10) Is the set of 2 X 2 diagonal matrices with real entries a subspace of the vector space of 2 x 2 matrices over R? Justify your answer.

Let ω be set of all 2×2 diagonal matrices

1.) zero vector ?

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Yes since $a_{ij} = 0$, when $i \neq j$

2.) close under + ?

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{R}$$

$$= \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}, \quad a+c,\ b+d \in \mathbb{R}$$

Yes, since $a_{ij} = 0$, when $i \neq j$

3.) closed under $\cdot$?

$$k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad k \in \mathbb{R}$$

$$= \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}, \quad ka,\ kb \in \mathbb{R}$$

Yes, since $a_{ij} = 0$, when $i \neq j$

$\therefore \omega$ is a subspace

12) On R^2 (all real ordered pairs), define the operation and multiplication by a real number as follows:

$$(x_1, y_1) + (x_2, y_2) = (2x_1x_2, y_1 + y_2)$$

$$a(x, y) = (ax, y)$$

a) Is + commutative?

$$(x_1, y_1) + (x_2, y_2) = (2x_1x_2, y_1 + y_2)$$

$$(x_2, y_2) + (x_1, y_1) = (2x_2x_1, y_2 + y_1) = (2x_1x_2, y_1 + y_2)$$

$\therefore$ Yes

$(x_1, y_1) + (x_2, y_2) = (2x_1 x_2, y_1 + y_2)$

b) Is there a 0 vector?

$(x,y) + (a,b) = (x,y)$

$(2xa, y+b) = (x,y)$

$2xa = x \qquad y + b = y$

$a = \frac{1}{2} \qquad b = 0$

$\vec{0} = (\frac{1}{2}, 0)$

c) Use the definition of additive inverse to find -(1,2)

$(1,2) + (a,b) = (\frac{1}{2}, 0)$

$(2a, 2+b) = (\frac{1}{2}, 0)$

$2a = \frac{1}{2} \qquad 2+b = 0$

$a = \frac{1}{4} \qquad b = -2$

$\therefore \ -(1,2) = (\frac{1}{4}, -2)$

d) Does 1v = v hold for all v ∈ V?

$1v = 1(x,y)$

$= (1x, y)$

$= (x,y)$

Yes $1v = v$

14) Let V = R^3, v1 = (1,0,2), v2 = (2,4,5) Determine whether v1, v2 are LI or not and fine the subspace spanned by {v1, v2} and describe it geometrically.

LI? Suppose $c_1(1,0,2) + c_2(2,4,5) = (0,0,0)$

$(c_1, 0, 2c_1) + (2c_2, 4c_2, 5c_2) = (0,0,0)$

$(c_1 + 2c_2, 0 + 4c_2, 2c_1 + 5c_2) = (0,0,0)$

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ 0 + 4c_2 &= 0 \\ 2c_1 + 5c_2 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} c_1 &= -2c_2 = 0 \\ c_2 &= 0 \end{aligned}$$

$\therefore \ v_1, v_2$ are LI

$c_1(1,0,2) + c_2(2,4,5) = (x,y,z)$

$(c_1, 0, 2c_1) + (2c_2, 4c_2, 5c_2) = (x,y,z)$

$(c_1 + 2c_2,\ 0 + 4c_2,\ 2c_1 + 5c_2) = (x,y,z)$

$$\begin{aligned} c_1 + 2c_2 &= x \\ 0 + 4c_2 &= y \\ 2c_1 + 5c_2 &= z \end{aligned} \Rightarrow \left[\begin{array}{cc:c} 1 & 2 & x \\ 0 & 4 & y \\ 2 & 5 & z \end{array}\right] \quad R2\left(\tfrac{1}{4}\right) \Rightarrow \left[\begin{array}{cc:c} 1 & 2 & x \\ 0 & 1 & \frac{y}{4} \\ 2 & 5 & z \end{array}\right]$$

