MATH 380, HOMEWORK 1, DUE SEPT 23

There are 8 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, such as Problem 1, you can use previous parts to prove subsequent ones and get the corresponding credit. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Consider the ring $A = \mathbb{Z}[\sqrt{-5}](:= \mathbb{Z}[x]/(x^2+5))$, *its elements can be thought of as expressions* $a + b\sqrt{-5}$ *for* $a, b \in \mathbb{Z}$ *and added and multiplied accordingly. This ring provides an example of a quadratic extension of* $\mathbb{Z}$ *that is a domain but not a UFD. The relevant decomposition is* $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. *Here the elements* $2, 3, 1 \pm \sqrt{-5}$ *are* <u>*irreducible*</u> *meaning that they cannot be represented as the product of two non-invertible elements. To see that these elements are indeed irreducible one considers the* <u>*norm*</u>, $N(a + b\sqrt{-5}) = a^2 + 5b^2$, *and uses that it is multiplicative. In the problem below you can use the fact that* $2, 3, 1 \pm \sqrt{-5}$ *are irreducible.*

Problem 1, 3pts. Consider the ideal $I = (2) \subset A = \mathbb{Z}[\sqrt{-5}]$.

a, 2pt) Prove that A/I is isomorphic to $\mathbb{F}_2[x]/(x^2)$, where $\mathbb{F}_2$ is the field with two elements.

b, 1pt) Find two elements of A that generate $\sqrt{I}$.

c*, 0pt) Explain why $\sqrt{I}$ cannot be generated by a single element.

Problem 2, 3pts total. *This problem defines the inverse limit of rings and studies its properties.*

Let $B_i, i \in \mathbb{Z}_{>0}$, be a collection of rings and, for $i < j$, let φ_{ij} be a homomorphism $B_j \to B_i$. Suppose that, for all $i < j < k$, we have $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$. Consider the subset in $\prod_{i>0} B_i$ consisting of all sequences (b_i) such that $b_i = \varphi_{ij}(b_j)$ for all $i < j$.

a, 1pt) Prove that this subset is a subring of $\prod_{i>0} B_i$. It is called the <u>inverse limit</u> of the sequence of the rings B_i and is denoted by $\varprojlim B_i$.

b, 1pt) Prove that, for each $j > 0$, the map $\varphi_j : \varprojlim B_i \to B_j$ defined by $(b_i) \mapsto b_j$ is a ring homomorphism satisfying $\varphi_j = \varphi_{jk} \circ \varphi_k$ for all $j < k$.

c, 1pt) *This is the universal property of the inverse limit* $\varprojlim B_i$ *and the homomorphisms* φ_j. Let A be a ring equipped with homomorphisms $\psi_i : A \to B_i$ satisfying $\psi_j = \varphi_{jk} \circ \psi_k$ for all $j < k$. Show that there exists a unique ring homomorphism $\psi : A \to \varprojlim B_i$ such that $\psi_j = \varphi_j \circ \psi$ for all j.

Problem 3, 4pts. *This problem describes a special case of the inverse limit of rings, the* <u>*completion*</u> *of a ring with respect to an ideal.*

Let A be a ring and I be its ideal. Set $B_j := A/I^j$ and let $\varphi_{jk} : A/I^k \to A/I^j$ for $j < k$ be the natural epimorphism. Persuade yourselves (not for credit) that this collection satisfies the assumptions of Problem 2. Set $\hat{A} := \varprojlim A/I^i$. This is the completion of interest.

a, 1pt) Prove that all homomorphisms $\varphi_j : \hat{A} \to A/I^j$ is surjective (for all $j > 0$).

An important example: when $A = \mathbb{Z}$ and $I = (p)$, where p is prime, the ring $\hat{A}$ is the ring of p-adic integers, it plays an important role in Algebraic Number theory and is likely to appear in that class.

b, 1pt) *We will concentrate on another important example: the ring of formal power series.* Let $A = B[x]$, where B is another ring, and $I = (x)$. Show that an element of $\hat{A}$ can be uniquely represented by a "formal power series", an infinite sum $\sum_{i=0}^{\infty} b_i x^i$, where $b_i \in B$ (unlike with polynomials, we do not require that the sum is finite). Write formulas for the sum and product of two formal power series $\sum_{i=0}^{\infty} a_i x^i$ and $\sum_{i=0}^{\infty} b_i x^i$ in $\hat{A}$. *For $B = \mathbb{R}$ or $\mathbb{C}$, power series should be familiar from Calculus or Real/Complex Analysis. Unlike there, we do not require our power series to converge anywhere – which is why they are called formal.* The common notation for the ring of formal power series $\hat{A}$ is $B[[x]]$.

c, 1pt) *The ring of formal power series is closely related to the ring of polynomials. But it behaves differently, in fact, in many respects, it is simpler. The same applies to the p-adic integers vs the integers. In this part we discuss invertible elements in* $B[[x]]$. Prove that $\sum_{i=0}^{\infty} b_i x^i$ is invertible in $B[[x]]$ if and only if b_0 is invertible in B.

d, 1pt) Let B be a field. Describe all possible ideals in $B[[x]]$.

