AFFINE HARISH-CHANDRA CENTER IN POSITIVE CHARACTERISTIC

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1. Introduction

1.1. Overview.

1.1.1. Let F be an algebraic group over a field $\mathbb{K}$ with Lie algebra $\mathfrak{f}$. Let $U(\mathfrak{f})$ denote the universal enveloping algebra of $\mathfrak{f}$. For many basic representation theoretic questions about $\mathfrak{f}$ one needs to understand the center $Z(U(\mathfrak{f}))$ of F. Note that if F is connected, and $\mathbb{K}$ has characteristic 0, the center $Z(U(\mathfrak{f}))$ coincides with the subalgebra $U(\mathfrak{f})^F$ of F-invariants in $U(\mathfrak{f})^F$.

Assume now the characteristic of $\mathbb{K}$ is $p > 0$. Then $U(\mathfrak{f})^F$ is still a subalgebra of $Z(U(\mathfrak{f}))$, usually called the *Harish-Chandra center* (shortly, HC center). One of the reasons why this central subalgerba is important is that it acts by endomorphisms on modules over the Harish-Chandra pair $(\mathfrak{f}, \underline{F})$ for every algebraic subgroup $\underline{F} \subset F$.

The subalgebra $U(\mathfrak{f})^F \subset Z(U(\mathfrak{f}))$ can be shown to be proper if F is not abelian. More precisely, there is a subalgebra in $Z(U(\mathfrak{f}))$ called the *p-center*. Denote it by $Z_{Fr}(U(\mathfrak{f}))$ (with "Fr" from Frobenius). It is a polynomial algebra in dim F-variables

such that $U(\mathfrak{f})$ is a free module over $Z_{Fr}(U(\mathfrak{f}))$ of rank $p^{\dim F}$. The subalgebra $Z_{Fr}(U(\mathfrak{f}))$ is F-stable, but the action of F on it is nontrivial if F is non-abelian, hence the p-center cannot be contain in the HC center. We will elaborate on the p-centers in the main body of the paper.

1.1.2. For a connected reductive group one can determine the HC center. In this case, we will denote the group by G instead of F. Let h denote the maximum of the Coxeter numbers for the simple factors of the Lie algebra $\mathfrak{g}$. To simplify the exposition we assume that $p > h$ (one can relax this restriction, see, e.g., [J, Sec. 6.4] or [R, Sec. 2.2]). Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and W be the Weyl group. Consider the ρ-shifted action $(w, x) \mapsto w \cdot x$ of W on $\mathfrak{h}^*$. Then we have the Harish-Chandra isomorphism: $U(\mathfrak{g})^G \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*]^{(W,\cdot)}$. This explains the name "Harish-Chandra center". Below we will need the following notation: let $r = \dim \mathfrak{h}$ and let $d_1, \ldots, d_r$ be the degrees of free homogeneous generators of $\mathbb{F}[\mathfrak{h}^*]^W$, for example, for $G = \mathrm{SL}_n$, we have $d_i = i + 1$ for $i = 1, \ldots, n - 1$.

In fact, one can also describe the entire center $Z(U(\mathfrak{g}))$ in this case. Namely, by a theorem of Veldkamp, [V], $Z(U(\mathfrak{g}))$ is identified with the tensor product of the HC center and the p-center over their intersection, the subalgebra of G-invariants in the p-center.

1.1.3. In this paper we want to determine the Harish-Chandra center for <u>affine</u> Kac–Moody algebras in characteristic p. Let us recall necessary definitions. Let G be a connected reductive group We assume that $\operatorname{char} \mathbb{K}$ is 0 or is bigger than h. To an element $\kappa \in S^2(\mathfrak{g}^*)^G$ one can assign a central $\hat{\mathfrak{g}}_\kappa$ of $\mathfrak{g} \otimes \mathbb{K}(\!(t)\!)$ by $\mathbb{K}$. We will use the notation $\mathbf{1}$ for the unit element in this central $\mathbb{K}$. Our convention is that the corresponding cocycle corresponds not to κ itself but to $\kappa - \frac{1}{2}\kappa_{\mathfrak{g}}$, where we write $\kappa_{\mathfrak{g}}$ for the Killing form. We refer to κ as the (shifted) level, $\kappa = 0$ is referred to as the *critical level*. The loop group $\mathcal{L}G$ acts on $\hat{\mathfrak{g}}_\kappa$ and hence on its universal enveloping algebra. It is customary to replace $U(\hat{\mathfrak{g}}_\kappa)$ with its quotient $U_\kappa(\hat{\mathfrak{g}}) := U(\hat{\mathfrak{g}}_\kappa)/(\mathbf{1} - 1)$, where 1 is the unit in $U(\hat{\mathfrak{g}}_\kappa)$.

1.1.4. However, the algebra $U_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$ is not expected to be particularly interesting – it is likely just the scalars, cf. [YZ] for a related result. To get something interesting one replace $U_\kappa(\hat{\mathfrak{g}})$ with its completion $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$, a topological algebra whose category of modules with discrete topology consists precisely of smooth representations of $U_\kappa(\hat{\mathfrak{g}})$. For this completion the description of the HC center is known when $\operatorname{char} \mathbb{K} = 0$ (and the HC center coincides with the entire center, [F, Proposition 4.3.8]). For now we state a very rough result, and give a more precise description later. To simplify the statement assume G is simple. Then if $\kappa \neq 0$, then the center consists of scalars. A famous theorem of Feigin and Frenkel, [FF], asserts that for $\kappa = 0$ the center is a suitable completion of the algebra of polynomials in infinitely many variables. More precisely, we have r (where $r = \operatorname{rk} \mathfrak{g}$) families of generators, $S_{i,j}$, where $j \in \mathbb{Z}$. For now, let us just mention that the PBW degree of $S_{i,j}$ equals d_i for all j.

1.1.5. The argument of the proof of the Feigin-Frenkel theorem is based on studying the center of a related and somewhat easier algebra: the affine vertex algebra $V_\kappa(\mathfrak{g})$. As a space (and a $\hat{\mathfrak{g}}_\kappa$-module) it is

$$\operatorname{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{K}\mathbf{1}}^{\hat{\mathfrak{g}}_\kappa} \mathbb{K},$$

where $\mathfrak{g}[[t]]$ acts on $\mathbb{K}$ by zero and $\mathbf{1} \in \mathbb{K}$ acts by 1. This space carries an additional structure: that of a vertex algebra so one can talk about the center, which, in fact, coincides with the $\mathfrak{g}[[t]]$-invariants. Therefore, it makes sense to talk about the Harish-Chandra center: the arc group $\mathcal{J}G$ acts on $V_\kappa(\mathfrak{g})$) and we can consider the invariants for this action. This is a commutative vertex subalgebra of $V_\kappa(\mathfrak{g})$. A version of the Feigin-Frenkel theorem on the level of vertex algebras can be stated as follows: for $\kappa = 0$, the center of $V_\kappa(\mathfrak{g})$ is the polynomial algebra in the elements $S_{i,j}$ for $j \leqslant -d_i$ (no completion is needed). It turns out that one can deduce the description of $Z(\widetilde{U}_\kappa(\hat{\mathfrak{g}}))$ from here, see, e.g., [F, Sec. 4.3].

1.1.6. The descriptions of the centers for $\widetilde{U}_0(\hat{\mathfrak{g}})$ and $V_0(\mathfrak{g})$ can be made much more precise and, in particular, coordinate-free. It was shown by Feigin and Frenkel that both are identified with the algebras of polynomial functions on space of *opers* for the Langlands dual group $\check{G}$ on suitable 1-dimensional schemes X over $\mathbb{K}$. An oper is a connection on a principal bundle over a curve with some additional structure. The center of $V_0(\mathfrak{g})$ (resp., $\widetilde{U}_0(\hat{\mathfrak{g}})$) is the algebra of functions on the $\check{G}$-opers on $\mathcal{D} := \operatorname{Spec}(\mathbb{K}[[t]])$ (resp., $\mathcal{D}^\times := \operatorname{Spec}(\mathbb{K}(\!(t)\!))$).

1.1.7. We now assume that $\operatorname{char}\mathbb{K} = p > 0$. The center of $V_0(\mathfrak{g})$ was studied in [ATV]. In [ATV, Theorem 1.3] the authors proved that a direct analog of the Feigin-Frenkel theorem for the HC center holds (in the case when $\mathfrak{g}$ is exceptional, they had to require that p is very large, in fact, the techniques of the present paper allow to remove that restriction). Moreover, they have proved that a direct analog of the Veldkamp's theorem holds for the entire center of $V_0(\mathfrak{g})$, roughly, the center is generated by the HC center and the p-center (the latter still makes sense in the affine setting, see [AW]).

1.1.8. The focus of this paper is that case when κ is non-degenerate. Here several new phenomena (very different from both the characteristic p story for semisimple Lie algebras and the characteristic 0 story for affine Lie algebras). We continue to assume that p is bigger than the Coxeter number of any simple factor of $\mathfrak{g}$. We show that the HC center of $V_\kappa(\mathfrak{g})$ is the polynomial algebra in the elements $\underline{S}_{i,j}$ with $j \leqslant -d_i$, where the PBW degree of $\underline{S}_{i,j}$ is pd_i. Informally, this center is "p times smaller" than the Feigin-Frenkel center. The situation with completed universal enveloping algebra is more complicated. Note that $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ carries the PBW filtration but it is not exhaustive. Let $\widehat{U}_\kappa(\hat{\mathfrak{g}})$ denote the union of filtered pieces. We show that $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$ is a suitable completion of the polynomial algebra in the variables $\underline{S}_{i,j}$ now for $j \in \mathbb{Z}$. We expect that $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$ is dense (with respect to the inverse limit topology) in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$.

1.1.9. We obtain a more precise description of the center (more on this in Section 1.4), but, at this point, we do not have a coordinate-free description except in one important special case. Suppose that κ is integral meaning that it sends $\mathfrak{g}_{\mathbb{F}_p} \times \mathfrak{g}_{\mathbb{F}_p}$ to $\mathbb{F}_p$. Here something quite surprising happens: the HC center is contained in the p-center (in particular, an analog of the Veldkamp theorem fails). In particular, the HC center of $V_\kappa(\mathfrak{g})$ is identified with the algebra of $(\mathcal{J}G)^{(1)}$-invariant functions on $\mathfrak{g}^{(1)} \otimes \omega_{\mathcal{D}^{(1)}}$. Here the superscript "(1)" indicates the Frobenius twist, and $\omega_?$ stands for the canonical bundle. We still expect an analog of the Veldkamp's theorem to hold for "irrational" κ.

1.1.10. We now briefly discuss motivations for our work. The Feigin-Frenkel theorem is one of the cornerstones of the categorical geometric Langlands program. Similarly, we expect that our description of the HC center will play an important role in the modular quantum categorical geometric Langlands program, which is currently in its infancy. The second, somewhat related motivation, comes from a more classical reporesentation theoretic problem: studying a category $\mathcal{O}$ Roughly, this is the category of modules over the Harish-Chandra pair $(\hat{\mathfrak{g}}, \mathsf{Iw})$, where Iw is the Iwahori subgroup in the Kac-Moody group corresponding to $\hat{\mathfrak{g}}$. One reason to care about this category is that it is supposed to be governed by a version of the double affine Hecke category (somewhat imprecisely, the category $\mathcal{O}$ categorifies the polynomial representation of the double affine Hecke algebra).

1.2. **Harish-Chandra isomorphism, revisited.** To explain our approach (and also a proof of the Feigin-Frenkel theorem) we start with sketching a proof of the Harish-Chandra isomorphism $U(\mathfrak{g})^G \xrightarrow{\mathbb{K}} [\mathfrak{h}^*]^{(W,\cdot)}$. It is quite similar to proofs typically found in basic Lie theory textbooks, but deviates in some aspects (for example, we do not use Verma modules or category $\mathcal{O}$).

In this section we assume that $\operatorname{char}\mathbb{K} = 0$ or is bigger than h, the maximum of the Coxeter numbers of the simple factors of $\mathfrak{g}$. We warn the reader that some of the steps outlined below are nearly obvious but we emphasize them because the proof of our main result essentially follows the pattern we outline and the analogs of these steps in our setting are very far from being obvious.

1.2.1. Choose a parabolic subgroup $P \subset G$ with Levi decomposition $P = L \ltimes N$. Let N^- denote the unipotent radical of the opposite parabolic P^-. This yields the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}$. It is easy to show that $[U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}]^P \xrightarrow{\sim} U(\mathfrak{l})^L$. Then we have the following algebra homomorphism

$$U(\mathfrak{g})^G \hookrightarrow U(\mathfrak{g})^P \to [U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}]^P \xrightarrow{\sim} U(\mathfrak{l})^L$$

that we denote by HC_L. Now choose a Borel subgroup B with Levi decomposition $H \ltimes \tilde{N}$, where $H \subset L$ and $N \subset \tilde{N}$. Then we also have the maps $\mathsf{HC}_H : U(\mathfrak{g})^G \to U(\mathfrak{h})^H$ and $\mathsf{HC}_H^L : U(\mathfrak{l})^L \to U(\mathfrak{h})$. Tracking the construction one arrives at the transitivity property

$$\mathsf{HC}_H = \mathsf{HC}_H^L \circ \mathsf{HC}_L. \tag{1.1}$$

1.2.2. The target for HC_H is very easy to determine: H acts trivially on $U(\mathfrak{h})$, so $U(\mathfrak{h})^H = \mathbb{K}[\mathfrak{h}^*]$.

1.2.3. The homomorphism HC_L preserves the PBW filtrations. The inclusions $\operatorname{gr}[U(\mathfrak{g})^G] \hookrightarrow S(\mathfrak{g})^G$ and $\operatorname{gr}[U(\mathfrak{l})^L] \hookrightarrow S(\mathfrak{l})^L$ intertwine the associated graded map $\operatorname{gr} HC_L$ with the Chevalley restriction map $\mathbb{K}[\mathfrak{g}^*]^G \to \mathbb{K}[\mathfrak{l}^*]^L$. The latter is injective, in particular, HC_L is injective.

Thanks to the injectivity, $U(\mathfrak{g})^G$ is identified with its image under HC_H. Now let $P_1, \ldots, P_r$ be the minimal parabolics containing a fixed Borel B, and let $L_1, \ldots, L_r$ be their Levi subgroups containing a fixed maximal torus H of B. Thanks to (1.1), we have

$$\operatorname{im}\mathsf{HC}_H \subset \bigcap_{i=1}^{r} \operatorname{im}\mathsf{HC}_H^{L_i}. \tag{1.2}$$

1.2.4. The determination of $\operatorname{im}\mathsf{HC}_H^{L_i}$ reduces to the case of $\mathfrak{g} = \mathfrak{sl}_2$. This case is easy and one sees that $\operatorname{im}\mathsf{HC}_H^{L_i} = \mathbb{K}[\mathfrak{h}^*]^{(s_i,\cdot)}$, where s_i is the simple reflection corresponding to L_i.

1.2.5. Clearly, $\bigcap_{i=1}^r \mathbb{K}[\mathfrak{h}^*]^{(s_i,\cdot)} = \mathbb{K}[\mathfrak{h}^*]^{(W,\cdot)}$ – the simple reflections s_i generate the group W. So, (1.2) implies $\operatorname{im}\mathsf{HC}_H \subset \mathbb{K}[\mathfrak{h}^*]^{(W,\cdot)}$.

1.2.6. It just remains to observe that $U(\mathfrak{g})^G$ is big enough: $\operatorname{gr} U(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{g})^G$ thanks to the symmetrization map (in positive characteristic we symmetrize free homogeneous generators of $S(\mathfrak{g})^G$, they sit in degrees less than p thanks to $p > h$).

1.3. **Sketch of proof for $\kappa = 0$.** In this section we will explain steps of a proof of the Feigin-Frenkel theorem, §1.1.4, and its positive characteristic version. Our proof works both in zero and positive characteristic (bigger than h) and deviates from the Feigin-Frenkel proof in several aspects, while still using many of the ideas of that proof. We follow the steps outlined in Section 1.2 and work with vertex algebras instead of the completed universal enveloping algebras. In the end of the section we explain how to pass from the former to the latter.

1.3.1. We have vertex algebra homomorphisms $\mathsf{HC}_{0,L} : V_0(\mathfrak{g})^{\mathcal{J}G} \to V_0(\mathfrak{l})^{\mathcal{J}L}$, where, recall, $\mathcal{J}G$ denotes the arc group of G. The construction is, however, significantly more complicated than in the semisimple case and is based on the so called *free field realization* map, see [F, Section 5] for the characteristic 0 case and [AW] for the characteristic p case. With some work one can show that the direct analog of (1.1) holds.

1.3.2. The target of $\mathsf{HC}_{0,H}$ is still easy to determine as the action of $\mathcal{J}H$ on $V_0(\mathfrak{h})$ is trivial. This commutative algebra is identified with the $\mathbb{K}[\mathcal{J}\mathfrak{h}^*]$, the algebra of functions on the arc space $\mathcal{J}\mathfrak{h}^*$ of $\mathfrak{h}^*$.

1.3.3. The homomorphism HC_H preserves the PBW filtrations. Similarly to §1.2.3, $\mathsf{HC}_{0,H}$ is injective. Further, a direct analog of (1.2) holds with the same argument.

1.3.4. Again, the determination of $\operatorname{im}\mathsf{HC}_{0,H}^{L_i}$ boils down to the case $\mathfrak{g} = \mathfrak{sl}_2$. Here $V_0(\mathfrak{g})^{\mathcal{J}G}$ is the polynomial algebra in the Sugawara modes $S_j, j \leqslant -2$, and it is to compute their images under $\mathsf{HC}_{0,H}^{L_i}$.

1.3.5. Now our task is to determine $\bigcap_{i=1}^r \operatorname{im}\mathsf{HC}_{0,H}^{L_i}$ getting an affine analog of §1.2.5. The desired answer is that this is the image of the algebra $\mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D})]$, i.e., the algebra of regular functions on the scheme of $\check{G}$-opers on the disc $\mathcal{D}$, under the Miura map. Proving this claim takes a significant portion of [F, Secs. 7,8.2] and uses several ingredients that are either quite complicated and do not carry easily to positive characteristic (screening operators) or just fail in characteristic p (the claim that the Miura map is the pullback under the projection from a principal $\check{N}$-bundle over $\mathsf{Op}_{\check{G}}(\mathcal{D})$). Fortunately, we found a much more elementary proof identifying the intersection with the image of the Miura map.

1.3.6. The remaining step in our plan is to show that $V_0(\mathfrak{g})^{\mathcal{J}G}$ is big enough so that the embedding $V_0(\mathfrak{g})^{\mathcal{J}G} \hookrightarrow \mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D})]$ is an isomorphism. The argument in the case of semisimple Lie algebras involved the symmetrization map, it completely fails in the affine setting as the symmetrization map is not continuous in the topology we use to complete. Instead, the proof that the center is big enough in the characteristic 0 case requires an identification of the Verma $\hat{\mathfrak{g}}_0$-module with zero highest weight and a suitable version of the Wakimoto module, see [F, Secs. 6.3, 8.1]. We do not expect that this identification continues to hold in characteristic p. Instead, to show that the center is sufficiently large, we can use a more elementary fact: the Verma $\hat{\mathfrak{g}})_0$-module with a very generic highest weight is isomorphic to the Wakimoto module with the same highest weight.

1.3.7. The isomorphism $V_0(\mathfrak{g})^{\mathcal{J}G} \hookrightarrow \mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D})]$ then yields a homomorphism

$$\mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D}^\times)] \to \widetilde{U}_0(\hat{\mathfrak{g}})^{\mathcal{L}G}.$$

That this is an isomorphism requires an argument involving the associated graded, [F, Sec. 4.3]. A crucial ingredient is the fact that $\operatorname{gr}[V_0(\mathfrak{g})^{\mathcal{J}G}] = \mathbb{K}[\mathcal{J}\mathfrak{g}^*]^{\mathcal{J}G}$.

1.4. **Approach and results of this paper.** Now suppose $\operatorname{char}\mathbb{K} = p > h$ and κ is nondegenerate. Our goal here is to describe $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$.

1.4.1. Completely analogously to §1.3.1 we get vertex algebra homomorphisms $\mathsf{HC}_{\kappa,L} : V_0(\mathfrak{g})^{\mathcal{J}G} \to V_0(\mathfrak{l})^{\mathcal{J}L}$. They satisfy the analog of the transitivity property, (1.1).

1.4.2. The first new challenge comes when determining $V_\kappa(\mathfrak{h})^{\mathcal{J}H}$: the action is no longer trivial. We prove that there is an embedding $\Psi : \mathbb{K}[\mathcal{J}\mathfrak{h}^*] \hookrightarrow \mathbb{K}[\mathcal{J}\mathfrak{h}^*]$ such that $V_\kappa(\mathfrak{h})^{\mathcal{J}H}$ is identified with its image. The linear generators of $\mathbb{K}[\mathcal{J}\mathfrak{h}^*]$ are sent to elements of PBW degree p.

1.4.3. Claims in §1.3.3 continue to hold with the same argument. From here we can already can handle the case when κ is integral. From now on we assume that $\mathfrak{g}$ is simple and κ is not integral.

1.4.4. Now we need to determine $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$ and its image under $\mathsf{HC}_{\kappa,H}$. This is significantly more challenging than in the case $\kappa = 0$ and requires several new ingredients. What we find is that $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$ is the polynomial algebra in certain elements of PBW degree $2p$. However, we have the following remarkable equality

$$\operatorname{im}\mathsf{HC}_{\kappa,H} = \Psi(\operatorname{im}\mathsf{HC}_{0,H}), \tag{1.3}$$

an equality of subalgebras in $V_\kappa(\mathfrak{h})^{\mathcal{J}H} \xrightarrow{\sim} \Psi\mathbb{K}[\mathcal{J}\mathfrak{h}^*]$.

1.4.5. Using (1.3) and §1.3.5 we identify $\bigcap_{i=1}^r \operatorname{im}\mathsf{HC}_{\kappa,H}^{L_i}$ with $\Psi(\mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D})])$ where we identify the algebra of the functions on the scheme of opers with its image under the Miura map.

1.4.6. Similarly to §1.3.6 we get an identification $V_\kappa(\mathfrak{g})^{\mathcal{J}G} \xrightarrow{\sim} \Psi(\mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D})])$.

1.4.7. Similarly to §1.3.7 we get a homomorphism

$$\Psi(\mathbb{K}[\mathsf{Op}_{\check{G}}(\mathcal{D}^\times)]) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \tag{1.4}$$

However, since the inclusion $\mathrm{gr}[V_0(\mathfrak{g})^{\mathcal{J}G}] \hookrightarrow \mathbb{K}[\mathcal{J}\mathfrak{g}^*]^{\mathcal{J}G}$ is not surjective in this case, the argument in [F, Sec. 4.3] no longer goes through and we do not know how to prove that (1.4) is an isomorphism, although we expect that it is. We do however prove that the restriction of (1.4) to the subalgebras of elements of finite degree is an isomorphism, which is done by studying the analog of $\mathsf{HC}_{\kappa,H}$ for the universal enveloping algebras.

1.5. **Organization of the paper.**

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2. Group ind-schemes and Lie algebras

2.1. **Introduction.**

2.1.1. We would like to discuss Lie algebras of group ind-schemes, possibly of infinite type, and their restricted Lie algebra structure in the case when the group is defined over a field of positive characteristic, as well as some basic properties thereof.

The particular Lie algebras of interest to us are the affine Kac–Moody algebra and the Virasoro algebra. These are Lie algebras of central extensions of the loop group and the group of automorphisms of a formal punctured disc, respectively. Both central extensions are pulled back from the Tate extension $GL^\flat(V)$ of the automorphisms $GL(V)$ of a suitable representation on a Tate vector space V. It is therefore natural to include in our discussion the case of $GL^\flat(V)$ itself. As $GL^\flat(V)$ is not a group ind-scheme of *countable type*, i.e., presentable as a countable filtered colimit of affine schemes along closed embeddings, this inclusion requires a little bit of care.

2.1.2. The basic idea is then the following. We will see that $GL^\flat(V)$ is in fact an ind-affine ind-scheme, albeit with a presentation of uncountable type. Therefore, in describing its algebra of functions, and coherent sheaves on it, we will need to work with those genuinely as pro-vector spaces; for countable ind-schemes, we recall that one can alternatively work with the perhaps more familiar language of topological vector spaces, complete and separated, admitting a countable basis for their topology. With this in mind, we first recall some basic assertions regarding pro-categories.

2.2. **Recollections on pro-completions.**

2.2.1. Let us review some relevant facts about the pro-completion of a category $\mathcal{C}$; as the pro-completion of $\mathcal{C}$ is the opposite of the ind-completion of the opposite category $\mathcal{C}^{op}$, one possible reference for the following material is Chapters 6 and 8 of [KS].

2.2.2. Let $\mathcal{C}$ be a category. Let us denote the pro-completion of $\mathcal{C}$ by $\mathrm{Pro}(\mathcal{C})$. Explicitly, objects of $\mathrm{Pro}(\mathcal{C})$ are formal cofiltered limits of objects of $\mathcal{C}$. We will denote such an object by $\varprojlim_\alpha M_\alpha$, where α runs over the essentially small cofiltered diagram category. Morphisms between pro-objects are computed as

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\varprojlim_\alpha M_\alpha, \varprojlim_\beta N_\beta) := \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(M_\alpha, N_\beta).$$

In particular, we have a tautological fully faithful embedding $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$, consisting of one-point diagrams.

It will be convenient in what follows to recall the following standard fact, cf. Corollary 6.1.14 of [KS]. Given objects $M^\wedge$ and $N^\wedge$ in $\mathrm{Pro}(\mathcal{C})$, and a morphism $\phi : M^\wedge \to N^\wedge$, one can always choose a cofiltered category I and presentations $M^\wedge \simeq \varprojlim M_i, N^\wedge \simeq \varprojlim N_i$, such that the morphism ϕ is induced by a system of compatible maps $\phi_i : M_i \to N_i$, for $i \in I$.

2.2.3. We next recall that if $\mathcal{C}$ is abelian, then $\mathrm{Pro}(\mathcal{C})$ is again abelian, and the tautological functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is exact, cf. Theorem 8.6.5 of [KS].

Explicitly, given a map $\phi : M^\wedge \to N^\wedge$ as above, if we compute kernels and cokernels levelwise, i.e.,

$$0 \to \ker \phi_i \to M_i \to N_i \to \operatorname{coker} \phi_i \to 0;$$

then the inverse limits $\varprojlim \ker \phi_i$ and $\varprojlim \operatorname{coker} \phi_i$ satisfy the required universal properties of the kernel and cokernel of ϕ, respectively, and the first isomorphism theorem in $\mathcal{C}$ then readily implies it in $\mathrm{Pro}(\mathcal{C})$. In particular, any exact sequence

$$0 \to A^\wedge \to B^\wedge \to C^\wedge \to 0$$

may be presented as a formal inverse limit of exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \tag{2.1}$$

for some cofiltered index category I, cf. Proposition 8.6.6(a) of [KS].

2.2.4. We next recall that $\mathcal{C}$ is (symmetric) monoidal, then $\mathrm{Pro}(\mathcal{C})$ inherits a (symmetric) monoidal structure. Explicitly, at the level of binary products, if we denote the product on $\mathcal{C}$ by $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(M, N) \mapsto M \otimes N$, then for $M^\wedge$ and $N^\wedge$ as above, we have that

$$M^\wedge \otimes N^\wedge \simeq \varprojlim M_\alpha \otimes N_\beta.$$

We recall two basic consequences. First, it follows that if $\mathcal{C}$ has monoidal unit $\mathbf{1}$, then the image of $\mathbf{1}$ along $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is again the monoidal unit.

Second, if $\mathcal{C}$ is abelian and monoidal, and the functor $- \otimes -$ is (left) right exact in the first (or second) variable, then it follows that the same exactness holds for $\mathrm{Pro}(\mathcal{C})$, e.g. by using (2.1).

2.2.5. The relevant consequences of the above discussion may then be summarized as follows.

Corollary 2.1. *Given a commutative ring* R*, the category* $\mathrm{Pro}(R\text{-Mod})$ *is naturally a symmetric monoidal* R*-linear abelian category, for which the tensor product is right exact in each factor. If* R *is a field, then the tensor product on* $\mathrm{Pro}(R\text{-Mod})$ *is moreover exact in each factor.*

2.3. **Restricted Lie algebras from symmetric monoidal categories.**

2.3.1. For the reader's convenience, we recall the definition of a restricted Lie algebra. Let R denote a commutative $\mathbb{F}_p$-algebra.

Definition 2.2. A *restricted Lie algebra* $\mathfrak{f}$ over R is a Lie algebra $\mathfrak{f}$

$$\mathfrak{f} \wedge \mathfrak{f} \to \mathfrak{f}, \qquad X \wedge Y \mapsto [X, Y] =: \mathrm{Ad}_X(Y), \qquad X, Y \in \mathfrak{f},$$

equipped with a further map of sets, the *restricted power map*,

$$(-)^{[p]} : \mathfrak{f} \to \mathfrak{f}, \qquad X \mapsto X^{[p]},$$

satisfying the following conditions.

(1) For any $X \in \mathfrak{f}$, one has the equality of R-linear endomorphisms of $\mathfrak{f}$

$$\mathrm{Ad}_X^p = \mathrm{Ad}_{X^{[p]}}.$$

(2) For any $r \in \mathsf{R}$, and $X \in \mathfrak{f}$, one has $(r \cdot X)^{[p]} = r^p \cdot X^{[p]}$.

(3) For any $X, Y \in \mathfrak{f}$, we have the equality

$$(X + Y)^{[p]} = X^{[p]} + \sum_{i=1}^{p-1} \sigma_i(X, Y) + Y^{[p]},$$

where for fixed i, $1 \leqslant i \leqslant p-1$, $i \cdot \sigma_i$ is the coefficient of t^{i-1} in

$$(\mathrm{Ad}_{t \cdot X + Y})^{p-1}(X) \in \mathfrak{f} \underset{\mathsf{R}}{\otimes} \mathsf{R}[t].$$

2.3.2. In what follows, we would like to discuss the restricted Lie algebra associated to a group ind-scheme, possibly of infinite type, as well as some basic properties thereof.

To do so, it suffices to more generally associate a Lie algebra to any Hopf algebra in a symmetric monoidal category $\mathcal{C}$, with the expected properties, which moreover inherits a restricted structure if $\mathcal{C}$ is $\mathbb{F}_p$-linear.

2.3.3. Let $\mathcal{C}$ be an additive monoidal category. In particular, the functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on Hom spaces, i.e., induces maps

$$\mathrm{Hom}(M_1, N_1) \underset{\mathbb{Z}}{\otimes} \mathrm{Hom}(M_2, N_2) \to \mathrm{Hom}(M_1 \otimes N_1, M_2 \otimes N_2).$$

2.3.4. Let A be an algebra in $\mathcal{C}$, and denote its multiplication map by

$$\mu : A \otimes A \to A.$$

Let us denote by A-Bimod the category of A-bimodules in $\mathcal{C}$. For an A-bimodule M, with corresponding action maps

$$\alpha_\ell : A \otimes M \to M, \qquad \alpha_r : M \otimes A \to M,$$

we recall that a derivation $d : A \to M$ is a map in $\mathcal{C}$ satisfying the Leibnitz rule

$$d \circ \mu = \alpha_\ell \circ (\mathrm{id}_A \otimes d) + \alpha_r \circ (d \otimes \mathrm{id}_A) \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, M). \tag{2.2}$$

Remark 2.3. Given a bimodule M, one can form the associated square zero extension $\epsilon : A \oplus M \to A$. Then it is straightforward to check that a derivation $A \to M$ is the same data as a map of algebras $\sigma : A \to A \oplus M$ splitting ϵ.

2.3.5. Given a bimodule M and derivations $d_1, d_2 : A \to M$, it is straightforward to see from (2.2) that their sum $d_1 + d_2$ is a derivation, as is $-d_1$. Let us denote by

$$\mathrm{Der}(A, M) \subset \mathrm{Hom}_{\mathcal{C}}(A, M)$$

the subgroup of derivations. If $\mathcal{C}$ is linear over a commutative ring R, then $\mathrm{Der}(A, M)$ is an R-submodule. We note in addition that, given a map $\phi : M_1 \to M_2$ of A-modules, the map $\mathrm{Hom}_{\mathcal{C}}(A, M_1) \to \mathrm{Hom}_{\mathcal{C}}(A, M_2)$ restricts to a map

$$\mathrm{Der}(A, M_1) \to \mathrm{Der}(A, M_2).$$

Remark 2.4. Suppose that A-Bimod admits cokernels. Then the functor of derivations $\mathrm{Der}(A, -)$ is corepresentable, i.e., there exists an A-bimodule Ω_A and a natural isomorphism of functors

$$\mathrm{Der}(A, -) \simeq \mathrm{Hom}_{A\text{-bimod}}(\Omega_A, -).$$

Namely, Ω_A can be presented as the cokernel of the alternating cyclic sum

$$\mu \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mu \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mu : A \otimes A \otimes A \otimes A \to A \otimes A \otimes A,$$

where we view $A^{\otimes 4}$ and $A^{\otimes 3}$ as A-bimodules via left multiplication on their leftmost tensor factors, and right multiplication on their rightmost tensor factors.

2.3.6. Recall that, given an associative algebra E, we may view it as a Lie algebra via the commutator $[e_1, e_2] = e_1 e_2 - e_2 e_1$. Moreover, if E is $\mathbb{F}_p$-linear, then we may further view it as a restricted Lie algebra, via setting $e^{[p]} := e^p$, where the latter denotes the multiplication of e with itself p times.

In particular, if $\mathcal{C}$ is $\mathbb{F}_p$-linear, then for any object M of $\mathcal{C}$, its endomorphisms $\mathrm{Hom}_{\mathcal{C}}(M, M)$ naturally form a restricted Lie algebra.

Lemma 2.5. *If $\mathcal{C}$ is $\mathbb{F}_p$-linear, and A is an algebra in $\mathcal{C}$, then its space of A-valued derivations*

$$\mathrm{Der}(A, A) \subset \mathrm{Hom}_{\mathcal{C}}(A, A)$$

is a restricted Lie subalgebra.

Proof. We must show that, given two derivations d, D, their commutator $dD - Dd$ is again a derivation, and that d^p is again a derivation. To see these, note that a derivation $\delta : A \to A$ is a map satisfying the identity

$$\delta \circ \mu = \mu \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta).$$

Using this, the verification of the asserted identities is straightforward. □

2.3.7. Suppose now that $\mathcal{C}$ is symmetric monoidal, so that we may speak of a commutative Hopf algebra $\mathcal{O}_{\mathsf{F}}$ in $\mathcal{C}$.[1] Write $\mathrm{Rep}(\mathsf{F})$ for its category of left comodules. Recall that $\mathrm{Rep}(\mathsf{F})$ is naturally symmetric monoidal, compatibly with the forgetful functor $\mathrm{Oblv} : \mathrm{Rep}(\mathsf{F}) \to \mathcal{C}$.

In particular, given an algebra object A in $\mathrm{Rep}(\mathsf{F})$, we may form its restricted Lie algebra of derivations. This comes equipped with a map of restricted Lie algebras

$$\mathrm{Oblv} : \mathrm{Der}_{\mathrm{Rep}(\mathsf{F})}(A, A) \hookrightarrow \mathrm{Der}_{\mathcal{C}}(A, A),$$

and explicitly consists of the subspace of derivations $d : A \to A$ in $\mathcal{C}$ which are maps of $\mathcal{O}_{\mathsf{F}}$-comodules.

Definition 2.6. For $\mathcal{O}_{\mathsf{F}}$ viewed as a comodule over itself via the right translation action, we set

$$\mathfrak{f} := \mathrm{Der}_{\mathrm{Rep}(\mathsf{F})}(\mathcal{O}_{\mathsf{F}}, \mathcal{O}_{\mathsf{F}}),$$

equipped with its natural structure of restricted Lie algebra. We call $\mathfrak{f}$ the *Lie algebra* of $\mathcal{O}_{\mathsf{F}}$.

2.3.8. The relationship between $\mathfrak{f}$ and the tangent space at the identity is as follows. Write 1 for the monoidal unit of $\mathcal{C}$ and $e : \mathcal{O}_{\mathsf{F}} \to 1$ for the augmentation map of algebras. Consider the associated induction and restriction functors

$$e^* : \mathcal{O}_{\mathsf{F}}\text{-Bimod} \rightleftarrows 1\text{-Bimod} \simeq \mathcal{C} : e_*,$$

so that in particular we may form the $\mathcal{O}_{\mathsf{F}}$-bimodule e_*1.

Lemma 2.7. *There is a canonical isomorphism*

$$\mathfrak{f} \simeq \mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1).$$

Proof. We will construct maps in either direction and check they are mutually inverse. Given an element X of $\mathfrak{f}$, we may consider the composition

$$\mathcal{O}_{\mathsf{F}} \xrightarrow{X} \mathcal{O}_{\mathsf{F}} \xrightarrow{e} 1;$$

this assignment yields a map $\mathfrak{f} \to \mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1)$. Conversely, given an element Z in $\mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1)$, one can produce an element of $\mathfrak{f}$ as follows. Write $\Delta : \mathcal{O}_{\mathsf{F}} \to \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}}$ for the comultiplication map, and attach to Z the composition

$$\mathcal{O}_{\mathsf{F}} \xrightarrow{\Delta} \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \xrightarrow{Z \otimes \mathrm{id}} e_*1 \otimes \mathcal{O}_{\mathsf{F}} \simeq \mathcal{O}_{\mathsf{F}}.$$

As Δ is a map of algebras, and $Z \otimes \mathrm{id}$ is a derivation of $\mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}}$-bimodules, the composition is an $\mathcal{O}_{\mathsf{F}}$-derivation, and each map is a morphism of $\mathcal{O}_{\mathsf{F}}$-comodules, this yields the desired map $\mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1) \to \mathfrak{f}$.

[1] In fact, likely the arguments in this section apply to general Hopf algebras in monoidal categories, but we did not check this in detail.

It is straightforward to see that the composition

$$\mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1) \to \mathfrak{f} \to \mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1)$$

is the identity, namely

$$e \circ (Z \otimes \mathrm{id}) \circ \Delta = Z \circ (\mathrm{id} \otimes e) \circ \Delta = Z \circ \mathrm{id} = Z.$$

To see that the composition $\mathfrak{f} \to \mathrm{Der}(\mathcal{O}_{\mathsf{F}}, e_*1) \to \mathfrak{f}$ is the identity, for an element X of $\mathfrak{f}$ one uses the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathsf{F}} \arrow[r, "X"] \arrow[d, "\Delta"'] & \mathcal{O}_{\mathsf{F}} \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"] & \mathcal{O}_{\mathsf{F}} \arrow[d, "\mathrm{id}"] \\
\mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \arrow[r, "X\otimes\mathrm{id}"] & \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \arrow[r, "e\otimes\mathrm{id}"] & \mathcal{O}_{\mathsf{F}}.
\end{tikzcd}$$

□

2.3.9. Let us now discuss the action of the Lie algebra $\mathfrak{f}$ on algebras in $\mathrm{Rep}(\mathsf{F})$.

Lemma 2.8. *For any algebra A in $\mathrm{Rep}(\mathsf{F})$, there is a canonical map of restricted Lie algebras*

$$\mathfrak{f} \to \mathrm{Der}_{\mathcal{C}}(A, A), \qquad X \mapsto \mathcal{L}_X.$$

Moreover, this assignment is functorial in A, i.e., given a map $\phi : A \to B$ of algebras in $\mathrm{Rep}(\mathsf{F})$, and any $X \in \mathfrak{f}$, the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\mathcal{L}_X"] \arrow[d, "\phi"'] & A \arrow[d, "\phi"] \\
B \arrow[r, "\mathcal{L}_X"] & B.
\end{tikzcd}$$

Proof. Consider the coaction map

$$\mathrm{coact}_A : A \to \mathcal{O}_{\mathsf{F}} \otimes A.$$

Note that this is a map of $\mathcal{O}_{\mathsf{F}}$-comodules, if we equip A with its previous action, and $\mathcal{O}_F \otimes A$ with the tensor product of the left translation action on $\mathcal{O}_F$ and trivial action on A. We will henceforth refer to these as the 'left' coactions on A and $\mathcal{O}_{\mathsf{F}} \otimes A$, respectively.

In addition, consider the commuting trivial 'right' coaction of $\mathcal{O}_{\mathsf{F}}$ on A, and the commuting 'right' coaction of $\mathcal{O}_{\mathsf{F}}$ on $A \otimes \mathcal{O}_{\mathsf{F}}$ given by the tensor product of coact_A and the right translation action on $\mathcal{O}_{\mathsf{F}}$. Then, with respect to these additional 'right' actions, coact_A is in fact a map of $\mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}}$-comodules.

Moreover, it is straightforward to see that coact_A exhibits A as the invariants of $\mathcal{O}_{\mathsf{F}} \otimes A$ with respect to the 'right' action.[2] Indeed, up to a standard change of variables, namely the automorphism of $\mathcal{O}_{\mathsf{F}} \otimes A$ given by the composition

$$\mathcal{O}_{\mathsf{F}} \otimes A \xrightarrow{\mathrm{id}\otimes\mathrm{coact}_A} \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \otimes A \xrightarrow{\mu\otimes\mathrm{id}} \mathcal{O}_{\mathsf{F}} \otimes A,$$

[2] I.e., as the equalizer of the 'right' coaction and the trivial coaction $\mathcal{O}_{\mathsf{F}} \otimes A \rightrightarrows \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \otimes A$, which in particular is therefore representable.

this follows from the assertion that, for any object M of $\mathcal{C}$, if one considers the tensor product of the left translation and trivial coaction of $\mathcal{O}_{\mathsf{F}}$ on $\mathcal{O}_{\mathsf{F}} \otimes M$, its invariants are given by the split monomorphism

$$M \simeq 1 \otimes M \xrightarrow{1 \otimes \mathrm{id}} \mathcal{O}_{\mathsf{F}} \otimes M.$$

With these preparations, for $X \in \mathfrak{f}$, i.e., a right invariant derivation $X : \mathcal{O}_{\mathsf{F}} \to \mathcal{O}_{\mathsf{F}}$, consider the derivation

$$X \otimes \mathrm{id} : \mathcal{O}_{\mathsf{F}} \otimes A \to \mathcal{O}_{\mathsf{F}} \otimes A.$$

By our assumption of right invariance, it follows $X \otimes \mathrm{id}$ commutes with the 'right' coaction of $\mathcal{O}_{\mathsf{F}}$ on $\mathcal{O}_{\mathsf{F}} \otimes A$, and in particular restricts to a derivation of its invariants, which is the desired map $\mathcal{L}_X : A \to A$. The claimed properties of this assignment follow from the construction. □

Let us record a formula for the Lie derivative $\mathcal{L}_X$.

Lemma 2.9. *For a commutative algebra A in* $\mathrm{Rep}(\mathsf{F})$*, and an element $X \in \mathfrak{f}$, the derivation $\mathcal{L}_X$ agrees with the composition*

$$A \xrightarrow{\mathrm{coact}} \mathcal{O}_{\mathsf{F}} \otimes A \xrightarrow{(e \circ X) \otimes \mathrm{id}} 1 \otimes A \simeq A.$$

Proof. This follows from the construction of $\mathcal{L}_X$ and the recollection that the composition

$$A \xrightarrow{\mathrm{coact}} \mathcal{O}_{\mathsf{F}} \otimes A \xrightarrow{e \otimes \mathrm{id}} 1 \otimes A \simeq A$$

is the identity. □

Let us also recover from the two preceding lemmasthe functoriality of Lie algebras with respect to maps of Hopf algebras.

Lemma 2.10. *The assignment $\mathcal{O}_{\mathsf{G}} \mapsto \mathfrak{g}$ naturally enhances to a functor from the category of commutative Hopf algebras in $\mathcal{C}$ to the category of restricted Lie algebras.*

Proof. We must show that a map $\phi : \mathcal{O}_{\mathsf{G}} \to \mathcal{O}_{\mathsf{F}}$ of commutative Hopf algebras induces a map of restricted Lie algebras

$$d\phi : \mathfrak{f} \to \mathfrak{g},$$

compatible with composition of maps, i.e., $d(\phi \circ \psi) = d\psi \circ d\phi$, $d(\mathrm{id}) = \mathrm{id}$.

Let us regard $\mathcal{O}_{\mathsf{G}}$ as an $\mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{G}}$-comodule, via the restriction of the left translation action to F and the right translation action. Lemma 2.8, applied to $\mathcal{C} = \mathrm{Rep}(\mathsf{G})$, then yields the desired map

$$\mathfrak{f} \to \mathrm{Der}_{\mathrm{Rep}(\mathsf{G})}(\mathcal{O}_{\mathsf{G}}, \mathcal{O}_{\mathsf{G}}) = \mathfrak{g}.$$

The compatibility with composition and the differential of the identity map follow from the construction. □

2.3.10. As a particular case of the above lemma, we may deduce the following. For a Hopf algebra $\mathcal{O}_{\mathsf{F}}$ in $\mathcal{C}$, let us write $\mathrm{Aut}(\mathcal{O}_{\mathsf{F}})$ for its group of Hopf algebra automorphisms.

Corollary 2.11. *There is a natural action of* $\mathrm{Aut}(\mathcal{O}_{\mathsf{F}})$ *on* $\mathfrak{f}$ *by restricted Lie algebra automorphisms.*

Remark 2.12. Here is a further special case, namely the adjoint action by *inner automorphisms*. First, note that the set of maps of algebras $\mathcal{F} := \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(\mathcal{O}_{\mathsf{F}}, 1)$ naturally forms a group, where the underlying binary product of two elements g, h is defined as the composition

$$\mathcal{O}_{\mathsf{F}} \xrightarrow{\Delta} \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \xrightarrow{g \otimes h} 1 \otimes 1 \simeq 1.$$

With respect to this, the augmentation $e : \mathcal{O}_{\mathsf{F}} \to 1$ is the identity element, and the inverse of g is given by $g \circ \tau$, where τ denotes the antipode map of $\mathcal{O}_{\mathsf{F}}$.

Given such a $g \in \mathcal{G}$, we may associate to it two maps in $\mathcal{C}$

$$\ell_g : \mathcal{O}_{\mathsf{F}} \xrightarrow{\Delta} \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \xrightarrow{g \otimes \mathrm{id}} 1 \otimes \mathcal{O}_{\mathsf{F}} \simeq \mathcal{O}_{\mathsf{F}}$$

$$r_g : \mathcal{O}_{\mathsf{F}} \xrightarrow{\Delta} \mathcal{O}_{\mathsf{F}} \otimes \mathcal{O}_{\mathsf{F}} \xrightarrow{\mathrm{id} \otimes g} \mathcal{O}_{\mathsf{F}} \otimes 1 \simeq \mathcal{O}_{\mathsf{F}}.$$

One may check that these give rise to an action of $\mathcal{F} \times \mathcal{F}$ on $\mathcal{O}_{\mathsf{F}}$ by algebra automorphisms, where (g, h) acts by $\ell_g \circ r_h = r_h \circ \ell_g$. In addition, one may check that one obtains an action of $\mathcal{F}$ on $\mathcal{O}_{\mathsf{F}}$ by Hopf algebra automorphisms, where g acts by conjugation, i.e., $\ell_g \circ r_{g^{-1}}$.

The corollary therefore implies we have an action of $\mathcal{F}$ on $\mathfrak{f}$ by restricted Lie algebra automorphisms.

2.3.11. Finally, let us note the compatibility of the previous constructions with change of the symmetric monoidal category. That is, consider a symmetric monoidal functor

$$\mu : \mathcal{C}_1 \to \mathcal{C}_2.$$

Let $\mathcal{O}_{\mathsf{F}}$ be a Hopf algebra in $\mathcal{C}_1$, and $\mu(\mathcal{O}_{\mathsf{F}})$ the corresponding Hopf algebra in $\mathcal{C}_2$. Let us write $\mathrm{Lie}(\mathcal{O}_{\mathsf{F}})$ and $\mathrm{Lie}(\mu(\mathcal{O}_{\mathsf{F}}))$ for the corresponding restricted Lie algebras.

Lemma 2.13. *There is a canonical map of restricted Lie algebras*

$$\mu : \mathrm{Lie}(\mathcal{O}_{\mathsf{F}}) \to \mathrm{Lie}(\mu(\mathcal{O}_{\mathsf{F}})),$$

equivariant for the map of groups $\mathrm{Aut}(\mathcal{O}_{\mathsf{F}}) \to \mathrm{Aut}(\mu(\mathcal{O}_{\mathsf{F}}))$ *acting by restricted Lie algebra automorphisms.*

Proof. The existence of the map $\mathrm{Lie}(\mathcal{O}_{\mathsf{F}}) \to \mathrm{Lie}(\mu(\mathcal{O}_{\mathsf{F}}))$ follows from noting that μ sends derivations of $\mathcal{O}_{\mathsf{F}}$ to derivations of $\mu(\mathcal{O}_{\mathsf{F}})$, i.e., functoriality induces a map of restricted Lie algebras

$$\mu : \mathrm{Der}(\mathcal{O}_{\mathsf{F}}) \to \mathrm{Der}(\mu(\mathcal{O}_{\mathsf{F}})),$$

and this preserves the subspace of invariant derivations. The compatibility with the actions by automorphisms again follows by functoriality. □

Remark 2.14. Note that, in the results and constructions of Section 2.3, if we pass from symmetric monoidal categories enriched over $\mathbb{F}_p$-modules to more general rings, everything save for the restricted structure still makes sense.

2.4. **The Tate extension as a restricted Lie algebra.**

2.4.1. In this subsection, we will apply the generalities of Subsection 2.3 to obtain the restricted Lie algebra structure on the Tate extension of the continuous endomorphisms of a Tate vector space; the formula appears in a canonical formulation in Section 2.4.14 and more explicitly in Corollary 2.23.

