

Exercises 1.1

2. Which of these are propositions? What are the truth values of those that are propositions?

a) Do not pass go.

No, it is not a proposition.

b) What time is it?

No, it is not a proposition.

c) There are no black flies in Maine.

Yes, it is a proposition. The truth value is False.

d) $4 + x = 5$.

No, it is not a proposition.

e) The moon is made of green cheese.

Yes, it is a proposition. The truth value is False.

f) $2n \geq 100$.

No, it is not a proposition.

4. What is the negation of each of these propositions?

a) Jennifer and Teja are friends.

Jennifer and Teja are not friends.

b) There are 13 items in a baker's dozen.

There are less or more than 13 items in a baker's dozen.

c) Abby sent more than 100 text messages every day.

Abby sent no more than 100 text messages every day.

d) 121 is a perfect square.

121 is not a perfect square.

6. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone B has the most RAM of these three smartphones.

True.

b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.

True.

c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.

False.

d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.

False.

e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

False.

Exercises 1.2

2. You can see the movie only if you are over 18 years old or you have the permission of a parent.

Express your answer in terms of m: "You can see the movie," e: "You are over 18 years old," and p: "You have the permission of a parent."

$$m \rightarrow (e \vee p)$$

4. To use the wireless network in the airport you must pay the daily fee unless you are a subscriber to the service. Express your answer in terms of w: "You can use the wireless network in the airport," d: "You pay the daily fee," and s: "You are a subscriber to the service."

$$w \rightarrow (d \vee s)$$

Exercises 1.3

6. Use a truth table to verify the first De Morgan law $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$$\therefore \neg(p \wedge q) \equiv \neg p \vee \neg q$$

8. Use De Morgan's laws to find the negation of each of the following statements.

a) Kwame will take a job in industry or go to graduate school.

Let $p \equiv$ "Kwame will take a job in industry," and $q \equiv$ "Kwame will go to graduate school," then "Kwame will take a job in industry or go to graduate school" $\equiv p \vee q$.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

To express in English, Kwame will not take a job in industry, and Kwame will not go to graduate school.

b) Yoshiko knows Java and calculus.

Let $p \equiv$ "Yoshiko knows Java," and $q \equiv$ "Yoshiko knows calculus," then "Yoshiko knows Java and calculus" $\equiv p \wedge q$.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

To express in English, Yoshiko does not know Java, or Yoshiko does not know calculus.

c) James is young and strong.

Let $p \equiv$ "James is young," and $q \equiv$ "James is strong," then James is young and strong" $\equiv p \wedge q$.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

To express in English, James is not young, or James is not strong.

d) Rita will move to Oregon or Washington

Let $p \equiv$ "Rita will move to Oregon," and $q \equiv$ "Rita will move to Washington," then "Rita will move to Oregon or Washington" $\equiv p \vee q$.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

To express in English, Rita will not move to Oregon and Rita will not move to Washington.

10. Show that each of these conditional statements is a tautology by using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

$\therefore [\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology.

b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$\therefore [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.

c) $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$\therefore [p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$	$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

$\therefore [(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ is a tautology.

12. Show that each conditional statement in Exercise 10 is a tautology without using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$

$[\neg p \wedge (p \vee q)] \rightarrow q$

$$\begin{aligned}
&\equiv \neg[\neg p \wedge (p \vee q)] \vee q \\
&\equiv \neg\neg p \vee \neg(p \vee q) \vee q \\
&\equiv p \vee q \vee \neg(p \vee q) \\
&\equiv (p \vee q) \vee \neg(p \vee q) ; \text{Negation Laws: } p \vee \neg p \equiv T \\
&\equiv T
\end{aligned}$$

$$\text{b) } [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

$$\begin{aligned}
&[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r) \\
&\equiv \neg [(p \rightarrow q) \wedge (q \rightarrow r)] \vee (p \rightarrow r) \\
&\equiv \neg (p \rightarrow q) \vee \neg(q \rightarrow r) \vee (p \rightarrow r) \\
&\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r) \\
&\equiv (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee (\neg p \vee r) \\
&\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)
\end{aligned}$$

$$\text{c) } [p \wedge (p \rightarrow q)] \rightarrow q$$

$$\begin{aligned}
&[p \wedge (p \rightarrow q)] \rightarrow q \\
&\equiv \neg [p \wedge (p \rightarrow q)] \vee q \\
&\equiv \neg [p \wedge (\neg p \vee q)] \vee q \\
&\equiv \neg p \vee \neg(\neg p \vee q) \vee q \\
&\equiv \neg p \vee \neg\neg p \wedge \neg q \vee q \\
&\equiv \neg p \vee p \wedge \neg q \vee q \\
&\equiv (\neg p \vee p) \wedge (\neg q \vee q) ; \text{Negation Laws: } p \vee \neg p \equiv T \\
&\equiv T \wedge T \\
&\equiv T
\end{aligned}$$

$$\text{d) } [(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$$

$$\begin{aligned}
&\equiv [(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r \\
&\equiv \neg [(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \vee r \\
&\equiv \neg(p \vee q) \vee \neg(p \rightarrow r) \vee \neg(q \rightarrow r) \vee r
\end{aligned}$$

$$\begin{aligned}
&\equiv \neg p \wedge \neg q \vee \neg(\neg p \vee r) \vee \neg(\neg q \vee r) \vee r \\
&\equiv \neg p \wedge \neg q \vee \neg\neg p \wedge \neg r \vee \neg\neg q \wedge \neg r \vee r \\
&\equiv \neg p \wedge \neg q \vee p \wedge \neg r \vee q \wedge \neg r \vee r \\
&\equiv \neg p \wedge \neg q \vee p \wedge \neg r \vee q \wedge (\neg r \vee r) \\
&\equiv \neg p \wedge \neg q \vee p \wedge \neg r \vee (q \wedge \top)
\end{aligned}$$

Exercises 1.4

6. Let $N(x)$ be the statement “ x has visited North Dakota,” where the domain consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$

Some students in your school have visited North Dakota.

b) $\forall x N(x)$

All students in your school have visited North Dakota.

c) $\neg \exists x N(x)$

No student in your school has visited North Dakota.

d) $\exists x \neg N(x)$

Some students in your school have not visited North Dakota.

e) $\neg \forall x N(x)$

Some students in your school have not visited North Dakota.

f) $\forall x \neg N(x)$

All students in your school have not visited North Dakota.

