CURRENT AND FUTURE PROJECTS

IVAN LOSEV

Below I describe three of my major current research projects. Each of the first three sections starts with an informal description of problems and results. Sections 1 and 2 continue with already known results and finish with precise results/conjectures and a brief description of techniques I plan to use to approach them.

In the last section I briefly mention some other current projects.

Contents

1. Modular representation theory of semisimple Lie algebras

With Roman Bezrukavnikov we work on character formulas for the irreducible representations of semisimple Lie algebras (e.g. $\mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F})$) over an algebraically closed field $\mathbb{F}$ of characteristic $p \gg 0$. Of course, the case when the base field is $\mathbb{C}$ (or any other algebraically closed field of characteristic 0) is very classical: the classification and computation of characters were due to Cartan, Killing, Weyl in the first half of the 20th century. Our work will compute the characters of the irreducible representations in characteristic $p \gg 0$ in terms of affine Kazhdan-Lusztig polynomials. The approach we plan to use, roughly speaking, is to relate representation categories in characteristic p to categories of equivariant perverse sheaves on various affine flag varieties (that are closely related to representations of affine Lie algebras, all in characteristic 0). The approach is inspired by my work on computing the characters for finite dimensional irreducible modules over finite W-algebras, [2].

1.1. **Basics on representations in positive characteristic.** Let us recall some basics of the representation theory of semisimple Lie algebras over $\mathbb{F}$. Let $G_{\mathbb{F}}$ be a semisimple algebraic group over $\mathbb{F}$ and let $\mathfrak{g}_{\mathbb{F}}$ be its Lie algebra. Let $\mathcal{U}_{\mathbb{F}}$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{F}}$ so a representation of $\mathfrak{g}_{\mathbb{F}}$ is the same as a representation of $\mathcal{U}_{\mathbb{F}}$. The algebra $\mathcal{U}_{\mathbb{F}}$ has two interesting central subalgebras. First, there is the so called Harish-Chandra center, it is the algebra $\mathcal{U}_{\mathbb{F}}^{G_{\mathbb{F}}}$ of $G_{\mathbb{F}}$-invariants. As in characteristic 0, it is isomorphic to the invariant algebra $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$ of the ρ-shifted action of the Weyl group W on $\mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. A special feature of the characteristic p case is another central subalgebra of $\mathcal{U}_{\mathbb{F}}$ called the p-center, it is $G_{\mathbb{F}}$-equivariantly isomorphic to $S(\mathfrak{g}_{\mathbb{F}}^{(1)})$, where the superscript "(1)" stands for the Frobenius twist. Note that $\mathcal{U}_{\mathbb{F}}$ is a free rank $p^{\dim\mathfrak{g}}$ module over $S(\mathfrak{g}_{\mathbb{F}}^{(1)})$. So for $\chi\in\mathfrak{g}_{\mathbb{F}}^{(1)*}$, we can consider the central reduction $\mathcal{U}_{\mathbb{F}}^{\chi}$, this is an $\mathbb{F}$-algebra of dimension $p^{\dim\mathfrak{g}}$. Every irreducible representation of $\mathcal{U}_{\mathbb{F}}$ factors through exactly one of the quotients $\mathcal{U}_{\mathbb{F}}^{\chi}$.

The study of the representation theory of $\mathcal{U}_{\mathbb{F}}^{\chi}$ easily reduces to the case when χ is nilpotent, this is what we are going to assume from now on. The algebra $\mathcal{U}_{\mathbb{F}}^{\chi}$ splits into the direct sum of infinitesimal blocks according to the Harish-Chandra center as follows:

$$\mathcal{U}_{\mathbb{F}}^{\chi}=\bigoplus_{\lambda\in\mathfrak{h}^*_{\mathbb{F}_p}/(W,\cdot)}\mathcal{U}_{\mathbb{F},\lambda}^{\chi}.$$

In fact, it is enough to understand the representation theory of the *principal block* $\mathcal{U}_{\mathbb{F},0}^{\chi}$.

