

MATH 380, HOMEWORK 3, DUE OCT 25

There are 8 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if a problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to your solutions to Homeworks 1,2 – or to the posted solutions. The text in *italic* below is meant to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 3pts. Let K be a field and $A \subset K$ be a subring containing an infinite field. Let B denote the integral closure of A in K . Prove that $B[x]$ is the integral closure of $A[x]$ in $K[x]$.

In fact, the claim is true even without the assumption that A contains an infinite field, but it is more complicated.

In particular, we see that if A is a normal domain, then $A[x]$ is also a normal domain.

Problem 2, 4pts total. Let A be a domain and Γ be a subgroup in the group of ring automorphisms of A (recall that an automorphism is an isomorphism $A \rightarrow A$; they do form a group). Set $A^\Gamma := \{a \in A \mid \gamma a = a, \forall \gamma \in \Gamma\}$.

1, 1pt) Prove that A^Γ is a subring of A .

2, 1pt) Suppose A is normal. Prove that A^Γ is normal.

3, 1pt) Suppose Γ is finite. Prove that A is integral over A^Γ .

4, 1pt) Suppose B is a normal domain, $K := \text{Frac}(B)$, L is a finite Galois extension of K with Galois group Γ , and A is the integral closure of B in L . Prove that $B = A^\Gamma$, the equality of subrings in L .

Problem 3, 3pts. Prove that the normalization of the domain $\mathbb{C}[x, y]/(x^2 + x^3 - y^2)$ is isomorphic to $\mathbb{C}[t]$, where t is an indeterminate.

Problem 4, 3pts. Consider the ring $A := \mathbb{C}[x, y]/(xy)$. Construct a ring isomorphism

$$A[(x+y)^{-1}] \xrightarrow{\sim} \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}].$$

A hint: what would the preimages in the left hand side of $(1, 0), (0, 1) \in \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]$ be?

A geometric picture: the locus $\{(x, y) \mid xy = 0, x + y \neq 0\}$ is the disjoint union of $\{(x, 0) \mid x \neq 0\}$ and $\{(0, y) \mid y \neq 0\}$. We will likely revisit this in our discussion of connections to Algebraic geometry later in the class.

Problem 5, 3pts total. *The ring $\mathbb{Z}[\sqrt{-5}]$ strikes back!* Consider the ring $A = \mathbb{Z}[\sqrt{-5}]$ and its ideal $I = (2, 1 + \sqrt{-5})$. Prove that the following localizations of I are rank one free modules over the corresponding localizations of A .

- a, 1pt) $I[2^{-1}]$ over $A[2^{-1}]$.
- b, 2pts) $I[3^{-1}]$ over $A[3^{-1}]$.

This is a part of a more general story. Let A be a Noetherian ring. We say that an A -module is locally free if there are elements $f_1, \dots, f_k \in A$ such that $(f_1, \dots, f_k) = A$ and $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$ -module for any $i = 1, \dots, k$. The problem shows that I is a locally free A -module. The significance of this class of modules will be explained in our discussion of connections with Algebraic geometry later in the class.

Problem 6, 3pts. *Prime ideals and localizations at complements of prime ideals.* Let A be a commutative ring and \mathfrak{p} a prime ideal in A . Show that the map $\mathfrak{q} \mapsto \mathfrak{q}_{\mathfrak{p}}$ defines a bijection between the set of prime ideals of A contained in \mathfrak{p} and the set of all prime ideals in $A_{\mathfrak{p}}$.

Problem 7, 2pts. *Direct sums vs localizations; to be used in one of the subsequent lectures.* Let I be an index set (finite or infinite), $M^i, i \in I$, be A -modules and S be a multiplicative subset of A . Produce a natural isomorphism $(\bigoplus_{i \in I} M^i)[S^{-1}] \xrightarrow{\sim} \bigoplus_{i \in I} (M^i[S^{-1}])$.

Problem 8, 6pts total. *Hom modules vs localizations, and how this helps to compute the modules of homomorphisms!* Let M, N be A -modules and S be a multiplicative subset in A .

1, 1pt) Prove that the map $\psi \mapsto \psi[S^{-1}] : \text{Hom}_A(M, N) \rightarrow \text{Hom}_{A[S^{-1}]}(M[S^{-1}], N[S^{-1}])$ is A -linear.

2, 1pt) Suppose that M is finitely presented meaning that there are k, ℓ and an A -linear map $\varphi : A^{\oplus k} \rightarrow A^{\oplus \ell}$ such that $M \cong A^{\oplus \ell} / \text{im } \varphi$. Prove that the A -linear map from part 1 factors into the composition of the natural homomorphism $\iota : \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N)[S^{-1}]$ and an isomorphism $\text{Hom}_A(M, N)[S^{-1}] \xrightarrow{\sim} \text{Hom}_{A[S^{-1}]}(M[S^{-1}], N[S^{-1}])$.

Hint: something from earlier homeworks should help...

3, 2pts) Now suppose that A is a Noetherian domain and $I \subset A$ is an ideal. Show that $\{\alpha \in \text{Frac}(A) \mid \alpha I \subset A\}$ (the containment in $\text{Frac}(A)$) is an A -submodule in $\text{Frac}(A)$. Further, use the previous two parts to identify the A -modules $\text{Hom}_A(I, A)$ and $\{\alpha \in \text{Frac}(A) \mid \alpha I \subset A\}$.

4, 1pt) Use part 3 to reprove the claim of Problem 8 in Homework 1: for the ideal $I = (x, y) \subset A = \mathbb{C}[x, y]$, we have an isomorphism of A -modules $\text{Hom}_A(I, A) \cong A$.

5, 1pt) Let $A = \mathbb{Z}[\sqrt{-5}]$, $I = (2, 1 + \sqrt{-5})$. Establish an A -module isomorphism $\text{Hom}_A(I, A) \cong I$.