2.4.2. We should emphasize that the formula is straightforward to guess directly. However, to deduce from it similar formulas for the restricted Lie algebras of the Kac–Moody and Virasoro groups, as we do in Section 2.5 below, we need to obtain the formula from the corresponding Tate extension of the group of automorphisms of a Tate vector space. This necessity accounts for the length of this subsection.

2.4.3. We begin by recalling some preliminary material on Tate modules; some useful references are Section 2 of [BBE] and Sections 3-5 of [D].

Let R be a commutative ring. Let P be a projective R-module, not necessarily of finite rank. Let us view P as an object of Pro(R-Mod) via the tautological embedding $\mathsf{R}\text{-Mod} \to \mathrm{Pro}(\mathsf{R}\text{-Mod})$.

Recall that its dual $P^* := \mathrm{Hom}_{\mathsf{R}\text{-Mod}}(P, \mathsf{R})$ canonically lifts to an object of Pro(R-Mod), in a more interesting way, as follows. We may write P as the direct limit of its finitely generated submodules $P = \varinjlim M_i$, and the expression

$$P^* \simeq \varprojlim \mathrm{Hom}(M_i, \mathsf{R})$$

provides the desired lift.

2.4.4. We recall three properties of this construction. First, given a collection of projective modules $P_i, i \in I$, then there is a canonical isomorphism in Pro(R-Mod)

$$(\oplus_i P_i)^* = \Pi_i(P_i^*).$$

This follows from noting that finitely generated submodules of the form $\oplus_i M_i$, where each M_i is a finitely generated submodule of P_i, and all but finitely many are zero, are cofinal among all finitely generated $M \subset P$.

Second, given a free module $P \simeq \oplus_i \mathsf{R}$, by cofinality we have similarly that $P^* \simeq \varprojlim_J \mathsf{R}^J$, where J runs over all finite subsets of I. Note the latter is simply the product $\mathsf{R}^{\Pi I}$ taken in Pro(R-Mod); in what follows, we always use $\mathsf{R}^{\Pi I}$ to denote this product, rather than image of the product in R-Mod.

Therefore, for a general P, if we write it as a summand of $\mathsf{R}^{\oplus I}$, then P^* is the corresponding summand of $\mathsf{R}^{\Pi I}$.

Finally, note that one has a canonical isomorphism of R-modules

$$\mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(P^*, \mathsf{R}) \simeq P;$$

this follows by taking summands from the case of a free module $P \simeq \mathsf{R}^{\oplus I}$.

2.4.5. We recall that an object V of Pro(R-Mod) is called an *elementary Tate module* if it is isomorphic to a direct sum $P \oplus Q^*$, for some projective R-modules P and Q.

We recall that a *Tate module* V is an object which can be realized as a direct summand of an elementary Tate module, equivalently of $\mathsf{R}^{\oplus I} \times \mathsf{R}^{\Pi J}$, for some sets I and J.

Remark 2.15. In fact, Drinfeld proved that for any Tate module R-module V, there exists a Nisnevich cover $\mathsf{R} \to \mathsf{R}'$ for which $V \otimes_{\mathsf{R}} \mathsf{R}'$ is elementary Tate; see Theorem 3.4 of [D] and Proposition 2.12 of [BBE].

The basic example of a Tate R-module, which explains its relevance to affine Lie algebras, is the following.

Example 2.16. The ring of Laurent series $\mathsf{R}(\!(t)\!)$, with its usual topology, is a Tate module in Pro(R-Mod). E.g, one may use the splitting

$$\mathsf{R}(\!(t)\!) \simeq \mathsf{R}[[t]] \oplus t^{-1}\mathsf{R}[t^{-1}];$$

the latter summand is free, and the former summand is the dual of the free module $\mathsf{R}(\!(t)\!)dt/\mathsf{R}[[t]]dt \simeq t^{-1}\mathsf{R}[t^{-1}]dt$, via the residue pairing.

2.4.6. Recall that Tate modules carry a natural involutive duality. Concretely, for an elementary Tate module $P \oplus Q^*$, its dual is the elementary Tate module $P^* \oplus Q$, and one passes to summands for the general case.

2.4.7. Given a commutative algebra S in Pro(R-Mod), we have the associated induction functor

$$\mathrm{Pro}(\mathsf{R}\text{-Mod}) \to S\text{-Mod}(\mathrm{Pro}(\mathsf{R}\text{-Mod})), \qquad M \mapsto M \underset{\mathsf{R}}{\otimes} S.$$

Given Tate R-modules V and W, consider the set-valued functor on commutative algebras in Pro(R-Mod) given by

$$\mathrm{Hom}(V,W)(S) := \mathrm{Hom}_{S\text{-Mod}}(V \underset{\mathsf{R}}{\otimes} S, W \underset{\mathsf{R}}{\otimes} S).$$

Similarly, we may consider the subfunctor of isomorphisms

$$\mathrm{Isom}(V,W)(S) := \mathrm{Isom}_{S\text{-Mod}}(V \underset{\mathsf{R}}{\otimes} S, W \underset{\mathsf{R}}{\otimes} S),$$

where the right hand side denotes the set of invertible homomorphisms between $V \underset{\mathsf{R}}{\otimes} S$ and $W \underset{\mathsf{R}}{\otimes} S$.

2.4.8. We would like to describe in what sense the previously introduced functors are representable. To do so, let us recall some standard definitions pertaining to ind-schemes. Let us write $\mathrm{Sch}_{\mathsf{R}}$ for the category of schemes over R. We refer to a general object of its ind-completion $\mathrm{IndSch}_{\mathsf{R}}^{ns} := \mathrm{Ind}(\mathrm{Sch}_{\mathsf{R}})$, i.e., a filtered colimit of R-schemes along arbitrary maps, as a *non-strict ind-scheme*, we emphasize that the filtered diagrams, here and below, need not be countable. If it is moreover isomorphic to a filtered colimit of R-schemes under closed embeddings, we refer to it as a *strict ind-scheme*, and denote by

$$\mathrm{IndSch}_{\mathsf{R}}^{s} \subset \mathrm{IndSch}_{\mathsf{R}}^{ns}$$

the corresponding full subcategory.

By a *non-strict ind-affine ind-scheme*, we mean an object of $\mathrm{IndSch}^{ns}_{\mathsf{R}}$ isomorphic to a filtered colimit of affine R-schemes under arbitrary maps; if the maps may be taken to be closed embeddings, we refer to it as a *strict ind-affine ind-scheme*. Let us denote the full subcategories by

$$\mathrm{IndAffSch}^{s}_{\mathsf{R}} \subset \mathrm{IndAffSch}^{ns}_{\mathsf{R}} \subset \mathrm{IndSch}^{ns}_{\mathsf{R}}.$$

We will follow the standard abuse of notation that strict (ind-affine) ind-schemes will be referred to simply as (ind-affine) ind-schemes, unless there is a risk of confusion.

2.4.9. Note that there is a tautological equivalence between non-strict ind-affine ind-schemes and the opposite of the pro-category of (commutative) R-algebras, i.e.,

$$\mathrm{IndAffSch}^{ns}_{\mathsf{R}} \simeq \mathrm{Pro}(\mathrm{CommAlg}(\mathsf{R}\text{-Mod}))^{\mathrm{op}}.$$

Explicitly, this assigns to a filtered diagram $(Z_i)_{i\in I}$ of affine schemes the cofiltered diagram of R-algebras obtained by passing to functions and their pullbacks $(\mathcal{O}_{Z_i})_{i\in I^{\mathrm{op}}}$. In particular, this exchanges ind-affine ind-schemes with cofiltered diagrams of R-algebras under surjective transition maps.

The following basic lemma will be of use to us.

Lemma 2.17. *The category* $\mathrm{IndAffSch}^{ns}_{\mathsf{R}}$ *is complete, i.e., every small limit of representable contravariant functors is again representable, and cocomplete, i.e., every small colimit of corepresentable covariant functors is again corepresentable.*

Proof. We equivalently must show that $\mathrm{Pro}(\mathrm{CommAlg}(\mathsf{R}\text{-Mod}))$ is complete and cocomplete.

We first recall that $\mathrm{CommAlg}(\mathsf{R}\text{-Mod})$ is complete and cocomplete. Indeed, for completeness, the formation of limits commutes with the forgetful functor to R-Mod. For cocompleteness, finite coproducts are given by tensor product over R, the formation of filtered colimits commutes with the forgetful functor to R-Mod, and the coequalizer of a diagram

$$f : A \to B \leftarrow A : g$$

is given by the quotient of B by the ideal generated by the elements $f(a)-g(a), a \in A$.

From here, we recall some generalities. Let $\mathcal{C}$ be a category. First, if $\mathcal{C}$ is closed under taking limits of finite diagrams, then $\mathrm{Pro}(\mathcal{C})$ is complete, i.e., closed under taking limits of arbitrary diagrams, cf. Proposition 6.1.18(iii) of [KS] for the dual statement for ind-completions. In addition, if $\mathcal{C}$ is cocomplete, then so is $\mathrm{Pro}(\mathcal{C})$, see Proposition 11.1 of [I]. In particular, if $\mathcal{C}$ is complete and cocomplete, the same holds for $\mathrm{Pro}(\mathcal{C})$. Applying this to $\mathcal{C} = \mathrm{CommAlg}(\mathsf{R}\text{-Mod})$, we are done. □

Finally, note that there is a tautological full embedding

$$\mathrm{Pro}(\mathrm{CommAlg}(\mathsf{R}\text{-Mod})) \hookrightarrow \mathrm{CommAlg}(\mathrm{Pro}(\mathsf{R}\text{-Mod})).$$

In particular, we may speak of a set-valued functor on commutative algebras S in $\mathrm{Pro}(\mathsf{R}\text{-Mod})$ being representable by a strict or non-strict ind-affine ind-scheme Z.

2.4.10. With those preparations in hand, we may state the following.

Lemma 2.18. *The functors* $\mathrm{Hom}(V,W)$ *and* $\mathrm{Isom}(V,W)$ *are representable by ind-affine ind-schemes.*

Proof. We will show that the case of $\mathrm{Isom}(V,W)$ follows from that of $\mathrm{Hom}(V,W)$. We will in turn reduce the latter, in a series of steps, to the case of V and W being the sum of a free module and the dual of a free module.

Step 1. We first consider the case of $\mathrm{Hom}(V,W)$, beginning with $V=\mathsf{R}$. For a free discrete module $W\simeq \oplus_I \mathsf{R}$, we have

$$\mathrm{Hom}(\mathsf{R},W)\simeq \varinjlim_{J} \Pi_J \mathbb{A}^1, \tag{2.3}$$

where J runs over all finite subsets of I, and to an inclusion $J\subset J'$ one associates the standard linear closed embedding $\Pi_J\mathbb{A}^1\hookrightarrow \Pi_{J'}\mathbb{A}^1$; in particular (2.3) exhibits $\mathrm{Hom}(\mathsf{R},W)$ as an ind-affine ind-scheme, namely ind-$\mathbb{A}^\infty$. For the dual of a free discrete module $W\simeq \Pi_I\mathsf{R}$ we have $\mathrm{Hom}(\mathsf{R},W)$ is the affine scheme $\Pi_I\mathbb{A}^1$, i.e., pro-$\mathbb{A}^\infty$.

Step 2. We next consider the case of $\mathrm{Hom}(V,W)$, for V and W of the form

$$V\simeq \mathsf{R}^{\oplus I}\times \mathsf{R}^{\Pi J},\qquad W\simeq \mathsf{R}^{\oplus K}\times \mathsf{R}^{\Pi L},$$

for sets I,J,K,L. Note we have a tautological isomorphism of functors

$$\mathrm{Hom}(\oplus_I\mathsf{R}\oplus\Pi_J\mathsf{R},\oplus_K\mathsf{R}\oplus\Pi_L\mathsf{R})\simeq \Pi_I(\oplus_K\mathsf{R})\times\Pi_{I\times L}\mathsf{R}\times\oplus_{J\times K}\mathsf{R}\times\oplus_J\Pi_L\mathsf{R}.$$

For the first term, note that ind-affine ind-schemes are closed under arbitrary products, e.g. by the argument of Proposition 11.1 of [I], hence $\Pi_I(\oplus_K\mathsf{R})$ is again ind-affine. We saw above that $\Pi_{I\times L}\mathsf{R}$ and $\oplus_{J\times K}\mathsf{R}$ are ind-affine, and for the final term we have $\oplus_J\Pi_L\mathsf{R}$ is the colimit of the finite products $\Pi_{J'\times L}\mathsf{R}$, where J' runs over all finite subsets of J, under the standard linear closed embeddings.

Step 3. We now consider the case of $\mathrm{Hom}(V,W)$, for general Tate modules V and W. Choose modules $V'\simeq\oplus_I\mathsf{R}\oplus\Pi_J\mathsf{R}$, $W'\simeq\oplus_K\mathsf{R}\oplus\Pi_L\mathsf{R}$ and embeddings $\iota_V:V\to V'$, $\iota_W:W\to W'$ with splittings $\pi_V:V'\to V$, $\pi_W:W'\to W$. It follows that $\mathrm{Hom}(V,W)$ is the equalizer of $\mathrm{Hom}(V',W')\rightrightarrows\mathrm{Hom}(V',W')$, where the two maps send $\phi\in\mathrm{Hom}(V',W')$ to ϕ and $\iota_W\circ\pi_W\circ\phi\circ\iota_V\circ\pi_V$, respectively.

As an ind-affine ind-scheme is in particular an inductive limit of separated schemes along closed embeddings, it follows that $\mathrm{Hom}(V,W)$ is a closed subfunctor of $\mathrm{Hom}(V',W')$, and in particular again an ind-affine ind-scheme, as desired.

Step 4. We finally consider the case of $\mathrm{Isom}(V,W)$. Note that $\mathrm{Isom}(V,W)$ is the subfunctor of $\mathrm{Hom}(V,W)\times\mathrm{Hom}(W,V)$ consisting of pairs (ϕ,ψ) satisfying $\phi\circ\psi=\mathrm{id}_W$, $\psi\circ\phi=\mathrm{id}_V$. Again by the separatedness of the appearing ind-schemes, it follows $\mathrm{Isom}(V,W)$ is a closed subfunctor, and hence again ind-affine. □

2.4.11. As a particular case of the preceding, it follows that, for a given Tate module V, its automorphism group

$$GL(V):=\mathrm{Isom}(V,V)$$

is a group ind-affine ind-scheme. In particular, it corresponds to a commutative Hopf algebra in Pro(R-Mod).[3]

2.4.12. Explicitly, if we write $GL(V)$ as a filtered colimit of affine R-schemes along closed embeddings $GL(V) \simeq \varinjlim_i \operatorname{Spec}(T_i)$, we have an isomorphism of algebras

$$\mathcal{O}_{GL(V)} \simeq \varprojlim T_i \in \mathrm{Pro}(\mathsf{R}\text{-Mod})$$

where the latter formal inverse limit of commutative algebras under algebra maps carries its natural commutative algebra structure. Namely, if we denote the multiplication and projection maps by

$$\mu : \varprojlim T_i \otimes \varprojlim T_i \to \varprojlim T_i, \quad \mu_i : T_i \otimes T_i \to T_i, \quad \pi_i : \varprojlim T_i \to T_i, \quad i \in I,$$

for any i the composite

$$\varprojlim T_i \otimes \varprojlim T_i \xrightarrow{\mu} \varprojlim T_i \xrightarrow{\pi_i} T_i$$

factors as

$$\varprojlim T_i \otimes \varprojlim T_i \xrightarrow{\pi_i \otimes \pi_i} T_i \otimes T_i \xrightarrow{\mu_i} T_i.$$

2.4.13. So, if R is an $\mathbb{F}_p$-algebra, we obtain from the discussion of Section 2.3 its restricted Lie algebra $\mathfrak{gl}(V)$, which we now explicitly identify.

Proposition 2.19. *Suppose that* R *is an* $\mathbb{F}_p$*-algebra. Then there is a canonical isomorphism of restricted Lie algebras*

$$\mathfrak{gl}_V \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V, V),$$

where the latter carries its tautological structure of restricted Lie algebra as in Section 2.3.6.

Proof. We begin with the identification as abelian groups. As in Lemma 2.7, we have

$$\mathfrak{gl}_V \simeq \mathrm{Der}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(\mathcal{O}_{GL(V)}, e_*\mathsf{R}). \tag{2.4}$$

As in Remark 2.3, (2.4) may identified with maps of algebras $\mathcal{O}_{GL(V)} \to \mathsf{R}[\epsilon]$ in Pro(R-Mod), where ϵ is square zero, lifting $e : \mathcal{O}_{GL(V)} \to \mathsf{R}$, i.e., automorphisms of $V \otimes \mathsf{R}[\epsilon]$ which reduce to id_V modulo ϵ. However, the latter are simply endomorphisms of the form $\mathrm{id}_V + \epsilon X$, for any element X in $\mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V, V)$, as desired.

Let us verify the obtained identification is one of restricted Lie algebras. Consider the natural action of $GL(V)$ on V, and in particular the induced map of restricted Lie algebras $\mathfrak{gl}_V \to \mathrm{Der}(\mathcal{O}_V)$. For $X \in \mathfrak{gl}_V$, consider the associated map

$$\operatorname{Spec}\mathsf{R}[\epsilon] \times V \to GL(V) \times V \to V. \tag{2.5}$$

Note that V is representable by the algebra $\operatorname{Sym} V^*$, i.e., the free commutative algebra in Pro(R-Mod) associated to its dual Tate module V^*, cf. Section 2.4.6, so in particular an S-point of V is the same data as a map $V^* \to S$ in Pro(R-Mod).

With this, the map (2.5) is given on S-points by the formula

$$(\eta, v) \mapsto v + \eta \cdot X(v),$$

[3]Note this commutative Hopf algebra has an underlying commutative algebra which is moreover *strict*, i.e., representable as an inverse limit of discrete R-algebras along surjective maps of algebras.

where $\eta : \mathsf{R}[\epsilon] \to S$, $v : V^* \to S$, and $\eta \cdot X(v)$ denotes the composition

$$V^* \simeq \mathsf{R} \otimes V^* \xrightarrow{-\epsilon \otimes X^*} \mathsf{R}[\epsilon] \otimes V^* \xrightarrow{\eta \otimes v} S \otimes S \to S,$$

and $X^* : V^* \to V^*$ denotes the dual of $X : V \to V$.

Consider within all vector fields on V the linear ones

$$\mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V^*, V^*) \hookrightarrow \mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V^*, \mathrm{Sym}\, V^*) \simeq \mathrm{Der}(\mathrm{Sym}\, V^*, \mathrm{Sym}\, V^*),$$

and note that the above composition is in fact a homomorphism of restricted Lie algebras. The identity (2.5) implies that the vector field $\mathcal{L}_X$ associated to X agrees with the linear vector field associated to $-X^*$, whence it follows that the identification $\mathfrak{gl}_V \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V, V)$ is one of restricted Lie algebras, as desired. □

2.4.14. Let us now consider the Tate extension of $GL(V)$, a group ind-affine ind-scheme which we denote by $GL^\flat(V)$:

$$1 \to \mathbb{G}_m \to GL^\flat(V) \to GL(V) \to 1. \tag{2.6}$$

A detailed construction of the underlying functor on R-algebras appears in Sections 2.10 and 2.11 of [BBE], cf. also Section 5 of [D].

Let us recall some salient aspects. First, given $V = V_1 \oplus V_2$, it is known that the pullback of the Tate extension for $GL(V)$ along the tautological map

$$GL(V_1) \hookrightarrow GL(V), \qquad \phi \mapsto \phi \oplus \mathrm{id}_{V_2},$$

yields the Tate extension for $GL(V_1)$, cf. Section 2.10(iii) of [BBE]. Therefore, we may restrict our discussion to the automorphisms of an elementary Tate module.

Recall that a submodule L of V is called a *c-lattice* if it may be realized as the summand Q^* with respect to some elementary decomposition $V = P \oplus Q^*$, i.e., is a summand of V such that V/L is a projective R-module. Given two c-lattices L_1, L_2, their sum $L_1 + L_2$ is again a c-lattice, and the quotients $(L_1 + L_2)/L_i$ are finite rank projective R-modules, for $i = 1, 2$. In particular, we may form the relative determinant line bundle

$$\mathrm{rel.det}(L_1, L_2) := \det(L_1 + L_2/L_1)^\vee \underset{\mathsf{R}}{\otimes} \det(L_1 + L_2/L_2).$$

With these preliminaries in hand, we may recall the functor of points description of $GL(V)^\flat$. Namely, for a commutative algebra S in R-Mod, an S-point of $GL(V)^\flat$ consists of an S-point of $GL(V)$, i.e., an automorphism of the corresponding S-Tate module

$$\phi : V \underset{\mathsf{R}}{\otimes} S \simeq V \underset{\mathsf{R}}{\otimes} S,$$

along with a trivialization of the relative determinant line

$$\tau : S \simeq \mathrm{rel.det}(Q^* \underset{\mathsf{R}}{\otimes} S, \phi(Q^* \underset{\mathsf{R}}{\otimes} S)).$$

The multiplicativity of relative determinant lines endows the set of S-points with a group structure. For a map $S \to T$, one sends a pair (ϕ, τ) as above to their base changes $(\phi \underset{S}{\otimes} T, \tau \underset{S}{\otimes} T)$, and this is a map of groups.

The forgetful map $(\phi, \tau) \mapsto \phi$ yields the desired sequence (2.6).

Lemma 2.20. *$GL(V)^\flat$ is representable by an ind-affine ind-scheme.*

Proof. It is known that for any R-scheme X and a map to $X \to GL(V)$, the pullback

$$X^\flat := GL^\flat(V) \underset{GL(V)}{\times} X,$$

equipped with its natural action of $\mathbb{G}_m$, is in fact a $\mathbb{G}_m$-torsor over X, cf. Section 5.4.3 of [D]. In particular, if we write $GL(V)$ as a filtered colimit of affine schemes under closed embeddings $(Z_i)_{i\in I}$, cf. Lemma 2.18, it follows that $GL^\flat(V)$ is the filtered colimit of affine schemes under closed embeddings $(Z_i^\flat)_{i\in I}$. □

In particular, as we did for $GL(V)$, we may again pass from $GL(V)^\flat$ to the corresponding commutative Hopf algebra in Pro(R-Mod) to obtain a canonical restricted structure on the Lie algebra of $GL(V)^\flat$.

2.4.15. With this in mind, let us review a canonical description of the corresponding Tate extension of Lie algebras

$$0 \to \mathsf{R} \to \mathfrak{gl}^\flat(V) \to \mathfrak{gl}(V) \to 0$$

cf. Section 2.13 of [BBE]. Again, it is enough, by the compatibility of Tate extensions and direct sums, to discuss the case of an elementary Tate module.

We first recall some definitions. An element $X \in \mathfrak{gl}(V)$ is called *discrete* if it has an open kernel. Equivalently, it factors through a usual R-module, i.e., admits a factorization $V \to I \to V$, where I is an object of R-Mod $\hookrightarrow$ Pro(R-Mod). It is straightforward to see that the set of discrete endomorphisms, which we denote by $\mathfrak{gl}_d(V) \hookrightarrow \mathfrak{gl}(V)$, is an ideal of $\mathfrak{gl}(V)$, stable under the adjoint action of $GL(V)$.

A subobject of a Tate module $L \hookrightarrow V$ is called *bounded* if for any map from V to a usual R-module the image of L is finitely generated image.

An element $X \in \mathfrak{gl}(V)$ is called bounded if it has bounded image. It is straightforward to see that the set of bounded endomorphisms, which we denote by $\mathfrak{gl}_c(V) \hookrightarrow \mathfrak{gl}(V)$, form an ideal of $\mathfrak{gl}(V)$, stable under the adjoint action of $GL(V)$.

In particular, the intersection $\mathfrak{gl}_f(V) := \mathfrak{gl}_c(V) \cap \mathfrak{gl}_d(V)$ is the Ad-invariant ideal of maps whose images are finitely generated R-modules. This admits a trace homomorphism

$$\mathrm{tr} : \mathfrak{gl}_f(V) \to \mathsf{R}, \quad \phi \mapsto \mathrm{tr}(\phi);$$

we recall an elementary definition of the latter in the next remark.

Remark 2.21. Concretely, if we exhibit V as a summand of $\mathsf{R}^{\oplus I} \times \mathsf{R}^{\Pi J}$, the trace of ϕ agrees with the trace of $\phi \oplus 0 \in \mathfrak{gl}_f(\mathsf{R}^{\oplus I} \times \mathsf{R}^{\Pi J})$. To define the latter, given $\psi \in \mathfrak{gl}_f(\mathsf{R}^{\oplus I} \times \mathsf{R}^{\Pi J})$, for any finite subsets $I' \subset I, J' \subset J$, consider the composition $\psi_{I',J'}$ given by

$$\mathsf{R}^{I' \sqcup J'} \hookrightarrow \mathsf{R}^{\oplus I} \times \mathsf{R}^{\Pi J} \xrightarrow{\psi} \mathsf{R}^{\oplus I} \times \mathsf{R}^{\Pi J} \twoheadrightarrow \mathsf{R}^{I' \sqcup J'}.$$

Then $\mathrm{tr}(\psi_{I',J'})$ is the usual trace of the endomorphism of a finitely generated free module. This is independent of I', J', for all sufficiently large finite subsets I', J', and for such subsets we have $\mathrm{tr}(\psi) = \mathrm{tr}(\psi_{I',J'})$.

Note that any element of ϕ of $\mathfrak{gl}(V)$ may be written (non-uniquely) as the sum of a bounded element and a discrete element.[4] We have a short exact sequenceof Lie algebras

$$0 \to \mathfrak{gl}_f(V) \to \mathfrak{gl}_d(V) \to \mathfrak{gl}(V)/\mathfrak{gl}_c(V) \to 0,$$

where the appearing maps are the tautological inclusion and projection. By taking the pushout of this along $\mathrm{tr} : \mathfrak{gl}_f(V) \to \mathsf{R}$, we obtain a central extension

$$0 \to \mathsf{R} \to (\mathfrak{gl}(V)/\mathfrak{gl}_c(V))^\flat \to \mathfrak{gl}(V)/\mathfrak{gl}_c(V) \to 0. \tag{2.7}$$

Finally, pulling this back along the projection $\mathfrak{gl}(V) \to \mathfrak{gl}(V)/\mathfrak{gl}_c(V)$ yields the Tate extension $\mathfrak{gl}^\flat(V)$.

As every appearing ideal in the construction is moreover restricted, this yields a canonical restricted structure on the Tate extension itself.

Lemma 2.22. *The restricted Lie algebra structure on $\mathfrak{gl}^\flat(V)$ viewed as the Lie algebra of $GL^\flat(V)$, see Lemma 2.20, agrees with the above construction.*

Proof. By the compatibility of Tate extensions and direct sums of Tate modules, we may reduce to the case of $V \simeq V^- \oplus V^+$, where $V^- := \mathsf{R}^{\oplus I}, V^+ := \mathsf{R}^{\Pi J}$.

Consider the object V^∞ of R-Mod $\hookrightarrow$ Pro(R-Mod) given by the colimit

$$V^\infty := \varinjlim V/L,$$

where L runs over all bounded submodules of V; explicitly a cofinal set of L's is provided by $R^{\oplus I'} \oplus V^+ \hookrightarrow V$, where I' runs over all finite subsets of I. Consider the associated space of endomorphisms

$$\mathrm{Hom}(V^\infty, V^\infty) := \varprojlim_i \varinjlim_j \mathrm{Hom}(V/L_i, V/L_j);$$

where each $\mathrm{Hom}(V/L_i, V/L_j)$ is the strict affine ind-scheme of Lemma 2.18. As the category Pro(CommAlg(R-Mod)) contains all limits and colimits, cf. Lemma 2.17, it follows that $\mathrm{Hom}(V^\infty, V^\infty)$ is again representable by an object of $\mathrm{IndAffSch}^{ns}_{\mathsf{R}}$, though it is no longer strict. As $\mathrm{IndAffSch}^{ns}_{\mathsf{R}}$ is in particular closed under taking equalizers, it follows that the invertible asymptotic endomorphisms

$$GL(V^\infty) \hookrightarrow \mathrm{Hom}(V^\infty, V^\infty) \times \mathrm{Hom}(V^\infty, V^\infty)$$

are again representable by a Hopf algebra in Pro(R-Mod). By an argument similar to that of Proposition 2.19, its restricted Lie algebra is given by

$$\mathfrak{gl}(V^\infty) := \varprojlim \varinjlim \mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V/L_i, V/L_j).$$

Consider, as in [BBE, Section 2.10], the Tate extension $GL^\flat(V^\infty)$ of $GL(V^\infty)$; according to *loc. cit.*, its pullback along $GL(V^-) \to GL(V^\infty)$ is canonically split, and moreover, if we write $GL(V^-)_f$ for the subgroup of endomorphisms which

[4]Indeed, for $V = \mathsf{R}^{\oplus I} \oplus R^{\Pi J}$, consider the projections p_-, p_+ to $\mathsf{R}^{\oplus I}$ and $\mathsf{R}^{\Pi J}$, respectively. Then $\phi = p_-\phi + p_+\phi$, and the two summands are bounded and discrete, respectively. For a general V, choose a split inclusion

$$\iota : V \leftrightarrows \mathsf{R}^{\oplus I} \oplus R^{\Pi J} : \pi.$$

Then we have

$$\phi = \pi\iota\phi\pi\iota = \pi(p_-\iota\phi\pi + p_+\iota\phi\pi)\iota = \pi p_-\iota\phi\pi\iota + \pi p_+\iota\phi\pi;$$

the two summands are again discrete and bounded, respectively.

are the identity on a summand of V^- of the form $R^{\oplus I'}$ for $I' \subset I$ a subset with finite complement,[5] the obtained map $GL(V^-)_f \to \mathbb{G}_m$ is simply the determinant. As the Tate extension $GL^\flat(V)$ is pulled back along the map $GL(V) \to GL(V^\infty)$, comparison with (2.7) yields the claim of the Lemma. □

2.4.16. Let us give an explicit description of the obtained restricted Lie algebra for an elementary Tate module with fixed decomposition $V = P \oplus Q^* =: V^- \oplus V^+$. Consider the associated decomposition

$$\mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V, V) \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V, V^-) \oplus \mathrm{Hom}_{\mathrm{Pro}(\mathsf{R}\text{-Mod})}(V, V^+),$$

and for an element $\phi \in \mathfrak{gl}(V)$ consider its corresponding expression $\phi = \phi_- + \phi_+$, where $\phi_\pm : V \to V^\pm$.

Corollary 2.23. *The Tate extension of $\mathfrak{gl}(V)$ for $V = V^+ \oplus V^-$ may be written as follows. As an R-module, we have*

$$\mathfrak{gl}^\flat(V) \simeq \mathfrak{gl}(V) \oplus \mathsf{R} \cdot 1.$$

As a Lie algebra, 1 is central, and for $\phi, \psi \in \mathfrak{gl}(V)$ we have

$$[\phi, \psi]_{\mathfrak{gl}^\flat(V)} = [\phi, \psi]_{\mathfrak{gl}(V)} - \mathrm{tr}(\phi_- \psi_+ - \psi_- \phi_+) \cdot 1.$$

For the restricted structure, we have $1^{[p]} = 1$, and for $\phi \in \mathfrak{gl}(V)$ we have

$$\phi^{[p]} = \phi^p + \mathrm{tr}((\phi_-)^p - (\phi^p)_-) \cdot 1. \tag{2.8}$$

Proof. Let V^∞ have the same meaning as in the proof of Lemma 2.22. Note that the map of Lie algebras $\mathfrak{gl}(V) \to \mathfrak{gl}(V^\infty)$ may be factored through the map of R-modules $\mathfrak{gl}(V) \to \mathfrak{gl}(V^-), \phi \mapsto \phi^-$, and via the further quotient map $\mathfrak{gl}(V^-) \to \mathfrak{gl}^\flat(V^\infty)$ this yields the splitting as R-modules $\mathfrak{gl}^\flat(V) \simeq \mathfrak{gl}(V) \oplus \mathsf{R} \cdot 1$. The obtained expression for the commutator follows (and is standard). The expression for the restricted power $\phi^{[p]}$ follows similarly, and the identity $1^{[p]} = 1$ comes from the identification of restricted Lie algebras $\mathsf{R} \simeq \mathrm{Lie}(\mathbb{G}_m)$. □

Remark 2.24. To make the formulae for the commutator and restricted structure in the Tate extension more closely resemble one another, one may note that

$$-\mathrm{tr}(\phi_- \psi_+ - \psi_- \phi_+) = \mathrm{tr}([\phi, \psi]^- - [\phi^-, \psi^-])).$$

2.5. **Kac–Moody and Virasoro I: the restricted structure.**

2.5.1. Let us now deduce the desired formulae for the restricted Lie algebra structures on the Kac–Moody and Virasoro Lie algebras.

[5] More intrinsically, this is the subgroup of automorphisms α of V^- such that there exists a direct sum decomposition $V^- = Q \oplus Q'$, where Q' is a finitely generated projective module, and α restricts to the identity on Q.

2.5.2. We begin with Kac–Moody. Let F be a linear algebraic group over R with Lie algebra $\mathfrak{f}$. Recall the loop group $F(\!(t)\!)$ is the group affine ind-scheme over R with S-points given by $F(S(\!(t)\!))$, for any R-algebra S. If we view $\mathcal{O}_{F(\!(t)\!)}$ as a commutative Hopf algebra in Pro(R-Mod), recall there is a canonical identification of its Lie algebra with $\mathfrak{f}(\!(t)\!) := \mathfrak{f} \underset{\mathsf{R}}{\otimes} \mathsf{R}(\!(t)\!)$.

Lemma 2.25. *The restricted Lie algebra structure on $\mathfrak{f}(\!(t)\!)$ is given by the formula*

$$(X \otimes f)^{[p]} = X^{[p]} \otimes f^p, \qquad X \in \mathfrak{f}, f \in \mathsf{R}(\!(t)\!).$$

Proof. Choose a closed embedding of F into the endomorphisms of a projective R-module V of finite rank, and consider the induced action of $F(\!(t)\!)$ on $V(\!(t)\!)$. This induces an injective homomorphism of restricted Lie algebras $\mathfrak{f}(\!(t)\!) \to \mathfrak{gl}(V(\!(t)\!))$. The formula now follows from Proposition 2.19. □

2.5.3. Suppose in addition we fix a representation $\rho : F \to GL(W)$, for a projective R-module W of finite rank. Consider the induced map $\rho : F(\!(t)\!) \to GL(W(\!(t)\!))$, and denote the pullback of the Tate extension by $F(\!(t)\!)^\flat$. This is a strict group ind-scheme of countable type, i.e., presentable as a countable filtered colimit of affine schemes along closed embeddings. In particular, we obtain its corresponding Lie algebra $\mathfrak{f}(\!(t)\!)^\flat$ that fits into the exact sequence

$$0 \to \mathsf{R} \to \mathfrak{f}(\!(t)\!)^\flat \to \mathfrak{f}(\!(t)\!) \to 0.$$

Proposition 2.26. *We have a splitting of R-modules $\mathfrak{f}(\!(t)\!)^\flat \simeq \mathfrak{f}(\!(t)\!) \oplus \mathsf{R} \cdot 1$, with respect to which the commutator is given by the formula*

$$[X \otimes f, Y \otimes g]_{\mathfrak{f}(\!(t)\!)^\flat} = [X, Y] \otimes fg - \mathrm{tr}_W(XY) \cdot \mathrm{Res} f dg \cdot 1,$$

and the restricted structure satisfies the identities $1^{[p]} = 1$, $(X \otimes t^i)^{[p]} = X^{[p]} \otimes t^{ip}$, for any $i \in \mathbb{Z}$.

Before giving the proof, we make some orienting remarks.

Remark 2.27. Note first that, by continuity, and compatibility of restricted structures and sums, cf. Definition 2.2(3), the above formulas determine the restricted Lie algebra structure uniquely.

Remark 2.28. Note in addition that, by inspection, the obtained extension $\mathfrak{f}(\!(t)\!)^\flat$ only depends, as a restricted Lie algebra, on the trace form

$$\mathrm{tr}_W \in \mathrm{Sym}^2(\mathfrak{f}^*)^F : \qquad X \otimes Y \mapsto \mathrm{tr}_W(XY).$$

We flag for the reader that, unlike in this preliminary discussion, beginning in Section 3.4.1 below we will incorporate a critical shift in our discussion of trace forms; we hope this does not cause confusion.

Remark 2.29. If $\mathfrak{f}$ is simple or one dimensional, as long as the characteristic of R is not too small relative to $\mathfrak{f}$, note that $\mathrm{Sym}^2(\mathfrak{f}^*)^F$ is a free rank one R-module. It follows that for two representations V and W with tr_V and tr_W both units in R, we have an identification of the underlying Lie algebras of the two central extensions,

$$\mathfrak{f}(\!(t)\!)^\flat_V \simeq \mathfrak{f}(\!(t)\!)^\flat_W, \qquad X \otimes f \mapsto X \otimes f, \qquad 1 \mapsto \frac{\mathrm{tr}_V}{\mathrm{tr}_W} \cdot 1.$$

Moreover, as tr_V and tr_W may be recovered from their restriction to the image of the cocharacter lattice of a maximal torus, it follows that the ration $\frac{\mathrm{tr}_V}{\mathrm{tr}_W}$ lies in $\mathbb{F}_p^\times$, whence the above identification is moreover one of restricted Lie algebras.

The same remarks apply, *mutatis mutandis*, to the Lie algebra of any split reductive group, where one considers each simple factor and the center as above. Finally, note that given a restricted Lie algebra $\mathfrak{a}$ and a finite set of restricted central extensions $\widetilde{\mathfrak{a}}_i, i \in I$, of $\mathfrak{a}$ by R, one has a natural restricted central extension of $\mathfrak{a}$ by $\mathsf{R}^{\oplus I}$,

$$0 \to \mathsf{R}^{\oplus I} \to \widetilde{\mathfrak{a}}_I \to \mathfrak{a} \to 0,$$

such that the pushout along each projection $\pi_i : \mathsf{R}^{\oplus I} \to \mathsf{R}, i \in I$, rescovers $\widetilde{\mathfrak{a}}_i$. So, in particular, in the reductive case, we may equally well discuss a Kac–Moody central extension where we include different central elements for different simple and central factors.

Proof of Proposition 2.26. We will apply Corollary 2.23 with respect to the splitting

$$W(\!(t)\!) = W \underset{\mathsf{R}}{\otimes} \mathsf{R}[[t]] \oplus W \underset{\mathsf{R}}{\otimes} t^{-1}\mathsf{R}[t^{-1}].$$

For the restricted structure, we apply the identity (2.8), i.e.,

$$\phi^{[p]} = \phi^p + \mathrm{tr}((\phi_-)^p - (\phi^p)_-) \cdot 1$$

to $\phi = X \otimes t^i \in \mathfrak{f}(\!(t)\!)$.

We saw in Lemma 2.25 that $\phi^{[p]} = X^{[p]} \otimes t^{ip}$. We claim that the cocycle term $\mathrm{tr}((\phi_-)^p - (\phi^p)_-) \cdot 1$ vanishes. Indeed, this this follows from considering their homogeneity with respect to the loop rotation grading on

$$W \underset{\mathsf{R}}{\otimes} \mathsf{R}[t^{\pm 1}].$$

Namely, for $i \neq 0$, it is clear that $\mathrm{tr}((\phi_-)^p - (\phi^p)_-)$ vanishes, as both appearing operators have degree $ip \neq 0$, hence have vanishing trace. For $i = 0$, the vanishing follows from noting that $(\phi_-)^n = (\phi^n)_-$, for any $n \in \mathbb{Z}$, and in particular for $n = p$. Therefore, we have $(\phi_-)^p - (\phi^p)_-$ vanishes, and in particular has vanishing trace. □

2.5.4. Let us now discuss the Virasoro case, beginning with some recollections on the Witt group and Lie algebra. Recall, cf. Section 2.4.8, that a non-strict ind-affine ind-scheme over R is an object in the opposite category of $\mathrm{Alg}(\mathrm{Pro}(\mathsf{R}\text{-Mod}))$. Let us write $\mathcal{D}^\times$ for the formal punctured disc, i.e., the ind-affine ind-scheme corresponding to $\mathsf{R}(\!(t)\!)$.

Consider the functor $\mathrm{End}(\mathcal{D}^\times)$ from R-algebras to sets given by the formula

$$\mathrm{End}(\mathcal{D}^\times)(S) := \mathrm{Hom}_{\mathrm{Alg}(\mathrm{Pro}(S\text{-Mod}))}(S(\!(t)\!), S(\!(t)\!)).$$

Explicitly, for fixed S, such a map is entirely determined by the image of t, which must be a Laurent series $\phi(t) = \sum a_i t^i \in S(\!(t)\!)$ with a_i nilpotent for $a_i \leqslant 0$. In particular $\mathrm{End}(\mathcal{D}^\times)$ is a strict ind-affine ind-scheme of countable type. We may further consider the automorphisms of the formal disc

$$\mathrm{Aut}(\mathcal{D}^\times) \hookrightarrow \mathrm{End}(\mathcal{D}^\times) \times \mathrm{End}(\mathcal{D}^\times),$$

consisting of pairs $(\phi(t), \psi(t))$ satisfying $\phi(\psi(t)) = \psi(\phi(t)) = t$. It follows that $\mathrm{Aut}(\mathcal{D}^\times)$ is a group ind-affine ind-scheme of countable type, and in particular we

form its commutative Hopf algebra of functions in Pro(R-Mod), and its restricted Lie algebra $\mathfrak{Witt}$.

Remark 2.30. One may also check that the first projection $\mathrm{Aut}(\mathcal{D}^\times) \to \mathrm{End}(\mathcal{D}^\times)$ exhibits it as the open subfunctor consisting of series $\phi(t) = \sum a_i t^i$ as before which moreover satisfy the condition that a_1 is a unit in S.

Consider $\mathsf{R}(\!(t)\!)\partial_t$, the restricted Lie algebra of vector fields on the formal punctured disc, i.e., derivations of $\mathsf{R}(\!(t)\!)$ in Pro(R-Mod). Explicitly, the commutator is the usual Lie bracket of vector fields, and the restricted structure sends a derivation to its pth power, cf. Section 2.3.6.

Proposition 2.31. *There is a canonical isomorphism of restricted Lie algebras between $\mathfrak{Witt}$ and $\mathsf{R}(\!(t)\!)\partial_t$. In particular, if for $i \in \mathbb{Z}$ we write $L'_i := -t^{i+1}\partial_t$, these satisfy the identities*

$$L_i'^{[p]} = \begin{cases} L'_{ip}, & (i,p) \neq 1, \\ 0 & (i,p) = 1. \end{cases}$$

Proof. The identification $\mathfrak{Witt} \simeq \mathsf{R}(\!(t)\!)\partial_t$ as abelian groups follows by considering $\mathsf{R}[\epsilon]$ points of $\mathrm{Aut}(\mathcal{D}^\times)$ extending the identity, cf. the beginning of the proof of Proposition 2.19.

If we write $\mathcal{D}^\times$ for '$\mathrm{Spec}\,\mathsf{R}(\!(t)\!)$', i.e., the object of $(\mathrm{CommAlg}(\mathrm{Pro}(\mathsf{R}\text{-Mod})))^{op}$ corresponding to $\mathsf{R}(\!(t)\!)$, we have a tautological action of $\mathrm{Aut}(\mathcal{D}^\times)$ on $\mathcal{D}^\times$. Moreover, for $X \in \mathsf{R}(\!(t)\!)\partial_t$, the associated derivation $\mathcal{L}_X$ of $\mathsf{R}(\!(t)\!)$ agrees with its standard action, cf. Lemma 2.9, from which the first claim of the proposition follows. The second statement concerning $L_i'^{[p]}$ then follows by applying L'_i, viewed as an endomorphism of $\mathsf{R}(\!(t)\!)$, p times to t. □

2.5.5. Let us now discuss the central extension. Consider the tautological linear action of $\mathrm{Aut}(\mathcal{D}^\times)$ on the Tate R-module $\mathsf{R}(\!(t)\!)$ by change of coordinates. Viewing this as a homomorphism $\mathrm{Aut}(\mathcal{D}^\times) \to GL(\mathsf{R}(\!(t)\!))$, we may pull back the Tate extension to obtain the Virasoro group ind-scheme $\mathrm{Aut}(\mathcal{D}^\times)^\flat$ that fits into the following short exact sequence of group ind-schemes

$$1 \to \mathbb{G}_m \to \mathrm{Aut}(\mathcal{D}^\times)^\flat \to \mathrm{Aut}(\mathcal{D}^\times) \to 1;$$

and in particular its restricted Lie algebra

$$0 \to \mathsf{R} \cdot 1 \to \mathfrak{Vir} \to \mathfrak{Witt} \to 0.$$

Proposition 2.32. *We have a splitting of R-modules $\mathfrak{Vir} \simeq \mathfrak{Witt} \oplus \mathsf{R} \cdot 1$, with respect to which the commutator is determined by the formula*

$$[L'_m, L'_n] = (m-n)L'_{m+n} - \delta_{m,-n} \cdot \binom{m+1}{3} \cdot 1,$$

where $\delta_{m,-n}$ denotes the Kronecker delta function, and the restricted structure is determined by the identities

$$1^{[p]} = 1, \qquad L_i'^{[p]} = \begin{cases} L'_{ip}, & (i,p) \neq 1, \\ 0 & (i,p) = 1. \end{cases}$$

.

Proof. This follows from applying Corollary 2.23 with respect to the splitting $\mathsf{R}(\!(t)\!) = \mathsf{R}[[t]] \oplus t^{-1}\mathsf{R}[t^{-1}]$. Let us describe the calculation of the cocycle for the commutator; the argument for the restricted structure proceeds similarly to that of Proposition 2.26.

Note the cocycle modifying the Lie bracket vanishes for $[L_m, L_n]$ unless $m = -n$ for degree reasons. For $m > 0$, note that $(L_m)_-(L_{-m})_+$ vanishes, hence we have

$$[L_m, L_{-m}] = 2m \cdot L_0 + \mathrm{tr}((L_{-m})_-(L_m)_+).$$

For degree reasons, the trace of $(L_{-m})_-(L_m)_+$ agrees with the trace of its restriction to the span of $t^{-1}, \ldots, t^{-m-1}$, where it agrees with the operator

$$(t^{-m+1}\partial_t) \circ (t^{m+1}\partial_t) = (t\partial_t)^2 + m \cdot t\partial_t.$$

In particular, its trace is given by

$$\sum_{i=1}^{m}(i^2 - i \cdot m) = -\binom{m+1}{3},$$

as desired. □

2.6. **Kac–Moody and Virasoro II: functor of points.**

2.6.1. So far, the methods used in this section produced the Kac–Moody and Virasoro algebras as restricted Lie algebras over R without directly producing their natural topologies; this occurred because, as far as we can tell, Pro($\mathsf{R}-$mod) is not naturally enriched over itself.

Let us now explain how to recover their topologies, or, more precisely, recover them as objects of algebraic geometry. We first recall some convenient terminology.

2.6.2. Recall that an R-*space* $\mathcal{Y}$ is an arbitrary functor from R-algebras to sets. This category is naturally symmetric monoidal with respect to the categorical product of functors, i.e.,

$$\mathcal{Y}_1 \times \mathcal{Y}_2(\mathsf{S}) := \mathcal{Y}_1(\mathsf{S}) \times \mathcal{Y}_2(\mathsf{S}).$$

Using this, one can speak of a group R-space $\mathcal{G}$; explicitly, this consists of the data of a group structure on each $\mathcal{G}(\mathsf{S})$, such that for each $\mathsf{S} \to \mathsf{T}$ the map $\mathcal{G}(\mathsf{S}) \to \mathcal{G}(\mathsf{T})$ is a group homomorphism.

Similarly, recall that one can speak of an R-module structure on an R-space $\mathcal{M}$, this is the datum of 'scaling' and 'addition' maps

$$\mathbb{A}^1_{\mathsf{R}} \times \mathcal{M} \to \mathcal{M} \qquad \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

satisfying the usual identities. Explicitly, an R-module structure on $\mathcal{M}$ is the data of an S-module structure on $\mathcal{M}(\mathsf{S})$, for any R-algebra S, such that for any map $\mathsf{S} \to \mathsf{T}$ the map $\mathcal{M}(\mathsf{S}) \to \mathcal{M}(\mathsf{T})$ is a map of S-modules.

Given an R-module $\mathcal{M}$, we may moreover speak of a restricted Lie algebra structure on $\mathcal{M}$; again this consists of S-linear restricted Lie algebra structures on each $M(\mathsf{S})$, such that for any map of R-algebras $\mathsf{S} \to \mathsf{T}$ the map $M(\mathsf{S}) \to M(\mathsf{T})$ is a S-linear map of restricted Lie algebras.

2.6.3. With these generalities in hand, we return to the problem of reconstructing the topologies on the Kac–Moody and Virasoro algebras, as well as their adjoint actions by the corresponding group ind-schemes.

Given a commutative R-algebra S, note we have a tautological symmetric monoidal functor

$$\mathrm{Pro}(\mathsf{R}-\mathrm{mod}) \to \mathrm{Pro}(\mathsf{S}-\mathrm{mod}), \qquad \varprojlim M_i \mapsto \varprojlim (M_i \otimes_{\mathsf{R}} \mathsf{S}).$$

In particular, given a commutative Hopf algebra $\mathcal{O}_{\mathsf{F}}$ in $\mathrm{Pro}(\mathsf{R}-\mathrm{mod})$, we have the corresponding base changed commutative Hopf algebra $\mathcal{O}_{\mathsf{F}} \otimes_{\mathsf{R}} \mathsf{S}$ in $\mathrm{Pro}(\mathsf{S}-\mathrm{mod})$. As in Lemma 2.13, this induces homomorphisms of abstract groups

$$\mathsf{F}(\mathsf{R}) \to \mathsf{F}(\mathsf{S})$$

and homomorphism of restricted Lie algebras

$$\mathfrak{f}(\mathsf{R}) \to \mathfrak{f}(\mathsf{S}),$$

which intertwines the adjoint actions of $\mathsf{F}(\mathsf{R})$ and $\mathsf{F}(\mathsf{S})$. It is straightforward to see that these assignments are compatible with maps of R-algebras $\mathsf{S} \to \mathsf{T}$, i.e., define a group R-space F and and restricted Lie algebra R-space $\mathfrak{f}$, along with an action of F on $\mathfrak{f}$ by Lie algebra automorphisms.

