Quantizations, lecture 7.

1) Restriction functor.

1) Restriction functor.

We consider the categories $HC(\mathfrak{g},K,\lambda)\subset HC(\mathfrak{g},\mathfrak{k})$ of HC $(\mathfrak{g},K,\lambda)$-modules & $(\mathfrak{g},\mathfrak{k})$-modules as in Sec 1.5 of Lec 6.

Let $\mathbb{O}\subset\mathfrak{g}^*(\simeq\mathfrak{g})$ be a nilpotent orbit & $e\in\mathfrak{g}$. In Lec 3 we have constructed a finite W-algebra $\mathcal{W}$, a filtered quantization of a transverse slice to S. Our goal in this section is to construct an exact functor

$$HC(\mathfrak{g},K,\lambda)\longrightarrow \mathcal{W}\text{-mod}$$

To slightly simplify the exposition we assume that $\lambda=0$ & $K\subset G$, and G is simply connected.

1.0) Reminder

Here we recall (a bit enhanced version) the construction of $\mathcal{W}$. We pick an $\mathfrak{sl}_2$-triple (e,h,f) and form the Slodowy slice $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$. We equip $\mathfrak{g}$ w. $\mathbb{G}_m$-action by $t.s = t^{-2}\gamma(t)s$, where γ is the composition of $\mathbb{G}_m \hookrightarrow SL_2$, $t \mapsto \operatorname{diag}(t, t^{-1})$, and the homomorphism $SL_2 \to G$ induced by the $\mathfrak{sl}_2$-triple

We also want an additional symmetry. Consider the Lie algebra anti-involution $\theta = -\sigma$ and assume e is fixed by θ. Then we can choose h & f s.t. $\theta(h) = -h$, $\theta(f) = f$. Observe that θ commutes w. $\mathbb{G}_m$-action. Note that S is $\mathbb{G}_m \ltimes \langle\theta\rangle$-stable.

Recall how the construction of $\mathcal{W}$ works. We consider the Rees algebra $\mathcal{U}_\hbar$ (for the doubled PBW filtration) & then complete it w.r.t. the maximal ideal $\mathfrak{m}_\hbar \subset \mathcal{U}_\hbar$ of $e \in \mathfrak{g}\ (\simeq \mathfrak{g}^*)$ getting a noncommutative algebra of formal power series, $\hat{\mathcal{U}}_\hbar$. Then we look at the space $\tilde{V} = \mathfrak{g}$ w. skew-symmetric form $\omega(x,y) = (e, [x,y])$ and its symplectic subspace $V := [\mathfrak{g}, f]$. V embeds into $\hat{\mathfrak{m}}_\hbar / \hat{\mathfrak{m}}_\hbar^2 = \mathfrak{g} \oplus \mathbb{C}\hbar$ and we lift this to an embedding $\iota: V \hookrightarrow \hat{\mathfrak{m}}_\hbar$ w. $[\iota(v), \iota(w)] = \hbar^2 \omega(u,v)$. We then consider the centralizer $\mathcal{W}_\hbar'$ of $\iota(V)$ in $\hat{\mathcal{U}}_\hbar$ & if $\widehat{\mathrm{Weyl}}_\hbar(V)$ denote the formal Weyl algebra of V (quantizing $\mathbb{C}[[V^*]]$) then ι gives rise to

$$(1)\quad \hat{W}_\hbar(V)\hat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{W}'_\hbar \xrightarrow{\sim} \hat{\mathcal{U}}_\hbar$$

We've mentioned that we can choose ι to be $\mathbb{C}_m$-equivariant. This gives a $\mathbb{C}_m$-action on $\mathcal{W}'_\hbar$. Let $\mathcal{W}_\hbar$ be the subalgebra of locally finite elements in $\mathcal{W}'_\hbar$ for this action. Then $\mathcal{W}:=\mathcal{W}_\hbar/(\hbar-1)$ is a filtered quantization of $\mathbb{C}[S]$.

