Lecture 14, Induced representations 1.

0) Lemma from last time.

1) Motivation & construction.

2) Frobenius reciprocity

Ref: [E], Secs 5.8 & 5.10.

0) Lemma from last time.

We start by proving Lemma from last time.

Lemma: Let $\varepsilon_1, \dots \varepsilon_n$ be roots of 1. If $z := \frac{\varepsilon_1 + \dots + \varepsilon_n}{n} \in \mathbb{Z}$, then either $\varepsilon_1 = \dots = \varepsilon_n$ or $\varepsilon_1 + \dots + \varepsilon_n = 0$.

Proof: Suppose $z \neq 0$. Let $f(x)$ be the minimal polynomial of z. By Lemma in Sec 1.1, $f(x) \in \mathbb{Z}[x]$. Let $z = z_1, z_2, \dots z_m$ be the conjugates of z = roots of $f(x)$. So

$$z_1 z_2 \dots z_n \,(= \pm f(0)) \in \mathbb{Z} \setminus \{0\} \qquad (\dagger)$$

We claim that each of z_i is of the form $\frac{\varepsilon_1' + \dots + \varepsilon_n'}{n}$, where ε_j' is a root of 1. First, by Proposition in Sec 2 (conjugates of sums are sums of conjugates), every conjugate of $\xi_1+\ldots+\xi_n$ is of the form $\xi_1'+\ldots+\xi_n'$. Exercise in Sec 1.1 of Lec 12 shows that conjugates of $\frac{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n}{n}$ are of the form $\frac{\varepsilon_1'+\varepsilon_2'+\ldots+\varepsilon_n'}{n}$

Now we use that the base field is $\mathbb{C}$ (and not a general algebraically closed field). Note that $|\xi_i'|=1$, so

$$|z_i| = \left|\frac{\varepsilon_1'+\varepsilon_2'+\ldots+\varepsilon_n'}{n}\right| \leqslant 1$$

w. equality iff $\xi_1'=\ldots=\xi_n'$. But, by (†), $|z_1\ldots z_m|\geqslant 1$. In particular, $|z|=1$, so $\xi_1=\ldots=\varepsilon_n$. □

1) Motivation & definition.

1.0) Motivation

Let G be a finite group, and $\mathbb{F}$ be an algebraically closed field of characteristic 0. Our primary question is to classify the irreducible representations & compute their characters. For this we need to have some way to construct (possibly reducible) representations of G. One could try to approach this inductively: if we have a subgroup $H\subset G$ & a representation (possibly reducible) U of H, we want to construct an "induced" representation of G out of (H,U).

One could hope that (for "right" choices of (H,U)) one recovers every irreducible representation of G inside of an induced representation. We will see that this is indeed the case when $G=S_n$ (this our topic after the break).

1.1) Construction.

Let $\mathbb{F}$ be any field, $H\subset G$ be finite groups & U be a finite dimensional representation of H.

Consider the set $Map(G,U)$ of all maps $G\to U$. It's an $\mathbb{F}$-vector space:

$$[\varphi_1+\varphi_2](g)=\varphi_1(g)+\varphi_2(g),\ [a\varphi_1](g)=a(\varphi_1(g)),$$
$$g\in G,\ \varphi_1,\varphi_2\in Map(G,U)$$

Let H act on G by right translations: $h.g=gh^{-1}$. Consider the subset of H-equivariant maps

$$Map_H(G,U)=\{\varphi: G\to U \mid \varphi(gh^{-1})=h_U\varphi(g)\ \forall g\in G,\ h\in H\}$$

We equip $Map_H(G,U)$ with the structure of a representation of G as follows: $[g_1.\varphi](g):=\varphi(g_1^{-1}g)$.

Lemma: $g_1.\varphi \in Map_H(G,U)$ if $\varphi \in Map_H(G)$ & $\varphi \mapsto g_1.\varphi$ is a representation of G in $Map_H(G,U)$.

Proof: $[g_1.\varphi](gh^{-1}) = \varphi(g_1^{-1}gh^{-1}) = [\varphi \in Map_H(G,U)] = h_U\varphi(g_1^{-1}g)$
$h_U[g_1.\varphi](g) \Rightarrow g_1.\varphi \in Map_H(G,U)$.

The map $\varphi \mapsto g_1.\varphi$ is linear $\forall g_1 \in G$ (exercise).

Now we check $g_1.[g_2.\varphi] = (g_1g_2).\varphi$.

$[g_1.[g_2.\varphi]](g) = [g_2.\varphi](g_1^{-1}g) = \varphi(g_2^{-1}g_1^{-1}g) = [g_1g_2.\varphi](g)$. $\square$

Definition: The representation of G in $Map_H(G,U)$ is called the induced representation (from the representation of H in U) and is denoted by $Ind_H^G U$.

