Define a function $T: \mathbb{R}^2 \to \mathbb{R}^2$ as follows: $T(x,y) = (2x-y, x+y)$

Let $B = \{(1,1), (1,2)\}$, $C = \{(3,1), (2,1)\}$ be ordered bases

a) Find $[T]_B^C$

$T(1,1) = (1,2) = c_1(3,1) + c_2(2,1)$

$T(1,2) = (0,3) = c_1(3,1) + c_2(2,1)$

skip 3 steps

$$\left[\begin{array}{cc:cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{cc:cc} 1 & 0 & -3 & -6 \\ 0 & 1 & 5 & 9 \end{array}\right]$$

$$[T]_B^C = \begin{bmatrix} -3 & -6 \\ 5 & 9 \end{bmatrix}$$

b) Use $T(x,y) = (2x-y, x+y)$ to compute $T(3,4)$

$T(3,4) = (2(3)-4, 3+4)$

$= (2,7)$

c) Use $[T(v)]_C = [T]_B^C [v]_B$ to compute $T(3,4)$

$[v]_B = (3,4) = c_1(1,1) + c_2(1,2)$

skip 3 steps

$$\left[\begin{array}{cc:c} 1 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right] \overset{RREF}{=} \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$[v]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[T(v)]_C = \begin{bmatrix} -3 & -6 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6-6 \\ 10+9 \end{bmatrix} = \begin{bmatrix} -12 \\ 19 \end{bmatrix}$$

$T(3,4) = -12(3,1) + 19(2,1)$

$= (-36,-12) + (38,19)$

$= (2,7)$

Show $\{(x, y, x+y) \} : x, y \in \mathbb{R} \}$ is a subspace of $\mathbb{R}^3$

a) $\vec{0}$?

$(0, 0, 0+0) \in \mathbb{R}^3$

since $0 \in \mathbb{R}$

$\therefore$ yes

b) closed under $+$?

$(x_1, y_1, x_1+y_1), (x_2, y_2, x_2+y_2) \in \mathbb{R}^3$

$(x_1, y_1, x_1+y_1) + (x_2, y_2, x_2+y_2)$

$= (x_1+x_2, y_1+y_2, x_1+x_2+y_1+y_2)$

$x_1+x_2, y_1+y_2 \in \mathbb{R}$

$\therefore$ Yes

c) closed under $\cdot$?

$k \in \mathbb{R} \quad (x, y, x+y) \in \mathbb{R}^3$

$k(x, y, x+y) = (kx, ky, k(x+y))$

$= (kx, ky, kx+ky)$

$kx, ky \in \mathbb{R}$

$\therefore$ Yes

$\therefore$ It is a subspace of $\mathbb{R}^3$

Is $(3,4,5)$ in span $\{(6,5,7), (3,1,2)\}$

$(3,4,5) = c_1(6,5,7) + c_2(3,1,2)$

$(6c_1, 5c_1, 7c_1) + (3c_2 + c_2, 2c_2) = (3,4,5)$

$(6c_1 + 3c_2, 5c_1 + c_2, 7c_1 + 2c_2) = (3,4,5)$

$$\begin{aligned} 6c_1 + 3c_2 &= 3 \\ 5c_1 + c_2 &= 4 \\ 7c_1 + 2c_2 &= 5 \end{aligned} \Rightarrow \left[\begin{array}{cc:c} 6 & 3 & 3 \\ 5 & 1 & 4 \\ 7 & 2 & 5 \end{array}\right] \overset{\text{RREF}}{\Rightarrow} \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

$\therefore (6,5,7) - (3,1,2) = (6-3, 5-1, 7-2)$

$= (3,4,5)$

Consider $S = \{(3,4,1),(1,4,5),(5,4,-3),(2,0,-4)\}$. Find a LI subset of S whose span is the same as the span of S

