# Lecture 21. Modified 11/19

1) Properties of left/right exact functors.
2) Localization vs base change.
3) Projective & flat modules.

Refs: [AM], Sections 2.9 & intro to 3; [E], A.3.2, 6.1, 6.3.

BONUSES: 1) 5-lemma
2) Injective modules.

## 1) Properties of left/right exact functors.

**Lemma:** Let $F: A\text{-Mod} \to B\text{-Mod}$ be left exact additive functor. Then

(a) $F$ sends injections to injections.

(b) $F$ sends every exact sequence $0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3$ to an exact sequence $0 \to F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3)$

(c) $F$ is exact $\Leftrightarrow$ $F$ sends surjections to surjections.

Proof: (a) $N \overset{\varphi_1}{\hookrightarrow} M$ can be included into SES

$$0 \to N \xrightarrow{\varphi_1} M \longrightarrow M' \to 0, \quad M' := M/\operatorname{im}\varphi_1.$$

$$\Big\downarrow F$$

$0 \to F(N) \xrightarrow{F(\varphi_1)} F(M) \longrightarrow F(M')$ -exact $\Rightarrow F(\varphi_1)$ is injective.

(b): $M_3' := \operatorname{im}\varphi_2 \subset M_3$: $\varphi_2' := \varphi_2$ viewed as a map to its image
$\iota: M_3' \hookrightarrow M_3$ – inclusion, so $\varphi_2 = \iota\circ\varphi_2'$.

$0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2'} M_3' \to 0$ is exact $\Rightarrow$

$$0 \to F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2')} F(M_3') \qquad (*)$$

is exact. Further, $\iota$ is injective $\Rightarrow$ [by (a)] $F(\iota)$ is injective
$F$ is a functor $\Rightarrow F(\varphi_2) = F(\iota)\circ F(\varphi_2')$. So $\ker F(\varphi_2) = \ker F(\varphi_2')$.
By this and $(*)$, $0 \to F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3)$ is exact.
(c) is exercise. □

Rem: There are direct analogs of this lemma for all other types of partial exactness. E.g. left exact functor
$F: A\text{-Mod}^{opp} \longrightarrow B\text{-Mod}$ sends $\forall$ exact sequence

$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ to exact sequence

$0 \to F(M_3) \longrightarrow F(M_2) \longrightarrow F(M_1)$ (exercise).

## 2) Localization vs base change.

A philosophical application of exactness: it helps to compute what a functor does to an object: Prob 7 in HW1 "computing" $\operatorname{Hom}_A(M,N)$ based on left exactness of Hom & our construction of $\otimes_A$ based on right exactness. Here's another, ideologically similar, application.

Let $A$ be a commutative ring, $S\subset A$ a multiplicative subset $\rightsquigarrow$ $A$-algebra $A[S^{-1}]$.

Thm: functors $\bullet[S^{-1}]$, $A[S^{-1}]\otimes_A \bullet : A\text{-Mod} \to A[S^{-1}]\text{-Mod}$ are isomorphic.

Proof: Step 1: construct $\eta: A[S^{-1}]\otimes_A \bullet \Rightarrow \bullet[S^{-1}]$.
Let $M$ be $A$-module, we need $A[S^{-1}]$-linear map
$\eta_M: A[S^{-1}]\otimes_A M \longrightarrow M[S^{-1}]$. Consider the map
$A[S^{-1}]\times M \longrightarrow M[S^{-1}]$, $(\frac{a}{s}, m) \mapsto \frac{am}{s}$.

Exercise: $\cdot$ This map is $A[S^{-1}]$-linear in the 1st argument & $A$-linear in the 2nd argument. So the proof of Thm in Sec 1 of Lec 20 gives $A[S^{-1}]$-linear map $\eta_M: A[S^{-1}]\otimes_A M \to M[S^{-1}]$, $\frac{a}{s}\otimes m \mapsto \frac{am}{s}$.

$\cdot$ The maps $\eta_M$ constitute a functor morphism.

Step 2: We claim $\eta_{A^{\oplus I}}$ is an isomorphism $\forall$ set $I$. By the construction of tensor products w. free modules, we have $A[S^{-1}]\otimes_A A^{\oplus I} \xrightarrow{\sim} A[S^{-1}]^{\oplus I}$, $\frac{a}{s}\otimes(a_i) \mapsto (\frac{aa_i}{s})$. Under this identification, $\eta_{A^{\oplus I}}$ becomes $A[S^{-1}]^{\oplus I} \to (A^{\oplus I})[S^{-1}]$, $(\frac{b_i}{s}) \mapsto \frac{(b_i)}{s}$. This is an isomorphism by Prob 7 in HW3.

