PARABOLIC WAKIMOTO MODULES AND APPLICATIONS

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We will follow [Fre07, §6.3], to define generalized Wakimoto modules, which gives a functorial way of constructing $\widehat{\mathfrak{g}}$-modules from $\widehat{\mathfrak{m}}$-modules for parabolic subalgebras $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{u} \subset \mathfrak{g}$.

1. Semi-infinite parabolic induction

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra with Borel subalgebra $\mathfrak{b}_+$ and Cartan subalgebra $\mathfrak{h}$ (we work in this generality since we want to apply our construction to Levi subalgebras of simple Lie algebras).

Wakimoto modules, constructed in [Wan24], are the images of Fock modules under a functor $\widetilde{U}_\kappa(\mathfrak{h})\text{−mod} \to \widetilde{U}_{\kappa+\kappa_c}(\mathfrak{g})\text{−mod}$.[1] We want to generalize the construction by replacing the Borel subalgebra $\mathfrak{b}$ with an arbitrary parabolic subalgebra $\mathfrak{p}$ and replacing the Cartan subalgebra $\mathfrak{h}$ with the Levi component $\mathfrak{m}$ of $\mathfrak{p}$. Let us first recall what a parabolic subalgebra is:

Definition 1.1. A *parabolic subalegbra* is a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that one of the following equivalent conditions hold:

- $\mathfrak{p}$ contains a Borel subalgebra of $\mathfrak{g}$; or
- the orthogonal complement of $\mathfrak{p}$ with respect to an invariant non-degenerate symmetric bilinear form[2] is its nilradical.

Example 1.2. $\mathfrak{b}_+$ and $\mathfrak{g}$ are parabolic subalgebras of $\mathfrak{g}$.

Each conjugacy class of parabolic subalgebras has a unique representative containing $\mathfrak{b}_+$: we call those parabolic subalgebras *standard*. Let Δ_s be the set of simple roots corresponding to $\mathfrak{b}_+ \subset \mathfrak{g}$. Then standard parabolic subalgebras of $\mathfrak{g}$ are classified by subsets of Δ_s: so $\mathfrak{b}_+$ corresponds to $\varnothing$ and $\mathfrak{g}$ corresponds to Δ_s. More generally, for a subset $S \subset \Delta_s$, the corresponding *standard parabolic subalgebra* $\mathfrak{p}_S \subset \mathfrak{g}$ is

$$\mathfrak{p}_S := \mathfrak{b}_+ \oplus \bigoplus_{\substack{\alpha>0 \\ \alpha\in\operatorname{span}\Delta_s}} \mathfrak{g}_\alpha.$$

The Levi component is then given by:

$$\mathfrak{m}_S := \mathfrak{h} \oplus \bigoplus_{\alpha\in\operatorname{span}\Delta_s} \mathfrak{g}_\alpha.$$

Analogous to the opposite Borel subalgebra, let

$$\mathfrak{p}_{S,-} := \mathfrak{b}_- \oplus \bigoplus_{\substack{\alpha<0 \\ \alpha\in\operatorname{span}\Delta_s}} \mathfrak{g}_\alpha$$

be the *opposite parabolic*.

[1] These are categories of smooth modules.

[2] When $\mathfrak{g}$ is semisimple we can use the Killing form, but for arbitrary reductive Lie algebras the Killing form may be degenerate.

Example 1.3. When $\mathfrak{g} = \mathfrak{sl}_n$, note that subsets of $\Delta_s = \{\alpha_1, \ldots, \alpha_{n-1}\}$ are parametrized by subsets $S = \{a_1, \ldots, a_k\}$ of $\{1, \ldots, n-1\}$. Then

$$\mathfrak{p}_S = \begin{pmatrix} M_{a_1\times a_1} & * & * & * \\ 0 & M_{(a_2-a_1)\times(a_2-a_1)} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & M_{(n-a_k)\times(n-a_k)} \end{pmatrix}$$

and

$$\mathfrak{p}_{S,-} = \begin{pmatrix} M_{a_1\times a_1} & & & \\ * & M_{(a_2-a_1)\times(a_2-a_1)} & & \\ * & * & \ddots & \\ * & * & * & M_{(n-a_k)\times(n-a_k)} \end{pmatrix}.$$