Suppose $c_1(3,4,1)+c_2(1,4,5)+c_3(5,4,-3)+c_4(2,0,-4)\} = (0,0,0)$

$(3c_1,4c_1,c_1)+(c_2,4c_2,5c_2)+(5c_3,4c_3,-3c_3)+(2c_4,0,-4c_4) = (0,0,0)$

$(3c_1+c_2+5c_3+2c_4,\ 4c_1+4c_2+4c_3+0,\ c_1+5c_2-3c_3-4c_4) = (0,0,0)$

$$\begin{aligned} 3c_1+c_2+5c_3+2c_4 &= 0\\ 4c_1+4c_2+4c_3+0 &= 0\\ c_1+5c_2-3c_3-4c_4 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 3 & 1 & 5 & 2\\ 4 & 4 & 4 & 0\\ 1 & 5 & -3 & 4 \end{bmatrix} \overset{RREF}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0\\ 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 = -2s-t$ Let $s=1$ $c_1 = -3$

$c_2 = s+t$ $t=1$ $c_2 = 2$

$c_3 = s$ $c_3 = 1$

$c_4 = t$ $c_4 = 1$

$\therefore \underbrace{-3(3,4,1)}_{\text{drop}}+2(1,4,5)+(5,4,-3)+(2,0,-4) = (0,0,0)$

(any non zero coeff vector can be dropped)

Suppose $c_1(1,4,5)+c_2(5,4,-3)+c_3(2,0,-4) = (0,0,0)$

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0\\ 4 & 4 & 0 & 0\\ 5 & -3 & -4 & 0 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 = \frac{1}{2}u$ let $u = 2$ $c_1 = 1$

$c_2 = -\frac{1}{2}u$ $c_2 = -1$

$c_3 = u$ $c_3 = 2$

$\therefore \underbrace{(1,4,5)}_{\text{drop}} - (5,4,-3) + 2(2,0,-4) = (0,0,0)$

Suppose $c_1(5,4,-3)+c_2(2,0,-4) = (0,0,0)$

$$\left[\begin{array}{cc|c} 5 & 2 & 0\\ 4 & 0 & 0\\ -3 & -4 & 0 \end{array}\right] \overset{RREF}{\Rightarrow} \left[\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} c_1 &= 0\\ c_2 &= 0 \end{aligned}$$

$\therefore \{(5,4,-3),(2,0,-4)\}$ is LI and span of S

Define operations on $\mathbb{R}^3$ as follows:

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1)$

$c(x, y, z) = (cx + c - 1, cy + c - 1, cz + c - 1)$

1.) Is $\mathbb{R}^3$ closed under $+$?

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1)$

$x_1 + x_2 + 1,\ y_1 + y_2 + 1,\ z_1 + z_2 + 1 \in \mathbb{R}$

$\therefore$ Yes

2.) Is $+$ commutative ?

$$
\begin{aligned}
(x_1, y_1, z_1) + (x_2, y_2, z_2) &= (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) \\
&= (x_2 + x_1 + 1, y_2 + y_1 + 1, z_2 + z_1 + 1) \\
&= (x_2, y_2, z_2) + (x_1, y_1, z_1)
\end{aligned}
$$

$\forall\ (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$

$\therefore$ Yes

3.) Is $+$ associative ?