Note that the group $Z_{G_{\mathbb{F}}^{(1)}}(\chi)$ acts on $\mathcal{U}_{\mathbb{F},0}^{\chi}$ by algebra automorphisms. Pick a maximal torus T in this group and consider its centralizer Z in $Z_{G_{\mathbb{F}}^{(1)}}(\chi)$. We can consider the category $\mathcal{U}_{\mathbb{F},0}^{\chi}$-mod^Z of Z-equivariant modules. Our aim is to describe this category (or more precisely, a closely related category in characteristic 0) in terms of perverse sheaves.

1.2. **Previous results by Bezrukavnikov and collaborators.** In [BMR] it was shown that $K_0(\mathcal{U}_{\mathbb{F},0}^{\chi}\text{-mod})\cong K_0(\operatorname{Coh}(\mathcal{B}_\chi))$, where we write $\mathcal{B}_\chi$ for the Springer fiber of χ. This equality upgrades to an equivalence of categories $D^b(\mathcal{U}_{\mathbb{F},0}^{\chi}\text{-mod})\cong D^b(\operatorname{Coh}(\mathcal{B}_\chi))$ (where, strictly speaking, we need to view $\mathcal{B}_\chi$ as a derived scheme). It also upgrades to the equivariant setting (where we consider actions by subgroups of $Z_{G_{\mathbb{F}}^{(1)}}(\chi)$).

In [BM], Bezrukavnikov and Mirkovic identified the basis of simple $\mathcal{U}_{\mathbb{F},0}^{\chi}$-mod-modules in $K_0(\operatorname{Coh}(\mathcal{B}_\chi))$. The description is as follows. We have a contracting $\mathbb{G}_m$-action on $\mathcal{B}_\chi$ and can consider the corresponding equivariant K_0-group $K_0(\operatorname{Coh}^{\mathbb{G}_m}(\mathcal{B}_\chi))$. It is a $\mathbb{Z}[q,q^{-1}]$-module specializing to $K_0(\operatorname{Coh}(\mathcal{B}_\chi))$ at $q=1$. Lusztig defined a canonical basis in $K_0(\operatorname{Coh}^{\mathbb{G}_m}(\mathcal{B}_\chi))$ consisting of elements that are invariant with respect to a suitable bar-involution and almost orthogonal with respect to a suitable symmetric form ("almost orthogonal" means that, for two basis elements, we have $(b,b')\in\delta_{b,b'}\in q\mathbb{Z}[q]$). Bezrukavnikov and Mirkovic have proved, in particular, that the basis of simples in $K_0(\operatorname{Coh}(\mathcal{B}_\chi))$ is obtained from Lusztig's canonical basis by specialization to $q=1$.

This is an impressive result, however, it doesn't allow to answer several basic questions about the representation theory of $\mathcal{U}_{\mathbb{F},0}^{\chi}$, for example, to compute the dimensions of irreducibles (or characters, if we consider a suitable equivariant category). A primary problem is that there's no algorithmic way to compute Lusztig's canonical basis in this case, because, for a general χ, there is no standard basis in K_0 with good upper triangularity properties.

To finish this section, let me explain another important observation of Bezrukavnikov. Let $G, \mathfrak{g}$ be the corresponding semisimple algebraic group and its Lie algebra over $\mathbb{C}$. Note that there is a natural bijection between nilpotent orbits in $\mathfrak{g}$ and in $\mathfrak{g}_{\mathbb{F}}^{(1)*}$. Pick an element e in the orbit corresponding to that of χ. Bezrukavnikov constructed a G-equivariant algebra $\mathcal{A}$ on $\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}$ with the following property: the reduction mod p of the fiber $\mathcal{A}_e$ is Morita equivalent to $\mathcal{U}^{\chi}_{\mathbb{F},0}$-mod. So we can consider $\mathcal{A}_e$-mod instead of $\mathcal{U}^{\chi}_{\mathbb{F},0}$-mod (and $\mathcal{A}_e$-mod^Z instead of $\mathcal{U}^{\chi}_{\mathbb{F},0}$-mod^Z). The point is that we want to use connections with perverse sheaves on affine flag varieties that behave much better in characteristic 0.

1.3. **Results and conjectures.** Recall that we write Z for the centralizer of the maximal torus in $Z_G(e)$. Our goal is to relate the category $\mathcal{A}_e$-mod^Z to a suitable category of perverse sheaves. Let us start with the case when $T=\{1\}$, nilpotent elements e with this property are called *distinguished*. From e one can produce a parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}$: one includes e into an $\mathfrak{sl}_2$-triple (e,h,f) and takes the sum of non-negative $[h,\cdot]$-eigenspaces for $\mathfrak{p}$.

