

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

20. KZ FUNCTOR, II: IMAGE

20.1. Some general facts about D_X -modules. Let X be a smooth algebraic variety and let D_X denote the sheaf of differential operators on X . Consider the subcategory $\text{Loc}(X) \subset D_X\text{-mod}$ consisting of all D_X -modules that are coherent sheaves on X . This is a Serre subcategory in $D_X\text{-mod}$. As we have seen in the previous lecture, all objects L in $\text{Loc}(X)$ are vector bundles. To specify a D_X -module structure on L means to provide a map $\text{Vect}_X \otimes L \rightarrow L$ or equivalently $\nabla : L \rightarrow L \otimes \Omega^1$, where Ω^1 is the sheaf of 1-forms. The compatibility condition between \mathcal{O}_X - and Vect_X -actions on L mean that ∇ is a connection. The condition that the map $\text{Vect}_X \otimes L \rightarrow L$ defines a Lie algebra action means that the connection ∇ is flat. So $\text{Loc}(X)$ is a category of vector bundles equipped with a flat connection (to be called, shortly, *flat bundles*).

There is an exact functor $M \mapsto M^\nabla$ of “taking flat sections” (i.e., sections σ with $\nabla\sigma = 0$) from $\text{Loc}(X)$ to the category $\pi_1(X)\text{-mod}$ of finite dimensional representations of the fundamental group $\pi_1(X)$. This functor is constructed as follows. One can show that any point $x \in X$ has a neighborhood U (in the usual complex-analytic topology), where a connection ∇ trivializes (we have seen a formal version of this statement in the previous lecture), this version follows from the existence and uniqueness of solutions of analytic differential equations. This means that $L_U \cong \underline{L} \otimes \mathcal{O}_D^{an}$, where \underline{L} is a vector space, and under this identification $\nabla(\ell \otimes f) = \ell \otimes df$. The space \underline{L} is uniquely recovered as the space of flat sections of L on U . So for all points $x' \in U$ we can canonically identify the fibers L_x and $L_{x'}$ (via their identification with \underline{L}). It follows that for two arbitrary points x, x' and a curve γ starting at x and ending at x' we can define a linear transformation $\mu_\gamma : L_x \rightarrow L_{x'}$ by identifying the fibers along the curve. Homotopic curves define the same transformation (that’s again some not very difficult fact from DE). In particular, for $x = x'$ we have a representation of $\pi_1(X)$ in L_x (the monodromy representation). We take this representation for L^∇ .

A similar construction, of course, can be done for complex analytic flat vector bundles. In that setting, the functor $M \mapsto M^\nabla$ is an equivalence of categories (take a representation V of $\pi_1(X)$ form a trivial $\pi_1(X)$ -equivariant bundle on the universal cover \tilde{X} of X with fiber V and the trivial connection and then push it forward to X). This construction is not algebraic and so does not produce an equivalence in the algebraic category that we need (it is unclear whether the bundle we get is algebraic, and two non-isomorphic algebraic flat bundles can become isomorphic as analytic flat bundles; for example, for $X = \mathbb{C}$, we have non-isomorphic connections $\nabla_1 = d, \nabla_2 = d + dz$ on \mathcal{O}_X , while the fundamental group is trivial).

This can be fixed if we restrict to a special class of flat bundles, those with *regular singularities*. Namely, consider a smooth proper variety \bar{X} containing X as an open subset (that always exists but is not unique, in general). Let Y_1, \dots, Y_r be all divisors in $\bar{X} \setminus X$. Pick a general point $x_i \in Y_i$ and consider its small analytic neighborhood U . Let $z = 0$ be the equation of Y_i in this neighborhood. Pick a transversal line to Y_i passing through x_i , we can view z as a coordinate on this line. We can restrict the bundle and the connection to the

