrow$ in a matrix form , since $T(1)=1-x, T(x)=1+x$

$[T]_B^C=[[T(1)]_B,[T(x)]_B]=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ Then $[T^{-1}]_C^B=[T]_B^{C-1}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$ Thus

$$T^{-1}(1)=\frac{1}{2}+\frac{1}{2}x,\ T^{-1}(x)=-\frac{1}{2}+\frac{1}{2}x\rightarrow T^{-1}(a+bx)$$
$$=aT^{-1}(1)+bT^{-1}(x)=a(\frac{1}{2}+\frac{1}{2}x)+b(-\frac{1}{2}+\frac{1}{2}x)=\frac{1}{2}(a-b)+\frac{1}{2}(a+b)x$$

Or:

$$[T^{-1}(a+bx)]_C^B=[T^{-1}]_C^B\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}a-\frac{1}{2}b\\\frac{1}{2}a+\frac{1}{2}b\end{bmatrix}$$

Since C={1,x}, $[T^{-1}(a+bx)]_C^B=\begin{bmatrix}\frac{1}{2}a-\frac{1}{2}b\\\frac{1}{2}a+\frac{1}{2}b\end{bmatrix}\rightarrow T(a+bx)=(\frac{1}{2}a-\frac{1}{2}b)+(\frac{1}{2}a+\frac{1}{2}b)x$

66)

Since the matrix is symmetric, it can be diagonalized using an orthogonal matrix.

a) It is easy to see that REF of $\lambda=2$ is $\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}$. Thus its eigenvectors are $\begin{bmatrix}-s-t\\s\\t\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+(t)\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Now apply Gram Schdmit: $v_1 = x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = x_2 - \frac{< x_2, v_1 >}{\|v_1\|^2} v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Divide each one by its magnitude we obtain $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$

Next, for $\lambda = 8$, REF is $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ Thus $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector and we obtain $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ after dividing by its magnitude. Thus

P= $\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and D= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

67)) Apply Gram Schmidt:

$v_1 = x_1 = 1$

$v_2 = x_2 - \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1$ and $< x_2, v_1 > = \int_0^1 1(x)dx = \frac{1}{2}$, $\|v_1\|^2 = < v_1, v_1 > = \int_0^1 1dx = 1$. Thus $v_2 = x - \frac{1}{2}$

$v_3 = x_3 - \frac{< x_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_3, v_2 >}{< v_2, v_2 >} v_2$ and $< x_3, v_1 >= \int_0^1 1(x)^2 dx = \frac{1}{3},\ < v_1, v_1 >= \int_0^1 1dx = 1$.

$< x_3, v_2 >= \int_0^1 x^2 (x - \frac{1}{2}) dx = \frac{1}{12},\ < v_2, v_2 >= \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$ Thus

$$v_3 = x^2 - \frac{1}{3}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

Thus $\{\mathbf{v}_1 = 1, \mathbf{v}_2 = x - \frac{1}{2}, \mathbf{v}_3 = x^2 - x + \frac{1}{6}\}$ is an orthogonal basis. We have computed

$\|v_1\| = 1,\ \|v_2\| = \sqrt{12} = 2\sqrt{3}$

Then use $\frac{v}{\|v\|}$ to find an orthogonal basis . It maybe helpful to use the formula

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$\|v_3\|^2 = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx = \int_0^1 [x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{1}{3}x^2 - \frac{1}{3}x] dx =$

$\frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} + \frac{1}{9} - \frac{1}{6} = \frac{36 + 60 + 5 - 90 + 20 - 30}{180} = \frac{1}{180}\ \|v_3\| \to \sqrt{180} = 6\sqrt{5}$

Thus

$\{\mathbf{u}_1 = 1, \mathbf{u}_2 = 2\sqrt{3}(x - \frac{1}{2}), \mathbf{u}_3 = 6\sqrt{5}(x^2 - x + \frac{1}{6})\}$ is an orthonormal basis for the subspace spanned by $\{1, x, x^2\}$

b) Using $f =< f, u_1 > u_1 + < f, u_2 > u_2 + < f, u_3 > u_3 =$

$< f, \mathbf{u}_1 >= \int_0^1 x^2 dx = \frac{1}{3},$

$< f, \mathbf{u}_2 >= 2\sqrt{3}\int_0^1 x^2 (x - \frac{1}{2}) dx = \frac{\sqrt{3}}{6}, < f, \mathbf{u}_3 >= 6\sqrt{5}\int_0^1 x^2 (x^2 - x + \frac{1}{6}) dx = 6\sqrt{5}[\frac{1}{5} - \frac{1}{4} + \frac{1}{18}] = \frac{6\sqrt{5}}{180}$

$x^2 = \frac{1}{3} + \frac{\sqrt{3}}{6} 2\sqrt{3}(x - \frac{1}{2}) + \frac{6\sqrt{5}}{180} 6\sqrt{5}(x^2 - x + \frac{1}{6})$ (check!)

