DIMENSIONS OF MODULAR IRREDUCIBLE REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

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Abstract. In this paper we classify and give Kazdhan-Lusztig character formulas for equivariantly irreducible representations of Lie algebras of reductive algebraic groups over a field of large positive characteristic. The equivariance is with respect to a group whose connected component is a torus. Character computation is done in two steps. First, we treat the case of distinguished p-characters: those that are not contained in a proper Levi. Here we essentially show that the category of equivariant modules we consider is a cell quotient of an affine parabolic category $\mathcal{O}$. For the general nilpotent p-character, we get character formulas by explicitly computing the duality operator on a suitable equivariant K-group.

Contents

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1. Introduction

The goal of this paper is to obtain character formulas for (equivariantly) irreducible representations of semisimple Lie algebras over algebraically closed fields of large enough postive characteristic. Below we write G for a connected reductive algebraic group over $\mathbb{C}$ and $\mathfrak{g}$ for its Lie algebra, both are defined over $\mathbb{Z}$. We pick a prime number $p \gg 0$ and choose an algebraically closed field $\mathbb{F}$ of characteristic p. We write $\mathfrak{g}_{\mathbb{F}}, G_{\mathbb{F}}$ for the $\mathbb{F}$-forms of $\mathfrak{g}, G$.

1.1. **Known results.** Recall that the universal enveloping algebra $\mathcal{U}_{\mathbb{F}} := U(\mathfrak{g}_{\mathbb{F}})$ has big center. Namely, we have the restricted pth power map $x \mapsto x^{[p]} : \mathfrak{g}_{\mathbb{F}}^{(1)} \to \mathfrak{g}_{\mathbb{F}}$, where the superscript "(1)" indicates the Frobenius twist so that $(ax)^{[p]} = ax^{[p]}$. Then we have an algebra embedding $S(\mathfrak{g}_{\mathbb{F}}^{(1)}) \to \mathcal{U}_{\mathbb{F}}$ with central image: on $\mathfrak{g}_{\mathbb{F}}^{(1)}$ it is given by $x \mapsto x^p - x^{[p]}$. We also have the so called Harish-Chandra center $\mathcal{U}_{\mathbb{F}}^{G_{\mathbb{F}}}$, as in characteristic 0 it is identified with $\mathbb{F}[\mathfrak{h}^*]^W$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, W denotes the Weyl group that acts on $\mathfrak{h}^*$ via the ρ-shifted action. By a theorem of Veldkamp, [V], the full center of $\mathcal{U}_{\mathbb{F}}$ is known to coincide with

$$S(\mathfrak{g}_{\mathbb{F}}^{(1)}) \otimes_{S(\mathfrak{g}_{\mathbb{F}}^{(1)})^{G_{\mathbb{F}}^{(1)}}} \mathcal{U}_{\mathbb{F}}^{G}.$$

For $\chi \in \mathfrak{g}_{\mathbb{F}}^{(1)*}, \lambda \in \mathfrak{h}_{\mathbb{F}}^*/(W, \cdot)$ we can define the central reduction $\mathcal{U}_{\lambda,\mathbb{F}}^{\chi}$ of $\mathcal{U}_{\mathbb{F}}$, this is a finite dimensional algebra. The most interesting case is when χ is nilpotent, the general case easily reduces to that one, see [KW]. If χ is nilpotent and $\mathcal{U}_{\lambda,\mathbb{F}}^{\chi} \neq \{0\}$, then $\lambda \in \mathfrak{h}_{\mathbb{F}_p}^*$. Then one can reduce the question about the dimensions of the irreducibles to the case when λ is regular, see, e.g., [BMR2]. This is what we are going to assume from now on. For $\chi' = g\chi$ for some $g \in G_{\mathbb{F}}$, then g gives an isomorphism $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}} \cong \mathcal{U}^{\chi'}_{\lambda,\mathbb{F}}$. So the algebra $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$ depends only on the $G_{\mathbb{F}}$-orbit of χ. The nilpotent $G_{\mathbb{F}}$-orbits in $\mathfrak{g}^{(1)*}_{\mathbb{F}}$ are in a natural bijection with the nilpotent G-orbits in $\mathfrak{g}$. We will write e for the nilpotent element of $\mathfrak{g}$ lying in the orbit corresponding that of χ.

Let us explain known results on the simple $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-modules. First of all, the number is known. Indeed, thanks to the results of [BMR2] (to be recalled in more detail below) there is a natural isomorphism $K_0(\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}) \xrightarrow{\sim} K_0(\mathcal{B}_e)$, where $\mathcal{B}_e$ denotes the Springer fiber of e and $K_0(\mathcal{B}_e)$ stands for the Grothendieck group of the coherent sheaves on $\mathcal{B}_e$. The rank of $K_0(\mathcal{B}_e)$ is known in all cases.

In [BM], the first named author and Mirkovic have determined the classes of simples in $K_0(\mathcal{B}_e)$. Namely, Lusztig, [Lu3], has defined a canonical basis in $K_0^{\mathbb{C}^\times}(\mathcal{B}_e)$ for a suitable contracting $\mathbb{C}^\times$-action on $\mathcal{B}_e$. The main result of [BM] is that the specialization of Lusztig's canonical basis to $q = 1$ (where q is the equivariant parameter) coincides with the basis of simple modules in $K_0(\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod})$. A problem with this canonical basis is that it is very implicit (with an exception of the case when e is principal in a Levi subalgebra, see [Lu2, Lu4],) so the result from [BM] does not allow to get the dimension formulas in the general case.

Before we proceed to our results on the dimensions and K_0-classes of the simple modules, let us explain what is known about their combinatorial classification. For this, let us recall that a nilpotent element $e \in \mathfrak{g}$ is called *distinguished* if it is not contained in any proper Levi subalgebra. Any nilpotent element e is distinguished in a Levi subalgebra of $\mathfrak{g}$. Namely, consider the maximal torus T_0 of the centralizer $Z_G(e)$. The Levi subalgebra we need is $\underline{\mathfrak{g}} := \mathfrak{g}^{T_0}$. Note that the group G_χ acts on $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$ by algebra automorphisms. In particular, a maximal torus $T_{0,\mathbb{F}} \subset G_\chi$ acts. We can consider the category $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^{T_0}$ of weakly T_0-equivariant $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-modules. Every simple $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-module has an equivariant lift unique up to a twist with a character of $T_{0,\mathbb{F}}$. So the set $\operatorname{Irr}(\mathcal{U}^{\chi}_{\lambda,\mathbb{F}})$ of irreducible $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-modules is in bijection with the quotient of $\operatorname{Irr}(\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^{T_0})$ by the free action of the character lattice $\mathfrak{X}(T_0)$. On the other hand, the simples in $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^{T_0}$ are in bijection with the T_0-equivariant simple objects in $\bigoplus_{\underline{\lambda}} \underline{\mathcal{U}}^{\chi}_{\underline{\lambda},\mathbb{F}}\text{-mod}$. Here $\underline{\mathcal{U}}$ indicates the enveloping algebra for $\underline{\mathfrak{g}}$, the summation is over all $\tilde{\lambda} \in \mathfrak{h}^*_{\mathbb{F}_p}/(\underline{W}, \cdot)$ that map to λ under the natural projection $\mathfrak{h}^*_{\mathbb{F}_p}/(\underline{W}, \cdot) \to \mathfrak{h}^*_{\mathbb{F}_p}/(W, \cdot)$. The bijection between the sets of simples works as follows: one fixes a generic one-parameter subgroup ν of T_0 and then takes the highest weight space of a simple T_0-equivariant $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-module to get a simple T_0-equivariant module in $\bigoplus_{\underline{\lambda}} \underline{\mathcal{U}}^{\chi}_{\underline{\lambda},\mathbb{F}}\text{-mod}$.

In the special case when e is principal in $\underline{\mathfrak{g}}$, each algebra $\underline{\mathcal{U}}^{\chi}_{\underline{\lambda},\mathbb{F}}$ is just $\mathbb{F}$ and so has a unique simple module. Therefore the simples in $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}$ are in bijection with $W/\underline{W}$ (the bijection depends on a choice of ν).

For the general χ, the situation is more difficult as there is no explicit labelling set for the simples in the case of a general distinguished element.

On the other hand, in [Lo], the second named author considered a problem that should be thought as a "finite" analog of the problem considered in the present paper (that is "affine"). The main result of [Lo] is the Kazhdan-Lusztig type formulas for the characters of certain equivariantly simple modules over finite W-algebras. An approach used in [Lo] was to relate the category of equivariant modules over the finite W-algebra to a suitable parabolic category

$\mathcal{O}$ over $\mathfrak{g}$. Note that the simple finite dimensional modules over the W-algebra associated to e with central character $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}/(W,\cdot)$ embed into $\operatorname{Irr}(\mathcal{U}^{\lambda}_{\chi,\mathbb{F}})$ so that the dimension multiplies by $p^{\dim Ge/2}$.

1.2. **Dimensions of equivariantly irreducible modules: distinguished case.** In this paper we give a combinatorial classification and compute dimensions (as well as characters and, even stronger, K_0-classes) of *equivariantly* irreducible $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-modules. Let us first explain our setting and our results in the case when e is distinguished. For simplicity, assume G is semisimple.

As we have mentioned in the previous section, the group G_χ acts on $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$ by automorphisms. Note that since e is distinguished, the reductive part of G_χ is finite. Denote this group by A. We will consider the category $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^A$ of A-equivariant $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-modules.

It turns out that there is a natural labelling set for the simple objects in $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^A$. To describe it, let us recall the parabolic subalgebra attached to e. Namely, we include e into an $\mathfrak{sl}_2$-triple (e,h,f). Then we can consider the parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}$, the sum of all eigenspaces for $\operatorname{ad}(h)$ with nonnegative eigenvalues. Let us write W^a for the affine Weyl group $W\ltimes\mathfrak{X}(T)$, where $\mathfrak{X}(T)$ denote the character lattice of T. The parabolic subgroup P defines a standard parabolic subgroup of W to be denoted by W_P. Note that W_P is also a standard parabolic subgroup of W^a. We write $W^{a,P}$ for the set of maximal length representatives of the right cosets for W_P so that $W^{a,P}\xrightarrow{\sim} W^a/W_P$. It is a standard fact that $W^{a,P}$ contains a left cell $\mathfrak{c}_P$ such that $W^{a,P}$ is the union of left cells that are less than or equal to $\mathfrak{c}_P$. Note that since e is distinguished, $\mathfrak{c}_P$ is finite, this follows, for example, from [Lu1]. We will see below that there is a natural bijection $\mathfrak{c}_P\xrightarrow{\sim}\operatorname{Irr}(\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^A)$.

Let us explain how to compute the dimension of the simple objects in $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^A$. From an element in $x\in W^{a,P}$ we can produce a dominant weight μ_x for $L_{\mathbb{F}}$ that maps to λ under the natural projection $\mathfrak{X}(T)\to\mathfrak{h}^*_{\mathbb{F}_p}/(W,\cdot)$. Namely, we have a unique element μ° in the p-alcove defined by $\langle\alpha_i^\vee,\bullet\rangle\leqslant -1, \langle\alpha_0^\vee,\bullet+\rho\rangle\geqslant -p$ (below we will call this p-alcove *anti-dominant*) that maps to λ (here the $\alpha_i^\vee$'s are the simple coroots and $\alpha_0^\vee$ is the minimal coroot). Consider the W^a-action on $\mathfrak{X}(T)$ given by $w.\mu := w\cdot\mu$ for $w\in W^\vee$ and $t_\theta.\mu:=\mu+p\theta$ for $\theta\in\mathfrak{X}(T)$. Note that for $x\in W^{a,P}$, the element $\mu_x := x^{-1}\cdot\mu^\circ$ is dominant for L. Let $d_L(\mu_x)$ denote the dimension of the finite dimensional L-module with highest weight μ_x, it is given by the Weyl dimension formula.

The next and final ingredient to state the dimension formula is the parabolic Kazhdan-Lusztig polynomials: to elements $x,y\in W^{a,P}$ we assign the corresponding Kazhdan-Lusztig polynomial $c^P_{x,y}(v)\in\mathbb{Z}[v^{\pm1}]$. Our convention, recalled in more detail in Section 8.3, is that $c^P_{x,y}(1)$ is the coefficient of the class of the standard object labelled by y in the parabolic affine category $\mathcal{O}$ in the simple object labelled by x. Note that for any given x, only finitely many of the polynomials $c^P_{x,y}$ are nonzero.

Theorem 1.1. *The dimension of the simple module in $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}\text{-mod}^A$ labelled by $x\in\mathfrak{c}_P$ equals*

$$\sum_{y\in W^{a,P}} c^P_{x,y}(1)(p^{\dim Ge/2}d_L(\mu_y)).$$

We can upgrade this theorem to computing the A-characters of the simple equivariant $\mathcal{U}^{\chi}_{\lambda,\mathbb{F}}$-modules. For this we need to replace $p^{\dim Ge/2}d_L(\mu_y)$ with the A-character of $U^0(\mathfrak{m}^-_{\mathbb{F}})\otimes V_L(\mu_y)$, where $\mathfrak{m}^-_{\mathbb{F}}$ is the maximal nilpotent subalgebra of the opposite parabolic of $\mathfrak{p}_{\mathbb{F}}$ and $V_L(\mu_y)$ is the irreducible L-module with highest weight μ_y. In fact, the strongest version of Theorem

1.1 has to do with K_0-classes: we will see that the class of the simple in $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod^A labelled by x is $\sum_{y\in W^{a,P}} c^P_{x,y}(1)[W^\chi_{\mathbb{F}}(\mu_y)]$, where $W^\chi_{\mathbb{F}}(\mu_y)$ is a certain induced module in $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod^A to be defined in Section 2.3.

Theorem 1.1 will be proved in Section 8.4, see Theorem 8.6. The main ingredient is a parabolic version of the main result of [B] that allows us to relate a characteristic 0 counterpart of $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod^A to a cell quotient of a suitable affine parabolic category $\mathcal{O}$.

1.3. **Dimensions of equivariantly irreducible modules: general case.** Now let e be arbitrary. As before we fix a maximal torus $T_{0,\mathbb{F}} \subset G_{\mathbb{F},\chi}$. We set $\underline{Q}_{\mathbb{F}}$ to be the centralizer of $T_{0,\mathbb{F}}$ in $G_{\mathbb{F},\chi}$. This is a reductive group whose connected component $T_{0,\mathbb{F}}$. Since p is large enough, $\underline{Q}_{\mathbb{F}}$ is linearly reductive. We consider the category $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod$^{\underline{Q}}$ of $\underline{Q}_{\mathbb{F}}$-equivariant $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-modules. Inside we consider the Serre subcategory of all objects M such that the Lie algebra $\mathfrak{k}_{\mathbb{F}}$ of $\underline{Q}_{\mathbb{F}}$ acts on the graded component M_θ by $\theta \bmod p$ for all $\theta \in \mathfrak{X}(T_{0,\mathbb{F}})$. Denote this subcategory by $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod$^{\underline{Q},0}$.

Let us write $\underline{G}_{\mathbb{F}}$ for the centralizer of $T_{0,\mathbb{F}}$ in $G_{\mathbb{F}}$. Inside, we have a parabolic subgroup $\underline{P}_{\mathbb{F}}$ constructed from e as in the previous section. Once we pick a generic one-parameter subgroup ν of $T_{0,\mathbb{F}}$, we can identify the set $\mathrm{Irr}(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q},0})$ with the set of pairs (u,x), where $u \in W$ is shortest in $uW_{\underline{G}}$ and x lies in the left cell $\mathfrak{c}_{\underline{P}}$ (note that $\underline{G}$ is not semisimple so $\mathfrak{c}_{\underline{P}}$ is not finite; however, it is preserved by the translations by elements of $\mathfrak{X}(\underline{G})$ and the quotient by this action is finite).

The choice of a generic one-parameter subgroup ν in $T_{0,\mathbb{F}}$ defines a parabolic subgroup $G^{\geqslant 0}_{\mathbb{F}} = \underline{G}_{\mathbb{F}} \ltimes G^{>0}_{\mathbb{F}}$. Inside we have the parabolic subgroup $P_{\mathbb{F}} := \underline{P}_{\mathbb{F}} \ltimes G^{>0}_{\mathbb{F}}$.

The first ingredient that we need to write the dimension formula is the *semiperiodic parabolic* Kazhdan-Lusztig polynomials to be denoted by $c^{P,\infty}_{x,y}(v)$. Here $x, y \in W^{a,P}$. These polynomials are constructed as follows. It turns out that once an element $\theta \in \mathfrak{X}(\underline{G})$ lying in the dominant Weyl chamber for G is large enough (meaning that its pairings with the simple coroots not in $\underline{G}$ are; how large depends on x, y) the polynomial $c^P_{xt_\theta,yt_\theta}(v)$ depends on x, y and not on θ. We denote this specialized polynomial by $c^{P,\infty}_{x,y}(v)$.