We have graded algebra automorphism $\tilde{\theta}$ of $\mathcal{U}_\hbar$: acting by θ on and by $\sqrt{-1}$ on $\hbar$. So we get an action of $\langle\tilde{\theta}\rangle\times\mathbb{C}_m$ on $\hat{\mathcal{U}}_\hbar$ ($\langle\tilde{\theta}\rangle\simeq\mathbb{Z}/4\mathbb{Z}$). There's also a compatible action on V. We can choose ι to be $\langle\theta\rangle\times\mathbb{C}_m$-equivariant.

2.1) Construction

Since G is simply connected the involution σ with $\mathfrak{g}^\sigma=\mathfrak{k}$ integrates to G & $K=G^\sigma$. Note that with our choice of (e,h,f) we have $\theta(h)=-h \Rightarrow \sigma(h)=h \Rightarrow \operatorname{im}\gamma\subset K$.

Now let M be a HC $(\mathfrak{g},K)$-module

Exercise: 1) $\exists$ K-stable good filtration on M

2) For such a filtration the action of $S(\mathfrak{g})$ on $\operatorname{gr} M$

factors through $S(\mathfrak{g}/\mathfrak{k})$

Now we can consider a functor of restriction to the slice

$$\mathrm{Coh}^K((\mathfrak{g}/\mathfrak{k})^*) \longrightarrow \mathrm{Coh}(S)$$

Premium exercise: prove that it is exact.

Our goal is to construct a quantum version of this functor. The construction is in several steps.

Step 1: Pass to a Rees module.
Take $M \in HC(\mathfrak{g}, K)$ & equip it with a K-stable good filtration. Form the Rees module $M_\hbar$. Note that $\mathfrak{k} \subset \mathcal{U}(\mathfrak{g})$ is in filtr. degree 2, & the action of $\mathfrak{k}$ preserves the filtration on M. So on $M_\hbar$ we have $x M_\hbar \subset \hbar^2 M_\hbar$ $\forall x \in \mathfrak{k}$

Now consider the subspace $\mathcal{U}_\hbar^{-1} = \{a \in \mathcal{U}_\hbar \mid \theta(a) = -a\}$. Note that $\mathfrak{k}, \hbar^2 \subset \mathcal{U}_\hbar^{-1}$.

Important exercise 1: $\mathfrak{k}, \hbar^2$ generate $\mathcal{U}_\hbar \mathcal{U}_\hbar^{-1}$ as a left ideal.

From here we deduce that

(2) $$\mathcal{U}_\hbar \mathcal{U}_\hbar^{-1} \mathcal{M}_\hbar \subset \hbar^2 \mathcal{M}_\hbar$$

Step 2: Completion.

Define the completion of $\mathcal{M}_\hbar$ by

$$\hat{\mathcal{M}}_\hbar := \hat{\mathcal{U}}_\hbar \otimes_{\mathcal{U}_\hbar} \mathcal{M}_\hbar$$

This is flat over $\mathbb{C}[[\hbar]]$ b/c $\mathfrak{m}_\hbar \subset \mathcal{U}_\hbar$ satisfies Artin-Rees.

Also consider the subspace $\hat{\mathcal{U}}_\hbar^{-1} = \{b \in \hat{\mathcal{U}}_\hbar \mid \theta(b) = -b\}$

Important exercise 2: $\mathcal{U}_\hbar^{-1}$ is dense in $\hat{\mathcal{U}}_\hbar^{-1}$.

From this & (2) we deduce

(3) $\hat{\mathcal{U}}_\hbar \hat{\mathcal{U}}_\hbar^{-1} \hat{\mathcal{M}}_\hbar \subset \hbar^2 \hat{\mathcal{M}}_\hbar$

Step 3: Decomposition:

Since (1) intertwines the actions of θ, we have

$\hat{W}_\hbar(V)^{-1} \hat{\mathcal{M}}_\hbar \subset \hbar^2 \hat{\mathcal{M}}_\hbar$

Consider $L := V^{-\theta} = V \cap \mathfrak{k}$, this a Lagrangian subspace. In particular $\mathbb{C}[[L, \hbar]]$ becomes a module over $\hat{W}_\hbar(V)$ via its identification w. $\hat{W}_\hbar(V)/\hat{W}_\hbar(V)\mathcal{L}$ so that $l\in\mathcal{L}$ acts as $\hbar^2\partial_l$.