line getting a flat bundle \tilde{L} on a punctured disk $D^\times = D \setminus \{0\}$. Shrinking the disk, we may assume that that \tilde{L} is a free module over $\mathcal{O}_{D^\times}^{an}$, let L_0 be a subspace of \tilde{L} with $\tilde{L} = L_0 \otimes \mathcal{O}_{D^\times}^{an}$. Then we can write the connection as $\nabla = d + \sum_{i=-j}^{+\infty} z^i dz A_i$ with $A_i \in \text{End}_{\mathbb{C}}(L_0)$ meaning that $\nabla(f \otimes \ell) = df \otimes \ell + f \sum_{i=-j}^{+\infty} z^i dz A_i \ell$. We say that the connection ∇ is regular along Y_i if there is L_0 such that $A_i = 0$ for $i < -1$. We say that ∇ is *regular* if it is regular along any divisor Y_i . According to Deligne, this is independent of the choice of a compactification \bar{X} .

Let $\text{Loc}_{rs}(X)$ denote the category of all flat bundles with regular singularities. Then, according to Deligne, the functor $M \mapsto M^\nabla$ is an equivalence $\text{Loc}_{rs}(X) \xrightarrow{\sim} \pi_1(X)\text{-mod}$. Also we would like to remark that $\text{Loc}_{rs}(X)$ is a Serre subcategory in $\text{Loc}(X)$.

Let us explain how to compute μ_γ for a loop γ around 0 in D . Fixing L_0 , we have $\nabla = d + A_{-1} \frac{dz}{z} + \dots$, where \dots means the regular part of the connection form that does not contribute to the monodromy (again, some fact from complex analysis) and so we may assume that it equals to 0. By the definition, μ_γ it is constructed as follows: we consider the differential equation $df + \frac{f}{z} dz \cdot A_{-1} = 0$, it has a multiple valued (matrix-valued) solution $f = z^{-A} = \exp(-A \ln(z))$. If we parameterize γ as usual: $\gamma(t) = \exp(2\pi\sqrt{-1}t)$, then $f(\exp(2\pi\sqrt{-1}t)) = \exp(-2\pi\sqrt{-1}A_{-1}t)$. Then $\mu_\gamma = f(1) = \exp(-2\pi\sqrt{-1}A_{-1})$.

For example, the monodromy of the connection (=differential equation) $d - \frac{dz}{2z}$ on the trivial bundle of rank 1 on \mathbb{C} is given by $\mu_\gamma = -1$ (and a multi-valued flat section is \sqrt{z}).

20.2. Cherednik case. Let us return back to the situation of interest. We have a complex reflection group W , its reflection representation \mathfrak{h} and a parameter $c : S \rightarrow \mathbb{C}$, where S is the subset of W consisting of all complex reflections. This gives rise to the rational Cherednik algebra H_c and its category \mathcal{O} . In the last lecture we have introduced the localization functor $\pi : \mathcal{O} \rightarrow \text{Loc}^W(\mathfrak{h}^{Reg})$, where the latter stands for the category of W -equivariant flat bundles on \mathfrak{h}^{Reg} .

We claim that $\text{im } \pi \subset \text{Loc}_{rs}^W(\mathfrak{h}^{Reg})$. Since $\text{Loc}_{rs}^W(\mathfrak{h}^{Reg})$ is a Serre subcategory in $\text{Loc}^W(\mathfrak{h}^{Reg})$, it is enough to show that the connection on $\pi(\Delta(E))$ has regular singularities (any simple is the quotient of $\Delta(E)$ and any other object has finite Jordan-Hölder series). We will do this in a simple example when $\dim \mathfrak{h} = 1$ and W is a cyclic group.

