Lecture 19

1) Adjoint functors, cont'd

2) Tensor-Hom adjunction

Ref: [R], Section 4.1, Hilton-Stammbach, Section 2.7.

BONUS: more on adjunction (see Sec 1.2)

1.0) Reminder

Let $\mathcal{C}, \mathcal{D}$ be categories & $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ be functors.

Definition: F is left adjoint to G if $\forall X \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D})$ $\exists$ bijection $\eta_{X,Y}: Hom_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} Hom_{\mathcal{C}}(X, G(Y))$ s.t.

(1) $\forall X, X' \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D}), X' \xrightarrow{\varphi} X$ the following is comm've:

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F(X), Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "? \circ F(\varphi)"] & Hom_{\mathcal{C}}(X, G(Y)) \arrow[d, "? \circ \varphi"] \\
Hom_{\mathcal{D}}(F(X'), Y) \arrow[r, "\eta_{X',Y}"] & Hom_{\mathcal{C}}(X', G(Y))
\end{tikzcd}$$

(2) $\forall Y, Y' \in Ob(\mathcal{D}), Y \xrightarrow{\psi} Y'; X \in Ob(\mathcal{C})$, the following is comm've

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F(X), Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "\psi \circ ?"] & Hom_{\mathcal{C}}(X, G(Y)) \arrow[d, "G(\psi) \circ ?"] \\
Hom_{\mathcal{D}}(F(X), Y') \arrow[r, "\eta_{X,Y'}"] & Hom_{\mathcal{C}}(X, G(Y'))
\end{tikzcd}$$

Rem: in this case, say G is right adjoint to F.

1.1) Examples.

Often we get interesting adjoint functors F/G starting from sometimes boring (e.g. forgetful or inclusion) functors, which is why we care about adjoint functors in this course.

Ex 1: Let A be commutative ring, $G := For : A\text{-}Mod \longrightarrow Sets$
$F := Free : Sets \longrightarrow A\text{-}Mod$, see Sec 1.1 in Lec 14, in particular, on the level of objects: $Free(I) = A^{\oplus I}$.

Claim: F is left adjoint to G

Hom Sets

• construct $\eta_{I,M} : Hom_A(A^{\oplus I}, M) \xrightarrow{\sim} M^{\times I} = Maps(I,M)$

$$\tau \xmapsto{\eta_{I,M}} [i \mapsto \tau(e_i)]$$

definition of $\eta_{I,M}$

• check commutative diagram (1): $\forall$ maps $\varphi: I' \to I$:

$$\begin{array}{ccc}
Hom_A(A^{\oplus I}, M) & \xrightarrow[\sim]{\eta_{I,M}} & Maps(I,M) \\
\big\downarrow {\scriptstyle ?\circ Free(\varphi)} & & \big\downarrow {\scriptstyle ?\circ\varphi} \\
Hom_A(A^{\oplus I'}, M) & \xrightarrow[\sim]{\eta_{I',M}} & Maps(I',M)
\end{array}$$

where $Free(\varphi)$ is the unique A-linear map $A^{\oplus I} \to A^{\oplus I'}$ with $[Free(\varphi)](e_i) := e_{\varphi(i)}$. Pick $\tau \in Hom_A(A^{\oplus I}, M)$. Then

$\downarrow \rightarrow$: $\tau \mapsto \left[\text{unique } \tau' : A^{\oplus I'} \to M \text{ s.t. } \tau'(e'_j) := \tau(e_{\varphi(j)})\right]$

$$\downarrow$$

$$[j \mapsto \tau(e_{\varphi(j)})]$$

$\rightarrow \downarrow$: $\tau \mapsto [i \mapsto \tau(e_i)] \mapsto [j \mapsto \tau(e_{\varphi(j)})]$

Check (2): for $\psi \in \mathrm{Hom}_A(M, M')$, the following is commutative.

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\sim"'] \arrow[d, "\psi \circ ?"] & \mathrm{Maps}(I, M) \arrow[d, "\psi \circ ?"] \\
\mathrm{Hom}_A(A^{\oplus I}, M') \arrow[r, "\sim"'] & \mathrm{Maps}(I, M')
\end{tikzcd}$$

where now ψ is viewed as map of sets

$\rightarrow \downarrow$: $\tau \mapsto [i \mapsto \tau(e_i)] \longmapsto [i \mapsto \psi(\tau(e_i))]$

$\downarrow \rightarrow$: $\tau \mapsto \psi \circ \tau \mapsto [i \mapsto \psi(\tau(e_i))]$

We've checked the adjunction.