$R1(-2)$	-2	-4	$-4x$
$R3$	2	5	z
New $R3$	0	1	$z-4x$

$$\Rightarrow \left[\begin{array}{cc:c} 1 & 2 & x \\ 0 & 1 & \frac{y}{4} \\ 0 & 1 & z-4x \end{array}\right]$$

$R2(-1)$	0	-1	$-\frac{y}{4}$
$R3$	0	1	$z-4x$
New $R3$	0	0	$z-4x-\frac{y}{4}$

$$\Rightarrow \left[\begin{array}{cc:c} 1 & 2 & x \\ 0 & 1 & \frac{y}{4} \\ 0 & 0 & z-4x-\frac{y}{4} \end{array}\right]$$

$$\therefore\ z - 4x - \frac{y}{4} = 0$$

15) For A= $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) Find the (2, 3) entry of the adjoint matrix

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} adj(A) &= (-1)^{2+3}[3(0) - (-2)(1)] \\ &= -1(2) \\ &= -2 \end{aligned}$$

18) a) Find a basis for the column space and row space for $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & 1 \end{bmatrix}$

RREF $\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

Row space : $\{(1,0,1,2), (0,1,-1,2), (0,0,1,-5)\}$ is a basis

column space : $\{(1,0,0), (0,1,-2), (1,-1,1)\}$ is a basis

b) Is it true that dim(row space) = dim(column space)? Justify your answer.

Yes, their leading ones are equal

20) Let V = R^3 and S = {(x1, x2, x3) ∈ R^3 : x1 + x2 + 2x3 =0}

a) Show S is a subspace of R^3

1.) zero vector ? $0 + 0 + 2(0) = 0$

Yes

2.) closed under + ? $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$

then $(x_1+y_1) + (x_2+y_2) + 2(x_3+y_3)$

$= (x_1+x_2+2x_3) + (y_1+y_2+2y_3)$

$= 0 + 0$

$= 0$

Yes

3.) closed under · ?

$K(x_1, x_2, x_3)$ $K \in \mathbb{R}$, $(x_1, x_2, x_3) \in S$

$= (Kx_1, Kx_2, Kx_3)$

Then $Kx_1 + Kx_2 + 2Kx_3$

$= K(x_1 + x_2 + 2x_3)$

$= K(0)$

$= 0$

Yes

∴ S is a subspace

b) Find a basis for S

(x_1, x_2, x_3) Rank = 1

Let $x_2 = s$, $x_3 = t$

$x_1 + x_2 + 2x_3 = 0$

$x_1 = -x_2 - 2x_3$

$= -s - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ t \end{bmatrix} + \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$\{(-1,1,0)\ (-2,0,1)\}$ is linear combination

22) Let V = M2(R) and S = {A $\in$ V : A = $\left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right]$

a) Show S is a subspace of V.

1.) $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ since second row entries are zero

$\therefore \vec{0} \in S$

2.) closed under $+$?

$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \in S$

$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$

since second row entries are zero

$\therefore$ S closed under $+$

3.) closed under $\cdot$?

$k \in \mathbb{R}$ $\quad \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in S$

$k \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & 0 \end{bmatrix}$ $\quad ka, kb \in \mathbb{R}$

since second row entries are zero

$\therefore$ S closed under $\cdot$

$\therefore$ S is a subspace of V

b) Prove that { $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ } is a basis for S

1.) LI?

suppose $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then $\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} = 0$ $\qquad c_1 = 0$

$c_2 = 0$

$\therefore$ they are LI

2) Span?

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in S$$

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$c_1 = a$, $c_2 = b$

$\therefore$ it span

$\therefore$ It is a subspace

24) Determine whether S={(1,3), (3,-1), (0,4)} is dependent or independent in R2. If the set is dependent, find a dependent relationship and find a LI subset of S that has the same span as S.