Now consider the Langlands dual group $G^\vee$ and its loop group $\mathcal{G}^\vee = G^\vee((t))$. Let $B^\vee\subset G^\vee$ be the Borel subgroup, $U^\vee\subset B^\vee$ be the maximal unipotent subgroup, and $P^\vee\subset G^\vee$ be the parabolic subgroup containing $B^\vee$ corresponding to $\mathfrak{p}\subset\mathfrak{g}$. Further, let $\mathcal{I}^\vee\subset\mathcal{G}^\vee$ be the Iwahori subgroup, the preimage of $B^\vee$ in $G^\vee[[t]]$ under the evaluation map $G^\vee[[t]]\twoheadrightarrow G^\vee$. Let $\mathcal{P}^\vee, \mathcal{I}^\vee_0$ be the similar preimages of $P^\vee, U^\vee$.

Then we can form the partial affine flag variety $\mathcal{F}l_P := \mathcal{G}^\vee/\mathcal{P}^\vee$ and consider the category $\mathrm{Perv}_{\mathcal{I}_0^\vee}(\mathcal{F}l_P)$ of $\mathcal{I}_0^\vee$-equivariant perverse sheaves on $\mathcal{F}l_P$. This category admits a filtration by Serre subcategories indexed by two-sided cells in the affine Weyl group (that by a result of Lusztig are in bijection with nilpotent orbits in $\mathfrak{g}$).

Here is our first result that, by now, is basically proved.

Theorem 1.1. *There is an exact quotient functor* $\mathrm{Perv}_{\mathcal{I}_0^\vee}(\mathcal{F}l_P)\twoheadrightarrow\mathcal{A}_e\text{-mod}_0^Z$ *whose kernel is spanned by the simples corresponding to the* G*-orbits in* $\overline{Ge}\setminus Ge$.

Note that there are standard objects in $\mathrm{Perv}_{\mathcal{I}_0^\vee}(\mathcal{F}l_P)$. Their images in $K_0(\mathrm{Coh}^Z(\mathcal{B}_e))$ should be easy to understand. For the category $\mathrm{Perv}_{\mathcal{I}_0^\vee}(\mathcal{F}l_P)$, the multiplicities of simples in standards are given by the values at $q=1$ of parabolic affine Kazhdan-Lusztig polynomials. This should lead to dimension formulas for the simples.

Let us proceed to the general case. Let L denote the centralizer of T in G, this is a Levi subgroup. We include it into a parabolic subgroup Q. Let M denote the unipotent radical of Q. Note that e is distinguished in $\mathfrak{l}$, let $\underline{\mathfrak{p}}\subset\mathfrak{l}$ be the corresponding parabolic subalgebra. Set $\mathfrak{p}=\underline{\mathfrak{p}}+\mathfrak{m}$, this is a parabolic subalgebra in $\mathfrak{g}$ contained in $\mathfrak{q}$. We are going to consider an appropriately defined category $\mathrm{Perv}_{\mathcal{I}_0^\vee}$ on the partial affine flag variety $\mathcal{G}^\vee/(\underline{\mathcal{P}}^\vee\ltimes M^\vee((t)))$ and establish an exact quotient functor $\mathrm{Perv}_{\mathcal{I}_0^\vee}\twoheadrightarrow\mathcal{A}_e\text{-mod}_0^Z$. Both categories should be standardly stratified in a suitable sense and the quotient functor should respect the filtration giving the quotient functor from Theorem 1.1 on the associated graded. This should allow us to express the multiplicities in $\mathcal{A}_e\text{-mod}_0^Z$ via suitable affine Kazhdan-Lusztig polynomials. Note that the very special case of $e=0$ (here Z is a maximal torus in G) was essentially done in [AB^{2}GM].