Step 3: here from Step 2 & exactness we deduce that $\eta_M$ is an isomorphism for all $M$. We can find surjective $A$-linear maps $A^{\oplus I} \twoheadrightarrow M$ & $A^{\oplus J} \twoheadrightarrow \ker[A^{\oplus I} \twoheadrightarrow M]$ for some sets $I, J$ leading to an exact sequence $A^{\oplus J} \xrightarrow{\varphi_1} A^{\oplus I} \xrightarrow{\varphi_2} M \longrightarrow 0$. Apply the functors $F_1 := A[S^{-1}]\otimes_A \bullet$, $F_2 := \bullet[S^{-1}]: A\text{-Mod} \to A[S^{-1}]\text{-Mod}$ to get the following diagram that is commutative b/c $\eta$ is a functor morphism:

$$\begin{tikzcd}
F_1(A^{\oplus J}) \arrow[r, "F_1(\varphi_1)"] \arrow[d, "\eta_{A^{\oplus J}}", "\wr"'] & F_1(A^{\oplus I}) \arrow[r, "F_1(\varphi_2)"] \arrow[d, "\eta_{A^{\oplus I}}", "\wr"'] & F_1(M) \arrow[r] \arrow[d, "\eta_M"] & 0 \\
F_2(A^{\oplus J}) \arrow[r, "F_2(\varphi_1)"] & F_2(A^{\oplus I}) \arrow[r, "F_2(\varphi_2)"] & F_2(M) \arrow[r] & 0
\end{tikzcd}$$

The rows are exact b/c both $F_1, F_2$ are right exact. Since the 1st & 2nd vertical maps are isomorphisms, we get commutative diagram

$$\begin{tikzcd}
F_1(A^{\oplus I})/\operatorname{im} F_1(\varphi_1) \arrow[r, "\sim"] \arrow[d, "\wr"] & F_1(M) \arrow[d, "\eta_M"] \\
F_2(A^{\oplus I})/\operatorname{im} F_2(\varphi_1) \arrow[r, "\sim"] & F_2(M)
\end{tikzcd}$$

which shows that $\eta_M$ is an isomorphism. $\square$

**Rem:** In fact, to prove $\eta_M$ is an isomorphism one only needs to know that in the above diagram $\eta_{A^{\oplus J}}$ is surjective (exercise). This is a special case of the so called 5-lemma, one of the bonuses for this lecture.

## 3) Projective & flat modules.

### 3.1) Projective modules.

Let $P$ be an $A$-module. We know that the functor $\operatorname{Hom}_A(P,\bullet): A\text{-Mod} \to A\text{-Mod}$ is left exact, Sec 3.3 of Lec 20.

**Def'n:** $P$ is **projective** if $\operatorname{Hom}_A(P,\bullet)$ is exact, equivalently, by

(c) of Lemma in Sec 1, sends surjections to surjections.

**Example:** $P = A^{\oplus I}$ is projective. Indeed, $\mathrm{Hom}_A(A^{\oplus I}, \cdot) \xRightarrow{\sim} \cdot^{\times I}$: $\mathrm{Hom}_A(A^{\oplus I}, M) \xrightarrow{\sim} M^{\times I}$ via $\psi \mapsto (\psi(e_i))_{i \in I}$.

In particular, for $\varphi: M \twoheadrightarrow N$, we have commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(A^{\oplus I}, M) & \xrightarrow{\varphi \circ ?} & \mathrm{Hom}_A(A^{\oplus I}, N) \\ \downarrow \wr & & \downarrow \wr \\ M^{\times I} & \xrightarrow{\varphi^{\times I}} & N^{\times I} \end{array}$$

Since $\varphi$ is surjective so is $\varphi^{\times I}$. Hence the top arrow is surjective as well finishing the proof.

**Thm:** TFAE

(1) $P$ is projective.

(2) $\forall$ $A$-linear surjection $\pi: M \twoheadrightarrow P$ $\exists$ $A$-linear $\iota: P \to M$ s.t. $\pi \circ \iota = id_P$ (say, $\pi$ "splits")

(3) $\exists$ $A$-module $P'$ s.t. $P \oplus P'$ is a free module.