Moreover, it is straightforward to see that the group R-space F is simply the group ind-scheme corresponding to $\mathcal{O}_{\mathsf{F}}$, and in particular is representable by an ind-scheme.

In particular, if $\mathfrak{f}$ is also representable by an ind-scheme over R, then we deduce that $\mathfrak{f}$ is naturally a restricted Lie algebra in ind-schemes, and F acts on $\mathfrak{f}$ by restricted Lie algebra automorphisms. Finally, we note that the representability of $\mathfrak{f}$ holds in particular for the Kac–Moody and Virasoro Lie algebras, as they are representable by Tate vector spaces.

To proceed, we further observe that if F is a central extension of group R-spaces

$$1 \to Z \to \mathsf{F} \to \overline{\mathsf{F}} \to 1,$$

then tautologically the adjoint action of $\mathsf{F}(\mathsf{S})$ on $\mathfrak{f}(\mathsf{S})$ factors through $\overline{\mathsf{F}}(\mathsf{S})$ for every R-algebra S, i.e., the adjoint action of F on $\mathfrak{f}$ factors through an action of the group R-space $\overline{\mathsf{F}}$.

The relevant corollaries of the preceding discussion are then the following.

Corollary 2.33. *The Kac–Moody Lie algebra, i.e., the restricted Lie algebra in ind-schemes over* R*, which assigns to a* R*-algebra* S *the restricted Lie algebra* $\mathfrak{f}(\!(t)\!)^\flat$ *over* S *as in Remark 2.29, carries a canonical action of the group ind-scheme* $F(\!(t)\!)$ *by restricted Lie algebra automorphisms.*

Corollary 2.34. *The Virasoro Lie algebra, i.e., the restricted Lie algebra in ind-schemes over* R*, which assigns to a* R*-algebra* S *the restricted Lie algebra* $\mathfrak{Vir}(\mathsf{S})$ *as in Section 2.5.5, carries a canonical action of the group ind-scheme* $\mathrm{Aut}(\mathcal{D}^\times)$ *by restricted Lie algebra automorphisms.*

3. Basics on vertex algebras

The goal of this section is to recall some basic definitions, constructions and results concerning vertex algebras over general rings.

3.1. **Vertex algebras.** Let R be a commutative associative unital ring.

3.1.1. The following definition of vertex algebras over R, as far as we know, is due to Borcherds and Ryba [BR].

Definition 3.1. A vertex algebra V over R is an R-module equipped with a map of R-modules

$$V \underset{\mathsf{R}}{\otimes} V \to V(\!(z)\!), \qquad a \otimes b \mapsto Y(a,z)b := \sum_n a_{(n)} b z^{-n-1},$$

(known as the *state-field correspondence*), and a vacuum vector

$$\mathsf{R} \to V, \qquad 1 \mapsto |\varnothing\rangle,$$

satisfying the following identities.

(1) For any $a \in V$, we have that $Y(|\varnothing\rangle, z) = \mathrm{id}$, i.e., that

$$|\varnothing\rangle_n = \begin{cases} \mathrm{id}_V, & \text{if } n = -1, \text{ and} \\ 0, & \text{otherwise}. \end{cases}$$

(2) For any $a \in V$, we have, in addition, that

$$Y(a,z)|\varnothing\rangle \in a + zV[[z]],$$

i.e. that $a_{(n)}|\varnothing\rangle = 0$ for $n \geqslant 0$, and $a_{(-1)}|\varnothing\rangle = a$.

(3) (Locality) For any a, b in V, there exists an integer $N \gg 0$ such that for any c in V we have

$$(z-w)^N \left(Y(a,z)Y(b,w)c - Y(b,w)Y(a,z)c\right) = 0$$

in $V[[z^{\pm 1}, w^{\pm 1}]]$ (the set of series in z, w that are infinite in both directions).

(4) (Translation) For $n \geqslant 0$, define the operator $T^{(n)} : V \to V$ by

$$T^{(n)} a := a_{(-n-1)}|\varnothing\rangle.$$

Then one has the identity

$$T^{(n)} Y(a,z) = \partial_z^{(n)} Y(a,z),$$

i.e., that for any $j \in \mathbb{Z}$ one has the equality of endomorphisms of V

$$(T^{(n)} a)_{(j)} = (-1)^n \binom{j}{n} a_{(j-n)}.$$

Definition 3.2. We say that V is *commutative*, if $a_{(n)} = 0$ for $n \geqslant 0$.

3.1.2. We note that the locality axiom, (3), is equivalent to the following formula, cf. [F, (2.3-7)]:

$$[Y(a,z), Y(b,w)] = \sum_{k \geqslant 0} Y(a_{(\ell)} b, w) \partial_w^{(\ell)} \delta(z-w), \tag{3.1}$$

where $\delta(z-w) = \sum_{m \in \mathbb{Z}} z^m w^{-m-1}$. Equivalently,

$$[a_{(m)}, b_{(n)}] = \sum_{\ell \geqslant 0} \binom{m}{k} (a_{(\ell)} b)_{(m+n-\ell)}. \tag{3.2}$$

We will also need the following identity that holds for $n<0$, cf. [F, (2.3-4)]:

$$Y(a_{(n)}b,z) =: [\partial_z^{(-n-1)}Y(a,z)]Y(b,z):, \tag{3.3}$$

where $:\bullet:$ indicates the normally ordered product.

Below in this section we will give some examples.

3.1.3. We will need to deal with filtrations on vertex algebras.

Definition 3.3. Let V be a vertex algebra over R. By a *filtration* on V we mean an exhaustive ascending R-module filtration $V=\bigcup_{i\geqslant 0}V_{\leqslant i}$ satisfying the following conditions:

(1) $|\varnothing\rangle\in V_{\leqslant 0}$,

(2) and for $a\in V_{\leqslant i}, b\in V_{\leqslant j}$, one has $a_{(n)}b\in V_{\leqslant i+j}$ for all $n\in\mathbb{Z}$.

If in (2), we have $a_{(n)}b\in V_{\leqslant i+j-1}$ for $n\geqslant 0$, then we say that the filtered vertex algebra is *almost commutative*.

3.1.4. We also have two notions of a grading on a vertex algebra, $V=\bigoplus_{i\in\mathbb{Z}}V_i$. We will consider two kind of gradings. An *energy grading* is defined as in [F, Section 2.2.2]: we require that $|0\rangle\in V_0$, while for $a\in V_i, b\in V_j$ we have $a_{(n)}b\in V_{i+j-n}$. We will also consider *naive gradings*, where $a_{(n)}b\in V_{i+j}$. For a filtered vertex algebra, it makes sense to speak about its associated graded, which carries a naive grading. The filtered vertex algebra is almost commutative if and only if its associated graded is commutative.

3.1.5. If V is a filtered vertex algebra over R, then we can form its Rees algebra $R_\hbar(V)$, a naively graded vertex algebra over $\mathsf{R}[\hbar]$. As usual, we have natural isomorphisms $R_\hbar(V)/(\hbar-1)\cong V, R_\hbar(V)/(\hbar)\cong \operatorname{gr}V$. Below we will often write $V^\hbar$ for $R_\hbar(V)$ and V^0 for $\operatorname{gr}V$.

3.1.6. We note that there is an obvious notion of a vertex algebra homomorphism, so vertex algebras form an R-linear category. Thanks to this it makes sense to speak about vertex subalgebras and vertex algebra ideals.

Also, for two vertex algebras V^1,V^2 we can form their tensor product $V^1\underset{\mathsf{R}}{\otimes}V^2$ of R-modules, and this tensor product carries a natural vertex algebra structure (with $Y(a^1\otimes a^2,z)=Y(a^1,z)\otimes Y(a^2,z)$).

Finally, we mention that there is an obvious base change construction: if V is a vertex algebra over R, and $\tilde{\mathsf{R}}$ is a commutative R-algebra, then $\tilde{\mathsf{R}}\otimes_{\mathsf{R}}V$ is naturally a vertex algebra over $\tilde{\mathsf{R}}$.

3.1.7. One can define the notion of a Poisson vertex algebra over R similarly to [ATV, Section 3.3]. By definition, a Poisson vertex algebra is a tuple $(V_0,|\varnothing\rangle,Y_+,Y_-)$ satisfying certain axioms. First, we require that $(V_0,\varnothing,Y_+)$ is a commutative vertex algebra, in particular, giving operators $T^{(n)}$ for all $n\geqslant 0$. The operation $Y_0: V\to z^{-1}\operatorname{End}(V_0)[z^{-1}]$ is a vertex algebra analog of the Poisson bracket. It is required to satisfy the following axioms:

(1) $Y_-(T^{(n)}a,z)=\partial^{(n)}Y_-(a,z)$.

(2) $Y_(a,z)b=\sum_{n=0}^\infty z^nT^{(n)}Y(b,-z)a$.

(3) If we write $Y_-(a,z)$ as $\sum_{n\geqslant 0} a_{(n)}z^{-n-1}$, then

$$[a_{(n)}, b_{(m)}] = \sum_{\ell=0}^{\infty} \binom{m}{\ell} (a_{(\ell)}b)_{(m+n-\ell)}$$

for all $m, n \geqslant 0$.

(4) $Y_-(a,z)(bc) = (Y_-(a,z)b)c + (Y_-(a,z)c)b$.

The main example for us is as follows. Let $(V, |\varnothing\rangle, Y)$ be an almost commutative filtered vertex algebra. We take $V_0 := \operatorname{gr} V$. For Y_+ (resp., Y_-) we take the top degree terms of $\sum_{n\geqslant 0} a_{(-n-1)}z^n$ (resp., $\sum_{n<0} a_{(-n-1)}z^n$). Note that Y_+ has degree 0 and Y_- has degree -1.

3.2. **Universal enveloping algebras.** In this section we will review the construction of the universal enveloping algebra associated to a vertex algebra V. The case when R is a characteristic 0 field can be found, e.g., in [F, Section 3.2], and the general case is easily adapted from there.

3.2.1. First, we construct the Lie algebra of Fourier coefficients F_V. Consider the $\mathsf{R}[t^{\pm 1}]$-module $V[t^{\pm 1}]$. For $n \in \mathbb{Z}_{\geqslant 0}$, define the endomorphism $\partial^{(n)}$ of $V[t^{\pm 1}]$ by

$$\partial^{(n)} = \sum_{j=0}^{n} T^{(j)} \partial_t^{(n-j)},$$

where $\partial_t^{(n-j)}$ is the divided power of the derivative with respect to t.

By the *Lie algebra of Fourier modes* one means

$$F_V := V[t^{\pm 1}] / \left(\sum_{n\geqslant 0} \operatorname{im} \partial^{(n)} \right)$$

We write $a_{[n]}$ for the image of at^n in F_V. Then F_V is the R-module with generators $a_{[n]}$ and relations $(T^{(n)}a)_{[j]} = (-1)^n \binom{j}{n} a_{[j-n]}$.

The bracket on F_V is uniquely recovered from (cf. [F, (3.2-2)])

$$[a_{[m]}, b_{[k]}] = \sum_{n\geqslant 0} \binom{m}{n} (a_{(n)}b)_{[m+k-n]}.$$

The proof that F_V is a Lie algebra with respect to this bracket repeats that of [F, Proposition 3.2.1].

3.2.2. Now assume that $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is an energy graded vertex algebra. We form the completed universal enveloping algebra $\widetilde{U}(F_V)$, the inverse limit

$$\varprojlim_{m\to\infty} U(F_V)/I_m,$$

where I_m is the left ideal in $U(F_V)$ spanned by the element $a_{[n]}$, where $a \in V_i$ with $n \geqslant m+i$. This is an associative algebra. The (completed) universal enveloping algebra $\widetilde{U}(V)$ is then defined as the quotient of $\widetilde{U}(F_V)$ by relations (3.4) below. To state this relation we introduce the notation $Y[a,z] := \sum_{n\in\mathbb{Z}} a_{[n]}z^{-n-1} \in F_V[[z^{-1}, z]]$. The relations are as follows:

$$Y[a_{-1}b, z] =: Y[a,z]Y[b,z] : . \tag{3.4}$$

We note that the assignment $V \mapsto \widetilde{U}(V)$ is a functor from the category of energy graded vertex algebras to the category of topological associative algebras. We also point out that for two vertex algebras V^1, V^2, we have a natural isomorphism of topological algebras (where in the source we have the completed tensor product):

$$\widetilde{U}(V^1) \underset{\mathsf{R}}{\widetilde{\otimes}} \widetilde{U}(V^2) \xrightarrow{\sim} \widetilde{U}(V^1 \underset{\mathsf{R}}{\otimes} V^2).$$

This is a direct check using the definitions and is left to the reader.

3.2.3. Now we discuss a variant of $\widetilde{U}(V)$ in the presence of filtrations. Suppose that V is, in addition, filtered, $V = \bigcup_j V_{\leqslant j}$, where the filtration is compatible with the energy grading in such a way that $V_{\leqslant j}$ is graded for all j. This gives rise to an exhaustive Lie algebra filtration on F_V (with $F_{V,\leqslant j}$ being the span of $a_{[n]}$ for $n \in \mathbb{Z}$ and $a \in V_j$) and hence to an ascending associative algebra filtration $\widetilde{U}(V)_{\leqslant i} \subset \widetilde{U}(V)$. We note that the latter is non-exhaustive, and we define $\widehat{U}(V)$ as $\bigcup_i \widetilde{U}(V)_{\leqslant i}$. This is a topological algebra but it is not complete. Instead, it is *filtered complete* in the sense of the following definition.

Definition 3.4. Let A be a topological associative algebra equipped with an exhaustive ascending algebra filtation $A = \bigcup_{i \geqslant 0} A_{\leqslant i}$. We say that A is *filtered complete* if the topology on each $A_{\leqslant i}$ is complete and separated. Similarly, if A is equipped with a grading, $A = \bigoplus_{i \geqslant 0} A_i$, then we say that A is *graded complete* if the topology on each A_i is complete and separated.

The naive grading on $\operatorname{gr} V$ induces a grading on $\widehat{S}(\operatorname{gr} V) := \widehat{U}(\operatorname{gr} V)$ and we have a natural epimorphism $\widehat{S}(\operatorname{gr} V) \twoheadrightarrow \operatorname{gr} \widehat{U}(V)$. The algebra $\widehat{S}(\operatorname{gr} V)$ is graded complete. Note that if the filtration on V is almost commutative, then so is the filtration on $\widehat{U}(V)$.

The grading on $R_\hbar(V)$ gives rise to a grading on $\widehat{U}^\hbar(V) := \widehat{U}(R_\hbar(V))$, and $\widehat{U}_\hbar(V)$ is identified with the Rees algebra of $\widehat{U}(V)$.

3.2.4. Suppose that $\varphi : V^1 \to V^2$ be a filtration preserving homomorphism between filtered vertex algebras. Set $\Phi := \widehat{U}(\varphi) : \widehat{U}(V^1) \to \widehat{U}(V^2)$. Here we explain how to recover V^i from $\widehat{U}(V^i)$ and φ from Φ. One can identify V^i with the quotient of $\widehat{U}(V^i)$ by the left ideal generated by the images of the elements $a_{[n]}$ for $a \in V^i$ with $n \geqslant 0$. The homomorphism Φ sends the image of $a_{[n]}$ to the image of $\varphi(a)_{[n]}$ for all n and hence induces a $\widehat{U}(V^1)$-module homomorphism $V^1 \to V^2$. It is easy to see that the induced homomorphism coincides with φ.

3.3. **Arc and loop spaces and groups.** Let X be a finite type affine scheme over R.

3.3.1. Recall that we have another affine R-scheme $\mathcal{J}X$, the *jet* or *arc space* of X, representing the functor $\mathsf{S} \mapsto X(\mathsf{S}[[t]])$. This is an affine scheme of infinite type. This scheme, by construction, is the inverse limit of the finite type schemes $\mathcal{J}_n X$, the nth jet schemes.

Note that $\mathsf{R}[\mathcal{J}X]$ carries a natural structure of a commutative vertex algebra. In order to see this, we first equip $\mathsf{R}[\mathcal{J}\mathbb{A}^n]$ with operators $T^{(\ell)}$ for $\ell \geqslant 0$. Namely, let $x^1, \ldots, x^n$ are the coordinates on $\mathbb{A}^n$. Then $\mathcal{J}\mathbb{A}^n$ is an infinite dimensional affine

space with coordinates x^i_j with $i = 1, \dots, n$, and $j < 0$. Then there is a unique collection of operators $T^{(\ell)} : \mathsf{R}[\mathcal{J}\mathbb{A}^n] \to \mathsf{R}[\mathcal{J}\mathbb{A}^n]$ subject to the following conditions

- $T^{(\ell)} x^i_j = (-1)^\ell \binom{j}{\ell} x^i_{j-\ell}$,
- $T^{(\ell)}(fg) = \sum_{j=0}^k (T^{(j)} f)(T^{(\ell-j)} g)$.

Now suppose that X is given in $\mathbb{A}^n$ by equations $F^1, \dots, F^\ell$. Then $\mathcal{J}X$ is given in $\mathcal{J}\mathbb{A}^n$ by equations $T^{(\ell)} F(x^1_{-1}, \dots, x^n_{-1})$. In particular, the operators $T^{(\ell)}$ descend to $\mathsf{R}[\mathcal{J}X]$. Moreover, they are independent of the choice of an embedding $X \hookrightarrow \mathbb{A}^n$.

Now we can define a vertex algebra structure on $\mathsf{R}[\mathcal{J}X]$: we set

$$Y(a, z)b = (\sum_{j=0}^{\infty} T^{(j)} a z^j) b.$$

Remark 3.5. Here is a coordinate independent way to think about the vertex algebra structure on $\mathsf{R}[\mathcal{J}X]$. For $a \in \mathsf{R}[\mathcal{J}X]$, we define $a_{-z} \in \mathsf{R}[\mathcal{J}X][[z]]$ as follows. Let $\tilde{\mathsf{R}}$ be an R-algebra. Then an $\tilde{\mathsf{R}}$-point of X is an $\tilde{\mathsf{R}}[[t]]$-point of X, denote it by $x(t)$. We define a_{-z} by $x(t) \mapsto a(x(t-z)) \in \tilde{R}[[z]]$. Equivalently, $a_{-z} = (\sum_{j=0}^{\infty} z^j T^{(j)}) a$, so $Y(a, z)$ is the multiplication by a_{-z}.

The assignments $X \mapsto \mathcal{J}X, \mathcal{J}_n X$ are easily seen to be functorial, and $\mathcal{J}(X \times Y)$ is naturally identified with $\mathcal{J}X \times \mathcal{J}Y$, the same for $\mathcal{J}_n$.

Remark 3.5 implies that $X \to \mathsf{R}[\mathcal{J}X]$ can be viewed as a functor to the category of commutative vertex algebras.

3.3.2. We now discuss arc groups.

Remark 3.6. If F is a group scheme over R, then $\mathcal{J}F$ and $\mathcal{J}_n F$ are also group schemes over R. It is straightforward to check that the coproduct $\mathsf{R}[\mathcal{J}F] \to \mathsf{R}[\mathcal{J}F] \underset{\mathsf{R}}{\otimes} \mathsf{R}[\mathcal{J}F]$ is a vertex algebra homomorphism. It follows that $\mathsf{R}[\mathcal{J}F]$ is a group object in the category of vertex algebra.

We will mostly care about the situation when F is a split connected reductive group G over R. We have the following lemma.

Lemma 3.7. *The following claims hold:*

(1) The scheme $\mathcal{J}_n G$ is smooth over $\mathrm{Spec}(\mathsf{R})$.

(2) The scheme $\mathcal{J}G$ is flat over $\mathrm{Spec}(\mathsf{R})$

Proof. We note that G is smooth over $\mathrm{Spec}(\mathsf{R})$ (to see this, one can, for example, cover G by translates of the open Bruhat cell, which is smooth as the product of an affine space and an algebraic torus). By the infinitesimal lifting property, $\mathcal{J}_n G$ is smooth over G, and (1) follows. (2) follows from (1) because $\mathsf{R}[\mathcal{J}G]$ is a filtered colimit of the algebras $\mathsf{R}[\mathcal{J}_n G]$. □

3.3.3. Now we proceed to loop spaces and loop groups. As before, let X be a finite type affine scheme. We write $\mathcal{L}X$ for the loop space of X. This is a strict ind-affine ind-scheme of countable type, so it is given by its algebra of functions,

that is a complete and separated topological algebra with a countable basis of open neighborhoods of zero that are ideals.

Lemma 3.8. *The algebra* $\mathsf{R}[\mathcal{L}X]$ *is naturally identified with the universal enveloping algebra* $\widetilde{U}(\mathsf{R}[\mathcal{J}X])$.

Proof. One first checks this for $X = \mathbb{A}^n$ and then shows that $\mathsf{R}[\mathcal{L}X]$ and $\widetilde{U}(\mathsf{R}[\mathcal{J}X])$ are quotients of $\mathsf{R}[\mathcal{L}\mathbb{A}^n] = \widetilde{U}(\mathsf{R}[\mathcal{J}\mathbb{A}^n])$. □

Of course, if $X = F$ is a group scheme over R, then $\mathcal{L}F$ is the same $F(\!(t)\!)$ from Section 2.5.2.

3.3.4. Assume now that $\mathsf{R}[X]$ is a Poisson algebra. The axioms for Y_- listed in Section 3.1.7 imply that there is at most one structure of a Poisson vertex algebra on $\mathsf{R}[JX]$ such that for $a, b \in \mathsf{R}[X] \subset \mathsf{R}[JX]$ we have $a_{(0)}b = \{a, b\}$. In fact, one can show that this structure always exists, but we will not need this.

3.4. **Affine vertex algebras.** We now proceed to the most important class of vertex algebras appearing in this paper.

3.4.1. Assume 2 is invertible in R. Let $\mathfrak{f}$ be a Lie algebra over R that is a finitely generated projective R. Fix a representation W of $\mathfrak{f}$, a finitely generated projective R-module. Let β_W denote the trace form for this representation (in particular, $\beta_{\mathfrak{f}}$ is the Killing form). Pick $\kappa \in \mathsf{R}$. We define the loop algebra $\mathfrak{f}(\!(t)\!)$ and its central extension $\hat{\mathfrak{f}} = \mathfrak{f}((t)) \oplus \mathsf{R}\mathbf{1} \oplus \mathsf{R}\mathbf{1}'$, where $\mathbf{1}, \mathbf{1}'$ are central and for $x_1 f_1, x_2 f_2 \in \mathfrak{f}(\!(t)\!)$ (with $x_i \in \mathfrak{f}, f_i \in \mathsf{R}(\!(t)\!)$) their bracket in $\hat{\mathfrak{f}}$ is given by

$$[x_1 f_1, x_2 f_2] := [x_1, x_2] f_1 f_2 - \mathrm{Res}_{t=0}(f_1 df_2)(\beta_W(x_1, x_2)\mathbf{1} - \frac{1}{2}\beta_{\mathfrak{f}}(x_1, x_2)\mathbf{1}'). \tag{3.5}$$

The algebra $\hat{\mathfrak{f}}$ is the *affine Lie algebra* associated to $\mathfrak{f}$ and κ.

Remark 3.9. We will be interested in the class that appeared in Section 2.5.3: when $\mathfrak{f}$ is the Lie algebra of an algebraic group F and W is a representation of F.

From $\hat{\mathfrak{f}}$ we can produce the *affine vertex algebra* to be denoted by $V_\kappa(\mathfrak{f})$. Namely, consider the $\mathfrak{f}[[t]] \oplus \mathsf{R}\mathbf{1}$-module R_κ, where $\mathfrak{f}[[t]]$ acts by 0, $\mathbf{1}$ acts by κ and $\mathbb{1}'$ acts by 1. Then the induced module $V_\kappa(\mathfrak{f})$ comes with a natural energy $\mathbb{Z}_{\leqslant 0}$-graded vertex algebra structure, cf. [F, Theorem 2.2.2] and [ATV, Section 3.4].

We can also consider the universal version $V_{\mathbf{1}}(\mathfrak{f})$, the induction of R from $\mathfrak{f}[[t]] \oplus \mathsf{R}\mathbf{1}'$, where $\mathbf{1}'$ acts by 1. This is a vertex algebra over $\mathsf{R}[\mathbf{1}]$. Specializing $\mathbf{1}$ to κ, we recover $V_\kappa(\mathfrak{f})$.

3.4.2. The standard PBW filtration on $U(\hat{\mathfrak{f}})$ gives rise to a filtration on $V_\kappa(\mathfrak{f})$ and this is easily seen to be a vertex algebra filtration. The associated graded is identified with $\mathsf{R}[\mathcal{J}\mathfrak{f}^*]$.

In the same way, we get a filtration on $V_{\mathbf{1}}(\mathfrak{f})$ (note that $\mathbf{1}$ is in filtration degree 1). The algebra $\mathrm{gr}\, V_{\mathbf{1}}(\mathfrak{f})$ can be interpreted as the algebra of regular functions on the space $\mathbb{A}^1 \times \mathcal{J}\mathfrak{f}^*$ that can be interpreted as the space of "λ-connections" $\{\lambda\partial + \mathfrak{f}^*[[t]]dt\}$.

3.4.3. Now we turn to the universal enveloping algebra of $V_\kappa(\mathbb{F})$. Consider the algebra $U_\kappa(\mathfrak{f}) := U(\hat{\mathfrak{f}})/(\mathbf{1}-\kappa, \mathbf{1}'-1)$, its completion $\widetilde{U}_\kappa(\hat{\mathfrak{f}})$ and its filtered part $\widehat{U}_\kappa(\hat{\mathfrak{f}})$. We note that the filtration on each filtered piece is complete and separated.

Similarly to [F, Lemma 3.2.2], we see that there is a natural topological algebra isomorphism

$$\widetilde{U}(V_\kappa(\mathfrak{f})) \xrightarrow{\sim} \widetilde{U}_\kappa(\hat{\mathfrak{f}}).$$

This isomorphism is compatible with the filtrations yielding a filtered algebra isomorphism

$$\widehat{U}(V_\kappa(\mathfrak{f})) \xrightarrow{\sim} \widehat{U}_\kappa(\hat{\mathfrak{f}}).$$

3.4.4. Now suppose that F is an algebraic group scheme over R with Lie algebra $\mathfrak{f}$. We write $F(\mathsf{R}(\!(t)\!))$ for the group of $\mathsf{R}(\!(t)\!)$-point of F. Suppose that β is F-invariant. Then the group $F(\mathsf{R}(\!(t)\!))$ acts on $\hat{\mathfrak{f}}$ by Lie algebra automorphisms via

$$g(t).x(t) = \mathrm{Ad}(g(t))x(t) + \mathrm{Res}_{t=0}(\beta_W \mathbf{1} - \frac{1}{2}\beta_{\mathfrak{f}}\mathbf{1}')([\partial_t g(t)]g(t)^{-1}, x(t)). \tag{3.6}$$

Here the first summand is for the adjoint action of $F(\mathsf{R}(\!(t)\!))$ and in the second we view $[\partial_t g(t)]g(t)^{-1}$ as an element of $\mathfrak{f}(\!(t)\!)$. Note that the action of $F(R((t)))$ extends to actions on $\widetilde{U}_\kappa(\hat{\mathfrak{f}})$ and $\widehat{U}_\kappa(\hat{\mathfrak{f}})$.

3.5. **Group actions and invariants.** Here we discuss an appropriate notion of an arc group action on a vertex algebra, the invariants for the action, and the interaction of taking the invariants and taking the universal enveloping algebra.

3.5.1. Let R be a commutative ring, V be a vertex algebra over R and F be a finite type affine algebraic group scheme over R. As we mentioned in Remark 3.6, $\mathsf{R}[\mathcal{J}F]$ is a group object in the category of vertex algebras.

Definition 3.10. By a *vertex $\mathcal{J}F$-action* on V we mean a vertex algebra homomorphism $\alpha : V \to V \underset{\mathsf{R}}{\otimes} \mathsf{R}[\mathcal{J}F]$ subject to the usual coassociativity and the counit axioms.

Now we reformulate this definition in terms of the action map (rather than the co-action).

Definition 3.11. (1) We say that the action of $\mathcal{J}F$ on an R-module V is *rational* if every element of V lies in a submodule V', where the action of $\mathcal{J}G$ factors through a rational representation of $\mathcal{J}_n F$.

(2) Let V be a vertex algebra. We say that a rational action of $\mathcal{J}F$ on V is a vertex action if for any point $g(t) \in \mathcal{J}F_{\tilde{\mathsf{R}}}$ (for an R-algebra $\tilde{\mathsf{R}}$) and any $a \in V$, we have

$$Y(g(t).a, z) = g(t-z)Y(a,z)g(t-z)^{-1}, \tag{3.7}$$

an equality in $\left(\mathrm{End}_{\tilde{\mathsf{R}}}(V \otimes_{\mathsf{R}} \tilde{\mathsf{R}})\right)[[z^{\pm 1}]]$.

The claim that the two definitions are equivalent is an easy consequence of Remark 3.5.

3.5.2. Recall that, over a characteristic 0 field, a rational representation of an algebraic group F in an algebra is by automorphisms if and only if the corresponding action of the Lie algebra $\mathfrak{f}$ is by derivations. We will need an analog of this claim in the vertex algebra setting.

Lemma 3.12. *Suppose that F is connected, and R is a characteristic 0 field. Suppose V is a vertex algebra equipped with a rational representation of $\mathcal{J}F$. Then the following conditions are equivalent:*

(1) The action of $\mathcal{J}F$ in V is a vertex action,

(2) for all $x \in \mathfrak{f}, m \in \mathbb{Z}_{\geqslant 0}, b \in V, n \in \mathbb{Z}$, we have

$$[xt^m, b_{(n)}] = \sum_{\ell \geqslant 0} \binom{m}{\ell} (xt^\ell.b)_{(m+n-\ell)}. \tag{3.8}$$

Proof. We note that it is enough to check (3.7) for the cases when g is constant, and $g(0) = 1$ separately. (3.7) for all constant g is equivalent to (3.8) for all x and $m = 0$. Now note that the elements g with $g(0) = 1$ are uniquely represented in the form $\exp(\xi)$ for $\xi \in t\mathfrak{f}[[t]]$. Now it is straightforward to check that (3.7) for all g satisfying $g(0) = 1$ is equivalent to (3.8) for all $x \in \mathfrak{f}$ and all $m > 0$. This completes the proof. □

Remark 3.13. Suppose there is a vertex algebra homomorphism $\varphi : V_\beta(\mathfrak{f}) \to V$. Then we get an action of $\mathfrak{f}[[t]]$ on V via $xt^m.b := \varphi(xt^{-1})_{(m)}b$. Then (3.8) follows from (3.2).

3.5.3. Now we discuss an example of this situation. Suppose R is a domain whose field of fractions has characteristic 0. Suppose that F is a smooth connected finite type affine group scheme over R. By the construction of $V_\kappa(\mathfrak{f})$ as the induced module, we have an action of F on the R-module $V_\kappa(\mathfrak{f})$. Note that this action is rational: observe that $V_\beta(\mathfrak{f})$ is $\mathbb{Z}_{\leqslant 0}$-graded via the energy grading. The action of $\mathcal{J}F$ preserves $\bigoplus_{j \geqslant j_0} V_\kappa(\mathfrak{f})^j$ for all j_0. Moreover, on the action on this R-submodule factors through a rational action of $\mathcal{J}_n F$ for n sufficiently large.

Lemma 3.14. *The action of $\mathcal{J}F$ on $V_\kappa(\mathfrak{f})$ is a vertex action.*

Proof. We need to show that the coaction map $V_\kappa(\mathfrak{f}) \to V_\kappa(\mathfrak{f}) \otimes \mathsf{R}[\mathcal{J}F]$ is a vertex algebra homomorphism. This amounts to checking some collection of polynomial equations. We split the check into three cases:

Case 1: R is a characteristic 0 field. We note that the action of $\mathfrak{f}[[t]]$ on $V_\beta(\mathfrak{f})$ arises as in Remark 3.13. Our claim follows from Lemma 3.12.

Case 2: General case. We note $V_\kappa(\mathfrak{f})$ is flat over R – this follows from the definition. By Lemma 3.7, $\mathsf{R}[\mathcal{J}\mathsf{R}]$ is also flat over R. So this case follows from the case of $\mathrm{Frac}(\mathsf{R})$. □

Similarly, the action of $\mathcal{J}F$ on $V_{\mathbf{1}}(\mathfrak{f})$ is a vertex action.

3.5.4. Now we get back to the general situation where we have action of $\mathcal{J}F$ on V with co-action map $\alpha : V \to V \underset{\mathsf{R}}{\otimes} \mathsf{R}[\mathcal{J}F]$. An element $v \in V$ is called *$\mathcal{J}F$-invariant* if $\alpha(v) = v \otimes 1$. It is an easy exercise that the invariants form a vertex subalgebra in V to be denoted by $V^{\mathcal{J}F}$.

Lemma 3.15. *The action of $\mathcal{J}F$ on V gives rise to an action of the loop group $\mathcal{L}F$ on the completed universal enveloping algebra $\tilde{U}(V)$. Moreover, the image of the natural homomorphism $\tilde{U}(V^{\mathcal{J}F}) \to \tilde{U}(V)$ lies in the subalgebra of invariants $\tilde{U}(V)^{\mathcal{L}F}$.*

Proof. We write $\widetilde{U}$ for the functor of taking universal enveloping algebras of vertex algebras. The co-action map $\alpha : V \to V \underset{\mathsf{R}}{\otimes} \mathsf{R}[\mathcal{J}F]$ gives rise to

$$\widetilde{U}(\alpha) : \widetilde{U}(V) \to \widetilde{U}(V \underset{\mathsf{R}}{\otimes} \mathsf{R}[\mathcal{J}F]) = \widetilde{U}(V) \underset{\mathsf{R}}{\widehat{\otimes}} \widetilde{U}(\mathsf{R}[\mathcal{J}F])).$$

By Lemma 3.8, $\widetilde{U}(\mathsf{R}[\mathcal{J}F])) = \mathsf{R}[\mathcal{L}F]$. It is easy to show that $\widetilde{U}(\alpha)$ is a co-action map. Let ι denote the inclusion map $V^{\mathcal{J}F} \hookrightarrow V$. It is easy to see that the image of $\widetilde{U}(\iota)$ consists of elements x satisfying $\widetilde{U}(x) = x \otimes 1$, which implies the claim of the lemma. □

3.5.5. Now suppose that V is a filtered vertex algebra. Equip $\mathsf{R}[\mathcal{J}F]$ with the trivial filtration, where all elements are of degree 0. The action of $\mathcal{J}F$ on V preserves the filtration if and only if the co-action map α is a homomorphism of filtered vertex algebras (where we take the trivial filtration on $\mathsf{R}[\mathcal{J}F]$). In this case the action of $\mathcal{L}F$ on $\widetilde{U}(V)$ preserves $\widehat{U}(V)$.

For example, if $V = V_\kappa(\mathfrak{f})$, then the action of $\mathcal{J}F$ preserves the filtration. We recover the actions of $\mathcal{L}F$ on $\widetilde{U}_\kappa(\hat{\mathfrak{f}}), \widehat{U}_\kappa(\hat{\mathfrak{f}})$ that on the level of R-points were discussed in Section 3.4.

3.6. **Chiral differential operators.** In this section we introduce another important class of vertex algebras, the algebras of chiral differential operators.

In the entire section we suppose that R satisfies the following conditions:

- R is a domain,
- and $\mathbb{K} := \mathrm{Frac}(\mathsf{R})$ is a characteristic 0 field.

We also assume that F is a smooth finite type affine group scheme over $\mathrm{Spec}(\mathsf{R})$.

Let W have the same meaning as in Remark 3.9. Form the Lie algebra $\hat{\mathfrak{f}}$, see Section 3.4.

3.6.1. *Construction.* Note that $\mathcal{J}\mathfrak{f} = \mathfrak{f}[[t]]$ acts on $\mathsf{R}[\mathcal{J}F]$ (in fact, in two different ways: via right – for the action from the left – and left invariant vector fields; we are now interested in the former action). So we can form the induced module

$$\mathsf{CDO}_\kappa(F) := \mathrm{Ind}_{\mathfrak{f}[[t]] \oplus \mathsf{R}\mathbf{1} \oplus \mathsf{R}\mathbf{1}'}^{\hat{\mathfrak{f}}} \mathsf{R}[\mathcal{J}F].$$

where $\mathbf{1}$ acts by κ and $\mathbf{1}'$ acts by 1. By definition, we have an R-linear identification

$$U(t^{-1}\mathfrak{f}[t^{-1}]) \otimes \mathsf{R}[\mathcal{J}F] \xrightarrow{\sim} \mathsf{CDO}_\kappa(F). \tag{3.9}$$

In particular, together with Lemma 3.7 this implies that $\mathsf{CDO}_\kappa(F)$ is flat as a module over R.

We need to equip $\mathsf{CDO}_\kappa(F)$ with a vertex algebra structure. Define the vacuum element in $\mathsf{CDO}_\kappa(F)$ as the image of $1\otimes 1$ under (3.9).

We will explain how to define the fields $Y(xt^{-1},z)$ for $x\in\mathfrak{f}$ and $Y(f,z)$ for $f\in\mathsf{R}[\mathcal{J}F]$, here we slightly abuse the notation and write xt^{-1} for the image of $(xt^{-1})\otimes 1$ under (3.9). To define $Y(xt^{-1},(z))$ is easy:

$$Y(xt^{-1},(z))=\sum_{i\in\mathbb{Z}}[xt^i]z^{-(i+1)},$$

these fields make sense for any smooth $\hat{\mathfrak{f}}$-module, including for $\mathsf{CDO}_\kappa(F)$. To define $Y(f,z)$ is more tricky. First, recall from Section 3.3 that $\mathsf{R}[\mathcal{J}F]$ is a commutative vertex algebra. We want it to be a vertex subalgebra, which defines $Y(f,z)$ on $\mathsf{R}[\mathcal{J}F]$. To extend $Y(f,z)$ to $\mathsf{CDO}_\beta(F)$, we note that thanks to (3.9) it's enough to specify the commutator

$$[Y(xt^{-1}z),Y(f,w)].$$

By formula (3.1), in a vertex algebra,

$$[Y(xt^{-1}z),Y(f,w)]=\sum_{\ell\geqslant 0}Y([xt^{-1}]_{(\ell)}f,w)\partial_w^{(\ell)}\delta(z-w). \tag{3.10}$$

Note that $[xt^{-1}]_{(\ell)}f$ is already defined (and equal to $xt^\ell.f$ for the $\mathcal{J}\mathfrak{f}$-module structure on $\mathsf{R}[\mathcal{J}F]$). So, (3.10) specifies $Y(f,z)$ uniquely.

Lemma 3.16. *There is a unique vertex algebra structure on $\mathsf{CDO}_\kappa(F)$ satisfying the properties listed above in this section.*

Proof. Using (3.9) and (3.3) allows to uniquely extend the assignment $a\mapsto Y(a,z)$ to any element $a\in\mathsf{CDO}_\kappa(F)$. It follows from [AG, Section 3.3] that $\mathsf{CDO}_\kappa(F)$ becomes a vertex algebra after changing the base to $\mathbb{K}$. Since $\mathsf{CDO}_\kappa(F)$ is flat over R (as was remarked after (3.9)), it follows that the axioms in Definition 3.1, so $\mathsf{CDO}_\kappa(F)$ is indeed a vertex algebra. □

Remark 3.17. Note that $\mathsf{R}\subset\mathsf{R}[\mathcal{J}F]$ (the constants) is an $\mathfrak{f}[[t]]$-submodule. This gives rise to an embedding $\iota_\ell: V_\kappa(\mathfrak{f})\hookrightarrow\mathsf{CDO}_\kappa(F)$ of $\hat{\mathfrak{f}}_\kappa$-modules. The construction of the vertex algebra structure on $\mathsf{CDO}_\kappa(F)$ implies that this is an embedding of vertex algebras.

3.6.2. *Localization and filtration.* Here we introduce a certain localization of $\mathsf{CDO}_\kappa(F)$ equip it with a (naive) filtration and study the associated graded.

Let F^0 be an open affine (say, principal) subset of F. Then $\mathcal{J}\mathfrak{f}$ still acts on $\mathsf{R}[\mathcal{J}F^0]$, so we can form the induced module

$$\mathrm{Ind}_{\mathfrak{f}[[t]]\oplus\mathsf{R}\mathbf{1}\oplus\mathsf{R}\mathbf{1}'}^{\hat{\mathfrak{f}}}\mathsf{R}[\mathcal{J}F^0].$$

Just as in Section 3.6.1, this induced module carries a natural vertex algebra structure. The resulting vertex algebra will be denoted by $\mathsf{CDO}_\kappa(F^0)$. Note that the natural $\hat{\mathfrak{f}}$-module embedding $\mathsf{CDO}_\kappa(F)\hookrightarrow\mathsf{CDO}_\kappa(F^0)$ is a vertex algebra embedding. The target should be thought of as the localization of $\mathsf{CDO}_\kappa(F)$.

The vertex algebra $\mathsf{CDO}_\kappa(F^0)$ is filtered by the degree of the differential operator. This is the filtration on the induced module coming from the PBW filtration on $U(\hat{\mathfrak{f}}_\kappa)$ (with $\mathsf{R}[\mathcal{J}F^0]$ in degree 0). The resulting filtered vertex algebra is easily seen to be almost commutative. Let $\mathsf{CDO}_\kappa(F)_{\leqslant m}$ denote the degree $\leqslant m$ piece.

Note that we have vertex algebra homomorphisms

$$\operatorname{gr} V_\kappa(\mathfrak{f}), \mathsf{R}[\mathcal{J}F^0] \hookrightarrow \operatorname{gr} \mathsf{CDO}_\kappa(F^0).$$

By considering the associated graded homomorphism of (3.9), we easily see that that the induced homomorphism of commutative vertex algebras $\operatorname{gr} V_\kappa(\mathfrak{f}) \otimes \mathsf{R}[\mathcal{J}F^0] \to \operatorname{gr} \mathsf{CDO}_\kappa(F^0)$ is an isomorphism. The source vertex algebra is nothing else but $\mathsf{R}[\mathcal{J}T^*F]$. So we get

$$\mathsf{R}[\mathcal{J}(T^*F^0)] \xrightarrow{\sim} \operatorname{gr} \mathsf{CDO}_\kappa(F^0). \tag{3.11}$$

Note that we can trivialize T^*F using left-invariant vector fields, then $\mathcal{J}(T^*F^0) \cong \mathcal{J}F^0 \times \mathcal{J}\mathfrak{f}^*$.

3.6.3. Similarly we can consider the universal version $\mathsf{CDO}_{\mathbf{1}}(F)$, it is induced from the $\mathfrak{f}[[t]] \oplus \mathsf{R}\mathbf{1}$-module R, and is a vertex algebra over $\mathsf{R}[\mathbf{1}]$. It admits a vertex algebra embedding ι_L from $V_{\mathbf{1}}(\mathfrak{f})$. It also admits a filtration induced from the PBW filtration on $U(\hat{\mathfrak{f}})$. And similarly to Section 3.6.2 we can consider the localization $\mathsf{CDO}_{\mathbf{1}}(F^0)$. The associated graded $\operatorname{gr} \mathsf{CDO}_{\mathbf{1}}(F^0)$ is the algebra of regular functions on $\mathcal{J}F^0 \times \{\lambda\partial + \mathfrak{f}^*[[t]]dt\}$.

3.6.4. *The case of graded unipotent groups.* Consider the case when $F := N$ is a unipotent group over R. We are interested in the structure of $\mathsf{CDO}(N) := \mathsf{CDO}_0(N)$ – we want to identify it with the vertex algebra known as the $\beta\gamma$-system (one could also call it the Weyl vertex algebra).

We are going to make simplifying assumptions:

(*) $\mathbb{G}_m$ acts on N by group automorphisms such that the induced grading on $\mathsf{R}[N]$ is negative (meaning $\mathsf{R}[N] = \mathsf{R} \oplus \bigoplus_{i<0} \mathsf{R}[N]^i$).

The main claim of this section, Proposition 3.18 should hold without (*) (and this is known over characteristic 0 fields) but the proof simplifies if we impose (*) and all groups we care about satisfy (*).

For example, suppose G is a split connected reductive group over R. If $P \subset G$ is a standard parabolic subgroup, we can take $N := \operatorname{Rad}_u(P)$ and take the $\mathbb{G}_m$-action corresponding to the co-character $2\rho^\vee$.

Choose free homogeneous generators $y^1, \dots, y^\ell$ of $\mathsf{R}[N]$. Let $y_1, \dots, y_\ell \in \mathfrak{n}$ denote the dual basis to $d_1y^1, \dots, d_1y^\ell$. Then $\mathsf{R}[\mathcal{J}N] = \mathsf{R}[y^i_n]$, where $i = 1, \dots, \ell$ and $n \leqslant 0$ (before the generators of the functions on jets were labelled by negative integers, but here it is convenient to shift the numbering by 1). The grading on $\mathsf{R}[N]$ gives rise to a grading on $\mathsf{CDO}(N)$ so that (3.9) is graded. This is a vertex algebra grading. The subalgebra $\mathsf{R}[\mathcal{J}N]$ is negatively graded (with $\deg T^{(i)} = -i$), while $V(\mathfrak{n})$ is positively graded.

Let $\partial_1, \dots, \partial_\ell$ be the constant vector fields of $N \cong \mathbb{A}^\ell$ corresponding to the free generators $y^1, \dots, y^\ell$ of $\mathsf{R}[N]$. Let ${}^Ry_1, \dots, {}^Ry_\ell$ be the right-invariant vector fields corresponding to $y_1, \dots, y_\ell$, equivalently, the images of $y_1, \dots, y_\ell$ under the homomorphism $\mathfrak{n} \to \mathsf{Vect}(N)$ corresponding to the action of N on itself from the

left. Then we can find unique elements $f_{ij} \in \mathsf{R}[N]$ satisfying $\partial_i = \sum_{j=1}^{\ell} f_{ij}\, {}^R y_j$ for all $i = 1, \dots, \ell$. Consider the following elements of $\mathsf{CDO}(N)$:

$$\partial_{i,-1} = \sum_{j=1}^{\ell} (f_{ij})_{(-1)} \iota_\ell(y_i t^{-1}). \tag{3.12}$$

Here we view f_{ij}'s as elements of $\mathsf{R}[\mathcal{J}N]$ via the inclusion $\mathsf{R}[N] \hookrightarrow \mathsf{R}[\mathcal{J}N]$.

Proposition 3.18. *The following equalities hold:*

(1) $[Y(y_0^i, z), Y(y_0^j, w)] = 0, \forall i, j.$

(2) $[Y(\partial_{i,-1}, z), Y(y_0^j, w)] = \delta_{ij}\delta(z-w), \forall i, j.$

(3) $[Y(\partial_{i,-1}, z), Y(\partial_{j,-1}, w)] = 0, \forall i, j.$

Over a characteristic 0 field, this is standard. We give a general proof for reader's convenience.

Proof. Part 1. This follows because $\mathsf{R}[\mathcal{J}N]$ is a commutative vertex algebra.

Part 2. We have

$$[Y(\partial_{i,-1}, z), Y(y_0^j, w)] = \sum_{k \geqslant 0} Y([\partial_i]_{(k)} y_0^j, w) \partial_w^{(k)} \delta(z-w)$$

Let's compute $[\partial_i]_{(k)} y_0^j$, i.e., the coefficient of z^{-n-1} in $Y(\partial_{i,-1}, z) y_0^j$. Thanks to (3.12), we have

$$Y(\partial_{i,-1}, z) y_0^j = \sum_{h=1}^{\ell} : Y(f_{ih,0}, z) Y(\iota_\ell(y_i t^{-1}), z) : y_0^j \tag{3.13}$$

Note that $f_{ih,(k)} y_0^j = 0$ for $k \geqslant 0$ because $\mathsf{R}[\mathcal{J}N]$ is a commutative vertex algebra. It turns out that $\iota_\ell(y_h t^{-1})_{(k)} y_0^j = 0$ for $k > 0$. In order to see this, we will need the following two observations:

(a) $y_0^j \in \mathsf{R}[N] \subset \mathsf{R}[\mathcal{J}N]$,

(b) and the endomorphism $\iota_\ell(y_h t^{-1})_{(k)}$ of $\mathsf{R}[\mathcal{J}N]$ is the action of the element $y_h t^k \in \mathfrak{n}[[t]]$ on the module $\mathsf{R}[\mathcal{J}N]$. The action of $\mathcal{J}N$ on $\mathsf{R}[N]$ factors through $\mathcal{J}N \twoheadrightarrow N$, hence elements of $t\mathfrak{n}[[t]]$ annihilate $\mathsf{R}[N]$.

Thanks to these observations, the right hand side of (3.13) simplifies to

$$\sum_{h=1}^{\ell} f_{ih,(-1)} \iota_L(y_h t^{-1})_{(k)} y_0^j = \partial_{i,(0)} y_0^j = \delta_{ij}$$

finishing the proof.

