Conditional Probability

David Armstrong

UCI

Conditional Probability

Conditional probability is the probability of an event occurring given that another event is already known to have occurred.

- Notationally this is stated as $A|B$, which is meant to define the event A occurring conditional (or given) on B having occurred.

- Of the people/items in event B, how many of them occur in event A?

- The event B has occurred and been observed. What remains random and yet to be realized is event A.

- $P(A|B)$ signifies the probability of event A occurring, conditional on the fact (or given) that event B has occurred.

Formulas

To compute conditional probabilities, a theorem knowns as Baye's theorem is used.

- $(P(B)\; P(A|B) = \left(\frac{P(A \cap B)}{\cancel{P(B)}}\right) \cancel{(P(B))}$ and $P(B|A) = \left(\frac{P(A \cap B)}{\cancel{P(A)}}\right) \cancel{(P(A))}$

$(P(A)$

Using the formula, one can see that:

- $P(A \cap B) = P(A|B)P(B)$ and $P(A \cap B) = P(B|A)P(A)$.

Remember that if A has two outcomes, then by the law of total probability

- $P(B) = P(A \cap B) + P(A^c \cap B)$.
  - In general $P(B) = \sum_{i=1}^{M} P(B \cap A_i)$.

Never assume events are independent

Conditional Probability: Independent Events

If the knowledge that B occurred has no bearing on the likelihood that A will occur (similarly if knowing A has no bearing on B occurring), this means that A and B are *independent*.

- Notationally this is $P(A|B) = P(A)$ (and also that $P(B|A) = P(B)$).
- As a result, $P(A \cap B) = P(A)P(B)$

mutually exclusive

$P(A \cap B) = 0$

Example - Assume we flip a coin twice. Let A be the event the first flip is heads and B be the event the second flip is heads. From our intuition, we know that the outcome of the second flip is independent of the first flip.

- What is the probability the second coin flip results in a heads?

$S = \{HH, HT, TH, TT\}$ $\quad P(B) = \frac{1}{2} = 0.5000$

$P(B) = \underset{HH}{P(A \cap B)} + \underset{TH}{P(A^c \cap B)} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

- What is the probability of the second flip being a heads, given the first flip is a heads?

  Notationally this is $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A \cap B)}{P(A \cap B) + P(A \cap B^c)}$.

  HH

  HH HT

  Law of Total Probability

  $= \dfrac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$

- Prove that the outcome of the first and second coin flip are independent.

$P(B|A) = P(B)$ Thus, the events are independent

$0.5000 \qquad 0.5000$

Contingency Table

Political Affiliation	Departement Applied To						Total
	ICS	Science	Law	Education	Liberal Studies	Arts	
Democrat	45	14	23	22	42	16	162
Republican	17	38	13	42	33	7	150
Total	62	52	36	64	75	23	312

Above is a *contingency table* of job applicant's political affiliation and the department applied to. Note that the two variables are categorical.

- What is the probability the applicant is a democrat and is applying to the science department?

$$P(D \cap S) = \frac{14}{312} = \boxed{0.0449}$$

- What is the probability an applicant is a republican?

$$P(R) = \frac{150}{312} = \boxed{0.4808}$$

- What is the probability an applicant applied to the arts department ?

$$P(A) = \frac{23}{312} = \boxed{0.0737}$$

- What is the probability that an applicant applied to the law department given that they were a republican?

$$P(L|R) = \frac{P(L \cap R)}{P(R)} = \frac{\frac{13}{312}}{\frac{150}{312}} = \frac{13}{312} \cdot \frac{312}{150} = \frac{13}{150} = \boxed{0.0867}$$

- What is the probability that an applicant applied to the ICS department or they were a democrat?

$$P(ICS \cup D) = P(ICS) + P(D) - P(I \cap D)$$

$$= \frac{62}{312} + \frac{162}{312} - \frac{45}{312} = \frac{179}{312} = \boxed{0.5737}$$

Contingency Table

Political Affiliation	Departement Applied To						Total
	ICS	Science	Law	Education	Liberal Studies	Arts	
Democrat	45	14	23	22	42	16	162
Republican	17	38	13	42	33	7	150
Total	62	52	36	64	75	23	312

- What is the probability an applicant is a democrat given they applied to the education department?

$$P(D|EDU) = \frac{P(D \cap EDU)}{P(EDU)} = \frac{\frac{22}{312}}{\frac{64}{312}} = \frac{22}{64} = \boxed{0.3438}$$

- What is the probability that an applicant was is a democrat?

$$P(D) = \frac{162}{312} = \boxed{0.5192}$$

- Compare the last two probabilities. Is an applicant's political party affiliation independent of which department they apply to?

$$P(D|EDU) \neq P(D)$$

Thus the events are dependent

Tree Diagrams

Tree diagrams are useful when we initially know conditional probabilities.

- We can use them to work backwards to understand the probability of events occuring at the same time (the intersection)
- We can use the intersections to calculate many other probabilities

Example: $P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$ $\quad P(X \cap Y) = P(X|Y)P(Y)$

Assume we have three events A, B, and C. Given we have seen any of these events, there is also an associated event of success or failure. Construct a tree diagram to calculate the intersections for the events.

$P(A)$

$P(S|A) \longrightarrow P(S \cap A) = P(S|A)P(A) =$

$P(S^c|A) \longrightarrow P(S^c \cap A) = P(S^c|A)P(A) =$

$P(B)$

$P(S|B) \longrightarrow P(S \cap B) = P(S|B)P(B) =$

$P(S^c|B) \longrightarrow P(S^c \cap B) = P(S^c|B)P(B) =$

$P(C)$

$P(S|C) \longrightarrow P(S \cap C) = P(S|C)P(C) =$

$P(S^c|C) \longrightarrow P(S^c \cap C) = P(S^c|C)P(C) =$

$P(S) = P(S \cap A) + P(S \cap B) + P(S \cap C)$

Tree Diagrams

Example.

Medical devices are tested before they go on the market to test their overally capabilitieis. A new bioassay type exam is developed to test for the presence of a certain type of disease. The exam will correctly return a positive test, $+$, when the person actually has the disease, D, with probability 0.95. The exam will incorrectly return a positive test when the person actually does not have the disease, D^c, with probability 0.02. The prevalence of the disease is 0.01.

$P(+|D) = 0.95$

$P(+|D^c) = 0.02$

$P(D) = 0.01$

Use a tree diagram to calculate the probability of getting a positive test on the medical exam.

$P(D)$ 0.01

- $P(+|D)$ 0.95 $\longrightarrow P(+\cap D) = (0.01)(0.95) = 0.0095$
- $P(-|D)$ 0.05 $\longrightarrow P(-\cap D) = (0.01)(0.05) = 0.0005$

$P(D^c)$ 0.99

- $P(+|D^c)$ 0.02 $\longrightarrow P(+\cap D^c) = (0.99)(0.02) = 0.0198$
- $P(-|D^c)$ 0.98 $\longrightarrow P(-\cap D^c) = (0.99)(0.98) = 0.9702$

$P(+) = P(+\cap D) + P(+\cap D^c) = 0.0095 + 0.0198 = 0.0293$

$P(-) = 1 - 0.0293 =$

Tree Diagram

Example:

Continued from previous page.

- What is the probability of having the disease given a positive medical result.

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{0.0095}{0.0293} = 0.3242$$

- What is the probability of having the disease?

$$P(D) = 0.01$$

- Are having the disease and getting a positive result independent events?

$P(D|+) \stackrel{?}{=} P(D)$ Thus the two events

$0.3242 \neq 0.01$ are dependent.

- If they were independent, then $P(D) = P(D|+)$.