LECTURE 3 (PART 1): MACDONALD POLYNOMIALS

CHRISTOPHER RYBA

ABSTRACT. These are notes for a seminar talk given at the MIT-Northeastern Double Affine Hecke Algebras and Elliptic Hall Algebras (DAHAEHA) seminar (Spring 2017).

CONTENTS

1. GOALS

The goal of this talk is to define Macdonald Polynomials and prove their existence (in most cases) using operators defined by minuscule coweights. In a different document we will discuss the Macdonald Conjecture, and its proof using Double Affine Hecke Algebras.

2. REVIEW OF NOTATION

2.1. **Root System and Weyl Groups.** We write R for an irreducible finite root system in a vector space V, equipped with inner product $(-,-)$. We write R^a for the associated affine root system. We employ the following notation:

- The set of positive roots of R is denoted R_+, and the set of negative roots is denoted R_-. Similarly we write R^a_+ and R^a_- for the positive and negative roots of R^a, respectively.
- We write $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ for a choice of simple roots in R. The coroot associated to α_i is $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$.
- We write α_0 for $\delta - \theta$ where θ is the longest root in R. In this way $\{\alpha_0, \alpha_1, \cdots, \alpha_n$ form a set of simple roots for R^a.
- The root lattice is $Q = \mathbb{Z}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The coroot lattice is $Q^\vee = \mathbb{Z}\{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_n^\vee\}$.
- The weight lattice is $P = \mathbb{Z}\{\omega_1, \omega_2, \cdots, \omega_n\}$, where ω_i is the i-th fundamental weight ($P = \{\lambda \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in R\}$). The dominant weights are $P_+ = \mathbb{N}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ ($P_+ = \{\lambda \mid (\lambda, \alpha^\vee) \in \mathbb{N} \forall \alpha \in R\}$). Similarly we have the coweight lattice is $P^\vee = \{\lambda \mid (\lambda, \alpha) \in \mathbb{Z} \forall \alpha \in R\}$, and the dominant coweights are $P^\vee = \{\lambda \mid (\lambda, \alpha) \in \mathbb{N} \forall \alpha \in R\}$.
- The half-sum of positive roots is $\rho = \sum_{\alpha \in R_+} \alpha$, and it is well known that $\rho = \sum_{i=1}^n \omega_i$.
- We write $W = \langle s_1, s_2, \cdots, s_n \rangle$ for the (finite) Weyl group associated to R, generated by the simple reflections s_i. We also write $W^a = \langle s_0, s_1, \cdots, s_n \rangle$ for the affine Weyl group. We have the isomorphism
$$W^a = W \ltimes t(Q^\vee)$$
Here, as in previous lectures, t indicates translation in $Q^\vee$, so this is a subgroup of the group of invertible affine maps on $Q^\vee$.

Date: March 3, 2017.

- The extended affine Weyl group is $W^{ae} = W \ltimes t(P^\vee)$ (a subgroup of the group of invertible affine linear maps on $P^\vee$).
- We write $\Omega \subset W^{ae}$ for the set of all length zero elements of W^{ae}. It is a subgroup which acts faithfully on the set of simple roots of R^a. Furthermore, Ω is isomorphic to $P^\vee/Q^\vee$ and is in bijection with minuscule weights (to be discussed later).
- We actually have $W^{ae} = \Omega \ltimes W^a$.
- We will work over the field $\mathbb{C}(q,t)$, which we write $\mathbb{C}_{q,t}$.
- Shortly after the beginning, we will specialise to the case where $t = q^k$ (where $k \in \mathbb{Z}_{\geq 0}$). The notation t appears in parts of the theory directly related to Hecke algebras.

