§7.3 & 8.1: Screening operators, cont'd & describing the center of the affine vertex algebra

Review of the big picture thus far:

We have worked hard to give a "free field realization" homomorphism of vertex algebras

$$\omega_{\kappa_c} : V_{\kappa_c}(\mathfrak{g}) \longrightarrow M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$$

where

- $M_{\mathfrak{g}} = \mathbb{C}[a_{\alpha,n}, a^*_{\alpha,m}]_{n<0,\, m\leq 0}$ is the Fock representation of the Weyl algebra $\mathcal{A}$.
- $V_0(\mathfrak{h})$ = commutative vertex algebra associated to $L\mathfrak{h}$

This is the vertex algebra version of the affine analogue of the map

$$\tilde{\rho} : U(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}} D(N_+).$$

Recall in the fin.dim. case, this $\tilde{\rho}$ can be used to describe the center, by showing $\tilde{\rho}(Z(\mathfrak{g}))$ lands in the first factor, and also in its W-invariants. (See Exercise 2.11(1) in Daishi's notes.)

Now back to the affine case. Frenkel sets up in §7.1 the following plan, which Daniil reviewed for us:

✓ Done by Daniil ✓

→ STEP 1: Show ω_{κ_c} is injective.

→ STEP 2: Show $\mathfrak{z}(\hat{\mathfrak{g}}) \subset V_{\kappa_c}(\mathfrak{g})$ maps to $V_0(\mathfrak{h}) \subset M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$

(related to the operator $\widetilde{S}_R$ constructed in Daniil's talk)

STEP 3: We'll construct the screening operators $\overline{S}_i$ $i=1,\dots,\ell$ from $W_{0,\kappa_c} = M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$ to some other modules, which commute with the action of $\hat{\mathfrak{g}}_{\kappa_c}$.

Our focus will be Steps 3 & 4

STEP 4 We'll show $\omega_{\kappa_c}(V_{\kappa_c}(\mathfrak{g}))$ is contained in

$$\bigcap_{i=1}^{\ell} \ker(\overline{S}_i)_{\mathfrak{g}} \Longrightarrow \boxed{\omega_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}})) \subset \bigcap_{i=1}^{\ell} \ker(\overline{V}_i[1]) \quad \overline{V}_i[1] \leftarrow \overline{S}_i|_{\pi_0}.} \quad (1)$$

We will also work toward Step 5, to be completed in Vasya's talk.

STEP 5 Show above inclusion (1) is equality.

STEP 6 use Miura opers to identify RHS of (1) with $\mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D)$.

Important ingredients from previous talks:

Daniil constructed a "screening operator of the first kind"

$$S_R : W_{0,\kappa} \longrightarrow W_{-2,\kappa}$$

along with other "screening operators of the second kind" $\widetilde{S}_R$.
We will extend this $\mathfrak{sl}_2$ statement to arbitrary $\mathfrak{g}$ with Steps 3 & 4 above as our goal.

Recall also that Kenta's talk gives the following:
For $\mathfrak{p} \subset \mathfrak{g}$ parabolic with Levi $\mathfrak{m}$, there is an exact functor between smooth modules for $\hat{\mathfrak{m}}$ and $\hat{\mathfrak{g}}$ sending

$$W_{\lambda, \kappa|_{\mathfrak{m}} + \kappa_c(\mathfrak{m})} \longmapsto W_{\lambda, \kappa + \kappa_c(\mathfrak{g})}.$$

(it comes from the parabolic free field realization, as in Thm 1.25 of Kenta's notes)

This will allow us to produce homomorphisms on the RHS from those on the LHS. $(*)$

7.3.1 (Goal here: use screening operators for $\widehat{sl}_2$ to build ones for $\hat{\mathfrak{g}}_\kappa$!)

For $i \in \{1, \dots, \ell\}$, let

$$sl_2^{(i)} = \langle e_i, h_i, f_i \rangle \subset \mathfrak{g}$$
$$\mathfrak{p}^{(i)} = \langle \mathfrak{b}_-, e_i \rangle \subset \mathfrak{g}$$
$$\mathfrak{m}^{(i)} = sl_2^{(i)} \oplus \mathfrak{h}_i^\perp = \text{Levi subalg of } \mathfrak{p}^{(i)}.$$

orthogonal complement of h_i in $\mathfrak{h}$

Recall semi-infinite parabolic induction:

$$\mathrm{Wak}^{\mathfrak{g}}_{\mathfrak{p}^{(i)}} : \left\{ \begin{matrix} \text{smooth reps of } \widehat{sl}_{2,k} \oplus \hat{\mathfrak{h}}^\perp_{i,\kappa_0} \\ \text{w/ } k \text{ and } \kappa_0 \text{ satisfying} \\ \text{some conditions} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{smooth} \\ \hat{\mathfrak{g}}_{\kappa+\kappa_c} \text{ modules.} \end{matrix} \right\}$$

