Lecture 17

1) Tensor products of modules cont'd

Refs: [AM], Section 2.7

BONUS: Tensor products over non-commutative rings

1.0) Recap. For commutative ring A & A-modules M_1, M_2, consider their tensor product: $M_1 \otimes_A M_2$ & a bilinear map $M_1 \times M_2 \to M_1 \otimes_A M_2$, $(m_1, m_2) \mapsto m_1 \otimes m_2$ w. the following univ. property:

$\forall$ A-bilinear map $\beta: M_1 \times M_2 \longrightarrow N$ $\exists!$ A-linear map $\tilde{\beta}: M_1 \otimes_A M_2 \to N$ s.t. $\beta(m_1, m_2) = \tilde{\beta}(m_1 \otimes m_2)$.

Thx to universal property, $M_1 \otimes_A M_2$ is unique up to an iso. More precisely, if $M_1 \otimes_A M_2$ w. $(m_1, m_2) \mapsto m_1 \otimes m_2$, $M_1 \otimes_A' M_2$ w. $(m_1, m_2) \mapsto m_1 \otimes' m_2$ are two tensor products, then $\exists!$ A-linear map $M_1 \otimes_A M_2 \to M_1 \otimes_A' M_2$ s.t. $m_1 \otimes m_2 \mapsto m_1 \otimes' m_2$ $\forall m_i \in M_i$ & is an isomorphism: the inverse arises from the universal property for $M_1 \otimes_A' M_2$ applied to $M_1 \times M_2 \to M_1 \otimes_A M_2$. Thx to the uniqueness part, the composition is the identity.

1.1) Existence.

Theorem: $M_1 \otimes_A M_2$ exists $\forall M_1, M_2 \in Ob(A\text{-Mod})$.

Construction/proof is in 2 steps.

Step 1: Assume $M_1 \simeq A^{\oplus I}$ for some set I. We claim that $A^{\oplus I} \otimes_A M$ exists & is identified w. $M^{\oplus I}$ w. $((a_i)_{i\in I}) \otimes m \longleftrightarrow (a_i m)_{i \in I}$.

Observe that the map $A^{\oplus I} \times M \longrightarrow M^{\oplus I}$, $((a_i), m) \mapsto (a_i m)$ is bilinear. We need to show universal property: $\forall$ bilinear map $\beta: A^{\oplus I} \times M \longrightarrow N$ $\exists!$ linear map $\tilde{\beta}: M^{\oplus I} \to N$ s.t. $\beta((a_i), m) = \tilde{\beta}((a_i m))$.

Define, for $i \in I$, a map $\beta_i : M \to N$, $\beta_i(m) := \beta(e_i, m)$, it's linear b/c β is linear in 2nd argument. Recall that (Prob 6 in HW1):

$$\begin{array}{ccc} \mathrm{Hom}_A(M^{\oplus I}, N) & \xrightarrow{\sim} & \mathrm{Hom}_A(M,N)^{\times I} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \tilde{\beta} & \longleftarrow\!\!\shortmid & (\beta_i)_{i\in I} \end{array}$$

definition of $\tilde{\beta}$ $\rightsquigarrow$

Check: $\tilde{\beta}((a_i m)) = \beta((a_i), m)$ $\quad (*)$

$$\tilde{\beta}((a_i m)) \underset{\text{by def'n of } \tilde{\beta}}{=} \sum_{i\in I} \beta_i(a_i m) \underset{\beta_i \text{ is linear}}{=} \sum a_i \beta_i(m) \underset{\text{by def'n of } \beta_i}{=} \sum a_i \beta(e_i, m) =$$

$= [\beta$ is linear in 1st argument$] = \beta((a_i), m)$.

Exercise: $\tilde{\beta}$ is a unique linear map $M^{\oplus I} \to N$ satisfying $(*)$.

This finishes the proof of existence of $A^{\oplus I} \otimes_A M$.

Step 2:

Let M_1', M_2 be A-modules s.t. $M_1' \otimes_A M_2$ exists. Let $K_1 \subset M_1'$ be an A-submodule $\rightsquigarrow$ $M_1 := M_1'/K_1$ & $\pi_1 : M_1' \twoheadrightarrow M_1$. Inside $M_1' \otimes_A M_2$ consider submodule $K := \mathrm{Span}_A(k_1 \otimes m_2 \mid k_1 \in K_1, m_2 \in M_2)$.

$\rightsquigarrow$ projection $\pi : M_1' \otimes_A M_2 \twoheadrightarrow M_1' \otimes_A M_2 / K$.

Claim: $M_1'\otimes_A M_2/K$ is the tensor product $M_1\otimes_A M_2$ & for $m_1=\pi_1(m_1')\in M_1$ & $m_2\in M_2 \Rightarrow m_1\otimes m_2 := \pi(m_1'\otimes m_2)$.

How this implies Thm: Pick M_1, M_2; know M_1 is a quotient of $A^{\oplus I}=: M_1'$ for some I; $M_1'\otimes_A M_2$ exists by Step 1 & $M_1\otimes_A M_2$ exists by Step 2.

