MAT 380, HOMEWORK 4, DUE OCT 31

There are 8 problems worth 26 points total. Your score for this homework is the minimum of the sum of the points you've got and 20.

All rings are contain 1. In problems 1-5 all rings are commutative.

Problem 1, 3pts. Let A be a ring, M be an A-module and $I \subset A$ be an ideal.

a, 2pts) Prove that $(A/I) \otimes_A M$ is naturally isomorphic to M/IM as A-modules.

b, 1pt) Prove that if M is flat, then $I \otimes_A M \cong IM$, an isomorphism of A-modules.

Problem 2, 4pts. This problem concentrates on flat modules. Let A be a ring.

a, 1pt) Prove that the direct sum of two flat A-modules is also flat.

b, 1pt) Prove that the tensor product of two flat A-modules is also flat.

c, 1pt) Prove that every projective A-module is flat.

d, 1pt) Let $A = \mathbb{Z}$. Prove that a finitely generated A-module M is flat if and only if M is free.

Problem 3, 2 pts. Let $A = \mathbb{Z}[\sqrt{-5}]$ and I be the ideal $(2, 1+\sqrt{-5})$. Prove that $I \otimes_A I \cong A$, an isomorphism of A-modules.

Problem 4, 5pts. Let A be a ring, M, N be A-modules and B be an A-algebra, i.e., a ring with a homomorphism from A. The goal of this problem is to compare $B \otimes_A \operatorname{Hom}_A(M, N)$ and $\operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$.

a, 1pt) Construct a natural B-module homomorphism $B \otimes_A \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$.

b, 2pts) Suppose B is flat as an A-module and M is *finitely presented*, i.e., there is an exact sequence $A^{\oplus \ell} \to A^{\oplus k} \to M \to 0$. Use the 5-lemma to prove that the homomorphism in the previous part is an isomorphism.

c, 1pt) Produce a counterexample to the statement in b) when B is not flat (e.g., for $A = \mathbb{Z}$).

d*, 1pt) Produce a counterexample to b) when B is flat but M is not finitely presented (e.g., for $A = \mathbb{Z}$).

Problem 5, 2pts. Let A be a ring, B be an A-algebra and C be a B-algebra. Prove that functors $C \otimes_A \bullet, C \otimes_B (B \otimes_A \bullet)$ (from A-Mod to C-Mod) are isomorphic.

The remaining problems deal with tensor products over rings that are not required to be commutative. They utilize the notions of left, right modules and bimodules over rings.

Let A be a ring (containing 1 but, perhaps, noncommutative). By a left A-module *one means an abelian group M equipped with a $\mathbb{Z}$-bilinear map $A \times M \to M, (a, m) \mapsto am$, satisfying $(a_1a_2)m = a_1(a_2m)$ and $1m = m$ for all $a_1, a_2 \in A, m \in M$. By a* right A-module *one means an abelian group M equipped with a $\mathbb{Z}$-bilinear map $M \times A \to M, (m, a) \mapsto ma$, satisfying $m(a_1a_2) = (ma_1)a_2$ and $m = m1$ for all $a_1, a_2 \in A, m \in M$.*

Note that when A is commutative, a left or right A-module is the same thing as an A-module.

Now let B be another ring. By an A-B-module one means an abelian group M equipped with structures of a left A-module and a right B-module such that $(am)b = a(mb)$ for all $a \in A, b \in B, m \in M$. For example, every ring A is an A-A-bimodule.

Problem 6, 4pts. Let A be a ring, M be a right A-module and N be a left A-module. For an abelian group L define $\mathrm{Bilin}_A(M \times N; L)$ to be the set of all $\mathbb{Z}$-bilinear maps $\mu\ M \times N \to L$ subject to the property $\mu(ma, n) = \mu(m, an)$ for all $m \in M, n \in N, a \in A$.

1, 1pt) Prove that $\mathrm{Bilin}_A(M \times N; \bullet)$ is a functor $\mathbb{Z}$-Mod $\to$ Sets.

2, 1pt) Prove that this functor is representable. The representing object is denoted by $M \otimes_A N$ and called the tensor product of M and N (*note that, in general, this is just an abelian group, not a left or right A-module*).

3, 1pt) Construct a natural element in $\mathrm{Bilin}_A(M \times N; M \otimes_A N)$. The image of (m, n) under this map will be denoted by $m \otimes n$. Prove that the elements $m \otimes n$ span the abelian group $M \otimes_A N$.

3, 1pt) Let A be commutative. Then $M \otimes_A N$ is the same abelian group as the tensor product defined in the lectures.

Problem 7, 3pts. Let A, B, C be rings, M be an A-B-bimodule, and N be a B-C-bimodule. Show that there is a unique A-C-bimodule structure on $M \otimes_B N$ satisfying $a(m \otimes n) = (am) \otimes n$ and $(m \otimes n)c = m \otimes (nc)$ for all $m \in M, n \in N, a \in A, c \in C$.

Problem 8, 3pts. This problem describes another important category, $\mathcal{R}$, whose objects are rings. For two rings A, B define the set of morphisms $\mathrm{Hom}_{\mathcal{R}}(A, B)$ to be the set of all B-A-bimodules (*actually, if we want this to be a set we need to consider isomorphism classes of bimodules, we ignore this issue*). For $M \in \mathrm{Hom}_{\mathcal{R}}(A, B), N \in \mathrm{Hom}_{\mathcal{R}}(B, C)$ define their composition $N \circ M$ by $N \otimes_B M$. Prove that we indeed get a category.