2-1 Probabilistic inference

(a) $P(E=1 \mid A=1) = \dfrac{P(A=1 \mid E=1) \cdot P(E=1)}{P(A=1)}$

numerator : $P(A=1 \mid E=1) = \sum_b (A=1, B=b \mid E=1) = \sum_b P(A=1 \mid B=b, E=1) \cdot P(B=b \mid E=1)$

given the assumption that B.E are independent

$\sum_b P(A=1 \mid B=b, E=1) \cdot P(B=b)$

$= 0.999 \times 0.29 + 0.001 \times 0.95$

$= 0.29066$

$P(E=1) = 0.002$

numerator : $0.29066 \times 0.002 = 0.00058132$

denominator : $P(A=1) = \sum_e \sum_b (A=1, B=b, E=e)$

$= \sum_e \sum_b P(A=1 \mid E=e, B=b) P(E=e \mid B=b)$

$= \sum_e \sum_b P(A=1 \mid E=e, B=b) P(E=e) P(B=b)$

$= \sum_e P(E=e) \cdot (P(B=1) P(A=1 \mid E=e, B=1) + P(B=0) P(A=1 \mid E=e, B=0))$

$= 0.002 \cdot (0.001 \times 0.95 + 0.999 \times 0.29) + 0.998 (0.001 \times 0.94 + 0.999 \times 0.001)$

$= 0.002516$

$P(E=1 \mid A=1) = \dfrac{0.00058132}{0.002516} \cong 0.231$ #

(b)

$$P(E=1 \mid A=1, B=0) = \frac{P(A=1 \mid E=1, B=0) \cdot P(E=1 \mid B=0)}{P(A=1 \mid B=0)}$$

denominator:

$$P(A=1 \mid B=0) = \sum_{e} P(A=1, E=e \mid B=0)$$

$$= \sum_{e} P(A=1 \mid E=e, B=0) \cdot P(E=e \mid B=0)$$

(given B E are independent

$$= \sum_{e} P(A=1 \mid E=e, B=0) \cdot P(E=e)$$

$$= 0.29 \times 0.002 + 0.001 \times 0.998$$

numerator

$$= 0.001578$$

$$P(A=1 \mid E=1, B=0) = 0.29$$

given that B, E are independent

$$P(E=1 \mid B=0) = P(E=1) = 0.02$$

$$0.29 \times 0.02 = 0.00058$$

$$P(E=1 \mid A=1, B=0) = \frac{0.00058}{0.001578} \cong 0.3676 \#$$

(c)

$$P(A=1 \mid M=1) = \frac{P(M=1 \mid A=1)\cdot P(A=1)}{P(M=1)}$$

denominator:

$$\begin{aligned} P(M=1) &= \sum_a P(M=1, A=a) \\ &= \sum_a P(M=1 \mid A=a)\cdot P(A=a) \\ &= 0.7 \times 0.0025 + 0.01 \times 0.9975 \quad \text{(given } P(A=1)=0.0025\text{)} \\ &= 0.011725 \end{aligned}$$

numerator:

$$\begin{aligned} P(M=1 \mid A=1)\cdot P(A=1) &= 0.7 \times 0.0025 \\ &= 0.00175 \end{aligned}$$

$$P(A=1 \mid M=1) \cong 0.149 \ \#$$

(d)

$$p(A=1 \mid M=1, J=0) = \frac{P(A=1, M=1, J=0)}{P(M=1, J=0)}$$

denominator :

$$p(M=1, J=0) = \sum_a (M=1, J=0, A=a)$$

$$= \sum_a P(A=a) \cdot p(M=1 \mid A=a) \cdot p(J=0 \mid M=1, A=a)$$

(given $P(J=0 \mid M=1, A=a)$ is conditional independence

$$= \sum_a P(A=a) \cdot p(M=1 \mid A=a) \cdot P(J=0 \mid A=a)$$

$$= 0.0025 \cdot 0.7 \cdot (1-0.9) + 0.9975 \cdot 0.01 \cdot (1-0.05)$$

$$\approx 0.00965$$

numerator :

$$= P(A=1) \cdot P(M=1 \mid A=1) \cdot p(J=0 \mid M=1, A=1)$$

(given $P(J=0 \mid M=1, A=a)$ is conditional independence

$$= p(A=1) \cdot p(M=1 \mid A=1) \cdot p(J=0 \mid A=1)$$

$$= 0.0025 \cdot 0.7 \cdot (1-0.9)$$

$$= 0.000175$$

$$p(A=1 \mid M=1, J=0) = \frac{0.000175}{0.00965} = 0.018 \, \#$$

(e)

$$p(A=1 \mid M=0) = \frac{P(M=0 \mid A=1) \cdot P(A=1)}{P(M=0)}$$

denominator :

