Lecture 3.

1) Prime ideals.
2) Modules & homomorphisms.

References: [AM], Chapter 1, Section 4; Chapter 2, Sections 1,4.

BONUS: Non-commutative counterparts, 3.

1) A is comm've unital ring.

Definitions: • $a \in A$ is a zero divisor if $a \neq 0$ & $\exists\, b \in A$ s.t. $b \neq 0$ but $ab = 0$.

• A is domain if A has no zero divisors.

• Ideal $\mathfrak{p} \subset A$ is prime if $\mathfrak{p} \neq A$ & $A/\mathfrak{p}$ is domain.

Lemma: TFAE (the following are equivalent)

i) $\mathfrak{p}$ is prime

ii) If $a, b \in A$ are s.t. $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ (note that "$\Leftarrow$" is automatic).

iii) If $I, J \subset A$ are ideals, $IJ \subseteq \mathfrak{p} \Rightarrow I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$.

Sketch of proof: $\pi: A \twoheadrightarrow A/\mathfrak{p}$, $a \mapsto a + \mathfrak{p}$.

i) $\Leftrightarrow$ ii): $ab \in \mathfrak{p} \Leftrightarrow \pi(a)\pi(b) = 0$.

ii) $\Rightarrow$ iii): $I \not\subset \mathfrak{p} \Leftrightarrow \exists\, a \in I \setminus \mathfrak{p}$; $\exists\, b \in J \setminus \mathfrak{p} \Rightarrow ab \in IJ \setminus \mathfrak{p}$.

iii $\Rightarrow$ ii): $I := (a)$, $J := (b)$. Then $I \not\subset \mathfrak{p} \Leftrightarrow a \notin \mathfrak{p}$; $IJ \subseteq \mathfrak{p} \Leftrightarrow ab \in \mathfrak{p}$. $\square$

Examples: • $\mathfrak{m} \subset A$ max'l $\Leftrightarrow A/\mathfrak{m}$ is field (so domain) $\Rightarrow \mathfrak{m}$ is prime.

• $\{0\} \subset A$ is prime $\Leftrightarrow A$ is domain.

• $A = \mathbb{Z}$. Every ideal is (n) for $n \in \mathbb{Z}$; (n) is prime $\Leftrightarrow \pm n$ is prime or $n=0$. So every prime is max'l or $\{0\}$.

• Same conclusion for $A = \mathbb{F}[x]$ if $\mathbb{F}$ is field.

• $A = \mathbb{F}[x,y]$, (x) is prime (but not maximal): $\mathbb{F}[x,y]/(x) \xrightarrow{\sim} \mathbb{F}[y]$ (domain but not field).

• The ideal $(xy) \subset \mathbb{F}[x,y]$ is not prime.

Remark: Let A is domain, $\mathfrak{p} = (p)$; $\mathfrak{p}$ is prime $\Leftrightarrow$ if $a,b \in A$ w. $ab \vdots p$, then $a \vdots p$ or $b \vdots p$, such p are called prime.

2.1) Definitions (of modules & homomorphisms).

A is a commutative ring.

Definitions:

1) By an A-module we mean abelian group M w. map $A \times M \to M$ (multiplication or action map) s.t. the following axioms hold:

$$\left.\begin{array}{ll} \bullet \text{ Associativity}: & (ab)m = a(bm) \in M \\ \bullet \text{ Distibutivity}: & (a+b)m = am+bm, \\ & a(m+m') = am+am' \in M \\ \bullet \text{ Unit}: & 1m = m \in M \end{array}\right\} \forall a,b \in A,\ m,m' \in M.$$

2) Let M,N be A-modules. A homomorphism (a.k.a A-linear map) is (abelian) group homomorphism $\varphi: M \to N$ s.t.

$$\forall a \in A,\ m \in M \Rightarrow \varphi(am) = a\varphi(m).$$

2.2) Examples.

0) $A=\mathbb{Z}$. Then $A\times M\to M$ can be recovered from $+$ in M, thx to unit & distributivity. So $\mathbb{Z}$-module = abelian group. And a $\mathbb{Z}$-module homomorphism is the same thing as group homomorphism.

1) If A is a field, then A-module = vector space over A, and homomorphism = linear map.

For the next examples & also below, we will need:

Observation: Let $\varphi: A\to B$ be a ring homomorphism.

I) If M is a B-module, then we can view M as A-module w. $A\times M\to M$ given by $(a,m)\mapsto \varphi(a)m$. Every B-linear map $M\to N$ is also A-linear.

II) If $\varphi: A\twoheadrightarrow B$ (surjective), then a B-module = A-module, where $\ker\varphi$ acts by 0 ($am=0$ $\forall m\in M, a\in\ker\varphi$). For a map between B-modules M & N to be A-linear is is equivalent to be B-linear.