Proof:

$(1) \Rightarrow (2)$: $\mathrm{Hom}_A(P, M) \xrightarrow{\pi \circ ?} \mathrm{Hom}_A(P,P)$ is surjective $\Longrightarrow$ $\exists$ $\iota \in \mathrm{Hom}_A(P, M)$ s.t. $\pi \circ \iota = id_P$, which is (2).

$(2) \Rightarrow (3)$: Pick $\iota: P \to M$ w. $\pi \circ \iota = id_P \Rightarrow \iota$ is injective so $P \simeq \mathrm{im}\, \iota$.

Exercise: $\pi \circ \iota = id_P \Rightarrow M = \ker \pi \oplus \operatorname{im} \iota$ $\left(m = (m - \iota(\pi(m)) + \iota(\pi(m))\right)$

We apply this to $\pi: M := A^{\oplus I} \twoheadrightarrow P$ to get (3) w. $P' = \ker \pi$.

$(3) \Rightarrow (1)$:

Exercise: let $M, M'$ be $A$-modules. TFAE:

(a) $\operatorname{Hom}_A(M \oplus M', \bullet)$ is exact

(b) $\operatorname{Hom}_A(M, \bullet)$, $\operatorname{Hom}_A(M', \bullet)$ are both exact.

Hint: compare to Example above.

Know $P \oplus P' \simeq A^{\oplus I} \Rightarrow \operatorname{Hom}_A(P \oplus P', \bullet)$ is exact (by Example above)
By (a) $\Rightarrow$ (b) of Exercise, $\operatorname{Hom}_A(P, \bullet)$ is exact, which is (1). $\square$

Rem: 1) There's a geometric reason to care about projective modules: they are algebro-geometric incarnation of vector bundles – an object of paramount importance in all geometric contexts.

2) One can ask whether every (say, finitely generated) projective $A$-module is actually free. This is true when $A$ is a PID (exercise). An interesting and important result in this direction: if $\mathbb{F}$ is a field, then any finitely generated projective $\mathbb{F}[x_1, \dots, x_n]$-module is free. This was conjectured by Serre and proved by Quillen & Suslin (both in 1976).

## 3.2) Flat modules

**Definition:** An $A$-module $F$ is *flat* if $F\otimes_A \bullet : A\text{-Mod} \longrightarrow A\text{-Mod}$ is exact ($\Leftrightarrow$ sends injections to injections)

**Examples:**

(I) $A^{\oplus I}$ is flat (a complete analog of Example in Sect. 3.1 b/c $A^{\oplus I}\otimes_A \bullet \xrightarrow{\sim} \bullet^{\oplus I}$).

(II) Projective $\Rightarrow$ flat: use (I) and argue as in the proof of (3) $\Rightarrow$ (1) of the theorem. Conversely, it's known that any finitely presented flat module is projective, we won't prove this.

(III) For a multiplicative subset $S$, $A[S^{-1}]$ is a flat $A$-module. Indeed, by Sec 3.3 of Lec 20, $\bullet[S^{-1}]$ is an exact functor, and by Section 2 of this lecture, $A[S^{-1}]\otimes_A \bullet \xrightarrow{\sim} \bullet[S^{-1}]$

(IV*) Let $I$ be an ideal in $A$. Consider the completion $\hat{A} = \varprojlim A/I^k$ (Prob 3 in HW1). If $A$ is Noetherian, then $\hat{A}$ is a flat $A$-module.

## BONUS 1: 5-lemma.

This important result sometimes allows to check whether a module homomorphism is an isomorphism.

**Theorem:** Suppose we have a commutative diagram of $A$-modules & their homomorphisms:

$$\begin{tikzcd}
M_1 \arrow[r, "\tau_1"] \arrow[d, "\varphi_1"] & M_2 \arrow[r, "\tau_2"] \arrow[d, "\varphi_2", "\wr"'] & M_3 \arrow[r, "\tau_3"] \arrow[d, "\varphi_3"] & M_4 \arrow[r, "\tau_4"] \arrow[d, "\varphi_4", "\wr"'] & M_5 \arrow[d, hook, "\varphi_5"] \\
M_1' \arrow[r, "\tau_1'"] & M_2' \arrow[r, "\tau_2'"] & M_3' \arrow[r, "\tau_3'"] & M_4' \arrow[r, "\tau_4'"] & M_5'
\end{tikzcd}$$

Assume both rows are exact, $\varphi_2, \varphi_4$ are isomorphisms, $\varphi_1$ is surjective and $\varphi_5$ is injective. Then $\varphi_3$ is an isomorphism.

