INTRODUCTION TO TYPE A CATEGORICAL KAC-MOODY ACTIONS.

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In these notes, we give a brief introduction to the theory of categorical actions of type A Kac-Moody algebras, as introduced in [CR] (in the case of $\mathfrak{sl}_2$) and [Rou] (a more general case). Intuitively speaking, these should be data of an exact category $\mathcal{C}$ and a collection of exact endofunctors E_i, F_i of $\mathcal{C}$ that descend to an action of a Kac-Moody algebra $\mathfrak{g}$ on $[\mathcal{C}] := K_0(\mathcal{C}) \otimes \mathbb{C}$. This is, however, too general to give an interesting theory. The actual definition is much subtler and will be given in Section 3 of the notes.

In Section 1 we give a brief reminder of the definition of a Kac-Moody algebra $\mathfrak{g}(I)$ associated to a graph I. Before giving the definition of a categorical $\mathfrak{g}(I)$-action this week we will provide, Section 2, an example of one via cyclotomic Hecke algebras (= Ariki-Koike algebras) that already appeared in Siddharth's talk. In Section 3 we finally give the definition of a categorical $\mathfrak{g}(I)$-action and study some of its immediate consequences. Finally, in Section 4 we will introduce Rickard complexes. These are categorifications of the action of simple reflections in a representation of $\mathfrak{g}(I)$, and we will show that these give a derived equivalence between certain 'weight' subcategories of $\mathcal{C}$.

1. Kac-Moody algebras.

We (very briefly) recall the definition of a Kac-Moody algebra associated to a simply laced graph I. For a more detailed account see, for example, [Et, K].

So let I be a simply laced graph. Denote by $V(I)$ the set of vertices of I. The *Cartan matrix* of I, $C(I)$, is the $|V(I)| \times |V(I)|$ matrix $(a_{ij})_{i,j\in V(I)}$ defined by $a_{ii} = 2$ for all i; $a_{ij} = -1$ if there is an edge $i - j$ in I; and $a_{ij} = 0$ if there is no edge between i and j in I.

Definition 1.1. *Let I be a simply laced graph. The* Kac-Moody algebra $\mathfrak{g}(I)$ *is the Lie algebra with generators e_i, f_i, h_i for $i \in V(I)$, known as the* Chevalley generators *of $\mathfrak{g}(I)$, and relations*

(KM1) $[h_i, e_j] = a_{ij}e_j$. $[h_i, f_j] = -a_{ij}f_j$.
(KM2) $[e_i, f_j] = \delta_{ij}h_j$.
(KM3) *If* $i \neq j$, $\mathrm{ad}_{e_i}^{(1-a_{ij})} e_j = 0$; $\mathrm{ad}_{f_i}^{(1-a_{ij})} f_j = 0$.

The relations $(KM3)$ *are known as the* Serre relations.

In these notes, we will always assume that I is one of the following graphs.

I	$\mathfrak{g}(I)$
$\circ - \circ - \cdots - \circ$	$\mathfrak{sl}_{n+1}$ (n vertices)
	$\widehat{\mathfrak{sl}}_n$ (n vertices)
$\cdots - \circ - \circ - \cdots$	$\mathfrak{gl}_\infty$

2. Cyclotomic Hecke algebras.

2.1. **Reminders.** Our first example of a categorical action will be given by the representation theory of cyclotomic Hecke algebras. Here we recall their definition and a few facts from Siddharth's talk.

Definition 2.1. *Let $\mathbb{F}$ be a commutative domain, and $q \in \mathbb{F}^\times$. The* affine Hecke algebra $\mathcal{H}^{\mathrm{aff}}_{\mathbb{F},q}(n)$ *is the unital associative $\mathbb{F}$-algebra generated by elements $T_1, \ldots, T_{n-1}, X_1^{\pm}, \ldots, X_n^{\pm}$ subject to the following relations:*

- *The subalgebra generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the finite Hecke algebra of type A.*
- *The subalgebra generated by $X_1^{\pm}, \ldots, X_n^{\pm}$ is isomorphic to the algebra $\mathbb{F}[X_1^{\pm}, \ldots, X_n^{\pm}]$ of Laurent polynomials in the variables X_i.*
- $T_iX_j = X_jT_i$ *if* $i \neq j, j-1$; $T_iX_iT_i = qX_{i+1}$.

Now choose $q_1, \ldots, q_m \in \mathbb{F}^\times$. The cyclotomic Hecke algebra *(or* Ariki-Koike algebra*) $H_{\mathbb{F},q,q_1,\ldots,q_m}(n)$ is the quotient of $\mathcal{H}^{\mathrm{aff}}_{\mathbb{F},q}(n)$ by the extra relation:*

$$\prod_{j=1}^{m}(X_1 - q_j) = 0.$$

We remark that to pass from our definition of the cyclotomic Hecke algebra $H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ to the one given in [Ven, Section 6], we just set $T_0 := \pi(X_1)$, where $\pi : \mathcal{H}_{\mathbb{F},q}(n) \to H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ is the canonical projection. We define the *Jucys-Murphy* elements $L_1, \dots, L_n \in H_{F,q,q_1,\dots,q_m}(n)$ by

$$L_i := \pi(X_i).$$

Note that an explicit formula for L_i is $L_i = q^{1-i}T_{i-1}\cdots T_1T_0T_1\cdots T_{i-1}$. It follows from [Ven, Section 4], that every symmetric polynomial in the variables $L_1, \dots, L_n$ belongs to the center of $H_{\mathbb{F},q,q_1,\dots,q_m}(n)$.

Siddharth has also constructed all the irreducible representations of $H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ in the case where $\mathbb{F}$ is a field and $q, q_1, \dots, q_m$ are generic. Let us remark that, here, 'generic' means that q is not a root of unity and $q_i/q_j \notin \{q^k : k \in \mathbb{Z}\}$ for every $i, j = 1, \dots m$ .We do not need an explicit construction of these representations. We just remark that they are indexed by m-multipartitions of n, say V_λ is the irreducible representation corresponding to $\lambda \vdash_m n$. Each V_λ has as a basis $\{v_{\mathfrak{t}}\}$, where $\mathfrak{t}$ runs over the set of all standard Young tableaux of shape λ. The action of the JM elements L_i on V_λ is given by

$$L_i v_{\mathfrak{t}} = q^{a-b}q_j v_{\mathfrak{t}}, \tag{1}$$

where the box with the number i appears in $\mathfrak{t}$ in column a and row b of $\lambda^{(j)}$.

2.2. **Induction and Restriction functors.** Note that we have a natural embedding $\iota : H(n-1) \to H(n)$, where we denote $H(k) := H_{\mathbb{F},q,q_1,\dots,q_m}(k)$. Recall from [Ven, Section 9] that $H(n)$ is free of rank $m^n n!$ over $\mathbb{F}$, with basis $X_n := \{L_1^{c_1}\cdots L_n^{c_n}T_w : w \in \mathfrak{S}_n, 0 \le c_i \le m-1\}$. This has the following easy consequence:

Proposition 2.2. *The algebra $H(n)$ is free as a left $H(n-1)$-module.*

Proof. By the definition of L_n and the relations on the affine Hecke algebra, it is clear that L_n commutes with $T_w, w \in \mathfrak{S}_{n-1}$. Thus, we have that

$$H(n) = \bigoplus_{0\le c_m<m}\bigoplus_{w\in\mathfrak{S}_n/\mathfrak{S}_{n-1}} H(n-1)L_m^{c_m}T_w,$$

where $w \in \mathfrak{S}_n/\mathfrak{S}_{n-1}$ runs over the the coset representatives of $\mathfrak{S}_{n-1}$ in $\mathfrak{S}_n$ of minimal length. □

The inclusion $\iota : H(n-1) \to H(n)$ also allows us to define induction and restriction functors for cyclotomic Hecke algebras. Namely, we define:

$$\begin{aligned}\operatorname{Res}_n^{n+1} : H(n+1)\text{-mod} &\to H(n)\text{-mod} \\ M &\mapsto \iota_n^*(M)\end{aligned} \qquad \begin{aligned}\operatorname{Ind}_n^{n+1} : H(n)\text{-mod} &\to H(n+1)\text{-mod}. \\ M &\mapsto H(n+1)\otimes_{H(n)} M\end{aligned}$$

By Proposition 2.2, both $\operatorname{Res}_n^{n+1}$ and $\operatorname{Ind}_n^{n+1}$ are exact functors. Moreover, it is clear that $(\operatorname{Ind}_n^{n+1}, \operatorname{Res}_n^{n+1})$ is an adjoint pair of functors. We also have the coinduction functor:

$$\begin{aligned}\operatorname{CoInd}_n^{n+1} : H(n)\text{-mod} &\to H(n+1)\text{-mod} \\ M &\mapsto \operatorname{Hom}_{H(n)}(H(n+1), M).\end{aligned}$$

This functor is right adjoint to $\operatorname{Res}_n^{n+1}$.

