

# Rational Cherednik Algebras of type A

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## 1 Rational Cherednik algebras

### 1.1 Smash-product algebras.

We are interested in filtered deformations of the algebra  $\mathbb{C}[\text{Sym}^n(\mathbb{C}^2)] = \mathbb{C}[(\mathbb{C}^2)^{\oplus n}]^{\mathfrak{S}_n} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ . Here and for the rest of these notes we denote  $\mathfrak{h} := \mathbb{C}^n$ . The deformations of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$  we are going to produce arise from deformations of a closely related noncommutative algebra, the smash-product  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$ . Let us define this algebra in a greater generality.

Let  $A$  be an associative algebra with an action of a finite group  $G$  by algebra automorphisms. The *smash product algebra*  $A \# G$  is, as a vector space, simply  $A \otimes \mathbb{C}G$ . The product is given on pure tensors by  $(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1 g_1(f_2) \otimes g_1 g_2$  and is extended bilinearly. Note that the assignment  $g \mapsto 1 \otimes g$  (resp.  $a \mapsto a \otimes 1$ ) identifies  $\mathbb{C}G$  (resp.  $A$ ) with a subalgebra of  $A \# G$ . Consider the trivial idempotent  $e = \frac{1}{|G|} \sum_{g \in G} g \in A \# G$  and the corresponding *spherical subalgebra*  $e(A \# G)e$ . Note that this is not a unital subalgebra of  $A \# G$ , but  $e$  is the unit in the spherical subalgebra. The connection of the smash product algebra with the algebra of invariants is given by the following result, whose proof is an exercise.

**Proposition 1.1** *The map  $a \mapsto ae = ea = eae$  from  $A^G$  to  $e(A \# G)e$  is an isomorphism of algebras.*

Let us now explain how deformations of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$  are related to deformations of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ . First of all, note that the smash-product  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$  is graded, with  $\mathfrak{S}_n$  on degree 0 and  $\mathfrak{h}, \mathfrak{h}^*$  on degree 1. Let  $\mathcal{A} = \bigcup_{n \geq 0} \mathcal{A}^{\leq n}$  be a filtered deformation of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$ , that is,  $\text{gr } \mathcal{A} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ . Note that it follows that  $\mathbb{C}\mathfrak{S}_n \subseteq \mathcal{A}^{\leq 0}$ , so we can consider the spherical subalgebra  $e\mathcal{A}e$ . This algebra inherits a filtration from  $\mathcal{A}$ ,  $(e\mathcal{A}e)^{\leq n} = e\mathcal{A}^{\leq n}e$ .

**Proposition 1.2** *We have  $\text{gr}(e\mathcal{A}e) = e(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n)e = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ .*

So we can get filtered deformations of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$  from those of the smash-product algebra. Even though this algebra is no longer commutative, a presentation by generators and relations is easier. Namely, we have:

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n = (T(\mathfrak{h} \oplus \mathfrak{h}^*) \# \mathfrak{S}_n) / (x \otimes y - y \otimes x, x, y \in \mathfrak{h} \oplus \mathfrak{h}^*).$$

Note that  $T(\mathfrak{h} \oplus \mathfrak{h}^*) \# \mathfrak{S}_n$  inherits a filtration from  $T(\mathfrak{h} \oplus \mathfrak{h}^*)$  (again, we put  $\mathbb{C}\mathfrak{S}_n$  in degree 0). So, to get a deformation of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$  we can correct the commutation relation on  $T(\mathfrak{h} \oplus \mathfrak{h}^*) \# \mathfrak{S}_n$  by  $[x, y] = \beta(x, y)$ , where  $\beta(x, y) \in (T(\mathfrak{h} \oplus \mathfrak{h}^*) \# \mathfrak{S}_n)^{\leq 1} = \mathbb{C}\mathfrak{S}_n \oplus (\mathfrak{h} \oplus \mathfrak{h}^*) \otimes \mathbb{C}\mathfrak{S}_n$ . This is what we're going to do to define rational Cherednik algebras. But before, let us look at another motivation via Dunkl operators.

### 1.2 Dunkl operators.

For  $i \neq j \in \{1, \dots, n\}$ , let  $s_{ij} \in \mathfrak{S}_n$  denote the transposition  $i \leftrightarrow j$ . For each reflection  $s_{ij} \in \mathfrak{S}_n$ , let  $P_{ij} \subseteq \mathfrak{h}$  denote the reflection hyperplane associated to  $s_{ij}$ , that is,  $P_{ij} = \{x_i = x_j\}$ , where  $x_i \in \mathfrak{h}^*$  is the standard  $i$ -th coordinate function. Let  $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \bigcup_{i < j} P_{ij}$ . Note that  $\mathfrak{h}^{reg}$  is the locus where the action of  $\mathfrak{S}_n$  is free, that it is Zariski open, and  $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \{\prod_{i < j} (x_i - x_j) = 0\}$ . Let  $\mathcal{D}(\mathfrak{h}^{reg})$  be the algebra of algebraic differential operators on  $\mathfrak{h}^{reg}$ . Note that  $\mathfrak{S}_n$  acts on  $\mathfrak{h}^{reg}$  and therefore it also acts on  $\mathcal{D}(\mathfrak{h}^{reg})$ .

**Definition 1.3** *For any  $i = 1, \dots, n$ ,  $t, c \in \mathbb{C}$ , the Dunkl operator is defined to be*

$$D_i = t \frac{\partial}{\partial x_i} - c \sum_{j \neq i} \frac{1}{x_i - x_j} (1 - s_{ij}) \in \mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n. \quad (1)$$

Note that for  $\sigma \in \mathfrak{S}_n$  we have  $\sigma D_i \sigma^{-1} = D_{\sigma(i)}$ . Also, for  $j \neq i$ :

$$[D_i, x_j] = cs_{ij},$$

and

$$[D_i, x_i] = t - \sum_{j \neq i} cs_{ij}.$$

Finally, we have the following important technical lemma.

**Lemma 1.4** *Dunkl operators  $D_i, D_j$  commute.*

*Proof.* The proof is left as an exercise for the reader.  $\square$

In our discussion, we will need a slightly modified version of the construction above. Namely, let  $\hbar$  be a variable and consider  $\text{Rees}_{\hbar}(\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n)$ , the Rees algebra with respect to the usual filtration on  $\mathcal{D}(\mathfrak{h}^{reg})$  by the order of a differential operator (and  $\mathfrak{S}_n$  is in filtration degree 0). Recall that this is  $\bigoplus_{n \geq 0} (\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n)^{\leq n} \hbar^n \subseteq \mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n[\hbar]$ . Then, let

$$D_i^{\hbar} = \hbar \frac{\partial}{\partial x_i} - c \sum_{i \neq j} \frac{1}{x_i - x_j} (1 - s_{ij}) \in \text{Rees}_{\hbar}(\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n).$$

Clearly, the relations above also hold in this setting if we replace  $t$  by  $\hbar$ . The reason why we pass to the Rees algebra is the following: note that for  $t \neq 0$ , setting  $\hbar = t$  we recover the notion of Dunkl operators above. However, this is no longer true for  $t = 0$ . Recall that  $\text{Rees}_{\hbar}(\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n) / (\hbar) = \text{gr}(\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n) = \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \# \mathfrak{S}_n$ . So we have:

$$D_i^0 = y_i - c \sum_{i \neq j} \frac{1}{x_i - x_j} (1 - s_{ij}) \in \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \# \mathfrak{S}_n,$$

where  $y_i \in \mathfrak{h}$  in the right hand side is dual to  $x_i \in \mathfrak{h}^*$ .