Then

$$\begin{aligned} \mathfrak{m}_S &= \{(x_0, \ldots, x_k) \in \mathfrak{gl}_{a_1} \times \cdots \times \mathfrak{gl}_{n-a_k} : \mathrm{tr}(x_0) + \cdots + \mathrm{tr}(x_k) = 0\} \\ &\simeq \mathfrak{sl}_{a_1} \times \cdots \times \mathfrak{sl}_{n-a_k} \times \mathbb{C}^{\oplus k}. \end{aligned}$$

First, we must extend relevant definitions from simple Lie algebras to reductive Lie algebras. The following is the generalization of the critical level:

Definition 1.4. Let $\mathfrak{g}$ be a reductive Lie algebra, which decomposes as $\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i \oplus \mathfrak{g}_0$ for some simple Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ and an abelian Lie algebra $\mathfrak{g}_0$. Then the *critical level* is $\kappa_c(\mathfrak{g}) := (\kappa_{i,c})_{i=0}^{s}$, where $\kappa_{0,c} = 0$ and $\kappa_{i,c}$ is the critical level for the simple Lie algebra $\mathfrak{g}_i$ for $1 \leq i \leq s$.

Given an invariant symmetric bilinear form κ on $\mathfrak{g}$, let $\widehat{\mathfrak{g}}_\kappa$ be the corresponding affine Kac-Moody algebra, as in [KL24]: it is given as a central extension

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}(\!(t)\!) \to 0$$

with commutation relation

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - \big(\kappa(A, B)\mathrm{Res} f dg\big)\mathbf{1}.$$

Let us now formally re-state our goal:

Goal 1.5. Let $\mathfrak{g}$ be a reductive Lie algebra, let κ be an invariant symmetric bilinear form on $\mathfrak{g}$, and let $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{u} \subset \mathfrak{g}$ be a parabolic subalgebra. Define an exact functor

$$\widetilde{U}_{\kappa|_{\mathfrak{m}}+\kappa_c(\mathfrak{m})}(\mathfrak{m})\text{–mod} \to \widetilde{U}_{\kappa+\kappa_c}(\mathfrak{g})\text{–mod}$$

such that the Wakimoto module with highest weight λ is sent to the Wakimoto module with highest weight λ.

1.1. **Finite-dimensional analog.** Let us first describe the finite-dimensional analog of Goal 1.5.

Definition 1.6. Let $\mathfrak{g}$ be a simple Lie algebra with standard parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{u}$. There is an exact functor, the *Vermatization functor*

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \colon \mathfrak{m}\text{–mod} \to \mathfrak{g}\text{–mod}.$$

Given a $\mathfrak{m}$-module V, we may view it as a $\mathfrak{p}$-module by extension by zero, i.e., by making $\mathfrak{u}$ act by zero, and we let

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V.$$

Now the $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ sends Verma modules to Verma modules:

Lemma 1.7. *For a weight $\lambda \in \mathfrak{h}^*$, let $V_{\mathfrak{m}}(\lambda)$ and $V_{\mathfrak{g}}(\lambda)$ be the Verma modules with highest weight λ of the Lie algebras $\mathfrak{m}$ and $\mathfrak{g}$, respectively. Then*

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V_{\mathfrak{m}}(\lambda) \simeq V_{\mathfrak{g}}(\lambda).$$

Proof. Follows from observing that $U(\mathfrak{p}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda$ is isomorphic to the inflation of the $\mathfrak{m}$-module $V_{\mathfrak{m}}(\lambda)$ to $\mathfrak{p}$, and because induction is transitive. □

Remark 1.8. When $\mathfrak{p} = \mathfrak{b}_+$, the above recovers the construction of Verma modules (i.e., $V_{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}_\lambda$).