suppose $c_1(1,3) + c_2(3,-1) + c_3(0,4) = (0,0)$

$(c_1, 3c_1) + (3c_2, -c_2) + (0, 4c_3) = (0,0)$

$(c_1 + 3c_2 + 0, 3c_1 - c_2 + 4c_3) = (0,0)$

$$\begin{aligned} c_1 + 3c_2 + 0 &= 0 \\ 3c_1 - c_2 + 4c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right] \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1.2 & 0 \\ 0 & 1 & -0.4 & 0 \end{bmatrix}$$

c_3 is free   let $t = 5$, then

$c_1 = -1.2t$ $c_1 = -6$

$c_2 = 0.4t$ $c_2 = 2$

$c_3 = t$ $c_3 = 5$

$\therefore$ S is LD

$\therefore -6(1,3) + 2(3,-1) + 5(0,4) = (0,0)$

To find LI set drop any vector where $c_i \neq 0$, Let's drop $(1,3)$

then $\{(3,-1), (0,4)\}$ and it is LI

since $(3,-1) = k(0,4)$ is impossible

27) Is S = {1+x, 2+x+x^2, 1-x} a basis for P2? Justify your answer

suppose $c_1(1+x) + c_2(2+x+x^2) + c_3(1-x) = 0$

$(c_1 + c_1x) + (2c_2 + c_2x + c_2x^2) + (c_3 - c_3x) = 0$

$(c_1 + 2c_2 + c_3) + (c_1x + c_2x - c_3x) + (0 + c_2x^2 + 0) = 0$

$(c_1 + 2c_2 + c_3) + (c_1 + c_2 - c_3)x + (0 + c_2 + 0)x^2 = 0$

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ c_1 + c_2 - c_3 &= 0 \\ 0 + c_2 + 0 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 0$

$\therefore$ S is LI

$\therefore$ S is a basis

28) Let S be the subset of M2(R) consisting of all upper triangular matrices. Show S is a subspace of M2(R) and find a basis for S.

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for all 2×2 upper triangular matrices.

LI ?

suppose $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad c_1 = 0 \quad c_2 = 0 \quad c_3 = 0$

$\therefore$ It is LI.

Span ? $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$

$$= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$a = c_1, \; b = c_2, \; c = c_3$ $\quad \therefore$ It span $\quad \therefore$ It is a basis

31) Let V = R2 and F= R. Define + and . on V as follows:

(a,b)+(c,d)=(2a+2c,2b+2d), k(a,b)=(2ka, 2kb).

a) Is the operation + commutative?

$(a,b)+(c,d) = (2a+2c, 2b+2d)$

$(c,d)+(a,b) = (2c+2a, 2d+2a)$ $\therefore$ Yes

b) Is + associative?

$(a,b)+((c,d)+(e,f)) = (a,b)+(2c+2e, 2d+2f)$

$= (2a+4c+4e, 2b+4d+4f)$

$((a,b)+(c,d))+(e,f) = (2a+2c, 2b+2d)+(e,f)$

$= (4a+4c+2e, 4b+4d+2f)$

Not equal

$\therefore$ Not associative

c) Is there a zero vector?

$(a,b)+(x,y) = (a,b)$

$(2a+2x, 2b+2y) = (a,b)$

$2a+2x = a$ $\qquad$ $2b+2y = b$

$2x = -a$ $\qquad$ $2y = -b$

$x = \frac{-a}{2}$ $\qquad$ $y = \frac{-b}{2}$

No zero vector $\qquad$ counter example

$(a,b)+(1,1) = (2a+2, 2b+2)$

$2a+2 = a$ $\qquad$ $2b+2 = b$

$2 = -a$ $\qquad$ $2 = -b$

$a = -2 \neq 1$ $\qquad$ $b = -2 \neq 1$

d) Does (c+d)v=cv+dv hold?

$(c+d)(a,b) = (2(c+d)a, 2(c+d)b)$

$= (2ca+2da, 2cb+2db)$

$c(a,b)+d(a,b) = (2ca, 2cb)+(2da, 2db)$

$= (4ca+4da, 4cb+4db)$

Not equal

$\therefore$ No

32) Determine whether S = {(1,3,1),(1,3,7),(2, 3,2)} is LI or LD in R3. If it is LD, find a dependency relationship and find a LI set of S that has the same span as S.