68) First find

$$[T]_B^C = [[T(1)]_C, [T(x)]_C, [T(x^2)]_C] = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Thus $[T(x)]_C = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$**. SInce** $C = \{3, 1+4x, 5+2x^2\}, T(x) = 3(3) + 0(1+4x) + 3(5+2x^2) = 24 + 6x^2$

c) $T(a+bx+cx^2)$

Use $[T(\mathbf{v})]_C = [T]_B^C[\mathbf{v}]_B$.

Since $B=\{1,x,x^2\},[a+bx+cx^2]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$$[T(\mathbf{v})]_C = [T]_B^C[\mathbf{v}]_B \to [T(a+bx+cx^2)]_C = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b+c \\ a+2c \\ a+3b+c \end{bmatrix}$$

Using $C=\{3,1+4x,5+2x^2\}, T(a+bx+cx^2) = 3(2a+3b+c)+(a+2c)(1+4x)+(a+3b+c)(5+2x^2)$
$=(12a+24b+10c)+(4a+8c)x+(2a+6b+2c)x^2$

d) A) Since T is 1-1, dim(ker)=0. Thus dimV=dim(ker)+dim(range)=dim(range). But dim(V)<dim(W). Thus dim(range)<dim(W). Therefore T is not onto.

b)Since T is onto, dimW=dim(range). In addition, dimV=dimW. Thus dimV=dim(ker)+dim(range) becomes dimW=dim(kerT) +dimW. Thus KerT={0}. Therefore T is 1-1

e)

Construct an isomorphism as follows:

a) LT: $T((a_1,b_1,c_1)+(a_2,b_2,c_2)) = T((a_1+a_2,b_1+b_2,c_1+c_2)) = \begin{bmatrix} a_1+a_2 & c_1+c_2 \\ c_1+c_2 & b_1+b_2 \end{bmatrix}$

$$= \begin{bmatrix} a_1 & c_1 \\ c_1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & c_2 \\ c_2 & b_2 \end{bmatrix} = T((a_1,b_1,c_1)) + T((a_2,b_2,c_2))$$

And

$$T(k(a,b,c)) = T((ka,kb,kc)) = \begin{bmatrix} ka & kc \\ kc & kb \end{bmatrix} = k\begin{bmatrix} a & c \\ c & b \end{bmatrix} = kT((a,b))$$

b) 1-1. Show the kernel is trivial: Suppose $T(a,b,c) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to (a,b,c)=(0,0,0)$. Thus it is 1-1.

c) The dim(ker)=0, n=3. Thus the dimension of the range is 3 by the rank-nullity theorem, which is the dimension of the 2x2 symmetric matrices, it is onto.

f)

a) Method 1: Separate the variables in $T(x_1, x_2)$

$T(x_1, x_2) = (x_1 - x_2, x_1 + x_2) = x_1(1,1) + x_2(-1,1)$. This expression shows the range is spanned by $\{(1,1), (-1,1)\}$. Since, $\{(1,1), (-1,1)\}$ is LI, it is basis.

Method 2: Find a matrix for T. Then the range is the column space of the matrix. Pick B=C={(1,0),(0,1)}. T(1,0)=(1,1), T(0,1)=(0,-1)

$\rightarrow [T]_B^C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rightarrow \text{RREF=}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. **Going back to the original matrix,** $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ **is basis for the column space. Going back to** $\mathbb{R}^2$ **{(1,1),(-1,1)} is a basis for the range.**

b) Place the vectors into the columns of a matrix and compute its determinant: $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$. Thus it is a basis.

c) First find c_1, c_2 such that $T(1,0) = c_1(1,-1) + c_2(0,1)$ and place it in the first column. Next find c_1, c_2 such that $T(1,1) = d_1(1,-1) + d_2(0,1)$ and place it in the second column.

$T(1,0) = (1-0, 1+0) = (1,1) = c_1(1,-1) + c_2(0,1) \rightarrow c_1 = 1, -c_1 + c_2 = 1 \rightarrow c_1 = 1, c_2 = 2$. Thus the first column is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Next $T(1,1) = (0,2) = d_1(1,-1) + d_2(0,1) \rightarrow d_1 = 0, -d_1 + d_2 = 2 \rightarrow d_1 = 0, d_2 = 2$

Thus the matrix is $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

d)

First express (2,3) in terms of B:

$$(2,3) = c_1(1,0) + c_2(1,1) \rightarrow (2,3) = (c_1 + c_2, c_3) \rightarrow c_1 + c_2 = 2, c_2 = 3 \rightarrow c_1 = -1, c_2 = 3$$

So $(2,3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_B$. $[T(2,3)]_C = [T]_B^C[\mathbf{v}]_B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Since C={(1,-1),(0,1)}, $[T(2,3)]_C = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \rightarrow -1(1,-1) + 4(0,1) = (-1,5)$

g) Since the matrix is symmetric, the matrix is diagonalizable using an orthogonal matrix. Its eigenvectors are

$\lambda = 3: \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ $\lambda = -6: \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$,. Next use Gram-Schdmit for the eigenvectors for $\lambda = 3$.

