Lecture 10.

1) Integral closure, cont'd.

2) Localization

BONUS: Proof of Thm (I) in Section 1.3.

Refs: [AM], Section 5.3, Chapter 3; [V], Section 9.3

1) A is comm've unital ring, B is an A-algebra. Recall (Section 2 of Lecture 9) the integral closure:

$\bar{A}^B = \{b \in B \mid b \text{ is integral over } A\}$, A-subalgebra in B.

1.1) Fraction fields.

Suppose A is a domain. We can form its fraction field $\operatorname{Frac}(A)$. By definition, it consists of fractions $\frac{a}{b}$ w. $a,b \in A$, $b \neq 0$.

In $\operatorname{Frac}(A)$ we have $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$. The fractions are added & multiplied in the usual way. A direct check shows that $\operatorname{Frac}(A)$ is a field.

Note that A has a ring embedding into $\operatorname{Frac}(A)$: $a \mapsto \frac{a}{1}$. Moreover, every embedding $\varphi: A \hookrightarrow L$ factors as $A \hookrightarrow \operatorname{Frac}(A) \hookrightarrow L$, where the latter is $\frac{a}{b} \mapsto \frac{\varphi(a)}{\varphi(b)}$.

Examples: • $\operatorname{Frac}(\mathbb{Z}) = \mathbb{Q}$.

• For a field $\mathbb{F}$, we have $\operatorname{Frac}(\mathbb{F}[x_1,\dots x_n]) = \mathbb{F}(x_1,\dots x_n)$, the field of rational functions in n variables.

• Let $d \in \mathbb{Z}$, not a complete square. Then $\mathbb{Z}[\sqrt{d}]$ is a domain & $\operatorname{Frac}(\mathbb{Z}[\sqrt{d}]) = \mathbb{Q}[\sqrt{d}]$.

1.1) Normal domains.

Let A be a domain.

Definition:

i) The **normalization** of A is $\overline{A}^{Frac(A)}$, integral closure of A in $Frac(A)$.

ii) A is **normal** if A coincides w. its normalization.

Special cases:

1) L is a field, $A\subset L$ is a subring. Claim: $\overline{A}^L$ is normal. Indeed, $\overline{A}^L$ is integr. closed in L & $Frac(\overline{A}^L)\subset L \Rightarrow$ $\overline{A}^L$ closed in $Frac(\overline{A}^L)$.

2) UFD $\Rightarrow$ normal: let A be UFD & $\frac{a}{b}\in Frac(A)$ w. coprime $a,b\in A$. Need to show: $\frac{a}{b}$ is integral over $A \Rightarrow$ $\frac{a}{b}\in A$ i.e. b is invertible. Let $f(x)=x^k+c_{k-1}x^{k-1}+\ldots+c_1x+c_0$ $(c_i\in A)$ be s.t. $f(\frac{a}{b})=0 \Rightarrow 0=b^kf(\frac{a}{b})=a^k+\sum_{i=0}^{k-1}c_ia^ib^{k-i}$ The sum is divisible by b. So $a^k \vdots b$. Since a & b are coprime, this implies that b is invertible.

Exercise: Let L be a field & $A_1,A_2\subset L$ be normal subrings. Prove that $A_1\cap A_2$ is normal. Prove the analogous claim for infinite intersections.

1.2) Examples of computation of integral closure.

Example 1: compute the integral closure of $A=\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$, $d\in\mathbb{Z}$ is a square-free number. We need to understand when $\beta\in\mathbb{Q}(\sqrt{d})$, $\beta=a+b\sqrt{d}$ $(a,b\in\mathbb{Q})$, is integral over $\mathbb{Z}$.

Lemma 1: TFAE

(i) β is integral over $\mathbb{Z}$,

(ii) $2a,\ a^2-b^2d\in\mathbb{Z}$.

Proof: We set $\bar{\beta}:=a-b\sqrt{d}$. Note that: $\beta+\bar{\beta}=2a$, $\beta\bar{\beta}=a^2-b^2d$, rational numbers. So $(x-\beta)(x-\bar{\beta})=x^2-2ax+(a^2-b^2d)$

So (ii) $\Rightarrow$ (i).

