Lecture 14.

1) Functors, cont'd.

2) Functor morphisms.

Ref: [R], Sec. 1.3, 1.4.

BONUS: Category equivalences.

1.1) More examples of functors.

1) More forgetful functors.

1a) Let A be a commutative ring. Then have the forgetful functor $For: A\text{-}Alg \longrightarrow A\text{-}Mod$, forgetting the multiplication.

1b) Let A, B be commutative rings & $\varphi: A \to B$ be a ring homom'm. Then can consider the pullback functor $\varphi^*: B\text{-}Mod \longrightarrow A\text{-}Mod$. It sends $M \in Ob(B\text{-}Mod)$ to M viewed as an A-module & $\psi \in Hom_B(M,N)$ to $\psi \in Hom_A(M,N)$. Forgets part of the action.

2) Let $\mathcal{C}$ be a category. For $X \in Ob(\mathcal{C})$ define the Hom functor $\mathcal{F}_X\ (:= Hom_{\mathcal{C}}(X, \bullet)): \mathcal{C} \longrightarrow Sets$.

On objects: $\mathcal{F}_X(Y) := Hom_{\mathcal{C}}(X,Y)$, a set.

On morphisms: (we'll use the notation $X \xrightarrow{\psi} Y$ to denote $\psi \in Hom_{\mathcal{C}}(X,Y)$)

$$Y_1 \xrightarrow{f} Y_2 \rightsquigarrow \text{map } \mathcal{F}_X(f): \begin{array}{ccc} Hom_{\mathcal{C}}(X,Y_1) & \to & Hom_{\mathcal{C}}(X,Y_2) \\ \psi & \longmapsto & f\circ\psi \end{array}$$

Check axioms: composition: $\mathcal{F}_X(g\circ f) = \mathcal{F}_X(g)\circ\mathcal{F}_X(f)$ for $Y_1 \xrightarrow{f} Y_2 \xrightarrow{g} Y_3$. For $\psi \in Hom_{\mathcal{C}}(X,Y_1)$ have

$[\mathcal{F}_X(g\circ f)](\psi) = (g\circ f)\circ\psi,$
$[\mathcal{F}_X(g)\circ\mathcal{F}_X(f)](\psi) = [\mathcal{F}_X(g)](f\circ\psi) = g\circ(f\circ\psi).$
By associativity axiom for morphisms, the two coincide.
The unit axiom is left as exercise.

2^{opp}) We can apply this construction to $\mathcal{C}^{opp} \rightsquigarrow$
$\mathcal{F}_X^{opp}: Y \longmapsto \mathrm{Hom}_{\mathcal{C}^{opp}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$
$f \in \mathrm{Hom}_{\mathcal{C}^{opp}}(Y_1,Y_2) = \mathrm{Hom}_{\mathcal{C}}(Y_2,Y_1) \rightsquigarrow$
$\mathcal{F}_X^{opp}(f): \mathrm{Hom}_{\mathcal{C}}(Y_1,X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y_2,X)$ - map of sets
$\psi \longmapsto \psi\circ f$
We can view $\mathcal{F}_X^{opp}$ as a functor $\mathcal{C} \longrightarrow Sets^{opp}$
(a traditional name: contravariant functor $\mathcal{C} \to Sets$)

3) Algebra constructions as functors:
3a) The "free" functor: Sets $\to$ A-Mod.
Let A be a ring. Want to define a functor Free: Sets $\to$ A-Mod
I, set, $\rightsquigarrow$ Free$(I) := A^{\oplus I}$
$f: I \to J \rightsquigarrow$ Free$(f): A^{\oplus I} \to A^{\oplus J}$ - the unique map sending the basis element e_i $(i\in I)$ to $e_{f(i)} \in A^{\oplus J}$.
Checking axioms of functor: exercise.

3b) Abelinization functor Ab: Groups $\to \mathbb{Z}$-Mod
Recall: in a group G have group commutator
$(g,h) = ghg^{-1}h^{-1}$ $((g,h)=1 \Leftrightarrow gh=hg)$

The derived subgr'p $(G,G):=$ subgroup generated by (g,h) $\forall g,h\in G$, it's normal.
Then $Ab(G):=G/(G,G)$ is abelian.
Notice: $\forall$ group homom. $f: G\to H \Rightarrow f((G,G))\subset (H,H)$
$\rightsquigarrow$ well-defined homom'm $Ab(f): Ab(G)\to Ab(H)$:
$$Ab(f)(g(G,G)):=f(g)(H,H).$$
Need to check: $Ab(f_2\circ f_1)=Ab(f_2)\circ Ab(f_1)$ follows from the constr'n.
$Ab(1_G)=1_{Ab(G)}$ – manifest.