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - (-\frac{3}{5})\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{8}{5} \end{bmatrix}.$$

Now divide each vector by its magnitude, we have an orthonormal basis $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

S is obtained by placing these vectors into columns, D by placing eigenvalues on the diagonal,

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix},\ S = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{1}{3} \end{bmatrix}$$

h)

a) Show LT: $T((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) = T((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) =$
$(a_1 + a_2 - (c_1 + c_2)) + ((b_1 + b_2) - (c_1 + c_2))x + (a_1 + a_2)x^2 = (a_1 - c_1) + (b_1 - c_1)x + a_1x^2$
$+(a_2 - c_2) + (b_2 - c_2)x + a_2x^2 = T(a_1 + b_1x + c_1x^2) + T(a_2 + b_2x + c_2x^2)$

and

$$T(k(a + bx + cx^2)) = (k(a - c) + (b - c)x + ax^2) = k\ T(a + bx + cx^2)$$

b) To find a basis for the range, $T(a + bx + cx^2) = (a - c) + (b - c)x + ax^2 = a(1 + x^2) + b(x) + c(-1 - x)$. Thus the polynomials $\{1 + x^2, x, -1 - x\}$ spans the range. Find a LI subset of $\{1 + x^2, x, -1 - x\}$.
$c_1(1 + x^2) + c_2x, + c_3(-1 - x) = \mathbf{0} \to c_1 = c_2 = c_3 = 0$. Thus $\{1 + x^2, x, -1 - x\}$ is a basis for the range.

Or alternatively, use a matrix to find a basis for the range:

$T(a+bx+cx^2)=(a-c)+(b-c)x+ax^2$
$\to T(1)=1+x^2$
$T(x)=x$
$T(x^2)=-1-x$

Using bases $B=\{1,x,x^2\},C=\{1,x,x^2\}$, $[T]_B^C=\begin{bmatrix}1&0&-1\\0&1&-1\\1&0&0\end{bmatrix}\to \text{RREF}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

In the matrix form, the range is the column space. Since there are three leading 1s, its columns are LI.
$\{\begin{bmatrix}1\\0\\1\end{bmatrix}_C,\begin{bmatrix}0\\1\\0\end{bmatrix}_C,\begin{bmatrix}-1\\-1\\0\end{bmatrix}_C\}$ is basis for the range. Going back to the polynomials, $\{1+x^2,x,-1-x\}$ is a basis for the range.

i)

Show Recall that a LT is 1-1 iff Ker T={0}

Suppose $T_2(T_1(\mathbf{v}))=0$. (need to show $\mathbf{v}=0$)

Since T_2 is 1-1, and $T_2(T_1(\mathbf{v}))=0$, $T_1(\mathbf{v})=0$ (since $T_1(\mathbf{v})$ is in the kernel of T_2). Now using 1-1 ness of T_1, $T_1(\mathbf{v})=0\to\mathbf{v}=0$

j) First find the eigenvalues: $\det(A-\lambda I)=(-2-\lambda)(\lambda+2)(\lambda+3)=0\to\lambda=-2,3$

For $\lambda=-2, A+2I=\begin{bmatrix}1&2&0\\2&4&0\\0&0&0\end{bmatrix}\to\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}$ are eigenvectors. Normally, you would perform Gram –Schdmit on these vectors, but here they are orthogonal already. Simply divide by their magnitudes to get $\to\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-2/\sqrt{5}\\0\\1/\sqrt{5}\end{bmatrix}$. For

$\lambda = 3, A-3I = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$. Then divide by its magnitude to get $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$. Thus $S =$

$$\begin{bmatrix} 0 & -2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 2/\sqrt{5} \\ 0 & 1/\sqrt{5} & 0 \end{bmatrix}$$

k) Define $T:\mathbf{R}^2 \to diag$ as follows: $T(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (from a typical two dimensional vector to a typical 2 x 2 diagonal matrix)

a) $T((a_1,b_1)+(a_2,b_2)) = T((a_1+a_2,+b_2) = \begin{bmatrix} a_1+a_2 & 0 \\ 0 & b_1+b_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$

$T((a_1,b_1)) + T((a_2,b_2))$

b) $T(k(a\ ,b\ )) = T((ka_1+kb) = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} = k\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = kT(a,b)$

1) Show 1-1

Show the kernel is trivial. Suppose $T(a,b) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $T(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to a = b = 0$

2) Show T is onto: by the rank-nullity theorem, $\dim(\mathbf{R}^2) = \dim(ketT) + \dim(rngT)$. But the dim(kerT)=0 by 2). Thus the dimension of the range is 2. But the dim of the codomain is also 2. Thus T is onto since the range and the codomain have the same dimension, it is onto.

