

Williamson

Lecture 1

\mathfrak{g} -complex s/simple Lie alg

$\supset \mathfrak{b}$ -Borel

$\supset \mathfrak{h}$ -Cartan

\mathcal{O} -category $\mathcal{O} \subset \mathfrak{g}\text{-mod}$

$\supset \mathcal{O}_0$ -princ. block

$\mathfrak{h} \cap W$ -Weyl gr-p

$\supset S$ -simple refl-ns

L_x -simple module w h.wt. $x \cdot 0$

$\nwarrow \Delta_x$ -Verma

$\nwarrow P_x$ -proj-ve cover

Kazhdan-Lusztig conj $[\Delta_x: L_y] = p_{xy}(1)$, p_{xy} -KL polyn-l
(Koszul-dual formuln)

∇_x -dual Verma $\Rightarrow [\nabla_x: L_y] = [\Delta_x: L_y]$

$\dim \text{Hom}(P_y, \nabla_x) = (P_y: \Delta_x)$ -mult. of Δ_x in Verma filt-n of P_y

Under $\pi_W \xrightarrow{\sim} [\mathcal{O}_\psi]$, $w \mapsto [\Delta_w]$

KL conj: $C'_y(1) \leftrightarrow [P_y]$, where C'_y is KL polynomial

KL conj. has 2 proofs: • via Beilinson-Bernstein localization:

$$U(\mathfrak{g})/(\mathbb{Z}^+)^{\text{-mod}} \xleftarrow[\Gamma(\cdot)]{\sim} D_{G/B}^{\text{-mod}}$$

$$D_{G/B}^{\text{-mod reg, loc}} \xleftrightarrow{\sim} \text{Per}(G/B)$$

\uparrow Riemann-Hilbert

\uparrow KL
 \downarrow KL polyn-s

• Soergel's approach

Soergel functor: $\mathcal{O} \xrightarrow{\mathbb{V}} \text{mod-} C$

$$\mathbb{V} = \text{Hom}(P_{w_0}, \cdot), \text{End}(P_{w_0}) = S(\mathfrak{h})/(S(\mathfrak{h})_+^W) = H^*(G^v/B^v) =: C$$

"invariant algebra"

$\text{Proj}_0 \subset \mathcal{O}_0$ -full additive subcat-ry of proj-ve objects

V is fully faithful on Proj .

We can describe combinatorially image of Proj combinatorially using Soergel modules

$\mathcal{O}_0 \hookrightarrow$ translation functors $\psi_s, s \in S$ (r.k.a wall-crossing functors)

Basic facts: 1) ψ_s are exact $[\Delta_x \psi_s] = [\Delta_{xs}] + [\Delta_x]$

so under $\mathbb{Z}W \xrightarrow{\sim} [\mathcal{O}_0]$

$$\begin{array}{ccc} \circlearrowright & & \circlearrowright \\ 1+s & & \psi_s \end{array}$$

2) If $x = st \dots u$ is reduced expression, then

$$\Delta_x \psi_x = P_x \oplus \bigoplus_{y < x} P_y^{\oplus m_y}$$

3) $V(M \psi_s) \xrightarrow{\sim} V(M) \otimes_{C^s} C$, $V(\Delta_x) = C$ (b/c $\text{soc } \Delta_x = L_{w_0}$)

$$\text{Gnd-n: } V(\Delta_x \psi_x) = C \otimes_{C^s} C \otimes_{C^t} C \otimes_{C^u} C = V(P_x) \oplus \bigoplus_{y < x} V(P_y)^{\oplus m_y}$$

\Rightarrow can characterize $V(P_x) =: D_x$

as a unique up to iso summand in $C \otimes_{C^s} C \otimes_{C^t} C$ which ~~is not~~ doesn't appear in smaller expressions

Def: full additive subcategory in $\text{mod-}C$ generated by summands of $C \otimes_{C^s} C \otimes_{C^t} C$ are Soergel modules

Not-n: $S\text{Mod}$

$$S_0 \text{ Proj} \xrightarrow{\sim} S\text{Mod}$$

$$\text{KL-conj} \iff [\text{Proj}_0] = [S\text{Mod}]$$

$$\mathbb{Z}W = [\mathcal{O}_0]$$

$$\text{KL-conj: } C'_x(1) \longleftrightarrow [D_x] \quad (*)$$

To conclude the proof, there are 2 possibilities

1: Soergel: identifiy D_x w. $IH^*(\overline{B^v \times B^v} / B^v)$ & apply decomposition theorem

2 (Elias-Williamson): an "algebraic" Hodge theory" for D_x to deduce equality. (*)

Problem: in related situations it's not clear how to define V

Example: $\text{Rep } G$, where G is reductive alg-c grp / $\overline{\mathbb{F}}_p$

Def: \mathcal{H} (= Soergel bimodules, a.k.a. Hecke category) = $\langle B_s | s \in S \rangle$
 $[R = S(\mathcal{H})]$

$\bigcap \otimes, [m], \text{Kar}$
 $R\text{-bimod}_{gr}$

$B_s := R \otimes_{R^s} R(1)$ (gen-r in deg-1) under taking tensor prod-s, grading shifts, direct sums & summands

So indecomposable Soergel bimodules are indec. summands of

$B_{\underline{s}} := B_{s_1} B_{s_2} \dots B_{s_n}$ (where we omit " \otimes_R ")

Prop: Krull-Schmidt theorem holds in $SMod$ and \mathcal{H}

Trivial Prop: $M \in G\text{-mod} \Rightarrow M \otimes_{\mathbb{C}} \mathbb{C} = MB_s$

Since $Proj_0 \xrightarrow[\sim]{\sim} SMod$, we deduce $Proj_0$ is a right \mathcal{H} -module cat-y.

w. B_s acting
 $\hookrightarrow U_s$.

