# Lecture 12.

1) Localization of modules, cont'd.
2) Local rings.

Ref: [AM], Sections 3, 3.1.

## 1.1) Universal property

Let $A$ be a commutative ring, $S\subset A$ a multiplicative subset. So we can form the localization $A[S^{-1}]$.
Now we discuss the universal property of localization of modules. Since $A[S^{-1}]$ is an $A$-algebra, every $A[S^{-1}]$-module is naturally an $A$-module.

**Thm:** Let $M$ be an $A$-module, $N$ be an $A[S^{-1}]$-module and $\varphi: M\to N$ be an $A$-linear map. Then $\exists!$ $A[S^{-1}]$-linear map $\varphi': M[S^{-1}]\to N$ making the following diagram commutative:

$$\begin{tikzcd} M \arrow[d, "\iota"'] \arrow[dr, "\varphi"] & \\ M[S^{-1}] \arrow[r, dashed, "\varphi'"] & N \end{tikzcd}$$

It's given by $\varphi'(\frac{m}{s}) = \frac{1}{s}\varphi(m)$.

Proof: essentially repeats the proof of Proposition in Section 1.2 of Lecture 9. Left as exercise. $\square$

## 1.2) Localization of homomorphisms.

Let $M,N$ be $A$-modules, $\varphi: M\to N$ be an $A$-linear map

Our goal is to construct an $A[S^{-1}]$-linear map

$$\varphi[S^{-1}]: M[S^{-1}] \longrightarrow N[S^{-1}].$$

**Lemma:** $\varphi[S^{-1}]$ given by $\frac{m}{s} \mapsto \frac{\varphi(m)}{s}$ is a well-defined map $M[S^{-1}] \to N[S^{-1}]$ & is $A[S^{-1}]$-linear.

Sketch of proof: Things to check:

- well-defined: $(m,s) \sim (n,t) \Leftrightarrow \exists\, u \in S \,|\, utm = usn \Rightarrow$

$\varphi(utm) = \varphi(usn) \Rightarrow$ [A-linear] $ut\varphi(m) = us\varphi(n) \Rightarrow \frac{\varphi(m)}{s} = \frac{\varphi(n)}{t}$ - checked.

- $A[S^{-1}]$-linear — exercise $\square$

**Example:** • $M = A^{\oplus k}$. **Claim:** $M[S^{-1}]$ is identified with $(A[S^{-1}])^{\oplus k}$.

Define maps in both directions:

$$(A^{\oplus k})[S^{-1}] \longrightarrow (A[S^{-1}])^{\oplus k}, \quad \frac{(a_1, \dots, a_k)}{s} \mapsto \left(\frac{a_1}{s}, \dots, \frac{a_k}{s}\right)$$

$$(A[S^{-1}])^{\oplus k} \longrightarrow (A^{\oplus k})[S^{-1}]: \left(\frac{a_1}{s_1}, \dots, \frac{a_k}{s_k}\right) \mapsto \frac{\left(\prod_{j \neq i} s_j a_i\right)_{i=1}^{k}}{\prod_j s_j}$$ ← common denominator

Exer: these two maps are mutually inverse.

- $\varphi: A^{\oplus k} \to A^{\oplus \ell} \longleftrightarrow$ matrix $\Phi = (a_{ij})_{i=1, j=1}^{\ell \quad k}$ (so that if elements of our modules are columns, $\varphi$ = multiplication by $\Phi$). Then $\varphi[S^{-1}]: A[S^{-1}]^{\oplus k} \longrightarrow A[S^{-1}]^{\oplus \ell}$ is given by the matrix $\left(\frac{a_{ij}}{1}\right)_{i=1, j=1}^{\ell \quad k}$.

Our next task is to relate $\ker \varphi[S^{-1}]$, $\operatorname{im} \varphi[S^{-1}]$ to $\ker \varphi$, $\operatorname{im} \varphi$.

