MAT 380, HOMEWORK 5, DUE NOV 14

There are 8 problems worth 26 points total. Your score for this homework is the minimum of the sum of the points you've got and 20.

A denotes a commutative ring containing 1.

Problem 1, 4pts. Let M_1, M_2 be A-modules.

1, 2pts) Is it always true that $S_A(M_1 \oplus M_2) \cong S_A(M_1) \otimes_A S_A(M_2)$, an isomorphism of A-algebras?

2, 2pts) Same for $T_A(M_1 \oplus M_2) \cong T_A(M_1) \otimes_A T_A(M_2)$.

Explain why or why not.

Problem 2, 4pts. This problem deals with exterior algebras. Let M be an A-module. Define the algebra $\Lambda_A(M)$, the *exterior algebra* of M by

$$\Lambda_A(M) := T_A(M)/J, \quad J := \mathrm{Span}_A(x(m \otimes m)y | x, y \in T_A(M), m \in M).$$

Further, set

$$J_j := \mathrm{Span}_A(x(m \otimes m)y | x, y \in T_A(M) \text{ are homogeneous with} \deg x + \deg y = j - 2, m \in M),$$

so that $J = \bigoplus_j J_j$ and $\Lambda_A(M) = \bigoplus_{j=0}^{\infty} \Lambda_A^j(M)$, where $\Lambda_A^j(M) = M^{\otimes j}/J_j$.

1, 1pt) Prove that for $x \in \Lambda_A^i(M), y \in \Lambda_A^j(M)$, we have $yx = (-1)^{ij}xy$ and $xy \in \Lambda_A^{i+j}(M)$.

2, 1pt) Assume that M is generated by elements $x_1, \ldots, x_k$. Show that, as an A-module, $\Lambda_A^j(M)$ is spanned by the elements $x_{i_1}x_{i_2}\ldots x_{i_j}$ with $i_1 < i_2 < \ldots < i_k$.

3, 1pt) Establish an A-algebra isomorphism $\Lambda_A(A) \cong A[x]/(x^2)$.

4, 1pt) Let M be a free A-module with basis $x_1, \ldots, x_k$. Show that $\Lambda_A^j(M)$ is a free A-module with basis $x_{i_1}x_{i_2}\ldots x_{i_j}$ with $i_1 < i_2 < \ldots < i_j$.

Problem 3, 2pts. This problem deals with general properties of Artinian modules. Let M be an A-module.

1, 1pt) Let M be Artinian and $\varphi : M \to M$ be an injective A-linear map. Show that φ is an isomorphism.

2, 1pt) Assume A is Artinian. Show that M is Artinian if and only if it is finitely generated.

Problem 4, 3pts. The goal of this problem is to give an example of an infinitely genetated Artinian module over $\mathbb{C}[x]$. *Consider the ring of Laurent polynomials* $\mathbb{C}[x^{\pm 1}]$ *consisting of formal sums* $\sum_{i=-m}^{n} a_i x^i$ *(we allow the powers of x to be both positive and negative) with addition and multiplication defined as for polynomials. In other words,* $\mathbb{C}[x^{\pm 1}]$ *is the localization of* $\mathbb{C}[x]$ *w.r.t.* $S := \{x^i | i \geqslant 0\}$. *We have a natural inclusion* $\mathbb{C}[x] \subset \mathbb{C}[x^{\pm 1}]$, *which gives rise to the quotient* $\mathbb{C}[x]$*-module* $\mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$.

1, 1pt) Show that every $\mathbb{C}[x]$-submodule of $\mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$ is generated by (the class of) x^i for some $i < 0$.

2, 1pt) Show that $\mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$ is an Artinian $\mathbb{C}[x]$-module.

3, 1pt) Show that $\mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$ is not finitely generated as a $\mathbb{C}[x]$-module.

Problem 5, 4pts. Assume that A is Noetherian and consider the formal power series algebra $A[[x]]$. Its elements are arbitrary sums $\sum_{i=0}^{\infty} a_i x^i$ and the addition and multiplication are defined similarly to that of polynomials.

1, 2pts) Let $I \subset A[[x]]$ be an ideal. For $j \geqslant 0$, define I_j as the set of all $a \in A$ such that there is an element $\sum_{i=j}^{\infty} a_i x^i \in I$ with $a_j = a$. Prove that I_j is an ideal and the chain $I_j, j \geqslant 0$, is ascending.

2, 2pts) Prove that $A[[x]]$ is Noetherian.

Problem 6, 2pts. Let A be Noetherian and M be a finitely generated A-module. Prove that $S_A(M)$ is a Noetherian ring.

Problem 7, 3pts. *This problem discusses an important construction called the blow-up algebra. What we blow-up is an ideal.*

Let A be a ring and $I \subset A$ be an ideal.

1 1pt) Consider the subset in $A[x]$ consisting of all polynomials $\sum_{i=0}^{n} a_i x^i$ with $a_j \in I^j$ for all j (we set $I^0 := A$). Prove that this subset is an A-subalgebra of $A[x]$. Denote it by $\mathrm{Bl}_I(A)$.

2, 1pt) Construct a surjective A-algebra homomorphism $S_A(I) \to \mathrm{Bl}_I(A)$.

3, 1pt) Prove that $\mathrm{Bl}_I(A)$ is Noetherian if A is.

Problem 8, 4pts. Let A be Noetherian, $I \subset A$ be an ideal, M be a finitely generated A-module, and $N \subset M$ a submodule.

1, 1pt) Equip $\bigoplus_{j=0}^{\infty} I^j M$ with a structure of a $\mathrm{Bl}_I(A)$-module so that $\bigoplus_{j=0}^{\infty}(N \cap I^j M)$ is a submodule.

2, 1pt) Prove that there is a finite collection of generators of the $\mathrm{Bl}_I(A)$-module $\bigoplus_{j=0}^{\infty}(N \cap I^j M)$ each lying in a single summand of the form $N \cap I^j M$ (for different generators you can have different j's).

3, 1pt) Prove that there is $d > 0$ such that $I^j N \supset N \cap I^{j+d} M$ for all j.

4, 1pt) Suppose now A is local with maximal ideal $\mathfrak{m}$. Prove that $\bigcap_{i=0}^{\infty} \mathfrak{m}^i M = \{0\}$. *This is a cancelled HW3 problem revisited.*