Uniform and Exponential Distribution

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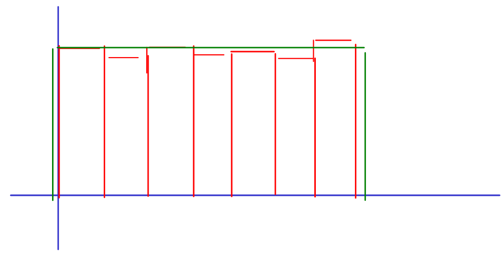
UCI

Uniform Distribution

The *uniform distribution* is a distribution for a continuous random variable that can take on any value in an interval [a,b], with uniform density.

includes a and b



- Denoted $X \sim Uniform(a, b)$
- $\mathbb{S}_X = [a, b]$.
- $f(x) = \dfrac{1}{b-a}$. pdf
  - We use this distribution for continuous variables where the values are all equally likely
- $\text{F}(X) = P(X \leq x) = \int_a^x f(x)dx = \dfrac{x-a}{b-a}$
- $\text{E}(X) = \dfrac{b+a}{2}$
- $\text{VAR}(X) = \dfrac{(b-a)^2}{12}$
- The parameters of the distribution are a (lower bound) and b (upper bound).

This is a valid probability density function.

- For all x in [a,b], $f(x) \geq 0$.
- $\int\limits_{\mathbb{S}_X} f(x)dx = \int\limits_a^b \dfrac{1}{b-a}dx = \dfrac{x}{b-a}\Bigg|_{x=a}^{x=b} = \dfrac{b}{b-a} - \dfrac{a}{b-a} = 1$

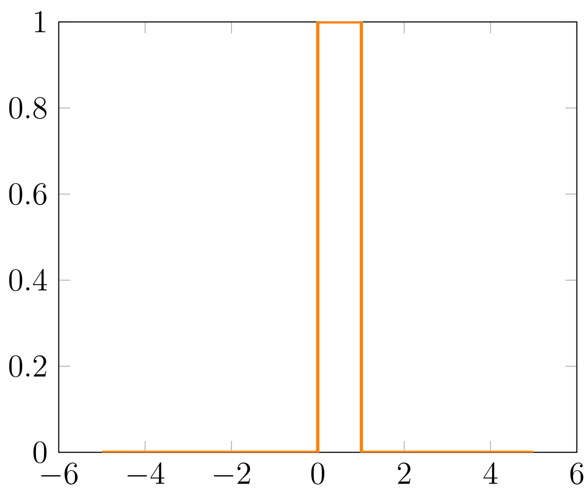
Uniform Distribution

Let X be a uniform continuous random variable on the interval $[a, b]$.

- $\mathbb{S}_X = [a, b]$.
- $f(x) = \dfrac{1}{b-a}$.

This is a valid probability density function.
Note: the CDF and pdf are only valid for x such that $a \leq x \leq b$

Ex: $X \sim Uniform(0, 1)$



Uniform Distribution - R Code

To get the area to the left of a Uniform(0,1) variable:
$punif(u, min = 0, max = 1)$

$P(X \leq x)$

To get the area to the right of a Uniform(0,1) variable:
$1 - punif(u, min = 0, max = 1)$

$P(X \geq x) = 1 - P(X \leq x)$

To get the area between two values say c and d (c < d):
$punif(d, min = 0, max = 1) - punif(c, min = 0, max = 1)$

$P(X \leq d) - P(X \leq c)$

To get the value of u related to the lower tail (α):
$qunif(\alpha, min = 0, max = 1)$ q^{th} quartile

To get the value of u related to the upper tail:
$qunif(1 - \alpha, min = 0, max = 1)$

Uniform Distribution - Expectation and Variance

Let X be a uniform continuous random variable on the interval (a, b).

- $E(X) = \int_{\mathbb{S}_X} x\, f(x)dx = \int_a^b x \frac{1}{b-a} dx$
  $= \frac{x^2}{2(b-a)}\Big|_{x=a}^{x=b} = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$

- The expected value of X is just the average of the two end points of the support.

