Free Field Realization Part 1. Finite-dimensional case

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Abstract

The notes are prepared for the seminar *Representations of affine Kac-Moody algebras at the critical level* at MIT in Spring 2024.

1 Overview

We are in the process of proving an isomorphism between $\mathfrak{z}(V_{\kappa_c}(\mathfrak{g}))$ and $\mathbb{C}[\mathrm{Op}_{{}^LG}(D)]$, which further implies an isomorphism $Z(\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})) \cong \mathbb{C}[\mathrm{Op}_{{}^LG}(D^\times)]$. The strategy for the proof is to embed both algebras inside $\mathbb{C}[\mathfrak{h}^*[[t]]dt]$.

We begin by reviewing the counterpart of the embedding $\mathfrak{z}(V_{\kappa_c}(\mathfrak{g})) \to \mathbb{C}[\mathfrak{h}^*[[t]]dt]$ in the finite-dimensional case, which coincides with the map used in the Harish-Chandra isomorphism, as well as constructions that are useful in the affine case.

2 Recollection about vector fields

Recall that if we have a group action $\alpha : G \times X \to G$, with G a Lie group, there is an induced Lie algebra homomorphism

$$\alpha_* : \mathfrak{g} \longrightarrow \mathrm{Vect}(X)$$

sending $z \in \mathfrak{g}$ to the vector field $\alpha_*(z)$ on X defined by

$$(\alpha_*(z)f)(x) \coloneqq \frac{d}{dt}\bigg|_{t=0} f(\exp(-tz)x).$$

Let us briefly review some properties of vector fields. For a smooth affine scheme X, the vector fields on X are precisely the derivations on the ring of functions $\mathbb{C}[X]$. For smooth affine schemes X and Y, we have a map $\mathrm{Vect}(X \times Y) \to \mathbb{C}[X] \otimes \mathrm{Vect}(Y) \oplus \mathbb{C}[Y] \otimes \mathrm{Vect}(X)$, or equivalently,

$$\mathrm{Der}(\mathbb{C}[X] \otimes \mathbb{C}[Y]) \longrightarrow (\mathbb{C}[X] \otimes \mathrm{Der}(\mathbb{C}[Y])) \oplus (\mathbb{C}[Y] \otimes \mathrm{Der}(\mathbb{C}[X])) \tag{1}$$

which sends $\varphi \longmapsto (\varphi|_{1\otimes\mathbb{C}[Y]}, \varphi|_{\mathbb{C}[X]\otimes 1})$. It is an isomorphism of Lie algebras with respect to the Lie bracket on RHS defined by $[f\varphi, g\psi] = f\varphi(g)\psi - g\psi(f)\varphi$ for $f \in \mathbb{C}[X]$, $\varphi \in \mathrm{Vect}(Y)$, $g \in \mathbb{C}[Y]$ and $\psi \in \mathrm{Vect}(X)$. Consequently, letting $D(X)$ be the algebra of differential operators on X, we can check that

$$D(X \times Y) = D(X) \otimes D(Y)$$

is an isomorphism.

Let G be an algebraic group G The Lie algebra of left-invariant (resp. right-invariant) vector fields $\mathfrak{g}_l$ (resp. $\mathfrak{g}_r$) are identified with $\mathfrak{g}$.

The action of G on itself induces a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathrm{Vect}(G)$ which is equivariant for the adjoint action of G on $\mathfrak{g}$ and the left G-action on $\mathrm{Vect}(G)$. It factors through an isomorphism

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}_r \subset \mathrm{Vect}(G)$$

mapping $x \in \mathfrak{g}$ to $-x_r$ where x_r is the corresponding right G-equivariant vector field.

Remark 2.1. If G is abelian, then we can identify x_l and x_r. In general, letting $\iota : G \to G$ be the inversion, $d\iota$ gives an isomorphism between $\mathfrak{g}_l$ and $\mathfrak{g}_r$.

Remark 2.2. The inclusion $\mathfrak{g}_l \subset D(G)^{G_l}$ lifts to an isomorphism $U(\mathfrak{g}_l) \cong D(G)^{G_l}$. Similarly, $D(G)^{G_r} \cong U(G_r)$.

3 Construction of algebra homomorphism $U(\mathfrak{g}) \to D(N_+) \otimes U(\mathfrak{h})$

Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ with Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and let $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ be Borel subalgebras.