Ex 2: G is the inclusion functor $\mathbb{Z}\text{-Mod} \hookrightarrow \text{Groups}$,
$F := Ab : \text{Groups} \longrightarrow \mathbb{Z}\text{-Mod}$, $G \mapsto G/(G,G)$, see Sec. 1.1 in Lecture 14.

Claim: F is left adjoint to G.

Notation: $\pi_G : G \twoheadrightarrow G/(G,G)$, $\pi_G(g) := g(G,G)$.

- Construct bijection $\eta_{G,M} : \mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M) \longrightarrow \mathrm{Hom}_{Groups}(G,M)$

$$\tau \longmapsto \tau \circ \pi_G$$

Exercise: $\eta_{G,M}$ is a bijection; hint M is commutative so $(M,M)=\{0\}$.

- Check comm've diagram (1): $\forall\, \varphi : G' \to G$, group homomorphism

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M) \arrow[r, "? \circ \pi_G"] \arrow[d, "? \circ Ab(\varphi)"] & \mathrm{Hom}_{Groups}(G,M) \arrow[d, "? \circ \varphi"] \\
\mathrm{Hom}_{\mathbb{Z}}(G'/(G',G'), M) \arrow[r, "? \circ \pi_{G'}"] & \mathrm{Hom}_{Groups}(G',M)
\end{tikzcd}$$

Exercise: Use the def'n of $Ab(\varphi)$: $Ab(\varphi)(g'(G',G')) = \varphi(g')(G,G)$ to show $\pi_G \circ \varphi = Ab(\varphi) \circ \pi_{G'}$ & deduce that the diagram is commutative.

Check comm've diagram (2): $\psi : M \to M'$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M) \arrow[r, "? \circ \pi_G"] \arrow[d, "\psi \circ ?"] & \mathrm{Hom}_{Groups}(G,M) \arrow[d, "\psi \circ ?"] \\
\mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M') \arrow[r, "? \circ \pi_G"] & \mathrm{Hom}_{Groups}(G,M')
\end{tikzcd}$$

Both compositions are $\psi \circ ? \circ \pi_G$, manifestly the same!

The adjunction is checked.

1.2) Bonuses

There are three bonus sections to this lecture. The first two contain some basic stuff for which we don't have time in the lecture, and the third is more advanced.

1) Given $G: \mathcal{D} \to \mathcal{C}$ its left adjoint F is unique up to a functor isomorphism (if it exists, which may fail, compare to Prob. 3c in HW5). The proof is fun w. commutative diagrams, which is Bonus 1 for this lecture.

2) There's an analogy between adjoint functors & adjoint linear maps (which motivates the former name). This is Bonus 2.

3) One can state the adjunction property in terms of functor morphisms $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$, $FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ called adjunction unit and counit. This is Bonus 3.

2) Tensor-Hom adjunction.

Setting: A, B are commutative rings, $\varphi: A \to B$ is a ring homomorphism $\rightsquigarrow$ pullback functor $\varphi^*: B\text{-Mod} \to A\text{-Mod}$ (partially forgetful), see Section 1.1 in Lec 14.

Goal: construct the left adjoint of this functor.

This turns out to be a special case of a more general result, the Tensor-Hom adjunction. To state this, we'll need a functor $A\text{-Mod} \to B\text{-Mod}$.

Let L be a B-module (so also an A-module) & M be an A-module $\rightsquigarrow$ A-module $L\otimes_A M$.

Lemma: $L\otimes_A M$ has a B-module structure extending its A-module structure and giving a functor $L\otimes_A \bullet : A\text{-Mod} \to B\text{-Mod}$.

Proof: (a): module structure. For $b\in B$, set $\mu_b : L\to L, \mu_b(l)=bl$. This is an A-linear map, moreover it's compatible w. A-algebra structure on B meaning that

$$\mu_1 = id_L,\ \mu_{b_1b_2}=\mu_{b_1}\circ\mu_{b_2},\quad \forall b_1,b_2\in B \qquad (i)$$

$$\mu_{b_1+b_2}=\mu_{b_1}+\mu_{b_2},\ \mu_{ab}=a\mu_b \quad \forall b, b_1, b_2\in B, a\in A \qquad (ii)$$

Consider the A-linear map $\mu_b\otimes id_M : L\otimes_A M \to L\otimes_A M$. Define the multiplication map

$$B\times (L\otimes_A M)\to L\otimes_A M: \quad (b,x)\mapsto (\mu_b\otimes id_M)(x) \qquad (I)$$

Using (ii) & the bilinearity of the tensor product of maps, Sec. 1.4 in Lec 17, we see that (I) is A-bilinear. Combining (i) w. the claim that the tensor product of maps preserves compositions, we see that (I) equips $L\otimes_A M$ w. a B-module structure.