3. Macdonald Polynomials

3.1. **Definition and Proof of Existence.** Recall that the Weyl group acts on the set of weights, P. We may therefore extend the action of W to the group algebra $\mathbb{C}_{q,t}[P]$. We will be concerned with elements of $\mathbb{C}_{q,t}[P]^W$, namely elements of the group algebra which are fixed by the Weyl group action. Note that $\mathbb{C}_{q,t}[P]^W$ is a linear subspace of $\mathbb{C}_{q,t}[P]$. Macdonald polynomials will form a basis of $\mathbb{C}_{q,t}[P]^W$. Note that an obvious basis of $\mathbb{C}_{q,t}[P]^W$ is given by the orbit sums $m_\lambda = \sum_{\mu \in W\lambda} e^\mu$ for $\lambda \in P_+$. By a standard theorem in Lie theory, there is a unique dominant weight in each Weyl group orbit on P, which shows that $\{m_\lambda \mid \lambda \in P_+\}$ is indeed a basis for $\mathbb{C}_{q,t}[P]^W$.

At this point, one might protest that it is unclear how these are polynomials. To answer this, recall that $P = \mathbb{Z}\{\omega_1, \omega_2, \cdots, \omega_n\}$ and therefore $\mathbb{C}_{q,t}[P]$ can be thought of as the algebra of Laurent polynomials in the variables $\omega_1, \omega_2, \cdots, \omega_n$ (with complex coefficients). To conform with standard notation, given $\lambda \in P$, we write e^λ instead of λ for the associated element in $\mathbb{C}_{q,t}[P]$ (this avoids ambiguity between additive and multiplicative notation).

Next, we introduce a bilinear form on $\mathbb{C}_{q,t}[P]$.

Definition 3.1. *If $f \in \mathbb{C}_{q,t}[P]$, write $[f]_0$ for the coefficient of e^0 in f, when expressed in the e^λ basis. Suppose that $f \mapsto \bar{f}$ is the involution of $\mathbb{C}_{q,t}[P]$ defined by $e^\lambda \mapsto e^{-\lambda}$. Let $\Delta_{q,t} = \prod_{\alpha\in R}\prod_{i=0}^{\infty} \frac{1-q^{2i}e^\alpha}{1-t^2q^{2i}e^\alpha}$ (consider this as a Laurent series in the variables q, t, having coefficients in $\mathbb{C}_{q,t}[P]$). Then, we define the bilinear form $\langle -, - \rangle_{q,t}$ on $\mathbb{C}_{q,t}[P]$ as follows:*

$$\langle f, g \rangle_{q,t} = \frac{1}{|W|}[f\Delta_{q,t}\bar{g}]_0$$

We will just be interested in the restriction of the bilinear form to $\mathbb{C}_{q,t}[P]^W$. Motivation for this construction will come later.

Note that if we define $\Delta^+_{q,t} = \prod_{\alpha\in R_+}\prod_{i=0}^{\infty} \frac{1-q^{2i}e^\alpha}{1-t^2q^{2i}e^\alpha}$, then $\Delta_{q,t} = \Delta^+_{q,t}\overline{\Delta^+_{q,t}}$. This will be convenient in what follows. We are now able to give a definition of Macdonald polynomials, although it will not be immediately clear that they exist.

Theorem 3.2. *For each $\lambda \in P_+$, there exists a unique $P_\lambda \in \mathbb{C}_{q,t}[P]^W$ such that:*

1. $P_\lambda = m_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu} m_\mu$
2. $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ *whenever* $\lambda \neq \mu$

Here, $\mu < \lambda$ means that $\lambda - \mu \in Q_+$.

We will prove this theorem (at least, in some special cases), but first we discuss it.

Remark 3.3. *Gram-Schmidt orthogonalisation cannot be applied here because $<$ is not a total order on P_+. However, it does imply uniqueness.*

To see why this is could be an interesting construction, we consider some examples.

Example 3.4. *Suppose that $t = 1$, so that $\Delta_{q,t} = 1$. Then $\langle f, g \rangle_{q,t} = [f\bar{g}]_0$, and it is easy to see that $P_\lambda = m_\lambda$ satisfy the statement of the theorem (and this does not depend on q).*

Now suppose that $t = q$. Then $\Delta^+_{q,t} = \prod_{\alpha\in R_+}(1-e^{\alpha})$ because the product telescopes. Let χ_λ be given by the Weyl Character Formula for $\lambda \in P_+$:

$$\chi_\lambda = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}$$

We calculate $\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t}$.

