Lecture 11.

1) Localization of rings, cont'd
2) Localization of modules.

Refs: [AM], Intro to Chapter 3.

BONUS: localization in non-commutative rings.

1.0) Definition:

Let A be a commutative ring, $S\subset A$ a multiplicative subset ($1\in S$ & $s,t\in S \Rightarrow st\in S$).

Definition: (i) Consider $A\times S$ (product of sets), equip it w. equivalence relation $\sim$ defined by

$$(*)\quad (a,s)\sim(b,t) \overset{def}{\Longleftrightarrow} \exists\, u\in S \mid uta=usb.$$

Exercise: Check that $\sim$ is indeed an equivalence relation.

(ii) $A[S^{-1}]:=$ the set of equivalence classes.

(iii) the class of (a,s) will be denoted by $\frac{a}{s}$.

Exercise: TFAE: • $A[S^{-1}]=\{\frac{0}{1}\}$.

• $0\in S$.

Rem: if S contains no zero divisors, then $(*)$ simplifies: $ta=sb$. But, in general, the latter equality fails to give an equivalence relation.

Addition & multiplication in $A[S^{-1}]$ are introduced by:

$$\frac{a_1}{s_1}+\frac{a_2}{s_2}:=\frac{s_2a_1+s_1a_2}{s_1s_2}, \quad \frac{a_1}{s_1}\,\frac{a_2}{s_2}:=\frac{a_1a_2}{s_1s_2}$$

Proposition: These operations are well-defined (the result depends only on $\frac{a_1}{s_1}, \frac{a_2}{s_2}$, not on $(a_1,s_1),(a_2,s_2)$) & equip $A[S^{-1}]$ w. structure of a commutative ring (w. unit $\frac{1}{1}$). Moreover, $\iota: A\to A[S^{-1}], a\mapsto \frac{a}{1}$ is a ring homomorphism.

Proof: omitted in order not to make everybody very bored...

Def'n: The ring $A[S^{-1}]$ is called the localization of A (w.r.t. S).

Rem: We view $A[S^{-1}]$ as an A-algebra via ι.

1.1) Examples.

0) $0\in S \iff A[S^{-1}]$ is the zero ring.

1) $S=\{$all invertible elements in $A\}$ is multiplicative; $A[S^{-1}]\simeq A$ b/c every equiv. class in $A\times S$ has a <u>unique</u> representative of the form $(a,1)$.

2) A is a domain, $S=A\setminus\{0\}$ is multiplicative; $A[S^{-1}]$ is the fraction field $\mathrm{Frac}(A)$.

Exercise: Let A be general & $S=\{$all non-zero divisors in $A\}$. Then every non-zero divisor in $A[S^{-1}]$ is invertible.

2) Let $f\in A \rightsquigarrow S:=\{f^n \mid n\geq 0\}$. The resulting localization is denoted by $A[f^{-1}]$.

Concrete example: $A = \mathbb{F}[x]$, $f = x$. Then $\mathbb{F}[x]_x = \left\{ \frac{f(x)}{x^n} \right\}$ is
is identified with the ring of Laurent polynomials
$\mathbb{F}[x^{\pm 1}] = \left\{ \sum_{i=-m}^{n} a_i x^i \right\}$.

Lemma: Have an A-algebra isomorphism $A[f^{-1}] \simeq A[t]/(tf-1)$.

Proof: As an A-algebra $A[f^{-1}]$ is generated by f^{-1}. So have a surjective homomorphism $\pi: A[t] \twoheadrightarrow A[f^{-1}]$, $g \mapsto g(\frac{1}{f})$. Clearly, $\pi(tf-1)=0$ so π factors through $\underline{\pi}: A[t]/I \twoheadrightarrow A[f^{-1}]$, $I := (tf-1)$.

It remains to show $\ker \underline{\pi} = 0$. Let $g = \sum_{i=0}^{n} a_i t^i \in \ker \pi \Leftrightarrow$ $g(\frac{1}{f}) = (\sum_{i=0}^{n} a_i f^{n-i})/f^n = 0$ in $A[f^{-1}] \Leftrightarrow \exists\, m \geq n$ s.t. $\sum_{i=0}^{n} a_i f^{m-i} = 0$ in A. Note that $f + I \in A[t]/I$ is invertible w. inverse $t + I$.
So $\sum_{i=0}^{n} a_i f^{m-i} = 0$ in $A \Rightarrow$ in $A[t]/I$ have $0 = \sum_{i=0}^{n} a_i (f+I)^{m-i} =$
$= \sum_{i=0}^{n} a_i (t+I)^{i-m} \Rightarrow \sum_{i=0}^{n} a_i (t+I)^i = 0 \Rightarrow g + I = 0 \Rightarrow \ker \underline{\pi} = \{0\}$. $\square$