Let G be the connected simply connected algebraic group corresponding to $\mathfrak{g}$, and let $N_\pm$ and $B_\pm$ be the unipotent and Borel subgroups corresponding to $\mathfrak{n}_\pm$ and $\mathfrak{b}_\pm$ respectively. There is an isomorphism of varieties $N_+ \times H \cong B_+$ sending $(n,t) \mapsto nt$. Note that $N_+ \times H$ admits the following actions:

- An N_+-action from the left: $(n', (n,t)) \mapsto (n'n, t)$
- An H-action from the left: $(t', (n,t)) \mapsto (t'nt'^{-1}, t't)$
- An H-action from the right: $((n,t), t'') \mapsto (n, tt'')$

Now the isomorphism 1 becomes

$$\mathrm{Vect}(N_+ \times H) \cong (\mathrm{Vect}(N_+) \otimes \mathbb{C}[H]) \oplus (\mathrm{Vect}(H) \otimes \mathbb{C}[N_+])) \otimes \mathbb{C}[N_+]).$$

Noting that right H-action is trivial on $\mathfrak{h}$ and $\mathrm{Vect}(N_+)$, while standard on $\mathbb{C}[H]$, it follows that

$$\mathrm{Vect}(N_+ \times H)^{H_r} \cong \mathrm{Vect}(N_+) \oplus (\mathbb{C}[N_+] \otimes \mathfrak{h}). \tag{2}$$

Remark 3.1. We can upgrade the isomorphism 2 to the level of algebra

$$D(N_+ \times H)^{H_r} = (D(N_+) \times D(H))^{H_r} = D(N_+) \otimes U(\mathfrak{h}).$$

Now consider the homogeneous space G/N_- with left G-action and right H-action, noting that these actions commute. There is an induced map of Lie algebras

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(G/N_-)^{H_r}.$$

By considering the restriction to the open B_+-orbit $B_+[1] \subset G/N_-$, it induces a map of Lie algebras

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^H = \mathrm{Vect}(N_+) \oplus (\mathbb{C}[N_+] \otimes \mathfrak{h}).$$

Remark 3.2. We first note that the map $U(\mathfrak{g}) \to D(B_+)$ induced by $\mathfrak{g} \to \mathrm{Vect}(B_+)$ preserves the filtrations with respect to the PBW filtration on $U(\mathfrak{g})$ and the order of differential operators on $D[B_+]$. Then, the associated graded $S(\mathfrak{g}) \longrightarrow \mathbb{C}[T^*B_+]$, where T^*B_+ denotes the cotangent bundle, can be described as the composition of the following maps:

1. The classical comoment map $S(\mathfrak{g}) \longrightarrow \mathbb{C}[T^*(G/N_-)]$
2. The restriction $\mathbb{C}[T^*(G/N_-)]] \longrightarrow \mathbb{C}[T^*B_+]$

The first map is injective because $T^*(G/N_-) \to \mathfrak{g}^*$ is dominant, while the second map is clearly injective.

4 Geometric realization of dual Verma modules

Definition 4.1. Let $\chi \in \mathfrak{h}^*$. Consider the one-dimensional representation $\mathbb{C}_\chi$ of $\mathfrak{b}_+$ on which $\mathfrak{h}$ acts by χ and $\mathfrak{n}_+$ acts by zero. The *Verma module* with highest weight $\chi \in \mathfrak{h}^*$ is the $\mathfrak{g}$-module defined by

$$M_\chi \coloneqq \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\chi.$$

Remark 4.2. The underlying $\mathfrak{n}_-$-module of M_χ is always isomorphic to $U(\mathfrak{n}_-)$, while its the $\mathfrak{h}$-module structure is the tensor product $U(\mathfrak{n}_-) \otimes \mathbb{C}_\chi$.

Remark 4.3. Noting that we have the weight decomposition $M_\chi = \bigoplus_{\mu \in \chi - Q_+} M_\chi[\mu]$, where Q_+ is the positive part of the root lattice of $\mathfrak{g}$, i.e., $Q_+ = \{\sum_i n_i \alpha_i : n_i \geq 0\}$. The *dual* $\mathfrak{g}$*-module* M_χ^* is the $\mathfrak{g}$-module

$$M_\chi^* \coloneqq \bigoplus_{\mu \in \chi - Q_+} M_\chi[\mu]^\vee$$

with the $\mathfrak{g}$-action defined by

$$(x \cdot \varphi)(m) = \varphi(-\tau(x) \cdot m)$$

for $x \in \mathfrak{g}$, $\varphi \in M_\chi^*$ and $m \in M_\chi$, where τ is the involutive automorpshim on $\mathfrak{g}$ such that $\tau(h_i) = -h_i$, $\tau(e_i) = f_i$, $\tau(f_i) = e_i$.

Exercise 4.4. The duality functor is exact and contravariant, and its square is the identity functor. Moreover, the duality functor preserves the formal character.

We now define a modified $\mathfrak{g}$-module structure on $\mathbb{C}[N_+]$ which depends on $\chi \in \mathfrak{h}^*$.

