DAY 3 EXERCISES

1. Modules over quantizations

Exercise 1.1 (Lemma 1.1 in the notes)**.** *Let* $M = \bigcup_{i \in \mathbb{Z}} M_{\leqslant i} = \bigcup_{i \in \mathbb{Z}} M_{\preceq i}$ *be two good filtrations. Then, there exist integers* a, b *such that* $M_{\preceq i+a} \subset M_{\leqslant i} \subset M_{\preceq i+b}$ *for all* $i \in \mathbb{Z}$*.*

Hint: First, show that M is finitely generated. Then show that if $M_{\leqslant i}$ is a good filtration on m, then there exists $m_1, \ldots, m_n \in M$ and integers $d_1, \ldots, d_n$ such that $M_{\leqslant i} = \mathcal{A}_{\leqslant i-d_1} m_1 + \cdots + \mathcal{A}_{\leqslant i-d_n} m_n$. The result now easily follows.

Exercise 1.2. *Let* M*,* N *be finitely generated modules over a filtered algebra* $\mathcal{A}$*. Assume that good filtrations are given on* M *and* N*. Show that* $\operatorname{Ext}^i_{\mathcal{A}}(M, N)$ *admits a filtration for all* i *and, moreover, that we have a natural inclusion* $\operatorname{gr} \operatorname{Ext}^i_{\mathcal{A}}(M, N) \hookrightarrow \operatorname{Ext}_A(\operatorname{gr} M, \operatorname{gr} N)$*.*

Hint: Work at the level of Rees algebras and modules.

2. Localization theorems

Exercise 2.1. *Finish the proof of Lemma 2.1 in the notes. That is, show that if* $\operatorname{Loc}^\theta_\lambda$ *is essentially surjective and* Γ *is exact, then abelian localization holds for* (θ, λ)*.*

Hint: To show that $M \to \Gamma(\operatorname{Loc}^\theta_\lambda(M))$ is an isomorphism, use the fact that we know that this is an isomorphism when M is a free $\mathcal{A}_\lambda$-module. Then you can show that $\mathcal{A}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} \Gamma(\mathcal{M}) \longrightarrow \mathcal{M}$ is an isomorphism (recall that the condition in the exercise is equivalent to saying that every $\mathcal{M} \in \operatorname{Coh}(\mathcal{A}^\theta_\lambda)$ is generated by its global sections and has no higher cohomology).