

DAY 1 EXERCISES

1. GEOMETRIC INVARIANT THEORY

Throughout this section, G is a reductive algebraic group acting on the affine variety X , and $\theta : G \rightarrow \mathbb{C}^\times$ is a character.

Exercise 1.1 (Lemma 1.3 in the notes). *Consider the G -action on $X \times \mathbb{C}$ given by $g(x, z) = (gx, \theta(g)z)$. Show that an element x is in $X^{\theta-ss}$ if and only if $\overline{G(x, 1)}$ does not intersect $X \times \{0\}$. Equivalently, if and only if there is no one parameter subgroup $\gamma : \mathbb{C}^\times \rightarrow G$ such that $\lim_{t \rightarrow 0} \gamma(t)x$ exists and $\theta(\gamma(t)) = t^m$ with $m > 0$.*

Hint: What is $\mathbb{C}[X \times \mathbb{C}]^G$?

Exercise 1.2 (Lemma 1.4 in the notes). *Show that the G orbit of a collection $(x_a, i_k) \in R := R(Q, v, w)$ is closed if and only if $i_k = 0$ for all $k \in Q_0$ and the representation $(x_a)_{a \in Q_1} \in R(Q, v, 0)$ is semisimple.*

Hint: That the G -orbit of (x_a, i_k) is closed implies that $i_k = 0$ and (x_a) is semisimple follows by considering appropriate 1-parametric subgroups. The other implication follows from the previous implication and the Hilbert-Mumford criterion.

Exercise 1.3 (Lemma 1.5 in the notes). *If $\theta_k > 0$ for all $k \in Q_0$, show that the subset $R^{\theta-ss}$ consists of all representations (x_a, i_k) such that the only x_a -stable collection of subspaces in $(\ker i_k)_{k \in Q_0}$ is the zero one.*

Hint: Assume that there is a nonzero x_a -stable collection of subspaces in $(\ker i_k)_{k \in Q_0}$. Use Exercise 1.1 to show that (x_a, i_k) is not θ -semistable. For the converse implication, use Exercise 1.1 again, together with the Hilbert-Mumford criterion and \mathbb{C}^\times -representation theory.

2. MOMENT MAPS AND HAMILTONIAN REDUCTION

Exercise 2.1. *Let G act on an affine variety X_0 . Lift this action to an action on $X := T^*X_0$. Show that $\mu^* : \mathfrak{g} \rightarrow \mathbb{C}[T^*X]$, $\xi \mapsto \xi_{X_0}$, is a comoment map for this action. You will need to use the following formula for the Poisson bracket on $\mathbb{C}[T^*X_0] = S_{\mathbb{C}[X_0]} \text{Vect}(X_0)$. For $f, g \in \mathbb{C}[X_0]$ and $\xi, \eta \in \text{Vect}(X_0)$ we have $\{f, g\} = 0$, $\{\xi, f\} = L_\xi f$ and $\{\xi, \eta\} = [L_\xi \eta]$. Here, L_ξ is the Lie derivative and $[\cdot, \cdot]$ stands for the commutator of vector fields.*

Exercise 2.2. *The kernel of $d_x \mu$ coincides with the ω -orthogonal complement of $T_x(Gx)$, and the image of $d_x \mu$ is the annihilator of $\mathfrak{g}_x := \{\xi \in \mathfrak{g} : \xi_{X, x} = 0\}$. In particular, μ is a submersion at x provided the stabilizer G_x is finite.*

Hint: Recall that the moment map $\mu : X \rightarrow \mathfrak{g}^*$ satisfies $\langle d_x \mu(\bullet), \xi \rangle = \omega_x(\bullet, \xi_x)$ for every $x \in X, \xi \in \mathfrak{g}$. From here the result follows easily.

Exercise 2.3. *Suppose that the action of G on $\mu^{-1}(\lambda)$ is free. Show that there is a unique symplectic form $\underline{\omega}$ on $X//_\lambda G$ satisfying $\pi^* \underline{\omega} = \iota^* \omega$, where $\pi : \mu^{-1}(\lambda) \rightarrow X//_\lambda G$ is the projection and $\iota : \mu^{-1}(\lambda) \rightarrow X$ is the inclusion.*

Hint: For each $x \in \mu^{-1}(\lambda)$, express $T_{\pi(x)}(X//_\lambda G)$ in terms of $T_x X$. The previous exercise may come in handy.

3. NAKAJIMA QUIVER VARIETIES

Exercise 3.1. (a) *Let U_1, U_2 be symplectic vector spaces and G an algebraic group acting on U_1, U_2 . Assume the actions are Hamiltonian with respective moment maps $\mu_1 : U_1 \rightarrow \mathfrak{g}^*$, $\mu_2 : U_2 \rightarrow \mathfrak{g}^*$. Show that the action of G on $U_1 \oplus U_2$ is Hamiltonian with moment map $\mu_1 \oplus \mu_2$.*

- (b) Let U be a symplectic vector space. Assume G_1, G_2 act on U in a Hamiltonian way, with moment maps μ_1, μ_2 , respectively. Moreover, assume that the actions of G_1 and G_2 commute. Then the $G_1 \times G_2$ -action on U is Hamiltonian with moment map $\mu = (\mu_1, \mu_2)$.

Exercise 3.2. Let $X, Y \in \text{End}(V)$ be such that $\text{rk}[X, Y] = 1$. Show that X and Y are simultaneously triangularizable.

Hint: You have to check that there is a proper subspace of V that is invariant under the actions of X and Y . Note that we may assume w.l.o.g. that $\ker(Y) \neq 0$. Then separate the cases $\ker(Y) \subseteq \ker([X, Y])$ and $\ker(Y) \not\subseteq \ker([X, Y])$.

Exercise 3.3. Let V be a finite dimensional vector space, and $X, Y \in \text{End}(V)$, $i \in V, j \in V^*$. Assume that $[X, Y] - ij = 0$ and that $\mathbb{C}\langle X, Y \rangle i = V$. Show that $j = 0$.

Hint: Use the previous exercise.