CSCI 190 Discrete Mathematics Applied to Computer Science
Exam 2

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<u>Read these instructions before proceeding</u>.

- Closed book. Closed notes. You can use calculator.
- You have **<u>80 minutes</u>** to complete this exam.
- No questions will be answered during the exam or immediately afterwards. Answer each question as best you can. Partial credit will be awarded for reasonable efforts. If a question contains an ambiguity or a misprint, then say so in your answer, providing the answer to a reasonable interpretation of the question; give your assumptions.
- Answer the problems on the blank spaces provided for each problem.
- **Box your answers**.

Q1 (8)	Q2 (10)	Q3 (8)	Q4 (10)	Q5 (8)	Q6 (12)	Q7 (10)	Q8 (8)	Q9 (8)	Q10 (10)	Q11 (8)	Total (100)

1. (8 pts)

Use the Principle of Mathematical Induction to prove that
$1 + 2 + 4 + 8 + \cdots + 2^n = (2^{n+1} - 1)$ for all $n \geq 0$.

① We use induction on n.

② Base case :
for $n = 0$

$P(0)$ is true since $2^0 = 2^{0+1} - 1$

$1 = 2^1 - 1$

$1 = 1$

③ Inductive step :

we assume that $P(k)$ is true for an arbitrary nonnegative integer k.

we assume $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$

$P(k)$ is true $\Rightarrow$ $P(k+1)$ is also true.

$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1}$

$\overset{IH}{=} (2^{k+1} - 1) + 2^{k+1}$

$= 2 \cdot 2^{k+1} - 1$

$= 2^{k+2} - 1 = 2^{(k+1)+1} - 1$

④ conclusion : Hence, by induction $[P(n)$ is true$]$ for all non-negative integers n.

2. ***(10 pts)*** **Give a recursive definition with initial condition(s).**

a) The function **f(n) = n!, n = 0, 1, 2, ...** (5 pts)

$f(n) = n!$, $n = 0, 1, 2, \ldots$

$n=0$, initial condition: $f(0) = 0! = 1$

$n=1$ $f(1) = 1! = 1$

$n=2$ $f(2) = 2! = 2$ ...

$n=k$, $f(k) = k! = k(k-1)!$

$= k \cdot f(k-1)$

Recusive definition

$f(n) = n \cdot f(n-1)$, $f(0) = 1$.

b) The Fibonacci numbers **1, 1, 2, 3, 5, 8, 13, ...** (5 pts)

$a_0 = 1$ $a_1 = 1$

$a_n = a_{n-1} + a_{n-2}$ where $n \geq 2$

each term of the sequence, after the first two, is the sum of the two previous terms.

$a_2 = a_{2-1} + a_{2-2} = a_1 + a_0 = 1+1 = 2$

$a_3 = a_{3-1} + a_{3-2} = a_2 + a_1 = 2+1 = 3$

$a_4 = a_{4-1} + a_{4-2} = a_3 + a_2 = 3+2 = 5$

$a_5 = a_{5-1} + a_{5-2} = a_4 + a_3 = 5+3 = 8$

$a_6 = a_{6-1} + a_{6-2} = a_5 + a_4 = 8+5 = 13$

& so forth.

3. ***(8 pts)***

a) **Find f(2) and f(3) if f(n) = 2f(n − 1) + 5, f(0) = 3.** (4 pts)

$f(0) = 3$

$f(1) = 2f(1-1) + 5$

$= 2f(0) + 5$

$= 2(3) + 5$

$= 6 + 5 = 11$

$f(2) = 2f(2-1) + 5$

$= 2f(1) + 5$

$= 2(11) + 5$

$= 22 + 5$

$= 27$

$f(3) = 2f(3-1) + 5$

$= 2f(2) + 5$

$= 2 \cdot 27 + 5$

$= 54 + 5$

$= 59$

b) **Find f(8) if f(n) = 2f(n/2) + 1, f(1) = 2.** (4 pts)