special case of $(*)$

The conditions on k & κ_0: $(\kappa - \kappa_c)(h_i, h_i) = 2(k+2)$

$$\kappa_0 = \kappa|_{\mathfrak{h}_i^\perp}.$$

For R a smooth $\widehat{sl}_2$-module of level k,

L a smooth $\hat{\mathfrak{h}}^\perp_\kappa$-module,

$M_{\mathfrak{g},\mathfrak{p}^{(i)}} \otimes R \otimes L$ is a smooth $\hat{\mathfrak{g}}_{\kappa+\kappa_c}$-module.

Letting R be the Wakimoto module $W_{\lambda,k}$ over sl_2,

L the Fock rep. $\pi^\kappa_{\lambda_0}$,

the corresponding $\hat{\mathfrak{g}}_\kappa$-module is isom. to $\boxed{W_{(\lambda,\lambda_0),\kappa+\kappa_c}}$

weight of $\mathfrak{g}$ built from λ & λ_0.

So we have:

Prop. Any intertwining operator $a: W_{\lambda_1,k} \longrightarrow W_{\lambda_2,k}$ over $\widehat{sl}_2$ gives an intertwining operator

$$\mathrm{Wak}^{\mathfrak{g}}_{\mathfrak{p}^{(i)}}(a) : W_{(\lambda_1,\lambda_0),\kappa+\kappa_c} \longrightarrow W_{(\lambda_2,\lambda_0),\kappa+\kappa_c} \text{ over } \hat{\mathfrak{g}}_{\kappa+\kappa_c}$$

for any weight λ_0 of $\mathfrak{h}_i^\perp$.

We will also need the following formula.

Recall from Sec 1.3 of Ivan's 1st CDO Note, the morphism

$$\iota : V(\mathfrak{n}_+) \longrightarrow CDO(N_+) \quad \left(\begin{array}{c}\text{Corresp. to the right action}\\ \text{of } N_+ \text{ on itself}\end{array}\right)$$

For all i, let $e^R_{i,-1}$ be the image of $e_{i,-1}$ under this map.

Def Let $e^R_i(z) = Y(e^R_{i,-1}, z)$, a field for the vertex alg. $CDO(N_+)$.
(Alternatively, in Frenkel: $e^R_i(z) = w^R(e_i(z))$, where
$w^R : L\mathfrak{n}_+ \longrightarrow \mathcal{A}^{\mathfrak{g}}_{\leq 1, loc}$ induced by right action of $\mathfrak{n}_+$ on N_+).

Exercise: Show that for a general coordinate system on N_+, we have

$$e^R_i(z) = a_{\alpha_i}(z) + \sum_{\beta \in \Delta_+} P^{R,i}_\beta (a^*_\alpha(z))\, a_\beta(z)$$

(c.f. thm 6.2.1 in Zeyu's notes!)

7.3.2 If $\mathfrak{h}$ is any abelian Lie alg. with nondegenerate inner prod. κ, we can identify $\mathfrak{h} \simeq \mathfrak{h}^*$ via κ. Then let $\hat{\mathfrak{h}}_\kappa$ be the Heisenberg Lie alg, π^κ_λ $\lambda \in \mathfrak{h}^* \simeq \mathfrak{h}$ its Fock reps. Then for any $\chi \in \mathfrak{h}^*$, we can define $V^\kappa_\chi(z) : \pi^\kappa_0 \to \pi^\kappa_\chi$ by

$$V^\kappa_\chi(z) = T_\chi \exp\left(-\sum_{n<0} \frac{\chi_n}{n} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{\chi_n}{n} z^{-n}\right).$$

Of course, we will use this definition for $\mathfrak{h}$ the Cartan as in previous sections.

Def For $\kappa \neq \kappa_c$, let

$$S_{i,\kappa} = e^R_i(z) V^{\kappa-\kappa_c}_{-\alpha_i}(z) : W_{0,\kappa} \longrightarrow W_{-\alpha_i,\kappa}$$

$\left(= Y_{W_{0,\kappa}, W_{-\alpha_i,\kappa}}(e^R_{i,-1}|-\alpha_i\rangle, z)\right.$ in the notation of Daniil's talk).

