6.2

Recall Let $T: V \to W$ be a LT

a) T is one to one iff $\ker T = \{0\}$

b) T is onto iff $\dim \operatorname{Rng} T = \dim W$

Recall ① The nullity of T is $\dim \ker T$

② The rank of T is $\dim \operatorname{Rng} T$

Rank – Nullity Theorem (General case)

Let $T: V \to W$ be a LT

Then Nullity + Rank $= \dim \underbrace{V}_{\text{domain}}$

i.e. $\dim \ker T + \dim \operatorname{Rng} T = \dim V$

eg. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a LT

show if T is one to one then T must be onto

Proof: Since T is 1-1, $\ker T = \{0\}$

$\Rightarrow \dim \ker T = 0$

$\dim \underbrace{\mathbb{R}^3}_{\text{domain}} = 3$

$\Rightarrow \dim \operatorname{Rng} T = 3$ by the RNT

$\Rightarrow \dim \operatorname{Rng} T = \dim \underbrace{\mathbb{R}^3}_{\text{codomain}}$

$\therefore T$ is onto

Quiz

6.1 6.2

6.3

eg. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a LT

Show that T cannot be 1 to 1

Proof by contradiction

Suppose T is 1 to 1

then $\ker T = \{0\}$

$\therefore \dim \ker T = 0$

By RNT, $\dim \operatorname{Rng} T = 3$

But this is impossible since $\dim \text{codomain} = 2$

$\therefore$ T is not 1 to 1 $\square$

Definition

Let $T: V \to W$ be a LT

If T is one to one and onto

then T is called an isomorphism

we say V and W are isomorphic

eg. Prove that P_2 and $\mathbb{R}^3$ are isomorphic

$\exists\, T: P_2 \to \mathbb{R}^3$ with T 1-1, ont

① Construct T

② show T is a LT

③ show T is 1-1

④ show T is onto

① Construct T

typical vector in V $\to$ typical vector in W

Define $T: P_2 \to \mathbb{R}^3$ as follows:

$T(a + bx + cx^2) = (a, b, c)$

② Show T is a LT

ⓐ T separate $+$

$$T(\underbrace{(a_1 + b_1 x + c_1 x^2)}_{v_1} + \underbrace{(a_2 + b_2 x + c_2 x^2)}_{v_2})$$

$$= T(\overset{a}{(a_1 + a_2)} + \overset{b}{(b_1 + b_2)}x + \overset{c}{(c_1 + c_2)}x^2)$$

$$= (a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2)$$

$$= (a_1, b_1, c_1) + (a_2, b_2, c_2)$$

$$= T(a_1 + b_1 x + c_1 x^2) + T(a_2 + b_2 x + c_2 x^2)$$

(b) A constant can be pulled out

$T(k(a+bx+cx^2))$

$= T(ka+kbx+kcx^2)$

$= (ka, kb, kc)$

$= k(a,b,c)$

$= kT(a+bx+cx^2)$

(3) show T is one to one

show $\ker T = \{\vec{0}\}$ suppose $T(\vec{v}) = \vec{0}$

Suppose $T(a+bx+cx^2) = (0,0,0)$

then $(a,b,c) = (0,0,0) \Rightarrow a=0, b=0, c=0$

$\therefore a+bx+cx^2 = \vec{0}$

$\therefore \ker T = \{\vec{0}\}$

(4) T is onto

show $\dim \mathrm{Rng}\, T = \dim \mathrm{Codomain}$

$\dim P_2 = 3$, $\dim \ker T = 0$

$\therefore \dim \mathrm{Rng}\, T = 3$ by RNT

Since $\dim \mathbb{R}^3$ is also 3, T is onto

$\therefore$ T is isomorphism

$\therefore$ P_2 and $\mathbb{R}^3$ are isomorphic

Thus P_2 and $\mathbb{R}^3$ are the same as vector spaces

eg.

$(1+3x+5x^2)$	$(1,3,5)$
$+ (7+11x+13x^2)$	$+ (7,11,13)$
$8+14x+18x^2$	$(8,14,18)$

eg. show the vector space of 2 x 2 upper triangular matrices is isomorphic to P_2

Define $T : \underset{\text{triangular}}{\overset{2\times 2}{\text{upper}}} \to P_2$

as $T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = a + bx + cx^2$

HW: show T is LT, 1-1, onto

6.3

Let $T: P_1 \to P_1$ be a LT

defined by $T(a+bx) = (a+b) + (a-b)x$

Let $B = \{7+x, 8+x\}$, $C = \{4+11x, 1+3x\}$

a) Find $[T]_B^C$

b) use (a) to find $T(-6-x)$

(a) $[T]_B^C = [[T(7+x)]_C, [T(8+x)]_C]$

$T(7+x) = \begin{matrix} a=7 \\ b=1 \end{matrix} = 8+6x = c_1(4+11x) + c_2(1+3x)$