8. Translate these statements into English, where $R(x)$ is “ x is a rabbit” and $H(x)$ is “ x hops” and the domain consists of all animals.

a) $\forall x (R(x) \rightarrow H(x))$

If it is a rabbit, then it hops.

b) $\forall x (R(x) \wedge H(x))$

Every animal is a rabbit, and it hops.

c) $\exists x (R(x) \rightarrow H(x))$

There exists an animal such that if it is a rabbit, then it hops.

d) $\exists x(R(x) \wedge H(x))$

There exists an animal such that it is a rabbit, and it hops.

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$

True.

b) $\exists x(x^4 < x^2)$

False.

c) $\forall x((-x)^2 = x^2)$

True.

d) $\forall x(2x > x)$

False.

Exercises 1.5

2. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\exists x \forall y(xy = y)$

There exists a real number x for every real number y such that x times y is equal to y .

b) $\forall x \forall y((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0)$

For every real number x and real number y , if x is equal or greater than zero and y is less than zero, then x minuses y is greater than zero.

c) $\forall x \forall y \exists z(x = y + z)$

For every real number x and real number y there exists a real number z such that x is equal to y pluses z .

6. Let $C(x, y)$ mean that student x is enrolled in class y , where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

a) $C(\text{Randy Goldberg}, \text{CS 252})$

Randy Goldberg is enrolled in CS 252.

b) $\exists x C(x, \text{Math 695})$

Some students in your school is enrolled in Math 695.

c) $\exists y C(\text{Carol Sitea}, y)$

Carol Sitea is enrolled in some classes.

d) $\exists x (C(x, \text{Math 222}) \wedge C(x, \text{CS 252}))$

Some students in your school enrolled in Math 222 and CS 252.

e) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$

There are two students both enrolled in the same classes.

f) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

There are two students at your school enrolled in exactly the same classes.

10. Let $F(x, y)$ be the statement “ x can fool y ,” where the domain consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Fred.

$\forall x F(x, \text{Fred})$

b) Evelyn can fool everybody.

$\forall y F(\text{Evelyn}, y)$

c) Everybody can fool somebody.

$\forall x \exists y F(x, y)$

d) There is no one who can fool everybody.

$\neg \exists x \forall y F(x, y)$

e) Everyone can be fooled by somebody.

$\exists x \forall y F(x, y)$

f) No one can fool both Fred and Jerry.

$\neg \exists x (F(x, \text{Fred}) \wedge F(x, \text{Jerry}))$

g) Nancy can fool exactly two people.

$\exists y \exists z ((y \neq z) \wedge F(\text{Nancy}, y) \wedge F(\text{Nancy}, z))$

h) There is exactly one person whom everybody can fool.

$\exists y (\forall x F(x, y) \wedge \forall z (\forall x F(x, z)) \rightarrow z=y)$

i) No one can fool himself or herself.

$$\neg \exists x F(x, x)$$

j) There is someone who can fool exactly one person besides himself or herself.

$$\exists x \exists y (x \neq y \wedge F(x, y) \wedge \forall z (F(x, z) \rightarrow z = y))$$

Exercises 1.6

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not a spider.

George is a spider.

\therefore George has eight legs.

p: "George does not have eight legs"

q: "George is not a spider"

First statement: $p \rightarrow q$

Second Statement: $\neg q$

The conclusion is: $\neg p$

4. What rule of inference is used in each of these arguments?

a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

$$p \wedge q$$

$$\therefore q$$

Simplification

b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

$$p \vee q$$

$$\neg p$$

$$\therefore q$$

Disjunctive syllogism

c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

P

$p \rightarrow q$

$\therefore q$

Modus ponens

d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.

p

$\therefore P \vee q$

Addition

e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Hypothetical syllogism

8. What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."

$M(x) = \text{"x is a man"}$

$I(x) = \text{"x is an island"}$

$\forall x M(x) \rightarrow \neg I(x)$

$M(\text{Manhattan}) \rightarrow \neg I(\text{Manhattan})$

Exercises 1.7

6. Use a direct proof to show that the product of two odd numbers is odd.

Assume m and n are integers and m and n are odd,

$m * n = (2k + 1) * (2h + 1)$ for some integers k and h

$m * n = 4kh + 2k + 2h + 1$

$$m * n = 2(2kh + k + h) + 1$$

$$m * n = 2r + 1 \quad \text{for } r = 2kh + k + h$$

$\therefore m * n$ is odd.

18. Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition.

Assume n is odd,

$$n = 2k + 1 \quad \text{for some integers } k$$

$$3n + 2 = 3(2k + 1) + 2$$

$$3n + 2 = 6k + 3 + 2$$

$$3n + 2 = 6k + 4 + 1$$

$$3n + 2 = 2(3k + 2) + 1$$

$$3n + 2 = 2r + 1 \quad \text{for } r = (3k + 2)$$

$\therefore 3n + 2$ is odd.