### 1.3 Rational Cherednik algebras of type A.

For  $c \in \mathbb{C}$ , let  $H_{\hbar,c}$  be the  $\mathbb{C}[\hbar]$ -subalgebra inside  $\text{Rees}_{\hbar}(\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n)$  generated by  $\mathfrak{h}^*$ ,  $\mathfrak{S}_n$ , and Dunkl operators  $D_i^{\hbar}$ ,  $i = 1, \dots, n$ .

**Proposition 1.5** *The algebra  $H_{\hbar,c}$  is the quotient of  $(\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# \mathfrak{S}_n)[\hbar]$  by the ideal generated by the following relations*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_j] = cs_{ij} \quad [y_i, x_i] = \hbar - \sum_{j \neq i} cs_{ij}. \quad (2)$$

For  $i, j = 1, \dots, n$ .

*Proof.* Denote the algebra defined in the proposition by  $H'$ . By the results of Subsection 1.2 it is clear that we have an epimorphism  $H' \rightarrow H_{\hbar,c}$  defined via  $x_i \mapsto x_i$ ,  $y_i \mapsto D_i^{\hbar}$ ,  $\mathfrak{S}_n \ni \sigma \mapsto \sigma$ . Let us show that this is injective. It is clear by its definition that  $H'$  is generated, over  $\mathbb{C}[\hbar]$ , by elements of the form  $(\prod_{i=1}^n x_i^{a_i}) (\prod_{i=1}^n y_i^{b_i}) \sigma$ . Note that the algebra  $\text{Rees}(\mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n)$  can be filtered by the order of a differential operator, and its associated graded is  $(\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \# \mathfrak{S}_n)[\hbar]$ . The symbols of the elements  $(\prod_{i=1}^n x_i^{a_i}) (\prod_{i=1}^n (D_i^{\hbar})^{b_i}) \sigma$  in  $(\mathbb{C}[T^*(\mathfrak{h}^{reg})] \# \mathfrak{S}_n)[\hbar] = (\mathbb{C}[\mathfrak{h}^{reg} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n)[\hbar]$  are clearly linearly independent, so the result follows.  $\square$

Now we specialize  $\hbar$  to be a complex number.

**Definition 1.6** *For  $t, c \in \mathbb{C}$ , the rational Cherednik algebra  $H_{t,c}$  associated to  $t, c$  is the quotient of the algebra  $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# \mathfrak{S}_n$  by the following relations:*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_j] = cs_{ij} \quad [y_i, x_i] = t - \sum_{j \neq i} cs_{ij}. \quad (3)$$

For  $i, j = 1, \dots, n$ . In other words,  $H_{t,c} = H_{\hbar,c} / (\hbar - t)$ .

So, for example,  $H_{0,0} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \# \mathfrak{S}_n$ , and  $H_{1,0} = \mathcal{D}(\mathbb{C}^n) \# \mathfrak{S}_n$ .

Note that, by definition, for every  $t \in \mathbb{C}^\times$ ,  $c \in \mathbb{C}$ , using Dunkl operators we get an injective homomorphism

$$\Theta_{t,c} : H_{t,c} \rightarrow \mathcal{D}(\mathfrak{h}^{reg}) \# \mathfrak{S}_n, \quad (4)$$

given by  $\Theta_{t,c}(x_i) = x_i$ ,  $\Theta_{t,c}(y_i) = D_i$ . For  $t = 0$ , we get

$$\Theta_{0,c} : H_{0,c} \rightarrow \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \# \mathfrak{S}_n, \quad (5)$$

given by similar formulae.

**Remark 1.7** From the relations (3) it's not hard to see that, if  $\lambda \in \mathbb{C}^\times$ , then we get a natural isomorphism  $H_{t,c} \rightarrow H_{\lambda t, \lambda c}$ . Hence, we have essentially two different cases:  $t = 0$  and  $t \neq 0$ .

The algebra  $H_{t,c}$  clearly inherits a filtration from the grading in  $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# \mathfrak{S}_n$ , where in the latter algebra  $\mathfrak{S}_n$  has degree 0. The next theorem tells us that  $H_{t,c}$  is indeed a filtered deformation of the smash-product algebra  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$ .

**Theorem 1.8 (PBW theorem for Rational Cherednik Algebras)**  $\text{gr}(H_{t,c}) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n$ .

*Proof.* It is easy to see that we have a map  $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \# \mathfrak{S}_n \twoheadrightarrow \text{gr}(H_{t,c})$ . From the commutation relations it is clear that any element of  $H_{t,c}$  can be written as a linear combination of monomials of the form  $(\prod_{i=1}^n x_i^{a_i}) (\prod_{i=1}^n y_i^{b_i}) \sigma$ , so the map is surjective. The PBW theorem, then, is equivalent to the claim that these monomials form a basis of  $H_{t,c}$ , and this follows from the proof of Proposition 1.5.  $\square$

Recall that  $e = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma$  is the trivial idempotent in  $\mathbb{C}\mathfrak{S}_n$ . Then, we get.

**Corollary 1.9** The spherical subalgebra  $eH_{t,c}e$  is a filtered deformation of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ .

