Introduction To Probability

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Statistics

- Statistics is the mathematical science of learning from data, and of measuring, controlling, and communicating uncertainty.
- It is concerned with developing methods for collecting and analyzing empirical data.
- In many fields of the physical and social sciences, empirical data will naturally have variability and randomness.
- Probability theory provides a substantial part of the underlying framework used to describe variability and randomness, and therefore provides a foundation for the tools developed in statistics.

Probability

- In a *frequency* framework, probability of an event (P) is defined to be the proportion of times the event is observed under repeated observation.

- Assume we conduct an experiment of flipping a coin n times. Let the number of heads, X, be recorded.

- The probability of getting a head from flipping the coin is $P = \lim_{n\to\infty} \frac{X}{n}$.

- It can be viewed as the long run average of the number of "success".

- If we flip the coin a very large number of times, the proportion of success' ($\frac{X}{n}$) will converge to the true probability of a single success. This is a loose statement of the *law of large numbers.*

- When the event cannot be repeated, it is a little difficult to intuitively view probability from a frequency standpoint.

  - An example is if it will rain on a specific day.

- As such, there is another interpretation of probability referred to as the *Bayesian* interpretation.

- In this framework, the probability of an event is the degree of one's belief (between 0 and 1) the event will occur.

  – Example: There is a 24% chance it will rain tomorrow.

  – Example: There is a 99% chance that a certain subject will recover from surgery.

Sample Space

For now, let us only concern ourselves with discrete and categorical outcomes.

- The set of all possible outcomes in a random experiment is the **Sample Space**, S.

- Determine the sample space for the following situations.

  - Example: Flip a coin once.

  - Example: Roll a die once.

  - Example: Flip a coin 3 times.

  - Example: Roll a pair of dice.

    $S = \{(1,1),(1,2),(1,3),...,(6,5),(6,6)\}$.

    or

    $S = \{(x,y) : x = 1,2,...,6\,, y = 1,2,...,6\}$.

Event Space

An **event**, A, is a subset of the sample space. Also known as a *sample point*.

Examples Determine the event space for the following situations.

- Assume you roll a die one time. Let the event A be the event that the number the die lands on is even.

- Assume you roll a die one time. Let the event A be the event that the number the die lands on is greater than 4.

- Assume you flip a coin three times. Let the event B be the event that more than 1 tail appears.

- Assume you flip a coin three times. Let the event B be the event that the first coin flip is a head.

Set Theory Basics

Let A and B be sets.

- $\varnothing$ is the **Empty Set**. A set that has no elements in it. I.e. $\varnothing = \{\}$.
- A is a **Subset** of B if $s \in A$ implies $s \in B$. That is to say whatever is in A is also in B.
  - Notationally this is presented as $A \subset B$.
  - Example: $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.
- The **Union** of A and B is denoted as $A \cup B$. When translating we say A or B.
  - If $s \in A$ or $s \in B$, then $s \in A \cup B$.
  - Example: Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$, and $B = \{3, 4, 5\}$
  - Find $A \cup B$
- The **Intersection** of A and B is denoted as $A \cap B$. It is the overlap of the two sets. When translating we say A and B.
  - If $s \in A \cap B$, then $s \in A$ and also $s \in B$.
  - Example: Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$, and $B = \{3, 4, 5\}$
  - Find $A \cap B$

Set Theory Basics

Let A and B be sets.

- The **Complement** of A is denoted as A^c. It is the collection of elements that are not in A. When translating we say not in A.
  - If $s \in A$, then $s \notin A^c$.
  - Example: Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$, and $B = \{3, 4, 5\}$
  - Find A^c
  - Note $(A^c)^c = A$ and $A \cup A^c = S$.
- $(A \cup B)^c = A^c \cap B^c$ $\quad$ $(A^c \cup B)^c = A \cap B^c$ $\quad$ $(A^c \cup B^c)^c = A \cap B$
  - Example: Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$, and $B = \{3, 4, 5\}$
  - Find B^c
  - Find $A^c \cap B^c$
  - Find $(A \cup B)^c$
- $(A \cap B)^c = A^c \cup B^c$ $\quad$ $(A^c \cap B)^c = A \cup B^c$ $\quad$ $(A^c \cap B^c)^c = A \cup B$

Set Theory Basics

$(A \cup B)^c = A^c \cap B^c$

- Example: Assume you roll a die one time. Let A be the event you roll a 1 or 2 on a die, and B is the event you roll a 3 or a 4.
  - The event you don't roll a 1 or a 2 NOR a 3 or a 4 is $(A \cup B)^c$.