Since if $3n + 2$ is odd, then n is odd is true, if $3n + 2$ is even, then n is even must also be true.

\therefore if n is an integer and $3n + 2$ is even, then n is even.

b) a proof by contradiction.

Assume $3n + 2$ is even and n is odd,

$$n = 2k + 1 \quad \text{for some integers } k$$

$$3n + 2 = 3(2k + 1) + 2$$

$$3n + 2 = 6k + 3 + 2$$

$$3n + 2 = 6k + 4 + 1$$

$$3n + 2 = 2(3k + 2) + 1$$

$$3n + 2 = 2r + 1 \quad \text{for } r = 3k + 2$$

$3n + 2$ is odd.

This contradicts the assumption that $3n + 2$ is even.

\therefore if $3n + 2$ is even, then n must also be even

24. Show that at least three of any 25 days chosen must fall in the same month of the year.

If there are at most two days falling in the same month, then we could have at most $2 * 12 = 24$ days, since there are 12 months. Since we have 25 days, at least three of them must fall in the same month.

28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

i) First: prove if $m = n$, then $m^2 = n^2$

$$m = n$$

$$m^2 = n^2$$

ii) Second: prove if $m = -n$, then $m^2 = n^2$

$$m = -n$$

$$m^2 = (-n)^2$$

$$m^2 = n^2$$

$\therefore m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Exercise 2.1

10. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$

True.

b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

True.

c) $\{\emptyset\} \in \{\emptyset\}$

False.

d) $\{\emptyset\} \in \{\{\emptyset\}\}$

True.

e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

True.

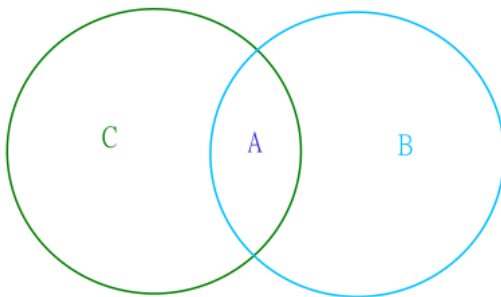
f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

True.

g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

False.

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.



24. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) \emptyset

$\emptyset = \{\emptyset, \{\emptyset\}\}$

It is not a power set.

b) $\{\emptyset, \{a\}\}$

It is a power set of $\{a\}$.

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

Since it contains three elements, it is not a power set.

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

It is a power set of $\{a, b\}$.

32. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$.

$$A \times B \times C = \{a, b, c\} \times \{x, y\} \times \{0, 1\}$$

$$A \times B \times C = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\} \times \{0, 1\}$$

$$A \times B \times C = \{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$$

b) $C \times B \times A$.

$$C \times B \times A = \{0, 1\} \times \{x, y\} \times \{a, b, c\}$$

$$C \times B \times A = \{(0, x), (0, y), (1, x), (1, y)\} \times \{a, b, c\}$$

$$C \times B \times A = \{(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)\}$$

c) $C \times A \times B$.

$$C \times A \times B = \{0, 1\} \times \{a, b, c\} \times \{x, y\}$$

$$C \times A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\} \times \{x, y\}$$

$$C \times A \times B = \{(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y)\}$$

d) $B \times B \times B$.

$$B \times B \times B = \{x, y\} \times \{x, y\} \times \{x, y\}$$

$$B \times B \times B = \{(x, x), (x, y), (y, x), (y, y)\} \times \{x, y\}$$

$$B \times B \times B = \{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$$

Exercise 2.2

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school

$$A \cap B$$

b) the set of sophomores at your school who are not taking discrete mathematics

$$A \cap \neg B$$

c) the set of students at your school who either are sophomores or are taking discrete mathematics

$$A \cup B$$

d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

$$\neg A \cup \neg B$$

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$.

$$A \cup B = \{a, b, c, d, e, f, g, h\}$$

b) $A \cap B$.

$$A \cap B = \{a, b, c, d, e\}$$

c) $A - B$.

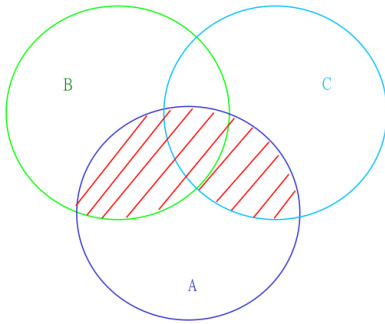
$$A - B = \{\emptyset\}$$

d) $B - A$.

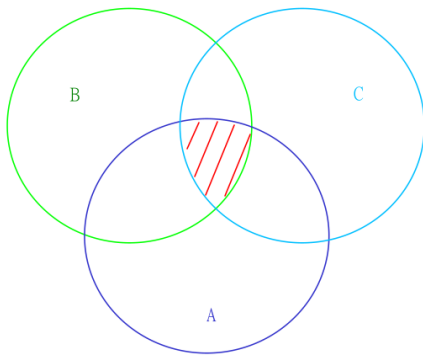
$$B - A = \{f, g, h\}$$

26. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

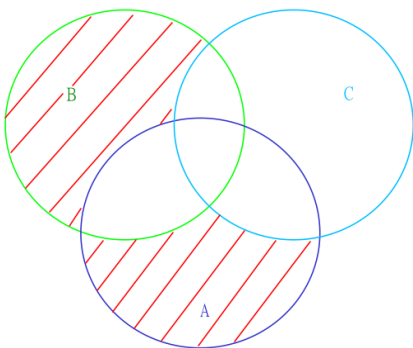
a) $A \cap (B \cup C)$



b) $A \cap B \cap C$



c) $(A - B) \cup (A - C) \cup (B - C)$



Exercise 2.3

8. Find these values.

a) [1.1]

= 1

b) [1.1]

= 2

c) $\lfloor -0.1 \rfloor$

$= -1$

d) $\lceil -0.1 \rceil$

$= 0$

e) $\lceil 2.99 \rceil$

$= 3$

f) $\lfloor -2.99 \rfloor$

$= -2$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

$= 1$

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

$= 2$

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

Yes, it is one-to-one.