Part 3. We need to show that the elements $\nu := \partial_{i,(k)} \partial_{j,-1}$ vanish for all $k \geqslant 0$. Thanks to part 2 of the proposition, $[Y(\partial_{i,-1}, z), Y(\partial_{j,-1}, w)] f = 0$ for all $f \in \mathsf{R}[\mathcal{J}N]$. It follows from an analog of (3.10) that

$$\nu_{(\ell)} f = 0, \forall \ell \geqslant 0. \tag{3.14}$$

Thanks to the filtration on $\mathsf{CDO}(N)$ by the order of differential operator (that turns $\mathsf{CDO}(N)$ into an almost commutative algebra), see Section 3.6.2, ν lies in

filtration degree 1. We claim that (3.14) implies that $\nu \in \mathsf{R}[\mathcal{J}N]$. Once we show this, we complete the proof that $\nu = 0$: by the construction ν has positive energy degree, while elements of $\mathsf{R}[\mathcal{J}N]$ have non-positive degree.

So, we need to show (3.14) implies $\nu \in \mathsf{R}[\mathcal{J}N]$. Since ν is in filtration degree 1, it is the sum of an element of $\mathsf{R}[\mathcal{J}N]$ and of a linear combination of elements of the form $\iota_L(y_i t^{-1})_{(k)} g_{ik}$, where $k < 0, g_{ik} \in \mathsf{R}[\mathcal{J}N]$. Note that one can express ${}^R y_i$ as a linear combination of ∂_j's with coefficients from $\mathsf{R}[N]$. Using this and an induction on k, one can express $\iota_L(y_i t^{-1})_{(k)} g_{ik}$ as a linear combination of an element in $\mathsf{R}[\mathcal{J}N]$ and elements $\partial_{i',(k')} g'_{i',k'}$ (where we abuse the notation and write $\partial_{i',(k')}$ for $(\partial_{i',-1})_{(k')}$). Now one can write a formula for $(\partial_{i',(k')} g'_{i',k'})_{(h)}$ for $h \geqslant 0$ (cf. (3.13)) and use part (2) to deduce that $\nu \in \mathsf{R}[\mathcal{J}N]$, hence finishing the proof. □

3.6.5. *Embedding $V_{-\kappa}(\mathfrak{f}) \hookrightarrow \mathsf{CDO}_\beta(F)$.* Our goal here is to produce an embedding

$$\iota_r : V_{-\kappa}(\mathfrak{f}) \to \mathsf{CDO}_\beta(F)$$

whose image commutes with that of ι_ℓ:

$$[Y(\iota_\ell(xt^{-1}), z), Y(\iota_r(yt^{-1}), w)] = 0.$$

We construct ι_r as the sum $\iota_1 + \iota_0$.

First, we explain the construction to ι_1, which is analogous to (3.12). Pick a basis $\{y_i\}$ for $\mathfrak{f}$. To construct ι_1, we express the left invariant vector field ${}^L y_i$ (=the image of y_i under the homomorphism associated to the F-action on itself from the right) as $\sum f_{ij}\, {}^R y_j$ and set

$$\iota_1(y_i t^{-1}) = \sum (f_{ij})_{(-1)} \iota_L(y_j t^{-1}), \text{ for } i = 1, \dots, \ell.$$

To define ι_0, we need a natural map $\mathfrak{f} \to \mathsf{R}[\mathcal{J}F]$. Let T be the derivation of $\mathsf{R}[\mathcal{J}F]$, which is a part of the vertex algebra structure. Consider the composition of T with the natural embedding $\mathsf{R}[F] \hookrightarrow \mathsf{R}[\mathcal{J}F]$. This is a derivation of $\mathsf{R}[F] \to \mathsf{R}[\mathcal{J}F]$, and hence it factors through a R-linear map $\eta : \Omega^1(F) \to \mathsf{R}[\mathcal{J}F]$.

We can trivialize $TF \cong F \times \mathfrak{f}$ using left-invariant vector fields. This gives rise to a trivialization $T^*F \cong F \times \mathfrak{f}^*$. Hence η restricts to $\mathfrak{f}^* \to \mathsf{R}[\mathcal{J}F]$ whose image consists of F-invariants for the action of F from the left. Next, we can view β' as a map $\mathfrak{f} \to \mathfrak{f}^*$. Hence we get a map $\mathfrak{f} \to \Omega^1(F)$

$$\mathfrak{f}t^{-1} \xrightarrow{t\cdot} \mathfrak{f} \xrightarrow{(*)} \mathfrak{f}^* \xrightarrow{\eta} \Omega^1(F) \to \mathsf{R}[\mathcal{J}F],$$

where (*) stands for the form $-\frac{1}{2}\beta_{\mathfrak{f}} - \kappa\beta_W$.

If we replace R with $\mathbb{K}$, the following result is a part of [AG, Theorem 3.7]. Once we know this over $\mathbb{K}$, the claim over R follows, cf. the proof of Lemma 3.16.

Proposition 3.19. *The map $\iota_r := \iota_0 + \iota_1 : \mathfrak{f}t^{-1} \to \mathsf{CDO}_\beta(F)$ gives a vertex algebra embedding $V_{-\kappa}(\mathfrak{f}) \to \mathsf{CDO}_\kappa(F)$ whose image commutes with that of $V_\kappa(\mathfrak{f})$.*

Remark 3.20. The homomorphism ι_r turns $\mathsf{CDO}_\kappa(F)$ into a $U_{-\kappa}(\hat{\mathfrak{f}})$-module. Note that $\iota_0(xt^{-1})_{(n)}$ vanishes on $\mathsf{R}[\mathcal{J}F]$ for $n \geqslant 0$, and $\iota_1(xt^{-1})_{(n)}$ is the action of $xt^n \in \mathfrak{f}[[t]]$ from the right. So we get a $U_{-\kappa}(\hat{\mathfrak{f}})$-module homomorphism

$$\mathrm{Ind}^{\hat{\mathfrak{f}}}_{\mathfrak{f}[[t]] \oplus \mathsf{R}\mathbf{1} \oplus \mathsf{R}\mathbf{1}'} \mathsf{R}[\mathcal{J}F] \to \mathsf{CDO}_\beta(F). \tag{3.15}$$

This homomorphism is an isomorphism one can see this by passing to associated graded algebras. Also one can equip the source with a vertex algebra structure as in Section 3.6.1, and (3.15) becomes an isomorphism of vertex algebras. Finally, notice that the resulting isomorphism $\mathsf{R}[\mathcal{J}(T^*F)] \xrightarrow{\sim} \operatorname{gr}\mathsf{CDO}_\kappa(F)$ is the same as (3.11).

We can extend ι_r to a homomorphism $V_{\mathbf{1}}(\mathfrak{f}) \to \mathsf{CDO}_{\mathbf{1}}(F)$. It is $\mathsf{R}[\mathbf{1}]$-semilinear with respect to the automorphism that sends $\mathbf{1}$ to $-\mathbf{1}$.

3.6.6. *Arc group actions.* The goal of this section is to establish two commuting vertex actions of $\mathcal{J}F$ on $\mathsf{CDO}_\kappa(F)$ and to identify the images of ι_L and ι_R.

We note that $\mathcal{J}F$ acts on $\mathsf{R}[\mathcal{J}F]$ from the right and the action commutes with that of $\mathfrak{f}[[t]] \oplus \mathsf{R}\mathbf{1}$. So we get a $\mathcal{J}F$-action on

$$\mathsf{CDO}_\kappa(F) = \operatorname{Ind}^{\hat{\mathfrak{f}}}_{\mathfrak{f}[[t]]\oplus\mathsf{R}\mathbf{1}\oplus\mathsf{R}\mathbf{1}'}\mathsf{R}[\mathcal{J}F].$$

The invariants of this $\mathcal{J}F$-action is exactly the image of ι_L.

Similarly, using the action of $\mathcal{J}F$ on $\mathsf{R}[\mathcal{J}F]$ from the left, we get another action of $\mathcal{J}F$ on $\mathsf{CDO}_\kappa(F)$ using the construction in Remark 3.20. Its invariants is the image of ι_R.

Lemma 3.21. *The following claims hold:*

(1) The two actions of $\mathcal{J}F$ commute.

(2) They are vertex actions.

Proof. To prove both statements we can base change to $\mathbb{K}$, cf. the proof of Lemma 3.16. To prove (1) we observe that the corresponding actions of $\mathfrak{f}[[t]]$ commute with each other, this is because they are parts of the commuting (see Proposition 3.19) actions of the vertex subalgebras $V_\kappa(\mathfrak{f})$ and $V_{-\kappa}(\mathfrak{f})$. The proof of (2) repeats the proof of Lemma 3.14. □

Note that both $\mathcal{J}F$-actions preserves the filtration on $\mathsf{CDO}_\kappa(F)$ by the order of differential operator. The isomorphism $\mathsf{R}[\mathcal{J}(T^*F)] \xrightarrow{\sim} \operatorname{gr}\mathsf{CDO}_\kappa(F)$ is $\mathcal{J}F \times \mathcal{J}F$-equivariant for the action of $\mathcal{J}F\times\mathcal{J}F$ on $\mathcal{J}(T^*F)$ induced from the action of $F\times F$ on T^*F.

Similarly, $\mathcal{J}F\times\mathcal{J}F$ acts on $\operatorname{gr}\mathsf{CDO}_{\mathbf{1}}(F)$. The identification $\operatorname{gr}\mathsf{CDO}_{\mathbf{1}}(F) \cong \mathcal{J}\mathbb{F}^0\times\{\lambda\partial + \mathfrak{f}^*[[t]]dt\}$ from Section 3.6.3 is $\mathcal{J}F \times \mathcal{J}F$-invariant (where the right action of $\mathcal{J}F$ on $\{\lambda\partial + \mathfrak{f}^*[[t]]dt\}$ is trivial, while the left action is by gauge transformations).

4. Free field realizations

As in the approach of Feigin and Frenkel, see [F], a crucial role in what we do is played by the free field realization of the affine vertex algebra $V_\kappa(\mathfrak{g})$ as well as parabolic analogs of the free field realization. These parabolic analogs are vertex algebra homomorphisms. In these section we construct them and establish some of their properties.

Here R is a commutative ring such that 2 is invertible in R. Here G is a split connected reductive group over R, P is its parabolic subgroup, $N \subset P$ is the

unipotent radical, and L is a Levi subgroup of P. We write $V_\kappa(\mathfrak{l})$ for the affine vertex algebra for $\mathfrak{l}$ and the restriction of κ to $\mathfrak{l}$.

The homomorphism in question is $V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l})$. It will be called the *parabolic free field realization map* and denote it by ffr_P. When P is a Borel subgroup, we drop "parabolic" from the name.

4.1. **Decomposition of $\mathsf{CDO}_\kappa(G^0)$.** We write N^- for the unipotent radical of the parabolic P^- opposite to P. Set

$$G^0 := NLN^- = (PN^- = NP^-) \subset G, \tag{4.1}$$

this is a principal open subset. It carries an action of P from the left and the action of P^- from the right.

Recall from Section 3.6.2 that we have the localization $\mathsf{CDO}_\kappa(G^0)$. Note that $\mathsf{R}[\mathcal{J}G^0]$ decomposes as $\mathsf{R}[\mathcal{J}N] \otimes \mathsf{R}[\mathcal{J}P^-]$.

First, we need analogs of subalgebras $D(N), D(P^-) \subset D(G^0)$ of usual differential operators in $\mathsf{CDO}(G^0)$. Consider the following vertex subalgebras in $\mathsf{CDO}(G^0)$:

1) The subalgebra generated by $\mathsf{R}[\mathcal{J}N]$ and the elements $\iota_L(xt^{-1})$ for all $x \in \mathfrak{n}$. As an R-submodule, it coincides with $\mathrm{Ind}_{\mathfrak{n}[[t]]}^{\mathfrak{n}((t))} \mathsf{R}[\mathcal{J}N]$, and as a vertex algebra, it is $\mathsf{CDO}(N)$.

2) The subalgebra generated by $\mathsf{R}[\mathcal{J}P^-]$ and the elements $\iota_R(xt^{-1})$ for $x \in \mathfrak{p}^-$. As an R-submodule it coincides with

$$\mathrm{Ind}_{\mathfrak{p}^-[[t]]\oplus \mathsf{R}\mathbf{1}}^{\hat{\mathfrak{p}}^-_\kappa} \mathsf{R}[\mathcal{J}P^-].$$

As a vertex algebra it is $\mathsf{CDO}_\kappa(P^-)$, where the form is pulled back from the restriction of κ to $\mathfrak{l}$.

From the construction it is easy to see that the vertex subalgebras

$$\mathsf{CDO}(N), \mathsf{CDO}_\kappa(P^-) \subset \mathsf{CDO}_\kappa(G^0)$$

commute.

So we get a vertex algebra homomorphism:

$$\mathsf{CDO}(N) \otimes \mathsf{CDO}_\kappa(P^-) \to \mathsf{CDO}_\kappa(G^0). \tag{4.2}$$

Lemma 4.1. *(4.2) is an isomorphism.*

Proof. Note that (4.2) is preserves the filtrations by the order of differential operator. The associated graded homomorphism coincides with the isomorphism $\mathsf{R}[\mathcal{J}(T^*N)] \otimes \mathsf{R}[\mathcal{J}(T^*P^-)] \xrightarrow{\sim} \mathsf{R}[\mathcal{J}(T^*G^0)]$. The latter comes from the decomposition $\mathcal{J}(T^*G^0) \cong \mathcal{J}(T^*N) \times \mathcal{J}(T^*P^-)$, which, in turns is induced by $T^*G^0 \cong T^*N \times T^*P^-$. □

Remark 4.2. We'll need equivariance properties of (4.2). First, note that $\mathcal{J}P^-$ acts from the right, and (4.2) is equivariant by the construction. Also, $\mathcal{J}P$ acts from the left: $\mathcal{J}L$ acts diagonally, while $\mathcal{J}N$ acts on the first factor only. Then (4.2) is $\mathcal{J}P$-equivariant.

Note that we also have an analog of (4.1) for $\mathsf{CDO}_{\mathbf{1}}(G^0)$: we can define the vertex subalgebra $\mathsf{CDO}_{\mathbf{1}}(P^-)$ whose specialization to $\mathbf{1} = \kappa$.

4.2. **Parabolic Free Field Realization Map.** We now construct a parabolic free field realization homomorphism

$$\mathsf{ffr}_P : V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l}) \tag{4.3}$$

using (4.2) and its equivariance properties from Remark 4.2. We will write $\mathsf{ffr}_{P,\kappa}$ if we want to indicate the dependence on κ.

4.2.1. Namely, consider the inclusion $\iota_L : V_\kappa(\mathfrak{g}) \to \mathsf{CDO}_\kappa(G)$ and compose it with the localization inclusion $\mathsf{CDO}_\kappa(G) \to \mathsf{CDO}_\kappa(G^0) \xrightarrow{\sim} \mathsf{CDO}(N) \otimes \mathsf{CDO}_\kappa(P^-)$. Denote the composition $V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes \mathsf{CDO}(P^-)$ by η.

The image of η is contained in the JP^--invariants. Note that the action of $\mathcal{J}P^-$ on $\mathsf{CDO}(N)$ is trivial, and the invariants in $\mathsf{CDO}_\kappa(P^-)$ is $V_\kappa(\mathfrak{p}^-)$. So we get an inclusion:

$$\theta : V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{p}^-).$$

Now note that we have a vertex algebra epimorphism:

$$\pi : V_\kappa(\mathfrak{p}^-) \twoheadrightarrow V_\kappa(\mathfrak{l})$$

induced by $\mathfrak{p}^- \twoheadrightarrow \mathfrak{l}$. So we get a vertex algebra homomorphism:

$$\mathsf{ffr}_P : V_\kappa(\mathfrak{g}) \xrightarrow{\theta} \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{p}^-) \xrightarrow{\mathrm{id}\otimes\pi} \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l}).$$

Remark 4.3. By the construction, ffr_P is $\mathcal{J}P$-equivariant. In particular, a choice of a maximal torus $H \subset L$ gives rise to gradings on $V_\kappa(\mathfrak{g}), \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l})$ by the root lattice, and ffr_P is graded for this choice of grading. Also, ffr_P is graded for energy grading by the construction.

4.2.2. Next, ffr_P is a filtered vertex algebra homomorphism. Now we discuss the associated graded of ffr_P to be denoted by ffr_P^0, as we will see just below, it is independent of κ. Consider the morphism $\mu : T^*P \to \mathfrak{g}^*$ that is the restriction of the moment map $T^*(G/N^-) \to \mathfrak{g}^*$ to the open subset $T^*P \subset T^*(G/N^-)$. It is L-invariant for the action of L from the right. Then we can consider the induced morphism $\underline{\mu} : T^*N \times \mathfrak{l}^* \to \mathfrak{g}^*$. Then $\mathsf{ffr}_P^0 = \mathcal{J}\underline{\mu}$.

We will write $\mathsf{ffr}_P^\hbar$ for the induced homomorphism between the Rees algebras $V_\kappa^\hbar(\mathfrak{g}) \to \mathsf{CDO}^\hbar(N) \otimes_{\mathsf{R}[\hbar]} V_\kappa^\hbar(\mathfrak{l})$.

4.2.3. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}$. We will need a formula for the images of the elements of the form xt^{-1} for $x \in \mathfrak{h}$. Let Φ_P^+ denote the subset of all roots whose root vectors lie in $\mathfrak{n}$. We note that there is an H-equivariant isomorphism $\mathfrak{n} \xrightarrow{\sim} N$ whose differential at $0 \in \mathfrak{n}$ is the identity. Choose root vectors $\partial_\alpha \in \mathfrak{n}, \alpha \in \Phi_P^+$ and dual basis vectors $y^\alpha \in \mathfrak{n}^* \subset \mathsf{R}[N]$. So we get elements $y_0^\alpha, \partial_{\alpha,-1} \in \mathsf{CDO}(N)$.

Lemma 4.4. *We have*

$$\mathsf{ffr}_P(xt^{-1}) = -\sum_{\alpha\in\Phi_+^P} \langle\alpha, x\rangle y_0^\alpha \partial_{\alpha,-1} \otimes |\varnothing\rangle + |\varnothing\rangle \otimes (xt^{-1}). \tag{4.4}$$

Proof. The analogous formula holds for ffr_P^0 by Section 4.2.2. So the difference between the two sides of (4.4) lies in $\mathsf{R}[\mathcal{J}N] \otimes |\varnothing\rangle \subset \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{h})$. Since ffr_P is graded both for root lattice grading and energy grading, Remark 4.3, the degree

of the difference for the root lattice grading is zero and its energy degree is -1. It follows that the difference is zero. $\square$

4.2.4. We also note that ffr_P can be extended to $\mathsf{ffr}_{P,\mathbf{1}} : V_{\mathbf{1}}(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes V_{\mathbf{1}}(\mathfrak{l})$ (so that ffr_P is obtained from this by setting $\mathbf{1} := \kappa$). This homomorphism is also filtered, let $\mathsf{ffr}^0_{P,\mathbf{1}}$ denote the associated graded homomorphism. We also have the intermediate maps: $\eta_{\mathbf{1}}, \theta_{\mathbf{1}}, \pi_{\mathbf{1}}$. They are filtered and their associated graded maps are going to be denoted by $\eta^0_{\mathbf{1}}, \theta^0_{\mathbf{1}}, \pi^0_{\mathbf{1}}$. We note that

$$\mathsf{ffr}^0_{P,\mathbf{1}} = (\mathrm{id} \otimes \pi^0_{\mathbf{1}}) \circ \theta^0_{\mathbf{1}}. \tag{4.5}$$

4.3. **Transitivity.** Let B be a Borel subgroup in P such that $B_L = P \cap L$ is a Borel subgroup in L. Choose a maximal torus $H \in B_L$ and consider the opposite Borel B^-. Let $\tilde{N} = R_u(B)$, $\tilde{N}^- = R_u(B^-)$. We set $\tilde{N}_L = \tilde{N} \cap L, \tilde{N}^-_L = \tilde{N}^- \cap L$ so that we have variety isomorphisms $N \times \tilde{N}_L \xrightarrow{\sim} \tilde{N}, \tilde{N}^-_L \times N^- \xrightarrow{\sim} \tilde{N}^-$.

We have the following vertex algebra homomorphisms:

$$\mathsf{ffr}_B : V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(\tilde{N}) \otimes V_\kappa(\mathfrak{h})$$
$$\mathsf{ffr}_P : V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l}).$$
$$\mathsf{ffr}_{B_L} : V_{\kappa'}(\mathfrak{l}) \to \mathsf{CDO}(N_L) \otimes V_\kappa(\mathfrak{h}).$$

Also note that we can identify $\mathsf{CDO}(\tilde{N})$ with $\mathsf{CDO}(N) \otimes \mathsf{CDO}(N_L)$ thanks to $N \times \tilde{N}_L \xrightarrow{\sim} \tilde{N}$ (cf. (4.2) and Lemma 4.1). The following claim is what we mean by the transitivity.

Proposition 4.5. *The following diagram is commutative:*

$$\begin{array}{ccc} V_\kappa(\mathfrak{g}) & \xrightarrow{\mathsf{ffr}_P} & \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l}) \\ \downarrow{\scriptstyle \mathsf{ffr}_B} & & \downarrow{\scriptstyle \mathrm{id} \otimes \mathsf{ffr}_{B_L}} \\ \mathsf{CDO}(\tilde{N}) \otimes V_\kappa(\mathfrak{h}) & \xrightarrow{\sim} & \mathsf{CDO}(N) \otimes \mathsf{CDO}(N_L) \otimes V_\kappa(\mathfrak{h}). \end{array}$$

Proof. We write L^0 for $N_L H N^-_L$ (the open Bruhat cell in L) and G^0 for NLN^-. Consider the inclusions:

$$\tilde{N}B^- = NL^0N^- \subset G^0 = NP^- \subset G$$

They give rise to localization homomorphisms of vertex algebras:

$$\mathsf{CDO}_\kappa(G) \to \mathsf{CDO}_\kappa(G^0) \to \mathrm{CDO}_\kappa(NL^0N^-).$$

In turn, these give rise to inclusions, see Section 4.2:

$$V_\kappa(\mathfrak{g}) \hookrightarrow \mathsf{CDO}_\kappa(G^0)^{JP^-} (= \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{p}^-)) \hookrightarrow$$
$$\mathsf{CDO}_\kappa(NL^0N^-)(= \mathsf{CDO}(\tilde{N}) \otimes V_\kappa(\mathfrak{b}^-) = \mathsf{CDO}(N) \otimes \mathsf{CDO}(\tilde{N}_L) \otimes V_\kappa(\mathfrak{b}^-)$$

The above homomorphism

$$\mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{p}^-) \to \mathsf{CDO}(N) \otimes \mathsf{CDO}(\tilde{N}_L) \otimes V_\kappa(\mathfrak{b}^-) \tag{4.6}$$

is the tensor product of the identity on $\mathrm{CDO}(N)$ with the homomorphism restricted from the localization homomorphism:

$$V_\kappa(\mathfrak{p}^-) = \mathsf{CDO}_\kappa(P^-)^{JP^-} \to \mathsf{CDO}_\kappa(N_L B^-)^{JB^-} = \mathsf{CDO}(N_L) \otimes V_\kappa(\mathfrak{b}^-). \tag{4.7}$$

Using the epimorphisms

$$V_{\kappa'}(\mathfrak{p}^-) \twoheadrightarrow V_\kappa(\mathfrak{l}), V_\kappa(\mathfrak{b}^-) \twoheadrightarrow V_\kappa(\mathfrak{b}_L^-)$$

from (4.7) we get

$$V_\kappa(\mathfrak{l}) = \mathsf{CDO}_\kappa(L)^{JL} \to \mathsf{CDO}_\kappa(L^0)^{JB_L^-} = \mathsf{CDO}(N_L) \otimes V_\kappa(\mathfrak{b}_L^-), \tag{4.8}$$

that also coincides with the localization homomorphism. The map ffr_{B_L} coincides with the composition of (4.8) with the tensor product of the identity on $\mathsf{CDO}(N_L)$ and the projection $V_\kappa(\mathfrak{b}_L^-) \twoheadrightarrow V_\kappa(\mathfrak{h})$.

It follows that we get the following commutative diagram.

$$\begin{tikzcd}
V_\kappa(\mathfrak{g}) \arrow[r, "(i)"] \arrow[dr, "\mathsf{ffr}_P"'] & \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{p}^-) \arrow[d, "(ii)"] \arrow[dr, "(iii)"] & \\
& \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l}) \arrow[dr, "\mathrm{id} \otimes \mathsf{ffr}_{B_L}"'] & \mathsf{CDO}(N) \otimes \mathsf{CDO}(\tilde{N}_L) \otimes V_\kappa(\mathfrak{b}^-) \arrow[d, "(iv)"] \\
& & \mathsf{CDO}(N) \otimes \mathsf{CDO}(\tilde{N}_L) \otimes V_\kappa(\mathfrak{h}) \arrow[d, "\cong"] \\
& & \mathsf{CDO}(\tilde{N}) \otimes V_\kappa(\mathfrak{h})
\end{tikzcd}$$

Here the maps are as follows. (i) is the restriction of the localization homomorphism that is used in the construction of ffr_P, see Section 4.2. (ii) and (iv) are the projection maps coming from epimorphisms $\mathfrak{p}^- \twoheadrightarrow \mathfrak{l}$ and $\mathfrak{b}^- \twoheadrightarrow \mathfrak{h}$, respectively. (iii) is (4.7). The square containing (ii),(iii),(iv) and $\mathrm{id} \otimes \mathsf{ffr}_{B_L}$ commutes because of the characterization of the latter explained after (4.8). And the composition of $(i), (iii), (iv)$ and the final isomorphism is ffr_B because the composition of (i) and (iii) is the restriction of the localization homomorphism (performed in two steps, first from G to G^0, then from G^0 to NL^0N^-). □

4.4. **Homomorphism between universal enveloping algebras.** In this section we will study the filitered associative algebra homomorphism $\mathsf{Ffr}_P := \widehat{U}(\mathsf{ffr}_P)$.

4.4.1. First, we are going to examine the structure of the algebras $\widehat{U}(\mathsf{CDO}(N)) \subset \widetilde{U}(\mathsf{CDO}(N))$. Recall the elements $y_0^i, \partial_{j,-1} \in \mathsf{CDO}(N)$, see Proposition 3.18. They give rise to elements $y^i)n, \partial_{j,m}$ with $n, m \in \mathbb{Z}$. The commutation relation between these elements is $[\partial_{j,m}, y_n^i] = \delta_{i,j}\delta_{n+m,0}$, the other brackets are zero. The energy degree of y_n^i and $\delta_{j,n}$ is n, while the filtration degree of the elements y_n^i is 0, and for the elements $\partial_{j,m}$ it is 1. The algebra $\widetilde{U}(\mathsf{CDO}(N))$ is the completed Weyl algebra, cf. [F, Section 5.3.3]. We will denote it by $\widetilde{D}(\mathcal{L}N)$. The algebra $\widehat{D}(\mathcal{L}N) := \widehat{U}(\mathsf{CDO}(N)) \subset \widetilde{D}(\mathcal{L}N)$ consists of all infinite sums of ordered monomials in the

elements $y^i_n, \partial_{j,m}$ written in the increasing order of energy degree (in degree 0 we write y's before ∂'s) subject to the following conditions:

- The degree in the elements ∂ is bounded,
- For any $N \in \mathbb{Z}$, there are only finitely many monomials that do not contain $y^i_n, \partial_{i,n}$ with $n > N$.

4.4.2. Note that the algebra $\widehat{U}(\mathcal{L}N) \otimes V_\kappa(\mathfrak{l}))$ (where the tensor product is taken over R) is the filtered completed tensor product $\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})$ defined as follows. Note that $\widehat{U}(\mathsf{CDO}(N) \otimes \widehat{U}_\kappa(\mathfrak{l})$ carries an exhaustive ascending algebra filtration. Then $\widehat{U}(\mathsf{CDO}(N)\widehat{\otimes}\widehat{U}_\kappa(\mathfrak{l})$ is the union of the completions of filtered pieces. Thanks to the PBW theorem, this subalgebra coincides with the union of filtered pieces in $\widetilde{U}(\mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l}))$.

4.4.3. We write Ffr_P for the homomorphism $\widehat{U}_\kappa(\hat{\mathfrak{g}}) \to \widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})$. Thanks to Remark 4.3, this homomorphism is $\mathcal{L}P$-equivariant. Thanks to Proposition 4.5 we have the following commutative diagram:

$$\begin{tikzcd} \widehat{U}_\kappa(\hat{\mathfrak{g}}) \arrow[r, "\mathsf{Ffr}_P"] \arrow[d, "\mathsf{Ffr}_B"] & \widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}}) \arrow[d, "\mathrm{id}\otimes\mathsf{Ffr}_{B_L}"] \\ \widehat{D}(\mathcal{L}\tilde{N})\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{h}}) \arrow[r, "\sim"] & \widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{D}(\mathcal{L}N_L)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{h}}). \end{tikzcd} \tag{4.9}$$

4.4.4. As for vertex algebras, we have the universal version of ffr_P, a homomorphism

$$\mathsf{Ffr}_{P,\mathbf{1}} : \widehat{U}_{\mathbf{1}}(\hat{\mathfrak{g}}) \to \widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_{\mathbf{1}}(\hat{\mathfrak{l}}).$$

We can also consider the associated graded homomorphism $\mathsf{Ffr}^0_{P,\mathbf{1}}$ and the Rees algebra homomorphism $\mathsf{Ffr}^\hbar_{P,\mathbf{1}}$.

5. ffr vs p-centers

The goal of this section is to describe the compatibility between the free field realization and certain p-centers of the algebras on both sides.

5.1. p-centers for restricted Lie algebras.

5.1.1. We first begin with some generalities on p-centers. To do so, we need to recall the absolute and relative Frobenius maps.

5.1.2. We begin with the absolute Frobenius. Recall that for an $\mathbb{F}_p$-algebra S, the absolute Frobenius map $\mathfrak{Fr}_\mathsf{S} : s \mapsto s^p$ is a map of algebras, and this is natural in S, i.e., for any map $\phi : \mathsf{S} \to \mathsf{T}$ of algebras, we have $\mathfrak{Fr}_\mathsf{T} \circ \phi = \phi \circ \mathfrak{Fr}_\mathsf{S}$.

As a consequence, given an $\mathbb{F}_p$-space $\mathcal{Y}$, for any S the tautological maps

$$\mathcal{Y}(\mathfrak{Fr}_\mathsf{S}) : \mathcal{Y}(\mathsf{S}) \to \mathcal{Y}(\mathsf{S})$$

yield a natural transformation from $\mathcal{Y}$ to itself, i.e., a map of spaces

$$\mathfrak{Fr}_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y};$$

we refer to this as the absolute Frobenius for $\mathcal{Y}$. Moreover, this is a monoidal natural transformation from the identity map of $\mathbb{F}_p$-spaces to itself. I.e., for any

map $\phi : \mathcal{Y} \to \mathcal{Z}$ of spaces, we again have $\mathrm{Fr}_{\mathcal{Z}} \circ \phi = \phi \circ \mathfrak{Fr}_{\mathcal{Y}}$, and any pair of spaces $\mathcal{Y}_1, \mathcal{Y}_2$, we have the identity

$$\mathfrak{Fr}_{\mathcal{Y}_1 \times \mathcal{Y}_2} = \mathfrak{Fr}_{\mathcal{Y}_1} \times \mathfrak{Fr}_{\mathcal{Y}_2}.$$

5.1.3. Let us turn to the relative Frobenius. Let R be a fixed $\mathbb{F}_p$-algebra. Given an R-space $\mathcal{Y}$, we may view it as an $\mathbb{F}_p$-space equipped with a map $\mathcal{Y} \to \operatorname{Spec} \mathsf{R}$. In particular, the tautological commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\mathfrak{Fr}_{\mathcal{Y}}"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \operatorname{Spec} \mathsf{R} \arrow[r, "\mathfrak{Fr}_{\mathsf{R}}"] & \operatorname{Spec} \mathsf{R} \end{tikzcd}$$

yields the relative Frobenius map of R-spaces

$$\mathrm{Fr}_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}^{(1)} := \mathcal{Y} \underset{\operatorname{Spec} \mathsf{R}}{\times} \operatorname{Spec} \mathsf{R}.$$

This is again natural and monoidal, i.e., given a map $\phi : \mathcal{Y} \to \mathcal{Z}$, in evident notation we have $\mathrm{Fr}_{\mathcal{Z}} \circ \phi = \phi^{(1)} \circ \mathrm{Fr}_{\mathcal{Y}}$, and with respect to the natural identification

$$(\mathcal{Y}_1 \times \mathcal{Y}_2)^{(1)} \simeq \mathcal{Y}_1^{(1)} \times \mathcal{Y}_2^{(2)}$$

we have $\mathrm{Fr}_{\mathcal{Y}_1 \times \mathcal{Y}_2} = \mathrm{Fr}_{\mathcal{Y}_1} \times \mathrm{Fr}_{\mathcal{Y}_2}$. In other words, the assignment $\mathcal{Y} \mapsto \mathcal{Y}^{(1)}$ is a symmetric monoidal functor $(-)^{(1)}$ from R-spaces to itself, and the map $\mathrm{Fr} : \mathcal{Y} \to \mathcal{Y}^{(1)}$ carries the datum of a natural transformation between monoidal functors $\mathrm{id} \to (-)^{(1)}$.

In what follows, we omit the subscript $\mathcal{Y}$ when discussing the relative Frobenius $\mathrm{Fr} : \mathcal{Y} \to \mathcal{Y}^{(1)}$, and similarly for the absolute Frobenius $\mathfrak{Fr} : \mathcal{Y} \to \mathcal{Y}$.

5.1.4. A useful consequence of the preceding discussion is the following.

Corollary 5.1. *Given a group R-space $\mathcal{G}$, its relative Frobenius twist $\mathcal{G}^{(1)}$ inherits a natural structure of group R-space, and $\mathrm{Fr} : \mathcal{G} \to \mathcal{G}^{(1)}$ is a map of group R-spaces.*

Proof. That $\mathcal{G}^{(1)}$ is again naturally a group R-space follows from the monoidality of $(-)^{(1)}$, and that Fr is a map of group R-spaces follows from the monoidality of the natural transformation Fr. □

A similar argument shows the following.

Corollary 5.2. *Given a restricted Lie algebra $\mathfrak{f}$ in R-spaces, its relative Frobenius twist $\mathfrak{f}^{(1)}$ inherits a natural structure of restricted Lie algebra in R-spaces, and $\mathrm{Fr} : \mathfrak{f} \to \mathfrak{f}^{(1)}$ is a map of restricted Lie algebra R-spaces.*

5.1.5. To discuss p-centers, the following setup will be convenient.

Given two restricted Lie algebras $\mathfrak{f}_1$ and $\mathfrak{f}_2$ in R-spaces, we have a natural associated R-space $\underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}_1, \mathfrak{f}_2)$. Namely, to a R-algebra S, we define $\underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}_1, \mathfrak{f}_2)(\mathsf{S})$ to be the set of all maps of restricted Lie algebra S-spaces

$$\phi : \mathfrak{f}_1 \underset{\operatorname{Spec} \mathsf{R}}{\times} \operatorname{Spec} \mathsf{S} \to \mathfrak{f}_2 \underset{\operatorname{Spec} \mathsf{R}}{\times} \operatorname{Spec} \mathsf{S}.$$

Given a map $\mathsf{S} \to \mathsf{T}$, the map $\underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}_1, \mathfrak{f}_2)(\mathsf{S}) \to \underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}_1, \mathfrak{f}_2)(\mathsf{T})$ is given by base change, i.e., assigns to ϕ as above the composition

$$\mathfrak{f}_1 \underset{\mathrm{Spec}\,\mathsf{R}}{\times} \mathrm{Spec}\,\mathsf{T} \simeq (\mathfrak{f}_1 \underset{\mathrm{Spec}\,\mathsf{R}}{\times} \mathrm{Spec}\,\mathsf{S}) \underset{\mathrm{Spec}\,\mathsf{S}}{\times} \mathrm{Spec}\,\mathsf{T}$$
$$\xrightarrow{\phi\times\mathrm{id}} (\mathfrak{f}_2 \underset{\mathrm{Spec}\,\mathsf{R}}{\times} \mathrm{Spec}\,\mathsf{S}) \underset{\mathrm{Spec}\,\mathsf{S}}{\times} \mathrm{Spec}\,\mathsf{T} \simeq \mathfrak{f}_1 \underset{\mathrm{Spec}\,\mathsf{R}}{\times} \mathrm{Spec}\,\mathsf{T}.$$

Given two (unrestricted) Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ in R-spaces, we have a similarly defined R-space $\underline{\mathrm{Hom}}_{\mathrm{Lie}_{\mathsf{R}}}(\mathfrak{l}_1, \mathfrak{l}_2)$, where one omits the restricted condition on the maps in the definition of the S-points.

Similarly, given two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in R-spaces, we have an analogously defined R-space $\underline{\mathrm{Hom}}_{\mathrm{Alg}_{\mathsf{R}}}(\mathfrak{A}_1, \mathfrak{A}_2)$, and given two modules $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in R-spaces, we have $\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathsf{R}}}(\mathfrak{M}_1, \mathfrak{M}_2)$.

5.1.6. Let us define $\mathrm{ResLie}_{\mathsf{R}}$ to be the category enriched in R-spaces with objects given by restricted Lie algebra R-spaces $\mathfrak{f}$, and morphisms from $\mathfrak{f}_1$ to $\mathfrak{f}_2$ given by $\underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}_1, \mathfrak{f}_2)$, with the evident composition rules.

We define the enriched categories $\mathrm{Lie}_{\mathsf{R}}, \mathrm{Alg}_{\mathsf{R}}$, and $\mathrm{Mod}_{\mathsf{R}}$ similarly.

5.1.7. By construction, we have evident forgetful functors

$$\begin{tikzcd} \mathrm{ResLie}_{\mathsf{R}} \arrow[dr] & & \mathrm{Alg}_{\mathsf{R}} \arrow[dl] \\ & \mathrm{Lie}_{\mathsf{R}} \arrow[d] & \\ & \mathrm{Mod}_{\mathsf{R}} & \end{tikzcd} .$$

Lemma 5.3. *The forgetful functor* $\mathrm{Oblv} : \mathrm{Alg}_{\mathsf{R}} \to \mathrm{Lie}_{\mathsf{R}}$ *admits a left adjoint, i.e.,*

$$U(-) : \mathrm{Lie}_{\mathsf{R}} \rightleftarrows \mathrm{Alg}_{\mathsf{R}} : \mathrm{Oblv}.$$

Proof. Given a R-algebra S, and Lie algebra R-space $\mathfrak{l}$, recall that $\mathfrak{l}(\mathsf{S})$ is a Lie algebra over S, and in particular we may form its S-linear enveloping algebra $U(\mathfrak{l}(\mathsf{S}))$. We set $U(\mathfrak{l})(S) := U(\mathfrak{l}(\mathsf{S}))$. Given a map $\mathsf{S} \to \mathsf{T}$, the composition

$$\mathfrak{l}(\mathsf{S}) \to \mathfrak{l}(\mathsf{T}) \to U(\mathfrak{l})(\mathsf{T})$$

is tautologically a S-linear map of Lie algebras, whence induces a unique map $U(\mathfrak{l})(\mathsf{S}) \to U(\mathfrak{l})(\mathsf{T})$, and it is straightforward to see this defines an algebra $U(\mathfrak{l})$ in R-spaces with the required universal property. □

5.1.8. In particular, given restricted Lie algebras in R-spaces, we can form their enveloping algebras, i.e., the functor

$$\mathrm{ResLie}_{\mathsf{R}} \xrightarrow{\mathrm{Oblv}} \mathrm{Lie}_{\mathsf{R}} \xrightarrow{U(-)} \mathrm{Alg}_{\mathsf{R}} \xrightarrow{\mathrm{Oblv}} \mathrm{Mod}_{\mathsf{R}}. \tag{5.1}$$

On the other hand, to describe the p-center, we would like to also consider the Frobenius twist of the underlying R-modules. Namely, we have a functor

$$\mathrm{Fr}^* : \mathrm{Mod}_{\mathsf{R}} \to \mathrm{Mod}_{\mathsf{R}},$$

defined as follows. For a module $\mathcal{M}$ in R-spaces, we have

$$(\mathrm{Fr}^*\mathcal{M})(\mathsf{S}) := \mathsf{S} \underset{\mathsf{S}}{\otimes} \mathcal{M}(\mathsf{S}),$$

where we are inducing along the absolute Frobenius $\mathfrak{Fr} : \mathsf{S} \to \mathsf{S}$. Given a map $\phi : \mathsf{S} \to \mathsf{T}$, with associated map $\mathcal{M}(\phi) : \mathcal{M}(\mathsf{S}) \to \mathcal{M}(\mathsf{T})$, we set

$$(\mathrm{Fr}^*\mathcal{M})(\phi) := \phi \otimes \mathcal{M}(\phi) : \mathsf{S} \underset{\mathsf{S}}{\otimes} \mathcal{M}(\mathsf{S}) \to \mathsf{T} \underset{\mathsf{T}}{\otimes} \mathcal{M}(\mathsf{T}).$$

For functoriality, note that we have a tautological map

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathsf{R}}}(\mathcal{M}_1, \mathcal{M}_2) \to \underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathsf{R}}}(\mathrm{Fr}^*\mathcal{M}_1, \mathrm{Fr}^*\mathcal{M}_2)$$

which on S-points sends a map ϕ to $\mathrm{id} \otimes \phi$.

In particular, we may also consider the functor

$$\mathrm{ResLie}_{\mathsf{R}} \xrightarrow{\mathrm{Oblv}} \mathrm{Mod}_{\mathsf{R}} \xrightarrow{\mathrm{Fr}^*} \mathrm{Mod}_{\mathsf{R}}. \tag{5.2}$$

Formation of the p-center may then be stated as follows.

Lemma 5.4. *There is a canonical natural transformation from* (5.2) *to* (5.1)*, i.e.,*

$$\mathfrak{Z}_p : \mathrm{Fr}^* \circ \mathrm{Oblv} \to \mathrm{Oblv} \circ U(-) \circ \mathrm{Oblv}.$$

Proof. Given a restricted Lie algebra R-space $\mathfrak{f}$, and an R-algebra S, we have a natural map

$$\mathsf{S} \underset{\mathsf{S}}{\otimes} \mathfrak{f}(\mathsf{S}) \to U(\mathfrak{f})(\mathsf{S}), \qquad 1 \otimes X \mapsto X^p - X^{[p]},$$

and it is straightforward to see this is a natural transformation. □

Corollary 5.5. *Given a restricted Lie algebra $\mathfrak{f}$ in R-spaces, the natural action of $\underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}, \mathfrak{f})$ on the p-center factors through the natural map*

$$\underline{\mathrm{Hom}}_{\mathrm{ResLie}_{\mathsf{R}}}(\mathfrak{f}, \mathfrak{f}) \to \underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathsf{R}}}(\mathrm{Fr}^*\mathfrak{f}, \mathrm{Fr}^*\mathfrak{f}).$$

Remark 5.6. Given an algebra $\mathfrak{A}$ in R-spaces, we can define its center $Z(\mathfrak{A})$, with S-points the elements $z \in \mathfrak{A}(\mathsf{S})$ which are central in $\mathfrak{A}(\mathsf{T})$ for every $\mathsf{S} \to \mathsf{T}$. With this, $\mathfrak{Z}_p$ factors for a fixed restricted Lie algebra $\mathfrak{f}$ as

$$\mathrm{Fr}^*\mathfrak{f} \to Z(U(\mathfrak{f})) \hookrightarrow U(\mathfrak{f}).$$

5.1.9. We would like to deduce from Corollary 5.5 the assertion that, if a group R-space $\mathcal{G}$ acts on a restricted Lie algebra R-space $\mathfrak{f}$, then the action of $\mathcal{G}$ on its p-center factors through the Frobenius twist $\mathcal{G}^{(1)}$.

To approach this, first note that given a R-space $\mathcal{Y}$, for a R-algebra $i : \mathsf{R} \to \mathsf{S}$, if we temporarily denote its S points by $\mathcal{Y}(\mathsf{S}, i)$, we have a canonical bijection

$$\mathcal{Y}^{(1)}(\mathsf{S}, i) \simeq \mathcal{Y}(\mathsf{S}, i \circ \mathfrak{Fr}).$$

From this, it easily follows that the Frobenius twist $\mathcal{M}^{(1)}$ of a module R-space is again naturally a module R-space, and similarly for restricted Lie algebras, Lie algebras, and algebras.

5.1.10. In general, there is no reason to expect an isomorphism $\mathrm{Fr}^*\mathcal{M} \simeq \mathcal{M}^{(1)}$. To address this, let us say a module R-space $\mathcal{M}$ is *quasicoherent* if for any map $\mathsf{S} \to \mathsf{T}$, the tautological map

$$\mathsf{T} \underset{\mathsf{S}}{\otimes} \mathcal{M}(\mathsf{S}) \to \mathcal{M}(\mathsf{T})$$

is an isomorphism of T-modules. Let us denote by $\mathrm{QCoh}_{\mathsf{R}}$ the full subcategory of $\mathrm{Mod}_{\mathsf{R}}$ consisting of quasicoherent module R-spaces.

Lemma 5.7. *The category* $\mathrm{QCoh}_{\mathsf{R}}$ *is preserved by the functors* $\mathcal{M} \mapsto \mathrm{Fr}^*\mathcal{M}$ *and* $\mathcal{M} \mapsto \mathcal{M}^{(1)}$*, and one has a canonical natural isomorphism*

$$\mathrm{Fr}^* \simeq (-)^{(1)} : \mathrm{QCoh}_{\mathsf{R}} \to \mathrm{QCoh}_{\mathsf{R}}.$$

Proof. Given a quasicoherent R-space $\mathcal{M}$ and R-algebra $i : \mathsf{R} \to \mathsf{S}$, we have

$$(\mathrm{Fr}^*\mathcal{M})(\mathsf{S}, i) = \mathsf{S} \underset{\mathsf{S}}{\otimes} \mathcal{M}(\mathsf{S}) = \mathcal{M}(\mathsf{S}, \mathfrak{Fr} \circ i) = \mathcal{M}^{(1)}(\mathsf{S}, i),$$

and it is straightforward to check these identifications yield a natural isomorphism of functors. □

Let us call a restricted Lie algebra R-space quasicoherent if its underlying module R-space is.

Corollary 5.8. *If a group* R*-space* $\mathcal{G}$ *acts on a quasicoherent restricted Lie algebra* R*-space* $\mathfrak{f}$*, then the action of* $\mathcal{G}$ *on the* p*-center of* $U(\mathfrak{f})$ *factors as the composition* $\mathrm{Fr} : \mathcal{G} \to \mathcal{G}^{(1)}$ *and the tautological action of the latter on* $\mathfrak{f}^{(1)} \simeq \mathrm{Fr}^*(\mathfrak{f})$.

5.1.11. Finally, as our restricted Lie algebra R-spaces of interest, namely Kac–Moody and Virasoro, are not quasicoherent restricted Lie algebras, we consider the following (minimal) modification.

Let us call a functor $F : I \to \mathrm{QCoh}_{\mathsf{R}}, i \mapsto \mathcal{M}_i$ a *pro-quasicoherent* module R-space if I is a countable codirected set, and for $i \to j$, the map $\mathcal{M}_i(\mathsf{S}) \to \mathcal{M}_j(\mathsf{S})$ is surjective for all R-algebras S. In what follows, we denote such an object as a formal inverse limit "$\varprojlim_i$"$\mathcal{M}_i$.

Pro-quasicoherent R-spaces naturally form a category $\mathrm{ProQCoh}_{\mathsf{R}}$ enriched in R-spaces, where we set

$$\underline{\mathrm{Hom}}_{\mathrm{ProQCoh}_{\mathsf{R}}}(\text{“}\varprojlim_{\alpha}\text{”}\mathcal{M}_\alpha, \text{“}\varprojlim_{\beta}\text{”}\mathcal{N}_\beta) := \varprojlim_{\beta} \varinjlim_{\alpha} \underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathsf{R}}}(\mathcal{M}_\alpha, \mathcal{N}_\beta).$$

We have a tautological conservative and faithful functor

$$\mathrm{ProQCoh}_{\mathsf{R}} \to \mathrm{Mod}_{\mathsf{R}}, \qquad \text{“}\varprojlim M_i\text{”} \mapsto \varprojlim M_i,$$

which sends a formal inverse limit to the actual limit computed in $\mathrm{Mod}_{\mathsf{R}}$. Explicitly, we have

$$(\varprojlim M_i)(\mathsf{S}) \simeq \varprojlim(M_i(\mathsf{S})),$$

i.e., the limit may be computed termwise. Note that the countability of the diagrams and surjectivity of the transition maps ensures this is indeed conservative and faithful. In particular, we may discuss the underlying R-space of a pro-quasicoherent R-space module.

5.1.12. We define the category of restricted Lie algebras in pro-quasicoherent R-spaces as the fiber product

$$\mathrm{ResLie}(\mathrm{ProQCoh}_{\mathsf{R}}) := \mathrm{ProQCoh}_{\mathsf{R}} \underset{\mathrm{Mod}_{\mathsf{R}}}{\times} \mathrm{ResLie}_{\mathsf{R}}.$$

In particular, given a pro-quasicoherent R-space "$\underleftarrow{\lim}$"$\mathcal{M}_i$, a restricted Lie algebra structure on it is the datum of a restricted Lie algebra structure on $\underleftarrow{\lim}\,\mathcal{M}_i$, such that the Lie bracket and restricted power maps lift to (unique) maps from "$\underleftarrow{\lim}$"$\mathcal{M}_i$ to itself. In particular, this implies that both structures are continuous with respect to pro-topologies on $\underleftarrow{\lim}\,\mathcal{M}_i(\mathsf{S})$, for all S.