Suppose $c_1(1,3,1) + c_2(1,3,7) + c_3(2,3,2) = (0,0,0)$

$(c_1, 3c_1, c_1) + (c_2, 3c_2, 7c_2) + (2c_3, 3c_3, 2c_3) = (0,0,0)$

$(c_1 + c_2 + 2c_3,\ 3c_1 + 3c_2 + 3c_3,\ c_1 + 7c_2 + 2c_3) = (0,0,0)$

$c_1 + c_2 + 2c_3 = 0$

$3c_1 + 3c_2 + 3c_3 = 0$ $\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 3 & 3 & 3 & 0 \\ 1 & 7 & 2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$c_1 + 7c_2 + 2c_3 = 0$

c_3 is free. Let $t = 2$, then

$\therefore$ It is LD

$c_1 = \frac{3}{2}$ $c_1 = 3$

$c_2 = -\frac{1}{2}$ $c_2 = -1$

$c_3 = t$ $c_3 = 2$

$3(1,3,1) - (1,3,7) + 2(2,3,2) = (0,0,0)$

Any vector can be dropped, let's drop $(1,3,1)$

$\{(1,3,7), (2,3,2)\}$ is LI

since $(1,3,7) = k(2,3,2)$ is impossible

37) Define + and • on R^2 over R as follows: (x1,y1) + (x2,y2) = (x1 + x2, y1*y2), k(x,y) = (kx,1)

a) Find the zero vector

$(a,b) + (x,y) = (x,y)$

$(a+x, by) = (x,y)$

$a + x = x$ $by = y$

$a = 0$ $b = 1$

$\therefore \vec{0} = (0,1)$

b) Find −(3,4)

$(3,4) + (a,b) = (0,1)$

$(3+a, 4b) = (0,1)$ $3 + a = 0$ $4b = 1$

$\therefore -(3,4) = (-3, \frac{1}{4})$ $a = -3$ $b = \frac{1}{4}$

40) Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ using the adjoint of A

$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

A is upper trianguar, then

$\det(A) = (1)(2)(2) = 4$

$C_{11} = (-1)^{1+1}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 1(2(2) - (0)(0)) = 4$

$C_{21} = (-1)^{2+1}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -1((1)(2) - (1)(0)) = -2$

$C_{12} = (-1)^{1+2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 0$

$C_{22} = (-1)^{2+2}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = (1)(2) - (1)(0) = 2$

$C_{13} = (-1)^{1+3}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 0$

$C_{23} = (-1)^{2+3}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$

$C_{31} = (-1)^{3+1}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = (1)(0) - (1)(2) = -2$

$C_{32} = (-1)^{3+2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$

$C_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = (1)(2) - (1)(0) = 2$

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)} = \frac{1}{4}\begin{bmatrix} 4 & -2 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

43) Let V = P2 and S = {ax + bx^2 $\in$ P2 : a,b $\in$ R} degree ≤ 2

a) Show that S is a subspace of P2.

i) $\vec{0} = 0x + 0x^2$

ii) closed under $+$ $\quad (ax + bx^2) + (cx + dx^2) = (a+c)x + (b+d)x^2 \in S$

iii) closed under $\cdot$ $\quad k \in \mathbb{R} \quad k(ax + bx^2) = kax + kbx^2 \in S$

b) Find a basis for S

$ax + bx^2 = a(x) + b(x^2)$

Thus $\{x, x^2\}$ is a basis. The dim is 2

49) Define an operation + and a scalar multiplication • on R^2 as follows:
(x1,y1) + (x2, y2) = (x1+x2,y1*y2), c(x,y) = (cx, c^2*y)

i) Show 0 zero vector = (0,1)

$(a,b) + (x,y) = (x,y)$

$(a+x, by) = (x,y)$

$a + x = x \qquad by = y$

$a = 0 \qquad b = 1 \qquad \therefore \vec{0} = (0,1)$

ii) Does -u exist for all u? Justify your answer.
(recall that -u is the vector such that u + (-u) = 0)

$(x,y) + (a,b) = (0,1)$

$(x+a, yb) = (0,1)$

$x + a = 0 \qquad yb = 1$

$a = -x \qquad b = \frac{1}{y} \qquad -u = (-x, \frac{1}{y})$, if $y \neq 0$

else if $y = 0$ then $-u$ does not exist

iii) Does (rs)v = r(sv) hold? Justify your answer.

