HW 9 6.3

Find $T(v)$ by using (a) the standard matrix and (b) the matrix relative to $B$ and $B'$

37) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (x+y, x, y)$, $v = (5,4)$,

$B = \{(1,-1), (0,1)\}$, $B' = \{(1,1,0), (0,1,1), (1,0,1)\}$

a) $T(v) = T(5,4) = (5+4, 5, 4) = (9,5,4)$

b) $[T]_B^{B'} [v]_B$

$T(1,-1) = (0,1,-1) = c_1(1,1,0) + c_2(0,1,1) + c_3(1,0,1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$T(0,1) = (1,0,1) = c_2(1,1,0) + c_2(0,1,1) + c_3(1,0,1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore [T]_B^{B'} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$[(5,4)]_B$

$(5,4) = c_1(1,-1) + c_2(0,1)$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ -1 & 1 & 4 \end{array}\right] \overset{RREF}{\Rightarrow} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 9 \end{bmatrix}$$

$$\therefore [(5,4)]_B = \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$[T(v)]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 5+0 \\ 0+0 \\ -5+9 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

$T(v_1) = 5(1,1,0) + 0(0,1,1) + 4(1,0,1)$

$\quad = (5,5,0) + (4,0,4)$

$\quad = (9,5,4)$

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4.) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x+y+z, 2z-x, 2y-z)$, $v = (4,-5,10)$

$B = \{(\overset{x}{2},\overset{y}{0},\overset{z}{1}), (\overset{x}{0},\overset{y}{2},\overset{z}{1}), (\overset{x}{1},\overset{y}{2},\overset{z}{1})\}$, $B' = \{(1,1,1), (1,1,0), (0,1,1)\}$

a) $T(4,-5,10) = (4+(-5)+10, 2(10)-4, 2(-5)-10) = (9, 16, -20)$

b) $[T]_B^{B'} [v]_B$

$T(2,0,1) = (3,0,-1) = c_1(1,1,1) + c_2(1,1,0) + c_3(0,1,1)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$T(0,2,1) = (3,2,3) = c_1(1,1,1) + c_2(1,1,0) + c_3(0,1,1)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$T(1,2,1) = (4,1,3) = c_1(1,1,1) + c_2(1,1,0) + c_3(0,1,1)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\therefore [T]_B^{B'} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & -1 & -2 \\ -3 & -1 & -3 \end{bmatrix}$$

$[(4,-5,10)]_B = (4,-5,10) = c_1(2,0,1) + c_2(0,2,1) + c_3(1,2,1)$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 0 & 2 & 2 & -5 \\ 1 & 1 & 1 & 10 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{25}{2} \\ 0 & 1 & 0 & \frac{37}{2} \\ 0 & 0 & 1 & -21 \end{array}\right]$$

$$\therefore [(4,-5,10)]_B = \begin{bmatrix} \frac{25}{2} \\ \frac{37}{2} \\ -21 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$[T(v)]_{B'} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & -1 & -2 \\ -3 & -1 & -3 \end{bmatrix} \begin{bmatrix} \frac{25}{2} \\ \frac{37}{2} \\ -21 \end{bmatrix} = \begin{bmatrix} 25 + 74 + (-126) \\ \frac{25}{2} + (-\frac{37}{2}) + 42 \\ -\frac{75}{2} + (-\frac{37}{2}) + 63 \end{bmatrix}$$

$$= \begin{bmatrix} -27 \\ 36 \\ 7 \end{bmatrix}$$

$$T(v_1) = -27(1,1,1) + 36(1,1,0) + 7(0,1,1)$$
$$= (-27,-27,-27) + (36,36,0) + (0,7,7)$$
$$= (9, 16, -20)$$

43.) Let $T: P_2 \to P_4$ be the linear transformation $T(p) = xp$.
Find the matrix for $T$ relative to the bases $B = \{1, x, x^2\}$ and
$B' = \{1, x, x^2, x^3\}$
find $[T]_B^C$

$$T(1) = x(1) = x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = x(x) = x^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(x^2) = x(x^2) = x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[T]_B^C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

45) Let $B = \{1, x, e^x, xe^x\}$ be a basis for a subspace $W$ of the space of continuous functions, and let $D_x$ be the differential operator on $W$. Find the matrix for $D_x$ relative to the basis $B$