$$\begin{aligned}
\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t} &= \frac{1}{|W|}[\chi_{\lambda_1}\Delta_{q,t}\overline{\chi_{\lambda_2}}]_0 \\
&= \frac{1}{|W|}[\left(\chi_{\lambda_1}\overline{\Delta^+_{q,t}}\right)\left(\Delta^+_{q,t}\overline{\chi_{\lambda_2}}\right)]_0 \\
&= \frac{1}{|W|}[\sum_{w_1\in W}\varepsilon(w_1)e^{w_1(\lambda_1+\rho)-\rho}\sum_{w_2\in W}\varepsilon(w_2)e^{-w_2(\lambda_2+\rho)+\rho}]_0 \\
&= \frac{1}{|W|}\sum_{w_1\in W}\sum_{w_2\in W}\varepsilon(w_1w_2)[e^{w_1(\lambda_1+\rho)-w_2(\lambda_2+\rho)}]_0
\end{aligned}$$

Note that the nonzero terms are precisely those for which $w_1(\lambda_1+\rho) = w_2(\lambda_2+\rho)$. This is equivalent to $\lambda_1+\rho = w_1^{-1}w_2(\lambda_2+\rho)$. Using the fact that each weight has a unique dominant weight in its orbit in the Weyl group, we see that this equation can only hold if $\lambda_1+\rho = \lambda_2+\rho$ (the latter is the unique dominant weight in its orbit). So we get zero unless $\lambda_1 = \lambda_2$. Furthermore, the terms which contribute 1 *are those for which $w_1^{-1}w_2$ fixes $\lambda_2+\rho$. Recall the length of a Weyl group element is equal to the number of positive roots that it maps to negative roots, so that*

$$w_1^{-1}w_2(\rho) = \frac{1}{2}\sum_{\alpha\in R_+} w_1^{-1}w_2(\alpha) \le \frac{1}{2}\sum_{\alpha\in R_+}\alpha = \rho$$

with equality if and only if $w_1^{-1}w_2$ is the identity element of W. Since $w_1^{-1}w_2(\lambda) \le \lambda$, we may add these two inequalities to find that the identity is the only element of W that fixes $\lambda_2+\rho$. The number of solutions (w_1, w_2) to $w_1^{-1}w_2 = \mathrm{Id}_W$ is clearly $|W|$, so we obtain

$$\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t} = \frac{|W|}{|W|} = 1$$

Finally, we write $\chi_\lambda = \sum_\mu a_{\lambda,\mu}m_\mu$, where the $a_{\lambda,\mu}$ correspond to the dimensions of weight spaces in the irreducible representation of the relevant simple Lie algebra of highest weight λ. Recall that the irreducible representation is generated by the action of the lower triangular part of the Lie algebra (usually written $\mathfrak{n}_-$) on a highest weight vector, which itself is unique up to scalar multiplication. This implies that only $\mu \le \lambda$ appear in the sum, and that $a_{\lambda,\lambda} = 1$. This proves that χ_λ satisfy the conditions of the theorem.

In light of the preceding example, it might be reasonable to view Macdonald polynomials as a deformation of characters of representations of simple Lie algebras.

Example 3.5. *Let $V = \mathbb{R}^n$ with standard basis e_i, and let $R = \{e_i - e_j \mid i \ne j\}$ so that R is a root system of type A_{n-1} and we may take $\alpha_i = e_i - e_{i+1}$. The associated simple Lie algebra is $\mathfrak{sl}_n$, and the Weyl group is $W = S_n$, which acts on V by permutation of coordinates. Since $\alpha = \alpha^\vee$ for all $\alpha \in R$, the weights and the coweights of R are the same. Let $\mathbb{C}_{q,t}[P]$ be presented by letting $x_i = \exp(e_i)$, so that $e^{\alpha_i} = \frac{x_i}{x_{i+1}}$. Then $\mathbb{C}_{q,t}[P]$ is realised as the space of Laurent polynomials in $x_1, x_2, \cdots, x_n$ of total degree zero. Then it is easily seen that the positive roots correspond to $\frac{x_i}{x_j}$ with $i < j$. It is also easy to see that if $\lambda \in P$ corresponds to $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$ (where necessarily the λ_i sum to zero), then the value of the fundamental weight ω_r applied to λ is $\omega_r(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. It is also easy to see that being dominant is equivalent to having the λ_i forming a weakly decreasing sequence, and λ is integral if and only if the λ_i are integers.*

We now prove theorem 3.2 in the case where $t = q^k$, for $k \in \mathbb{Z}_{\ge 0}$, and when there are minuscule weights associated to the root system R (so R cannot be G_2, F_4, E_8). Although these restrictions are not required, they mitigate technical difficulties. The specialisation of the parameter t is the case relevant to the Macdonald conjecture, so not much will be lost to us. The reader who is interested in greater generality is directed to [Mac00].

