Lecture 25. (updated 12/9)

1) Modules, geometrically.
2) Modules over local rings & Nakayama lemma.

Ref: [AM], Section 2.5.

1) Motivation:

One can say that the main players in Algebra are rings, their homomorphisms & their modules (ideals are also very important but they can be recovered as kernels of homomorphisms; they can also be regarded as modules). Counterparts of rings in Algebraic geometry are affine (Sect. 1.3 of Lec 24) or more general (Bonus to Lec 24) varieties. Counterparts of homomorphisms are polynomial maps between varieties. Counterparts of modules are "quasicoherent sheaves". They are obtained from modules and carry a lot of geometric information about a variety. We won't explain the concept of a sheaf here, while extremely important, it goes beyond the scope of this class.

We will concentrate on "locally free" modules.

Definition (compare to Prob 5 in HW3): let A be a ring and M be a finitely generated module. We say that M is locally free if $\exists\ f_1,\dots,f_k \in A$ s.t. $(f_1,\dots,f_k) \ni A$ & $M[f_i^{-1}]$ is free over $A[f_i^{-1}]$ $\forall i=1,\dots,k$.

For example, free $\Rightarrow$ locally free but not vice versa: the ideal $(2, 1+\sqrt{-5})$ is a locally free $\mathbb{Z}[\sqrt{-5}]$-module (Prob 5 in HW3) but not free (Prob 4, HW6).

Now suppose $\mathbb{F}$ is an algebraically closed field, $X \subset \mathbb{F}^n$ an algebraic subset & $A = \mathbb{F}[X]$. As was discussed in Sec 2.1 of Lec 24, the localization $A[f^{-1}]$ can be interpreted as the algebra of polynomial functions on the "principal open subset" $X_f = \{\alpha \in X \mid f(\alpha) \neq 0\}$. The $A[f^{-1}]$-module $M[f^{-1}]$ is viewed as the "restriction" of M from X to X_f, which explains "locally free."

Exercise: Show that the condition $(f_1, \dots, f_k) = A$ is equivalent to $X = \bigcup_{i=1}^{k} X_{f_i}$ (hint: reduce this to the last exercise of Sec 1.4 in Lec 22.

Smooth digression: A motivation for considering locally free modules comes from the "usual" geometry of manifolds.

A C^∞-manifold is glued from little balls in $\mathbb{R}^n$, coordinate neighborhoods (think about the 2-dimensional sphere S^2, for example). For a manifold, we can talk about objects such as vector fields (a collection of tangent vectors at each point that smoothly depend on the point), differential forms, metrics, etc. These are of primary importance for geometry. Let's discuss vector fields.

Let M be a manifold (e.g. S^2), $U \subset M$ be an open subset, also a manifold. Let Vect(U) be the collection of all vector fields on U. This carries the natural structure of a module over $C^\infty(U)$, the algebra of C^∞-functions on U: we multiply a vector field by a function pointwise and get a new vector field. One can view it like this: there's a geometric object, the "tangent bundle" and Vect(U) is its module of "C^∞-sections" on U. This "bundle" is "locally free": if U has coordinates $x_1, \dots x_n$, then Vect(U) is a free $C^\infty(U)$-module with basis $\partial_{x_1}, \dots, \partial_{x_n}$. It may fail to be free, for $\mathbb{S}^2$ this follows from the hairy ball theorem.

This is, essentially, the setting we have emulated above in the algebraic world with the definition of a locally free module and its algebro-geometric interpretation.

Now we get to our main result about these modules: under mild assumptions locally free is the same as projective ($\mathrm{Hom}_A(P, \cdot)$ is exact (see Sec 3.1 of Lec 21).

Definition/reminder from Prob 8 of HW3: An A-module M is **finitely presented** if $\exists$ k & surjective A-module homomorphism $A^{\oplus k} \twoheadrightarrow M$ w. finitely generated kernel.

For example, if A is Noetherian, then fin. presented $\Leftrightarrow$ fin. generated.

Recall that for a prime (e.g. maximal) ideal $\mathfrak{p} \subset A$, the subset $A \setminus \mathfrak{p} \subset A$ is multiplicative & we write $\cdot_{\mathfrak{p}}$ for $\cdot[(A\setminus\mathfrak{p})^{-1}]$.