## 1.4 Rational Cherednik algebra as universal deformation.

In this subsection, we explain how a closely related algebra to  $H_{t,c}$  satisfies a certain universality property with respect to deformations. Take elements  $\bar{x}_i = x_i - \frac{1}{n} \sum_{j=1}^n x_j$ ,  $\bar{y}_i = y_i - \frac{1}{n} \sum_{j=1}^n y_j \in H_{t,c}$ . Note that  $\sum \bar{x}_i = 0 = \sum \bar{y}_i$ . Let  $H_{t,c}^+$  be the subalgebra of  $H_{t,c}$  generated by  $\bar{x}_i, \bar{y}_i$  and  $\mathfrak{S}_n$ . Note that we can present the algebra  $H_{t,c}^+$  by generators and relations as the quotient of  $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# \mathfrak{S}_n$  by the relations:

$$\sum_{i=1}^n x_i = 0, \quad \sum_{i=1}^n y_i = 0, \quad [x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_j] = cs_{ij} - \frac{t}{n}, \quad [y_i, x_i] = t \frac{n-1}{n} - \sum_{j \neq i} cs_{ij}. \quad (6)$$

It follows from the theory developed above that  $H_{t,c}^+$  is a filtered deformation of the algebra  $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$ , where  $\mathfrak{h}^+ = \{(a_1, \dots, a_n) \in \mathbb{C}^n : \sum a_i = 0\}$  is the reflection representation of  $\mathfrak{S}_n$ . Take any deformation of the form  $H_\kappa := T(\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*) \# \mathfrak{S}_n / ([x, y] = \kappa(x, y))$ , where  $T(\bullet)$  denotes tensor algebra and  $\kappa(x, y)$  is a map  $\kappa : \bigwedge^2(\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*) \rightarrow \mathbb{C}\mathfrak{S}_n$ ,  $\kappa(x, y) = \sum_{\sigma \in \mathfrak{S}_n} \kappa_\sigma(x, y) \sigma$ . Let us see why  $\kappa$  must be of the form (6). We will see that some  $\kappa_\sigma$  are identically 0. First, it is an exercise to see that if  $\text{gr } H_\kappa = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$ , then  $\kappa$  must be  $\mathfrak{S}_n$ -equivariant (otherwise the condition fails already in filtration degree 1). Note that

$$[\kappa(u, v), w] = \sum_{\sigma \in \mathfrak{S}_n} \kappa_\sigma(u, v) (\sigma(w) - w) \sigma, \quad (7)$$

this follows easily from the definition of the smash-product algebra. Now, the Jacobi identity in  $H_\kappa$  tells us that we must have:

$$[\kappa(u, v), w] + [\kappa(v, w), u] + [\kappa(w, u), v] = 0$$

for any  $u, v, w \in \mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$ . Since  $\text{gr } H_\kappa = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$ , the map  $(\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*) \otimes \mathbb{C}\mathfrak{S}_n \rightarrow H_\kappa$  is injective. Then, by (7), for every  $\sigma \in \mathfrak{S}_n$ ,  $u, v, w \in \mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$  we have:

$$\kappa_\sigma(u, v) (\sigma(w) - w) + \kappa_\sigma(v, w) (\sigma(u) - u) + \kappa_\sigma(w, u) (\sigma(v) - v) = 0. \quad (8)$$

It follows that, if  $\text{rank}(\sigma - 1) > 2$ , then  $\kappa_\sigma$  must be identically 0. Since the action of  $\mathfrak{S}_n$  on  $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$  is lifted from an action on  $\mathfrak{h}^+$ , it follows that  $\kappa_\sigma$  must be identically zero unless  $\sigma = s_{ij}$  for some  $i, j$  or  $\sigma = 1$ .

Consider the case  $\sigma = s_{ij}$ . Note that, if  $k \neq i, j$  then  $s_{ij}(x_k) = x_k$ . It follows from (8) that  $\kappa_{s_{ij}}(x_k, \bullet) = 0$ . Similarly,  $\kappa_{s_{ij}}(y_k, \bullet) = 0$ . Now,  $2x_j = x_j - x_i - \sum_{k \neq i, j} x_k$ , and similarly for  $x_i$ . From here it follows that  $\kappa_{s_{ij}}(x_j, x_i) = 0$ . Using a similar formula for the  $y$ 's, one has that  $\kappa_{s_{ij}}(y_i, x_i) = -\kappa_{s_{ij}}(y_i, x_j)$ .

Now, note that  $\kappa_1$  is a  $\mathfrak{S}_n$ -invariant skew-symmetric form on  $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$ . By irreducibility of  $\mathfrak{h}^+$  and Schur's lemma, there is, up to scaling, a unique such form, namely the canonical symplectic form on  $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^* = T^*(\mathfrak{h}^+)$ . Finally, the  $\mathfrak{S}_n$ -equivariance of the form  $\kappa$  forces us to have relations of the form (6).

The discussion above makes us see that the algebra  $H_{\hbar, \bar{c}}^+$ , where  $\hbar, \bar{c}$  are formal variables in degree 2 (so that  $H_{\hbar, \bar{c}}^+$  is graded), should be a 'universal' deformation of  $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$ . It turns out that is the case. First, let us explain what we mean by a universal deformation. Recall that a graded deformation of a  $\mathbb{Z}_{\geq 0}$ -graded algebra  $A$  of degree  $n$  over a vector space  $P$ , is a free, graded  $S(P)$ -algebra  $\mathcal{A}$  where  $P$  sits in  $\mathcal{A}$  in degree  $n$ , and such that  $\mathcal{A}/\mathcal{A}P = A$ .

**Definition 1.10** *A universal graded deformation of a  $\mathbb{Z}_{\geq 0}$ -graded algebra  $A$  of degree 2 is a graded deformation  $\mathcal{A}_{un}$  over a vector space  $P_{un}$ , where  $P_{un}$  sits inside  $\mathcal{A}_{un}$  in degree 2, such that, for any other deformation  $\mathcal{A}'$  of  $A$  over a vector space  $P'$  where  $P'$  sits in degree 2, there exists a unique linear map  $P_{un} \rightarrow P'$  such that the deformations  $\mathcal{A}'$  and  $S(P) \otimes_{S(P_{un})} \mathcal{A}$  are equivalent (via a unique equivalence).*

In the next subsection, we'll see that  $H_{\hbar, \bar{c}}^+$  is indeed a universal deformation of  $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$ .

## 1.5 Hochschild cohomology.

Let  $A$  be an associative algebra, and let  $M$  be an  $A$ -bimodule. The space of Hochschild  $n$ -cochains on  $M$ ,  $C^n(A, M)$ , is the space of  $\mathbb{C}$ -linear maps  $A^{\otimes n} \rightarrow M$ . We have a map  $d : C^n(A, M) \rightarrow C^{n+1}(A, M)$  given by the formula:

$$\begin{aligned} df(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = & f(a_1 \otimes \cdots \otimes a_n) a_{n+1} - f(a_1 \otimes \cdots \otimes a_n a_{n+1}) + f(a_1 \otimes \cdots \otimes a_{n-1} a_n \otimes a_{n+1}) + \cdots \\ & + (-1)^n f(a_1 a_2 \otimes \cdots \otimes a_n) + (-1)^{n+1} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}). \end{aligned}$$

It is an exercise to see that  $d^2 = 0$ , so that we have a complex  $C^0(A, M) \rightarrow C^1(A, M) \rightarrow \cdots$ . The cohomology of this complex is called the Hochschild cohomology and is denoted by  $\overline{\text{HH}}^i(A, M)$ . We denote  $\overline{\text{HH}}^i(A) := \overline{\text{HH}}^i(A, A)$ . Note that, using the standard resolution of the  $A$ -bimodule  $A$ , we can see that, actually,  $\overline{\text{HH}}^i(A, M) = \text{Ext}_{A\text{-Bimod}}^i(A, M)$ .