77) $T(2,0)=2+2x=c_1(1+x)+c_2(2x)\to c_1=2, c_2=0$ (the first column)

$T(0,3)=3+6x=c_1(1+x)+c_2(2x)\to c_1=3, c_2=\frac{3}{2}$ (the second column)

$T_B^C=\begin{pmatrix}2 & 3\\ 0 & \frac{3}{2}\end{pmatrix}$ Use the formula $\mathbf{w}=T\mathbf{v}\leftrightarrow[\mathbf{w}]_C=[T]_B^C[\mathbf{v}]_B$ to find T(4, 3): first express $(4,3)$ using B:

$(4,3)=2(2,0)+1(0,3)\to(4,3)=\begin{bmatrix}2\\1\end{bmatrix}_B$

Then $T\Big|_B^C[\mathbf{v}]_B=\begin{bmatrix}2 & 3\\ 0 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}7\\ \frac{3}{2}\end{bmatrix}=[\mathbf{w}]_C$

Finally using the basis C, $\mathbf{w}=7(1+x)+\frac{3}{2}(2x)=7+10x$

78)

a) Recall that nondefective matrices can be diagonalized using eigenvectors. The columns of S are eigenvectors and the diagonal elements of D are the corresponding eigenvalues. Eigenvalues are 0, 2, 3 and their eigenvectors are

$\begin{bmatrix}1\\4\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\0\end{bmatrix},$

b) Det(A)=det(D)=(0)(2)(3)=0

79) No, it is not positive definite. For $f(x)=x,\ <f,f>=\int_{-1}^{0}x(x)(x)dx=-\frac{1}{4}$

80)

a. We know that {1,x} is a basis for P_1. Thus P_1 is a 2 dimensional vector space. Since {3x, -1+3x} is a set with 2 vectors, it suffice to show the set is LI. But the set is LI since one is not a scalar multiple of the other. Therefore it is a basis.

b. Express 3x and -1+3x in terms of {x,1+ x}

Suppose $\begin{matrix}3x=c_1x+c_2(1+x)\\ -1+3x=d_1x+d_2(1+x)\end{matrix}$

This expression simplifies as $\begin{matrix}0+3x=c_2+(c_1+c_2)x\\ -1+3x=d_2+(d_1+d_2)x\end{matrix}$

By equating coefficients, $\begin{matrix}(c_1+c_2)=3, c_2=0\\ (d_1+d_2)=3, d_2=-1\end{matrix}$

Or using one big matrix, $\begin{matrix}3x=c_1x+c_2(1+x)\\ -1+3x=d_1x+d_2(1+x)\end{matrix}$

$$\to\left[\begin{array}{cc|cc}0 & 1 & 0 & -1\\ 1 & 1 & 3 & 3\end{array}\right]\to \text{RREF}=\left[\begin{array}{cc|cc}1 & 0 & 3 & 4\\ 0 & 1 & 0 & -1\end{array}\right]$$

So the transition matrix is $\begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}$

c. $P_{C\to B} = (P_{B\to C})^{-1} = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} \\ 0 & -1 \end{bmatrix}$)

d. Using {3x,-1+3x}, $-1+6x = c_1(3x)+c_2(-1+3x)$. Using common sense, $c_1 = c_2 = 1$. (or construct a matrix from $-1+6x = c_1(3x)+c_2(-1+3x)$ as $\left[\begin{array}{cc:c} 0 & -1 & -1 \\ 3 & 3 & 6 \end{array}\right] \to \text{RREF} = \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

So $-1+6x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B$

e. $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_C = [P]_B^C \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$. So $-1+6x = \begin{bmatrix} 7 \\ -1 \end{bmatrix}_C$ (we found $-1+6x = 7(x)+(-1)(1+x)$).

81)

4 points each) Define operations $+,\cdot$ on $\mathbf{R}^2$ as follows:

$$(x_1, y_1)+(x_2, y_2) = (x_1^2 x_2^2, y_1 + y_2)$$
$$r(x, y) = (x+r, y+r)$$

a) Is $+$ associative?

$$((x_1, y_1)+(x_2, y_2))+(x_3, y_3) = (x_1^2 x_2^2, y_1+y_2)+(x_3, y_3) = ((x_1^2 x_2^2)^2 x_3^2, y_1+y_2+y_3)$$
$$\text{but}$$
$$(x_1, y_1)+((x_2, y_2)+(x_3, y_3)) = (x_1, y_1)+(x_2^2 x_3^2, y_2+y_3) = (x_1^2(x_2^2 x_3^2), y_1+y_2+y_3)$$

Thus the + operation is not associative.

b) Is $r(s\mathbf{v}) = (rs)\mathbf{v}$ for $r.s \in \mathbf{R}, v \in \mathbf{R}^2$?

$$r(s\mathbf{v}) = r(s(x, y)) = r(x+s, y+s) = (x+s+r, y+s+r)$$

But

$$(rs)\mathbf{v}) = rs(x, y)) = (x+rs, y+rs)$$

Thus $r(s\mathbf{v}) = (rs)\mathbf{v}$ does not hold.

Is there a zero vector?