Proposition 2.3. *There is an isomorphism of functors* $\operatorname{Ind}_n^{n+1} \cong \operatorname{CoInd}_n^{n+1}$.

Proof. Let $M \in H(n)$-mod. Since $H(n+1)$ is a free left $H(n)$-module, there is a natural isomorphism $\operatorname{Ind}_n^{n+1}(M) = \operatorname{Hom}_{H(n)}(H(n+1), M) \xrightarrow{\cong} \operatorname{Hom}_{H(n)}(H(n+1), H(n)) \otimes_{H(n)} M$. Thus, we only need to show that the $H(n+1)$-$H(n)$-bimodule $\operatorname{Hom}_{H(n)}(H(n+1), H(n))$ is isomorphic (as a bimodule) to $H(n+1)$. This follows because $H(n+1), H(n)$ are symmetric algebras. Indeed, we have:

$$\begin{aligned}\operatorname{Hom}_{H(n)}(H(n+1), H(n)) &\xrightarrow{\cong} \operatorname{Hom}_{H(n)}(H(n+1), H(n)^*) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{F}}(H(n)\otimes_{H(n)} H(n+1), \mathbb{F}) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{F}}(H(n+1), \mathbb{F}) \\ &\xrightarrow{\cong} H(n+1).\end{aligned}$$

□

2.3. **Action on K_0.** Since Res and Ind are exact functors, they descend to maps

$$[\mathrm{Res}_n^{n+1}] : [H(n+1)\text{-mod}] \longleftrightarrow [H(n)\text{-mod}] : [\mathrm{Ind}_n^{n+1}].$$

In the generic case, an explicit formula for the maps [Res] and [Ind] is easy to find. Recall that $[H(n)\text{-mod}]$ has basis $\{[V_\lambda] : \lambda \vdash_m n\}$. By the construction of the representations V_λ, we have that:

$$[\mathrm{Res}_{n-1}^n][V_\lambda] = \sum_{x \in \mathrm{rem}(\lambda)} [V_{\lambda - \{x\}}], \tag{2}$$

where $\mathrm{rem}(\lambda)$ denotes the set of removable boxes of λ. By adjunction and Frobenius reciprocity, we have that

$$[\mathrm{Ind}_{n-1}^n][V_\mu] = \sum_{x \in \mathrm{add}(\mu)\}} [V_{\mu \cup \{x\}}], \tag{3}$$

where $\mathrm{add}(\mu)$ denotes the set of addable boxes of μ. We would like to have similar formulas for [Ind] and [Res] in the general case, not just the generic one.

To do this, we introduce a specialization map. Consider $\mathbb{F}[\mathbf{t}]_{(\mathbf{t}-1)}$, the localization of $\mathbb{F}[\mathbf{t}]$ at the ideal $(\mathbf{t}-1)$, and its completion, $\mathbb{S} := \mathbb{F}[\mathbf{t}]^\wedge_{(\mathbf{t}-1)}$. Let $\mathbb{K}$ be the fraction field of $\mathbb{S}$. Note that $\mathbb{S}$ is a complete discrete valuation ring, with residue field $\mathbb{F}$. Let $\mathbf{q} := q\mathbf{t}^m$, and, for $i = 1, \ldots, n$, $\mathbf{q}_i := q_i \mathbf{t}^{i-1}$. We will consider the cyclotomic Hecke algebras $H_\mathbb{S} := H_{\mathbb{S},\mathbf{q},\ldots,\mathbf{q}_m}$ and $H_\mathbb{K} := H_{\mathbb{K},\mathbf{q},\ldots,\mathbf{q}_m}$. We have that $H_\mathbb{F} = H_\mathbb{S} \otimes_\mathbb{S} \mathbb{F}$, and $H_\mathbb{K} = H_\mathbb{S} \otimes_\mathbb{S} \mathbb{K}$.

By our choice of parameters, the algebra $H_\mathbb{K}$ is semisimple. Indeed, it is clear that $\mathbf{q}$ is not a root of unity and that $\mathbf{q}_i/\mathbf{q}_j$ is not a power of $\mathbf{q}$. So we have the simple modules $V_\lambda^\mathbb{K}$. We introduce the specialization map $d : K_0(H_\mathbb{K}\text{-mod}) \longrightarrow K_0(H_\mathbb{F}\text{-mod})$ as follows. Let $M \in H_\mathbb{K}\text{-mod}$. Pick an $\mathbb{S}$-latice in M of maximal rank, say L. So L is a $H_\mathbb{S}$-submodule of M with $M = L \otimes_\mathbb{S} \mathbb{K}$. The class $[L \otimes_\mathbb{S} \mathbb{F}] \in K_0(H_\mathbb{F})$ depends only on $[M]$, not on the choice of representative of this class, nor on the choice of a lattice L. This gives us our specialization map.

We claim that $d : [H_\mathbb{K}\text{-mod}] \longrightarrow [H_\mathbb{F}\text{-mod}]$ is surjective. To see this, we check that its dual map d^* is injective. This is where we use that we are working with Grothendieck groups with coefficients in a field, rather than just a domain. Recall that, if A is a finite dimensional $\mathbb{F}$-algebra, then the dual to $[A\text{-mod}]$ is $[A\text{-proj}]$: a pairing is given by $(M, P) \mapsto \dim_\mathbb{F} \mathrm{Hom}_A(P, M)$. So we have to check that $d^* : [H_\mathbb{F}\text{-proj}] \longrightarrow [H_\mathbb{K}\text{-proj}]$ is injective. In other words, we have to check that any projective $H_\mathbb{F}$-module has a unique, up to isomorphism, deformation to a projective $H_\mathbb{K}$-module. Existence of this deformation is an easy consequence of Hensel's lemma. Uniqueness follows from the fact that $\mathrm{Ext}^1_{H_\mathbb{F}}(P, P) = 0$ for any projective $H_\mathbb{F}$-module. So d is surjective. We define

$$[V_\lambda] := d[V_\lambda^\mathbb{K}].$$

Then, $\{[V_\lambda] : \lambda \vdash_m n\}$ generates $[H_\mathbb{F}\text{-mod}]$. We remark that this is not, in general, a basis. But we can give the action of $[\mathrm{Res}], [\mathrm{Ind}]$ on $[V_\lambda]$. Indeed, it is an easy consequence of the definitions that the diagrams

$$\begin{tikzcd}
{[H_\mathbb{K}(n)\text{-mod}]} \arrow[r, "d"] \arrow[d, "{[\mathrm{Res}^n_{n-1}]}"'] & {[H_\mathbb{F}(n)\text{-mod}]} \arrow[d, "{[\mathrm{Res}^n_{n-1}]}"] \\
{[H_\mathbb{K}(n-1)\text{-mod}]} \arrow[r, "d"] & {[H_\mathbb{F}(n-1)\text{-mod}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[H_\mathbb{K}(n)\text{-mod}]} \arrow[r, "d"] & {[H_\mathbb{F}(n)\text{-mod}]} \\
{[H_\mathbb{K}(n-1)\text{-mod}]} \arrow[r, "d"] \arrow[u, "{[\mathrm{Ind}^n_{n-1}]}"] & {[H_\mathbb{F}(n-1)\text{-mod}]} \arrow[u, "{[\mathrm{Ind}^n_{n-1}]}"']
\end{tikzcd}$$

commute. It follows that formulas (2), (3) are valid in general, not just in the generic case.

