The De Concini-Kac Form of $U_q(\mathfrak{g})$

- v an indeterminate $/\mathbb{C}$
- $k = \mathbb{C}(v)$ field of fractions of $R = \mathbb{C}[v, v^{-1}]$
- $\mathfrak{g}$ a complex s/s Lie algebra
- $\Pi \subset \Phi^+ \subset \Phi$ roots of $\mathfrak{g}$.
- $d_\alpha = (\alpha, \alpha)/2 \in \{1, 2, 3\}$.

As in Jantzen $U_k = U_k(\mathfrak{g})$ is the k-algebra with gens $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ (for $\alpha \in \Pi$) and relations

$$\boxed{\begin{aligned} K_\alpha K_\alpha^{-1} &= 1 = K_\alpha^{-1} K_\alpha \\ K_\alpha K_\beta &= K_\beta K_\alpha \\ K_\alpha E_\beta K_\alpha &= v^{(\alpha,\beta)} E_\beta \\ K_\alpha F_\beta K_\alpha^{-1} &= v^{-(\alpha,\beta)} F_\beta \\ E_\alpha F_\beta - F_\beta E_\alpha &= \delta_{\alpha\beta} [K_\alpha; 0] \\ &\vdots \end{aligned}}$$

Recall that for any $a \in \mathbb{Z}$ we have

$$[K_\alpha; a] = \frac{K_\alpha v_\alpha^a - K_\alpha^{-1} v_\alpha^{-a}}{v_\alpha - v_\alpha^{-1}}$$

where $v_\alpha = v^{d_\alpha}$.

Definition

The De Concini-Kac form of U_k is the R-subalgebra $U_R = U_R(\mathfrak{g})$ generated by $E_\alpha, F_\alpha, K_\alpha^{\pm}, [K_\alpha; 0]$ (for $\alpha \in \Pi$).

Remark

As in Jantzen we can work more generally with a lattice $\mathbb{Z}\Phi \subset \Gamma \subset \Lambda$. However, for simplicity we will just considered this simply connected type with $\Gamma = \mathbb{Z}\Phi$.

Remark

The algebra U_R appears implicitly in Lemma 5.12 of [J].

We have $U_k \cong k \otimes U_R$.

Given $\varepsilon \in \mathbb{C}^\times$ let $\mathbb{C}_\varepsilon$ be the R-mod with $v \cdot c = \varepsilon c$ for all $c \in \mathbb{C}$. Then we define

$$U_\varepsilon = \mathbb{C}_\varepsilon \otimes_R U_R,$$

which is a $\mathbb{C}$-algebra.

In this setup we can take $\varepsilon = \pm 1$. If this is the case then for each $\alpha \in \Pi$ we have $K_\alpha \in Z(U_\varepsilon)$ and $K_\alpha^2 = 1$. Moreover, if $H_\alpha = [K_\alpha; 0]$ then

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} H_\alpha.$$

Proposition

We have $U_1 / \langle K_\alpha - 1 \mid \alpha \in \Pi \rangle$ is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Proof ($\mathfrak{g} = \mathfrak{sl}_2$): Just consider the case of $\mathfrak{sl}_2$. We drop the subscripts α. To see that $HE - EH = 2E$ apply the formulae:

- $v^2 [K; 2] = [K; 0] + v(v^2+1) K$
- $[K; a] E = E [K; a+2]$ any $a \in \mathbb{Z}$.

Similarly for $HF - FH = -2F$. $\square$

Recall that we have the following commutator formulas

- $[E_\alpha, F_\alpha^s] = [s]_\alpha F_\alpha^{s-1} [K_\alpha; d_\alpha(1-s)]$
- $[E_\alpha^s, F_\alpha] = [s]_\alpha [K_\alpha; d_\alpha(1-s)] E_\alpha^{s-1}$

for all $s > 0$. As usual $[a] = \dfrac{v^a - v^{-a}}{v - v^{-1}}$ for all $a \in \mathbb{Z}$ and $[a]_\alpha = [a]_{v = v_\alpha}$.

Let $U_R^\pm \subseteq U_{\mathcal{R}}^\pm$ be the R-subalgebras generated by E_α, in the $+$ case, and F_α, in the $-$ case, with $\alpha \in \Pi$. Similarly $U_R^0 \subset U_{\mathcal{R}}^0$ is the R-subalgebra gen. by $K_\alpha^{\pm 1}$ and $[K_\alpha;0]$ with $\alpha \in \Pi$. Let $U_\varepsilon^\pm$ and U_ε^0 be the images in U_ε.

Remark

The quotient map $U_1 \to U_1/\langle K_\alpha - 1 \mid \alpha \in \Pi \rangle$ maps the subalgebras $U_1^\pm$ isomorphically onto the universal enveloping algebra of a Lie algebra. Hence $U_1^\pm$ has no zero divisors. This can be used to conclude that U_R has no zero divisors.

