Exercise 5.1

4. Let P (n) be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n + 1)/2)^2$ for the positive integer n.

a) What is the statement P (1)?

n = 1, p(1)

p(1): $1^3 = (1(1 + 1)/2)^2$

b) Show that P (1) is true, completing the basis step of the proof.

$1^3 = (1(1 + 1)/2)^2$

$1 = (1(2)/2)^2$

$1 = 1^2$

$1 = 1$

∴ P(1) is true.

c) What is the inductive hypothesis?

$1^3 + 2^3 + \ldots + k^3 = (k(k + 1)/2)^2$

d) What do you need to prove in the inductive step?

We show that for every positive k, p(k) ⟶ p(k + 1) is true.

p(k + 1): $1^3 + 2^3 + \ldots + k^3 + (k + 1)^3 = ((k + 1)(k + 2)/2)^2$

e) Complete the inductive step, identifying where you use the inductive hypothesis.

$P(k + 1) = 1^3 + 2^3 + \ldots + k^3 + (k + 1)^3$

$= (k(k + 1)/2)^2 + (k + 1)^3$

$= k^2(k + 1)^2/2^2 + (k + 1)^2$

$= (k + 1)^2\ (k^2/4 + (k + 1)$

$= (k + 1)^2\ [(k^2 + 4k + 4)/4]$

$= (k + 1)^2\ [(k + 2)^2/4]$

$= ((k + 1)(k + 2)/2)^2$

∴ p(k + 1) is true.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.

a)

Let p(n) be $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$

$n = 1$, $p(1)$: $1 \cdot 1! = (1 + 1)! - 1$

$1 \cdot 1! = (1 + 1)! - 1$

$1 = 2 - 1$

$1 = 1$

$\therefore$ p(1) is true.

b)

$p(k)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$

c)

Show that p(k + 1) is also true.

$p(k + 1) = 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)! = (k + 2)! - 1$

d)

$$
\begin{aligned}
p(k + 1) &= 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)! \\
&= (k + 1)! - 1 + (k + 1) \cdot (k + 1)! \\
&= 1 \cdot (k + 1)! + (k + 1) \cdot (k + 1)! - 1 \\
&= (1 + (k + 1)) \cdot (k + 1)! - 1 \\
&= (k + 2) \cdot (k + 1)! - 1 \\
&= (k + 2)! - 1
\end{aligned}
$$

$\therefore$ p(k + 1) is true.

We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

Exercise 5.2

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

Basis step: We know that the first three dominoes fall, so p(1), p2), p(3) are true.

Inductive step: Every three dominoes fall then the next three dominoes will fall.

If n ⩾ 3, p(k) is true for each k ⩽ n.

4. Let P (n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P (n) is true for n ⩾ 18.

a) Show statements P(18), P (19), P (20), and P (21) are true, completing the basis step of the proof.

p(18) is true because 18 cents of postage can be formed by one 4-cent stamp and two 7-cent stamps.

p(19) is true because 19 cents of postage can be formed by three 4-cent stamps and one 7-cent stamps.

p(20) is true because 20 cents of postage can formed by five 4-cent stamps.

p(21) is true because 21 cents of postage can formed by three 7-cent stamps.

b) What is the inductive hypothesis of the proof?

We can form j cents of postage for all j with 18 ⩽ j ⩽ k, where is greater than 21.

c) What do you need to prove in the inductive step?

We can form k + 1 cents of postage using just 4-cent stamps and 7-cent stamps.

d) Complete the inductive step for k ⩾ 21.

Because k ⩾ 21, we know that P (k − 3) is true so that we can form k − 3 cents of postage. By putting one more 4-cent stamp on the envelope, we have formed k + 1 cents of postage.

e) Explain why these steps show that this statement is true whenever n ⩾ 18.

We have completed both the basis step and the inductive step, so by the principle of strong induction,

the statement is true for every integer n ⩾ 18.