The algebra in this case is the quotient of $\mathbb{C}\langle x, y \rangle \# \mathbb{Z}_\ell$, where we write $\mathbb{Z}_\ell = \mathbb{Z}/\ell\mathbb{Z}$, by $[y, x] = 1 - 2 \sum_{i=1}^{\ell-1} c_i \gamma^i$, where γ is the generator of \mathbb{Z}_ℓ acting on \mathfrak{h} by $\eta := \exp(2\pi\sqrt{-1}/\ell)$. Let E_i be the irreducible \mathbb{Z}_ℓ -module, where s acts by η^i . Then $\Delta(E_i) = \mathbb{C}[x]$, where x acts by multiplication by x , s acts on x^j by η^{i-j} . Finally, y acts by the Dunkl operator $\partial_x + \sum_{j=1}^{\ell-1} \frac{2c_j}{(1-\eta^{-j})x} (s^j - 1)$. We trivialize $\pi(\Delta_c(E_i)) = \mathcal{O}_{\mathfrak{h}^{Reg}} \otimes E_i$. The Dunkl operator annihilates E_i and so ∂_x acts on E_i by

$$-\sum_{j=1}^{\ell-1} \frac{2c_j}{(1-\eta^{-j})x} (\eta^{ij} - 1).$$

Therefore the connection is given by

$$\nabla = d - \sum_{j=1}^{\ell-1} \frac{2c_j}{(1-\eta^{-j})} (\eta^{ij} - 1) \frac{dx}{x}.$$

We compactify $\mathfrak{h}^{Reg} = \mathbb{C}^\times$ by embedding it into \mathbb{P}^1 in a standard way. The 1-form in the connection has poles of order 1 at both 0 and ∞ . So it has regular singularities.

Problem 20.1. Write the connection on $\pi(\Delta(E))$ and show that it lies in $\text{Loc}_{rs}(\mathfrak{h}^{\text{Reg}})$ in the general case.

To apply the flat sections functor we want a usual category of local systems rather than the equivariant one. It turns out that we can replace $\text{Loc}^W(\mathfrak{h}^{\text{Reg}})$ with $\text{Loc}(\mathfrak{h}^{\text{Reg}}/W)$. Namely, recall that the W -action on $\mathfrak{h}^{\text{Reg}}$ is free.

Problem 20.2. Let $\rho : \mathfrak{h}^{\text{Reg}} \rightarrow \mathfrak{h}^{\text{Reg}}/W$ be the quotient morphism. Show that the functors $M \mapsto \rho_*(M)^W$ and ρ^* define mutually (quasi)-inverse equivalences between the following pairs of categories:

- (1) $\mathcal{O}_{\mathfrak{h}^{\text{Reg}}}\text{-mod}^W$ and $\mathcal{O}_{\mathfrak{h}^{\text{Reg}}/W}\text{-mod}$.
- (2) $\text{Coh}^W(\mathfrak{h}^{\text{Reg}})$ and $\text{Coh}(\mathfrak{h}^{\text{Reg}}/W)$.
- (3) $D_{\mathfrak{h}^{\text{Reg}}} \# W\text{-mod}$ and $D_{\mathfrak{h}^{\text{Reg}}/W}\text{-mod}$.
- (4) $\text{Loc}^W(\mathfrak{h}^{\text{Reg}})$ and $\text{Loc}(\mathfrak{h}^{\text{Reg}}/W)$.

It is not really difficult to see (and quite easy to believe) that under the identification $\text{Loc}^W(\mathfrak{h}^{\text{Reg}}) \cong \text{Loc}(\mathfrak{h}^{\text{Reg}}/W)$ we have $\text{Loc}_{rs}^W(\mathfrak{h}^{\text{Reg}}) = \text{Loc}_{rs}(\mathfrak{h}^{\text{Reg}}/W)$ (a hint: $\frac{dt^r}{t^r} = r \frac{dt}{t}$).

Let \tilde{KZ} denote the composition of $\pi : \mathcal{O}_c \rightarrow \text{Loc}_{rs}(\mathfrak{h}^{\text{Reg}}/W)$ and the flat section functor $\text{Loc}_{rs}(\mathfrak{h}^{\text{Reg}}/W) \rightarrow \pi_1(\mathfrak{h}^{\text{Reg}}/W)\text{-mod}_{f.d.}$. At this point we know that \tilde{KZ} is an equivalence of $\mathcal{O}/\mathcal{O}^{\text{tor}}$ and $\text{im } \tilde{KZ} \subset \pi_1(\mathfrak{h}^{\text{Reg}}/W)\text{-mod}_{f.d.}$ and this equivalence is fully faithful on projectives.