Now assume that  $A$  is  $\mathbb{Z}$ -graded, and that  $M$  is a  $\mathbb{Z}$ -graded  $A$ -bimodule. We can modify the construction above to get a notion of graded Hochschild cohomology as follows. Define  $C^n(A, M)^m = \{f \in C^n(A, M) : f((A^{\otimes n})^i) \subseteq M^{i+m}\}$ . Note that  $d(C^n(A, M)^m) \subseteq C^{n+1}(A, M)^m$ . So we can define  $\text{HH}^\bullet(A, M)^m$  to be the cohomology of the complex  $d : C^n(A, M)^m \rightarrow C^{n+1}(A, M)^m$ , and define  $\text{HH}^\bullet(A, M) := \bigoplus_m \text{HH}^\bullet(A, M)^m$ . Note, however, that in general  $\overline{\text{HH}}^i(A, M) \neq \text{HH}^i(A, M)$ . The following result is well-known in deformation theory.

**Theorem 1.11** *Assume that*

- $\text{HH}^2(A)^{-2}$  *is finite dimensional.*
- $\text{HH}^2(A)^i = 0$  *for*  $i < -2$ .
- $\text{HH}^3(A)^i = 0$  *for*  $i < -3$ .

*Then, modulo uniqueness of the map in the universal property, a universal graded deformation of  $A$  exists, with  $P_{un} = (\text{HH}^2(A)^{-2})^*$ .*

For a proof of Theorem 1.11 see, for example, [3]. A condition that guarantees uniqueness of the map in the universal property for a universal graded deformation is that  $\text{HH}^1(A)^i = 0$  for  $i \leq -2$ . We will not check this. Instead, we will see uniqueness in the case of interest for us by more elementary methods.

So we need to compute  $\text{HH}^2(\overline{A})^i$ ,  $i \leq -2$  and  $\text{HH}^3(\overline{A})^j$ ,  $j < -3$  for  $\overline{A} = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$ . Let us sketch how to do this. Let  $A = \mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*]$ . For  $\sigma \in \mathfrak{S}_n$ , define the  $A$ -bimodule  $A\sigma$  as follows: as a left  $A$ -module,  $A\sigma$  is just  $A$ . The right multiplication is twisted by the action of  $\sigma$ :  $(a_1\sigma)a_2 = (a_1\sigma(a_2))\sigma$ . Note that  $\mathfrak{S}_n$  acts on the direct sum  $\bigoplus_\sigma A\sigma$ ,  $\gamma(a\sigma) = \gamma(a)\gamma\sigma\gamma^{-1}$ . Then, we get a  $\mathfrak{S}_n$ -action on  $\bigoplus_\sigma \text{HH}^i(A, A\sigma)$ . We state without proof the following result.

**Proposition 1.12** *We have an isomorphism of graded vector spaces  $\mathrm{HH}^i(A \# \mathfrak{S}_n) \cong (\bigoplus_{\sigma \in \mathfrak{S}_n} \mathrm{HH}^i(A, A\sigma))^{\mathfrak{S}_n}$ .*

So we need to compute  $\mathrm{HH}^i(A, A\sigma)$  for  $\sigma \in \mathfrak{S}_n$ . For  $\sigma \in \mathfrak{S}_n$ , pick a basis  $v_1, \dots, v_{2(n-1)}$  of  $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$  such that  $\sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_{2(n-1)})$ . Note that every number  $\sigma_i$  is a root of 1, so we can think of  $\sigma_i$  as a member of a cyclic group acting on  $\mathbb{C}v_i$ . Then, we get:

$$A\sigma := \bigotimes_{i=1}^{2(n-1)} \mathbb{C}[v_i]\sigma_i.$$

And, by the Künneth formula,

$$\mathrm{HH}^\bullet(A, A\sigma) = \bigotimes \mathrm{HH}^\bullet(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i),$$

where the homological degree on the left hand side equals the sum of the homological degrees on the right hand side. So we have reduced the problem to computing the Hochschild cohomology  $\mathrm{HH}^\bullet(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i)$ . Now we can use the standard Koszul resolution of  $\mathbb{C}[v_i]$  to compute these cohomology. Since this resolution has length 1, one immediately gets that  $\mathrm{HH}^j(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i) = 0$  for  $j \geq 2$ , regardless of the value of  $\sigma_i$ . If  $\sigma_i = 1$ , then  $\mathrm{HH}^0(\mathbb{C}[v_i], \mathbb{C}[v_i]) = \mathbb{C}[v_i]$  with its usual grading, and  $\mathrm{HH}^1(\mathbb{C}[v_i], \mathbb{C}[v_i]) = \mathbb{C}[v_i]$ , with its grading shifted by 1. If  $\sigma_i \neq 1$ , then  $\mathrm{HH}^\bullet(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i)$  is 1-dimensional, concentrated in degree 1, and  $\mathrm{HH}^1(\mathbb{C}[v_i], \mathbb{C}[v_i]\sigma_i)^{-1} = \mathbb{C}$ .

Since the action of  $\mathfrak{S}_n$  on  $\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*$  is lifted from an action on  $\mathfrak{h}^+$ , any  $\sigma \in \mathfrak{S}_n$  has an even number of eigenvalues different from 1. From here it follows easily that  $\mathrm{HH}^2(A, A\sigma)^j = 0$  for  $j < -2$ , and  $\mathrm{HH}^3(A, A\sigma)^j = 0$  for  $j < -3$ . It also follows that, unless  $\sigma = s_{ij}$  for some  $i, j$ , every element in  $\mathrm{HH}^\bullet(A, A\sigma)$  has homological degree at least 4. Then, only reflections are important when computing  $\mathrm{HH}^2(\overline{A})^{-2}$ . In fact,  $\dim \left( \bigoplus_{i,j} \mathrm{HH}^2(A, As_{ij}) \right)^{\mathfrak{S}_n} = 2$  see, for example, [3, Exercise 7.10]. In the computation, a very important fact one uses is that  $\mathfrak{h}^+$  is an irreducible  $\mathfrak{S}_n$ -module.

So  $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$  does have a universal deformation over a vector space of dimension 2. It follows from our computations in the previous subsection that this deformation must coincide with  $H_{\hbar, \bar{c}}^+$ . Finally, note that, since  $V$  and  $\mathbb{C}\mathfrak{S}_n$  generate  $H_{\hbar, \bar{c}}^+$  over  $\mathbb{C}[\hbar, \bar{c}]$ , any self-equivalence of  $H_{\hbar, \bar{c}}^+$  as a deformation of  $\mathbb{C}[\mathfrak{h}^+ \oplus (\mathfrak{h}^+)^*] \# \mathfrak{S}_n$  must be the identity. Hence,  $H_{\hbar, \bar{c}}^+$  is a universal deformation.