Exercise 5.3

2. Find f (1), f (2), f (3), f (4), and f (5) if f (n) is defined recursively by f (0) = 3 and for n = 0, 1, 2, . . .

a) f (n + 1) = −2f (n).

f(0) = 3

f(1) = -2f(0) = -2 * 3 = -6

f(2) = -2f(1) = -2 * -6 = 12

f(3) = -2f(2) = -2 * 12 = -24

$f(4) = -2f(3) = -2 * -24 = 48$

$f(5) = -2f(4) = -2 * 48 = -96$

b) $f(n + 1) = 3f(n) + 7$.

$f(0) = 3$

$f(1) = 3f(0) + 7 = 3 * 3 + 7 = 16$

$f(2) = 3f(1) + 7 = 3 * 16 + 7 = 55$

$f(3) = 3f(2) + 7 = 3 * 55 + 7 = 172$

$f(4) = 3f(3) + 7 = 3 * 172 + 7 = 523$

$f(5) = 3f(4) + 7 = 3 * 523 + 7 = 1576$

c) $f(n + 1) = f(n)^2 – 2 f(n) – 2$.

$f(0) = 3$

$f(1) = f(0)^2 – 2 f(0) – 2 = 3^2 – 2 * 3 – 2 = 9 – 6 – 2 = 1$

$f(2) = f(1)^2 – 2 f(1) – 2 = 1^2 – 2 * 1 – 2 = 1 – 2 – 2 = -3$

$f(3) = f(2)^2 – 2 f(2) – 2 = (-3)^2 – 2(-3) – 2 = 9 + 6 – 2 = 13$

$f(4) = f(3)^2 – 2 f(3) – 2 = 13^2 – 2 * 13 – 2 = 169 – 26 – 2 = 141$

$f(5) = f(4)^2 – 2 f(4) – 2 = 141^2 – 2 * 141 – 2 = 19881 – 282 – 2 = 19597$

d) $f(n + 1) = 3^{f(n)/3}$.

$f(0) = 3$

$f(1) = 3^{f(0)/3} = 3^{3/3} = 3$

$f(2) = 3^{f(1)/3} = 3^{3/3} = 3$

$f(3) = 3^{f(2)/3} = 3^{3/3} = 3$

$f(4) = 3^{f(3)/3} = 3^{3/3} = 3$

$f(5) = 3^{f(4)/3} = 3^{3/3} = 3$

4. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1, 2, \ldots$

a) $f(n + 1) = f(n) – f(n – 1)$.

$f(0) = f(1) = 1$

$f(2) = f(1) – f(0) = 1 – 1 = 0$

$f(3) = f(2) – f(1) = 0 – 1 = -1$

$f(4) = f(3) – f(2) = -1 – 0 = -1$

$f(5) = f(4) – f(3) = -1 – (-1) = 0$

b) $f(n + 1) = f(n)f(n − 1)$.

$f(0) = f(1) = 1$

$f(2) = f(1) * f(0) = 1 * 1 = 1$

$f(3) = f(2) * f(1) = 1 * 1 = 1$

$f(4) = f(3) * f(2) = 1 * 1 = 1$

$f(5) = f(4) * f(3) = 1 * 1 = 1$

c) $f(n + 1) = f(n)^2 + f(n − 1)^3$.

$f(0) = f(1) = 1$

$f(2) = f(1)^2 + f(0)^3 = 1^2 + 1^3 = 1 + 1 = 2$

$f(3) = f(2)^2 + f(1)^3 = 2^2 + 1^3 = 4 + 1 = 5$

$f(4) = f(3)^2 + f(2)^3 = 5^2 + 2^3 = 25 + 8 = 33$

$f(5) = f(4)^2 + f(3)^3 = 33^2 + 5^3 = 1089 + 125 = 1214$

d) $f(n + 1) = f(n)/f(n − 1)$.

$f(0) = f(1) = 1$

$f(2) = f(1) / f(0) = 1 / 1 = 1$

$f(3) = f(2) / f(1) = 1 / 1 = 1$

$f(4) = f(3) / f(2) = 1 / 1 = 1$

$f(5) = f(4) / f(3) = 1 / 1 = 1$

8. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n − 2$.