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

No, it is not one-to-one, because $f(a) = f(b)$.

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

No, it is not one-to-one, because $f(a) = f(d)$.

14. Determine whether $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ is onto if

a) $f(m, n) = 2m - n$.

Yes, it is onto.

b) $f(m, n) = m^2 - n^2$.

No, it is not onto.

c) $f(m, n) = m + n + 1$.

Yes, it is onto.

d) $f(m, n) = |m| - |n|$.

Yes, it is onto.

e) $f(m, n) = m^2 - 4$.

No, it is not onto.

20. Give an example of a function from \mathbb{N} to \mathbb{N} that is

a) one-to-one but not onto.

$$f(n) = 3n$$

b) onto but not one-to-one.

$$f(n) = n$$

c) both onto and one-to-one (but different from the identity function).

$$f(n) = n^2$$

d) neither one-to-one nor onto.

$$f(n) = n^2$$

Exercise 2.4

2. What is the term a_8 of the sequence $\{a_n\}$ if a_n equals

a) 2^{n-1} ?

$$2^{8-1} = 2^7 = 128$$

b) 7?

$$7$$

c) $1 + (-1)^n$?

$$1 + (-1)^8 = 0$$

d) $-(-2)^n$?

$$-(-2)^8 = -256$$

4. What are the terms a_0 , a_1 , a_2 , and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $(-2)^n$?

$$a_0 = (-2)^0 = 1$$

$$a_1 = (-2)^1 = -2$$

$$a_2 = (-2)^2 = 4$$

$$a_3 = (-2)^3 = -8$$

b) 3?

$$a_0 = 3$$

$$a_1 = 3$$

$$a_2 = 3$$

$$a_3 = 3$$

c) $7 + 4^n$?

$$a_0 = 7 + 4^0 = 8$$

$$a_1 = 7 + 4^1 = 11$$

$$a_2 = 7 + 4^2 = 23$$

$$a_3 = 7 + 4^3 = 71$$

d) $2^n + (-2)^n$?

$$a_0 = 2^0 + (-2)^0 = 2$$

$$a_1 = 2^1 + (-2)^1 = 0$$

$$a_2 = 2^2 + (-2)^2 = 8$$

$$a_3 = 2^3 + (-2)^3 = 0$$

6. List the first 10 terms of each of these sequences.

a) the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term

$$a_1 = 10$$

$$a_2 = 10 - 3 = 7$$

$$a_3 = 7 - 3 = 4$$

$$a_4 = 4 - 3 = 1$$

$$a_5 = 1 - 3 = -2$$

$$a_6 = -2 - 3 = -5$$

$$a_7 = -5 - 3 = -8$$

$$a_8 = -8 - 3 = -11$$

$$a_9 = -11 - 3 = -14$$

$$a_{10} = -14 - 3 = -17$$

b) the sequence whose n^{th} term is the sum of the first n positive integers

$$a_1 = 1$$

$$a_2 = 1 + 3 = 4$$

$$a_3 = 1 + 3 + 5 = 9$$

$$a_4 = 1 + 3 + 5 + 7 = 16$$

$$a_5 = 1 + 3 + 5 + 7 + 9 = 25$$

$$a_6 = 1 + 3 + 5 + 7 + 9 + 11 = 36$$

$$a_7 = 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$$

$$a_8 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64$$

$$a_9 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$$

$$a_{10} = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100$$

c) the sequence whose n^{th} term is $3^n - 2^n$

$$a_1 = 3^1 - 2^1 = 1$$

$$a_2 = 3^2 - 2^2 = 5$$

$$a_3 = 3^3 - 2^3 = 19$$

$$a_4 = 3^4 - 2^4 = 65$$

$$a_5 = 3^5 - 2^5 = 211$$

$$a_6 = 3^6 - 2^6 = 665$$

$$a_7 = 3^7 - 2^7 = 2059$$

$$a_8 = 3^8 - 2^8 = 6305$$

$$a_9 = 3^9 - 2^9 = 19171$$

$$a_{10} = 3^{10} - 2^{10} = 58025$$

d) the sequence whose n^{th} term is $\lfloor \sqrt{n} \rfloor$

$$a_1 = \lfloor \sqrt{1} \rfloor = 1$$

$$a_2 = \lfloor \sqrt{2} \rfloor = 1$$

$$a_3 = \lfloor \sqrt{3} \rfloor = 1$$

$$a_4 = \lfloor \sqrt{4} \rfloor = 2$$

$$a_5 = \lfloor \sqrt{5} \rfloor = 2$$

$$a_6 = \lfloor \sqrt{6} \rfloor = 2$$

$$a_7 = \lfloor \sqrt{7} \rfloor = 2$$

$$a_8 = \lfloor \sqrt{8} \rfloor = 2$$

$$a_9 = \lfloor \sqrt{9} \rfloor = 3$$

$$a_{10} = \lfloor \sqrt{10} \rfloor = 3$$

e) the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms

$$a_1 = 1$$

$$a_2 = 5$$

$$a_3 = 1 + 5 = 6$$

$$a_4 = 5 + 6 = 11$$

$$a_5 = 6 + 11 = 17$$

$$a_6 = 11 + 17 = 28$$

$$a_7 = 17 + 28 = 45$$

$$a_8 = 28 + 45 = 73$$

$$a_9 = 45 + 73 = 118$$

$$a_{10} = 73 + 118 = 191$$

f) the sequence whose n^{th} term is the largest integer whose binary expansion (defined in Section 4.2) has n bits (Write your answer in decimal notation.)