2. Modular representation theory of rational Cherednik algebras

Another project with Bezrukavnikov that I'm working on right now is on the representation theory of rational Cherednik algebras for the symmetric groups S_n (and of more general quantized Nakajima quiver varieties) in characteristic $p \gg 0$. The rational Cherednik algebras, H_c, are filtered deformations of the smash-product $\mathbb{F}[V]\# S_n$, where $\mathbb{F} := \overline{\mathbb{F}}_p$ for $p \gg 0$ and V is a vector space with basis $x_1, y_1, \ldots, x_n, y_n$. Here $c \in \mathbb{F}$ is a parameter. Their representation theory is closely related to the geometry of the Hilbert scheme of n points in $\mathbb{A}^2$, the algebraic variety whose points are codimension n ideals in the polynomial algebra $\mathbb{F}[x, y]$ (in the same way as the representation theory of semisimple Lie algebras is related to the geometry of cotangent bundles to flag varieties).

Here's one of the main reasons why the representation theory of rational Cherednik algebras over fields of large positive characteristic is important. Passing from $\mathbb{C}$ to a field of characteristic $p \gg 0$ "affinizes" the representation theory. For example, the multiplicities in interesting categories of representations of $\mathfrak{g}_{\mathbb{C}}$ (such as the Bernstein-Gelfand-Gelfand category $\mathcal{O}$) are expressed via finite Kazhdan-Lusztig polynomials, while over fields of positive characteristic one needs to use affine Kazhdan-Lusztig polynomials (as explained, for example, in the previous section). For rational Cherednik algebras over $\mathbb{C}$, categories $\mathcal{O}$ still make sense, and, in the S_n-case, the multiplicities are expressed via affine Kazhdan-Lusztig polynomials, [R, 3]. So one would expect that in characteristic $p \gg 0$ one will need "double affine Kazhdan-Lusztig polynomials" to describe the multiplicities. Presently, these polynomials are not defined (to do so is one of my longer term goals), while they are expected to play an important role in several representation theoretic context, for example, in the study of the representation theory of p-adic loop groups.

We plan to relate categories of finite dimensional H_c-modules to perverse sheaves on suitable moduli spaces. This, in particular, will establish a Koszul graded lift for the categories of H_c-modules, which can be used to compute the characters of simple modules.

Our approach is based on interactions between two types of categorical structures:

- perverse (in the sense of Chuang and Rouqier) derived equivalences,
- categorical actions of Heisenberg algebras and of elliptic Hall algebras.

2.1. **Background.** The algebra H_c is defined as the quotient of $\mathbb{F}\langle x_1, y_1, \ldots, x_n, y_n\rangle \# S_n$ by the relations

$$[x_i, x_j] = [y_i, y_j] = 0, [y_i, x_j] = c(ij) \text{ for } i \neq j, [y_i, x_i] = 1 - c\sum_{j\neq i}(ij).$$

When we want to indicate the dependence on n we write $H_c(n)$ instead of H_c.

We are interested in the case when $c \in \mathbb{F}_p$. Similarly to the Lie algebra case it makes sense to speak about the p-center: $\mathbb{F}[V^{(1)}]^{S_n}$ is a central subalgebra of H_c, where the superscript "(1)" indicates the Frobenius twist. We consider the category $\mathcal{A}_c$ of all finitely generated H_c-modules, its subcategory $\mathcal{A}_c^0$ of all finite dimensional modules supported at $0 \in V^{(1)}/S_n$, the derived category $\mathcal{C}_c := D^b(\mathcal{A}_c)$ and its subcategory $\mathcal{C}_c^0 := \{M \in \mathcal{C}_c | H_*(M) \in \mathcal{A}_c^0\}$. It is not difficult to show that the simple objects in $\mathcal{A}_c^0$ are labelled by the irreducible representations of S_n, i.e., by Young diagrams with n boxes. For example, the category $\mathcal{A}_0^0$ is equivalent to the category of S_n-equivariant finite dimensional $\mathbb{F}[x_1, y_1, \ldots, x_n, y_n]$-modules supported at 0, the simples are the irreducible S_n-modules with zero action by x_i, y_i.

Assume, for simplicity, that $n!$ divides $p - 1$.