Centre of $U_\varepsilon(\mathfrak{sl}_2)$

$\boxed{\text{Assume now that } \varepsilon^2 \neq 1}$

We have $Z(U_R)$ is generated by the quantum Casimir element

$$C = FE + (v - v^{-1})^{-1}[K;1].$$

Specialising $v \to \varepsilon$ gives an element of $Z(U_\varepsilon)$.

Assume ε is a primitive ℓ^{th} root of unity. Let ℓ' be the least positive integer satisfying $[\ell']_{v=\varepsilon} = 0$. Then

$$\ell' = \begin{cases} \ell/2 & \text{if } \ell \text{ is even} \\ \ell & \text{if } \ell \text{ is odd.} \end{cases}$$

Proposition

When $\mathfrak{g} = \mathfrak{sl}_2$ we have $E^{\ell'}, F^{\ell'}, K^{\ell'} \in Z(U_\varepsilon)$.

Proof: For $K^{\ell'}$ this follows straight from the defining relations. For $E^{\ell'}$ and $F^{\ell'}$ use the prior commutator formulas together with $[\ell']_{v=\varepsilon} = 0$. $\square$

Remark

We actually have $Z(U_\varepsilon)$ is gen. by C, $E^{\ell'}$, $F^{\ell'}$, $K^{\pm \ell'}$.

The algebra gen. by $E^{\ell'}, F^{\ell'}, K^{\pm\ell'}$, is the analogue of the p-centre and that gen. by c is the Harish-Chandra centre.

The Centre In General

> We assume $\varepsilon \in \mathbb{C}^{\times}$ is a primitive ℓ^{th} root of unity with ℓ odd and $\ell > d_\alpha$ for all $\alpha \in \Pi$

In this case $Z(U_\varepsilon)$ has been described by De Concini-Kac-Processi (DKP). The case where ℓ is even is treated by Beck.

Lemma
The elements $E_\alpha^\ell, F_\alpha^\ell$ with $\alpha \in \Pi$ and K_β^ℓ with $\beta \in \mathbb{Z}\Phi$ are in the centre $Z(U_\varepsilon)$.

Let $\mathcal{B}$ be the braid group. The action of $\mathcal{B}$ on U_R stabilises the subalgebra U_R. Hence we get an action of $\mathcal{B}$ on U_ε. We define

$$Z_0 \subseteq Z(U_\varepsilon)$$

to be the smallest $\mathcal{B}$-invariant subalgebra containing the elements

- $y_\alpha = (\varepsilon^{d_\alpha} - \varepsilon^{-d_\alpha})^\ell F_\alpha^\ell$ with $\alpha \in \Pi$
- $z_\alpha = K_\alpha^\ell$ with $\alpha \in \mathbb{Z}\Phi$.

This is DKP's analogue of the p-centre.

We have a subalgebra

$$Z_0^0 \subseteq Z_0$$

with basis z_α $(\alpha \in \mathbb{Z}\Phi)$.

Fix a reduced expression $w_0 = s_{\alpha_1} \cdots s_{\alpha_N}$ of the longest element. For $1 \leq k \leq N$ let

$$\boxed{y_k = T_{\alpha_1} \cdots T_{\alpha_{k-1}} (y_{\alpha_k})}$$

We have a Z_0^0-subalgebra $Z_0^- \subseteq Z_0$ gen. by the elements $y_1, \dots, y_N$.

Recall that we have a bijection $\rho : \Pi \to \Pi$ given by $\rho(\alpha) = -w_0(\alpha)$. This gives a new reduced expression $w_0 = s_{\rho(\alpha_1)} \cdots s_{\rho(\alpha_N)}$ of w_0. For $1 \leq k \leq N$ we set

$$\bar{y}_k = T_{\rho(\alpha_1)} \cdots T_{\rho(\alpha_{k-1})} (y_{\rho(\alpha_k)}),$$

and $x_k = T_{w_0}(\bar{y}_k)$, where $T_{w_0} = T_{\alpha_1} \cdots T_{\alpha_N}$. If $Z_0^+ = T_{w_0}(Z_0^-)$ then DKP have shown,

Theorem

The algebra Z_0 is the tensor product $Z_0^- \otimes Z_0^0 \otimes Z_0^+$ and as a Z_0^0-algebra it is the poly. ring

$$Z_0^0[y_k, x_k \mid 1 \leq k \leq N].$$

For the Harish-Chandra centre recall that we have an iso.

$$Z(U_k) \longrightarrow (U_{k,ev}^0)^W$$

Specialising we get an inj. hom.

$$(U_{\varepsilon,ev}^0)^W \longrightarrow Z(U_\varepsilon)$$

whose image is denoted by Z_1.