$a_1 = 4(1) – 2 = 4 – 2 = 2$

$a_2 = 4(2) – 2 = 8 – 2 = 6$

$a_3 = 4(3) – 2 = 12 – 2 = 10$

$a_{n+1} = 4(n + 1) – 2 = 4n + 4 – 2 = 4n – 2 + 4 = a_n + 4$

$\therefore\ a_{n+1} = a_n + 4$ for $n \geqslant 1$ and $a_1 = 2$

b) $a_n = 1 + (−1)^n$.

$a_1 = 1 + (-1)^1 = 0$

$a_2 = 1 + (-1)^2 = 2$

$a_3 = 1 + (-1)^3 = 0$

$a_{n+1} = 1 + (-1)^{n+1} = 1 + (-1)^{n+1} = 1 + (-1)^1 (-1)^n = 1 - (-1)^n = 1 - [-1 + 1 + (-1)^n] = 1 + 1 - [1 + (-1)^n] = 2 - a_n$

$\therefore\ a_{n+1} = 2 - a_n$ for $n \geqslant 1$ and $a_1 = 0$

c) $a_n = n(n + 1)$.

$a_1 = 1(1 + 1) = 1(2) = 2$

$a_2 = 2(2 + 1) = 2(3) = 6$

$a_3 = 3(3 + 1) = 3(4) = 12$

$a_{n+1} = (n + 1)[(n + 1) + 1] = (n + 1)(n + 2) = n(n + 1) + 2(n + 1) = a_n + 2(n + 1)$

$\therefore\ a_{n+1} = a_n + 2(n + 1)$ for $n \geqslant 1$ and $a_1 = 2$

d) $a_n = n^2$.

$a_1 = 1^2 = 1$

$a_2 = 2^2 = 4$

$a_3 = 3^2 = 9$

$a_{n+1} = (n + 1)^2 = n^2 + 2n + 1 = a_n + 2n + 1$

$\therefore\ a_{n+1} = a_n + 2n + 1$ for $n \geqslant 1$ and $a_1 = 1$

Exercise 5.4

2. Trace Algorithm 1 when it is given n = 6 as input. That is, show all steps used by Algorithm 1 to find 6!, as is done in Example 1 to find 4!.

6! = 6 * factorial(6 – 1)

5! = 5 * factorial(5 – 1)

4! = 4 * factorial(4 – 1)

3! = 3 * factorial(3 – 1)

2! = 2 * factorial(2 – 1)

1! = 1 * factorial(1 – 1)

0! = 0 then return 1

1! = 1 * 1 = 1

2! = 2 * 1 = 2

3! = 3 * 2 = 6

4! = 4 * 6 = 24

5! = 5 * 24 = 120

6! = 6 * 120 = 720

8. Give a recursive algorithm for finding the sum of the first n positive integers.

Procedure sum of positive number(n : positive integer)

If n = 1, return n

Else return n + sum of positive number(n – 1)

Exercise 6.1

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

27 * 37 = 999

8. How many different three-letter initials with none of the letters repeated can people have?

26 * 25 * 24 = 15600

30. How many license plates can be made using either three uppercase English letters followed by three digits, or four uppercase English letters followed by two digits?

Three uppercase letters followed by three digits: $26^3 * 10^3$ = 17576000

Four uppercase letters followed by two digits: $26^4 * 10^2$ = 45697600

40. How many subsets of a set with 100 elements have more than one element?

Let A be a subset of a set with 100 elements.

$2^{100} – 101 = 1.2676506 * 10^{30}$

44. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

(10 * 9 * 8 * 7) / 4 = 5040 / 4 = 1260

Exercise 6.2

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

By using the pigeonhole principle, there n objects are placed into k boxes, then there is at least one box containing at least ⌈ n/k ⌉ objects. ⌈ 30 students/26 letters ⌉ = ⌈ 1.15 ⌉ = 2.

∴ at least two students have last names that begin with the same letter.

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

She wants to have at least three balls of the same color and she has two different colors of balls, so

3 + 2 = 5

∴ she must select five balls to be sure of having at least three balls of the same color.

b) How many balls must she select to be sure of having at least three blue balls?

