CSCI 190 -- Homework 2

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Exercises from Book:

- Chapter 5
  - 5.1 – 4, 6

4. **a)** Plugging in $n = 1$ we have that $P(1)$ is the statement $1^3 = [1 \cdot (1+1)/2]^2$.
b) Both sides of $P(1)$ shown in part **(a)** equal 1.
c) The inductive hypothesis is the statement that

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 .$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can prove

$$[1^3 + 2^3 + \cdots + k^3] + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 .$$

e) Replacing the quantity in brackets on the left-hand side of part **(d)** by what it equals by virtue of the inductive hypothesis, we have

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\left(\frac{k^2+4k+4}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2 ,$$

as desired.
f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

6. The basis step is clear, since $1 \cdot 1! = 2! - 1$. Assuming the inductive hypothesis, we then have

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1 , \end{aligned}$$

as desired.

  - 5.2 – 2, 4

2. Let $P(n)$ be the statement that the n^{th} domino falls. We want to prove that $P(n)$ is true for all positive integers n. For the basis step we note that the given conditions tell us that $P(1)$, $P(2)$, and $P(3)$ are true. For the inductive step, fix $k \geq 3$ and assume that $P(j)$ is true for all $j \leq k$. We want to show that $P(k+1)$ is true. Since $k \geq 3$, $k-2$ is a positive integer less than or equal to k, so by the inductive hypothesis we know that $P(k-2)$ is true. That is, we know that the $(k-2)^{\text{nd}}$ domino falls. We were told that "when a domino falls, the domino three farther down in the arrangement also falls," so we know that the domino in position $(k-2)+3 = k+1$ falls. This is $P(k+1)$.

Note that we didn't use strong induction exactly as stated in the text. Instead, we considered all the cases $n = 1$, $n = 2$, and $n = 3$ as part of the basis step. We could have more formally included $n = 2$ and $n = 3$ in the inductive step as a special case. Writing our proof this way, the basis step is just to note that the first domino falls, so $P(1)$ is true. For the inductive step, if $k = 1$ or $k = 2$, then we are already told that the second and third domino fall, so $P(k+1)$ is true in those cases. If $k > 2$, then the inductive hypothesis tells us that the $(k-2)^{\text{nd}}$ domino falls, so the domino in position $(k-1)+2 = k+1$ falls.

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Exercises from Book:

4. a) $P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps. $P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp. $P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps. $P(21)$ is true, because we can form 20 cents of postage with three 7-cent stamps.

b) The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \leq j \leq k$, where we assume that $k \geq 21$.

c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 4-cent and 7-cent stamps.

d) We want to form $k+1$ cents of postage. Since $k \geq 21$, we know that $P(k-3)$ is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

- 5.3 – 2, 4, 8

2. a) $f(1) = -2f(0) = -2 \cdot 3 = -6$, $f(2) = -2f(1) = -2 \cdot (-6) = 12$, $f(3) = -2f(2) = -2 \cdot 12 = -24$, $f(4) = -2f(3) = -2 \cdot (-24) = 48$, $f(5) = -2f(4) = -2 \cdot 48 = -96$

b) $f(1) = 3f(0) + 7 = 3 \cdot 3 + 7 = 16$, $f(2) = 3f(1) + 7 = 3 \cdot 16 + 7 = 55$, $f(3) = 3f(2) + 7 = 3 \cdot 55 + 7 = 172$, $f(4) = 3f(3) + 7 = 3 \cdot 172 + 7 = 523$, $f(5) = 3f(4) + 7 = 3 \cdot 523 + 7 = 1576$

c) $f(1) = f(0)^2 - 2f(0) - 2 = 3^2 - 2 \cdot 3 - 2 = 1$, $f(2) = f(1)^2 - 2f(1) - 2 = 1^2 - 2 \cdot 1 - 2 = -3$, $f(3) = f(2)^2 - 2f(2) - 2 = (-3)^2 - 2 \cdot (-3) - 2 = 13$, $f(4) = f(3)^2 - 2f(3) - 2 = 13^2 - 2 \cdot 13 - 2 = 141$, $f(5) = f(4)^2 - 2f(4) - 2 = 141^2 - 2 \cdot 141 - 2 = 19{,}597$

d) First note that $f(1) = 3^{f(0)/3} = 3^{3/3} = 3 = f(0)$. In the same manner, $f(n) = 3$ for all n.