Observation: Suppose O_0 or $Proj_0$ has structure of a right \mathcal{H} -module cat-y s.t. the morphisms $id \rightarrow B_s, B_s \rightarrow id, B_s \rightarrow B_s B_s, B_s B_s \rightarrow B_s$ are induced by adj-s (move in Lecture 2), then we have an equivalence of \mathcal{H} -modules ~~O_0~~ $\xleftarrow{\sim} Proj_0 \xleftarrow{\sim} \mathbb{C} \otimes_R \mathcal{H}$

Rmk: It seems hard to produce \mathcal{H} -action directly, but using KLR alg-s & Brundan's results this can be done for $gl_n^+(\mathbb{C})$

Part 2: \mathcal{H} by generators & relations (w. B Elias)

(W, S) is Coxeter gr-p, $W \curvearrowright \mathcal{H}$ is refl-n rep-n

$\alpha_s^\vee \in \mathcal{H}$ -coreot, $\alpha_s \in \mathcal{H}^*$ -root

\mathcal{H}/K , K is a field (or \mathbb{Z})

Def: $(I \subset S)$ is finitary if $W_I = \langle I \rangle$ is finite

Recall: H (Hecke algebra) / $\mathbb{Z}[v^{\pm 1}]$

have gen-rs $h_s, s \in S$, & rel-ns

$$h_s^2 = (v^{-1} - v)h_s + 1 \quad (\text{rk 1 rel-n})$$

$$\underbrace{h_s h_t}_{m_{st}} = \underbrace{h_t h_s}_{m_{st}}, \quad m_{st} = \text{ord}(st), \text{ for } \{s, t\} \subset S \text{ is finitary}$$

\mathcal{H}_S = full subcat-y of $\mathcal{L}\text{-bimod}_g$ w. objects $B_x = B_s B_t \dots B_u$ for all expressions x and morphism spaces graded by degree.

$\mathcal{H} \cong \text{Kar}(\mathcal{H}_S)$ "additive graded Karoubi envelope."

Will describe \mathcal{H}_S by generators & relations

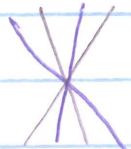
Consider monoidal cat-y \mathcal{D}_S gen-d as \otimes -cat. by $S \in S$ meaning that objects are sequences x :

|||| $tstus$

Morphisms: linear comb-ns of colored decorated planar graphs w. following local generators

rank 1 $\left\{ \begin{array}{c} \phi \\ \downarrow \\ s \end{array} \right. \bullet \quad \wedge \quad \begin{array}{c} t \\ \wedge \\ t \end{array} \in \text{Hom}(t, tt)$

rank 0 $\boxed{f} \quad f \in R = S(k^*)$

rank 2  "2m_{st} valent vertex for $\{s, t\} \in S$ binary


Monoidal cat-y: w.r.t horizontal (\otimes) and vertical (composition)

~~Diagrammatic~~ concatenations.

Relations: $\boxed{1} = \phi$, $\boxed{f} \boxed{g} = \boxed{fg}$: rk 0

rk 1: \downarrow_t is Frobenius object $\wedge_\bullet = |$, $\wedge = \wedge$ etc


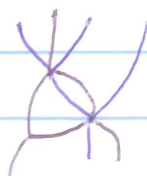
any diagram w. 1 color only depends on thickening up to iso and

 = 0 (where $\cap = \wedge$)

and: Polynomial sliding $| \boxed{f} = \boxed{sf} | + \downarrow | \boxed{sf}$



$\partial_s f$ -divided difference operator, $\partial_s f = \frac{f - sf}{\alpha_s}$

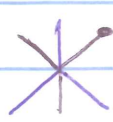
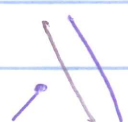
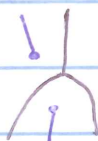
Rank 2: "2 color associativity"

eg  =  ($m_{st}=3$)

 =  ($m_{st}=2$)

Jones-Wenzl

 = 

 =  +  ($m_{st}=3$)

Rank 3: Zamolodchikov rel-ns

In Soergel bimodules:

$D_{BS} \rightarrow \mathcal{H}_{BS}$


$\uparrow \mapsto [B_t \rightarrow R(1), f \otimes g \mapsto fg \text{ (deg 1)}]$

$\downarrow \mapsto [R(-1) \rightarrow B_t, 1 \mapsto \frac{1}{2}(\alpha \otimes 1 + 1 \otimes \alpha)]$

$\boxplus \mapsto [R \xrightarrow{f} R \text{ (deg = deg } f)]$

$\wedge \mapsto [B_s^2 \rightarrow B_s(-1) \text{ (} f \otimes g \otimes h \mapsto f \otimes_2(g) \otimes h)]$

$\vee \mapsto [B_s \rightarrow B_s^2(-1) \text{ (} f \otimes g \mapsto f \otimes_1 g)]$

 $\left[\begin{array}{c} B_{st} = R \otimes_{R^{st}} R(3) \\ B_s B_t B_s \longrightarrow B_t B_s B_t \end{array} \right]$

Thm (Elias-Khovanov $W = S_n$, Libedinsky-Elias for dihedral groups,
Elias-Williamson, in gen-l)

$$\mathcal{D}_{BS} \xrightarrow{\sim} \mathcal{H}_{BS}, \text{ hence } \text{Kar}(\mathcal{D}_{BS}) \xrightarrow{\sim} \mathcal{H} \quad (\text{in char } 0)$$

or if \mathcal{H} is refl-n faithful rep-n
& char $\neq 2$ | \uparrow faithful & refl-n in \mathcal{H} =
abstr. refl-n.)