We define the category $\mathrm{Lie}(\mathrm{ProQCoh}_{\mathsf{R}})$ similarly, so in particular we have forgetful functors

$$\mathrm{ResLie}(\mathrm{ProQCoh}_{\mathsf{R}}) \to \mathrm{Lie}(\mathrm{ProQCoh}_{\mathsf{R}}) \to \mathrm{ProQCoh}_{\mathsf{R}}.$$

5.1.13. We have a natural functor of taking the completed enveloping algebra

$$\widetilde{U}(-) : \mathrm{Lie}(\mathrm{ProQCoh}_{\mathsf{R}}) \to \mathrm{Alg}_{\mathsf{R}}, \qquad \mathfrak{f} \mapsto \widetilde{U}(\mathfrak{f}).$$

Explicitly, on objects, if the underlying pro-quasicoherent R-space of $\mathfrak{f}$ is "$\underleftarrow{\lim}$"$\mathfrak{f}_\alpha, \alpha \in A$, for fixed $\beta \in A$ consider the 'compact open' subspace

$$\mathfrak{k}_\beta := \text{“}\underset{\alpha \geqslant \beta}{\underleftarrow{\lim}}\text{”} \ker(\mathfrak{f}_\alpha \to \mathfrak{f}_\beta).$$

Then on S-points, we set

$$(\widetilde{U}(\mathfrak{f}))(\mathsf{S}) := \underleftarrow{\lim}\, U(\mathfrak{f}(\mathsf{S}))/U(\mathfrak{f}(\mathsf{S})) \cdot \mathfrak{k}_\alpha(\mathsf{S}),$$

with its natural structure as a S-algebra, and with the natural functoriality along maps $\mathfrak{f}(\mathsf{S}) \to \mathfrak{f}(\mathsf{T})$. The definition of what $\widetilde{U}(-)$ does to morphisms, i.e. the map

$$\underline{\mathrm{Hom}}_{\mathrm{Lie}(\mathrm{ProQCoh}_{\mathsf{R}})}(\mathfrak{f}_1, \mathfrak{f}_2) \to \underline{\mathrm{Hom}}_{\mathrm{Alg}_{\mathsf{R}}}(\widetilde{U}(\mathfrak{f}_1), \widetilde{U}(\mathfrak{f}_2))$$

is similarly induced by applying the functoriality of taking the completed enveloping algebra of a given topological Lie algebra $\mathfrak{f}(\mathsf{S})$.

In particular, when discussing p-centers, we may again may consider the composition

$$\mathrm{ResLie}(\mathrm{ProQCoh}_{\mathsf{R}}) \xrightarrow{\mathrm{Oblv}} \mathrm{Lie}(\mathrm{ProQCoh}_{\mathsf{R}}) \xrightarrow{\widetilde{U}(-)} \mathrm{Alg}_{\mathsf{R}} \xrightarrow{\mathrm{Oblv}} \mathrm{Mod}_{\mathsf{R}}. \tag{5.3}$$

On the other hand, we may consider the functor

$$\mathrm{ResLie}(\mathrm{ProQCoh}_{\mathsf{R}}) \xrightarrow{\mathrm{Oblv}} \mathrm{Mod}_{\mathsf{R}} \xrightarrow{(-)^{(1)}} \mathrm{Mod}_{\mathsf{R}}. \tag{5.4}$$

Proposition 5.9. *There exists a canonical natural transformation from* (5.4) *to* (5.3)*, i.e.,*

$$\xi_p : (-)^{(1)} \circ \mathrm{Oblv} \to \mathrm{Oblv} \circ \widetilde{U}(-) \circ \mathrm{Oblv}.$$

Proof. Suppose that $\mathfrak{f} \simeq$ "$\underleftarrow{\lim}$"$\mathfrak{f}_\alpha$ is a restricted Lie algebra in pro-quasicoherent R-spaces.

We first note that $(\mathrm{Oblv}(\mathfrak{f}))^{(1)}$ is computed as follows. Using that Frobenius twist commutes with limits of R-spaces, and that each $\mathfrak{f}_\alpha$ is quasicoherent, for a R-algebra S, we have

$$\begin{aligned}(\mathrm{Oblv}(\mathfrak{f}))^{(1)}(\mathsf{S}) = (\varprojlim \mathfrak{f}_\alpha)^{(1)}(\mathsf{S}) &\simeq (\varprojlim \mathfrak{f}_\alpha^{(1)})(\mathsf{S}) \\ &\simeq \varprojlim(\mathfrak{f}_\alpha^{(1)}(\mathsf{S})) \\ &\simeq \varprojlim \mathsf{S} \underset{\mathsf{S}}{\otimes} \mathfrak{f}_\alpha(\mathsf{S}).\end{aligned}$$

To prove the proposition, it therefore suffices to check that the composition

$$\mathsf{S} \underset{\mathsf{S}}{\otimes} \mathfrak{f}(\mathsf{S}) \xrightarrow{\mathfrak{z}_p} U(\mathfrak{f}(\mathsf{S})) \to \widetilde{U}(\mathfrak{f}(\mathsf{S}))$$

is continuous, i.e., factors through $\varprojlim \mathsf{S} \underset{\mathsf{S}}{\otimes} \mathfrak{f}_\alpha(\mathsf{S})$. I.e., for fixed β, we must show the composition

$$\mathsf{S} \underset{\mathsf{S}}{\otimes} \mathfrak{f}(\mathsf{S}) \xrightarrow{\mathfrak{z}_p} U(\mathfrak{f}(\mathsf{S})) \to U(\mathfrak{f}(\mathsf{S}))/U(\mathfrak{f}(\mathsf{S}))\cdot\mathfrak{k}_\beta, \qquad 1\otimes X \mapsto X^p - X^{[p]} \bmod U(\mathfrak{f}(\mathsf{S}))\cdot\mathfrak{k}_\beta.$$

factors through $\mathsf{S} \underset{\mathsf{S}}{\otimes} \mathfrak{f}_\gamma(\mathsf{S})$, for some γ, i.e., annihilates $1 \otimes \mathfrak{k}_\gamma$. If $\gamma \geqslant \beta$, then $X^p \in U(\mathfrak{f}(\mathsf{S})) \cdot \mathfrak{k}_\beta$, for any $X \in \mathfrak{k}_\gamma$. By the continuity of the restricted power map, for all sufficiently large γ' we have $X^{[p]} \in \mathfrak{k}_\beta$ for all $X \in \mathfrak{k}_{\gamma'}$, and this γ' may be chosen uniformly in S. In particular, taking such a γ' to be greater than β, we see the composition annihilates $1 \otimes \mathfrak{k}_{\gamma'}$, as desired. □

Remark 5.10. Note that again, for a fixed $\mathfrak{f}$, the map $\xi_p : \mathfrak{f}^{(1)} \to \widetilde{U}(\mathfrak{f})$ factors through the center of the latter, i.e., $Z(\widetilde{U}(\mathfrak{f}))$. Indeed, this follows by continuity from the analogous assertion for uncompleted enveloping algebras, cf. Remark 5.6.

5.1.14. Let us explicitly state the desired consequence of Proposition 5.9 which will be used going forwards.

Corollary 5.11. *Suppose $\mathfrak{f}$ is a pro-quasicoherent restricted Lie algebra in R-spaces, arising as the restricted Lie algebra of an ind-affine group ind-scheme F over R. Then we have a canonical p-center map of R-spaces*

$$\xi_p : \mathfrak{f}^{(1)} \to Z(\widetilde{U}(\mathfrak{f})).$$

Moreover, if $\mathcal{G}$ is a group R-space acting on $\mathfrak{f}$ by restricted Lie algebra automorphisms, then the map ξ_p is $\mathcal{G}$-equivariant, where $\mathcal{G}$ acts on $\mathfrak{f}^{(1)}$ via the relative Frobenius $\mathrm{Fr} : \mathcal{G} \to \mathcal{G}^{(1)}$, and the tautological action of the latter group R-space on $\mathfrak{f}^{(1)}$.

In particular, one may take $\mathfrak{f}$ to be a Kac–Moody or Virasoro Lie algebra, as in Section 2.6 and $\mathcal{G}$ to be the loop group $F(\!(t)\!)$ or $\mathrm{Aut}(\mathcal{D}^\times)$, respectively.

5.2. **Poisson structures.** Assume that R is an algebra over $\mathbb{F}_p$ with $p > 2$ (and $p > 3$ if $\mathfrak{g}$ contains a component of type G_2).

The goal of this section is to show the equality between two Poisson brackets on the p-center of $\widehat{U}_\kappa(\hat{\mathfrak{g}})$: one coming from the identification with $\widehat{S}_{\mathsf{AS}(\kappa)}(\hat{\mathfrak{g}}^{(1)})$ and the other coming from a deformation.

Assume for now that $\mathsf{R} = \mathbb{F}_p$. We can form the algebra $\hat{\mathfrak{g}}_{\mathbb{Z}_p}$ using the same K_0-class of representation W as was used to form $\hat{\mathfrak{g}}_{\mathbb{F}_p}$. In particular, $\widehat{U}(\hat{\mathfrak{g}}_{\mathbb{Z}_p})/(p)$ is identified with $\widehat{U}(\hat{\mathfrak{g}}_{\mathbb{F}_p})$ as an algebra over $\mathbb{F}_p[\mathbf{1}, \mathbf{1}']$.

For x, y in the center of $\widehat{U}(\hat{\mathfrak{g}}_{\mathbb{F}_p})$, consider their lifts $\breve{x}, \breve{y}$ in $U(\hat{\mathfrak{g}}_{\mathbb{Z}_p})$. Then the image of $\{x, y\} := \frac{1}{p!}[\tilde{x}, \tilde{y}]$ is a well-defined element of the center of $\widehat{U}(\hat{\mathfrak{g}}_{\mathbb{F}_p})$. This gives a Poisson bracket. For a general $\mathbb{F}_p$-algebra R, we get a Poisson bracket by base change.

On the other hand, $\widehat{S}(\hat{\mathfrak{g}}^{(1)})$ comes with a natural Poisson bracket, as the symmetric algebra of a Lie algebra.

Lemma 5.12. *The embedding ι from $\widehat{S}(\hat{\mathfrak{g}}^{(1)})$ to the center of $\widehat{U}(\hat{\mathfrak{g}})$ is Poisson.*

Proof. It is enough to show that $\iota(\{x, y\}) = \{\iota(x), \iota(y)\}$ when x is an element of the form $x = e_{\alpha,(n)}$ or x is of the form $h_{(n)}$ for a central element $h \in \mathfrak{g}$. Here $n \in \mathbb{Z}$. The reason is that these elements generate the Lie algebra $\hat{\mathfrak{g}}$.

Case 1. First, assume $x = e_{\alpha,(n)}$. The claim that ι is equivariant from Proposition 5.11 implies that $x^{(p)}.\iota(y) = \iota(\{x, y\})$. It remains to show that $x^{(p)}.\iota(y) = \{\iota(x), \iota(y)\}$. Let $\breve{y}$ be a lift of $\iota(y)$ to $\widehat{U}(\hat{\mathfrak{g}}_{\mathbb{Z}_p})$ and $\breve{x} := e_{\alpha,(n)}$. Then $x^{(p)}.\iota(y)$ is the image of $\breve{x}^{(p)}.\breve{y} = \frac{1}{p!}\operatorname{ad}(\breve{x})^p\breve{y}$. So we need to show that the image in $\widehat{U}(\hat{\mathfrak{g}}_{\mathbb{F}_p})$ of

$$\frac{1}{p!}\left(\operatorname{ad}(\breve{x})^p - \operatorname{ad}(\breve{x}^p)\right)\tilde{y}$$

is zero, equivalently, that the latter element is divisible by p. The element is equal to $\sum_{i=1}^{p-1}\frac{(-1)^i}{(p-i)!i!}\breve{x}^{p-i}\breve{y}x^i$ and its image is

$$\sum_{i=1}^{p-1}\frac{(-1)^i}{(p-i)!i!}x^{p-i}\iota(y)x^i = \left(\sum_{i=1}^{p-1}\frac{(-1)^i}{(p-i)!i!}\right)x^p\iota(y),$$

the last equality holds because $\iota(y)$ is central. It remains to note that $\sum_{i=1}^{p-1}\frac{(-1)^i}{(p-i)!i!} = 0$ in $\mathbb{Z}[\frac{1}{(p-1)!}]$: when we multiply this element by $p!$ we get $(1-1)^p$.

Case 2. Let $x = h_{(n)}$ for central $h \in \mathfrak{g}_{\mathbb{F}_p}$. By symmetry, we can also assume that $y = f_{(m)}$ for central $f \in \mathfrak{g}_{\mathbb{F}_p}$. We can pick lifts $\breve{x} := \breve{h}_{(n)}, \breve{y} := \breve{f}_{(m)}$ for central $\breve{h}, \breve{f} \in \mathfrak{g}_{\mathbb{Z}_p}$. Set $\breve{c} := \breve{\beta}_W(\breve{h}, \breve{f})\mathbf{1} \in \hat{\mathfrak{g}}_{\mathbb{Z}_p}$. The element $\{x, y\}$ is zero unless $m + n = 0$, on the other hand $[\breve{h}^p_{(n)} - \breve{h}_{(np)}, \breve{f}^p_{(m)} - \breve{f}_{(mp)}]$ is easily seen to be divisible by p^2 if $m+n \neq 0$, so $\{\iota(x), \iota(y)\} = 0$ in this case too. It remains to consider the case when $m + n = 0$. Here $\{x, y\} = nc$ so $\iota(\{x, y\}) = n(c^p - c)$. Now we examine

$$[\breve{h}^p_{(n)} - \breve{h}_{(np)}, \breve{f}^p_{(m)} - \breve{f}_{(mp)}] = [\breve{h}^p_{(n)}, \breve{f}^p_{(n)}] + np\breve{c}.$$

Note that $[\breve{h}^p_{(n)}, \breve{f}^p_{(n)}]$ is congruent to $\operatorname{ad}(h_{(n)})^p(\breve{f}^p_{(n)}) = p!n^p\breve{c}^p$. And $\frac{1}{p!}(p!n^p\breve{c}^p + np\breve{c})$ is congruent to $n(c^p - c)$ finishing the proof. □

5.3. **p-centers of $V_\kappa(\mathfrak{g})$ and $\mathsf{CDO}_\kappa(F)$.** Let F be a smooth algebraic group over R, where R is a perfect field containing $\mathbb{F}_p$. Fix representations $W_1, \ldots, W_k$ of F and form the corresponding group ind-scheme $F(\!(t)\!)^\flat$, see Section 2.5.2 so that we

can talk about the p-center of $U(\mathfrak{f}(\!(t)\!))^\flat)$, see Section 5.1.5. In particular, this gives rise to the p-center in $U_\kappa(\hat{\mathfrak{f}})$.

5.3.1. Recall the p-center of $V_\kappa(\mathfrak{f})$ following [AW, Section 2.3] and [ATV, Section 3.3]. Note that we can view $V_\kappa(\mathfrak{f})$ as a quotient of $U_\kappa(\hat{\mathfrak{f}})$, where 1 goes to $|\varnothing\rangle$. By definition, the p-center of $V_\kappa(\mathfrak{f})$, to be denoted by $\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{f}))]$, it is the image of the p-center in $V_\kappa(\mathfrak{f})$. This is a central vertex subalgebra, see [ATV, Lemma 3.7].

5.3.2. Now we proceed to defining and studying the p-center of $\mathsf{CDO}_\kappa(F)$.

Also we consider the subalgebra $\mathsf{R}[(\mathcal{J}F)^{(1)}] \subset \mathsf{R}[\mathcal{J}F]$.

Lemma 5.13. *The following claims hold:*

(1) The subalgebras $\mathsf{R}[(\mathcal{J}F)^{(1)}], \iota_L(\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{f})))$ *of* $\mathsf{CDO}_\kappa(F)$ *are central.*

(2) These subalgebras freely generate the center of $\mathsf{CDO}_\kappa(F)$.

Sketch of proof. (1): Note that, as a vertex algebra, $\mathsf{CDO}_\kappa(F)$ is generated by $\mathsf{R}[\mathcal{J}F]$ and $V_\kappa(F)$. Every endomorphism xt^n for $x \in \mathfrak{f}$ and $n \geqslant 0$ of $\mathsf{R}[\mathcal{J}F]$ annihilates $\mathsf{R}[(\mathcal{J}F)^{(1)}]$. It follows that the elements of $\mathsf{R}[(\mathcal{J}F)^{(1)}]$ are centralized by $V_\kappa(\mathfrak{f})$ and $\mathsf{R}[\mathcal{J}F]$, so are central.

Since $\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{f}))$ is central in $V_\kappa(\mathfrak{f})$, it remains to show that $a_{(n)}f = 0$ for all $a \in \mathfrak{z}_{Fr}(V_\kappa(\mathfrak{f}))$ and $f \in \mathsf{R}[\mathcal{J}F]$. We note that $\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{f}))$ is a commutative vertex algebra that is generated, as an associative algebra, by the elements $(xt^i)^p - x^{[p]}t^{ip}, i < 0$. So we need to show that

$$[(xt^i)^p - x^{[p]}t^{ip}]_{(n)}f = 0, \forall n \geqslant 0.$$

This expression can be computed using [AW, Lemma 2.3]: the operator $[(xt^i)^p - x^{[p]}t^{ip}]_{(n)}$ is 0 if n is coprime to p and is equal to $\binom{n/p-i-1}{-i-1}([xt^{n/p}]^p - x^{[p]}t^n)$ else. Note that [AW, Lemma 2.3] does this computation inside the vertex algebra $V_\kappa(\mathfrak{f})$ but the same argument works for all of its modules, including $\mathsf{R}[\mathcal{J}F]$. Since $\mathsf{R}[\mathcal{J}F]$ is a rational representation of the pro-algebraic group, the elements $([xt^{n/p}]^p - x^{[p]}t^n)$ act on $\mathsf{R}[\mathcal{J}F]$ by zero.

(2): We note that since the naive filtration on $\mathsf{CDO}_\kappa(F)$ is almost commutative, the associated graded $\operatorname{gr}\mathsf{CDO}_\kappa(F)$ carries a natural Poisson vertex algebra structure, see Section 3.1.7. Recall the identification $\operatorname{gr}\mathsf{CDO}_\kappa(F) \cong \mathsf{R}[\mathcal{J}(T^*F)]$, a special case of (3.11). We claim that the restriction of the bracket on $\operatorname{gr}\mathsf{CDO}_\kappa(F)$ to $\mathsf{R}[T^*F] \subset \mathsf{R}[\mathcal{J}(T^*F)]$ is the standard bracket. These easily follows from the identities $\iota_\ell(xt^{-1})_{(0)}\iota_\ell(yt^{-1}) = \iota_\ell([x,y]t^{-1})$ (that follows because ι_ℓ is an embedding of vertex algebras) and $\iota_\ell(xt^{-1})_{(0)}f = x.f$ for $x \in \mathfrak{f}, f \in \mathsf{R}[F]$ that is a consequence of (3.10).

We will show that the Poisson center of $\mathsf{R}[\mathcal{J}(T^*F)]$ coincides with $\mathsf{R}[(\mathcal{J}(T^*F))^{(1)}]$, this will imply the claim of the lemma.

We can cover F by open affines such that each of these open affines admits an etale morphism to $\mathbb{A}^n$, where n is the dimension of X over $\operatorname{Spec}(\mathsf{R})$. Let F^0 be one of these open affines. We have an isomorphism

$$\mathcal{J}(T^*F^0) \cong F^{0,(1)} \times_{\mathbb{A}^{n,(1)}} \mathcal{J}(T^*\mathbb{A}^n). \tag{5.5}$$

Note that $\mathsf{R}[\mathcal{J}(T^*\mathbb{A}^n)]$ carries a Poisson vertex algebra structure whose restriction to $\mathsf{R}[T^*\mathbb{A}^n]$ is the usual one (for example, as the associated graded of the chiral differential operators on the additive group $\mathbb{A}^n$). There is a unique extension of the Poisson structure from $\mathsf{R}[\mathcal{J}(T^*\mathbb{A}^n)]$ to $\mathsf{R}[F^{0,(1)} \times_{\mathbb{A}^{n,(1)}} \mathcal{J}(T^*\mathbb{A}^n)]$ such that $\mathsf{R}[F^{0,(1)}]$ is central. Now (5.5) gives two Poisson vertex algebra structures on $\mathsf{R}[\mathcal{J}(T^*F^0)]$, whose restrictions to $\mathsf{R}[T^*F^0]$ coincide with the usual bracket. By Section 3.3.4, they coincide.

Let X be a smooth affine finite type R-scheme. Note that the action of $\mathbb{G}_m$-action on T^*X by fiberwise dilations equips $\mathsf{R}[\mathcal{J}(T^*X)]$ with an energy grading. The graded pieces are finitely generated modules over $\mathsf{R}[X^{(1)}]$. In case $\mathsf{R}[\mathcal{J}(T^*X)]$ carries a Poisson vertex algebra structure such that the bracket on $\mathsf{R}[T^*X]$ is the standard one, the Poisson structure is compatible with the grading, and the Poisson center is graded. Notice that, for each m, the equations giving the degree m component of the Poisson center in $\mathsf{R}[F^{0,(1)} \times_{\mathbb{A}^{n,(1)}} \mathcal{J}(T^*\mathbb{A}^n)]$ is obtained by pulling back the degree m component in the Poisson center of $\mathsf{R}[\mathcal{J}(T^*\mathbb{A}^n)]$. This reduces our task to showing that the Poisson center of $\mathsf{R}[\mathcal{J}(T^*\mathbb{A}^n)]$ coincides with $\mathsf{R}[\mathcal{J}(T^*\mathbb{A}^n)^{(1)}]$. This is verified by a direct check and is left as an exercise. □

5.3.3. Now we are going to give a description of the center $\mathfrak{z}(\mathsf{CDO}_\kappa(G))$ as a commutative energy graded vertex algebra with an arc group action.

Set $\mathsf{AS}(\kappa) := \kappa^p - \kappa$. Recall the commutative vertex algebra $\operatorname{gr}\mathsf{CDO}_{\mathbf{1}}(F)$. This algebra is a deformation of $\mathsf{R}[\mathcal{J}(T^*F)]$ over $\mathsf{R}[\mathbf{1}]$. Consider the Frobenius twist $(\operatorname{gr}\mathsf{CDO}_{\mathbf{1}}(F))^{(1)}$, an algebra over $\mathsf{R}[\mathbf{1}]^{(1)}$, and its specialization to $\mathbf{1}$ to $\mathsf{AS}(\kappa)$, to be denoted by $\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{J}(T^*F)^{(1)}]$. The $\mathcal{J}F \times \mathcal{J}F$-action on $\operatorname{gr}\mathsf{CDO}_{\mathbf{1}}(F)$, see Section 3.6.6 gives rise to an action of $(\mathcal{J}F)^{(1)} \times (\mathcal{J}F)^{(1)}$ on $\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{J}(T^*F)^{(1)}]$.

It is an easy exercise to show that a vertex action of an arc group on a vertex algebra preserves the center. In particular, the group $\mathcal{J}F \times \mathcal{J}F$ acts on $\mathfrak{z}(\mathsf{CDO}_\kappa(F))$.

Note that both $\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{J}(T^*F)^{(1)}]$ and $\mathfrak{z}(\mathsf{CDO}_\kappa(F))$ come with natural energy gradings.

Lemma 5.14. *There is a $\mathcal{J}F \times \mathcal{J}F$-equivariant energy graded vertex algebra isomorphism*

$$\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{J}(T^*F)^{(1)}] \xrightarrow{\sim} \mathfrak{z}(\mathsf{CDO}_\kappa(F)),$$

where the action of $\mathcal{J}F \times \mathcal{J}F$ on the source is pulled back from the action of $(\mathcal{J}F)^{(1)} \times (\mathcal{J}F)^{(1)}$.

Sketch of proof. This is a consequence of Lemma 5.13 and the easy analogous statements for the affine vertex algebras and the functions on jets. □

5.4. **Main result.** We are going to show that $\mathsf{ffr}_P : V_\kappa(\mathfrak{g}) \to \mathsf{CDO}(N) \otimes V_\kappa(\mathfrak{l})$ restricts to a map between the p-centers and describe the restriction.

Namely, consider the homomorphism

$$\mathsf{ffr}_P^{0,(1)} : \mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g})) \to \mathsf{R}[(\mathcal{J}T^*N)^{(1)}] \otimes \mathfrak{z}_{Fr}(V_\kappa(\mathfrak{l})). \tag{5.6}$$

Proposition 5.15. *The restriction of the homomorphism* $\mathsf{ffr}_{P,\kappa}$ *to* $\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g}))$ *coincides with the composition of* $\mathsf{ffr}^{0,(1)}_{P,\mathsf{AS}(\kappa)}$ *and the embedding of the target of (5.6) into* $\mathsf{CDO}(N)\otimes V_\kappa(\mathfrak{l})$.

Proof. The proof tracks the construction of ffr_P. The restriction of the embedding $V_\kappa(\mathfrak{g})\to\mathsf{CDO}_\kappa(G^0)$ sends $\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g}))=\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{J}\mathfrak{g}^*]$ to the copy of $\mathsf{R}_{\mathsf{AS}(\kappa)}[(\mathcal{J}\mathfrak{g}^*)^{(1)}]$ inside $\mathfrak{z}(\mathsf{CDO}_\kappa(G^0))=\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{J}(T^*G^0)^{(1)}]$. The restriction of η_κ is nothing else but $\eta^{0,(1)}_{\mathsf{AS}(\kappa)}$. It follows that the homomorphism $\theta_\kappa: V_\kappa(\mathfrak{g})\to\mathsf{CDO}(N)\otimes V_\kappa(\mathfrak{p}^-)$ restricts to

$$\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g}))\to\mathfrak{z}(\mathsf{CDO}(N))\otimes\mathfrak{z}(\mathsf{CDO}(P^-))^{\mathcal{J}P^-}. \tag{5.7}$$

Under the identifications

$$\mathsf{R}_{\mathsf{AS}(\kappa)}[(\mathcal{J}\mathfrak{g}^*)^{(1)}]\xrightarrow{\sim}\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g})),$$
$$\mathfrak{z}(\mathsf{CDO}(P^-))^{JP^-}\xrightarrow{\sim}\mathsf{R}[\mathcal{J}(T^*N)^{(1)}]\otimes\mathsf{R}_{\mathsf{AS}(\kappa)}[(\mathcal{J}\mathfrak{p}^{-,*})^{(1)}].$$

(5.7) becomes $\theta^{0,(1)}_{\mathsf{AS}(\kappa)}$. Next, $\pi_\kappa: V_\kappa(\mathfrak{p}^-)\twoheadrightarrow V_\kappa(\mathfrak{l})$ restricts to a homomorphism $\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{p}^-))\twoheadrightarrow\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{l}))$, which is nothing else but $\pi^{0,(1)}_{\mathsf{AS}(\kappa)}$. We see that $\mathsf{ffr}_{P,\kappa}$ restricts to a homomorphism between p-centers. Moreover, thanks to (4.5) and the preceding proof, this restriction coincides with $\mathsf{ffr}^{0,(1)}_{P,\mathsf{AS}(\kappa)}$. □

5.5. **Map between p-centers of universal enveloping algebras.** Our goal in this section is to get an analog of Proposition 5.15 for the universal enveloping algebras.

Lemma 5.16. *Let* $V=V_\kappa(\mathfrak{l})$ *or* $\mathsf{CDO}(N)$ *and* $\mathfrak{z}_{Fr}(V)$ *denote the p-center in the first case and the center in the 2nd case. Then the natural map* $\widehat{U}(\mathfrak{z}_{Fr}(V))\to\widehat{U}(V)$ *is an embedding whose image is the p-center of* $\widehat{U}(V)$.

Proof. The proof easily follows from description of PBW bases. For example, the $\mathfrak{z}_{Fr}(\mathsf{CDO}(N))$ is the algebra of polynomials in the elements $(y^i_n)^p,\partial^p_{j,m}$ (with $n\geqslant 0, m>0$). The algebra $\widehat{U}(\mathfrak{z}_{Fr}(\mathsf{CDO}(N)))$ is then the algebra of infinite sums of the monomials in the elements $(y^i_n)^p,\partial^p_{j,m}$ for $n,m\in\mathbb{Z}$. The homomorphism $\widehat{U}(\mathfrak{z}_{Fr}(\mathsf{CDO}(N)))\to\widehat{U}(\mathsf{CDO}(N))$ is continuous and sends such a monomial to the analogous monomial. So it is injective. □

We note that the image of $\widehat{U}(\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g})))$ in $\widehat{U}_\kappa(\hat{\mathfrak{g}})$ is nothing else but the completion of the p-center in $U_\kappa(\hat{\mathfrak{g}})$.

Note also that $\widetilde{U}(\mathfrak{z}_{Fr}(V_\kappa(\mathfrak{g})))$ is identified with $\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{L}(\mathfrak{g}^*)^{(1)}]$. Let $\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{L}(\mathfrak{g}^*)^{(1)}]_f$ denote the filtered part (=the union of filtered pieces) of $\mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{L}(\mathfrak{g}^*)^{(1)}]$. This filtered algebra is identified with the p-center in $\widehat{U}_\kappa(\hat{\mathfrak{g}})$.

Using this lemma and Proposition 5.15, we deduce the following corollary.

Corollary 5.17. *The restriction of the homomorphism*

$$\mathsf{Ffr}_{P,\kappa}:\widehat{U}_\kappa(\hat{\mathfrak{g}})\to\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})$$

to the p-center coincides with the composition of

$$\mathsf{Ffr}^{0,(1)}_{P,\mathsf{AS}(\kappa)} : \mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{L}(\mathfrak{g}^*)^{(1)}]_f \to \mathsf{R}[\mathcal{L}(T^*N)^{(1)}]_f \widehat{\otimes} \mathsf{R}_{\mathsf{AS}(\kappa)}[\mathcal{L}(\mathfrak{l}^*)^{(1)}]_f$$

and the embedding of the target to $\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})$.

6. Affine Harish-Chandra homomorphism

6.1. **Construction.** The goal of this section is to prove the following

Proposition 6.1. *The homomorphism*

$$\mathsf{Ffr}_P : \widehat{U}_\kappa(\hat{\mathfrak{g}}) \to \widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})$$

restricts to $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \to \widehat{U}_\kappa(\hat{\mathfrak{l}})^{\mathcal{L}L}$.

For $L = H$, this map should be thought of as an affine analog of the Harish-Chandra homomorphism, hence the name of the section. The restriction of ffr_P will be denoted by HC_L.

Proof. Recall, Section 4.4.3, that ffr_P is $\mathcal{L}P$-equivariant, and so restricts to a map between $\mathcal{L}P$-invariants. Since $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \hookrightarrow \widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}P}$, the claim in the previous paragraph follows from the next lemma. □

Lemma 6.2. *The algebra of invariants* $[\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})]^{\mathcal{L}P}$ *coincides with* $\widehat{U}_\kappa(\hat{\mathfrak{l}})^{\mathcal{L}L}$ *(embedded into the 2nd factor).*

Proof. The proof is in several steps.

Step 1. Here we prove that $\widehat{D}(\mathcal{L}N)^{\mathcal{L}N}$ coincides with the image of $\widehat{U}(\mathcal{L}\mathfrak{n})$ under the embedding induced by ι_r (this image consists of invariants by the construction of ι_r). It is enough to show this claim after passing to the associated graded algebra. There $\mathsf{R}[\mathcal{L}(T^*N)]_f = \mathsf{R}[\mathcal{L}N]\widehat{\otimes}\mathsf{R}[\mathcal{L}\mathfrak{n}^*]_f$ with $\mathcal{L}N$ acting on the first factor by left translations. Take an element of degree d for the naive grading (by degree in $\mathcal{L}\mathfrak{n}^*$). It can be uniquely written as a converging sum $\sum_b f_b \otimes b$, where b runs over the monomial topological basis in the degree d component of $\mathsf{R}[\mathcal{L}\mathfrak{n}^*]$ and $f_b \in \mathsf{R}[\mathcal{L}N]$. Let α denote the co-action map for the action of $\mathcal{L}N$ on $\mathsf{R}[\mathcal{L}N]$. Then the co-action map for the action of $\mathcal{L}N$ on $\mathsf{R}[\mathcal{L}(T^*N)]_f$ sends $\sum_b f_b \otimes b$ to $\sum_b \alpha(f_b) \otimes b$. This element equals $\sum_b (1 \otimes f_b) \otimes b$ if and only if $\alpha(f_b) = 1 \otimes f_b$. This is only possible if f_b is in R, yielding our claim.

Step 2. Observe that there is a one-parameter subgroup $\mathbb{G}_m \to Z(L)$ that acts on $\mathfrak{n}$ (and hence $\mathcal{L}\mathfrak{n}$) with positive weights. The invariants in $\widehat{U}(\mathcal{L}\mathfrak{n})$ for this action therefore coincide with R. We conclude that $\widehat{D}(\mathcal{L}N)^{\mathcal{L}N \rtimes \mathbb{G}_m} = \mathsf{R}$.

Step 3. Note that the subgroup $\mathcal{L}N \rtimes \mathbb{G}_m$ acts trivially on $\widehat{U}_\kappa(\hat{\mathfrak{l}})$. By the previous step,

$$[\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{l}})]^{\mathcal{L}P} \subset \widehat{U}_\kappa(\hat{\mathfrak{l}}),$$

which easily implies the claim of the lemma. □

6.1.1. We also note that, for the same reason, the homomorphism

$$\mathsf{Ffr}^0_P : \widehat{S}_\kappa(\hat{\mathfrak{g}}) \to \mathsf{R}[\mathcal{L}(T^*N)]_f \widehat{\otimes} \widehat{S}_\kappa(\hat{\mathfrak{l}})$$

restricts to

$$\mathsf{HC}^0_L : \widehat{S}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \to \widehat{S}_\kappa(\hat{\mathfrak{l}})^{\mathcal{L}L}.$$

6.1.2. We have similar (and easier) constructions on the level of vertex algebras, for example, we have $\mathsf{HC}_{L,\kappa} : V_\kappa(\mathfrak{g})^{\mathcal{J}G} \to V_\kappa(\mathfrak{l})^{\mathcal{J}L}$.

6.2. **Properties.** Here we study the properties of the maps HC_L and HC^0_L.

6.2.1. Let HC^L_H denote the Harish-Chandra homomorphism $\widehat{U}_\kappa(\hat{\mathfrak{l}})^{\mathcal{L}L} \to \widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$.

It follows from (4.9) that

$$\mathsf{HC}_{H,\kappa} = \mathsf{HC}^L_{H,\kappa} \circ \mathsf{HC}_{L,\kappa}. \tag{6.1}$$

Similarly we have

$$\mathsf{HC}^0_{H,\kappa} = \mathsf{HC}^{L,0}_{H,\kappa} \circ \mathsf{HC}^0_{L,\kappa}. \tag{6.2}$$

6.2.2. Here we describe the map $\mathsf{HC}^0_{L,0} : \mathsf{R}[\mathcal{L}\mathfrak{g}^*]^{\mathcal{L}G}_f \to \mathsf{R}[\mathcal{L}\mathfrak{l}^*]^{\mathcal{L}L}_f$. The embedding $\mathfrak{l}^* \hookrightarrow \mathfrak{g}^*$ gives rise to an embedding $\mathcal{L}\mathfrak{l}^* \hookrightarrow \mathcal{L}\mathfrak{g}^*$.

Lemma 6.3. *The homomorphism* $\mathsf{HC}^0_{L,0}$ *coincides with the restriction to* $\mathcal{L}\mathfrak{l}^*$.

Proof. $\mathsf{HC}^0_{L,0}$ coincides with the composition of $\mathsf{ffr}^0_B : \mathsf{R}[\mathcal{L}\mathfrak{g}^*]_f \to \mathsf{R}[\mathcal{L}(T^*N)]_f \widehat{\otimes} \mathsf{R}[\mathcal{L}\mathfrak{l}^*]_f$ and the projection to the 2nd factor (tensored with the augmentation homomorphism $\mathsf{R}[\mathcal{L}(T^*N)]_f \to \mathsf{R}$). This composition is obtained by applying the functor $\widehat{U}$ from the analogous vertex algebra homomorphism $\mathsf{R}[\mathcal{J}\mathfrak{g}^*] \to \mathsf{R}[\mathcal{J}\mathfrak{l}^*]$. It follows from Section 4.2.2 that the latter is the restriction for the embedding $\mathcal{J}\mathfrak{l}^* \hookrightarrow \mathcal{J}\mathfrak{g}^*$. This implies the claim of the lemma. □

6.2.3. Our goal here is to prove the following lemma.

Lemma 6.4. *Suppose* 3 *is invertible in* R *if* $\mathfrak{g}$ *has a component of type* G_2. *The homomorphisms* $\mathsf{HC}_{L,\kappa}$ *and* $\mathsf{HC}^0_{L,\kappa}$ *are injective.*

Proof. Note that $\operatorname{gr} \widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \hookrightarrow \mathsf{R}[\mathcal{L}\mathfrak{g}^*]^{\mathcal{L}G}$ and similarly for $\mathfrak{l}$. Then $\operatorname{gr} \mathsf{HC}_{L,\kappa}$ is the restriction of $\mathsf{HC}^0_{L,0}$. And thanks to (6.2), it is enough to show that $\mathsf{HC}^0_{H,0}$ is injective.

By Lemma 6.3, HC^0_H is the restriction to $\mathcal{L}\mathfrak{h}^*$. A standard argument, cf. the proof of [F, Proposition 4.3.4] reduces the proof of the injectivity of this map to showing the injectivity of the restriction map $\mathsf{R}[\mathcal{J}\mathfrak{g}^*]^{\mathcal{J}G} \to \mathsf{R}[\mathcal{J}\mathfrak{h}^*]$. The injectivity of the latter is equivalent to the claim that the maps $\mathsf{R}[\mathcal{J}_n\mathfrak{g}^*]^{\mathcal{J}_nG} \to \mathsf{R}[\mathcal{J}_n\mathfrak{h}^*]$ are injective for all n. This will follow once we show that the morphism $\mathcal{J}_nG \times^{\mathcal{J}_nH} \mathcal{J}_n\mathfrak{h}^* \to \mathcal{J}_n\mathfrak{g}^*$ is dominant. Since both schemes are defined over $\mathbb{Z}[2^{-1}]$ and are flat, it is enough to assume that R is an algebraically closed field of characteristic different from 2. There the claim follows by computation of the differential at a point $(1, x)$ with $x \in (\mathfrak{h}^*)^{reg}$: the roots are nonzero on $\mathfrak{h}$. □

6.2.4. For the same reason, $\mathsf{HC}_{L,\kappa} : V_\kappa(\mathfrak{g})^{\mathcal{J}G} \to V_\kappa(\mathfrak{l})^{\mathcal{J}L}$ is injective.

6.3. **The Harish-Chandra center in abelian case.** Here we describe the algebra $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$, where R is an algebra over $\mathbb{F}_p$.

6.3.1. First, assume that R is a general commutative ring. Choose an energy graded basis $b_\alpha, \alpha \in A$, and order the indexes so that, for $\alpha < \beta$, the energy of b_α does not exceed the energy of B_β. Then we have a filitered $\mathsf{R}[\mathbf{1}]$-module isomorphism $S(\hat{\mathfrak{h}})/(\mathbf{1}'-1) \xrightarrow{\sim} U(\hat{\mathfrak{h}})/(\mathbf{1}'-1)$ that sends an ordered monomial in b_α's to the same ordered monomial.

Lemma 6.5. *This isomorphism extends by continuity to $\widehat{S}_\kappa(\hat{\mathfrak{h}}) \xrightarrow{\sim} \widehat{U}_\kappa(\hat{\mathfrak{h}})$ and this extension is $\mathcal{L}H$-equivariant.*

Proof. The claim about the extension is manifest. To prove that the resulting isomorphism is $\mathcal{L}T$-equivariant, we argue as follows. Note that for an R-point $\mathcal{L}T$, we have $g.b_\alpha = b_\alpha + f_\alpha(g)\mathbf{1}$ with $f_\alpha(g) \in \mathsf{R}$. Plugging the elements $b_\alpha + f_\alpha(g)\mathbf{1}$ into an ordered monomial instead of b_α and distributing, we get the sum of still ordered monomials. The same argument works over any extension of R. It follows that the isomorphism $\widehat{S}_\kappa(\hat{\mathfrak{h}}) \xrightarrow{\sim} \widehat{U}_\kappa(\hat{\mathfrak{h}})$ is $\mathcal{L}H$-equivariant. □

6.3.2. Now we are ready to describe $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$.

Proposition 6.6. *Assume that R is an $\mathbb{F}_p$-algebra and κ is invertible in R. Then the elements of the form $(xt^j)^p - \kappa^{p-1}(xt^{jp})$ with $x \in \mathfrak{h}_{\mathbb{F}_p}$ and $j \in \mathbb{Z}$ lie in $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$. Moreover, any invariant element in $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}T}_{\leqslant j}$ (the PBW filtration term) is uniquely written as the converging sum of degree $\leqslant j/p$ polynomial in the elements $(xt^j)^p - \kappa^{p-1}(xt^{jp})$.*

Proof. The claim easily reduces to the case when $\dim H = 1$, which is what we are going to assume. The proof is in several steps.

Step 1. We use Lemma 6.5 to see that the claim of the proposition is equivalent to its direct analog for $\widehat{S}_\kappa(\hat{\mathfrak{h}})$. Indeed, note that the image in $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}_{\leqslant j}$ of an ordered monomial in elements $(xt^j)^p - \kappa^{p-1}(xt^{jp}) \in \widehat{S}_\kappa(\hat{\mathfrak{h}})$ is also the same ordered monomial but now $(xt^j)^p - \kappa^{p-1}(xt^{jp})$ are viewed as elements of $\widehat{U}_\kappa(\hat{\mathfrak{h}})$.

Step 2. In this and the subsequent steps we prove the complete analog of the claim of proposition for $\widehat{S}_\kappa(\hat{\mathfrak{h}})$, where we can assume that $\kappa = 1$. We write b_n for t^n viewed as an element of $\hat{\mathfrak{h}}$. The action of $\mathcal{L}H$ on b_n is given by $g.b_n = b_n + \chi_n(g)$, where $\chi_n(g) = \mathrm{Res}_{t=0}(t^n g^{-1} dg)$. One can then check that $b_n^p - b_{np}$ is $\mathcal{L}H$-invariant, for example, using Step 1 and Proposition 5.11.

Step 3. We claim that the elements χ_n (viewed as elements of $\mathsf{R}[\mathcal{L}H]$ are topologically linearly independent: the equality $\sum_{n \geqslant n_0} \alpha_n \chi_n = 0$ for some n_0 and $\alpha_n \in \mathsf{R}$ implies $\alpha_n = 0$. Assume the contrary: there are elements α_n with $\sum_{n \geqslant n_0} \alpha_n \chi_n = 0$. Let $f(t) = \sum_{n \geqslant N} \alpha_n t^n$. Note that we have $\mathrm{Res}_{t=0}(f(t)g(t)^{-1}dg(t))$ for all $g(t) \in \mathsf{R}'(\!(t)\!)^\times$ for all possible R-algebras R'.

Step 4. Take $\mathsf{R}' := \mathsf{R}[x]/(x^N)$ for a sufficiently large integer N. Consider $g(t) = t - x$. Then $dg(t) = 1$, while $g(t)^{-1} = t^{-1}(1 - t^{-1}x)^{-1} = t^{-1}\sum_{i=0}^{N-1} t^{-i}x^i$. From $\mathrm{Res}_{t=0}(f(t)g(t)^{-1}dg(t)) = 0$ we deduce that $\sum_{i=0}^{N-1} \alpha_n x^n = 0$ in R'. This shows

that $\alpha_n = 0$ for $0 \leqslant n < N$. Similarly, considering $g(t) = t^{-1} - x$, we see that $\alpha_n = 0$ for $-N < n \leqslant 0$. We conclude that all $\alpha_n = 0$. This finishes the proof of the claim in Step 3 that the characters χ_n are topologically linearly independent.

Step 5. Let V be the (topological) span of monomials in the elements b_i, where each element has power strictly less than p. Note that V is $\mathcal{L}H$-stable. We claim $V^{\mathcal{L}H}$ consists of scalars. Namely, let f be an element in V and let f' be its top degree homogeneous component, of degree, say d. Then the top degree homogeneous component in $gf - f$ is

$$\sum_i \chi_i(g)\partial_i f', \tag{6.3}$$

where we write ∂_i for the derivative with respect to b_i. Note that since f is converging, for each monomial F of degree $d-1$, there is $n_0(F)$ such that F does not appear in $\partial_i f'$ for $i < n_0(F)$.

Since the characters χ_i are topologically linearly independent, we see that $\partial_i f' = 0$ for all i. Since $f' \in V$, this implies $f' = 0$.

Step 6. We note that, for each k, $\widehat{S}_1(\hat{\mathfrak{h}})_{\leqslant kp-1}$ is the completed tensor product of V and the completion of $\mathbb{F}[b_i^p - b_{ip}]_{\leqslant k-1}$. The action of $\mathcal{L}H$ respects this decomposition and is trivial on the completion of $\mathbb{F}[b_i^p - b_{ip}]_{\leqslant k-1}$. It follows from Step 5 that the subspace of $\mathcal{L}H$-invariants in $\widehat{S}_1(\hat{\mathfrak{h}})_{\leqslant kp-1}$ equals to the completion of $\mathbb{F}[b_i^p - b_{ip}]_{\leqslant k-1}$. This finishes the proof. □

6.3.3. An analogous (and easier) argument shows that, for an invertible element $\kappa \in \mathsf{R}$, the algebra $V_\kappa(\mathfrak{h})^{\mathcal{J}\mathfrak{h}}$ is the algebra of polynomials in $(xt^j)^p - \kappa^{p-1}(xt^{jp})$ with $j < 0$ and x running through a basis in $\mathfrak{h}_{\mathbb{F}_p}$. One just needs to modify Step 4 considering $g(t) = 1 - tx$.

6.4. **Case of integral κ.** In this section R is a perfect field of positive characteristic p. We assume that p is bigger than the Coxeter number for any almost simple normal subgroup of G.

Recall the $\mathcal{L}G$-equivariant inclusion $\iota : \widehat{S}_{\mathsf{AS}(\kappa)}(\hat{\mathfrak{g}}^{(1)}) \hookrightarrow \widehat{U}_\kappa(\hat{\mathfrak{g}})$.

Here is the main result.

Theorem 6.7. *The following claims are true:*

(1) $\operatorname{gr} \widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \subset \widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}}$*, the containment of subalgebras of* $\widehat{S}(\mathcal{L}\mathfrak{g})^{\mathcal{L}G}$*.*

(2) If $\kappa \in \mathbb{F}_p \setminus \{0\}$*, then*

$$\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} = \iota\left(\widehat{S}\left((\mathcal{L}\mathfrak{g})^{(1)}\right)^{(\mathcal{L}G)^{(1)}}\right).$$

6.4.1. Recall the injective (by Lemma 6.4) homomorphism

$$\mathsf{HC}^0_{H,0} : \widehat{S}(\mathcal{L}\mathfrak{g})^{\mathcal{L}G} \hookrightarrow \widehat{S}(\mathcal{L}\mathfrak{h}),$$

it is given by restriction from $\mathcal{L}\mathfrak{g}^*$ to $\mathcal{L}\mathfrak{h}^*$, see Lemma 6.3. Consider its Frobenius twist $\mathsf{HC}^{0,(1)}_{H,0}$. Below we embed $\mathsf{R}[(\mathcal{L}\mathfrak{h}^*)^{(1)}]_f$ into $\mathsf{R}[(\mathcal{L}\mathfrak{h}^*)]_f$ via Fr^*. We will need the following lemma.

Lemma 6.8. *We have*

$$\operatorname{im}(\mathsf{HC}^{0,(1)}_{H,0}) = \operatorname{im}\mathsf{HC}^0_{H,0} \cap \mathsf{R}[(\mathcal{L}\mathfrak{h}^*)^{(1)}]_f. \tag{6.4}$$

Proof. We have the inclusion of the left hand side into the right hand side, we need to show that it is the equality. The proof is in several steps.

