Lecture 2

1) Operations with ideals.
2) Maximal ideals.

Ref's: [AM], Chapter 1, Sections 3 and 6.

BONUS: Non-commutative counterparts 2.

1) Setting: A is commutative ring, pick ideals $I, J \subset A$.

Def: The sum $I+J := \{a+b \mid a \in I, b \in J\} \subset A$,

The product $IJ := \{\sum_{i=1}^{k} a_i b_i \mid k \in \mathbb{Z}_{\geq 0}, a_i \in I, b_i \in J\}$,

The ratio $I:J := \{a \in A \mid aJ \subset I\}$,

The radical $\sqrt{I} := \{a \in A \mid \exists\, n \in \mathbb{Z}_{>0} \text{ w. } a^n \in I\}$.

Proposition: $I \cap J, I+J, IJ, I:J, \sqrt{I}$ are ideals.

Proof for $\sqrt{I}$: Need to check

$$\left.\begin{array}{l}(0)\ \sqrt{I} \neq \varnothing \\ (1)\ a \in A, b \in \sqrt{I} \Rightarrow ab \in \sqrt{I} \\ (2)\ a, b \in \sqrt{I} \Rightarrow a+b \in \sqrt{I}\end{array}\right\} \Rightarrow \sqrt{I} \text{ is abelian subgroup.}$$

(0) $\Leftarrow \sqrt{I} \supset I$.

b/c A is commutative

(1): $b \in \sqrt{I} \Rightarrow \exists\, n$ w. $b^n \in I \Rightarrow (ab)^n = a^n b^n \in I \Rightarrow ab \in \sqrt{I}$.

(2) $a, b \in \sqrt{I} \Rightarrow \exists\, n$ w. $a^n, b^n \in I$

$$(a+b)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} a^i b^{2n-i} \in I \Rightarrow a+b \in \sqrt{I}$$

again, use that A is comm've $\qquad$ $\in I$ if $i \geq n$ $\qquad$ $\in I$ if $i \leq n$ $\qquad \square$

Example (generators): $I=(f_1,\dots f_n)$, $J=(g_1,\dots g_m)$. Then:

- $I+J=(f_1,\dots f_n, g_1,\dots g_m)$: $0\in I,J \Rightarrow f_i, g_j \in I+J \Rightarrow$
$(f_1,\dots f_n, g_1,\dots g_m)\subset I+J$; $I+J\subset(f_1,\dots f_n, g_1,\dots g_m)$ is manifest.

Exercise: Show that $IJ=(f_ig_j \mid i=1,\dots n, j=1,\dots m)$

Rem: For $I\cap J$, $I:J$, $\sqrt{I}$ - generators may be tricky...

Example: $A=\mathbb{Z}$, $I=(a)$. Want to compute $\sqrt{I}$:
$a=p_1^{\alpha_1}\dots p_k^{\alpha_k}$, p_i primes, $\alpha_i\in\mathbb{Z}_{>0}$
$b\in\sqrt{I} \Leftrightarrow b^n \vdots a$ for some $n \Leftrightarrow b \vdots p_1\dots p_k \Leftrightarrow \sqrt{(a)}=(p_1\dots p_k)$.
divisible by

Exercise: for general A, I, show $\sqrt{\sqrt{I}}=\sqrt{I}$.

2) Maximal ideals:

2.1) Definition & examples.

Def: An ideal $\mathfrak{m}\subset A$ is maximal if:

- $\mathfrak{m}\neq A$.
- If $\mathfrak{m}'$ another ideal s.t. $\mathfrak{m}\subseteq\mathfrak{m}'\subsetneq A$, then $\mathfrak{m}'=\mathfrak{m}$.

i.e. maximal = maximal w.r.t. inclusion among ideals $\neq A$.