She wants to have at least three blue balls and she also has 10 red balls, so

3 + 10 = 13

∴ she must select 13 balls to be sure of having at least three blue balls.

8. Show that if f is a function from S to T, where S and T are finite sets with |S| > |T |, then there are elements s_1 and s_2 in S such that f (s_1) = f (s_2), or in other words, f is not one-to-one.

18. Suppose that there are nine students in a discrete mathematics class at a small college.

a) Show that the class must have at least five male students or at least five female students.

We have nine students and students are either male or female, so

$\lceil 9/2 \rceil = \lceil 4.5 \rceil = 5$

∴ the class must have at least five male students or at least five female students.

b) Show that the class must have at least three male students or at least seven female students.

Since we have nine students,

if we have at least three male students, then we can have at most six female students.

$9 - 3 = 6$

If we have at least seven female students, then we can have at most two male students.

$9 - 7 = 2$

∴ the class must have at least three male students or at least seven female students.

Exercise 6.3

4. Let S = {1, 2, 3, 4, 5}.

a) List all the 3-permutations of S.

$P(5, 3) = 5!/(5 - 3)! = 5!/2! = 5 * 4 * 3 = 60$

123, 124, 125, 132, 134, 135, 142, 143, 145, 152, 153, 154,

213, 214, 215, 231, 234, 235, 241, 243, 245, 251, 253, 254,

312, 314, 315, 321, 324, 325, 341, 342, 345, 351, 352, 354,

412, 413, 415, 421, 423, 425, 431, 432, 435, 451, 452, 453,

512, 513, 514, 521, 523, 524, 531, 532, 534, 541, 542, 543

b) List all the 3-combinations of S.

$C(5, 3) = 5!/(3!2!) = (5 * 4)/2 = 10$

123, 124, 125, 134, 135, 145, 234, 235, 245, 345

6. Find the value of each of these quantities.

a) C(5, 1)

$= \frac{5!}{1!4!} = \frac{5}{1} = 5$

b) C(5, 3)

$= \frac{5!}{3!2!} = \frac{5*4}{2} = 10$

c) C(8, 4)

$= \frac{8!}{4!4!} = \frac{8*7*6*5}{4*3*2} = 70$

d) C(8, 8)

$= \frac{8!}{8!0!} = 1$

e) C(8, 0)

$= \frac{8!}{0!8!} = 1$

f) C(12, 6)

$= \frac{12!}{6!6!} = \frac{12*11*10*9*8*7}{6*5*4*3*2} = 924$

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

$P(6, 6) = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! = 720$

12. How many bit strings of length 12 contain

a) exactly three 1s?

$C(12, 3) = \frac{12!}{3!9!} = \frac{12*11*10}{3*2} = 220$

b) at most three 1s?

$C(12, 3) + C(12, 2) + C(12, 1) + C(12, 0) = \frac{12!}{3!9!} + \frac{12!}{2!10!} + \frac{12!}{1!11!} + \frac{12!}{0!12!} = 220 + 66 + 12 + 1 = 299$

c) at least three 1s?

Total length of bit string with length of 12 is 2^{12} = 4096.

N(at least three 1s) = Total length of bit string - N(at most three 1s)

= 4096 – 299

= 3797

d) an equal number of 0s and 1s?

N(an equal number of 0s and 1s) = N(exactly six 1s)

$$= C(12, 6) = \frac{12!}{6!6!} = \frac{12*11*10*9*8*7}{6*5*4*3*2} = 924$$

Exercise 6.4

2. Find the expansion of $(x + y)^5$

a) using combinatorial reasoning, as in Example 1.

$(x + y)^5$

$= (x+y)(x+y)(x+y)(x+y)(x+y)$

$= (x^2 + 2xy + y^2)(x^3 + 3x^2y + 3xy^2 + y^3)$

$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

b) using the binomial theorem.

