Lecture 15

1) Yoneda lemma.
2) Objects representing functors.
Ref: [R], Sections 2.1, 2.2.

1.1) Statement. The Yoneda lemma gives a powerful tool to compute the morphisms between certain functors.

Theorem: Let $\mathcal{C}$ be a category, $X \in Ob(\mathcal{C}) \rightsquigarrow$ the Hom functor $\mathcal{F}_X : \mathcal{C} \longrightarrow Sets$. Then the following are true:
i) Let F be another functor $\mathcal{C} \to Sets$. The functor morphisms $\mathcal{F}_X \Rightarrow F$ are in bijection w. $F(X)$.
ii – Yoneda lemma) for $F = \mathcal{F}_{X'}$, for $X' \in Ob(\mathcal{C})$, the bijection in i) sends $g \in \mathcal{F}_{X'}(X) = Hom_{\mathcal{C}}(X', X)$ to $\eta^g : \mathcal{F}_X \Rightarrow \mathcal{F}_{X'}$.

1.2) Proof.
Step 1: from $a \in F(X)$ construct $\eta^a : \mathcal{F}_X \Rightarrow F$. I.e $\forall\, Y \in Ob(\mathcal{C})$ need a map ($\mathcal{F}_X, F$ land in Sets) $\eta^a_Y : \mathcal{F}_X(Y) = Hom_{\mathcal{C}}(X,Y) \to F(Y)$ so that the maps η^a_Y satisfy the axioms of functor morphism.

Step 1.1: construct $\eta^a_Y(\psi)$ for $\psi \in Hom_{\mathcal{C}}(X,Y)$. Note that F gives a map of sets $F(\psi): F(X) \to F(Y)$. Set

(1) $\eta^a_Y(\psi) = [F(\psi)](a)$.

Step 1.2: check that (η^a_Y) satisfies the axiom:
$\forall\ Y_1 \xrightarrow{f} Y_2$ the diagram

$$\begin{tikzcd}
Hom_{\mathcal{C}}(X, Y_1) \arrow[r, "f\circ ?"] \arrow[d, "\eta^a_{Y_1}"] & Hom_{\mathcal{C}}(X, Y_2) \arrow[d, "\eta^a_{Y_2}"] \\
F(Y_1) \arrow[r, "F(f)"] & F(Y_2)
\end{tikzcd}$$

is commutative. Pick $\psi \in Hom_{\mathcal{C}}(X, Y_1)$

$\longrightarrow\downarrow$: $\quad \psi \mapsto \eta^a_{Y_2}(f\circ\psi) = [(1)] = [F(f\circ\psi)](a)$

$\downarrow\longrightarrow$: $\quad \psi \mapsto F(f)(\eta^a_{Y_1}(\psi)) = F(f)(F(\psi)(a))$

The equality of final expressions follows from $F(f\circ\psi) = F(f)\circ F(\psi)$.
Conclusion: η^a is a functor morphism $\mathcal{F}_X \Rightarrow F$.

Step 2: For $F = \mathcal{F}_{X'}$, $a \mapsto \eta^a$ recovers the previous construction:
$a = g \in \mathcal{F}_{X'}(X) = Hom_{\mathcal{C}}(X', X)$, $\psi \in Hom_{\mathcal{C}}(X, Y)$

$$\eta^a_Y(\psi) = [(1)] = \mathcal{F}_{X'}(\psi)(a) = [\text{def'n of } \mathcal{F}_{X'}(\psi)] = \psi\circ a = \eta^g(\psi)$$

for η^g defined previously.

It remains to establish part (i) of Theorem.

Step 3: From $\eta: \mathcal{F}_X \Rightarrow F$ construct $a_\eta \in F(X)$; η is a collection of $\eta_Y: Hom_{\mathcal{C}}(X, Y) \to F(Y)$. Pick $\eta_X: Hom_{\mathcal{C}}(X, X) \to F(X)$. Set

(2) $\quad a_\eta := \eta_X(1_X)$.

It remains to show $a \mapsto \eta^a$, $\eta \mapsto a_\eta$ are mutually inverse.

Step 4: $a_{\eta^a} = a$: $a_{\eta^a} = \eta^a_X(1_X) = [F(1_X)](a) = [F(1_X) = 1_{F(X)}]$
$= 1_{F(X)}(a) = a$.

Step 5: $\eta^{a_\eta} = \eta \Leftrightarrow \forall\, Y \in Ob(\mathcal{C})$ have $\eta^{a_\eta}_Y = \eta_Y$, equality of maps $Hom_{\mathcal{C}}(X,Y) \longrightarrow F(Y)$. Pick $\psi \in Hom_{\mathcal{C}}(X,Y)$. Then
$\eta^{a_\eta}_Y(\psi) = [F(\psi)](a_\eta) = F(\psi)(\eta_X(1_X))$.
We will apply the commutative diagram of maps, that is a part of functor morphism axioms for $\eta: \mathcal{F}_X \Rightarrow F$, $X \xrightarrow{\psi} Y$:

$$\begin{tikzcd}
Hom_{\mathcal{C}}(X,X) \arrow[r, "\psi\circ ?"] \arrow[d, "\eta_X"] & Hom_{\mathcal{C}}(X,Y) \arrow[d, "\eta_Y"] \\
F(X) \arrow[r, "F(\psi)"] & F(Y)
\end{tikzcd}$$

to $1_X \in Hom_{\mathcal{C}}(X,X)$.