And let
$$S_{i,\kappa} = \int S_{i,\kappa}(z)\,dz : W_{0,\kappa} \longrightarrow W_{-\alpha_i,\kappa}.$$
"the i^{th} screening operator of the first kind"

By Proposition 2.4 in Daniil's talk, $S_{i,\kappa}$ is induced by the screening operator S_κ for the ith $\widehat{sl}_2$ subalgebra, with κ satisfying $(\kappa-\kappa_c)(h_i,h_i) = 2(\kappa+2)$. It also implies:

Proposition $S_{i,\kappa}$ is an intertwining operator between $W_{0,\kappa}$ and $W_{-\alpha_i,\kappa}$ for each $i = 1,\dots,\ell$.

We won't use this next result, but it's useful for intuition as a step toward our main result later:

Proposition For generic κ, $V_\kappa(\mathfrak{g})$ is equal to the intersection of the kernels of $S_{i,\kappa}$ $i = 1,\dots,\ell$.

7.3.3 We now approach defining screening operators of the second kind for $\hat{\mathfrak{g}}$. To do so, we'll need to make sense of $(e^R_i(z))^\gamma$ for $\gamma \in \mathbb{C}$.

First, fixing i, we can choose coordinates in N_+ s.t. $e^R_i(z) = a_{\alpha_i}(z)$ (we can get this naturally if we define Wakimoto modules over $\hat{\mathfrak{g}}$ via semiinfinite parabolic induction from the ith $\widehat{sl}_2$). Concretely, we choose coords $\{y_\alpha\}_{\alpha\in\Delta_+}$ on N_+ such that $\rho^R(e_i) = \partial/\partial y_{\alpha_i}$ where $\rho^R: \mathfrak{n}_+ \to D_{\le 1}(N_+)$ corresp. to $\underset{\text{right action}}{N_+ \circlearrowleft \mathfrak{n}_+}$.

Now, recall the Friedan-Martinec-Shenker bosonization of the Weyl algebra generated by $a_{\alpha_i,n}, a^*_{\alpha_i,n}$ $n\in\mathbb{Z}$.

Def Let
$$\widetilde{W}^{(i)}_{0,0,\kappa} = \mathrm{Wak}^{\mathfrak{g}}_{p^{(i)}}(\widetilde{W}_{0,\gamma,\kappa})$$

which is a $\hat{\mathfrak{g}}_\kappa$-module containing $W_{\lambda,\kappa}$ if $\gamma = 0$, via homomorphism $V_\kappa(\mathfrak{g}) \xrightarrow{w_{\kappa_c}} M_\mathfrak{g} \otimes \Pi_0 \longrightarrow M^{(i)}_\mathfrak{g} \otimes \widetilde{W}_{0,0,\kappa}$.

More generally, by replacing the Fock representation Π_0 with Π_γ $\gamma \in \mathbb{C}^{\prime}$ and π^κ_0 with π^κ_λ, we get a modified $\hat{\mathfrak{g}}$-action giving rise to a module $\widetilde{W}^{(i)}_{\lambda,\gamma,\kappa}$ for all λ, γ.

Def Let $\beta = \frac{1}{2}(\kappa - \kappa_c)(h_i, h_i)$ and define
$$\widetilde{S}_{i,\kappa}(z) = (e^R_i(z))^{-\beta} V_{\check{\alpha}_i}(z) : \widetilde{W}_{0,0,\kappa} \longrightarrow \widetilde{W}_{-\beta,\beta\check{\alpha}_i,\kappa}.$$

(well-defined as a map $\Pi_0 \to \Pi_{-\beta}$ by def of F-M-S bosonization)

$\check{\alpha}_i = h_i \in \mathfrak{h}$
ith coroot of $\mathfrak{g}$

Let $\widetilde{S}_{i,\kappa} = \int \widetilde{S}_{i,\kappa}(z)\,dz$.

As for $S_{i,\kappa}$, $\widetilde{S}_{i,\kappa}$ is induced by $\widetilde{S}_\mathcal{K}$ for the ith $\mathfrak{sl}_2$, where $\mathcal{K} = (\kappa - \kappa_c)(h_i, h_i) = 2(\mathcal{K}+2)$

Prop. The operator $\widetilde{S}_{i,\kappa}$ is an intertwining operator of $\hat{\mathfrak{g}}_\kappa$-modules
$$\widetilde{W}^{(i)}_{0,0,\kappa} \longrightarrow \widetilde{W}^{(i)}_{-\beta,\beta\check{\alpha}_i,\kappa}$$

Prop. For generic κ, $V_\kappa(\mathfrak{g})$ is the intersection of the kernels of $\widetilde{S}_{i,\kappa} : W_{0,\kappa} \longrightarrow \widetilde{W}^{(i)}_{-\beta, \beta\check{\alpha}_i, \kappa}$ $i = 1, \dots, \ell$.

Again, we won't use or prove this.

Exercise. Show that this intersection is a vertex subalgebra of $W_{0,\kappa}$.