MAT 380, HOMEWORK 2, DUE OCT 1

There are 13 problems worth 30 points total. Your score for this homework is the minimum of the sum of the points you've got and 20.

All rings contain 1.

Problem 1, 2pts. This problem constructs a category out of an oriented graphs and looks at functors between such categories. Namely, let Γ be an oriented graph. We write $V(\Gamma)$ for the set of vertices. For two vertices x, y, we write $E(x, y)$ for the set of edges from x to y and $P(x, y)$ for the set of oriented paths from x to y. Note that $P(x, x)$, by definition, includes the trivial (=empty) path.

1, 1pt) Prove that the data of: $V(\Gamma)$ as objects, $P(x, y)$ as morphisms from x to y and concatenation of paths as composition defines a category. What is the unit morphism of an object x? Denote this category by $\mathcal{C}(\Gamma)$.

2, 1pt) Prove that giving a functor $\mathcal{C}(\Gamma_1) \to \mathcal{C}(\Gamma_2)$ amounts to giving maps $\varphi : V(\Gamma_1) \to V(\Gamma_2)$ and $\varphi_{x,y} : E(x, y) \to P(\varphi(x), \varphi(y))$ for each $x, y \in \Gamma_1$.

Problem 2, 2pts. From a finite group G, we can construct its *group ring* $\mathbb{Z}G$ (that is a noncommutative ring, in general): its elements are all formal linear combinations $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{Z}$ and the multiplication is uniquely determined by the rule that the elements of $G \subset \mathbb{Z}G$ are multiplied as in the group G.

1, 1pt) From a group homomorphism $\varphi : G \to H$, construct a ring homomorphism $\varphi' : \mathbb{Z}G \to \mathbb{Z}H$ in a natural way (and check that you indeed get a ring homomorphism).

2, 1pt) Show that you get a functor $\mathsf{Groups} \to \mathsf{Rings}$ (of taking group rings) in this way.

Problem 3, 2pts. The goal of this problem is to construct a functor $\mathsf{Sets} \to \mathsf{Rings}^{opp}$. To a set X we assign the ring $\mathrm{Map}(X, \mathbb{Z})$ of all maps $X \to \mathbb{Z}$ with pointwise operations. To a map $\varphi : X \to Y$ we assign a map $\varphi^* : \mathrm{Map}(Y, \mathbb{Z}) \to \mathrm{Map}(X, \mathbb{Z})$ by $\psi \mapsto \psi \circ \varphi$, it is a ring homomorphism (you don't need to check that). Prove that this indeed defines a functor $\mathsf{Sets} \to \mathsf{Rings}^{opp}$.

Problem 4, 2pts. Carefully define a functor $\mathsf{Rings} \to \mathsf{Groups}$ sending a (generally, noncommutative) ring A to its group of invertible elements, to be denoted by $A^\times$.

Problem 5, 4pts. This problem examines endomorphisms (=morphisms of a functor to itself) for forgetful functors.

1, 2pts) Let A be a commutative ring. Consider the forgetful functor $F : A\text{-Mod} \to \mathsf{Sets}$. Identify the set $\mathrm{End}_{Fun}(F)$ of functor endomorphisms of F with A. What does the composition of endomorphisms correspond to in A?

2, 2pts) Compute the endomorphisms of the forgetful functor $F : \mathsf{Groups} \to \mathsf{Sets}$.

Problem 6, 2pts. Here we investigate formal properties of functor morphisms.

1, 1pt) Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. By F', G' we denote the same functors viewed as functors $\mathcal{C}^{opp} \to \mathcal{D}^{opp}$. Identify $\mathrm{Hom}_{Fun}(F, G)$ and $\mathrm{Hom}_{Fun}(G', F')$.

2, 1pt) Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}, G_1, G_2 : \mathcal{D} \to \mathcal{E}$ be functors and $\eta : F_1 \to F_2, \mu : G_1 \to G_2$ be functor morphisms. Define a morphism $F_2 \circ F_1 \to G_2 \circ G_1$ from η, μ in a natural way.

Problem 7, 2pts. What is the object of Rings representing the forgetful functor $\mathsf{Rings} \to \mathsf{Sets}$? Justify your answer.

Problem 8, 2pts. Suppose that in a category $\mathcal{C}$ the product $X_1 \times X_2$ exists for all $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$.

1, 1pt) For $\psi_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i), i = 1, 2$, define a morphism $\psi_1 \times \psi_2 :\in \mathrm{Hom}_{\mathcal{C}}(X_1 \times X_2, Y_1 \times Y_2)$.

2, 1pt) Prove that $\bullet \times ?$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Problem 9, 3pts. Let $\mathsf{CommRings}$ denote the category of commutative rings. Let F denote the forgetful functor $\mathsf{CommRings} \to \mathsf{Sets}$.

1, 2pts) Prove that there is a left adjoint functor to F and that it sends X to the ring of polynomials $\mathbb{Z}[x|x \in X]$ (*for example, when* $X = \{1, \dots, n\}$ *we get* $\mathbb{Z}[x_1, \dots, x_n]$).

2, 1pt) How does the answer in part 1) changes if we replace $\mathsf{CommRings}$ with Rings?

Problem 10, 3pts. Recall that in every group G we have its derived subgroup (G, G) generated by the elements of the form $ghg^{-1}h^{-1}$ for all $g, h, \in G$. It is a normal subgroup.

1, 2pt) Show that the inclusion functor $\mathbb{Z}$-Mod $\hookrightarrow$ Groups (of abelian groups into all groups) has left adjoint and that this adjoint sends G to $G/(G, G)$.

2, 0pt – bonus) Answer a similar question for the inclusion $\mathsf{CommRings} \hookrightarrow \mathsf{Rings}$.

3^*, 1pt) Show that the inclusion functor $\mathbb{Z}$-Mod $\hookrightarrow$ Groups does not admit a right adjoint functor.

Problem 11, 2pts. Prove that the functors from Problems 2 and 4 are adjoint to each other (on a suitable side).

Problem 12, 2pts. Consider the category Sets_{fin} of all finite sets. Prove that it is not equivalent to its opposite category.

Problem 13, 2pts. Let $\mathbb{F}$ be a field and let A be a finite dimensional commutative algebra over $\mathbb{F}$. We write A-mod for the category of finitely generated A-modules. Construct a category equivalence A-mod $\to A$-mod^{opp}.