## 1.6 Spherical RCA.

We return to the study of the rational Cherednik algebra  $H_{t,c}$ . Denote the spherical subalgebra  $eH_{t,c}e$  by  $B_{t,c}$ . Consider the  $H_{t,c} - B_{t,c}$ -bimodule  $H_{t,c}e$ . An exercise is to show that  $\mathrm{End}_{H_{t,c}}(H_{t,c}e) = B_{t,c}^{opp}$ . It is clear that we have a map  $H_{t,c} \rightarrow \mathrm{End}_{B_{t,c}}(H_{t,c}e)$ . It turns out that this map is an isomorphism.

**Theorem 1.13 (Double centralizer property)**  $H_{t,c} \cong \mathrm{End}_{B_{t,c}}(H_{t,c}e)$ .

Let us outline an strategy to prove Theorem 1.13. First, we prove it in the associated graded case  $t, c = 0$ . We prove injectivity first, which is easier, and then surjectivity of the natural map  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n \rightarrow \mathrm{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$ . After this, we can give a filtration to  $\mathrm{End}_{B_{t,c}}(H_{t,c}e)$  that will reduce the general case to the associated graded one.

Injectivity of the double centralizer map in the associated graded case is straightforward. Surjectivity is harder. To prove it, one observes that surjectivity would hold if the action of  $\mathfrak{S}_n$  on  $\mathfrak{h} \oplus \mathfrak{h}^*$  were free. Of course, this is not the case here. But we can restrict to a subset of codimension 2 where this holds, and the codimension claim implies that the map is surjective. We provide details in the Appendix, see Subsection 3.1.

Recall that for  $t = 0$ , we have the Dunkl embedding  $\Theta_{0,c} : H_{0,c} \rightarrow \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \# \mathfrak{S}_n$ . Passing to spherical subalgebras, we get an inclusion of  $B_{0,c}$  in  $\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^{\mathfrak{S}_n}$ . In particular, we see that the algebra  $B_{0,c}$  is commutative. So it follows that the Poisson bracket  $\{\bullet, \bullet\}_{0,c}$  induced in  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$  is identically 0. Also note that the bracket  $\{\bullet, \bullet\}_{t,c}$  induced by  $B_{t,c}$  depends linearly on  $(t, c)$ . It follows that  $\{\bullet, \bullet\}_{t,c} = \{\bullet, \bullet\}_{t,0}$ . Finally, it is easy to see from the relations (2) that  $\{\bullet, \bullet\}_{t,0} = t\{\bullet, \bullet\}$ , where the latter bracket is the usual bracket on  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ .

Denote by  $Z_{t,c}$  the center of the rational Cherednik algebra  $H_{t,c}$ . For any values of the parameters  $(t, c)$ , we have a natural map  $Z_{t,c} \rightarrow B_{t,c}$  given by  $m \mapsto em$ . Using the double centralizer theorem we can prove that, for  $t = 0$ , this is actually an isomorphism.

**Theorem 1.14 (Satake isomorphism)** *The natural homomorphism  $Z_{0,c} \rightarrow B_{0,c}$  is an isomorphism.*

*Proof.* We provide an inverse homomorphism. Since  $B_{0,c}$  is commutative, for any  $b \in B_{0,c}$  multiplication by  $b$  provides an endomorphism of the right  $B_{0,c}$ -module  $H_{0,c}e$ , and this endomorphism commutes with any other endomorphism. By the double centralizer property, this endomorphism corresponds to a unique element  $\varphi(b) \in Z_{0,c}$ . It is easy to check that this is inverse to  $Z_{0,c} \rightarrow B_{0,c}$ ,  $m \mapsto em$ .  $\square$

We remark that there is a natural Poisson bracket on  $B_{0,c}$ . Namely, it is easy to see that this algebra is the quasiclassical limit of  $B_{\hbar,c}$ , the spherical subalgebra of the algebra  $H_{\hbar,c}$  introduced in Definition 1.5. Then, we can define the Poisson bracket as  $\{a, b\} := \frac{1}{\hbar}[\bar{a}, \bar{b}] \bmod \hbar$ , where  $\bar{a}, \bar{b}$  are lifts of  $a, b$  to  $H_{\hbar,c}$ .

## 1.7 $\mathfrak{sl}_2$ -actions on the rational Cherednik algebra.

To finish this section, let us mention some  $\mathfrak{sl}_2$  actions on the rational Cherednik algebra  $H_{t,c}$  and its spherical subalgebra that will be of importance later. First, assume  $t \neq 0$ , so we may as well assume  $t = 1$ . Consider the following elements in  $H_{1,c}$ :

$$\mathbf{E} := -\frac{1}{2} \sum_i x_i^2, \quad \mathbf{F} := \frac{1}{2} \sum_i y_i^2, \quad \mathbf{h} := \sum_i \frac{x_i y_i + y_i x_i}{2}.$$

The following can be seen via a direct calculation:

**Proposition 1.15** *The elements  $(\mathbf{E}, \mathbf{F}, \mathbf{h})$  form an  $\mathfrak{sl}_2$ -triple in  $H_{1,c}$ , where the Lie bracket is the usual commutator.*

Moreover, the induced  $\mathfrak{sl}_2$  action on  $H_{1,c}$  is locally finite, this follows from our calculations below, see Section 3, where the element  $\mathbf{h}$  is of special importance. Note that  $[e, \mathbf{E}] = 0$ ,  $[e, \mathbf{F}] = 0$ ,  $[e, \mathbf{h}] = 0$ , this is an exercise. It follows that the spherical subalgebra  $eH_{1,c}e$  contains the  $\mathfrak{sl}_2$  triple  $(\mathbf{E}e, \mathbf{F}e, \mathbf{h}e)$ , and the induced  $\mathfrak{sl}_2$  action on  $eH_{1,c}e$  is locally finite so, in particular, it integrates to an action of  $\mathrm{SL}_2$ .

We would like to get an  $\mathfrak{sl}_2$ -triple in the spherical subalgebra  $B_{0,c}$ . We've seen in the previous subsection that this is a Poisson algebra, with the Poisson bracket induced by the commutator in  $B_{\hbar,c}$ . We have the following result.