Step 1. Let $F_1, \ldots, F_r$ be free homogeneous generators of $\mathsf{R}[\mathfrak{g}^*]^G$. We then can consider the elements $F_{i,n} \in \mathsf{R}[\mathcal{J}(\mathfrak{g}^*)]^{\mathcal{J}G}$ for $i = 1, \ldots, r$ and $n < 0$. By [ATV, Theorem 4.4], the elements $F_{i,n}$ are free generators of $\mathsf{R}[\mathcal{J}(\mathfrak{g}^*)]^{\mathcal{J}G}$. Then the same argument as in the proof of [F, Proposition 4.3.4] shows that

$$\mathsf{R}[\mathcal{L}(\mathfrak{g}^*)]^{\mathcal{L}G} = \varprojlim_{n\to+\infty} \mathsf{R}[(t^{-n}\mathfrak{g}^*[[t]])]^{\mathcal{J}G}$$

Step 2. Thanks to this, we can deduce (6.4) from its jet analog, and the statements about jets follows from that for nth jets (for all n). In other words, let res_n denote the embedding $\mathsf{R}[\mathcal{J}_n(g^*)]^{\mathcal{J}_nG} \to \mathsf{R}[\mathcal{J}_n(\mathfrak{h}^*)]$ (we have seen that this map is an embedding in the proof of Lemma 6.4). We need to show that

$$\operatorname{im}(\mathsf{res}^{(1)}_n) = \operatorname{im}(\mathsf{res}_n) \cap \mathsf{R}[(\mathcal{J}_n\mathfrak{h}^*)^{(1)}]. \tag{6.5}$$

Step 3. We note that since $\mathsf{res}^{(1)}_\ell$ is injective as well, the left hand side is a normal domain. The right hand side is integral over the left hand side because $\mathsf{R}[\mathcal{J}_n\mathfrak{g}^*]^{\mathcal{J}_nG}$ is integral over $\mathsf{R}[(\mathcal{J}_n\mathfrak{g}^*)^{(1)}]^{(\mathcal{J}_nG)^{(1)}}$ (the latter contains the pth powers of the elements of the former). It remains to show that the right hand side has the same fraction field as the left hand side. Consider the element $\delta \in \mathsf{R}[\mathfrak{g}^*]^G \subset \mathsf{R}[\mathcal{J}_n(\mathfrak{g}^*)]^{\mathcal{J}G_n}$, the product of squares of all roots, and localize both sides of (6.5) at $\mathsf{res}_n(\delta^p)$. Note that the action map $G \times^{N_G(H)} \mathfrak{h}^{*,reg} \to \mathfrak{g}^{*,reg}$ is an isomorphism, hence the same is true for $\mathcal{J}_nG \times^{N_G(H)\mathcal{J}_nH} \mathcal{J}_n\mathfrak{h}^{*,reg} \to \mathcal{J}_n\mathfrak{g}^{*,reg}$. So the localization of the left hand side of (6.5) is $\left(\mathsf{R}[(\mathcal{J}_n\mathfrak{h}^*)^{(1)}][\mathsf{res}_n(\delta^p)^{-1}]\right)^W$, while the localization of the right hand side is $\mathsf{R}[\mathcal{J}_n\mathfrak{h}^*][\mathsf{res}_n(\delta^p)^{-1}] \cap \mathsf{R}[(\mathcal{J}_n\mathfrak{h}^*)^{(1)}][\mathsf{res}_n(\delta^p)^{-1}]$. It is easy to see that these two subalgebras are the same. □

6.4.2. Now we are ready to prove the theorem.

Proof of Theorem 6.7. Let $\mathcal{Z}$ denote $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$ and $\mathcal{Z}_0 := \iota(\widehat{S}\left((\mathcal{L}\mathfrak{g})^{(1)}\right)^{(\mathcal{L}G)^{(1)}})$. When $\kappa \in \mathbb{F}_p \setminus \{0\}$, we have $\mathcal{Z}_0 \subset \mathcal{Z}$. Both claims of the theorem will then follow from

$$\operatorname{gr}\mathcal{Z} \subset \widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}} \tag{6.6}$$

Note that $\operatorname{gr}\mathcal{Z} \subset \widehat{S}(\mathcal{L}\mathfrak{g})^{\mathcal{L}G}$. Thanks to Lemma 6.4, $\mathsf{HC}^0_{H,0}$ is injective. It follows that

$$\operatorname{gr}(\operatorname{im}\mathsf{HC}_{H,\kappa}) \subset \operatorname{im}\mathsf{HC}^0_{H,0}. \tag{6.7}$$

From Proposition 6.6 it follows that $\operatorname{gr}\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H} = \widehat{S}((\mathcal{L}\mathfrak{h})^{(1)})$. So

$$\operatorname{gr}(\operatorname{im}\mathsf{HC}_{H,\kappa}) \subset \widehat{S}((\mathcal{L}\mathfrak{h})^{(1)}). \tag{6.8}$$

Combining (6.7) and (6.8) we see that that $\mathsf{HC}^0_{H,0}(\operatorname{gr}\mathcal{Z})$ is contained in

$$\mathsf{HC}^0_{H,0}(\widehat{S}(\mathcal{L}\mathfrak{g})^{\mathcal{L}G} \cap \widehat{S}\left((\mathcal{L}\mathfrak{h})^{(1)}\right)).$$

Lemma 6.8, the letter coincides with $\mathsf{HC}^0_{H,0}[\widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}}$. Again using that $\mathsf{HC}^0_{0,H}$ is injective, we arrive at (6.6). □

6.4.3. A complete analog of Theorem 6.7 holds for vertex algebras, in particular $\operatorname{gr} V_\kappa(\mathfrak{g})^{\mathcal{J}G} \subset \mathsf{R}[(\mathcal{J}\mathfrak{g}^*)^{(1)}]^{(\mathcal{J}G)^{(1)}}$. The proof is the same using an analog of Proposition 6.6 explained in Section 6.3.3.

Lemma 6.9. *Suppose* $\operatorname{gr}[V_\kappa(\mathfrak{g})^{\mathcal{J}G}] = \mathsf{R}[(\mathcal{J}\mathfrak{g}^*)^{(1)}]^{(\mathcal{J}G)^{(1)}}$. *Consider the filtered algebra homomorphism* $\psi : \widehat{U}(V_\kappa(\mathfrak{g})^{\mathcal{J}G}) \to \widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$, *cf. Lemma 3.15. Then*

(1) this homomorphism is an isomorphism,

(2) and $\operatorname{gr}\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} = \widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}}$.

Proof. Thanks to Step 1 in the proof of Lemma 6.8, we have $\mathsf{R}[\mathcal{J}\mathfrak{g}^*]^{\mathcal{J}G} \cong \mathsf{R}[\mathcal{J}(\mathfrak{g}^*//G)]$ and $\widehat{S}(\mathcal{L}\mathfrak{g})^{\mathcal{L}G} = \mathsf{R}[\mathcal{L}(\mathfrak{g}^*//G)]_f$. Thanks to $\operatorname{gr}(V_\kappa(\mathfrak{g})^{\mathcal{J}G}) = \mathsf{R}[(\mathcal{J}(\mathfrak{g}^{*,(1)}//G^{(1)})]$, we have that

$$\mathsf{R}[(\mathcal{L}\mathfrak{g}^*//G)^{(1)}]_f \xrightarrow{\sim} \widehat{U}(V_\kappa(\mathfrak{g})^{\mathcal{J}G}).$$

The composition of this isomorphism, $\operatorname{gr}\psi$, and the inclusion of $\operatorname{gr}\psi$ with the inclusion $\operatorname{gr}\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \hookrightarrow \operatorname{gr}\widehat{U}_\kappa(\hat{\mathfrak{g}})$ is the inclusion $\mathsf{R}[(\mathcal{L}\mathfrak{g}^*//G)^{(1)}]_f \hookrightarrow \widehat{S}(\mathcal{L}\mathfrak{g})$, its image is $\widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}}$.

In particular, $\operatorname{gr}\psi$ is injective. And from (1) of Theorem 6.7 and the description of the composition in the previous paragraph we conclude that $\operatorname{gr}\psi$ is an isomorphism. □

7. The case of $\hat{\mathfrak{sl}}_2$

Here we suppose that 2 is invertible in R. Take $G = \mathrm{SL}_2$.

7.1. **Sugawara elements.** Let e, h, f be the standard basis elements of $\mathfrak{g}$. Let $(\cdot,\cdot)$ be the invariant symmetric bilinear form on $\mathfrak{g}$ normalized in such a way that $(h,h) = 2$. This is the form we use to form $\hat{\mathfrak{g}}$.

Pick a basis x_1, x_2, x_3 of (the $\mathbb{Z}[\frac{1}{2}]$-form of) $\mathfrak{g}$ and the dual basis x^1, x^2, x^3 with respect to $(\cdot,\cdot)$. For example, we can take $x_1 = f, x_2 = h, x_3 = e$, then $x^1 = f, x^2 = h/2, x^3 = e$. For $x \in \mathfrak{g}$, we write x_n for $xt^n \in \hat{\mathfrak{g}}$.

7.1.1. Consider the Sugawara elements

$$L_n := \frac{1}{2}\sum_{i=1}^{3}\sum_{j\in\mathbb{Z}} :x_{i,n+j}x^i_{-j}: \in \widehat{U}_\kappa(\hat{\mathfrak{g}}), \tag{7.1}$$

where $:\bullet:$ denotes the normally ordered product. Equivalently,

$$L_n = \frac{1}{2}\sum_{i=1}^{3}\sum_{j\in\mathbb{Z}} x_{i,n+j}x^i_{-j}, \text{ for } n \neq 0,$$

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{3} x_i x^i + 2\sum_{i=1}^{3}\sum_{j>0} x_{i,-j}x^i_j\right).$$

7.1.2. We will need the following well-known property of the elements L_i:

$$[L_i, x_n] = -\kappa n x_{i+n}, \forall x \in \mathfrak{g}_{\mathsf{R}}, i, n \in \mathbb{Z}. \tag{7.2}$$

(7.2) implies the following identity. Note that for a R-point g of $\mathcal{L}G$ and a R-linear derivation ∂ of $\mathsf{R}((t))$, we can view $g^{-1}\partial g$ as an R-point of $\mathcal{L}\mathfrak{g}$. For example, for $g = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix}$, and $\partial = t^n \frac{d}{dt}$, we have $g^{-1}\partial g = \kappa f_n$. Then

$$\mathrm{Ad}(g^{-1})L_i = L_i - \kappa g^{-1}(t^{i+1}\frac{d}{dt})g. \tag{7.3}$$

7.1.3. We can consider the analogs of Sugawara elements $\underline{L}_i \in \widehat{S}_{\mathsf{AS}(\kappa)}(\hat{\mathfrak{g}})$. They are given by

$$\underline{L}_n := \frac{1}{2}\sum_{i=1}^{3}\sum_{j\in\mathbb{Z}} x_{i,n+j}x^i_{-j}. \tag{7.4}$$

Further, we have the direct analog of (7.3), where we replace L_i with $\underline{L}_i$.

7.2. **Construction of central elements.** Let R be an $\mathbb{F}_p$-algebra. We note that (7.3) implies the following claim.

Lemma 7.1. *The elements $Y_i := L_i^p$ for i not divisible by p and $Y_i := L_i^p - \kappa^{p-1}L_{ip}$ for i divisible by p are central in $\widehat{U}_\kappa(\hat{\mathfrak{g}})$.*

Now we proceed to another sequence of central elements.

Proposition 7.2. *For all n, $X_n := \kappa^{p-1}\iota(\underline{L}_n) - (\kappa^{p-1}-1)Y_n \in \widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$.*

Proof. The proof is in several steps. We note that it is enough to prove the claim for $\mathsf{R} = \mathbb{F}_p[\kappa]$, where κ is an indeteminate. The claim for this ring will follow from the claim for its field of fractions, and then from the claim for the algebraic closure. So we can and will assume that κ is invertible in an algebraically closed field R.

Step 1. Let us outline the strategy of the proof. Note that by the construction both Y_n and $\underline{L}_n$ are G-invariant. In the next two steps we will see that $\kappa^{p-1}\iota(\underline{L}_n) - (\kappa^{p-1}-1)Y_n$ is invariant with respect to the element $g := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Then we will see that this implies that it is $\mathcal{J}G$-invariant, and then that it is $\mathcal{L}G$-invariant.

Step 2. Here we compute $\mathrm{Ad}(g^{-1})Y_n$ for $g = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. By (7.3) and the computation preceding this equation, we have $\mathrm{Ad}(g)^{-1}(L_n) = L_n - \kappa f_{n+1}$ for all n.

Now we compute $\mathrm{Ad}(g^{-1})L_n^p = (\mathrm{Ad}\, g^{-1})L_n)^p = (L_n - \kappa f_{n+1})^p$. Recall that for elements x, y of a Lie algebra over $\mathbb{F}_p$, the element $(x+y)^p - x^p - y^p$ in the universal enveloping algebra is actually a degree p Lie polynomial, denote it by $\sigma(x, y)$, cf. Definition 2.2. If the elements

(*) $\mathrm{ad}(x)^j y$ and $\mathrm{ad}(x)^k y$ commute for all k, j,

then $\sigma(x,y)$ has degree $p-1$ in x and degree 1 in y. An easy check shows that this component coincides $\mathrm{ad}(x)^{p-1}y$. We see that

$$(L_n - f_{n+1})^p = L_n^p - \kappa\,\mathrm{ad}(L_n)^{p-1} f_{n+1} - \kappa^p f_{n+1}^p = \\ L_n^p - \kappa^p (f_{n+1}^p - \kappa^p \left(\prod_{j=0}^{p-1}(jn+1)\right) f_{pn+1}.$$

We note that if n is not divisible by p, then the last summand is equal to 0, otherwise the coefficient of f_{pn+1} is equal to κ^p. In both cases, we have

$$\mathrm{Ad}(g)^{-1} Y_n = Y_n - \kappa^p f_{n+1}^p. \tag{7.5}$$

Step 3. By Section 7.1.3, we have

$$\mathrm{Ad}(g)^{-1}\underline{L}_n = \underline{L}_n - (\kappa^p - \kappa) f_{n+1}. \tag{7.6}$$

Note that $\iota(f_{n+1}) = f_{n+1}^p$. Combining (7.5) and (7.6), we see that $\kappa^{p-1}(\iota(\underline{L}_n)) - (\kappa^{p-1}-1)Y_n$ is g-invariant.

Step 4. We claim that any G- and g-invariant element $a \in \widehat{U}_\kappa(\hat{\mathfrak{g}})_{\leqslant i}$ (for some i) is $\mathcal{J}G$-invariant. Recall that $V := \widehat{U}_\kappa(\hat{\mathfrak{g}})_{\leqslant i}$ is a complete topological space whose base of neighborhoods of 0 is formed by $U_N := \widehat{U}_\kappa(\hat{\mathfrak{g}})_{\leqslant i-1}\mathfrak{g}(\geqslant N), N > 0$, where we write $\mathfrak{g}(\geqslant N)$ for the span of x_n with $n \geqslant N$. Each U_N is $\mathcal{J}G$-stable. Moreover, V/U_N is a rational representation of $\mathcal{J}G$, in particular, $a_N := a + U_N$ lies in a subspace where the action of $\mathcal{J}G$ factors through $\mathcal{J}_{d(N)}G$ for some $d(N) > 0$. Note that a is $\mathcal{J}G$-invariant if and only if a_N is $\mathcal{J}_{d(N)}G$-invariant. It remains to notice that, as an abstract group, $\mathcal{J}_nG$ is generated by G and g for all n (recall that R is an algebraically closed field). We conclude that a is $\mathcal{J}G$-invariant.

Step 5. We now claim that any $\mathcal{J}G$-invariant element in V is $\mathcal{L}G$-invariant. Consider the affine Grassmanian Gr, the fpqc quotient $\mathcal{L}G/\mathcal{J}G$. In Lemma 7.3 we will see that the algebra of global functions on Gr is R. Since Gr is the fpqc quotient, the subalgebra of $\mathcal{J}G$-invariants in $\mathsf{R}[\mathcal{L}G]$ is R.

Now suppose that $v \in V$ is a $\mathcal{J}G$-invariant element. Let $\alpha : V \to V\widetilde{\otimes}\mathsf{R}[\mathcal{L}G]$ denote the coaction map. Let $\mathcal{B}$ be a topological basis of V compatible with the energy grading. As any element of $V\widetilde{\otimes}\mathsf{R}[\mathcal{L}G]$ is uniquely written as a converging sum $\sum_{b\in\mathcal{B}} b \otimes f_b$ for $f_b \in \mathsf{R}[\mathcal{L}G]$, we see that all f_b are $\mathcal{J}G$-invariant. So $f_b \in \mathsf{R}$ for all b, hence v is $\mathcal{L}G$-invariant. □

Lemma 7.3. *The algebra of global functions on* Gr *is* R.

Proof. The ind-scheme Gr is ind-projective: it is a countable union of finite type projective schemes: $\mathsf{Gr} = \bigcup_{i\geqslant 0} Z_i$. For Z_i we can take the scheme of the following form: we have a vector space V_i of dimension $2d_i$ equipped with a nilpotent linear operator. For Z_i we take the subscheme of the Grassmanian of d_i-dimensional subspaces in V_i that are stable with respect to the operator. In particular, Z_i is connected. On the other hand, according to [BL, Theorem 6.4], Gr is an integral ind-scheme, i.e., is the countable union of integral reduced finite type schemes. We note that [BL] assumes that the base field is $\mathbb{C}$, however, the proof works over R as well. Now according to [BL, Lemma 6.3], $\mathsf{Gr} = \bigcup_{i\geqslant 0} Z_{i,red}$, where the subscript

"red" means the reduced scheme. We have $\mathsf{R}[\mathsf{Gr}] = \varprojlim_j \mathsf{R}[Z_{i,red}]$. Each $\mathsf{R}[Z_{i,red}]$ is R finishing the proof. □

7.3. **Generators of the Harish-Chandra center.** Suppose that R is a perfect field of characteristic $p > 2$. The following is the main result of this section.

Theorem 7.4. *The algebra $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G}$ is the filtered complete polynomial algebra in the elements X_n.*

Proof. By (1) of Theorem 6.7, we have a graded algebra embedding $\widehat{U}_\kappa(\hat{\mathfrak{g}})^{\mathcal{L}G} \hookrightarrow \widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}}$. By a special case of the claim in Step 1 in the proof of Lemma 6.8, the algebra $\widehat{S}((\mathcal{L}\mathfrak{g})^{(1)})^{(\mathcal{L}G)^{(1)}}$ is a graded complete polynomial algebra in homogeneous generators

$$\frac{1}{2}\sum_{i=1}^{3}\sum_{j\in\mathbb{Z}} x_{i,n+j}^p (x_{-}^i)^p. \tag{7.7}$$

Notice that (7.7) is also the principal symbol of the element X_n. This is because the elements Y_n and $\iota(\underline{L}_n)$ are of PBW degree $2p$ and their principal symbols are (7.7). The claim about the principal symbols of the X_n's implies the claim of the theorem in the standard fashion. □

7.4. **Poisson bracket on the centers of $\widehat{U}_\kappa(\hat{\mathfrak{g}}), \widehat{U}_\kappa(\hat{\mathfrak{h}})$.** We continue to assume that R is a perfect field of characteristic $p > 2$.

As was explained in Section 5.2, the centers of $\widehat{U}_\kappa(\hat{\mathfrak{g}}), \widehat{U}_\kappa(\hat{\mathfrak{h}})$ carry natural Poisson structures. The purpose of this section is to compute the brackets between the elements X_n, X_m. This will be needed for the computation of $\mathsf{HC}(X_n)$ in Section 7.5.

7.4.1. We start with easier computations.

Lemma 7.5. *The following claims are true:*

(1) $\{\iota(\underline{L}_m), X_n\} = 0$ *for all* $n, m \in \mathbb{Z}$

(2) $\{\iota(\underline{L}_m), \iota(\underline{L}_n)\} = (\kappa^p - \kappa)(m-n)\iota(\underline{L}_{n+m})$.

Proof. (1): Since X_n is $\mathcal{L}G$-invariant, see Proposition 7.2, the argument of the proof of Lemma 5.12 implies that $\{\iota(y), X_n\} = 0$ for y of the form e_k or $f_k, k \in \mathbb{Z}$. These two families topologically generate the Poisson algebra $\widehat{S}_{\mathsf{AS}(\kappa)}(\hat{\mathfrak{g}}^{(1)})$ and so $\{\iota(y), X_n\} = 0$ for all $y \in \widehat{S}_{\mathsf{AS}(\kappa)}(\hat{\mathfrak{g}}^{(1)})$, finishing the proof of (1).

(2): By Lemma 5.12, ι is a Poisson homomorphism. So we need to prove that

$$\{\underline{L}_m, \underline{L}_n\} = \mathsf{AS}(\kappa)(m-n)\underline{L}_{n+m}.$$

This formula is a straightforward consequence of its direct analog in $\widehat{S}(\hat{\mathfrak{g}})$. That formula follows from the $\mathfrak{sl}_2$-case of [K, (12.8.8)] by passing to the associated graded. This finishes the proof. □

7.4.2. It remains to compute the bracket between the elements Y_n, Y_m. Note that if $\kappa = 0$, then the elements $L_i \in \widehat{U}_\kappa(\hat{\mathfrak{g}})$ are $\mathcal{L}G$-invariant for all o. In particular, $\{Y_m, Y_n\} = 0$. So, assume $\kappa \neq 0$. Then we can consider the elements $\kappa^{-1}L_i$ and $\kappa^{-p}Y_i$.

Thanks to [K, (12.8.8)], the elements $\kappa^{-1}L_i$ satisfy the following commutation relation

$$[\kappa^{-1}L_m, \kappa^{-1}L_n] = (m-n)\kappa^{-1}L_{n+m} + \delta_{m+n,0}\frac{3(\kappa-2)}{2\kappa}\binom{m+1}{3}. \tag{7.8}$$

Recall the Virasoro algebra $\mathfrak{Vir}$ from Section 2.5.4, it has a topological basis $L'_i, 1$. In particular, $L'_i \mapsto \kappa^{-1}L_i, 1 \mapsto -\frac{3(\kappa-2)}{2\kappa}$ defines a homomorphism $\psi : \widehat{U}(\mathfrak{Vir}) \to \widehat{U}_\kappa(\hat{\mathfrak{g}})$. We note that this homomorphism also makes sense over $\mathbb{Z}_p$ (where we use a version of $\widehat{U}$, where κ is an invertible indeterminate).

Both $\widehat{S}(\mathfrak{Vir}^{(1)})$ and the center $Z(\widehat{U}(\mathfrak{Vir}))$ are Poisson algebras similarly to Section 5.2.

Lemma 7.6. *The inclusion $\iota : \widehat{S}(\mathfrak{Vir}^{(1)}) \hookrightarrow Z(\widehat{U}(\mathfrak{Vir}))$ is Poisson.*

Proof. The proof is similar to that of Lemma 5.13. It reduces to showing that the inclusion intertwines the brackets with $\iota(L'_i)$. The proof is in several steps.

Until the further notice assume that $i \geqslant 0$. Recall, Proposition 5.11, that ι is $\mathrm{Aut}(\mathcal{D}^\times)$-equivariant, and, in particular, $\mathrm{Aut}(\mathcal{D})$-equivariant. Note that $\mathrm{Aut}(\mathcal{D})$ is a pro-algebraic group, it is the inverse limit of the groups $\mathrm{Aut}(\mathcal{D})_i$: as functor this group sends a R-algebra S to the group of S-linear automorphisms of $S[t]/(t^{i+1})$.

We claim that the bracket with $\iota(L'_i)$ in $Z(U(\mathfrak{Vir}))$ is the same as the action of $L'_i \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{D}))$. For this we will show that the latter action is induced from $\frac{1}{p!}(\mathrm{ad}(\breve{L}'_i)^p - \mathrm{ad}(\breve{L}'^{[p]}_i))$.

First, suppose that we are given an algebraic group F over $\mathbb{Z}_p$ and let $\breve{\xi} \in \mathfrak{f}_{\mathbb{Z}_p}$. Then $\frac{1}{p!}(\breve{\xi}^p - \breve{\xi}^{[p]})$ is an element of the distribution algebra $\mathrm{Dist}_1(F_{\mathbb{Z}_p})$. We will abuse the notation and write $\xi^{(p)}$ for the image of this element in $\mathrm{Dist}_1(F)$, here ξ is the image of $\breve{\xi}$. Under the homomorphism $\mathsf{Fr}_* : \mathrm{Dist}_1(F) \to \mathrm{Dist}_1(F^{(1)})$ induced by the Frobenius morphism $\mathsf{Fr} : F \to F^{(1)}$, the element $\xi^{(p)}$ goes to ξ viewed as an element of $\mathfrak{f}^{(1)}$.

Consider now the restriction of ι to $S(\mathfrak{Vir}^{(1)})_{\leqslant j}$ to $Z(U(\mathfrak{Vir}))_{\leqslant pj}$. For $N \in \mathbb{Z}_{>0}$, let I_N be the left ideal in $U(\mathfrak{Vir})$ generated by the elements L'_i for $i \geqslant N$. Then ι induces a homomorphism

$$S(\mathfrak{Vir}^{(1)})_{\leqslant j}/(S(\mathfrak{Vir}^{(1)})_{\leqslant j} \cap I_N) \to Z(U(\mathfrak{Vir}))_{\leqslant pj}/(Z(U(\mathfrak{Vir}))_{\leqslant pj} \cap I_N). \tag{7.9}$$

Both sides are rational representations of the pro-algebraic group

$$\mathrm{Aut}(\mathcal{D}) = \varprojlim_{k\to\infty} \mathrm{Aut}(\mathcal{D})_i.$$

Applying the previous paragraph to $F := \mathrm{Aut}(\mathcal{D})_i$, we see that (7.9) intertwines the actions of the operators $L'^{(p)}_i$. It follows that $\iota : S(\mathfrak{Vir}^{(1)}) \hookrightarrow Z(U(\mathfrak{Vir}))$ intertwines the actions of $L'^{(p)}_i$.

The action of $L_i'^{(p)}$ on $S(\mathfrak{Vir}^{(1)})$ is by $\{L_i', \cdot\}$. Similarly to the proof of Lemma 5.12, the action of $L_i'^{(p)}$ on $Z(U(\mathfrak{Vir}))$ is by $\{\iota(L_i'), \cdot\}$.

It remains to show that for all $i < 0, a \in S(\mathfrak{Vir}^{(1)})$ we have $\{\iota(L_i'), a\} = \{\iota(L_i'), \iota(a)\}$. For this, consider the "polynomial" version of the Virasoro algebra, to be denoted by $\underline{\mathfrak{Vir}}$, one where the elements $L_i', 1$ form a genuine, not a topological, basis. Note that this algebra has an automorphism sending L_m' to $-L_{-m}', 1$ to -1. Now we use the case of $i \geqslant 0$ to deduce that $\{\iota(L_i'), a\} = \{\iota(L_i'), \iota(a)\}$ for all $a \in S(\underline{\mathfrak{Vir}}^{(1)})$. By continuity, we get the same equality for all $a \in \widehat{S}(\mathfrak{Vir}^{(1)})$. □

7.4.3. Now we are ready to compute $\{X_m, X_n\}$.

Corollary 7.7. *We have*

$$\{Y_m, Y_n\} = \kappa^p \left((m-n) Y_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{2} (\kappa^{p-1} - 1) \right),$$
$$\{X_m, X_n\} = (1 - \kappa^{p-1}) \kappa^p \left((m-n) X_{m+n} - \delta_{m+n,0} \frac{m^3 - m}{2} (1 - \kappa^{p-1})^2 \right).$$

Proof. To prove (1) we use the notation from Section 7.4.2. (1) follows from the observation that $Y_i = \kappa^p \psi(\iota(L_i'))$ for all i and the formulas $\{\iota(L_i'), \iota(L_j')\} = \iota([L_i', L_j'])$ and

$$\psi(\iota(1)) = \psi(1^p - 1) = -\frac{3(\kappa - 2)^p}{2\kappa^p} + \frac{3(\kappa - 2)}{2\kappa} = \frac{3(1 - \kappa^{p-1})}{\kappa^p}.$$

Now we proceed to (2). By (1) of Lemma 7.5, we have

$$\{X_m, X_n\} = \kappa^{2(p-1)} \{\iota(\underline{L}_m, \underline{L}_n)\} - (\kappa^{p-1} - 1)^2 \{Y_m, Y_n\}.$$

The first summand is computed in (2) of Lemma 7.5, while the 2nd is computed using (1). Simplifying the resulting expression, we get (2). □

7.4.4. Set $\tilde{h}_m := h_m^p - \kappa^{p-1} h_{mp}$, this is an element in $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$. Moreover, the latter is a filtered complete algebra of polynomials in the variables $\tilde{h}_n$, this is a special case of Proposition 6.6. We proceed to computing $\{\tilde{h}_m, \tilde{h}_m\}$ in the center of $\widehat{U}_\kappa(\hat{\mathfrak{h}})$. The following lemma is proved similarly to Case 2 in the proof of Lemma 5.12.

Lemma 7.8. *We have* $\{\tilde{h}_m, \tilde{h}_n\} = \delta_{m+n,0} 2m\kappa^p (1 - \kappa^{p-1})$.

7.4.5. Set $X_n' := \frac{1}{4} \left(\sum_{j \in \mathbb{Z}} \tilde{h}_{j+n} \tilde{h}_{-j} \right)$. We have the following standard corollary of Lemma 7.8.

Corollary 7.9. *We have the following equalities*

$$\{X_m', \tilde{h}_j\} = \kappa^p (\kappa^{p-1} - 1) j \tilde{h}_{j+m},$$
$$\{X_m', X_n'\} = \kappa^p (1 - \kappa^{p-1})(m-n) X_{m+n}'.$$

7.5. **Formula for $\mathsf{HC}(X_n)$.**

Proposition 7.10. *We have*

$$\mathsf{HC}(X_n) = \frac{1}{4}\left(\sum_{j\in\mathbb{Z}} \tilde{h}_{j+n}\tilde{h}_{-j}\right) - \frac{1}{2}(1-\kappa^{p-1})(n+1)\tilde{h}_n.$$

Proof. Note that HC is a Poisson algebra homomorphism because ffr_H restricts to the Poisson homomorphism from $\widehat{U}_\kappa(\hat{\mathfrak{g}})$ to the center of $\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_\kappa(\hat{\mathfrak{h}})$ and ffr deforms to the algebras over $\mathbb{Z}_p$.

Step 1. Note that $\mathsf{HC}(X_n) - X'_n$ lies in the span of the constant and the element $\tilde{h}_n$ by the PBW degree and energy degree reasons (the constant can be nonzero only if $n=0$). So, $\mathsf{HC}(X_n) = X'_n + \alpha_n\tilde{h}_n + \delta_{n,0}\beta$ for some $\alpha_n \in \mathsf{R}$. Note that X_0 acts by 0 on the $\widehat{U}_\kappa(\hat{\mathfrak{g}})$-module $V_\kappa(\mathfrak{g})$. From the compatibility between Ffr^B_κ and ffr^B_κ explained in Section 3.2.4, we see that $\mathsf{HC}(X_0)$ acts by zero on $\mathsf{CDO}(N)\otimes V_\kappa(\mathfrak{h})$. Hence $\beta = 0$.

Step 2. We show that we have $\alpha_n = n\alpha + \underline{\alpha}$ for some $\alpha, \underline{\alpha} \in \mathsf{R}$. This is standard: we use Corollaries 7.7 and 7.9 to compare the coefficients of $\tilde{h}_{n+m}$ in $\mathsf{HC}(\{X_m, X_n\})$ and in $\{X'_m + \alpha_m\tilde{h}_m, X'_n + \alpha_n\tilde{h}_n\}$.

Step 3. Now we look at the constant terms in

$$\mathsf{HC}(\{X_m, X_{-m}\}) = \{X'_m + (m\alpha + \underline{\alpha})\tilde{h}_m, X'_{-m} + (\underline{\alpha} - m\alpha)\tilde{h}_{-m}\}.$$

Using Corollaries 7.7, 7.9, and Lemma 7.8 we conclude that

$$m^3\alpha^2 - m\underline{\alpha}^2 = (m^3 - m)\frac{1}{4}(1-\kappa^{p-1})^2.$$

This gives the required formula for β and shows that $\alpha = \pm\underline{\alpha} = \pm\frac{1}{2}(1-\kappa^{p-1})$.

Step 4. To prove that $\alpha' = \alpha = -\frac{1}{2}(1-\kappa^{p-1})$ we argue as follows. The coefficients ± 1 are independent of the choice of κ. So in the proof we can assume that $\kappa = 0$. Here $X_n = Y_n = L_n^p$. Note that L_n is already in the Harish-Chandra center. It is well-known, see e.g. [F, Proposition 6.2.2], that

$$\mathsf{HC}(L_n) = \left(\frac{1}{4}\sum_{j\in\mathbb{Z}} h_{j+n}h_{-j}\right) - \frac{1}{2}(n+1)h_n.$$

Taking the pth power (note that h_i's pairwise commute), we see that the coefficients of $n\tilde{h}_n$ and $\tilde{h}_n$ are both $-\frac{1}{2}$. This completes the proof. □

8. Verma and Wakimoto modules

8.1. **Construction.**

8.1.1. First, we introduce algebras whose modules we are going to study. Set $\mathsf{R} := \mathbb{Z}_p$ for $p > 2$. Let G be a split connected reductive group, $H \subset B$ be a maximal torus and Borel subgroup. We fix a parabolic subgroup $P \subset G$ containing B and its Levi decomposition $P = L \ltimes N$, where L contains H. Let $B = H \ltimes \tilde{N}$ be the Levi decomposition for B. We write B_L for $B \cap L$.

Consider the algebra $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}}) := R_{\hbar}(\widehat{U}_{\mathbf{1}}(\hat{\mathfrak{g}}))$. This is an algebra over $\mathsf{R}[\mathbf{1},\hbar]$. Its specialization to $\mathbf{1}=\kappa, \hbar=1$ is $\widehat{U}_{\kappa}(\hat{\mathfrak{g}})$, while its specialization to $\kappa=\hbar=0$ is $\mathsf{R}[\mathcal{L}(\mathfrak{g}^*)]_f$.

The algebra $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$ carries the following compatible gradings. First, there is the naive grading, where the degrees of $\hbar, \mathbf{1}$ and $\hat{\mathfrak{g}}$ are equal to 1. Second consider the root lattice Λ of $\mathfrak{g}$ and the imaginary root δ. Form the affine root lattice $\Lambda^a := \Lambda \oplus \mathbb{Z}\delta$.

Now let V be a topological R-module that is complete and separated with respect to its topology. By a *topological Λ^a-grading* on V we mean the following data:

- A system of neighborhoods $U_N, N>0$, of 0 with $V = \varprojlim_{N\to\infty} V/U_N$,
- and a collection of Λ^a-gradings on V/U_N for each N such that $V/U_{N+1} \to V/U_N$ is graded.

Each graded piece of the algebra $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$ for the naive grading carries a topological Λ^a-grading induced from the Λ^a-grading on $\hat{\mathfrak{g}}$ (with $\mathbf{1}$ and $\hbar$ in degree 0). In what follows we will abuse the terminology and say that the (graded complete) algebra $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$ is Λ^a-graded.

Similarly, we can consider the $\mathsf{R}[\mathbf{1},\hbar]$-algebra $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}})$. It also carries compatible naive and Λ^a-gradings.

Next, we consider the algebra $\widehat{D}^{\hbar}(\mathcal{L}N)$, the Rees algebra of $\widehat{D}(\mathcal{L}N)$. Let $\Phi^+_P, \partial_\alpha, y^\alpha$ for $\alpha \in \Phi^+_P$ have the same meaning as in Section 4.2.3. The algebra $\widehat{D}_{\hbar}(\mathcal{L}N)$ has topological generators $\partial_{\alpha,n}, y^{\beta}_m$ with $\alpha,\beta \in \Phi^+_P$ and $n,m \in \mathbb{Z}$ with relations $[\partial_{\alpha,n}, y^{\beta}_m] = \delta_{m+n,0}\delta_{\alpha,\beta}\hbar$ and all other brackets equal to 0. This algebra also carries naive and Λ^a-gradings. For the naive grading, we have $\deg y^{\beta}_m = 0, \deg\hbar = \deg\partial_{\alpha,n} = 1$. The Λ^a-degrees are as follows: $\deg\hbar = 0, \deg y^{\beta}_m = -\beta + m\delta, \deg\partial_{\alpha,n} = \alpha + n\delta$.

Consider the graded complete tensor product $\widehat{D}^{\hbar}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}})$. Recall, Section 4.4.4, that we have a naive graded $\mathsf{R}[\mathbf{1},1]$-linear algebra homomorphism

$$\mathsf{Ffr}^{\hbar}_{P,\mathbf{1}} : \widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}}) \to \widehat{D}^{\hbar}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}}).$$

It is also Λ^a-graded.

8.1.2. Now we define certain left ideals in the algebras $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}}), \widehat{D}^{\hbar}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}})$ and consider the quotients by these left ideals.

Consider the Iwahori subgroup Iw, the preimage of B^- under $\mathcal{J}G \twoheadrightarrow G$ (we emphasize that we need to consider B^- instead of B here). We choose the system of simple affine roots I^a for this choice of Iwahori, and choose positive affine roots accordingly. Let $I \subset I^a$ be the system of simple roots for $\mathfrak{g}$.

Suppose A is a (graded complete) Λ^a-graded algebra. Consider the left ideal $I_{>0}(A)$ generated by the graded components of degrees in the span of positive roots.

Consider the left ideal $I^{G,\hbar}_{\mathbf{1}} = I_{>0}(\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}}))$. Recall that the loop group $\mathcal{L}G$ acts on $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$. The ideal $I^{G,\hbar}_{\mathbf{1}}$ is stable under Iw. Form the quotient $\Delta^{G,\hbar}_{\mathbf{1}} := \widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})/I^{1}_{\hbar,\mathbf{1}}$, the universal Verma module. It is acted on by Iw and is a rational representation

of this pro-algebraic group. It carries a naive graded and also a (genuine) Λ^a-grading. For $\lambda \in \Lambda^a$, we write $\Delta^{G,\hbar}_{\mathbf{1}}[\lambda]$ for the λ-graded component. It is a free $\mathsf{R}[\mathbf{1},\hbar,\mathfrak{h}^*]$-module of rank equal to the affine Kostant partition function of $-\lambda$ (=the number of ways to represent $-\lambda$ as the sum of positive affine roots). We also note that we have a commuting action of $\mathsf{R}[\mathbf{1},\hbar,\mathfrak{h}^*]$-action on $\Delta^G_{\mathbf{1},\hbar}$, it comes from the observation that $I^{G,\hbar}_{\mathbf{1}}\mathsf{R}[\mathbf{1},\hbar,\mathfrak{h}^*] \subset I^{G,\hbar}_{\mathbf{1}}$. This action is compatible with the naive grading, preserves the Λ^a-grading, and is Iw-invariant.

Similarly, we consider the left ideal

$$I^{P,\hbar}_{\mathbf{1}} = I_{>0}(\widehat{D}^{\hbar}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}})).$$

The group $\mathcal{L}P$ acts on $\widehat{D}^{\hbar}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}})$ by automorphisms and $I^{P,\hbar}_{\mathbf{1}}$ is stable with respect to the intersection $\mathsf{Iw}_P := \mathsf{Iw} \cap \mathcal{L}P$. Set

$$\mathsf{W}\Delta^{P,\hbar}_{\mathbf{1}} := (\widehat{D}^{\hbar}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{l}}))/I^{P,\hbar}_{\mathbf{1}}.$$

We can view $\mathsf{W}\Delta^{P,\hbar}_{\mathbf{1}}$ as a $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$-module via the homomorphism ffr_P. This module carries compatible naive and Λ^a-grading. Notice that

$$\mathsf{W}\Delta^{P,\hbar}_{\mathbf{1}} = [\widehat{D}_{\hbar}(\mathcal{L}N)/I_{>0}(\widehat{D}_{\hbar}(\mathcal{L}N))] \otimes_{\mathsf{R}[\hbar]} \Delta^{L,\hbar}_{\mathbf{1}}.$$

From here we see that the weight λ component $\mathsf{W}\Delta^P_{\mathbf{1},\hbar}[\lambda]$ is a free $\mathsf{R}[\mathbf{1},\hbar]$-module of the same rank as $\Delta^G_{\mathbf{1},\hbar}[\lambda]$. Similarly to the previous paragraph, $\mathsf{R}[\mathbf{1},\hbar,\mathfrak{h}^*]$ acts on $\mathsf{W}\Delta^P_{\mathbf{1},\hbar}$ from the right.

When $P = B$, we write $\mathsf{Wak}_{\hbar,\mathbf{1}}$ for $\mathsf{W}\Delta^B_{\hbar,\mathbf{1}}$.

8.1.3. Let $F \subset \mathsf{Iw}_P \rtimes \mathbb{G}_m$ be an algebraic subgroup. This group acts on $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$, where the $\mathbb{G}_m$ acts by loop rotations. A possible choice is $F := H \times \mathbb{G}_m$. We can consider the categories $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})\text{-mod}^F_{gr}$ of naive graded $\widehat{U}_{\mathbf{1},\hbar}(\hat{\mathfrak{g}})$-modules equipped with a rational action of F that preserves the naive grading and makes the module structure map F-equivariant. Similarly, we can consider the category $\widehat{U}_{\mathbf{1},\hbar}(\hat{\mathfrak{l}})\text{-mod}^F_{gr}$. We have the *Wakimotization* functor

$$\mathsf{Wak}^P : \widehat{U}_{\mathbf{1},\hbar}(\hat{\mathfrak{l}})\text{-mod}^F_{gr} \to \widehat{U}_{\mathbf{1},\hbar}(\hat{\mathfrak{l}})\text{-mod}^F_{gr},$$
$$M \mapsto (\widehat{D}_{\hbar}(\mathcal{L}N)/I_{>0}(\widehat{D}_{\hbar}(\mathcal{L}N))) \otimes_{\mathsf{R}[\hbar]} M.$$

Note that $\mathsf{Wak}^P(\Delta^{L,\hbar}_{\mathbf{1}}) \xrightarrow{\sim} \mathsf{W}\Delta^{P,\hbar}_{\mathbf{1}}$.

8.2. **Main result and reductions.**

8.2.1. We start by introducing some notation. Our base ring is $\mathsf{R} = \mathbb{Z}_p$. Set $\mathfrak{h}^{da} := \mathrm{Spec}(\mathsf{R}[\mathbf{1},\hbar,\mathfrak{h}^*])$, an affine space. For a field $\mathbf{k}$ we write $\mathfrak{h}^{da}(\mathbf{k})$ for the set of $\mathbf{k}$-points of $\mathfrak{h}^{da}$. Fix a point $\mathsf{x} = (\kappa,\nu,\eta) \in \mathfrak{h}^{da}$ (with $\kappa,\nu \in \mathbf{k}, \eta \in \mathfrak{h}^*(\mathbf{k})$). We write $\Delta_{\mathsf{x}}, \mathsf{W}\Delta^P_{\mathsf{x}}, \mathsf{Wak}_{\mathsf{x}}$ for the specializations of the corresponding modules to x.

Note that by the construction we have the unique $\widehat{U}^{\hbar}_{\mathbf{1}}(\hat{\mathfrak{g}})$-linear homomorphism $\varphi^P : \Delta^{G,\hbar}_{\mathbf{1}} \to \mathsf{W}\Delta^{P,\hbar}_{\mathbf{1}}$ sending the coset of 1 to the coset of 1. This homomorphism is $\mathsf{Iw}_P \rtimes \mathbb{G}_m$-equivariant and graded for the naive grading. We write $\varphi^P[\lambda]$ for the specialization of φ^P to the components of degree λ, and φ^P_{x} for the specialization of φ to the point x.

8.2.2. Recall that the algebra $\mathsf{R}[\mathfrak{h}^{da}]$ acts on $\Delta^{G,\hbar}_{\mathbf{1}}, \mathsf{W}\Delta^{P,\hbar}_{\mathbf{1}}$ from the right. The homomorphism φ^P is not equivariant for this action, instead, it is semi-linear with respect to a certain automorphism of $\mathsf{R}[\mathfrak{h}^{da}]$. Since $\hbar$ and $\mathbf{1}$ are central, φ^P is $\mathsf{R}[\mathbf{1},\hbar]$-linear. Set $\rho_P := \frac{1}{2}\sum_{\alpha\in\Phi^+_P}\alpha$.

Lemma 8.1. *For any $x \in \mathfrak{h}, v \in \Delta^{G,\hbar}_{\mathbf{1}}$, we have $\varphi^P(mx) = \varphi^P(m)(x + 2\hbar\langle\rho_P, x\rangle)$.*

Proof. It is enough assume that $m := 1 + I^{G,\hbar}_{\mathbf{1}}$. Note that, by Lemma 4.4,

$$\mathsf{Ffr}_P(x) = (x - \sum_{\alpha\in\Phi^+_P}\langle\alpha, x\rangle y^\alpha\partial_\alpha)+?,$$

where ? lies in the left ideal generated by elements of positive energy. Note that $y^\alpha\partial_\alpha = \partial_\alpha y^\alpha - 1$ in $\widehat{D}(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}}\widehat{U}_{\mathbf{1}}(\hat{\mathfrak{l}})$, and y^α kills $\varphi^P(m)$. In the Rees algebras, we need to replace 1 with $\hbar$. So

$$\varphi^P(mx) = \varphi^P(xm) = \mathsf{Ffr}_P(x)\varphi^P(m) = \varphi^P(m)(x + 2\hbar\langle\rho_P, x\rangle)$$

finishing the proof. □

Define the R-linear automorphism $\mathtt{t}_P$ of $\mathfrak{h}^{da} = \mathsf{R}\mathbf{1} \oplus \mathsf{R}\hbar \oplus \mathfrak{h}^*$ by $\mathtt{t}_P(\kappa, \nu, \eta) = (\kappa, \nu, \eta + 2\nu\rho_P)$. So, for any $x \in \mathsf{R}[\mathfrak{h}^{da}]$, we have $\varphi^P(mx) = \varphi^P(m)\mathtt{t}^*_P(x)$.

In particular, φ^P_{x} is a homomorphism $\Delta^G_{\mathsf{x}} \to \mathsf{W}\Delta^P_{\mathtt{t}_P(\mathsf{x})}$.

8.2.3. We remark that thanks to the transitivity property (4.9), we have

$$\varphi^B = \mathsf{Wak}^P(\varphi^{B_L}) \circ \varphi^P. \tag{8.1}$$

8.2.4. Note that we can view the hyperplane $\hbar = 1$ in $\mathfrak{h}^{da}$ as the affine Cartan space (over R). Kac and Kazhdan in [KK] specified a collection of hyperplanes $H_{\beta,m}$ in the affine Cartan space. Let us now explain what we need to know about these hyperplanes. These hyperplanes are of the form $\{\eta|(\eta,\beta) + (\rho,\beta) - n(\beta,\beta) = 0\}$, where $(\cdot,\cdot)$ is an affine Weyl group invariant pairing, β is an indecomposable positive root, n is a positive integer, ρ is a suitable element. Note that with a suitable choice of $(\cdot,\cdot)$ (namely with the usual normalization, where all short coroots have length 2) all these hyperplanes are defined over R. Let $\tilde{H}_{\beta,m}$ denote the unique linear hyperplane in $\mathfrak{h}^{da}$ whose intersection with $\hbar = 1$ is $H_{\beta,m}$. Here are the two properties of these hyperplanes established in [KK] that we are going to need. Suppose $\operatorname{char}\mathbf{k} = 0$ and $\mathsf{x} \in \mathfrak{h}^{da}(\mathbf{k})$ is of the form $(\kappa, 1, \eta)$. Since $\tilde{H}_{\beta,m}$ is defined over R, it makes sense to speak of x lying in $\tilde{H}_{\beta,m}$, we will just write $\mathsf{x} \in \tilde{H}_{\beta,m}$ in this case. Then the following is true.

(*) x does not lie on any of the hyperplanes $\tilde{H}_{\beta,m}$. Then the $\widehat{U}_\kappa(\hat{\mathfrak{g}})$-module Δ^G_{x} is irreducible.

8.2.5. Now we state the main result of this section.

Definition 8.2. We say that x is *P-generic* if from $\mathsf{x} \in \tilde{H}_{\beta,m}$ it follows that β is a root of $\hat{\mathfrak{l}}$.

Theorem 8.3. *Suppose that x is P-generic. Then $\varphi^P_{\mathsf{x}} : \Delta^{\mathfrak{g}}_{\mathsf{x}} \to \mathsf{W}\Delta^P_{\mathtt{t}_P(\mathsf{x})}$ is an isomorphism.*

8.2.6. In the rest of the section we will reduce the proof to the following result (that will be proved in Section 8.3).

Proposition 8.4. *Suppose* $\mathsf{x} = (0, 0, \eta)$. *Further suppose that the only root hyperplanes in* $\mathfrak{h}^*(\mathbf{k})$ *containing* η *correspond to roots of* $\mathfrak{l}$. *Finally assume that* $\operatorname{char} \mathbf{k} = 0$ *or* $P = B$. *Then* $\varphi^P_{\mathsf{x}} : \Delta^G_{\mathsf{x}} \to \mathsf{W}\Delta^P_{\mathsf{t}_P(\mathsf{x})}$ *is an isomorphism.*

Proof of Theorem 8.3 modulo Proposition 8.4. The proof is in several steps.

Step 1. First, suppose that x does not lie on $(\hbar)$, (p), or any of $\tilde{H}_{\beta,m}$. Property (a) in Section 8.2.4 implies that Δ^G_x is simple. Moreover, by the construction in Section 8.1.2, the Λ^a-graded characters of $\Delta^{\mathfrak{g}}_{\mathsf{x}}$ and $\mathsf{W}\Delta^P_{\mathsf{t}_P(\mathsf{x})}$ are the same. It follows that φ^P_{x} is an isomorphism.

Step 2. Note that $\varphi^P[\lambda] : \Delta^G[\lambda] \to \mathsf{W}\Delta^P[\lambda]$ is a $\mathsf{R}[\mathfrak{h}^{da}]$-semi-linear homomorphism between $\mathsf{R}[\mathfrak{h}^{da}]$-modules of the same ring. By Step 1, $\varphi^P[\lambda]$ is an isomorphism generically. It follows that the locus in $\mathfrak{h}^{da}$, where this homomorphism fails to be an isomorphism is exactly a divisor, denote it by $D[\lambda]$. Applying Step 1 again, we see that the irreducible components of $D[\lambda]$ are of the form $(p), (\hbar)$ or $\tilde{H}_{\beta,m}$. Note that thanks to (8.1), the map φ^P_{x} is injective provided φ^B_{x} is. Proposition 8.4 now shows that (p) and $(\hbar)$ are not possible.

Step 3. It remains to show that $\tilde{H}_{\beta,m}$, where β is not a root of $\hat{\mathfrak{l}}$, cannot a component in any $D[\lambda]$. The intersection of $\tilde{H}_{\beta,m}$ with $\mathfrak{h}^* \subset \mathfrak{h}^{da}$ is a root hyperplane that is given by a root of $\mathfrak{g}$ but not of $\mathfrak{l}$. We arrive at a contradiction with Proposition 8.4. □

8.3. **Specializations with $\kappa = \nu = 0$.**

Proof of Proposition 8.4. The proof is in several steps. Note that it is enough to consider the case when $\mathbf{k}$ is algebraically closed. Below we work over $\mathbf{k}$.