$(x + y)^5$

$= \sum_{k=0}^{n=5} \binom{n}{k} (1) x^{5-0} y^0$

$= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$

$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

6. What is the coefficient of x^7 in $(1 + x)^{11}$?

$\binom{11}{7} 1^{(11-7)}x^7 = \binom{11}{7} 1^4x^7 = \binom{11}{7}x^7 = C(11, 7)x^7 = \frac{11!}{7!4!}x^7 = \frac{11*10*9*8}{4*3*2}x^7 = 330x^7$

∴ the coefficient of x^7 is 330.

8. What is the coefficient of x^8y^9 in the expansion of

$(3x + 2y)^{17}$?

$\binom{17}{9} (3x)^{(17-9)}(2y)^9 = \binom{17}{9}(3x)^8(2y)^9 = C(17, 9)(3x)^8(2y)^9 = \frac{17!}{9!8!}6561x^8512y^9 = \frac{17*16*15*14*13*12*11*10}{8*7*6*5*4*3*2}3359232x^8y^9 = (24310)3359232x^8y^9 = 81662929920x^8y^9$

∴ the coefficient of x^8y^9 is 81662929920.

12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \leqslant k \leqslant 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

$\binom{10+1}{k}$, $0 \leqslant k \leqslant 11$

$\binom{11}{0}$ $\binom{11}{1}$ $\binom{11}{2}$ $\binom{11}{3}$ $\binom{11}{4}$ $\binom{11}{5}$ $\binom{11}{6}$ $\binom{11}{7}$ $\binom{11}{8}$ $\binom{11}{9}$ $\binom{11}{10}$ $\binom{11}{11}$

1 11 55 165 330 462 462 330 165 55 11 1

Exercise 7.1

2. What is the probability that a fair die comes up six when it is rolled?

Probability(roll a six) = $\frac{1}{6}$

8. What is the probability that a five-card poker hand contains the ace of hearts?

C(52, 5) = $\frac{52!}{5!47!} = \frac{52*51*50*49*48}{5*4*3*2}$ = 132546960

C(51, 4) = $\frac{51!}{4!47!} = \frac{51*50*49*48}{4*3*2}$ = 12744900

$\frac{C(51,4)}{C(52,5)} = \frac{12744900}{132546960}$ = 0.09615

30. What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?

C(6, 5) = $\frac{6!}{5!1!} = \frac{6}{1}$ = 6

C(34, 1) = $\frac{34!}{1!33!} = \frac{34}{1}$ = 34

C(40, 6) = $\frac{40!}{6!34!} = \frac{40*39*38*37*36*35}{6*5*4*3*2}$ = 3838380

$\frac{C(6,5)C(34,1)}{C(40,6)} = \frac{6*34}{3838380}$ = 0.000053147

Exercise 7.2

2. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.

p(1) = p(2) = p(4) = p(5) = p(6) = $\frac{1}{7}$

p(3) = $\frac{2}{7}$

6. What is the probability of these events when we randomly select a permutation of {1, 2, 3}?

Total permutation of {1, 2, 3} is P(3, 3) = $\frac{3!}{(3-3)!}$ = 3! = 6

a) 1 precedes 3.

There are three premutation that 1 precedes 3.

p(1 precedes 3) = $\frac{3}{6}$ = $\frac{1}{2}$

b) 3 precedes 1.

There are three premutation that 3 precedes 1.

p(3 precedes 1) = $\frac{3}{6}$ = $\frac{1}{2}$

c) 3 precedes 1 and 3 precedes 2.

There are two premutation that 3 precedes 1 and 3 precedes 2.

p(3 precedes 1 2) = $\frac{2}{6}$ = $\frac{1}{3}$

12. Suppose that E and F are events such that p(E) = 0.8 and p(F) = 0.6. Show that p(E ∪ F) ⩾ 0.8 and p(E ∩ F) ⩾ 0.4.

p(E ∪ F) = p(E) + p(F) - p(E ∩ F)

p(E ∪ F) = 0.8 + 0.6 - p(E ∩ F)

Since probabilities are always between 0 and 1,

0 ≤ p(E ∩ F) ≤ 0.6

p(E ∪ F) = 0.8 + 0.6 – 0.6

∴ p(E ∪ F) ⩾ 0.8

p(E ∩ F) = 0.8 + 0.6 - p(E ∪ F)

We know that $p(E \cup F) = 0.8$

$p(E \cap F) = 0.8 + 0.6 - 0.8$

$\therefore\ p(E \cap F) \geqslant 0.6$

24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

Let A be the event that exactly four heads appear, and let B be the event that the flip came up tails.