Special case: U is the one-dimensional trivial representation. The $Map_H(G,U) = \{\varphi \in Fun(G,\mathbb{F}) \mid \varphi(gh^{-1}) = \varphi(g)\}$.
So $Map_H(G,U)$ is identified w. $Fun(G/H,\mathbb{F})$, and it's an identification of representations of G (exercise). For example, for $H=\{e\}$ we recover the regular representation $\mathbb{F}G$.

1.2) Basic properties.

Let $g_1,\dots,g_\ell$ ($\ell=|G/H|$) be representatives of all right H-cosets. Note that $\forall g\in G$, $\exists!\ i=1,\dots,\ell,\ h\in H$ s.t. $g=g_ih^{-1}$.

Lemma: The map $\varphi\mapsto(\varphi(g_1),\dots,\varphi(g_\ell)): \operatorname{Map}_H(G,U)\to U^{\oplus\ell}$ is an isomorphism (of vector spaces).

Proof: The inverse sends $\underline{u}:=(u_1,\dots,u_\ell)$ to $\varphi_{\underline{u}}: G\to U$ defined by $\varphi(g_ih^{-1})=h_u u_i$. To check details is an exercise $\square$

2) Frobenius reciprocity

Given a representation V of G we can restrict it to H, denote the resulting representation by $\operatorname{Res}_H^G V$. The operation of restriction is closely related to induction.

2.1) Main result.

Theorem (Frobenius reciprocity) For (finite dimensional) representations U of H and V of G we have a natural isomorphism

$$\operatorname{Hom}_G(V,\operatorname{Ind}_H^G U)\xrightarrow{\sim}\operatorname{Hom}_H(\operatorname{Res}_H^G V,U)\qquad(1)$$

Proof:

- First, we construct a map in (1). Define the map

$$ev: \mathrm{Ind}_H^G U = \mathrm{Map}_H(G,U) \longrightarrow U,\ ev(\varphi) := \varphi(e)$$

We claim that it is a homomorphism of representations of H:

$ev(h.\varphi) = [h.\varphi](e) = \varphi(h^{-1}) = [\varphi \text{ is equivariant}] = h_U.\varphi(e)$.

Now our map (1) is $\psi \mapsto ev \circ \psi$. Since ev & ψ are homomorphisms of representations of H, so is their composition. Hence we indeed get a well-defined linear map as above.

- Now, we are going to construct an inverse. Let $\eta \in \mathrm{Hom}_H(V,U)$. We are going to define $\psi_\eta : V \longrightarrow \mathrm{Map}_H(G,U)$. We do this by $[\psi_\eta(v)](g) := \eta(g_V^{-1} v)$. We need to check that
  - $\psi_\eta(v) \in \mathrm{Map}_H(G,U) \Leftrightarrow [\psi_\eta(v)](gh^{-1}) = h_U([\psi_\eta(v)](g))$.

  $[\psi_\eta(v)](gh^{-1}) = \eta(h_V g_V^{-1} v) = h_U\, \eta(g_V^{-1} v) = h_U([\psi_\eta(v)](g))$.
  - ψ_η is G-equivariant: $\psi_\eta(g'_V v) = g'_{\mathrm{Map}_H(G,U)}\, \psi_\eta(v)$, $\forall g' \in G$.

  $\psi_\eta(g'_V v)(g) = \eta(g_V^{-1} g'_V v)$

  $[g'_{\mathrm{Map}_H(G,U)}\, \psi_\eta(v)](g) = [\psi_\eta(v)](g'^{-1} g) = \eta(g_V^{-1} g'_V v)$ - checked.
  - We need to show that $\eta \mapsto \psi_\eta$ & $\psi \mapsto ev \circ \psi$ are inverse to each other: $ev \circ \psi_\eta = \eta$ & $\psi_{ev\circ\psi} = \psi$.

$ev \circ \varphi_\eta(v) = [\varphi_\eta(v)](e) = \eta(v)$ ✓.

$[\varphi_{ev\circ\varphi}(v)](g) = ev(\varphi(g_V^{-1}v)) = [\varphi(g_V^{-1}v)](e) = [\varphi \text{ is } G\text{-equiv't}]$
$= [\varphi(v)](g) \Rightarrow \varphi_{ev\circ\varphi} = \varphi$ $\square$

2.2) Application to computation.

Let $G = S_n$ & λ be a partition of n, i.e. the presentations $n = \lambda_1 + \ldots + \lambda_k$ w. $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k \geqslant 0$. The subgroup S_λ, by definition, consists of all permutations σ s.t. σ preserves each of the k subsets $\{1, 2, \ldots, \lambda_1\}$, $\{\lambda_1 + 1, \ldots \lambda_1 + \lambda_2\}, \ldots, \{\lambda_1 + \ldots + \lambda_{k-1} + 1, \ldots n\}$. E.g. for $n = 4$ we get the following subgroups:

$\lambda = (4)$ (one part): $S_\lambda = S_4$.