Furthermore, from $y \in W^{a,P}$ we can produce the dominant weight μ_y of L as before. Let $\hat{d}_L(\mu_y)$ denote the $\underline{Q}$-character of the corresponding irreducible L-module. Finally, consider the character ch_{M^-} of the action of $\underline{Q}_{\mathbb{F}}$ on $U^0(\mathfrak{m}^-_{\mathbb{F}})$, where $\mathfrak{m}^-_{\mathbb{F}}$ is the maximal nilpotent subalgebra of the parabolic opposite to $\mathfrak{p}_{\mathbb{F}}$.

Theorem 1.2. *The $\underline{Q}_{\mathbb{F}}$-character of the simple module in $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod$^{\underline{Q},0}$ labelled by $u\underline{x}$, where $u \in W$ is shortest in $uW_{\underline{G}}$ and $\underline{x} \in \mathfrak{c}_{\underline{P}}$, equals*

$$\sum_{y\in W^{a,P}} c^{P,\infty}_{u\underline{x},y}(1)\mathsf{ch}_{M^-}\hat{d}_L(\mu_y).$$

Note that, unlike in Theorem 1.1, the sum in the right hand side is no longer finite. However, a priori, it is easy to see that it converges in a suitable topology on $K_0(\mathrm{Rep}(\underline{Q}_{\mathbb{F}}))$. And Theorem 1.2 upgrades to an equality of classes in $K_0(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q},0})$, just like Theorem 1.1.

However, the proof of Theorem 1.2 is quite different from that of Theorem 1.1. Namely, we construct a contravariant duality functor on $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod$^{\underline{Q}}$ that fixes all simple objects in this category. Then we compute a graded lift of this duality functor in a suitable graded lift of a category closely related to $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$-mod$^{\underline{Q},0}$. Below in the paper we will speculate on a categorical nature of Theorem 1.2.

1.4. **Applications to characteristic 0 representation theory.** One can, in principle, use Theorem 1.2 to compute the dimensions of equivariantly irreducible representations of finite W-algebras. In more detail, to $e \in \mathfrak{g}$ one assigns the finite W-algebra $\mathcal{W}$ and to $\lambda \in \mathfrak{h}^*$ one assigns the central reduction $\mathcal{W}_\lambda$ of $\mathcal{W}$.

In [BL], we have related the irreducible finite dimensional representations of $\mathcal{W}_\lambda$ to those of $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$. Assume λ is rational. One has natural bijections between the sets $\operatorname{Irr}(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod})$ for different p provided they are sufficiently large and the residues modulo the "denominator" of λ is fixed. Under these bijections, the dimensions of the irreducibles are polynomials in p. Then $\operatorname{Irr}_{fin}(\mathcal{W}_\lambda)$ embeds into $\operatorname{Irr}(\mathcal{U}^\chi_{\lambda,\mathbb{F}})$ as the subset of all representation with the degree of dimension polynomial equal to $\frac{1}{2}\dim Ge$. Equivalently, [BL, Theorem 1.1], $\operatorname{Irr}_{fin}(\mathcal{W}_\lambda)$ consists of all simples whose K_0-class lies in a certain two-sided cell component of $K_0(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod})$ for the integral Weyl group $W_{[\lambda]}$ of λ.

Thanks to Theorem 1.1 we get a formula for the dimensions of equivariantly irreducible representations of $\mathcal{W}_\lambda$. However, it would be desirable to get a formula in terms of finite not affine Kazhdan-Lusztig polynomials, as that of [Lo] in the case when λ is integral. Knowing the dimensions of the irreducible $\mathcal{W}_\lambda$-modules should lead to a solution of other problems in the Lie representation theory over $\mathbb{C}$, for example, to formulas for Goldie ranks of primitive ideals, a precise relation between the Goldie ranks and the dimensions was conjectured in [LP] (at least, when $\mathfrak{g}$ is classical).

1.5. **Content of the paper.** The subsequent sections of the paper can be roughly separated into two groups.

Sections 2-5 are preparatory. While some of them contain new results, those results are technical ramifications of well-known ones. In Section 2 we recall some basic facts and constructions related to the modular representation theory of semisimple Lie algebras. In particular, there we introduce categories $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^Q_-$, the main object of study in this paper as well as "standard modules" in these categories that we call χ-Weyl modules. In Section 3 we recall the derived localization theorem in positive characteristic proved in [BMR2]. In Section 4 we recall the tilting bundles on the Springer and Grothendieck simulataneous resolutions constructed in [BM] and their properties. And then in Section 5 we recall results of [B] on an equivalence between the coherent and constructible categorifications of affine Hecke algebras.

Sections 6-8 are the main part of the paper. In Section 6 we generalize the results of [B] to the parabolic setting. This is the main ingredient in the proof of Theorem 1.1. In Section 7 we introduce and study a contravariant duality functor on the category $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^Q_-$ that is an important ingredient in the proof of Theorem 1.2. Then in Section 8 we prove Theorems 1.1 and 1.2.

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1.6. **List of notation.** Here we provide the list of common notation used in the paper. The notation is listed alphabetically with Roman letters and then Greek letters.

A	$:= Z_G(e)/Z_G(e)^\circ$,
$\mathcal{A}$	$:= \operatorname{End}_{\tilde{\mathcal{N}}}(\mathcal{T})$,
$\mathcal{A}_\mathfrak{h}$	$:=\operatorname{End}_{\tilde{\mathfrak{g}}}(\mathcal{T}_\mathfrak{h})$,
$\mathcal{A}_P$	$:= \operatorname{End}_{\tilde{\mathcal{N}}_P}(\mathcal{T}_P)$,
B	a Borel subgroup of G,

$\mathcal{B}$	$:= G/B$, the flag variety of G,
$\underline{\mathcal{B}}$	the flag variety of $\underline{G}$,
$\mathcal{B}_e$	the Springer fiber for a nilpotent element $e \in \mathfrak{g}$,
Br	the braid group of W,
Br^a	the braid group of W^a,
C_x	the Kazhdan-Lusztig basis element in $\mathcal{H}^a_G$ labelled by $x \in W^a$,
$\mathfrak{c}_P$	the left cell in W^a containing $w_{0,P}$,
$\mathrm{Coh}_Y(X)$	the category of coherent sheaves on a scheme X that are supported on Y set-theoretically,
$\mathbb{D}$	the contravariant duality functor of $\mathcal{U}^{\chi}_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}}$ defined by (7.1),
$\mathbb{D}_{coh}$	the contravariant duality functor of $D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))$ from Section 7.5,
$D^b_H(X)$	the H-equivariant bounded constructible derived category of a variety (or an ind-variety) X,
D^μ_X	the sheaf of μ-twisted differential operators on a smooth algebraic variety X,
$\tilde{D}_{\mathcal{B}}$	the sheaf $\upsilon_* D_{G/U}$, where $\upsilon : G/U \to G/B$ is the natural projection,
(e, h, f)	an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$,
$\mathcal{F}l$	$:= G^\vee((t))/I^\vee$,
$\mathcal{F}l_P$	$:= G^\vee((t))/J^\vee$, where $\mathcal{J}^\vee$ is the parahoric subgroup of $G^\vee((t))$ corresponding to $P \subset G$,
G	a connected reductive algebraic group over $\mathbb{C}$,
$G^\vee$	the Langlands dual group of G,
$\mathfrak{g}^i$	$\{x \in \mathfrak{g} \mid \nu(t)x = t^i x, \forall t \in T_0\}$,
$\mathfrak{g}^{\geqslant 0}$	$\bigoplus_{i \geqslant 0} \mathfrak{g}^i$,
$\underline{\mathfrak{g}}$	$:= \mathfrak{g}^0$,
$\mathfrak{g}(i)$	$:= \ker(\mathrm{ad}\, h - i)$,
$\underline{G}^{\geqslant 0}$, $\underline{G}$	the connected subgroups of G with Lie algebras $\mathfrak{g}^{\geqslant 0}, \underline{\mathfrak{g}}$,
$\mathfrak{g}_{\mathfrak{h}}$	$:= \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$,
$\tilde{\mathfrak{g}}$	the Grothendieck resolution of $\mathfrak{g}_{\mathfrak{h}}$,
$\mathfrak{h}$	a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$,
$\mathcal{H}_W$	the Hecke algebra of W,
$\mathcal{H}^a_G$	the affine Hecke algebra of a reductive algebraic group G; it is associated to $W \ltimes \mathfrak{X}(T)$,
H_x	the standard basis element in $\mathcal{H}^a_G$ labelled by $x \in W^a$,
$\underline{Q}$	the centralizer of T_0 in Q,
$K^H_0(X)$	$:= K_0(\mathrm{Coh}^H(X))$,
$I^\vee$	the Iwahori subgroup of $G^\vee((t))$,
I°	the kernel of $I^\vee \twoheadrightarrow T^\vee$,
L	the Levi subgroup of P containing T,
M	the unipotent radical of P,
M^-	the unipotent radical of the parabolic opposite to P,
$\mathcal{N}$	the nilpotent cone in $\mathfrak{g}$,
$\tilde{\mathcal{N}}$	$:= T^*\mathcal{B}$,
$\tilde{\mathcal{N}}_P$	$:= T^*\mathcal{P}$,

$\mathcal{O}(\mu)$	the line bundle on $\mathcal{B}$ (and related varieties) corresponding to $\mu \in \mathfrak{X}(T)$,
$\mathbb{O}$	$:= Ge$,
P	a parabolic subgroup of G containing B,
$\mathcal{P}$	$:= G/P$,
Q	$:= Z_G(e,h,f)$,
S	the Slodowy slice $e + \mathfrak{z}_{\mathfrak{g}}(f)$,
$\mathsf{St}_{\mathfrak{h}}$	$:= \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$,
St_B	$:= \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$,
St_0	$:= \tilde{\mathcal{N}} \times^L_{\mathfrak{g}} \tilde{\mathcal{N}}$,
St_P	$:= \tilde{\mathfrak{g}} \times^L_{\mathfrak{g}} \tilde{\mathcal{N}}_P$,
T	the maximal torus of B with Lie algebra $\mathfrak{h}$,
T_0	a maximal torus in Q,
T_w	the element of Br_a corresponding to $w \in W$ or the wall-crossing functor for this element,
$\mathcal{U}$	$:= U(\mathfrak{g})$,
$\mathcal{U}^\chi_{\mathbb{F}}$	the p-central reduction of $\mathcal{U}_{\mathbb{F}}$,
$\mathcal{U}^\chi_{(\lambda),\mathbb{F}}$	the infinitesimal block in $\mathcal{U}^\chi_{\mathbb{F}}$ corresponding to a HC character λ.
$\underline{\mathcal{U}}$	$:= U(\underline{\mathfrak{g}})$,
$\mathcal{V}^\chi_\mu$	the splitting bundle introduced in Section 3.3,
W	the Weyl group of G,
W^a	$:= W \ltimes \mathfrak{X}(T)$,
W_P	the parabolic subgroup of W corresponding to P,
$W^{a,P}$	the set of longest coset representatives in $xW_P \subset W^a, x \in W^a$,
$W^{P,-}$	the set of shortest coset representatives in $wW_P, w \in W$,
$W^\chi_{\mathbb{F}}(\mu)$	the χ-Weyl module corresponding to a dominant weight μ of L,
w_0	the longest element of W,
$w_{0,P}$	the longest element of W_P,
$\mathfrak{X}(H)$	the character group of an algebraic group H,
Z	$:= G \times^B \mathfrak{m}$,
γ	the one-parameter subgroup of G corresponding to h,
$\Delta^P(\mu)$	the parabolic Verma module for P with highest weight μ,
$\underline{\Delta}^\chi$	the parabolic induction functor $\underline{\mathcal{U}}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}} \to \mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}}$,
$\alpha_1,\ldots,\alpha_r$	the simple roots of $\mathfrak{g}$,
α_0	the root of $\mathfrak{g}$ such that $\alpha_0^\vee$ is maximal,
η	the projection $\mathcal{F}l \to \mathcal{F}l_P$,
ι	the embedding $Z \hookrightarrow \tilde{\mathcal{N}}$,
μ_x	$:= x^{-1}\cdot\mu^\circ$ for μ° in the anti-dominant p-alcove,
ν	a generic one-parameter subgroup of T_0,
ϖ	the projection $Z \twoheadrightarrow \tilde{\mathcal{N}}_P$,
ρ	half the sum of positive roots,
K_0	the Springer resolution morphism $\tilde{\mathcal{N}} \to \mathcal{N}$ or $\tilde{\mathfrak{g}} \to \mathfrak{g}_{\mathfrak{h}}$,
σ	the standard antiinvolution of $\mathfrak{g}$: $\sigma(e_i) = f_i$,
ς	$:= \mathrm{Ad}(n)\sigma$, defined in Section 7.2,
τ	the derived equivalences from Theorem 5.1,
τ_P	the derived equivalence from Theorem 6.1,

χ the element in $\mathfrak{g}_{\mathbb{F}}^{(1)*}$ corresponding to $(e,\cdot)\in\mathfrak{g}^*$.

2. Basics on modular representations

2.1. **Notation and content.** Let G be a connected reductive algebraic group over $\mathbb{C}$ and $\mathfrak{g}$ be its Lie algebra. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. We fix a nilpotent orbit $\mathbb{O}\subset\mathfrak{g}$ and pick an element $e\in\mathbb{O}$. We include e into an $\mathfrak{sl}_2$-triple (e,h,f). Let us write $\mathfrak{g}(i)$ for $\ker(\mathrm{ad}(h)-i)$. We write A for $Z_G(e)/Z_G(e)^\circ$. We also write T_0 for a maximal torus in $Q:=Z_G(e,h,f)$. Further, we write $\mathcal{U}$ for $U(\mathfrak{g})$.

Pick a generic one-parameter subgroup $\nu:\mathbb{G}_m\to T_0$. We set $\mathfrak{g}^i:=\{x\in\mathfrak{g}|\nu(t)x=t^ix,\forall t\in\mathbb{G}_m\}$. We will also write $\underline{\mathfrak{g}}$ for $\mathfrak{g}^0$ and $\underline{\mathcal{U}}$ for $U(\underline{\mathfrak{g}})$. Set $\mathfrak{g}^{\geqslant 0}:=\bigoplus_{i\geqslant 0}\mathfrak{g}^i$, this is a parabolic subalgebra in $\mathfrak{g}$ with Levi subalgebra $\underline{\mathfrak{g}}$. Let $G^{\geqslant 0},\underline{G}$ denote the corresponding subgroups of G.

Set $\underline{\mathfrak{p}}:=\bigoplus_{j\geqslant 0}(\underline{\mathfrak{g}}\cap\mathfrak{g}(j)),\mathfrak{l}=\underline{\mathfrak{g}}\cap\mathfrak{g}(0),\mathfrak{p}:=\underline{\mathfrak{p}}\oplus\mathfrak{g}^{>0}$. Then $\mathfrak{p}$ is a parabolic subalgebra in $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}$. Let $L\subset\underline{P}\subset P$ denote the corresponding connected subgroups of G. Further, let M denote the unipotent radical of P and M_- stand for the unipotent radical of the opposite parabolic.

It is known that e is even in $\underline{\mathfrak{g}}$, i.e., the eigenvalues of h in $\underline{\mathfrak{g}}$ are even.

We pick a Borel subgroup $B\subset G$ containing M and a maximal torus $T\subset L\cap B$. Let $\alpha_1,\ldots,\alpha_r$ denote the corresponding simple roots and let α_0 be a root such that $\alpha_0^\vee$ is maximal. Let $\mathfrak{X}$ denote the character lattice of T and W denote the Weyl group. By W_P (or W_L) we denote the parabolic subgroup of W corresponding to P. We consider the (extended) affine Weyl group $W^a:=W\ltimes\mathfrak{X}$. Let $W^{a,P}$ (or $W^{a,L}$) denote the subset of all $x\in W^a$ such that x is longest in xW_P. Finally, let Br^a denote the braid group associated to W^a.

Now fix a prime number $p\gg 0$ and set $\mathbb{F}:=\overline{\mathbb{F}}_p$. We can assume that e,h,f are defined over a finite localization of $\mathbb{Z}$ hence all the objects introduced above in this section are defined over that localization. So they can be base-changed to $\mathbb{F}$, we will indicate this with the subscript $\mathbb{F}$: $G_{\mathbb{F}},\mathfrak{g}_{\mathbb{F}}$, etc. Let $\chi\in\mathfrak{g}_{\mathbb{F}}^{(1)*}$ be the element corresponding to $(e,\cdot)\in\mathfrak{g}^*$. Here, as usual, the superscript (1) denotes the Frobenius twist.

We will need to p-alcoves in $\mathfrak{h}^*_{\mathbb{Z}}$. By the *dominant* p-alcove we mean the locus of $\mu\in\mathfrak{h}^*_{\mathbb{Z}}$ such that $\langle\mu+\rho,\alpha_i^\vee\rangle\geqslant 0$ for all $i=1,\ldots,r$, and $\langle\mu+\rho,\alpha_0^\vee\rangle\leqslant p$. By the *antidominant* p-alcove we mean the locus of $\mu\in\mathfrak{h}^*_{\mathbb{Z}}$ such that $\langle\mu+\rho,\alpha_i^\vee\rangle\leqslant 0$ for all $i=1,\ldots,r$, and $\langle\mu+\rho,\alpha_0^\vee\rangle\geqslant -p$. Note that each of the roots $\alpha_i,i=0,\ldots,r$, defines a codimension 1 face in the (anti)dominant p-alcove.

