Lecture 23.

1) Algebraic subsets vs radical ideals
2) Prime ideals & irreducibility.

Refs: [E], Section 1.6, [V], Section 9.4.

BONUSES: 1) Is this ideal radical?
2) Tensor products of algebras vs products of algebraic subsets.

1) Algebraic subsets vs radical ideals

1.1) Definitions:

Below $\mathbb{F}$ denotes an alg. closed field. Let A be commutative ring.

Definition: An ideal $I \subset A$ is *radical* if $I = \sqrt{I}$.

Def'n: For ideal $I \subset \mathbb{F}[x_1, \ldots x_n]$, define $V(I) := \{\alpha \in \mathbb{F}^n \mid f(\alpha) = 0 \ \forall f \in I\}$

Note that if $I = (f_1, \ldots f_k)$ - and any ideal has this form b/c $\mathbb{F}[x_1, \ldots x_n]$ is Noetherian - then $V(I) = V(f_1, \ldots f_k)$.

By Lemma in Sect 1.1 of Lec 22, $V(\sqrt{I}) = V(I)$.

Definition: • A subset $X \subset \mathbb{F}^n$ is *algebraic* if $X = V(I)$ for some ideal $I \subset \mathbb{F}[x_1, \ldots x_n]$, equiv. $X = V(f_1, \ldots f_k)$ for some $f_1, \ldots f_k \in \mathbb{F}[x_1, \ldots x_n]$.

• For $X \subseteq \mathbb{F}^n$ set $I(X) := \{f \in \mathbb{F}[x_1, \ldots x_n] \mid f|_X = 0\}$, this is a radical ideal in $\mathbb{F}[x_1, \ldots x_n]$ (exercise).

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• $\mathbb{F}[X] := \mathbb{F}[x_1, \dots x_n]/I(X)$, the algebra of polynomial functions on X. Note that it has no nonzero nilpotent elements.

An element of $\mathbb{F}[X]$ can be viewed as a function $X \to \mathbb{F}$: for $f \in \mathbb{F}[x_1, \dots x_n]$, the restriction $f|_X$ only depends on $f + I(X) \in \mathbb{F}[X]$. Further, let $\pi: \mathbb{F}[x_1, \dots x_n] \twoheadrightarrow \mathbb{F}[X]$ denote the projection, it sends f to $f|_X$, the restriction.

1.2) Basic properties:

Corollary (of Nullstellensatz): the maps $I \mapsto V(I)$ & $X \mapsto I(X)$ are inclusion-reversing & mutually inverse bijections between:

$\{$radical ideals in $\mathbb{F}[x_1, \dots x_n]\}$

$\{$algebraic subsets in $\mathbb{F}^n\}$

Proof: By construction, both $I \mapsto V(I)$ & $X \mapsto I(X)$ reverse inclusions (exercise). It remains to check that

i) $I = I(V(I))$: if $I = (f_1, \dots f_k)$, then $V(I) = V(f_1, \dots f_k) \Rightarrow I(V(I)) =$
$= \{f \mid f \text{ is } 0 \text{ on } V(f_1, \dots f_k)\} = [\text{Nullstellensatz}] = \sqrt{I} = [I \text{ is radical}] = I$.

ii) $\forall$ algebraic subsets $X \subseteq \mathbb{F}^n \Rightarrow X = V(I(X))$: note $X = V(J)$ for some ideal J. Can replace J w. $\sqrt{J}$ & assume J is radical. Hence $V(I(V(J)) = [\text{by i}), I(V(J)) = J] = V(J)$. This finishes the proof $\square$

Now we discuss the behavior of the bijections in the corollary under intersections (of ideals & of algebraic subsets)

Lemma: Let $X, Y \subseteq \mathbb{F}^n$ be algebraic subsets.

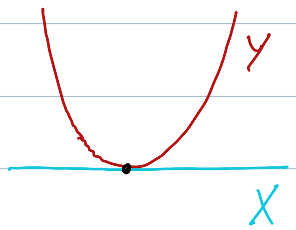
(a) $X \cup Y$ is algebraic w. $I(X \cup Y) = I(X) \cap I(Y)$.

