Discrete Random Variables

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Discrete Random Variables

- *Variable*: A quantity that may take different values.
- *Random variable*: A variable that may assume different values with certain probabilities.
  - One way to think of it as a function that assigns a real number to each outcome in the sample space.
  - A *discrete* random variable is one who can only be discrete values. Integers
  - For now, we will focus on bounded discrete random variables.
- Example of a discrete random variable:
  - Flip a coin 3 times.
    Then $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
  - Let X be a random variable that is equal to the number of heads in 3 flips of a coin.
  - X can be 0,1,2, or 3.

The Support

- Denote the *support* of X as $\mathbb{S}_X$.

- The *support* of X is the space of values which X has a positive probability of occurring.

  - Notationally, $X : S \to \mathbb{S}_X$

- An example is flipping a coin once.

- $S = \{T, H\}$. Categorical

  - If we let X be equal to the number of heads (this is the same as setting heads to equal 1 and tails to equal 0).
  - Then $\mathbb{S}_X = \{0, 1\}$. quantitatied

- In the example on the previous slide with 3 coin flips, $\mathbb{S}_X = \{0, 1, 2, 3\}$.

- The concept of the support of a random variable is an important one.

- Once the appropriate random variable is specified, we can focus only on the support of it as opposed to the entire sample space.

- Example: If we flip a coin 100 times and interested in the number of heads seen.

  - What does X represent?

    x = # of heads in 100 coin flips

  - How many elements are in the sample space?

    $n = 2^{100} = 1.2677 \times 10^{30}$

  - What is the support of the random variable X?

    $S_x = \{0, 1, 2, 3, 4, \ldots, 100\}$

    $n = 101$

Example of the Support

Assume we flip a coin 3 times and let X be the number of heads.

- What does $X = 0$ represent?

No heads flipped in 3 coin tosses

- What is the probability all three of the flips land in tails?

$$P(X=0) = \boxed{\binom{3}{0}} \overset{T}{\left(\frac{1}{2}\right)} \overset{T}{\left(\frac{1}{2}\right)} \overset{T}{\left(\frac{1}{2}\right)} = \frac{1}{8} = 0.125$$

- What is the probability that $X = 1$

$$P(X=1) = \binom{3}{1} \overset{H}{\left(\frac{1}{2}\right)} \overset{T}{\left(\frac{1}{2}\right)} \overset{T}{\left(\frac{1}{2}\right)} = \frac{3}{8} = 0.375$$

- What is the probability that $X = 2$

$$P(X=2) = \binom{3}{2} \overset{H}{\left(\frac{1}{2}\right)} \overset{H}{\left(\frac{1}{2}\right)} \overset{T}{\left(\frac{1}{2}\right)} = \frac{3}{8} = 0.375$$

- What is the probability that $X = 3$

$$P(X=3) = \binom{3}{3} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{1}{8} = 0.125$$

- Is this a valid distribution?

$\Sigma = 1$

Yes, it is a valid distribution.

Probability Mass Function (pdf)

The *probability distribution* of X assigns a number to all values x in $\mathbb{S}_X$ such that:

- $0 \leq P(X = x) \leq 1$
- $\sum_{x \in S_x} P(X = x) = 1$ valid

Notationally we state $f(x) = P(X = x)$.

With discrete random variables, $f(x)$ is termed the *probability mass function (p.m.f.)*.

- From here on out, we will refer to X as the random variable.
- We will denote x as the values that X can be.
- For example flipping a coin 3 times, and setting X to be the number of heads.
  - X is the random variable.
  - X can be set equal to x where $x = 0, 1, 2$, or 3.

Cumulative Distribution Function (cdf)

- With the p.m.f. of a discrete random variable, we can compute quantities such as $P(X < a)$ or $P(a \leq X < b)$ for some set constants of a and b.

- The *cumulative distribution function* (cdf) of a random variable at value X is $P(X \leq x)$.

- Notationally this is $F(x) = P(X \leq x)$
  - It is the sum of all probabilities which have $X \leq x$.

Properties of discrete random variables

Let a and b be a values of x

- $P(X \leq a) = 1 - P(X > a)$.
- $P(X > a) = P(X \geq a + 1)$ $\quad P(X > 6) = P(X \geq 7)$
- $P(X < a) = P(X \leq a - 1)$. $\quad P(X < 5) = P(X \leq 4)$
- Something that looks counter intuitive, but holds true for discrete distributions.
  - $P(a \leq X \leq b) = P(X \leq b) - P(X < a)$.

  - $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$.

Expectation - Discrete Random Variables

The p.m.f. completely determines the probability distribution of a discrete random variable.

- The *expectation* of X can be viewed as the mean or average of X.
- Within a frequentist framework, it can be seen as the average of X across many trials of the experiment.
- The *expectation* of X is denoted as $\mathrm{E}(X)$.

weighted average

- $\mathrm{E}(X) = \sum_{x \in \mathbb{S}_X} xP(X = x) = \sum_{x \in \mathbb{S}_X} xf(x)$.
- Can be viewed as averaging over all possible X values while weighting each possible value by its probability.

Properties of expectations.

- If a and b are constants and X is a random variables, then:

  $\mathrm{E}(a + bX) = E(\text{a}) + E(\text{bX}) = a + b\mathrm{E}(X)$.