(b) Functor: we have explained what $L\otimes_A ?$ does to objects. For an A-linear map $\psi: M\to M'$ define

$$\psi_L: L\otimes_A M \to L\otimes_A M' \quad \psi_L := id_L\otimes\psi$$

This is an A-bilinear map by Section 1.4 in Lec 17.

The claim that it's B-bilinear boils down to

$(\mu_b\otimes id_M)\circ(id_L\otimes\psi) = (id_L\otimes\psi)\circ(\mu_b\otimes id_M)$. Both sides are $\mu_b\otimes\psi$.

And then $\varphi \mapsto \varphi_L$ is compatible w. units & compositions by Sec 1.4 in Lec 17. This finishes the check that we have a functor. $\square$

Observation: $\mathrm{Hom}_B(L,\bullet)$ can be viewed as a functor $B\text{-Mod} \to B\text{-Mod}$. This is because for a B-linear homomorphism $\tau: N \to N'$, the map $\tau \circ ?: \mathrm{Hom}_B(N,N') \to \mathrm{Maps}(\mathrm{Hom}_B(L,N), \mathrm{Hom}_B(L,N'))$ is B-linear (Prob. 7, a), in HW1)

Thm (Tensor-Hom adjunction): the functor $L\otimes_A ?: A\text{-Mod} \to B\text{-Mod}$ is left adjoint to $\varphi^*\mathrm{Hom}_B(L,\bullet): B\text{-Mod} \to A\text{-Mod}$

This result will be proved next time.

Cor: The functor $B\otimes_A \bullet: A\text{-Mod} \to B\text{-Mod}$ (names: base change, induction) is left adjoint of $\varphi^*: B\text{-Mod} \to A\text{-Mod}$.
Proof: uses the Thm & the next exercise
Exercise: The functor $\mathrm{Hom}_B(B,\bullet): B\text{-Mod} \to B\text{-Mod}$ is isomorphic to the identity functor $\square$

BONUS 1: uniqueness of adjoints. Modified 11/17/21

Proposition: Let $G:\mathcal{D}\to\mathcal{C}$ be a functor. If its left adjoint exists, then it's unique up to a functor isom'm.

Proof: Suppose $F^1, F^2: \mathcal{C}\to\mathcal{D}$ are both left adj't to $G:\mathcal{D}\to\mathcal{C}$ $\rightsquigarrow$ $\eta^i_{X,Y}: Hom_{\mathcal{D}}(F^i(X),Y)\xrightarrow{\sim} Hom_{\mathcal{C}}(X,G(Y))$ that make (1) & (2) comm've $\rightsquigarrow$

$\eta_{X,Y} := (\eta^2_{X,Y})^{-1}\circ\eta^1_{X,Y}: Hom_{\mathcal{D}}(F^1(X),Y)\longrightarrow Hom_{\mathcal{D}}(F^2(X),Y)$

which makes analogs of (1) & (2) comm've (stack diagrams to η^1 & $(\eta^2)^{-1}$ horizontally.

(1) $\forall\ X'\xrightarrow{\varphi} X$:

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F^1(X),Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "?\circ F^1(\varphi)"] & Hom_{\mathcal{D}}(F^2(X),Y) \arrow[d, "?\circ F^2(\varphi)"] \\
Hom_{\mathcal{D}}(F^1(X'),Y) \arrow[r, "\eta_{X',Y}"] & Hom_{\mathcal{D}}(F^2(X'),Y)
\end{tikzcd}$$

(2) $\forall\ Y\xrightarrow{\psi} Y'$

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F^1(X),Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "\psi\circ ?"] & Hom_{\mathcal{D}}(F^2(X),Y) \arrow[d, "\psi\circ ?"] \\
Hom_{\mathcal{D}}(F^1(X),Y') \arrow[r, "\eta_{X,Y'}"] & Hom_{\mathcal{D}}(F^2(X),Y')
\end{tikzcd}$$

Fix X, look at (2): it tells us that $\eta_{X,?}$ is a functor morphism (and hence isomorphism – b/c each $\eta_{X,Y}$ is bijection) between $Hom_{\mathcal{D}}(F^1(X),\bullet)$ & $Hom_{\mathcal{D}}(F^2(X),\bullet)$. By Yoneda Lemma, have the unique isomorphism $\tau_X \in Hom_{\mathcal{C}}(F^2(X), F^1(X))$ s.t. $\eta_{X,?}(\bullet) = \bullet \circ \tau_X$.

