

Lecture 5.

- 1) Noetherian rings & modules
- 2) Hilbert's Basis theorem.
- 3) Noetherian modules over Noetherian rings.

References: [AM], Chapter 6, intro to Chapter 7; [E], Section 1.4.

BONUS: • Non-Noetherian rings in Complex Analysis.
• Why Hilbert cared.

- 1) A is commutative ring.

1.1) Main definitions & examples.

Definition:

- i) An A -module M is **Noetherian** if \mathfrak{f} submodule of M (including M) is finitely generated.
- ii) A is a **Noetherian ring** if it's Noetherian as a module over itself, i.e. every ideal is finitely generated.

Examples:

- 0) Every field \mathbb{F} is Noetherian ring (ideals in \mathbb{F} are $\{0\}, \mathbb{F} = (1)$),
- 1) $A = \mathbb{Z}$ is Noetherian: b/c \mathfrak{f} ideal is generated by one element.

Non-example: $\mathbb{F}[x_i]_{i \in I}$ for infinite I .

1.2) Equivalent characterizations of Noetherian modules.

Definition: M is A -module. • By an **ascending chain (AC)** of submodules of M we mean: collection $(N_i)_{i \in \mathbb{N}_{>0}}$ of submodules of

M s.t. $N_i \subseteq N_{i+1} \forall i > 0 : N \subseteq N_2 \subseteq N_3 \subseteq \dots$

- We say that the AC $(N_i)_{i \in \mathbb{N}_{>0}}$ terminates if $\exists k > 0$ s.t. $N_j = N_k \forall j > k$.

Proposition : For an A -module M TFAE:

- 1) M is Noetherian.
- 2) \nexists AC of submodules of M terminates.
- 3) \nexists nonempty set of submodules of M has a maximal element w.r.t. inclusion.

Proof: (2) \Leftrightarrow (3) is easy & general statement about posets.
(exercise).

(1) \Rightarrow (2): AC $(N_i)_{i > 0} : N \subseteq N_2 \subseteq \dots \rightsquigarrow N = \bigcup_{i > 0} N_i$ is a submodule (compare to proof of \exists of max. ideal, 2.2 of Lec 2). This N is fin. gen'd so $\exists m_1, \dots, m_e \in N$ w. $N = \text{Span}_A(m_1, \dots, m_e)$. Now $m_i \in N_{k(i)}$ for some $k(i) \Rightarrow m_1, \dots, m_e \in N_k$ for $k = \max\{k(i)\}$ b/c $N_{k(i)} \subset N_k$ thx to AC condition $\Rightarrow N = \text{Span}_A(m_1, \dots, m_e) = N_k$ so AC (N_i) terminates at N_k .

(2) \Rightarrow (1). Know: \nexists AC of submodules terminates. Let N be a submodule that is not fin generated: construct N_i 's by induction: pick $m_1 \in N \rightsquigarrow N_1 = \text{Span}_A(m_1) = Am_1$. Now suppose we've constructed $m_1, \dots, m_i \in N$ & $N_i = \text{Span}_A(m_1, \dots, m_i)$. N is not fin. gen. $\Rightarrow N \neq N_i \Rightarrow \exists m_{i+1} \in N \setminus N_i$, set $N_{i+1} = \text{Span}_A(m_1, \dots, m_{i+1}) \supset N_i$. So $(N_i)_{i > 0}$ is AC, doesn't terminate. Contradiction. \square

2.1) Hilbert basis theorem.

It turns out that there are a lot of Noetherian rings, in fact most rings we are dealing with are Noetherian. The following is a basic result in this direction.

Thm (Hilbert): If A is Noetherian, then $A[x]$ is Noetherian.

Proof: Let $I \subset A[x]$ be an ideal. Assume it's not finitely generated. We construct a sequence of elements $f_1, f_2, \dots \in I$ as follows: f_1 is an element of I with minimal possible degree. Once f_1, \dots, f_{k-1} are constructed, we choose $f_k \in I \setminus (f_1, \dots, f_{k-1})$ (this set is nonempty b/c $I \neq (f_1, \dots, f_{k-1})$) - again of minimal possible degree.

For $k > 0$, define $a_k \in A$ & $n_k \in \mathbb{Z}_{\geq 0}$ from $f_k = a_k x^{n_k} + \text{lower deg. terms}$.

By the construction, $n_1 \leq n_2 \leq \dots \leq n_k \dots$. Now let $I_k = (a_1, \dots, a_k) \subset A$, $k > 0$.