By a *stability interval* we mean a connected component of $\mathbb{R}\setminus\{a/b | a, b \in \mathbb{Z}, GCD(a, b) = 1, 2 \leqslant b \leqslant n\}$. It turns out that, for $c \in \mathbb{Z}$, we have an equivalence $\mathcal{A}_c \cong \mathcal{A}_{c+1}$ if $c/(p-1), (c+1)/(p-1)$ lie in the same stability interval. So we can talk about categories

$\mathcal{A}_I^0 \subset \mathcal{A}_I$ for a stability interval I. We have the *localization equivalence* $\mathrm{Loc}_I : \mathcal{C}_I \xrightarrow{\sim} D^b(\mathrm{Coh}\, X)$, where $X = \mathrm{Hilb}_n(\mathbb{A}^2)$, which is a conical symplectic resolution of V/S_n. For two adjacent stability intervals $I = (a/b, a'/b'), I' = (a'/b', a''/b'')$, the *wall-crossing functor* $\mathfrak{WC}_{I\to I'} := \mathrm{Loc}_{I'}^{-1} \circ \mathrm{Loc}_I$ is right t-exact, and restricts to $\mathcal{C}_I^0 \xrightarrow{\sim} \mathcal{C}_{I'}^0$.

It follows from results of [1] that the functor $\mathfrak{WC}_{I\to I'}$ is *perverse*, which, informally, means some upper-triangularity properties with respect to suitable filtrations by Serre subcategories.

The category $\mathcal{A}_I^0$ admits two graded lifts. The first, "naive", graded lift can be defined purely algebraically. The algebra H_c is graded with $\deg x_i = 1, \deg y_i = -1, \deg S_n = 0$ so we can consider the category $\tilde{\mathcal{A}}_I$ of graded objects in $\mathcal{A}_I$ and the corresponding category $\tilde{\mathcal{A}}_I^0$. This grading comes from the Hamiltonian $\mathbb{G}_m$-action on X.

The second graded lift is more tricky. Consider the contracting $\mathbb{G}_m$-action on X and form the derived category $D_0^b(\mathrm{Coh}^{\mathbb{G}_m} X)$, where the subscript 0 indicates the subcategory of all objects with homology supported on the zero fiber of the Hilbert-Chow morphism $X \to V/S_n$. Bezrukavnikov proved that there are compatible graded lifts $\mathcal{A}_I^{0,gr}, \mathcal{C}_I^{0,gr}$ of $\mathcal{A}_I^0, \mathcal{C}_I^0$ such that Loc_I upgrades to an equivalence $\mathcal{C}_I^{0,gr} \xrightarrow{\sim} D_0^b(\mathrm{Coh}^{\mathbb{G}_m} X)$. For example, the category $\mathcal{A}_0^{0,gr}$ is equivalent to the category of S_n-equivariant finite dimensional graded $\mathbb{F}[x_1, y_1, \ldots, x_n, y_n]$-modules.

We can also consider the similarly defined graded lift $\tilde{\mathcal{A}}_I^{0,gr}$.

We want to relate the categories $\mathcal{A}_I^0$ to categories of perverse sheaves on suitable moduli spaces. Let us explain what moduli spaces we need. Now fix a curve C. Given r and n and a real number $\tau \in (\frac{n}{r}, \frac{n}{r-1})$ one can consider the moduli space $M_\tau = M_\tau(C)$ of *τ-stable pairs* of the form $\mathcal{O} \xrightarrow{\phi} \mathcal{E}$, where $\mathcal{E}$ is a vector bundle on C of rank r and degree n. Then M_τ is a smooth projective variety exactly when τ^{-1} is not a singular number, and $M_{\tau_1} = M_{\tau_2}$ provided τ_1^{-1}, τ_2^{-1} are in the same stability interval. Thus we get a collection of smooth projective moduli spaces indexed by stability intervals; we set $M_I := M_\tau$, $\tau^{-1} \in I$. Then we can consider the category $\mathrm{Perv}(M_I)$ of perverse sheaves on M_I.

2.2. **Results and conjectures.** The following conjecture is due to Bezrukavnikov.

Conjecture 2.1. *For each stability interval, there is a full embedding* $\mathcal{A}_I^0 \hookrightarrow \mathrm{Perv}(M_I)$.

The case of the stability interval $I_0 = (-1/n, 1/n)$ is easy to understand: $M_{I_0} = C^n/S_n$ and the image of $\mathcal{A}_{I_0}^0$ is spanned by the components of the direct image of the constant sheaf on C^n. The basic idea is then to transfer the full embeddings to the adjacent intervals, say moving to the right. For this we want to establish a perverse equivalence between (some full subcategories of) $\mathrm{Perv}(M_I)$ and $\mathrm{Perv}(M_{I'})$, where I, I' are adjacent stability intervals. We want to show, by induction, that the perverse equivalence are "the same" on the Cherednik and on the perverse side. For this we plan to use Heisenberg algebra actions on the relevant derived categories.