Lemma (equivalent characterization): TFAE:

(1) $\mathfrak{m}$ is maximal

(2) $A/\mathfrak{m}$ is a field

Proof: We claim that both (1) & (2) are equivalent to:
(3) The only two ideals in $A/\mathfrak{m}$ are $\{0\}$ & $A/\mathfrak{m}$.

(1) $\Leftrightarrow$ (3): b/c of bijection $\{$ideals in A containing $\mathfrak{m}\} \xleftrightarrow{\sim}$ $\{$ideals in $A/\mathfrak{m}\}$, Remark in Section 3.2 of Lecture 1.
(3) $\Leftrightarrow$ (2): Remark & exercise in the end of Section 3.1 of Lecture 1. $\square$

Examples (of maximal ideals)

1) $A=\mathbb{Z}$, so every ideal is of the form $(a):=a\mathbb{Z}$ for $a\in\mathbb{Z}$. (a) is maximal $\Leftrightarrow$ a is prime. Indeed, the inclusion $(a)\subseteq(b)$ is equivalent to $b \vdots a$.

2) $A=\mathbb{F}[x]$ ($\mathbb{F}$ is field), (f) is maximal $\Leftrightarrow$ f is irreducible, for the same reason as in the previous example. For example, for $\mathbb{F}=\mathbb{C}$ (or any alg. closed field), the maximal ideals are exactly $(x-\alpha)$ for $\alpha\in\mathbb{F}$.

3) $A=\mathbb{F}[x_1,\dots x_n]$
$\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{F}^n \rightsquigarrow \mathfrak{m}_\alpha:=\{f\in\mathbb{F}[x_1,\dots x_n] \mid f(\alpha)=0\}$ is an ideal (exercise). We claim it's maximal $\Leftrightarrow$ $A/\mathfrak{m}_\alpha$ is a field. Consider the composition $\mathbb{F} \xrightarrow[\text{deg 0 polyn's}]{} A \twoheadrightarrow A/\mathfrak{m}_\alpha$. We claim it's an isom'm, equivalently, that every element $f\in A$ is uniquely written as $\beta+f_0$ w. $\beta\in\mathbb{F}$ & $f_0\in\mathfrak{m}_\alpha$. For this, set $y_i=x_i-\alpha_i$. Then $f\in A$

is uniquely written as a polynomial in $y_1,\dots,y_n$. For β we take the constant term & $f_0 := f-\beta \in \mathfrak{m}_\alpha$.

Note that once we know that $\mathbb{F} \xrightarrow{\sim} A/\mathfrak{m}_\alpha$, the homomorphism $A \twoheadrightarrow A/\mathfrak{m}_\alpha$ is the evaluation homomorphism: $f \mapsto f(\alpha)$.

In fact, this way we get <u>all</u> max. ideals in $\mathbb{F}[x_1,\dots,x_n]$, if $\mathbb{F}$ is algebraically closed. This claim will be proved much later when we discuss Nullstellensatz.

2.2) Existence.

Proposition: Every nonzero (commutative) ring has at least one maximal ideal.

Proof is based on Zorn's lemma from Set theory ($\Leftrightarrow$ axiom of choice)

Definitions: Let X be a set.

- A partial order $\leqslant$ on X is a binary relation s.t.

$$\left.\begin{array}{l} - \ x \leqslant x, \\ - \ x\leqslant y \ \& \ y \leqslant x \Rightarrow x=y \\ - \ x \leqslant y \ \& \ y \leqslant z \Rightarrow x \leqslant z \end{array}\right\} \forall x,y,z \in X$$

- $Y \subseteq X$ is linearly ordered (under $\leqslant$) if $\forall x,y \in Y$ have $x \leqslant y$ or $y \leqslant x$.
- poset = a set equipped with partial order.

Example: $X := \{\text{ideals } I \subset A \mid I \neq A\}$, $\leqslant := \subseteq$

Zorn lemma: Let X be a poset. Suppose that:

(*) $\forall$ linearly ordered subset $Y \subseteq X$ $\exists$ an upper bound in X, i.e. $x \in X$ s.t. $y \leq x$ $\forall y \in Y$.

