1. (8 pts)

Use the Principle of Mathematical Induction to prove that $1 + 2 + 4 + 8 + \cdots + 2^n = (2^{n+1} - 1)$ for all $n \geqslant 0$.

$n = 0$

$2^0 = 2^{0+1} - 1$

$1 = 2 - 1$

$1 = 1$

$f(0)$ is true

Assume that $f(k)$ is true,

show that $f(k+1)$ is also true.

$1 + 2 + 4 + 8 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1$

$$\begin{aligned} 1 + 2 + 4 + 8 + \ldots + 2^k + 2^{k+1} &= (2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= (2^{k+1} + 2^{k+1}) - 1 \\ &= 2(2^{k+1}) - 1 \\ &= 2^{k+1+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

$\therefore$ $f(n)$ is true for all $n \geqslant 0$.

2. (10 pts) Give a recursive definition with initial condition(s).

a) The function f(n) = n!, n = 0, 1, 2, ... (5 pts)

$a_0 = 0! = 1$

$a_1 = 1! = 1 = 1 f(0)$

$a_2 = 2! = 2 = 2 f(1)$

$a_3 = 3! = 6 = 3 f(2)$

$a_4 = 4! = 24 = 4 f(3)$

$a_{n+1} = (n+1)! = (n+1) n!$

$\therefore$ $a_{n+1} = (n+1) a_n$

for $n \geqslant 0$ and $a_0 = 1$

b) The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... (5 pts)

$a_0 = 1$

$a_1 = 1$

$a_2 = a_1 + a_0 = 1 + 1 = 2$

$a_3 = a_2 + a_1 = 2 + 1 = 3$

$a_4 = a_3 + a_2 = 3 + 2 = 5$

$a_{n+1} = a_n + a_{n-1}$

$\therefore$ $a_{n+1} = a_n + a_{n-1}$

for $n \geqslant 2$ and

$a_0 = 1$, $a_1 = 1$

3. (8 pts)

a) Find f(2) and f(3) if f(n) = 2f(n − 1) + 5, f(0) = 3. (4 pts)

$f(1) = 2f(0)+5 = 2(3)+5 = 11$

$f(2) = 2f(1)+5 = 2(11)+5 = 27$

$f(3) = 2f(2)+5 = 2(27)+5 = 59$

b) Find f(8) if f(n) = 2f(n/2) + 1, f(1) = 2. (4 pts)

$f(2) = 2f(1)+1 = 2(2)+1 = 5$

$f(4) = 2f(2)+1 = 2(5)+1 = 11$

$f(8) = 2f(4)+1 = 2(11)+1 = 23$

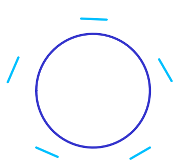
4 (10 pts)

a) Consider a bit string of length 14. How many begin with 10 and end with 11? (5 pts)

$1\,0\,\underbrace{\_\,\_\,\_\,\_\,\_\,\_\,\_\,\_\,\_\,\_}_{10 \text{ bits}}\,1\,1$ $\qquad 2^{10} = 1024$

bit = {0,1}

b) How many ways are there to seat 5 people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor? (5 pts)



$(5-1)!$

$= 4! = 24$

5. (8 pts)

Explain how the Pigeonhole Principle can be used to show that among any 31 integers, at least four must have the same last digit.

0-9 = 10 digits $\qquad \lceil \frac{31}{10} \rceil = \lceil 3.1 \rceil = 4$

total = 32 integers

6. (12 pts)

a) How many ways are there to select 6 students from a class of 25 to serve on a committee? (4 pts)

$$C(25,6) = \frac{25!}{6!(25-6)!} = \frac{25!}{6!19!} = \frac{25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20}{6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$= 177100$$

b) How many ways are there to select 6 students from a class of 25 to hold six different executive positions on a committee? (4 pts)

$$C(25,6)\cdot P(6,6) = \frac{25!}{6!19!}\cdot\frac{6!}{(6-6)!} = 177100\cdot\frac{6!}{0!}$$