The zero vector is (a,b) such that (x,y)+(a,b)=(x,y), (a,b)+(x,y)=(x,y) for all x and y.

$$(x, y)+(a,b) = (x, y) \to (x^2a^2, y+b) = (x, y) \to x^2a^2 = x, y+b = y \to a = \frac{1}{\sqrt{x}}, b = 0 \text{ (if x is positive)}$$

But (a,b) must work for all x and y, but here a depends on x. So there is no zero vector.

82) (2 points each)

a) True/False Let V be a vector space. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V and if $\mathbf{w} \neq \mathbf{v}_2$, then $\{\mathbf{w}, \mathbf{v}_2\}$ is also a basis for V.

False: if w=3v₂, {3v₂,v₂} is LD.

b) True/False Let V be a vector space. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans V then it is impossible that $\{\mathbf{v}_1\}$ spans V.

False : if $\{v_1,v_2\}$ is LD, then Span $\{v_1,v_2\}$ may be Span$\{v_1\}$

c) True/False If T is a linear transformation, then T(1)=1

False: $T:\mathbf{R} \to \mathbf{R}$ defined by T(v)=3v is a LT and T(1)=3.

d) Find a matrix of a LT $T:\mathbf{R}^2 \to \mathbf{R}^2$ that send (x,y) to (-x,y). (reflection about the y-axis)

Find the image of the standard basis vectors:

T(1,0)=(-1,0), T(0,1)=(0,1), so using the standard bases, the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

83)

Method1:

find the matrix of T with respect to the standard ordered base B={1,x,x²}, $C=\{1, x, x^2, x^3\}$

$T(a+bx+cx^2)=(a-b)+(a+b+2c)x+(a+2c)x^2+(3a+b+4x)x^3$, so

$$T(1)=a=1,b=0,c=0 \to 1+x+x^2+3x^2 \to T[(1)]_c=\begin{bmatrix}1\\1\\1\\3\end{bmatrix}, T(x)=a=0,b=1,c=0 \to -1+x+x^3,$$

$$\to T[(x)]_c=\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}, T(x^2)=a=0,b=0,c=1 \to \begin{bmatrix}0 & 2\\2 & 4\end{bmatrix} 2x+2x^2+4x^3 \to T[(x^2)]_c=\begin{bmatrix}0\\2\\2\\4\end{bmatrix},$$

So $[T]_B^C=\begin{bmatrix}1 & -1 & 0\\1 & 1 & 2\\1 & 0 & 2\\3 & 1 & 4\end{bmatrix}$

Use TI to compute RREF (add the zero vector to the last column to do the operation in TII 84)

RREF is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$

Since the null space is trivial, the kerT={0}.

A basis for the column space is $[T]_B^C\{\begin{bmatrix}1\\1\\1\\3\end{bmatrix}_C, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}_C, \begin{bmatrix}0\\2\\2\\4\end{bmatrix}_C\}$ Going back to P_3

$\{1+x+x^2+3x^3, -1+x+x^3, 2x+2x^2+4x^3\}$ is a basis for the range of T. ,

Alternatively, using definition,

$$KerT = \{(a,b,c): T(a+bx+cx^2) = (a-b)+(a+b+2c)x+(a+2c)x^2+(3a+b++4c)x^3 = 0+x+x^2+x^3$$

From here, we get $\begin{matrix} a-b=0 \\ a+b+2c=0 \\ a+2c=0 \\ 3a+b+4c=0 \end{matrix}$, which is the same as

$$[T]_B^C = \begin{bmatrix}1 & -1 & 0\\1 & 1 & 2\\1 & 0 & 2\\3 & 1 & 4\end{bmatrix} \to \text{RREF=}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} \to a=0, b=0, c=0.$$

Thus Ker T={**0**}

For a basis for the range, first find the spanning set for the range as

$$T(a+bx+cx^2) = (a-b)+(a+b+2c)x+(a+2c)x^2+(3a+b++4c)x^3 =$$
$$a(1+x+x^2+3x^3)+b(-1+x+x^3)+c(2x+2x^2+4x^3)$$

Then find range of T is $\text{Span}\{1+x+x^2+3x^3, -1+x+x^3, 2x+2x^2+4x^3\}$

To find a basis for the range, first find a dependency relation and drop a vector with nonzero coefficient. It can be shown that (this was covered in Exam 2) $\{1+x+x^2+3x^3, -1+x+x^3, 2x+2x^2+4x^3\}$ is LI. Therefore $\{1+x+x^2+3x^3, -1+x+x^3, 2x+2x^2+4x^3\}$ is a basis for the range of T.

84)

Let $T: P_1 \to P_1$ be defined by $T(ax+b) = (a+b)x + (2a-b)$

a) Show T is a bijection. You may assume T is a LT

You only need to show T is 1-1. Then by the rank nullity theorem, Rank=2=0=2. This would imply T is onto since the dimension of the range = the dimension of the codomain.

To show T is 1-1, $T(ax+b) = (a+b)x+(2a-b) = (0,0) \to a+b=0, 2a-b=0 \to a=b=0$

So T is 1-1.

b) Find the inverse of T.

