Lecture 6.

1) Properties of Noetherian modules.
2) Artinian modules & rings.
3) Finite length modules.

References: [AM], Chapter 6, Chapter 7, introduction.

1) The following proposition was stated in Sec 3 of Lec 5.

Proposition: Let M be A-module, $N\subset M$ be a submodule
TFAE (1) M is Noetherian
(2) Both N, M/N are Noetherian.

Proof: (1) $\Rightarrow$ (2): M is Noetherian $\Rightarrow N$ is Noeth'n (tautology)
Check M/N is Noetherian by verifying that $\forall$ AC of submod's of M/N terminates. Let $\pi: M \twoheadrightarrow M/N$, $m\mapsto m+N$.
Let $(\underline{N}_i)_{i\geqslant 0}$ be an AC of submodules in M/N, $N_i=\pi^{-1}(\underline{N}_i)$
$\underline{N}_i\subset \underline{N}_{i+1} \Rightarrow N_i\subset N_{i+1}$ so $(N_i)_{i\geqslant 0}$ form an AC of submodules of M, it must terminate: $\exists\, k\geqslant 0 \mid N_j=N_k\ \forall j\geqslant k$. But $\underline{N}_i=\pi(N_i)$ so $\underline{N}_j=\pi(N_j)=\pi(N_k)=\underline{N}_k$. So $(\underline{N}_i)_{i\geqslant 0}$ terminates.

(2) $\Rightarrow$ (1): Have $(N_i)_{i\geqslant 0}$ is an AC of submodules in M. Want to show it terminates. Then $(N_i\cap N)_{i\geqslant 0}$ is AC in N & $(\pi(N_i))_{i\geqslant 0}$ is AC in M/N. We know that both terminate: $\exists\, k\geqslant 0$ s.t. $N_j\cap N=N_k\cap N$ & $\pi(N_j)=\pi(N_k)$ $\forall j\geqslant k$.
Want to check: $N_j=N_k$ (so (N_i) terminates):

$n \in N_j \rightsquigarrow \pi(n) \in \pi(N_j) = \pi(N_k)$ so $\exists\ n' \in N_k \mid \pi(n') = \pi(n)$
$\Leftrightarrow \pi(n-n') = 0 \Leftrightarrow n-n' \in N$. But $n, n' \in N_j$ (b/c $n' \in N_k \subset N_j$) $\Rightarrow$
$n-n' \in N_j \Rightarrow n-n' \in N \cap N_j = N \cap N_k \Rightarrow n = n' + (n-n') \in N_k$ b/c both summands are in N_k. $\square$

2.1) Artinian modules.

Noetherian $\Leftrightarrow$ satisfies AC condition.

Definition: Let M be A-module. A descending chain (DC) of submodules is $(N_i)_{i \geq 0}$ s.t. $N_k \supseteq N_{k+1}$ $\forall k \geq 0$.

Definition: M is an Artinian A-module if $\forall$ DC of submodules terminates (DC condition)

Example: $A = \mathbb{F}$ (a field). Claim: Artinian $\Leftrightarrow$ finite dim'l.
$\Leftarrow$ is clear b/c dimensions decrease in DC's.
$\Rightarrow$ let $\dim M = \infty \Leftrightarrow M$ has basis, e_i, $i \in I$, where I is infinite. Since I is infinite $\exists$ subsets $I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \ldots$ (infinite chain of subsets). Define $M_j = \operatorname{Span}_{\mathbb{F}}(e_i \mid i \in I_j)$ – a DC of subspaces that doesn't terminate.

2.2) Basic properties.

The first result (together with its proof is analogous to Proposition in 1.2 of Lec 5).

Proposition 1: For A-module M TFAE:

1) M is Artinian

2) $\forall$ nonempty set of submodules of M has a <u>minimal</u> el·t (w.r.t. $\subset$)

Proposition 2: M is A-module, $N \subset M$ is an A-submodule. TFAE: 1) M is Artinian.

2) Both N & M/N are Artinian.

Proofs: repeat those in Noeth'n case (exercise).

2.2) Artinian rings.

Definition: A ring A is Artinian if it's Artinian as A-module.

Examples: 1) Any field is Artinian.

2) Let $\mathbb{F}$ be a field, A be an $\mathbb{F}$-algebra s.t. $\dim_{\mathbb{F}} A < \infty$. Then A is Artinian ring (b/c A-submodule is a subspace).

3) $A = \mathbb{Z}/n\mathbb{Z}$ Artinian (b/c it's a finite set so every DC of subsets terminates)

4) Let A be a domain. Then A is Artinian $\Rightarrow$ A is a field. Indeed, let $a \in A$ be noninvertible & nonzero.

