Lecture 23, 04/14.

1) GIT of Quot.

Ref: [HL], Secs 1.7, 2.2, 4.4.

1) GIT of Quot.

Our goal is to produce moduli spaces of (semi)stable bundles as GIT-quotients for the action of $PGL(V)$ on the projective scheme $Q := Quot_C^{P,V}$ (see Lec 22). A key step here is to relate the (semi) stability of vector bundles (Sec 1.1 of Lec 21) to the GIT semistability (Sec 2 of Lec 22). In this lecture we will analyze the latter. The notion of (semi)stability depends on a choice of a $PGL(V)$-linearized (very) ample line bundle on Q. Our first step is to construct such - and we'll need to know something about the construction of Q for this.

We'll analyze (semi) stability using the Hilbert-Mumford Thm (Sec 2.2 of Lec 22). For this we need to do two things:

- For $\gamma: \mathbb{C}^\times \to PGL(V)$, we give a geometric interpretation of the limit $\lim_{t\to 0} \gamma(t)q$ for $q \in Q$.
- we then compute the integer $\mu^{\mathcal{H}}(q,\gamma)$ & apply this computation to study the $\mathcal{H}$-(semi)stability of points of Q.

1.1) Line bundle

1.1.1) Construction of bundle

Let $\pi_Q : Q\times C \twoheadrightarrow Q$, $\pi_C : Q\times C \twoheadrightarrow C$ denote the projections & $\mathcal{O}_C(1)$ be a very ample line bundle on C. Then $\pi_C^*\mathcal{O}(1)$ is very ample for the scheme $Q\times C$ over Q.

Recall that Q comes w. a sheaf $\mathcal{F}^{un}$ s.t. for $\mathcal{F}_q := \mathcal{F}^{un}|_{\{q\}\times C}$; we have $\chi(\mathcal{F}_q(m)) = P(m)$, where (in our case) the polynomial P is given by $P(m) = N + rd_0 m$, $N := d + (1-g)r$; $d_0 = \deg \mathcal{O}_C(1)$. Set $\mathcal{F}^{un}(m) := \mathcal{F}^{un}\otimes \pi_C^*\mathcal{O}(1)^{\otimes m}$

Lemma: For m sufficiently large, $\pi_{Q*}(\mathcal{F}^{un}(m))$ is a vector bundle of $rk = P(m)$. Its fiber at $q\in Q$ is $H^0(\mathcal{F}_q(m))$.

Proof: By Serre's thm, $R^i\pi_{Q*}(\mathcal{F}^{un}(m)) = 0$ $\forall i>0$ if m is sufficiently large. Note that $\forall$ $q\in Q$ we have the projection formula $\mathbb{C}_q \otimes^L_{\mathcal{O}_Q} R\pi_{Q*}(\mathcal{F}^{un}(m)) \overset{(*)}{=} R\pi_{Q*}(\pi_Q^*\mathbb{C}_q \otimes^L_{\mathcal{O}_{Q\times C}} \mathcal{F}^{un}(m)) = [\mathcal{F}^{un}$ is flat over $Q] = R\Gamma(\mathcal{F}_q(m))^{(**)}$. $(*)$ is in cohomological deg ≤ 0, while $(**)$ is in cohomological deg ≥ 0. So both are in deg 0 & of dim $= P(m)$. $\square$

We define the line bundle on Q:

(1) $\mathcal{H}_m := \Lambda^{top}(\pi_{Q*}(\mathcal{F}^{un}(m)))$,

its fiber at q is $\mathcal{H}_{m,q} = \Lambda^{top} H^0(\mathcal{F}_q(m))$. The main result of Sec 1.1 is

Proposition: $\mathcal{H}_\ell$ is very ample for $\ell \gg 0$.

1.1.2) Sketch of construction of Q.

Notation: for $\mathcal{M} \in Coh(C)$, $P_{\mathcal{M}} \in \mathbb{Q}[t]$ is its Hilbert polynomial.

To prove the proposition we need to sketch a construction of Q. Recall that Q parameterizes quotients, $\mathcal{F}$, of $V \otimes \mathcal{O}_C$ (where V is a fixed vector space of $\dim = N$) w. $P_{\mathcal{F}} = P$. Let $\mathcal{G}$ denote the kernel of $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$.

