7.1 Eigenvalues and Eigenvectors

Notation : λ = lambda

Definition : Let A be an $n \times n$ matrix

If $\exists\ \lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$, $v \neq \vec{0}$

λ may be 0

satisfying $Av = \lambda v$, then we say

λ is an eigenvalue of A and v is

an eigenvector of A corresponding to λ

P 356 # 4

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

verify $\lambda = 5$ is an eigenvalue of A and $v = (1, 2, -1)$

is a corresponding eigenvector

Need to verify

$$Av \stackrel{?}{=} \lambda v$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \stackrel{?}{=} (5) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

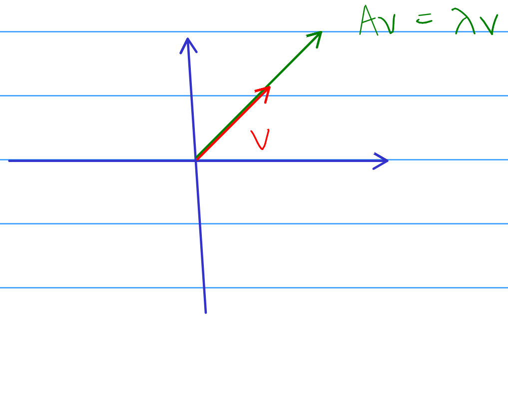
$$\begin{bmatrix} -2+4+3 \\ 2+1+7 \\ -1-4+0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$$

$\therefore$ Yes

In picture

$Av = \lambda v$



In CS: condition number

To find λ and v

$$Av = \lambda v$$
$$\lambda v - Av = \vec{0}$$
$$\lambda I_n v - Av = \vec{0}$$
$$(\lambda I_n - A)v = \vec{0}$$

This is a homogeneous system of linear equations

v is nonzero solution

Homogeneous: nontrivial solutions exist iff $\lambda I_n - A$ is not invertible

$\therefore \det(\lambda I_n - A) = 0$

$\det(\lambda I_n - A)$ is a polynomial of degree n

This polynomial is called the characteristic polynomial of A

Thus to find eigenvalues and their eigenvectors

1. Solve $\det(\lambda I_n - A) = 0$ to find eigenvalues
2. For each λ, find (LI) nontrivial solution to $(\lambda I_n - A)v = \vec{0}$

Definition: Let A be an $n \times n$ matrix

If λ is an eigenvalue to A:

1. the algebraic multiplicity of λ is the exponent of λ in $\det(\lambda I_n - A)$
2. the geometric multiplicity of λ is the number of LI eigenvectors corresponding to λ

Quiz
wed
7.1/7.2

② we say A is nondefective
if algebraic multiplicity = geometric multiplicity
for each eigenvalue λ of A
Otherwise A is defective

Theorem: Let A be an $n \times n$ matrix
A is nondefective iff A has n LI eigenvectors

eg. Let $A = \begin{bmatrix} 7 & -8 & 6 \\ 8 & -9 & 6 \\ 0 & 0 & -1 \end{bmatrix}$

Find eigenvalues of A and corresponding eigenvectors.
Determine if A is defective or nondefective

$\lambda I_n - A$

$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 7 & -8 & 6 \\ 8 & -9 & 6 \\ 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} \lambda-7 & 8 & -6 \\ -8 & \lambda+9 & -6 \\ 0 & 0 & \lambda+1 \end{bmatrix}$

$$\begin{aligned}
\det(\lambda I_n - A) &= (\lambda+1)((\lambda-7)(\lambda+9) - (-8)(8)) \\
&= (\lambda+1)(\lambda^2 + 2\lambda - 63 + 64) \\
&= (\lambda+1)(\lambda^2 + 2\lambda + 1) \\
&= (\lambda+1)(\lambda+1)(\lambda+1) \\
&= (\lambda+1)^3
\end{aligned}$$

$(\lambda+3)^3 = 0$

$\lambda = -1$ algebraic multiplicity $= 3$

eigenvectors

$(\lambda I_n - A) = \vec{0}$

$$\begin{bmatrix} \lambda-7 & 8 & -6 \\ -8 & \lambda+9 & -6 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

$-1+1$

Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\begin{bmatrix} -8 & 8 & -6 \\ -8 & 8 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\downarrow$

skip 3 steps

$$\left[\begin{array}{ccc|c} -8 & 8 & -6 & 0 \\ -8 & 8 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{\text{RREF}}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = t - \frac{3}{4}s$