3c) Localization of modules is a functor: $S\subset A$ multiplicative $\rightsquigarrow \bullet[S^{-1}]: A\text{-Mod}\longrightarrow A[S^{-1}]\text{-Mod}$, a functor (on the level of objects: Sec 2 of Lec 11 on the level of objects, Sec. 1.2 of Lec 12 on the level of morphisms, very important exercise checks the axioms.)

2) Functor morphisms.

Motto: A functor morphism relative to functor is as group homomorphism relative to groups & as morphism relative to object.

2.0) Motivation:
Objects in a category $\mathcal{C}$ considered on their own is something amorphous (no structure) & often wild – too many to form a set. What gives them structure is morphisms. In particular, they allow to talk about isomorphic objects & often we can understand objects up to isomorphism.

Functors $F: \mathcal{C} \to \mathcal{D}$ are less amorphous but often even wilder, for example, every object $X \in Ob(\mathcal{C})$ gives rise to a functor $\mathcal{F}_X : \mathcal{C} \to Sets$. The functors $\mathcal{F}_X, \mathcal{F}_Y$ would be the "same" if the objects X, Y are the same - not a good notion. Instead we should be talking about "isomorphic" functors (to be defined) so that isomorphic objects give rise to isomorphic functors. And for this - and other things - we need the notion of a functor morphism (a.k.a. a natural transformation of functors).

2.1) Definition: Let $\mathcal{C}, \mathcal{D}$ be categories & $F, G: \mathcal{C} \to \mathcal{D}$ be functors.

Def'n: A functor morphism $\eta: F \Rightarrow G$ is

(Data) $\forall X \in Ob(\mathcal{C})$, a morphism $\eta_X \in Hom_{\mathcal{D}}(F(X), G(X))$ s.t.

(axiom) $f \in Hom_{\mathcal{C}}(X, Y) \rightsquigarrow F(f) \in Hom_{\mathcal{D}}(F(X), F(Y)), G(f) \in Hom_{\mathcal{D}}(G(X), G(Y))$. Want: $\forall X, Y \in Ob(\mathcal{C}), f \in Hom_{\mathcal{C}}(X, Y)$, the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

Rem: in many (but not all) examples, η_X is "natural" meaning it's "uniform" & "independent of additional choices". Hence the name "natural transformation."

2.2) Important example.

Let $X, X' \in Ob(\mathcal{C}) \rightsquigarrow$ functors

$$\mathcal{F}_X := Hom_{\mathcal{C}}(X, \bullet),\ \mathcal{F}_{X'} := Hom_{\mathcal{C}}(X', \bullet) : \mathcal{C} \longrightarrow Sets.$$

Goal: from $g \in Hom_{\mathcal{C}}(X', X)$ produce a functor morphism $\eta^g : F_X \Rightarrow F_{X'}$.

i.e. for each $Y \in Ob(\mathcal{C})$ we need to define a map

$$\eta^g_Y : Hom_{\mathcal{C}}(X, Y) \longrightarrow Hom_{\mathcal{C}}(X', Y) : \qquad X' \xrightarrow{g} X \xrightarrow{\psi} Y$$
$$\psi \longmapsto \psi \circ g$$

Now we need to check the axiom (commutative diagram)

$\forall f \in Hom_{\mathcal{C}}(Y_1, Y_2)$, $F_X(f) = f \circ ?$, $F_{X'}(f) = f \circ ?$

$$\left.\begin{array}{ccc} \psi \in Hom_{\mathcal{C}}(X, Y_1) & \xrightarrow[f\circ ?]{} & Hom_{\mathcal{C}}(X, Y_2) \\ \Big\downarrow \eta_{Y_1}(?) = ? \circ g & & \Big\downarrow \eta_{Y_2}(?) = ? \circ g \\ Hom_{\mathcal{C}}(X', Y_1) & \xrightarrow{f \circ ?} & Hom_{\mathcal{C}}(X', Y_2) \end{array}\right\} \text{is commutative}$$

$\downarrow \longrightarrow : \psi \mapsto \psi \circ g \mapsto f \circ (\psi \circ g)$

$\qquad\qquad\qquad\qquad\qquad \| \longleftarrow$ b/c composition in a category is associative.

$\longrightarrow \downarrow : \psi \mapsto f \circ \psi \mapsto (f \circ \psi) \circ g$

We've checked: η^g is a functor morphism.

2.3) Remarks.

(i) Have the identity morphism $id : F \Rightarrow F$.

(ii) Can take compositions of functor morphisms

$\tau : G \Rightarrow H,\ \eta : F \Rightarrow G \rightsquigarrow \tau \circ \eta : F \Rightarrow H,\ (\tau \circ \eta)_X = \tau_X \circ \eta_X$

Exercise: check that $(\tau \circ \eta)$ is indeed a functor morphism.

Very importantly, in Example 2.2.

$$\eta^{g'} \circ \eta^{g} = \eta^{g \circ g'}$$ (note that the order of g, g' is reversed b/c we compose on the right).