**Proposition 1.16** *The elements  $(\mathbf{E}, \mathbf{F}, \mathbf{h})$  form an  $\mathfrak{sl}_2$ -triple in  $B_{0,c}$ , where the Lie bracket is the natural Poisson bracket on  $B_{0,c}$ . The induced  $\mathfrak{sl}_2$ -action on  $B_{0,c}$  is locally finite, and it integrates to an  $\mathrm{SL}_2$  action.*

## 2 Representation theory at $t = 0$ .

### 2.1 Irreducible representations of $H_{0,c}$ .

Note that the Satake isomorphism is filtered, where both  $Z_{0,c}$  and  $B_{0,c}$  have the inherited filtration from the one on  $H_{0,c}$ . By Corollary 1.9, it follows that  $\mathrm{gr} Z_{0,c} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ , so  $\mathrm{gr} H_{0,c}$  is a finitely generated module over  $\mathrm{gr} Z_{0,c}$ . It follows that  $H_{0,c}$  is finitely generated over  $Z_{0,c}$ , so every irreducible representation of  $H_{0,c}$  is finite dimensional.

By Schur's lemma, the center  $Z_{0,c}$  acts on every irreducible  $H_{0,c}$ -module by a character. For a central character  $\chi : Z_{0,c} \rightarrow \mathbb{C}$ , let  $(\chi)$  be the ideal in  $H_{0,c}$  generated by the kernel of  $\chi$ .

**Theorem 2.1** *Any irreducible representation of  $H_{0,c}$  has dimension  $n!$ , and is isomorphic to the regular representation as an  $\mathfrak{S}_n$ -module.*

We divide the proof of the preceding theorem in two parts. First, we show that any irreducible representation of  $H_{0,c}$  has dimension  $\leq n!$ . This is a consequence of the Amitsur-Levitski identity for  $H_{0,c}$ , which we show first. After that, we show that any irreducible representation of  $H_{0,c}$  must be a multiple of the regular representation of  $\mathfrak{S}_n$ .

Recall that we have the Dunkl embedding  $H_{0,c} \hookrightarrow \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \# \mathfrak{S}_n$ . Note that the latter algebra may be embedded in  $\mathbb{C}(x_1, \dots, x_n, y_1, \dots, y_n) \# \mathfrak{S}_n$ . But this algebra is isomorphic to the algebra of  $n! \times n!$  matrices over  $\mathbb{C}(x_1, \dots, x_n, y_1, \dots, y_n)^{\mathfrak{S}_n}$ . It follows that any polynomial identity satisfied by  $\mathrm{Mat}_{n!}(\mathbb{C}(x_1, \dots, x_n, y_1, \dots, y_n)^{\mathfrak{S}_n})$  is also satisfied by  $H_{0,c}$ . In particular, we have the following.

**Theorem 2.2 (Amitsur-Levitzki Identity)** *For any  $N \times N$  matrices  $A_1, \dots, A_{2N}$  with entries in a commutative ring  $R$ , we have*

$$\sum_{\sigma \in \mathfrak{S}_{2N}} \text{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(2N)} = 0.$$

**Lemma 2.3** *Any simple representation of  $H_{0,c}$  has dimension  $\leq n!$ .*

*Proof.* Let  $M$  be a simple representation of  $H_{0,c}$  of dimension  $m$ . By the Density theorem, the matrix algebra  $\text{Mat}_m(\mathbb{C})$  must be a quotient of  $H_{0,c}$ . Then, the Amitsur-Levitzki identity (with  $N = n!$ ) must be satisfied by  $\text{Mat}_m(\mathbb{C})$ . But this identity is not valid in any matrix algebra of size larger than  $n!$ . Then,  $m \leq n!$ .  $\square$

**Lemma 2.4** *Any irreducible representation of  $H_{0,c}$  is isomorphic to a multiple of the regular representation of  $\mathfrak{S}_n$ .*

*Proof.* We show that the trace of every element  $1 \neq \sigma \in \mathfrak{S}_n$  at a finite dimensional representation of  $H_{0,c}$  is 0. Take  $j \in \{1, \dots, n\}$ , with  $\sigma(j) = i \neq j$ . Then, in  $H_{0,c}$  we have:

$$[y_j, x_i s_{ij} \sigma] = x_i (y_j s_{ij} \sigma - s_{ij} \sigma y_i) + [y_j, x_i] s_{ij} \sigma = 0 + c s_{ij} s_{ij} \sigma = c \sigma.$$

So that  $\sigma$  is a commutator and therefore has trace 0 on every finite dimensional  $H_{0,c}$  representation. It follows that any such representation is a multiple of the regular representation of  $\mathfrak{S}_n$ .  $\square$

Now, thanks to the Dunkl embedding,  $H_{0,c}$  is a finitely generated Polynomial Identity (P.I.) algebra of degree  $n!$  (that is, all polynomial relations of the algebra of matrices of size  $n!$  are satisfied on  $H_{0,c}$ ) all whose irreducible modules have the same dimension  $n!$ . Recall that an algebra is said to be an *Azumaya algebra of degree  $N$*  if, for every maximal ideal  $\chi$  of its center  $Z(A)$ , the completion  $A^\wedge_\chi$  is a matrix algebra of size  $N$  over  $Z(A)^\wedge_\chi$ . See [4, Section 2] for more on (sheaves of) Azumaya algebras. The next important result tells us that  $H_{0,c}$  is an Azumaya algebra of degree  $n!$ .

**Theorem 2.5** *Let  $A$  be a finitely generated P.I. algebra of degree  $N$ , that is, all polynomial relations in the algebra of matrices of size  $N$  are satisfied in  $A$ . Then,  $A$  is an Azumaya algebra if and only if every irreducible representation of  $A$  has dimension  $N$ .*

For a proof of Theorem 2.5, see [1, Theorem 8.3]. Then,  $H_{0,c}$  is an Azumaya algebra of degree  $n!$ . The following result follows.

**Corollary 2.6** *For every central character  $\chi$ ,  $H_{0,c}/(\chi)$  is a matrix algebra of size  $n!$ , and therefore  $H_{0,c}$  has a unique irreducible representation with central character  $\chi$ .*

## 2.2 Generalized Calogero-Moser space.

**Definition 2.7** *For  $c \neq 0$ , the generalized Calogero-Moser space is  $V = \text{Spec}(Z_{0,c}) = \text{Spec}(B_{0,c})$ .*

In the next subsection, we will see that the word ‘generalized’ is superfluous.

Note that, since  $\text{gr } B_{0,c} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$ , the generalized Calogero-Moser space is reduced. Also note that  $B_{0,c}$  admits a graded quantization: the spherical subalgebra  $B_{\hbar,c}$  of the rational Cherednik algebra  $H_{\hbar,c}$  (that is, we replace ‘ $t$ ’ by a variable), so that  $B_{0,c}$  has a natural structure of a Poisson algebra. Our goal for this section is to prove the following.

**Theorem 2.8** *The generalized Calogero-Moser space is smooth.*

Since  $V = \text{Spec}(B_{0,c})$ , Theorem 2.8 is equivalent to the statement that the global dimension of  $B_{0,c}$  is finite. On the other hand, we know that the global dimension of  $H_{0,c}$  is finite: its associated graded is  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\# \mathfrak{S}_n}$ , which has finite global dimension, and for any filtered algebra  $\mathcal{A}$ , its global dimension cannot exceed the global dimension of  $\text{gr } \mathcal{A}$ . Hence, Theorem 2.8 is deduced from the following result.