As calculated in (c) $p(M=0) = 1 - p(M=1) = 0.988$

nominator :

$$p(M=0 \mid A=1) \cdot p(A=1)$$

$$= (1-0.7) \cdot 0.0025$$

$$= 0.00075$$

$$p(A=1 \mid M=0) = \frac{0.00075}{0.988} \cong 0.000759$$

(f)

$$P(A=1 \mid M=0, B=1) = \frac{P(M=0 \mid A=1, B=1) \cdot P(A=1 \mid B=1)}{P(M=0 \mid B=1)}$$

numerator:

$P(M=0 \mid A=1, B=1) = P(M=0 \mid A=1) = 0.3$ By: d-seperation

$$P(A=1 \mid B=1) = \sum_e P(A=1, E=e \mid B=1)$$

$$= \sum_e \frac{P(B=1) P(E=e \mid B=1) \cdot P(A=1 \mid E=e, B=1)}{P(B=1)}$$

$$= \sum_e P(E=e) \cdot P(A=1 \mid E=e, B=1)$$

$$= 0.002 \times 0.95 + 0.998 \times 0.94$$

$$= 0.94$$

denominator:

$$P(M=0 \mid B=1) = \sum_a P(M=0, A=a \mid B=1)$$

$$= \sum_a \frac{P(B=1) \cdot P(A=a \mid B=1) \cdot P(M=0 \mid A=a, B=1)}{P(B=1)}$$

$$= 0.94 \times 0.3 + 0.06 \times 0.99$$

$$= 0.3414$$

$$P(A=1 \mid M=0, B=1) = \frac{0.94 \times 0.3}{0.3414} \cong 0.826 \ \#$$

Result (b) v.s (a)

$p(E=1 \mid A=1, B=0)$ versus $p(E=1 \mid A=1)$

$\downarrow$ 0.3676 $\qquad$ $\downarrow$ 0.231

=> knowing there was no burglar should increase the likelihood that the alarm was triggered by an earthquake. Without the information in case A, the alarm could have been caused by either earthquake or burglar, but in case b, the only plausible cause left is the earthquake. Thus, the probability in case b should be higher than case A

Result (d) v.s (c)

$p(A=1 \mid M=1, J=0)$ versus $p(A=1 \mid M=1)$

$\downarrow$ 0.0018 $\qquad$ $\downarrow$ 0.149

if John didn't call but Mary did, it introduces some doubts — maybe the alarm didn't go off and only Mary make a mistake. Thus, probability in case (d) should be lower than case (c)

Result (f) v.s (e)

$p(A=1 \mid M=0, B=1)$ Versus $p(A=1 \mid M=0)$

$\downarrow$ $\qquad$ $\downarrow$ 0.000759

If we know that a burglary occurred, the probabilily that the alarm went off should increase, regardless of whether Marry called. Thus, the probability of (f) should higher than (e)

2-2 Probabilistic reasoning

(a)

$$P(D=i \mid S_1=1, S_2=1, \cdots S_K=1) = \frac{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=i) \cdot P(D=i)}{P(S_1=1, S_2=1, \cdots S_K=1)}$$

$$r_K = \frac{\dfrac{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=0) \cdot P(D=0)}{P(S_1=1, S_2=1, \cdots S_K=1)}}{\dfrac{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=1 \ P(D=1)}{P(S_1=1, S_2=1, \cdots S_K=1)}} = \frac{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=0) \cdot P(D=0)}{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=1) \cdot P(D=1)}$$

$$\because \ P(D=0) = P(D=1) = \frac{1}{2}$$

$$\therefore \ r_K = \frac{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=0)}{P(S_1=1, S_2=1, \cdots S_K=1 \mid D=1)}$$

=> From Believe Network we can see that $S_1, S_2 \cdots S_K$ are conditionally independence given D

$$\Rightarrow P(S_1=1, S_2=1, \cdots S_K=1 \mid D=0) = P(S_1=1 \mid D=0) \cdot P(S_2=1 \mid D=0) \cdots P(S_K=1 \mid D=0)$$

$$= 1 \cdot \frac{f(1)}{\cancel{f(2)}} \cdot \frac{\cancel{f(2)}}{f(3)} \cdots \frac{f\cancel{(K-1)}}{f(K)}$$

