MAT 380, HOMEWORK 3, DUE OCT 15

There are 11 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20.

All rings are commutative and contain 1.

Problem 1, 2pts. Let A be a ring and N_1, N_2, N_3 be A-modules. Let $\tau_1 : N_1 \to N_2, \tau_2 : N_2 \to N_3$ be A-linear maps with $\tau_2 \circ \tau_1 = 0$.

1) Assume that $0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_3)$ is exact for all $M \in A$-Mod. Prove that $0 \to N_1 \to N_2 \to N_3$ is exact.

2) Assume that $0 \to \mathrm{Hom}_A(N_3, M) \to \mathrm{Hom}_A(N_2, M) \to \mathrm{Hom}_A(N_1, M)$ is exact for all A-modules M. Show that $N_1 \to N_2 \to N_3 \to 0$ is exact. *Hint: this is harder than part 1. Start by showing that τ_2 is surjective. Note that both parts are "if and only if" statements: for part 1 this was established in Lecture 8, for part 2 this was left as an exercise.*

Problem 2, 2pts. Let A, B be rings and $F : A\text{-Mod} \to B\text{-Mod}, G : B\text{-Mod} \to A\text{-Mod}$ be additive functors such that G is left adjoint to F. Show that G is right exact and F is left exact. *Hint: use problem 1.*

Problem 3, 4pts. Let $A := \mathbb{Z}[\sqrt{-5}]$ and I be the ideal $(2, 1 + \sqrt{-5})$. The goal of this problem is to show that I is a projective A-module. Consider the A-linear map $\tau : A \oplus A \mapsto I$ given by $(a, b) \mapsto 2a + (1 + \sqrt{-5})b$. Note that it is surjective.

1, 1pt) Find $\alpha, \beta \in \mathbb{Q}[\sqrt{-5}]$ such that $\alpha c, \beta c \in A$ for all $c \in I$ and $2\alpha + (1 + \sqrt{-5})\beta = 1$.

2, 1pt) Construct an A-linear map $\iota : I \to A \oplus A$ satisfying $\tau \circ \iota = \mathrm{id}_I$.

3, 1pt) Deduce that I is projective.

4, 1pt) Show that there is an A-module isomorphism $I \oplus I \cong A \oplus A$.

Problem 4, 3pts. Let $A = \mathbb{Z}[x]$ and I be the ideal $(2, x)$. The goal of this problem is to show that I is not projective. Consider the surjective map $\tau : A \oplus A \mapsto I, (a, b) \mapsto 2a + xb$.

1, 1pt) Prove that any A-module homomorphism $I \to A$ is of the form $c \mapsto \alpha c$ for $\alpha \in A$. *Hint: perhaps, you will need to use that A is a unique factorization domain. And the map τ may be helpful too.*

2, 1pt) Show that there is no A-module homomorphism $\iota : I \to A \oplus A$ such that $\tau \circ \iota = \mathrm{id}_I$.

3, 1pt) Deduce that I is not projective.

Problem 5, 2pts. This problem studies duality for projective modules.

1, 1pt) Let A be a ring and M be a finitely generated projective A-module. Show that $\mathrm{Hom}_A(M, A)$ is also a projective A-module.

2, 1pt) Let A, I be as in Problem 3. Show that there is an isomorphism of A-modules $I \cong \mathrm{Hom}_A(I, A)$.

Problem 6, 2pts. Let A be a domain and $S \subset A$ be a subset that can be localized: closed under multiplication, containing 1 and not containing 0. Let $\mathrm{Frac}(A) := A_{A \setminus 0}$ be the fraction field of A.

1, 1pt) Show that $B := \{\frac{a}{s} | a \in A, s \in S\}$ is a subring of $\mathrm{Frac}(A)$.

2, 1pt) Construct an isomorphism $A_S \xrightarrow{\sim} B$.

Problem 7, 3pts. Let $A := \mathbb{C}[x, y]/(xy)$.

1, 1pt) Explicitly describe the set S of all nonzero divisors in A.

2, 1pt) Identify A_S with $\mathrm{Frac}(\mathbb{C}[x]) \oplus \mathrm{Frac}(\mathbb{C}[y])$. Show that the natural homomorphism $A \to A_S$ is injective.

3, 1pt) Prove that $S' := \{x^k | k \geqslant 0\}$ can be localized. Describe $A_{S'}$ and compute the kernel of the natural homomorphism $A \to A_{S'}$.

Problem 8, 3pts. This problem examines the behavior of projective modules under localization. *We are dealing with finitely generated projective modules for simplicity, but the results can be generalized to arbitrary projective modules in a straightforward way.* Let A be a ring and $S \subset A$ be a subset that can be localized.

1, 1pt) Let M, N be A-modules. Construct a natural isomorphism $(M \oplus N)_S \xrightarrow{\sim} M_S \oplus N_S$ of A_S-modules. *A remark: we will prove this result in a more general situation of tensor products. But you are supposed to prove this part directly.*

2, 1pt) Let M be a finitely generated projective A-module. Prove that M_S is a finitely generated projective A_S-module.

3, 1pt) Let A be a domain and $I \subset A$ be an ideal. Prove that if I is free as an A-module, then I is a principal ideal. *Hint – you need to use some localization. In particular, this problem implies that the ideal I from Problem 3 is a projective module, which is not free.*

Problem 9, 2pts. This problems deals with rings known as "completions". Let A be a ring and $\mathfrak{m}$ be its maximal ideal. So we can form the product ring $\prod_{k=1}^{\infty} A/\mathfrak{m}^k$, its elements are sequences (a_k) with $a_k \in A/\mathfrak{m}^k$.

1, 1pt) Let $\widehat{A}$ be the subset of $\prod_{k=1}^{\infty} A/\mathfrak{m}^k$ consisting of all sequence (a_k) such that, for each $j \in \mathbb{Z}_{>0}$, we have $a_{j+1} \bmod \mathfrak{m}^j = a_j$. Show that $\widehat{A}$ is a subring.

2, 1pt) Show that the subset $\{(a_i) | a_1 = 0\} \subset \widehat{A}$ is the unique maximal ideal in $\widehat{A}$. *So $\widehat{A}$ is local.*

Special cases of this: when $A = \mathbb{Z}$ and $\mathfrak{m} = (p)$, then $\widehat{A}$ is the ring of p-adic integers $\mathbb{Z}_p$. When $A = \mathbb{F}[x]$, where $\mathbb{F}$ is a field, and $\mathfrak{m} = (x)$, then $\widehat{A}$ is the ring of formal power series $\mathbb{F}[[x]]$.

Problem 10, 2pts. Let A be a ring and $\mathfrak{p}$ be a prime ideal. Let $\iota : A \to A_{\mathfrak{p}}$ be the natural homomorphism. Show that the map $J \mapsto \iota^{-1}(J)$ gives a bijection between the following two sets:

1) the set of all prime ideals J in $A_{\mathfrak{p}}$,

2) and the set of all prime ideals in A that are contained in $\mathfrak{p}$.

Problem 11, 2pts. Let A be a local ring with maximal ideal $\mathfrak{m}$. Let M be an A-module. Suppose that $\bigcap_{i=1}^{\infty} \mathfrak{m}^i M$ is finitely generated (*for a big class of rings, called* Noetherian, *this will be the case as long M is finitely generated. This class of rings will be studied later in the class*). Show that $\bigcap_{i=1}^{\infty} \mathfrak{m}^i M = \{0\}$.