Then $\exists$ a maximal element $z \in X$ (i.e. $x \in X$ & $z \leq x \Rightarrow z = x$).

Note that both the condition & the conclusion are essentially vacuous for finite sets.

Proof of Proposition: $X, \leq$ are as in Example. Want to show (*): let Y be linearly ordered subset of X, being linearly ordered in our case means: $\forall I, J \in Y$ have $I \subseteq J$ or $J \subseteq I$. Set $\tilde{I} := \bigcup_{I \in Y} I$. We claim this is an ideal, $\neq A$ (note: unlike the intersection, the union of ideals may fail to be an ideal).

Need to show:

(i) $\tilde{I}$ is an ideal $\Leftarrow a+b \in \tilde{I}$ as long as $a, b \in \tilde{I}$.

Check: $a, b \in \tilde{I} = \bigcup_{I \in Y} I \Rightarrow \exists I, J \in Y$ s.t. $a \in I$, $b \in J$.

Can assume $I \subseteq J \Rightarrow a, b \in J \Rightarrow a+b \in J \subseteq \tilde{I}$. This shows (i).

(ii) $\tilde{I} \neq A \overset{\tilde{I} \text{ is an ideal}}{\Longleftrightarrow} 1 \notin \tilde{I}$

But $1 \notin I$ for every $I \in Y \Rightarrow \tilde{I} = \bigcup_{I \in Y} I \not\ni 1$

Apply Zorn's lemma to finish the proof of Proposition. $\square$

Remark: Why do we care about ideals:

Reason 0: Can be used to produce new examples of rings (quotient construction from Section 3.2 of Lecture 1.)

Reason 1: If A is a "ring of alg. integers" then we have unique factorization for non-zero ideals. - will not discuss. Should be discussed in MATH 373.

Reason 2: Connection to geometry (3rd part of the class) $Z \subseteq \mathbb{C}^n \rightsquigarrow I_Z := \{f \in \mathbb{C}[x_1,\dots x_n] \mid f|_Z = 0\}$ - an ideal.

BONUS: Non-commutative counterparts part 2.

B1) Proper generalizations or what we discussed in this lecture will be for two-sided ideals. For two such ideals I, J it still makes sense to talk about $I \cap J$, $I+J$, IJ, $I:J$ - those are still 2-sided ideals. For $\sqrt{I}$ the situation is more interesting: the definition we gave doesn't produce an ideal (look at $I=\{0\}$ in $Mat_2(\mathbb{C})$). Under some addit'l assumptions, still can define a 2-sided ideal. We'll explain this for $I=\{0\}$, for the general case just take the preimage of $\sqrt{\{0\}} \subset A/I$ under $A \twoheadrightarrow A/I$.

Definition: A two-sided ideal $I \subset A$ is called <u>nilpotent</u> if $\exists\, n \in \mathbb{Z}_{>0} \mid I^n = \{0\}$.

Exercise: The sum of two nilpotent ideals is a nilpotent ideal.

Under additional assumption: A is "Noetherian" for 2-sided ideals - this condition for commutative rings will be studied later in the class) there's an automatically unique maximal nilpotent ideal. We take this ideal for $\sqrt{\{0\}}$!

B2) Now we discuss maximal ideals.

Definition: A ring A is called <u>simple</u> if it has only 2 two-sided ideals, $\{0\}$ & A.

Exercise: $Mat_n(\mathbb{F})$ is simple for any field $\mathbb{F}$.

Premium exercise: $Weyl_1 = \mathbb{F}\langle x, y\rangle / (xy - yx - 1)$ is simple if $\operatorname{char} \mathbb{F} = 0$ & not simple if $\operatorname{char} \mathbb{F} > 0$.

A two-sided ideal $m \subset A$ is maximal if A/m is simple