$$= \frac{1}{f(K)} = \frac{1}{2^K + (-1)^K}$$

$$\Rightarrow P(S_1=1, S_2=1, \cdots S_K=1 \mid D=1) = P(S_1=1 \mid D=1) \cdot P(S_2=1 \mid D=1) \cdots P(S_K=1 \mid D=1)$$

$$= \left(\frac{1}{2}\right)^K$$

$$\Rightarrow r_K = \frac{\dfrac{1}{2^K + (-1)^K}}{\left(\frac{1}{2}\right)^K} = \frac{2^K}{2^K + (-1)^K} \quad \#$$

if $2^k > 2^k + (-1)^k$ => doctor diagnoses the patient with $D=0$

=> $(-1)^k < 0$

$k =$ odd numbers

otherwise $2^k < 2^k + (-1)^k$ => doctor diagnoses the patient with $D=0$

=> $(-1)^k > 0$

$k =$ even numbers

Conclusion:

The doctor diagnoses the patient with $D=0$ form of the disease on odd number of days in a month. The doctor diagnoses the patient with $D=1$ form of the disease on even number of days in a month.

(b)

With k getting larger, in which more symptoms are observed, $r(k)$ will gradually coverge to 1, which mean it will be harder to distinguish $D=0$ and $D=1$, so it will be less certain.

2-3 True or false

1. $p(B|G,C) = p(B|G)$: False #

evidence : G

path: $B \rightarrow D \rightarrow G \leftarrow E \leftarrow C$

2. $p(F,G|D) = p(F|D)\,p(G|D)$: True #

evidence : D

path : $F \leftarrow D \rightarrow G$

path : $F \leftarrow D \leftarrow B \leftarrow A \rightarrow G$

3. $p(A,C) = P(A)P(C)$: True #

4. $p(D|B,F,G) = p(D|B,F,G,A)$: False #

evidence : B, F, G

path : $D \leftarrow B \leftarrow A$ block

path : $D \rightarrow G \leftarrow A$ unblock

5. $p(F,H) = P(F)\cdot P(H)$: True #

path : $F \leftarrow D \rightarrow G \leftarrow E \rightarrow H$ block

path : $F \leftarrow D \leftarrow B \leftarrow A \rightarrow G \leftarrow E \rightarrow H$ block

6. $p(D,E|F,H) = P(D|F)\,p(E|H)$: True #

$$\Rightarrow \frac{P(D,E,F,H)}{P(F,H)} = \frac{P(F)P(H|F)P(E|H,F)P(D|E,H,F)}{P(F)P(H|F)} = P(E|H,F)P(D|E,H,F)$$

7. $p(F,C|G) = p(F|G)\,p(C|G)$: False

path : $F \leftarrow D \rightarrow G \leftarrow E \leftarrow C$ $\rightarrow$ unblock

$F \leftarrow D \leftarrow B \leftarrow A \rightarrow G \leftarrow E \leftarrow C$ $\rightarrow$ unblock

8. $p(D,E,G) = p(D)p(E)p(G|D,E)$: True #

$\Rightarrow p(D) \cdot p(E|D) \cdot p(G|D,E)$

9. $p(H|C) = p(H|A,B,C,D,F)$: True #

path: $H \leftarrow E \rightarrow G \leftarrow D \rightarrow F$

10.

$p(H|A,C) = p(H|A,C,G)$: False #

path: $G \leftarrow E \leftarrow H$

2-4 More on believe network

1. $p(F|H) = p(F|C,H)$: False #

   path: $F \rightarrow H \leftarrow C$

2. $p(E|A,B) = p(E|A,B,F)$: true #

   path: $E \rightarrow G \leftarrow F \leftarrow B$

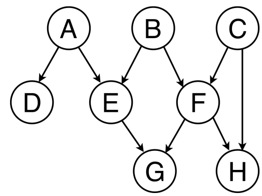
3. $p(E,F|B,G) = p(E|B,G)\,p(F|B,G)$: False #

$$\Rightarrow \frac{P(E,F,B,G)}{P(B,G)} = \frac{P(B)\,P(G|B)\,P(E|B,G)\,P(F|B,G,E)}{P(B)\,P(G|B)} = P(E|B,G)\,P(F|B,G,E)$$