Theorem

The natural product map

$$Z_0 \otimes_{Z_0 \cap Z_1} Z_1 \longrightarrow Z(U_\varepsilon)$$

is an isomorphism.

Poisson Structures

Let A be a $\mathbb{C}$-algebra. A poisson bracket on A is a bilinear map

$$\{-,-\} : A \times A \to A$$

is a Lie bracket satisfying

$$\{ab, c\} = a\{b,c\} + \{a,c\}b.$$

for all $a, b, c \in A$.

Remark: As the Lie bracket is alternating some authors, such as DKP, consider the bracket to be defined on the 2nd exterior power $\Lambda^2(A)$.

We get a natural Poisson bracket on $A \otimes A$ by setting

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}$$

If A is a Hopf algebra then A is said to be a Poisson Hopf algebra if the coproduct $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to \mathbb{C}$ are hom. of Poisson algebras.

Let $\mathfrak{m} = \operatorname{Ker}(\varepsilon)$ be the augmentation ideal. Then

$$\{\mathfrak{m}, \mathfrak{m}\} \subset \mathfrak{m}$$

So $\{-,-\}$ induces a Lie bracket on $\mathfrak{m}/\mathfrak{m}^2$.

Let $\sigma : A \otimes A \to A \otimes A$ be the permutation map $\sigma(a \otimes b) = b \otimes a$ and let $\delta = \Delta - \sigma\Delta$. If $\mathfrak{m} = \operatorname{Ker}(\varepsilon)$ then $\delta(\mathfrak{m}) \subset \mathfrak{m} \otimes \mathfrak{m}$. Thus we get a map

$$\delta : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \otimes \mathfrak{m}/\mathfrak{m}^2$$

This is a 1-cocycle of the Lie algebra m/m^2. Moreover, the cotrangradient map

$$\delta^* : (m/m^2)^* \otimes (m/m^2)^* \longrightarrow (m/m^2)^*$$

defines a Lie bracket on the dual space. The pair $(m/m^2, \delta)$ is an example of a Lie bialgebra.

Example

Assume $(H_\alpha, X_{\pm\beta} \mid \alpha \in \Pi, \beta \in \Phi^+)$ is a Chevalley basis of $\mathfrak{g}$. Then one has a standard Lie bialgebra structure $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ given by

$$\delta(H_\alpha) = 0 \quad \text{and} \quad \delta(X_{\pm\alpha}) = d_\alpha X_{\pm\alpha} \otimes H_\alpha$$

any $\alpha \in \Pi$. This is most easily proved using the notion of a Manin triple which is a reformulation of the notion of Lie bialgebra.

The Poisson Structure on Z_0

Let $\phi : U_R \to U_\varepsilon$ be the map s.t. $\phi(u) = 1 \otimes u$. Following DKP we define a bracket on Z_0 by setting

$$\{a, b\} = \frac{(\hat{a}\hat{b} - \hat{b}\hat{a})}{\ell(v^\ell - v^{-\ell})}$$

where $\hat{a}, \hat{b} \in U_R$ satisfy $\phi(\hat{a}) = a$ and $\phi(\hat{b}) = b$.

The subalgebra $Z_0 \subseteq U_\varepsilon$ is a Hopf subalgebra and the bracket makes this a Poisson Hopf algebra. The normalisation of the bracket is chosen as it relates to the standard example given above.

An infinite dimensional group

If $b \in U_R$ is an element with $[b,a] \in [\ell] U_R$ for all $a \in U_R$ then the image of b in U_ε is central! However, if $D_b : U_R \to U_R$ is the map given by

$$D_b(a) = [b/[\ell], a]$$

then the specialisation gives a derivation of U_ε.

Recall the divided powers

$$E_\alpha^{(s)} = E_\alpha^s / [s]_{d_\alpha}!$$

for any $\alpha \in \Pi$ and integer $s \geq 0$. Applying the above remarks we get derivations $\underline{e}_\alpha$ and $\underline{f}_\alpha$ of U_ε given by specialising

$$D = [E_\alpha^{(\ell)}, -] \quad \text{and} \quad T_\alpha D T_\alpha^{-1}$$

respectively.

We want to exponentiate these derivations to get automorphisms of U_ε. But for this we must enlarge U_ε.

Let $\hat{Z}_0$ be the algebra of formal power series in the elements $E_\alpha^\ell, F_\alpha^\ell$ ($\alpha \in \Phi^+$) and $K_\alpha^{\pm \ell}$ ($\alpha \in \Delta$) that define holomorphic functions

$$\mathbb{C}^{2N} \times (\mathbb{C}^\times)^n \longrightarrow \mathbb{C}.$$

Proposition

If $\hat{U}_\varepsilon = U_\varepsilon \otimes_{Z_0} \hat{Z}_0$ then the series $\exp(t\underline{e}_\alpha)$ and $\exp(t\underline{f}_\alpha)$, for $\alpha \in \Pi$, converge to well defined automorphisms of $\hat{U}_\varepsilon$.

