Lecture 18.

1) Tensor products of modules, finished.
2) Tensor products of algebras.
3) Adjoint functors.

Refs: [AM], Sect. 2.11, [R], Section 4.1, [HS], Section 2.7.

1.1) "Algebra properties" of tensor products.

Theorem: Let M_1, M_2, M_3 be A-modules. Then:

1) There is a unique isomorphism $(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A (M_2 \otimes_A M_3)$ s.t. $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$. (i.e. tensor product is associative).

2) $\exists!$ isom'm $M_1 \otimes_A M_2 \xrightarrow{\sim} M_2 \otimes_A M_1$ w. $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$.

3) $\exists!$ isom'm $M_1 \otimes_A (M_2 \oplus M_3) \xrightarrow{\sim} M_1 \otimes_A M_2 \oplus M_1 \otimes_A M_3$ w. $m_1 \otimes (m_2, m_3) \mapsto (m_1 \otimes m_2, m_1 \otimes m_3)$

4) $\exists!$ unique isom'm $A \otimes_A M \xrightarrow{\sim} M$ s.t. $a \otimes m \mapsto am$.

Proof: (1) We want an A-linear map

$\tilde{\beta}: (M_1 \otimes_A M_2) \otimes_A M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$, $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$

i.e. want a bilinear map $\beta: (M_1 \otimes_A M_2) \times M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$

$(m_1 \otimes m_2, m_3) \mapsto m_1 \otimes (m_2 \otimes m_3)$

Fix $m_3 \rightsquigarrow$ a linear map $M_2 \to M_2 \otimes_A M_3$, $m_2 \mapsto m_2 \otimes m_3$. Define $\beta_{m_3}: M_1 \otimes_A M_2 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$ to be the tensor product of id_{M_1} & $[m_2 \mapsto m_2\otimes m_3]$ so $\beta_{m_3}(m_1\otimes m_2) = m_1\otimes(m_2\otimes m_3)$

Note that β_{m_3} depends linearly on m_3

$\rightsquigarrow$ bilinear map $\beta: (M_1\otimes_A M_2)\times M_3 \longrightarrow M_1\otimes_A(M_2\otimes_A M_3)$,

$\beta(x, m_3) := \beta_{m_3}(x) \rightsquigarrow \tilde{\beta}$ as needed.

But $\tilde{\beta}$ is an isom'm: have $M_1\otimes_A(M_2\otimes_A M_3)\to(M_1\otimes_A M_2)\otimes M_3$

$m_1\otimes(m_2\otimes m_3)\mapsto(m_1\otimes m_2)\otimes m_3$. It's inverse of $\tilde{\beta}$ b/c this is so on generators. $\square$ of (1).

(2) - commutativity - is an exercise & (4) - unit - follows from our construction.

Proof of (3) - distributivity: consider the projection $\pi_i: M_2\oplus M_3 \longrightarrow M_i$, $i=2,3$; & inclusion $\iota_i: M_i \hookrightarrow M_2\oplus M_3$

$\rightsquigarrow id_{M_1}\otimes\pi_i: M_1\otimes_A(M_2\oplus M_3) \rightleftarrows M_1\otimes_A M_i : id_{M_1}\otimes\iota_i$

$$(id_{M_1}\otimes\pi_2, id_{M_1}\otimes\pi_3): M_1\otimes_A(M_2\oplus M_3) \rightleftarrows M_1\otimes_A M_2\oplus M_1\otimes_A M_3 : (id_{M_1}\otimes\iota_2, id_{M_1}\otimes\iota_3)$$

$$id_{M_1}\otimes\iota_2(x) + id_{M_1}\otimes\iota_3(y) \longleftarrow (x,y)$$

Exercise: check that these maps are mutually inverse.

2) Tensor product of algebras.

2.1) Construction.

Let A be a commutative ring, B, C be A-algebras $\rightsquigarrow B\otimes_A C$, A-module.

Proposition: $\exists!$ A-algebra str're on $B\otimes_A C$ s.t. $(b_1\otimes c_1)\cdot(b_2\otimes c_2) = b_1b_2\otimes c_1c_2$ (w. unit $1\otimes 1$).

Proof: Uniqueness will follow b/c elements $b\otimes c$ span $B\otimes_A C$ & a bilinear map is uniquely determined by images of generators.

