# Discrete Random Variables

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# Discrete Random Variables

- *Variable*: A quantity that may take different values.
- *Random variable*: A variable that may assume different values with certain probabilities.
  - One way to think of it as a function that assigns a real number to each outcome in the sample space.
  - A *discrete* random variable is one who can only be discrete values.
  - For now, we will focus on bounded discrete random variables.
- Example of a discrete random variable:
  - Flip a coin 3 times.
    Then $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
  - Let $X$ be a random variable that is equal to the number of heads in 3 flips of a coin.
  - $X$ can be 0,1,2, or 3.

## The Support

- Denote the *support* of $X$ as $\mathbb{S}_X$.

- The *support* of $X$ is the space of values which $X$ has a positive probability of occurring.
  - Notationally, $X : S \to \mathbb{S}_X$

- An example is flipping a coin once.
- $S = \{T, H\}$.
  - If we let $X$ be equal to the number of heads (this is the same as setting heads to equal 1 and tails to equal 0).
  - Then $\mathbb{S}_X = \{0, 1\}$.
- In the example on the previous slide with 3 coin flips, $\mathbb{S}_X = \{0, 1, 2, 3\}$.
- The concept of the support of a random variable is an important one.
- Once the appropriate random variable is specified, we can focus only on the support of it as opposed to the entire sample space.
- Example: If we flip a coin 100 times and interested in the number of heads seen.
  - What does $X$ represent?
  - How many elements are in the sample space?
  - What is the support of the random variable $X$?

## Example of the Support

Assume we flip a coin 3 times and let $X$ be the number of heads.

- What does $X = 0$ represent?
- What is the probability all three of the flips land in tails?
- What is the probability that $X = 1$
- What is the probability that $X = 2$
- What is the probability that $X = 3$
- Is this a valid distribution?

# Probability Mass Function (pdf)

The *probability distribution* of $X$ assigns a number to all values $x$ in $\mathbb{S}_X$ such that:

- $0 \leq P(X = x) \leq 1$
- $\sum\limits_{x \in S_x} P(X = x) = 1$

Notationally we state $f(x) = P(X = x)$.

With discrete random variables, $f(x)$ is termed the *probability mass function (p.m.f.)*.

- From here on out, we will refer to $X$ as the random variable.
- We will denote $x$ as the values that $X$ can be.
- For example flipping a coin 3 times, and setting $X$ to be the number of heads.
  - $X$ is the random variable.
  - $X$ can be set equal to $x$ where $x = 0, 1, 2,$ or 3.

## Cumulative Distribution Function (cdf)

- With the p.m.f. of a discrete random variable, we can compute quantities such as $P(X < a)$ or $P(a \leq X < b)$ for some set constants of $a$ and $b$.

- The *cumulative distribution function* (cdf) of a random variable at value $X$ is $P(X \leq x)$.

- Notationally this is $F(x) = P(X \leq x)$
  - It is the sum of all probabilities which have $X \leq x$.

Properties of discrete random variables

Let $a$ and $b$ be a values of $x$

- $P(X \leq a) = 1 - P(X > a)$.

- $P(X > a) = P(X \geq a + 1)$

- $P(X < a) = P(X \leq a - 1)$.

- Something that looks counter intuitive, but holds true for discrete distributions.

  - $P(a \leq X \leq b) = P(X \leq b) - P(X < a)$.

  - $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$.

# Expectation - Discrete Random Variables

The p.m.f. completely determines the probability distribution of a discrete random variable.

- The *expectation* of $X$ can be viewed as the mean or average of $X$.
- Within a frequentist framework, it can be seen as the average of $X$ across many trials of the experiment.
- The *expectation* of $X$ is denoted as $\mathrm{E}(X)$.
- $\mathrm{E}(X) = \sum\limits_{x \in \mathbb{S}_X} xP(X = x) = \sum\limits_{x \in \mathbb{S}_X} xf(x)$.
- Can be viewed as averaging over all possible $X$ values while weighting each possible value by its probability.

