MAT 380, HOMEWORK 1, DUE SEPT 17

There are 10 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20.

All rings are assumed to be commutative and contain 1.

Problem 1, 2pts total. Let A be a ring. Let $e \in A$ satisfy $e^2 = e$.

a, 1pt) Show that both $Ae := \{ae|a \in A\}$ is closed under multiplication and that e is the unit element in Ae (*here, "the unit element"="the identity element"="the element* 1*", and* <u>*not*</u> *the same as "an invertible element"*). Show the analogous statements for $A(1-e)$ and $1-e$.

b, 1pt) Prove that the map $a \mapsto (ae, a(1-e))$ is a ring isomorphism $A \to Ae \oplus A(1-e)$.

Problem 2, 4pts total. Consider the ring $A = \mathbb{Z}[\sqrt{-5}](:= \mathbb{Z}[x]/(x^2+5))$, its elements can be thought as expressions $a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$ and added and multiplied accordingly. Consider the following ideals $I_1 := (2), I_2 := (3), I_3 := (1+\sqrt{-5}), I_4 := (1-\sqrt{-5}), I_{13} := I_1 + I_3, I_{23} := I_2 + I_3, I_{14} = I_1 + I_4, I_{24} := I_2 + I_4$.

1, 1pt) Describe the quotients of A by the ideals $I_{13}, I_{23}, I_{14}, I_{24}$.

2, 1pt) Describe the quotients of A by the ideals I_1, I_2, I_3, I_4.

(*In almost all cases of these two parts your answer should be the direct sum of residue rings, i.e., the rings of the form* $\mathbb{Z}/n\mathbb{Z}$*, while in one case you should get a quotient ring of the ring of polynomials with coefficients in* $\mathbb{Z}/2\mathbb{Z}$).

3, 1pt) Which of these eight ideals are maximal, prime, radical (i.e., coincide with their radicals)?

4, 1pt) Prove that $I_1 = I_{13}I_{14}, I_2 = I_{23}I_{24}, I_3 = I_{13}I_{23}, I_4 = I_{14}I_{24}$.

Problem 3, 2pts. Let n be a positive integer and $n = p_1^{k_1} \ldots p_\ell^{k_\ell}$ be its factorization into primes, where $p_1, \ldots, p_k$ are pairwise different primes and $k_1, \ldots, k_\ell \in \mathbb{Z}_{>0}$. Compute $\sqrt{(n)}$, the radical of the ideal (n).

Problem 4, 4pts total. Let $\varphi : A \to B$ be a ring homomorphism and let J be an ideal in B. Set $I := \varphi^{-1}(J)$, this is an ideal in A.

1, 1pt) Let $J = \sqrt{J}$. Is it always true that $I = \sqrt{I}$?

2, 1pt) Let J be maximal. Is it always true that I is maximal?

3, 1pt) Is it always true that $B\varphi(I) \subset J$?

4, 1pt) Is it always true that $J \subset B\varphi(I)$?

If you think a statement is true, provide a proof. If you think it is false, provide a counterexample.

Problem 5, 4pts total. The goal of this problem is to describe the maximal ideals in $\mathbb{C}[x_1,\ldots,x_n]$: they are in bijection with $\mathbb{C}^n$. First, we observe that a point $(a_1,\ldots,a_n)\in\mathbb{C}^n$ indeed defines a maximal ideal in $\mathbb{C}[x_1,\ldots,x_n]$, namely,

$$\{f(x_1,\ldots,x_n)\in\mathbb{C}[x_1,\ldots,x_n]|f(a_1,\ldots,a_n)=0\}.$$

The claims below imply that there are no other maximal ideals.

1, 1pt) Let $I\subset\mathbb{C}[x_1,\ldots,x_n]$ be an ideal. Show that the vector space $\mathbb{C}[x_1,\ldots,x_n]/I$ is at most countably dimensional.

2, 2pts) Now let I be maximal and write $\bar{x}_i$ for the image of x_i in $\mathbb{C}[x_1,\ldots,x_n]/I$. Prove that, for each i, either $\bar{x}_i$ is in $\mathbb{C}$ or the elements $(\bar{x}_i-a)^{-1}$ with $a\in\mathbb{C}$ are linearly independent (*for an infinite collection being linearly independent means that every finite subcollection is linearly independent*).

3, 1pt) assuming parts 1 and 2, deduce that there are $a_1,\ldots,a_n\in\mathbb{C}$ such that I has the form $\{f(x_1,\ldots,x_n)|f(a_1,\ldots,a_n)=0\}$.

Problem 6, 2pts total. Let A be a ring, M be an A-module, and $m\in M$.

1, 1pt) We define the subset $\mathrm{Ann}_A(m):=\{a\in A|am=0\}$ ("Ann" stands for the "annihilator"). Prove that $\mathrm{Ann}_A(m)$ is an ideal in A.

2, 1pt) We define the subset $\mathrm{Ann}_A(M):=\{a\in A|am=0,\forall m\in M\}$. Assume that M is generated by elements $m_1,\ldots,m_k\in M$. Prove that $\mathrm{Ann}_A(M)=\bigcap_{i=1}^{k}\mathrm{Ann}_A(m_i)$.

Problem 7, 2pts. Let A be a ring. Prove that the following two statements are equivalent (both implications are 1pt).

a) A is a field.

b) Every finitely generated A-module is free (i.e., the direct sum of several copies of the regular A-module A).

Problem 8, 2pts. Let A be a ring, $I\subset A$ be an ideal, and M be an A-module. Construct a natural map $\mathrm{Hom}_A(A/I,M)\to\{m\in M|Im=\{0\}\}$ and prove that it is an isomorphism.

Problem 9, 3pts. Let $\mathbb{F}$ be a field, and A be an $\mathbb{F}$-algebra (=a ring with a fixed ring homomorphism $\mathbb{F}\to A$). This, in particular, makes A itself and any A-module into vector spaces over $\mathbb{F}$. Assume that A is finite dimensional over $\mathbb{F}$.

1, 1pt) Prove that every finitely generated A-module is also finite dimensional over $\mathbb{F}$.

2, 1pt) For an A-module M, endow $M^*(=\mathrm{Hom}_{\mathbb{F}}(M,\mathbb{F}))$ with a natural A-module structure.

3, 1pt) Prove that the A-modules M^* and $\mathrm{Hom}_A(M,A^*)$ are naturally isomorphic.

Problem 10, 2pts. Let A be a principal ideal domain and $M=A^{\oplus k}$, a free A-module. Let $m=(a_1,\ldots,a_k)\in M$ and $a\in A$ be such that the ideal of A generated by $a_1,\ldots,a_k$ is (a). Prove that there is an A-module isomorphism $\varphi:M\to M$ mapping m to $(a,0,0,\ldots,0)$. *The case of $k=2$ is worth one point.*