(iii)* If $\mathcal{C}$ is small ($Ob(\mathcal{C})$ is a set), then the collection of functor morphisms $F \Rightarrow G$, denoted $Hom_{Fun}(F, G)$, is a set $\rightsquigarrow$ category of functors $Fun(\mathcal{C}, \mathcal{D})$: objects are functors, morphisms = morphisms of functors.

(iv): (i) & (ii) allow to talk about functor isomorphisms;
$\eta : F \Rightarrow G$ is an isom'm $\Leftrightarrow$ $\forall\ Y \in Ob(\mathcal{C})$ have that $\eta_Y \in Hom_{\mathcal{D}}(F(Y), G(Y))$ is an isom'm:
$\eta^{-1} : G \Rightarrow F$ is given $(\eta^{-1})_Y = (\eta_Y)^{-1}$

BONUS: Category equivalences.

Our question here: when are two categories the "same"? Turns out, functor isomorphisms play an important role in answering this question.

Before we address this we should discuss an easier question: when are two sets the same? Well, they are literally the same if they consist of the same elements. But this definition is quite useless: sets arising from different constructions won't be the same in this sense. Of course, we use isomorphic instead of being literally the same.

Now back to categories. Again, being the same is useless. How about being isomorphic? Turns out, this is not useful as well. Let's see why. Let $\mathcal{C}, \mathcal{D}$ be categories. We say that $\mathcal{C}, \mathcal{D}$ are isomorphic if there are functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ such that $FG = Id_{\mathcal{D}}$, $GF = Id_{\mathcal{C}}$. The issue is: two functors obtained by different constructions are never the same (compare to sets). The solution: replace "equal" w. "isomorphic" (as functors).

Definition: • We say F, G as above are <u>quasi-inverse</u> if $FG \xRightarrow{\sim} Id_{\mathcal{D}}$, $GF \xRightarrow{\sim} Id_{\mathcal{C}}$ (isomorphic).

• We say $\mathcal{C}, \mathcal{D}$ are <u>equivalent</u> if there are quasi-inverse functors (called <u>equivalences</u>) $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$.

Now we are going to state a general result

Definitions: A functor $F: \mathcal{C} \to \mathcal{D}$ is called

- fully faithful if $\forall X, X' \in Ob(\mathcal{C}) \Rightarrow$
  $f \mapsto F(f)$ is a bijection $Hom_{\mathcal{C}}(X,X') \xrightarrow{\sim} Hom_{\mathcal{D}}(F(X), F(X'))$
- essentially surjective if $\forall Y \in Ob(\mathcal{D})$ $\exists X \in Ob(\mathcal{C})$

such that $F(X)$ is isomorphic to Y.

Thm: A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence $\Leftrightarrow$ F is fully faithful & essentially surjective.

We won't prove this, but we will give an example - that illustrates how the proof works in general.

Example: Consider the category $\mathcal{D} = \mathbb{F}\text{-}Vect_{fd}$ of finite dimensional vector spaces over $\mathbb{F}$ and its full subcategory $\mathcal{C}$ w. objects $\mathbb{F}^n$ $(n \geq 0)$. We claim that the inclusion functor $F: \mathcal{C} \hookrightarrow \mathcal{D}$ is an equivalence. It's fully faithful by def'n and the claim that it's essentially surjective.

Now we produce a quasi-inverse functor, G. In each $V \in Ob(\mathcal{D})$ we fix a basis, which leads to an isomorphism $\eta_V: V \xrightarrow{\sim} \mathbb{F}^n$. We define $G(V)$ as $\mathbb{F}^n$. For a linear map $f: U \to V$ (w. $\dim U = m$, $\dim V = n$) we set $G(f) := \eta_V^{-1} \circ f \circ \eta_U$.

Exercise: Check G is a functor.

Now we are going to simplify our life a bit an assume that $\eta_{\mathbb{F}^n}: \mathbb{F}^n \xrightarrow{\sim} \mathbb{F}^n$ is the identity.

Exercise: $GF: \mathcal{C} \longrightarrow \mathcal{C}$ is the identity functor (not just isomorphic to it).

Now we produce a functor isomorphism $\eta: \mathrm{Id}_{\mathcal{D}} \overset{\sim}{\Rightarrow} FG$

So we need to have $\eta_V: V \to \mathbb{F}^{\dim V}$ and this is the isomorphism from above.

Exercise: prove that η is indeed a functor morphism

Then η is an isomorphism of functors. So F is indeed a category equivalence.

A different exercise: prove that the duality functor $\bullet^*$ is an equivalence $\mathbb{F}\text{-}\mathrm{Vect}_{fd} \longrightarrow \mathbb{F}\text{-}\mathrm{Vect}_{fd}^{opp}$.