MATH 380 FINAL, DUE DEC 22

There are four problems, worth 20 points each so that the maximal score is 80. You can refer to the lecture notes and solutions to homeworks, either written by you or by the instructor. You are not allowed to collaborate with other students in this class or seek external help. It is your responsibility to make sure that your solutions are detailed enough. Partial credit is given.

Below all rings are commutative.

Problem 1. Consider the $\mathbb{C}$-algebra A with basis of formal symbols e^α where α runs over nonnegative rational numbers, and product defined on the basis elements by $e^\alpha e^\beta = e^{\alpha+\beta}$ (and extended to arbitrary elements by bilinearity). Prove that

a, 7pts) A is a domain.

b, 7pts) Every finitely generated ideal in A is principal.

c, 6pts) A is not Noetherian.

Problem 2. Let f be a homogeneous degree d element of $\mathbb{C}[x,y]$. Set $A := \mathbb{C}[x,y]/(f)$ and view A as a $\mathbb{C}[x]$-algebra via the composition $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x,y] \twoheadrightarrow A$. Prove that

a, 10pts) A is a finitely generated $\mathbb{C}[x]$-module if and only if the coefficient of y^d in f is nonzero.

b, 5pts) Prove that if a) holds, then A is a free $\mathbb{C}[x]$-module of rank d.

c, 5pts) Assume now that f is irreducible so that A is a domain, and we can consider the integral closure $\bar{A}$ of A in $\mathrm{Frac}(A)$. Prove that $\bar{A}$ is a free $\mathbb{C}[x]$-module of rank d. *You are allowed to use that $\bar{A}$ is a finite A-algebra, see the bonus to Lecture 10.*

Problem 3. Let A be a Noetherian domain and M be a finitely generated A-module. Prove that the following conditions are equivalent:

a) There is a nonzero element $a \in A$ such that $aM = \{0\}$,

b) $\mathrm{Frac}(A) \otimes_A M = \{0\}$,

c) $\mathrm{Hom}_A(M, A) = \{0\}$.

Problem 4. Let A be a local ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated A-module. Assume that $\bigcap_{i=1}^\infty \mathfrak{m}^i = \{0\}$ (*this is, in fact, true for every local Noetherian ring*). Show that the following three conditions are equivalent:

a) M is free.

b) There are A-module generators $m_1, \ldots, m_k \in M$ and elements $\varphi_1, \ldots, \varphi_k \in \mathrm{Hom}_A(M, A)$ such that the elements $\varphi_i(m_j) \in A$ satisfy $\varphi_i(m_j) - \delta_{ij} \in \mathfrak{m}$ for all i, j, where δ_{ij} is the Kronecker δ.

c) The A-linear map $\mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, A/\mathfrak{m})$ induced by $A \twoheadrightarrow A/\mathfrak{m}$ is surjective.