(b) $X \cap Y$ is algebraic with $I(X \cap Y) = \sqrt{I(X) + I(Y)}$

Example: $n=2$, $X = V(y)$, $Y = V(y - x^2)$

$I(X) = (y)$, $I(Y) = (y - x^2)$

$X \cap Y = \{(0,0)\}$, $I(X) + I(Y) = (y - x^2, y) = (x^2, y)$ – not radical



Proof of Lemma

(a) $I := I(X)$, $J := I(Y)$ – radical ideals. Observe that:

- $I \cap J$ is radical (exercise).
- for $I = (f_1, \dots, f_k)$, $J = (g_1, \dots, g_\ell) \Rightarrow X \cup Y = \{a \in \mathbb{F}^n \mid f_i g_j (a) = 0 \ \forall i, j\}$

Since $(f_i g_j \mid i = 1, \dots, k, j = 1, \dots, \ell) = IJ \Rightarrow X \cup Y = V(IJ)$.

- $(I \cap J)^2 \subset IJ \subset I \cap J$, so $\sqrt{IJ} = I \cap J$ & $V(IJ) = V(I \cap J)$.

(b) $X \cap Y = V(f_1, \dots, f_k, g_1, \dots, g_\ell)$ & $(f_1, \dots, f_k, g_1, \dots, g_\ell) = I + J$. So

$X \cap Y = V(I + J) \Rightarrow I(X \cap Y) = \sqrt{I + J}$ $\square$

Exercise: if $X \cap Y = \phi$, then $\mathbb{F}[X \amalg Y] = \mathbb{F}[X] \times \mathbb{F}[Y]$.

Finally, let us mention products.

Proposition: Let $X \subset \mathbb{F}^n$, $Y \subset \mathbb{F}^m$ be algebraic subsets. Then $X \times Y \subset \mathbb{F}^{n+m}$ is an algebraic subset & $\mathbb{F}[X\times Y] = \mathbb{F}[X] \otimes \mathbb{F}[Y]$.

This will be proved in Bonus 2.

2) Prime ideals & irreducibility.

Reminder on prime ideals: A is commutative ring, $I \subset A$ ideal. Say I is prime (Lec 3, Sect 1) if one of equiv't conditions hold:

1) A/I is domain
2) $a_1, a_2 \notin I \Rightarrow a_1 a_2 \notin I$.
3) if $I_1, I_2 \subset A$ are ideals & $I_1 I_2 \subset I \Rightarrow I_1$ or $I_2 \subset I$.

In particular, prime $\Rightarrow$ radical.

Recall that $\mathbb{F}$ is an algebraically closed field.

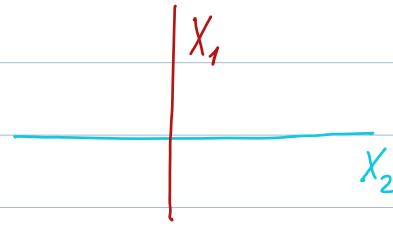
Question: find a geometric characterization of algebraic subsets in $\mathbb{F}^n$ corresponding to prime ideals.

2.1) Irreducible algebraic subsets.

Definition: an alg. subset X in $\mathbb{F}^n$ is called

- **irreducible**: if $X = X_1 \cup X_2$, where $X_i \subset \mathbb{F}^n$ is algebraic, implies $X = X_i$ for some i.
- **reducible**, else.

Example: Set $X = V(x_1 x_2) \subset \mathbb{F}^2$. It's reducible: $X = X_1 \cup X_2$, where $X_1 = V(x_1)$, $X_2 = V(x_2)$



Proposition: TFAE

(a) X is irreducible.

(b) $I(X) \left(= \{f \in \mathbb{F}[x_1,\dots x_n] \mid f|_X = 0\}\right)$ is prime.

(c) $\mathbb{F}[X] \left(= \mathbb{F}[x_1,\dots x_n]/I(X)\right)$ is a domain.

Proof: (b) $\Leftrightarrow$ (c) is standard.

(a) $\Rightarrow$ (b): assume that $I(X)$ isn't prime, i.e. $\exists f_i \in \mathbb{F}[x_1,\dots x_n] \setminus I(X)$ s.t. $f_1 f_2 \in I(X)$; $X_i := \{\alpha \in X \mid f_i(\alpha) = 0\}$, $i = 1,2$. Then $X_i \subsetneq X$ (properly b/c $f_i \notin I(X)$, i.e. $f_i|_X \neq 0$), is an alg'c subset & $X_1 \cup X_2 = \{\alpha \in X \mid (f_1 f_2)(\alpha) = 0\} = [f_1 f_2 \in I(X)] = X$. Contradiction w. X being irreducible.