$\downarrow\longrightarrow$: $1_X \mapsto F(\psi)(\eta_X(1_X))$

equal expressions.

$\longrightarrow\downarrow$: $1_X \mapsto \psi\circ 1_X = \psi \mapsto \eta_Y(\psi)$

So $\eta^{a_\eta}_Y(\psi) = F(\psi)(\eta_X(1_X)) = \eta_Y(\psi) \Rightarrow \eta^{a_\eta} = \eta$. $\square$

Rem: We have seen in Section 2.3 of Lecture 14, that the bijection $Hom_{\mathcal{C}}(X',X) \xrightarrow{\sim} \{$functor morphisms $\mathcal{F}_X \Rightarrow \mathcal{F}_{X'}\}$ is compatible w. compositions: $\eta^{gg'} = \eta^{g'}\eta^{g}$. In particular, g is an isomorphism $\Leftrightarrow \eta^g$ is.

2) Objects representing functors.

2.1) Definition & uniqueness.

Let $\mathcal{C}$ be a category & $F: \mathcal{C} \longrightarrow Sets$ be a functor.

Definition: We say that $X \in Ob(\mathcal{C})$ represents F if there is a functor isomorphism $\mathcal{F}_X \xRightarrow{\sim} F$. We say F is representable if it's represented by an object.

Lemma: If there is an object representing F, then it's unique up to an isomorphism.

Proof: Suppose we have two representing objects, X, X': $\tau: \mathcal{F}_X \xRightarrow{\sim} F$, $\tau': \mathcal{F}_{X'} \xRightarrow{\sim} F$. Consider the isomorphism $\tau^{-1} \circ \tau'$: $\mathcal{F}_{X'} \xRightarrow{\sim} \mathcal{F}_X$. By Yoneda lemma, $\exists!\ g \in Hom_{\mathcal{C}}(X, X')$ s.t $\tau^{-1} \circ \tau' = \eta^g$. By Remark in Sect. 1.2, g is an isomorphism. □

Remarks: 1) The proof shows more than what was claimed: if we fix $\tau: \mathcal{F}_X \xRightarrow{\sim} F$, $\tau': \mathcal{F}_{X'} \xRightarrow{\sim} F$, then $\exists!$ isomorphism $X \xrightarrow[\sim]{g} X'$ s.t. the following is commutative:

$$\begin{tikzcd} & F & \\ \mathcal{F}_{X'} \arrow[ur, Rightarrow, "\tau'"] \arrow[rr, Rightarrow, "\eta_g"] & & \mathcal{F}_X \arrow[ul, Rightarrow, "\tau"'] \end{tikzcd}$$

2) The lemma above (which is an immediate consequence of Yoneda) is useful: often it's easier to construct a functor than an object. But if the functor is representable, it gives rise to a unique (up to isomorphism) object. We'll see this in action when

we discuss tensor products of modules next time.
3) Not every $F: \mathcal{C} \to Sets$ is representable, we'll see an example in the next lecture.

2.2) Example & application

Application: For $F = \mathcal{F}_X$, $G = \mathcal{F}_{X'}$, we can compute the set $\{F \Rightarrow G\}$ as $Hom_{\mathcal{C}}(X', X)$ & the monoid $\{F \Rightarrow F\}$ as $Hom_{\mathcal{C}}(X, X)$ w. <u>opposite</u> composition.

Question: Let F: Groups $\longrightarrow$ Sets be the forgetful functor. Find an object representing F & use it to compute the monoid $\{F \Rightarrow F\}$ of functor endomorphisms of F.

Solution: We are looking for a group X s.t. we have bijections $Hom_{Groups}(X, G) \xrightarrow[\eta_G]{\sim} G$ with (η_G) satisfying the axiom of functor morphism.

$X := \mathbb{Z}$:
$$\eta_G : Hom_{Groups}(\mathbb{Z}, G) \xrightarrow{\sim} F(G) = G$$
$$\varphi \longmapsto \varphi(1)$$

η_G is a bijection: a homomorphism from $\mathbb{Z}$ is uniquely determined by the image of 1.

Check η is a functor isomorphism; iso is b/c η_G is a bijection

Morphism: $\forall\, \tau: G \to H$, for any group homomorphism τ, the diagram

$$\begin{tikzcd} Hom_{Groups}(\mathbb{Z}, G) \arrow[r, "\tau \circ ?"] \arrow[d, "?(1)"] & Hom_{Groups}(\mathbb{Z}, H) \arrow[d, "?(1)"] \\ G \arrow[r, "\tau"] & H \end{tikzcd}$$

is commutative – exercise.

Now we deterimine the monoid structure.

By the above, $\mathrm{Hom}_{\mathrm{Groups}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\sim} F(\mathbb{Z}) = \mathbb{Z}$ as a set, a homomorphism corresponding to $n \in \mathbb{Z}$ is $x \mapsto nx$. The composition of $x \mapsto nx$ & $x \mapsto mx$ is $x \mapsto nmx$ (the order doesn't matter, it's commutative). So the monoid of functor endomorphisms of F is $\mathbb{Z}$ w.r.t. the multiplication.

Exercise/question: What is the map $\eta_G : G \to G$ corresponding to $n \in \mathbb{Z}$?