CSCI 190 Discrete Mathematics Applied to Computer Science
Exam 2

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Read these instructions before proceeding.

- Closed book. Closed notes. You can use calculator.
- You have **80 minutes** to complete this exam.
- No questions will be answered during the exam or immediately afterwards. Answer each question as best you can. Partial credit will be awarded for reasonable efforts. If a question contains an ambiguity or a misprint, then say so in your answer, providing the answer to a reasonable interpretation of the question; give your assumptions.
- Answer the problems on the blank spaces provided for each problem.
- **Box your answers**.

Q1 (8)	Q2 (8)	Q3 (8)	Q4 (8)	Q5 (6)	Q6 (12)	Q7 (10)	Q8 (8)	Q9 (8)	Q10 (8)	Q11 (8)	Q12 (8)	Total (100)

1. (8 pts)
Use the Principle of Mathematical Induction to prove that
$1 + 4 + 16 + 64 + \cdots + 4^n = (4^{n+1} - 1)/3$ for all $n \geq 0$.

$n = 0$

$4^0 = \frac{(4^{0+1} - 1)}{3}$

$1 = \frac{4-1}{3}$

$1 = 1$

for $f(0)$ is true

Assume that $f(k)$ is true,
show that $f(k+1)$ is also true.

$1 + 4 + 16 + 64 + \ldots 4^k + 4^{k+1} = \left(\frac{4^{k+1} - 1}{3}\right)$

$= (4^0 + 4^1 + 4^2 + 4^3 + \ldots + 4^k) + 4^{k+1}$

$= \frac{4^{k+1} - 1}{3} + 4^{k+1}$

$= \frac{4^{k+1} - 1}{3} + \frac{3(4^{k+1})}{3}$

$= \frac{4(4^{k+1}) - 1}{3} = \frac{4^{k+2} - 1}{3}$

$\boxed{\therefore f(n) \text{ is true for all } n \geq 0}$

2. (8 pts) **Give a recursive definition with initial condition(s).**

a) The function **f(n) = (2n)!, n = 0, 1, 2, ...** (4 pts)

$a_0 = 0! = 1$ $\qquad a_{n+1} = (n+1)! = (n+1)\, n!$

$a_1 = 1! = 1$

$a_2 = 2! = 2 \cdot 1 = 2$ $\qquad \boxed{\therefore\ a_{n+1} = (n+1)\, n! \ \text{ for } n \geq 0 \text{ and } a_0 = 1}$

$a_3 = 3! = 3 \cdot 2 = 6$

b) The Fibonacci numbers **1, 1, 2, 3, 5, 8, 13, ...** starting from f(0) (4 pts)

$a_0 = 1$ $\qquad a_{n+1} = a_n + a_{n-1}$

$a_1 = 1$

$a_2 = a_1 + a_0 = 1+1 = 2$ $\qquad \boxed{\therefore\ a_{n+1} = a_n + a_{n-1} \ \text{ for } n \geq 2 \text{ and } a_0 = 1 \text{ and } a_1 = 1}$

$a_3 = a_2 + a_1 = 2+1 = 3$

3. (8 pts)

a) **Find f(2) and f(3) if f(n) = 3f(n − 1) + 5, f(0) = 1**. (4 pts)

$f(0) = 1$

$f(1) = 3f(1-1)+5 = 3(1)+5 = 8$

$\boxed{\begin{array}{l} f(2) = 3f(2-1)+5 = 3(8)+5 = 29 \\ f(3) = 3f(3-1)+5 = 3(29)+5 = 92 \end{array}}$

b) **Find f(8) if f(n) = 2f(n/2) + 2, f(1) = 2.** (4 pts)

$f(1) = 2$

$f(2) = 2f(\frac{2}{2})+2 = 2(2)+2 = 6$

$f(4) = 2f(\frac{4}{2})+2 = 2(6)+2 = 14$

$\boxed{f(8) = 2f(\frac{8}{2})+2 = 2(14)+2 = 30}$

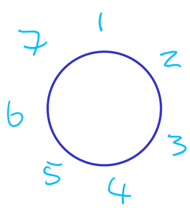
4 (8 pts)

a) **Consider a bit string of length 15. How many begin with 100 and end with 010?** (4 pts)

$1\,0\,0\,\underbrace{\_\,\_\,\_\,\_\,\_\,\_\,\_\,\_\,\_}_{9 \text{ bits}}\,0\,1\,0$ $\qquad \text{bit} = \{0, 1\}$

$\boxed{2^9 = 512}$

b) **How many ways are there to seat 7 people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?** (4 pts)



$(7-1)! = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

$= \boxed{720}$

5. (6 pts)
Explain how the Pigeonhole Principle can be used to show that among any 21 integers, at least three must have the same last digit.

$$\left\lceil \frac{21}{10} \right\rceil = \lceil 2.1 \rceil = \boxed{3}$$

6. (12 pts)

a) **How many ways are there to select 4 students from a class of 30 to serve on a committee?** (4 pts)

$$C(30,4) = \frac{30!}{4!\,26!} = \frac{\overset{15}{\cancel{30}} \cdot 29 \cdot \overset{7}{\cancel{28}} \cdot \overset{9}{\cancel{27}}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = \boxed{27405}$$

b) **How many ways are there to select 5 students from a class of 35 to hold six different executive positions on a committee?** (4 pts)

$$C(30,4) \cdot P(5,5) = \frac{30!}{4!\,26!} \cdot \frac{5!}{(5-5)!} = 27405 \cdot (5 \cdot 4 \cdot 3 \cdot 2)$$
$$= \boxed{3288600}$$

c) **How many bit strings of length 14 have equal numbers of 0's and 1's?** (4 pts)

$$C(14,7) \cdot C(7,7) = \frac{14!}{7!\,7!} \cdot \frac{7!}{7!\,0!} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$
$$= \boxed{3432}$$

