Quantizations, lecture 6

1) Modules over quantizations

1.1) Good filtrations

Let A be a $\mathbb{Z}_{\geqslant 0}$-graded finitely generated Poisson algebra w. $\deg\{\cdot,\cdot\}=-d$ & let $\mathcal{A}$ be its filtered quantization. Our meta-goal is to understand finitely generated modules over $\mathcal{A}$ (in reality we only can answer partial questions for certain $\mathcal{A}$) A basic tool to study $\mathcal{A}$-modules is the notion of good filtration

Definition: Let $\mathcal{M}$ be an $\mathcal{A}$-module. By an $\mathcal{A}$-module filtration on $\mathcal{M}$ we mean an ascending filtration $\mathcal{M}=\bigcup_{j\in\mathbb{Z}}\mathcal{M}_{\leqslant j}$ s.t. $\mathcal{A}_{\leqslant i}\mathcal{M}_{\leqslant j}\subset\mathcal{M}_{\leqslant i+j}$. Note that $\operatorname{gr}\mathcal{M}$ becomes a $\operatorname{gr}\mathcal{A}=A$-module. We say that the filtration is good if $\operatorname{gr}\mathcal{M}$ is finitely generated over A & $\mathcal{M}_{\leqslant i}=\{0\}$ $\forall\, i\ll 0$.

Example: The filtration on $\mathcal{A}$ itself is good.

Lemma: TFAE

a) M admits a good filtration

b) M is finitely generated

Proof:

a) $\Rightarrow$ b) is an exercise. For b) $\Leftarrow$ a) choose generators $m_1, \dots m_k$ of M & $d_1, \dots d_k \in \mathbb{Z}$. Set $M_{\leq i} = \sum_{j=1}^{k} \mathcal{A}_{\leq i - d_j} m_j$. This is a good filtration (exercise) $\square$

As the lemma shows a good filtration is not unique (in fact, the proof describes all good filtrations). However, there are invariants of $\operatorname{gr} M$ that do not depend on the choice. For example, consider the Grothedieck semi-group, where elements are $[\mathcal{F}]$ for $\mathcal{F} \in A\text{-mod}_{fg}$ & the relations are $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ $\forall$ SES $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$.

Proposition: $[\operatorname{gr} M]$ is independent of the choice of a good filtration.

See Sec 2.3 in Chriss-Ginzburg

1.2) Associated varieties & Gabber's theorem

In particular, $\mathrm{Supp}(\mathrm{gr}\, M) \subset \mathrm{Spec}(A)$ (a reduced subscheme by definition) depends only on $[\mathrm{gr}\, M]$, hence is independent of the choice of good filtration. This is known as the associated variety of M, and is denoted by $V(M)$.

Gabber's thm puts an important restriction on $V(M)$. Let $I \subset A$ be the annihilator of $\mathrm{gr}(M)$, hence $\sqrt{I}$ is the defining ideal of $V(M)$.

Thm (Gabber) $\{\sqrt{I}, \sqrt{I}\} \subset \sqrt{I}$.

Rem: Here's a geometric meaning of this condition. Let X be a smooth symplectic affine variety. Then a radical ideal $J \subset \mathbb{C}[X]$ satisfies $\{J, J\} \subset J \Leftrightarrow$ it's subvariety of zeroes, Y, is coisotropic meaning that $T_y Y \subset T_y X$ is a coisotropic subspace $\forall\, y \in Y^{reg}$. The condition $\{J, J\} \subset J$ makes sense for every Poisson algebra, so we can define the notion of a coisotropic subvariety in any Poisson scheme.

1.3) Holonomic modules

Let X be a smooth symplectic variety. A subvariety Y in X is called isotropic if $T_yY \subset T_yX$ is isotropic $\forall y \in Y^{reg}$. A subvariety that is both isotropic & coisotropic is called lagrangian (equivalently, $T_yY \subset T_yX$ is lagrangian $\forall y \in Y^{reg}$)

Assume now that X is affine (but still smooth & symplectic) & $\mathbb{C}[X]$ is $\mathbb{Z}_{\geq 0}$-graded (so we are in the setting of Sec 1.1). Then by a holonomic $\mathcal{A}$-module we mean a finitely generated $\mathcal{A}$-module w. lagrangian associated variety. This is mostly considered in the case when $X = T^*X_0$ (so that $\mathcal{A}$ is an algebra of TDO). The notion of holonomic modules can be easily extended to sheaves of TDO (over non-affine X_0). In a way, these are the smallest possible modules & ones we usually study.