Remark 2.4. *We would like to make some comments about the elements $[V_\lambda] \in [H(n)\text{-mod}]$. The algebra $H(n)$ has the structure of a* cellular algebra*, cf.* [GL]*. This means that there is a basis of $H(n)$ that satisfies some upper triangularity conditions. This basis is indexed by pairs of standard tableaux of the same shape λ, where λ is an m-multipartition of n. It follows from the theory of cellular algebras that $H(n)$ comes equipped with a set of representations $C_\lambda, \lambda \vdash_m n$, called* cell modules*. For each λ, the representation C_λ has a natural bilinear form, φ^λ, whose radical is an $H(n)$-submodule. This bilinear form may be zero. The set $\{D_\lambda := C_\lambda/\operatorname{rad}\varphi^\lambda : \varphi^\lambda \neq 0\}$ forms a complete list of irreducible $H(n)$-modules. Moreover, D_λ is the unique irreducible quotient of C_λ and, if D_μ appears as a composition factor of C_λ, then $\mu \trianglerighteq \lambda$ under the dominance ordering. We have that $[V_\lambda] = [C_\lambda]$ and the transition matrix from the basis $\{C_\lambda\}$ to the basis $\{D_\lambda\}$ is upper unitriangular. This is another way to see that the specialization map is surjective. We can also look at cell modules from the point of view of rational Cherednik algebras. Let $\mathbb{H}$ be a cyclotomic rational Cherednik algebra whose category $\mathcal{O}$ maps to $H(n)$*-mod *under the KZ functor. Then, we have that $C_\lambda = KZ(\Delta(\lambda))$,* *where* $\Delta(\lambda)$ *is the Verma module for* $\mathbb{H}$*, see* [CGG]. *The upper unitriangularity of the transition matrix also follows from here.*

2.4. **i-Induction and i-Restriction.** We will need refined versions of the induction and restriction functors. From now on, we assume that $\mathbb{F}$ is an algebraically closed field. Let $L_n \in H_{\mathbb{F}}(n)$ be the n-th JM element. It is a direct consequence of the definition that L_n centralizes the subalgebra $H_{\mathbb{F}}(n-1)$. So, for every $H_{\mathbb{F}}(n)$-module M, we can think of L_n as a $H_{\mathbb{F}}(n-1)$-endomorphism of $\mathrm{Res}^n_{n-1}(M)$. For $a \in \mathbb{F}$, let $(\mathrm{Res}^n_{n-1})_a(M)$ be the a-generalized eigenspace for the action of L_n on M. This construction is functorial and we have $\mathrm{Res}^n_{n-1} = \bigoplus_{a\in\mathbb{F}}(\mathrm{Res}^n_{n-1})_a$. Clearly, $(\mathrm{Res}^n_{n-1})_a$ is an exact functor, so it induces a map on K_0. By adjointness, we have a decomposition $\mathrm{Ind}^n_{n-1} = \bigoplus_{a\in\mathbb{F}}(\mathrm{Ind}^n_{n-1})_a$ such that, for each $a \in \mathbb{F}$, the functor $(\mathrm{Res}^n_{n-1})_a$ is left adjoint to $(\mathrm{Ind}^n_{n-1})_a$.

Proposition 2.5. *For every* $a \in \mathbb{F}$*, the functors* $(\mathrm{Res}^n_{n-1})_a, (\mathrm{Ind}^n_{n-1})_a$ *are biadjoint.*

Proof. For every k, denote $\mathscr{L}_k = L_1 + \cdots + L_k \in H(k)$. Since this is a symmetric polynomial in the JM elements, it actually belongs to the center of $H(k)$. So we have a decomposition

$$H(k)\text{-mod} = \bigoplus_{b\in\mathbb{F}}(H(k)\text{-mod})_b,$$

where $(H(k)\text{-mod})_b$ consists of those modules on which $\mathscr{L}_k$ acts with generalized eigenvalue b. Now, let $M \in (H(n)\text{-mod})_b$. Then, since $\mathscr{L}_n = \mathscr{L}_{n-1} + L_n$, we have that $(\mathrm{Res}^n_{n-1})_a(M)$ is the projection of the $H(n-1)$-module $\mathrm{Res}^n_{n-1}(M)$ to $(H(n-1)\text{-mod})_{b-a}$. Since $((\mathrm{Res}^n_{n-1})_a, (\mathrm{Ind}^n_{n-1})_a)$ is an adjoint pair, it follows that for $N \in (H(n-1)\text{-mod})_b$, $(\mathrm{Ind}^n_{n-1})_a(N)$ is the projection of $\mathrm{Ind}^n_{n-1}(M)$ to $(H(n)\text{-mod})_{b+a}$. The result now follows since $\mathrm{Res}^n_{n-1}, \mathrm{Ind}^n_{n-1}$ are biadjoint. ☐

We can give the action of $(\mathrm{Res}^n_{n-1})_a$, $(\mathrm{Ind}^n_{n-1})_a$ in the generic case. This follows from (1). To express this, we introduce some notation. Let $\lambda \vdash_m n$ be an m-multipartition. Assume that the box $\square$ is column a, row b of $\lambda^{(i)}$. We define the *content* of $\square$ to be:

$$\mathrm{cont}(\square) := q^{a-b}q_i$$

Then, we have the following identity:

$$(\mathrm{Res}^n_{n-1})_a(V_\lambda) = \bigoplus_{\substack{x\in\mathrm{rem}(\lambda)\\ \mathrm{cont}(x)=a}} V_{\lambda-\{x\}}. \tag{4}$$

And, by Frobenius reciprocity, we get

$$(\mathrm{Ind}^n_{n-1})_a(V_\mu) = \bigoplus_{\substack{x\in\mathrm{add}(\mu)\\ \mathrm{cont}(x)=a}} V_{\mu\cup\{x\}}.$$

In the general case, we can only get similar formulas at the level of the Grothendieck group. We again use specialization maps. Recall the notation $\mathbb{S}, \mathbb{K}, \mathbf{q}, \mathbf{q}_1, \dots, \mathbf{q}_m$ from Subsection 2.3. We remark that all eigenvalues of L_n on an $H_{\mathbb{K}}(n)$-module are, actually, in $\mathbb{S}$. This follows from semisimplicity of $H_{\mathbb{K}}(n)$ and (1). For an element $\mathbf{a} \in \mathbb{S}$, we denote by a its projection to $\mathbb{F}$.

Proposition 2.6. *The following diagram commutes:*

$$\begin{tikzcd} {[H_{\mathbb{K}}(n)\text{-mod}]} \arrow[r, "d"] \arrow[d, "\sum_{\mathbf{a}\mapsto a}[\mathrm{Res}^n_{n-1}]_{\mathbf{a}}"'] & {[H_{\mathbb{F}}(n)]\text{-mod}} \arrow[d, "{[\mathrm{Res}^n_{n-1}]_a}"] \\ {[H_{\mathbb{K}}(n-1)\text{-mod}]} \arrow[r, "d"] & {[H_{\mathbb{F}}(n-1)\text{-mod}]} \end{tikzcd} \tag{5}$$

Proof. It is enough to show that, for a multipartition $\lambda \vdash_m n$, $d\left(\sum_{\mathbf{a}\mapsto a}[\mathrm{Res}^n_{n-1}]_{\mathbf{a}}[V^{\mathbb{K}}_\lambda]\right) = [\mathrm{Res}^n_{n-1}]_a[V_\lambda]$. Recall that $V^{\mathbb{K}}_\lambda$ has a $\mathbb{K}$-basis $\{v_{\mathfrak{t}} : \mathfrak{t}$ is a standard tableau of shape $\lambda\}$. Then, as an $\mathbb{S}$-latice L we can take the $H_{\mathbb{S}}(n)$-module generated by $\{v_{\mathfrak{t}}\}$. From here, the result follows. ☐

For the rest of this section, we will make the following assumption on parameters:

(†) *There exists* $\ell \in \mathbb{Z}_{>0}$ *such that* $q = \sqrt[\ell]{1}$ *is a primitive* ℓ*-root of* 1*, and for every* $i = 1, \dots, r$*,* $q_i = q^{k_i}$ *for some* $k_i \in \mathbb{Z}/\ell\mathbb{Z}$.