$(rs)(x,y) = (rsx, (rs)^2 y) \qquad r(s(x,y)) = r(sx, s^2 y)$

$= (rsx, r^2 s^2 y) \qquad = (rsx, r^2 s^2 y)$

$\therefore$ It holds

51) Let V be a vector space.

a) Show that the set {0,V2,V3} is LD for any vectors V2, V3 in V.

$3(0) + (0)V_2 + (0)V_3 = \vec{0}$

b) Show that if {W1,W2} is LI, then {W1 - W2,W1 + W2} is also LI

suppose $c_1(w_1 - w_2) + c_2(w_1 + w_2) = (0,0)$

$(c_1 w_1 - c_1 w_2) + (c_2 w_1 + c_2 w_2) = (0,0)$

$(c_1 w_1 + c_2 w_1) + (-c_1 w_2 + c_2 w_2) = (0,0)$

$(c_1 + c_2) w_1 + (-c_1 + c_2) w_2 = (0,0)$

since $\{w_1, w_2\}$ is LI, then $c_1 + c_2 = -c_1 + c_2 = 0$ $\therefore$ They are LI

56) Consider the vector space R^3

a) Describe geometrically the subspace of R^3 spanned by the vectors {(1,0,-3),(0,2,-4)}

Suppose $c_1(1,0,-3) + c_2(0,2,-4) = (x,y,z)$

$(c_1, 0, -3c_1) + (0, 2c_2, -4c_2) = (x,y,z)$

$(c_1+0,\ 0+2c_2,\ -3c_1-4c_2) = (x,y,z)$

$c_1 + 0 = x$

$0 + 2c_2 = y$

$-3c_1 - 4c_2 = z$

$$\Rightarrow \begin{bmatrix} 1 & 0 & x \\ 0 & 2 & y \\ -3 & -4 & z \end{bmatrix}$$

R1(3)	3	0	3x
R3	-3	-4	z
New R3	0	-4	3x+z

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 2 & y \\ 0 & -4 & 3x+z \end{bmatrix}$$

R2(2)	0	4	2y
R3	0	-4	3x+z
New R3	0	0	3x+2y+z

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 2 & y \\ 0 & 0 & 3x+2y+z \end{bmatrix} \quad R2\left(\tfrac{1}{2}\right)$$

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & \frac{y}{2} \\ 0 & 0 & 3x+2y+z \end{bmatrix}$$

$3x+2y+z = 0$

b) Add a vector to {(1,0,-3),(0,2,-4)} and extend the set to a basis for R^3. You want to add a vector of the form (2,1,z) to the set. Use part a) to find the number z you must avoid so that {(1,0,-3),(0,2,-4),(2,1,z)} is a basis for R^3.

$(2,1,z) \Rightarrow 3x + 2y + z = 0$

$3(2) + 2(1) + z = 0$

$8 + z = 0$

$z = -8$

$\therefore$ must be avoid $z = -8$

58) Let V be a vector space, v1,v2,...,vn be vectors in V

a) Explain why if {v1,v2} is LD, then {v1,v2,v3} is also LD. You may provide a formal proof, or give a brief explanation as to why the result holds.

Since not all vectors nt necessaryly used in a relationship

so $\{v_1, v_2, v_3\}$ is also LD because $\{v_1, v_2\}$ is LD

b) Give an example of a nonzero vectors v1,v2,v3 such that {v1,v2,v3} is LD but {v1,v2} is LI

$\{(1,0), (0,1), (1,1)\}$ is LD in $\mathbb{R}^2$

$\{(1,0), (0,1)\}$ is LI in $\mathbb{R}^2$

59) Define an operation +,* on R^2 over the real numbers as follows:
(x1,y1) + (x2,y2) = (x1+x2,2y1+2y2),a(x,y) = (a^2*x,a^2*x), a R

a) Is + associative? Justify your answer.