*Proof:* Let's prove $\varphi_3$ is surjective. The proof is by "diagram chase." Pick $m_3' \in M_3'$. We want to show $m_3' \in \operatorname{im} \varphi_3$.

Set $m_4' := \tau_3'(m_3')$, $m_4 := \varphi_4^{-1}(m_4')$, $m_5 := \tau_4(m_4)$. Note that, since bottom row is exact, $\tau_4'(m_4') = \tau_4'(\tau_3'(m_3)) = 0$. Since the rightmost square is commutative, $\varphi_5(m_5) = \varphi_5(\tau_4(m_4)) = \tau_4'(\varphi_4(m_4)) = \tau_4'(m_4') = 0$. Since $\varphi_5$ is injective, $m_5 = 0$. So $m_4 \in \ker \tau_4 = \operatorname{im} \tau_3$.

Pick $m_3 \in M_3$ w. $\tau_3(m_3) = m_4$. Since the 2nd square from the right is commutative, $m_4' = \varphi_4(\tau_3(m_3)) = \tau_3'(\varphi_3(m_3))$. So $\tau_3'(m_3' - \varphi_3(m_3)) = 0$ $\Rightarrow m_3' - \varphi_3(m_3) \in \ker \tau_3' = \operatorname{im} \tau_2'$. Take $m_2' \in M_2'$ w. $\tau_2'(m_2') = m_3' - \varphi_3(m_3)$.
Set $m_2 = \varphi_2^{-1}(m_2')$. Since the 2nd square from the left is commutative, we get $\varphi_3(\tau_2(m_2)) = \tau_2'(\varphi_2(m_2)) = \tau_2'(m_2') = m_3' - \varphi_3(m_3)$.

So $m_3' = \varphi_3(\tau_2(m_2)) + \varphi_3(m_3) \in im\, \varphi_3$.

The proof that $\varphi_3$ is injective is similar and is left as an exercise $\square$

BONUS 2: injective modules.

Let $A$ be a (comm've unital) ring.

Definition: An $A$-module $I$ is injective if $Hom_A(\cdot, I)$: $A\text{-}Mod^{opp} \longrightarrow A\text{-}Mod$ is exact (equivalently, for an inclusion $N \hookrightarrow M$ the induced homomorphism

$$Hom_A(I, M) \longrightarrow Hom_A(I, N) \text{ is surjective}).$$

The definition looks very similar to that of projective modules, however the properties of injective & projective modules are very different! Projective modules – especially finitely generated ones – are nice, but injective modules are quite ugly, they are almost never finitely generated.

The simplest ring is $\mathbb{Z}$. Let's see what being injective means for $\mathbb{Z}$.

Definition: An abelian group $M$ is divisible if $\forall\, m \in M, a \in \mathbb{Z}$ $\exists\, m' \in M$ s.t $am' = m$.

Example: The abelian group $\mathbb{Q}$ is divisible. So is $\mathbb{Q}/\mathbb{Z}$.

Proposition 1: For an abelian group $M$ TFAE:

(a) $M$ is injective

(b) $M$ is divisible

Sketch of proof: (a) $\Rightarrow$ (b): apply

$$N \hookrightarrow M \Rightarrow \mathrm{Hom}_A(I,M) \twoheadrightarrow \mathrm{Hom}_A(I,N) \qquad (*)$$

to $M = \mathbb{Z}$, $N = a\mathbb{Z}$.

(b) $\Rightarrow$ (a) is more subtle. The first step is to show that if $(*)$ holds for $N \subset M$, then it holds for $N + \mathbb{Z}m \subset M$ $\forall\, m \in M$. So $(*)$ holds for all fin. gen'd submodules $N \subset M$. Then a clever use of transfinite induction yields $(*)$ for all submodules of $M$. $\square$

We can get examples of injective modules for more general rings as follows. Note that for an abelian group $M$, the group $\mathrm{Hom}_{\mathbb{Z}}(A,M)$ is an $A$-module

Proposition 2: If $M$ is injective as an abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A,M)$ is an injective $A$-module.

Finally, using this proposition one can show that every $A$-module embeds into an injective one (the corresponding statement for projectives – that every module admits a surjection from a projective module – is easy b/c every free module is proj've).