It follows from (4) and commutativity of the diagram (5) that, under the assumption (†), the functors $(\mathrm{Res}^n_{n-1})_a$ vanish unless $a = q^i$ for some $i = 0, \ldots, \ell-1$. In this setting, we define the i-restriction and i-induction functors:

$$i\text{-}\mathrm{Res}^n_{n-1} := (\mathrm{Res}^n_{n-1})_{q^i}, \quad i\text{-}\mathrm{Ind}^n_{n-1} := (\mathrm{Ind}^n_{n-1})_{q^i},$$

so that $\mathrm{Res}^n_{n-1} = \bigoplus_{i=0}^{\ell-1} i\text{-}\mathrm{Res}^n_{n-1}$, $\mathrm{Ind}^n_{n-1} = \bigoplus_{i=0}^{\ell-1} i\text{-}\mathrm{Ind}^n_{n-1}$. Using (4) and (5) again, it is easy to see the action of i-Ind and i-Res at the level of the (complexified) Grothendieck groups. For a box $\square$ of the Young diagram of a multipartition $\lambda \vdash_m n$, we define the *ℓ-content* of $\square$ to be:

$$\mathrm{cont}_\ell(\square) = a - b + k_i \mod \ell,$$

where the box $\square$ is on column a and row b of $\lambda^{(i)}$. Then, we have

$$[i\text{-}\mathrm{Res}^n_{n-1}][V_\lambda] = \sum_{\substack{x \in \mathrm{rem}(\lambda) \\ \mathrm{cont}_\ell(x) = i}} [V_{\lambda - \{x\}}], \quad [i\text{-}\mathrm{Ind}^n_{n-1}][V_\mu] = \sum_{\substack{x \in \mathrm{add}(\mu) \\ \mathrm{cont}_\ell(x) = i}} [V_{\mu \cup (x)}]. \tag{6}$$

2.5. **Categorification functors.** We denote $H(n) := H_{\mathbb{F}}(n)$, and throughout this Subsection we assume (†).

Now let $\mathcal{C} := \bigoplus_{n \geq 0} H(n)$-mod, where, by definition, $H(0)$-mod is just the category of $\mathbb{F}$-vector spaces. Let $E := \bigoplus \mathrm{Res}^{n+1}_n$ (we define Res^0_{-1} to be just the zero functor) and $F := \bigoplus \mathrm{Ind}^{n+1}_n$ be endofunctors of the category $\mathcal{C}$. We have seen that:

- The endofunctors E and F are biadjoint.
- There exists an endomorphism $L := \bigoplus L_n$ of E that yields a decomposition $E = \bigoplus_{i \in \mathbb{Z}/\ell\mathbb{Z}} E_i$ into generalized eigenfunctors. By adjointness, this induces a decomposition $F = \bigoplus_{i \in \mathbb{Z}/\ell\mathbb{Z}} F_i$ such that each pair E_i, F_i consists of biadjoint endofunctors, cf. Proposition 2.5.
- Let $f_i := [F_i] : [\mathcal{C}] \to [\mathcal{C}], e_i := [E_i] : [\mathcal{C}] \to [\mathcal{C}]$ be the induced maps in the complexified Grothendieck group of $\mathcal{C}$. Using (6) we can see that e_i, f_i, $i = 1, \ldots, \ell$ induce an action of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_\ell$ on $[\mathcal{C}]$. Indeed, $[\mathcal{C}]$ is a quotient of the level m Fock space of $\widehat{\mathfrak{sl}}_\ell$ that has as a basis the set of all m-multipartitions.

Note that we do not have categorical analogues of the Chevalley generators $h_i, i \in I$. However, there is a decomposition of $\mathcal{C}$ lifting the decomposition of $[\mathcal{C}]$ into weight spaces. This is induced by the action of the center of the algebra $\mathcal{H}^{\mathrm{aff}}_q(n)$ on $H(n)$-modules. Recall that the center of $\mathcal{H}^{\mathrm{aff}}_q(n)$ is Z_n, the space of symmetric Laurent polynomials on $x_1, \ldots, x_n$. Since $H(n)$ is a quotient of $\mathcal{H}^{\mathrm{aff}}_q(n)$, this induces a decomposition by central characters:

$$H(n)\text{-mod} = \bigoplus_\chi (H(n)\text{-mod})_\chi.$$

Identifying central characters of $\mathcal{H}^{\mathrm{aff}}_q(n)$ with points in $(\mathbb{F}^\times)^n/\mathfrak{S}_n$ we have, by (†), that $(H(n)\text{-mod})_\chi = 0$ unless $\chi \in \mathcal{I}^n/\mathfrak{S}_n$, where $\mathcal{I} = \{q, q^2, \ldots, q^{\ell-1}\}$. For such χ, let

$$\mathrm{wt}(\chi) = \sum_{i=0}^{\ell-1} m_i \alpha_i,$$

where α_i is the simple root of $\widehat{\mathfrak{sl}}_\ell$ corresponding to the Chevalley generators e_i, f_i, h_i; and m_i is the multiplicity of q^i on χ. Note that χ is uniquely determined by $\mathrm{wt}(\chi)$. Let $\varpi := \sum_{i=1}^m \omega_{k_i}$, where $\omega_0, \ldots, \omega_{\ell-1}$ are the fundamental weights of $\widehat{\mathfrak{sl}}_\ell$.

Proposition 2.7. *For $\chi \in \mathcal{I}^n/\mathfrak{S}_n$, $\widehat{\mathfrak{sl}}_\ell$ acts on $[H(n)\text{-mod}]_\chi$ with weight $\varpi + \mathrm{wt}(\chi)$.*

There is an extra piece of structure we have not seen yet. Namely, consider the endofunctor $E^2 : \mathcal{C} \to \mathcal{C}$, that sends $M \in H(n)$-mod to $\mathrm{Res}^{n-1}_{n-2} \mathrm{Res}^n_{n-1}(M) = \mathrm{Res}^n_{n-2}(M)$. Note that the element $T_{n-1} \in H(n)$ centralizes the subalgebra $H(n-2)$, so we can consider it as an endomorphism of $\mathrm{Res}^n_{n-2}(M)$. Then, similarly to above, we can consider an endomorphism T of E^2, given by $T = \bigoplus T_{n-1}$. We remark that the endomorphism T satisfies the relations

$$(1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E) \in \mathrm{End}(E^3), \tag{7}$$

$$(T + 1_E)(T - q 1_E) = 0 \in \mathrm{End}(E^2), \tag{8}$$

Let us explain the notation. For $M \in \mathcal{C}$, $(1_E T) \circ (T 1_E) \circ (1_E T)_M : E^3 M \longrightarrow E^3 M$ is the endomorphism given by $E(T_M) \circ T_{EM} \circ E(T_M)$, where $T_M : E^2 M \longrightarrow E^2 M$ and $T_{EM} : E^2 EM \longrightarrow E^2 EM$. The other notation is similar.

Then, (7) is nothing more than the braid relation $T_{n-1}T_{n-2}T_{n-1} = T_{n-2}T_{n-1}T_{n-2}$, while (8) is nothing else but the Hecke relation $(T_{n-1}+1)(T_{n-1}-q) = 0$. We also have a relation between the endomorphisms $L \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$,

$$T \circ (L1_E) \circ T = q1_E L \in \mathrm{End}(E^2), \tag{9}$$

which is just different way to say that $T_{n-1}L_{n-1}T_{n-1} = qL_n$.

Remark 2.8. *Throughout this subsection we assumed the condition (†) on the parameters. Similar results are obtained if we assume that $q_i = q^{k_i}$ for some $k_i \in \mathbb{Z}$, and $q \in \mathbb{F}^\times$ is not a root of unity. In this case, we obtain an action of $\mathfrak{gl}_\infty$ on $[\mathcal{C}]$. More generally, let $S = \{q_1, \ldots, q_m\}$. We have an equivalence relation on S: $q_i \sim q_j$ if $q_i/q_j \in q^{\mathbb{Z}}$. This yields a partition $S = \bigsqcup_{k=1}^{j} S_k$. Then, we obtain an action of $\widehat{\mathfrak{sl}_\ell}^{j}$ (if q is a primitive ℓ-root of unity) or of $\mathfrak{gl}_\infty^j$ (if q is not a root of unity) on $\mathcal{C}$. This follows from, for example,* [Ari, Theorem 13.30].

3. Categorical actions.

3.1. **Definition.** Let $\mathcal{C}$ be an abelian, $\mathbb{F}$-linear category, and let I be a graph as in Section 1 of these notes. A *categorical $\mathfrak{g}(I)$-action* on $\mathcal{C}$ consists of the following data:

- A pair or biadjoint endofunctors $E, F : \mathcal{C} \to \mathcal{C}$. We fix an adjunction (E, F), meaning that we fix a counit $\varepsilon : EF \longrightarrow 1_{\mathcal{C}}$, and a unit $\eta : 1_{\mathcal{C}} \longrightarrow FE$ of adjunction.
- Endomorphisms $L \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$
- An element $q \in \mathbb{F}^\times$. Associated to q, we have a labeling of the vertices of I, given as follows:

I	$q = 1,\ e = \mathrm{char}\,\mathbb{F}$	$q \neq 1, e =$ multiplicative order of q
$\circ$ —— $\circ$ —— $\cdots$ —— $\circ$	1 —— 2 —— $\cdots$ —— n , $e = 0, e > n$	q —— q^2 —— $\cdots$ —— q^n , $e > n$
$\circ$ $\circ$ —— $\circ$ —— $\cdots$ —— $\circ$	0 1 —— 2 —— $\cdots$ —— $e-1$	1 q —— q^2 —— $\cdots$ —— q^{e-1}
$\cdots$ —— $\circ$ —— $\circ$ —— $\cdots$	$\cdots$ —— n —— $n+1$ —— $\cdots$, $e = 0$	$\cdots$ —— q^n —— q^{n+1} —— $\cdots$, $e = \infty$

These data are subject to the following conditions.