Proof of Claim. Exercise: $m_1\otimes m_2$ is well-defined (independent of choice of m_1') & gives a bilinear map $M_1\times M_2\longrightarrow M_1'\otimes_A M_2/K$.

Now we only need to check univ'l property: $\forall$ bilinear $\beta: M_1\times M_2\to N$ $\exists!$ linear $\tilde\beta: M_1'\otimes_A M_2/K\longrightarrow N$ s.t. $\beta(m_1,m_2)=\tilde\beta(m_1\otimes m_2)$.

Define $\beta': M_1'\times M_2\longrightarrow N$ by $\beta'(m_1',m_2)=\beta(\pi_1(m_1'),m_2)$ so β' is bilinear $\rightsquigarrow \exists!\ \tilde\beta': M_1'\otimes_A M_2\longrightarrow N$ s.t. $\tilde\beta'(m_1'\otimes m_2) = \beta'(m_1',m_2)$. Note that $\tilde\beta'(k_1\otimes m_2)=\beta'(k_1,m_2)=\beta(0,m_2)=0$ so $\tilde\beta'(K)=0$. So $\exists!\ \tilde\beta: M_1'\otimes_A M_2/K\to N$ s.t. $\tilde\beta'=\tilde\beta\circ\pi$. This is precisely the cond'n $\tilde\beta(m_1\otimes m_2)=\beta(m_1,m_2)$ $\square$

1.2) Examples.

1) Tensor product of free modules: $A^{\oplus I}\otimes_A A^{\oplus J}=$[Step 1] $=(A^{\oplus J})^{\oplus I}\simeq A^{\oplus(I\times J)}$ w. basis $e_i\otimes e_j$ $(i\in I, j\in J)$.

2) $A=\mathbb{F}[x,y]$ ($\mathbb{F}$ is field), ideal $I=(x,y)$, want to compute $I\otimes_A I$, i.e. to present it by generators & relations.

Step 1:

Need to present I as a quotient of a free module:
$A^{\oplus 2} \twoheadrightarrow I$, $(a,b) \mapsto (ax+by)$.
Compute the kernel $K_1 = \{(a,b) \mid ax = -by\} = $ [Prob 8 of HW1] $=$
$\mathrm{Span}_A((y,-x))$.

Step 2: $I \otimes_A I = A^{\oplus 2} \otimes_A I / \mathrm{Span}_A(g(y,-x) \otimes m = (y,-x) \otimes \underbrace{gm}_{\text{also element of } I} \mid m \in I, g \in A) = I^{\oplus 2} / \mathrm{Span}_A((ym, -xm) \mid m \in I)$.

Now we want to present $I \otimes_A I$ by generators & relations
$I = A^{\oplus 2} / \mathrm{Span}_A((y,-x))$ (the 1st generator of $I \leftrightarrow e_1 \in A^{\oplus 2}$, the 2nd gen'r $\leftrightarrow e_2 \in A^{\oplus 2}$)

the numerator is $I^{\oplus 2}$

$$I \otimes_A I = \frac{\left(A^{\oplus 4} / \mathrm{Span}_A((y,-x,0,0), (0,0,y,-x))\right)}{\{(ym, -xm) \mid m \in I\}} =$$

elements in the numerator i.e. cosets

$=$ [I is generated by 2 elements, the images of $e_1, e_2 \in A^{\oplus 2}$ under $A^{\oplus 2} \twoheadrightarrow I$. So $(ye_1, -xe_1) = (y, 0, -x, 0)$, $(ye_2, xe_2) = (0, y, 0, -x)$ are lifts of the generators of the denominator to $A^{\oplus 4}$]
$= A^{\oplus 4} / \mathrm{Span}_A((y,-x,0,0), (0,0,y,-x), (y,0,-x,0), (0,y,0,-x))$.

1.3) Generators of tensor product.

Not every element of $M_1 \otimes_A M_2$ has the form $m_1 \otimes m_2$ (we'll call such elements elementary tensors or tensor monomials.)

Exercise: $A = \mathbb{F}$ (a field), $M_1 = \mathbb{F}^{\oplus k}$, $M_2 = \mathbb{F}^{\oplus \ell}$. By Example 1,

$M_1 \otimes_{\mathbb{F}} M_2 = \{ \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} a_{ij}\, e_i \otimes e_j \} \cong \{ k \times \ell\text{-matrices} \}$. Show that under this identification, tensor monomials correspond to rk 1 matrices.

However, $m_1 \otimes m_2$ span (generate) A-module $M_1 \otimes_A M_2$.

Lemma: If $M_k = \mathrm{Span}_A(m_k^i \mid i \in I_k)$ $k=1,2$, then
$M_1 \otimes_A M_2 = \mathrm{Span}_A(m_1^i \otimes m_2^j \mid i \in I_1, j \in I_2)$.