- If X and Y are random variables, then $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$.
  - As a result if a and b are constants, then $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
  - As a further result, let X_i be random variables and a_i's be constants.
  - $\mathrm{E}\left(\sum_i a_i X_i\right) = \sum_i a_i \mathrm{E}(X_i)$

Expectation - Discrete Random Variables

Functions of X are also random variables.

- We can set $h(X)$ to be a function of X.
  - Example: $h(X) = X^2$.
  - In the number of heads in 3 coin flips, $\mathbb{S}_X = \{0, 1, 2, 3\}$
  - The support of X^2 will be $\mathbb{S}_{X^2} = \{0, 1, 4, 9\}$
- We can take expectation of these functions without having to first find the distribution of $h(X)$ first.
  - $\mathrm{E}(h(X)) = \sum_{x \in \mathbb{S}_X} h(x) P(X = x) = \sum_{x \in \mathbb{S}_X} h(x) f(x)$.
- Just like with X, $\mathrm{E}(h(X))$ can be viewed as averaging over all possible $h(X)$ values while weighting each possible value by the probability of X.

Variance - Discrete Random Variables

Now we come to another quantity that describes the probability distribution of X, known as the *variance*.

- The *variance* of X is defined to be the average squared deviation from the mean.
- $\text{Var}(X)=\text{E}[(X-\text{E}(X))^2]=\text{E}[(X-\mu)^2]$.

  pop mean
- $\text{Var}(X)=\text{E}(X^2)-[\text{E}(X)]^2$
- We denote the variance of X as σ^2. (SIGMA)^2
- Since it is the expected value of a squared random variable, $\sigma^2 > 0$.
- $\sigma = \sqrt{\text{Var}(X)}$ is known as the *standard deviation* of X.
  - σ - The typical distance of the datapoints to the mean.

Properties

- If c is a constant, then $\text{Var}(c)=0$
- If c is a constant, then $\text{Var}(cX)=c^2\text{Var}(X)$

If X and Y are independent random variables.

- $\text{Var}(X+Y)=\text{Var}(X)+\text{Var}(Y)$

As a result, if X_i's are independent random variables and c_i's are constants, then:

- $\text{Var}(\sum_i c_i X_i) = \sum_i c_i^2 \text{Var}(X_i)$

Discrete Random Variables: Distribution

Returning to the example of flipping a coin 3 times. A distribution table for X, the number of heads, can be constructed below. $S_x = \{0, 1, 2, 3\}$

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	Total
$P(X = x)$	0.125	0.375	0.375	0.125	1
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

- Calculate $f(2)$ $= P(X = 2)$

  $= 0.375$

- Calculate $F(2)$. $= P(X \leq 2)$

  $= P(X=0) + P(X=1) + P(X=2)$

  $= 0.125 + 0.375 + 0.375 = 0.875$

  OR $= 1 - P(X > 2) = 1 - P(X = 3) = 0.875$

- What is the expectation of X?

$$E(X) = \sum_{x \in \mathbb{S}_X} x f(x) = 0(0.125) + 1(0.375) + 2(0.375) + 3(0.125)$$

$$= 1.5$$

- What is the expectation of $5 - 3X$?

$$E(5 - 3X) = E(5) - 3E(X)$$

$$= 5 - 3(1.5)$$

$$= 5 - 4.5 = 0.5$$

Discrete Random Variables: Distribution

Returning to the example of flipping a coin 3 times. A distribution table for X, the number of heads, can be constructed below.

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	Total
$P(X = x)$	0.125	0.375	0.375	0.125	1

- Let $h(X) = X^2$. What is the expectation of $h(x)$?

$$E(X^2) = \Sigma x^2 f(x)$$
$$= 0^2(0.125) + 1^2(0.375) + 2^2(0.375) + 3^2(0.125)$$
$$= 3$$

- Calculate the Variance of X

$$VAR(X) = E(X^2) - (E(X))^2$$
$$= 3 - (1.5)^2 = 0.75$$

- What is the Variance of $5 - 3X$?

$$VAR(5-3X) = VAR(5) + (-3)^2 VAR(X)$$
$$= 0 + 9\,VAR(X) = 0 + 9(0.75) = 6.75$$

- What is the standard deviation of X?

$$\sigma_X = \sqrt{VAR(X)}$$
$$= \sqrt{0.75} = 0.8660$$

Discrete Random Variables: Classes

Example: Say X is the number of days a student is registered to take classes at University of California Irvine.

x	1	2	3	4	5
$f(x)$	0.10	0.30	0.25	0.25	0.10

- What is the expectation of X?

$$E(X) = \sum_{x \in S_x} x f(x)$$
$$= 1(0.1) + 2(0.3) + 3(0.25) + 4(0.25) + 5(0.1)$$
$$= 2.95$$

- What is the expectation of $5 + 3X$?

$$E(5+3X) = E(5) + 3E(X)$$
$$= 5 + 3(2.95)$$
$$= 13.85$$

- Let $h(X) = X^2$. What is the expectation of $h(x)$?

$$E(X^2) = \sum x^2 f(x)$$
$$= 1^2(0.1) + 2^2(0.3) + 3^2(0.25) + 4^2(0.25) + 5^2(0.1)$$
$$= 10.05$$

- What is the Variance of X?

$$VAR(X) = E(X^2) - (E(X))^2$$
$$= 10.05 - (2.95)^2$$
$$= 1.3475$$

$$\sigma_x = \sqrt{VAR(X)} = \sqrt{1.3475} = 1.1608$$