4. a) $f(2) = f(1) - f(0) = 1 - 1 = 0$, $f(3) = f(2) - f(1) = 0 - 1 = -1$, $f(4) = f(3) - f(2) = -1 - 0 = -1$, $f(5) = f(4) - f(3) = -1 - 1 = 0$

b) Clearly $f(n) = 1$ for all n, since $1 \cdot 1 = 1$.

c) $f(2) = f(1)^2 + f(0)^3 = 1^2 + 1^3 = 2$, $f(3) = f(2)^2 + f(1)^3 = 2^2 + 1^3 = 5$, $f(4) = f(3)^2 + f(2)^3 = 5^2 + 2^3 = 33$, $f(5) = f(4)^2 + f(3)^3 = 33^2 + 5^3 = 1214$

d) Clearly $f(n) = 1$ for all n, since $1/1 = 1$.

8. Many answers are possible.

a) Each term is 4 more than the term before it. We can therefore define the sequence by $a_1 = 2$ and $a_{n+1} = a_n + 4$ for all $n \geq 1$.

b) We note that the terms alternate: 0, 2, 0, 2, and so on. Thus we could define the sequence by $a_1 = 0$, $a_2 = 2$, and $a_n = a_{n-2}$ for all $n \geq 3$.

c) The sequence starts out 2, 6, 12, 20, 30, and so on. The differences between successive terms are 4, 6, 8, 10, and so on. Thus the n^{th} term is $2n$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n$. Together with the initial condition $a_1 = 2$, this defines the sequence recursively.

d) The sequence starts out 1, 4, 9, 16, 25, and so on. The differences between successive terms are 3, 5, 7, 9, and so on—the odd numbers. Thus the n^{th} term is $2n - 1$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n - 1$. Together with the initial condition $a_1 = 1$, this defines the sequence recursively.

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Exercises from Book:

- 5.4 – 2, **8**

2. First, we use the recursive step to write $6! = 6 \cdot 5!$. We then use the recursive step repeatedly to write $5! = 5 \cdot 4!$, $4! = 4 \cdot 3!$, $3! = 3 \cdot 2!$, $2! = 2 \cdot 1!$, and $1! = 1 \cdot 0!$. Inserting the value of $0! = 1$, and working back through the steps, we see that $1! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 = 6$, $4! = 4 \cdot 3! = 4 \cdot 6 = 24$, $5! = 5 \cdot 4! = 5 \cdot 24 = 120$, and $6! = 6 \cdot 5! = 6 \cdot 120 = 720$.

8. The sum of the first n positive integers is the sum of the first $n - 1$ positive integers plus n. This trivial observation leads to the recursive algorithm shown here.

procedure *sum of first*(n : positive integer)
if $n = 1$ **then return** 1
else return *sum of first*$(n - 1) + n$

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Exercises from Book:

- Chapter 6
  - 6.1 – 2, 8, 30, 40, 44

2. By the product rule there are $27 \cdot 37 = 999$ offices.

8. There are 26 choices for the first initial, then 25 choices for the second, if no letter is to be repeated, then 24 choices for the third. (We interpret "repeated" broadly, so that a string like RWR, for example, is prohibited, as well as a string like RRW.) Therefore by the product rule the answer is $26 \cdot 25 \cdot 24 = 15{,}600$.

30. $26^3 10^3 + 26^4 10^2 = 63{,}273{,}600$

40. We know that there are 2^{100} subsets in all. Clearly 101 of them do not have more than one element, namely the empty set and the 100 sets consisting of 1 element. Therefore the answer is $2^{100} - 101 \approx 1.3 \times 10^{30}$.

44. If we ignore the fact that the table is round and just count ordered arrangements of length 4 from the 10 people, then we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ arrangements. However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so this overcounts by a factor of 4. (For example, the sequence Mary–Debra–Cristina–Julie gives the same circular seating as the sequence Julie–Mary–Debra–Cristina.) Therefore the answer is $5040/4 = 1260$.

  - 6.2 – 2, 4, 8, 18

2. This follows from the pigeonhole principle, with $k = 26$.

4. We assume that the woman does not replace the balls after drawing them.
a) There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains three pigeons. By the generalized pigeonhole principle, the answer is 5. If five balls are selected, at least $\lceil 5/2 \rceil = 3$ must have the same color. On the other hand four balls is not enough, because two might be red and two might be blue. Note that the number of balls was irrelevant (assuming that it was at least 5).
b) She needs to select 13 balls in order to insure at least three blue ones. If she does so, then at most 10 of them are red, so at least three are blue. On the other hand, if she selects 12 or fewer balls, then 10 of them could be red, and she might not get her three blue balls. This time the number of balls did matter.