Let $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) \in \mathbb{R}^3$

a) $(x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3))$

$= (x_1, y_1, z_1) + (x_2 + x_3 + 1, y_2 + y_3 + 1, z_2 + z_3 + 1)$

$= (x_1 + x_2 + x_3 + 2, y_1 + y_2 + y_3 + 2, z_1 + z_2 + z_3 + 2)$

b) $((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3)$

$= (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) + (x_3, y_3, z_3)$

$= (x_1 + x_2 + x_3 + 2, y_1 + y_2 + y_3 + 2, z_1 + z_2 + z_3 + 2)$

$\therefore$ Yes

4.) $\vec{0} = (a, b, c)$

$(a, b, c) + (x, y, z)$

$= (a + x + 1, b + y + 1, c + z + 1)$

$a + x + 1 = x \qquad b + y + 1 = y \qquad c + z + 1 = z$

$a = -1 \qquad b = -1 \qquad c = -1$

$\therefore \vec{0} = (-1, -1, -1)$

5.) additive inverse

$(x,y,z) + (a,b,c) = (-1,-1,-1)$

$(x+a+1, y+b+1, z+c+1) = (-1,-1,-1)$

$x+a+1=-1 \qquad y+b+1=-1 \qquad z+c+1=-1$

$a=-x-2 \qquad b=-y-2 \qquad c=-z-2$

$\therefore -(x,y,z) = (-x-2, -y-2, -z-2)$

6.) Is v closed under scalar multiplication

let k $\mathbb{R}$ $\quad (x,y,z) \in \mathbb{R}^3$

$k(x,y,z) = (kx+k-1, ky+k-1, kz+k-1)$

$kx+k-1, ky+k-1, kz+k-1 \in \mathbb{R}$

$\therefore$ Yes

7.) Is · distributive

$\forall k \in \mathbb{R}, (x_1,y_1,z_1), (x_2,y_2,z_2) \in \mathbb{R}^3$

a) $k((x_1,y_1,z_1) + (x_2,y_2,z_2))$

$= k(x_1+x_2+1, y_1+y_2+1, z_1+z_2+1)$

$= (k(x_1+x_2+1)+k-1, k(y_1+y_2+1)+k-1, k(z_1+z_2+1)+k-1)$

$= (kx_1+kx_2+2k-1, (ky_1+ky_2+2k-1, kz_1+kz_2+2k-1)$

b) $k(x_1,y_1,z_1) + k(x_2,y_2,z_2)$

$= (kx_1+k-1, ky_1+k-1, kz_1+k-1) + (kx_2+k-1, ky_2+k-1, kz_2+k-1)$

$= (kx_1+k-1+kx_2+k-1+1, ky_1+k-1+ky_2+k-1, kz_1+k-1+kz_3+k-1)$

$= (kx_1+kx_2+2k-1, ky_1+ky_2+2k-1, kz_1+kz_2+2k-1)$

$\therefore$ Yes

8.) Is · distributive

$\forall k \in \mathbb{R}, (x,y,z) \in \mathbb{R}^3$

a) $(c+d)(x,y,z)$

$= ((c+d)x+c+d-1, (c+d)y+c+d-1, (c+d)z+c+d-1)$

$= (cx+dx+c+d-1, cy+dy+c+d-1, cz+dz+c+d-1)$

b) $c(x,y,z) + d(x,y,z)$

$= (cx + c - 1, cy + c - 1, cz + c - 1) + (dx + d - 1, dy + d - 1, dz + d - 1)$

$= (cx + dx + c + d - 1 - 1 + 1, cy + dy + c + d - 1 - 1 + 1, cz + dz + c + d - 1 - 1 + 1)$

$= (cx + dx + c + d - 1, cy + dy + c + d - 1, cz + dz + c + d - 1)$

$\therefore$ Yes

9.) is $\cdot$ associative ?

$\forall\ c, d \in \mathbb{R},\ (x,y,z) \in \mathbb{R}^3$

a) $(cd)(x,y,z)$

$= (cdx + cd - 1, cdy + cd - 1, cdz + cd - 1)$

b) $c(d(x,y,z))$

$= c(dx + d - 1, dy + d - 1, dz + d - 1)$

$= (cdx + cd - c + c - 1, cdy + cd - c + c - 1, cdz + cd - c + c - 1)$

$= (cdx + cd - 1, cdy + cd - 1, cdz + cd - 1)$

$\therefore$ Yes

10.) scalar identity

$1 \in \mathbb{R},\ (x,y,z) \in \mathbb{R}^3$

$1(x,y,z) = (x + 1 - 1, y + 1 - 1, z + 1 - 1)$

$= (x,y,z)$

$\therefore$ Yes

$\therefore\ \mathbb{R}^3$ is a vector space under the defined operations

$$B = \{ \overset{v_1}{(0,1,2)}, \overset{v_2}{(2,0,0)}, \overset{v_3}{(1,1,1)} \}$$