We now describe the content of this section. In Section 2.2 we recall basics on modular representations of $\mathfrak{g}$ and introduce the main category of study in this paper, $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}}$. In Section 2.3 we introduce the χ-Weyl modules that should be thought as "standard" objects in the latter category. Finally, in Section 2.4 we recall translation functors between the categories $\mathcal{U}^\chi_{(?),\mathbb{F}}\text{-mod}^{\underline{Q}}$ and also an affine braid group action on $D^b(\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}})$ (in the case when $\lambda+\rho$ is regular).

2.2. **Central reductions and equivariant modules.** Pick $\lambda\in\mathfrak{h}^*_{\mathbb{F}}/W$. Let $\mathfrak{m}_\lambda^{HC}$ denote the maximal ideal of λ in the Harish-Chandra center $\mathcal{U}_{\mathbb{F}}^{G_{\mathbb{F}}}\cong\mathbb{F}[\mathfrak{h}^*]^W$ (where we consider the dot-action of W on $\mathfrak{h}^*$) and $\mathfrak{m}_\chi^{p\text{-cen}}$ be the maximal ideal of χ in the p-center $S(\mathfrak{g}_{\mathbb{F}}^{(1)})$. We can form the central reductions

$$\mathcal{U}_{\lambda,\mathbb{F}}:=\mathcal{U}_{\mathbb{F}}/\mathcal{U}_{\mathbb{F}}\mathfrak{m}_\lambda^{HC},\quad \mathcal{U}^\chi_{\mathbb{F}}:=\mathcal{U}_{\mathbb{F}}/\mathcal{U}_{\mathbb{F}}\mathfrak{m}_\chi^{p\text{-cen}},\quad \mathcal{U}^\chi_{\lambda,\mathbb{F}}:=\mathcal{U}_{\mathbb{F}}/\mathcal{U}_{\mathbb{F}}(\mathfrak{m}_\lambda^{HC}+\mathfrak{m}_\chi^{p\text{-cen}}). \tag{2.1}$$

We note that $\mathcal{U}^\chi_{\lambda,\mathbb{F}} \neq 0 \Rightarrow \lambda \in \mathfrak{h}^*_{\mathbb{F}_p}/W$. In particular, the $\mathbb{F}[\mathfrak{h}^*]^W$-module $\mathcal{U}^\chi_{\mathbb{F}}$ is supported on the finite set $\mathfrak{h}^*_{\mathbb{F}_p}/W$, where we write $\mathfrak{h}^*_{\mathbb{F}_p}$ for the set of $\mathbb{F}_p$-points of $\mathfrak{h}^*_{\mathbb{F}}$. We write $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}$ for the direct summand of $\mathcal{U}^\chi_{\mathbb{F}}$ corresponding to λ so that

$$\mathcal{U}^\chi_{\mathbb{F}} = \bigoplus_{\lambda\in\mathfrak{h}^*_{\mathbb{F}_p}/W} \mathcal{U}^\chi_{(\lambda),\mathbb{F}}.$$

The algebra $\mathcal{U}^\chi_{\lambda,\mathbb{F}}$ is the quotient of $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}$ by a nilpotent ideal.

Now let $\underline{Q}_{\mathbb{F}}$ be an algebraic subgroup of $Q_{\mathbb{F}}$. The group $Q_{\mathbb{F}}$ acts on $\mathcal{U}^\chi_{\mathbb{F}}$. We consider (weakly) $\underline{Q}_{\mathbb{F}}$-equivariant $\mathcal{U}^\chi_{\mathbb{F}}$-modules, i.e., $\mathcal{U}^\chi_{\mathbb{F}}$-modules V equipped with a rational $\underline{Q}_{\mathbb{F}}$-action such that the module map $\mathcal{U}^\chi_{\mathbb{F}} \otimes_{\mathbb{F}} V \to V$ is $\underline{Q}_{\mathbb{F}}$-equivariant. The category of these modules is denoted by $\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}$. Similarly, we can consider the categories $\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}}, \mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}}$.

The choice of $\underline{Q}$ we need is $Z_Q(T_0)$. In particular, $\underline{Q}^\circ = T_0$. Note that $\underline{Q}_{\mathbb{F}}$ acts on $\operatorname{Irr}(\mathcal{U}^\chi_{\mathbb{F}})$ and the action factors through the component group $\underline{Q}_{\mathbb{F}}/\underline{Q}^\circ_{\mathbb{F}}$ because $\operatorname{Irr}(\mathcal{U}^\chi_{\mathbb{F}})$ is a finite set. When e is distinguished in $\mathfrak{g}$, i.e., $T_0 = \{1\}$, the group $\underline{Q}$ is finite and coincides with A. In general, the projection $\underline{Q}_{\mathbb{F}} \to A$ is not surjective. Note that the order of $\underline{Q}_{\mathbb{F}}/\underline{Q}^\circ_{\mathbb{F}}$ is uniformly bounded with respect to p, in particular, $\underline{Q}_{\mathbb{F}}$ is linearly reductive.

The following lemma is standard.

Lemma 2.1. *Let U, V be irreducible objects in $\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}, \mathcal{U}^\chi_{\mathbb{F}}\text{-mod}$, respectively. Then the following hold:*

(1) $\operatorname{Hom}_{\mathfrak{g}_{\mathbb{F}}}(V, U)$ *is an irreducible $\underline{Q}_{\mathbb{F}}$-module or zero.*
(2) *The module U is completely reducible over $\mathcal{U}^\chi_{\mathbb{F}}$ and all irreducible $\mathcal{U}^\chi_{\mathbb{F}}$-modules that appear in M are in the same $\underline{Q}/\underline{Q}^\circ$-orbit.*

2.3. **χ-Weyl modules.** Now we are going to produce some examples of modules in $\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}}$. These modules are obtained by induction from a Levi subalgebra.

The induction functor we consider will map from $U^0(\mathfrak{l}_{\mathbb{F}})\text{-mod}^{\underline{Q}}$, the category of weakly $\underline{Q}_{\mathbb{F}}$-equivariant modules over the p-central reduction $U^0(\mathfrak{l}_{\mathbb{F}})$, to $\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}$. Note that the p-central reduction $U^0(\mathfrak{p}_{\mathbb{F}})$ has a natural $P_{\mathbb{F}}$-equivariant epimorphism onto $U^0(\mathfrak{l}_{\mathbb{F}})$. In particular, for an object of $U^0(\mathfrak{l}_{\mathbb{F}})\text{-mod}^{\underline{Q}}$ we can consider its inflation to a $U^0(\mathfrak{p}_{\mathbb{F}})$-module, this inflation is $\underline{Q}_{\mathbb{F}}$-equivariant.

The embedding $U(\mathfrak{p}_{\mathbb{F}}) \hookrightarrow U(\mathfrak{g}_{\mathbb{F}})$ gives rise to $U^0(\mathfrak{p}_{\mathbb{F}}) \hookrightarrow \mathcal{U}^\chi_{\mathbb{F}}$, which is $\underline{Q}_{\mathbb{F}}$-equivariant. Note that $\mathfrak{m}_{-,\mathbb{F}}$ is $\underline{Q}_{\mathbb{F}}$-stable and we have a $\underline{Q}_{\mathbb{F}}$-equivariant linear isomorphism $\mathcal{U}^\chi_{\mathbb{F}} \xrightarrow{\sim} U^\chi(\mathfrak{m}_{-,\mathbb{F}}) \otimes U^0(\mathfrak{p}_{\mathbb{F}})$. The induction functor we need is

$$\underline{\Delta}^\chi := \mathcal{U}^\chi_{\mathbb{F}} \otimes_{U^0(\mathfrak{p}_{\mathbb{F}})} \bullet : U^0(\mathfrak{l}_{\mathbb{F}})\text{-mod}^{\underline{Q}} \to \mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}.$$

We will be interested in certain induced modules that we call χ-Weyl modules. Namely, pick a dominant weight μ for L. Then we can consider the Weyl module $W_{L,\mathbb{F}}(\mu)$ over $L_{\mathbb{F}}$, it can be defined as $\Gamma(\mathcal{O}_{P_{\mathbb{F}}/B_{\mathbb{F}}}(\mu^*))^*$, where μ^* is the dual highest weight. Its character is given by the Weyl character formula. Clearly, we can view $W_{L,\mathbb{F}}(\mu)$ as a $\underline{Q}_{\mathbb{F}}$-equivariant module over $U^0(\mathfrak{l}_{\mathbb{F}})$.

Definition 2.2. The χ-*Weyl module* labelled by μ is, by definition, $W^\chi_{\mathbb{F}}(\mu) := \underline{\Delta}^\chi(W_{L,\mathbb{F}}(\mu))$. This is an object in $\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}$.

Let us establish some properties of the modules $W^\chi_{\mathbb{F}}(\mu)$. The following lemma is straightforward.

Lemma 2.3. *In the notation above, we have the following:*

(1) *The HC central character of $W^\chi_{\mathbb{F}}(\mu)$ is the image of $W \cdot \mu$ modulo p.*

(2) *Let us write $\hat{d}(\mu)$ for the $\underline{Q}$-character of $W_{L,\mathbb{F}}(\mu)$ and $\mathsf{ch}_{\mathfrak{m}^-}$ for the $\underline{Q}$-character of $U^\chi(\mathfrak{m}^-)$. Then the $\underline{Q}$-character of $W^\chi_{\mathbb{F}}(\mu)$ equals $\mathsf{ch}_{\mathfrak{m}_-}\hat{d}(\mu)$.*

Other properties will be proved as they are needed.

2.4. **Translation functors and braid group action.** First, we discuss translation equivalences between categories $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}}, \mathcal{U}^\chi_{(\lambda'),\mathbb{F}}\text{-mod}^{\underline{Q}}$, where λ, λ' are regular in $\mathfrak{h}^*_{\mathbb{F}_p}/W$.

Let μ° denote the element of $\mathfrak{h}^*_{\mathbb{Z}}$ representing λ lying in the anti-dominant p-alcove. Note that, for $w \in W^{a,P}$, the element $\mu_w := w^{-1} \cdot \mu^\circ$ is dominant for L.

Lemma 2.4. *There is an equivalence $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}} \xrightarrow{\sim} \mathcal{U}^\chi_{(\lambda'),\mathbb{F}}\text{-mod}^{\underline{Q}}$ that sends $W^\chi_{\mathbb{F}}(\mu_w)$ to $W^\chi(\mu'_w)$.*

Proof. The construction of an equivalence is standard – via translation functors. Namely, let $\mu'^\circ - \mu^\circ = w\kappa$, where κ is dominant and $w \in W$. Let V denote the irreducible $G_{\mathbb{F}}$-module with highest weight κ. The equivalence $\mathsf{T}_{\lambda' \leftarrow \lambda} : \mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}} \xrightarrow{\sim} \mathcal{U}^\chi_{(\lambda'),\mathbb{F}}\text{-mod}^{Z}$ is given by $\mathrm{pr}_{\lambda'}(V \otimes \bullet)$, where we write $\mathrm{pr}_{\lambda'}$ for the projection to $\mathcal{U}^{(\lambda')}_{\chi,\mathbb{F}}\text{-mod}^{\underline{Q}}$. The functor $\mathsf{T}_{\lambda' \leftarrow \lambda}$ sends the parabolic Verma module $\Delta^P_{\mathbb{F}}(\mu_w)$ to the parabolic Verma module $\Delta^P_{\mathbb{F}}(\mu'_w)$ and is $S(\mathfrak{g}^{(1)}_{\mathbb{F}})$-linear. It follows that $\mathsf{T}_{\lambda' \leftarrow \lambda} W^\chi_{\mathbb{F}}(\mu_w) = W^\chi_{\mathbb{F}}(\mu'_w)$. □

Now let us proceed to the braid group action.

Suppose that $\lambda + \rho$ is regular. The affine braid group Br^a acts on $D^b(\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}})$ (from the right) by the so called wall-crossing functors, see [BMR1, Section 2.1]. Namely, for $i = 0, \ldots, r$, let μ_i be such that μ_i lies on exactly one wall of the anti-dominant p-alcove, and this wall corresponds to the simple affine root α_i. Then T_i is given by the complex $\mathrm{id} \to \mathsf{T}_{\mu^\circ \leftarrow \mu_i} \mathsf{T}_{\mu_i \leftarrow \mu^\circ}$, where the second term is in homological degree 0.

We denote the wall-crossing functor corresponding to $T_x \in \mathrm{Br}^a$ for $x \in W^a$ again by T_x. This functor is right t-exact. The translation equivalences $\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}} \xrightarrow{\sim} \mathcal{U}^\chi_{(\lambda'),\mathbb{F}}\text{-mod}^{\underline{Q}}$ intertwine the actions of Br^a. The Br^a-action on the category induces an action of W^a on $K_0(\mathcal{U}^\chi_{(\lambda),\mathbb{F}}\text{-mod}^{\underline{Q}})$.

Lemma 2.5. *For $x \in W^a$, let x_- denote the shortest element in xW_P and x_+ be the longest element in xW_P. Then*

$$[W^\chi_{\mathbb{F}}(\mu^\circ)]x = (-1)^{\ell(x)-\ell(x_-)}[W^\chi_{\mathbb{F}}(\mu_{x_+})].$$

Additional properties of the braid group action will be established or recalled as needed.

3. Derived localization in positive characteristic

3.1. **Notation and content.** The notation $G, (e,h,f), \nu, \chi, W^a, \mathrm{Br}^a$ has the same meaning as in Section 2.1. Let $\underline{Q}$ stand for the centralizer of (e,h,f) in $\underline{G}$. Also recall the one-parameter subgroup $\gamma : \mathbb{G}_m \to G$ associated to h.

Let $\mathcal{B}$ denote the flag variety for $\mathfrak{g}$, i.e., $\mathcal{B} := G/B$, where B is a Borel subgroup. We write U for the unipotent radical of B. We write v for the projection $G/U \to G/B$.

Let $\mathfrak{b}$ be the Lie algebra of B and $\mathfrak{n}$ its nilpotent radical. We also pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let T be the maximal torus in B corresponding to $\mathfrak{h}$ and W be the Weyl group.

We consider the cotangent bundle $\tilde{\mathcal{N}} := T^*\mathcal{B}(= G \times^B \mathfrak{n})$. It is a resolution of singularities for the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. We can also consider the Grothendieck simultaneous resolution

$\tilde{\mathfrak{g}} := G \times^B \mathfrak{b}$ of $\mathfrak{g}_{\mathfrak{h}} := \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$. The scheme $\tilde{\mathfrak{g}}$ is smooth over $\mathfrak{h}$ and its fiber over 0 is $\tilde{\mathcal{N}}$. We write K_0 for the Springer morphism $\tilde{\mathcal{N}} \to \mathcal{N}$ and also for $\tilde{\mathfrak{g}} \to \mathfrak{g}_{\mathfrak{h}}$.

As before, we can reduce all objects introduced above mod $p \gg 0$. We will write $\mathcal{B}_\chi$ for the corresponding Springer fiber, it consists of all Borel subalgebras $\mathfrak{b}^{(1)}_{\mathbb{F}} \subset \mathfrak{g}^{(1)}_{\mathbb{F}}$ such that χ vanishes on $[\mathfrak{b}^{(1)}_{\mathbb{F}}, \mathfrak{b}^{(1)}_{\mathbb{F}}]$. We view $\mathcal{B}_\chi$ as a subvariety of $\tilde{\mathcal{N}}^{(1)}_{\mathbb{F}}$.

We set $(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}} := \tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}} \times_{\mathfrak{h}^{(1)*}_{\mathbb{F}}} \mathfrak{h}^*_{\mathbb{F}}$. Define $(\mathfrak{g}^{(1)}_{\mathbb{F}})_{\mathfrak{h}}$ similarly.

We now describe the content of this section. In Section 3.2 we recall results about derived localization in positive characteristic from [BMR2]. Then we describe splitting bundles for Azumaya algebras that arise in the derived localization theorem in Section 3.3, also following [BMR2]. Finally, in Section 3.4 we discuss equivariant structures on the splitting bundles. This has not appeared in the literature but is standard.

3.2. **Derived localization equivalence.** Let $\lambda \in \mathfrak{h}^*_{\mathbb{F}_p}/W$ be such that $\lambda + \rho$ is regular (recall that we consider the ρ-shifted action of W). Pick $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$ such that $W \cdot \mu$ mod p coincides with λ. Let t_μ denote the translation by μ in $\mathfrak{h}^*_{\mathbb{F}}$. Note that it intertwines the Artin-Schreier map $\mathfrak{h}^*_{\mathbb{F}} \to \mathfrak{h}^{*(1)}_{\mathbb{F}}$.