$f(1) = 2$

$f(2) = 2f(\frac{2}{2}) + 1$

$= 2f(1) + 1$

$= 2 \cdot 2 + 1$

$= 4 + 1 = 5$

$f(4) = 2f(\frac{4}{2}) + 1$

$= 2f(2) + 1$

$= 2 \cdot 5 + 1 = 10 + 1$

$= 11$

$f(8) = 2f(\frac{8}{2}) + 1$

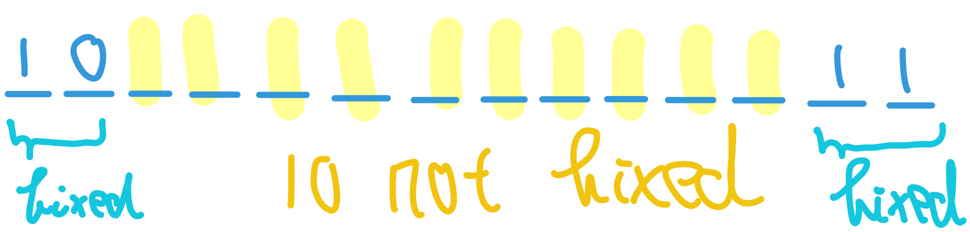
$= 2f(4) + 1$

$= 2 \cdot 11 + 1$

$= 22 + 1 = 23$

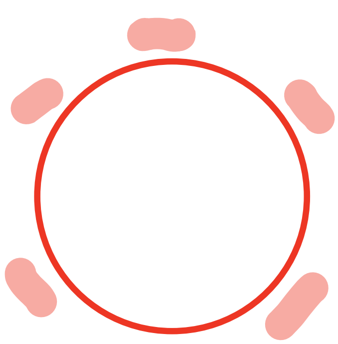
4 (10 pts)

a) **Consider a bit string of length 14. How many begin with 10 and end with 11?** (5 pts)



$2^{10} = 1024$

b) **How many ways are there to seat 5 people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?** (5 pts)



$(5-1)!$

$= 4! = 24$

5. ***(8 pts)***
Explain how the Pigeonhole Principle can be used to show that among any 31 integers, at least four must have the same last digit.

Integers 0 to 9

$n=10$

$$\left\lceil \frac{31}{10} \right\rceil = \lceil 3.1 \rceil = 4$$

6. ***(12 pts)***

a) **How many ways are there to select 6 students from a class of 25 to serve on a committee?** (4 pts)

$$_{25}C_6 = \frac{25!}{6!(25-6)!} = \frac{25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20\cdot \cancel{19!}}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{19!}} = 177{,}100$$

b) **How many ways are there to select 6 students from a class of 25 to hold six different executive positions on a committee?** (4 pts)

$$_{25}C_6 \cdot 6! = 177100 \cdot 720 = 127{,}512{,}000$$

c) **How many bit strings of length 10 have equal numbers of 0's and 1's?** (4 pts)

1's = 5
0's = 5

$$\left[_{10}C_5\right]\cdot {_5C_5} = \left[\frac{10!}{5!(10-5)!}\right]\cdot 1 = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \cancel{5!}}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{5!}} = 252$$

7. ***(10 pts)***

a) **Use the Pascal's Tringle to expand** $\boldsymbol{(x+y)^7}$**.** (5 pts)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$(x+y)^7$

$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

$\Rightarrow (x+y)^7$

b) **Find the coefficient of** x^4y^6 **in the expansion of** $(3x+2y)^{10}$**.** (5 pts)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$a=3x$
$b=2y$
$n=10$

$$\Rightarrow (3x+2y)^{10} = \sum_{k=0}^{10} \binom{10}{k} (3x)^{10-k} (2y)^k$$

$$k=6: \binom{10}{6}(3x)^{10-6}(2y)^6 = \frac{10!}{6!(10-6)!}(3x)^4(2y)^6$$

$$= 1088640\, x^4y^6$$

8. ***(8 pts)***
(a) **What is the probability that a card chosen from an ordinary deck of 52 cards is an ace or a king or a queen?.** (4 pts)

ace = 4
king = 4
queen = 4

$$\frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52} = \frac{3}{13} = 0.23 \text{ OR } 23.08\%$$