20.3. Braid groups. Our goal is to describe $\pi_1(\mathfrak{h}^{\text{Reg}}/W)$. The most classical case here is $W = \mathfrak{S}_n$, where the fundamental group is the usual braid group B_n given by the generators T_1, \dots, T_{n-1} subject to the relations: $T_i T_j = T_j T_i$, $|i - j| > 1$, and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. The generators T_i are obtained as follows. Pick a point p close to the hyperplane $x_i = x_{i+1}$. Then take a small semi-circle γ_i connecting p and $s_i p$. Then T_i corresponds to the image of γ_i under the quotient morphism.

This has a classical generalization to the case of an arbitrary Coxeter group W : we pick a fundamental chamber for W , let $\alpha_1, \dots, \alpha_m$ be the equations of its walls. Then $B_W := \pi_1(\mathfrak{h}^{\text{Reg}}/W)$ is given by the generators T_1, \dots, T_m subject to the relations of the form $T_i T_j T_i \dots = T_j T_i T_j \dots$, where the number of factors in each part is the integer m_{ij} such that the angle between α_i, α_j is $\pi - \frac{\pi}{m_{ij}}$ (in particular, if α_i, α_j are orthogonal, then T_i and T_j commute). For example, in type B_n , the group B_W is generated by the T_0, \dots, T_{n-1} such that T_1, \dots, T_{n-1} satisfy the braid relations in type A, T_0 commutes with T_i for $i > 0$, and $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$.

This can be generalized to the case of arbitrary complex reflection groups. We are not going to provide the construction of B_W in this case. Let us only mention that the group B_W is generated by elements T_i corresponding to some hyperplanes $\ker \alpha_s$. In the case of $W = G(\ell, 1, n) = \mathfrak{S}_n \ltimes \mathbb{Z}_\ell^n$ (this is the case of most interest for us) the group B_W is the same as for $\ell = 2$ (and the hyperplanes that give rise to the generators T_0, \dots, T_{n-1} are $x_1 = 0, x_1 = x_2, \dots, x_{n-1} = x_n$). The generator T_i corresponds to an arc of angle $\frac{2\pi}{\ell}$ around $x_1 = 0$. In particular (and this is very easy to see), for $n = 1$, the braid group is just \mathbb{Z} .

20.4. Hecke algebras. It turns out that all $\mathbb{C}B_W$ -modules of the form $\tilde{KZ}(M)$ for $M \in \mathcal{O}_c$ factor through a certain quotient of $\mathbb{C}B_W$ depending on c , the Hecke algebra $\mathcal{H}_c(W)$. To motivate the definition we compute $\tilde{KZ}(\Delta(E_i))$ for the cyclic group. The corresponding representation of B_W is in the fiber of $L := \pi(\Delta(E_i))$ at some point, i.e., in the one-dimensional space E_i .

In our example, a connection on $\mathfrak{h}^{Reg} = \mathbb{C}^\times$ is on the trivial bundle of rank 1 and has $A_{-1} = -k_i$, where

$$(1) \quad k_i := \sum_{j=1}^{\ell-1} \frac{2c_j}{1-\eta^{-j}} (\eta^{ij} - 1).$$

We notice that $k_0 = 0$.

So the monodromy will be given by $\exp(2\pi\sqrt{-1}k_i)$. However, we need not the monodromy for this flat bundle, but rather the monodromy for the induced flat bundle on $\mathbb{C}^\times/\mathbb{Z}_\ell$. The fiber of the induced bundle in $\rho(1) \in \mathbb{C}^\times/\mathbb{Z}_\ell$ can be identified with each of the fibers of L at the points η^j , the induced identification $L_1 \rightarrow L_{\eta^j}$ is given by s^j , i.e., by the multiplication with η^{ij} . A preimage of the parameterized unit circle γ in $\mathbb{C}^\times/\mathbb{Z}_\ell$ is the curve $\exp(2\pi\sqrt{-1}t/\ell)$ that connects 1 to η . So $\mu_\gamma = \eta^{-i} \exp(2\pi\sqrt{-1}k_i/\ell) = q_i$, where

$$(2) \quad q_i = \exp(2\pi\sqrt{-1}(k_i - i)/\ell)$$

So if T is the generator of $B_{\mathbb{Z}_\ell} = \mathbb{Z}$ corresponding to γ , we see that the action of $\mathbb{C}\mathbb{Z}$ on $\pi(\Delta_c(E_i))$ is zero on the element $\prod_{i=0}^{\ell-1}(T - q_i)$.