Proof. Firstly, note that the product in the definition of $\Delta^+_{q,t}$ telescopes:

$$\begin{aligned}
\Delta^+_{q,t} &= \prod_{\alpha\in R_+}\prod_{i=0}^{\infty}\frac{1-q^{2i}e^{\alpha}}{1-t^2q^{2i}e^{\alpha}} \\
&= \prod_{\alpha\in R_+}\prod_{i=0}^{\infty}\frac{1-q^{2i}e^{\alpha}}{1-q^{2k}q^{2i}e^{\alpha}} \\
&= \prod_{\alpha\in R_+}\prod_{i=0}^{k-1}(1-q^{2i}e^{\alpha})
\end{aligned}$$

In particular, we obtain a finite expression. Next, if $\pi\in P^\vee$ we define $T_\pi(e^\lambda)=q^{2(\pi,\lambda)e^\lambda}$ (and extend linearly), where we may have to include fractional powers of q in our ring. Now let us write

$$D_\pi(f)=\sum_{w\in W} w\left(\frac{T_\pi(\Delta^+_{q,t}(f))}{\Delta^+_{q,t}}\right)$$

In the case where π is a minuscule coweight (i.e. $0\le(\lambda,\alpha)\le 1$ for all positive roots $\alpha\in R_+$), this simplifies as follows.

$$D_\pi(f)=\left(\sum_{w\in W} w\right)\left(\prod_{\alpha\in R_+,(\pi,\alpha)=1}\frac{1-q^{2k}e^{\alpha}}{1-e^{\alpha}}\right)T_\pi(f) \tag{1}$$

This is clearly the symmetrisation of some rational function, whose denominator is a product of distinct terms of the form $(1-e^\alpha)$. It is certainly W-invariant. In particular, let $\delta=\prod_{\alpha\in R_+}\frac{1}{e^{\alpha/2}-e^{-\alpha/2}}$ be the Weyl denominator, which is antisymmetric (implicitly we are now working in $e^\rho\mathbb{C}_{q,t}[P]$). To see this, recall that the action of $s_i\in W$ permutes the positive roots except for α_i which it maps to $-\alpha_i$. Thus s_i acts by multiplication by -1 on $\frac{1}{e^{\alpha_i/2}-e^{-\alpha_i/2}}$, and permutes the other factors of δ. Thus $\delta T_\pi(f)$ is antisymmetric with respect to the W-action, and is also a polynomial (we have removed the denominators). In particular, for any s_i, the coefficient of e^λ must be minus the coefficient of $e^{s_i(\lambda)}$. This means that no e^λ fixed by s_i can occur in $\delta T_\pi(f)$, so $\delta T_\pi(f)$ is a linear combination of $e^\lambda-e^{s_i(\lambda)}$. Now observe that

$$e^\lambda-e^{s_i(\lambda)}=e^\lambda-e^{\lambda-(\lambda,\alpha_i^\vee)\alpha_i}=e^\lambda(1-e^{-(\lambda,\alpha_i^\vee)\alpha_i})=(1-e^{-\alpha_i})(1+e^{-\alpha_i}+\cdots+e^{-((\lambda,\alpha_i^\vee)-1)\alpha_i})$$

This is therefore divisible by $e^{\alpha_i/2}-e^{-\alpha_i/2}$ for each simple root α_i. As a result, the same is true of $\delta D_\pi(f)$. Since each root is in the orbit of a simple root, by applying the action of a suitable element of W we find that $e^{\alpha/2}-e^{-\alpha/2}$ divides $\delta D_\pi(f)$ for all positive roots α. It is not difficult to check that these are coprime in the UFD $\mathbb{C}_{q,t}[P/2]$. Hence, $\delta D_\pi(f)$ is divisible by $\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})=\delta$ (in the sense of polynomial divisibility). We conclude that $D_\pi(f)$ is actually a polynomial (rather than a rational function), which preserves $\mathbb{C}_{q,t}[P]^W$.