$T(8+x) = \begin{matrix} a=8 \\ b=1 \end{matrix} = 9+7x = c_1(4+11x) + c_2(1+3x)$

skipped 3 steps

skipped 3 steps

$\left[\begin{array}{cc:c} 4 & 1 & 8 \\ 11 & 3 & 6 \end{array}\right]$ $\left[\begin{array}{cc:c} 4 & 1 & 9 \\ 11 & 3 & 7 \end{array}\right]$

$$\left[\begin{array}{cc:cc} 4 & 1 & 8 & 9 \\ 11 & 3 & 6 & 7 \end{array}\right] \xRightarrow{\text{RREF}} \begin{array}{cccc} & & c_1 & c_1 \\ & & & \\ & & c_2 & c_2 \end{array}\left[\begin{array}{cc:cc} 1 & 0 & 18 & 20 \\ 0 & 1 & -64 & -71 \end{array}\right]$$

$\therefore [T]_B^C = \begin{bmatrix} 18 & 20 \\ -64 & -71 \end{bmatrix}$

b) Use $[T(v_1)]_C = [T]_B^C [v]_B$

$[T(-6-x)]_C = [T]_B^C [-6-x]_B$

Step ① find $[-6-x]_B$

② find $[T(-6-x)]_C$

③ find $T(-6-x)$

① Find $[-6-x]_B$

$-6-x = c_1(7+x) + c_2(8+x)$

Skipped 3 steps

$$\left[\begin{array}{cc:c} 7 & 3 & -6 \\ 1 & 1 & -1 \end{array}\right] \xRightarrow{\text{RREF}} \begin{array}{c} \left[\begin{array}{cc:c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{l} c_1 \\ c_2 \end{array}$$

$$\Rightarrow [-6-x]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

② Find $[T(-6-x)]_C$

$$[T(-6-x)]_C = \begin{bmatrix} 18 & 20 \\ -64 & -71 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ 57 \end{bmatrix}$$

③ Find $T(-6-x) = -16(4+11x) + 57(1+3x)$

$$= (-64-176x) + (57+171x)$$

$$= -7-5x$$

Check $T(-6-x)$

$a=-6$
$b=-1$

$= -7-5x$

Composition

Let V_1, V_2, V_3 be vector spaces

If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are LTs,

then their composition written $T_2 \circ T_1$, is defined as follows:

$$T_2 \circ T_1 : V_1 \to V_3$$

with $T_2 \circ T_1 (v) = T_2(T_1(v))$

$$\begin{tikzcd} V_1 \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & V_2 \arrow[r, "T_2"] & V_3 \end{tikzcd}$$

Theorem: $T_2 \circ T_1$ is also a LT

proof: HW

eg. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a LT defined by

$$T_1(x,y) = (x-3y,\ 4x+5y)$$

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be a LT defined by

$$T_2(x,y) = (7x+9y,\ 11x+15y)$$

Composite $T_2 \circ T_1$

$T_2 \circ T_1(x,y)$

$= T_2(T_1(x,y))$

$\downarrow$ apply

$= T_2(x-3y,\ 4x+5y)$

$= (7(x-3y)+9(4x+5y),\ 11(x-3y)+15(4x+5y))$

$= (7x-21y+36x+45y,\ 11x-33y+60x+75y)$

$= (43x+24y,\ 71x+42y)$

Theorem

Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be LTs.

Let B be a basis for V_1, C be a basis for V_2,

and D be a basis for V_3

Then $[T_2 \circ T_1]^D_B = [T_2]^D_C [T_1]^C_B$

$$\begin{tikzcd} \underset{B}{V_1} \arrow[rr, bend right, "T_2 \circ T_1"'] & \underset{C}{V_2} & \underset{D}{V_3} \end{tikzcd}$$

eg. $T_1(x,y) = (x-3y,\ 4x+5y)$

$T_2(x,y) = (7x+9y,\ 11x+15y)$

Let $B = C = D = \{(1,0),\ (0,1)\}$

$\Rightarrow [T_1]^C_B = [[T(0,1)]_C,\ [T(0,1)]_C]$

$T(1,0) = \begin{matrix} x=1 \\ y=0 \end{matrix} = (1,4) \Rightarrow [T(1,0)]_C = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$T(0,1) = \begin{matrix} x=0 \\ y=1 \end{matrix} = (-3,5) \Rightarrow [T(0,1)]_C = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}$

and $[T_2]_C^D = \begin{bmatrix} 7 & 9 \\ 11 & 15 \end{bmatrix}$

$$\begin{aligned} [T_2 \circ T_1]_B^D &= [T_2]_C^D \cdot [T_1]_B^C \\ &= \begin{bmatrix} 7 & 9 \\ 11 & 15 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7\cdot1 + 9\cdot4 & 7(-3)+9(5) \\ 11(1)+15(4) & 11(-3)+15(5) \end{bmatrix} \\ &= \begin{bmatrix} 43 & 24 \\ 71 & 42 \end{bmatrix} \end{aligned}$$

To find $T_2 \circ T_1(x,y)$:

use $[T_2 \circ T_1(x,y)]_D = [T_2 \circ T_1]_B^D \cdot [(x,y)]_B$

$[T_2 \circ T_1]_B^D \, [(x,y)]_B$

$= \begin{bmatrix} 43 & 24 \\ 71 & 42 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 43x+24y \\ 71x+42y \end{bmatrix}$

$\therefore (T_2 \circ T_1(x,y)) = (43x+24y)(1,0)+(71x+42y)(0,1)$

$= (43x+24y, 71x+42y)$