$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) = (x_1, y_1) + (x_2+x_3, 2y_2+2y_3)$

$= (x_1+x_2+x_3, 2y_1+4y_2+4y_3)$

$((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1+x_2, 2y_1+2y_2) + (x_3, y_3)$

$= (x_1+x_2+x_3, 4y_1+4y_2+2y_3)$

Not equal

$\therefore$ No

b) Is there a zero vector? Justify your answer.

$(a, b) + (x, y) = (x, y)$

$(a+x, 2b+2y) = (x, y)$

$a + x = x \qquad 2b + 2y = y$

$a = 0 \qquad 2b = -y$

$b = -\frac{y}{2}$

$\therefore$ No zero vector

c) Does r(sv) = (rs)v hold for r,s ∈ R, v ∈ R^2?

$r(s(x,y)) = r(s^2x, s^2y) \qquad (rs)(x,y) = ((rs)^2x, (rs)^2y)$

$= (r^2s^2x, r^2s^2y) \qquad = (r^2s^2x, r^2s^2y)$

$\therefore$ It holds

63) a) Use DEFINITION to show {(1,3,1),(0,1,0)} is a LI set in R^3 (you cannot say that are LI since one is not a multiple of the other

Suppose $c_1(1,3,1) + c_2(0,1,0) = (0,0,0)$

$(c_1, 3c_1, c_1) + (0, c_2, 0) = (0,0,0)$

$$\begin{aligned} c_1 + 0 &= 0 \\ 3c_1 + c_2 &= 0 \\ c_1 + 0 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \overset{REF}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since rank of A is 2, It has only trivial solution. Thus it is LI

b) Find any vector that is not in the span of {(1,3,1),(0,1,0)}

$c_1(1,3,1) + c_2(0,1,0)$ $\quad c_1 = 1 \quad 3c_1 + c_2 = 0 \quad c_1 = 1$

but $(1,0,0)$ contradict with it

$(1,0,0)$ is not in the span of $\{(1,3,1), (0,1,0)$

c) Extend the basis found in a) to a basis in R^3

Since in b) $(1,0,0)$ is not in the span

Thus $(1,3,1), (0,1,0), (1,0,0)$ is a basis for $\mathbb{R}^3$

81) Define operations +, on R^2 as follows: (x1,y1) + (x2,y2) = (x1^2*x2^2,y1+y2)
r(x,y) = (x + r,y + r)

a) Is + associative?

$(x_1, y_1) + ((x_2,y_2) + (x_3,y_3)) = (x_1,y_1) + (x_2^2 x_3^2, y_2 + y_3)$

$= (x_1^2 x_2^4 x_3^4, y_1 + y_2 + y_3)$

$((x_1,y_1) + (x_2,y_2)) + (x_3,y_3) = (x_1^2 x_2^2, y_1 + y_2) + (x_3, y_3)$

$= (x_1^4 x_2^4 x_3^2, y_1 + y_2 + y_3)$

No equal

$\therefore$ No

b) Is r(sv) = (rs)v for r,s R, v R^2?

$r(s(x,y)) = r(x+s, y+s)$ $\qquad (rs)(x,y) = (x + rs, y + rs)$

$= (x+s+r, y+s+r)$

$\therefore$ No

c) Is there a zero vector?

$(a,b) + (x,y) = (x,y)$

$(a^2x^2, b+y) = (x,y)$

$a^2x^2 = x \qquad b+y = y$

$a^2 = \frac{1}{x} \qquad b = 0$

$a = \frac{1}{\sqrt{x}}$

$\therefore$ No

Counter example:

$(2,2) + (x,y) \qquad (0,0) + (x,y)$

$= (4x^2, y+2) \qquad = (0^2x^2, y+0)$

$= (0,y)$

82) a) True/False. Let V be a vector space. If {(v1,v2)} is a basis for V and if w ≠ v2, then {w,v2} is also a basis for V.