$$w_1 = v_1 = (0,1,2)$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$\langle v_2, w_1 \rangle = \langle (2,0,0), (0,1,2) \rangle = (0,0,0) = 0$$

$$\langle w_1, w_1 \rangle = \langle (0,1,2), (0,1,2) \rangle = (0,1,4) = 5$$

$$w_2 = (2,0,0) - \frac{0}{5}(0,1,2) = (2,0,0)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\langle v_3, w_1 \rangle = \langle (1,1,1), (0,1,2) \rangle = (0,1,2) = 3$$

$$\langle w_1, w_1 \rangle = 5$$

$$\langle v_3, w_2 \rangle = \langle (1,1,1), (2,0,0) \rangle = (2,0,0) = 2$$

$$\langle w_2, w_2 \rangle = \langle (2,0,0), (2,0,0) \rangle = (4,0,0) = 4$$

$$w_3 = (1,1,1) - \frac{3}{5}(0,1,2) - \frac{2}{4}(2,0,0)$$

$$= (1,1,1) - (0, \frac{3}{5}, \frac{6}{5}) - (1,0,0)$$

$$= (0, \frac{2}{5}, -\frac{1}{5})$$

$\therefore \{ w_1, w_2, w_3 \} = \{ (0,1,2), (2,0,0), (0, \frac{2}{5}, -\frac{1}{5}) \}$

is orthogonal basis

$$\langle w_3, w_3 \rangle = \langle (0, 2/5, -1/5), (0, 2/5, -1/5) \rangle$$

$$= (0, 4/25, 1/25) = \frac{5}{25} = 1/5$$

orthonormal basis $= \{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|} \}$

$$= \{ \sqrt{\tfrac{1}{5}}(0,1,2), \sqrt{\tfrac{1}{4}}(2,0,0), \sqrt{5}(0, \tfrac{2}{5}, -\tfrac{1}{5}) \}$$

$$= \{ (0, \tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}), (1,0,0), (0, \tfrac{2\sqrt{5}}{5}, -\tfrac{\sqrt{5}}{5}) \}$$

Let $T: P_2 \to P_2$ be defined by

$$T(a+bx) = (a-b+c) + (2a-2b-c)x + (3a-3b-3c)x^2$$

a) Find a basis for Ker T

suppose $T(a+bx) = (0+0x+0x^2)$

$$(0,0,0) = (a-b+c) + (2a-2b-c)x + (3a-3b-3c)x^2$$

$$\begin{aligned} a-b+c &= 0 \\ 2a-2b-c &= 0 \\ 3a-3b-3c &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -2 & -1 & 0 \\ 3 & -3 & -3 & 0 \end{array}\right] \xRightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$a = t$ let $t = 1$ $a = 1$

$b = t$ $b = 1$

$c = 0$ $c = 0$

$\therefore \ker T = \{(1,1,0)\}$

b) Find a basis for Rng T

Rng T = columns containing leading one

$\therefore \text{Rng}\,T = \{1+2x+3x^2,\ 1-x-3x^2\}$

c) Is T invertible ?

$\ker T = \{(1,1,0)\} \neq \{(0,0,0)\}$

$\therefore$ T is not 1-1

T must be 1-1 and onto to be invertible

$\therefore$ T is not invertible

Prove that P_1 and $\mathbb{R}^2$ are isomorphic

a) define a function

b) show the function defined is a LT

c) show the function defined is bijection

a) define a function

$T: P^1 \to \mathbb{R}^2$ defined by $T(a+bx) = (a,b)$

b) show the function defined is a LT

i) show $T((a_1+b_1x)+(a_2+b_2x)) = T(a_1+b_1x) + T(a_2+b_2x)$

$T((a_1+b_1x)+(a_2+b_2x))$

$= T((a_1+a_2)+(b_1+b_2)x)$

$= (a_1+a_2, b_1+b_2)$

$= (a_1, b_1) + (a_2, b_2)$

$= T(a_1+b_1x) + T(a_2+b_2x)$

$\therefore$ yes

ii) show $T(k(a+bx)) = kT(a+bx)$

$T(k(a+bx))$

$= T(ka+kbx)$

$= (ka, kb)$

$= k(a,b)$

$= kT(a+bx)$

$\therefore$ Yes

$\therefore$ T is a LT

c) show the function defined is bijection

i) show T is 1-1

Suppose $T(a+bx) = (0,0)$

$$T(1) = 0 \qquad \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$T(x) = 0$$