Properties of expectations.

- If $a$ and $b$ are constants and $X$ is a random variables, then:

  $\mathrm{E}(a + bX) = E(\mathrm{a}) + E(\mathrm{bX}) = a + b\mathrm{E}(X)$.

- If $X$ and $Y$ are random variables, then $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$.
  - As a result if $a$ and $b$ are constants, then $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
  - As a further result, let $X_i$ be random variables and $a_i$'s be constants.
  - $\mathrm{E}\left(\sum\limits_i a_i X_i\right) = \sum\limits_i a_i \mathrm{E}(X_i)$

# Expectation - Discrete Random Variables

Functions of $X$ are also random variables.

- We can set $h(X)$ to be a function of $X$.
  - Example: $h(X) = X^2$.
  - In the number of heads in 3 coin flips, $\mathbb{S}_X = \{0, 1, 2, 3\}$
  - The support of $X^2$ will be $\mathbb{S}_{X^2} = \{0, 1, 4, 9\}$
- We can take expectation of these functions without having to first find the distribution of $h(X)$ first.
  - $\mathrm{E}(h(X)) = \sum_{x \in \mathbb{S}_X} h(x)P(X = x) = \sum_{x \in \mathbb{S}_X} h(x)f(x)$.
- Just like with $X$, $\mathrm{E}(h(X))$ can be viewed as averaging over all possible $h(X)$ values while weighting each possible value by the probability of $X$.

# Variance - Discrete Random Variables

Now we come to another quantity that describes the probability distribution of $X$, known as the *variance*.

- The *variance* of $X$ is defined to be the average squared deviation from the mean.
- $\text{Var}(X)$=$\text{E}[(X - \text{E}(X))^2]$=$\text{E}[(X - \mu)^2]$.
- $\text{Var}(X)$= $\text{E}(X^2) - [\text{E}(X)]^2$
- We denote the variance of $X$ as $\sigma^2$.
- Since it is the expected value of a squared random variable, $\sigma^2 > 0$.
- $\sigma = \sqrt{\text{Var}(X)}$ is known as the *standard deviation* of $X$.
  - $\sigma$ - The typical distance of the datapoints to the mean.

### Properties

- If $c$ is a constant, then $\text{Var}(c)$= 0
- If $c$ is a constant, then $\text{Var}(cX) = c^2\text{Var}(X)$

If $X$ and $Y$ are independent random variables.

- $\text{Var}(X + Y)$=$\text{Var}(X)$+$\text{Var}(Y)$

As a result, if $X_i$'s are independent random variables and $c_i$'s are constants, then:

- $\text{Var}(\sum_i c_i X_i) = \sum_i c_i^2 \text{Var}(X_i)$

# Discrete Random Variables: Distribution

Returning to the example of flipping a coin 3 times. A distribution table for $X$, the number of heads, can be constructed below.

| | $x = 0$ | $x = 1$ | $x = 2$ | $x = 3$ | Total |
|---|---|---|---|---|---|
| $P(X = x)$ | | | | | |

- Calculate $f(2)$

- Calculate $F(2)$.

- What is the expectation of $X$?

  $$\mathrm{E}(X) = \sum_{x \in \mathbb{S}_X} x f(x)$$

- What is the expectation of $5 - 3X$?

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|---|---|---|---|---|---|
| $P(X = x)$ | | | | | |

- Let $h(X) = X^2$. What is the expectation of $h(x)$?

- Calculate the Variance of $X$

- What is the Variance of $5 - 3X$?

- What is the standard deviation of $X$?

# Discrete Random Variables: Classes

Example: Say $X$ is the number of days a student is registered to take classes at University of California Irvine.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x)$ | 0.10 | 0.30 | 0.25 | 0.25 | 0.10 |

- What is the expectation of $X$?

- What is the expectation of $5 + 3X$?

- Let $h(X) = X^2$. What is the expectation of $h(x)$?

- What is the Variance of $X$?