8. This is just a restatement of the pigeonhole principle, with $k = |T|$.

18. a) If not, then there would be 4 or fewer male students and 4 or fewer female students, so there would be $4 + 4 = 8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.
b) If not, then there would be 2 or fewer male students and 6 or fewer female students, so there would be $2 + 6 = 8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

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Exercises from Book:

- 6.3 – 4, 6, 10, 12

4. There are 10 combinations and 60 permutations. We list them in the following way. Each combination is listed, without punctuation, in increasing order, followed by the five other permutations involving the same numbers, in parentheses, without punctuation.

123 (132 213 231 312 321) 124 (142 214 241 412 421) 125 (152 215 251 512 521)
134 (143 314 341 413 431) 135 (153 315 351 513 531) 145 (154 415 451 514 541)
234 (243 324 342 423 432) 235 (253 325 352 523 532)
245 (254 425 452 524 542) 345 (354 435 453 534 543)

6. **a)** $C(5,1) = 5$ **b)** $C(5,3) = C(5,2) = 5 \cdot 4/2 = 10$ **c)** $C(8,4) = 8 \cdot 7 \cdot 6 \cdot 5/(4 \cdot 3 \cdot 2) = 70$
d) $C(8,8) = 1$ **e)** $C(8,0) = 1$ **f)** $C(12,6) = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7/(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 924$

10. $P(6,6) = 6! = 720$

12. **a)** To specify a bit string of length 12 that contains exactly three 1's, we simply need to choose the three positions that contain the 1's. There are $C(12,3) = 220$ ways to do that.
b) To contain at most three 1's means to contain three 1's, two 1's, one 1, or no 1's. Reasoning as in part (a), we see that there are $C(12,3)+C(12,2)+C(12,1)+C(12,0) = 220+66+12+1 = 299$ such strings.
c) To contain at least three 1's means to contain three 1's, four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, 10 1's, 11 1's, or 12 1's. We could reason as in part (b), but we would have too many numbers to add. A simpler approach would be to figure out the number of ways not to have at least three 1's (i.e., to have two 1's, one 1, or no 1's) and then subtract that from 2^{12}, the total number of bit strings of length 12. This way we get $4096 - (66 + 12 + 1) = 4017$.
d) To have an equal number of 0's and 1's in this case means to have six 1's. Therefore the answer is $C(12,6) = 924$.

- 6.4 – 2, 6, 8, 12

2. **a)** When $(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, a term in the third sum, a term in the fourth sum, and a term in the fifth sum are added. Terms of the form x^5, x^4y, x^3y^2, x^2y^3, xy^4 and y^5 arise. To obtain a term of the form x^5, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^5 term in the product has a coefficient of 1. (We can think of this coefficient as $\binom{5}{5}$.) To obtain a term of the form x^4y, an x must be chosen in four of the five sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 4-combinations of five objects, namely $\binom{5}{4} = 5$. Similarly, the number of terms of the form x^3y^2 is the number of ways to pick three of the five sums to obtain x's (and consequently take a y from each of the other two factors). This can be done in $\binom{5}{3} = 10$ ways. By the same reasoning there are $\binom{5}{2} = 10$ ways

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Exercises from Book:

- 6.4 – 6, 8, 12

6. $\binom{11}{7}1^4 = 330$

8. $\binom{17}{9}3^8 2^9 = 24310 \cdot 6561 \cdot 512 = 81{,}662{,}929{,}920$

12. We just add adjacent numbers in this row to obtain the next row (starting and ending with 1, of course):

$$1 \quad 11 \quad 55 \quad 165 \quad 330 \quad 462 \quad 462 \quad 330 \quad 165 \quad 55 \quad 11 \quad 1$$

- Chapter 7
  - 7.1 – 2, 8, 30

2. The probability is $1/6 \approx 0.17$, since there are six equally likely outcomes.

8. We saw in Example 11 of Section 6.3 that there are $C(52,5)$ possible poker hands, and we assume by symmetry that they are all equally likely. In order to solve this problem, we need to compute the number of poker hands that contain the ace of hearts. There is no choice about choosing the ace of hearts. To form the rest of the hand, we need to choose 4 cards from the 51 remaining cards, so there are $C(51,4)$ hands containing the ace of hearts. Therefore the answer to the question is the ratio

$$\frac{C(51,4)}{C(52,5)} = \frac{5}{52} \approx 9.6\% \,.$$

The problem can also be done by subtracting from 1 the answer to Exercise 9, since a hand contains the ace of hearts if and only if it is not the case that it does not contain the ace of hearts.