Thm: Let M be a finitely presented A-module. TFAE:

(a) M is projective.

(b) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m} \subset A$.

If A is Noetherian, then (a) & (b) are equivalent to

(c) A is locally free.

We'll prove this in the next lecture.

Rem: In the C^∞-setting, the most naive way to think about "vector bundles" is to view them as a collection of vector spaces, one for each point of the manifold. For the "tangent bundle," these are the tangent spaces.

We have a similar picture for modules. Let $X \subset \mathbb{F}^n$ be an algebraic subset, $A = \mathbb{F}[X]$ & M be a finitely generated A-module. Recall that the points of X are in bijection w. the maximal ideals of A (the 2nd corollary in 1.4 of Lec 22). Let $x \in X$ & $\mathfrak{m} \subset A$ be the corresponding maximal ideal. Recall that $A/\mathfrak{m} = \mathbb{F}$. By the fiber of M at x we mean the vector space $M/\mathfrak{m}M$.

2) Modules over local rings & Nakayama lemma.

Let A be a local ring, i.e. a commutative ring w. unique maximal ideal $\mathfrak{m}$. For example, if B is a commutative ring & $\mathfrak{p} \subset B$ is a prime ideal, then $B_{\mathfrak{p}}$ is local.

We'll be interested in some results about modules over local rings.

2.1) Nakayama lemma

This is the most fundamental result about modules over local rings.

Thm (Nakayama lemma). Let M be a finitely generated A-module. If $\mathfrak{m}M = M$, then $M = \{0\}$.

The proof is based on the Cayley-Hamilton lemma (Lemma 1.2 from Sec. 1.2 of Lecture 9).

Lemma: Let B be a commutative ring, $I \subset B$ be an ideal, M a fin. generated B-module & $\varphi: M \to M$ a B-linear map. Suppose that $\varphi(M) \subset IM$. Then $\exists f \in A[x]$, $f(x) = x^k + a_1 x^{k-1} + \ldots + a_k$ w. $a_\ell \in I^\ell$, $\ell = 1, \ldots, k$, s.t. $f(\varphi) = 0$.

Proof of Thm: We apply Lemma to $B := A$, $I := \mathfrak{m}$, $\varphi = id$: the condition $M = \mathfrak{m}M$ means $\varphi(M) \subset IM$. We conclude $0 = f(\varphi) = (1 + a_1 + \ldots + a_k)\varphi$. Set $a = a_1 + \ldots + a_k \in \mathfrak{m}$. Note that
$1 + a \notin \mathfrak{m} \Leftrightarrow (1+a) \not\subset \mathfrak{m} \Leftrightarrow$ [$\mathfrak{m}$ is the unique max. ideal] $(1+a) = A$ $\Leftrightarrow 1+a$ is invertible. So $(1+a)\varphi = 0 \Rightarrow \varphi = 0$. But $\varphi = id_M$, so $M = \{0\}$. $\square$

Corollary: Let M be a finitely generated A-module & $m_1, \ldots m_k \in M$. Let $\overline{m}_1, \ldots, \overline{m}_k$ be the images of $m_1, \ldots, m_k$ in $M/\mathfrak{m}M$. If $\overline{m}_1, \ldots, \overline{m}_k$ span the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$, then $m_1, \ldots, m_k$ span A-module M.

Proof: Set $N := \mathrm{Span}_A(m_1, \dots m_k)$. Note that the composed map $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is surjective $\Leftrightarrow$ $M = N + \mathfrak{m}M$ $\Leftrightarrow$ the map $\mathfrak{m}M \hookrightarrow M \twoheadrightarrow M/N$ is surjective $\Leftrightarrow$ $\mathfrak{m}(M/N) = M/N$. The A-module M/N is finitely generated. Applying the Nakayama lemma, we get $M/N = \{0\} \Leftrightarrow M = N$. $\square$

2.2) Projective modules over the local ring.

We will use the Nakayama lemma to prove the following result in the next lecture.

Thm: Every finitely generated projective module over a local ring A is free.