The group $\mathcal{G} \leq \mathrm{Aut}(\hat{U}_\varepsilon)$ generated by $\exp(t\underline{e}_\alpha)$ and $\exp(t\underline{f}_\alpha)$ is infinite dimensional and acts on $Z(U_\varepsilon)$.

Theorem

(i) The specialisation of the Harish-Chandra hom. gives an isomorphism

$$Z(U_\varepsilon)^{G} \longrightarrow (U^{\circ}_{ev,\varepsilon})^{W}$$

(ii) $Z_1 = Z(U_\varepsilon)^{G}$

(iii) The orbits of G on $\mathrm{Spec}(Z_0)$ are the symplectic leaves coming from the Poisson structure.

The Lusztig Form of $U_q(\mathfrak{g})$

We now let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Lusztig's integral form of the quantum group is the $\mathcal{A}$-subalgebra $U_{\mathcal{A}} \subseteq U_{\mathbb{K}}$ generated by the elements

$$E_\alpha^{(s)}, \; F_\alpha^{(s)}, \; K_\alpha^{\pm 1}, \quad \text{with } \alpha \in \Pi \text{ and } s \in \mathbb{Z}.$$

Denote by $U_{\mathcal{A}}^+$ and $U_{\mathcal{A}}^-$ the $\mathcal{A}$-subalgebras of $U_{\mathbb{K}}$ generated by all $E_\alpha^{(s)}$ and $F_\alpha^{(s)}$ respectively.

A Basis of $U_{\mathcal{A}}$

For any $\alpha \in \Pi$ and $a, n \in \mathbb{Z}$ with $n \geq 0$ set

$$\begin{bmatrix} K_\alpha ; a \\ n \end{bmatrix} = \prod_{c=1}^{n} \frac{K_\alpha v_\alpha^{a-c+1} - K_\alpha^{-1} v_\alpha^{-(a-c+1)}}{v_\alpha^{c} - v_\alpha^{-c}}$$

Taking $n=1$ we get $[K_\alpha ; a]$.

We now follow Calder's talks. Fix a reduced expression $\underline{w}_0 = (s_{\alpha_1}, \dots, s_{\alpha_N})$ of the longest element $w_0 = s_{\alpha_1} \cdots s_{\alpha_N}$.

For $1 \leq j \leq N$ let $\beta_j = w_j(\alpha_j)$, where $w_j = s_{\alpha_1} \cdots s_{\alpha_{j-1}}$, and let

$$E_j^{(s)} = T_{w_j}(E_{\alpha_j}^{(s)}) \quad \text{and} \quad F_j^{(s)} = T_{w_j}(F_{\alpha_j}^{(s)})$$

for any $s \in \mathbb{Z}$. Recall $\Phi^+ = \{\beta_1, \dots, \beta_N\}$ is now totally ordered.

Theorem

Pick a reduced expression $\underline{w}_0$ of w_0. Then $U_{\mathcal{A}}$ has an $\mathcal{A}$-basis given by terms

$$\prod_{j=1}^{N} F_{N+1-j}^{(n_j)} \prod_{\alpha \in \Pi} \left(K_\alpha^{\delta_\alpha} \begin{bmatrix} K_\alpha ; 0 \\ t_\alpha \end{bmatrix} \right) \prod_{j=1}^{N} E_j^{(m_j)}$$

with $n_j, m_j, t_\alpha \geq 0$ and $\delta_\alpha \in \{0,1\}$. Moreover,

$$U_{\mathcal{A}}^+ = \mathrm{Span}_{\mathcal{A}} \left\{ \prod_{j=1}^{N} E_j^{(n_j)} \right\} \quad \text{and} \quad U_{\mathcal{A}}^- = \mathrm{Span}_{\mathcal{A}} \left\{ \prod_{j=1}^{N} F_{N+1-j}^{(m_j)} \right\}.$$

Remarks (on Proof)

- The main step is to prove that we have a homomorphism $\mathcal{B} \to \mathrm{Aut}(U_{\mathcal{R}})$, which we saw in Calder's talks.
- One then observes that $\mathcal{B}$ preserves the algebra $U_{\mathcal{A}}$. From this one can obtain the bases of $U_{\mathcal{A}}^{\pm}$.
- In the original papers Lusztig proves the statement for a certain choice of reduced expression $\underline{w}_0$ with favourable properties. An argument of Dyer-Lusztig shows you can then deduce the result for all choices. In Lusztig's book an argument for any choice of reduced expression is given.