If we can show that D_π is triangular with respect to the e^λ basis, and is self-adjoint with respect to $\langle -,-\rangle_{q,t}$ with distinct eigenvalues, then the theorem will follow. This is because triangularity allows us to restrict to the finite dimensional subspace spanned by m_μ for $\mu\le\lambda$, whence distinct eigenvalues guarantee diagonalisability. Finally, self adjointness (and distinctness of eigenvalues) implies the eigenvectors are orthogonal. The Macdonald polynomials will be the eigenvectors of this operator.

To see the self-adjoint property, recall that $\Delta_{q,t}$ is W-invariant, as is the m_λ basis:

$$\begin{aligned}
\frac{1}{|W|}[D_\pi(m_\lambda)\Delta_{q,t}\bar{e^{\mu}}]_0 &= \frac{1}{|W|}[\sum_{w\in W} w\left(\frac{T_\pi(\Delta^+_{q,t}m_\lambda)}{\Delta^+_{q,t}}\right)\Delta_{q,t}\overline{m_\mu}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W}\left(\frac{T_\pi(\Delta^+_{q,t}m_\lambda)}{\Delta^+_{q,t}}\right)\Delta_{q,t}\overline{m_{w^{-1}\mu}}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} T_\pi(\Delta^+_{q,t}m_\lambda)\overline{\Delta^+_{q,t}}\overline{m_\mu}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} \Delta^+_{q,t}m_\lambda T_{-\pi}(\overline{\Delta^+_{q,t}}\overline{m_\mu})]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} m_\lambda\Delta^+_{q,t}\overline{T_\pi((\Delta^+_{q,t}m_\mu))}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\sum_{w\in W}\frac{\overline{T_\pi((\Delta^+_{q,t}m_\mu))}}{\overline{\Delta^+_{q,t}}}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\sum_{w\in W} w^{-1}\frac{\overline{T_\pi(\Delta^+_{q,t}m_\mu)}}{\overline{\Delta^+_{q,t}}}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\overline{D_\pi(m_\mu)}]_0
\end{aligned}$$

We calculate the leading order term, and in doing so, observe triangularity. For this, we use a deformed version of the Weyl characters χ_λ. Throughout we consider everything as formal series of the form $c_\lambda e^\lambda + \sum_{\mu<\lambda} c_\mu e^\mu$ (the c_λ being constants), where we refer to $c_\lambda e^\lambda$ as the leading term.

$$\chi^{(k)}_\lambda = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+k\rho)}}{\prod_{\alpha\in R_+}(e^{k\alpha/2}-e^{-k\alpha/2})}$$

This is a formal series rather than a polynomial. The numerator has leading order term $e^{\lambda+k\rho}$, and the denominator has leading order term $e^{k\rho}$, so it is easy to see that the $\chi^{(k)}_\lambda$ (for $\lambda \geq 0$) have leading term e^λ and so are related to the e^λ by a triangular matrix. Thus, it will be enough to consider $D_\pi(\chi^{(k)}_\lambda)$ to prove triangularity and calculate the eigenvalues. So, we write

$$D_\pi(\chi^{(k)}_\lambda) = \sum_{w'\in W}\left[w'\left(\prod_{\alpha\in R_+,(\pi,\alpha)=1}\frac{1-q^{2k}e^\alpha}{1-e^\alpha}\right)\right]\left[w'(T_\pi(\chi^{(k)}_\lambda))\right]$$

We calculate the leading order term of each set of square brackets separately. This means we will write it as a constant times e^μ plus terms indexed by weights lower than μ in the dominance order. We first consider the action of w' on each factor of

$$\left(\prod_{\alpha\in R_+,(\pi,\alpha)=1}\frac{1-q^{2k}e^\alpha}{1-e^\alpha}\right) = \left(\prod_{\alpha\in R_+}\frac{1-q^{2k(\pi,\alpha)}e^\alpha}{1-e^\alpha}\right)$$

