9.1 Relations and Their Properties

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3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

Q3

a) This relation is not reflexive, since it does not include, for instance (1, 1). It is not symmetric, since it includes, for instance, (2, 4) but not (4, 2). It is not antisymmetric since it includes both (2, 3) and (3, 2), but 2 f- 3. It is transitive. To see this we have to check that whenever it includes (a,b) and (b,c), then it also includes (a, c). We can ignore element 1 since it never appears. If (a, b) is in this relation, then by inspection we see that a must be either 2 or 3. But (2, c) and (3, c) are in the relation for all c -:/=- 1; thus (a, c) has to be in this relation whenever (a, b) and (b, c) are. This proves that the relation is transitive. Note that it is very tedious to prove transitivity for an arbitrary list of ordered pairs.

b) This relation is reflexive, since all the pairs (1, 1), (2, 2), (3, 3), and (4, 4) are in it. It is clearly symmetric, the only nontrivial case to note being that both (1, 2) and (2, 1) are in the relation. It is not antisymmetric because both (1, 2) and (2, 1) are in the relation. It is transitive; the only nontrivial cases to note are that since both (1, 2) and (2, 1) are in the relation, we need to have (and do have) both (1, 1) and (2, 2) included as well.

c) This relation clearly is not reflexive and clearly is symmetric. It is not antisymmetric since both (2, 4) and (4,2) are in the relation. It is not transitive, since although (2,4) and (4,2) are in the relation, (2,2) is not.

d) This relation is clearly not reflexive. It is not symmetric, since, for instance, (1, 2) is included but (2, 1) is not. It is antisymmetric, since there are no cases of (a, b) and (b, a) both being in the relation. It is not transitive, since although (1, 2) and (2, 3) are in the relation, (1, 3) is not.

e) This relation is clearly reflexive and symmetric. It is trivially antisymmetric since there are no pairs (a, b) in the relation with a -:/=- b. It is trivially transitive, since the only time the hypothesis (a, b) E R /\ (b, c) E R is met is when a= b= c.

f) This relation is clearly not reflexive. The presence of (1,4) and absence of (4, 1) shows that it is not symmetric. The presence of both (1, 3) and (3, 1) shows that it is not antisymmetric. It is not transitive; both (2, 3) and (3, 1) are in the relation, but (2, 1) is not, for instance.

9. Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.

Each of the properties is a universally quantified statement. Because the domain is empty, each of them is vacuously true.

9.3 Representing Relations

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

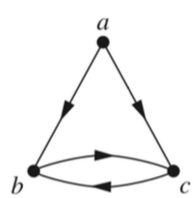
a) Since the (1,1)th entry is a 1, (1,1) is in the relation. Since (1,2)th entry is a 0, (1,2) is not in the relation. Continuing in this manner, we see that the relation contains (1, 1), (1, 3), (2, 2), (3, 1), and (3, 3).

7. Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

a) Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not at positions (1, 3) and (3, 1). Finally, the Boolean square of this matrix is itself, so the relation is transitive.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

23.



We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: {(a, b), (a, c), (b, c), (c, b)}.

9.5 Equivalence Relations

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

In each case, we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair (1, 1) is missing. It is also not transitive, since the pairs (0, 2) and (2, 3) are there, but not (0, 3).

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs (1, 3) and (3, 2) are present, but not (1, 2).

e) This relation would be an equivalence relation were the pair (2, 1) present. As it is, its absence makes the relation neither symmetric nor transitive.