This is an ascending chain of ideals in A . Since A is Noetherian, it must terminate. So $a_{m+1} \in (a_1, \dots, a_m) \Leftrightarrow a_{m+1} = \sum_{i=1}^m b_i a_i$ for some m .

$$\begin{aligned} \text{Set } g_{m+1} &= f_{m+1} - \sum_{i=1}^m b_i x^{n_{m+1}-n_i} f_i = && \in I \text{ b/c } f_1, \dots, f_{m+1} \in I \text{ & } n_{m+1} - n_i \geq 0 \\ &= (a_{m+1} - \sum_{i=1}^m b_i a_i) x^{n_{m+1}} + \text{lower deg. terms} && \Rightarrow \deg g_{m+1} < \deg f_{m+1} \end{aligned}$$

So by the choice of f_{m+1} - of minimal degree - $g_{m+1} \in (f_1, \dots, f_m)$. Since $g_{m+1} \in I$ we have $f_{m+1} \in (f_1, \dots, f_m)$, contradiction \square

2.2) Finitely generated algebras.

Now we proceed to a generalization of the basis theorem.

Definition: Let B be an A -algebra. Then B is **finitely generated** (as an A -algebra) if $\exists b_1, \dots, b_k \in B$ s.t. $\forall b \in B \exists F \in A[x_1, \dots, x_k]$ s.t. $b = F(b_1, \dots, b_k)$

Hence $\Phi: A[x_1, \dots, x_k] \rightarrow B$, $F \mapsto F(b_1, \dots, b_k)$, is surjective. So B is finitely generated over $A \Leftrightarrow$ it's the quotient of $A[x_1, \dots, x_k]$ for some k .

Corollary: Let A be Noetherian & B be a finitely generated A -algebra. Then B is a Noetherian ring.

Proof: Use Hilbert's Thm K times to see that $A[x_1, \dots, x_k]$ is Noetherian. Let $I \subset B$ be ideal, need to show it's fin. gen'd. $J := \Phi^{-1}(I) \subset A[x_1, \dots, x_k]$ is ideal so $J = (F_1, \dots, F_e)$. But then $I = \Phi(J) = (\Phi(F_1), \dots, \Phi(F_e))$ is finitely generated \square

Since fields, \mathbb{F} , & \mathbb{Z} are Noetherian rings, any finitely generated \mathbb{F} - or \mathbb{Z} -algebra is also Noetherian.

In fact, as we will see later, many constructions (e.g. localization and completion) produce Noetherian rings from Noetherian rings. This is why Noetherian rings are so widespread.

3) Noetherian modules over Noetherian rings.

The following result to be proved next time compares the property of being Noetherian for M & its subs & quotients.

Proposition: Let M be A -module, $N \subset M$ be a submodule

TFAE (1) M is Noetherian

(2) Both $N, M/N$ are Noetherian.

We now proceed to characterizing Noetherian modules over Noetherian rings. In general, Noetherian \Rightarrow fin.gen'd. But, when A is Noetherian, we also have \Leftarrow .

Corollary: Let A be Noetherian. Then if fin. gen'd A -module M is Noetherian.

Proof: By 2.1 of Lec 4, M is a quotient of $A^{\oplus k}$. By (1) \Rightarrow (2) of Proposition, it's enough to show $A^{\oplus k}$ is Noetherian. Since A is Noetherian, it's enough to check that the direct sum of 2 Noetherian modules, say M_1, M_2 , is Noetherian - then we'll be done by induction. Note that we have inclusion $M_1 \hookrightarrow M_1 \oplus M_2$:

$m_1 \mapsto (m_1, 0)$ & projection $M_1 \oplus M_2 \rightarrow M_2$, $(m_1, m_2) \mapsto m_2$ whose kernel is the image of M_1 so $(M_1 \oplus M_2)/M_1 \cong M_2$. We use (2) \Rightarrow (1) of Proposition to conclude $M_1 \oplus M_2$ is Noetherian. \square

BONUS I: Non-Noetherian rings in Complex analysis.

Most of the rings we deal with in Commutative algebra are Noetherian. Here is, however, a very natural example of a non-Noetherian ring that appears in Complex analysis.

Complex analysis studies holomorphic (a.k.a complex analytic or complex differentiable functions). Let $Hol(\mathbb{C})$ denote the set of holomorphic functions on \mathbb{C} . These can be thought as power series that absolutely converge everywhere.

$Hol(\mathbb{C})$ has a natural ring structure - via addition & multi-

plication of functions.