So let us now discuss categorical Heisenberg actions. Such an action consists of one endo-functor B_1 of a triangulated category with several additional structures. An exact list of these structures is not known at this point. But two of them are pretty clear and agreed upon: there should be compatible actions of S_k on B^k for all $k > 1$ and B_1 should admit a right adjoint functor isomorphic to the left adjoint up to a homological shift. For example, in the case of categories $\mathcal{O}$ for the rational Cherednik algebras over complex reflection groups $G(\ell, 1, n)$, a categorical Heisenberg action was constructed in [SV]. The construction is based on the Bezrukavnikov-Etingof restriction functors that only make sense for categories $\mathcal{O}$ and so does not extend, say, to the categories of all modules.

With Bezrukavnikov we found an alternative construction of the Heisenberg action on the category $\bigoplus_{n=0}^{\infty} D^b(H_c(n)\text{-mod})$, where we consider the $\mathbb{C}$-algebras $H_c(n)$ and $c \notin (-1,0)$. The functor B_1 is exact and sends the summand $H_c(\bullet)\text{-mod}$ to $H_c(\bullet+b)\text{-mod}$, where b stands for the denominator of c. The construction uses the realization of the spherical subalgebras $eH_c(n)e$ (where $e \in \mathbb{C}S_n$ is the averaging idempotent) as quantum Hamiltonian reduction and it defines B_1 as the tensor product with a suitable (Harish-Chandra) bimodule. It should extend to the setting of quantizations of Nakajima quiver varieties for arbitrary affine type quivers.

And this Heisenberg action is compatible with the wall-crossing functors in a certain sense. More precisely, in characteristic 0 we also have wall-crossing functors $\mathfrak{WC}_{c\to c'}$: $D^b(H_c(n)\text{-mod}) \xrightarrow{\sim} D^b(H_{c'}\text{-mod})$, where $c, c' \notin (-1,0)$ have different signs and integral difference. These functors are defined as derived tensor products with certain *wall-crossing* bimodules. The compatibility with Heisenberg action is that, roughly, the filtrations by Serre subcategories making the equivalences $\mathfrak{WC}_{c\to c'}$ perverse both have the form $\operatorname{im} B_1^0 \supset \operatorname{im} B_1 \supset \operatorname{im} B_1^2 \supset \ldots$.

We can reduce the bimodules defining $B_1, \mathfrak{WC}_{c\to c'}$ to characteristic $p \gg 0$. We get a "partial" Heisenberg action on $\bigoplus_{i=0}^{N} \mathcal{C}^0_{I_n(a/b)}$, where by $I_n(a/b)$ we mean the stability interval (for the given n) such that a/b is contained in I or is the right end-point. And the derived tensor product with the reduction of the wall-crossing bimodule coincides with the wall-crossing functor from the previous section. So there is still a compatibility between the wall-crossing functor and the categorical Heisenberg action in characteristic p.

To prove Conjecture 2.1 we plan to construct these categorical structures on the perverse sheaf side as well. There is a candidate for the categorical Heisenberg action: the functor B_1 should a "Hall product functor" that, roughly speaking, comes from taking extensions with the stable vector bundle of slope a/b. Various properties, such as the t-exactness on suitable perverse sheaves, are non-trivial but we hope to check those. Further, we plan to construct a perverse equivalence between suitable subcategories of the constructible derived categories on $M_I(n), M_{I'}(n)$, where a/b is the right end-point of I and the left end-point of I' compatible with these categorical Heisenberg actions. In this way, once we have an embedding $\mathcal{A}^0_I(n) \hookrightarrow \operatorname{Perv}(M_I(n))$ that is closed under the categorical Heisenberg action corresponding to the point a/b – meaning, in particular, that the Heisenberg functors B_1 intertwine the embeddings $\mathcal{A}^0_I(n-b) \hookrightarrow \operatorname{Perv}(M_I(n-b))$ and $\mathcal{A}^0_I(n) \hookrightarrow \operatorname{Perv}(M_I(n))$ – we will get an embedding $\mathcal{A}^0_{I'}(n) \hookrightarrow \operatorname{Perv}(M_{I'}(n))$. One of the problems we encounter is to show that

(*) the Heisenberg functor B_1' corresponding to the right end-point a'/b' of I' sends the image of $\mathcal{A}_{I'}(n-b')$ to the image of $\mathcal{A}_{I'}(n)$.