- Example: Let A be the event someone has blue eyes and B be the event they are a computer science major.
  - The event that someone is not blue eyed nor a computer science major is $(A \cup B)^c$

Probability Theory Basics

A **probability distribution** is the rule that assigns a number ($P(\cdot)$) to each possible outcome in the sample space ($s \in S$), with the following conditions.

- $0 \leq P(s) \leq 1$ for all $s \in S$
- $\sum\limits_{s \in S} P(s) = 1$
- As an example, assume you rolla die one time. Each event in $S = \{1, 2, 3, 4, 5, 6\}$ has probability of $1/6$.
  - Note: The sum of all the probabilities is equal to 1 $\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right)$.
  - Thus, we call this a valid probability distribution.

Let A and B be events in the sample space S.

- $P(S) = 1$.
- If $A \subset B$ then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  - In the equation above, solve for $P(A \cap B)$.
- $P(A) + P(A^c) = 1$
  - As a result: $P(A) = 1 - P(A^c)$.
- $P(B) = P(B \cap A) + P(B \cap A^c)$.

Venn Diagrams

EXAMPLE: Let A and B be events in the sample space S.
Draw a venn-diagram for each probability:

$P(A \cap B)$ $\qquad$ $P(A^c \cap B)$

$P(B)$ $\qquad$ $P(B^c)$

Venn Diagrams

EXAMPLE: Let A and B be events in the sample space S. Draw a venn-diagram for each probability:

$P(A \cap B^c)$

$P(A \cup B)$

$P(A \cup B)^c$

A_1, A_2, A_3 Partition S

Probability Theory Basics

Example: Assume we sample UCI Information and Computer Science students. Let $P(A) = 0.7$ where A is the event someone is an Undergrad. Let $P(B) = 0.8$ where B is the event someone is a computer science major. And let $P(A \cap B) = 0.6$.

- What is the probability someone is a grad student?

- What is the probability someone is an undergrad or a computer science major?

- What is the probability someone is a grad student and a computer science major?

- What is the probability someone is a computer science grad student?

Mutually Exclusive Events

Let A and B be sets.

- We say that sets (or events) are **Mutually Exclusive** if the two sets (or events) cannot occur at the same time.
- Notationally this is $A \cap B = \varnothing$.
  - Example: Assume you roll one die once. Let A be the event that the number showing on the die is odd and B be the event that the number is a 2. Find $A \cap B$
- We say events A and A^c form a *partition* of the sample space if they are mutually exclusive and if $A \cup A^c = S$.

Let $A_1, A_2, A_3, ...$ be sets. (You can also think of sets $A, B, C, ...$).

- We say that sets (or events) are mutually exclusive if the intersection between any of these two sets is the null set.
- Notationally $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Mutually Exclusive Events

Let A and B be events in the sample space S.

- $P(\varnothing) = 0$.
- Note that this means that A and B are mutually exclusive if and only if $P(A \cap B) = 0$.

Let sets $A_1, A_2, A_3, ..., A_M$ (or can think of sets $A, B, C, ..., M$) be mutually exclusive events.

- $P(A_1 \cup A_2 \cup A_3 \cup ... \cup A_M) = P(A_1) + P(A_2) + P(A_3) + ... + P(A_M)$.
- If $A_1, A_2, A_3, ...A_M$ form a partition of the sample space, then $P(A_1 \cup A_2 \cup A_3 \cup ... \cup A_M) = P(S) = 1$.