Bonus lecture

1) More on slice theorems

Ref: [L1].

1.1) Compact setting

We are in the setting of Sec 1.4 of Lec 16 (and use the notation from there) but we drop the assumption that $\mu(m)=0$. Set $\xi:=\mu(m)$. Set $L:=\mathrm{Stab}_K(\xi)$, this is a compact Levi subgroup (in particular, it is connected).

Our goal is to reduce the study of the action of K on M in a neighborhood of m to a suitable Hamiltonian action of L via the so called cross-section theorem (due to Guillemin-Sternberg).

Let $\mathfrak{l}^{*,reg}:=\{\alpha\in\mathfrak{l}^*\mid \mathrm{Stab}_K(\alpha)\subset L\}$, in particular, $\xi\in\mathfrak{l}^{*,reg}$. Further, let $\mathfrak{l}^\perp$ denote the unique L-stable complement to $\mathfrak{l}$ in $\mathfrak{k}$ so that $\mathfrak{l}^*\simeq(\mathfrak{k}/\mathfrak{l}^\perp)^*\hookrightarrow\mathfrak{k}^*$. We set $N:=\mu^{-1}(\mathfrak{l}^{*,reg})$.

Exercise: Let $n\in N$. Then the following claims hold:

1) $\{x_{M,n}\mid x\in\mathfrak{l}^\perp\}\subset T_nM$ is a symplectic subspace

2) T_nN coincides with both of the following:

$d_n\mu^{-1}(\mathfrak{l}^*)$ & $\{x_{M,n}\mid x\in\mathfrak{l}^\perp\}^\angle$

in particular, it is a symplectic subspace of dimension $\dim M - \dim \mathfrak{l}^{\perp}$.

3) N is a smooth symplectic manifold w. Hamiltonian action of L & moment map $\mu|_N$.

4) The manifold $K\times^L N$ acquires a unique symplectic structure s.t. the natural map $K\times^L N \to M$ is a symplectomorphism. In particular, the action of K on $K\times^L N$ is Hamiltonian w. moment map sending $L.(k,n)$ to $k.n$.

Describing a K-stable neighborhood of m in M reduces to studying an L-stable neighborhood of m in N. Note that $\xi=\mu(m)$ lies in $(\mathfrak{k}^*)^L$ and so $\mu(m)-\xi$ is also a moment map, see Sec 1.3 in Lec 16. This new moment map sends m to 0 and so we have reduced to the case when the image of the point in question is 0.

1.2) Reductive setting

Now we change the setup: we consider a reductive group G acting on a smooth variety X. We can talk about algebraic symplectic forms on X and hence about Hamiltonian G-actions. A recent reason to be interested in such actions is the relative Langlands program of Ben-Zvi, Sakellaridis & Venkatesh which includes, roughly speaking, a duality between varieties w. Hamiltonian actions of reductive groups.

Our goal in this note is to understand a local structure of Hamiltonian G-varieties (near a closed orbit).

First, the approach in Sec 1.1 carries over to the reductive group setting (extended exercise). So we can assume that we have $x\in X$ w. Gx closed & $\mu(x)$ <u>nilpotent</u>. One can state and prove a "symplectic slice theorem", cf. Sec 1.4 in Lec 16, see [L1], describing $\pi^{-1}(U)$ for a neighborhood U (in the <u>complex</u> topology) of $\pi(x)$ in $X/\!\!/G$ in terms of the triple $\mathrm{Stab}_G(x)=:H$, $\mu(x)$ & the H-representation $T_xX/T_x(Gx)$. In what follows we will only construct an example: the equivariant Slodowy slice, it corresponds to the situation when $H=\{1\}$, $\mu(x)\in\mathfrak{g}^*$ is nilpotent & $T_x(Gx)\subset T_xX$ is a coisotropic subspace (meaning that $T_x(Gx)^\perp\subset T_x(Gx)$).

Let $e\in\mathfrak{g}$ be a nilpotent element. We can include it into an $\mathfrak{sl}_2$-triple (e,h,f) & form a transverse slice $S:=e+\ker(\operatorname{ad}f)\subset\mathfrak{g}$, see Sec 2.3.1 in Lec 10. Recall also that $\mathfrak{g}$ comes w. a $\mathbb{C}^\times$-action, $t\cdot x=\gamma(t)^{-1}t^2x$, that preserves S and contracts it to e.

Now choose a non-degenerate G-invariant symmetric form $(\cdot,\cdot)$ on $\mathfrak{g}$ that allows us to identify $\mathfrak{g}$ w. $\mathfrak{g}^*$ and hence T^*G w. $G\times\mathfrak{g}$. So we can consider $G\times S$ as a subvariety in $G\times\mathfrak{g}=T^*G$.

Important exercise: $G\times S\subset G\times\mathfrak{g}$ is a symplectic subvariety. Hint: first show that the restriction of the symplectic form to $G\times S$ is non-degenerate at $(1,e)$. Then use the actions of G & $\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by $t.(g,s)=(g\gamma(t), t\cdot s)$.

Remark: 1) A more general class of "model varieties" is obtained as follows: take a reductive subgroup $H\subset Z_G(e,h,f)$ & a symplectic representation V of H. Then similarly to what was done in Sec 1.4 of Lec 16 we can form the Hamiltonian reduction $[(G\times S)\times V]///_0 H$, where $///_0 H$ means the Hamiltonian reduction at 0.

2) Of special interest is the case when e is a principal nilpotent element (see Sec 2.3.2 of Lec 10). In this case, one can show that for $x\in\mathfrak{g}$ TFAE:

- $Gx\cap S\neq\emptyset$ &
- x is regular meaning that $\dim Z_G(x)=\operatorname{rk}\mathfrak{g}$.

Let $\mathfrak{g}^{reg}$ denote the locus of regular elements in $\mathfrak{g}$, it's exactly the image of $\mu: G\times S\to\mathfrak{g}$. The fiber $\mu^{-1}(s)$ for $s\in S$ is nothing else as $Z_G(s)\times\{s\}\subset G\times S$. In other words $G\times S$ is the pullback of the "commuting scheme" $Z=\{(g,x)\mid Ad(g)x=x\}$ (over $\mathfrak{g}$) to $\mathfrak{g}^{reg}$ (known as the universal centralizer).