Need to show existence. The product map $B\times B\to B$ is A-bilinear $\rightsquigarrow$ $\mu_B: B\otimes_A B\to B$, $b_1\otimes b_2\mapsto b_1b_2$. Similarly, have $\mu_C: C\otimes_A C\to C$ $\rightsquigarrow$

$$\begin{tikzcd}
 & & \mu_B\otimes\mu_C: (B\otimes_A B)\otimes_A (C\otimes_A C) \arrow[r] & B\otimes_A C \\
x\otimes y \arrow[r, phantom, "\in"] & {} & (B\otimes_A C)\otimes_A (B\otimes_A C) \arrow[u, "\wr", "\text{assoc. \& commut. of } \otimes"'] \arrow[ur, "(b_1\otimes c_1)\otimes(b_2\otimes c_2)\mapsto(b_1b_2)\otimes(c_1c_2)"'] & \\
(x,y) \arrow[u, mapsto] \arrow[r, phantom, "\in"] & {} & (B\otimes_A C)\times(B\otimes_A C) \arrow[u] \arrow[uur, "\text{our multiplication map}"'] &
\end{tikzcd}$$

So we've shown existence of a bilinear product map. Associativity & unit axioms can be checked on tensor monomials, e.g. here is a part of unit axiom:

$(1\otimes 1)(b\otimes c) = (1\otimes b)\otimes(1\otimes c) = b\otimes c.$ $\square$

Rem: B, C are commutative $\Rightarrow$ so is $B\otimes_A C$.

2.2) Coproduct.

Theorem: Let B, C be commutative. Then $B\otimes_A C$ is the coproduct of B & C in A-CommAlg (cat'y of commutative A-algebras),

i.e. the functors $\mathrm{Hom}_{A\text{-Alg}}(B\otimes_A C, \cdot)$, $\mathrm{Hom}_{A\text{-Alg}}(B,\cdot)\times \mathrm{Hom}_{A\text{-Alg}}(C,\cdot)$: A-CommAlg $\longrightarrow$ Sets are isomorphic, equiv'y we have algebra homomorphisms $\iota^B: B\to B\otimes_A C$, $\iota^C: C\to B\otimes_A C$ s.t. $\forall$ alg. homom's $\varphi^B: B\to D$, $\varphi^C: C\to D$, where D is a commutative A-algebra, $\exists!$ A-alg. homom. $\varphi: B\otimes_A C\to D$

making the following diagram commutative:

$$\begin{tikzcd}
B \arrow[r, "\iota^B"] \arrow[dr, bend right, "\varphi^B"'] & B\otimes_A C \arrow[d, dashed, "\varphi"] & C \arrow[l, "\iota^C"'] \arrow[dl, bend left, "\varphi^C"] \\
 & D &
\end{tikzcd}$$

Proof: Constr'n of ι^B, ι^C: $\iota^B(b) = b\otimes 1$, $\iota^C(c) = 1\otimes c$. Commutativity $\Rightarrow \varphi(b\otimes 1) = \varphi^B(b)$, $\varphi(1\otimes c) = \varphi^C(c) \Rightarrow \varphi(b\otimes c) = [b\otimes c = (b\otimes 1)(1\otimes c)]$ $= \varphi(b\otimes 1)\varphi(1\otimes c) = \varphi^B(b)\varphi^C(c)$, in turn implying commutativity.

The map $B\times C \to D$, $(b,c)\mapsto \varphi^B(b)\varphi^C(c)$ is A-bilinear, so $\exists!$ A-linear map $\varphi: B\otimes_A C \to D$ w. $\varphi(b\otimes c) = \varphi^B(b)\varphi^C(c)$

What remains to check is: φ respects ring multipl'n (unit is clear), enough to do this on tensor monomials

$$\varphi(b_1\otimes c_1 \cdot b_2\otimes c_2) = \varphi(b_1b_2\otimes c_1c_2) = \varphi^B(b_1b_2)\varphi^C(c_1c_2) =$$
$$= \varphi^B(b_1)\varphi^B(b_2)\varphi^C(c_1)\varphi^C(c_2) = [D \text{ is comm've}] = (\varphi^B(b_1)\varphi^C(c_1))\cdot (\varphi^B(b_2)\varphi^C(c_2)) = \varphi(b_1\otimes c_1)\varphi(b_2\otimes c_2)$$ $\square$

Example: $B = A[x_1,\dots,x_k]/(F_1,\dots F_{k'})$, $C = A[y_1\dots y_\ell]/(G_1,\dots G_{\ell'})$.

Then $B\otimes_A C = A[x_1,\dots x_k, y_1\dots y_\ell]/(\underbrace{F_1,\dots F_{k'}}_{\text{on } x_1\dots x_k}, \underbrace{G_1,\dots, G_{\ell'}}_{\text{on } y_1\dots y_\ell})$, denote the right hand side by D.