Then we can consider the sheaf $\tilde{D}_{\mathcal{B}_{\mathbb{F}}} := \upsilon_*(D_{G_{\mathbb{F}}/U_{\mathbb{F}}})^{T_{\mathbb{F}}}$. We have $\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) = \mathcal{U}_{\mathbb{F},\mathfrak{h}} := \mathcal{U}_{\mathbb{F}} \otimes_{\mathbb{F}[\mathfrak{h}^*]^W} \mathbb{F}[\mathfrak{h}^*]$, and $R^i\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) = 0$ for $i > 0$. We can view $\tilde{D}_{\mathcal{B}_{\mathbb{F}}}$ as an Azumaya algebra on $(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}}$. We write $\mathbb{F}[\mathfrak{h}^*]^{\wedge_\mu}$ for the completion of $\mathbb{F}[\mathfrak{h}^*]$ at μ. We set

$$(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})^{\wedge_\mu}_{\mathfrak{h}} := (\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}} \times_{\mathfrak{h}^*_{\mathbb{F}}} \operatorname{Spec}(\mathbb{F}[\mathfrak{h}^*]^{\wedge_\mu}), \tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}} := \mathsf{t}^*_\mu \left(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}|_{(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})^{\wedge_\mu}_{\mathfrak{h}}} \right).$$

Then $\tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}}$ is an Azumaya algebra on $(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})^{\wedge_0}_{\mathfrak{h}}$.

So it makes sense to consider the derived global section functor

$$R\Gamma^\mu : D^b(\operatorname{Coh}(\tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}})) \to D^b(\mathcal{U}^{\wedge_\lambda}_{\mathbb{F}}\text{-mod}),$$

where we write $\mathcal{U}^{\wedge_\lambda}_{\mathbb{F}}$ for $\mathcal{U}_{\mathbb{F}} \otimes_{\mathbb{F}[\mathfrak{h}^*]^W} (\mathbb{F}[\mathfrak{h}^*]^W)^{\wedge_\lambda}$ with the second factor being the completion at λ.

The following is [BMR2, Theorem 3.2].

Theorem 3.1. *The functor $R\Gamma^\mu$ is an equivalence.*

3.3. **Splitting bundle.** It turns out that $\mathcal{U}^{\wedge_{-\rho}}_{\mathfrak{h},\mathbb{F}}$ is an Azumaya algebra on $(\mathfrak{g}^{(1)}_{\mathbb{F}})^{\wedge_0}_{\mathfrak{h}}$. Moreover, $\tilde{D}^{\wedge_{-\rho}}_{\mathcal{B}_{\mathbb{F}}} = K_0^* \mathcal{U}^{\wedge_{-\rho}}_{\mathfrak{h},\mathbb{F}}$, see [BMR2, Proposition 5.2.1]. The Azumaya algebras $\tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}}$ and $D^{\wedge_{-\rho}}_{\mathcal{B}_{\mathbb{F}}}$ are Morita equivalent via the $\tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}}$-$\tilde{D}^{\wedge_{-\rho}}_{\mathcal{B}_{\mathbb{F}}}$-bimodule $\operatorname{Fr}_{\mathfrak{h},*}\left(\mathcal{O}(\mu+\rho) \otimes \tilde{D}^{\wedge_{-\rho}}_{\mathcal{B}_{\mathbb{F}}}\right)$. Here we write $\operatorname{Fr}_{\mathfrak{h}}$ for the morphism $\tilde{\mathfrak{g}}_{\mathbb{F}} \to (\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}}$ given by $(\operatorname{Fr}, \operatorname{id})$. Note that $\mathcal{O}(\mu+\rho)$ is a G-equivariant line bundle when $\mu + \rho \in \mathfrak{X}(T)$.

From here we deduce that the restriction $\tilde{D}^{\wedge_{\mu,\chi}}_{\mathcal{B}_{\mathbb{F}}}$ of $\tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}}$ to

$$(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})^{\wedge_\chi}_{\mathfrak{h}} := (\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}} \times_{\mathfrak{g}^{(1)*}_{\mathbb{F}}} \mathfrak{g}^{(1)*\wedge_\chi}_{\mathbb{F}}$$

splits. Note that since the Artin-Schreier map $\mathfrak{h}^*_{\mathbb{F}} \to \mathfrak{h}^{*(1)}_{\mathbb{F}}$ is unramified over 0, we have a natural identification of $(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})^{\wedge_\chi}_{\mathfrak{h}}$ with $\tilde{\mathfrak{g}}^{(1)\wedge_\chi}_{\mathbb{F}}$.

Recall that a splitting bundle of an Azumaya algebra is unique up to a twist with a line bundle. A choice of a splitting bundle $\mathcal{V}$ then gives rise to an abelian equivalence

$$\operatorname{Coh}_{\mathcal{B}_\chi}\left(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}\right) \xrightarrow{\sim} \operatorname{Coh}_{\mathcal{B}_\chi}\left(\tilde{D}^{\wedge_\mu}_{\mathcal{B}_{\mathbb{F}}}\right), \quad \mathcal{F} \mapsto \mathcal{V} \otimes \mathcal{F}. \tag{3.1}$$

Here we write $\mathrm{Coh}_{\mathcal{B}_\chi}$ for the category of all coherent sheaves that are set theoretically supported on $\mathcal{B}_\chi$.

We have full embeddings

$$D^b\left(\mathrm{Coh}_{\mathcal{B}_\chi}(\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)})\right) \hookrightarrow D^b(\mathrm{Coh}\left((\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)})^{\wedge_0}_{\mathfrak{h}}\right), D^b(\mathcal{U}_{\mathbb{F}}\text{-mod}^\chi_\lambda) \hookrightarrow D^b(\mathcal{U}_{\mathbb{F}}^{\wedge_\lambda}\text{-mod}),$$

where we write $\mathcal{U}_{\mathbb{F}}\text{-mod}^\chi_\lambda$ for the category of $\mathcal{U}_{\mathbb{F}}$-modules with generalized HC character λ and generalized p-character χ. Hence we also have the derived equivalence

$$D^b\left(\mathrm{Coh}_{\mathcal{B}_\chi}(\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)})\right) \xrightarrow{\sim} D^b(\mathcal{U}_{\mathbb{F}}\text{-mod}^\chi_\lambda), \quad \mathcal{F} \mapsto R\Gamma(\mathcal{V}\otimes\mathcal{F}). \tag{3.2}$$

Let us now explain our choice of a splitting bundle. A splitting bundle $\overline{\mathcal{V}}^\chi_{-\rho}$ for $\mathcal{U}_{\mathfrak{h},\mathbb{F}}^{\wedge_{\chi,-\rho}}$ on $(\mathfrak{g}_{\mathbb{F}}^{(1)})^{\wedge_\chi}_{\mathfrak{h}}$ is unique up to an isomorphism. We can take

$$\mathcal{V}^\chi_\mu := \left(\mathrm{Fr}_{\mathfrak{h}*}(\mathcal{O}(\mu-(p-1)\rho)\otimes \tilde{D}_{\mathcal{B}_{\mathbb{F}}}^{\wedge_{-\rho}})\right)^{\wedge_\chi} \otimes_{K_0^*(\mathcal{U}_{\mathfrak{h},\mathbb{F}}^{\wedge_{-\rho}})^{\wedge_\chi}} K_0^*\overline{\mathcal{V}}^\chi_{-\rho}. \tag{3.3}$$

Finally, let us discuss a compatibility with braid group actions. The braid group B^a acts (via right actions) on both $D^b(\mathrm{Coh}_{\mathcal{B}_\chi}(\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)}))$ (see [BR]) and $D^b(\mathcal{U}_{\mathbb{F}}\text{-mod}^\chi_\lambda)$. The following proposition was proved in [R, Section 5.4].

Proposition 3.2. *Let $\mu+\rho$ lie inside the dominant alcove and $\rho\in\mathfrak{X}(T)$. The functor $R\Gamma(\mathcal{V}^\chi_\mu(\rho)\otimes\bullet)$ is Br^a-equivariant.*

3.4. **Equivariance of the splitting bundle.** Here we introduce a $\underline{Q}_{\mathbb{F}}$-equivariant structure on $\mathcal{V}^\chi_\mu$ and discuss its properties.

We start by treating the case of $\chi=0$. Note that the fiber of $\overline{\mathcal{V}}^0_{-\rho}$ at 0 carries a natural $G_{\mathbb{F}}$-action, in fact, it is the Steinberg $G_{\mathbb{F}}$-module.

Lemma 3.3. *There is an extension of the $G_{\mathbb{F}}$-equivariant structure from $\left(\overline{\mathcal{V}}^0_{-\rho}\right)_0$ to $\underline{\mathcal{V}}^0_{-\rho}$.*

Proof. The obstruction for the existence of a $G_{\mathbb{F}}$-equivariant structure lies in the cohomology group $H^1_{G_{\mathbb{F}}}(1+\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal in $\mathbb{F}[(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}]^{\wedge_0}$. Note that $H^1_{G_{\mathbb{F}}}(1+\mathfrak{m})=\{0\}$ will follow once we know $H^1_{G_{\mathbb{F}}}(\mathbb{F}[(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}])=0$ because $\mathbb{F}[(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}]$ is a positively graded algebra.

First of all, we claim that

$$H^i_{G_{\mathbb{F}}^{(1)}}(\mathbb{F}[(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}])=0, \forall i>0. \tag{3.4}$$

Note that $\mathbb{F}[(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}]=\mathbb{F}[\mathfrak{h}]\otimes_{\mathbb{F}[\mathfrak{h}^{(1)}]}\mathbb{F}[\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)}]$, Since $\mathbb{F}[\mathfrak{h}]$ is free over $\mathbb{F}[\mathfrak{h}^{(1)}]$, (3.4) will follow if we check that $H^i_{G_{\mathbb{F}}^{(1)}}(\mathbb{F}[\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)}])=0$ for $i>0$. Note that

$$\mathbb{F}[\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)*}] = R\,\mathrm{Ind}_{B_{\mathbb{F}}^{(1)}}^{G_{\mathbb{F}}^{(1)}}\mathbb{F}[\mathfrak{b}_{\mathbb{F}}^{(1)}].$$

So we reduce to checking that $H^i_{B_{\mathbb{F}}^{(1)}}(\mathbb{F}[\mathfrak{b}_{\mathbb{F}}^{(1)*}])=0$ for all $i>0$. This follows from the weight considerations for the $T_{\mathbb{F}}$-action.

Using (3.4) we deduce that $H^1_{G_{\mathbb{F}}}(\mathbb{F}[\mathfrak{g}_{\mathbb{F}}^{(1)*}]) = H^1_{G_1}(\mathbb{F}[\mathfrak{g}_{\mathbb{F}}^{(1)*}])$, where we write G_1 for the Frobenius kernel. But in the cohomology group we have a trivial G_1-representation. By the weight reasons, the 1st self-extensions of the trivial one-dimensonal G_1-module vanish. □

Now we proceed to the case of general χ. For technical reasons we need to work with a one-parameter version of $\mathcal{V}_\mu^\chi$ that we going to introduce now. We note that $\mathcal{U}_{-\rho,\mathbb{F}}$ splits on $\mathbb{F}\chi$. The splitting bundle is $\underline{\Delta}^{\mathbb{F}\chi}(W_{L,\mathbb{F}}((p-1)\rho)\otimes\mathbb{F}[z])$, where we write z for a coordinate on $\mathbb{F}\chi$ and $\underline{\Delta}^{\mathbb{F}\chi}$ for the induction functor whose fiber at $a\in\mathbb{F}$ is $\underline{\Delta}^{a\chi}$. We note that $\underline{\Delta}^{\mathbb{F}\chi}(W_{L,\mathbb{F}}((p-1)\rho)\otimes\mathbb{F}[z])$ carries a natural $\underline{Q}_{\mathbb{F}}$-action. Besides, it also has an action of $\mathbb{F}^\times$ via γ, note that this action rescales z.

The following lemma shows that we can extend $\underline{\Delta}^{\mathbb{F}\chi}(W_L((p-1)\rho)\otimes\mathbb{F}[t])$ to a $\underline{Q}_{\mathbb{F}}\times\mathbb{F}^\times$-equivariant splitting bundle for the restriction of $\mathcal{U}_{\mathfrak{h},\mathbb{F}}^{\wedge-\rho}$ to

$$(\mathfrak{g}_{\mathbb{F}}^{*(1)})_{\mathfrak{h}}^{\wedge\mathbb{F}\chi}:=(\mathfrak{g}_{\mathbb{F}}^{*(1)})_{\mathfrak{h}}\times_{\mathfrak{g}_{\mathbb{F}}^{(1)*}}\operatorname{Spec}(\mathbb{F}[\mathfrak{g}^{*(1)}]^{\mathbb{F}\chi}).$$

Lemma 3.4. *Let R be an $\mathbb{F}$-algebra, $\mathfrak{m}\subset R$ be an ideal such that $R/\mathfrak{m}=\mathbb{F}[z]$ and R is complete in the $\mathfrak{m}$-adic topology. Let B be an Azumaya R-algebra. Let $\tilde{Q}_{\mathbb{F}}$ be an algebraic group with the following properties:*

- *$\tilde{Q}_{\mathbb{F}}^\circ$ is a torus,*
- *$\tilde{Q}_{\mathbb{F}}/\tilde{Q}_{\mathbb{F}}^\circ$ is of order coprime to p.*

Suppose that $\tilde{Q}_{\mathbb{F}}$ acts on R by $\mathbb{F}$-linear automorphisms and the algebra B is $\tilde{Q}_{\mathbb{F}}$-equivariant. We also suppose that z gets rescaled with a nontrivial character. Further, suppose that we have a $\tilde{Q}_{\mathbb{F}}$-equivariant isomorphism $\mathsf{B}/\mathsf{B}\mathfrak{m}\cong\operatorname{End}_{\mathbb{F}[z]}(V_z)$ for some finite rank free $\mathbb{F}[z]$-module V_z with a rational $\tilde{Q}_{\mathbb{F}}$-action. Then the following claims hold:

1. *There is a finite rank free R-module $\tilde{V}_z$ with a pro-rational $\tilde{Q}_{\mathbb{F}}$-action such that $\tilde{V}_z/\mathfrak{m}\tilde{V}_z\cong V_z$ and $\operatorname{End}_R(\tilde{V}_z)\cong\mathsf{B}$, $\tilde{Q}_{\mathbb{F}}$-equivariant isomorphisms.*
2. *Suppose we are given a free $R/(z)$-module $\tilde{V}$ with a rational $\tilde{Q}_{\mathbb{F}}$-action and isomorphisms $\mathsf{B}/(z)\xrightarrow{\sim}\operatorname{End}_{R/(z)}(\tilde{V})$ and $\tilde{V}/\mathfrak{m}\tilde{V}\xrightarrow{\sim}V_z/zV_z$ that are compatible. Then we can find $\tilde{V}_z$ as in (1) that comes with a $\tilde{Q}_{\mathbb{F}}$-equivariant isomorphism $\tilde{V}_z/z\tilde{V}_z\xrightarrow{\sim}\tilde{V}$ compatible with the isomorphism $\mathsf{B}/(z)\xrightarrow{\sim}\operatorname{End}_{R/(z)}(\tilde{V})$.*

Proof. Let us prove (1). Suppose that we have constructed a lift $\tilde{V}_k$ of V_z to $R/\mathfrak{m}^k$ with a rational $\tilde{Q}_{\mathbb{F}}$-action and a $\tilde{Q}_{\mathbb{F}}$-equivariant isomorphism $\operatorname{End}_{R/\mathfrak{m}^k}(\tilde{V}_k)\xrightarrow{\sim}\mathsf{B}/\mathsf{B}\mathfrak{m}^k$. Note that the set of lifts $(\tilde{V}_{k+1},\iota_{k+1})$ of any given isomorphism $\iota_k:\operatorname{End}_{R/\mathfrak{m}^k}(\tilde{V}_k)\xrightarrow{\sim}\mathsf{B}/\mathsf{B}\mathfrak{m}^k$ to $\operatorname{End}_{R/\mathfrak{m}^{k+1}}(\tilde{V}_{k+1})\xrightarrow{\sim}\mathcal{B}/\mathcal{B}\mathfrak{m}^{k+1}$ is an affine bundle over $\mathbb{A}^1$. If ι_k is $\tilde{Q}_{\mathbb{F}}$-equivariant, then $\tilde{Q}_{\mathbb{F}}$ acts on the affine bundle of lifts by affine transformations. The fixed point locus is also an affine bundle. But any affine bundle over $\mathbb{A}^1$ can be trivialized, which implies the existence of $\tilde{Q}_{\mathbb{F}}$-equivariant lift ι_{k+1}.

The argument above also proves (2). $\square$

Let $\underline{\mathcal{V}}_{-\rho}^{\mathbb{F}\chi}$ denote the resulting $\tilde{Q}_{\mathbb{F}}\times\mathbb{F}^\times$-equivariant splitting bundle for the restriction of $\mathcal{U}_{\mathbb{F}}^{\wedge-\rho}$ to $(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}^{\wedge\mathbb{F}\chi}$. This gives rise to a splitting bundle for the restriction of $\tilde{D}_{\mathcal{B}_{\mathbb{F}}}^{\wedge\mu}$ to

$$(\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}^{\wedge\mathbb{F}\chi}:=(\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}\times_{(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}}(\mathfrak{g}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}^{\wedge\mathbb{F}\chi}.$$

This splitting bundle will be denoted by $\mathcal{V}_\mu^{\mathbb{F}\chi}$ and given by the formula analogous to (3.3). Its restriction to $(\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)})_{\mathfrak{h}}^{\wedge\chi}$ is $\mathcal{V}_\mu^\chi$. By the construction, it follows that the $\underline{Q}_{\mathbb{F}}$-equivariant structure on $\mathcal{V}_\mu^0$ coincides with the restriction of the $G_{\mathbb{F}}$-equivariant structure.

