6.4

Recall 4.7

Transition matrix

Let V be a vector space

B and C be ordered bases for V

Transition matrix from B to C, $\underbrace{[P]_B^C}_{\text{not used in the book}}$ or $P_{C\leftarrow B}$

is a matrix satisfying

$[v]_C = [P]_B^C [v]_B \quad \forall \in V$

Recall

If $B = \{v_1, v_2, \ldots, v_n\}$

then $[P]_B^C = [\,[v_1]_C, [v_2]_C, \ldots, [v_n]_C\,]$

eg. Let $V = \mathbb{R}^2$, $B = \{(2,1), (3,1)\}$, $C = \{(1,2), (2,3)\}$

① find $[(9,4)]_B$

② find $[(9,4)]_C$

③ find $[P]_B^C$

④ verify $[(9,4)]_C = [P]_B^C [(9,4)]_B$

① $[(9,4)]_B = ?$

Express $(9,4)$ using B

$(9,4) = c_1(2,1) + c_2(3,1)$

skip 3 steps

$$\left[\begin{array}{cc:c} 2 & 3 & 9 \\ 1 & 1 & 4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right] \qquad \begin{array}{l} c_1 = 3 \\ c_2 = 1 \end{array}$$

$$[(9,4)]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

② $[(9,4)]_C = ?$

Express $(9,4)$ using C

$(9,4) = c_1(1,2) + c_2(2,3)$

skip 3 steps

$$\left[\begin{array}{cc:c} 1 & 2 & 9 \\ 2 & 3 & 4 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{cc:c} 1 & 0 & -19 \\ 0 & 1 & 14 \end{array}\right] \qquad \begin{array}{l} c_1 = -19 \\ c_2 = 14 \end{array}$$

$$[(9,4)]_C = \begin{bmatrix} -19 \\ 14 \end{bmatrix}$$

③ find $[P]_B^C$

$$[P]_B^C = [\,[(2,1)]_C, [(3,1)]_C\,]$$

$$(2,1) = c_1(1,2) + c_2(3,1)$$

$$(3,1) = c_1(1,2) + c_2(3,1)$$

skip 3 steps

$$\left[\begin{array}{cc:cc} 1 & 3 & 2 & 3 \\ 2 & 1 & 1 & 1 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{cc:cc} 1 & 0 & -4 & -7 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

$$[P]_B^C = \begin{bmatrix} -4 & -7 \\ 3 & 5 \end{bmatrix}$$

④ verify

$$\begin{bmatrix} -4 & -7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12-7 \\ 9+5 \end{bmatrix} = \begin{bmatrix} -19 \\ 14 \end{bmatrix}$$

Back to 6.4

Recall Let A, B be $n \times n$ matrices

A and B are similar matrices

if $\exists P$ with $\underbrace{B = P^{-1}AP}$

$$B = \square^{-1} A \square \text{ form}$$

Theorem

Let A be a vector space, B and C bases for V

$\uparrow$ B' is used in the book

Let $T: V \to V$ be a LT

same

Then $[T]_B^B$ and $[T]_C^C$ are similar matrices

Proof:

$$\begin{tikzcd}
\underset{B}{V} \arrow[r, bend left, "T"] \arrow[r, phantom, "{[T]_B^B}"'] & \underset{B}{V} \arrow[d, "\text{transition matrix}"] \\
\underset{C}{V} \arrow[u, "\text{transition matrix}"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "{[T]_C^C}"] & \underset{C}{V}
\end{tikzcd}$$

$[T]_C^C = [P]_B^C [T]_B^B [P]_C^B$

But $[P]_B^C = ([P]_C^B)^{-1}$

$\therefore [T]_C^C = ([P]_C^B)^{-1} [T]_B^B [P]_C^B$

$\therefore [T]_B^B$ and $[T]_C^C$ are similar

P. 334 #14

Let $A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$ $[T]_B^B$ $\quad B = \{(1,1), (-2,3)\}$