First find the matrix of T with B=C={1,x}

$T(1) = -1 + x,\ T(x) = 2 + x$, so $[T] = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$. Then $[T]^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

So $T^{-1}(1) = \frac{1}{3}x - \frac{1}{3},\ T^{-1}(x) = \frac{1}{3}x + \frac{2}{3} \rightarrow T^{-1}(ax+b) = aT^{-1}(x) + bT^{-1}(1) = (\frac{1}{3}a + \frac{1}{3}b)x + (\frac{2}{3}a - \frac{1}{3}b)$

c) Find the matrix of T with respect to $B = \{2x, 4\}$ and $C = \{x+1, x-1\}$

First find $[T(2x)]_C$ and $[T(4)]_C$, that is, express T(2x) and T(4) in terms of $C = \{x+1, x-1\}$

$$T(2x) = 2x + 4 = c_1(x+1) + c_2(x-1) \rightarrow c_1 + c_2 = 2, c_1 - c_2 = 4 \rightarrow c_1 = 3, c_2 = -1$$
$$T(4) = 4x - 4 = d_1(x+1) + d_2(x-1) \rightarrow d_1 + d_2 = 4, d_1 - d_2 = -4 \rightarrow d_1 = 0, d_2 = 4$$

So $[T]\big|_B^C = \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$

d) Use the matrix found in a) to find $T(2x+8)$

We need to use $[T(\mathbf{v})]_C = [T]_B^C[\mathbf{v}]_B$

First find $[\mathbf{v}]_B$. We need to express 2x+8 in terms of $B = \{2x, 4\}$: $2x + 8 = c_1(2x) + c_2(4) \rightarrow c_1 = 1, c_2 = 2$

So $2x + 8 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_B$. $T(2x+8) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_C = [T(\mathbf{v})]_C = \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Finally, $[T(\mathbf{v})]_C = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \rightarrow T(2x+8) = 3(x+1) + 7(x-1) = 10x - 4$

85) (6 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbf{R}^3$. Show if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is LD, then $\{2\mathbf{v}_1, 3\mathbf{v}_2\}$ is LD.

Pf: Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is LD, $\mathbf{v}_1 = c\mathbf{v}_2$ for some constant c. Then $2\mathbf{v}_1 = \frac{2c}{3}(3\mathbf{v}_2)$. Since $2\mathbf{v}_1$ is a scalar multiple of $3\mathbf{v}_2$, $\{2\mathbf{v}_1, 3\mathbf{v}_2\}$ is LD.

86) Since A is similar to B, there is a nonsingular matrix S such that $A = S^{-1}BS$. Then $A^2 = A(A) = (S^{-1}BS)(S^{-1}BS) = S^{-1}B^2S$. Thus A^2 is similar to B^2.

87) First observe that since $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ are eigenvectors for the eigenvalue 4, $A\begin{bmatrix}1\\3\end{bmatrix}=4\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}4\\12\end{bmatrix}$ and

$A\begin{bmatrix}2\\1\end{bmatrix}=4\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}8\\4\end{bmatrix}$. Express $\begin{bmatrix}7\\6\end{bmatrix}$ as a linear combination of $c_1\begin{bmatrix}1\\3\end{bmatrix}+\ c_2\begin{bmatrix}2\\1\end{bmatrix}$. Using common sense, or solving

$\left[\begin{matrix}1 & 2\\3 & 1\end{matrix}\ \vdots\ \begin{matrix}7\\6\end{matrix}\right]$, $\begin{bmatrix}7\\6\end{bmatrix}=1\begin{bmatrix}1\\3\end{bmatrix}+\ 3\begin{bmatrix}2\\1\end{bmatrix}$

Then $A\begin{bmatrix}7\\6\end{bmatrix}=A(\begin{bmatrix}1\\3\end{bmatrix}+3\begin{bmatrix}2\\1\end{bmatrix})=A(\begin{bmatrix}1\\3\end{bmatrix})+3A\begin{bmatrix}2\\1\end{bmatrix}=$. Since, $A\begin{bmatrix}1\\3\end{bmatrix}=4\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}4\\12\end{bmatrix}$ and $A\begin{bmatrix}2\\1\end{bmatrix}=4\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}8\\4\end{bmatrix}$,

$$A\begin{bmatrix}7\\6\end{bmatrix}=A(\begin{bmatrix}1\\3\end{bmatrix}+3\begin{bmatrix}2\\1\end{bmatrix})=\begin{bmatrix}4\\12\end{bmatrix}+3\begin{bmatrix}8\\4\end{bmatrix}=\begin{bmatrix}28\\24\end{bmatrix}$$

