

Conditional Probability

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Conditional Probability

Conditional probability is the probability of an event occurring **given** that another event is already known to have occurred.

- Notationally this is stated as $A|B$, which is meant to define **the event A occurring conditional (or given) on B having occurred**.
- Of the people/items in event B , how many of them occur in event A ?
- The event B has occurred and been observed. What remains random and yet to be realized is event A .
- $P(A|B)$ signifies the probability of event A occurring, conditional on the fact (or given) that event B has occurred.

Formulas

To compute conditional probabilities, a theorem known as Baye's theorem is used.

$$\bullet P(A|B) = \left(\frac{P(A \cap B)}{P(B)} \right) \quad \text{and} \quad P(B|A) = \left(\frac{P(A \cap B)}{P(A)} \right)$$

Using the formula, one can see that:

$$\bullet P(A \cap B) = P(A|B)P(B) \quad \text{and} \quad P(A \cap B) = P(B|A)P(A).$$

Remember that if A has two outcomes, then by the law of total probability

$$\bullet P(B) = P(A \cap B) + P(A^c \cap B).$$

$$\text{-- In general } P(B) = \sum_{i=1}^M P(B \cap A_i).$$

Never assume events are independent

Conditional Probability: Independent Events

If the knowledge that B occurred has no bearing on the likelihood that A will occur (similarly if knowing A has no bearing on B occurring), this means that A and B are independent.

- Notationally this is $P(A|B) = P(A)$ (and also that $P(B|A) = P(B)$).
- As a result, $P(A \cap B) = P(A)P(B)$

mutually exclusive

$$P(A \cap B) = 0$$

Example - Assume we flip a coin twice. Let A be the event the first flip is heads and B be the event the second flip is heads. From our intuition, we know that the outcome of the second flip is independent of the first flip.

- What is the probability the second coin flip results in a heads?

$$S = \{ \text{HH}, \text{HT}, \text{TH}, \text{TT} \}$$

$$P(B) = \frac{1}{2} = 0.5000$$

$$P(B) = P(A \cap B) + P(A^c \cap B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

- What is the probability of the second flip being a heads, given the first flip is a heads?

$$\text{Notationally this is } P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \cap B)}{P(A \cap B) + P(A \cap B^c)}$$

Law of Total Probability

$$= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$$

- Prove that the outcome of the first and second coin flip are independent.

$$P(B|A) = P(B) \quad \text{Thus, the events are independent}$$

0.5000 0.5000

Contingency Table

Political Affiliation	Departement Applied To						Total
	ICS	Science	Law	Education	Liberal Studies	Arts	
Democrat	45	14	23	22	42	16	162
Republican	17	38	13	42	33	7	150
Total	62	52	36	64	75	23	312

Above is a *contingency table* of job applicant's political affiliation and the department applied to. Note that the two variables are categorical.

- What is the probability the applicant is a democrat and is applying to the science department?

$$P(D \cap S) = \frac{14}{312} = 0.0449$$

- What is the probability an applicant is a republican?

$$P(R) = \frac{150}{312} = 0.4808$$

- What is the probability an applicant applied to the arts department ?

$$P(A) = \frac{23}{312} = 0.0737$$

- What is the probability that an applicant applied to the law department given that they were a republican?

$$P(L|R) = \frac{P(L \cap R)}{P(R)} = \frac{\frac{13}{312}}{\frac{150}{312}} = \frac{13}{312} \cdot \frac{312}{150} = \frac{13}{150} = 0.0867$$

- What is the probability that an applicant applied to the ICS department or they were a democrat?

$$\begin{aligned}
 P(ICS \cup D) &= P(ICS) + P(D) - P(ICS \cap D) \\
 &= \frac{62}{312} + \frac{162}{312} - \frac{45}{312} = \frac{179}{312} = 0.5737
 \end{aligned}$$

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- What is the probability an applicant is a democrat given they applied to the education department?

$$P(D|EDU) = \frac{P(D \cap EDU)}{P(EDU)} = \frac{\frac{22}{312}}{\frac{64}{312}} = \frac{22}{64} = 0.3438$$

- What is the probability that an applicant was is a democrat?

$$P(D) = \frac{162}{312} = 0.5192$$

- Compare the last two probabilities. Is an applicant's political party affiliation independent of which department they apply to?

$$P(D|EDU) \neq P(D)$$

thus the events are dependent

Tree Diagrams

Tree diagrams are useful when we initially know conditional probabilities.

- We can use them to work backwards to understand the probability of events occurring at the same time (the intersection)
- We can use the intersections to calculate many other probabilities

Example:

Assume we have three events A, B, and C. Given we have seen any of these events, there is also an associated event of success or failure. Construct a tree diagram to calculate the intersections for the events.

Tree Diagrams

Example.

Medical devices are tested before they go on the market to test their overall capabilities. A new bioassay type exam is developed to test for the presence of a certain type of disease. The exam will correctly return a positive test, $+$, when the person actually has the disease, D , with probability 0.95. The exam will incorrectly return a positive test when the person actually does not have the disease, D^c , with probability 0.02. The prevalence of the disease is 0.01.

Use a tree diagram to calculate the probability of getting a positive test on the medical exam.

Tree Diagram

Example:

Continued from previous page.

- What is the probability of having the disease given a positive medical result.
- What is the probability of having the disease?
- Are having the disease and getting a positive result independent events?
- If they were independent, then $P(D) = P(D|+)$.