(Cat1) There is a decomposition $E = \bigoplus_{i \in I} E_i$ into generalized eigenfunctors for the action of L. The eigenvalues are supposed to correspond to the labeling of the vertices of I as above. This induces the corresponding decomposition $F = \bigoplus_{i \in I} F_i$, where F_i is right adjoint to E_i. Let $f_i = [F_i], e_i = [E_i]$ be the corresponding operators on the complexified Grothendieck group $[\mathcal{C}]$. We require that $e_i, f_i (i \in I)$ define an integrable $\mathfrak{g}(I)$-action on $[\mathcal{C}]$. Moreover, if $[\mathcal{C}] = \bigoplus_\lambda V_\lambda$ is the weight decomposition of $[\mathcal{C}]$, we require that there exists a category decomposition

$$\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda,$$

where $[\mathcal{C}]_\lambda = V_\lambda$. Note that this implies that the classes of simple objects of $\mathcal{C}$ are weight vectors.

(Cat2) We have the equality $(1_E T) \circ (T1_E) \circ (1_E T) = (T1_E) \circ (1_E T) \circ (T1_E) \in \mathrm{End}(E^3)$.

(Cat3) There are equalities;
 (i) $(T - 1_E) \circ (T + q1_E) = 0$ in $\mathrm{End}(E^2)$.
 (ii) Inside $\mathrm{End}(E^2)$ we have:

$$T \circ (L1_E) \circ T = \begin{cases} q1_E L & \text{if } q \neq 1 \\ 1_E L - T & \text{if } q = 1. \end{cases} \tag{10}$$

Note that (Cat2) is an abstraction of (7), while (Cat3)(i) is an abstraction of (8). The case $q \neq 1$ in (Cat3)(ii) comes from (9), while the case $q = 1$ comes from the definition of degenerate affine Hecke algebras, see e.g. [Kle, Chapters 3 and 4].

Remark 3.1. *In light of the decomposition $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$, it follows similarly to Proposition 2.5 that F_i is also left adjoint to E_i.*

Let us give an important consequence of axioms (Cat2), (Cat3).

Proposition 3.2. *Let $\mathcal{C}$ be endowed with a categorical $\mathfrak{g}(I)$-action. Let $U \in \mathcal{C}$, and $n > 0$. Then, $E^n(U)$ has a natural left $\mathcal{H}_q^{\mathrm{aff}}(n)$-module structure, while $F^n(U)$ has a natural right $\mathcal{H}_q^{\mathrm{aff}}(n)$-module structure.*

Proof. Let us define a map $\mathcal{H}_q^{\mathrm{aff}}(n) \to \mathrm{End}(E^n(U))$. For $i = 1, \dots, n-1$, let $T_i \mapsto 1_{E^{i-1}} T 1_{E^{n-i-1}}$, and, for $j = 1, \dots, n$, map $X_j \mapsto 1_{E^{i-1}} L 1_{E^{n-i}}$. Axioms (Cat2), (Cat3) tells us that this map is actually an algebra morphism. Then, $E^n(U)$ is a left $\mathcal{H}_q^{\mathrm{aff}}(n)$-module. The statement for F follows from an isomorphism $\mathrm{End}(E^n) \xrightarrow{\cong} \mathrm{End}(F^n)^{\mathrm{opp}}$ that is given by adjointness. □

3.2. **Categorical $\mathfrak{sl}_2$-actions.** We single out the definition of a categorical $\mathfrak{sl}_2$-action. Of course, for a categorical $\mathfrak{g}(I)$-action we should expect that each pair of functors (E_i, F_i) gives a categorical $\mathfrak{sl}_2$-action. This is the case. A *categorical* $\mathfrak{sl}_2$*-action* on an abelian category $\mathcal{C}$ is the data of:

- A pair (E, F) of adjoint functors, with F isomorphic to a right adjoint of E, such that the actions of $e = [E]$, $f = [F]$ on $[\mathcal{C}]$ give an integrable $\mathfrak{sl}_2$-action on $[\mathcal{C}]$, where the classes of simple objects in $\mathcal{C}$ are weight vectors.
- Elements $q \in \mathbb{F}^\times, a \in \mathbb{F}$ (with $a \neq 0$ if $q \neq 1$) and morphisms $L \in \mathrm{End}(E), T \in \mathrm{End}(E^2)$ satisfying (Cat2), (Cat3), and such that the morphism $(L - a) \in \mathrm{End}(E)$ is locally nilpotent.

Note that we do not ask for the scalar $a \in \mathbb{F}$ to respect the labelings of the graph I (which, in this case, is a single point) for the definition of a categorical action. This is because the scalar a can be adjusted. Namely, when $q \neq 1$ we can replace L by αL. This changes a into αa. When $q = 1$, we can replace L by $L + \alpha 1_E$. This changes a to $a + \alpha$.

Also, we replaced the weight decomposition $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$ with the requirement that classes of simples are weight vectors. This is, however, an equivalent condition. Fix a categorical $\mathfrak{sl}_2$-action on $\mathcal{C}$. Let $[\mathcal{C}]_\lambda$ be a weight space of $[\mathcal{C}]$. Denote by $\mathcal{C}_\lambda$ the full subcategory of $\mathcal{C}$ consisting of objects whose class is in $[\mathcal{C}]_\lambda$.

Proposition 3.3. *We have* $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$.

Proof. It is enough to show that, if S_1 and S_2 are simple objects of $\mathcal{C}$ such that $[S_1], [S_2]$ belong to different weight spaces, then $\mathrm{Ext}^1(S_1, S_2) = 0$. In order to do so, let us introduce some notation. For a simple object S, $h_+(S) := \max\{i : e[S] \neq 0\}, h_-(S) := \max\{i : f[S] \neq 0\}$. Since S_1, S_2 lie in different weight spaces, there exist $\varepsilon \in \{\pm\}, i, j = \{1, 2\}$ such that $h_\epsilon(S_i) > h_\epsilon(S_j)$. Let us assume that $\epsilon = +$, the other case is totally analogous. Now let M be an object of $\mathcal{C}$ with composition factors S_1, S_2. Then, $E^{h_+(S_i)}(M) \cong E^{h_+(S_i)}(S_i)$. It follows that all composition factors of $F^{h_+(S_i)} E^{h_+(S_i)}(M)$ are in the same weight space as S_i. Now, by biadjunction:

$$\mathrm{Hom}_{\mathcal{C}}(F^{h_+(S_i)} E^{h_+(S_i)} M, M) \cong \mathrm{Hom}_{\mathcal{C}}(E^{h_+(S_i)}(M), E^{h_+(S_i)}(M)) \cong \mathrm{Hom}(M, F^{h_+(S_i)} E^{h_+(S_i)}(M)),$$

since the middle space is nonzero, all these spaces are nonzero. Thus, M has both a subobject and a factor in the same weight space as S_i. It follows that $M = S_i \oplus S_j$. We are done. □

We remark that a categorical $\mathfrak{g}(I)$-action gives $|I|$ categorical $\mathfrak{sl}_2$-actions. Indeed, each pair of functors (E_i, F_i) gives a categorical $\mathfrak{sl}_2$-action. Indeed, it is obvious that L restricts to an endomorphism of E_i. The fact that T restricts to an endomorphism of E_i^2 follows from the relations on the affine Hecke algebra. More specifically, it follows from the relation in $\mathcal{H}_q^{\mathrm{aff}}(2)$:

$$T_1(x_2 - a)^n - (x_1 - a)^n T_1 = \begin{cases} (q-1)x_2 \sum_{j=0}^{n-1} (x_1 - a)^{n-1-j}(x_2 - a)^j & q \neq 1 \\ \sum_{j=0}^{n-1} (x_1 - a)^{n-1-j}(x_2 - a)^j & q = 1 \end{cases}$$

which is just a special case of [Ven, Lemma 2.3]. We leave it as an exercise to check that this implies that T restricts to an endomorphism of E_i^2.