4. $p(F|B,C,G,H) = p(F|B,C,E,G,H)$: False #

   path: $F \leftarrow B \rightarrow E$ blocked

   $F \rightarrow G \leftarrow E$ unblocked



5. $p(A,B|D,E,F) = p(A,B|D,E,F,G,H)$: False #

$\Rightarrow P(A,B|D,E,F) = P(B|D,E,F)\cdot P(A|B,D,E,F)$

$\Rightarrow p(A,B|D,E,F) = p(B|D,E,F,G,H)\cdot p(A|B,D,E,F,G,H)$

6. $p(D,E,F) = p(D)\,p(E|D)\,p(F|E)$: False #

$\Rightarrow p(D,E,F) = p(D)\cdot p(E|D)\cdot p(F|D,E)$

path: $F \rightarrow G \leftarrow E \leftarrow A \rightarrow D$

$F \leftarrow B \rightarrow E \leftarrow A \rightarrow D$

7

$P(A|F) = P(A)$: True #

path: $A \rightarrow E \leftarrow B \rightarrow F$

path: $A \rightarrow E \rightarrow G \leftarrow F$

8. $P(E,F) = P(E)P(F)$ **False** #

$P(E) \cdot P(F|E) = P(E)P(F)$

$\Rightarrow P(F|E) = P(F)$

$P \rightarrow B \rightarrow F$

9.

$P(D|A) = P(D|A,E)$: True #

path: $D \leftarrow A \rightarrow E$

10.

$P(B,C) = P(B) \cdot P(C)$: True #

path: $B \rightarrow F \leftarrow C$

2-5 conditional independence

X, Y : {month, puddle} E : {rain, sprinkler}

X, Y : {month, puddle} E : {rain, sprinkler, fall}

X, Y : {month, fall} E : {sprinkler, rain}

X, Y : {month, fall} E : {sprinkler, rain, puddle}

X, Y : {month, fall} E : {sprinkler, puddle}

X, Y : {month, fall} E : {rain, puddle}

X, Y : {month, fall} E : {puddle}

X, Y : {rain, fall} E : {puddle}

X, Y : {rain, fall} E : {puddle, sprinkler}

X, Y : {rain, fall} E : {puddle, month}

X, Y : {rain, fall} E : {puddle, sprinkler, month}

X, Y : {sprinkler, rain} E : {month}

X, Y : {sprinkler, fall} E : {puddle}

X, Y : {sprinkler, fall} E : {puddle, rain}

X, Y : {sprinkler, fall} E : {puddle, month}

X, Y : {sprinkler, fall} E : {puddle, rain, month}

2-6 Noisy-OR

(a) $\boxed{<}$ #

$1-(\frac{1}{2})^0\cdot(\frac{1}{2})^0$

$P(Z=1 \mid X=0, Y=0) = 1-1 = 0$

$P(Z=1 \mid X=0, Y=1) = 1-(1-P_y)^1 = P_y$

(b) $\boxed{<}$ #

$P(Z=1 \mid X=1, Y=0) = P_x$

$P(Z=1 \mid X=0, Y=1) = P_y$

(c) $\boxed{<}$ #

$P(Z=1 \mid X=1, Y=0) = P_x$

$P(Z=1 \mid X=1, Y=1) = 1-(1-p_x)(1-p_y)$

$= 1-(1-p_y-p_x+p_xp_y)$

$= p_y+p_x-p_xp_y$

$p_y - p_xp_y$

$= p_y(1-p_x)$

$\Rightarrow p_y+p_x-p_xp_y-p_x = p_y(1-p_x) > 0$

(d) $\boxed{<}$ #

$$P(X=1 \mid Z=1) = \frac{P(Z=1 \mid X=1)\cdot P(X=1)}{P(Z=1)}$$

numerator:

$P(Z=1 \mid X=1) = \sum_y P(Z=1, Y=y \mid X=1)$

$$= \sum_y \frac{P(X=1)\cdot P(Y=y \mid X=1)\cdot P(Z=1 \mid Y=y, X=1)}{P(X=1)}$$

$= \sum_y P(Y=y)P(Z=1 \mid Y=y, X=1)$

$= P_y[1-(1-P_x)(1-P_y)] + (1-P_y)P_x$

$= P_y(P_y+P_x-P_xP_y) + P_x - P_yP_x$

$= P_y^2 - P_xP_y^2 + P_x$

numerator : $P_x P_y^2 - P_x^2 P_y^2 + P_x^2$

denominator :

$$p(Z=1) = \sum_x \sum_y P(Z=1, X=x, Y=y)$$

$$= \sum_x \sum_y P(Y=y) \cdot P(X=x \mid Y=y) \cdot P(Z=1 \mid X=x, Y=y)$$

$$= \sum_x \sum_y P(Y=y) \cdot P(X=x) \cdot P(Z=1 \mid X=x, Y=y)$$

$$= (1-P_y)\cdot(1-P_x)\cdot 0 + P_y \cdot (1-P_x) \cdot P_y + (1-P_y)\cdot P_x \cdot P_x + P_x \cdot P_y \cdot (P_x + P_y - P_x P_y)$$