$v_2 = t$

$v_3 = s$

$$\Rightarrow \begin{bmatrix} t - 3/4 s \\ t \\ s \end{bmatrix}$$

$$\begin{bmatrix} t - 3/4 s \\ t \\ s \end{bmatrix} \xRightarrow{\text{separate the variables}} \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -3/4 s \\ 0 \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix}$$

let $t = 1$ and $s = 4$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} \right\}$ are LI eigenvectors

corresponding to $\lambda = -1$

$\Rightarrow$ geometric multiplicity of λ is 2

	algebraic	geometric
$\lambda = -1$	3	2

$\Rightarrow$ A is defective

eg. $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

$$\det(\lambda I_n - A)$$

$$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \boxed{\lambda-1 \quad 1 \quad -2} \\ -1 \quad \lambda+1 \quad -2 \\ -1 \quad 1 \quad \lambda-2 \end{bmatrix} \right)$$

$= (\lambda-1)((\lambda+1)(\lambda-2) - (1)(-2))$ $\rightarrow (\lambda-1)((\lambda^2-\lambda-2)+2))$

$+1(-1(\lambda-2) - (-1)(-2))$ $\rightarrow (-\lambda+2-2) = -\lambda$

$-2(-1(1) - (-1)(\lambda+1))$ $\rightarrow -2(-1+\lambda+1) = -2\lambda$

$= (\lambda-1)(\lambda^2-\lambda) - \lambda$

$= (\lambda-1)\lambda(\lambda-1) - \lambda$

$= \lambda((\lambda-1)(\lambda-1) - 1)$

$= \lambda(\lambda^2 - 2\lambda + 1 - 1)$

$= \lambda(\lambda^2 - 2\lambda)$

$= \lambda^2(\lambda-2)$

$\lambda^2 = 0$ $\qquad$ $\lambda - 2 = 0$

$\lambda = 0$ $\qquad$ $\lambda = 2$

algebraic multiplicity $= 2$ $\qquad$ algebraic multiplicity $= 1$

Eigenvectors

① $\lambda = 0$

$(\lambda I_n - A) = \begin{bmatrix} -1 & 1 & -2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$

Let $\lambda = 0$

$(\lambda I_n - A)v = \vec{0}$

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Skip some steps

$$\left[\begin{array}{ccc:c} -1 & 1 & -2 & 0 \\ -1 & 1 & -2 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = t - 2s$

$v_2 = t$

$v_3 = s$

$\Rightarrow \begin{bmatrix} t-2s \\ t \\ s \end{bmatrix}$

$$= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ 0 \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$, $s = 1$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are LI eigenvectors

corresponding to $\lambda = 0$

geometric multiplicity = 2

(2) $\lambda = 2$

$\lambda I_n - A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix}$

Let $\lambda = 2$

$(\lambda I_n - A)v = \vec{0}$

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

skip some steps

$$\left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = t$

$v_2 = t \quad = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$v_3 = t$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is LI eigenvector corresponding to $\lambda = 2$

geometric multiplicity $= 1$

	algebraic	geometric
$\lambda = 0$	2	2
$\lambda = 1$	1	1

A is nondefective

A has 3 LI eigenvalues:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Theorem : Let A be an $n \times n$ matrix

If λ is an eigenvalue of A,

then $\left\{ \begin{array}{l} \text{eigenvector of } A \\ \text{corresponding to } \lambda \end{array} \right\}$ $\cup$ $\{ \vec{0} \}$ union

is a subspace of $\mathbb{R}^n$. This subspace is called the eigenspace of λ

Proof : HW

$S =$

① $\vec{0} \in S$

② close under $+$

③ close under $\cdot$

Possible Final Question

Let $T: V \to W$ be a LT

Then T is 1-1 iff $\ker T = \{\vec{0}\}$

p iff q : If p then q

and If q then p

① Prove that if T is 1-1 then $\ker T = \{\vec{0}\}$

Recall : $f: A \to B$ is 1-1 if : if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Suppose $T(v) = \vec{0}$

need to show $v = \vec{0}$

we know $T(\vec{0}) = \vec{0}$

$\therefore T(v) = T(\vec{0})$

since T is 1-1, $v = \vec{0}$

$\therefore \ker T = \{\vec{0}\}$ □

② Prove that if $\ker T = \{\vec{0}\}$ then T is 1-1

Suppose $T(v_1) = T(v_2)$

need to show $v_1 = v_2$

But $T(v_1) = T(v_2)$

iff $T(v_1) - T(v_2) = \vec{0}$

iff $T(v_1 - v_2) = \vec{0}$

Then $v_1 - v_2 \in \ker T$

since $\ker T = \{\vec{0}\}$ by assumption,

$v_1 - v_2 = \vec{0}$

$\therefore v_1 = v_2$ □