3.3. **Divided powers.** In this subsection, we will see a result that partly explains the need for conditions (Cat2), (Cat3) in the definition of a categorical action.

An important role in the representation theory of the algebra $\mathfrak{g}(I)$ is played by the divided powers, $e_i^{(n)} := e_i^n/n!, f_i^{(n)} := f_i^n/n!$, see e.g. Subsection 4.1. So we would like to have categorified versions of these. Of course, we cannot divide a functor by a natural number: for a functor E, the expression $E/n!$ only makes sense if we can find functor E' with $E \cong n! \cdot E'$. Then we can define $E/n! = E'$.

It turns out that, for the functors E_i, F_i in the definition of a categorical action this is the case. To see this, fix a categorical $\mathfrak{sl}_2$-action on $\mathcal{C}$. Recall that for every $U \in \mathcal{C}$, $E^n(U)$ is a module over the affine Hecke algebra $\mathcal{H}_q^{\mathrm{aff}}(n)$. Even more is true: in the definition of a categorical $\mathfrak{sl}_2$-action we have a scalar $a \in \mathbb{F}$ such that $(L - a)$ is a locally nilpotent endomorphism of E, so $E^n(U)$ actually belongs to a special subcategory of $\mathcal{H}_q^{\mathrm{aff}}(n)$-mod that we have already seen in Siddharth's talk, [Ven, Part C], and that we recall now. Let $\mathfrak{m}_n \subseteq \mathbb{F}[x_1^\pm, \dots, x_n^\pm]$ be the ideal generated by $(x_i - a)$, $i = 1, \dots, n$. Let $\mathfrak{n}_n := \mathfrak{m}_n \cap \mathbb{F}[x_1^\pm, \dots, x_n^\pm]^{\mathfrak{S}_n} \subseteq \mathcal{H}_q^{\mathrm{aff}}(n)$. We remark that $\mathfrak{n}_n$ is contained in the center of $\mathcal{H}_q^{\mathrm{aff}}(n)$.

Then, let $\mathcal{N}_n$ be the category of $\mathcal{H}_q^{\mathrm{aff}}(n)$-modules with locally nilpotent action of $\mathfrak{n}_n$. Clearly, $E^n(U) \in \mathcal{N}_n$.

To procceed further, we need to recall some notation from [Ven, Part C]. Recall that, for the finite Hecke algebra $\mathcal{H}_q^{\mathrm{fin}}(n)$ (that we identify with the subalgebra of $\mathcal{H}_q^{\mathrm{aff}}(n)$ generated by $T_1, \ldots, T_{n-1}$) we have two distinguished one-dimensional characters: the trivial character triv, defined by $\mathrm{triv}(T_i) = q$ for $i = 1, \ldots, n-1$ (so $\mathrm{triv}(T_w) = q^{\ell(w)}$ for $w \in \mathfrak{S}_n$); and the sign character defined by $\mathrm{sign}(T_i) = -1$ for $i = 1, \ldots, n-1$, so $\mathrm{sign}(T_w) = (-1)^{\ell(w)}$ for $w \in \mathfrak{S}_n$. Then, for $\tau \in \{\mathrm{triv}, \mathrm{sign}\}$, we define

$$c_n^\tau := \sum_{w \in \mathfrak{S}_n} q^{-\ell(w)} \tau(T_w) T_w \in \mathcal{H}_q^{\mathrm{fin}}(n),$$

so that $c_n^{\mathrm{triv}} = \sum_{w \in \mathfrak{S}_n} T_w$, while $c_n^{\mathrm{sign}} = \sum_{w \in \mathfrak{S}_n} (-q)^{\ell(w)} T_w$. For a projective left $\mathcal{H}_q^{\mathrm{fin}}(n)$-module M, $c_n^\tau M = \{m \in M : T_w m = \tau(T_w) m \text{ for every } w \in \mathfrak{S}_n\}$.

Since every module $M \in \mathcal{N}_n$ is a projective (actually, free) $\mathcal{H}_q^{\mathrm{fin}}(n)$-module, we have that the canonical map

$$c_n^\tau \mathcal{H}_q^{\mathrm{aff}}(n) \otimes_{\mathcal{H}_q^{\mathrm{aff}}(n)} E^n(U) \longrightarrow c_n^\tau E^n(U)$$

is an isomorphism. It follows from the equivalence of categories proved in [Ven, Part C] that the map

$$\mathcal{H}_q^{\mathrm{aff}}(n) c_n^\tau \otimes_{Z_n} c^\tau E^n(U) \to E^n(U)$$

is an isomorphism, where Z_n denotes the algebra of symmetric Laurent polynomials in $x_1, \ldots, x_n$. In particular, if we denote $E^{(\tau,n)}(U) := c_n^\tau E(U)$ we have that:

$$E^n(U) \cong n! E^{(\tau,n)}(U).$$

By construction, this isomorphism is functorial. We denote by $E^{(n)}$ any of the isomorphic functors $E^{(\mathrm{triv},n)}, E^{(\mathrm{sign},n)}$. Since $E^n \cong n! \cdot E^{(n)}$, this is a categorification of the divided power $e^{(n)}$. We remark that, by biadjointness, we also have divided powers $F^{(n)}$. Moreover, we have a morphism $(\mathcal{H}_q^{\mathrm{aff}}(n))^{\mathrm{opp}} \to \mathrm{End}(F^n)$, and the functors $F^{(n)}$ are constructed similarly to $E^{(n)}$.

4. Categorification of simple reflections.

4.1. Action of the Weyl group. Let V be an integrable $\mathfrak{sl}_2$-module. There exists an action of SL_2 on V lifting the action of $\mathfrak{sl}_2$. In particular, we have an action of the matrix $s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp(-f)\exp(e)\exp(-f)$ on V. This action restricts to isomorphisms $V_{-\lambda} \to V_\lambda$, where λ is a weight of V. It is known that for $v \in V_{-\lambda}$ we have:

$$s(v) = \sum_{r=\max(0,-\lambda)}^{h_-(v)} (-1)^r e^{(\lambda+r)} f^{(r)}(v) \tag{11}$$

We would like to have a categorification of (11). We have already seen how to categorify the divided powers $e^{(\lambda+r)}, f^{(r)}$. The fact that the expression of s is an alternating sum tells us that we should have a complex of functors of the form:

$$\Theta_\lambda = \longrightarrow E^{(\lambda+r)} F^{(r)} \xrightarrow{d^{-r}} E^{(\lambda+r-1)} F^{(r-1)} \longrightarrow$$

This is what we are going to do in this section.

4.2. Rickard's complexes. Take an $\mathfrak{sl}_2$-categorification $\mathcal{C}$, and let $\lambda \in \mathbb{Z}$ be a weight of $[\mathcal{C}]$. We will define a complex of functors

$$\Theta_\lambda : \mathrm{Comp}(\mathcal{C}_{-\lambda}) \to \mathrm{Comp}(\mathcal{C}_\lambda).$$

(here, $\mathrm{Comp}(\mathcal{C}_\lambda)$ is the category of complexes over $\mathcal{C}_\lambda$) categorifying (11). First, let $(\Theta_\lambda)^{-r}$ be the restriction of the functor $E^{(\mathrm{sign},\lambda+r)} F^{(\mathrm{triv},r)}$ (of course, we could have just written $E^{(\lambda+r)} F^{(r)}$, but the choice of characters will be important when doing calculations, see e.g. Lemma 4.1) to $\mathcal{C}_{-\lambda}$. Let $\varepsilon : EF \to \mathrm{id}_\mathcal{C}$ be the (fixed) counit of adjunction. We have the natural transformation:

$$f : E^{\lambda+r} F^r \xrightarrow{\mathbf{1}_{E^{\lambda+r-1}} \varepsilon \mathbf{1}_{F^{r-1}}} E^{\lambda+r-1} F^{r-1}$$