$\lambda = (3,1)$: $S_\lambda = S_3 = \{\sigma \in S_4 \mid \sigma(4) = 4\}$.

$\lambda = (2,2)$: $S_\lambda = \{e, (12), (34), (12)(34)\} \simeq S_2 \times S_2$.

$\lambda = (2,1,1)$: $S_\lambda = \{e, (12)\}$.

$\lambda = (1,1,1,1)$: $S_\lambda = \{e\}$.

In the general case, the representations of the form $\mathrm{Ind}_{S_\lambda}^{S_n} \mathrm{triv}$ play an important role in classifying the irreducible representations of S_n. Let's compute them for $n = 4$.

Below $\mathbb{F}$ has characteristic 0 (and is alg. closed, although this is not important)

We consider the case $\lambda = (2,2)$: here we use the Frobenius reciprocity to show the following general result

Lemma: Let $H \subset G$ be finite groups. The multiplicity of an irreducible representation U of G in $\mathrm{Ind}_H^G \mathrm{triv} = \mathrm{Fun}(G/H, \mathbb{F})$ is $\dim U^H$ (H-invariants).

Proof:

The multiplicity of U in $\mathrm{Ind}_H^G \mathrm{triv}$ is

$$\dim \mathrm{Hom}_G(U, \mathrm{Ind}_H^G \mathrm{triv})$$

By Frobenius reciprocity, this dimension is that of $\mathrm{Hom}_H(U, \mathrm{triv})$, i.e. the multiplicity of the trivial representation in U, which is $\dim U^H$ $\square$

Let's compute $\dim U^H$ for $U = \mathrm{triv}, \mathbb{F}_0^4, \mathrm{sgn}$

- for $U = \mathrm{triv}$, $\dim U^H = \dim U = 1$.
- for $U = \mathbb{F}_0^4$, $U^H = [H = S_2 \times S_2,\ U = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 + x_3 + x_4 = 0\}]$

$= \{(x_1,x_1,x_2,x_2) \mid x_1+x_2=0\} \simeq \mathbb{F}$, so $\dim U^H = 1$.

- on $U = \mathrm{sgn}$ every permutation acts by -1, so $U^H = \{0\}$.

So we see that triv & $\mathbb{F}_0^4$ both occur in $\mathrm{Ind}_{S_2\times S_2}^{S_4}\mathrm{triv}$ w. multiplicity 1, while sgn doesn't occur. By Lemma in Sec. 1.2, $\dim \mathrm{Ind}_{S_2\times S_2}^{S_4}\mathrm{triv} = 6$. We have $\mathrm{Ind}_{S_2\times S_2}^{S_4}\mathrm{triv} = \mathrm{triv}\oplus \mathbb{F}_0^4 \oplus ?$, so $\dim ? = 2$. And neither triv nor sgn can occur in ?. So $? = V_2$

We can also show that V_2 occurs w. multiplicity 1 as above.

2.3) More on Frobenius reciprocity.

This section requires MATH 380. Our goal here is to understand the Frobenius reciprocity better and more conceptually.

Consider the following situation: let A be an associative algebra and B be its subalgebra. We have the functor Res_B^A from the category of A-modules (to be denoted by A-Mod) to B-Mod of restriction (i.e. only remembering the action of elements of B).

This functor has left adjoint, the induction functor, Ind_B^A

defined by $A\otimes_B \bullet$, and right adjoint, the coinduction functor Coind_B^A given by $\mathrm{Hom}_B(A,\bullet)$. Here A is viewed as a left B-module and the action of A on $\mathrm{Hom}_B(A,N)$ comes from the right multiplication on A: $a\varphi(a'):=\varphi(a'a)$.

Now consider the case of $A=\mathbb{F}G$ and $B=\mathbb{F}H$ for finite groups $H\subset G$. The induction functor $\mathrm{Ind}_H^G: \mathbb{F}H\text{-mod}\longrightarrow \mathbb{F}G\text{-mod}$ can be equivalently defined as $\mathrm{Coind}_{\mathbb{F}H}^{\mathbb{F}G}$ (note that the map $g\mapsto g^{-1}: G\to G$ intertwines actions from the left & from the right). However, in our situation, more is true:

$$\mathrm{Ind}_{\mathbb{F}H}^{\mathbb{F}G}\simeq \mathrm{Coind}_{\mathbb{F}H}^{\mathbb{F}G}.$$

So, in the notation of Sec 2.1, we have a natural isomorphism:

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G U, V)\xrightarrow{\sim}\mathrm{Hom}_H(U, \mathrm{Res}_H^G V).$$

Premium exercise: Let $K\subset H\subset G$ be subgroups (all finite). Establish a natural (i.e. functor) isomorphism

$$\mathrm{Ind}_H^G\circ \mathrm{Ind}_K^H\xrightarrow{\sim}\mathrm{Ind}_K^G.$$