$$= 177100\cdot 720 = 127512000$$

c) How many bit strings of length 10 have equal numbers of 0's and 1's? (4 pts)

$$C(10,5)\cdot C(5,5) = \frac{10!}{5!5!}\cdot\frac{5!}{5!0!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2}$$

$$= 252$$

7. (10 pts)

a) Use the Pascal's Tringle to expand $(x + y)^7$. (5 pts)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$$x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

b) Find the coefficient of x^4y^6 in the expansion of $(3x + 2y)^{10}$. (5 pts)

$$\binom{10}{6}(3x)^{10-6}(2y)^6 = C(10,6)\,(3x)^4\,(2y)^6$$

$$= \frac{10!}{6!\,4!}(81x^4)(64y^6)$$

$$= 210\,(81x^4)(64y^6)$$

$$= 1088640\,x^4y^6$$

8. (8 pts)

(a) What is the probability that a card chosen from an ordinary deck of 52 cards is an ace or a king or a queen? (4 pts)

total = 52
aces = 4
kings = 4
queens = 4

p(ace or king or queen)

$$= \frac{4+4+4}{52} = \frac{12}{52} = \frac{3}{13} = 0.23$$

(b) What is the probability that two cards chosen from an ordinary deck of 52 cards are both kings? (4 pts)

total = 52
kings = 4

$$p(\text{both kings}) = \frac{4}{52}\cdot\frac{3}{51}$$

$$= 0.0045$$

9. (8 pts) Suppose you have a class with <u>40 students</u> — <u>14 freshmen</u>, <u>16 sophomores</u>, and <u>10 juniors</u>.

a) You pick two students at random, one at a time. What is the probability that both are juniors? (4 pts)

$$p(\text{both juniors}) = \frac{10}{40}\cdot\frac{9}{39}$$

$$= 0.0577$$

b) You pick two students at random, one at a time. What is the probability that the second student is a freshman, given that the first is a sophomore? (4 pts)

$$P(\text{freshman} \mid \text{sophomore}) = \frac{P(\text{freshman} \cap \text{sophomore})}{P(\text{sophomore})}$$

$$= \frac{\frac{14}{39} \cdot \frac{16}{40}}{\frac{16}{40}}$$

$$= \frac{14}{39} = 0.39$$

10. (10 pts)

a) In a certain lottery game you choose a set of six numbers out of 45 numbers. Find the probability that none of your numbers match the six winning numbers. (4 pts)

$$\frac{\binom{39}{6}}{\binom{45}{6}} = \frac{C(39,6)}{C(45,6)} = \frac{\frac{39!}{6!\,33!}}{\frac{45!}{6!\,39!}} = \frac{3262623}{8145060}$$

$$\simeq 0.401$$

b) An experiment consists of picking at random a bit string of length four. Consider the following events:

E_1: the bit string chosen begins with 01; $\quad 01\_\,\_$

E_2: the bit string chosen ends with 10. $\quad \_\,\_10$

Determine whether E_1 and E_2 are independent. Show your work. (6 pts)

$$\text{total} = 2^4 = 16$$

$$P(E_1) = \frac{2^2}{2^4} = \frac{4}{16} = \frac{1}{4}$$

$$P(E_2) = \frac{2^2}{2^4} = \frac{4}{16} = \frac{1}{4}$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

$$= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

$$P(E_1 \cap E_2) = \{0110\}$$

$$= \frac{1}{16}$$

$\therefore$ E_1 and E_2 are independent

11. (8 pts) Four coins are tossed.

a) List the elements in the sample space. (4 pts)

HHHH TTTT HTHT THTH

HHHT HHTH HTHH THHH

TTTH TTHT THTT HTTT

HHTT HTTH TTHH THHT

total = 16

b) Find the probability that exactly three heads show. (4 pts)

$p(\text{exactly 3 heads}) = \frac{4}{16} = \frac{1}{4}$