EXERCISE SHEET. MICROLOCALIZATION

1. Completed Rees algebra

Let $\mathcal{A}$ be a $\mathbb{Z}$-filtered algebra. Set $R_\hbar(\mathcal{A}) := \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{\leqslant i}\hbar^i \subset \mathcal{A}[\hbar^{\pm 1}]$.

a) Show that $R_\hbar(\mathcal{A})$ is a graded $\mathbb{C}[\hbar]$-subalgebra in $\mathcal{A}[\hbar^{\pm 1}]$ (the degree is with respect to $\hbar$). Identify $R_\hbar(\mathcal{A})/(\hbar)$ with $\operatorname{gr}\mathcal{A}$ and $R_\hbar(\mathcal{A})/(\hbar - a)$ with $\mathcal{A}$ for $a \neq 0$.

We also consider the completed Rees algebra $R^\wedge_\hbar(\mathcal{A}) = \prod_{i\in\mathbb{Z}} \mathcal{A}_{\leqslant i}\hbar^i$ so that $R^\wedge_\hbar(\mathcal{A})$ is complete and separated in the $\hbar$-adic topology and carries a $\mathbb{C}^\times$-action by $\mathbb{C}$-algebra automorphisms with $t.\hbar = t\hbar$. This action is rational on all quotients mod $\hbar^k$.

Now let $\mathcal{A}_\hbar$ be a $\mathbb{C}[[\hbar]]$-algebra that is complete and separated in the $\hbar$-adic topology that comes equipped with a $\mathbb{C}^\times$-action by $\mathbb{C}$-algebra automorphisms that is rational on all quotients $\mathcal{A}_\hbar/(\hbar^k)$ and satisfying $t.\hbar = t\hbar$. If $A := \mathcal{A}_\hbar/(\hbar)$ is commutative and finitely generated, we will call $\mathcal{A}_\hbar$ a *graded formal quantization* of A. We define $\mathcal{A}_{\hbar,fin}$ as the span of all elements $a \in \mathcal{A}_\hbar$ with $t.a = t^i a$ for some $i \in \mathbb{Z}$.

b) Prove that $\mathcal{A}_{\hbar,fin}$ is a graded subalgebra of $\mathcal{A}_\hbar$ that is dense in the $\hbar$-adic topology and satisfies $\mathcal{A}_{\hbar,fin}/(\hbar) = A$.

c) Prove that $\mathcal{A}_{\hbar,fin}/(\hbar - 1)$ is a filtered quantization of A.

d) Prove that the maps $\mathcal{A} \mapsto R^\wedge_\hbar(\mathcal{A})$ and $\mathcal{A}_\hbar \mapsto \mathcal{A}_{\hbar,fin}/(\hbar - 1)$ are mutually inverse bijections between filtered quantizations and graded formal quantizations.

2. (Micro)localization for formal quantizations

Let $\mathcal{A}_\hbar$ be a formal quantization of A (we do not require the presence of $\mathbb{C}^\times$-actions/gradings, A is just required to be a finitely generated commutative algebra). We are going to sheafify $\mathcal{A}_\hbar$ in the Zariski topology on $\operatorname{Spec}(A)$.

a) Let $f \in A$ be a nonzero divisor and let $\hat{f} \in \mathcal{A}_k := \mathcal{A}_\hbar/(\hbar^k)$ be a lift of f. Show that $[\hat{f}, \cdot]^k = 0$ and deduce from here that every left fraction by $\hat{f}$ is also a right fraction. Show that the localization $\mathcal{A}_k[\hat{f}^{-1}]$ (defined by the same universality property as in the commutative case) makes sense and is independent of the choice of the lift. We will denote this localization by $\mathcal{A}_k[f^{-1}]$

b) Show that the algebras $\mathcal{A}_k[f^{-1}]$ form an inverse system. Further show that $\mathcal{A}_\hbar[f^{-1}] := \varprojlim_{k\to\infty} \mathcal{A}_k[f^{-1}]$ is a formal quantization of $A[f^{-1}]$.

c) Establish a natural homomorphism $\mathcal{A}_\hbar[f^{-1}] \to \mathcal{A}_\hbar[(fg)^{-1}]$.

d) Show that $\mathcal{A}_\hbar$ naturally sheafifies to a sheaf $\mathcal{D}_\hbar$ on $\operatorname{Spec}(A)$. Show that $\Gamma(\mathcal{D}_\hbar) = \mathcal{A}_\hbar$.

Note that if $\mathcal{A}_\hbar$ is graded, then $\mathcal{A}_\hbar[f^{-1}]$ is graded provided f is $\mathbb{C}^\times$-semiinvariant. So we can get the microlocalization of $\mathcal{A}_{\hbar,fin}/(\hbar - 1)$ by taking the sheaf $\mathcal{D}_{\hbar,fin}/(\hbar - 1)$ that makes sense in the conical topology.

e) Work out the details.

3. Coherent modules over formal quantizations

Let $\mathcal{D}_\hbar$ be a formal quantization of a Poisson scheme X. We say that a $\mathcal{D}_\hbar$-module $M_\hbar$ is coherent if it is complete and separated in the $\hbar$-adic topology and $M_\hbar/\hbar M_\hbar$ is a coherent sheaf on X.

a) Suppose that there is an open covering $X = \bigcup_i X^i$ such that $M_\hbar|_{X^i}$ is coherent. Then $M_\hbar$ is coherent.

Now suppose that X comes with a $\mathbb{C}^\times$-action as before and $\mathcal{D} := \mathcal{D}_{\hbar,fin}/(\hbar - 1)$.

b) Show that a filtered $\mathcal{D}$-module M is coherent with a good filtration if and only if $R_\hbar^\wedge(M)$ is a coherent $\mathcal{D}_\hbar$-module.

4. Microlocalization of modules

Let $\mathcal{A}$ be a filtered quantization of A and $f \in A$. Our goal here is to define the localization functor $M \mapsto M[f^{-1}]$.

a) Assume that M is equipped with a good filtration. Emulate the procedure in Exercise 2 to define $M[f^{-1}]$ and check that this space has a natural $\mathcal{A}[f^{-1}]$-module structure. Furthermore check that there is a natural isomorphism $\mathcal{A}[f^{-1}] \otimes_{\mathcal{A}} M \xrightarrow{\sim} M[f^{-1}]$ so that $M[f^{-1}]$ is independent of the choice of a filtration.

b) Check that the functor $M \mapsto M[f^{-1}]$ is exact.

c) Check that M sheafifies in the conical topology on $X := \mathrm{Spec}(A)$.

5. Quotient categories

Let, in the notation of the previous exercise, X^0 be an open $\mathbb{C}^\times$-stable subset of X. Let $\mathcal{A}|_{X^0}$ denote the restriction of the microlocalization of $\mathcal{A}$ to X^0, a quantization of X^0.

Show that $\mathrm{Coh}(\mathcal{A}|_{X^0})$ is a quotient category of $\mathcal{A}$-mod by the full subcategory of modules supported on $X \setminus X^0$.

Recall the general definition. Let $\mathcal{C}$ be an abelian category and $\mathcal{C}_0$ its Serre subcategory. By definition, the objects in the quotient category $\mathcal{C}/\mathcal{C}_0$ are the same as in $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}/\mathcal{C}_0}(M, N)$ is the direct limit of $\mathrm{Hom}_{\mathcal{C}}(M^0, N/N_0)$, where $M/M^0, N_0 \in \mathcal{C}_0$.