- $E(X^2) = \int_{\mathbb{S}_X} x^2\, f(x)dx = \int_a^b x^2 \frac{1}{b-a} dx$
  $= \frac{x^3}{3(b-a)}\Big|_{x=a}^{x=b} = \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} = \frac{(b^3 - a^3)}{3(b-a)}$

Show that VAR(X)=$E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{12}$.

$$\frac{(b^3 - a^3)}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 = \frac{\cancel{(b-a)}(b^2 + ba + a^2)}{3\cancel{(b-a)}} - \frac{b^2 + 2ba + a^2}{4}$$

$$= \frac{4b^2 + 4ba + 4a^2}{12} - \frac{3b^2 + 6ba + 3a^2}{12}$$

$$= \frac{b^2 - 2ba + a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

Uniform Distribution - CDF

Let X be a uniform continuous random variable on the interval (a, b).

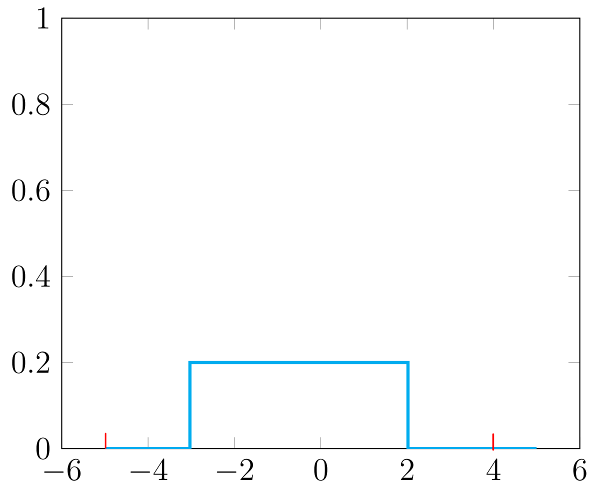
The cumulative distribution function is as follows:

- $F(x) = P(X < x) = \int\limits_{\tilde{x}<x} f(\tilde{x})d\tilde{x} = \int\limits_{a}^{x} \frac{1}{b-a} d\tilde{x}$
  $= \frac{\tilde{x}}{(b-a)}\Bigg|_{\tilde{x}=a}^{\tilde{x}=x} = \frac{x-a}{b-a}$

Thus for any x in (a,b), $F(x) = \dfrac{x-a}{b-a}$.

Note that if $x \leq a$ then F(x)=0 and if $x \geq b$ then F(x)=1.

Example: $X \sim Uniform(-3, 2)$



$P(X \leq -5) = 0$

$P(X \geq -5) = 1$

$P(X \geq 4) = 0$

$P(X \leq 4) = 1$

Example - Car Intersections $X \sim \text{Uniform}(a=0, b=13)$

Say you are waiting for the first car to cross an intersection. The waiting time is assumed to be any number between 0 and 13 minutes. The waiting time X is a uniform random variable on the interval [0,13].

- What is the probability that you wait less than 5 minutes?

$$P(X<5) = P(X \leq 5) = \frac{x-a}{b-a} = \frac{5-0}{13-0} = 0.3846$$

punif(5, 0, 13)

- How much time do you expect to wait?

$$E(X) = \frac{b-a}{2} = \frac{13-0}{2} = 6.5 \text{ mins}$$

- Find the variance for the waiting time.

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(13-0)^2}{12} = \frac{169}{12} = 14.0833 \text{ min}^2$$

$$\sigma = \sqrt{Var(X)} = \sqrt{14.0833} = 3.7528$$

Exponential Distribution

Now we come to what is known as the *exponential distribution.*
It is used to model the the time between events in a Poisson process.

$\frac{60 \text{ car}}{1 \text{ hr}}$

Let X be a random variable that follows an exponential distribution.

- $X \sim Exponential(\lambda)$
- The probability density function of X (pdf) is $f(x) = \lambda e^{-\lambda x}$.

$\frac{1 \text{ car}}{1 \text{ min}}$ (1 events)

- The expectation is $E(X) = \dfrac{1}{\lambda}$.