88)

a. For Gauss Jordan, obtain RREF

$$\left[\begin{matrix}1 & 2 & 3 & 4\\1 & 3 & 2 & 5\end{matrix}\ \vdots\ \begin{matrix}10\\8\end{matrix}\right]\rightarrow(R1(-1)+R2=new\,R2)=\left[\begin{matrix}1 & 2 & 3 & 4\\0 & 1 & -1 & 1\end{matrix}\ \vdots\ \begin{matrix}10\\-2\end{matrix}\right]$$

$$(R2(-1)+R1=new\,R1)=\left[\begin{matrix}1 & 0 & 5 & 2\\0 & 1 & -1 & 1\end{matrix}\ \vdots\ \begin{matrix}14\\-2\end{matrix}\right]$$

x_3 and x_4 are free variables, so $x_3=s$, $x_4=t$. Then $x_2=x_3-x_4-2$, $x_1=-5x_3-2x_4+14$

b. $\begin{matrix}x_1+2x_2=5\\x_1+3x_2=-5\end{matrix}$ ca be expressed as $\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5\\-5\end{bmatrix}$. Next find the inverse of $\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}$ by Gauss Jordan or adjoint. $\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}^{-1}=\begin{bmatrix}3 & -2\\-1 & 1\end{bmatrix}$. $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3 & -2\\-1 & 1\end{bmatrix}\begin{bmatrix}5\\-5\end{bmatrix}=\begin{bmatrix}25\\-10\end{bmatrix}$

89) Consider a set $A=\{(1,2,3),(1,0,1)\}$ in $\mathbf{R}^3$

a) A is LI since (1,2,3) is not a scalar multiply of (1,0,1).

b) $\mathbf{R}^3$ is. 3 dimensional, so add one vector

c) Pick any 3 vectors that are not in the span of A.

To find the span of A, $\left[\begin{matrix}1 & 1\\2 & 0\\3 & 1\end{matrix}\ \vdots\ \begin{matrix}x\\y\\z\end{matrix}\right]\rightarrow\left[\begin{matrix}1 & 1\\0 & -2\\0 & -2\end{matrix}\ \vdots\ \begin{matrix}x\\-2x+y\\-3x+z\end{matrix}\right]\rightarrow\left[\begin{matrix}1 & 1\\0 & 1\\0 & -2\end{matrix}\ \vdots\ \begin{matrix}x\\x-\frac{1}{2}y\\-3x+z\end{matrix}\right]\rightarrow\left[\begin{matrix}1 & 1\\0 & 1\\0 & 0\end{matrix}\ \vdots\ \begin{matrix}x\\x-\frac{1}{2}y\\-x-y+z\end{matrix}\right]$

For this to have a solution, -x-y+z=0, which is the span of A. So pick three points that are not on the plane –x-y+z=0. So add (0,0,1), (0,0,2), or (0,0,3) .

90) A) $A^3 = I_n \to A \cdot A^2 = A^2 \cdot A = I_n$. Since A^2 cancels A, $A^{-1} = A^2$ B) Since

$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = B(I_n)B^{-1} = BB^{-1} = I_n$, and

$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B = B^{-1}B = I_n$, $(AB)^{-1} = B^{-1}A^{-1}$

91)

a. Positive definiteness: $< v, v > = < (a,b),(a,b) > = a^2 + 3b^2 \geq 0$ and

$(a,b)\cdot(a,b) = a^2 + 3b^2 = 0 \; iff \;\; a = b = 0 \;\; iff \;\; (a,b) = (0,0)$

b. Symmetry: $< (a,b),(c,d) > = ac + 3bd = ca + 3db = < (c,d),(a,b) >$

c. Linearity:

$< k_1(a,b),(c,d)) > = < (ka,kb),(c,d) > = ka = kac + 3kcd = k(ac + 3bd) = k < (a,b),(c,d) >$

And
$< (a,b)+(c,d),(e,f) > = < (a+c,b+d),(e,f) > = (a+c)e + 3(b+d)f = (ae+3bf) + (ce+3df)$
$= < (a,b),(e,f) > + < (a,b),(e,f) >$

Therefore, it is an inner product.

$\|(2,4)\| = \sqrt{< (2,4),(2,4) >} = \sqrt{2^2 + 3(4)^2} = 2\sqrt{13}$

92) Since λ is an eigenvalue of a matrix A, there is a vector v such that $A\mathbf{v} = \lambda\mathbf{v}$. Then

$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$. Therefore λ^2 is an eigenvalue of a matrix A^2.

93) Apply Gram Schmidt:

$v_1 = x_1 = 1$

$v_2 = x_2 - \frac{< x_2, v_1 >}{\|v_1\|^2} v_1$ and $< x_2, v_1 > = \int_0^1 2(1)(2x)dx = 2,\; \|v_1\|^2 = < v_1, v_1 > = \int_0^1 2(1)dx = 2$. Thus

$v_2 = 2x - \frac{2}{2}(1) = 2x - 1$

In addition, $\|v_2\|^2 = < v_2, v_2 > = \int_0^1 2(2x-1)^2 dx = \frac{2}{3}$

Thus $\{\mathbf{v}_1 = 1, \mathbf{v}_2 = 2x - 1\}$ is an orthogonal basis. We have computed $\|v_1\| = \sqrt{2},\; \|v_2\| = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

Thus

$\{\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}, \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{3}{\sqrt{6}}(2x-1)\}$ is an orthonormal basis for the subspace spanned by $\{1, 2x\}$

Use $f = < f, u_1 > u_1 + < f, u_2 > u_2$:

Since $< f, \mathbf{u}_1 > = \int_0^1 2(4x+3)\frac{1}{\sqrt{2}}dx = 5\sqrt{2}$, $< f, \mathbf{u}_2 > = \int_0^1 2(\frac{3}{\sqrt{6}}(2x-1))(4x+3)dx = \frac{2\sqrt{6}}{3}$,

So $4x + 3 = 5(\sqrt{2}(\frac{1}{\sqrt{2}})) + \frac{2\sqrt{6}}{3}(\frac{3}{\sqrt{6}}(2x-1))$ (check!)

