MATH 380, HOMEWORK 5, DUE NOV 18

There are 7 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-4. The text in italic below is meant to to be comments to a problem but not a part of it.

The first two problems are about tensor products, the next three are about adjoint functors, and the last two are about interactions between tensor products and adjoint functors!

Problem 1, 4pts total + bonus. Let A be a (commutative) ring, $I \subset A$ be an ideal, M be an A-module.

1, 1pt) Identify $(A/I) \otimes_A M$ with M/IM.

2, 1pt) Construct a natural surjective A-linear map $I \otimes_A M \to IM$.

3, 2pt) Let $A = \mathbb{F}[x, y]$, where $\mathbb{F}$ is a field, $I = M = (x, y)$. Show that the A-linear map in 2) is not injective.

4, 0pt) *This is harder.* Moreover, show that the kernel of that map is finite dimensional over $\mathbb{F}$. *So it is torsion, and we see that the tensor product of two torsion-free modules may fail to be torsion free.*

Problem 2, 4pts. *Tensor products of finite fields.* Let p be a prime integer. For its power q we write $\mathbb{F}_q$ for the field with q elements, this is an $\mathbb{F}_p$-algebra. Pick positive integers k, ℓ and set $q = p^k, q' = p^\ell$.

1, 2pts) Assume k, ℓ are coprime. Prove that there is a ring isomorphism $\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_{q'} \cong \mathbb{F}_{p^{k\ell}}$ (*hint: what can you say about the quotient of this tensor product by one of its maximal ideals?*).

2, 2pts) Prove that there is a ring homomorphism $\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_q \cong \mathbb{F}_q^k$ (the direct product of k copies of $\mathbb{F}_q$, *hint: use an exercise from Lecture 18 notes and some facts from the Fields and Galois theory course*).

Problem 3, 3pts + 2 bonuses. *More examples of adjoint functors.*

a, 1pts) Show that the polynomial ring functor $\mathsf{Sets} \to \mathsf{CommRings}$ from a) of Problem 2 in HW4 is left adjoint to the forgetful functor $\mathsf{CommRings} \to \mathsf{Sets}$.

b, 2pts) Show that the invertible elements functor from b) of Problem 2 in HW4 is right adjoint to the group ring functor from c) of that problem.

c, 0pts) Show that the inclusion functor $\mathbb{Z}$-Mod $\hookrightarrow \mathsf{Groups}$ *(that, per Lecture 19 has a left adjoint functor, the abelinization)* has no right adjoint functor.

d, 0pts) Describe the left adjoint of the inclusion $\mathsf{CommRings} \to \mathsf{Rings}$.

Problem 4, 3pts. *Compositions of adjoint functors.* Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}, F' : \mathcal{D} \to \mathcal{E}, G' : \mathcal{E} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that F is left adjoint to G and F' is left adjoint to G'. Prove that $F'F$ is left adjoint to GG'.

Problem 5, 3pts. *Endomorphisms of adjoint functors.* Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with bijection

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

1, 2pts) Let τ be a functor endomorphism of G. Let $\tau_{X,Y}$ denote the map

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y)), \psi \mapsto \tau_Y \circ \psi.$$

and let $\tau'_{X,Y} := \eta^{-1}_{X,Y} \circ \tau_{X,Y} \circ \eta_{X,Y}$ denote the corresponding map

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

Show that there is a unique functor endomorphism τ' of F such that $\tau'_{X,Y}$ sends $\varphi \in \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ to $\varphi \circ \tau'_X$. *Hint: Yoneda!*

2, 1pts) Consider the monoids $\mathrm{End}_{Fun}(F)$ and $\mathrm{End}_{Fun}(G)^{opp}$, where in the latter we reverse the order of multiplication. Establish a monoid isomorphism $\mathrm{End}_{Fun}(F) \xrightarrow{\sim} \mathrm{End}_{Fun}(G)^{opp}$.

Problem 6, 4pts total. *Tensor algebra of a module. This problem discusses a left adjoint functor to the forgetful functor* A-Alg $\to$ A-Mod. Let A be a commutative ring and M be an A-module. Let $M^{\otimes i}$ denote the i-fold tensor product $M \otimes_A M \otimes_A \ldots \otimes_A M$ (with $M^{\otimes 0} = A, M^{\otimes 1} = M$).

Consider the A-module $T_A(M) := \bigoplus_{i=0}^{\infty} M^{\otimes i}$. We define a graded algebra structure on $T_A(M)$ as follows: for $u \in M^{\otimes i}, v \in M^{\otimes j}$ their product is $u \otimes v$ in $M^{\otimes i} \otimes_A M^{\otimes j}$, which, as we know, is identified with $M^{\otimes(i+j)}$. This equips $T_A(M)$ with the structure of a graded associative A-algebra with unit $1 \in A$. *Check this, not for credit. Recall that graded algebras were discussed in HW2. Note that $T_A(M)$ is not commutative.*

1, 1pt) Let M be a free A-module with basis $x_i, i \in \mathcal{I}$. Identify $T_A(M)$ with the algebra $A\langle x_i \rangle_{i \in \mathcal{I}}$ of noncommutative polynomials in the variables x_i (recall that in that algebra we have a basis formed by words in the alphabet $x_i, i \in \mathcal{I}$, and the multiplication of basis elements is the concatenation of words).

2, 1pt) Let $\varphi : M \to N$ be an A-module homomorphism. Produce a graded algebra homomorphism $T_A(\varphi) : T_A(M) \to T_A(N)$.

3, 1pt) Show that T_A is a functor A-Mod $\to$ A-Alg.

4, 1pts) Show that the functor T_A is left adjoint to the forgetful functor A-Alg $\to$ A-Mod.

The point of the tensor algebra construction is that many interesting algebras arise as quotients: symmetric and exterior algebras of vector spaces (or modules), their deformations, Clifford and Weyl algebras associated to bilinear symmetric and skew-symmetric forms, the universal enveloping algebras important for the representation theory of semisimple Lie algebras, to name a few. The next problem deals with the symmetric algebra, it is important for Commutative algebra.

Problem 7, 6pts total. *Symmetric algebra of a module. This problem discusses a left adjoint functor to the forgetful functor* A-CommAlg $\to$ A-Mod. Let A, M be as in the previous problem. Consider the two-sided ideal $I_M \subset T_A(M)$ generated by the elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1 \in T_A(M)$ for all $m_1, m_2 \in M$ (by definition, this means that I_M is the A-linear span of the elements of the form $\alpha(m_1 \otimes m_2 - m_2 \otimes m_1)\beta$ for $m_1, m_2 \in M, \alpha, \beta \in T_A(M)$). Set $S_A(M) := T_A(M)/I_M$.

1, 1pt) Show that $S_A(M)$ is a graded <u>commutative</u> A-algebra with the i-th graded component being the image of $M^{\otimes i} \subset S_A(M)$ in $S_A(M)$.

2, 1pt) Let $\varphi : M \to N$ be an A-module homomorphism. Produce a graded algebra homomorphism $S_A(\varphi) : S_A(M) \to S_A(N)$.

3, 1pt) Show that S_A is a functor A-Mod $\to A$-CommAlg.

4, 1pt) Show that the functor S_A is left adjoint to the forgetful functor A-CommAlg $\to$ A-Mod.

5, 1pt) Let M be a free module with basis $x_i, i \in \mathcal{I}$. Identify $S_A(M)$ with the algebra $A[x_i]_{i\in\mathcal{I}}$ of usual polynomials.

6, 1pt) Assume A is a Noetherian ring and M is a finitely generated A-module. Prove that $S_A(M)$ is a Noetherian ring.