We will use the following algebro-geometric fact ("Boundedness")

Fact: Let $\mathcal{M} \in Coh(C)$ & $\underline{P} \subset \mathbb{Q}[t]$ be finite. Then $\exists\, \ell_{\mathcal{M},\underline{P}} \in \mathbb{Z}$ such that for $\forall$ subs & quotients $\mathcal{M}'$ of $\mathcal{M}$ w. $P_{\mathcal{M}'} \in \underline{P}$ & $\forall k \geqslant 0$ & $\ell \geqslant \ell_{\mathcal{M},\underline{P}}$, the following hold

(i) $H^i(\mathcal{M}'(\ell+k-i)) = 0$, $\forall i > 0$ (a.k.a. $\mathcal{M}'$ is "$\ell+k$-regular")

(ii) $\mathcal{M}'(\ell+k)$ is generated by global sections

(iii) $H^0(\mathcal{M}'(\ell)) \otimes H^0(\mathcal{O}_C(k)) \twoheadrightarrow H^0(\mathcal{M}'(\ell+k))$
multiplication

For quotients, this follows by combining Lem 1.7.6, Lem 1.7.2 in [HL]. The case of subs follows from the case of quotients. Note that for an individual sheaf $\mathcal{M}'$, Fact just amounts to Serre's thm.

Apply this to $\mathcal{M}' \subset \mathcal{M} = V \otimes \mathcal{O}_C$ w. $P_{\mathcal{M}'} = NP_{\mathcal{O}_C} - P (= P_{\mathcal{G}})$. Let $\ell_0 = \ell_{\mathcal{M}, \{P_{\mathcal{M}'}\}}$. Note that we recover $\mathcal{F}$ from $H^0(\mathcal{G}(\ell)) \subset H^0(V \otimes \mathcal{O}_C(\ell))$ for all $\ell \geqslant \ell_0$: we first recover $\mathcal{G}(\ell)$ as the image of the composition

$$H^0(\mathcal{G}(\ell)) \otimes \mathcal{O}_C \hookrightarrow H^0(V \otimes \mathcal{O}_C(\ell)) \otimes \mathcal{O}_C \twoheadrightarrow V \otimes \mathcal{O}_C(\ell)$$

which then recovers $\mathcal{G} \subset V \otimes \mathcal{O}_C$ by twisting w. $\mathcal{O}(-\ell)$ and then, finally, $\mathcal{F}$ as $(V \otimes \mathcal{O}_C)/\mathcal{G}$. This gives rise to the inclusion from the set of quotients $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$ w. $P_{\mathcal{F}} = P$ to the Grassmannian

$$Gr_\ell := Gr(V \otimes H^0(\mathcal{O}_C(\ell)), P(\ell))$$

of subspaces of <u>codim</u> $= P(\ell)$. The image is locally closed: it consists of all subspaces $U \subset V \otimes H^0(\mathcal{O}_C(\ell))$ s.t the image of $U \otimes H^0(\mathcal{O}_C(k))$ in $V \otimes H^0(\mathcal{O}_C(\ell + k))$ has codim $P(\ell + k)$ $\forall k \geqslant 0$. The subscheme defined by those conditions is, by defn, the Quot scheme $Quot_C^{P,V}$. Then one can check that it has a universal sheaf as required and that it's proper (hence closed) using the valuative criterion.

1.1.3) Sketch of proof of Proposition

Let $\mathcal{W}$ denote the tautological rk $P(\ell)$ bundle on Gr_ℓ. Then $\Lambda^{top}\mathcal{W}$ is a very ample line bundle – it corresponds to the Plücker embedding. The pullback of $\mathcal{W}$ is nothing else but $\pi_{Q*}(\mathcal{F}^{un}(m))$

– we don't check this carefully but notice that the fibers of both bundles at q are $V \otimes H^0(\mathcal{O}_C(\ell))/G_q(\ell)$, hence $\mathcal{H}_\ell$ is indeed very ample.

1.1.4) Linearizability.

We note $GL(V)$ naturally acts on $H^0(V \otimes \mathcal{O}_C(\ell)) = V \otimes H^0(\mathcal{O}_C(\ell))$ (via the action on the first factor) inducing

$$PGL(V) \longrightarrow PGL(V \otimes H^0(\mathcal{O}_C(\ell))) \text{ \& similarly for } SL\text{'s}.$$

So, the bundles $\mathcal{W}$ and $\Lambda^{top}\mathcal{W}$ are $SL(V)$-equivariant and the center of $SL(V)$ acts by characters. Clearly, the embedding $Q \hookrightarrow Gr_\ell$ is $PGL(V)$-equivariant. It follows that $\mathcal{H}_\ell$ is $SL(V)$-equivariant with $Z(SL(V))$ acting by a character. So $\mathcal{H}_\ell^{\otimes N}$ is $PGL(V)$-equivariant.

