MICROLOCALIZATION EXERCISES

1. Completed Rees algebra

a) Show that $\mathrm{R}_\hbar(\mathcal{A})$ is a graded $\mathbb{C}[\hbar]$-subalgebra in $\mathcal{A}[\hbar^{\pm 1}]$. Identify $\mathrm{R}_\hbar(\mathcal{A})/(\hbar)$ with $\operatorname{gr}\mathcal{A}$ and $\mathrm{R}_\hbar(\mathcal{A})/(\hbar-a)$ with $\mathcal{A}$ for $a \in \mathbb{C}^\times$.
Hint: This is straighforward. Identifying $\mathrm{R}_\hbar(\mathcal{A})/(\hbar)$ with $\operatorname{gr}\mathcal{A}$ is easier if you do it by graded components.

b) Prove that $\mathcal{A}_{\hbar,fin}$ is a graded subalgebra of $\mathcal{A}_\hbar$ that is dense in the $\hbar$-adic topology and satisfies $\mathcal{A}_{\hbar,fin}/(\hbar) = A$.
Hint: All of this follows from the fact that $\mathbb{C}^\times$ acts rationally on $\mathcal{A}_\hbar/(\hbar^k)$.

c) Prove that $\mathcal{A}_{\hbar,fin}/(\hbar-1)$ is a filtered quantization of A.
Hint: Note that the filtration on $\mathcal{A}_{\hbar,fin}/(\hbar-1)$ is induced by the grading on $\mathcal{A}_{\hbar,fin}$. A very important fact here is that $t.\hbar = t\hbar$, that is, $\hbar$ is of degree 1.

d) Prove that the maps $\mathcal{A} \mapsto \mathrm{R}_\hbar^\wedge(\mathcal{A})$ and $\mathcal{A}_\hbar \mapsto \mathcal{A}_{\hbar,fin}/(\hbar-1)$ are mutually inverse equivalences between filtered quantizations and graded formal quantizations.

2. (Micro)localization for formal quantizations

a) Let $f \in A$ be not a zero divisor, and let $\hat{f} \in \mathcal{A}_k := \mathcal{A}_\hbar/(\hbar^k)$ be a lift of f. Show that $[\hat{f}, \cdot]^k = 0$ and deduce from here that every left fraction by $\hat{f}$ is also a right fraction. Show that the localization $\mathcal{A}_k[\hat{f}^{-1}]$ makes sense and is independent of the choice of the lift. We will denote this localization by $\mathcal{A}_k[f^{-1}]$.
Hint: The fact that $\operatorname{ad}_{\hat{f}}^k = 0$ follows because $\mathcal{A}_k/(\hbar)$ is commutative. The fact that the localization is independent of the choice of lift is a consequence of (a suitable adaptation of) the following commutative algebra lemma: the sum of an invertible element and a nilpotent element is again invertible.

b) Show that $\mathcal{A}_k[\hat{f}^{-1}]$ form an inverse system. Further, show that $\mathcal{A}[f^{-1}] := \varprojlim_{k\to\infty} \mathcal{A}_k[f^{-1}]$ is a formal quantization of $A[f^{-1}]$.
Hint: Use universal properties.

c) Establish a natural homomorphism $\mathcal{A}[f^{-1}] \to \mathcal{A}[(fg)^{-1}]$.
Hint: Use universal properties.

d) Show that $\mathcal{A}_\hbar$ naturally sheafifies to a sheaf $\mathcal{D}_\hbar$ on $\operatorname{Spec}(A)$. Show that $\Gamma(\mathcal{D}_\hbar) = \mathcal{A}_\hbar$.
Hint: Reduce modulo $\hbar^k$ for all k.

e) Note that if $\mathcal{A}_\hbar$ is graded then $\mathcal{A}[f^{-1}]$ is graded provided f is $\mathbb{C}^\times$-semiinvariant. So we can take the microlocalization of $\mathcal{A}_{\hbar,fin}/(\hbar-1)$ by taking the sheaf $\mathcal{D}_{\hbar,fin}/(\hbar-1)$ that makes sense in the conical topology. Work out the details.
Hint: As in Exercise 1, assume that the weight of $\hbar$ is 1. Then all quotient maps modulo $\hbar$ are graded. Note that the condition of f being semiinvariant means that it is homogeneous under the grading, so all non-commutative localizations are graded. So there is a $\mathbb{C}^\times$-action on $\mathcal{A}_\hbar[f^{-1}]$. Now you have to check that sections in $\mathcal{A}_\hbar[g_i^{-1}]_{\hbar,fin}$ that agree on intersections glue together to a locally finite section.

3. Coherent modules over formal quantizations

a) Suppose that there is an open covering $X = \bigcup_i X^i$ such that $M_\hbar|_{X^i}$ is coherent. Then $M_\hbar$ is coherent.
Hint: This is an exercise on inverse limits of sheaves.

b) Show that a filtered $\mathcal{D}$-module M is coherent with a good filtration if and only if $\mathrm{R}_\hbar^\wedge M$ is a coherent $\mathcal{D}_\hbar$-module.

4. Microlocalization of modules

a) Assume that M is equipped with a good filtration. Emulate the procedure in Exercise 2 to define $M[f^{-1}]$ and check that this space has a natural $\mathcal{A}_\hbar[f^{-1}]$-module structure. Furthermore check that there is a natural isomorphism $\mathcal{A}_\hbar[f^{-1}] \otimes_{\mathcal{A}_\hbar} M \longrightarrow M[f^{-1}]$ so that $M[f^{-1}]$ is independent of the choice of a filtration.
Hint: To check the properties, first check them at the level of (completed) Rees algebras/modules.

b) Check that the functor $M \to M[f^{-1}]$ is exact.
Hint: Again work with Rees algebras. Use that the functor is exact modulo $\hbar$.

c) Check that M sheafifies in the conical topology on $\mathrm{Spec}(A)$.
Hint: Imitate what is done in Exercise 2.

5. Quotient categories

Show that $\mathrm{Coh}(\mathcal{A}|_{X^0})$ is a quotient category of $\mathcal{A}$-mod by the full subcategory of modules supported on $X \setminus X^0$.
Hint: Use Lemma 1.3 in Lecture 3.