$C = \{(1,-1), (0,1)\}$

$[v]_C = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

ⓐ find $[P]_C^B$

ⓑ use $[P]_C^B$ and A to find $[v]_B$ and $[T(v)]_B$

ⓒ Find P^{-1} and $[T]_C^C$

ⓓ find $[T(v)]_C$ in two ways

a) $[P]_C^B = [\,[(1,-1)]_B\,,\,[(0,1)]_B\,]$

$$\left.\begin{aligned}(1,-1) &= c_1(1,1) + c_2(-2,3)\\ (0,1) &= d_1(1,1) + d_2(-2,3)\end{aligned}\right]$$ don't skip this

skip 3 steps

$$\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 1 & 3 & -1 & 1 \end{array}\right] \overset{\text{RREF}}{\Rightarrow} \left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{1}{5} \end{array}\right]$$

decimal → fraction

Math > D → F (#1)

$$\therefore [P]_C^B = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

b) $[v]_B = [P]_C^B [v]_C$

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$[T(v)]_B = [T]_B^B [v]_B$

(general formal : $[T(v)]_C = [T]_B^C [v]_B$)

$$[T(v)]_B = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$$

c) $$([P]_C^B)^{-1} = [P]_B^C = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$[T]_C^C$

$$\begin{tikzcd} C \arrow[r] \arrow[d, "①"'] & C \\ B \arrow[r, "②"'] & B \arrow[u, "③"'] \end{tikzcd}$$

method 1

$$[T]_C^C = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -14 \end{bmatrix}$$

$[T(v)]_C = [P]_B^C [T(v)]_B$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -14 \end{bmatrix}$$

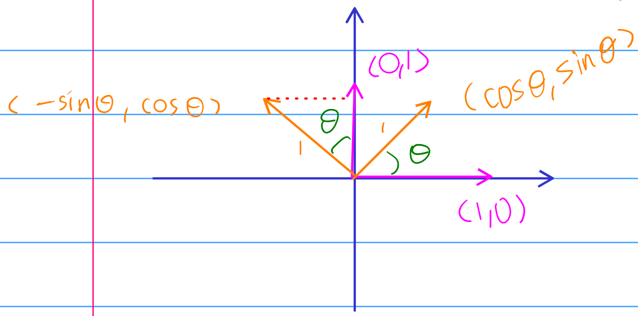
6.5 Application of Linear Transformations

Recall: In $\mathbb{R}^2$, any LT T can be represented in $[T]_B^C$

with $B = C = \{(1,0), (0,1)\}$

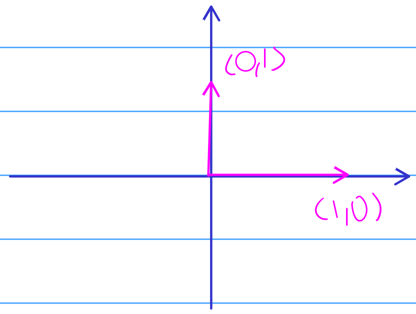
$[T]_B^C = [\,[T(1,0)]_C, [T(0,1)]_C\,]$

① 2 dimensional rotation by angle θ



$$\Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

② Reflection about the x-axis

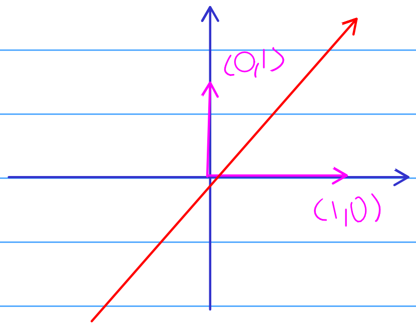


$T(1,0) = (1,0)$

$T(0,1) = (0,-1)$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

③ Reflection about the line $y = x$



$T(1,0) = (0,1)$

$T(0,1) = (1,0)$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

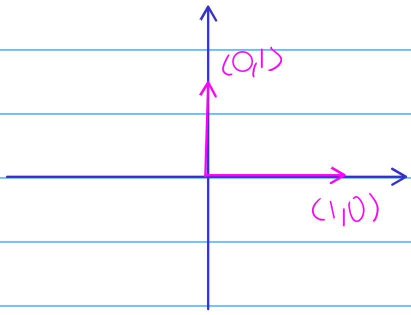
eg. Find the matrix of the following:

first rotate an object $60°$ counterclockwise

then reflect is about the y-axis

$$[T_1] = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$