

## DAY 1 EXERCISES

### 1. GEOMETRIC INVARIANT THEORY

Throughout this section,  $G$  is a reductive algebraic group acting on the affine variety  $X$ , and  $\theta : G \rightarrow \mathbb{C}^\times$  is a character.

**Exercise 1.1** (Lemma 1.3 in the notes). *Consider the  $G$ -action on  $X \times \mathbb{C}$  given by  $g(x, z) = (gx, \theta(g)z)$ . Show that an element  $x$  is in  $X^{\theta-ss}$  if and only if  $\overline{G(x, 1)}$  does not intersect  $X \times \{0\}$ . Equivalently, if and only if there is no one parameter subgroup  $\gamma : \mathbb{C}^\times \rightarrow G$  such that  $\lim_{t \rightarrow 0} \gamma(t)x$  exists and  $\theta(\gamma(t)) = t^m$  with  $m > 0$ .*

Hint: What is  $\mathbb{C}[X \times \mathbb{C}]^G$ ?

**Exercise 1.2** (Lemma 1.4 in the notes). *Show that the  $G$  orbit of a collection  $(x_a, i_k) \in R := R(Q, v, w)$  is closed if and only if  $i_k = 0$  for all  $k \in Q_0$  and the representation  $(x_a)_{a \in Q_1} \in R(Q, v, 0)$  is semisimple.*

Hint: That the  $G$ -orbit of  $(x_a, i_k)$  is closed implies that  $i_k = 0$  and  $(x_a)$  is semisimple follows by considering appropriate 1-parametric subgroups. The other implication follows from the previous implication and the Hilbert-Mumford criterion.

**Exercise 1.3** (Lemma 1.5 in the notes). *If  $\theta_k > 0$  for all  $k \in Q_0$ , show that the subset  $R^{\theta-ss}$  consists of all representations  $(x_a, i_k)$  such that the only  $x_a$ -stable collection of subspaces in  $(\ker i_k)_{k \in Q_0}$  is the zero one.*

Hint: Assume that there is a nonzero  $x_a$ -stable collection of subspaces in  $(\ker i_k)_{k \in Q_0}$ . Use Exercise 1.1 to show that  $(x_a, i_k)$  is not  $\theta$ -semistable. For the converse implication, use Exercise 1.1 again, together with the Hilbert-Mumford criterion and  $\mathbb{C}^\times$ -representation theory.

### 2. MOMENT MAPS AND HAMILTONIAN REDUCTION

**Exercise 2.1.** Let  $G$  act on an affine variety  $X_0$ . Lift this action to an action on  $X := T^*X_0$ . Show that  $\mu^* : \mathfrak{g} \rightarrow \mathbb{C}[T^*X]$ ,  $\xi \mapsto \xi_{X_0}$ , is a comoment map for this action. You will need to use the following formula for the Poisson bracket on  $\mathbb{C}[T^*X_0] = S_{\mathbb{C}[X_0]} \text{Vect}(X_0)$ . For  $f, g \in \mathbb{C}[X_0]$  and  $\xi, \eta \in \text{Vect}(X_0)$  we have  $\{f, g\} = 0$ ,  $\{\xi, f\} = L_\xi f$  and  $\{\xi, \eta\} = [\xi, \eta]$ . Here,  $L_\xi$  is the Lie derivative and  $[\cdot, \cdot]$  stands for the commutator of vector fields.

**Exercise 2.2.** The kernel of  $d_x\mu$  coincides with the  $\omega$ -orthogonal complement of  $T_x(Gx)$ , and the image of  $d_x\mu$  is the annihilator of  $\mathfrak{g}_x := \{\xi \in \mathfrak{g} : \xi_{X,x} = 0\}$ . In particular,  $\mu$  is a submersion at  $x$  provided the stabilizer  $G_x$  is finite.

Hint: Recall that the moment map  $\mu : X \rightarrow \mathfrak{g}^*$  satisfies  $\langle d_x\mu(\bullet), \xi \rangle = \omega_x(\bullet, \xi_x)$  for every  $x \in X, \xi \in \mathfrak{g}$ . From here the result follows easily.

**Exercise 2.3.** Suppose that the action of  $G$  on  $\mu^{-1}(\lambda)$  is free. Show that there is a unique symplectic form  $\underline{\omega}$  on  $X//_\lambda G$  satisfying  $\pi^*\underline{\omega} = \iota^*\omega$ , where  $\pi : \mu^{-1}(\lambda) \rightarrow X//_\lambda G$  is the projection and  $\iota : \mu^{-1}(\lambda) \rightarrow X$  is the inclusion.

Hint: For each  $x \in \mu^{-1}(\lambda)$ , express  $T_{\pi(x)}(X//_\lambda G)$  in terms of  $T_x X$ . The previous exercise may come in handy.

### 3. NAKAJIMA QUIVER VARIETIES

**Exercise 3.1.** (a) Let  $U_1, U_2$  be symplectic vector spaces and  $G$  an algebraic group acting on  $U_1, U_2$ . Assume the actions are Hamiltonian with respective moment maps  $\mu_1 : U_1 \rightarrow \mathfrak{g}^*$ ,  $\mu_2 : U_2 \rightarrow \mathfrak{g}_*$ . Show that the action of  $G$  on  $U_1 \oplus U_2$  is Hamiltonian with moment map  $\mu_1 \oplus \mu_2$ .

- (b) Let  $U$  be a symplectic vector space. Assume  $G_1, G_2$  act on  $U$  in a Hamiltonian way, with moment maps  $\mu_1, \mu_2$ , respectively. Moreover, assume that the actions of  $G_1$  and  $G_2$  commute. Then the  $G_1 \times G_2$ -action on  $U$  is Hamiltonian with moment map  $\mu = (\mu_1, \mu_2)$ .

**Exercise 3.2.** Let  $X, Y \in \text{End}(V)$  be such that  $\text{rk}[X, Y] = 1$ . Show that  $X$  and  $Y$  are simultaneously triangularizable.

Hint: You have to check that there is a proper subspace of  $V$  that is invariant under the actions of  $X$  and  $Y$ . Note that we may assume w.l.o.g. that  $\ker(Y) \neq 0$ . Then separate the cases  $\ker(Y) \subseteq \ker([X, Y])$  and  $\ker(Y) \not\subseteq \ker([X, Y])$ .

**Exercise 3.3.** Let  $V$  be a finite dimensional vector space, and  $X, Y \in \text{End}(V)$ ,  $i \in V, j \in V^*$ . Assume that  $[X, Y] - ij = 0$  and that  $\mathbb{C}\langle X, Y \rangle i = V$ . Show that  $j = 0$ .

Hint: Use the previous exercise.