$(a) \supsetneq (a^2) \supsetneq (a^3) \supsetneq \ldots$ a DC of ideals that doesn't terminate. Indeed, if $(a^k) = (a^{k+1})$ then $\exists\, b \in A$ s.t. $a^k = ba^{k+1} \Leftrightarrow$ $(1-ab)a^k = 0 \Leftrightarrow [a=0$ or $1=ab]$ - contradiction.

Thm: Every Artinian ring is Noetherian.
For proof, see [AM], Prop 8.1 - Thm 8.5 (comments: nilradical $= \sqrt{0} =$
$= \bigcap$ all prime ideals by Prop. 1.8, Jacobson radical $= \bigcap$ all max. ideals).

3) Finite length modules.

Thm motivates us to consider modules that are both Noetherian (AC condition) & Artinian (DC condition) so satisfy ("AC/DC" condition). They admit an equivalent characterization.

Definition: Let M be an A-module.

i) Say that M is **simple** if $\{0\} \neq M$ are the only two submodules of M.

ii) Let M be arbitrary. By a **filtration** (by submodules) on M we mean $\{0\} = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$ (finite AC of submodules).

iii) A **Jordan-Hölder** (JH) **filtr'n** is a filtr'n $\{0\} = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_k = M$ s.t. M_i / M_{i-1} is simple $\forall i$ (so a JH filtr'n is "tightest possible")

iv) M has **finite length** if a JH filtr'n exists.

Example: 1) When $A = \mathbb{F}$ is a field, an A-module M is simple $\Leftrightarrow \dim_{\mathbb{F}} M = 1$.

2) Let $A = \mathbb{Z}$ & consider the A-module $M = \mathbb{Z}/4\mathbb{Z}$. It's

JH filtration is $M_0 = \{0\}$, $M_1 = 2\mathbb{Z}/4\mathbb{Z}$, $M_2 = M$.

Proposition: For an A-module M TFAE:

1) M is Artinian & Noetherian.
2) M has finite length.

Proof: 2) $\Rightarrow$ 1): M has fin. length $\rightsquigarrow$ JH filtr'n $\{0\} = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_k = M$. We prove by induction on i that M_i is Artinian & Noetherian.

Base: $i=1$: M_1 is simple $\Rightarrow$ Artinian & Noetherian.

Step: $i-1 \rightsquigarrow i$: M_{i-1} is Art'n & Noeth'n, so is M_i/M_{i-1} b/c it's simple. $\Rightarrow$ by Prop in Sec 1 M_i is Noetherian & by Prop 2 in 2.1, M_i is Artinian.

Use this for $i=k \rightsquigarrow M_i = M$ is Artinian & Noeth'n. So 2) $\Rightarrow$ 1).

1 $\Rightarrow$ 2): M is Artinian & Noetherian. Want to produce a JH filtr'n. By induction: $M_0 = \{0\}$.

Suppose we've constr'd $M_i \subset M$. Need M_{i+1}.

Note: M/M_i is Artinian & therefore $\forall$ nonempty set of submodules has a min. el't. Assume $M_i \neq M$. Consider the set of all <u>nonzero</u> submodules of M/M_i. It's $\neq \emptyset$ so has a min'l element, N. This N must be simple. Now take M_{i+1} to be the preimage of N under $M \twoheadrightarrow M/M_i$. So $M_{i+1}/M_i \simeq N$, simple.

We've got is an AC $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots$, it must terminate b/c M is Noeth'n. By constr'n it can only terminate at $M_i = M$. So we've got a JH filtration $\square$

Exercise: We can classify simple modules as follows: a map $\mathfrak{m} \mapsto A/\mathfrak{m}$ defines a bijection between the set of maximal ideals in A and the set of simple A-modules (up to isomorphism).

Up next: classification questions.

Motivation: for a field $\mathbb{F}$, we can completely classify finite dimensional $\mathbb{F}$-vector spaces: $\forall$ such V $\exists\, k \in \mathbb{Z}_{\geq 0}$ s.t. $V \simeq \mathbb{F}^{\oplus k}$; this k is uniquely recovered from V: $k = \dim V$.

Q: Can we generalize this to finitely gen'd modules over a ring?

A: Yes, but only in very rare – yet important – cases. We can do so for principal ideal domains such as $\mathbb{Z}$ & $\mathbb{F}[x]$ but not for more complicated rings – for example $\mathbb{Z}[x]$ is already hopeless.