Definition 4.5. For $\chi \in \mathfrak{h}^*$, let us write $\mathrm{ev}_\chi : U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}$. The modified $\mathfrak{g}$-module structure on $\mathbb{C}[N_+]$ is defined by the composition

$$U(\mathfrak{g}) \longrightarrow D(N_+) \otimes U\mathfrak{h} \xrightarrow{-\otimes \mathrm{ev}_\chi} D(N_+),$$

noting that $\mathbb{C}[N_+]$ is naturally a $D(N_+)$-module. The resulting $\mathfrak{g}$-module is denoted by $\mathbb{C}[N_+]_\chi$.

Theorem 4.6. *There is an isomorphism between* $\mathfrak{g}$*-modules*

$$\mathbb{C}[N_+]_\chi \cong M_\chi^*.$$

Proof. We will prove the dual statement $\mathbb{C}[N_+]_\chi^* \cong M_\chi$. We first note that the Killing form identifies $\mathfrak{n}_+^*$ with $\mathfrak{n}_-$. The exponential map identified N_+ with $\mathfrak{n}_+$, and it is H-equivariant. It follows that the character of the dual $\mathfrak{g}$-module $\mathbb{C}[N_+]_\chi^*$ can be computed as

$$\begin{aligned} \mathrm{char}\, \mathbb{C}[N_+]_\chi^* = \mathrm{char}\, \mathbb{C}[N_+]_\chi &= e^\chi \sum_\lambda \mathbb{C}[N_+]_\lambda e^\lambda \\ &= e^\chi \sum_\lambda S(\mathfrak{n}_+^*)_\lambda e^\lambda \\ &= e^\chi \sum_\lambda S(\mathfrak{n}_-)_\lambda e^\lambda \\ &= \mathrm{char}\, M_\chi. \end{aligned}$$

Note that there is a pairing $\langle -, - \rangle : U(\mathfrak{n}_+) \times \mathbb{C}[N_+] \to \mathbb{C}$ given by

$$\langle \alpha, f \rangle \coloneqq (\alpha \cdot f)(1)$$

for $\alpha \in U(\mathfrak{n}_+)$ and $f \in \mathbb{C}[N_+]$, where we view α as a left-invariant differential operator on N_+ via the identification $U(\mathfrak{n}_+) \cong D(N_+)^{N_+}$. Note the following properties of the pairing:

- The pairing is non-degenerate in th first argument. Indeed, if we have $(\alpha \cdot f)(1) = 0$ for all $f \in \mathbb{C}[N_+]$, then by the left-invariance α it follows that $(\alpha \cdot f)(n) = 0$ for all $f \in \mathbb{C}[N_+]$ and $n \in N_+$, hence $\alpha = 0$.
- The pairing is H-equivariant, so we can split it into pairings between the individual weight spaces. It follows that the induced pairing between the individual weight spaces is perfect.
- The pairing is $U(\mathfrak{n}_+)$-equivariant, i.e., $\langle \alpha\alpha', f \rangle = \langle \alpha', \alpha' f \rangle$ for $\alpha, \alpha' \in U(\mathfrak{n}_+)$ and all $f \in \mathbb{C}[N_+]$.

As a consequence, there is an isomorphism $U(\mathfrak{n}_+) \cong \mathbb{C}[N_+]^*$ as right $U(\mathfrak{n}_+)$-modules, and so there is an isomorphism $\mathbb{C}[N_+]^* \cong U(\mathfrak{n}_-)$ as left $U(\mathfrak{n}_-)$-modules via the anti-isomorphism. We can see that $\mathbb{C}[N_+]^*_\chi$ is a free $U(\mathfrak{n}_-)$-module generated by its highest weight vector. We conclude that $\mathbb{C}[N_+]^*_\chi \cong M_\chi$ as $\mathfrak{g}$-modules. □

Exercise 4.7. 1. The algebra homomorphism $U(\mathfrak{g}) \to D(B_+)^{H_r} = D(N_+) \otimes U(\mathfrak{h})$ restricts to an embedding $\iota : U(\mathfrak{g})^G \to U(\mathfrak{h})$.

Hint. The left-B_+-invariant part $(D(B_+)^{H_r})^{B_+}$ is precisely the left H-invariant part of $(D(B_+)^{H_r})^{N_+}$. The latter invariants coincide with $U(\mathfrak{n}_+) \otimes U(\mathfrak{h})$.

2. From Theorem 4.6, conclude that an element z of $U(\mathfrak{g})^G$ acts on M^*_χ by the scalar $\mathrm{ev}_\chi(\iota(z))$.

3. Conclude that ι coincides with the embedding used to construct the Harish-Chandra isomorphism.

Hint. The center $Z(U(\mathfrak{g}))$ acts on M_χ and M^*_χ by the same scalars. To see this, note that the action is by scalars on both $\mathfrak{g}$-modules and recall that they have the same irreducible constituents.