1.2) Computing the limit.

Let $q = [V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}] \in Q$. Let $\gamma : \mathbb{C}^\times \longrightarrow PGL(V)$ be a 1-parameter subgroup. Our goal in this section is to compute $\lim_{t \to 0} \gamma(t) q$. We start with a baby case.

Consider $Gr(V,p)$: p-dimensional <u>quotients</u> of V. Consider a lift $\tilde{\gamma} : \mathbb{C}^\times \longrightarrow GL(V)$ of γ so that for $F \in Gr(V,p)$ we have $\tilde{\gamma}(t).F = \gamma(t).F$. As usual, consider the decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n(\tilde{\gamma})$

w. $V_n(\tilde{\gamma}) = \{v \in V \mid \tilde{\gamma}(t)v = t^n v\}$, and a filtration $V_{\geq m}(\tilde{\gamma}) = \bigoplus_{n \geq m} V_n(\tilde{\gamma})$
The associated graded $gr^{\tilde{\gamma}} V$ for this filtration is identified w. V.
On F we have the induced filtration: $F_{\geq m}(\tilde{\gamma})$ is defined as the image of $V_{\geq m}(\tilde{\gamma})$. Let $gr^{\gamma} F$ denote the associated graded, it's naturally a quotient of $gr^{\tilde{\gamma}} V = V$ (note that different choices of $\tilde{\gamma}$ lead to shifts in the filtration but do not change $gr^{\gamma} F$ as a quotient of V). For $v \in V$ we write v^m for its component in $V_m(\tilde{\gamma})$.

Lemma: $\lim_{t \to 0} \gamma(t).F = gr^{\gamma} F$

Proof:

Let $G = \ker[V \twoheadrightarrow F]$. Choose a basis $v^1, \dots v^r$ of G compatible w. the filtration $G \cap V_{\geq m}(\tilde{\gamma})$: i.e., if $m_1, \dots m_r$ are maximal integers s.t. $v^i \in G \cap V_{\geq m_i}(\tilde{\gamma})$, then v^i w. $m_i \geq m$ form a basis in $G \cap V_{\geq m}(\tilde{\gamma})$. Then $\tilde{\gamma}(t) v^i = t^{m_i} v^i_{m_i} +$ higher powers of t. So the elements $v^i_{m_i}$ form a basis in $\ker[V \twoheadrightarrow \lim_{t \to 0} \gamma(t).F]$, therefore, $\lim_{t \to 0} \gamma(t).F = gr^{\gamma} F$

$\square$

Consider the filtration $V_{\geq m}(\tilde{\gamma}) \otimes \mathcal{O}_C$ on $V \otimes \mathcal{O}_C$ and induced filtration $\mathcal{F}_{\geq m}(\tilde{\gamma})$ on $\mathcal{F}_{\geq m}$.

Proposition: $\lim_{t \to 0} [V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}] = [V \otimes \mathcal{O}_C \twoheadrightarrow gr^{\gamma} \mathcal{F}]$

A proof (see [HL], Lem 4.4.3) is based on a Rees construction: from the filtration on $V\otimes\mathcal{O}_C \to \mathcal{F}$ we produce an element of $\underline{\mathrm{Quot}}_C^{P,V}(\mathbb{A}^1)$ whose specialization to $t\in\mathbb{C}^\times$ is $\gamma(t).[V\otimes\mathcal{O}_C\to\mathcal{F}]$ and therefore the specialization to 0, the associated graded $[V\otimes\mathcal{O}_C \twoheadrightarrow gr^\gamma\mathcal{F}]$ is the required limit.

1.3) Computing $\mu^{\mathcal{H}_\ell}(q,\gamma)$ & applications to (semi)stability.

We choose ℓ so that conclusions of Fact in Sec 1.1.2 hold for all quotients of $V\otimes\mathcal{O}_C$ with Hilbert polynomial P.

Let $\gamma: \mathbb{C}^\times \to SL(V)$ be a 1-parameter subgroup. Let $q=[V\otimes\mathcal{O}_C \twoheadrightarrow \mathcal{F}]\in Q$. Then $q' := \lim_{t\to 0}\gamma(t).q = [V\otimes\mathcal{O}_C \twoheadrightarrow gr^\gamma\mathcal{F}]$. We want to compute the integer r s.t. $\gamma(t)$ acts on the fiber $\mathcal{H}_{\ell,q'}$ by t^r, this r is what we denote by $\mu^{\mathcal{H}_\ell}(q,\gamma)$. Note that there's a difference of conventions with [HL] (who consider right group actions resulting in a different choice of sign). Note also that (semi) stability for $PGL(V)$ is equivalent for (semi) stability for its cover $SL(V)$.