Step 1. Consider the finite analog of ffr^0_P. Let $\mathfrak{n}^{-,\perp}$ denote the annihilator of $\mathfrak{n}^-$ in $\mathfrak{g}^*$. Consider the action of G on $T^*(G/U^-) = (G \times \mathfrak{p}^-)/N^-$, where N^- acts by $n.(g, x) = (gn^{-1}, \operatorname{Ad}^*(n)x)$. The moment map $T^*(G/N^-) \to \mathfrak{g}$ sends the orbit $[g, x]$ to $\operatorname{Ad}^*(g)x$. It is L-invariant and so descends to $G \times^{P^-} \mathfrak{n}^{-,\perp}$. Restrict it to the open subset $N \times \mathfrak{n}^{-,\perp}$ and identify $\mathfrak{n}^{-,\perp}$ with $\mathfrak{n}^* \oplus \mathfrak{l}^*$. The resulting map $N \times \mathfrak{n}^* \times \mathfrak{l}^* \to \mathfrak{g}^*$ is therefore given by $(n, x, y) \to \operatorname{Ad}^*(n)(x + y), n \in N, x \in \mathfrak{n}^*, y \in \mathfrak{l}^*$. Denote this map by μ.

Step 2. We need to understand $\mu^* : \mathfrak{g} \to \mathbf{k}[N] \otimes (\mathfrak{n} \oplus \mathfrak{l})$ more explicitly. Choose a one-parameter subgroup $\varrho : \mathbb{G}_m \to Z(L)$ such that its centralizer is $\mathfrak{l}$ and its eigen-characters on $\mathfrak{n}$ are positive. So we have decompositions $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{n}_i$ and $\mathfrak{n}^- = \bigoplus_{i<0} \mathfrak{n}^-_i$. Under the assumption of the proposition, the action of L on N linearizes leading to an L-equivariant isomorphism $N \xrightarrow{\sim} \mathfrak{n}$ that can be chosen so that its differential at 1 is the identity. Let $\mathfrak{m}$ denote the augmentation ideal in $\mathbf{k}[N]$. We write $\mathfrak{m}_i$ for its degree i component with respect to the $\mathbb{G}_m$-action, note that $\mathfrak{m}_i \neq 0 \Rightarrow i < 0$.

The map μ^* sends $x \in \mathfrak{n}_i$ to the corresponding action vector field on N, we have

$$\mu^*(x) \in x + \sum_{j>i} \mathfrak{m}_{i-j} \otimes \mathfrak{n}_j. \tag{8.2}$$

Next, for $y \in \mathfrak{l}$, the image $\mu^*(y)$ is the sum of y and the action vector field for the adjoint action of L on $\mathfrak{n}$, so

$$\mu^*(y) \in y + \sum_{i>0} \mathfrak{n}_i \otimes \mathfrak{n}_i^*. \tag{8.3}$$

Finally, consider $w \in \mathfrak{n}^-_{-i}$. Then for the weight reasons, we have

$$\mu^*(w) \in \theta(w) + (\mathfrak{m}^2)_{-i} \otimes \mathfrak{l} \oplus \bigoplus_{j>0} \mathfrak{m}_{-i-j} \otimes \mathfrak{n}_j. \tag{8.4}$$

Here θ is an L-equivariant linear map $\mathfrak{n}^-_{-i} \to \mathfrak{n}^-_{-i} \otimes \mathfrak{l}$, where we embed $\mathfrak{n}^-_{-i}$ into $\mathfrak{m}$ via its identification with $\mathfrak{n}_i^*$ via the invariant form.

Step 3. For $y \in \mathfrak{l}^*$, let $\theta_y : \mathfrak{n}_i^- \to \mathfrak{n}_i^-$ denote the pairing of θ with y. We can find $F \in \mathbf{k}[\mathfrak{l}^*]^L$ such that the zeroes of F in $\mathfrak{h}^* \subset \mathfrak{l}^*$ is exactly the union of the root hyperplanes defined by roots that are not of $\mathfrak{l}$. We claim that if $F(y)$, then θ_y is an isomorphism. In order to see this Identify $\mathfrak{n}^*$ with $\mathfrak{n}^-$ and $\mathfrak{l}^*$ with $\mathfrak{l}$ using the invariant form. The differential $d_{(1,x,y)}\mu$ at the point (ξ, x', y') for $\xi \in \mathfrak{n}, x, x' \in \mathfrak{n}^-, y, y' \in \mathfrak{l}$ is given by $x' + y' + [\xi, x + y]$. The condition that $F(y) \neq 0$ is equivalent to the operator $\xi \mapsto [\xi, x]$ being invertible on $\mathfrak{n}_i$. This is equivalent to θ_y being an isomorphism.

Step 4. Recall the description of $\mathsf{ffr}^0_P : \mathsf{R}[\mathcal{L}(\mathfrak{g}^*)]_f \to \mathsf{R}[\mathcal{L}(T^*N)]_f \widehat{\otimes} \mathsf{R}[\mathcal{L}(\mathfrak{l}^*)]_f$ that stems from the construction and the description of the jet analog in Section 4.2.2. Namely, ffr^0_P is the pullback under the morphism $\mathcal{L}\mu : \mathcal{L}(N \times \mathfrak{n}^* \times \mathfrak{l}^*) \to \mathcal{L}(\mathfrak{g}^*)$. To compute the image of $x_n := xt^n \in \mathcal{L}\mathfrak{g}$ under ffr_P we form the infinite series $\sum_n x_n z^{-n-1}$, expand $\mu^*(x)$ as a polynomial in the elements $\partial_\alpha, y^\alpha$ and $y \in \mathfrak{l}$. Then $\sum_n \mathsf{ffr}^0_P(x_n) z^{-n-1}$ is obtained from that polynomial by replacing each y with $\sum_n (yt^n) z^{-n-1}$, each ∂_α with $\sum_n \partial_{\alpha,n} z^{-n-1}$ and each y^α with $\sum_n y^\alpha_n z^{-n}$. Then we expand the result in the powers of z.

Step 5. We will show that $\Delta^G_{\mathsf{x}} \twoheadrightarrow \mathsf{W}\Delta^P_{\mathsf{x}}$. We will use the following version of the graded Nakayama lemma. Suppose that A, B are two positively graded commutative $\mathbf{k}$-algebras (meaning that they are graded by $\mathbb{Z}_{\geqslant 0}$ and the degree 0 components are $\mathbf{k}$). Let $\varphi : A \to B$ be a graded algebra homomorphism and y_i, where i is in some indexing set I, are homogeneous elements of A of positive degree. Then if the induced homomorphism

$$A/\operatorname{Span}_A(y_i | i \in I) \to B/\operatorname{Span}_B(\varphi(y_i) | i \in I)$$

is surjective, then φ is surjective.

Step 6. Note that both $\Delta^{\mathfrak{g}}_{\mathsf{x}}, \mathsf{W}\Delta^P_{\mathsf{x}}$ are commutative $\mathbf{k}$-algebras: we have $\Delta^G_{\mathsf{x}} \cong S(t^{-1}\mathfrak{b}^-[t^{-1}]) \oplus \tilde{\mathfrak{n}}[t^{-1}])$, where $\tilde{\mathfrak{n}}$ is the maximal nilpotent subalgebra of $\mathfrak{b}$. Further,

$$\mathsf{W}\Delta^P_{\mathsf{x}} = \mathbf{k}[\partial_{\alpha,n}]_{n \leqslant 0} \otimes_{\mathbf{k}} \mathbf{k}[y^\alpha_n]_{n<0} \otimes_{\mathbf{k}} S(t^{-1}\mathfrak{b}^-_L[t^{-1}]) \oplus \tilde{\mathfrak{n}}_L[t^{-1}]).$$

The map φ^P_{x} is the homomorphism of quotients induced by ffr^0_P. Note that both algebras $\Delta^{\mathfrak{g}}_{\mathsf{x}}$ and $\mathsf{W}\Delta^P_{\mathsf{x}}$ are graded by the span of positive affine roots, we turn this grading into a $\mathbb{Z}_{\geqslant 0}$-grading by using the height of affine roots.

Step 7. We first apply the procedure in Step 5 to the elements $\mathfrak{n}[t^{-1}] \subset \Delta^{\mathfrak{g}}_{\mathsf{x}}$. From (8.2) and the description of ffr^0_P in Step 4, we easily conclude that

$$\mathrm{Span}_{\mathsf{W}\Delta^P_{\mathsf{x}}}(\mathsf{ffr}^0_P(\mathfrak{n}[t^{-1}])) = \mathrm{Span}_{\mathsf{W}\Delta^P_{\mathsf{x}}}(\partial_{\alpha,i} | i \leqslant 0).$$

We reduce to proving that the induced homomorphism

$$S(t^{-1}\mathfrak{b}^-[t^{-1}] \oplus \tilde{\mathfrak{n}}_L[t^{-1}]) \to \mathbf{k}[y^\alpha_n]_{n<0} \otimes_{\mathbf{k}} S(t^{-1}\mathfrak{b}^-_L[t^{-1}]) \oplus \tilde{\mathfrak{n}}_L[t^{-1}])$$

is surjective. Now we similarly use (8.3) to reduce the proof of surjectivty to showing that the following homomorphism of quotients is surjective:

$$S(t^{-1}\mathfrak{n}^-[t^{-1}]) \to \mathbf{k}[y^\alpha_n]_{n<0} = S(t^{-1}\mathfrak{n}^-[t^{-1}]). \tag{8.5}$$

Thanks to (8.4), the map in question sends w_{-n} for $w \in \mathfrak{n}^-_i$ and $n \geqslant 0$ to a polynomial of the form $\theta_\eta(w)_{-n}+?$. Here, recall $\mathsf{x} = (0, 0, \eta)$, and "?" stands for the polynomial in the elements $w'_{-n'}$ for $w' \in \mathfrak{n}^-$ and $n' < n$. Thanks to Step 3, θ_η is a bijection $\mathfrak{n}^-_i \to \mathfrak{n}^-_i$ for all i. Now an easy induction on n shows that (8.5). □

9. Intertwining operators for categorical representations

9.1. **Preliminaries.**

9.1.1. In this section, all algebraic groups are taken to be smooth. Let K be an algebraic group over a field k, and denote its Lie algebra by

$$[-,-] : \mathfrak{k} \otimes \mathfrak{k} \to \mathfrak{k}.$$

Denote by $U^\hbar(\mathfrak{k})$ its Rees enveloping algebra, i.e., the tensor algebra over $k[\hbar]$ with generators $\mathfrak{k}$, modulo the commutator relations

$$X \cdot Y - Y \cdot X = \hbar \cdot [X, Y], \qquad X, Y \in \mathfrak{k}.$$

9.1.2. For later use, let us recall that if $\mathfrak{k}$ is equipped with a restricted Lie algebra structure, which we denote by $X \mapsto X^{[p]}$, then this equips $U^\hbar(\mathfrak{k})$ with a p-center. Namely, we have a well-defined linear map

$$\mathfrak{k}^{(1)} \to U^\hbar(\mathfrak{k}), \qquad X \mapsto X^p - \hbar^{p-1} \cdot X^{[p]},$$

whose image consists of central elements. These freely generate a central subalgebra

$$Z^\hbar_p(\mathfrak{k}) \hookrightarrow U^\hbar(\mathfrak{k})$$

whose spectrum, if we write $\mathbb{A}^1_\hbar := \mathrm{Spec}\, k[\hbar]$, is therefore given by

$$(\mathfrak{k}^*)^{(1)} \times \mathbb{A}^1_\hbar.$$

9.1.3. Consider an algebra A over $k[\hbar]$, equipped with a *strong action* of K, i.e., an $\hbar$-linear action of K on A by algebra automorphisms,

$$\alpha : A \to \mathcal{O}_K \underset{k}{\otimes} A$$

along with an $\hbar$-linear map of algebras

$$\phi : U^\hbar(\mathfrak{k}) \to A$$

satisfying the following condition: for any $X \in \mathfrak{k}$, if we write $d\alpha_X : A \to A$ for the derivation of A obtained by differentiating the action of K, we have the equality of derivations

$$\hbar \cdot d\alpha_X(-) = [\phi(X), -] : A \to A.$$

Algebras (A, α, ϕ) form a category in an evident manner; this is simply the category of algebra objects in Harish-Chandra bimodules for K.

9.1.4. An example of primary interest for us is the following.

Example 9.1. If X is a smooth affine variety over k equipped with an action of K, write $D^\hbar(X)$ for its Rees algebra of differential operators. Then an action of K on X equips $D^\hbar(X)$ with a strong action of K.

In addition, we recall that $D^\hbar(X)$ has a p-center $Z_p^\hbar(X) \hookrightarrow D^\hbar(X)$. Namely, for each vector field V, with restricted power $V^{[p]}$, one has the corresponding central element

$$V^p - \hbar^{p-1} V^{[p]} \in D^\hbar(X),$$

and these, along with p^{th} powers of functions, generate the p-center, so that canonically one has

$$\operatorname{Spec} Z_p^\hbar(X) \simeq (T^*X)^{(1)} \times \mathbb{A}^1_\hbar.$$

We recall that in this case, as the map from $\mathfrak{k}$ to vector fields on X is a morphism of restricted Lie algebras, the quantum comoment map $\phi : U^\hbar(\mathfrak{k}) \to D^\hbar(\mathfrak{k})$ restricts to a map $\phi : Z_p^\hbar(\mathfrak{k}) \to Z_p^\hbar(X)$. Moreover, on spectra, if we write $\mu : T^*X \to \mathfrak{k}^*$ for the moment map, ϕ corresponds to

$$\mu^{(1)} \times \mathrm{id} : (T^*X)^{(1)} \times \mathbb{A}^1_\hbar \to (\mathfrak{k}^*)^{(1)} \times \mathbb{A}^1_\hbar.$$

9.1.5. Given an algebra A as above, we recall a datum of *weak equivariance* on an A-module M is an $\hbar$-linear action of K on M

$$\beta : M \to \mathcal{O}_K \underset{k}{\otimes} M$$

such that the multiplication map

$$A \underset{k[\hbar]}{\otimes} M \to M$$

is a map of K-modules. We denote the usual category of weakly equivariant A-modules by $A{-}\mathrm{mod}^{K,w}$. Note that if we write $Z(A)$ for the center of A, then $A\text{-}\mathrm{Mod}^{K,w}$ is naturally a $Z(A)^K$-linear category.

9.1.6. We recall a datum of *strong equivariance* is a datum of weak equivariance satisfying the further condition that, for any $X \in \mathfrak{k}$, if we write $d\beta_X : M \to M$ for the endomorphism of M obtained by differentiating the action of K on M, we have an equality of endomorphisms

$$\hbar \cdot d\beta_X(-) = \phi(X) \cdot - : M \to M.$$

We denote the corresponding full subcategory of strongly equivariant A-modules by

$$\mathrm{Oblv} : A{-}\mathrm{mod}^K \to A{-}\mathrm{mod}^{K,w}.$$

and note that it is again naturally $Z(A)^K$-linear.

Lemma 9.2. *Restriction defines an equivalence*

$$\mathrm{Oblv} : A \underset{k[\hbar]}{\otimes} U^\hbar(\mathfrak{k}){-}\mathrm{mod}^K \xrightarrow{\sim} A{-}\mathrm{mod}^{K,w}.$$

In particular, $A{-}\mathrm{mod}^{K,w}$ *is naturally* $U^\hbar(\mathfrak{k})^K$*-linear.*

Proof. The inverse functor is constructed as follows. For (M, β) an object of $A-\mathrm{mod}^{K,w}$ the formula

$$X \mapsto \hbar \cdot d\beta_X - \phi(X)$$

defines an action of $U^\hbar(\mathfrak{k})$ on M commuting with the action of A, and tautologically (M, β) is strongly equivariant with respect to this extended action. □

9.1.7. Note that if we have a homomorphism of groups $S \to K$, then the strong action of K on A induces a strong action of S. Moreover, let us fix $\mathring{S}$ normal in S, with quotient $Q = S/\mathring{S}$, and write the Lie algebra of the latter as $\mathfrak{q}$. Then, in evident notation, we can consider the intermediate category of modules which are only strongly equivariant for $\mathring{S}$,

$$A-\mathrm{mod}^S \hookrightarrow A-\mathrm{mod}^{\mathring{S},Q,w} \hookrightarrow A-\mathrm{mod}^{S,w},$$

and this category is naturally linear over $U^\hbar(\mathfrak{q})^Q$.

9.1.8. Given a map $(A, \alpha, \phi) \to (A', \alpha', \phi')$ of algebras with strong actions, note that we have associated induction and restriction functors

$$\mathrm{ind} : A-\mathrm{mod}^{\mathring{S},Q,w} \rightleftarrows A'-\mathrm{mod}^{\mathring{S},Q,w} : \mathrm{Oblv},$$

compatible with the forgetting to the nonequivariant adjunction

$$\mathrm{ind} : A-\mathrm{mod} \rightleftarrows A'-\mathrm{mod} : \mathrm{Oblv}.$$

9.1.9. The following standard fact will be useful to us. The natural action of $K \times K$ on K by left and right translations yields a strong action of $K \times K$ on $D^\hbar(K)$. Let us equip

$$A \underset{k[\hbar]}{\otimes} D^\hbar(K)$$

with a strong $K^\ell \times K^r := K \times K$ action, where the action of K^ℓ is the tensor product of the action on A and the left translation action on $D^\hbar(K)$, and the action of K^r is the right translation action on $D^\hbar(K)$.

Lemma 9.3. *There is a canonical equivalence, functorial in A*

$$A-\mathrm{mod} \simeq (A \underset{k[\hbar]}{\otimes} D^\hbar(K))-\mathrm{mod}^{K^\ell}.$$

Moreover, for any $\mathring{S}$ and Q as above, this induces an equivalence

$$A-\mathrm{mod}^{\mathring{S},Q,w} \simeq (A \underset{k[\hbar]}{\otimes} D^\hbar(K))-\mathrm{mod}^{K^\ell \times (\mathring{S}^r, Q^r, w)}.$$

Proof. Consider the augmentation module $k[\hbar]$ of $U^\hbar(\mathfrak{k}^\ell)$, and note that equipping it with the trivial action of K makes it naturally strongly equivariant, i.e.,

$$k[\hbar] \in U^\hbar(\mathfrak{k}^\ell)\text{-}\mathrm{Mod}^{K^\ell},$$

wherein it corepresents the functor of K^ℓ-invariants.

Consider the induced object

$$(A \otimes D^\hbar(K)) \underset{U^\hbar(\mathfrak{k}^\ell)}{\otimes} k[\hbar] \in (A \underset{k[\hbar]}{\otimes} D^\hbar(K))-\mathrm{mod}^{K^\ell}.$$

We claim it is a projective generator, i.e., that the functor of K^ℓ-invariants is exact and conservative. But this follows from considering the forgetful functor

$$\mathrm{Oblv} : (A \underset{k[\hbar]}{\otimes} D^\hbar(K))\text{−mod}^{K^\ell} \to \mathcal{O}_K\text{-Mod}^{K^\ell,w} \simeq \mathrm{Vect}.$$

It remains to identify its endomorphisms with A^{op}, but this follows from recalling that

$$A \underset{k}{\otimes} \mathcal{O}_K \xrightarrow{\sim} (A \otimes D^\hbar(K)) \underset{U^\hbar(\mathfrak{k}^\ell)}{\otimes} k[\hbar]$$

whence upon passing to K^ℓ-invariants

$$A \simeq (A \underset{k}{\otimes} \mathcal{O}_K)^{K_\ell} \simeq ((A \otimes D^\hbar(K)) \underset{U^\hbar(\mathfrak{k}^\ell)}{\otimes} k[\hbar])^{K^\ell}.$$

Explicitly, as the obtained equivalence is the Rees algebra version of quantum Hamiltonian reduction, and the obtained action of K^r and quantum comoment map $U^\hbar(\mathfrak{k}^r) \to A$ agree with the original ones, the claim about the residual K^r-action follows. □

9.2. **Localization.**

9.2.1. We now specialize the preceding discussion to $K = M$, a split connected reductive group, and k a field of characteristic $p > h$, where h denotes the maximum of the Coxeter numbers of all the simple factors of M. We will need an $\hbar$-linear, families version of the localization theorem in positive characteristic, cf. [BMR], albeit only for suitably generic infinitesimal characters. We were unable to locate the needed formulation in the literature, so we now set it up.

9.2.2. Fix a Borel subgroup with Levi quotient

$$T \leftarrow B \to M,$$

i.e., if we write N for the unipotent radical of B, T is the universal Cartan B/N. Write $\mathfrak{t}$ for the Lie algebra of T, and $\check{\Phi} \subset \mathfrak{t}$ for the coroots.

9.2.3. For an algebra A with a strong action of M, we can consider the category of weak highest weight modules

$$A\text{−mod}^{N,T,w},$$

which is naturally linear over $U^\hbar(\mathfrak{t})^T = U^\hbar(\mathfrak{t})$.

Definition 9.4. Let us call a $U^\hbar(\mathfrak{t})$-module M *generic* if the elements

$$\check{\alpha} - \hbar \cdot n, \quad \text{for } \check{\alpha} \in \check{\Phi}, n \in \mathbb{F}_p,$$

act invertibly on M.

Denote the corresponding localization of of $U^\hbar(\mathfrak{t})$ by $U^\hbar(\mathfrak{t})^\circ$, and the corresponding full subcategory of generic objects by

$$j_* : A\text{−mod}^{N,T,w,\circ} \hookrightarrow A\text{−mod}^{N,T,w};$$

note the inclusion j_* admits a left adjoint j^*.

9.2.4. For later use, let us note that one can detect the genericity of a module M using only the p-center of $U^\hbar(\mathfrak{t})$. Namely, recall the canonical identification

$$\operatorname{Spec} Z_p^\hbar(\mathfrak{t}) \simeq (\mathfrak{t}^*)^{(1)} \times \mathbb{A}^1_\hbar.$$

In particular, we may consider the open subset

$$(\mathfrak{t}^*_{rs})^{(1)} \times \mathbb{A}^1_\hbar \hookrightarrow (\mathfrak{t}^*)^{(1)} \times \mathbb{A}^1_\hbar,$$

where $\mathfrak{t}^*_{rs}$ is the regular semisimple locus, i.e., the complement of the root hyperplanes. Denote the corresponding localization of $Z_p^\hbar(\mathfrak{t})$ by $Z_p^\hbar(\mathfrak{t})^\circ$.

Lemma 9.5. *A module M over $U^\hbar(\mathfrak{t})$ is generic if and only if the action of $Z_p^\hbar(\mathfrak{t})$ factors through $Z_p^\hbar(\mathfrak{t})^\circ$, i.e., we have a canonical isomorphism of algebras*

$$U^\hbar(\mathfrak{t})^\circ \simeq U^\hbar(\mathfrak{t}) \underset{Z_p^\hbar(\mathfrak{t})}{\otimes} Z_p^\hbar(\mathfrak{t})^\circ.$$

Proof. Note the identity, for each $\check{\alpha} \in \check{\Phi}$,

$$\prod_{n \in \mathbb{F}_p} (\check{\alpha} - \hbar \cdot n) = \check{\alpha}^p - \hbar^{p-1} \cdot \check{\alpha};$$

the same identity holds with $\check{\alpha}$ and $\hbar$ replaced by any two commuting elements in an $\mathbb{F}_p$-algebra. In particular, inverting the product on the left-hand side is equivalent to inverting one p-central element, namely the right-hand side. □

9.2.5. Note that, as for any torus, if we write Λ for the character lattice of T, we have an equivalence of cocomplete abelian categories

$$U^\hbar(\mathfrak{t})\text{−mod}^{T,w} \simeq \underset{\lambda \in \Lambda}{\oplus} U^\hbar(\mathfrak{t})\text{−mod}.$$

Explicitly, this corresponds to the set of projective generators $U^\hbar(\mathfrak{t})_\lambda$, $\lambda \in \Lambda$, where $U^\hbar(\mathfrak{t})_\lambda$ denotes the Rees enveloping algebra with an action of T purely by the character λ. In particular, we have

$$\operatorname{Hom}(U^\hbar(\mathfrak{t})_\lambda, U^\hbar(\mathfrak{t})_\lambda) \simeq U^\hbar(\mathfrak{t}), \qquad \lambda \in \Lambda.$$

If we set $U^\hbar(\mathfrak{t})^\circ_\lambda := j^* U^\hbar(\mathfrak{t})_\lambda$, it follows these compactly generate the generic category, and by inspection their endomorphisms yield

$$U^\hbar(\mathfrak{t})\text{−mod}^{T,w,\circ} \simeq \underset{\lambda \in \Lambda}{\oplus} U^\hbar(\mathfrak{t})^\circ\text{−mod}.$$

9.2.6. Consider the parabolic induction functor

$$\operatorname{pind} : U^\hbar(\mathfrak{t})\text{−mod}^{T,w} \to U^\hbar(\mathfrak{m})\text{−mod}^{N,T,w}, \qquad M \mapsto U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{b})}{\otimes} M.$$

We note this functor is $U^\hbar(\mathfrak{t})$-linear, and in particular restricts to a functor

$$\operatorname{pind} : U^\hbar(\mathfrak{t})\text{−mod}^{T,w,\circ} \to U^\hbar(\mathfrak{m})\text{−mod}^{N,T,w,\circ} \tag{9.1}$$

In particular, we have the generic universal Verma modules

$$\Delta^\circ_\lambda := \operatorname{pind}(U^\hbar(\mathfrak{t})^\circ_\lambda), \qquad \lambda \in \Lambda \in U^\hbar(\mathfrak{m})\text{-Mod}^{N,T,w,\circ}.$$

Note that by functoriality, we have a map $U^\hbar(\mathfrak{t})^\circ \to \operatorname{End}(\Delta^\circ_\lambda)$. This is an isomorphism, as follows by noting that any endomorphism of Δ°_λ must in particular respect the Λ grading, and in particular send the canonical generator of Δ°_λ to an

element of the highest weight space, i.e., $U^\hbar(\mathfrak{t})^\circ_\lambda \hookrightarrow \Delta^\circ_\lambda$. In particular, the action of the Harish-Chandra center $U^\hbar(\mathfrak{m})^M$ on Δ°_λ factors through this, i.e.,

$$U^\hbar(\mathfrak{m})^M \to \mathrm{End}(\Delta^\circ_\lambda) \simeq U^\hbar(\mathfrak{t})^\circ;$$

of course the analogous result holds before base changing to the generic locus as well.

Remark 9.6. In fact, one may show that (9.1) is equivalence of categories; we do not need this, so omit the straightforward proof.

9.2.7. Consider the tautological induction and restriction functors

$$\mathrm{Ind} : U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{m})\text{−mod} \rightleftarrows D^\hbar(M)\text{−mod} : \mathrm{Oblv}.$$

Passing to weak highest weight objects with respect to right translations, and further localizing to the generic part, we obtain

$$\mathrm{Ind} : U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{m})\text{−mod}^{N^r,T^r,w,\circ} \rightleftarrows D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} : \mathrm{Oblv}.$$

Precomposing with the tautological adjunction

$$U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod} \rightleftarrows U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{m})\text{−mod}^{N^r,T^r,w,\circ}$$

afforded by the bimodule Δ°_0, we obtain the adjunction relevant for localization, namely

$$\mathrm{Loc} : U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod} \rightleftarrows D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} : \mathrm{Loc}^R. \tag{9.2}$$

9.2.8. The desired localization in families, over the generic locus, now reads as follows.

Theorem 9.7. *The functors in* (9.2) *are mutually inverse equivalences of abelian categories.*

Proof. For ease of reading, we denote the enhanced and usual flag varieties by

$$\widetilde{\mathcal{B}} := M/N \xrightarrow{\pi} M/B =: \mathcal{B},$$

and break the argument into several steps. We will first show that Loc^R is exact and faithful.

Step 1. Note that the exactness and faithfulness of Loc^R may be checked at the level of the further composition, which we denote by Γ,

$$\Gamma : D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} \xrightarrow{\mathrm{Loc}^R} U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod} \xrightarrow{\mathrm{Oblv}} \mathrm{Vect}.$$

Let us begin by recalling an alternative description of Γ. Note that we have a natural forgetful functor

$$\mathrm{Oblv} : D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} \to \mathrm{QCoh}(\mathcal{B}), \tag{9.3}$$

which passes from an $(N^r, T^r, w, \circ)$-equivariant $D^\hbar(M)$-module to its underlying B^r-equivariant quasicoherent sheaf on M. With this, Γ then canonically identifies with the composition

$$D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} \xrightarrow{\mathrm{Oblv}} \mathrm{QCoh}(\mathcal{B}) \xrightarrow{\Pi_*} \mathrm{Vect},$$

where $\Pi : \mathcal{B} \to \mathrm{pt}$ is the projection to a point. That is, Γ simply calculates the global sections on $\mathcal{B}$ of the underlying quasicoherent sheaf.

To proceed, recall the projection $\pi : \widetilde{\mathcal{B}} \to \mathcal{B}$. Write $\mathcal{D}^\hbar$ for the sheaf of Rees differential operators on $\widetilde{\mathcal{B}}$, and consider the associated sheaf of algebras $\pi_*(\mathcal{D}^\hbar)$ on $\mathcal{B}$, along with its subsheaf of right invariant operators

$$\mathcal{R} := \pi_*(\mathcal{D}^\hbar)^{T^r}.$$

By construction, we may factor the functor (9.3) as

$$D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} \xrightarrow{\mathrm{Oblv}^{enh}} \mathcal{R}\text{−mod}(\mathrm{QCoh}(\mathcal{B})) \to \mathrm{QCoh}(\mathcal{B}), \tag{9.4}$$

and hence obtain a similar factorization for Γ.

Step 2. Note that we have natural maps of algebras

$$\mathcal{O}_{\mathcal{B}} \to \mathcal{R} \quad \text{and} \quad \phi : U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t}) \to \Pi_*(\mathcal{R}).$$

We recall that ϕ is not injective. To describe the kernel, let us write $\mathcal{T}(\widetilde{\mathcal{B}})$ for the tangent sheaf of $\widetilde{\mathcal{B}}$, and let us consider the corresponding restricted Lie algebroid on $\mathcal{B}$, namely

$$\widetilde{\mathcal{T}} := \pi_*\mathcal{T}(\widetilde{\mathcal{B}})^{T^r}.$$

Consider the tautological short exact sequence of restricted Lie algebroids

$$0 \to \mathfrak{b}_{univ} \to \mathcal{O}_{\mathcal{B}} \otimes (\mathfrak{m} \oplus \mathfrak{t}) \to \widetilde{\mathcal{T}} \to 0.$$

If we pass to their Rees enveloping sheaves of algebras, we obtain the local counterpart to ϕ, namely a surjective map

$$\mathcal{U}^\hbar(\mathcal{O}_{\mathcal{B}} \otimes (\mathfrak{m} \oplus \mathfrak{t})) \twoheadrightarrow \mathcal{U}^\hbar(\widetilde{\mathcal{T}}) \simeq \mathcal{R},$$

and moreover deduce that its kernel is the two sided ideal generated by the augmentation ideal of $\mathcal{U}^\hbar(\mathfrak{b}_{univ})$.

Similarly, passing further to the p-centers of their Rees enveloping sheaves of algebras, we obtain a surjective map

$$\mathcal{Z}_p^\hbar(\mathcal{O}_{\mathcal{B}} \otimes (\mathfrak{m} \oplus \mathfrak{t})) \twoheadrightarrow \mathcal{Z}_p^\hbar(\widetilde{\mathcal{T}}) \tag{9.5}$$

and deduce that its kernel is the ideal generated by the augmentation ideal of $\mathcal{Z}_p^\hbar(\mathfrak{b}_{univ})$. We also note that explicitly, on spectra, if we write $\widetilde{\mathfrak{m}}^* \to \mathfrak{m}^*$ for the Grothendieck alteration, the map (9.5) corresponds to the tautological closed embedding

$$(\mathcal{B} \times \mathfrak{m}^* \times \mathfrak{t}^*)^{(1)} \times \mathbb{A}^1_\hbar \hookleftarrow (\widetilde{\mathfrak{m}}^*)^{(1)} \times \mathbb{A}^1_\hbar : \iota^{(1)} \times \mathrm{id}, \tag{9.6}$$

where $\iota : \widetilde{\mathfrak{m}}^* \to \mathcal{B} \times \mathfrak{m}^* \times \mathfrak{t}^*$ is the usual closed embedding. Namely, recall that $\widetilde{\mathfrak{m}}^*$ consists of pairs $(\xi, \mathfrak{b}')$, where $\xi \in \mathfrak{m}^*$ and $\mathfrak{b}'$ is a Borel subalgebra whose nilradical $\mathfrak{n}'$ is annihilated by ξ; ι sends such a pair $(\xi, \mathfrak{b}')$ to the triple $(\mathfrak{b}', \xi, \xi')$ where ξ' denotes the induced map $\mathfrak{t} \simeq \mathfrak{b}'/\mathfrak{n}' \xrightarrow{\xi} k$.

Step 3. Let us now show the claimed exactness and faithfulness of Loc^R. Namely, note that Γ factors as

$$\begin{aligned} D^\hbar(M)\text{−mod}^{N^r,T^r,w,\circ} &\xrightarrow{\mathrm{Oblv}^{enh}} \mathcal{R}\text{−mod}(\mathrm{QCoh}(\mathcal{B})) \\ &\xrightarrow{\mathrm{Fr}_*^{enh}} \mathcal{Z}_p^\hbar(\widetilde{\mathcal{T}})\text{−mod}(\mathrm{QCoh}(\mathcal{B}^{(1)}) \xrightarrow{\Pi_*} \mathrm{Vect}; \end{aligned}$$

here Fr_* denotes pushforward along the relative Frobenius, the superscript 'enh' on Fr_* simply remembers the residual action of the p-center of $\mathcal{R}$, and Π_* again denotes pushforward to a point.

To finish, we note that Oblv^{enh} and Fr_*^{enh} are tautologically exact and faithful. Moreover, by our assumption of genericity on our $D^\hbar(M)$-module, as a quasicoherent sheaf on the relative spectrum of $\mathcal{Z}_p^\hbar(\widetilde{\mathcal{T}})$, it is pushed forward from the open set

$$(\widetilde{\mathfrak{m}}^*)^{(1)} \times \mathbb{A}^1_\hbar \underset{(\mathcal{B}\times\mathfrak{m}^*\times\mathfrak{t}^*)^{(1)}\times\mathbb{A}^1_\hbar}{\times} (\mathcal{B}\times\mathfrak{m}^*\times\mathfrak{t}^*_{rs})^{(1)} \times \mathbb{A}^1_\hbar = (\widetilde{\mathfrak{m}}^*_{rs})^{(1)} \times \mathbb{A}^1_\hbar.$$

But it is standard that over the regular semisimple locus the Grothendieck alteration $\widetilde{\mathfrak{m}}^*_{rs} \to \mathfrak{m}_{rs}$ is a finite Weyl group torsor, and in particular finite over affine, whence affine, which shows the exactness and faithfulness of the final pushforward to a point.

Step 4. We next claim that the unit map $\mathrm{id} \to \mathrm{Loc}^R \circ \mathrm{Loc}$, when applied to the generator

$$U^\circ := U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ,$$

is an isomorphism. However, we note that the adjunction (9.2) is obtained by base changing to the generic locus a similar adjunction

$$\mathrm{Loc} : U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t}) \rightleftarrows D^\hbar(M)\text{–mod}^{N,T^r,w} : \mathrm{Loc}^R.$$

By the commutation of both appearing functors with filtered colimits, it is therefore enough to verify the analogous claim for

$$U := U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t}).$$

Note that U, viewed in its natural way as an algebra in $\mathrm{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$, is $\mathbb{Z}^{\geqslant 0}$-graded and torsion free, as is $\mathrm{Loc}^R \circ \mathrm{Loc}(U)$. Therefore, it is enough to check the assertion holds at the central fiber, i.e., at $\hbar = 0$, where it is the standard fact that the projection

$$\widetilde{\mathfrak{m}}^* \to \mathfrak{m}^* \underset{\mathfrak{m}^*//M}{\times} \mathfrak{t}^*$$

exhibits the latter as the affinization of the former.

Step 5. We next claim that the unit map $\mathrm{id} \to \mathrm{Loc}^R \circ \mathrm{Loc}$ is an isomorphism for any U°-module. However, by writing it as the cokernel of a map between two free U°-modules, this follows from the previous step, the exactness of Loc^R, and its commutation with arbitrary direct sums.

Step 6. We now claim we are done on general grounds. Namely, we are considering an adjunction between categories

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

where we know (i) F is fully faithful, and (ii) G reflects isomorphisms, i.e., a map $d \to d'$ is an isomorphism if and only if $G(d) \to G(d')$ is an isomorphism; for functors between abelian categories this follows from faithfulness, which we know holds for Loc^R. Let us see this implies F and G are mutually inverse equivalences.

Indeed, it remains to see that G is fully faithful, i.e., that for object d of $\mathcal{D}$ we have the counit map $FG(d) \to d$ is an isomorphism. It is enough to see that the map $GFG(d) \to G(d)$ is an isomorphism. But for any adjunction the composition

$$G(d) \to GFG(d) \to G(d)$$

is an isomorphism, and the first arrow $G(d) \to GFG(d)$ is an isomorphism by the fully faithfulness of F, whence the second is an isomorphism as well, as desired. □

9.3. **Intertwining.**

9.3.1. We retain the notation of the previous subsection. Let us first record the following useful consequence of Theorem 9.7.

Proposition 9.8. *For any algebra A equipped with a strong action of M, there is a canonical equivalence, functorial in A and linear over $Z(A)^M$*

$$A\text{−mod}^{N,T,w,\circ} \simeq (A \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ)\text{−mod}^{M,w}.$$

Proof. Recall from Lemma 9.3 the canonical identification

$$A\text{−mod} \simeq (A \underset{k[\hbar]}{\otimes} D^\hbar(M))\text{−mod}^{M^\ell},$$

and in particular the induced identification

$$A\text{−mod}^{N,T,w,\circ} \simeq (A \underset{k[\hbar]}{\otimes} D^\hbar(M))\text{−mod}^{M^\ell,N^r,T^r,w,\circ}.$$

Note next that Theorem 9.7 identifies the latter with

$$\simeq (A \underset{k[\hbar]}{\otimes} U^\hbar(\mathfrak{m}) \underset{U(\hbar(\mathfrak{m}))^M}{\otimes} U^\hbar(\mathfrak{t})^\circ)\text{−mod}^{M^\ell},$$

which we may by Lemma 9.2 we may rewrite in the desired form, namely

$$\simeq (A \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ)\text{−mod}^{M,w}.$$

□

Remark 9.9. By a standard unwinding of definitions, the equivalence of Proposition 9.8 is explicitly given as follows. Starting with an object c of $A\text{−mod}^{N,T,w,\circ}$, we need to produce an object

$$c' \in (A \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ)\text{−mod}^{M,w}.$$

First, we twist the T-weak equivariant structure on c by tensoring it with the determinant line $c \mapsto \det(\mathfrak{m}/\mathfrak{b})^* \otimes c$; this preserves the condition of genericity. Then, we apply the relative averaging functor from B weak invariants to M weak invariants

$$c' := \mathrm{Av}_w^{B,M}(\det(\mathfrak{m}/\mathfrak{b})^* \otimes c);$$

by functoriality, c' is an object $A\text{−mod}^{M,w}$ on which the action of $U^\hbar(\mathfrak{m})^M$ has been extended to $U^\hbar(\mathfrak{t})^\circ$, i.e., is an object of $(A \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ)\text{−mod}^{M,w}$, as desired.

9.3.2. Let us now use the previous proposition to construct the desired intertwining operators. As preparation, denote the abstract Weyl group of M by W, and recall that one has a natural dot action of W on $U^\hbar(\mathfrak{t})$ such that the Harish-Chandra map $U^\hbar(\mathfrak{m})^M \hookrightarrow U^\hbar(\mathfrak{t})$, cf. Section 9.2.6, factors through an isomorphism

$$U^\hbar(\mathfrak{m})^M \simeq U^\hbar(\mathfrak{t})^W \hookrightarrow U^\hbar(\mathfrak{t}).$$

Explicitly, if we denote the usual linear action of W on $\mathfrak{t}$ by $(w,h) \mapsto w(h)$, and we write $\rho \in \mathfrak{t}^*$ for the one half of the image in $\mathfrak{t}^*$ of the sum of the positive roots, the dot action of W on $U^\hbar(\mathfrak{t})$ on generators is given by

$$\check{\lambda} \mapsto w \overset{\hbar}{\cdot} \check{\lambda} := w(\check{\lambda}) + \hbar \cdot \langle w(\check{\lambda}) - \check{\lambda}, \rho \rangle, \qquad \check{\lambda} \in \mathfrak{t}.$$

On the spectrum $\operatorname{Spec} U^\hbar(\mathfrak{t}) \simeq \mathfrak{t}^* \times \mathbb{A}^1_\hbar$, the induced action takes the form

$$w \overset{h}{\cdot} (\lambda, h) = (w(\lambda + h\rho) - h\rho, h).$$

Corollary 9.10. *For any algebra A equipped with a strong action of M, there is a canonical action of the Weyl group on its generic weak highest weight modules*

$$W_M \circlearrowright A\text{−mod}^{N,T,w,\circ}, \qquad \mathcal{M} \mapsto w \star \mathcal{M}, \qquad \mathcal{M} \in A\text{−mod}^{N,T,w,\circ}, w \in W_M.$$

enjoying the following properties.

(1) This action is $Z(A)^M$-linear.

(2) This action is $U^\hbar(\mathfrak{t})$-semilinear. That is, for any $\mathcal{M}$ as above, and $\phi \in U^\hbar(\mathfrak{t})$, we have that applying $w \star -$ to $\phi : \mathcal{M} \to \mathcal{M}$ gives

$$w \overset{\hbar}{\cdot} \phi : w \star \mathcal{M} \to w \star \mathcal{M}.$$

(3) This action is functorial in the algebra A. That is, for any map $A \to B$ of algebras with strong actions, cf. Section 9.1.8, the associated adjunction

$$\operatorname{ind} : A\text{−mod}^{N,T,w,\circ} \rightleftarrows B\text{−mod}^{N,T,w,\circ} : \operatorname{Oblv}$$

is one of W_M-equivariant functors.

Proof. By the previous proposition, it is enough to produce an action of W on

$$A \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ,$$

viewed as an algebra with a strong M-action, functorially in the argument A. However, as the dot action of W_M on $U^\hbar(\mathfrak{t})$ fixes $U^\hbar(\mathfrak{m})^M$ and permutes the affine hyperplanes $\check{\alpha} - \hbar \cdot n = 0$, $\check{\alpha} \in \check{\Phi}, n \in \mathbb{F}_p$, we have an induced action of W_M on the above tensor product of the desired form. The claimed properties follow from the construction. □

9.3.3. We finish with one relevant calculation of the intertwining operators.

Proposition 9.11. *Consider the generic universal Verma module*

$$\Delta^\circ_{-2\rho} \in U^\hbar(\mathfrak{m})\text{−mod}^{N,T,w,\circ}$$

of weak highest weight -2ρ , cf. Section 9.2.6. Then we have isomorphisms

$$w \star \Delta^\circ_{-2\rho} \simeq \Delta^\circ_{-2\rho}, \qquad w \in W.$$

Proof. For ease of reading, we again break the argument into steps.

Step 1. Consider as in Theorem 9.7 the functor

$$\mathrm{Loc}^R : D^\hbar(M)^{N,T,w,\circ} \to U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod}. \tag{9.7}$$

It follows from the definition that this functor is naturally $U^\hbar(\mathfrak{t})^\circ$-linear, where on the left-hand side the action comes from $(T, w, \circ)$-equivariance, which we henceforth refer to as the right monodromy action, and on the right-hand side the action comes from the central subalgebra

$$U^\hbar(\mathfrak{t})^\circ \hookrightarrow U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ.$$

Similarly, it is straightforward that (9.7) is a W-equivariant functor, where on the left hand side we have the action of Corollary 9.10, which we denote by $- \star w$, and on the right hand side we have the action induced by the dot action on $U^\hbar(\mathfrak{t})^\circ$, which we again denote by $- \star w$.

Step 2. Consider on the left hand side of (9.7) the universal delta D-module δ, i.e., the pushforward along

$$U^\hbar(\mathfrak{t})^\circ\text{−mod} \simeq D^\hbar(B)^{N,T,w,\circ} \to D^\hbar(M)^{N,T,w,\circ}$$

of the object $U^\hbar(\mathfrak{t})^\circ$ itself. Note that we have a natural forgetful functor

$$\mathrm{Oblv} : U^\hbar(\mathfrak{m})\text{−mod}^{N,T,w,\circ} \to U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod},$$

and that one has a canonical isomorphism

$$\mathrm{Loc}^R(\delta) \simeq \mathrm{Oblv}(\Delta^\circ_{-2\rho}) \otimes \ell,$$

where ℓ is the line $\ell \simeq \det(\mathfrak{m}/\mathfrak{b})$.

Step 3. On the other hand, let us pass to weak highest vectors, i.e., consider the functor

$$\mathrm{Loc}^R : D^\hbar(M)^{N\times N,T\times T,w,\circ} \to U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod}^{N,T,w,\circ}. \tag{9.8}$$

This is now by definition, cf. Corllary 9.10, a W-equivariant functor, for the left monodromy action $w \star -$ on the left hand side, and the usual W-action $w \star -$ on the right hand side.

If we equip δ with its standard datum of left $(N, T, w, \circ)$-equivariance, we similarly have

$$\mathrm{Loc}^R(\delta) \simeq \Delta^{\circ,enh}_{-2\rho} \otimes \ell,$$

where the superscript 'enh' denotes the lift of the generic Verma module

$$\Delta^\circ_{-2\rho} \in U^\hbar(\mathfrak{m})\text{−mod}^{N,T,w,\circ}$$

to an object of $U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod}^{N,T,w,\circ}$ afforded by its endomorphisms as a $U^\hbar(\mathfrak{m})^{N,T,w,\circ}$-module, cf. Section 9.2.6.

Step 4. Consider the inversion map $\mathrm{inv} : M \simeq M, m \mapsto m^{-1}$, and the induced involution

$$\mathrm{inv}_* : D^\hbar(M)^{N\times N,T\times T,w,\circ} \simeq D^\hbar(M)^{N\times N,T\times T,w,\circ}.$$

As one has a canonical isomorphism $\mathrm{inv}_*(\delta) \simeq \delta$, it follows in particular that one has canonical isomorphisms $w \star \delta \simeq \delta \star w$, i.e., the actions of left and right intertwining

functors canonically coincide. In particular, upon applying $\ell^\vee \otimes \mathrm{Loc}^R$, we now obtain a compatible system of isomorphisms

$$w \star \Delta^{\circ,enh}_{-2\rho} \simeq \Delta^{\circ,enh}_{-2\rho} \star w, \quad w \in W.$$

Step 5. Consider the forgetful functor

$$\mathrm{Oblv} : U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ\text{−mod}^{N,T,w,\circ} \to U^\hbar(\mathfrak{m})\text{−mod}^{N,T,w}$$

given by restriction along $U^\hbar(\mathfrak{m}) \hookrightarrow U^\hbar(\mathfrak{m}) \underset{U^\hbar(\mathfrak{m})^M}{\otimes} U^\hbar(\mathfrak{t})^\circ$. Note that in particular the functor is invariant under the right monodromy action, so that in particular one has canonical equivalences $\mathrm{Oblv}(- \star w) \simeq \mathrm{Oblv}(-)$, for $w \in W$.

Combining the previous observations, we deduce deduce compatible isomorphisms

$$\begin{aligned} w \star \Delta^\circ_{-2\rho} &\simeq w \star \mathrm{Oblv}(\Delta^{\circ,enh}_{-2\rho}) \\ &\simeq \mathrm{Oblv}(w \star \Delta^{\circ,enh}_{-2\rho}) \\ &\simeq \mathrm{Oblv}(\Delta^{\circ,enh})_{-2\rho} \star w) \\ &\simeq \mathrm{Oblv}(\Delta^{\circ,enh}_{-2\rho}) \simeq \Delta^\circ_{-2\rho}, \end{aligned}$$

as desired.

□

Example 9.12. For a closed point $(\lambda, h) \in \mathrm{Spec}\, U^\hbar(\mathfrak{t})^\circ \hookrightarrow \mathfrak{t}^* \times \mathbb{A}^1_\hbar$, consider the associated one dimensional module $k_{(\lambda,h)}$ for $U^\hbar(\mathfrak{t})$, and associated Verma module

$$M^h_\lambda := U^h(\mathfrak{g}) \underset{U^h(\mathfrak{b})}{\otimes} k(\lambda, h) \in U^h(\mathfrak{g})\text{−mod}^N.$$

On the other hand, consider the generic universal Verma Δ°_Θ of weak highest weight $\Theta \in \Lambda$. Let us view this as a $U^\hbar(\mathfrak{t}) \otimes U^\hbar(\mathfrak{t})$-module, where the two commuting actions come from $U^\hbar(\mathfrak{t}) \hookrightarrow U^\hbar(\mathfrak{m})$ and the (N, T, w)-equivariance on Δ°_Θ, respectively. Let us denote these two actions by

$$X \mapsto X^\ell, X^r \in \mathrm{End}_k(\Delta^\circ_\Theta), \quad \text{for } X \in \mathfrak{t},$$

respectively, and the corresponding subalgebras of $\mathrm{End}_k(\Delta^\circ_\Theta)$ by $U^\hbar(\mathfrak{t})^\ell$ and $U^\hbar(\mathfrak{t})^r$. Then by definition the two actions satisfy the integrality condition

$$X^\ell + X^r = \langle X, \Theta \rangle \cdot \hbar \cdot \mathrm{id}.$$

In particular, if we specialize the action of X^r to a closed point $(\mu, h) \in \mathrm{Spec}\, U^\hbar(\mathfrak{t})^\circ$, we obtain

$$k_{(\mu,h)} \underset{U^\hbar(\mathfrak{t})^r}{\otimes} \Delta^\circ_\Theta \simeq M^h_{-\mu+h\cdot\Theta}.$$

Using a special case of the preceding for $\Theta = -2\rho$ and Proposition 9.11, we obtain

$$k_{(\mu,h)} \underset{U^\hbar(\mathfrak{t})^r}{\otimes} w \star (\Delta^\circ_{-2\rho}) \simeq k_{(\mu,h)} \underset{U^\hbar(\mathfrak{t})^r}{\otimes} \Delta^\circ_{-2\rho} \simeq M^h_{-\mu-2h\rho}. \tag{9.9}$$

On the other hand, by Corollary 9.10, we may rewrite this as

$$\simeq w \star (k_{w(\mu+h\rho)-h\rho,h} \underset{U^\hbar(\mathfrak{t})^r}{\otimes} \Delta^\circ_{-2\rho}) \simeq w \star M^h_{-w(\mu+h\rho)-h\rho}.$$

Noting that $w\overset{h}{\cdot}(-w(\mu+h\rho)-h\rho) = -\mu-2h\rho$, we therefore obtain $w\star M^h_x \simeq M^h_{w\overset{h}{\cdot}x}$.