For us, this is a motivation of a Hecke algebra $\mathcal{H}_c(W)$ in the general case. Namely, for each hyperplane H of the form $\ker \alpha_s$ the point-wise stabilizer W_H of H in W is a cyclic group of order say ℓ_H . Pick a generator, s , in this group that acts on $\mathfrak{h}/\ker \alpha_s$ by η . Then define the number $k_{H,i}$ similarly to (1) where $c_j := c_{s^j}$. Then define $q_{H,i}$ by (2). Denote by \underline{q} the collection of numbers $q_{H,i}$ (the number of a priori different q 's is the same as the number of c 's). Then define $\mathcal{H}_{\underline{q}}(W)$ as the quotient of $\mathbb{C}B_W$ by the relations

$$(3) \quad (T_H - 1) \prod_{j=1}^{\ell_H-1} (T_H - q_{H,j}).$$

We write $\mathcal{H}_c(W)$ for $\mathcal{H}_{\underline{q}}(W)$, where \underline{q} is defined by c as above.

Let us consider the case when $\tilde{W} = \mathfrak{S}_n$. Here we have only one conjugacy class of hyperplanes H with $\ell_H = 2$. We have $k_{H,1} = -2c$ and $q_{H,1} = -\exp(2\pi\sqrt{-1}c)$. If we set $q := -q_{H,1}$ and rescale all generators T_i by -1 , then we will recover the classical Hecke algebra $\mathcal{H}_q(n)$ of type A : the quotient of $\mathbb{C}B_n$ by the relations $(T_i - q)(T_i + 1) = 0$. For other Weyl groups and constant functions c we recover the usual Iwahori-Hecke algebras (with the same additional relations) that appear in Lie theory (say in the study of the representations of finite reductive groups). For non-constant c we recover their straightforward generalization (Hecke algebras with unequal parameters).

We will need the case when $W = G(\ell, 1, n)$. In this case we have two conjugacy classes of H : the class containing $x_1 = x_2$ and the class containing $x_1 = 0$. Let $q_0 = 1, \dots, q_{n-1}$ be the parameters corresponding to the second class. Then the Hecke algebra $\mathcal{H}_c(W)$ is quotient of $\mathbb{C}B_W$ by the relations $(T_i - q)(T_i + 1) = 0$ for $i = 1, \dots, n-1$ and $\prod_{i=0}^{\ell-1}(T_0 - q_i)$. If we drop the last relation, we will get the affine Hecke algebra $\mathcal{H}_q^{aff}(n)$ of type A . The Hecke algebra $\mathcal{H}_c(W) = \mathcal{H}_q^{aff}(n)/\prod_{j=0}^{\ell-1}(T_0 - q_j)$ is called the cyclotomic Hecke algebra of type A (and level ℓ).

Of course, $\mathcal{H}_0(W) = \mathbb{C}W$. Conjecturally, $\dim \mathcal{H}_c(W) = |W|$ for all c and all W . This is known in both examples considered above. We will assume this as a hypothesis.

20.5. Image of KZ.

- Theorem 20.1.** (1) For any $M \in \mathcal{O}_c$ the action of $\mathbb{C}B_W$ on $\tilde{\text{KZ}}(M)$ factors through $\mathcal{H}_c(W)$.
(2) The induced functor $\text{KZ} : \mathcal{O}_c \rightarrow \mathcal{H}_c(W)\text{-mod}$ is essentially surjective (so giving rise to an equivalence $\mathcal{O}_c/\mathcal{O}_c^{\text{tor}} \cong \mathcal{H}_c(W)\text{-mod}$).