Lemma 3.5. *We have the following:*

1. *The restriction of $\mathcal{V}_\mu^0$ to $\mathcal{B}_{\mathbb{F}}^{(1)}$ is $G_{\mathbb{F}}$-equivariantly isomorphic to $\operatorname{Fr}_{\mathcal{B},*}\mathcal{O}(\mu)$.*

(2) *The class of* $\mathcal{V}^\chi_\mu$ *in* $K^Q_0(\mathcal{B}_\chi)$ *coincides with the pull-back of* $[\mathrm{Fr}_{\mathcal{B},*}\mathcal{O}(\mu)]$ *under the inclusion* $\mathcal{B}_\chi \hookrightarrow \mathcal{B}^{(1)}_{\mathbb{F}}$.

Proof. Let us prove (1). Note that we can replace $G_{\mathbb{F}}$ with a cover. So we can assume that $G_{\mathbb{F}}$ is the product of a torus and a simply connected semisimple group. It is sufficient to consider the case of a torus and the case of a simply connected semisimple group separately. The torus case is trivial.

Let $G_{\mathbb{F}}$ be semisimple and simply connected. Both $\mathcal{V}^0_\mu|_{\mathcal{B}^{(1)}_{\mathbb{F}}}$ and $\mathrm{Fr}_{\mathcal{B},*}\mathcal{O}(\mu)$ are $G_{\mathbb{F}}$-equivariant splitting bundles for $D^\mu_{\mathcal{B}_{\mathbb{F}}}|_{\mathcal{B}^{(1)}_{\mathbb{F}}}$. So they differ by a twist with a $G_{\mathbb{F}}$-equivariant line bundle $\mathcal{L}$ on $\mathcal{B}^{(1)}_{\mathbb{F}}$. Note that every line bundle on $\mathcal{B}^{(1)}_{\mathbb{F}}$ has a unique $G_{\mathbb{F}}$-equivariant structure.

The abelian group $K_0(\mathcal{B}^{(1)}_{\mathbb{F}})$ is torsion free. Therefore a class of a vector bundle is not a zero divisor in the ring $K_0(\mathcal{B}^{(1)}_{\mathbb{F}})$. Moreover, $\mathrm{Pic}(\mathcal{B}^{(1)}_{\mathbb{F}})$ embeds into $K_0(\mathcal{B}^{(1)}_{\mathbb{F}})$. So in order to check that $\mathcal{L}$ is equivariantly trivial it suffices to show that the classes of $\mathcal{V}^0_\mu|_{\mathcal{B}^{(1)}_{\mathbb{F}}}$ and $\mathrm{Fr}_{\mathcal{B},*}\mathcal{O}(\mu)$ in the usual (i.e., non-equivariant) K_0-group are the same. This is true for $\mu=-\rho$: it is easy to see that in that case both bundles are $\mathcal{O}_{\mathcal{B}^{(1)}_{\mathbb{F}}}(-\rho)^{\oplus p^{\dim\mathcal{B}}}$. And for an arbitrary element μ, we have

$$[\mathcal{V}^0_\mu|_{\mathcal{B}^{(1)}_{\mathbb{F}}}] = [\mathcal{O}_{\mathcal{B}^{(1)}_{\mathbb{F}}}(-\rho)^{\oplus p^{\dim G/B}}][\mathcal{O}(\frac{\mu+\rho}{p})] \tag{3.5}$$

in $K_0(\mathcal{B}^{(1)}_{\mathbb{F}})$. By $[\mathcal{O}(\frac{\mu+\rho}{p})]$ we mean the pth root of $[\mathcal{O}(\mu+\rho)]$, the latter class is unipotent so its pth root makes sense in $K_0(\mathcal{B}^{(1)}_{\mathbb{F}})\otimes_{\mathbb{Z}}\mathbb{Q}$. On the other hand, we have $[\mathrm{Fr}_{\mathcal{B},*}\mathcal{O}(p\mu'-\rho)] = \mathcal{O}(\mu'-\rho)^{\oplus\dim\mathcal{B}}$ for any $\mu'\in\mathfrak{X}(T)$. It follows that $[\mathrm{Fr}_{\mathcal{B},*}\mathcal{O}(\mu)]$ equals to the right hand side of (3.5). This finally implies an isomorphism $\mathcal{V}^0_\mu|_{\mathcal{B}^{(1)}_{\mathbb{F}}}\cong \mathrm{Fr}_{\mathcal{B},*}\mathcal{O}(\mu)$.

Let us prove (2). Thanks to the existence of $\mathcal{V}^{\mathbb{F}\chi}_\mu$, in the proof we can replace $\mathcal{V}^\chi_\mu$ with $\mathcal{V}^0_\mu|_{\mathcal{B}_\chi}$. Now the claim follows from (1). □

4. Tilting bundle on $\tilde{\mathfrak{g}}$

4.1. **Notation and content.** In this section, our base field is $\mathbb{C}$. The notation $G,(e,h,f),\rho,\chi,\gamma,\mathfrak{g},\mathcal{B},\tilde{\mathcal{N}},\tilde{\mathfrak{g}}$ has the same meaning as in Section 3.1.

Let P denote a parabolic subgroup of G containing B. Set $\mathcal{P}:=G/P, \tilde{\mathcal{N}}_P:=T^*\mathcal{P}$. We will write Z for $G\times^B\mathfrak{m}$, ι for the natural inclusion $Z\hookrightarrow\tilde{\mathcal{N}}$ and ϖ for the natural projection $Z\twoheadrightarrow\tilde{\mathcal{N}}_P$. We assume G is semisimple and simply connected.

We consider the following versions of the Steinberg varieties: $\mathsf{St}_{\mathfrak{h}}:=\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}}$, $\mathsf{St}_B:=\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathcal{N}}$, $\mathsf{St}_0:=\tilde{\mathcal{N}}\times^L_{\mathfrak{g}}\tilde{\mathcal{N}}$. The first two are genuine varieties, while the last one is a derived scheme. This makes sense because the codimensions of $\mathsf{St}_{\mathfrak{h}}$ in $\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}$ and of St_B in $\tilde{\mathfrak{g}}\times\tilde{\mathcal{N}}$ are equal to $\dim\mathfrak{g}$ – so the derived schemes are the same as usual schemes. Moreover, these schemes are generically reduced hence reduced. On the other hand, the codimension of St_0 in $\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}$ is less than $\dim\mathfrak{g}$.

Set $\mathbb{O}:=Ge$. We write $\mathcal{B}_e$ for the Springer fiber $\tilde{\mathfrak{g}}\times^L_{\mathfrak{g}}\{e\}$ that we view with its natural derived scheme structure. Consider the Slodowy slice $S=e+\mathfrak{z}_{\mathfrak{g}}(f)$, it is transverse to $\mathbb{O}$. The reductive group $Q:=Z_G(e,h,f)$ acts on S. We consider the $\mathbb{C}^\times$-action on S given by $t.s:=t^{-2}\gamma(t)s$. It commutes with Q.

The purpose of this section is to recall some results from [BM]. In Section 4.2 we record some generalities of tilting generators for categories of coherent sheaves. Then in Section 4.3 we recall some properties of a tilting generator for $\tilde{\mathfrak{g}}$ constructed and studied in [BM].

4.2. **Generalities on tilting generators.** Let X be a smooth Calabi-Yau algebraic variety (or scheme) with a projective morphism to an affine variety. Recall that a vector bundle $\mathcal{T}$ on X is called a *tilting generator* (for $\operatorname{Coh}(X)$ or simply for X) if it has no higher self-extensions and the algebra $\operatorname{End}(\mathcal{T})$ has finite homological dimension. Note that $R\Gamma(\mathcal{T}\otimes\bullet): D^b(\operatorname{Coh}(X))\to D^b(\operatorname{End}(\mathcal{T})\text{-mod})$ is an equivalence, [BK]. It defines a new t-structure on $D^b(\operatorname{Coh}(X))$, where $\mathcal{T}^*$ is a projective generator and $R\Gamma(\mathcal{T}\otimes\bullet)$ is t-exact.

Definition 4.1. We say that this t-structure is defined by $\mathcal{T}^*$.

Set $\mathcal{A}:=\operatorname{End}(\mathcal{T})$.

Now let Y be an affine smooth algebraic variety and let $X\to Y$ be a projective morphism. Suppose that $\mathcal{A}$ is flat over $\mathbb{C}[Y]$. The derived scheme $X\times^L_Y X$ comes with a vector bundle $\mathcal{T}\otimes\mathcal{T}^*$ so that

$$R\Gamma(\mathcal{T}\otimes\mathcal{T}^*,\bullet): D^b(\operatorname{Coh}(X\times^L_Y X))\to D^b(\mathcal{A}\otimes_{\mathbb{C}[Y]}\mathcal{A}^{opp}\text{-mod}) \tag{4.1}$$

is an equivalence. Note that it maps $\mathcal{T}^*\otimes\mathcal{T}$ to $\mathcal{A}\otimes_{\mathbb{C}[Y]}\mathcal{A}^{opp}$.

Here is a basic property of this equivalence. Note that $D^b(\operatorname{Coh}(X\times^L_Y X))$ is a monoidal category with respect to convolution of coherent sheaves.

Lemma 4.2. *The equivalence (4.1) is monoidal with respect to the convolution on* $D^b(\operatorname{Coh}(X\times^L_Y X))$ *and the tensor product of bimodules on* $D^b(\mathcal{A}\otimes_{\mathbb{C}[Y]}\mathcal{A}^{opp}\text{-mod})$.

There is also an obvious module (and bimodule) analogs of this lemma.

4.3. **Bezrukavnikov-Mirkovic tilting bundle.** We will need a $G\times\mathbb{C}^\times$-equivariant vector bundle $\mathcal{T}_{\mathfrak{h}}$ on $\tilde{\mathfrak{g}}$ with remarkable properties that was constructed in [BM].

Here are two crucial properties of this bundle established in [BM, Section 2.5] that we will need:

Lemma 4.3. *The following claims are true:*

(1) *The bundle $\mathcal{T}_{\mathfrak{h}}$ is a tilting generator and* $\operatorname{End}(\mathcal{T}_{\mathfrak{h}})$ *is flat over $\mathfrak{g}$. Moreover, $\mathcal{T}:=\mathcal{T}_{\mathfrak{h}}|_0$ is a tilting generator for $\tilde{\mathcal{N}}$.*
(2) *The bundle $\mathcal{T}_{\mathfrak{h}}$ is defined over a finite localization of $\mathbb{Z}$.*

We write $\mathcal{A}_{\mathfrak{h}}$ for $\operatorname{End}(\mathcal{T}_{\mathfrak{h}})$.

Thanks to (1) of Lemma 4.3, we have the following derived equivalence

$$R\Gamma(\mathcal{T}_{\mathfrak{h}}\otimes\bullet): D^b(\operatorname{Coh}(\tilde{\mathfrak{g}}))\xrightarrow{\sim} D^b(\mathcal{A}_{\mathfrak{h}}\text{-mod}),$$

as well as the similarly defined equivalences between categories of equivariant objects with respect to algebraic subgroups of G.

We record some related equivalences that we will need below. Set $\mathcal{A}:=\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{h}^*]}\mathbb{C}_0, \mathcal{T}:=\mathcal{T}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{h}^*]}\mathbb{C}_0$.

First of all, since $\mathsf{St}_{\mathfrak{h},\mathfrak{h}},\mathsf{St}_{\mathfrak{h},0}$ are complete intersections in $\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}\times\tilde{\mathcal{N}}$, respectively, we have the following derived equivalences:

$$D^b(\operatorname{Coh}^G(\mathsf{St}_B))\xrightarrow{\sim} D^b(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}^{opp}\text{-mod}^G), \tag{4.2}$$

$$D^b(\operatorname{Coh}^G(\mathsf{St}_{\mathfrak{h}}))\xrightarrow{\sim} D^b(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G) \tag{4.3}$$

Note that (4.3) is an equivalence of monoidal categories and then (4.2) is an equivalence of module categories. We have one more equivalence of module categories.

$$D^b(\mathrm{Coh}^{Z_G(e)}_{\mathcal{B}_e}(\tilde{\mathfrak{g}})) \xrightarrow{\sim} D^b(\mathcal{A}_{\mathfrak{h}}\text{-mod}^{Z_G(e)}_e). \tag{4.4}$$

Moreover, we can replace the equivariance with respect to $Z_G(e)$ with that with respect to any algebraic subgroup of $Z_G(e)$.

Finally, let us explain a Koszulity property. Consider the restriction $\mathcal{A}_{\mathfrak{h}}|_S := \mathbb{C}[S]\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}$. This algebra is acted on by Q. The following is one of the main results of [BM], see Section 5.5 there.

Theorem 4.4. *There is a Koszul grading on $\mathcal{A}_{\mathfrak{h}}|_S$ compatible with the grading on $\mathbb{C}[S]$.*

Remark 4.5. Note that the Koszul grading on $\mathcal{A}_{\mathfrak{h}}|_S$ comesfrom a $\mathbb{C}^\times$-equivariant structure on the restriction of $\mathcal{T}$ to $S\times_{\mathfrak{g}}\tilde{\mathfrak{g}}$. The group $Q\times\mathbb{C}^\times$ acts on $\mathcal{A}_{\mathfrak{h}}|_S$ because $\mathcal{T}|_{S\times_{\mathfrak{g}}\tilde{\mathfrak{g}}}$ is $Q\times\mathbb{C}^\times$-equivariant. We claim that we can choose the grading to be $Q\times\mathbb{C}^\times$-stable. Note that the descending filtration $(\mathcal{A}_{\mathfrak{h}}|_S)_{\geqslant d}$ on $\mathcal{A}_{\mathfrak{h}}|_S$ coming from a Koszul grading is canonical. It is by $\mathbb{C}[S]$-submodules and is preserved by the action of $Q\times\mathbb{C}^\times$. For each $d>0$, the group of $\mathbb{C}[S]$-linear automorphisms of $\mathcal{A}_{\mathfrak{h}}|_S/(\mathcal{A}_{\mathfrak{h}}|_S)_{\geqslant d}$ that are the identity on the associated graded is algebraic. It is normalized by $Q\times\mathbb{C}^\times$. So we can choose a grading on $\mathcal{A}_{\mathfrak{h}}|_S/(\mathcal{A}_{\mathfrak{h}}|_S)_{\geqslant d}$ that

- splits the filtration,
- is compatible with the grading on $\mathbb{C}[S]$,
- and is $Q\times\mathbb{C}^\times$-stable.

We can do this in a compatible way for all d. This gives rise to a $Q\times\mathbb{C}^\times$-stable Koszul grading on $\mathcal{A}_{\mathfrak{h}}|_S$.

Since $\mathcal{T}_{\mathfrak{h}}$ is defined over a finite localization of $\mathbb{Z}$, we can reduce it mod p for p large enough. We will view the reduction, to be denoted by $\mathcal{T}_{\mathfrak{h},\mathbb{F}}$, as a vector bundle on $\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}$. This vector bundle is still a tilting generator. Now we discuss a connection between $\mathcal{T}_{\mathfrak{h},\mathbb{F}}$ and the splitting bundles considered in Section 3.

Lemma 4.6. *The indecomposable summands of $\mathcal{T}_{\mathfrak{h},\mathbb{F}}$ restricted to the formal neighborhood of $\mathcal{B}^{(1)}_\chi$ are precisely the indecomposable summands of the bundle $\mathcal{V}^\chi_0(\rho)$, where $\mathcal{V}^\chi_0$ is defined by (3.3).*

Proof. According to [BM, Section 1.5.1], the following two conditions determine a tilting generator $\mathcal{V}$ on $\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}\times_{\mathfrak{g}^{(1)}_{\mathbb{F}}}\mathfrak{g}^{(1)\wedge\chi}_{\mathbb{F}}$ uniquely (up to changing the multiplicities of the indecomposable summands):

- Braid positivity: the right action of the affine braid monoid on $D^b_{\mathcal{B}^{(1)}_\chi}(\mathrm{Coh}(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}))$ is by right t-exact functors with respect to the t-structure given by $\mathcal{V}^*$.
- Normalization: $\mathcal{O}$ is a direct summand in $\mathcal{V}$, equivalently, $R\Gamma$ is exact in the t-structure defined by $\mathcal{V}$.