Below we suppress γ from notations like $V_{\geqslant n}(\gamma)$ & $gr_n^\gamma\mathcal{F}$. Further, we write $\mathcal{F}_n$ instead $gr_n\mathcal{F}$.

The fiber $\mathcal{H}_{\ell,q'}$ is $\Lambda^{top}H^0(gr\,\mathcal{F}(\ell))$, see (1') in Sec 1.1 so $\mu^{\mathcal{H}_\ell}(q,\gamma)$ is the sum of weights of the γ-action on $H^0(gr\,\mathcal{F}(\ell))$.

The action is induced by the action on γ on $gr\,\mathcal{F}$, where on $\mathcal{F}_n$ it acts by $t \mapsto t^n$. Note that since $gr\,\mathcal{F}$ is a quotient of $V\otimes\mathcal{O}_C$ w. Hilbert polynomial P, we have (since ℓ is large enough, Sec 1.2.2)

$$H^1(gr\,\mathcal{F}(\ell))=0 \Rightarrow \dim H^0(\mathcal{F}_n(\ell)) = P_{\mathcal{F}_n}(\ell) \text{ leading to:}$$

(2) $$\mu^{\mathcal{H}_\ell}(q,\gamma) = \sum_n n P_{\mathcal{F}_n}(\ell)$$

Using this computation and Sec 2.2 of Lec 22 we can deduce the following result describing $\mathcal{H}_\ell$-semistable points

Theorem: A point $q=[V\otimes\mathcal{O}_C \twoheadrightarrow \mathcal{F}]$ is $\mathcal{H}_\ell$-semistable (resp. $\mathcal{H}_\ell$-stable) iff $\forall$ nontrivial subspaces $V' \subsetneq V$ we have, for the image $\mathcal{F}'$ of $V'\otimes\mathcal{O}_C$, we have

(3) $$P_{\mathcal{F}'}(\ell) \geqslant (\text{resp.} >) \frac{\dim V'}{\dim V} P(\ell)$$

Proof: Note that since γ takes values in $SL(V)$, so

(4) $$\sum_n n\cdot \dim V_n = 0.$$

By Thm in Sec 2.2, q is $\mathcal{H}_\ell$-semistable $\Leftrightarrow \mu^{\mathcal{H}_\ell}(q,\gamma)\geqslant 0\ \forall\gamma$. Let $k\in\mathbb{Z}_{>0}$ be s.t. $V_{\geqslant -k}=V$. Then from (4) we get

$$0=\sum_{n\geqslant -k}(n+k)\dim V_n - k\sum_n \dim V_n = \sum_{n\geqslant 0}\dim V_{\geqslant n-k} - kN \Rightarrow$$

(5) $$\frac{1}{N}\sum \dim V_{\geqslant n-k} = k$$

Similarly,

$$\mu^{\mathcal{H}_\ell}(q,\gamma) = \sum_{n\geqslant 0} n P_{\mathcal{F}_{n-k}}(\ell) - k\sum_n P_{\mathcal{F}_n}(\ell) = \sum_{n\geqslant 0} P_{\mathcal{F}_{\geqslant n-k}}(\ell) - kP(\ell).$$

Assume (3) holds. Then

$$P_{\mathcal{F}_{\geq n-k}}(\ell) \geq \frac{\dim V_{\geq n-k}}{N} P(\ell).$$

So

$$\mu^{\mathcal{H}_e}(q,\gamma) \geq \sum_{n \geq 0} \frac{\dim V_{\geq n-k}}{N} P(\ell) - k P(\ell) = [(5)] = 0.$$

Conversely, suppose $\mu^{\mathcal{H}_e}(q,\gamma) \geq 0 \ \forall \gamma$. Pick γ as follows. Let $N' = \dim V'$, then take $\gamma(t) = (\underbrace{t^{N-N'}, \dots, t^{N-N'}}_{N' \text{ times}}, t^{-N'}, \dots, t^{-N'})$. For this γ, the condition that $\mu^{\mathcal{H}_e}(q,\gamma) \geq 0$ is equivalent to (3), exercise.

The stable case is handled similarly, exercise. □

Corollary: If $[V \otimes \mathcal{O}_C \xrightarrow{\varphi} \mathcal{F}]$ is a $\mathcal{H}_e$-semistable point, then $H^0(\varphi)$ is injective.

Proof: exercise.