We would like for f to restrict to a natural transformation $E^{(\mathrm{sign},\lambda+r)} F^{(\mathrm{triv},r)} \to E^{(\mathrm{sign},\lambda+r-1)} F^{(\mathrm{triv},r-1)}$. This is indeed the case. To see this, take $M \in \mathcal{C}_\lambda$. Note that we have:

$$
\begin{aligned}
E^{(\mathrm{sign},\lambda+r)}M = &\ c^{\mathrm{sign}}_{\lambda+r}(E^{\lambda+r}M) \\
= &\ c^{\mathrm{sign}}_{[\mathfrak{S}_{\lambda+r}/\mathfrak{S}_{[1,\lambda+r-1]}]}c^{\mathrm{sign}}_{[1,\lambda+r-1]}(E^{\lambda+r}M) \\
\subseteq &\ c^{\mathrm{sign}}_{[1,\lambda+r-1]}(E^{\lambda+r}M) \\
= &\ E^{(\mathrm{sign},\lambda+r-1)}EM
\end{aligned} \tag{12}
$$

so that $E^{(\mathrm{sign},\lambda+r)} \subseteq E^{(\mathrm{sign},\lambda+r-1)}E$. Similarly, $F^{(\mathrm{triv},r)} \subseteq FF^{(\mathrm{triv},r-1)}$. Thus, f restricts to a map $d^{-r} : (\Theta_\lambda)^{-r} \to (\Theta_\lambda)^{-r+1}$. Set

$$\Theta_\lambda := \quad \longrightarrow (\Theta_\lambda)^{-r} \xrightarrow{d^{-r}} (\Theta_\lambda)^{-r+1} \longrightarrow$$

Lemma 4.1. *Θ_λ is a complex.*

Proof. We have to check that $d^{-r+1}d^{-r} = 0$. By its definition, $d^{-r+1}d^{-r}$ is the restriction to $E^{(\mathrm{sign},\lambda+r)}F^{(\mathrm{triv},r)}$ of the map

$$E^{\lambda+r}F^r \xrightarrow{\mathbf{1}_{E^{\lambda+r-2}}\,\varepsilon_2\,\mathbf{1}_{F^{r-2}}} E^{\lambda+r-2}F^{r-2}$$

where ε_2 is the composition $EEFF \xrightarrow{\mathbf{1}_E\,\varepsilon\,\mathbf{1}_F} EF \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{C}}$. Now, similarly to (12) we can show that $E^{(\mathrm{sign},\lambda+r)}F^{(\mathrm{triv},r)} \subseteq E^{\lambda+r-2}E^{(\mathrm{sign},2)}F^{(\mathrm{triv},2)}F^{\lambda+r-2}$. Thus, to check that $d^{-r+1}d^{-r} = 0$, it is enough to check that ε_2 vanishes on $E^{(\mathrm{sign},2)}F^{(\mathrm{triv},2)} \subseteq E^2F^2$. In other words, we need to check that the composition

$$E^2F^2 \xrightarrow{c_2^{\mathrm{sign}}c_2^{\mathrm{triv}}} E^2F^2 \xrightarrow{\varepsilon_2} \mathrm{id}_{\mathcal{C}}$$

vanishes, where in the first arrow c_2^{sign} acts on E^2, while c_2^{triv} acts on F^2. Recall, however, that the action of $\mathcal{H}_q^{\mathrm{aff}}(2)$ on F^2 is given via an isomorphism $\mathrm{End}(E^2) \xrightarrow{\simeq} \mathrm{End}(F^2)^{\mathrm{opp}}$ that is constructed using the adjunction between E^2 and F^2. Note that $\varepsilon_2 : E^2F^2 \longrightarrow \mathrm{id}_{\mathcal{C}}$ is the counit of adjunction. It follows that the diagram

$$\begin{tikzcd}
 & E^2F^2 \arrow[dr, "\varepsilon_2"] & \\
E^2F^2 \arrow[ur, "c_2^{\mathrm{sign}}c_2^{\mathrm{triv}}"] \arrow[dr, "(c_2^{\mathrm{sign}}c_2^{\mathrm{triv}})\,\mathbf{1}_{F^2}"'] & & \mathrm{id}_{\mathcal{C}} \\
 & E^2F^2 \arrow[ur, "\varepsilon_2"'] &
\end{tikzcd}$$

commutes. Since $c_2^{\mathrm{sign}}c_2^{\mathrm{triv}} = 0$, we have that $d^{-r+1}d^{-r} = 0$. Thus, Θ_λ is a complex. $\square$

We are going to see the following.

Theorem 4.2. *The complex Θ_λ gives an equivalence $D^b(\mathcal{C}_{-\lambda}) \to D^b(\mathcal{C}_\lambda)$.*

The strategy to prove Theorem 4.2 is roughly as follows. First, in Subsection 4.3 we will prove some general vanishing results on $D^b(\mathcal{C})$, where, remember, $\mathcal{C}$ carries a categorical $\mathfrak{sl}_2$-action. After that, Subsection 4.4 we are going to introduce so-called 'minimal' categorifications. The strategy to show that Θ_λ is an equivalence is to reduce the claim to these minimal categorifications, where calculations can be carried out by hand. We show this reduction in Subsection 4.5.

4.3. **General results.** We show a couple of vanishing results in the derived category. These will be helpful when showing that Θ_λ provides a derived equivalence.

Lemma 4.3. *Let $\mathcal{C}$ be a category with a categorical $\mathfrak{sl}_2$-action. Let $C \in D^b(\mathcal{C})$ be such that $\mathrm{Hom}_{D^b(\mathcal{C})}(E^iT, C[j]) = 0$ for all $j \in \mathbb{Z}$, $i \geq 0$, and $T \in \mathcal{C}$ a simple object with $FT = 0$. Then, $C = 0$.*

Proof. Let $C \in D^b(\mathcal{C})$ and assume $C \neq 0$. Let n be minimal with $H^n(C) \neq 0$, and let S be a simple subobject of $H^n(C)$. Define $i := h_-(S) = \max\{j : F^jS\} \neq 0$, and let T be a simple subobject of $F^i(S)$. Clearly, $FT = 0$. Moreover, by adjunction $\mathrm{Hom}_{\mathcal{C}}(E^iT, S) = \mathrm{Hom}_{\mathcal{C}}(T, F^iS) \neq 0$, so $\mathrm{Hom}_{D^b(\mathcal{C})}(E^iT, C[n]) \neq 0$. We are done. $\square$

Lemma 4.4. *In the setting of the previous lemma, let $\mathcal{C}'$ be another category, and $G = (G^i)$ be a complex of exact functors $\mathcal{C} \to \mathcal{C}'$ with exact right adjoints, $G^{i,\vee}$. Assume, moreover, that for every $M \in \mathcal{C}$ (resp. $M' \in \mathcal{C}'$) $G^i(M) = 0$ (resp. $G^{i,\vee}(M') = 0$) for large enough i.*
Assume that $G(E^iT)$ is acyclic for every $i \geq 0$ and every simple object $T \in \mathcal{C}$ such that $FT = 0$. Then, $G(C)$ is acyclic for every $C \in \mathrm{Comp}(\mathcal{C})$.

Proof. Since every functor G^i has a right adjoint, there exists a right adjoint $G^\vee$ to G. Then, for every $C, D \in D^b(\mathcal{C})$:

$$\mathrm{Hom}_{D^b(\mathcal{C})}(C, G^\vee G(D)) \cong \mathrm{Hom}_{D^b(\mathcal{C}')}(GC, GD),$$

so by our assumptions it follows that $\mathrm{Hom}_{D^b(\mathcal{C})}(E^iT, G^\vee GD) = 0$ for every $D \in D^b(\mathcal{C})$, every $i \geq 0$ and every simple object $T \in \mathcal{C}$ with $FT = 0$. By the previous lemma, $\mathrm{Hom}_{D^b(\mathcal{C})}(C, G^\vee GD) = 0$ for all $C \in D^b(\mathcal{C})$. Thus, for all C, $\mathrm{Hom}_{D^b(\mathcal{C}')}(GC, GC) = 0$. We are done. □

4.4. **Minimal categorifications.** For $n \geq 0$, let V_n be the irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$. Thus, the weights of V_n are $-n, -n+2, \ldots, n-2, n$, and each weight space is one-dimensional. Here, we will introduce a category $\mathcal{A}(n)$ with a categorical action of $\mathfrak{sl}_2$ satisfying $[\mathcal{A}(n)] \cong V_n$.