The restriction of $\mathcal{T}_{\mathfrak{h},\mathbb{F}}$ satisfies these two properties by the construction, see [BM, Section 1.5.1]. We need to show that $\mathcal{V}^\chi_0(\rho)$ does. The braid positivity follows because $R\Gamma(\mathcal{V}^\chi_0(\rho)\otimes\bullet)$ intertwines the braid group actions and the action on the category of $\mathcal{U}_{\mathbb{F}}\text{-mod}^\chi_0$ is by right t-exact functors, Proposition 3.2. The normalization follows because $R\Gamma$ is, up to a t-exact category equivalence, the translation functor to $-\rho$, [BMR1, Section 2.2.5]. $\square$

Finally, let us discuss a tilting bundle on $\tilde{\mathcal{N}}_P$. Set

$$\mathcal{T}_P := \varpi_*\iota^*(\mathcal{T}(-\rho)). \tag{4.5}$$

The following claim was established in [BM, Sections 4.1, 4.2].

Lemma 4.7. *The complex $\mathcal{T}_P$ is a vector bundle in homological degree* 0*. Moreover, it is a tilting generator for $\tilde{\mathcal{N}}_P$.*

5. Constructible realization

5.1. **Notation and content.** We continue to work over $\mathbb{C}$. The notation $G, \mathfrak{g}, B, T, \rho, \mathfrak{b}, \mathfrak{h}, \mathcal{B}$, $\mathcal{N}$ has the same meaning as in Section 3.1, and $\mathsf{St}_{\mathfrak{h}}, \mathsf{St}_B, \mathsf{St}_0$ have the same meaning as in Section 4.1. As in Section 2.1, we write W^a for the affine Weyl group $W \ltimes \mathfrak{X}(T)$.

Let $G^\vee$ be the Langlands dual group of G. Consider the Cartan and Borel subalgebras $\mathfrak{h}^\vee \subset \mathfrak{b}^\vee \subset \mathfrak{g}^\vee$ so that $\mathfrak{h}^\vee = \mathfrak{h}^*$ and the positive roots for $\mathfrak{b}^\vee$ are the positive coroots for $\mathfrak{b}$. Let $I^\vee$ be the Iwahori subgroup of $G^\vee$, the preimage of $B^\vee \subset G^\vee$ under the projection $G^\vee[[t]] \twoheadrightarrow G^\vee$. Consider the affine flag variety $\mathcal{F}l$ for $G^\vee$, $\mathcal{F}l = G^\vee((t))/I^\vee$. Let I° denote the pro-unipotent radical of $I^\vee$, the kernel of $I^\vee \twoheadrightarrow T^\vee$, where $T^\vee$ denotes the maximal torus in $G^\vee$ corresponding to $\mathfrak{h}^\vee$.

We now proceed to describing the content of this section whose purpose is to review some constructions and results from [B]. In Section 5.2 we recall the main result from [B] on an equivalence between two geometric categorifications of an affine Hecke algebra. Then in Section 5.3 we discuss the compatibility of this equivalence with t-structures.

5.2. **Derived equivalence.** We need to relate the equivariant coherent derived categories for the versions of the Steinberg varieties introduced in Section 4.1 to equivariant constructible derived categories for the affine flag variety $\mathcal{F}l$.

On the coherent side, we consider the categories $D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}})), D^b(\mathrm{Coh}^G(\mathsf{St}_B))$, and $D^b(\mathrm{Coh}^G(\mathsf{St}_0))$. The first category consists of the complexes of coherent sheaves set theoretically supported at the preimage of $\mathcal{N}$ in $\mathsf{St}_{\mathfrak{h}}$. Note that $D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$, $D^b(\mathrm{Coh}^G(\mathsf{St}_0))$ are tensor categories (with respect to convolution of coherent sheaves), and $D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ is a bimodule category with a left action of $D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$ and a right action of $D^b(\mathrm{Coh}^G(\mathsf{St}_0))$.

We consider the constructible equivariant derived categories $D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ), D^b_{I^\circ}(\mathcal{F}l)$, and $D^b_{I^\vee}(\mathcal{F}l)$, where the first category is that of monodromic equivariant constructible sheaves with unipotent monodromy. Again, $D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ), D^b_{I^\vee}(\mathcal{F}l)$ are tensor categories with respect to the !-convolution, while $D^b_{I^\circ}(\mathcal{F}l)$ is a bimodule category.

Theorem 5.1 (Theorem 1 in [B])**.** *We have tensor equivalences*

$$\tau : D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ) \xrightarrow{\sim} D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}})), D^b_{I^\vee}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_0))$$

and a bimodule equivalence

$$\tau : D^b_{I^\vee}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_B)).$$

Remark 5.2. Note that we have functors given by partially forgetting equivariance

$$D^b_{I^\vee}(\mathcal{F}l) \to D^b_{I^\circ}(\mathcal{F}l) \to D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ).$$

We also have inclusions of derived schemes $\mathsf{St}_0 \hookrightarrow \mathsf{St}_B \hookrightarrow \mathsf{St}_{\mathfrak{h}}$. They give rise to push-forward functors

$$D^b(\mathrm{Coh}^G(\mathsf{St}_0)) \to D^b(\mathrm{Coh}^G(\mathsf{St}_B)) \to D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}})).$$

These functors intertwine the equivalences τ by the construction in [B].

5.3. **Perverse equivalence.** On the derived category $D^b_{I^\circ}(\mathcal{F}l)$ we have the usual perverse t-structure with heart $\mathrm{Perv}_{I^\circ}(\mathcal{F}l)$ consisting of perverse sheaves. We want to compare it with the t-structure on $D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ coming from (4.2). We assume that G is semisimple and simply connected.

In fact, equivalence (4.2) is compatible with certain filtrations on the categories indexed by nilpotent orbits in $\mathfrak{g}$. Let us explain what filtrations we consider and state the corresponding result.

Let us start with $D^b(\mathrm{Coh}^G(\mathsf{St}_B))$. The Steinberg variety St_B maps to $\mathcal{N}$ via $\mathsf{St}_B \to \tilde{\mathcal{N}} \to \mathcal{N}$. For a nilpotent orbit $\mathbb{O} \subset \mathfrak{g}$ let $\mathsf{St}_{\leqslant\mathbb{O}}$ denote the preimage of $\overline{\mathbb{O}}$ in $\mathsf{St}_{\mathfrak{h},0}$. Then we can consider the full subcategory $D^b_{\leqslant\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_B))$ of all complexes with cohomology supported on $\mathsf{St}_{\leqslant\mathbb{O}}$. We also can consider the quotient category

$$D^b_{\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_B)) := D^b_{\leqslant\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h},0}))/D^b_{<\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h},0})).$$

We have a similarly defined filtration

$$D^b_{\leqslant\mathbb{O}}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}).$$

The derived global section functor restricts to

$$D^b_{\leqslant\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_B)) \xrightarrow{\sim} D^b_{\leqslant\mathbb{O}}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod})$$

and so gives an equivalence

$$D^b_{\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h},0})) \xrightarrow{\sim} D^b_{\mathbb{O}}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}^*]} \mathcal{A}^{opp}\text{-mod}).$$

The target category has a natural t-structure whose heart is the subquotient category $\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}^*]} \mathcal{A}^{opp}\text{-mod}_{\mathbb{O}}$.

Let us proceed to a filtration on $D^b_{I^\circ}(\mathcal{F}l)$. Recall that the simples in $\mathrm{Perv}_{I^\circ}(\mathcal{F}l)$ are indexed by the elements of the affine Weyl group W^a. We have the two-sided cell filtration on W^a. By a result of Lusztig, [Lu1], the two-sided cells in W^a are in a natural one-to-one correspondence with the nilpotent orbits in $\mathfrak{g}$. So we can consider the category $D^b_{I^\circ,\leqslant\mathbb{O}}(\mathcal{F}l)$ of all objects with perverse homology in the Serre subcategory $\mathrm{Perv}_{I^\circ,\leqslant\mathbb{O}}(\mathcal{F}l)$ spanned by the simples from two-sided cells corresponding to orbits contained in $\overline{\mathbb{O}}$. Again we have the quotient $D^b_{I^\circ,\mathbb{O}}(\mathcal{F}l)$ and its t-structure $\mathrm{Perv}_{I^\circ,\mathbb{O}}(\mathcal{F}l)$.

Theorem 5.3 (Theorem 54 in [B]). *The equivalence*

$$\tau : D^b_{I^\circ}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_B))$$

preserves the filtrations by nilpotent orbits. Moreover, for the induced equivalence

$$\tau_{\mathbb{O}} : D^b_{I^\circ,\mathbb{O}}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G_{\mathbb{O}}(\mathsf{St}_B))$$

we have that $\tau[\frac{1}{2}\operatorname{codim}_{\mathcal{N}}\mathbb{O}]$ is t-exact (with respect to t-structures with hearts $\mathrm{Perv}_{I^\circ,\mathbb{O}}(\mathcal{F}l)$ and $\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}_{\mathbb{O}}$).

Remark 5.4. Note that the simples in the hearts of t-structures of

$$D^b_{I^\vee}(\mathcal{F}l), D^b_{I^\circ}(\mathcal{F}l), D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ)$$

are the same, the functors in Remark 5.2 are t-exact and give the identity on the simple objects. The same holds for the categories $D^b(\mathrm{Coh}^G(\mathsf{St}_0)), D^b(\mathrm{Coh}^G(\mathsf{St}_B)), D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$. In particular,

$$\tau : D^b_{I^\vee}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_0)), D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ) \xrightarrow{\sim} D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$$

are perverse as well.

Remark 5.5. So we can characterize the image of the perverse t-structure in

$$D^b(\mathrm{Coh}^G_{\mathbb{O}}(\mathsf{St}_B)) \cong D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G).$$

Namely, define the *perverse* t-structure on $D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G)$ as follows. For an orbit $\mathbb{O}$ and $M \in D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G)$ we can consider the pullback $M|_{\mathbb{O}}$. Define $D^{b,\leqslant 0}_{perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G)$ as the full subcategory of $D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G)$ consisting of all objects M such that $H^i(M|_{\mathbb{O}}) = 0$ for $i < \frac{1}{2}\operatorname{codim}_{\mathcal{N}} \mathbb{O}$. Thanks to Theorem 5.3, this is precisely the image of $D^{b,\leqslant 0}_{I^\circ}(\mathcal{F}l)$. It follows, in particular, that $D^{b,\leqslant 0}_{perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G)$ is the negative part of a t-structure. One can also see that it is the negative part of a t-structure by using the construction of perverse coherent sheaves from [AB].

6. Equivalence in parabolic setting

6.1. **Notation and content.** The meaning of $G, \mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \rho, B, T, \tilde{\mathfrak{g}}, G^\vee, \mathfrak{g}^\vee, \mathfrak{b}^\vee, \mathfrak{h}^\vee, \mathcal{N}, \mathsf{St}_{\mathfrak{h}}, \mathsf{St}_B, \mathsf{St}_0, I^\vee, I^\circ, \mathcal{F}l$ is the same as in Section 5.1. The notation $P, \mathcal{P}, \tilde{\mathcal{N}}_P, Z, \iota, \varpi$ has the same meaning as in Section 4.1. We assume that G is semisimple and simply connected.

We write τ for each of the derived equivalences of Theorem 5.1. Let $\mathcal{T}_{\mathfrak{h}}$ denote the Bezrukavnikov-Mirkovic tilting bundle on $\tilde{\mathfrak{g}}$, $\mathcal{T}$ be its specialization to $0 \in \mathfrak{h}$, $\mathcal{A}_{\mathfrak{h}} := \operatorname{End}(\mathcal{T})$, and $\mathcal{A} := \operatorname{End}(\mathcal{T})$. Recall that in the end of Section 4.3 we have introduced the tilting bundle $\mathcal{T}_P := \varpi_* \iota^*(\mathcal{T}(-\rho))$. We write $\mathcal{A}_P$ for the endomorphism algebra of $\mathcal{T}_P$.

We also consider the parabolic version of the Steinberg variety, the derived scheme $\mathsf{St}_P := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}_P$ as well as the derived scheme $\widehat{Z} := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} Z$. We write $\tilde{\iota}, \tilde{\varpi}$ for the induced morphisms $\widehat{Z} \hookrightarrow \mathsf{St}_B$ and $\widehat{Z} \twoheadrightarrow \mathsf{St}_P$.

Let $P^\vee$ denote the parabolic subgroup of $G^\vee$ that contains $B^\vee$ and corresponds to P. We write $J^\vee \subset G^\vee((t))$ for the preimage of $P^\vee$ in $G^\vee[[t]]$, and $\mathcal{F}l_P$ for $G^\vee((t))/J^\vee$. Let η denote the projection $\mathcal{F}l \twoheadrightarrow \mathcal{F}l_P$ with fibers $P^\vee/B^\vee$.

This section is organized as follows. In Section 6.2 we state the main results of this section that are generalizations of Theorems 5.1 and Theorem 5.3 to the parabolic setting as well as their corollary that is a crucial ingredient in the proof of Theorem 1.1. The subsequent sections are devoted to proving these results, their content is described in more detail below.

6.2. **Statements of results.** Now we proceed to stating the main result of this section. We first construct the functors $\varphi_1 : D^b(\mathrm{Coh}^G(\mathsf{St}_P)) \to D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ and $\varphi_2 : D^b_{I^\circ}(\mathcal{F}l_P) \to D^b_{I^\circ}(\mathcal{F}l)$. The latter is given by $\varphi_2(\bullet) := \eta^*[\dim P/B]$ so that, in particular, it is Verdier self-dual. Both $D^b_{I^\circ}(\mathcal{F}l_P), D^b_{I^\circ}(\mathcal{F}l)$ are module categories over $D^b_{un}(I^\circ \backslash G^\vee((t))/I^\circ)$ and the functor φ_2 is an equivariant functor.

Now let us describe the functor φ_1. Let ρ_L denote half the sum of positive roots in $\mathfrak{l}$.

$$\varphi_1(\bullet) := \tilde{\iota}_*(\tilde{\varpi}^*(\bullet)(-\rho, \rho - 2\rho_L)) : D^b(\mathrm{Coh}^G(\mathsf{St}_P)) \to D^b(\mathrm{Coh}^G(\mathsf{St}_B)), \tag{6.1}$$

where the expression in round brackets means the twist with the line bundle $\mathcal{O}(-\rho, \rho - 2\rho_L)$ on St_B. Both categories are module categories over $D^b(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$ however the functor φ_1 is not equivariant due to the $(-\rho)$-twist in the first copy of $\tilde{\mathfrak{g}}$. It becomes equivariant if we redefine the action of $D^b(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$ on $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ by conjugating it with $\mathcal{O}(\rho)$:

$$\mathcal{F} *_\rho \mathcal{G} := (p_{13})_* \left(p_{12}^*(\mathcal{F}) \otimes p_{23}^*(\mathcal{G}(-\rho))\right)(\rho).$$

Here are the main results of this section. The first one should be thought as a parabolic analog of Theorem 5.1.

Theorem 6.1. *We have an equivalence* $\tau_P : D^b_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Coh}^G \mathsf{St}_P)$ *of triangulated categories making the following diagram commutative*

$$\begin{tikzcd} D^b_{I^\circ}(\mathcal{F}l_P) \arrow[r, "\sim", "\tau_P"'] \arrow[d, "\varphi_2"] & D^b(\mathrm{Coh}^G(\mathsf{St}_P)) \arrow[d, "\varphi_1"] \\ D^b_{I^\circ}(\mathcal{F}l) \arrow[r, "\sim", "\tau"'] & D^b(\mathrm{Coh}^G(\mathsf{St}_B)) \end{tikzcd}$$

mapping $\underline{\mathbb{C}}_{P^\vee/P^\vee}$ *to* $\mathcal{O}_{Z^{diag}}$ *and intertwining the actions of*

$$D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ) \xrightarrow{\sim} D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$$

(where the action on $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ *is twisted by* $\mathcal{O}(\rho)$*, as above) and, in particular, the actions of* Br^a*.*

Our second result is an analog of Theorem 5.3. As in the case of $P = B$, we have the cell filtration on $D^b_{I^\circ}(\mathcal{F}l_P)$ and the nilpotent orbit filtration on $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$. We shift the numeration by $\dim P/B$ so that the filtration degree 0 quotient functor for $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ is isomorphic to the restriction to $\mathbb{O}$. As usual, we consider the perverse t-structure on $D^b_{I^\circ}(\mathcal{F}l_P)$. The t-structure on $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ is given by $\mathcal{T}_{\mathfrak{h}}(-\rho)^* \otimes \mathcal{T}_P(-2\rho)$ and the heart is $\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G$. We write $\mathcal{N}_P$ for the image of $\tilde{\mathcal{N}}_P$ in $\mathcal{N}$.

Theorem 6.2. *The equivalence*

$$\tau_P : D^b_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_P))$$

preserves the filtrations by nilpotent orbits (that are contained in the image $\mathcal{N}_P$ *of* $\tilde{\mathcal{N}}_P$ *in* $\mathcal{N}$*). Moreover, for the induced equivalence*

$$\tau_{\mathbb{O}'} : D^b_{I^\circ,\mathbb{O}'}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Coh}^G_{\mathbb{O}'}(\mathsf{St}_P))$$

we have that $\tau[\frac{1}{2}\operatorname{codim}_{\mathcal{N}_P}\mathbb{O}']$ *is t-exact (with respect to t-structures with hearts* $\mathrm{Perv}_{I^\circ,\mathbb{O}'}(\mathcal{F}l_P)$ *and* $\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}_{\mathbb{O}'}$*).*

Theorems 6.1 and 6.2 will be proved simultaneously. The proof goes as follows. We will introduce and study left adjoint functors ψ_1, ψ_2 of φ_1, φ_2 in Section 6.3. We will describe compositions $\varphi_j\psi_j$ and $\psi_j\varphi_j$. In particular, we will see that $\varphi_j\psi_j$ is given by convolving on the right with certain objects. In Section 6.4 we will see that the equivalence τ intertwines those objects and use this to deduce that τ intertwines the full Karoubian subcategories generated by the images of φ_1, φ_2 (below these subcategories will be called *full images*). In Section 6.5 we will combine results of the preceding sections and see that τ restricts to an equivalence of abelian categories between $\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P)$ and the heart of the perverse t-structure on $D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$. We will use this equivalence to prove Theorems 6.1 and 6.2.