Claim: (1) now tells us that τ is a functor morphism (hence, an isomorphism). Indeed, in (1):

$$\longrightarrow\downarrow \; : \quad ?\circ(\tau_X \circ F^2(\varphi))$$

(*) $\|$ – b/c (1) is comm've

$$\downarrow\longrightarrow \; : \quad ?\circ(F^1(\varphi)\circ\tau_{X'})$$

Note that in (1) we can choose any Y and then any $? \in Hom_{\mathcal{D}}(F^1(X), Y)$. Take $Y = F^1(X)$ & $? = id_{F^1(X)}$. Then (*) above tells us that $\tau_X \circ F^2(\varphi) = F^1(\varphi)\circ\tau_{X'}$.

i.e.
$$\begin{tikzcd}
F^2(X') \arrow[r, "\tau_{X'}"] \arrow[d, "F^2(\varphi)"] & F^1(X') \arrow[d, "F^1(\varphi)"] \\
F^2(X) \arrow[r, "\tau_X"] & F^1(X)
\end{tikzcd}$$
is commutative

So τ is indeed a functor (iso)morphism $\square$

BONUS 2: adjoint functors vs adjoint linear maps.

Let $\mathbb{F}$ be a field, U, V be $\mathbb{F}$-vector spaces, and $\alpha: U \to V$ be a linear map. It gives rise to the adjoint linear map $\alpha^*: V^* \to U^*$ (Prob 2d in HW4)

The following table summarizes analogies between the adjoint linear maps & adjoint functors.

Linear map story	Functor story
Fin. dim. vector space V	Category $\mathcal{C}$
Dual space V^*	Category $\mathcal{C}^{opp}$
Map $V^* \times V \to \mathbb{F}$ (pairing)	Functor $\mathcal{C}^{opp} \times \mathcal{C} \longrightarrow Sets$ $Hom_{\mathcal{C}}(\bullet, ?)$
Cond'n for $A^*: W^* \to V^*$ being adj't to $A: V \to W$:	Condition for functors being adjoint:
$\langle A^*\beta, v\rangle = \langle \beta, Av\rangle$ $\forall \beta \in W^*, v \in V$.	Isom'm of functors $\mathcal{C}^{opp} \times \mathcal{D} \to Sets$ $Hom_{\mathcal{D}}(F(\bullet), ?) \xrightarrow{\sim} Hom_{\mathcal{C}}(\bullet, G(?))$

Note: this is just an analogy...

BONUS 3: adjunction unit & counit.

Let $F: \mathcal{C} \to \mathcal{D}$ be left adjoint to $G: \mathcal{D} \to \mathcal{C}$. We claim that this gives rise to functor morphisms: the adjunction unit $\varepsilon: \mathrm{Id}_{\mathcal{C}} \Longrightarrow GF$ & counit $\eta: FG \Longrightarrow \mathrm{Id}_{\mathcal{D}}$.

We construct ε and leave η as an exercise.

Consider $X_1, X_2 \in Ob(\mathcal{C})$. Then we have the bijection

$$\eta_{X_1, F(X_2)}: \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X_1, GF(X_2))$$

Note that F gives rise to a map $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2))$

Composing this map w. the bijection $\eta_{X_1, F(X_2)}$ we get

$$\varepsilon_{X_1, X_2}: \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_1, GF(X_2)).$$

Now we can argue as in the proof of Proposition 1.3 to see that $\exists!\ \varepsilon: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ s.t. $\varepsilon_{X_1, X_2}(\varphi) = \varepsilon_{X_2} \circ \varphi$.

A natural question to ask is: for two functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ & functor morphisms $\varepsilon: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$, $\eta: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ when is F left adjoint to G (& ε, η unit & counit).

Very Premium Exercise: TFAE

a) F is left adjoint to G w. unit ε & counit η

b) The composed morphisms $F \Rightarrow FGF \Rightarrow F$, $G \Rightarrow GFG \Rightarrow G$ induced by ε, η (cf. Problem 8 in HW3) are the identity endomorphisms (of F & G).