This where we are going to use a "categorical elliptic Hall algebra action" (at this point, it is unclear what precisely one should mean by that). The elliptic Hall algebra is an associative algebra with generators labelled by nonzero pairs of integers. The generators with the same ratio satisfy the Heisenberg relations and there are also certain relations for different slopes. We plan to find categorical analogs of these relations and use them to prove (*). Along the way, we hope to better understand the axiomatics for categorical Heisenberg and elliptic Hall algebra actions.

Let us describe some important prospective applications of Conjecture 2.1. First of all, it should imply that the categories $\mathcal{A}_I^{0,gr}$ are Koszul. Using this and the perversity properties of the wall-crossing functors we plan to find character formulas for the simple modules. More precisely, we can identify $K_0(\mathcal{A}_I^{0,gr})$ with $K_0(\mathrm{Coh}^{\mathbb{G}_m} X)$, where, recall $X = \mathrm{Hilb}_n(\mathbb{A}^2)$. The $\mathbb{Z}[q, q^{-1}]$-module $K_0(\mathrm{Coh}^{\mathbb{G}_m} X)$ has the fixed point basis (the fixed points are taken for the action of $\mathbb{G}_m^2$ on X, they are naturally labelled by the partitions on n). One can ask to express the basis of simple objects of $\mathcal{A}_I^{0,gr}$ via the fixed point basis. For example, when $I = I_0$, the coefficients are the modified Macdonald polynomials of Haiman (specialized at $q = t$, to get the non-specialized Macdonald polynomials one needs to consider $\tilde{\mathcal{A}}_{I_0}^{0,gr}$). Using the Koszulity property for the categories $\mathcal{A}_I^{0,gr}$ and the perversity of the wall-crossing functors, we plan to relate the bases of simples for adjacent stability intervals I, I' proving a conjecture of Bezrukavnikov and Okounkov. This will give us an inductive (w.r.t. the stability intervals) way to compute the classes of simples in $\mathcal{A}_I^{0,gr}$.

Let me also add that there are more general conjectures describing categories of modules over quantizations of finite and affine type Nakajima quiver varieties in terms of perverse sheaves on certain moduli spaces generalizing M_I. To prove these conjectures is our longer term goal.

3. Application to $n!$ theorem

The $n!$ theorem of Mark Haiman, [H1], is one of the most significant achievements in the areas of the combinatorics of symmetric polynomials and of the geometry of $\mathrm{Hilb}_n(\mathbb{A}^2)$. On the combinatorial level, it says that the vector space spanned by the derivatives of a certain polynomial $\Delta_\lambda(x_1, y_1, \ldots, x_n, y_n)$ associated to Young diagrams λ with n boxes and generalizing the Vandermond determinant has dimension $n!$. On the algebro-geometric level it says that the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ coincides with an a priori much more complicated object, the S_n-equivariant Hilbert scheme $\mathrm{Hilb}^{S_n}(\mathbb{A}^{2n})$. Haiman's proof (the only proof up to this moment) is very hard as it is based on an extremely involved combinatorial argument. Haiman himself in his survey [H2] says that "I will be the first to admit that it is rather unsatisfactory from a conceptual point of view".

I plan to give a conceptual proof of the $n!$ theorem based on the results conjectured in Section 2. Consider the composition of wall-crossing functors $\mathfrak{WC}_{0\to 1} : \mathcal{C}_{I_0}^0 \to \mathcal{C}_{I_1}^0$, where I_1 is the stability interval $(1 - 1/n, 1 + 1/n)$, this is a right t-exact functor. For example, when $n = 3$, the relevant stability intervals are $I_0 = (-1/3, 1/3), I = (1/3, 1/2), I' = (1/2, 2/3), I_1 = (2/3, 4/3)$ and $\mathfrak{WC}_{0\to 1} = \mathfrak{WC}_{I'\to I_1} \circ \mathfrak{WC}_{I\to I'} \circ \mathfrak{WC}_{I_0\to I}$.