Below we will need a corollary of these two theorems. Recall that $\mathbb{O}$ denotes the dense orbit in $\mathcal{N}_P$. Inside $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ consider the full subcategory $D^b_{<\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_P))$ of all objects whose cohomolgy are supported on the preimage of $\partial\mathbb{O}$ in St_P. Set

$$D^b_{\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_P)) := D^b(\mathrm{Coh}^G(\mathsf{St}_P))/D^b_{<\mathbb{O}}(\mathrm{Coh}^G(\mathsf{St}_P)).$$

Recall that we write $\mathcal{B}_e$ for the Springer fiber of e with its natural derived scheme structure (as of subscheme in $\tilde{\mathfrak{g}}$). Note that the preimage of $\mathbb{O}$ in St_P is naturally identified with

$G\times^{Z_P(e)}\mathcal{B}_e$ (an isomorphism of derived schemes). So

$$D^b_{\mathbb{O}_P}(\mathrm{Coh}^G(\mathsf{St}_P))\xrightarrow{\sim} D^b(\mathrm{Coh}^{Z_P(e)}\mathcal{B}_e), \tag{6.2}$$

Consider the t-structure on $D^b(\mathrm{Coh}^{Z_P(e)}\mathcal{B}_e)$ given by $\mathcal{T}(-\rho)^*$. Its heart is $\mathcal{A}_{\mathfrak{h},e}$-$\mathrm{mod}^{Z_P(e)}$. Here and below we write $\mathcal{A}_{\mathfrak{h},e}$ for the fiber of $\mathcal{A}_{\mathfrak{h}}$ at e.

Inside $\mathcal{B}_e$ consider the (ordinary) subvariety $\mathcal{B}_{\mathfrak{m}}$ consisting of all Borel subalgebras containing $\mathfrak{m}$. It is naturally identified with P/B.

Corollary 6.3. *The quotient functor $D^b_{I^\circ}(\mathcal{F}l_P)\twoheadrightarrow D^b(\mathrm{Coh}^{Z_P(e)}\mathcal{B}_e)$ has the following properties:*

1. *it is t-exact,*
2. *it intertwines the actions of $D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ)\xrightarrow{\sim} D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$ (hence the* Br^a*-actions),*
3. *it maps $\underline{\mathbb{C}}_{P^\vee/P^\vee}$ to $\mathcal{O}_{\mathcal{B}_{\mathfrak{m}}}$.*

Remark 6.4. Above we have assumed that G is simply connected, which is needed for $\rho\in\mathfrak{X}(T)$. We will need to weaken this assumption to the case when we have a character ρ' of T that coincides with ρ on the coroots: this holds when our group is a Levi of a simply connected semisimple group, which is precisely the situation we need. Results in this section continue to hold with easy modifications, for example, in the definition of φ_1 we need to twist with $(-\rho',\rho'-2\rho_L)$.

6.3. **Adjoint functors.** In this section we are going to introduce and study left adjoint functors of φ_1,φ_2 to be denoted by ψ_1,ψ_2, respectively.

Let us start with ψ_2, which is easier. The left adjoint of $\eta^!=\varphi_2[\dim P/B]$ is $\eta_!$ so that $\psi_2=\eta_![\dim P/B]$.

Lemma 6.5. *The following claims are true.*

1. *We have $\psi_2\varphi_2\cong \mathrm{id}\otimes H^*(P^\vee/B^\vee,\mathbb{C})[2\dim P/B]$, where we view $H^*(P^\vee/B^\vee,\mathbb{C})$ as the complex with zero differential, $H^*(P^\vee/B^\vee,\mathbb{C})=\bigoplus_i H^i(P^\vee/B^\vee,\mathbb{C})[-i]$.*
2. *We have $\varphi_2\psi_2\cong \bullet*\underline{\mathbb{C}}_{P^\vee/B^\vee}[2\dim P/B]$.*

Proof. We have $\psi_2\varphi_2=\eta_!\eta^!$, which implies (1). The proof of (2) is similar. □

The functor φ_1 admits a left adjoint functor as well. Namely, $\tilde\iota^*$ is the left adjoint functor to $\tilde\iota_*$. Also the relative canonical bundle of $Z\to\tilde{\mathcal{N}}_P$ is $\mathcal{O}(-2\rho_L)$ hence, by the Serre duality, ϖ_* is left adjoint of $\varpi^*(\bullet)(-2\rho_L)[\dim P/B]$. We conclude that the left adjoint ψ_1 of φ_1 is given by

$$\psi_1(\bullet):=\tilde\varpi_*\left(\tilde\iota^*(\bullet)(\rho,-\rho)\right)[\dim P/B].$$

Lemma 6.6. *The following claims are true.*

1. *We have $\psi_1\varphi_1\cong\mathrm{id}\otimes H^*(P/B,\mathbb{C})[2\dim P/B]$.*
2. *We have $\varphi_1\psi_1\cong\bullet*\mathcal{O}_{Z\times_{T^*(G/P)}Z}(-\rho,\rho-2\rho_L)[\dim P/B]$.*

Proof. Let us prove part (1). Note that

$$\psi_1\circ\varphi_1(\mathcal{G})=\tilde\varpi_*\left(\tilde\iota^*\circ\tilde\iota_*\left[\tilde\varpi^*\mathcal{G}\right](-2\rho_L)\right)\right)[-\dim P/B].$$

The composition $\iota^*\circ\iota_*(\bullet)$ is tensoring with $\mathcal{O}_Z\otimes^L_{\mathcal{O}_{\tilde{\mathcal{N}}}}\mathcal{O}_Z$, where we view Z as a closed subscheme of $\tilde{\mathcal{N}}$ via ι. Tensoring this complex by $\mathcal{O}(-2\rho_L)$ and applying the shift by $\dim P/B$ we get $R\mathcal{E}nd_{\mathcal{O}_{\tilde{\mathcal{N}}}}(\mathcal{O}_Z)[2\dim P/B]$. We have $\varpi_*R\mathcal{E}nd_{\mathcal{O}_{\tilde{\mathcal{N}}}}(\mathcal{O}_Z)\cong\mathcal{O}_{\tilde{\mathcal{N}}_P}\otimes H^*(P/B,\mathbb{C})$.

This boils down to proving $R\mathcal{E}nd_{\mathcal{O}_{\tilde{\mathcal{N}}}}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B}) \cong H^*(\mathcal{B}, \mathbb{C})$ as a G-module. The latter is a consequence of the Hodge theorem.

To prove part (2), we first note that

$$\varphi_1\psi_1(\bullet) = [\tilde{\iota}_* \circ \tilde{\varpi}^* \circ \tilde{\varpi}_* \circ \tilde{\iota}^*(\bullet(\rho, -\rho))]\,(-\rho, \rho - 2\rho_L)[\dim P/B]$$

Consider the variety $Z \times_{T^*(G/P)} Z$ and let $\kappa_i : Z \times_{T^*(G/P)} Z \to T^*\mathcal{B}$ denote the projection to the ith factor composed with the inclusion $\iota : Z \hookrightarrow T^*\mathcal{B}$. Let $\tilde{\kappa}_i$ denote the induced morphism $\tilde{\mathfrak{g}} \times^L_{\mathfrak{g}} (Z \times_{T^*(G/P)} Z) \to \mathsf{St}_B$. Note that

$$\tilde{\iota}_* \circ \tilde{\omega}^* \circ \tilde{\omega}_* \circ \tilde{\iota}^* = \tilde{\kappa}_{2*} \circ \tilde{\kappa}_1^*,$$

so we get

$$\begin{aligned}\varphi_1\psi_1(\mathcal{F}) &= (\tilde{\kappa}_{2*} \circ \tilde{\kappa}_1^*(\mathcal{F}(\rho, -\rho)))\,(-\rho, \rho - 2\rho_L)[\dim P/B] = \\ &(\tilde{\kappa}_{2*} \circ \tilde{\kappa}_1^*(\mathcal{F}(0, -\rho)))\,(0, \rho - 2\rho_L)[\dim P/B] = \\ &\mathcal{F} * \mathcal{O}_{Z \times_{T^*(G/P)} Z}(-\rho, \rho - 2\rho_L)[\dim P/B]\end{aligned}$$

This proves part (2). □

6.4. **Coincidence of full images.** Let $\operatorname{Fim}\varphi_j$ denote the Karoubian envelope of the full subcategory in $D^b_{I^\circ}(\mathcal{F}l)$ (for $j = 2$) or $D^b(\operatorname{Coh}^G(\mathsf{St}_B))$ (for $j = 1$) generated by the objects in the image of φ_j.

A crucial step in the proof of the claim that τ intertwines $\operatorname{Fim}\varphi_2$ with $\operatorname{Fim}\varphi_1$ is the following lemma.

Lemma 6.7. *The image of* $\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P^\vee/B^\vee]$ *in* $D^b(\operatorname{Coh}^G(\mathsf{St}_0))$ *is* $\mathcal{O}_{Z \times_{T^*(G/P)} Z}(-\rho, \rho - 2\rho_L)$.

We remark that the same result holds for $\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P^\vee/B^\vee]$ viewed as an object of $D^b_{I^\circ}(\mathcal{F}l)$ and $\mathcal{O}_{Z \times_{T^*(G/P)} Z}(-\rho, \rho - 2\rho_L)$ viewed as an object in $D^b(\operatorname{Coh}^G(\mathsf{St}_B))$. This is because the three versions of τ coincide on the simple perverse sheaves, this follows from Remark 5.4.

Proof. The proof of this lemma is in two steps.

Step 1. First, consider the case when $P = G$. Here we need to prove that $\tau(\underline{\mathbb{C}}_{\mathcal{B}^\vee}[\dim \mathcal{B}]) = \mathcal{O}_{\mathcal{B}\times\mathcal{B}}(-\rho, -\rho)$. We have

$$R\Gamma\left((\mathcal{T} \otimes \mathcal{T}^*) \otimes \mathcal{O}_{\mathcal{B}\times\mathcal{B}}(-\rho, -\rho)\right) = R\Gamma(\mathcal{T}(-\rho)) \otimes R\Gamma(\mathcal{T}^*(-\rho)).$$

Thanks to Lemma 4.6, we have a Morita equivalence $(\mathcal{A}_{\mathfrak{h},0})_\mathbb{F} \cong \mathcal{U}^0_{(0),\mathbb{F}}$. Using that lemma, we also see that, under that Morita equivalence the object $R\Gamma(\mathcal{T}(-\rho)) \otimes R\Gamma(\mathcal{T}^*(-\rho))$ becomes $R\Gamma(\mathcal{V}^0_0) \otimes R\Gamma((\mathcal{V}^0_0)^*(-2\rho))$. By (1) of Lemma 3.5, the first factor is $\Gamma(\mathcal{O}_{\mathcal{B}_\mathbb{F}}) = \mathbb{F}$, the trivial $G_\mathbb{F}$-module. By the Serre duality, the second factor is $\mathbb{F}^*[-\dim \mathcal{B}]$.

On the other hand, by Theorem 5.3, $\tau(\underline{\mathbb{C}}_{\mathcal{B}^\vee}[\dim \mathcal{B}])$ is also a simple G-equivariant $\mathcal{A}$-bimodule shifted by $\dim \mathcal{B}$. So we need to show is that the K_0-classes of $\tau(\underline{\mathbb{C}}_{\mathcal{B}^\vee}[\dim \mathcal{B}])$ and $\mathcal{O}_{\mathcal{B}\times\mathcal{B}}(-\rho, -\rho)$ coincide. Under the standard identification of $K_0(D^b_{I^\circ}(\mathcal{F}l))$ with $\mathbb{Z}W^a$, the class of $\underline{\mathbb{C}}_{\mathcal{B}^\vee}[\dim \mathcal{B}]$ is $\sum_{w\in W}(-1)^{\ell(w_0 w)}w$. So we need to show that the class of $\mathcal{O}_{\mathcal{B}\times\mathcal{B}}$ in $K_0^G(\mathsf{St}_B)$ (or equivalently $K_0^G(\mathsf{St}_0)$) is $\sum_{w\in W}(-1)^{\ell(w)}w$ (it is sufficient to prove this up to a sign). For this we consider the action of $K_0^G(\mathsf{St}_0)$ on $K_0^T(T^*\mathcal{B})$ by convolution (where the convolution is ρ-twisted as before the statement of Theorem 6.1). After localizing $K_0^T(\mathrm{pt})$, the module $K_0^T(T^*\mathcal{B})$ gets the fixed point basis. The elements of this basis are naturally indexed by elements of W^a. The elements of W correspond to the skyscraper sheaves at the fixed points with trivial T-actions. In this basis the action of W^a is by left multiplications. Convolving $\mathcal{O}_{\mathcal{B}\times\mathcal{B}}(-\rho,-\rho)$ with the skyscraper sheaf at $1B$ and decomposing with respect to the fixed point basis we get $\sum_{w\in W}(-1)^{\ell(w)}w$. This finishes the proof of $\tau(\underline{\mathbb{C}}_{\mathcal{B}^\vee}[\dim\mathcal{B}])\cong\mathcal{O}_{\mathcal{B}\times\mathcal{B}}(-\rho,-\rho)$.

Step 2. In this step we consider the case of general P.

Note that we have a functor $\xi_P: D^b(\mathrm{Coh}^L(\mathsf{St}_{L,0}))\to D^b(\mathrm{Coh}^G(\mathsf{St}_0))$, where $\mathsf{St}_{L,0}$ is the analog of St_0 for L. Indeed, consider the derived scheme S of pairs $x\in\mathfrak{n}$ and $\mathfrak{b}'\in\mathcal{B}$ with $x\in\mathfrak{b}'$ (with its natural derived scheme structure). Note that $D^b(\mathrm{Coh}^G(\mathsf{St}_0))$ is naturally identified with $D^b(\mathrm{Coh}^B(\mathsf{S}))$. Consider the derived subscheme $\mathsf{S}_{\mathfrak{m}}\subset\mathsf{S}$ consisting of all pairs $(x,\mathfrak{b}')\in\mathsf{S}$ such that $\mathfrak{m}\subset\mathfrak{b}'$. It projects to S_L, an analog of S for L. The functor $D^b(\mathrm{Coh}^L(\mathsf{St}_{L,0}))\to D^b(\mathrm{Coh}^G(\mathsf{St}_0))$ we need is the pull-push functor via $\mathsf{S}_L\leftarrow\mathsf{S}_{\mathfrak{m}}\to\mathsf{S}$.

By the construction of the homomorphism from Br^a to $D^b(\mathrm{Coh}^G(\mathsf{St}_0))$ (which is completely analogous to the construction of the homomorphism to $D^b(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$ given in [BR]), ξ_P is equivariant for the action of the braid group Br_P for W_P. The functor ξ_P maps the structure sheaf of the diagonal to the structure sheaf of the diagonal.It also maps $\mathcal{O}_{P/B\times P/B}(-\rho,\rho-2\rho_L)$ to $\mathcal{O}_{Z\times_{T^*(G/P)}Z}(-\rho,\rho-2\rho_L)$.

Let us write τ_L for the equivalence $D^b_{I_L^\vee}(\mathcal{F}l_L)\xrightarrow{\sim}D^b(\mathrm{Coh}^L(\mathsf{St}_{L,0}))$, where $\mathcal{F}l_L$ is the affine flag variety for $L^\vee$ and $I_L^\vee$ is the standard Iwahori subgroup in $L^\vee((t))$. By Step 1, $\tau_L(\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P/B])=\mathcal{O}_{P/B\times P/B}(-\rho,\rho-2\rho_L)$ (this twist is the same as by $(-\rho_L,-\rho_L)$, we need to replace L with a cover for ρ_L to be a character of the maximal torus).