Proof: $N := M_1 \otimes_A M_2 / \mathrm{Span}_A(m_1^i \otimes m_2^j)$ $\rightsquigarrow$ linear map $\tilde{\beta}: M_1 \otimes_A M_2 \twoheadrightarrow N \rightsquigarrow$ bilinear map $\beta: M_1 \times M_2 \to N$
$\beta(m_1^i, m_2^j) = \tilde{\beta}(m_1^i \otimes m_2^j) = 0$. Since m_1^i span M_1, m_2^j span M_2
$\beta(m_1^i, m_2^j) = 0$ $\forall i,j \Rightarrow \beta = 0$. By the universal property of the tensor product, this implies $\tilde{\beta} = 0$. So $N = \{0\}$ □

1.4) Tensor products of linear maps & functoriality.

Let M_1, M_1', M_2, M_2' be A-modules & $\varphi_i : M_i' \to M_i$ be A-linear maps.

Goal: define A-linear map $\varphi_1 \otimes \varphi_2 : M_1' \otimes_A M_2' \to M_1 \otimes_A M_2$.

Consider:
$$\begin{array}{ccc} M_1' \times M_2' & \longrightarrow & M_1 \otimes_A M_2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (m_1', m_2') & \longmapsto & \varphi_1(m_1') \otimes \varphi_2(m_2') \end{array}$$

Exercise: This map is A-bilinear.

So it gives rise to an A-linear map $M_1' \otimes_A M_2' \longrightarrow M_1 \otimes_A M_2$ denoted by $\varphi_1 \otimes \varphi_2$: $\varphi_1 \otimes \varphi_2 (m_1' \otimes m_2') = \varphi_1(m_1') \otimes \varphi_2(m_2')$ on the generators $m_1' \otimes m_2'$ of $M_1' \otimes_A M_2'$.

Properties of tensor products of maps:

• $id_{M_1}\otimes id_{M_2} = id_{M_1\otimes_A M_2}$.

• Compositions: $M_1'' \xrightarrow{\varphi_1'} M_1' \xrightarrow{\varphi_1} M_1$, $M_2'' \xrightarrow{\varphi_2'} M_2' \xrightarrow{\varphi_2} M_2$
$(\varphi_1\varphi_1')\otimes(\varphi_2\varphi_2') = (\varphi_1\otimes\varphi_2)(\varphi_1'\otimes\varphi_2')$ b/c they coincide on generators $m_1''\otimes m_2''$ of $M_1''\otimes_A M_2''$.

So: we have tensor product functor
$$A\text{-Mod}\times A\text{-Mod} \longrightarrow A\text{-Mod}$$

Important exercise: Prove that $(\varphi_1,\varphi_2)\mapsto \varphi_1\otimes\varphi_2$:
$\mathrm{Hom}_A(M_1',M_1)\times \mathrm{Hom}_A(M_2',M_2) \longrightarrow \mathrm{Hom}_A(M_1'\otimes_A M_2', M_1\otimes_A M_2)$
is A-bilinear (hint: check on generators of $M_1'\otimes_A M_2'$)

We will mostly be interested in situation when fix one of the modules: $M_1 := L \rightsquigarrow$ functor $L\otimes_A \bullet : A\text{-Mod}\to A\text{-Mod}$
$M \mapsto L\otimes_A M$
$\varphi: M'\to M \mapsto id_L\otimes\varphi: L\otimes_A M' \longrightarrow L\otimes_A M$.
This is a functor $A\text{-Mod}\longrightarrow A\text{-Mod}$.

BONUS 1: Tensor products over noncommutative rings.
Let A be a comm've ring & R be an A-algebra (associative & but perhaps non-commutative). Recall that it makes sense to talk about left & right R-modules & also about bimodules. Also (compare to Bonus of Lec 3)

for two left R-modules M_1, M_2, the Hom set $\mathrm{Hom}_R(M_1, M_2)$ is only an A-module, not an R-module.

As for tensor products, we can tensor left R-modules w. right R-modules. Namely, let M be a left R-module & N be a right R-module. For an A-module L consider the set $\mathrm{Bilin}_R(N \times M, L)$ consisting of all A-bilinear maps $\varphi: N \times M \to L$ s.t. in addition $\varphi(nr, m) = \varphi(n, rm)$ $\forall\, r \in R, n \in N, m \in M$.

Definition: $N \otimes_R M \in \mathrm{Ob}(A\text{-Mod})$ represents the functor $\mathrm{Bilin}_R(N \times M, \bullet): A\text{-Mod} \longrightarrow \mathrm{Sets}$.

Important exercise: If R is comm've, then this definition gives the same as the definition in Lec 15.

To construct $N \otimes_R M$ we can use the same construction as we did in the lecture. Alternatively, $N \otimes_R M$ is the quotient of $N \otimes_A M$ by the A-submodule $\mathrm{Span}_A(nr \otimes m - n \otimes rm \mid n \in N, m \in M, r \in R)$.

Now suppose we have 2 more A-algebras, S and T. Let N be an S-R-bimodule & M be an R-T-bimodule.

Important exercise: $\exists!$ S-T-bimodule str've on $N \otimes_R M$ s.t. $s(n \otimes m) = sn \otimes m$, $(n \otimes m)t = n \otimes mt$.