Find $[D_x]_B^C$

① $D_x(1) = x = c_1 + c_2x + c_3e^x + c_4xe^x$

$c_1 = 0,\ c_2 = 1,\ c_3 = 0,\ c_4 = 0$

$$[D_x(1)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

② $D_x(x) = \frac{x^2}{2} = c_1 + c_2x + c_3e^x + c_4xe^x$

$c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = 0$

$$[D_x(x)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

③ $D_x(e^x) = e^x = c_1 + c_2x + c_3e^x + c_4xe^x$

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 1 \quad c_4 = 0$

$$[D_x(e^x)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

④ $D_x(xe^x) = e^x + xe^x = c_1 + c_2x + c_3e^x + c_4xe^x$

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 1 \quad c_4 = 1$

| | |
|---|---|
| $x$ | $e^x$ |
| $1$ | $e^x$ |
| $0$ | $e^x$ |

$$[D_x(xe^x)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore\ [D_x]_B^B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

47) Use the matrix from ex 45 to evalute $D_x[4x-3xe^x]$

$$[4x-3xe^x]_B = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}$$

$$[D_x]_B^B [v]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ -3 \end{bmatrix}$$

$\therefore D_x(4x-3xe^x) = 4 - 3e^x - 3xe^x$

49) Let $B = \{1, x, x^2, x^3\}$ be a basis for $P_3$, and $T: = P_3 \to P_4$ be the linear transformation represented by $T(x^k) = \int_0^x t^k \, dt$

a) Find the matrix A for T with respect to B and the standard basis for $P_4$.

Let $x=1$, $k=0$

$$T(1) = \int_0^x t^0 \, dt = \int_0^x 1 \, dt = t \Big|_0^x = x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x = x$, $k=1$

$$T(x) = \int_0^x t^1 \, dt = \frac{t^2}{2} \Big|_0^x = \frac{x^2}{2} = \frac{1}{2}x^2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

Let $x = x^2$, $k=2$

$$T(x^2) = \int_0^x t^2 \, dx = \frac{t^3}{3} \Big|_0^x = \frac{1}{3}x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

Let $x = x^3$, $k=3$

$$T(x^3) = \int_0^x t^3 \, dx = \frac{t^4}{4} \Big|_0^x = \frac{1}{4}x^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

b) Use $A$ to integrate $p(x) = 8 - 4x + 3x^3$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -2 \\ 0 \\ \frac{3}{4} \end{bmatrix}$$

$$0 + 8x - 2x^2 + 0x^3 + \frac{3}{4}x^4$$

$$= 8x - 2x^2 + \frac{3}{4}x^4$$

6.2

Find the kernel of the linear transformation

1.) $T : R^3 \to R^3$, $T(x,y,z) = (0,0,0)$

$T(p(x)) = 0$

$(x,y,z) = (0,0,0)$

$\{ \mathbb{R}^3 \}$

5.) $T : P_3 \to R$, $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + a_2$

$T(a_0 + a_1x + a_2x^2 + a_3x^2) = (0,0,0,0)$

$T(p(x)) = 0$

$a_1 + a_2 = 0$

$a_1 = -a_2$

$\{ a_0 - a_2x + a_2x^2 + a_3x^2 \ ; \ a_0, a_2, a_3 \in \mathbb{R} \}$

7.) $T : P_2 \to P_1$, $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$

$T(p(x)) = 0$

$a_1 + 2a_2x = 0$

$a_1 = -2a_2x$

$a_0 - 2a_2x^2 + a_2x^2$

$\{ a_0 - a_2x^2 \ ; \ a_0, a_2 \in \mathbb{R} \}$

Define the linear transformation $T$ by $T(x) = Ax$. Find (a) the kernel of $T$ and (b) the range of $T$

13) $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

(a) $\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$ $\overset{RREF}{\Rightarrow}$ $\left[\begin{array}{ccc:c} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$x = -4t$ Let $t = 1$ $x = -4$

$y = -2t$ $y = -2$

$z = t$ $z = 1$

$\ker T = \{ (-4,-2,1) \}$

b) $A^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rang $T = \mathbb{R}^2$

15.) $A = \begin{bmatrix} 1 & 3 \\ -1 & -3 \\ 2 & 2 \end{bmatrix}$

a) $\left[\begin{array}{cc:c} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x = 0$, $y = 0$

$\ker T = \{(0,0)\}$, No basis

b) $A^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rang K $= \{(1,-1,0), (0,0,1)\}$

63.) Define $T: M_{n,n} \to M_{n,n}$ by $T(A) = A - A^T$. Show that the kernel of $T$ is the set of $n \times n$ symmetric matrices.

$T(A) = 0$

$A - A^T = 0$

$A = A^T$

$\therefore \ker T = \{A : A = A^T\}$