$$a_1 = 1$$

$$a_2 = 2$$

$$a_3 = 4$$

$$a_4 = 8$$

$$a_5 = 16$$

$$a_6 = 32$$

$$a_7 = 64$$

$$a_8 = 128$$

$$a_9 = 256$$

$$a_{10} = 512$$

g) the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on

$$a_1 = 1$$

$$a_2 = 1 + 1 = 2$$

$$a_3 = 2 * 1 = 2$$

$$a_4 = 2 + 2 = 4$$

$$a_5 = 4 * 2 = 8$$

$$a_6 = 8 + 3 = 11$$

$$a_7 = 11 * 3 = 33$$

$$a_8 = 33 + 4 = 37$$

$$a_9 = 37 * 4 = 148$$

$$a_{10} = 148 + 5 = 153$$

h) the sequence whose n^{th} term is the largest integer k such that $k! \leq n$

$$a_1 = 1, 1! \leq 1$$

$$a_2 = 2, 2! \leq 2$$

$$a_3 = 2, 2! \leq 3$$

$$a_4 = 2, 2! \leq 4$$

$$a_5 = 2, 2! \leq 5$$

$$a_6 = 3, 3! \leq 6$$

$$a_7 = 3, 3! \leq 7$$

$$a_8 = 3, 3! \leq 8$$

$$a_9 = 3, 3! \leq 9$$

$$a_{10} = 3, 3! \leq 10$$

Exercise 2.6

2. Find $A + B$, where

$$\text{a) } A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix} B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 + (-1) & 0 + 3 & 4 + 5 \\ -1 + 2 & 2 + 2 & 2 + (-3) \\ 0 + 2 & -2 + (-3) & -3 + 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix} B = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -1 + (-3) & 0 + 9 & 5 + (-3) & 6 + 4 \\ -4 + 0 & -3 + (-2) & 5 + (-1) & -2 + 2 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix}$$

4. Find the product AB , where

$$\text{a) } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(0) + 0(1) + 1(-1) & 1(1) + 0(-1) + 1(0) & 1(-1) + 0(0) + 1(1) \\ 0(0) + (-1)(1) + (-1)(-1) & 0(1) + (-1)(-1) + (-1)(0) & 0(-1) + (-1)(0) + (-1)(1) \\ -1(0) + 1(1) + 0(-1) & -1(1) + 1(-1) + 0(0) & -1(-1) + 1(0) + 0(1) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix} B = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$$

$AB =$

$$\begin{bmatrix} 1(1) + (-3)(-1) + 0(-3) & 1(-1) + (-3)(0) + 0(-2) & 1(2) + (-3)(3) + 0(0) & 1(3) + (-3)(-1) + 0(2) \\ 1(1) + 2(-1) + 2(-3) & 1(-1) + 2(0) + 2(-2) & 1(2) + 2(3) + 2(0) & 1(3) + 2(-1) + 2(2) \\ 2(1) + 1(-1) + (-1)(-3) & 2(-1) + 1(0) + (-1)(-2) & 2(2) + 1(3) + (-1)(0) & 2(3) + 1(-1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 4 & -1 & -7 & 6 \\ -7 & -5 & 8 & 5 \\ 4 & 0 & 7 & 3 \end{bmatrix}$$

$$\text{c) } A = \begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix} B = \begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 & -3 & -4 & -1 \\ 24 & -7 & 20 & 29 & 2 \\ -10 & 4 & -17 & -24 & -3 \end{bmatrix}$$

26. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Find

a) $A \vee B$.

$$A \vee B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

b) $A \wedge B$.

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

c) $A \odot B$.

$$A \odot B = \begin{bmatrix} (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 0) \end{bmatrix} = \begin{bmatrix} 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Exercise 3.1

4. Describe an algorithm that takes as input a list of n integers and produces as output the largest difference obtained by subtracting an integer in the list from the one following it.

`procedure max difference(a_1, a_2, \dots, a_n : integers)`

`max := $a_2 - a_1$`

`for i := 2 to n`

`if $(a_{i+1} - a_i) > \text{max}$`

`max := $a_{i+1} - a_i$`

`return max {max is the largest difference}`

12. Describe an algorithm that uses only assignment statements that replaces the triple (x, y, z) with (y, z, x) . What is the minimum number of assignment statements needed?