(b) $\Rightarrow$ (a): assume X is reducible: $X = X_1 \cup X_2$ w. $X_i \subsetneq X$ alg'c subset, define $I_i := I(X_i) \supsetneq I(X)$ ($\supsetneq$ is b Corollary in Sect. 1.2). By Lemma in Sec. 1.2, $I(X) = I_1 \cap I_2$, so $I(X) \supset I_1 I_2$. Since $I(X)$ is prime $\Rightarrow$ say $I(X) \supset I_1 \Leftrightarrow$ [by the same Corollary] $X \subset V(I_1) = X_1$. Contradiction w. $X_1 \subsetneq X$. $\square$

Examples: 1) $\mathbb{F}^n$ is irreducible b/c $\mathbb{F}[\mathbb{F}^n] = \mathbb{F}[x_1,\dots x_n]$ is domain.

2) Let $f \in \mathbb{F}[x_1,\dots x_n]/(f)$. Decompose $f = f_1^{n_1} \cdots f_k^{n_k}$, where f_i's are irreducible. Then $V(f) \subset \mathbb{F}^n$ is irreducible $\Leftrightarrow k = 1$.

2.2) Irreducible components.

Theorem: Let X be an algebraic subset in $\mathbb{F}^n$. Then

a) $\exists$ irreducible algebraic subsets $X_1, \dots X_k$ s.t. $X = \bigcup_{i=1}^{k} X_i$.

b) For $X_1, \dots X_k$ we can take maximal (w.r.t. inclusion) irreducible algebraic subsets contained in X.

Note, that (b) recovers $X_1, \dots X_k$ uniquely.

Def'n: These $X_1, \dots X_k$ (from b)) are called irreducible components of X.

Example: Irreducible components of $V(x_1 x_2)$ are $V(x_1)$ & $V(x_2)$. More generally, for $f = f_1^{n_1} \dots f_k^{n_k}$, the irreducible components of $V(f)$ are $V(f_1), \dots, V(f_k)$.

Proof of Theorem:

a) Assume the contrary: $\exists$ $X \neq$ finite union of irreducibles $\Leftrightarrow$ the set $\mathfrak{A}$ of all such X's is $\neq \emptyset$. $\rightsquigarrow$ nonempty set $\{I(X) \mid X \in \mathfrak{A}\}$. Since $\mathbb{F}[x_1, \dots x_n]$ is Noetherian, every nonempty set of ideals has maximal (w.r.t $\subset$) element. Pick $X' \in \mathfrak{A}$ s.t. $I(X')$ is maximal in $\{I(X) \mid X \in \mathfrak{A}\}$ $\Longleftrightarrow$ X' is minimal in $\mathfrak{A}$ w.r.t. $\subset$. But X' is reducible b/c $X' \in \mathfrak{A}$ $\Longleftrightarrow$ $X' = X^1 \cup X^2$ w. $X^i \subsetneq X'$ $\Rightarrow$ [X' is min'l in $\mathfrak{A}$] $X^i \notin \mathfrak{A}$ $\rightsquigarrow$ $X^i = \bigcup_j X^i_j$ (finite unions of irreducibles) $\rightsquigarrow$ $X' = \bigcup_j X^1_j \cup \bigcup_j X^2_j$ – contradicts $X' \in \mathfrak{A}$.

b) $X = \bigcup_{i=1}^{k} X_i$, where assume that none of X_i's is contained in another. Need to show: if $Y \subset X$ max'l irreducible $\Rightarrow Y = X_i$ (for autom. unique i). To prove this, we observe

$Y = \bigcup_{i=1}^{k} (Y \cap X_i)$; since Y is irreducible $\Rightarrow Y = Y \cap X_i$ for some $i \Rightarrow Y \subset X_i$, by since Y is maximal, $Y = X_i$. $\square$

Corollary (alg'c formulation of Thm): Let $I \subset \mathbb{F}[x_1, \dots x_n]$ be radical ideal. Then $I = \bigcap_{i=1}^{k} I_i$, where I_i is prime; and we can recover I_i's uniquely if we assume they are minimal (w.r.t $\subseteq$) w. $I \subset I_i$.

Remark: the same statement is true if $\mathbb{F}[x_1, \dots, x_n]$ w. arbitrary Noetherian ring (premium exercise). There's a suitable generalization to arbitrary ideals: primary decomposition, [AM], Ch. 4 & 7.1.

BONUS: 1) Is this ideal radical?

We've talked about various properties of ideals (being radical/prime) and rings (being a normal domain). We work w. the ring $\mathbb{F}[x_1, \dots x_n]$, where $\mathbb{F}$ is a field, its ideals & quotients. Usually, the ideals are specified by their generators. So we can ask the following questions:

I) Given $F_1, \dots F_k \in \mathbb{F}[x_1, \dots x_n]$ is the ideal, can we determine whether $(F_1, \dots F_k)$ is radical or prime?