The notion of isotropic subvarieties (and holonomic modules) can be somewhat generalized. Namely, let X be a Poisson variety. We say that X has finitely many symplectic leaves if it admits a stratification by locally closed Poisson subvarieties that are smooth and symplectic (and whose connected components are called symplectic leaves). For example, the nilpotent cone $\mathcal{N}$ has finitely many leaves (that are G-orbits). A result of Kaledin says that every singular symplectic variety has finitely many leaves.

Definition: Let X be a Poisson variety w. finitely many leaves: $\mathcal{L}_1, \dots \mathcal{L}_k$. We say that $Y \subset X$ is isotropic if $Y \cap \mathcal{L}_i$ is isotropic $\forall i$.

Thanks to this definition we can extend the notion of holonomic modules to quantizations of varieties w. finitely many leaves.

Example (of isotropic subvarieties) Let X be a Poisson variety with finitely many leaves & let θ be an anti-Poisson involution of X (meaning that for local functions f, g on X we have $\theta(\{f,g\}) = -\{\theta(f), \theta(g)\}$). Then X^θ is both isotropic & coisotropic (exercise; hint - do the smooth symplectic case first).

1.4) Excellent filtrations

Warning: this is not a common terminology.

We have seen that good filtrations are not unique. We would like a good filtration that is:

a) Defined canonically (possibly under some normalization conditions)

b) Has good properties.

Definition: Let A be a $\mathbb{Z}_{\geqslant 0}$-graded Poisson algebra w. $\deg\{\cdot,\cdot\} = -d$ $(d \in \mathbb{Z}_{>0})$, $\mathcal{A}$ its filtered quantization & M an $\mathcal{A}$-module. By an excellent filtration on M we mean a good filtration $M = \bigcup_j M_{\leqslant j}$ s.t.

- $I = \mathrm{Ann}_A(\mathrm{gr}\, M)$ is radical
- if $a \in A/I$ is not a zero divisor in A/I, then it's not a zero divisor in $\mathrm{gr}\, M$ (i.e. $\mathrm{gr}\, M$ has positive depth).

Note that a shift of an excellent filtration is again excellent, so some normalization condition is needed to pin it down uniquely.

Examples: 1) The most important class of holonomic $\mathcal{D}$-modules are "regular" ones (that we won't define). In 1981 Kashiwara & Kawai equipped such modules w. an excellent filtration depending on a real number.

2) Let $\mathfrak{g}$ be a semisimple Lie algebra and σ a Lie algebra involution of $\mathfrak{g}$. Set $\mathfrak{k} := \mathfrak{g}^{\sigma}$ (e.g. $\mathfrak{g} = \mathfrak{sl}_n$, $\sigma(x) = -x^t$ $\Rightarrow \mathfrak{k} = \mathfrak{so}_n$). By a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module we mean a finitely generated $U(\mathfrak{g})$-module M s.t. the action of $\mathfrak{k}$ on M is locally finite: $\dim U(\mathfrak{k})m < \infty$ $\forall\, m \in M$. A result of I.L. from 23 is that every <u>irreducible</u> Harish-Chandra module admits an excellent filtration. With suitable normalization conditions, it is unique.

1.5) More on Harish-Chandra modules.

First, consider an equivariant version. Let G, K be connected algebraic groups w. $Lie(G)=\mathfrak{g}$ & homomorphism $K\to G$ inducing $Lie(K)\xrightarrow{\sim}\mathfrak{k}$. Fix $\lambda\in(\mathfrak{k}^*)^K$. By a HC $(\mathfrak{g},K,\lambda)$-module we mean a HC $(\mathfrak{g},\mathfrak{k})$-module s.t. the action of $\mathfrak{k}$ given by $\xi\cdot m=\xi m-\langle\lambda,\xi\rangle m$ integrates to K. Here are some motivations for this definition.

Exercise: Every irreducible HC $(\mathfrak{g},\mathfrak{k})$-module is a HC

Remark: From $(\mathfrak{g},K)$ we can recover a real s/simple Lie group $G_{\mathbb{R}}$ (we skip the construction: but here are two examples – for $\mathfrak{g}=\mathfrak{sl}_n$, $K=SO_n$ we get $G_{\mathbb{R}}=SL_n(\mathbb{R})$, while for $K=Spin_n$ we get a 2-fold cover of $SL_n(\mathbb{R})$, which is not an algebraic group.

Assume $\lambda=0$. Then HC $(\mathfrak{g},K)$-modules (we omit λ from the notation) are closely related to representations of $G_{\mathbb{R}}$ in Banach spaces. This is the origin of HC modules.