`procedure interchange (x, y, z : integers)`

`t := x`

`x := y`

`y := z`

`z := t`

end procedure

34. Use the bubble sort to sort 6, 2, 3, 1, 5, 4, showing the lists obtained at each step.

bubble sort

first pass

{6, 2, 3, 1, 5, 4}

{2, 6, 3, 1, 5, 4}

{2, 3, 6, 1, 5, 4}

{2, 3, 1, 6, 5, 4}

{2, 3, 1, 5, 6, 4}

{2, 3, 1, 5, 4, 6}

second pass

{2, 3, 1, 5, 4, 6}

{2, 3, 1, 5, 4, 6}

{2, 1, 3, 5, 4, 6}

{2, 1, 3, 5, 4, 6}

{2, 1, 3, 4, 5, 6}

third pass

{2, 1, 3, 4, 5, 6}

{1, 2, 3, 4, 5, 6}

{1, 2, 3, 4, 5, 6}

{1, 2, 3, 4, 5, 6}

fourth pass

{1, 2, 3, 4, 5, 6}

{1, 2, 3, 4, 5, 6}

fifth pass

{1, 2, 3, 4, 5, 6}

{1, 2, 3, 4, 5, 6}

Exercise 3.2

2. Determine whether each of these functions is $O(x^2)$.

a) $f(x) = 17x + 11$

$$f(x) = 17x + 11$$

$$= 17x + x, \text{ where } x \geq 11$$

$$= 18x \leq 18x^2, \text{ for } \forall x > 11$$

$$C = 18, k = 11$$

b) $f(x) = x^2 + 1000$

$$f(x) = x^2 + 1000$$

$$\leq x^2 + x^2, \text{ where } x \geq 1000$$

$$= 2x^2, \forall x > 1000$$

$$C = 2, K = 1000$$

c) $f(x) = x \log x$

$$f(x) = x \log x$$

$$= x * x, \text{ where } x = \log x$$

$$= x^2, \forall x$$

$$C = 1, K = 0$$

d) $f(x) = x^4/2$

$$f(x) = x^4/2$$

$$\leq C x^2$$

$$C \geq x^2/2$$

\therefore constant C does not exist

e) $f(x) = 2^x$

$$f(x) = 2^x$$

since x is the largest growth of 2^x

\therefore constant C does not exist

f) $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

$$\begin{aligned}
 f(x) &= \lfloor x \rfloor \cdot \lceil x \rceil \\
 &\leq x(x+1) \\
 &= x^2 + x \\
 &\leq x * 2x \\
 &= 2x^2, \text{ for } \forall x > 1 \\
 C &= 2, K = 1
 \end{aligned}$$

24. Suppose that you have two different algorithms for solving a problem. To solve a problem of size n , the first algorithm uses exactly $n^2 2^n$ operations and the second algorithm uses exactly $n!$ operations. As n grows, which algorithm uses fewer operations?

As n grows, the first algorithm uses fewer operations. In fact, when n is equal or greater than 8, the first algorithm starts outperforming the second algorithm. Here are the operations used by each of the algorithms.

$$8^2 2^8 = 16384$$

$$8! = 40320$$

Exercise 3.3

2. Give a big-O estimate for the number additions used in this segment of an algorithm.

$t := 0$

for $i := 1$ to n

 for $j := 1$ to n

$t := t + i + j$

$$\begin{aligned}
 \sum_{i=1}^n * \sum_{j=1}^n 2 + \sum_{i=1}^n 1 &= \sum_{i=1}^n (2n) + n \\
 &= n * 2n \\
 &= 2n^2
 \end{aligned}$$

The number of additions used is $O(n^2)$

16. What is the largest n for which one can solve within a day using an algorithm that requires $f(n)$ bit operations, where each bit operation is carried out in 10^{-11} seconds, with these functions $f(n)$?

$$\text{Total number of operations} = 10^5 / 10^{-11} = 10^{16}$$

a) $\log n$

$$f(n) = \log n \leq 10^{16}$$

$$\log_2 n \leq 10^{16}$$

$$n \leq 2^{10^{16}}$$

The largest n is $2^{10^{16}}$.

b) $1000n$

$$f(n) = 1000n \leq 10^{16}$$

$$n \leq 10^{16} / 1000$$

$$n \leq 10^{13}$$

The largest n is 10^{13} .

c) n^2

$$f(n) = n^2 \leq 10^{16}$$

$$n^2 \leq 10^{16 * \frac{1}{2}}$$

$$n^2 \leq 10^8$$

The largest n is 10^8 .

d) $1000n^2$

$$f(n) = 1000n^2 \leq 10^{16}$$

$$n^2 \leq 10^{16} / 1000$$

$$n^2 \leq 10^{13}$$

$$n \leq 10^{13/2}$$

The largest n is $10^{13/2}$.

e) n^3

$$f(n) = n^3 \leq 10^{16}$$

$$n^3 \leq 10^{16}$$

$$n \leq 10^{16/3}$$

The largest n is $10^{16/3}$.

f) 2^n

$$f(n) = 2^n \leq 10^{16}$$

$$n \log 2 \leq 16 \log 10$$

$$n \leq 16 \log 10 / \log 2$$

$$n \leq \log_2 10^{16}$$

The largest n is $\log_2 10^{16}$.

$$g) 2^{2^n}$$

$$f(n) = 2^{2^n} \leq 10^{16}$$

$$2n \log 2 \leq 16 \log 10$$

$$n \leq \log_2 10^{16} / 2$$

The largest n is $\log_2 10^{16} / 2$.

$$h) 2^{2^n}$$

$$f(n) = 2^{2^n} \leq 10^{16}$$

$$2^n \log 2 \leq 16 \log 10$$

$$2^n \leq 16 \log 10 / \log 2$$

$$n \log 2 \leq 16 \log 10 / \log 2$$

$$n \leq \log_2 (\log_2 10^{16})$$

18. How much time does an algorithm take to solve a problem of size n if this algorithm uses $2n^2 + 2^n$ operations, each requiring 10^{-9} seconds, with these values of n ?

a) 10

$$f(10) = (2(10)^2 + 2^{10}) * 10^{-9} = 1.224 * 10^{-6}$$

b) 20

$$f(20) = (2(20)^2 + 2^{20}) * 10^{-9} = 1.05 * 10^{-3}$$

c) 50

$$f(50) = (2(50)^2 + 2^{50}) * 10^{-9} = 1.13 * 10^6$$

d) 100

$$f(100) = (2(100)^2 + 2^{100}) * 10^{-9} = 1.27 * 10^{21}$$

Exercise 4.1

6. Show that if a , b , c , and d are integers, where $a \neq 0$, such that $a \mid c$ and $b \mid d$, then $ab \mid cd$.