Let $M$ be an $A$-module, $M' \subset M$ $A$-submodule, let $i: M' \hookrightarrow M$

be the inclusion. Then $i[S^{-1}]: M'[S^{-1}] \to M[S^{-1}]$ is injective: for $m, n \in M'$ the equality $\frac{m}{s} = \frac{n}{t}$ in $M'$ is equivalent to that in $M$, while $i[S^{-1}](\frac{m}{s}) = \frac{m}{s}$. So we can view $M'[S^{-1}]$ as a submodule in $M[S^{-1}]$ via this injective map.

**Proposition:** Let $\varphi: M \to N$ be an $A$-linear map.

i) $\ker(\varphi[S^{-1}]) = (\ker\varphi)[S^{-1}]$

ii) $\operatorname{im}(\varphi[S^{-1}]) = (\operatorname{im}\varphi)[S^{-1}]$

Proof:

i) First, we check $\ker(\varphi[S^{-1}]) \subset (\ker\varphi)[S^{-1}]$

$\ker(\varphi[S^{-1}]) = \{\frac{m}{s} \in M[S^{-1}] \mid \varphi[S^{-1}](\frac{m}{s}) = 0 \Leftrightarrow$ [def'n of $\varphi[S^{-1}]$] $\frac{\varphi(m)}{s} = 0 \Leftrightarrow \exists\, u \in S \mid u\varphi(m) = 0 \Leftrightarrow um \in \ker\varphi\}$

$\subseteq [\frac{um}{us} = \frac{m}{s}] \subseteq (\ker\varphi)[S^{-1}]$.

Now $(\ker\varphi)[S^{-1}] = \{\frac{m}{s} \mid \varphi(m) = 0\} \subset \ker(\varphi[S^{-1}])$, finishing (i).

(ii) $\operatorname{im}(\varphi[S^{-1}]) = \{\varphi[S^{-1}](\frac{m}{s}) = \frac{\varphi(m)}{s}\} = (\operatorname{im}\varphi)[S^{-1}]$. $\square$

**Corollary** of Prop'n: Let $M$ be $A$-module, $M' \subset M$ be an $A$-submodule. Then there's a natural $A[S^{-1}]$-module isomorphism $(M/M')[S^{-1}] \xrightarrow{\sim} M[S^{-1}]/M'[S^{-1}]$.

Proof: Apply Proposition to $\varphi: M \twoheadrightarrow M/M'$, $m \mapsto m + M'$.
Then $\operatorname{im}(\varphi[S^{-1}]) = (\operatorname{im}\varphi)[S^{-1}] = (M/M')[S^{-1}]$
$\ker(\varphi[S^{-1}]) = (\ker\varphi)[S^{-1}] = M'[S^{-1}]$, $M[S^{-1}]/M'[S^{-1}] \xrightarrow{\sim} (M/M')[S^{-1}]$ $\square$

Very important exercise:

- $\mathrm{id}[S^{-1}]: M[S^{-1}] \to M[S^{-1}]$ is the identity map.
- for $\varphi: M \to N$, $\varphi': N \to P$ have $(\varphi' \circ \varphi)[S^{-1}] = \varphi'[S^{-1}] \circ \varphi[S^{-1}]$.

## 1.3) Submodules in $M[S^{-1}]$.

Let $M$ be an $A$-module. For $A$-submodule $N \subset M \Rightarrow N[S^{-1}]$ is an $A[S^{-1}]$-submodule. On the other hand, for an $A[S^{-1}]$-submodule $N' \subset M[S^{-1}]$, consider $\iota^{-1}(N') \subset M$, where $\iota: M \to M[S^{-1}]$, $m \mapsto \frac{m}{1}$.

Proposition: The maps $N \mapsto N[S^{-1}]$ & $N' \mapsto \iota^{-1}(N')$ are mutually inverse bijections between:

$\{A[S^{-1}]\text{-submodules } N' \subset M[S^{-1}]\}$ &

$\{A\text{-submodules } N \subset M \mid \underbrace{sm \in N \text{ for } s \in S, m \in M \Rightarrow m \in N}_{(\dagger)}\}$

Proof: Step 1: $\iota^{-1}(N')$ is a submodule satisfying $(\dagger)$; $N'$ is an $A[S^{-1}]$-submodule, hence also $A$-submodule in $M[S^{-1}]$ & $\iota$ is $A$-linear. Therefore, $\iota^{-1}(N')$ is $A$-submodule. Check $(\dagger)$: $sm \in \iota^{-1}(N') \Leftrightarrow \iota(sm) \in N' \Leftrightarrow \frac{s}{1}\iota(m) \in N' \Leftrightarrow [\frac{s}{1}$ is invertible in $A[S^{-1}]] \Leftrightarrow \iota(m) \in N' \Leftrightarrow m \in \iota^{-1}(N')$.