On the other hand, note that $\beta\mapsto\bar{\beta}$ is a homomorphism $\mathbb{Z}[\sqrt{d}]\to\mathbb{Z}[\sqrt{d}]$. So for $f(x)\in\mathbb{Z}[x]$ we have $f(\bar{\beta})=\overline{f(\beta)}$. So if $f(\beta)=0$, then $f(\bar{\beta})=0$. In particular, if β is integral over $\mathbb{Z}$, then $\bar{\beta}$ is integral. By Proposition 1 of Section 2 of Lecture 9, $\beta+\bar{\beta},\ \beta\bar{\beta}\in\mathbb{Q}$ are integral over $\mathbb{Z}$. But $\mathbb{Z}$ is UFD, hence normal. So elements of $\mathbb{Q}$ integral over $\mathbb{Z}$ are integers. (ii) follows. $\square$

Exercise (Number thy): If $d\equiv 2$ or $3 \bmod 4$, then (ii) $\Leftrightarrow$ $a,b\in\mathbb{Z}$;
if $d\equiv 1 \bmod 4$, then (ii) $\Leftrightarrow$ either $a,b\in\mathbb{Z}$ or $a,b\in\mathbb{Z}+\frac{1}{2}$.

Corollary: i) $\mathbb{Z}[\sqrt{d}]$ is normal $\Leftrightarrow$ $d\equiv 2$ or $3 \bmod 4$.
If $d\equiv 1 \bmod 4$, then the normal'n of $\mathbb{Z}[\sqrt{d}]$ is $\{a+b\sqrt{d}\mid a,b\in\mathbb{Z} \text{ or } a,b\in\mathbb{Z}+\frac{1}{2}\}$.

(ii) $\mathbb{Z}[\sqrt{-5}]$ is normal but not UFD.

Example 2: Consider the ring $A = \mathbb{C}[x,y]/(y^2 - f(x))$, $f(x) \in \mathbb{C}[x]$.

Exercise: This ring is a domain $\Leftrightarrow$ $f(x)$ is not a complete square

Fact (going beyond what we cover in this class) A is normal $\Leftrightarrow$ $f(x)$ has no repeated roots ($\Leftrightarrow$ the curve $\{(x,y) \mid y^2 = f(x)\}$ is "non-singular" – compute the Jacobian vector of $(x,y) \mapsto y^2 - f(x)$).

We consider the special case $f(x) = x^3$ and describe the normalization in this case. Let's write $\bar{x}, \bar{y}$ for the images of x, y in A.

Exercise: the elements $\bar{x}^k, \bar{x}^k\bar{y}$ for $k \geq 0$ form a basis in A (for any f, in fact).

Lemma 2: The integral closure of A in $K := \operatorname{Frac}(A)$ coincides w. $\mathbb{C}[\underline{y}/\underline{x}]$ (and so is isomorphic to the polynomial algebra). In particular, A isn't normal.

Proof: We have $\bar{y}^2 = \bar{x}^3$ in A. So for $t := \bar{y}/\bar{x} \in K$ we have $t^2 = \bar{x}$ meaning t is integral over A. Let $\tilde{A} := \mathbb{C}[t] \subset K$ (note that t is transcendental over $\mathbb{C}$). We have $\bar{x} = t^2, \bar{y} = t^3$ so $A \subset \tilde{A}$. But for any domain $A' \subset K$ have $\operatorname{Frac}(A') \subset K$. Since $A \subset \tilde{A}$ have $K = \operatorname{Frac}(\tilde{A})$. Since $\tilde{A}$ is generated by an element integral over A, its integral over A. As we've seen in Section 1.1, $\tilde{A}$ is normal. So $\tilde{A}$ is the normalization of A and A is not normal $\square$

1.3) Finiteness of integral closures.

Let A be a domain, $K = \mathrm{Frac}(A)$, $K \subset L$ finite field ext'n.

Q: Is $\bar{A}^L$ finite over A?

A: It's complicated...

Theorem: Assume that <u>one</u> of the following holds:

(I) A is Noeth'n & normal, char $K = 0$.

(II) A is a fin. gen'd algebra over a field or over $\mathbb{Z}$.

Then $\bar{A}^L$ is finite over A.

Proof under (I) will appear as a bonus.

Example: let $A = \mathbb{Z}$, L is a finite extension of $\mathbb{Q}$. The ring $\bar{A}^L$ is called the ring of algebraic integers in L (crucially important for Alg. Number th'y). Both (I) & (II) apply, so Thm $\Rightarrow$ $\bar{A}^L$ is finite over $\mathbb{Z}$, i.e. is a fin. gen'd abelian group.

Fact/Premium exercise: As an abelian group, $\bar{A}^L \simeq \mathbb{Z}^n$, where $n = \dim_{\mathbb{Q}} L$.

(Hint: observe that since $\bar{A}^L$ is a domain, $n\alpha = 0$ for $l \in \mathbb{Z}, \alpha \in \bar{A}^L$ implies $n=0$ or $\alpha = 0$, so $\bar{A}^L \simeq \mathbb{Z}^n$ for some n. Then observe that $\forall \beta \in L$ $\exists$ $k \in \mathbb{Z} \setminus \{0\}$ s.t. $k\beta \in \bar{A}^L$.)