(b) **What is the probability that two cards chosen from an ordinary deck of 52 cards are both kings?.** (4 pts)

1st king $= P(\text{king}) = \frac{4}{52}$

2nd king $= P(\text{king}) = \frac{3}{51}$

$$P(\text{both king}) = \frac{4}{52}\cdot\frac{3}{51}$$

$$= 0.45\%$$

9. (8 pts) Suppose you have a class with 40 students — 14 freshmen, 16 sophomores, and 10 juniors.

a) **You pick two students at random, one at a time. What is the probability that both are juniors?** (4 pts)

$$\frac{{}_{10}C_2}{{}_{40}C_2} = \frac{\frac{10!}{2!(10-2)!}}{\frac{40!}{2!(40-2)!}} = \frac{\frac{10\cdot 9\cdot \cancel{8!}}{2\cdot 1\cdot \cancel{8!}}}{\frac{40\cdot 39\cdot \cancel{38!}}{2\cdot 1\cdot \cancel{38!}}} = \frac{\frac{90}{2}}{\frac{1560}{2}} = \frac{45}{780} \approx 0.0577 \approx 5.77\%$$

b) **You pick two students at random, one at a time. What is the probability that the second student is a freshman, given that the first is a sophomore?** (4 pts)

$$P(\text{sophmore} \mid \text{freshmen}) = \frac{P(\text{sophmore} \cap \text{freshmen})}{P(\text{freshmen})} = \frac{P(\text{sophmore})\cdot P(\text{freshmen})}{P(\text{freshmen})} = \frac{\frac{16}{40}\cdot\frac{14}{39}}{\frac{14}{39}} = \frac{16}{40} = 0.4$$

10. (10 pts)

a) **In a certain lottery game you choose a set of six numbers out of 45 numbers. Find the probability that none of your numbers match the six winning numbers.** (4 pts)

$$\frac{\binom{45-6}{6}}{\binom{45}{6}} = \frac{\binom{39}{6}}{\binom{45}{6}} = \frac{\left[\frac{39!}{6!(39-6)!}\right]}{\left[\frac{45!}{6!(45-6)!}\right]} = \frac{\left[\frac{39\cdot 38\cdot 37\cdot 36\cdot 35\cdot 34\cdot \cancel{33!}}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{33!}}\right]}{\left[\frac{45\cdot 44\cdot 43\cdot 42\cdot 41\cdot 40\cdot \cancel{39!}}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{39!}}\right]} = \frac{3262623}{8145060} \approx 0.401$$

b) **An experiment consists of picking at random a bit string of length four. Consider the following events:**

E_1: the bit string chosen begins with 01;
E_2: the bit string chosen ends with 10.

Determine whether E_1 and E_2 are independent. Show your work. (6 pts)

01 _ _ $\quad P(E_1) = \frac{2^2}{2^4} = \frac{4}{16} = \frac{1}{4}$

_ _ 10 $\quad P(E_2) = \frac{2^2}{2^4} = \frac{4}{16} = \frac{1}{4}$

$P(E_1 \cap E_2) = P(E_1)\cdot P(E_2)$

0111	0110
0110	1010
0101	1110
0100	0010

$P(E_1)\cdot P(E_2)$
$= \frac{1}{4}\cdot\frac{1}{4}$
$= \frac{1}{16}$ equal

$P(E_1 \cap E_2) = \frac{1}{16}$
Hence, E_1 & E_2 are independent.

11. (8 pts) Four coins are tossed.

a) **List the elements in the sample space.** (4 pts)

HHHH, HHHT, HHTH, HHTT, HTHH, HTHT,
HTTH, HTTT, THHH, THHT, THTH, THTT
TTHH, TTHT, TTTH, TTTT
Total # of outcomes = 16

b) **Find the probability that exactly three heads show.** (4 pts)

{THHH HTHH HHTH HHHT}
there are 4 out of 16 ways.

$P(\text{exactly 3 heads show}) = \frac{4}{16} = \frac{1}{4} = 0.25 = 25\%$