2) Modules vs linear algebra

i) $A=\mathbb{F}[x]$ ($\mathbb{F}$ is field)

By Observation I applied to $\mathbb{F}\to\mathbb{F}[x]$, every $\mathbb{F}[x]$-module is $\mathbb{F}$-module = vector space; $xm = Xm$ for an $\mathbb{F}$-linear operator $M\to M$; from X we can recover $\mathbb{F}[x]$-module str're

$$f(x)m = [f(X): M\to M] = f(X)m.$$

So $\mathbb{F}[x]$-module = $\mathbb{F}$-vector space w. a linear operator.
An $\mathbb{F}[x]$-module homomorphism $\varphi: M \to N$ is the same thing as a linear map $\varphi: M \to N$ s.t. $X_N \circ \varphi = \varphi \circ X_M$, where $X_M: M \to M$, $X_N: N \to N$ are operators coming from x.

ii) $A = \mathbb{F}[x_1, \dots x_n]$. An A-module = vector space w. n operators $X_1, \dots X_n$ (coming from $x_1, \dots x_n$) s.t. $X_i X_j = X_j X_i \ \forall i, j$.

iii) $A = \mathbb{F}[x_1, \dots x_n]/(G_1, \dots G_k)$, $G_i \in \mathbb{F}[x_1, \dots x_n]$. Use of Observation II w. $\mathbb{F}[x_1, \dots x_n] \overset{\varphi}{\twoheadrightarrow} A$ shows that A-module $= \mathbb{F}[x_1, \dots x_n]$-module where $\ker\varphi$ acts by 0 = $\mathbb{F}$-vector space w. n commuting operators $X_1, \dots X_n$ s.t. $G_i(X_1, \dots X_n) = 0$ as operators $M \to M$ $\forall i = 1, \dots k$.

3) Any ring B is a module over itself (via multiplication $B \times B \to B$). This is often called the regular module.

4) Combining 3) w. Observation I, we see that for a ring homom. $\varphi: A \to B$, B becomes an A-module. So A-algebra (Sect. 2 of Lec 1) is an A-module. The product on B is A-bilinear $\Leftrightarrow$ A-linear in each argument:
$$(b_1 + b_2)b' = b_1 b' + b_2 b', \quad (ab)b' = a(bb')$$
$$b(b_1' + b_2') = bb_1' + bb_2', \quad b(ab') = a(bb')$$

Equivalent definition of an A-algebra: This an A-module B w. a ring structure s.t. the multiplication map is A-bilinear.

Once M is an A-algebra with this definition, we get a ring homomorphism $A \to M: a \mapsto a1$ (1 unit in M).

Exercise: Persuade yourself that the two definitions are indeed equivalent.

2.3) Constructions with modules (& homomorphisms).

I) Direct sums & products.

M_1, M_2 A-modules $\rightsquigarrow$

$M_1 \oplus M_2$ (direct sum) $= M_1 \times M_2$ (direct product) = product $M_1 \times M_2$ as abelian groups w. $a(m_1, m_2) := (am_1, am_2)$.

More generally, for a set I (possibly infinite) & modules $M_i, i \in I$ define direct product $\prod_{i \in I} M_i = \{(m_i)_{i \in I} \mid m_i \in M_i\}$ w. componentwise operations.

Direct sum: $\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} \mid \text{only fin. many } m_i \neq 0\}$

Have A-module inclusion:

$$\bigoplus_{i \in I} M_i \hookrightarrow \prod_{i \in I} M_i$$

which is an isomorphism $\Leftrightarrow$ I is finite.

II) Hom module: let M, N be A-modules

As a set $\mathrm{Hom}_A(M,N) = \{\text{A-linear maps } M \to N\}$

Claim: $\mathrm{Hom}_A(M,N)$ has a natural A-module structure.

Need to define addition & multipl'n by elements of A.

$\varphi, \varphi' \in \mathrm{Hom}_A(M,N)$, $a \in A$

$$[\varphi + \varphi'](m) := \varphi(m) + \varphi'(m) \in N$$
$$[a\varphi](m) := a\varphi(m) \quad \in N$$

Lemma: 1) $\varphi+\varphi'$, $a\varphi$ are A-linear maps.

2) The operations $+, \cdot$ turn $\mathrm{Hom}_A(M,N)$ into A-module.

Partial proof: $[a\varphi](bm) = b[a\varphi](m)$.

$[a\varphi](bm) = a(\varphi(bm)) = ab\,\varphi(m) = [ab=ba] = b(a\varphi(m)) = b[a\varphi](m)$.