2) Localization.

Localizations of rings generalize the fraction fields of domains. To construct a localization of A we need a suitable subset in A.

Definition: A subset $S \subset A$ is *multiplicative* if

- $1 \in S$
- $s, t \in S \Rightarrow st \in S$

Examples:

0) $S = \{$all invertible elements in $A\}$ is multiplicative.

1) If A is a domain, then $S = A \setminus \{0\}$ is multiplicative.

1') For general A, $S := \{$all nonzero divisors $\in A\}$ is multiplicative.

2) For $f \in A$, the subset $S := \{f^n \mid n \geqslant 0\}$ is multiplicative.

2') For $f_1, \dots, f_k \in A \rightsquigarrow S := \{f_1^{n_1} \dots f_k^{n_k} \mid n_i \geqslant 0\}$ is multiplicative.

3) Let $\mathfrak{p} \subset A$ be prime ideal $\Rightarrow S := A \setminus \mathfrak{p}$ is multiplicative
$1 \in S \Leftrightarrow 1 \notin \mathfrak{p}$; $\quad s, t \in S \Leftrightarrow s, t \notin \mathfrak{p} \Rightarrow st \notin \mathfrak{p}$ (the last implication uses that $\mathfrak{p}$ is prime) $\Rightarrow st \in S$.

BONUS: Proof of Theorem in Section 1.3 under assumption (I).

Proof: Let $\dim_K L = n$. Every element $\alpha \in L$ gives the K-linear operator, say, m_α, on L via multiplication. So for $\alpha \in L$ it makes sense to speak about $tr(\alpha) := tr(m_\alpha) \in K$.

Step 1: We claim that for $\alpha \in \bar{A}^L$ we have $tr(\alpha) \in A$. Let $f(x) \in A[x]$ be a monic polynomial w $f(\alpha) = 0$. Choose an algebraic extension $\tilde{L}$ of L where $f(x)$ decomposes into linear factors. All eigenvalues of m_α are roots of $f(x)$, hence are integral over A. Therefore $tr(\alpha)$ - the sum of eigenvalues - is integral over A. But $tr(\alpha) \in K$ and, since A is normal, we see $tr(\alpha) \in A$.

Step 2: For $\alpha, \beta \in L$ define $(\alpha, \beta) := tr(\alpha\beta)$. This is a symmetric K-bilinear form $L \times L \to K$. We claim that since $\operatorname{char} K = 0$, this bilinear form is nondegenerate. More precisely, for $u \in L \setminus \{0\}$ $\exists\, m \geq 0$ s.t. $(u, u^{m-1}) = tr(u^m) \neq 0$. Let $u_1 = u, u_2, \ldots u_n$ be the eigenvalues of m_u counted w. multiplicities. Then $tr(u^m) = \sum_{i=1}^{n} u_i^m$. If the r.h.s.'s are 0 for all m, then, thx to $\operatorname{char} \tilde{L} = \operatorname{char} K = 0$, we get $u_1 = \ldots = u_n = 0$, which is impossible since $u \neq 0$.

Step 3: Thx to the exercise in Section 1.3, we can find a K-basis $l_1, \ldots l_n$ of L w $l_i \in \bar{A}^L$. Let $l^1, \ldots, l^n$ be the dual basis w.r.t. $(\cdot, \cdot)$, i.e $tr(l_i l^j) = \delta_{ij}$, it exists b/c $(\cdot, \cdot)$ is nondegenerate. Let $M := \operatorname{Span}_A (l^1, \ldots, l^n)$. Note that, for $\alpha \in \bar{A}^L$ we have $\alpha = \sum_{i=1}^{n} (\alpha, l_i) l^i$. We have $(\alpha, l_i) = tr(\alpha l_i) \in A$ b/c $\alpha l_i \in A$. So $\alpha \in M \Rightarrow \bar{A}^L \subset M$. But A is Noetherian, and M is manifestly finitely generated A-module. Hence $\bar{A}^L$ is a finitely generated A-module and we are done. $\square$

Corollary: Theorem is also true under assumption (II) if A is a finitely generated $\mathbb{F}$-algebra & char $\mathbb{F}=0$.

Sketch of proof: Thanks to the Noether normalization lemma we can replace A w. $\mathbb{F}[x_1,\dots x_m]$ for some m, this doesn't change $\bar{A}^L$ (exercise). Now we are in the situation of (I) of the theorem. $\square$

For a proof of (II) w. a finitely generated algebra over an arbitrary field, see [E], Section 13.3.