The action of w' on

$$\frac{1-q^{2k(\pi,\alpha)}e^\alpha}{1-e^\alpha}$$

is

$$\frac{1-q^{2k(\pi,\alpha)}e^{w'(\alpha)}}{1-e^{w'(\alpha)}}$$

If $w'(\alpha)$ is negative root, the leading term is just 1, otherwise we may write this as

$$\frac{q^{2k(\pi,\alpha)} - e^{-w'(\alpha)}}{1 - e^{-w'(\alpha)}}$$

whence the leading term is clearly $q^{2k(\pi,\alpha)}$. Thus the total contribution to the leading order term is $q^{(\pi,2k\nu)}$, where ν is the sum of all positive roots α such that $w'(\alpha)$ is also positive.

Upon applying T_π to $\chi_\lambda^{(k)}$, we get

$$T_\pi(\chi_\lambda^{(k)}) = \frac{\sum_{w\in W} \varepsilon(w) q^{2(\pi,w(\lambda+k\rho))} e^{w(\lambda+k\rho)}}{\prod_{\alpha\in R_+}(q^{(\pi,k\alpha)}e^{k\alpha/2} - q^{(\pi,-k\alpha)}e^{-k\alpha/2})}$$

If the Weyl group element w' is applied to this expression, we get

$$w'(T_\pi(\chi_\lambda^{(k)})) = \frac{\sum_{w\in W} \varepsilon(w) q^{2(\pi,w(\lambda+k\rho))} e^{w'(w(\lambda+k\rho))}}{\prod_{\alpha\in R_+} w'(q^{(\pi,k\alpha)}e^{k\alpha/2} - q^{(\pi,-k\alpha)}e^{-k\alpha/2})}$$

We expand the denominator as a series of the form e^μ plus lower order terms. To extract the leading order term from $\prod_{\alpha\in R_+} w'(q^{(\pi,k\alpha)}e^{k\alpha/2} - q^{(\pi,-k\alpha)}e^{-k\alpha/2})$, we first note that we pick up a sign for each α mapped to a negative root, thus obtaining $\varepsilon(w')$. We pick the term corresponding to the positive root in each $w'(\alpha)/2, -w'(\alpha)/2$ pair, obtaining the following:

$$\varepsilon(w')e^{k\rho}q^{(\pi,k\nu')}$$

Here $\nu' = \sum_{\alpha\in R_+}\sigma(w'(\alpha))\alpha$, where $\sigma(\alpha)$ is the sign of a root. Clearly $2\nu - \nu' = 2\rho$. So our leading term so far is $q^{(\pi,2k\rho)}/\varepsilon(w')e^{k\rho}$. The leading term of the numerator arises when $w = (w')^{-1}$, when we get $\varepsilon((w')^{-1})q^{2(\pi,(w')^{-1}(\lambda+k\rho))}e^{\lambda+k\rho}$. Taking the product of these (and noting that $\varepsilon(w') = \varepsilon((w')^{-1})$), we obtain

$$q^{2(\pi,(w')^{-1}(\lambda+k\rho))}q^{(\pi,k(2\rho))}e^{\lambda}$$

Finally, we sum over $w' \in W$ to get the coefficient

$$q^{2(\pi,k\rho)}\sum_{w\in W} q^{2(\pi,w(\lambda+k\rho))}$$

These are not necessarily distinct for distinct $\lambda \in P_+$. However, one can find a suitable coweight π in types A and B (and E_6 and E_7). In type D it is possible to find a linear combination of the operators D_π with this property. We demonstrate the case of type A below. $\square$

Example 3.6. *Suppose that R is the root system A_{n-1} as before. Then the positive roots are $e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \le i < j \le n$. All coefficients are zero or one, so it is clear that the fundamental weights ω_i are minuscule coweights (note that $\alpha = \alpha^\vee$ for all $\alpha \in R$). Additionally $\rho = (n-1, n-3, \cdots, 1-n)$.*