Rest of proof is an exercise. $\square$

Example: 1) Let $M=A$. Then $\mathrm{Hom}_A(A,N) \xrightarrow{\sim} N$

Exercise: Prove that maps $\mathrm{Hom}_A(A,N)\longrightarrow N$, $\varphi \mapsto \varphi(1)$ & $N\to \mathrm{Hom}_A(A,N)$, $n\mapsto \varphi_n : \varphi_n(a) = an$ are mutually inverse.

2) $\mathrm{Hom}_A(M_1\oplus M_2, N) \xrightarrow{\sim} \mathrm{Hom}_A(M_1,N)\times \mathrm{Hom}_A(M_2,N)$

$$\varphi \mapsto (\varphi|_{M_1}, \varphi|_{M_2}),$$ where $M_1 \hookrightarrow M_1\oplus M_2$ via $m_1\mapsto (m_1,0)$, and similarly for M_2.

Inverse maps $(\varphi_1,\varphi_2)\in \mathrm{Hom}_A(M_1,N)\times\mathrm{Hom}_A(M_2,N)$ goes to $\varphi: M_1\oplus M_2\to N$ given by $\varphi(m_1,m_2) := \varphi_1(m_1)+\varphi_2(m_2)$

Exercise: Prove that these two maps are mutually inverse.

3) There is a direct analog of this example for $M_1\oplus\ldots\oplus M_k$.

E.g. $\mathrm{Hom}_A(A^{\oplus k}, N) \xrightarrow{\sim} \mathrm{Hom}_A(A,N)^{\times k} \xrightarrow{\sim} N^{\times k}$

$$\varphi \longmapsto (\varphi(e_1),\ldots \varphi(e_k))$$

where $e_i = (0,\ldots 1,\ldots 0)$ (1 in the ith place). The inverse map is given by $\underline{n} = (n_1,\ldots n_k)\mapsto \varphi_{\underline{n}} : (a_1,\ldots a_k)\mapsto \sum a_i n_i$.

BONUS: Noncommutative counterparts, part 3.

B1) Prime & completely prime ideals: For a comm've ring A

& an ideal $\mathfrak{p} \subset A$ we have two equivalent conditions:

- For $a, b \in \mathfrak{p}$: $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$
- For ideals $I, J \subset A$: $IJ \subset \mathfrak{p} \Rightarrow I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.

For noncommutative A and a two-sided ideal $\mathfrak{p}$, these conditions are no longer equivalent.

Definition: Let A be a ring and $\mathfrak{p} \subset A$ be a two-sided ideal.

- We say $\mathfrak{p}$ is <u>prime</u> if for two-sided ideals $I, J \subset \mathfrak{p}$, have $IJ \subset \mathfrak{p} \Rightarrow I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.
- We say $\mathfrak{p}$ is <u>completely prime</u> if for $a, b \in A$, have $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}, b \in \mathfrak{p}$.

completely prime $\Rightarrow$ prime but not vice versa.

Exercise: 1) $\{0\} \subset Mat_n(\mathbb{F})$ is prime but not completely prime (if $n > 1$).

2) $\{0\} \subset Weyl_1 \left(= \mathbb{F}\langle x, y\rangle / (yx - xy - 1)\right)$ is completely prime.

B2) Modules over noncommutative rings. Here we have left & right modules & also bimodules. Let A be a ring.

Definition: · A <u>left A-module</u> M is an abelian group w. multiplication map $A \times M \to M$ subject to the same axioms as in the commutative case.

· A <u>right A-module</u> is a similar thing but with multiplication map $M \times A \to M$ subject to associativity $((ma)b = m(ab))$, distributivity & unit axioms.

· An A-bimodule is an abelian group M equipped

w. left & right A-module structures s.t. we have another associativity axiom: $(am)b = a(mb)\ \forall a,b \in A$.

When A is commutative, there's no difference between left & right modules and any such module is also a bimodule. Note also that for two a priori different rings A, B we can talk about A-B-bimodules

Example: 1) A is an A-bimodule.

2) $\mathbb{F}^n$ (the space of columns) is a left $Mat_n(\mathbb{F})$-module, while its dual $(\mathbb{F}^n)^*$ (the space of rows) is a right $Mat_n(\mathbb{F})$-module. None of these has a bimodule structure.

Exercise: Construct a left $Weyl_1$-module structure on $\mathbb{F}[x]$ (hint: y acts as $\frac{d}{dx}$).

Remark: let M, N be left A-modules. In general, $Hom_A(M,N)$ is not an A-module, it's just an abelian group. If M is an A-B-bimodule, then $Hom_A(M,N)$ gets a natural left B-module structure (exercise: how?). Similarly, if N is an A-C-bimodule, then $Hom_A(M,N)$ is a right C-module. And if M is an A-B-bimodule, and N is an A-C-bimodule, then $Hom_A(M,N)$ is a B-C-bimodule.