$\lambda = \frac{1}{E(X)}$

- The variance is $VAR(X) = \dfrac{1}{\lambda^2}$
- The cumulative distribution function is $F(x) = 1 - e^{-\lambda x}$.
- $P(a < X < b) = F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$
- λ is the parameter where $\lambda > 0$.
- The support of X is $\mathbb{S}_X = [0, \infty)$

Exponential Distribution - R Code

To get the area to the left of a Exponential(1) variable: $\lambda = 1$
$pexp(e, rate = 1)$ $\leq$

To get the area to the right of a Exponential(1) variable:
$1 - pexp(e, rate = 1)$ $\geq$

To get the area between two values say c and d (c < d):
$pexp(d, rate = 1) - pexp(c, rate = 1)$

To get the value of e related to the lower tail (α):
$qexp(\alpha, rate = 1)$

To get the value of e related to the upper tail:
$qexp(1 - \alpha, rate = 1)$

Exponential Distribution

Let X be a random variable that follows an exponential distribution with parameter $\lambda > 0$.

$$\begin{aligned}
E(X) &= \int_0^\infty x f(x) dx \\
&= \int_0^\infty \lambda x e^{-\lambda x} dx \\
&= [-x e^{-\lambda x}]|_{x=0}^{x=\infty} + \int_0^\infty e^{-\lambda x} dx \\
&= (0-0) + \frac{1}{\lambda} x e^{-\lambda x}]|_{x=0}^{x=\infty} \\
&= 0 + \left(0 + \frac{1}{\lambda}\right) \\
&= \frac{1}{\lambda}
\end{aligned}$$

The parameter λ can be viewed as the expected time until the next event is observed.

Similarly we can show that $E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = 2\left(\frac{1}{\lambda}\right)^2$

As a result, $VAR(X) = \frac{1}{\lambda^2}$.

Exponential Distribution

Let X be a random variable that follows an exponential distribution with parameter $\lambda > 0$.

The cumulative distribution function (cdf) F(x) is as follows.

$$
\begin{aligned}
F(x) &= \int_0^x f(u)du \\
&= \int_0^x -\lambda e^{-\lambda u} du \\
&= -e^{-\lambda u}\Big|_{u=0}^{u=x} \\
&= -e^{-\lambda x} - (-1) = 1 - e^{-\lambda x}
\end{aligned}
$$

As a result: $P(a < X < b) = F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$.

Exponential Distribution

Let X follows an exponential distribution with a constant parameter $\lambda > 0$.

A property of this distribution is called its *memoryless* property.

- Notationally this is $P(X > x + t | X > x) = P(X > t)$ where $x, t > 0$.
- For example $P(X > 45 | X > 35) = P(X > 35 + 10 | X > 35) = P(X > 10)$

Given the wait time X for the next event is greater than x, the probability that the time X is greater than $x + t$ is just equal to the unconditional probability that X is greater than t.

$X \sim \text{Exponential}(\lambda = \frac{1}{10}) \frac{\text{CAR}}{\text{min}}$ $\quad Y \sim \text{Poisson}(\lambda = \frac{6}{1}) \frac{\text{CAR}}{\text{hrs}}$

Ex: The wait time (in minutes) to observe the next car that crosses an intersection follows an exponential distribution with $\lambda = \dfrac{1}{10}$.

a. What is the expected wait time for the next car to cross the intersection?

$$E(X) = \frac{1}{\lambda} = \frac{1}{(\frac{1}{10})} = 10 \text{ min}$$

b. What is the variance of the wait time for the next car to cross the intersection?

$$Var(X) = \frac{1}{\lambda^2} = \frac{1}{(\frac{1}{10})^2} = 100 \text{ min}^2$$

c. Find the probability that the wait time is less than 5 minutes.

$$P(X < 5) = P(X \leq 5) = \text{pexp}(5, \tfrac{1}{10}) = 0.3935$$

d. What is the probability that the wait time is between 4 and 6 minutes?

$$P(4 < X < 6) = P(X < 6) - P(X \leq 4)$$
$$= \text{pexp}(6, \tfrac{1}{10}) - \text{pexp}(4, \tfrac{1}{10})$$
$$= 0.1215$$

e. Now say we know (or that we are given, or condition on) that the wait time is more than 4 minutes. What is the probability that the wait time is more than 6 minutes?

$$P(X > 6 \mid X > 4) = P(X > 2)$$
$$= 1 - P(X \leq 2)$$
$$= 1 - \text{pexp}(2, \tfrac{1}{10})$$
$$= 0.8187$$