We are given (a constant independent of λ multiplied by) $\sum_{w\in W} q^{2(\pi,w(\lambda+k\rho))}$, which is equivalent to knowing the multiset of values $(w^{-1}(\pi), \lambda + k\rho)$ as w ranges across W. In general, this does not determine λ, but for certain π, it does. For example, in type A_3, we may choose $\lambda_1 = (4, 0, -2, -2)$ and $\lambda_2 = (2, 2, 0, -4)$, so that $\lambda_1 + k\rho = (3k + 4, k, -k - 2, -3k - 2)$ and $\lambda_2 + k\rho = (3k + 2, k + 2, -k, -3k - 4)$. Then we see that $\pi_2 = (1/2, 1/2, -1/2, -1/2)$ is a fundamental (co)weight. But, one can check that in both cases, the multiset of values of $(w^{-1}(\pi_2), \lambda_i + k\rho)$ is $\{4k + 4, 2k + 2, 2, -2, -2k - 2, -4k - 4\}$. However, if π_1 is used instead, it is easy to restrict λ from the multiset $(w^{-1}(\pi_1), \lambda + k\rho)$. For example, we may add any multiple of $(1, 1, \cdots, 1)$ to π_1 without affecting the inner product, allowing us to assume $\pi_1 = (1, 0, 0, \cdots, 0)$. Thus, the inner product gives the coordinates of the vector $\lambda + k\rho$. Since these are strictly decreasing, they determine $\lambda + k\rho$, and hence λ.

Proposition 3.7. *The D_π operators commute.*

Proof. Let D be the operator with distinct eigenvalues that was used to construct the Macdonald Polynomials. For π a minuscule coweight, consider $D_\pi + cD$, where $c \in \mathbb{C}_{q,t}$. Since D has distinct eigenvalues, this linear combination has district eigenvalues for generic c. This means that this linear combination of operators is diagonalisable, and as before, its eigenvectors are the Macdonald polynomials. Since the Macodonald polynomials are unique, this means that D and $D_\pi + cD$ are diagonalisable with the same eigenbasis. We conclude that D_π is diagonalisable, with Macdonald polynomials as eigenvectors. This means that in the basis of Macdonald polynomials, the D_π are diagonal operators, and hence commute. □

Example 3.8. *Continuing with R being of type A_{n-1} as in example 3.5, we recall that the positive roots correspond to $\frac{x_i}{x_j}$ with $i < j$. In this setting, T_{ω_r} can be taken to send x_i to $q^2 x_i$ if $i \le r$ and to x_i otherwise. In this way $\frac{x_i}{x_{i+1}}$ is unchanged unless $i = r$, in which case it is multiplied by q^2, which is the correct action. In fact, this makes it easy to write down explicit formulae for D_{ω_r} (acting on $\mathbb{C}_{q,t}[P]^W$) in terms of the "shift operators" T_i. If $I \subset \{1, 2, \cdots, n\}$, we write $T_I = \prod_{i \in I} T_i$ (the order of composition is unimportant since these operators clearly commute). Using equation 1, we have*

$$\begin{aligned} D_{\omega_r} &= \sum_{w \in S_n} w \left(\prod_{1 \le i \le r < j \le n} \frac{1 - q^{2k} \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} T_{\{1,2,\cdots,r\}} \right) \\ &= \sum_{w \in S_n} w \left(\prod_{1 \le i \le r < j \le n} \frac{x_j - q^{2k} x_i}{x_j - x_i} T_{\{1,2,\cdots,r\}} \right) \\ &= r!(n-r)! \sum_{I \subset \{1,2,\cdots,n\}, |I| = r} \left(\prod_{i \in I, j \notin I} \frac{x_j - q^{2k} x_i}{x_j - x_i} \right) T_I \end{aligned}$$

Here we have used the fact that S_n acts transitively on r-element subsets of $\{1, 2, \cdots, n\}$ with stabiliser of size $r!(n-r)!$. It is easy to see that these commute.

Remark 3.9. *The proof in* [Mac00] *begins by introducing the concept of a "quasi-minuscule weight".*

References

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 2nd edition, 1995.

[Mac00] I. G. Macdonald. Orthogonal polynomials associated with root systems. *Séminaire Lotharingien de Combinatoire*, 45:B45a, 2000.