10. Endomorphisms of the universal Verma module

10.1. **Main result.** In this section R is a perfect characteristic p with $p > h$. We assume $\kappa \neq 0$.

Let $\mathfrak{h}^a$ denote the affine subspace in $\mathfrak{h}^{da}$ defined by the equation $\hbar = 1$. Following Section 8.1.2, consider the module $\Delta^G_{\kappa\hbar} := \Delta^{G,\hbar}_{\mathbf{1}}/(\mathbf{1}-\kappa\hbar)\Delta^{G,\hbar}_{\mathbf{1}}$ over R. This is an object of the following category: Λ^a-graded and Iw-equivariant $\widehat{U}^\hbar_\kappa(\hat{\mathfrak{g}})$-$\mathsf{R}[\mathfrak{h}^*,\hbar]$-bimodules. Denote this category by $\mathcal{O}(\mathfrak{h}^*)$. Our goal is to construct a homomorphism from a certain algebra, to be denoted by ${}^I\mathcal{A}^\hbar$ to $\mathrm{End}_{\mathcal{O}(\mathfrak{h}^*)}(\Delta^G_{\kappa\hbar})$. The construction is based on Theorem 8.3.

10.1.1. Now we construct the algebra ${}^I\mathcal{A}$. Here I is the set of simple roots for $\mathfrak{g}$. Recall the elements $\tilde{h}_n := h^p_n - \kappa^{p-1}h_{pn}, h \in \mathfrak{h}, n \in \mathbb{Z}$. Their significance is that for the simple coroot basis $h^i, i \in I$, of $\mathfrak{h}$, the algebra $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$ is the filtered complete algebra of polynomials in the elements $\tilde{h}^i_n$, see Proposition 6.6. Define the algebra $\mathcal{A}$ as the algebra of polynomials in the variables $\tilde{h}^i_n$ with $n \leqslant 0$, a quotient of $\widehat{U}_\kappa(\hat{\mathfrak{h}})^{\mathcal{L}H}$. It carries a filtration, where the degree of $\tilde{h}^i_n$ is p. Let $\mathcal{A}^\hbar$ denote the Rees algebra of $\mathcal{A}$.

We construct ${}^I\mathcal{A}^\hbar$ as the intersection $\bigcap_{i\in I}{}^i\mathcal{A}^\hbar$ of certain subalgebras ${}^i\mathcal{A}^\hbar$ of $\mathcal{A}^\hbar$. The subalgebra ${}^i\mathcal{A} \subset \mathcal{A}$ is defined as follows. Let L_i be the minimal Levi in G corresponding to $i \in I$. Then ${}^i\mathcal{A}$ is the image of $\widehat{U}_\kappa(\hat{\mathfrak{l}}_i)^{\mathcal{L}L_i}$ in $\mathcal{A}$ under the Harish-Chandra homomorphism. Explicitly, this subalgebra is described as follows. Let X^i_n denote the elements from Proposition 7.2 for the $\mathfrak{sl}_2$-subalgebra $[\mathfrak{l}_i,\mathfrak{l}_i]$. For $n \leqslant 0$, we set $\underline{X}^i_n$ to be the image of $\mathsf{HC}(X^i_n)$ in $\mathcal{A}$. Explicitly, thanks to Proposition 7.10, we have

$$\underline{X}^i_n = \frac{1}{4}\left(\sum_{j=n}^{0}\tilde{h}^i_{j+n}\tilde{h}^i_{-j}\right) - \frac{1}{2}(1-\kappa^{p-1})(n+1)\tilde{h}^i_n. \tag{10.1}$$

For ${}^i\mathcal{A}$ we take the subalgebra of $\mathcal{A}$ generated by the elements $\tilde{h}^k_n$ with $k \neq i$ and $n \leqslant 0$ as well as the elements $\underline{X}^i_n$. Equivalently, thanks to Theorem 7.4 and Proposition 7.10, ${}^i\mathcal{A}$ is the image of $\widehat{U}_\kappa(\hat{\mathfrak{l}}_i)^{\mathcal{L}L_i}$.

And for ${}^i\mathcal{A}^\hbar$ we take the Rees algebra of ${}^i\mathcal{A}$ with its filtration restricted from $\mathcal{A}$. So ${}^i\mathcal{A}^\hbar$ is a graded subalgebra of $\mathcal{A}^\hbar$.

10.1.2. Note that $\mathcal{A}^\hbar$ naturally acts on $\mathsf{Wak}_{\kappa\hbar}$ by $\widehat{U}^\hbar_\kappa(\hat{\mathfrak{g}})$-$\mathsf{R}[\mathfrak{h}^*,\hbar]$-linear endomorphisms. In particular, the subalgebra ${}^I\mathcal{A}^\hbar \subset \mathcal{A}^\hbar$ acts on $\mathsf{Wak}_{\kappa\hbar}$. The following is the main result of this section.

Theorem 10.1. *There is a unique algebra homomorphism ${}^I\mathcal{A}^\hbar \to \mathrm{End}_{\mathcal{O}(\mathfrak{h}^*)}(\Delta^G_{\kappa\hbar})$ such that the homomorphism $\varphi^B_{\kappa\hbar} : \Delta^G_{\kappa\hbar} \to \mathsf{Wak}_{\kappa\hbar}$ is ${}^I\mathcal{A}^\hbar$-semi-linear: for any $m \in \Delta^G_{\kappa\hbar}$ and $a \in {}^I\mathcal{A}^\hbar$, we have $\varphi^B_{\kappa\hbar}(ma) = \varphi^B_{\kappa\hbar}(m)\mathsf{t}_P(a)$.*

10.1.3. We now explain the steps of the proof. For this we need to introduce some notation. Let P_i be the minimal parabolic containing B with Levi L_i. We consider the hyperplane $\mathfrak{h}^{da}_{\kappa\hbar}$ given by equation $\mathbf{1} = \kappa\hbar$ and the complement to the union of

hyperplanes of the form $\tilde{H}_{\beta,m}$, see Section 8.2.4, with $\beta = \alpha + n\delta$, where α is as follows:

- all possible α; this complement will be denoted by $\mathfrak{h}^{da}_{\kappa\hbar}(\varnothing)$.
- all $\alpha \neq \alpha_i$; this complement will be denoted by $\mathfrak{h}^{da}_{\kappa\hbar}(i)$. all non-simple α; this complement will be denoted by $\mathfrak{h}^{da}_{\kappa\hbar}(I)$.

We write $\Delta^{G,\varnothing}_{\kappa\hbar}$ for $\Delta^{G}_{\kappa\hbar} \otimes_{\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}]} \mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(\varnothing)]$. The notations $\Delta^{G,i}_{\kappa}, \mathsf{Wak}^{\varnothing}_{\kappa\hbar}, \mathsf{W}\Delta^{P_i,i}_{\kappa\hbar}$, etc., have similar meaning. Thanks to Theorem 8.3 we have the following isomorphisms

$$\varphi^{B}_{\kappa\hbar} : \Delta^{G,\varnothing}_{\kappa\hbar} \xrightarrow{\sim} \mathsf{Wak}^{B,\varnothing}_{\kappa\hbar}, \tag{10.2}$$

$$\varphi^{P_i}_{\kappa\hbar} : \Delta^{G,i}_{\kappa\hbar} \xrightarrow{\sim} \mathsf{W}\Delta^{P_i,i}_{\kappa\hbar}. \tag{10.3}$$

Using this, we will construct an action of ${}^{I}\mathcal{A}^{\hbar}$ on $\Delta^{G,I}_{\kappa\hbar}$ by $\widehat{U}^{\hbar}_{\kappa}(\hat{\mathfrak{g}}) \otimes \mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(I)]$-linear (this is clear) and Iw-equivariant (this is less clear) endomorphisms. Then we will show that this action preserves the $\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}]$-lattice $\Delta^{G}_{\kappa\hbar}$.

10.1.4. Finally we state a corollary of Theorem 10.1 combined with the transitivity property (8.1). Note that ${}^{i}\mathcal{A}^{\hbar}$ acts on $\mathsf{W}\Delta^{P_i}_{\kappa\hbar}$ by endomorphisms.

Corollary 10.2. *The homomorphism $\varphi^{P_i}_{\kappa\hbar} : \Delta^{G}_{\kappa\hbar} \to \mathsf{W}\Delta^{P_i}_{\kappa\hbar}$ is ${}^{I}\mathcal{A}^{\hbar}$-semi-linear.*

10.2. **Action on $\Delta^{G,I}$.** Recall, Lemma 6.2, that

$$[\widehat{D}(\mathcal{L}N)\widehat{\otimes}\widehat{U}_{\kappa}(\hat{\mathfrak{l}})]^{\mathcal{L}P} \xrightarrow{\sim} \widehat{U}_{\kappa}(\hat{\mathfrak{l}})^{\mathcal{L}L}.$$

This is a filtered algebra isomorphism. Applying this to $P = P_i$, we see that $\widehat{U}_{\kappa\hbar}(\hat{\mathfrak{l}}_i)^{\mathcal{L}L_i}$ acts on any Iw_{P_i}-equivariant module over $\widehat{D}^{\hbar}(\mathcal{L}N_i)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}_{\kappa\hbar}(\hat{\mathfrak{l}}_i)$, in particular, on $\mathsf{Wak}\Delta^{P_i}_{\kappa\hbar}$. The action is by $\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}]$-linear automorphisms, so extends to $\mathsf{Wak}\Delta^{P_i,i}_{\kappa\hbar}$. Note that the resulting action factors through ${}^{i}\mathcal{A}^{\hbar}$.

In particular, we can carry the action of ${}^{i}\mathcal{A}$ to $\Delta^{G,i}_{\kappa\hbar}$ using isomorphism (10.3). Note that $\Delta^{G,I}_{\kappa\hbar}$ coincides with $\bigcap_{i\in I}\Delta^{G,i}_{\kappa\hbar}$, where the intersection is taken in $\Delta^{G,\varnothing}_{\kappa\hbar}$. It follows that ${}^{I}\mathcal{A}^{\hbar}$ acts on $\Delta^{G,I}_{\kappa\hbar}$. The action is by $\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(I)]$-linear endomorphisms that are equivariant for the action of Iw_{P_i} for all $i \in I$. Note that, thanks to (8.1), the condition that $\varphi^{B} : \Delta^{G,\varnothing}_{\kappa\hbar} \to \mathsf{Wak}^{B,\varnothing}_{\kappa\hbar}$ is ${}^{I}\mathcal{A}^{\hbar}$-equivariant recovers the action of ${}^{I}\mathcal{A}^{\hbar}$ on the source module uniquely.

Lemma 10.3. *The action of ${}^{I}\mathcal{A}^{\hbar}$ on $\Delta^{G,I}_{\kappa\hbar}$ is by Iw-equivariant endomorphisms.*

Proof. A $\hat{U}_{\kappa\hbar}(\hat{\mathfrak{g}}) \otimes \mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(I)]$-linear endomorphism of $\Delta^{G,I}_{\kappa\hbar}$ is Iw- (resp., Iw_{P_i}-) equivariant Note that Iw is a pro-algebraic group, and its representation in $\Delta^{G,I}_{\kappa\hbar}$ is rational. For $m \geqslant 0$, let Iw_m denote the preimage of B in $\mathcal{J}_n G$. Define $\mathsf{Iw}_{P_i,m}$ similarly. So $\mathsf{Iw}_{P_i,m} \subset \mathsf{Iw}_m$ are algebraic groups, they are connected, and $\mathsf{Iw} = \varprojlim_m \mathsf{Iw}_m, \mathsf{Iw}_{P_i} = \varprojlim_m \mathsf{Iw}_{P_i,m}$. We need to show that any element in a rational representation of Iw that is invariant under $\mathsf{Iw}_{P_i,m}$ for all i, is also Iw_m-invariant. This will follow if we show that the subalgebras $\mathrm{Dist}_1(\mathsf{Iw}_{P_i,m})$ for $i \in I$ generate the algebra $\mathrm{Dist}_1(\mathsf{Iw}_m)$.

Note that $\mathcal{J}_m\tilde{N}^- \subset \mathsf{Iw}_m$. The product map $\mathcal{J}_m\tilde{N}^- \times (\mathsf{Iw}_m \cap \mathcal{J}_m B) \to \mathsf{Iw}_m$ is an open embedding. Since $\mathcal{J}_m B \subset \mathsf{Iw}_{P_i,m}$ for all i, it suffices to show that the

subalgebras $\mathrm{Dist}_1(\mathcal{J}_m\tilde{N}_i^-)$ for $i \in I$ generate $\mathrm{Dist}_1(\mathcal{J}_m\tilde{N}-)$. First, of all, the claim on the level of Lie algebras is clear. We reduce to showing that the subalgebras $\mathrm{Dist}_1([\mathcal{J}_m\tilde{N}_i^-]^{(1)})$ generate $\mathrm{Dist}_1([\mathcal{J}_m\tilde{N}-]^{(1)})$. This follows once we observe that the algebraic groups $\mathcal{J}_m\tilde{N}_i^-, \mathcal{J}_m\tilde{N}^-$ are defined over $\mathbb{F}_p$ as are the embeddings $\mathcal{J}_m\tilde{N}_i^- \hookrightarrow \mathcal{J}_m\tilde{N}^-$, while the Frobenius homomorphisms $\mathcal{J}_m\tilde{N}_i^- \to (\mathcal{J}_m\tilde{N}_i^-)^{(1)}$ and $\mathcal{J}_m\tilde{N}^- \to (\mathcal{J}_m\tilde{N}^-)^{(1)}$ are epimorphisms (because these algebraic groups are smooth over $\mathrm{Spec}(\mathsf{R})$). □

11. Description of ${}^I\mathcal{A}$

Let R be a perfect characteristic p field. Assume $\kappa \in \mathsf{R} \setminus \mathbb{F}_p$.

11.1. **Main result.**

11.1.1. Let $\mathcal{A}^{\hbar,\natural}$ denote the algebra of polynomials in the variables $h_n^{i,\natural}, n \leqslant 0$ and $\hbar^\natural$. We have $\mathsf{R}[\hbar] \otimes_{\mathsf{R}[\hbar^\natural]} \mathcal{A}^{\natural,\hbar} \hookrightarrow \mathcal{A}^\hbar$ via

$$h_n^{i,\natural} \mapsto ((h_n^i)^p - (\kappa\hbar)^{p-1}h_{pn}^i)/(1-\kappa^{p-1}), \hbar^\natural \mapsto \hbar^p.$$

Note that $\mathcal{A}^{\hbar,\natural}$ is (naively) graded with $\deg h_n^{i,\natural} = \deg \hbar^\natural = 1$ and the isomorphism with $\mathcal{A}^\hbar$ rescales the degrees p times.

Define a homogeneous degree 2 element in $\underline{X}_n^{i,\natural} \in \mathcal{A}^{\hbar,\natural}$ by

$$\underline{X}_n^{i,\natural} := \frac{1}{4}\left(\sum_{j=n}^{0} h_{j+n}^{i,\natural}h_{-j}^{i,\natural}\right) - \frac{1}{2}(n+1)\hbar^\natural h_n^{i,\natural}. \tag{11.1}$$

Note that under the homomorphism $\mathcal{A}^{\hbar,\natural} \to \mathcal{A}^\hbar$, the element $X_n^{i,\natural}$ is sent to the homogenous analog of $\mathsf{HC}(X_n^i)$, see Section 7.5.

Consider the $\mathsf{R}[\hbar^\natural]$-subalgebra ${}^i\mathcal{A}^{\hbar,\natural}$ generated by the elements $X_n^{i,\natural}$ and $h_n^{j,\natural}$ for $j \neq i$. Then $\mathsf{R}[\hbar] \otimes_{\mathsf{R}[\hbar^\natural]} {}^i\mathcal{A}^{\hbar,\natural} \xrightarrow{\sim} {}^i\mathcal{A}^\hbar$. We remark that ${}^i\mathcal{A}^{\hbar,\natural}/(\hbar^\natural) \hookrightarrow {}^i\mathcal{A}^{\hbar,\natural}/(\hbar^\natural)$.

11.1.2. Set ${}^i\mathcal{A}^\natural = {}^i\mathcal{A}^{\hbar,\natural}/(\hbar^\natural - 1)$. We notice that, over a characteristic 0 field, by results of [F], the analog of $\bigcap_{i\in I} {}^i\mathcal{A}^\natural$ is the algebra on the space of $\check{G}$-opers on $\mathcal{D}^\times$ with regular singularities. Here $\check{G}$ is the Langlands dual group of adjoint type. We will see that this is essentially the case in our situation as well.

11.1.3. Choose a Borel subgroup $\check{B} \subset \check{G}$. Let $\rho : \mathbb{G}_m \to \check{G}$ be the one-parameter subgroup corresponding to the sum of fundamental weights. Let $\check{\mathfrak{g}} = \bigoplus_{i\in\mathbb{Z}} \check{\mathfrak{g}}_i$ be the corresponding grading. Let $\check{f} \in \check{\mathfrak{g}}_{-1}$ be the sum of roots vectors corresponding to negative simple roots.

For $k \in \mathbb{Z}$, define the subscheme $\widetilde{\mathsf{Op}}_k^\hbar \subset \mathrm{Spec}(\mathsf{R}[\hbar^\natural]) \times \mathcal{L}\check{\mathfrak{g}}$ as

$$\{\lambda\partial + f + \sum_{i\geqslant 0} F_i | F_i \in t^{-k(i+1)}\check{\mathfrak{g}}_i[[t]]\},$$

here $\hbar^\natural$ is the function sending the element in the brackets to λ. We will write $\widetilde{\mathsf{Op}}_k^\hbar$ when we want to indicate the dependence on $\check{G}$. The fiber of $\widetilde{\mathsf{Op}}_k^\hbar$ at $\hbar^\natural = 1$ will be denoted by $\widetilde{\mathsf{Op}}_k$.

Define the subgroup scheme $\mathcal{J}_k\check{N} \subset \mathcal{L}\check{N}$ as

$$\{\exp(\sum_{i>0} x_i)|x_i \in t^{-ki}\check{\mathfrak{g}}_i[[t]]\}.$$

Note that since $p > h$, the morphism $\exp : \mathfrak{n} \to N$ is well-defined. We also remark that

$$\widetilde{\mathsf{Op}}_k^{\hbar} \subset \widetilde{\mathsf{Op}}_{k+1}^{\hbar}, \mathcal{J}_k\check{N} \subset \mathcal{J}_{k+1}\check{N}, \forall k.$$

Define the actions of $\mathbb{G}_m$ and $\mathcal{J}_k\check{N}$ on $\widetilde{\mathsf{Op}}_k^{\hbar}(\check{\mathfrak{g}})$ by

$$\begin{aligned} n.(\lambda\partial + F) &= \lambda\zeta + \mathrm{Ad}(n)F - \lambda(\zeta.n)n^{-1}, n \in \mathcal{J}_kN \\ z.(\lambda\zeta + F) &= \lambda z^{-1}\partial + z^{-1}\rho(z)^{-1}F. \end{aligned} \tag{11.2}$$

In particular, note that f is $\mathbb{G}_m$-stable. These two actions combine into an action of $\mathbb{G}_m \ltimes \mathcal{J}\check{N}$, where $\mathbb{G}_m$ acts on $\mathcal{J}\check{N}$ by $\mathrm{Ad}(\rho)^{-1}$.

11.1.4. Now we introduce a "deformed jet" version of the Kostant-Slodowy slice, a space of opers.

Consider the $\mathfrak{sl}_2$-triple $(\check{e}, \rho, \check{f})$ and the affine subspace

$$\mathsf{Op}_k^{\hbar} := \{\lambda\partial + f + \sum_{i\geqslant 1} F_i | F_i \in t^{-(i+1)k}\mathfrak{z}_{\check{\mathfrak{g}}}(\check{e})[[t]]\},$$

where $\mathfrak{z}_{\check{\mathfrak{g}}}(\check{e})$ stands for the centralizer of $\check{e}$ in $\check{\mathfrak{g}}$. Then we have the following result.

Lemma 11.1. *Suppose $p > h$. Then the multiplication map*

$$\mathcal{J}_k\check{N} \times \mathsf{Op}_k^{\hbar} \to \widetilde{\mathsf{Op}}_k^{\hbar} \tag{11.3}$$

is an isomorphism.

Proof. We note that, under our conditions on p, the multiplication map $\check{N} \times \check{S} \to \check{f} + \check{\mathfrak{b}}$ is an isomorphism, [R, Proposition 3.2.1]. By passing to arc spaces we see that $\mathcal{J}\check{N} \times \mathcal{J}\check{S} \to \mathcal{J}(\check{f} + \check{\mathfrak{b}})$ is an isomorphism. From here we see that $\mathcal{J}_k\check{N} \times \mathcal{J}_k\check{S} \to \mathcal{J}_k(\check{f} + \check{\mathfrak{b}})$ is an isomorphism for all k. Namely, we have an automorphism of t^{ρ} of $\mathcal{L}\check{\mathfrak{g}}$ that on $\mathcal{L}\check{\mathfrak{g}}_i$ is given by the multiplication by t^i. Applying $t^{-k\rho}$ to $\mathcal{J}_k\check{S}, \mathcal{J}_k(\check{f} + \check{\mathfrak{b}}), \mathcal{J}_k\check{N}$ we get $t^{-k}\mathcal{J}\check{S}, t^{-k}\mathcal{J}(\check{f} + \check{\mathfrak{b}}) \subset \mathcal{L}\check{\mathfrak{g}}$ and $\mathcal{J}\check{N} \subset \mathcal{L}\check{N}$. The claim that the multiplication morphism is an isomorphism follows.

Now note that the $\mathbb{G}_m$-actions on $\mathcal{J}_k\check{N}, \mathsf{Op}_k^{\hbar}, \widetilde{\mathsf{Op}}_k^{\hbar}$ gives positive gradings on the algebras of functions. So, a version of the graded Nakayama lemma tells us that once the multiplication map (11.3) is an isomorphism over $0 \in \mathrm{Spec}(\mathsf{R}[\hbar])$, it is an isomorphism. □

We remark that the fibers at $\hbar = 1$ of the affine schemes $\mathsf{Op}_0^{\hbar}$ and $\mathsf{Op}_1^{\hbar}$ can be interpreted as the space of $\check{G}$-opers on $\mathcal{D}$ and the space of $\check{G}$-opers on $\mathcal{D}^{\times}$ with regular singularities, respectively.

11.1.5. It turns out that there is a (naively) graded algebra homomorphism $\mathsf{M}_k : \mathsf{R}[\mathsf{Op}_k^{\hbar}] \to \mathsf{R}[t^{-k}\mathcal{J}\check{\mathfrak{h}}]$ called the *Miura map*. This will be recalled in Section 11.2. Our main result is as follows.

Proposition 11.2. M_1 *is injective and its image coincides with* ${}^I\mathcal{A}^{\hbar,\natural}$.

11.2. **Miura map.**

11.2.1. We start by recalling how M_k is defined for arbitrary k. Consider the embedding

$$\lambda\partial + F \mapsto \lambda\partial + \check{f} + F : \mathrm{Spec}(\mathsf{R}[\hbar^\natural]) \times t^{-k}\mathcal{J}(\check{\mathfrak{h}}) \hookrightarrow \widetilde{\mathsf{Op}}_k^\hbar. \tag{11.4}$$

The pullback map $\mathsf{R}[\widetilde{\mathsf{Op}}_k^\hbar] \to \mathsf{R}[t^{-k}\mathcal{J}\check{\mathfrak{h}}]$ is a (naively) graded algebra homomorphism. Let M_k be the composition of this pullback map with the embedding

$$\mathsf{R}[\mathsf{Op}_k^\hbar] \xrightarrow{\sim} \mathsf{R}[\widetilde{\mathsf{Op}}_k^\hbar]^{\mathcal{J}\check{N}} \hookrightarrow \mathsf{R}[\widetilde{\mathsf{Op}}_k^\hbar].$$

11.2.2. A crucial example to understand is for $\check{G} = \mathrm{PGL}_2$. We can write an element of $t^{-k}\mathcal{J}(\check{h})$ as $\mathrm{diag}(a(t), -a(t))$, where $a(t)$ is a formal Laurent series $\sum_{j=-k}^\infty a_j t^j$. The function h_n with $n \in \mathbb{Z}$ sends $a(t)$ to $2a_{-n}$. Under the map to $\mathsf{Op}_k^\hbar$, the element $\lambda\partial + \mathrm{diag}(a(t), -a(t))$ goes to

$$\lambda\partial + \begin{pmatrix} 0 & b(t) \\ 1 & 0 \end{pmatrix}, b(t) := a(t)^2 + \lambda\partial a(t).$$

The free generator x_n of $\mathsf{R}[\zeta + \psi + \mathcal{J}\check{S}]$ sends $b(t) = \sum_{j=0}^\infty b_j t^j$ to b_{-n}. So, for $k = 1$, $\mathsf{M}_1(x_n)$ is exactly the element $\underline{X}_n^\natural$. In particular, $\mathrm{im}\,\mathsf{M}_1 = {}^I\mathcal{A}^{\hbar,\natural}$.

11.2.3. We finish this section with the following lemma.

Lemma 11.3. *For general* $\check{\mathfrak{g}}$ *we have* $\mathrm{im}\,\mathsf{M}_1 \subset {}^I\mathcal{A}^{\hbar,\natural}$.

Proof. Consider the minimal Levi subgroup $\check{L}_i$ and its maximal unipotent subgroup $\check{N}_i$. Let $f_i, \check{S}_i$ be the analogs of $\check{f}, \check{S}$ for L_i. We have the analog of M for L_i, denote it by M^i. Thanks to Section 11.2.2, for $k = 1$, we have

$$\mathrm{im}\,\mathsf{M}^i = {}^i\mathcal{A}^{\hbar,\natural}. \tag{11.5}$$

Note that we have a $\mathcal{J}_1 N_i$-equivariant embedding

$$\widetilde{Op}_k^\hbar(\check{L}_i) \hookrightarrow \widetilde{Op}_k^\hbar(\check{G}), \lambda\partial + \check{f}_i + F \mapsto \lambda\partial + \check{f} + F. \tag{11.6}$$

The embedding (11.4) factors into the composition of its analog for $H \subset L_i$ and (11.6). It follows that M factors as $\mathsf{M}^i \circ \varphi$, where φ is an algebra homomorphism $\mathsf{R}[\mathsf{Op}_1^\hbar(\check{G})] \to \mathsf{R}[\mathsf{Op}_1^\hbar(\check{L}_i)]$. The claim of the lemma now follows from (11.5). □

11.3. **Proof of the main result.**

Proof of Proposition 11.2. The proof is in several steps.

Step 1. Thanks to Lemma 11.3, we have $\mathrm{im}\,\mathsf{M}_1 \subset {}^I\mathcal{A}^\natural$. Recall that M_1 is $\mathsf{R}[\hbar^\natural]$-linear and graded. Its specialization at $\hbar^\natural = 0$ is a homomorphism

$$\mathsf{R}[\mathcal{J}_1\check{S}] \to \mathsf{R}[t^{-1}\mathcal{J}\check{\mathfrak{h}}]. \tag{11.7}$$

This homomorphism can be understood as follows. We note that the restriction of the quotient morphism $\check{\mathfrak{g}} \to \check{\mathfrak{g}}//\check{G}$ to $\check{S}$ is an isomorphism under our restrictions on p, see [R, Theorem 3.2.2]. From here we conclude that the pullback under the composition

$$\check{\mathfrak{h}} \hookrightarrow \check{f} + \check{\mathfrak{b}} \twoheadrightarrow (\check{f} + \check{\mathfrak{b}})/\check{N} \xrightarrow{\sim} \check{S} \xrightarrow{\sim} \check{\mathfrak{g}}//\check{G}$$

is the Chevalley restriction map to be denoted by res.

As in the proof of Lemma 11.1, we can identify $\mathcal{J}_1\check{S}$ with $\mathcal{J}\check{S}$ and $t^{-1}\mathcal{J}\check{\mathfrak{h}}$ with $\mathcal{J}\check{\mathfrak{h}}$. Under this identification, the homomorphism (11.7) is nothing else but $\mathcal{J}(\mathsf{res})$. As we have seen in the proof of Lemma 6.4, the map $\mathcal{J}(\mathsf{res})$ is injective. It follows that M_1 is injective. It remains to show that

$$\operatorname{im}\mathsf{M}_1 = {}^I\mathcal{A}^{\natural,\hbar}. \tag{11.8}$$

Step 2. Let $W_i \cong S_2$ denote the Weyl group of $\check{L}_i$ and W denote the Weyl group of $\check{G}$. The quotient morphisms $\check{\mathfrak{h}} \to \check{\mathfrak{h}}/W_i, \check{\mathfrak{h}} \to \check{\mathfrak{h}}/W$ give rise to the embeddings $\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W_i)], \mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)] \hookrightarrow \mathsf{R}[\mathcal{J}\check{\mathfrak{h}}]$. Note that $\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)] \subset \mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W_i)]$. Note also that the image of M_1 in $\mathsf{R}[\mathcal{J}\check{\mathfrak{h}}]$ contains $\operatorname{im}\mathcal{J}(\mathsf{res})$, while the image of ${}^i\mathcal{A}^{\hbar,\natural}$ in $\mathsf{R}[\mathcal{J}\check{\mathfrak{h}}][\hbar^\natural]$ is $\hbar^\natural$-saturated (meaning that the cokernel has no $\hbar^\natural$-torsion) and modulo $\hbar^\natural$ this image is equal to $\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W_i)]$. So we reduce (11.8) to checking that

$$\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)] = \bigcap_{i\in I}\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W_i)]. \tag{11.9}$$

Step 3. Let $(\check{\mathfrak{h}}/W_i)^{reg}$ denote the locus in $\check{\mathfrak{h}}/W_i$, where the natural morphism between quotients $\check{\mathfrak{h}}/W_i \to \check{\mathfrak{h}}/W$ is etale. This is a principal open subset. Note that for a smooth finite type scheme X over $\operatorname{Spec}(\mathsf{R})$, the morphism $\mathcal{J}X \to X$ is flat. Therefore we have the natural morphisms $\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W_i)] \to \mathsf{R}[\mathcal{J}((\check{\mathfrak{h}}/W_i)^{reg})] \to \mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}^{reg})]$ are embeddings. So (11.9) will follow from

$$\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)] = \bigcap_{i\in I}\mathsf{R}[\mathcal{J}((\check{\mathfrak{h}}/W_i)^{reg})]. \tag{11.10}$$

Step 4. Note that if $Y \to X$ is an etale morphism of finite type smooth schemes over $\operatorname{Spec}(\mathsf{R})$, then $\mathcal{J}Y \xrightarrow{\sim} Y \times_X \mathcal{J}X$. Applying this to $X = \check{\mathfrak{h}}/W, Y = (\check{\mathfrak{h}}/W_i)^{reg}$ or $Y = \check{\mathfrak{h}}^{reg}$, we see that

$$\mathsf{R}[\mathcal{J}((\check{\mathfrak{h}}/W_i)^{reg})] = \mathsf{R}[(\check{\mathfrak{h}}/W_i)^{reg}] \otimes_{\mathsf{R}[\check{\mathfrak{h}}/W]} \mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)],$$
$$\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}^{reg})] = \mathsf{R}[\check{\mathfrak{h}}^{reg}] \otimes_{\mathsf{R}[\check{\mathfrak{h}}/W]} \mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)].$$

Since $\mathsf{R}[\mathcal{J}(\check{\mathfrak{h}}/W)]$ is a flat $\mathsf{R}[\check{\mathfrak{h}}/W]$-module, tensoring with this module preserves the intersections of submodules, so (11.10) will follow from

$$\mathsf{R}[\check{\mathfrak{h}}/W] = \bigcap_{i\in I}\mathsf{R}[(\check{\mathfrak{h}}/W_i)^{reg}]. \tag{11.11}$$

For the latter, note that the right hand side is contained in $\mathsf{R}[\check{\mathfrak{h}}^{reg}/W]$ and consists of all elements there that do not have poles on $(\check{\mathfrak{h}}/W)\setminus(\check{\mathfrak{h}}^{reg}/W)$. Now (11.11). $\square$

Remark 11.4. The same argument shows that, for any $k \geqslant 0$, the homomorphism $\mathsf{M}_0 : \mathsf{R}[\mathsf{Op}_0^{\hbar}] \to \mathsf{R}[\mathcal{J}\check{h}][\hbar^\natural]$ is injective and its image is the intersection of the images of the homomorphisms M_0^i.

Remark 11.5. Over a characteristic 0 field, one can interpret M_1 (specialized to $\hbar^\natural = 1$) as follows. One identifies $\mathcal{A}^\natural$ with the algebra of functions on the space of generic Miura opers with regular singularities. This endows $\operatorname{Spec}(\mathcal{A}^\natural)$ with a free action of $\check{N}$ and the natural map $\operatorname{Spec}(\mathcal{A}^\natural) \to \operatorname{Spec}(\mathsf{Op}_1^1)$ is a quotient for this action. This construction fails in characteristic p: for example, the map from the

space of generic Miura opers to the space of opers can be shown to be invariant for $\check{N}[[t^p]]$.

12. Computation of $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$

In this section we assume that R is a perfect field of characteristic $p > h$. We further assume that $\kappa \notin \mathbb{F}_p$. The goal of this section is to compute the commutative vertex algebra $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$. Recall the embedding $\mathsf{HC}_H : V_\kappa(\mathfrak{g})^{\mathcal{J}G} \hookrightarrow V_\kappa(\mathfrak{h})^{\mathcal{J}H}$ of commutative vertex algebras. We are going to describe the image. For this observe that we have a surjective algebra homomorphism $\mathcal{A} \twoheadrightarrow V_\kappa(\mathfrak{h})^{\mathcal{J}H}$ sending $\tilde{h}_n$ to $\tilde{h}_n$ for $n < 0$ and to zero else. Under the isomorphism $\mathcal{A}^\natural \xrightarrow{\sim} \mathcal{A}$, the quotient $V_\kappa(\mathfrak{h})^{\mathcal{J}H}$ gets identified with the polynomial algebra $\mathcal{A}_0^\natural$ in the variables $h_n^{i,\natural}$ with $n < 0$. The following is the main result of this section (and of the paper):

Theorem 12.1. *Under the isomorphism $\mathcal{A}_0^\natural \cong V_\kappa(\mathfrak{h})^{\mathcal{J}H}$, the subalgebra $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$ is sent to* $\operatorname{im} \mathsf{M}_0$.

Here is a scheme of proof of this theorem that will be given below in this section.

Step 1: Recall, Theorem 10.1, that ${}^I\mathcal{A}^\hbar$ acts on $\Delta^G_{\kappa\hbar}$ by graded endomorphisms. The vertex algebra $V^\hbar_\kappa(\mathfrak{g})$ is a quotient of $\Delta^G_{\kappa\hbar}$. We will see that the action is in fact by $\mathcal{J}G$-equivariant endomorphisms. This will give rise to a graded $\mathsf{R}[\hbar]$-algebra homomorphism ${}^I\mathcal{A}^\hbar \to V^\hbar_\kappa(\mathfrak{g})^{\mathcal{J}G}$.

Step 2. We will analyze the specialization of the homomorphism ${}^I\mathcal{A}^\hbar \to V^\hbar_\kappa(\mathfrak{g})^{\mathcal{J}G}$ at $\hbar = 0$. We will see that it is surjective. That will allow us to determine the energy graded character of $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$.

Step 3. Using this we will finish the proof of the theorem.

12.1. **Homomorphism ${}^I\mathcal{A}^\hbar \to V^\hbar_\kappa(\mathfrak{g})^{\mathcal{J}G}$.** We have an associative algebra isomorphism $V^\hbar_\kappa(\mathfrak{g})^{\mathcal{J}G} \xrightarrow{\sim} \operatorname{End}(V^\hbar_\kappa(\mathfrak{g}))$, where by End we mean $\widehat{U}^\hbar_\kappa(\hat{\mathfrak{g}})$-linear and $\mathcal{J}G$-equivariant endomorphisms. See [F, (3.3-3)] for a completely analogous statement.

Note that $V^\hbar_\kappa(\mathfrak{g})$ is a quotient of $\Delta^G_{\kappa\hbar}$ viewed as an object of $\mathcal{O}(\mathfrak{h}^*)$. Recall, Theorem 10.1 that there is a unique algebra homomorphism ${}^I\mathcal{A}^\hbar \to \operatorname{End}_{\mathcal{O}(\mathfrak{h}^*)}(\Delta^G_{\kappa\hbar})$ such that the homomorphism $\varphi^B_\kappa : \Delta^{\mathfrak{g}}_\kappa \to \mathsf{Wak}^B_\kappa$ is ${}^I\mathcal{A}^\hbar$-semi-linear (with respect to the automorphism t_B of $\mathcal{A}^\hbar$).

The following is the main result of this section.

Proposition 12.2. *For every $a \in {}^I\mathcal{A}$, the following claims hold:*

1. *The endomorphism $\theta(a)$ of $\Delta^G_{\kappa\hbar}$ descends to a $\widehat{U}^\hbar_\kappa(\hat{\mathfrak{g}})$-linear endomorphism of $V^\hbar_\kappa(\mathfrak{g})$, to be denoted by $\underline{\theta}(a)$.*
2. *The endomorphism $\underline{\theta}(a)$ is $\mathcal{J}G$-equivariant.*

Proof. We start by proving (1). The proof is in several steps.

Step 1. Let v denote a generator in the highest weight space $\Delta^G_{\kappa\hbar}[0]$. The quotient $V^\hbar_\kappa(\mathfrak{g})$ of $\Delta^G_{\kappa\hbar}[0]$ is given by relations $xv = 0$ for all $x \in \mathfrak{h}$ and $e_i v = 0$ for all $i \in I$. Let $\check{\varpi}_i$ denote the fundamental coweight labelled i. We will show that the action of ${}^I\mathcal{A}^\hbar$ on $\Delta^G_{\kappa\hbar}$ preserves the submodules generated by $\check{\varpi}_i v$ and $e_i v$.

Step 2. Let P be the minimal parabolic P_i, and L and N be its Levi subgroup and the maximal unipotent subgroup. Let v' and v'' be generators of $\widehat{D}_\hbar(\mathcal{L}N)/I_{>0}(\widehat{D}_\hbar(\mathcal{L}N))][0]$ and $\Delta^L_{\kappa\hbar}[0]$ so that $v'\otimes v''$ is a generator of $\mathsf{Wak}^P_{\kappa\hbar}$. We can assume that $\varphi^P(v)=v'\otimes v''$. One can get a formula for $\mathsf{ffr}_P(e_it^{-1})$ similarly to Lemma 4.4: $\mathsf{ffr}_P(e_it^{-1})$ is represented as the sum of two summands, one in $\mathsf{CDO}(N)$, the other in $V_\kappa(\mathfrak{l})$. Consequently, one sees that $e_i(v'\otimes v'')=v'\otimes e_iv''$.

Step 3. For $z\in\mathsf{R}$, we write $\Delta^G_{\kappa\hbar}(z)$ and $\mathsf{Wak}^P_{\kappa\hbar}(z)$ for the quotients of $\Delta^G_{\kappa\hbar},\mathsf{Wak}^P_{\kappa\hbar}$ by the image of $\check{\varpi}_i-z\hbar$ under the right action (and we write $\Delta^G_{\kappa\hbar,i}(z)$ if we want to indicate the dependence on i). Then φ^P induces $\varphi^P(0):\Delta^G_{\kappa\hbar}(0)\to\mathsf{Wak}^P_{\kappa\hbar}(0)$. Under the action of $\widehat{U}^\hbar_\kappa(\hat{\mathfrak{g}})$, the vector $e_iv\in\Delta^G_{\kappa\hbar}(0)$ generates a copy of $\Delta^G_{\kappa\hbar}(2)$. Similarly, under the action of $\widehat{D}^\hbar(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^\hbar_\kappa(\hat{\mathfrak{l}})$, the vector $v'\otimes e_iv''\in\mathsf{Wak}^P_{\kappa\hbar}(0)$ generates a copy of $\mathsf{Wak}^P_{\kappa\hbar}(2)$.

Step 4. We claim that

(*) ${}^I\mathcal{A}^\hbar$ preserves $\Delta^G_{\kappa\hbar}(2)\subset\Delta^G_{\kappa\hbar}(0)$.

Once we know (*) for all i, the claim of (1) follows. Indeed, the action of ${}^I\mathcal{A}^\hbar$ preserves the submodule $\Delta^G_{\kappa\hbar}\mathfrak{h}$. And $V^\hbar_\kappa(\mathfrak{g})$ is the quotient of $\Delta^G_{\kappa\hbar}/\Delta^G_{\kappa\hbar}\mathfrak{h}$ by the sum of images of $\Delta^G_{\kappa\hbar,i}(2)$ over all i.

Step 5. We proceed to proving (*). Note that $\varphi^P(0)$ restricts to a homomorphism $\Delta^G_{\kappa\hbar}(2)\to\mathsf{Wak}^P_{\kappa\hbar}(2)$ that sends a generator to a generator. So this restriction, to be denoted by $\varphi^P(2)$ is also induced by φ^P (possibly after multiplying by an invertible element of R). The action of ${}^i\mathcal{A}^\hbar$ on $\mathsf{Wak}^P_{\kappa\hbar}(0)$, by the construction, comes from the action of the center of $\widehat{D}^\hbar(\mathcal{L}N)\widehat{\otimes}_{\mathsf{R}[\hbar]}\widehat{U}^\hbar_\kappa(\hat{\mathfrak{l}})$. So it preserves the submodule $\mathsf{Wak}^P_{\kappa\hbar}(2)$. We note that ${}^i\mathcal{A}^\hbar$ is stable under the automorphism t_P. It follows that $\mathsf{Wak}^P_{\kappa\hbar}(2)$ is also stable under the action of ${}^I\mathcal{A}^\hbar$.

Step 6. Let $\mathfrak{h}^{da}_{\kappa\hbar}(2)$ be the hyperplane in $\mathfrak{h}^{da}_{\kappa\hbar}$ defined by $\check{\varpi}_i=2\hbar$. Let x be the generic point of this hyperplane (i.e., $\mathbf{k}=\mathrm{Frac}(\mathfrak{h}^{da}_{\kappa\hbar}(2))$ and the homomorphism $\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(2)]\to\mathbf{k}$ is the canonical one). We make two observations. First

(*) $\Delta^G_{\kappa\hbar}(0)/\Delta^G_{\kappa\hbar}(2)$ is free over $\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(2)]$.

This is because this module is parabolically induced from the $U^\hbar(\mathfrak{l})$-module $\mathsf{R}[\mathfrak{h}^{da}_{\kappa\hbar}(2)]$ (where $\mathbf{1}$ acts by $\kappa\hbar$). Second, the point x is P-generic in the sense of Definition 8.2. It follows from Theorem 8.3 that

- $\varphi^P(2):\Delta^G_{\mathsf{x}}\to\mathsf{Wak}^P_{\mathsf{x}}$ is an isomorphism.

It follows from (**) that ${}^I\mathcal{A}^\hbar$ preserves Δ^G_{x} (viewed as a submodule in a suitable localization of $\Delta^G_{\kappa\hbar}(0)$). Thanks to (*), we see that $\Delta^G_{\kappa\hbar}(2)=\Delta^G_{\mathsf{x}}\cap\Delta^G_{\kappa\hbar}(0)$. Hence we see that $\Delta^G_{\kappa\hbar}(2)$ is fixed by ${}^I\mathcal{A}^\hbar$ finishing the proof of (1).

Now we proceed to (2). Note that $\underline{\theta}(a)$ is an Iw-equivariant endomorphism of $V^\hbar_\kappa(\mathfrak{g})$. We need to show that it is $\mathcal{J}G$-equivariant. Note that since $V^\hbar_\kappa(\mathfrak{g})$ is a cyclic $\widehat{U}_{\kappa\hbar}(\hat{\mathfrak{g}})$-module, the space of its Iw- (resp., $\mathcal{J}G$-) equivariant endomorphisms is identified with $V^\hbar_\kappa(\mathfrak{g})^{\mathsf{Iw}}$ (resp., $V^\hbar_\kappa(\mathfrak{g})^{\mathcal{J}G}$). We need to show that $V_\kappa(\mathfrak{g})^{\mathcal{J}G}=V_\kappa(\mathfrak{g})^{\mathsf{Iw}}$. Since $V_\kappa(\mathfrak{g})$ is a rational representation of $\mathcal{J}G$, our claim amounts to the classical claim that any B^--invariant vector in a rational representation of G is G-invariant. $\square$

12.2. **The graded character of $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$.** The goal of this section is to prove the following result.

Proposition 12.3. *The energy graded characters of* $\mathsf{R}[(\mathcal{J}\mathfrak{g}^*)^{(1)}]^{(\mathcal{J}G)^{(1)}}$ *and* $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$ *are the same.*

Proof. The proof is in several steps.

Step 1. Set ${}^I\mathcal{A}^0 := {}^I\mathcal{A}^\hbar/(\hbar)$. This algebra acts by endomorphisms on $\Delta_0^G := S(\mathfrak{b}^- \oplus t^{-1}\mathfrak{g}[t^{-1}])$. Moreover, the generic point of $\mathfrak{h}^* \subset \mathfrak{h}^{da}$ is also generic in the sense of Definition 8.2, so the homomorphism $\varphi^B : \Delta_0^G \to \mathsf{Wak}_0^B$ is generically injective. The action of ${}^I\mathcal{A}^0$ on Δ_0^G is uniquely characterized by the condition that the homomorphism $\Delta_0^G \to \mathsf{Wak}_0^B$ is ${}^I\mathcal{A}^0$-equivariant.

Step 2. Note that $S(\mathfrak{g}[t^{-1}]^{(1)})^{(\mathcal{J}G)^{(1)}}$ acts on Δ_0^G by Iw-equivariant endomorphisms. Consider the Chevalley restriction map $S(\mathfrak{g}[t^{-1}]^{(1)})^{(\mathcal{J}G)^{(1)}} \to S(\mathfrak{h}[t^{-1}]^{(1)})$. This gives an action of $S(\mathfrak{g}[t^{-1}]^{(1)})^{(\mathcal{J}G)^{(1)}}$. The homomorphism $\Delta_0^G \to \mathsf{Wak}_0^B$ is equivariant for the action of $S(\mathfrak{g}[t^{-1}]^{(1)})^{(\mathcal{J}G)^{(1)}}$. Proposition 11.2 and its proof show that ${}^I\mathcal{A}^0$ embeds into $S(\mathfrak{h}[t^{-1}]^{(1)})$ and the image coincides with that of the Chevalley restriction map. This shows that ${}^I\mathcal{A}^0$ acts on Δ_0^G via its isomorphism with $S(\mathfrak{g}[t^{-1}]^{(1)})^{(\mathcal{J}G)^{(1)}}$.

Step 3. It follows from Step 2 that ${}^I\mathcal{A}^0$ acts on $\operatorname{gr} V_\kappa(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}])$ via its natural homomorphism to $S((t^{-1}\mathfrak{g}[t^{-1}])^{(1)})^{(\mathcal{J}G)^{(1)}}$. This homomorphism is surjective. On the other hand, the homomorphism ${}^I\mathcal{A}^0 \to S(t^{-1}\mathfrak{g}[t^{-1}])$ factors through $\operatorname{gr} V_\kappa(\mathfrak{g})$. This gives rise to an inclusion $S((t^{-1}\mathfrak{g}[t^{-1}])^{(1)})^{(\mathcal{J}G)^{(1)}} \hookrightarrow \operatorname{gr} V_\kappa(\mathfrak{g})$. This inclusion is graded by energy. On the other hand, Section 6.4.3 yields an energy graded inclusion in the opposite direction. The claim of the lemma follows. □

12.3. **Completion of proof of Theorem 12.1.**

Proof of Theorem 12.1. By Lemma 6.4 and (6.1), we see that HC_H embeds $V_\kappa(\mathfrak{g})^{\mathcal{J}G}$ into $\bigcap_{i\in I} \operatorname{im} \mathsf{HC}_{L_i}$, where L_i is the minimal Levi associated to i. Under the isomorphism of $V(\mathfrak{h})^{\mathcal{J}H}$ and $\mathcal{A}_0^\natural$, the subalgebra $\operatorname{im} \mathsf{HC}_{L_i}$ is identified with $\operatorname{im} \mathsf{M}_0^i$. By Remark 11.4, $\bigcap_{i\in I} \operatorname{im} \mathsf{M}_0^i = \operatorname{im} \mathsf{M}_0$. The image of $\operatorname{im} \mathsf{M}_0$ under the isomorphism $\mathcal{A}_0^\natural \cong V_\kappa(\mathfrak{h})^{\mathcal{J}H}$ has the same energy graded character as $S((t^{-1}\mathfrak{g}[t^{-1}])^{(1)})^{(\mathcal{J}G)^{(1)}}$. By Proposition 12.3, $V_\kappa(\mathfrak{g})$ has the same energy graded character. The claim of the theorem follows. □

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