6.3 Matrices for Linear Transformation

Def: Let $T: V \to W$ be a LT

The inverse of T, written T^{-1},

is a function $T^{-1}: W \to V$

satisfying

① $T^{-1} \circ T(v) = v \quad \forall v \in V$

and

② $T \circ T^{-1}(w) = w \quad \forall w \in W$

Theorem: Let $T: V \to W$ be a LT

then T^{-1} exists iff T is one to one and onto

Theorem: Let $T: V \to W$ be a LT

T^{-1}, if exists, is also a LT

Proof: HW

To find T^{-1}

Observe that $[T^{-1} \circ T]_B^B = I_n$

Let $T: V \to W$

Let B be a basis for V

C be a basis for W

Then $[T^{-1} \circ T]_B^B = I_n$

But $[T^{-1} \circ T]_B^C = [T^{-1}]_C^B \cdot [T]_B^C$

Thus $[T^{-1}]_C^B \cdot [T]_B^C = I_n$

$\therefore [T^{-1}]_C^B = ([T]_B^C)^{-1}$

Thus :

To find T^{-1} :

① Pick bases B and C

② Compute $[T]_B^C$

③ Find $([T]_B^C)^{-1}$

Use x^{-1} in TI 84

④ Go back to T^{-1} $\leftarrow$ there are two ways

eg. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a LT

defined by $T(x,y) = (x+y, 2x+y)$

① Determine if T is invertible

② Find T^{-1} if exists

① show T is 1-1 and onto

ⓐ show T is 1-1 (start with suppose $T(v) = \vec{0}$)

Suppose $T(x,y) = (0,0)$

$(x+y, 2x+y) = (0,0)$

$x+y=0 \qquad \Rightarrow \qquad x=0$

$2x+y=0 \qquad\qquad\quad y=0$

$\therefore \ker T = \{(0,0)\}$

$\therefore T$ is 1-1

ⓑ show T is onto

use Rn T

dim Domain = dim Ker T + dim Rng T

and dim Domain = 2 , dim Ker T = 0

$\therefore$ dim Rng T = 2

since dim codomain is also 2

T is onto

$\therefore$ T is invertible

② To find T^{-1}

Let $B = C = \{(1,0),(0,1)\}$

$[T]_B^C = [[T(1,0)]_C, [T(0,1)]_C]$

$[T(1,0)]_C = \begin{matrix} x=1 \\ y=0 \end{matrix} = [(1,2)]_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$[T(0,1)]_C = \begin{matrix} x=0 \\ y=1 \end{matrix} = [(1,1)]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow [T^{-1}]_C^B = ([T]_B^C)^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

To find $T^{-1}(x,y)$

Method 1 :

$$\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$\uparrow$ $\uparrow$

$T^{-1}(1,0)$ $T^{-1}(0,1)$

$$\begin{aligned} T^{-1}(x,y) &= T^{-1}(x(1,0)+y(0,1)) \\ &= xT^{-1}(1,0)+yT^{-1}(0,1) \\ &= x(-1,2)+y(1,-1) \\ &= (-x,2x)+(y,-y) \\ &= (-x+y,\ 2x-y) \end{aligned}$$

Method 2

$$[T^{-1}(x,y)]_C = [T^{-1}]_C^B \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -x+y \\ 2x-y \end{bmatrix}$$

$$T^{-1}(x,y) = (-x+y,\ 2x-y)$$

eg. Let $T: P_1 \to P_1$ be a LT

Defined by $T(a+bx) = (a+4b) + (3a+11b)x$

Find $T^{-1}(a+bx)$

Let $B = C = \{1, x\}$

Find $[T]_B^C = [[T(1)]_C, [T(x)]_C]$

$[T(1)]_C = \begin{matrix} a=1 \\ b=0 \end{matrix} = [1+3x]_C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$[T(x)]_C = \begin{matrix} a=0 \\ b=1 \end{matrix} = [4+11x]_C = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & 4 \\ 3 & 11 \end{bmatrix}$

$\Rightarrow [T^{-1}]_C^B = \begin{bmatrix} 1 & 4 \\ 3 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} -11 & 4 \\ 3 & -1 \end{bmatrix}$

To find $T^{-1}(a+bx)$

Method 1

$$\begin{array}{cc} \begin{bmatrix} -11 & 4 \\ 3 & -1 \end{bmatrix} & \begin{matrix} \text{coeff of } x \\ \text{coeff of } x \end{matrix} \\ \begin{matrix} \uparrow & \uparrow \\ T^{-1}(1) & T^{-1}(x) \end{matrix} & \end{array}$$

$$T^{-1}(a+bx) = T^{-1}(a) + T^{-1}(bx)$$

$$= aT^{-1}(1) + bT^{-1}(x)$$

$$= a(-11+3x) + b(4-x)$$

$$= (-11a+3ax) + (4b-bx)$$

$$= (-11a+4b,\ 3ax-bx)$$

$$= (-11a+4b) + (3a-b)x$$

Method 2

Use $[T^{-1}(a+bx)]_C = [T^{-1}]_C^B [a+bx]_C$

$$= \begin{bmatrix} -11 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} -11a+b \\ 3a-b \end{bmatrix}$$

$T^{-1}(a+bx) = (-11a+b) + (3a-b)x$

6.4 Transition Matrices and similarity

Def: Let A and B be $\underbrace{n \times n}_{\text{square}}$ matrices

We say A and B are similar and write $A \sim B$ if $\exists P$

A is similar to B

such that $\boxed{B = P^{-1}AP}$ must memorize

Theorem:

① Reflexive Property

$A \sim A$

Proof: (need to show $\exists P$ with $A = P^{-1}AP$

Let $P = I_n$

then $A = I_n^{-1} A I_n$

$\therefore A \sim A$

② Symmetric Property

If $A \sim B$ then $B \sim A$

Proof: since $A \sim B$, $\exists P$

with $B = P^{-1}AP$

To show $B \sim A$, you need to find a matrix $\square$

with $A = \square^{-1} B \square$

since $B = P^{-1}AP$,

$PBP^{-1} = PP^{-1}APP^{-1}$

$A = PBP^{-1}$

$= (P^{-1})^{-1} B P^{-1}$

$\therefore B \sim A$

③ Transitive Property

If $A \sim B$ and $B \sim C$ then $A \sim C$

Proof: HW

Theorem: Let A and B be $n \times n$ matrices

If $A \sim B$ the $\det(A) = \det(B)$

Proof: HW

Recall Transition matrix (sec 4.7)

Let V be a vector space,

B, C be ordered bases for V

then the transition matrix from B to C,

written $[P]_B^C$, or $P_{C \leftarrow B}$ is a matrix

satisfying $[v]_C = [P]_B^C [v]_B$

Recall sec 4.7

Let $B = \{v_1, v_2, \ldots, v_n\}$

then $[P]_B^C = [[v_1]_C, [v_2]_C, \ldots, [v_n]_C]$

observation

$[P]_B^C = [T]_B^C$ with $T(v) = v$

Theorem: Let V be a vector space

B and C be ordered bases

then $[T]_B^B \sim [T]_C^C$

In fact, $[T]_C^C = ([P]_C^B)^{-1} [T]_B^B [P]_C^B$