Step 2: $\iota^{-1}(N[S^{-1}]) = N$: $\iota^{-1}(N[S^{-1}]) = \{m \in M \mid \iota(m) \in N[S^{-1}] \Leftrightarrow \frac{m}{1} = \frac{n}{s}$ for some $n \in N, s \in S \Leftrightarrow \exists\, u \in S \mid usm = un \in N \Leftrightarrow [(\dagger)]$ $m \in N\} = N$.

Step 3: $(\iota^{-1}(N'))[S^{-1}] = N'$: $(\iota^{-1}(N'))[S^{-1}] = \{\frac{n}{s} \mid \frac{n}{1} \in N' \Leftrightarrow$ $[\frac{s}{1}$ is invertible$] \Leftrightarrow \frac{n}{s} \in N'\} = N'$. $\square$

In particular, if $I \subset A$ is an ideal, then $I[S^{-1}] \subset A[S^{-1}]$ is an ideal.

**Corollary:** Suppose $M$ is a Noetherian (resp. Artinian) $A$-module. Then $M[S^{-1}]$ is a Noetherian (resp. Artinian) $A$-module. In particular, if $A$ is a Noetherian (resp. Artinian) ring, then so is $A[S^{-1}]$.

Proof (of $M$ is Noetherian $\Rightarrow M[S^{-1}]$ is; everything else is exercise.) Let $N_1' \subset N_2' \subset \dots \subset N_i' \subset \dots$ be an AC of submodules in $M[S^{-1}]$. Set $N_i := \iota^{-1}(N_i')$. Then $N_1 \subset N_2 \subset \dots$ is AC of submodules in $M$ so $\exists k$ s.t. $N_k = N_i$ $\forall i \geq k$. By Proposition, $N_i' = N_i[S^{-1}] \Rightarrow N_k' = N_i'$. So the AC in $M[S^{-1}]$ terminates, hence $M[S^{-1}]$ is Noetherian. $\square$

**2) Local rings.** Recall, Lecture 10, that if $\mathfrak{p} \subset A$ is prime ideal, then $S := A \setminus \mathfrak{p}$ is multiplicative. We write $A_\mathfrak{p}$ for $A[(A \setminus \mathfrak{p})^{-1}]$.

Consider the ideal $\mathfrak{p}_\mathfrak{p} := \mathfrak{p}[(A\setminus\mathfrak{p})^{-1}] \subset A_\mathfrak{p}$.

**Proposition:** $\mathfrak{p}_\mathfrak{p}$ is the unique maximal ideal of $A_\mathfrak{p}$.
Proof: Pick an ideal $I' \subsetneq A_\mathfrak{p}$. Need to show $I' \subseteq \mathfrak{p}_\mathfrak{p}$.
By Prop'n in Section 1.3, this is equivalent to $\iota^{-1}(I') \subseteq \mathfrak{p}$.

By the same Proposition, the ideal $I = \iota^{-1}(I')$ satisfies

$$sa \in I \text{ for } s \notin \mathfrak{p} \Rightarrow a \in I \qquad (\heartsuit)$$

Assume $I \not\subset \mathfrak{p} \Leftrightarrow S \cap I \neq \emptyset$. Pick $s \in S \cap I$, $a = 1$, so $sa \in I$ but $a \notin I$. This contradicts $(\heartsuit)$. $\square$

Definition: A commutative ring $B$ is local if it has a unique maximal ideal.

Example: $A_{\mathfrak{p}}$ is local.

When we talk about connections to Algebraic geometry we'll explain the meaning of the name "local."

Local rings are important because they have nice properties that general rings do not, while some questions about general rings can be reduced to those of local rings — by passing from $A$ to $A_{\mathfrak{p}}$. We'll talk more about this in our discussion of connections to Alg. geometry.