7. (10 pts)

a) **Use the Pascal's Tringle to expand *(x + y)*** $(x+y)^5$. Show your work. (5 pts)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\boxed{x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5}$$

b) **Find the coefficient of x^4y^6 in the expansion of $(4x - 2y)^{10}$.** (5 pts)

$$\binom{10}{6}(4x)^{10-6}(2y)^6 = C(10,6)(4x)^4(2y)^6$$
$$= \frac{10!}{6!\,4!}(256x^4)(64y^6)$$
$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2}(16384)(x^4y^6)$$
$$= 210(16384)(x^4y^6) = \boxed{3440640\,x^4y^6}$$

8. (8 pts)

(a) **What is the probability that a card chosen from an ordinary deck of 52 cards is a Queen or Jack?.** (4 pts)

total = 52
queen = 4
jack = 4

$$p(\text{Queen or Jack}) = \frac{4+4}{52} = \frac{8}{52} = \frac{2}{13} = \boxed{0.1538}$$

(b) **What is the probability that a fair coin lands Heads exactly 3 times out of 4 flips?** (4 pts)

$$\frac{\binom{4}{3}}{2^4} = \frac{\left(\frac{4!}{3!\,1!}\right)}{16} = \frac{4}{16} = \frac{1}{4} = \boxed{0.25}$$

9. (8 pts) Suppose you have a class with 30 students — 10 freshmen, 12 sophomores, and 8 juniors.

a) **You pick three students at random, one at a time. What is the probability that all three are junior?** (4 pts)

total = 30

$$p(\text{all three are juniors}) = \frac{8}{30}\cdot\frac{7}{29}\cdot\frac{6}{28} = \boxed{0.0138}$$

b) **You pick two students at random, one at a time. What is the probability that the second student is a freshman, given that the first is a sophomore?** (4 pts)

$$p(\text{freshman} \mid \text{sophomore}) = \frac{p(\text{freshman} \cap \text{sophomore})}{p(\text{sophomore})} = \frac{\frac{10}{29}\cdot\cancel{\frac{12}{30}}}{\cancel{\frac{12}{30}}} = \frac{10}{29} = \boxed{0.3448}$$

10. (8 pts)

a) **In a certain lottery game you choose a set of six numbers out of 33 numbers. Find the probability that none of your numbers match the six winning numbers.** (4 pts)

$$\frac{\binom{27}{6}}{\binom{33}{6}} = \frac{C(27,6)}{C(33,6)} = \frac{\left(\frac{27!}{6!\,21!}\right)}{\left(\frac{33!}{6!\,27!}\right)} = \frac{\left(\frac{27\cdot26\cdot25\cdot24\cdot23\cdot22}{6\cdot5\cdot4\cdot3\cdot2}\right)}{\left(\frac{33\cdot32\cdot31\cdot30\cdot29\cdot28}{6\cdot5\cdot4\cdot3\cdot2}\right)} = \frac{296010}{1107568} = \boxed{0.2673}$$

b) **An experiment consists of picking at random a bit string of length five. Consider the following events:**

E_1: the bit string chosen begins with 0; 0 _ _ _ _ total = 2^5 = 32

E_2: the bit string chosen ends with 1. _ _ _ _ 1

Determine whether E_1 and E_2 are independent. Show your work. (4 pts)

$$P(E_1) = \frac{2^4}{2^5} = \frac{16}{32} = \frac{1}{2}$$

$$P(E_2) = \frac{2^4}{2^5} = \frac{16}{32} = \frac{1}{2}$$

$$P(E_1 \cap E_2) = \{0\,\_\,\_\,\_\,1\} = \frac{2^3}{2^5} = \frac{8}{32} = \boxed{\frac{1}{4}}$$

$$P(E_1 \cap E_2) = P(E_1)\cdot P(E_2) = \frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}}$$

$\therefore$ E_1 and E_2 are independent

11. (8 pts) Three coins are tossed.

a) **List the elements in the sample space.** (4 pts)

HHH HHT HTH THH
TTT TTH THT HTT

total $= 2^3 = 8$

b) **Find the probability that at least two heads show.** (4 pts)

$p(\text{at least two heads}) = \frac{4}{8} = \frac{1}{2} = \boxed{0.5}$

12. (8 pts)

Describe the following sequence recursively. Include initial condition of $\mathbf{a_1}$.

a_n **= the number of bit strings of length n with at least two consecutive zeros.**

$a_0 = 0$

a) $\mathbf{a_1}$ **=** 0 (2 pts)

b) $\mathbf{a_2}$ **=** $a_1 + a_0 + 2^0 = 0 + 0 + 1 = 1$ (2 pts)

c) $\mathbf{a_n}$ **=** $a_{n-1} + a_{n-2} + 2^{n-2}$ (4 pts)