Fact: Let $\mathcal{N}_\hbar$ a complete & separated $\hat{W}_\hbar(V)$-module s.t. $\hbar$ is not a zero divisor in $\mathcal{N}_\hbar$. If $l\mathcal{N}_\hbar \subset \hbar^2\mathcal{N}_\hbar$ $\forall\, l\in\mathcal{L}$, then $\mathcal{N}_\hbar$ decomposes as the completed tensor product of $\mathbb{C}[[\mathcal{L},\hbar]]$ & $\mathcal{N}_\hbar^{\mathcal{L}} := \{n\in\mathcal{N}_\hbar \mid ln=0\}$

Main computation: consider the Weyl algebra of 2-dim. space w. generators x & ∂ (so that $[\partial,x]=\hbar^2$). Then $\forall\, v\in\mathcal{N}_\hbar$ we have that $\sum_{i=0}^{n}\frac{1}{i!}x^i(\partial/\hbar^2)^i v$ is a well-defined element of $\mathcal{N}_\hbar$ annihilated by ∂.

Fact applies to $\hat{\mathcal{M}}_\hbar$ b/c $\mathcal{L}\in\hat{W}_\hbar(V)^{-1}$.
Note that $\hat{\mathcal{M}}_\hbar^{\mathcal{L}}\subset\mathcal{M}_\hbar$ is a $W_\hbar'$-submodule

Step 4: Decompletion

Since $\operatorname{im}\gamma\subset K$, γ acts on every $HC(\mathfrak{g},K)$-module preserving any K-stable good filtration. Twisting the usual $\mathbb{C}_m$-

action by γ, we get a $\mathbb{G}_m$-action that extends to $\hat{\mathcal{M}}_\hbar$.

Note that $\hat{\mathcal{M}}_\hbar^L$ is $\mathbb{G}_m$-stable. We take the subspace of locally finite vectors $\subset \hat{\mathcal{M}}_\hbar^L$ (a graded $\mathcal{W}_\hbar$-submodule) & mod out $\hbar - 1$ getting a filtered $\mathcal{W}$-module to be denoted $\mathcal{M}_\dagger$. Note that $\mathcal{M}_\dagger$ comes equipped w. natural filtration.

2.2) Properties

1) $\mathcal{M}_\dagger$ as a module over $\mathcal{W}$ is independent of the choice of a good filtration on $\mathcal{M}$. Moreover, $\mathcal{M} \mapsto \mathcal{M}_\dagger$ is a functor. Both claims are based on the following easy exercise:

Exercise: Let $\mathcal{A}$ be a filtered quantization, $\mathcal{M}, \mathcal{N}$ $\mathcal{A}$-modules w. good filtrations $\mathcal{M} = \bigcup\limits_i \mathcal{M}_{\leq i}$, $\mathcal{N} = \bigcup\limits_j \mathcal{N}_{\leq j}$. Let $\varphi: \mathcal{M} \to \mathcal{N}$ be an $\mathcal{A}$-linear map. Then $\exists\, k \in \mathbb{Z}$ w. $\varphi(\mathcal{M}_{\leq i}) \subset \mathcal{N}_{\leq i+k}$ $\forall\, i \in \mathbb{Z}$.

2) For the filtration on $\mathcal{M}_\dagger$ coming from the construction we have $\operatorname{gr} \mathcal{M}_\dagger \simeq (\operatorname{gr} \mathcal{M})|_S$. This & Premium exercise in Sec. 2.1 show, in particular, that $\mathcal{M} \mapsto \mathcal{M}_\dagger$ is an exact functor.