In the proof of (1) we will need the following lemma.

Lemma 20.2. If c is generic (=away from countably many hyperplanes), then \mathcal{O}_c is a semisimple category with simples $\Delta_c(E)$.

Proof. We have seen that if $L(E')$ is a composition factor of $\Delta(E)$ different from the irreducible quotient $L(E)$ or if there is a nontrivial extension of $\Delta(E')$ by $\Delta(E)$, then $c_{E'} - c_E \in \mathbb{Z}_{>0}$. So to prove the statements of the lemma it is enough to show that, for c generic, $c_{E'} - c_E$ cannot be a nonzero integer. Recall that

$$c_E := \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in S} \frac{2}{1 - \lambda_s} c_s s|_E.$$

When $c = 0$, we have $c_E = 0$ for all E . Since c_E is an affine function in c , our claim follows. \square

Proof of Theorem 20.1. It is enough to prove (1) when M is projective. As we have seen any indecomposable projective is a direct summand of the module $\Delta_m(E) := H_c \otimes_{S(\mathfrak{h})\#W} E \otimes S(\mathfrak{h})/(\mathfrak{h}^m)$, so it is enough to prove (1) for those modules. The connection on $\pi(\Delta_m(E))$ and hence the operators corresponding to the generators T_H on $\tilde{\text{KZ}}(\Delta_m(E))$ depend continuously on c . So it is enough to prove (1) when c is generic. Here $\Delta_m(E)$ splits into the direct sum of Verma modules and so it is enough to prove the claim for those. This is done by a direct computation similar to what we have done before (the result will be reproved later independently).

Problem 20.3. Show that the action of $\mathbb{C}B_W$ on $\tilde{\text{KZ}}(\Delta_c(M))$ factors through $\mathcal{H}_c(W)$.

Let us proceed to (2). Since the functor KZ is exact it is isomorphic to a functor $\text{Hom}_{\mathcal{O}}(P_{\text{KZ}}, \bullet)$ for some projective P_{KZ} , where the action of $\mathcal{H} := \mathcal{H}_c(W)$ on $\text{KZ}(?)$ comes from a homomorphism $\phi : A := \mathcal{H}^{\text{opp}} \rightarrow B := \text{End}_{\mathcal{O}}(P_{\text{KZ}})^{\text{opp}}$. What we need to show is that ϕ is an isomorphism.

We will do it in two steps. In this lecture we will show that ϕ is surjective. In the next lecture, we will compute the dimension of B and see that it equals $|W|$. According to our hypothesis, $\dim A = |W|$. This will imply that ϕ is an isomorphism.

We claim that the surjectivity of ϕ is equivalent to the claim that im KZ is closed under taking subobjects. Indeed, the latter is equivalent to a similar statement on the pull-back functor ϕ^* , i.e., to the claim that any A -submodule in a B -module is a B -submodule. In particular, $\text{im } \phi$ is an A -submodule in B containing 1 so it has to be B . Therefore ϕ is surjective.

Since $\tilde{\text{KZ}}$ is obtained by composing the localization functor π with a category equivalence, to check that im KZ is closed under taking subobjects, it suffices to show a similar claim for $\text{im } \pi$.

So let us take $M \in \mathcal{O}_c$ and take a subobject $N \subset M[\delta^{-1}]$. We may assume that M is torsion free by modding out the torsion part. So $M \subset M[\delta^{-1}]$. The subobject $M \cap N$ of M satisfies $\pi(M) = N$. Indeed, for any $n \in N$ there is $k \in \mathbb{Z}_{>0}$ such that $\delta^k n \in M$ and hence $\delta^k n \in M \cap N$. We conclude that $(M \cap N)[\delta^{-1}] = N$. \square

REFERENCES

- [GGOR] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, *On the category \mathcal{O} for rational Cherednik algebras*, Invent. Math., **154** (2003), 617-651.