As usual, the answer is both Yes & No.

Yes: for given n, k (& $F_1, \dots F_k$) there are algorithms (often implemented in Computer Algebra software) that allow to answer these and related questions. The main approach is via Gröbner bases. For more on them, see [E], Chapter 15.

No: if we care about the situation where we have a family of ideals with varying n, k.

Here's a famous example. Consider the space of pairs of square matrices, $\mathrm{Mat}_n(\mathbb{C})^2 \simeq \mathbb{C}^{2n^2}$. We have n^2 quadratic polynomials in these $2n^2$ variables – the entries of the matrix commutator $[A,B] = AB - BA$. For example, for $n=2$ we have

$$\left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\right] = \begin{pmatrix} x_{12}y_{21} - y_{12}x_{21} & x_{11}y_{12} + x_{12}y_{22} - y_{11}x_{12} - y_{12}x_{22} \\ x_{21}y_{11} + x_{22}y_{21} - y_{21}x_{11} - y_{22}x_{21} & y_{12}x_{21} - x_{12}y_{21} \end{pmatrix}$$

In fact, as this example indicates, the n^2 polynomials we get are linearly dependent – $\mathrm{tr}[A,B] = 0$. In any case, let I be the ideal generated by these polynomials so that
$V(I) = \{(A,B) \in \mathrm{Mat}_n(\mathbb{C})^2 \mid AB = BA\}$, a.k.a. the commuting variety.

Open problem: is I radical?

BONUS 2: Tensor product of algebras vs product of algebraic subsets.

Proof of Proposition in Section 1.2:
Let $I(X) = (f_1, \dots, f_k) \subset \mathbb{F}[x_1, \dots x_n]$, $I(Y) = (g_1, \dots g_\ell) \subset \mathbb{F}[y_1, \dots y_m]$.
$X \times Y = \{(\alpha, \beta) \in \mathbb{F}^n \times \mathbb{F}^m = \mathbb{F}^{n+m} \mid f_i(\alpha) = 0, g_j(\beta) = 0 \ \forall i = 1, \dots k, j = 1, \dots, \ell\}$

-algebraic subset. Recall (Example in Sec. 2.2 of Lec 18) that $\mathbb{F}[X]\otimes_{\mathbb{F}}\mathbb{F}[Y]=\mathbb{F}[x_1,\dots,x_n,y_1,\dots,y_n]/(f_1,\dots f_k,g_1,\dots g_\ell)$. Note that $f_1,\dots f_k$, $g_1,\dots,g_\ell$ vanish on $X\times Y\subset\mathbb{F}^{n+m}$, equivalently, $f_1,\dots f_k,g_1,\dots g_\ell\in I(X\times Y)$.

So we have a surjective algebra homomomorphism

$$\pi: \mathbb{F}[X]\otimes_{\mathbb{F}}\mathbb{F}[Y]=\mathbb{F}[x_1,\dots x_n,y_1,\dots y_m]/(f_1,\dots f_k,g_1,\dots g_\ell)\twoheadrightarrow\mathbb{F}[x_1,\dots x_n,y_1,\dots y_m]/I(X\times Y)$$
$$=\mathbb{F}[X\times Y]$$

Exercise: $\pi(F\otimes G)(\alpha,\beta)=F(\alpha)G(\beta)$.

It remains to show π is injective. Let F_r, $r\in R$, be an $\mathbb{F}$-basis in $\mathbb{F}[X]$; G_s, $s\in S$, $\mathbb{F}$-basis in $\mathbb{F}[Y]$, so $F_r\otimes G_s$ form an $\mathbb{F}$-basis in $\mathbb{F}[X]\otimes_{\mathbb{F}}\mathbb{F}[Y]$. Need to show that

$$\pi\Big(\sum_{r,s}a_{rs}F_r\otimes G_s\Big)=0 \text{ (equality of functions on } X\times Y)$$
$\Rightarrow a_{rs}=0$.

Fix $\beta\in Y$.

Then the function $\sum_{r,s}a_{rs}G_s(\beta)F_r: X\to\mathbb{F}$ is zero

$\sum_{r,s}a_{rs}G_s(\beta)F_r\in\mathbb{F}[X]\Rightarrow\forall r\ \sum_s a_{rs}G_s(\beta)=0$ b/c F_r's form a vector space basis in $\mathbb{F}[X]$.

Now we can vary β: $\sum_s a_{rs}G_s=0\Rightarrow a_{rs}=0$. $\square$