7.2 Diagonalization

Definition: Let A be an $n \times n$ matrix

A is said to be diagonalizable

if $\exists$ a matrix P and a $\underbrace{\text{diagonal matrix}}_{a_{ij}=0 \text{ whenever } i \neq j}$ D

with $P^{-1}AP = D$

Thus A is diagonalizable $\Rightarrow$ A is similar to a diagonal matrix

eg. Is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ diagonalizable ?

Yes, Let $P = I_3$

$I_3^{-1} A I_3 = A$

$\hookrightarrow$ diagonal matrix

Theorem: Let A be an $n \times n$ matrix

A is diagonalizable iff A is nondefective $\longrightarrow$ A has n LI eigenvectors

Proof: ① show if A is diagonalizable then A is nondefective

Special matrix multiplication

① $A \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ Av_1 & Av_2 & \cdots & Av_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$

② $\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}_{\text{Diagonal}} = \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix}$

eg. $\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot5+2\cdot6 & 1\cdot7+2\cdot8 \\ 3\cdot5+4\cdot6 & 3\cdot7+4\cdot8 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix} \right]$

$v_1 \quad v_2$

eg. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot5 & 2\cdot6 \\ 3\cdot5 & 4\cdot6 \end{bmatrix} = \left[5\begin{bmatrix} 1 \\ 3 \end{bmatrix}, 6\begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$

Proof: Since A is diagonalizable, $\exists$ P and D with $P^{-1}AP = D$, where D is a diagonal matrix

Let $P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

Observe that

$P^{-1}AP = D$

iff $AP = PD$

ie, $A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

using special matrix multiplications,

$\begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix}$

By equating columns,

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

$\vdots$

$Av_n = \lambda_n v_n$

and $v_1 \neq \vec{0}$, $v_2 \neq \vec{0}$, ..., $v_n \neq \vec{0}$

since P is invertible

$\therefore$ $\lambda_1, \lambda_2, \dots \lambda_n$ are eigenvalues of A

and $v_1, v_2, \dots, v_n$ are eigenvectors

since P is invertible $\{v_1, v_2, \cdots, v_n\}$ is LI

proving A is nondefective

② Show that if A is nondefective, then A is diagonalizable

Proof: Since A is nondefective, A has n LI eigenvectors

Let $v_1, v_2, \dots, v_n$ be LI eigenvectors and

$\lambda_1, \lambda_2, \dots, \lambda_n$ be their eigenvalues (need to find P and D with $P^{-1}AP = D$)

Using definitions of eigenvalues/vectors,

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_1 v_2$

$\vdots$

$Av_n = \lambda_n v_n$

Thus $\begin{bmatrix} | & | & & | \\ Av_1, & Av_2, & \dots, & Av_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 v_1, & \lambda_2 v_n, & \dots, & \lambda_n v_n \\ | & | & & | \end{bmatrix}$

Using special matrix multiplications

$$A\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Let $P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

We get $AP = PD$

D is a diagonal matrix and P is invertible since the column of P are LI

$\therefore$ $P^{-1}AP = D$, proving A is diagonalizable $\square$

eg. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Determine if A is diagonalizable or not.
If it is diagonalizable, find P and D
with $P^{-1}AP = D$

① Eigenvalues

$\det(\lambda I_3 - A)$

$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right)$

$= \det\left(\begin{bmatrix} \lambda-1 & -2 & 0 \\ -2 & \lambda-1 & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} \right)$

$= (\lambda-3)((\lambda-1)^2 - 4)$

$= (\lambda-3)(\lambda^2 - 2\lambda + 1 - 4)$

$= (\lambda-3)(\lambda^2 - 2\lambda - 3)$

$= (\lambda-3)^2(\lambda+1)$

$(\lambda-3)^2 = 0$ $\quad\quad$ $\lambda+1 = 0$

$\lambda = 3$ $\quad\quad$ $\lambda = -1$

alg = 2 $\quad\quad$ alg = 1

$\lambda = 3$

$\lambda I_n - A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ $v_1 = t$, $v_2 = t$, $v_3 = s$ $\Rightarrow \begin{bmatrix} t \\ t \\ s \end{bmatrix}$

$= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $t = 1$, $s = 1$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ are LI eigenvectors for $\lambda = 3$

② $\lambda = -1$

$$\lambda I_n - A = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} v_1 = -t \\ v_2 = t \\ v_3 = 0 \end{matrix} \Rightarrow \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{Let } t = 1$$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is LI eigenvector for $\lambda = -1$

	alg	geo
$\lambda = 3$	2	2
$\lambda = -1$	1	1

$\therefore$ A is nondefective

since A is nondefective

A is diagonalizable

To find P:

Place the eigenvectors into columns (in any order)

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then it can be verified that $P^{-1}AP = D$

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{-1} & 0 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

$\downarrow \quad \downarrow \quad \downarrow$

$\lambda = \boxed{3} \quad \lambda = \boxed{-1} \quad \lambda = \boxed{3}$

7.3 Orthogonal Matrices

observation

define an inner product on $\mathbb{R}^n$ as dot product

then $\langle v, w \rangle = v^T w$ when v, w are column vectors

Dot product

eg. $\langle (1,3), (5,7) \rangle = \cancel{\ }$

$\left\langle \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix} \right\rangle = [1, 3] \begin{bmatrix} 5 \\ 7 \end{bmatrix} = 1 \cdot 5 + 3 \cdot 7 = 26$

Definition: Let O be an $n \times n$ matrix

O is orthogonal if $O^T O = I_n$

This means $O^T = O^{-1}$

eg. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix

check: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem: If O is an orthogonal matrix then

$\det(O) = 1$ or -1

Proof: HW

Important for CS

Theorem: Let $V = \mathbb{R}^n$ use dot product as an inner product

Then $\forall v \in \mathbb{R}^n$, $\|Ov\| = \|v\|$

Orthogonal matrix is norm preserving

Proof: HW

Theorem : A matrix is orthogonal iff its columns form an orthonormal set

Proof : HW