$\ker T = \{(0,0)\}$

$\therefore$ T is 1-1

ii) show T is onto

$\dim \text{Domain} = \dim \ker T + \dim \text{Rng} T$

$2 = 0 + \dim \text{Rng} T$

$\dim \text{Rng} T = 2$

$\dim$ Codomain is also 2

$\therefore$ T is onto

Consider a set $S = \{(1,0,2), (0,2,1)\}$ in $\mathbb{R}^3$

a) show S is LI

b) Find a equation of the plane spanned by S

c) use the equation found in b) to extend S to a basis for $\mathbb{R}^3$

a) show S is LI

suppose $c_1(1,0,2) + c_2(0,2,1) = (0,0,0)$

$(c_1, 0, 2c_1) + (0, 2c_2, c_2) = (0,0,0)$

$(c_1 + 0, 0 + 2c_2, 2c_1 + c_2) = (0,0,0)$

$$\begin{aligned} c_1 + 0 &= 0 \\ 0 + 2c_2 &= 0 \\ 2c_1 + c_2 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 0$

$c_2 = 0$

$\therefore$ T is LI

b) suppose $c_1(1,0,2)+c_2(0,2,1)=(x,y,z)$

skip 3 steps

$$\left[\begin{array}{cc:c} 1 & 0 & x \\ 0 & 2 & y \\ 2 & 1 & z \end{array}\right] \Rightarrow \begin{array}{lccc} R1(-2) & -2 & 0 & -2x \\ R3 & 2 & 1 & z \\ \hline \text{New } R3 & 0 & 1 & -2x+z \end{array} \Rightarrow \left[\begin{array}{cc:c} 1 & 0 & x \\ 0 & 2 & y \\ 0 & 1 & -2x+z \end{array}\right]$$

$$R2\left(\tfrac{1}{2}\right) \Rightarrow 0 \quad 1 \quad \tfrac{1}{2}y \Rightarrow \left[\begin{array}{cc:c} 1 & 0 & x \\ 0 & 1 & \frac{1}{2}y \\ 0 & 1 & -2x+z \end{array}\right] \Rightarrow \begin{array}{lccc} R2(-1) & 0 & -1 & -\frac{1}{2}y \\ R3 & 0 & 1 & -2x+z \\ \hline & 0 & 0 & -2x-\frac{1}{2}y+z \end{array}$$

$$\left[\begin{array}{cc:c} 1 & 0 & x \\ 0 & 1 & \frac{1}{2}y \\ 0 & 0 & -2x-\frac{1}{2}y+z \end{array}\right]$$