Suppose $c = a * s$ and $d = b * t$ for some integers s and t

$$c * d = a * s * b * t$$

$$c * d = (a * b)(s * t)$$

$$\therefore ab \mid cd$$

10. What are the quotient and remainder when

a) 44 is divided by 8?

$$44 = 8 * 5 + 4$$

$$\text{Quotient} = 5$$

$$\text{Remainder} = 4$$

b) 777 is divided by 21?

$$777 = 21 * 37 + 0$$

$$\text{Quotient} = 37$$

$$\text{Remainder} = 0$$

c) -123 is divided by 19?

$$-123 = 19 * -7 + 10$$

$$\text{Quotient} = -7$$

$$\text{Remainder} = 10$$

d) -1 is divided by 23?

$$-1 = 23 * -1 + 22$$

$$\text{Quotient} = -1$$

$$\text{Remainder} = 22$$

e) -2002 is divided by 87?

$$-2002 = 87 * -24 + 86$$

$$\text{Quotient} = -24$$

$$\text{Remainder} = 86$$

f) 0 is divided by 17?

$$0 = 17 * 0 + 0$$

$$\text{Quotient} = 0$$

$$\text{Remainder} = 0$$

g) 1,234,567 is divided by 1001?

$$1234567 = 1001 * 1233 + 334$$

$$\text{Quotient} = 1233$$

$$\text{Remainder} = 334$$

h) -100 is divided by 101?

$$-100 = 101 * -1 + 1$$

$$\text{Quotient} = -1$$

$$\text{Remainder} = 1$$

28. Decide whether each of these integers is congruent to 3 modulo 7.

a) 37

$$37 \equiv 3 \pmod{7}?$$

$$37 \bmod 7 = 2 \neq 3$$

\therefore No.

b) 66

$$66 \equiv 3 \pmod{7}?$$

$$66 \bmod 7 = 3$$

\therefore Yes.

c) -17

$$-17 \equiv 3 \pmod{7}$$

$$-17 \bmod 7 = 4 \neq 3$$

\therefore No.

d) -67

$$-67 \equiv 3 \pmod{7}$$

$$-67 \bmod 7 = 3$$

∴ Yes.

Exercise 4.2

2. Convert the decimal expansion of each of these integers to a binary expansion.

a) 321

$$321 - 256 = 65$$

$$65 - 64 = 1$$

$$1 - 1 = 0$$

$$(1\ 0100\ 0001)_2$$

b) 1023

$$1023 - 512 = 511$$

$$511 - 256 = 255$$

$$255 - 128 = 127$$

$$127 - 64 = 63$$

$$63 - 32 = 31$$

$$31 - 16 = 15$$

$$15 - 8 = 7$$

$$7 - 4 = 3$$

$$3 - 2 = 1$$

$$1 - 1 = 0$$

$$(11\ 1111\ 1111)_2$$

c) 100632

$$100632 - 65536 = 35096$$

$$35096 - 32768 = 2328$$

$$2328 - 2048 = 280$$

$$280 - 256 = 24$$

$$24 - 16 = 8$$

$$8 - 8 = 0$$

$$(1\ 1000\ 1001\ 0001\ 1000)_2$$

4. Convert the binary expansion of each of these integers to a decimal expansion.

a) $(1\ 1011)_2$

$$16 + 8 + 2 + 1 = 27$$

b) $(10\ 1011\ 0101)_2$

$$512 + 128 + 32 + 16 + 4 + 1 = 693$$

c) $(11\ 1011\ 1110)_2$

$$512 + 256 + 128 + 32 + 16 + 8 + 4 + 2 = 958$$

d) $(111\ 1100\ 0001\ 1111)_2$

$$16384 + 8192 + 4096 + 2048 + 1024 + 16 + 8 + 4 + 2 + 1 = 30875$$

6. Convert the binary expansion of each of these integers to an octal expansion.

a) $(1111\ 0111)_2$

$$11\ 110\ 111 = (367)_8$$

b) $(1010\ 1010\ 1010)_2$

$$101\ 010\ 101\ 010 = (5252)_8$$

c) $(111\ 0111\ 0111\ 0111)_2$

$$111\ 011\ 101\ 110\ 111 = (73567)_8$$

d) $(101\ 0101\ 0101\ 0101)_2$

$$101\ 010\ 101\ 010\ 101 = (52525)_8$$

8. Convert $(\text{BADFACED})_{16}$ from its hexadecimal expansion to its binary expansion.

$$B = 1011, A = 1010, D = 1101, F = 1111, A = 1010, C = 1100, E = 1110, D = 1101$$

$$(\text{BADFACED})_{16} = (1011\ 1010\ 1101\ 1111\ 1010\ 1100\ 1110\ 1101)_2$$

22. Find the sum and product of each of these pairs of numbers. Express your answers as a base 3 expansion.

a) $(112)_3, (210)_3$

Sum:

112

+210

1022

Product:

112

*210

0

1120

+22400

101220

b) $(2112)_3, (12021)_3$

Sum:

2112

+12021

21210

Product:

2112

*12021

2112

42240

4224000

+21120000

111020122

c) $(20001)_3, (1111)_3$

Sum:

20001

+1111

21112

Product:

20001

+1111

20001

200010

2000100

+20001000

22221111

d) $(120021)_3, (2002)_3$

Sum:

120021

+2002

122100

Product:

120021

*2002

$$\begin{array}{r} 240042 \\ +240042000 \\ \hline \end{array}$$

1011122112

Exercise 4.3

2. Determine whether each of these integers is prime.

a) 19

The prime number $\leq \sqrt{19}$ are 2 and 3.

