

INVARIANT THEORY, HW4

Problem 1 (5pts). Recall that $\mathcal{D}_n^+ \in \mathrm{Mat}_k^+(\mathbb{C})$ denotes the scheme theoretic intersection of the determinantal scheme \mathcal{D}_n of all matrices of rank $\leq n$ with the subspace $\mathrm{Mat}_k^+(\mathbb{C})$ of symmetric $k \times k$ -matrices. Assuming the scheme \mathcal{D}_n^+ is reduced and normal, prove that $(\mathbb{C}^n)^{\oplus k} // \mathrm{O}_n(\mathbb{C}) \xrightarrow{\sim} \mathcal{D}_n^+$.

Problem 2 (5pts). Show that $\mathrm{Mat}_n^+(\mathbb{C}) // \mathrm{O}_n(\mathbb{C})$ (the quotient under the action $g.A := gAg^{-1}$) is the polynomial algebra in the polynomials $F_i, i = 1, \dots, n$, given by $F_i(A) := \mathrm{Tr}(A^i)$.

Problem 3 (5pts). Consider the Grassmannian cone $\mathrm{CGr}(k, n) \subset \Lambda^k \mathbb{C}^n$ of decomposable wedges with the reduced scheme structure. Define $f \in \mathbb{C}[\mathrm{CGr}(k, n)]$ that sends $\sum_{\alpha} c_{\alpha} e_{\alpha_1} \wedge e_{\alpha_2} \wedge \dots \wedge e_{\alpha_k}$, where the summation is taken over all increasing sequences $\alpha = (\alpha_1 < \alpha_2 < \dots < \alpha_k)$, to $c_{(1, 2, \dots, k)}$. Show that $\mathbb{C}[\mathrm{CGr}(k, n)]^U$ (where U is the subgroup of the unitriangular matrices in GL_n) is the polynomial algebra in the single variable f .¹

Extra-credit problem. Here we describe a canonical G -algebra with given algebra of U -invariants. We also show that for any G -algebra, there is a G -equivariant filtration whose associated graded is this canonical algebra.

a) Let A be a G -algebra, i.e., a commutative associative unital algebra equipped with a locally finite algebraic action of G . For $\lambda \in \mathfrak{X}^+$, let A_{λ} denote the isotypic component of $V(\lambda)$ in A . We say that A is \mathfrak{X}^+ -graded if $A = \bigoplus_{\lambda} A_{\lambda}$ is an algebra grading. Show that the categories of \mathfrak{X}^+ -graded G -algebras is equivalent to the category of \mathfrak{X}^+ -graded algebras via $A \mapsto A^U$. Moreover, show that a quasi-inverse functor is given by $B \mapsto (\mathbb{C}[G/U] \otimes B)^T$, where the action of T on $V(\mu) \otimes A_{\lambda}$ is by $t \mapsto \mu(t)\lambda(t)^{-1}$.

b) Now let A be any G -algebra. Define an ascending $\mathbb{Z}_{\geq 0}$ -filtration $A_{\leq i}$ on A by $A_{\leq i} := \bigoplus_{\langle \lambda, \rho^{\vee} \rangle \leq i} A_{\lambda}$. Show that it is an algebra filtration and that $\mathrm{gr} A = (\mathbb{C}[G/U] \otimes A^U)^T$. In particular, $\mathrm{gr} A$ is finitely generated.

¹This shows that $\mathrm{CGr}(k, n)$ is a normal variety. In fact, one can also show that it is reduced with its natural scheme structure given by the Plücker equations.