$\therefore \quad -2x-\frac{1}{2}y+z=0$

c) use the equation found in b) to extend S to a basis for $\mathbb{R}^3$

add a vector (x,y,z) with $-2x-\frac{1}{2}y+z \neq 0$

$-2x-\frac{1}{2}y+z=0$

$-2(0)-\frac{1}{2}(0)+1 = 1$

$\therefore \ \{(1,0,2),(0,2,1),(0,0,1)\}$ is a basis for $\mathbb{R}^3$

Let A and B be $n \times n$ matrices

If $A \sim B$ then $\det(A)=\det(B)$

since $A \sim B$, $A, B \ \exists \ P$

suppose $B = P^{-1}AP$

$\det(B) = \det(P^{-1}AP)$

$\det(B) = \det(P^{-1})\det(A)\det(P)$

$\det(B) = \dfrac{1}{\det(P)} \det(A)\det(P)$

$\det(B) = \dfrac{1}{\cancel{\det(P)}} \cancel{\det(P)} \det(A)$

$\det(B) = \det(A)$

Let $T: V \to W$ be a LT

Then T is 1-1 iff $\ker T = \{\vec{0}\}$

a) Prove if T is 1-1 then $\ker T = \{\vec{0}\}$

suppose $T(v) = \vec{0}$

we know $T(\vec{0}) = \vec{0}$

$\therefore T(v) = T(\vec{0})$

since T is 1-1, $v = \vec{0}$

$\therefore \ker T = \{\vec{0}\}$

b) Prove if $\ker T = \{\vec{0}\}$ then T is 1-1

suppose $T(v_1) = T(v_2)$

iff $T(v_1) - T(v_2) = \vec{0}$

iff $T(v_1 - v_2) = \vec{0}$

then $v_1 - v_2 \in \ker T$

since $\ker T = \{\vec{0}\}$ by assumption,

$v_1 - v_2 = \vec{0}$

$\therefore v_1 = v_2$

Find the characteristic equation and the eigenvalues and eigenvectors

a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda I_3 - A$

$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix}$

$\det(\lambda I_3 - A)$

$= (\lambda - 2)^3$

$\lambda - 2 = 0$

$\lambda = 2$

Let $\lambda = 2$

$\begin{bmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{RREF}{\Rightarrow} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$v_1 = t$
$v_2 = 0$
$v_3 = s$
$\Rightarrow \begin{bmatrix} t \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $t = 1$, $s = 1$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ are LI eigenvectors corresponding to $\lambda = 2$

	al	ge
$\lambda = 2$	3	2

$\therefore$ A is defective

b) $A = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$

$\lambda I_3 - A$

$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} \lambda & 3 & -5 \\ 4 & \lambda-4 & 10 \\ 0 & 0 & \lambda-4 \end{bmatrix}$

$\det(\lambda I_3 - A)$

$= (-1)^{3+3}(\lambda-4)(\lambda(\lambda-4)-3(4))$

$= (\lambda-4)(\lambda^2-4\lambda-12)$

$= (\lambda-4)(\lambda-6)(\lambda+2)$ characteristic equation

$\lambda-4=0 \quad \lambda-6=0 \quad \lambda+2=0$

$\lambda=4 \quad \lambda=6 \quad \lambda=-2$

Let $\lambda = 4$

$\begin{bmatrix} \lambda & 3 & -5 \\ 4 & \lambda-4 & 10 \\ 0 & 0 & \lambda-4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -5 \\ 4 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$v_1 = -\frac{5}{2}t$

$v_2 = 5t \Rightarrow \begin{bmatrix} -\frac{5}{2}t \\ 5t \\ t \end{bmatrix} \Rightarrow t\begin{bmatrix} -\frac{5}{2} \\ 5 \\ 1 \end{bmatrix}$

$v_3 = t$

Let $t = 2$ $\quad v_1 = -5$

$v_2 = 10 \quad \left\{ \begin{bmatrix} -5 \\ 10 \\ 2 \end{bmatrix} \right\}$ is a LI eigenvector corresponding to $\lambda = 4$

$v_3 = 2$

Let $\lambda = 6$

$\begin{bmatrix} \lambda & 3 & -5 \\ 4 & \lambda-4 & 10 \\ 0 & 0 & \lambda-4 \end{bmatrix} = \begin{bmatrix} 6 & 3 & -5 \\ 4 & 2 & 10 \\ 0 & 0 & 2 \end{bmatrix} \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$v_1 = -\frac{1}{2}t$

$v_2 = t$ $\Rightarrow \begin{bmatrix} -\frac{1}{2}t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

$v_3 = 0$

Let $t = 2$ $\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$ is a LI eigenvector corresponding to $\lambda = 6$

Let $\lambda = -2$

$$\begin{bmatrix} \lambda & 3 & -5 \\ 4 & \lambda-4 & 10 \\ 0 & 0 & \lambda-4 \end{bmatrix} = \begin{bmatrix} -2 & 3 & -5 \\ 4 & -6 & 10 \\ 0 & 0 & -6 \end{bmatrix} \overset{RREF}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = \frac{3}{2}t$