Will show isomorphism of functors:

$$\mathrm{Hom}_{A\text{-}Alg}(D,\cdot),\quad \mathrm{Hom}_{A\text{-}Alg}(B,\cdot)\times \mathrm{Hom}_{A\text{-}Alg}(C,\cdot)$$

$\mathcal{F}_B := \mathrm{Hom}_{A\text{-}Alg}(B,\cdot)$, define another functor $\mathcal{F}_B' : A\text{-}CommAlg \to Sets$:
$\mathcal{F}_B'$: comm've alg'a $E \mapsto \{(e_1,\dots,e_k)\in E^k \mid F_i(e_1\dots e_k)=0,\ i=1,\dots k'\}$
for $\varphi: E\to E' \rightsquigarrow \mathcal{F}_B'(\varphi): \mathcal{F}_B'(E)\to \mathcal{F}_B'(E'), (e_1\dots e_k)\mapsto(\varphi(e_1),\dots,\varphi(e_k))$.
-well-defined map b/c $\varphi(e_1)\dots\varphi(e_k)$ satisfy rel'ns of $e_1\dots e_k$.
Then $\mathcal{F}_B \xRightarrow{\sim} \mathcal{F}_B'$ (b/c a homom'm from B is uniquely determined by images of generators $x_1,\dots x_k$ & exists as long as the images satisfy relations, $F_1,\dots,F_{k'}$).
Similarly, we have $\mathcal{F}_C \xRightarrow{\sim} \mathcal{F}_C'$, $\mathcal{F}_D \xRightarrow{\sim} \mathcal{F}_D'$. From descriptions of $B, C, \mathcal{D}$ by generators & rel'ns see $\mathcal{F}_D' \xRightarrow{\sim} \mathcal{F}_B' \times \mathcal{F}_C'$.
This completes the example.

Exercise: Let G_i^B be the image of $G_i \in A[x_1,\dots x_\ell]$ in $B[x_1,\dots x_\ell]$. Note the $B\otimes_A C$ is a B-algebra via ι^B. Show that

$$B\otimes_A C \simeq B[x_1,\dots x_\ell]/(G_1^B,\dots,G_{\ell'}^B)$$

3) Adjoint functors.
Let $\mathcal{C},\mathcal{D}$ be cat's, $F:\mathcal{C}\to\mathcal{D}$, $G:\mathcal{D}\to\mathcal{C}$ be functors.

Definition: F is left adjoint to G if
$\forall X\in Ob(\mathcal{C}), Y\in Ob(\mathcal{D})\ \exists$ bijection $\eta_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ s.t.
(1) $\forall\ X, X'\in Ob(\mathcal{C}), Y\in Ob(\mathcal{D}), X' \xrightarrow{\varphi} X$ the following is comm've:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "?\circ F(\varphi)"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \arrow[d, "?\circ\varphi"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X'),Y) \arrow[r, "\eta_{X',Y}"] & \mathrm{Hom}_{\mathcal{C}}(X',G(Y))
\end{tikzcd}$$

(2) $\forall\, Y, Y' \in Ob(\mathcal{D})$, $Y \xrightarrow{\psi} Y'$; $X \in Ob(\mathcal{C})$, the following is comm've

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "\psi\circ ?"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \arrow[d, "G(\psi)\circ ?"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X),Y') \arrow[r, "\eta_{X,Y'}"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y'))
\end{tikzcd}$$

Remarks: 1) Fix X & consider composition of functors

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{F_X = \mathrm{Hom}_{\mathcal{C}}(X,\bullet)} Sets$$

If F is left adj't to G, then $F(X)$ represents this comp'n i.e. $\mathrm{Hom}_{\mathcal{D}}(F(X),\bullet)$ is isom'c to this comp'n via $\eta_{X,?}$, which is a functor isomorphism by diagram (2).

2*) Can view $\mathrm{Hom}_{\mathcal{C}}(\bullet,?)$ as a functor $\mathcal{C}^{opp}\times\mathcal{C} \longrightarrow Sets$

Similarly for $\mathcal{D} \rightsquigarrow$ compositions $\mathcal{C}^{opp}\times\mathcal{D} \longrightarrow Sets$

$\mathrm{Hom}_{\mathcal{D}}(F(\bullet),?)$, $\mathrm{Hom}_{\mathcal{C}}(\bullet, G(?))$

Diagrams (1) & (2) combine to show that [F is left adj't to G] $\iff$ the two functors above are isomorphic (via $\eta_{\bullet,?}$)