19 is not divisible by 2 or 3.

\therefore 19 is prime.

b) 27

The prime number $\leq \sqrt{27}$ are 2, 3, and 5.

27 is divisible by 3.

\therefore 27 is not prime.

c) 93

The prime number $\leq \sqrt{93}$ are 2, 3, 5, and 7.

93 is divisible by 3.

\therefore 93 is not prime.

d) 101

The prime number $\leq \sqrt{101}$ are 2, 3, 5, and 7.

101 is not divisible by any number above.

\therefore 101 is prime.

e) 107

The prime number $\leq \sqrt{107}$ are 2, 3, 5, and 7.

107 is not divisible by any number above.

\therefore 107 is prime.

f) 113

The prime number $\leq \sqrt{113}$ are 2, 3, 5, and 7.

113 is not divisible by any number above.

\therefore 113 is prime.

4. Find the prime factorization of each of these integers.

a) 39

$$39 = 3 \times 13$$

b) 81

$$81 = 3^4$$

c) 101

$$101 = 101$$

d) 143

$$143 = 11 \times 13$$

e) 289

$$289 = 17^2$$

f) 899

$$899 = 29 \times 31$$

16. Determine whether the integers in each of these sets are pairwise relatively prime.

a) 21, 34, 55

$$\gcd(21, 34) = 1$$

$$\gcd(21, 55) = 1$$

$$\gcd(34, 55) = 1$$

\therefore 21, 34, and 55 are pairwise relatively prime.

b) 14, 17, 85

$$\gcd(14, 17) = 1$$

$$\gcd(14, 85) = 1$$

$$\gcd(17, 85) = 17$$

\therefore 14, 17, and 85 are not pairwise relatively prime.

c) 25, 41, 49, 64

$$\gcd(25, 41) = 1$$

$$\gcd(25, 49) = 1$$

$$\gcd(25, 64) = 1$$

$$\gcd(41, 49) = 1$$

$$\gcd(41, 64) = 1$$

$$\gcd(49, 64) = 1$$

\therefore 25, 41, 49, and 64 are pairwise relatively prime.

d) 17, 18, 19, 23

$$\gcd(17, 18) = 1$$

$$\gcd(17, 19) = 1$$

$$\gcd(17, 23) = 1$$

$$\gcd(18, 19) = 1$$

$$\gcd(18, 23) = 1$$

$$\gcd(19, 23) = 1$$

\therefore 17, 18, 19, and 23 are pairwise relatively prime.

24. What are the greatest common divisors of these pairs of integers?

a) $2^2 \cdot 3^3 \cdot 5^5, 2^5 \cdot 3^3 \cdot 5^2$

$$\gcd(2^2 \cdot 3^3 \cdot 5^5, 2^5 \cdot 3^3 \cdot 5^2) = 2^2 \cdot 3^3 \cdot 5^2$$

b) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$

$$\gcd(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}) = 2 \cdot 3 \cdot 11$$

c) $17, 17^{17}$

$$\gcd(17, 17^{17}) = 17$$

d) $2^2 \cdot 7, 5^3 \cdot 13$

$$\gcd(2^2 \cdot 7, 5^3 \cdot 13) =$$

e) $0, 5$

$$\gcd(0, 5) = 5$$

f) $2 \cdot 3 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 5 \cdot 7$

$$\gcd(2 \cdot 3 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 5 \cdot 7) = 2 \cdot 3 \cdot 5 \cdot 7$$

26. What is the least common multiple of each pair in Exercise 24?

a) $2^2 \cdot 3^3 \cdot 5^5, 2^5 \cdot 3^3 \cdot 5^2$

$\text{lcm}(2^2 \cdot 3^3 \cdot 5^5, 2^5 \cdot 3^3 \cdot 5^2) = 2^5 \cdot 3^3 \cdot 5^5$

b) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$

$\text{lcm}(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}) =$

c) $17, 17^{17}$

$\text{lcm}(17, 17^{17}) = 17^{17}$

d) $2^2 \cdot 7, 5^3 \cdot 13$

$\text{lcm}(2^2 \cdot 7, 5^3 \cdot 13) = 2^2 \cdot 5^3 \cdot 7 \cdot 13$

e) $0, 5$

Since zero is not a positive integer, $\text{lcm}(0, 5)$ is undefined.

f) $2 \cdot 3 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 5 \cdot 7$

$\text{lcm}(2 \cdot 3 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 5 \cdot 7) = 2 \cdot 3 \cdot 5 \cdot 7$

Exercise 4.6

2. Encrypt the message STOP POLLUTION by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

a) $f(p) = (p + 4) \bmod 26$

WXST TSPPYXMSR

b) $f(p) = (p + 21) \bmod 26$

NOJK KJGGPODJI

c) $f(p) = (17p + 22) \bmod 26$

QHAR RABBYHCAJ

4. Decrypt these messages that were encrypted using the Caesar cipher.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

$$f(p) = (p + 3) \bmod 26$$

a) EOXH MHDQV

BLUE JEANS

b) WHVW WRGDB

TEST TODAY

c) HDW GLP VXP

EAT DIM SUM