Fix $q \in \mathbb{F}^\times, a \in \mathbb{F}$ with $a \neq 0$ if $q \neq 1$. Recall the definition of $\mathfrak{n}_n$ from Subsection 3.3. Since $\mathfrak{n}_n$ is contained in the center of $\mathcal{H}(n)^{\mathrm{aff}}_q$, the quotient $\overline{\mathcal{H}(n)} = \mathcal{H}(n)^{\mathrm{aff}}_q/\mathfrak{n}_n\mathcal{H}(n)^{\mathrm{aff}}_q$ is an algebra. For $0 \leq i \leq n$, let $B_{i,n}$ be the image of the subalgebra $\mathcal{H}(i)^{\mathrm{aff}}_q$ inside $\overline{\mathcal{H}(n)}$. Clearly, we have $B_{i,n} \subseteq B_{i+1,n}$. Moreover, similarly to [Ven, Proposition 10.17], we have an isomorphism

$$B_{i+1,n} = B_{i,n} \otimes \bigoplus_{w \in [\mathfrak{S}_i \backslash \mathfrak{S}_{i+1}], 0 \leq a \leq n-i-1} \mathbb{F}x_{i+1}^a T_w$$

so that, in particular $B_{i+1,n}$ is a free $B_{i,n}$-module of rank $(i+1)(n-i)$. We remark that the algebras $B_{i,n}$ are symmetric, cf. [CR, 3.3.2].

Now, for a weight λ of V_n, define $(\mathcal{A}(n))_\lambda := B_{(\lambda+n)/2,n}$-mod, and let $\mathcal{A}(n) = \bigoplus_{\lambda \text{ weight of } V_n} (\mathcal{A}(n))_\lambda = \bigoplus_{i=0}^n B_{i,n}$-mod. Recall, [Ven, Section 6], that Siddharth has proved that the algebras $B_{i,n}$ admit a unique irreducible module, so that $[(\mathcal{A}(n))_\lambda]$ is 1-dimensional. Now let $F = \bigoplus \mathrm{Res}^{n+1}_n, E = \bigoplus \mathrm{Ind}^{n+1}_n$. These functors are biadjoint since the algebras $B_{i,n}$ are symmetric. Moreover, we have

$$EF(B_{i,n}) = B_{i,n} \otimes_{B_{i-1,n}} B_{i,n} \cong i(n-i+1)B_{i,n}, \qquad FE(B_{i,n}) \cong B_{i+1,n} = (i+1)(n-i)B_{i,n},$$

it follows that $(ef - fe)[B_{i,n}] = (2i-n)[B_i]$. So $[\mathcal{A}(n)] \cong V_n$ as representations of $\mathfrak{sl}_2$. Similarly to the case of cyclotomic Hecke algebras, let $T \in \mathrm{End}(F^2)$ be multiplication by (the image inside $B_{i,n}$ of) T_{i-1}, and $L \in \mathrm{End}(F)$ be multiplication by (the image inside $B_{i,n}$ of) X_i. It is now easy to see that these data define an $\mathfrak{sl}_2$-categorification. We call it the *minimal categorification* of V_n.

Let us see in what sense the categorifications $\mathcal{A}(n)$ are minimal. We have been using throughout this text that for every $M \in \mathcal{C}$, there is an affine Hecke algebra action on $E^n(M)$. It turns out that, when M is a simple object we can do better. The following result shows that the algebras $B_{i,n}$ appear very naturally in the context of categorifications. It is a part of [CR, Proposition 5.20].

Proposition 4.5. *Let S be a simple object of $\mathcal{C}$. Let $n := h_+(S) = \max\{j : E^jS \neq 0\}$. Then, for every $i \leq n$, the map $\mathcal{H}^{\mathrm{aff}}_q(i) \longrightarrow \mathrm{End}(E^iS)$ induces an isomorphism $B_{i,n} \xrightarrow{\cong} \mathrm{End}_{\mathcal{C}}(E^iS)$.*

We state without proof the following result.

Lemma 4.6 (Lemma 5.23, [CR])**.** *Let $S \in \mathcal{C}$ be simple, and assume that $FS = 0$. Let $n = h_+(S)$. Let $i < n$, and denote $B_i := B_{i,n}$. The following composition:*

$$E^iS \otimes_{B_i} B_{i+1} \xrightarrow{\eta_{E^iS} \otimes 1} FE^{i+1}S \otimes_{B_i} B_{i+1} \longrightarrow FE^{i+1}S$$

is an isomorphism, where the last arrow stands for the action map, and $\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow FE$ is the unit of adjunction.

Let us see the consequences of Lemma 4.6. We use the notation of that result. By Proposition 4.5 (iii), for each $i = 0, \ldots, n$, we have a fully faithful functor

$$\begin{gathered} B_i\text{-mod} \to \mathcal{C} \\ M \mapsto E^iS \otimes_{B_i} M \end{gathered}$$

Note that we have canonical isomorphisms of functors $E(E^iS \otimes_{B_i} \bullet) \xrightarrow{\cong} E^{i+1}S \otimes_{B_i} \bullet \xrightarrow{\cong} E^{i+1}S \otimes_{B_{i+1}} B_{i+1} \otimes_{B_i} \bullet$ that make the following diagram commute:

$$\begin{tikzcd}
B_i\text{-mod} \arrow[r, "E^iS\otimes_{B_i}\bullet"] \arrow[d, "\mathrm{Ind}_i^{i+1}"'] & \mathcal{C} \arrow[d, "E"] \\
B_{i+1}\text{-mod} \arrow[r, "E^{i+1}S\otimes_{B_{i+1}}\bullet"] & \mathcal{C}
\end{tikzcd}$$

On the other hand, the isomorphism given in Lemma 4.6 makes the following diagram commute.

$$\begin{tikzcd}
B_i\text{-mod} \arrow[r, "E^iS\otimes_{B_i}\bullet"] & \mathcal{C} \\
B_{i+1}\text{-mod} \arrow[r, "E^{i+1}S\otimes_{B_{i+1}}\bullet"] \arrow[u, "\mathrm{Res}_i^{i+1}"] & \mathcal{C} \arrow[u, "F"']
\end{tikzcd}$$

In other words, we have the following important result.

Theorem 4.7. *The construction given above gives a fully faithful functor $R_S : \mathcal{A}(n) \to \mathcal{C}$. This functor is a morphism of $\mathfrak{sl}_2$-categorifications.*

4.5. **Reduction to minimal categorifications.** We use the results from Subsections 4.3, 4.4 to show that to prove Theorem 4.2 in general it is enough to prove it for the minimal categorifications $\mathcal{A}(n)$.

First of all, since every term in the complex Θ_λ admits a right adjoint, the complex of functors Θ_λ admits a right adjoint, $\Theta_\lambda^\vee$. Let $\varepsilon : \Theta_\lambda\Theta_\lambda^\vee \longrightarrow \mathrm{id}$ be the counit of adjunction. We must show that this is, actually, an equivalence. To do this, let Z be the cone of ε. To prove Theorem 4.2, we must show that Z is acyclic. To do this it is enough, by Lemma 4.4, to check that $Z(E^iS)$ is acyclic whenever $S \in \mathcal{C}$ is a simple object with $FS = 0$. Let $n = h_+(S)$, and consider the fully faithful functor $R_S : \mathcal{A}(n) \to \mathcal{C}$. The functor R_S commutes with $\Theta_\lambda, \Theta_\lambda^\vee$ and Z since it is a morphism of categorifications. Then, assuming the result valid for the minimal categorifications, we have $Z(E^iS) = Z(R_SB_i) = R_S(Z(B_i)) = 0$. So ε is an equivalence. Similarly, we can check that the unit of adjunction $\eta : \mathrm{id} \to \Theta_\lambda^\vee\Theta_\lambda$ is an equivalence. Then, we have an equivalence $D^b(\mathcal{C}_{-\lambda}) \xrightarrow{\cong} D^b(\mathcal{C}_\lambda)$. The proof of the equivalence $K^b(\mathcal{C}_{-\lambda}) \xrightarrow{\cong} K^b(\mathcal{C}_\lambda)$ is similar.

The proof of Theorem 4.2 for minimal categorifications $\mathcal{A}(n)$ can be carried out by hand. This is done in [CR, 6.3]. We remark that the key step here is to show that in this case, the homology of Θ_λ is concentrated in degree ℓ, where $\ell = \frac{n-\lambda}{2}$. After knowing this, the argument goes as follows: the complex Θ_λ is given by a tensor product by a complex of $(B_{n-\ell}, B_\ell)$-bimodules that are projective as $B_{n-\ell}$- and as B_ℓ-modules. The homology of the complex is then given by tensoring with a bimodule M that is projective on both sides (because it is concentrated in the lowest degree at which the complex is nonzero). The result then follows by recalling that the functor categorifies the simple reflection s (so that it induces an isomorphism at the level of Grothendieck groups), and Morita theory. Note that here we obtain an abelian, rather than derived, equivalence.

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