$$= P_y^2 - P_x P_y^2 + P_x^2 - P_x^2 P_y + P_x^2 P_y + P_x P_y^2 - P_x^2 P_y^2$$

$$= P_y^2 + P_x^2 - P_x^2 P_y^2$$

so

$$p(X=1 \mid Z=1) = \frac{P_x P_y^2 - P_x^2 P_y^2 + P_x^2}{P_y^2 + P_x^2 - P_x^2 P_y^2}$$

$$p(X=1 \mid Z=1) - p(X=1) = \frac{\cancel{P_x P_y^2} - P_x^2 P_y^2 + P_x^2}{P_y^2 + P_x^2 - P_x^2 P_y^2} - \frac{\cancel{P_x P_y^2} + P_x^3 - P_x^3 P_y^2}{P_y^2 + P_x^2 - P_x^2 P_y^2}$$

$$= \frac{P_x^2(1-P_y^2) - P_x^3(1-P_y^2)}{P_y^2 + P_x^2 - P_x^2 P_y} = \frac{(P_x^2 - P_x^3)(1-P_y^2)}{P_y^2 + P_x^2 - P_x^2 P_y} > 0$$

(e) $\boxed{=}$ #

Given the Belief Network, X, Y should be independent

(f) $\boxed{>}$

$$p(X=1 \mid Y=1, Z=1) = \frac{P(X=1, Y=1, Z=1)}{P(Y=1, Z=1)} = \frac{P(Y=1)\cdot P(X=1 \mid Y=1) \cdot P(Z=1 \mid X=1, Y=1)}{P(Y=1, Z=1)}$$

numerator :

$$p_y \cdot p_x \cdot (p_x + p_y - p_x p_y) = P_x^2 P_y + P_x p_y^2 - P_x^2 P_y^2$$

denominator :

$$p(Y=1, Z=1) = \sum_{x=x} (Y=1, X=x, Z=1)$$

$$= \sum_{x=x} P(Y=1)\cdot P(X=x \mid Y=1)\cdot P(Z=1 \mid X=x, Y=1)$$

$$= P_y \cdot P_x \cdot (P_x + P_y - P_x P_y) + P_y \cdot (1-P_x)\cdot P_y$$

$$= P_x^2 P_y + \cancel{P_y^2 P_x} - P_x^2 P_y^2 + P_y^2 - \cancel{P_y^2 P_x}$$

$$= P_x^2 P_y - P_x^2 P_y^2 + P_y^2$$

$$p(X=1 \mid Y=1, Z=1) = \frac{P_x^2 P_y + P_x P_y^2 - P_x^2 P_y^2}{P_x^2 P_y - P_x^2 P_y^2 + P_y^2}$$

$$p(X=1 \mid Z=1) \quad \frac{P_x P_y^2 - P_x^2 P_y^2 + P_x^2}{P_y^2 + P_x^2 - P_x^2 P_y^2}$$

$$p(X=1 \mid Z=1) - p(X=1 \mid Y=1, Z=1) = \frac{\cancel{P_x P_y^2} - \cancel{P_x^2 P_y^2} + P_x^2 - P_x^2 P_y - \cancel{P_x P_y^2} + \cancel{P_x^2 P_y^2}}{P_x^2 + P_y^2 - P_x^2 P_y^2}$$

$$= \frac{P_x (P_x - P_y^2)}{P_x^2 + P_y^2 - P_x^2 P_y^2} > 0$$

(g) $\boxed{<}$ $\neq$

$$p(X=1)\, p(Y=1)\, P(Z=1)$$

$$= p_x \cdot p_y \cdot (P_x^2 + P_y^2 - P_x^2 P_y^2)$$

$$p(X=1, Y=1, Z=1)$$

$$= P(X=1)\cdot P(Y=1, X=1)\cdot P(Z=1 \mid X=1, Y=1)$$

$$= P_x \cdot P_y \cdot (P_x + P_y - P_x P_y)$$

$$p(X=1, Y=1, Z=1) - p(X=1)\, p(Y=1)\, P(Z=1) = p_x p_y\,(P_x + P_y - P_x P_y - P_x^2 - P_y^2 + P_x^2 P_y^2)$$

$$= p_x p_y\,(P_x(1-p_x) + P_y(1-p_y) + P_x P_y (P_x P_y - 1))$$

$$\Rightarrow \quad P_x(1-p_x) + P_y(1-p_y) + P_x P_y (P_x P_y - 1) = P_x (1 - P_y^2)(1-P_x) > 0$$