Recall that [Kiy24] gives a geometric construction of dual Verma modules using vector fields on G/N_-, where N_- is the unipotent radical of the opposite Borel subalgebra B_-. The construction admits a straightforward generalization to the parabolic setting: let $P_\pm = M \ltimes U_\pm \subset G$ be subgroups whose Lie algebras are $\mathfrak{p}_\pm = \mathfrak{m} \ltimes \mathfrak{u}_\pm \subset \mathfrak{g}$. Then analogously to [Kiy24, §2] there is a map of Lie algebras

$$\mathfrak{g} \to \mathrm{Vect}(G/U_-)^{M_r},$$

where M_r acts on G/U_- from the right.[3] Now as in Daishi's talk, $P_+U_-/U_- \subset G/U_-$ is Zariski open, and restricting to the locus gives a homomorphism of algebras

$$\varphi^G_{P_+}\colon U(\mathfrak{g}) \to D(P_+)^M \simeq D(U_+) \otimes U(\mathfrak{m}), \tag{1.9}$$

where the second isomorphism follows from the isomorphism of varieties $P_+ \simeq U_+ \times M$. Now:

Lemma 1.10. *Let V be a $\mathfrak{m}$-module, with structure morphism $\varphi\colon U(\mathfrak{m}) \to \mathrm{End}(V)$. Then the modified $\mathfrak{g}$-module structure on $\mathbb{C}[U_+] \otimes V$ is defined by*

$$U(\mathfrak{g}) \to D(U_+) \otimes U(\mathfrak{m}) \xrightarrow{1\otimes\varphi} D(U_+) \otimes \mathrm{End}(V) \to \mathrm{End}(\mathbb{C}[U_+] \otimes V),$$

noting that $\mathbb{C}[U_+]$ is naturally a $D(U_+)$-module. Then the $\mathfrak{g}$-module $\mathbb{C}[U_+] \otimes V^\vee$ is isomorphic to the dual Vermatization $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V)^\vee$.[4]

We hope to see Lemma 1.7 from the geometric perspective:

Proposition 1.11. *Let $P_+ = M \ltimes U_+ \subset G$ be a standard parabolic subgroup. There is a commutative diagram*

$$\begin{tikzcd}
U(\mathfrak{g}) \arrow[rr, "\varphi^G_{B_+}"] \arrow[d, "\varphi^G_{P_+}"] & & D(N_+) \otimes U(\mathfrak{h}) \arrow[d, "\simeq"] \\
D(U_+) \otimes U(\mathfrak{m}) \arrow[rr, "\mathrm{id}_{D(U_+)} \otimes \varphi^M_{B_+\cap M}"] & & D(U_+) \otimes \big(D(N_+ \cap M) \otimes U(\mathfrak{h})\big).
\end{tikzcd}$$

Here, the homomorphisms $U(\mathfrak{g}) \to D(N_+) \otimes U(\mathfrak{h})$ and $U(\mathfrak{g}) \to D(U_+) \otimes U(\mathfrak{m})$ are as in (1.9), and the right vertical isomorphism follows from the multiplication isomorphism[5] *$U_+ \times (N_+ \cap M) \simeq N_+$.*

Proof. Indeed, the following diagram commutes:

$$\begin{tikzcd}
D(G)^{G_r} \arrow[r, hook] & D(G/U_-)^{M_r} \arrow[r, hook] \arrow[d] & D(G/N_-)^{H_r} \arrow[d] \\
& D(P_+)^{M_r} \arrow[r, hook] & D(P_+/(P_+ \cap N_-))^{H_r}
\end{tikzcd} \tag{1.12}$$

where the vertical homomorphisms are restricting along open immersions $P_+ \subset G/U_-$ and $P_+/(P_+\cap N_-) \subset G/N_-$. The first horizontal homomorphism $D(G)^{G_r} \hookrightarrow D(G/U_-)^{M_r}$ is obtained as follows: any $\sigma \in D(G)^{G_r}$ is an operator $\sigma\colon \mathbb{C}[G] \to \mathbb{C}[G]$ which is G_r-invariant, hence it sends $(U_-)_r$-invariant functions to $(U_-)_r$-invariant functions. In fact, for any $(U_-)_r$-invariant open subset X of

[3]the action is well-defined because M normalizes U_-.