Why the name completion? This is a special case of the completion of a topological (abelian) group. A related procedure is used to get $\mathbb{R}$ from $\mathbb{Q}$. Namely, we have a topology on A, where the ideals I^j by definition form a base of neighborhoods of zero. For example, we can define the limit of a sequence (a_i) to be $a \in A$ if for all $j > 0$ there is $n > 0$ with $a_i - a \in I^j$ for all $i > n$. We can define the notion of a Cauchy sequence in A in a similar fashion. There is the usual equivalence relation on the set of Cauchy sequences. The ring $\hat{A}$ is identified with the set of equivalence classes.

Problem 4, 5pts total. Let $\varphi : A \to B$ be a ring homomorphism and let J be an ideal in B. Set $I := \varphi^{-1}(J)$.

a, 1pt) Prove I is an ideal in A.

b, 1pt) Let J be prime. Is it always true that I is prime?

c, 1pt) Let J be maximal. Is it always true that I is maximal?

d, 1pt) Is it always true that $B\varphi(I) \subset J$?

e, 1pt) Is it always true that $J \subset B\varphi(I)$?

If you think a statement is true, provide a proof. If you think it is false, provide a counterexample.

Problem 5, 2pts total. Let A be a ring, M be an A-module, and $m \in M$.

1, 1pt) We define the subset $\mathrm{Ann}_A(m) := \{a \in A | am = 0\} \subset A$ ("Ann" stands for the "annihilator"). Prove that $\mathrm{Ann}_A(m)$ is an ideal in A.

2, 1pt) We define the subset $\mathrm{Ann}_A(M) := \{a \in A | am = 0, \forall m \in M\}$. Assume that M is generated by elements $m_1, \ldots, m_k \in M$. Prove that $\mathrm{Ann}_A(M) = \bigcap_{i=1}^{k} \mathrm{Ann}_A(m_i)$.

The following two problems deal with properties of Hom modules. They may seem to be ad-hoc for now but many statements there have some meaning from the point of view of Category theory and will be revisited later in the class. The first problem examines the interplay between Hom modules and direct sums/ products.

Problem 6, 3pts total. Let I be a (possibly, infinite) set and M_I be a collection of A-modules indexed by i. Let N be another A-module.

a, 1pt) Construct a natural A-module isomorphism

$$\operatorname{Hom}_A(\bigoplus_{i\in I} M_i, N) \xrightarrow{\sim} \prod_{i\in I} \operatorname{Hom}_A(M_i, N).$$

b, 1pt) Construct a natural A-module isomorphism

$$\operatorname{Hom}_A(N, \prod_{i\in I} M_i) \xrightarrow{\sim} \prod_{i\in I} \operatorname{Hom}_A(N, M_i).$$

c, 1pt) Suppose N is finitely generated. Construct a natural isomorphism

$$\operatorname{Hom}_A(N, \bigoplus_{i\in I} M_i) \xrightarrow{\sim} \bigoplus_{i\in I} \operatorname{Hom}_A(N, M_i).$$

The next problem ultimately provides an important tool to compute Hom modules in some way.

Problem 7, 4pts total. Let L, M, N be A-modules. Then we have the composition map $\operatorname{Hom}_A(L, M) \times \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(L, N), (\varphi, \psi) \mapsto \psi \circ \varphi$.

a, 1pt) Prove that the composition map is A-bilinear, i.e., if we fix one of the arguments φ, ψ, then we get an A-linear map in the other argument.

Now consider four A-modules, M_1, M_2, M_3, N. Suppose we have A-linear maps $\varphi_1 : M_1 \to M_2, \varphi_2 : M_2 \to M_3$. Suppose that φ_2 is surjective, while $\operatorname{im} \varphi_1 = \ker \varphi_2$. Consider the maps, linear by part (a),

$$\tilde{\varphi}_1 : \operatorname{Hom}_A(M_2, N) \to \operatorname{Hom}_A(M_1, N), \psi_1 \mapsto \psi_1 \circ \varphi_1,$$
$$\tilde{\varphi}_2 : \operatorname{Hom}_A(M_3, N) \to \operatorname{Hom}_A(M_2, N), \psi_2 \mapsto \psi_2 \circ \varphi_2.$$

b, 1pt) Prove that $\tilde{\varphi}_2$ is injective.

c, 1pt) Prove that $\operatorname{im} \tilde{\varphi}_2 = \ker \tilde{\varphi}_1$.

d, 1pt) Suppose that, in the previous notation, $M_1 = A^{\oplus k}, M_2 = A^{\oplus \ell}$. So the map φ_1 is the multiplication by a matrix, denote it by $T = (t_{ij})$. Construct an isomorphism of A-modules between $\operatorname{Hom}_A(M_3, N)$ and the submodule of $N^{\oplus \ell}$ consisting of all ℓ-tuples $(n_1, \dots, n_\ell)$ such that $\sum_{j=1}^{\ell} t_{ij} n_j = 0$ for all $i = 1, \dots, k$.

The last part of Problem 7 can be used to describe the Hom module in some cases.

Problem 8, 3pts total. Now let $A = \mathbb{C}[x, y]$. Consider the ideal $I = (x, y)$.

a, 2pt) Establish an isomorphism $\operatorname{Hom}_A(I, A) \cong A$ of A-modules. *Hint: A is a unique factorization domain.*

b, 1pt) Explain what the homomorphism $I \to A$ corresponding to $a \in A$ does to an element of I.