Example (of computation): $A = \mathbb{C}[x,y]/(xy)$, $f = x$ (zero divisor)

$(\mathbb{C}[x,y]/(xy))[x^{-1}] = [\text{Lemma}] = (\mathbb{C}[x,y]/(xy))[t]/(tx-1)$

$= \mathbb{C}[x,y,t]/(xy, tx-1) = [\, t(xy) = y(tx-1) + y \Rightarrow y \in (xy, tx-1)$

so $(xy, tx-1) = (y, tx-1)\,] = \mathbb{C}[x,y,t]/(y, tx-1) \simeq \mathbb{C}[x,t]/(tx-1)$

$[\text{Lemma}] \simeq \mathbb{C}[x]_x = [\text{concrete example}] = \mathbb{C}[x^{\pm 1}]$.

1.2) Universal property of $A[S^{-1}]$

Recall the ring homomorphism $\iota: A \to A[S^{-1}]$, $a \mapsto \frac{a}{1}$ (Prop'n in Sec 1.1). Note that $\iota(s) = \frac{s}{1}$ is invertible in $A[S^{-1}]$.

Proposition: Let $\varphi: A \to B$ be a ring homomorphism s.t. $\varphi(s) \in B$ is invertible $\forall\, s \in S$. Then the following hold:

1) $\exists!$ ring homom'm $\varphi': A[S^{-1}] \to B$ that makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[d, "\iota"'] \arrow[dr, "\varphi"] & \\ A[S^{-1}] \arrow[r, dashed, "\varphi'"] & B \end{tikzcd}$$

2) φ' is given by $\varphi'(\frac{a}{s}) = \varphi(a)\varphi(s)^{-1}$.

Sketch of proof:

<u>Existence:</u> need to show that formula in 2) indeed gives a well-defined ring homomorphism.

Well-defined: need to check $\frac{a}{s} = \frac{b}{t} \Rightarrow \varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$.

Indeed: $\frac{a}{s} = \frac{b}{t} \Leftrightarrow \exists\, u \in S$ s.t. $uta = usb \Rightarrow \varphi(u)\varphi(t)\varphi(a) = \varphi(u)\varphi(s)\varphi(b)$. But $\varphi(u), \varphi(t), \varphi(s)$ are invertible. It follows that $\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$. So φ' is well-defined.

Exercise – on addition & multiplication of fractions. Check that φ' is a ring homomorphism.

<u>Uniqueness:</u> φ' makes diagram comm've $\Leftrightarrow \varphi'(\frac{a}{1}) = \varphi(a)\ \forall a \in A$
$\Rightarrow \varphi'(\frac{s}{1}) = \varphi(s)$ - invertible $\Rightarrow \varphi'(\frac{1}{s}) = \varphi(s)^{-1} \Rightarrow$
$\varphi'(\frac{a}{s}) = \varphi'(\frac{a}{1})\varphi'(\frac{1}{s}) = \varphi(a)\varphi(s)^{-1}$. $\square$

Corollary: Let A be a domain, $S \subset A$ be multiplicative subset s.t $0 \notin S$. Then $A[S^{-1}] \simeq \{\frac{a}{s} \in \mathrm{Frac}(A) \mid s \in S\}$, a ring isom'm

Proof: The inclusion $\varphi: A \hookrightarrow \mathrm{Frac}(A)$ satisfies $\varphi(s)$ is invertible. Consider the resulting ring homomorphism $\varphi': A[S^{-1}] \to \mathrm{Frac}(A)$. It's image coincides w. $\{\frac{a}{s} \mid s \in S\}$. It remains to show φ' is injective: $\varphi(\frac{a}{s}) = 0 \Leftrightarrow \varphi(a)\varphi(s)^{-1} = 0 \Leftrightarrow \varphi(a) = 0 \Leftrightarrow a = 0 \Leftrightarrow \frac{a}{s} = 0$ □

Exercise: Let $f_1, \dots f_k \in A$ & $S = \{f_1^{n_1} \dots f_k^{n_k} \mid n_i \geq 0\}$. Then we have a ring isomorphism $A[S^{-1}] = A[(f_1 f_2 \dots f_k)^{-1}]$.

2) Localization of modules.