$v_2 = t$ $\Rightarrow \begin{bmatrix} \frac{3}{2}t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$

$v_3 = 0$

Let $t = 2$ $\left\{\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}\right\}$ is a LI eigenvector corresponding to $\lambda = -2$

	al	ge
$\lambda = 4$	1	1
$\lambda = 6$	1	1
$\lambda = -2$	1	1

$\therefore$ A is nondefective

$\therefore$ A has 3 LI eigenvectors:

$$\left\{\begin{bmatrix} -5 \\ 10 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}\right\}$$

since A is nondefective

A is diagonalizable

If $\det(A) = 4$, $\det(B) = 2$, A, B are 4×4 matrix

compute $\det(2A^{-1}B^2)$

$$\det(2A^{-1}B^2)$$
$$= 2^4 \det(A^{-1}B^2)$$
$$= 16 \det(A^{-1}) \det(B \cdot B)$$
$$= 16 \frac{1}{\det(A)} \det(B) \det(B)$$
$$= 16 \left(\frac{1}{4}\right)(2)(2)$$
$$= 16$$

suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$

compute $\begin{vmatrix} 3a+3b & 3b & 3c \\ -d-e & -e & -f \\ g+h & h & i \end{vmatrix}$

$$= 3 \begin{vmatrix} a+b & b & c \\ -d-e & -e & -f \\ g+h & h & i \end{vmatrix}$$
$$= (-1)3 \begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix}$$
$$= -3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - 3 \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}$$
$$= -3(5) - 3(0)$$
$$= -15$$

Compute det (A)

$$A = \begin{bmatrix} 0 & 6 & 3 \\ 2 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 6 & 3 \\ 2 & 2 & 4 \\ 3 & 6 & 3 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 2 & 4 \\ 0 & 6 & 3 \\ 3 & 6 & 3 \end{vmatrix}$$

$$= -1(2) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 3 & 6 & 3 \end{vmatrix} = -2(6) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 3 & 6 & 3 \end{vmatrix}$$

$$= -12(3) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \end{vmatrix}$$

R1 (-1)	-1	-1	-2
R3	1	2	1
New R3	0	1	-1

$$= -36 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 \end{vmatrix}$$

R2(-1)	0	-1	$-\frac{1}{2}$
R3	0	1	-1
	0	0	$-\frac{3}{2}$

$$= -36 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} \end{vmatrix} = -36\left(-\frac{3}{2}\right) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 54\,(1)(1)(1) = 54$$

Let $M = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Find $(\text{adj } M)_{13}$.

$$\begin{bmatrix} 11 & 21 & 31 \\ 12 & 22 & 32 \\ 13 & 23 & 33 \end{bmatrix}$$

$$M_{13} = (-1)^{1+3}\,(0(3) - (1)(2))$$
$$= 0 - 2$$
$$= -2$$

Use fourier expansion theorem to find $[w]_B$

$w = (4,-3)$

$B = \{ (\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}), (-\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}) \}$

$[w]_B = \underbrace{\langle w, w_1 \rangle}_{c_1} w_1 + \underbrace{\langle w, w_2 \rangle}_{c_2} w_2$

$c_1 = \langle (4,-3), (\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}) \rangle = (\frac{4\sqrt{3}}{3}, -\sqrt{6}) = \frac{4\sqrt{3}}{3} - \sqrt{6}$

$c_2 = \langle (4,-3), (-\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}) \rangle = (-\frac{4\sqrt{6}}{3}, -\sqrt{3}) = -\frac{4\sqrt{6}}{3} - \sqrt{3}$

Thus $[w]_B = \{ (\frac{4\sqrt{3}}{3} - \sqrt{6})(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}), (-\frac{4\sqrt{6}}{3} - \sqrt{3})(-\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}) \}$

