MATH 380, HOMEWORK 2, DUE OCT 7

There are 8 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homework 1, which is useful for some of the problems – assuming you've solved the related problems there correctly. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

The first problem is about factorization into irreducibles in Noetherian domains. Its purpose is to illustrate what being Noetherian tells us about the structure of a ring. It also shows that not all of the three equivalent conditions of being Noetherian are created equal – from the point of view of a particular problem – I don't know solutions that would use finite generation of ideals, the AC condition gives a longish solution, while the most elegant solution is produced by using that every set of ideals has a maximal element.

Problem 1, 4pts total. Let A be a domain. Recall that a nonzero element $a \in A$ is irreducible if, first, it is not invertible, and, second, $a = a_1 a_2$ implies that one of a_i's is invertible. *Do not confuse irreducible and prime elements: being prime is a stronger condition, unless A is a UFD, in which case the two are equivalent.* Prove that if A is Noetherian, then every element a decomposes into the product of irreducible elements and an invertible element.

The next two problems have to do with an important concept of a graded ring. We don't have time to discuss this in class – but graded rings are VERY important so we cannot bypass them completely.

Definition 1. *Let A be a ring. By a grading (or, more precisely, $\mathbb{Z}_{\geqslant 0}$-grading) on A we mean a direct sum decomposition $A = \bigoplus_{i=0}^{\infty} A_i$ of abelian groups such that $A_i A_j \subset A_{i+j}$ for all i and j, and $1 \in A_0$. More generally, if A is a B-algebra, then by an algebra grading on A we mean the direct sum decomposition as above, where all A_i are B-submodules with the same condition on the multiplication and the unit. A graded ring (or algebra) is a ring (or algebra) equipped with a grading.*

In particular, A_0 is a subring, and A is an A_0-algebra.

Example 1. $A = B[x_1, \ldots, x_n]$, *where A_i is the span of degree i monomials. This can be generalized by assigning arbitrary positive integer degrees to the variables $x_1, \ldots, x_n$.*

The following problem gives an easy – but quite general construction of new graded algebras from existing ones. By a homogeneous element in a graded ring (or algebra) A we mean an element of some A_i. By a homogeneous ideal in A we mean an ideal I satisfying $I = \bigoplus_{i=0}^{\infty}(I \cap A_i)$.

Problem 2, 3 pts total. Let $\tilde{A}$ be a graded ring and I its homogeneous ideal.

1, 1pt) Equip the quotient $\tilde{A}/I$ with a grading.

2, 1pt) Let $a_1, \ldots, a_k \in \tilde{A}$ be homogeneous elements. Prove that $(a_1, \ldots, a_k)$ is a homogeneous ideal.

3, 1pt) Assume $\tilde{A}$ is a Noetherian ring. Show that I is generated by finitely many homogeneous elements.

One remarkable feature of graded rings is that being Noetherian for such rings is closely related to being finitely generated – which is far from being the case for more general rings. This was already observed by Hilbert in his work described in the bonus to Lecture 5.

Problem 3, 3 pts. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded ring. Assume A_0 is Noetherian. Prove that the following three conditions are equivalent.

a) A is Noetherian.

b) As an A_0-algebra, A is generated by a finite collection of homogeneous elements.

c) The ideal $A_{>0} := \bigoplus_{i=1}^{\infty} A_i$ in A is generated (as an ideal) by a finite collection of homogeneous elements.

Problem 4, 4pts. Let A be a Noetherian ring. Show that the formal power series ring $A[[x]]$ is Noetherian as well.

Hint: look at the proof of the Basis theorem. Of course, a formal power series doesn't have degree. But what does it have?

In fact, the completion of ANY Noetherian ring is still Noetherian.

The next problem gives an example of a $\mathbb{C}[x]$-module that is Artinian but not Noetherian.

Problem 5, 3pts total. Let $\mathbb{C}[x^{\pm 1}]$ denote the ring of Laurent polynomials, i.e. expressions $\sum_{i=-m}^{n} a_i x^i$ (with $a_i \in \mathbb{C}$) with addition and multiplication analogous to those for the usual polynomials. Note that it contains $\mathbb{C}[x]$ as a subring, hence it is a $\mathbb{C}[x]$-module. Consider the quotient $\mathbb{C}[x]$-module $M := \mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$.

1, 1pt) Describe all possible submodules of M.

2, 1pt) Prove that M is an Artinian $\mathbb{C}[x]$-module.

3, 1pt) Prove that M is not Noetherian.

Recall from Linear Algebra that if V is a finite dimensional vector space, then for a linear operator $\psi : V \to V$ begin injective or being surjective implies being an isomorphism. It turns out that suitable parts of this statement survive for Noetherian and Artinian modules over general rings.

Problem 6, 4pts total. Let M be an A-module and $\psi : M \to M$ be an A-linear map.

1, 1pt) Suppose that M is Noetherian and ψ is surjective. Prove that ψ is an isomorphism. *Hint: consider a sequence of submodules* $\ker \psi^n \subset M$.

2, 1pt) Suppose that M is Artinian and ψ is injective. Prove that ψ is an isomorphism.

3, 1pt) Now suppose that M has finite length and $\psi : M \to M$ is an arbitrary A-linear map. Define the submodules $M_0 := \bigcup_n \ker \psi^n$ and $M_1 := \bigcap_n \operatorname{im} \psi^n$. Show that $M = M_0 \oplus M_1$, the direct sum of submodules.

4, 1pt) Furthermore, prove that M_0, M_1 are ψ-stable, the restriction $\psi|_{M_1}$ is an isomorphism $M_1 \to M_1$, while there is $d \in \mathbb{Z}_{>0}$ such that $(\psi|_{M_0})^d = 0$.

Problem 7, 3pts. Let $\mathbb{F}$ be a field. Classify the finitely generated modules over the formal power series ring $\mathbb{F}[[x]]$.

Problem 8, 3pts. Let A be a PID and M be a finitely generated A-module. So we have $M \cong A^{\oplus k} \oplus \bigoplus_{i=1}^{\ell} A/(p_i^{d_i})$. In terms of this decomposition, find a necessary and sufficient condition for M to have finite length.