30. In order to specify a winning ticket, we must choose five of the six numbers to match ($C(6,5) = 6$ ways to do so) and one number from among the remaining 34 numbers not to match ($C(34,1) = 34$ ways to do so). Therefore there are $6 \cdot 34 = 204$ winning tickets. Since there are $C(40,6) = 3{,}838{,}380$ tickets in all, the answer is $204/3838380 = 17/319865 \approx 5.3 \times 10^{-5}$, or about 1 chance in $19{,}000$.

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Exercises from Book:

- 7.2 – 2, 6, 12, 24, 26

2. We are told that $p(3) = 2p(x)$ for each $x \neq 3$, but it is implied that $p(1) = p(2) = p(4) = p(5) = p(6)$. We also know that the sum of these six numbers must be 1. It follows easily by algebra that $p(3) = 2/7$ and $p(x) = 1/7$ for $x = 1, 2, 4, 5, 6$.

6. We can exploit symmetry in answering these.

a) Since 1 has either to precede 3 or to follow it, and there is no reason that one of these should be any more likely than the other, we immediately see that the answer is $1/2$. We could also simply list all 6 permutations and count that 3 of them have 1 preceding 3, namely 123, 132, and 213.

b) By the same reasoning as in part **(a)**, the answer is again $1/2$.

c) The stated conditions force 3 to come first, so only 312 and 321 are allowed. Therefore the answer is $2/6 = 1/3$.

12. Clearly $p(E \cup F) \geq p(E) = 0.8$. Also, $p(E \cup F) \leq 1$. If we apply Theorem 2 from Section 7.1, we can rewrite this as $p(E) + p(F) - p(E \cap F) \leq 1$, or $0.8 + 0.6 - p(E \cap F) \leq 1$. Solving for $p(E \cap F)$ gives $p(E \cap F) \geq 0.4$.

24. There are 16 equally likely outcomes of flipping a fair coin five times in which the first flip comes up tails (each of the other flips can be either heads or tails). Of these only one will result in four heads appearing, namely $THHHH$. Therefore the answer is $1/16$.

26. Intuitively the answer should be yes, because the parity of the number of 1's is a fifty-fifty proposition totally determined by any one of the flips (for example, the last flip). What happened on the other flips is really rather irrelevant. Let us be more rigorous, though. There are 8 bit strings of length 3, and 4 of them contain an odd number of 1's (namely 001, 010, 100, and 111). Therefore $p(E) = 4/8 = 1/2$. Since 4 bit strings of length 3 start with a 1 (namely 100, 101, 110, and 111), we see that $p(F) = 4/8 = 1/2$ as well. Furthermore, since there are 2 strings that start with a 1 and contain an odd number of 1's (namely 100 and 111), we see that $p(E \cap F) = 2/8 = 1/4$. Then since $p(E) \cdot p(F) = (1/2) \cdot (1/2) = 1/4 = p(E \cap F)$, we conclude from the definition that E and F are independent.

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Exercises from Book:

- Chapter 8
  - 8.1: 8

8. This is very similar to Exercise 7, except that we need to go one level deeper.

a) Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit string of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

b) There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

  - 8.3: 8, 10, 14, 16

8. **a)** $f(2) = 2 \cdot 5 + 3 = 13$ **b)** $f(4) = 2 \cdot 13 + 3 = 29$, $f(8) = 2 \cdot 29 + 3 = 61$
c) $f(16) = 2 \cdot 61 + 3 = 125$, $f(32) = 2 \cdot 125 + 3 = 253$, $f(64) = 2 \cdot 253 + 3 = 509$
d) $f(128) = 2 \cdot 509 + 3 = 1021$, $f(256) = 2 \cdot 1021 + 3 = 2045$, $f(512) = 2 \cdot 2045 + 3 = 4093$, $f(1024) = 2 \cdot 4093 + 3 = 8189$

10. Since f increases one for each factor of 2 in n, it is clear that $f(2^k) = k + 1$.

14. If there is only one team, then no rounds are needed, so the base case is $R(1) = 0$. Since it takes one round to cut the number of teams in half, we have $R(n) = 1 + R(n/2)$.

16. The solution of this recurrence relation for $n = 2^k$ is $R(2^k) = k$, for the same reason as in Exercise 10.