MATH 380, HOMEWORK 4, DUE NOV 4

There are **7 problems worth 27 points total.** Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-3 and to posted solutions. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative, unless stated otherwise, and contain 1.

Problem 1, 3pts total. *A functor from sets to algebras.* Let $\mathbb{F}$ be a field. To a set X we assign the algebra $\mathbb{F}[X]$ of all functions $X \to \mathbb{F}$ with pointwise addition and multiplication. To a map of sets $\varphi : X \to Y$ we assign a map $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X]$ given by $[\varphi^*(g)](x) = g(\varphi(x))$ for all $g \in \mathbb{F}[Y]$ and $x \in X$.
   1, 1pt) Prove that φ^* is an $\mathbb{F}$-algebra homomorphism.
   2, 2pt) Check that $X \mapsto \mathbb{F}[X]$ and $\varphi \mapsto \varphi^*$ defines a functor $\mathsf{Sets}^{opp} \to \mathbb{F}\text{-}\mathsf{Alg}$.

   We may see this construction again in our discussion of connections to Algebraic geometry.

Problem 2, 4pts total. *Algebraic constructions as functors.* Show that the following constructions are naturally functors (*each part is 1pt, in parts a-c you are responsible for explaining what the functor does on morphisms*).

   a) Sending a set I to the ring of polynomials $\mathbb{Z}[x_i]_{i \in I}$ *(in fact, we can replace $\mathbb{Z}$ with any ring)* is a functor $\mathsf{Sets} \to \mathsf{CommRings}$.

   b) Sending a ring R to the group $R^\times$ of the invertible elements in R is a functor $\mathsf{Rings} \to \mathsf{Groups}$.

   c) Sending a group G to its <u>group ring</u> $\mathbb{Z}G$ (a free modules with basis e_g labelled by elements of G and multiplication uniquely determined by $e_g e_h = e_{gh}$) is a functor from $\mathsf{Groups} \to \mathsf{Rings}$. *This construction is important for Representation theory of groups.*

   d) Sending a vector space V over $\mathbb{F}$ to its dual $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and sending a linear map $A : U \to V$ to the unique linear map $A^* : V^* \to U^*$ such that $[A^*\alpha](u) := \alpha(Au)$ ($\forall u \in U, \alpha \in V^*$) gives a functor $\mathbb{F}\text{-Vect} \to \mathbb{F}\text{-Vect}^{opp}$, where $\mathbb{F}\text{-Vect}$ is the category of vector spaces over $\mathbb{F}$.

Problem 3, 3pts. (*Double dual*). Consider the full subcategory $\mathbb{F}\text{-Vect}_{fd}$ in $\mathbb{F}\text{-Vect}$ of all finite dimensional vector spaces. Prove that the endo-functor $\bullet^{**}$ of $\mathbb{F}\text{-Vect}_{fd}$ is isomorphic to the identity endo-functor.

Problem 4, 3pts total. *Functor morphisms for compositions of functors.* Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, $F, F' : \mathcal{D} \to \mathcal{C}$ and $G, G' : \mathcal{E} \to \mathcal{D}$ be functors, $\kappa : F \Rightarrow F'$ and $\eta : G \Rightarrow G'$ be functor morphisms.

1, 2pt) Explain how κ gives rise to functor morphisms $FG \Rightarrow F'G$ and $FG' \Rightarrow F'G'$ and how η gives rise to functor morphisms $FG \Rightarrow FG'$ and $F'G \Rightarrow F'G'$.

2, 1pt) Establish a commutative diagram involving the functors and functor morphisms from part 1.

Notation: for an object $X \in \mathrm{Ob}(\mathcal{C})$, we write $\mathcal{F}_X$ for the Hom functor $\mathcal{C} \to \mathsf{Sets}$ defined by X. For two functors $F, G : \mathcal{C} \to \mathcal{D}$ we write $\mathrm{Hom}_{Fun}(F, G)$ for the (usually) set of functor morphisms $F \Rightarrow G$ and $\mathrm{End}_{Fun}(F, F)$ for the monoid of functor endomorphisms.

Problem 5, 4pts total. *Naturality of the bijection in the proof of* $\mathrm{Hom}_{Fun}(F_X, F) \xrightarrow{\sim} F(X)$. Recall the bijection $\mathrm{Hom}_{Fun}(\mathcal{F}_X, F) \xrightarrow{\sim} F(X)$ from Lecture 15, denote it by $\sigma_{X,F}$.

a, 2pt) Let G be another functor $\mathcal{C} \to \mathsf{Sets}$ and $\eta : F \Rightarrow G$ be a functor morphism. Prove that there is the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{Hom}_{Fun}(\mathcal{F}_X, F) & \xrightarrow{\ \sigma_{X,F}\ } & F(X) \\
\downarrow{\scriptstyle \eta \circ ?} & & \downarrow{\scriptstyle \eta_X} \\
\mathrm{Hom}_{Fun}(\mathcal{F}_X, G) & \xrightarrow{\ \sigma_{X,G}\ } & G(X)
\end{array}
$$

b, 2pt) Let Y be an object of $\mathcal{C}$ and $f : X \to Y$ be a morphism. Let f^* denote the corresponding element of $\mathrm{Hom}_{Fun}(\mathcal{F}_Y, \mathcal{F}_X)$. Prove that the following diagram is commutative.

$$
\begin{array}{ccc}
\mathrm{Hom}_{Fun}(\mathcal{F}_X, F) & \xrightarrow{\ \sigma_{X,F}\ } & F(X) \\
\downarrow{\scriptstyle ? \circ f^*} & & \downarrow{\scriptstyle F(f)} \\
\mathrm{Hom}_{Fun}(\mathcal{F}_Y, F) & \xrightarrow{\ \sigma_{Y,F}\ } & F(Y)
\end{array}
$$

Problem 6, 6pts total. *Representing objects and endomorphisms of functors.* Construct objects representing the forgetful functors F below and use this to determine the monoid $\mathrm{End}_{Fun}(F)$ (i.e. describe the set together with composition).

a, 3pts) $F : A\text{-}\mathrm{Mod} \to \mathsf{Sets}$, where A is a commutative ring.

b, 3pts) $F : \mathsf{Rings} \to \mathsf{Sets}$.

Problem 7, 4pts. *Functoriality of product.* Let $\mathcal{C}$ be a category where every two objects have the product. Explain how the product can be viewed as a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.