Proposition: $\text{Hol}(\mathbb{C})$ is not Noetherian

Proof: We'll produce an AC of ideals: $I_j = \{f(z) \in \text{Hol}(\mathbb{C}) \mid f(2\pi\sqrt{-1}k) = 0 \text{ for integer } k \geq j\}$, $j \in \mathbb{Z}_{\geq 0}$. It's easy to check that all of these are indeed ideals. It is also clear that they form an AC (when we increase j we relax the condition on zeroes). We claim that $I_j \not\subseteq I_{j+1}$, hence this AC doesn't terminate & $\text{Hol}(\mathbb{C})$ is not Noetherian. Equivalently, we need to show that, for each j , there $f(z) \in \text{Hol}(\mathbb{C})$ such that $f_j(2\pi\sqrt{-1}k) = 0 \text{ for } k > j$ while $f_j(2\pi\sqrt{-1}j) \neq 0$.

Consider the function $f(z) = e^z - 1$. This function is periodic with period $2\pi\sqrt{-1}$. Also $f(z) = \sum_{i=1}^{\infty} \frac{1}{i!} z^i$. So $z=0$ is an order 1 zero of $f(z)$. Since $2\pi\sqrt{-1}$ is a period, every $2\pi\sqrt{-1}k$ ($k \in \mathbb{Z}$) is an order 1 zero. We set $f_j(z) = (e^z - 1)/(z - 2\pi\sqrt{-1}j)$. This function is still holomorphic on the entire \mathbb{C} , we have $f_j(2\pi\sqrt{-1}j) \neq 0$ & $f_j(2\pi\sqrt{-1}k) = 0$ for $k \neq j$. \square

BONUS II: Why did Hilbert care about the Basis theorem?

Hilbert was interested in Invariant theory, one of the central branches of Mathematics of the 19th century. Let G be a group acting on fin. dim \mathbb{C} -vector space V by linear transformations, $(g, v) \mapsto gv$. He want to understand when two vectors v_1, v_2 lie in the same orbit.

Definition: A function $f: V \rightarrow \mathbb{C}$ is invariant if f is constant on orbits: $f(gv) = f(v) \quad \forall g \in G, v \in V$.

Exercise: $v_1, v_2 \in V$ lie in the same orbit $\Leftrightarrow f(v_1) = f(v_2)$ \forall invariant function f . (we say: G -invariants separate G -orbits).

Unfortunately, all invariant functions are completely out of control. However, we can hope to control polynomial functions. Those are functions that are written as polynomials in coordinates of v in a basis (if we change a basis, then coordinates change via a linear transformation, so if a function is a polynomial in one basis, then it's a polynomial in every basis). The \mathbb{C} -algebra of polynomial functions will be denoted by $\mathbb{C}[V]$, if $\dim V = n$, then a choice of basis identifies $\mathbb{C}[V]$ with $\mathbb{C}[x_1, \dots, x_n]$.

By $\mathbb{C}[V]^G$ we denote the subset of G -invariant functions in $\mathbb{C}[V]$.
Exercise: It's a subring of $\mathbb{C}[V]$.

Example 1: Let $V = \mathbb{C}^n$, $G = S_n$, the symmetric group, acting on V by permuting coordinates. Then $\mathbb{C}[V]^G$ consists precisely of symmetric polynomials.

Example 2: Let $V = \mathbb{C}^n$ & $G = \mathbb{C}^\times$ ($= \mathbb{C} \setminus \{0\}$ w.r.t. multiplication).

Let G act on V by rescaling the coordinates: $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. We have $f(x_1, \dots, x_n) \in \mathbb{C}[V]^G \Leftrightarrow f(tx_1, \dots, tx_n) = f(x_1, \dots, x_n) \quad \forall t \in \mathbb{C}^\times, x_1, \dots, x_n \in \mathbb{C}$. This is only possible when f is constant.

As Example 2 shows polynomial invariants may fail to separate orbits. However, to answer our original question, it's still worth

to study polynomial invariants.

Premium exercise: When G is finite, the polynomial invariants still separate G -orbits.

Now suppose we want to understand when, for $v_1, v_2 \in V$, we have $f(v_1) = f(v_2)$ if $f \in \mathbb{C}[V]^G$. It's enough to check this for generators f of the \mathbb{C} -algebra $\mathbb{C}[V]^G$. So a natural question is whether this algebra is finitely generated.

Hilbert proved this for "reductive algebraic" groups G - he didn't know the term but this is what his proof uses.

Finite groups are reductive algebraic and so are $GL_n(\mathbb{C})$, the group of all nondegenerate matrices, $SL_n(\mathbb{C})$, matrices of determinant 1, $O_n(\mathbb{C})$, orthogonal matrices, and some others (for these infinite groups one needs to assume that their actions are "reasonable" - in some precise sense). Later, mathematicians found examples, where the algebra of invariants are not finitely generated (counterexamples to Hilbert's 14th problem).

Basis theorem is an essential ingredient in Hilbert's proof of finite generation. For more details on this see [E], 1.4.1 & 1.5; 1.3 contains some more background on Invariant theory.