2.1) Definition: A, S as before. Let M be an A-module. Define it's localization $M[S^{-1}]$ as the set of equivalence classes $M \times S/\sim$ w. $\sim$ defined by:

(*) $(m, s) \sim (n, t) \overset{def}{\Longleftrightarrow} \exists\, u \in S \mid utm = usn$

Equiv. class of (m, s) will be denoted by $\frac{m}{s}$.

Proposition: $M[S^{-1}]$ has a natural $A[S^{-1}]$-module str're (w. addition of fractions) & $A[S^{-1}] \times M[S^{-1}] \to M[S^{-1}]$ given by $\frac{a}{s} \frac{m}{t} := \frac{am}{st}$.

Proof: the same price as the ring structure on $A[S^{-1}]$.

□

Remark: $M[S^{-1}]$ is an $A[S^{-1}]$-module, ring homom'm $\iota: A \to A[S^{-1}]$ gives an A-module structure on $M[S^{-1}]$: $a\frac{m}{s} = \frac{am}{s}$.

The map $M \xrightarrow{\iota} M[S^{-1}]$, $m \mapsto \frac{m}{1}$, is A-module homomorphism ($\iota: A \to A[S^{-1}]$ is a special case).

Important exercise: For $\iota: M \to M[S^{-1}]$, $\ker \iota = \{m \in M \mid \exists\, s \in S$ s.t. $sm = 0\}$. – use (*).

Example: Suppose M is an A-module where all elements of S act by invertible operators. Then $\iota: M \to M[S^{-1}]$ is bijective: every equivalence class contains a unique element of the form $(m,1)$ – compare to Example 1 in Sec. 1.

BONUS: Localization in noncommutative rings.

When we define the ring structure on A_S it's important that the elements of S commute w. all elements of A. Otherwise, assume for simplicity that all elements of S are invertible. We are trying to multiply right fractions as^{-1} and bt^{-1} and get a right fraction. We get $as^{-1}bt^{-1}$ - and we are stuck...

How to do localization in noncommutative rings was discovered by Ore (who was a faculty at Yale 1927-1968)

Let S be a subset of a (noncommutative) ring A such that $0 \notin S$, $1 \in S$; $s,t \in S \Rightarrow st \in S$ as before. There are so called Ore conditions that guarantee that there is a localization A_S consisting of right, equivalently, of left fractions. Namely if S doesn't contain zero divisors we need to require:

(O1) $\forall\, a \in A, s \in S \;\exists\, b \in A, t \in S$ s.t $ta = bs$ (think, $as^{-1} = t^{-1}b$).

+ its mirror analog (left $\leftrightarrow$ right)

When S contains zero divisors we also should require:

(O2) if $sa = 0$ for $a \in A, s \in S$, then $\exists\, t \in S$ w. $at = 0$

- and its mirror condition.

In fact, (O2) allows to reduce to the case when there are no zero divisors in S: $J := \{a \in A \mid \exists\, s \in S \text{ s.t. } sa = 0\}$ is a two-sided ideal thx to (O2)+its mirror, so we replace A w. A/J, and S with its image in A/J. So we can just assume there are no zero divisors in S & (O1) and its mirror.

The we can define the set A_S of equivalence classes in A_S: $(a,s)\sim(a',s')$ (think $as^{-1}=a's'^{-1}$): we find b,t w $ta=bs$ (think $as^{-1}=t^{-1}b$) and declare $(a,s)\sim(a',s')$ if $ta'=bs'$.

Here we already see that everything becomes more painful: even to see that this doesn't depend on the choice of b,t requires a check. And there's more of this. Eventually, one gets the localization A_S consisting of right fractions (equivalently left) fractions) w. natural ring structure. It has a universal property similar to what we have in the commutative case.

Checking the Ore conditions is hard. And they are not always satisfied. For example, they aren't satisfied when $A=\mathbb{F}\langle x,y\rangle$ is a free $\mathbb{F}$-algebra & $S=A\setminus\{0\}$.

Still, they are satisfied in a number of examples. Namely, recall that A is <u>prime</u> if for any two-sided ideals I,J we have $IJ=\{0\}\Rightarrow I\neq\{0\}$ or $J\neq\{0\}$. We say A is <u>Noetherian</u> if all left & right ideals are finitely generated.

Theorem (Goldie) Let A be a prime Noetherian ring. Then the set S of all non-zero divisors in A satisfies the Ore conditions. The localization A_S is of the form $\mathrm{Mat}_n(D)$, where $n>0$ & D is a skew-field (a.k.a. divison ring).

In particular, A has no zero divisors $\Leftrightarrow n=1$.

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