One can show that Haiman's isomorphism $\mathrm{Hilb}_n(\mathbb{A}^2) \cong \mathrm{Hilb}^{S_n}(\mathbb{A}^{2n})$ is equivalent to the claim that there is only one simple object L (corresponding to either trivial or sign representation of S_n, depending on a normalization) such that $H_0(\mathfrak{WC}_{0\to 1} L) \neq 0$. I plan to prove this using the Koszulity of the categories of modules over the algebras H_c and the perversity of the wall-crossing functors. It is possible that the Koszul property can be circumvented by using the theory of categorical stable envelopes of Halpern-Leistner, Maulik and Okounkov, and the current work of Bezrukavnikov and Okounkov relating these categorical stable envelopes to Verma modules in characteristic p.

I expect that techniques explained in Section 2.2 should also apply to a number of other problems in the combinatorics of symmetric polynomials such as proving the so called ∇-conjecture and finding a conceptual proof of the LLT positivity of Macdonald polynomials.

4. Other projects

4.1. **Modular categories O.** I am currently finishing a paper studying modular analogs of categories $\mathcal{O}$ over quantizations of symplectic resolutions (in characteristic $p \gg 0$). This includes, for example, the categories $\tilde{\mathcal{A}}_I^0$ over the rational Cherednik algebras mentioned in Section 2.1. The main result of the paper relates these modular categories to the categories $\mathcal{O}$ in characteristic 0 introduced in this generality in [BLPW] (via establishing certain filtrations on the former so that the associated graded are reductions to characteristic p of the latter). This confirms expectations by physicists (Gaiotto and collaborators).

4.2. **Dimensions of irreducible representations over quantized quiver varieties.** With Andrei Negut, we work on a project to describe categories of finite dimensional representations of quantizations of quantized quiver varieties (for quivers of finite and affine type under some mild restrictions on the quantization parameter) and to compute the dimensions of irreducibles. The part describing the categories is already done.

4.3. **Indexes and Goldie ranks.** With Ivan Panin, we work on a project that bounds the Goldie ranks of primitive quotients of $U(\mathfrak{g})$ by Goldie ranks of suitable equivariant Azumaya algebras on certain homogeneous spaces. We also compute the latter. Together with results of Section 1 this should allow to compute the Goldie ranks of primitive ideals in $U(\mathfrak{g})$ at least for classical Lie algebras $\mathfrak{g}$, which is an old open question in Lie representation theory (solved in [2] for primitive ideals with integral central character).

4.4. **Long wall-crossing bijections for rational Cherednik algebras.** With my student Seth Shelley-Abrahamson we are working on computing the bijection induced by the long wall-crossing functors for categories $\mathcal{O}$ over rational Cherednik algebras associated to the complex reflection groups $G(\ell, 1, n) = S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$ (in the special case of S_n this problem was solved in [4], the bijection is an extension of the Mullineux involution from the modular representation theory of symmetric groups). Our approach is based on studying the interaction of this wall-crossing bijection with crystals that appear in I. Frenkel's level rank duality for affine type A Lie algebras. The motivation for the project is that this bijection is basically the same as the wall-crossing bijection for categories $\mathcal{O}$ over quantizations of Gieseker moduli spaces. The latter should play an important role in studying modular representation theory of the quantizations of Nakajima quiver varieties for affine type quivers.

4.5. **Deligne categories and categorical products of Fock spaces.** This is a joint project with Pavel Etingof that is basically finished and we plan to start writing a paper shortly. We prove that the abelian envelope of the Deligne category of representations of $U_q(\mathfrak{gl}_t)$ (where t is a formal) is a categorical tensor product of categorified highest weight and lowest weight Fock spaces. We then compute the multiplicities for this category by relating it to parabolic affine categories $\mathcal{O}$.

4.6. **The representation theory of quantized Gieseker moduli spaces.** With my student Vasily Krylov (who is currently a master student at HSE, Moscow) we are working on a project continuing my work [5]. We plan to finish computation of supports for the simple objects in categories $\mathcal{O}$ over quantizations of Gieseker moduli spaces (this is basically done by now). We also plan to compute the character of the unique irreducible finite dimensional representation (that should be given by a suitably generalized q-Catalan number).

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