**Proposition 2.9** *The algebras  $H_{0,c}$  and  $B_{0,c}$  are Morita equivalent.*

*Proof.* We know that, for every  $\xi \in V$ , the unique irreducible representation of  $H_{0,c}$  with central character  $\chi$  is isomorphic to the regular representation of  $\mathfrak{S}_n$ . Then,  $e$  is a rank 1 idempotent in  $H_{0,c}/(\chi)$ , so  $H_{0,c}e$  is vector bundle on  $V$ . Hence,  $H_{0,c}e$  is a projective module over  $B_{0,c}$ , and the latter algebra is Morita equivalent to  $\text{End}_{B_{0,c}}(H_{0,c}e)$ . By the Double Centralizer Property, we are done.  $\square$

Now we see that  $V$  is actually symplectic. Note that the set of points where  $\bigwedge^{\text{top}} \Pi$  vanishes has codimension at least 2, where  $\Pi$  is the Poisson bivector. This is so because it also holds in the degeneration  $\text{Sym}_n(\mathbb{C}^2)$  of  $V$ . Since  $V$  is smooth and  $\bigwedge^{\text{top}} \Pi$  is a regular function, this implies the following.

**Corollary 2.10** *The generalized Calogero-Moser space  $V$  is symplectic.*

## 2.3 Calogero-Moser space.

In this subsection, we introduce the so-called Calogero-Moser space, and prove that it is isomorphic to the generalized Calogero-Moser space of the previous subsection.

Let  $G = \mathrm{PGL}_n(\mathbb{C})$ , and  $\mathcal{M} = T^* \mathrm{Mat}_n(\mathbb{C})$ . Using the trace form, we may identify  $\mathcal{M} = \mathrm{Mat}_n(\mathbb{C}) \oplus \mathrm{Mat}_n(\mathbb{C})$ . Also, note that  $\mathrm{Lie}(G) = \mathfrak{sl}_n(\mathbb{C})$ . The group  $G$  acts on  $\mathcal{M}$  by simultaneous conjugation. This action is Hamiltonian, with moment map  $\mu : (X, Y) \mapsto [X, Y]$ , cf. [2, Section 4.4]. Let  $\mathbb{O} := \{A \in \mathfrak{sl}_n : \mathrm{rank}(A + I) = 1\}$ , where  $I$  denotes the identity matrix. Note that this is a single conjugacy class, namely  $\mathbb{O} = GL_n \cdot \mathrm{diag}(n-1, -1, -1, \dots, -1)$ .

**Definition 2.11** *The Calogero-Moser space  $\mathcal{C}_n$  is the scheme  $\mu^{-1}(\mathbb{O})//G$ .*

In other words,  $\mathcal{C}_n$  is the Hamiltonian reduction of  $\mathcal{M}$  at the orbit  $\mathbb{O}$ , so  $\mathcal{C}_n = \mathrm{Spec}((\mathbb{C}[\mathcal{M}]/\mathbb{C}[\mathcal{M}]_{\mu^* I_{\mathbb{O}}}^{\mathrm{ad} \mathfrak{g}})^{\mathrm{ad} \mathfrak{g}})$ , where  $I_{\mathbb{O}}$  is the ideal in  $S_{\mathfrak{g}}$  corresponding to the closed coadjoint orbit  $\mathbb{O}$ .

**Proposition 2.12** *The action of  $G$  on  $\mu^{-1}(\mathbb{O})$  is free.*

*Proof.* Let  $(X, Y) \in \mu^{-1}(\mathbb{O})$ . So  $(X, Y)$  determines a representation of the quiver with one vertex and two loops. This representation is simple because no proper subcollection of  $(-1, -1, \dots, n-1)$  adds up to 0. Details of this are left as an exercise. From here, the result follows by Schur's lemma.  $\square$

It follows, see [2, Theorem 4.4], that  $\mathcal{C}_n$  is a smooth symplectic variety, of dimension  $\dim(\mathcal{M}) - 2 \dim(\mathrm{PGL}_n) + \dim(\mathbb{O}) = 2n$ . The space  $\mathcal{C}_n$  is closely related to a Hamiltonian reduction considered by Barbara and Yi. Let  $\mathcal{M}' = T^*(\mathrm{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n)$ . Recall that  $G' = \mathrm{GL}_n$  acts on  $\mathcal{M}'$  in a Hamiltonian way, with moment map  $\mu' : T^*(\mathrm{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n) \rightarrow \mathfrak{gl}_n$ ,  $\mu'(X, Y, i, j) = [X, Y] + ij$ . Barbara and Yi considered the Hamiltonian reduction of  $\mathcal{M}'$  at 0,  $\mu'^{-1}(0)//G'$ . In the next result, we see that  $\mathcal{C}_n$  is the Hamiltonian reduction of  $\mathcal{M}'$  at  $-I$  (which is a single orbit on  $\mathfrak{gl}_n$ ).

**Proposition 2.13** *The reductions  $\mu'^{-1}(-I)//G'$  and  $\mu^{-1}(\mathbb{O})//G'$  are naturally identified.*

*Proof.* We have a natural projection  $\rho : \mathcal{M}' \rightarrow \mathcal{M}$ ,  $(X, Y, i, j) \mapsto (X, Y)$ . It is clear that  $\rho(\mu'^{-1}(-I)) = \mu^{-1}(\mathbb{O})$ . Now, if  $(X, Y, i, j), (X, Y, i', j') \in \rho^{-1}(X, Y)$  where  $(X, Y) \in \mu^{-1}(\mathbb{O})$ , then there exists a unique  $t \in \mathbb{C}^\times$  such that  $i' = ti$ ,  $j' = t^{-1}j$ . It follows that  $\rho$  identifies  $\mu^{-1}(\mathbb{O}) = \mu'^{-1}(-I)//\mathbb{C}^\times$ , where  $\mathbb{C}^\times$  acts on  $\mu^{-1}(\mathbb{O})$  as the center of  $\mathrm{GL}_n$ . From here, the result follows.  $\square$

The description of  $\mathcal{C}_n$  as a Hamiltonian reduction of the variety  $\mathcal{M}'$  is useful because the moment map  $\mu'$  is flat. On the other hand, we will need to see  $\mathcal{C}_n$  as a Hamiltonian reduction of  $\mathcal{M}$  when looking at the connection of  $\mathcal{C}_n$  with  $V$ , the generalized Calogero-Moser space from the previous subsection.

