Permutations and Combinatorics

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Permutations and Combinatorics

Equally likely outcomes.

- Say all events in the sample space are equally likely.
- Let X denote the number of ways event A can occur.
- Let n denote the number of possible outcomes (number of elements in the sample space).
- Then $P(A) = \dfrac{X}{n}$.

$S = \{ ?, \# \}$

How do we figure out what n is equal to in a large space?

- *Permutation*: A permutation is an arrangement of objects in a definite order.

  particular order

- *Combination*: A combination is a selection of objects without regard to order,

  order

- Assume we have 4 people: Amy, Bruce, Chad, and Dina, and we are going to select two of them to go on a trip.
  - *Permutation*: In a permutation, the order matters. So the sets {Amy, Bruce} is not the same as {Bruce, Amy}.
    - You can think of {Amy, Bruce} as picking Amy first and then Bruce, while {Bruce, Amy} is picking Bruce first and then Amy.
  - *Combination*: In a combination, the order does not matter, The set {Chad, Dina} is the same as {Dina, Chad}.
  - We will have more ways to create a permutation than a combination (due to order mattering).

Permutations

Permutation.

- In how many ways can we select r many objects from a total of n many to choose from?

- The formula for this is $\mathbb{P}_{n,r} = \dfrac{n!}{(n-r)!} = n * (n-1) * ... * (n-r+1)$

  n factorial

  - The notation n! (read as n-factorial) is computed as:
    $n! = n * (n-1) * (n-2) * ... * 2 * 1.$ $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
  - Also $(n-r)! = (n-r) * (n-(r-1)) * ... * 2 * 1.$
  - And so $\dfrac{n!}{(n-r)!} = n * (n-1) * ... * (n-r+1).$

- We can think of a permutation as positioning r many objects, selected from n many in total, into slots.

  - The first slot will have n many options to pick from, the second slot will have $n-1$ objects to choose from,..., and the r-th slot will have $n-(r-1) = n-r+1$ many objects to choose from.

There is no direct code in R to compute this. We can make our own function.

```
perm <- function(n, r){
       return(factorial(n)/factorial(n - r))
       }
```

Combinations

- Now think of the case where order does not matter.
- In the previous example, this would mean that the duo {Amy, Bruce} is the same as {Bruce, Amy}.
- This is a combination. In statistics we say n choose r.
- If we select r many objects from a total of n possible objects, where order does not matter.
- The formula is $\mathbb{C}_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Similar to the permutation formula, this accounts for the notion that order does not matter among the r many selected (hence the division by $r!$).

We can use the choose function in R to calculate the number of calculations.

```
choose(n,r)
```

choose is a function in R

Permutations and Combinatorics

Example: Say we have 5 people: Audry, Bruce, Colin, Daniel, and Emily.

- How many ways can we select 2 people if the order matters?

permutation

$$P_{5,2} = \frac{n!}{(n-r)!} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}} = \boxed{20}$$

- How many ways can we select 3 people if the order matters?

$$P_{5,3} = \frac{n!}{(n-r)!} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot 4\cdot 3\cdot \cancel{2!}}{\cancel{2!}} = \boxed{60}$$

- How many ways can we select 2 people if the order does not matter?

$$C_{5,2} = \frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot \cancel{3!}}{2\cdot 1\cdot \cancel{3!}} = \frac{20}{2} = \boxed{10}$$

- How many ways can we select 3 people if the order does not matter?

$$C_{5,3} = \frac{n!}{r!(n-r)!} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!} = \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot 2\cdot 1} = \frac{20}{2} = \boxed{10}$$

Counting

Permutations and Combinatorics are a type of selection process where the objects selected are not replaced. Once selected, the object is removed from the remaining possible objects to be selected.

- Assume we have n many total objects, and want to create a grouping of r many of them, where order matters and objects are selected with replacements (i.e. in each group, an objects can occupy several places).
    - An example is creating a password using only lower case letters.
    - Each password spot (which character, going from left to right) can be one of 26 objects (a,b,c,...,x,y, or z).
    - it is possible use a single letter numerous times. For example abcda or aacde or aaaaaa.
    - Note: The ordering of the objects matters. For example abcdef and fedcba are different passwords.

Counting

Say we have n many total objects, and want to create a grouping of r many of them, where order matters and objects are selected with replacements (i.e. in each group, an objects can occupy several places).

- The formula for this is $\prod_{i=1}^{r} n = n^r$

Example: Lets say you want to create a password that is 5 characters long, using only lower case letters.

- What does n equal?

$n = 26$

- What does r equal?

$r = 5$

- How many possible passwords are there that are 5 lowercase letters long?

product $\rightarrow \prod_{i=1}^{5} 26 = 26^5 = 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 11881376$

$P(\text{guess}) = \frac{1}{11881376}$

- How many ways can we create a password that has two numbers?

$\prod_{i=1}^{2} 10 = 10^2 = 100$

- What is the total number of ways we can create a password with 5 lowercase letters followed by two numbers?

$\left(\prod_{i=1}^{5} 26\right)\left(\prod_{i=1}^{2} 10\right) = 1188137600$