# DAY 1 EXERCISES

## 1. Geometric Invariant Theory

Throughout this section, $G$ is a reductive algebraic group acting on the affine variety $X$, and $\theta : G \to \mathbb{C}^\times$ is a character.

**Exercise 1.1** (Lemma 1.3 in the notes). *Consider the $G$-action on $X \times \mathbb{C}$ given by $g(x, z) = (gx, \theta(g)z)$. Show that an element $x$ is in $X^{\theta-ss}$ if and only if $\overline{G(x, 1)}$ does not intersect $X \times \{0\}$. Equivalently, if and only if there is no one parameter subgroup $\gamma : \mathbb{C}^\times \to G$ such that $\lim_{t\to 0} \gamma(t)x$ exists and $\theta(\gamma(t)) = t^m$ with $m > 0$.*

**Exercise 1.2** (Lemma 1.4 in the notes). *Show that the $G$ orbit of a collection $(x_a, i_k) \in R := R(Q, v, w)$ is closed if and only if $i_k = 0$ for all $k \in Q_0$ and the representation $(x_a)_{a\in Q_1} \in R(Q, v, 0)$ is semisimple.*

**Exercise 1.3** (Lemma 1.5 in the notes). *If $\theta_k > 0$ for all $k \in Q_0$, show that the subset $R^{\theta-ss}$ consists of all representations $(x_a, i_k)$ such that the only $x_a$-stable collection of subspaces in $(\ker i_k)_{k\in Q_0}$ is the zero one.*

## 2. Moment maps and Hamiltonian reduction

**Exercise 2.1.** *Let $G$ act on an affine variety $X_0$. Lift this action to an action on $X := T^*X_0$. Show that $\mu^* : \mathfrak{g} \to \mathbb{C}[T^*X]$, $\xi \mapsto \xi_{X_0}$, is a comoment map for this action. You will need to use the following formula for the Poisson bracket on $\mathbb{C}[T^*X_0] = S_{\mathbb{C}[X_0]} \operatorname{Vect}(X_0)$. For $f, g \in \mathbb{C}[X_0]$ and $\xi, \eta \in \operatorname{Vect}(X_0)$ we have $\{f, g\} = 0$, $\{\xi, f\} = L_\xi f$ and $\{\xi, \eta\} = [\xi, \eta]$. Here, $L_\xi$ is the Lie derivative and $[\cdot, \cdot]$ stands for the commutator of vector fields.*

**Exercise 2.2.** *The kernel of $d_x\mu$ coincides with the $\omega$-orthogonal complement of $T_x(Gx)$, and the image of $d_x\mu$ is the annihilator of $\mathfrak{g}_x := \{\xi \in \mathfrak{g} : \xi_{X,x} = 0\}$. In particular, $\mu$ is a submersion at $x$ provided the stabilizer $G_x$ is finite.*

**Exercise 2.3.** *Suppose that the action of $G$ on $\mu^{-1}(\lambda)$ is free. Show that there is a unique symplectic form $\underline{\omega}$ on $X/\!/\!/_\lambda G$ satisfying $\pi^*\underline{\omega} = \iota^*\omega$, where $\pi : \mu^{-1}(\lambda) \to X/\!/\!/_\lambda G$ is the projection and $\iota : \mu^{-1}(\lambda) \to X$ is the inclusion.*

## 3. Nakajima quiver varieties

**Exercise 3.1.**
(a) *Let $U_1, U_2$ be symplectic vector spaces and $G$ an algebraic group acting on $U_1, U_2$. Assume the actions are Hamiltonian with respective moment maps $\mu_1 : U_1 \to \mathfrak{g}^*$, $\mu_2 : U_2 \to \mathfrak{g}_*$. Show that the action of $G$ on $U_1 \oplus U_2$ is Hamiltonian with moment map $\mu_1 \oplus \mu_2$.*
(b) *Let $U$ be a symplectic vector space. Assume $G_1$, $G_2$ act on $U$ in a Hamiltonian way, with moment maps $\mu_1, \mu_2$, respectively. Moreover, assume that the actions of $G_1$ and $G_2$ commute. Then the $G_1 \times G_2$-action on $U$ is Hamiltonian with moment map $\mu = (\mu_1, \mu_2)$.*

**Exercise 3.2.** *Let $X, Y \in \operatorname{End}(V)$ be such that $\operatorname{rk}[X, Y] = 1$. Show that $X$ and $Y$ are simultaneously triangularizable.*

**Exercise 3.3.** *Let $V$ be a finite dimensional vector space, and $X, Y \in \operatorname{End}(V)$, $i \in V, j \in V^*$. Assume that $[X, Y] - ij = 0$ and that $\mathbb{C}\langle X, Y\rangle i = V$. Show that $j = 0$.*