$p(A) = \frac{1}{2^5} = \frac{1}{32}$

$p(B) = \frac{1}{2}$

$P(A|B) = \frac{1}{32} / \frac{1}{2} = \frac{2}{32} = \frac{1}{16}$

26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

If E and F are independent, $p(E) * p(F) = p(E \cap F)$

The number of bit strings of length 3 is $2^3 = 8$

$p(E) = \{001, 010, 100, 111\} = \frac{4}{8} = \frac{1}{2}$

$p(F) = \{100, 101, 110, 111\} = \frac{4}{8} = \frac{1}{2}$

$p(E) * p(F) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$

$p(E \cap F) = \{100, 111\} = \frac{2}{8} = \frac{1}{4}$

$\therefore$ E and F are independent.

Exercise 8.1

8. a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

$a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}, n > 3$

b) What are the initial conditions?

$a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 1, 2^0 = 1$

c) How many bit strings of length seven contain three consecutive 0s?

$a_0 = 0$

$a_1 = 0$

$a_2 = 0$

$a_3 = 1$

$a_4 = a_{4-1} + a_{4-2} + a_{4-3} + 2^{4-3} = a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3$

$a_5 = a_{5-1} + a_{5-2} + a_{5-3} + 2^{5-3} = a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8$

$a_6 = a_{6-1} + a_{6-2} + a_{6-3} + 2^{6-3} = a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20$

$a_7 = a_{7-1} + a_{7-2} + a_{7-3} + 2^{7-3} = a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47$

Exercise 8.3

8. Suppose that f (n) = 2f (n/2) + 3 when n is an even positive integer, and f (1) = 5. Find

a) f (2).

f(2) = 2 f(2/2) + 3 = 2 f(1) + 3 = 2 (5) + 3 = 13

b) f (8).

f(4) = 2 f(4/2) + 3 = 2 f(2) + 3 = 2 (13) + 3 = 29

f(8) = 2 f(8/2) + 3 = 2 f(4) + 3 = 2 (29) + 3 = 61

c) f (64).

f(16) = 2 (61) + 3 = 125

f(32) = 2 (125) + 3 = 253

f(64) = 2 (253) + 3 = 509

d) f (1024).

f(128) = 2 (509) + 3 = 1021

f(256) = 2 (1021) + 3 = 2045

f(512) = 2 (2045) + 3 = 4093

f(1024) = 2 (4093) + 3 = 8189

10. Find f (n) when $n = 2^k$, where f satisfies the recurrence relation f (n) = f (n/2) + 1 with f (1) = 1.

$f(2^k) = f(2^k/2) + 1$

$f(2^k) = f(2^{k-1}/2) + 2$

$f(2^k) = f(2^{k-2}/2) + 3$

$f(2^k) = f(2^{k-(k-1)}/2) + 1 + k - 1$

$\therefore$ $n = 2^k$ is $k + 1$

14. Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are $n/2$ games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.

$n = 2^k$

if $k = 1$, $n = 2$, so

$f(2) = 1$

$k = 2$, $n = 4$,

$f(2^2) = 2$

$k = 3$, $n = 8$,

$f(2^3) = 3$

$\therefore$ $f(n) = f(n/2) + 1$, $n = 2^k$

16. Solve the recurrence relation for the number of rounds in the tournament described in Exercise 14.

$f(n) = [\ f(2^{k-2}/2) + 1\ ] + 2$

$f(n) = f(2^2) + (k - 2)$

$f(n) = [f(2^1/2) + 1\ ] + (k - 1)$

$f(n) = f(2^0) + k$

$f(n) = \log_2 n$

$\therefore$ $\log_2 n$