**Corollary 2.14** *The Calogero-Moser space  $\mathcal{C}_n$  is connected.*

*Proof.* Recall that the moment map  $\mu'$  is flat. This was proved by Barbara in her lecture. It then follows that we have a filtration on  $\mathbb{C}[\mu'^{-1}(-I)//G'] = [\mathbb{C}[\mathcal{M}']/\mathbb{C}[\mathcal{M}']_{\mu'^* \mathcal{I}}]^{G'}$ , where  $\mathcal{I}$  is the ideal in  $S$  corresponding to the closed orbit  $-I$ , whose associated graded is  $\mathbb{C}[\mu^{-1}(0)//G']$ . Barbara proved that we have an isomorphism  $\mu^{-1}(0)//G \cong \mathrm{Sym}^n(\mathbb{C}^2)$ , which is connected. It follows that  $\mathrm{gr}(\mathbb{C}[\mathcal{C}_n])$  doesn't have zero divisors, hence  $\mathcal{C}_n$  is irreducible. Since we know it's smooth, it must be connected.  $\square$

In the next lecture, we are going to show that, in fact,  $\mathcal{C}_n = \mathrm{Spec}(B_{0,c})$ .

## 3 Appendix

### 3.1 Proof of the Double Centralizer Property

*Injectivity of the natural map  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n \rightarrow \text{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$ .* Note that  $\mathfrak{h}^{reg} \times \mathfrak{h}^*$  is open and dense in  $\mathfrak{h} \oplus \mathfrak{h}^*$ , and the action of  $\mathfrak{S}_n$  here is free. Let  $\sum_{\sigma \in \mathfrak{S}_n} f_{\sigma} \sigma$  be in the kernel of  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n \rightarrow \text{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$ . Then, for every  $g \in \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ ,  $\sum f_{\sigma} \sigma(g) = 0$ . Now pick  $v \in \mathfrak{h}^{reg} \times \mathfrak{h}^*$ . Since  $\mathfrak{S}_n$  acts freely on  $v$ , for any collection of complex numbers  $\{z_{\sigma}\}_{\sigma \in \mathfrak{S}_n}$  we can find a regular function  $g$  such that  $g(\sigma^{-1}v) = z_{\sigma}$ , so  $\sum f_{\sigma}(v)z_{\sigma} = 0$ . Then,  $f_{\sigma}(v) = 0$  for every  $\sigma \in \mathfrak{S}_n$ . But  $\mathfrak{h}^{reg} \times \mathfrak{h}^*$  is dense in  $\mathfrak{h} \oplus \mathfrak{h}^*$ , so  $f_{\sigma} = 0$  for every  $\sigma \in \mathfrak{S}_n$ .

*Surjectivity of the natural map  $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# \mathfrak{S}_n \rightarrow \text{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$ .* First, we make the following observation: *If a group  $\Gamma$  acts freely on a smooth affine variety  $X$ , then the homomorphism  $\mathbb{C}[X] \# \Gamma \rightarrow \text{End}_{\mathbb{C}[X]^{\Gamma}}(\mathbb{C}[X])$  is an isomorphism.* This follows because both algebras are locally free over  $\mathbb{C}[X]^{\Gamma}$  of rank  $|\Gamma|^2$ , so to show bijectivity, it is enough to show injectivity fiberwise, and this can be done as in the previous step of this proof.

Now, since  $\mathfrak{S}_n$  acts on  $\mathbb{C}^{2n} = T^*\mathfrak{h}$  by symplectomorphisms, the fixed point locus of any permutation  $\sigma \in \mathfrak{S}_n$  has codimension at least 2. So the codimension of  $\mathbb{C}^{2n} \setminus \mathbb{C}^{2n,reg}$  is at least 2. For a point  $v \in \mathbb{C}^{2n,reg}$ , we can find an invariant function  $f_v \in \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}$  with  $f_v(v) \neq 0$ ,  $f|_{\mathbb{C}^{2n} \setminus \mathbb{C}^{2n,reg}} = 0$ . Now let  $\mathbb{C}_v^{2n,reg} := \{x : f(x) \neq 0\}$ . This is an affine open set on  $\mathbb{C}^{2n,reg}$ ,  $\mathfrak{S}_n$ -stable since  $f_v$  is  $\mathfrak{S}_n$ -invariant. Cover  $\mathbb{C}^{2n,reg}$  by a finite number of sets of the for  $\mathbb{C}_v^{2n,reg}$ , say  $\mathbb{C}^{2n,reg} = \bigcup V_i$ , with  $f_i$  the function used to define  $V_i$ . An observation here is that  $\text{End}_{\mathbb{C}[V_i]^{\mathfrak{S}_n}}(\mathbb{C}[V_i])$  is just the localization of  $\text{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$  at  $f_i$ . In particular, we have a morphism  $\iota_i : \text{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]) \rightarrow \text{End}_{\mathbb{C}[V_i]^{\mathfrak{S}_n}}(\mathbb{C}[V_i])$ . Since  $f_i$  is  $\mathfrak{S}_n$ -invariant,  $\iota_i$  is injective.

Let  $f \in \text{End}_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathfrak{S}_n}}(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*])$ . Consider  $\iota_i(f)$ . Since the action of  $\mathfrak{S}_n$  on  $V_i$  is free, there exist  $f_i^{\sigma} \in \mathbb{C}[V_i]$  such that  $\iota_i(f) = \sum_{\sigma \in \mathfrak{S}_n} f_i^{\sigma} \sigma$ . We claim that, for every  $\sigma, i, j$ ,  $f_i^{\sigma}|_{V_i \cap V_j} = f_j^{\sigma}|_{V_i \cap V_j}$ . This follows, again, because the action of  $\mathfrak{S}_n$  on  $V_i \cap V_j$  is free. So the  $f_i^{\sigma}$  glue to form a regular function  $f^{\sigma}$  on  $\mathbb{C}^{2n,reg}$ . Since the codimension of  $\mathbb{C}^{2n} \setminus \mathbb{C}^{2n,reg}$  is at least 2,  $f^{\sigma}$  is actually regular in  $\mathbb{C}^{2n} = \mathfrak{h} \oplus \mathfrak{h}^*$ . We are done with surjectivity.

Since  $H_{t,c}e$  is a finitely generated  $eH_{t,c}e$ -module, we can equip it with a filtration compatible with that of  $eH_{t,c}e$  making  $\text{gr}(H_{t,c}e)$  a finitely generated  $\text{gr}(eH_{t,c}e)$ -module, and we can use this filtration to equip  $\text{End}_{eH_{t,c}e}(H_{t,c}e)$  with a filtration such that  $\text{gr } H_{t,c} \rightarrow \text{gr } \text{End}_{eH_{t,c}e}(H_{t,c}e)$  is precisely the isomorphism  $H_{0,0} \rightarrow \text{gr } \text{End}_{eH_{0,0}e}(H_{0,0}e)$ . It follows that the original morphism  $H_{t,c} \rightarrow \text{End}_{eH_{t,c}e}(H_{t,c}e)$  is an isomorphism.

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