94) Since A is nondefective, there is a diagonal matrix D and a nonsingular matrix P such that $P^{-1}AP = D$. By taking the determinant of both sides, $\det(P^{-1}AP) = \det(D)$. But

$\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(P)^{-1}\det(A)\det(P) = \det(A)$. And since D is a diagonal matrix,

$\det(D)$ is the product of its diagonal entries, which are the eigenvalues of A. Thus $\det(A)$ is the product of its eigenvalues.

95) Since $A \sim B$, there is a nonsingular matrix S_1 such that $B = S_1^{-1}AS_1$. Since $B \sim C$, there is a nonsingular matrix S_2 such that $C = S_2^{-1}BS_2$. Since S_1 and S_2 are nonsingular, S_1S_2 is also nonsingular. Since $(S_1S_2)^{-1} = S_2^{-1}S_1^{-1}$, $C = S_2^{-1}BS_2 = C = S_2^{-1}S_1^{-1}AS_1S_2 = (S_1S_2)^{-1}A(S_1S_2)$. Thus $A \sim C$

96) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a LT given by $T(x,y) = (2x+y, x+y)$. Let $B = \{(3,4),(4,5)\}$ and $C = \{(2,-1),(1,-1)\}$ be bases for $\mathbb{R}^2$

a. Find $[T]_B^B$

Solution: $[T]_B^B = [[T(3,4)]_B, [T(4,5)]_B]$

$T(3,4) = (10,7) = c_1(3,4) + c_2(4,5)$

$T(4,5) = (13,9) = d_1(3,4) + d_2(4,5)$

Solving them simultaneously, $\begin{bmatrix} 3 & 4 & \vdots & 10 & 13 \\ 4 & 5 & \vdots & 7 & 9 \end{bmatrix} \to \text{RREF} \begin{bmatrix} 1 & 0 & \vdots & -22 & -29 \\ 0 & 1 & \vdots & 19 & 25 \end{bmatrix}$

Thus $[T]_B^B = \begin{bmatrix} -22 & -29 \\ 19 & 25 \end{bmatrix}$

b. Find $[P]_C^B$ the transition matrix from C to B.

Solution: $[P]_C^B = [[(2,-1)]_B, [(1,-1)]_b]$

$(2,-1) = c_1(3,4) + c_2(4,5)$

$(1,-1) = d_1(3,4) + d_2(4,5)$

Solving them simultaneously,

$[P]_C^B = \begin{bmatrix} -14 & -9 \\ 11 & 7 \end{bmatrix}$

c. Use similarity to find $[T]_C^C$.

$$[T]_C^C = ([P]_C^B)^{-1}[T]_B^B[P]_C^B = \begin{bmatrix} 4 & 1 \\ -5 & -1 \end{bmatrix}$$

97) A) Apply the subspace test. I) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ since $a_{12} = a_{21} = 0$ ii) let $\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} \in S$. Then $\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, + \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix} \in S$ since $a_{12} = a_{21}$ iii) If $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in S$, then $k\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ kb & kc \end{bmatrix} \in S$ since $a_{12} = a_{21}$. Therefore the set of 2x2 symmetric matrices are a subspace of $M_2(\mathbf{R})$

B) Claim: $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is a basis for the set of 2 x 2 symmetric matrices

i) Show LI: Suppose $c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}0&0\\0&1\end{bmatrix}+c_3\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\to\begin{bmatrix}c_1&c_3\\c_3&c_2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\to c_1=c_2=c_3=0$,

so LI

ii) Show spans: Pick a typical 2 x2 symmetric matrix $\begin{bmatrix}a&b\\b&c\end{bmatrix}$. Then $\begin{bmatrix}a&c\\c&b\end{bmatrix}=a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&0\\0&1\end{bmatrix}+c\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

Thus $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ spans 2 x 2 sym

Therefore $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is a basis for 2x2 symmetric matrices.

a. $M_2(\mathbf{R})$ is a 4 dimensional vector space. Since the set of 2x2 symmetric matrices is 3 dimensional, one more vector that is not in the set of 2x2 symmetric matrices needs to be added. Then the resulting set would still be LI, thus a basis since it contains 4 vectors. $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ is not symmetric, so

$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix}\right\}$ is LI, so is a basis for $M_2(\mathbf{R})$.