[4]Here, as usual, letting $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution and given $M = \oplus_\mu M_\mu$, we let $M^\vee = \bigoplus_\mu M_\mu^*$ with $\langle x \cdot n, m\rangle = \langle n, -\tau(x)m\rangle$ for $n \in M^\vee, m \in M$.

[5]An isomorphism of varieties; not of groups!

G, there is an operator $\sigma\colon \mathbb{C}[X]^{U_-,r} \to \mathbb{C}[X]^{U_-,r}$. In other words, since $\mathbb{C}[X/U_-] = \mathbb{C}[X]^{U_-,r}$, it defines an endomorphism of sheaves $\widetilde{\sigma}\colon \mathcal{O}_{G/U_-} \to \mathcal{O}_{G/U_-}$, which can be shown to be a differential operator. Note that we need $\widetilde{\sigma}$ to be an endomorphism of the sheaf $\mathcal{O}_{G/U_-}$, and not just $\mathbb{C}[G/U_-]$, since G/U_- may not be affine, e.g., $\mathrm{SL}_2/N_- \simeq \mathbb{A}^2\backslash\{(0,0)\}$. Moreover, since σ is G_r-invariant $\widetilde{\sigma}$ must be M_r-invariant, hence $\widetilde{\sigma} \in D(G/U_-)^{M_r}$. All other horizontal maps are constructed in a similar fashion.

Now we have the isomorphisms $U(\mathfrak{g}) \simeq D(G)^{G_r}$ and $D(P_+)^{M_r} \simeq D(U_+) \otimes U(\mathfrak{m})$, so (1.12) can be re-written as

$$\begin{tikzcd}
U(\mathfrak{g}) \arrow[r] \arrow[rrr, bend left, "\varphi^G_{B_+}"] \arrow[dr, "\varphi^G_{P_+}"'] & D(G/U_-)^{M_r} \arrow[r] \arrow[d] & D(G/N_-)^{H_r} \arrow[r] \arrow[d] & D(N_+) \otimes U(\mathfrak{h}) \arrow[d, "\simeq"] \\
 & D(U_+) \otimes U(\mathfrak{m}) \arrow[r] \arrow[rr, bend right, "1\otimes\varphi^M_{B_+\cap M}"'] & D(P_+/(P_+ \cap N_-))^{H_r} \arrow[r] & D(U_+) \otimes D(N_+ \cap M) \otimes U(\mathfrak{h}),
\end{tikzcd}$$

which is the desired commtativity. Here the homomorphism $D(G/N_-)^{H_r} \to D(N_+) \otimes U(\mathfrak{h})$ is the composition of the restriction to the open Bruhat cell $D(G/N_-)^{H_r} \to D(B_+)^{H_r}$, together with the standard isomorphism $D(B_+)^{H_r} \simeq D(N_+) \otimes U(\mathfrak{h})$ from [Kiy24]. $\square$

Remark 1.13. Proposition 1.11 implies Lemma 1.7.

References

[Fre07] Edward Frenkel, *Langlands correspondence for loop groups*, Cambridge Studies in Advanced Mathematics, vol. 103, Cambridge University Press, Cambridge, 2007. MR 2332156

[Kiy24] Daishi Kiyohara, *Free field realization*, seminar notes 2024.

[KL24] Ivan Karpov and Ivan Losev, *Invariants of jets and the center for* $\widehat{\mathfrak{sl}}_2$, seminar notes 2024.

[Wan24] Zeyu Wang, *Wakimoto modules*, seminar notes 2024.

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