False. If $w = 2v_2$ then $\{w, v_2\} = \{2v_2, v_2\}$, $\{2v_2, v_2\}$ is LD

b) True/False. Let V be a vector space. If {v1,v2} spans V then it is impossible that {v1} spans V.

False. If $\{v_1, v_2\}$ is LD, then $\{v_1\}$ may span V

89) Consider a set A = {(1,2,3),(1,0,1)} in R^3

a) Explain why the set A is LI without doing any row operations.

A is LI since $(1,2,3) \neq k(1,0,1)$, $k \in \mathbb{R}$

b) How many vectors must be added to extend A to a basis for R^3? Justify your answer.

One vector, since $\mathbb{R}^3$ is 3 dimension

c) You would like to extend A to a basis for R^3. Find three vectors not in A that must be avoided in order to extend A to a basis for R^3. Justify your answer.

Suppose $c_1(1,2,3) + c_2(1,0,1) = (0,0,0)$

$(c_1, 2c_1, 3c_1) + (c_2, 0, c_2) = (0,0,0)$

$(c_1+c_2, 2c_1+0, 3c_1+c_2) = (0,0,0)$

$c_1 + c_2 = 0$

$2c_1 + 0 = 0$

$3c_1 + c_2 = 0$

$\Rightarrow \begin{bmatrix} 1 & 1 & x \\ 2 & 0 & y \\ 3 & 1 & z \end{bmatrix}$

R1(-2)	-2	-2	-2x
R2	2	0	y
New R2	0	-2	-2x+y

$\begin{bmatrix} 1 & 1 & x \\ 0 & -2 & -2x+y \\ 3 & 1 & z \end{bmatrix}$

R1(-3)	-3	-3	-3x
R3	3	1	z
New R3	0	-2	-3x+z

$\begin{bmatrix} 1 & 1 & x \\ 0 & -2 & -2x+y \\ 0 & -2 & -3x+z \end{bmatrix}$

$R_2(-1)$ 0 2 $2x-y$

R_3 0 -2 $-3x+z$

New R_3 0 0 $-x-y+z$

$$\begin{bmatrix} 1 & 1 & x \\ 0 & -2 & -2x+y \\ 0 & 0 & -x-y+z \end{bmatrix}$$

$R_2(-\frac{1}{2})$ 0 1 $x-\frac{1}{2}y$

$$\begin{bmatrix} 1 & 1 & x \\ 0 & 1 & x-\frac{1}{2}y \\ 0 & 0 & -x-y+z \end{bmatrix}$$

$-x-y+z=0$

$(0,0,1)$, $(0,0,2)$, $(0,0,3)$

96) Let S be the set of 2 x 2 symmetric matrices

a) Show S is a subspace of M2(R)

i) zero vector ?

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ since $a_{12} = a_{21}$

ii) Is closed under + ?

$\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix} \in S$ since $a_{12} = a_{21}$

iii) Is closed under $\cdot$?

$k\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $k \in \mathbb{R}$

$= \begin{bmatrix} ka & kb \\ kb & kc \end{bmatrix} \in S$ since $a_{12} = a_{21}$

$\therefore$ S is a subspace

b) Find a basis for S and prove that the set you found is indeed a basis.

Let $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for the set of 2 x 2 symmetric matrices

i) LI ? suppose $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix} + \begin{bmatrix} 0 & c_3 \\ c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_3 \\ c_3 & c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

$\therefore$ Yes

ii) Span ? $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ where $a = c_1$

$b = c_2$

$c = c_3$

Thus $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ span 2×2 symmetric matrices

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis for symmetric matrices

c) Extend the basis found in b) to a basis for M2(R). You must carefully justify (not necessarily by computations) all the steps.

M_2 is a 4 dim vector space, since the set of 2×2 symmetric matrices is 3 dim

add one more vector that is not symmetric is needed.

$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is LI

$\therefore$ it is a basis