Let $V = C[0,1]$ and define $\langle f,g \rangle = \int_0^1 f(x)g(x)\,dx$

Let $v = x^2$, $w = x$

find proj w v

$\text{proj}_w v = \frac{\langle v,w \rangle}{\langle w,w \rangle} w$

a) $\langle v,w \rangle = \langle x^2, x \rangle = \int_0^1 x^2 x\,dx$

$= \int_0^1 x^3\,dx$

$= \frac{x^4}{4} \Big|_0^1$

$= \frac{1}{4}$

b) $\langle w,w \rangle = \langle x,x \rangle = \int_0^1 x\,x\,dx$

$= \int_0^1 x^2\,dx$

$= \frac{x^3}{3} \Big|_0^1$

$= \frac{1}{3}$

$\text{proj}_w v = \frac{\frac{1}{4}}{\frac{1}{3}} x = \frac{3}{4} x$

Let $T : P_2 \to P_1$ be a function defined by $T(p(x)) = p'(x)$, $p(x) \in P_2$ (T is the differential operator)

a) show T is a LT

i) show $T(P_1(x) + P_2(x)) = T(P_1(x)) + T(P_2(x))$

$$T(P_1(x) + P_2(x)) = (P_1(x) + P_2(x))'$$
$$= P_1(x)' + P_2(x)'$$
$$= T(P_1(x)) + T(P_2(x))$$

ii) show $T(k\,p(x)) = kT(p(x))$

$$T(kP(x)) = (kP(x))'$$
$$= k(P(x)')$$
$$= k(T(P(x)))$$

$\therefore$ T is a LT

b) Find $[T]_B^C$ using ordered bases $B = \{2, 16x, 8x^2\}$, $C = \{4, 2x\}$

$$T(2) = 0 = 0 + 0x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(16x) = 16 = 16 + 0(2x) = 4(4) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$T(8x^2) = 16x = 0(4) + 16x = 8(2x) = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$$[T]_B^C = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Define an inner product on $C[0,1]$ as $\langle f,g\rangle = \int_0^1 f(x)\,g(x)\,dx$

a) Find $\|6x\|$

$$\|6x\| = \sqrt{\langle 6x, 6x\rangle} = \sqrt{\int_0^1 6x \cdot 6x\,dx}$$
$$= \sqrt{\int_0^1 36x^2\,dx}$$
$$= \sqrt{36\int_0^1 x^2\,dx}$$
$$= \sqrt{36\left(\frac{x^3}{3}\Big|_0^1\right.}$$
$$= \sqrt{36\left(\frac{1}{3}\right)}$$
$$= \sqrt{12}$$
$$= 2\sqrt{3}$$

b) Determine if $f(x) = x$, $g(x) = x - \frac{2}{3}$ are orthogonal

$$\langle f, g\rangle = \int_0^1 x\left(x - \tfrac{2}{3}\right)dx$$
$$= \int_0^1 \left(x^2 - \tfrac{2}{3}x\right)dx$$
$$= \left(\frac{x^3}{3} - \frac{x^2}{3}\Big|_0^1\right.$$
$$= \frac{1}{3} - \frac{1}{3}$$
$$= 0$$

$\therefore$ f and g are orthogonal

c) Find an orthogonal basis for the subspace of $C[0,1]$ spanned by $\{\underset{v_1}{2}, \underset{v_2}{x+1}\}$

$$w_1 = v_1 = 2$$
$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$\langle v_2, w_1\rangle = \int_0^1 (x+1)(2)\,dx$$
$$= \int_0^1 (2x+2)\,dx$$
$$= \left(x^2 + 2x\Big|_0^1\right.$$
$$= 1+2$$
$$= 3$$

$$\langle w_1, w_1 \rangle = \int_0^1 2(2)\,dx$$
$$= \int_0^1 4\,dx$$
$$= 4x \Big|_0^1$$
$$= 4$$

$$v_2 = (x+1) - \frac{3}{4}(2)$$
$$= x + 1 - \frac{3}{2}$$
$$= x - \frac{1}{2}$$

$\therefore$ orthogonal basis $= \{ 2, x - \frac{1}{2} \}$