The simple labelled by $x\in W^a$ in $\mathrm{Perv}_{I^\vee}(\mathcal{F}l)$ is the image of a unique (up to rescaling) nonzero homomorphism $T_x^{-1}\underline{\mathbb{C}}_{B^\vee/B^\vee}\to T_x\underline{\mathbb{C}}_{B^\vee/B^\vee}$. It was proved in [B] that τ map simples in $\mathrm{Perv}_{B^\vee}(G^\vee/B^\vee)$ to coherent sheaves. So $\tau(\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P/B])$ is the image of a unique nonzero morphism $T_{w_{0,L}}^{-1}\tau(\underline{\mathbb{C}}_{B^\vee/B^\vee})\to T_{w_{0,L}}\tau(\underline{\mathbb{C}}_{B^\vee/B^\vee})$. The functor ξ_P is exact and faithful on the heart of the usual t-structure of coherent sheaves. This is because it is the composition of the pull-back under a locally trivial fibration and the push-forward under a closed embedding. The functor ξ_P is Br_{W_P}-equivariant hence maps $T_{w_{0,L}}^{-1}\tau_L(\underline{\mathbb{C}}_{B^\vee/B^\vee}),T_{w_{0,L}}\tau_L(\underline{\mathbb{C}}_{B^\vee/B^\vee})$ to $T_{w_{0,L}}^{-1}\tau(\underline{\mathbb{C}}_{B^\vee/B^\vee}),T_{w_{0,L}}\tau(\underline{\mathbb{C}}_{B^\vee/B^\vee})$, respectively. It follows that it maps $\tau_L(\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P/B])$ to $\tau(\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P/B])$. This finishes the proof. □

Corollary 6.8. *The equivalence* $\tau: D^b_{I^\circ}(\mathcal{F}l)\xrightarrow{\sim}D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ *intertwines* $\mathrm{Fim}\,\varphi_2$ *with* $\mathrm{Fim}\,\varphi_1$.

Proof. Recall that τ is equivariant with respect to the action of $D^b_{I^\vee}(\mathcal{F}l)\cong D^b(\mathrm{Coh}^G(\mathsf{St}_0))$ by convolutions on the right. By Lemma 6.7 that, under the equivalence, $D^b_{I^\vee}(\mathcal{F}l)\xrightarrow{\sim}D^b(\mathrm{Coh}^G(\mathsf{St}_0))$, the object $\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P^\vee/B^\vee]$ goes to $\mathcal{O}_{Z\times_{T^*(G/P)}Z}(-\rho,\rho-2\rho_L)\}$. It follows from Lemma 6.5, that $\mathrm{Fim}\,\varphi_2$ is the Karoubian envelope of

$$\{\mathcal{F}*\underline{\mathbb{C}}_{P^\vee/B^\vee}[\dim P/B]|\mathcal{F}\in D^b_{I^\circ}(\mathcal{F}l)\}.$$

Similarly, it follows from Lemma 6.6 that $\mathrm{Fim}\,\varphi_1$ is the Karoubian envelope of

$$\{\mathcal{G}*\mathcal{O}_{Z\times_{T^*(G/P)}Z}(-\rho,\rho-2\rho_L)|\mathcal{G}\in D^b(\mathrm{Coh}^G(\mathsf{St}_B))\}.$$

This finishes the proof. □

6.5. **Abelian equivalence.** Consider the t-structure on $D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ given by $\mathcal{T}^*\otimes\mathcal{T}$ and the structure on $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ given by $\mathcal{T}(-\rho)^*\otimes\mathcal{T}_P(-2\rho)[\dim P/B]$. The following lemma is a straightforward corollary of Proposition in [BM, Section 4.1].

Lemma 6.9. *The functor $\varphi_1 : D^b(\mathrm{Coh}^G(\mathsf{St}_P)) \to D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ is t-exact with respect to these t-structures.*

We will need some ideas from the proof of the proposition in [BM, Section 4.1] for our arguments below. It is enough to prove the lemma after forgetting the G-equivariant structure, reducing mod $p \gg 0$ and restricting to the formal neighborhoods of zero. Up to a Morita equivalence the hearts become the categories of $G_{\mathbb{F}}$-equivariant modules over (the completions of)

$$\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}, \Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{B}_{\mathbb{F}}})^{opp}.$$

Since p is sufficiently large, we have a surjective homomorphism $\Gamma(D_{\mathcal{B}_{\mathbb{F}}}) \twoheadrightarrow \Gamma(D_{\mathcal{P}_{\mathbb{F}}})$. It follows from [BM, Section 4.1] that φ_1 is the pull-back under the corresponding epimorphism

$$\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}, \Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{B}_{\mathbb{F}}})^{opp}. \tag{6.3}$$

Now consider the same t-structure on $D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ and the homologically shifted t-structure on $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$: consider the t-structure given by $\mathcal{T}(-\rho)^* \otimes \mathcal{T}_P(-2\rho)$. The corresponding tilting bundles give identifications

$$D^b(\mathrm{Coh}^G(\mathsf{St}_B)) \cong D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G), D^b(\mathrm{Coh}^G(\mathsf{St}_P)) \cong D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G).$$

On $D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G), D^b(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$ we have perverse bimodule t-structures, see Remark 5.5.

Proposition 6.10. *The functor φ_1 is t-exact with respect to the perverse bimodule t-structures.*

Proof. By the construction, φ_1 preserves the supports of bimodules in $\mathcal{N}$. Using the descriptions of subcategories $D_p^{b,\leqslant 0}erv$ and the exactness of φ_1 in the usual t-structures, we see that φ_1 is left t-exact in the perverse bimodule t-structures (note that we have shifted the t-structure on the source category of φ_1).

Let us prove that φ_1 is right t-exact. For this it is sufficient to work with the nonequivariant categories. Also, as above, we can replace $\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}$ with $\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}$ and $\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}$ with $\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{B}_{\mathbb{F}}})^{opp}$. The functor φ_1 becomes the pullback under (6.3).

Now we need a description of the subcategories $D_{perv}^{b,\geqslant 0}$. Consider the functor

$$R\operatorname{Hom}_{\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})}(\bullet, \Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})) : D^b(\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}\text{-mod}) \to D^b(\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})^{opp} \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}\text{-mod}).$$

Note that we have

$$\begin{aligned} &\mathcal{F} \in D^{b,\geqslant 0}(\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}) \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}\text{-mod}) \Leftrightarrow \\ &R\operatorname{Hom}_{\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})}(\mathcal{F}, \Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})) \in D^{b,\leqslant 0}(\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})^{opp} \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}\text{-mod}). \end{aligned} \tag{6.4}$$

The functor $R\operatorname{Hom}_{\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})}(\bullet, \Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}}))$ intertwines the pullback under (6.3) with the pullback under the analogously defined epimorphism

$$\Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})^{opp} \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{P}_{\mathbb{F}}})^{opp}, \Gamma(\tilde{D}_{\mathcal{B}_{\mathbb{F}}})^{opp} \otimes_{\mathbb{F}[\mathfrak{g}^{(1)}]} \Gamma(D_{\mathcal{B}_{\mathbb{F}}})^{opp}.$$

The latter is left t-exact in the perverse bimodule t-structure for the same reasons as φ_1. The claim that φ_1 is right t-exact now follows from (6.4). □

Let $\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$ denote the heart of the perverse t-structure. We view it as a full subcategory of $D^b(\mathrm{Coh}^G(\mathsf{St}_P))$.

We will need the following lemma.

Lemma 6.11. *Every object in* $\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$ *has finite length.*

Proof. There are finitely many simples in $\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$. The set of simples splits according to the G-orbits in $\overline{\mathbb{O}}$. Let $M \in \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$. Let $\mathbb{O}'$ be the minimal orbit such that a simple object corresponding to this orbit appears as a subquotient of M. Note that the multiplicity of the simples corresponding to this orbit is bounded above by the generic rank of $H^i(M)$, where we take the cohomology in the usual t-structure and $i = \frac{1}{2}\operatorname{codim}_{\mathcal{N}} \mathbb{O}'$. Using this observation combined with the induction on $\dim \mathbb{O}'$, we complete the proof of the lemma. □

The main result of this section is the following proposition.

Proposition 6.12. *The following claims are true.*

(1) *The functors*

$$\varphi_1 : \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G) \to D^b(\mathrm{Coh}^G(\mathsf{St}_B)), \varphi_2 : \mathrm{Perv}_{I^\circ}(\mathcal{F}l_P) \to D^b_{I^\circ}(\mathcal{F}l)$$

are full embeddings.

(2) *The equivalence*

$$\tau : D^b_{I^\circ}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_B))$$

intertwines $\varphi_2(\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P))$ *and* $\varphi_1(\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G))$.

Proof. We start by proving 1). We will consider the case of φ_1, the other case is similar. Note that

$$\mathrm{Hom}_{D^b(\mathrm{Coh}^G(\mathsf{St}_B))}(\varphi_1\mathcal{F}, \varphi_1\mathcal{G}) = \mathrm{Hom}_{D^b(\mathrm{Coh}^G(\mathsf{St}_B))}(\psi_1\varphi_1\mathcal{F}, \mathcal{G}).$$

Recall, Lemma 6.6, that $\psi_1\varphi_1 = \mathrm{id} \otimes H^*(P/B, \mathbb{C})[2\dim P/B]$. Since $\mathcal{F}, \mathcal{G}$ lie in the heart of a t-structure, we have $\mathrm{Hom}_{D^b(\mathrm{Coh}^G(\mathsf{St}_B))}(\mathcal{F}[i], \mathcal{G}) = 0$ for $i > 0$. It follows that

$$\mathrm{Hom}_{D^b(\mathrm{Coh}^G(\mathsf{St}_B))}(\psi_1\varphi_1\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{D^b(\mathrm{Coh}^G(\mathsf{St}_B))}(\mathcal{F}, \mathcal{G}),$$

which finishes the proof of 1).

Now we prove 2). Proposition 6.9 says that φ_1 is t-exact with respect to the perverse bimodule t-structures. And φ_2 is t-exact as well. So we have

$$\begin{aligned} &\varphi_1(\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)) = \mathrm{Fim}\,\varphi_1 \cap \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}^{opp}\text{-mod}^G), \\ &\varphi_2(\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P)) = \mathrm{Fim}\,\varphi_2 \cap \mathrm{Perv}_{I^\circ}(\mathcal{F}l). \end{aligned} \tag{6.5}$$

By Theorem 5.3, τ is t-exact with respect to the perverse t-structures. Combining Corollary 6.8 with (6.5), we get 2). □

So we get an equivalence $\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$ that we denote by τ_P.

6.6. **Completion of proofs of main results.** First, we need to produce a functor $\tau_P : D^b_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ that makes the commutative diagram in Theorem 6.1 commutative. Note that $D^b(\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P)) \xrightarrow{\sim} D^b_{I^\circ}(\mathcal{F}l_P)$ by a theorem of Beilinson. Also, since $\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)$ is the heart of a t-structure on $D^b(\mathrm{Coh}^G \mathsf{St}_P)$, the embedding $\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G) \to D^b(\mathrm{Coh}^G \mathsf{St}_P)$ extends to a triangulated functor $D^b(\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)) \to D^b(\mathrm{Coh}^G(\mathsf{St}_P))$. We denote the composed functor

$$D^b_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathcal{A}_P^{opp}\text{-mod}^G)) \to D^b(\mathrm{Coh}^G(\mathsf{St}_P))$$

by τ_P. That the diagram of Theorem 6.1 is commutative follows directly from the construction of τ_P.

Lemma 6.13. *The functor τ_P is an equivalence.*

Proof. We need to show that τ_P is fully faithful and essentially surjective.

The functor φ_2 is faithful. On the other hand, it is the composition $\tau^{-1}\varphi_1\tau_P$. So τ_P is faithful.

Now we show τ_P is full. By Lemma 6.11, $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ is finite dimensional. So we need to show that $\dim\mathrm{Hom}(\mathcal{F},\mathcal{G}) = \dim\mathrm{Hom}(\tau_P\mathcal{F},\tau_P\mathcal{G})$. We have an isomorphism

$$\mathrm{Hom}(\varphi_1\tau_P\mathcal{F},\varphi_1\tau_P\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\varphi_2\mathcal{F},\varphi_2\mathcal{G})$$

given by τ^{-1}. Since $\psi_i\varphi_i \cong \mathrm{id}\otimes H^*(P/B,\mathbb{C})[2\dim P/B]$, we then get an isomorphism

$$\mathrm{Hom}(\tau_P\mathcal{F}\otimes H^*(P/B,\mathbb{C}),\tau_P\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}\otimes H^*(P/B,\mathbb{C}),\mathcal{G}).$$

Since $\dim\mathrm{Hom}(\mathcal{F}[i],\mathcal{G}) \leqslant \mathrm{Hom}(\tau_P\mathcal{F}[i],\tau_P\mathcal{G})$ for all i (τ_P is faithful), we see that

$$\dim\mathrm{Hom}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}(\tau_P\mathcal{F},\tau_P\mathcal{G}).$$

Now let us show that τ_P is essentially surjective. Note that for each $\mathcal{F}'\in D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ there are $i>j$ such that $\mathcal{F}'$ lies in the intersection of $\leqslant i$ and $\geqslant j$ subcategories for the perverse bimodule t-structure. Now we prove that $\mathcal{F}'$ lies in the image of τ_P by induction on $i-j$. □

Theorem 6.2 follows. To finish the proof of Theorem 6.1, it remains to show two things: that $\tau_P : D^b_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ is equivariant for the action of $D^b_{un}(I^\circ\backslash G^\vee((t))/I^\circ) \cong D^b_{\mathcal{N}}(\mathrm{Coh}^G(\mathsf{St}_{\mathfrak{h}}))$ and that this equivalence maps $\underline{\mathbb{C}}_{P^\vee/P^\vee}$ to $\mathcal{O}_{Z^{diag}}$.

Lemma 6.14. *The equivalence $\tau_P : D^b_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_P))$ is equivariant.*

Proof. Consider the negative parts $D^{b,\leqslant 0}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G)$ and $D^{b,\leqslant 0}_{perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G)$. It is straightforward from the definition of the perverse bimodule t-structure that

$$D^{b,\leqslant 0}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G)\otimes^L_{\mathcal{A}_{\mathfrak{h}}} D^{b,\leqslant 0}_{perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G) \subset D^{b,\leqslant 0}_{perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G).$$

We can carry the action of $D^{b,\leqslant 0}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G)$ to $D^b_{I^\circ}(\mathcal{F}l)$ using the equivalence τ.

So we get the action of $\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G$ on

$$\mathrm{Perv}_{I^\circ}(\mathcal{F}l), \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}^{opp}\text{-mod}^G), \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G)$$

by right t-exact functors. The full embedding $\varphi_1 : \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G) \hookrightarrow \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}^{opp}\text{-mod}^G)$ is equivariant by the construction. Since $\varphi_2 : D^b_{I^\circ}(\mathcal{F}l_P)\hookrightarrow D^b_{I^\circ}(\mathcal{F}l)$ is $D^b(\mathrm{Coh}^G\mathsf{St}_{\mathfrak{h}})$-equivariant, we see that $\varphi_2(\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P))\subset\mathrm{Perv}_{I^\circ}(\mathcal{F}l)$ is closed under the action of $\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G$. So $\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P)$ becomes an $\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G$-module category and

$$\tau_P : \mathrm{Perv}_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} \mathrm{Perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G)$$

is $\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G$-equivariant. This implies that $\tau_P : D^{b,\leqslant 0}_{I^\circ}(\mathcal{F}l_P) \xrightarrow{\sim} D^{b,\leqslant 0}_{perv}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_P^{opp}\text{-mod}^G)$ is $D^{b,\leqslant 0}(\mathcal{A}_{\mathfrak{h}}\otimes_{\mathbb{C}[\mathfrak{g}]}\mathcal{A}_{\mathfrak{h}}^{opp}\text{-mod}^G)$-equivariant, From here one easily deduces the claim of the lemma. □

Lemma 6.15. *We have $\tau_P(\underline{\mathbb{C}}_{P^\vee/P^\vee}) \cong \mathcal{O}_{Z^{diag}}$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
D^b_{I^\circ}(\mathcal{F}l_P) \arrow[rr, "\sim", "\tau_P"'] & & D^b(\mathrm{Coh}^G(\mathsf{St}_P)) \\
D^b_{I^\circ}(\mathcal{F}l) \arrow[u, "\psi_2"] \arrow[rr, "\sim"] & & D^b(\mathrm{Coh}^G(\mathsf{St}_B)) \arrow[u, "\psi_1"']
\end{tikzcd}$$

Consider the object $\underline{\mathbb{C}}_{1B^\vee} \in D^b_{I^\circ}(\mathcal{F}l)$. We have $\psi_2(\underline{\mathbb{C}}_{1B^\vee}) = \underline{\mathbb{C}}_{1P^\vee}[\dim P/B]$. Furthermore, under the equivalence $D^b_{I^\circ}(\mathcal{F}l) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\mathsf{St}_B))$ the object $\underline{\mathbb{C}}_{1B^\vee}$ goes to $\mathcal{O}_{\tilde{\mathcal{N}}_{diag}}$, where we write $\tilde{\mathcal{N}}_{diag}$ for the diagonal in St_B, this is a special case of Lemma 6.7.

So we need to prove that $\psi_1(\mathcal{O}_{\tilde{\mathcal{N}}^{diag}}) = \mathcal{O}_{Z^{diag}}[\dim P/B]$, equivalently,

$$\tilde{\varpi}_* \circ \tilde{\iota}^*(\mathcal{O}_{\tilde{\mathcal{N}}^{diag}}(\rho, -\rho)) = \mathcal{O}_{Z^{diag}}.$$

First, note that $\mathcal{O}_{\tilde{\mathcal{N}}^{diag}}(\rho, -\rho) = \mathcal{O}_{\tilde{\mathcal{N}}^{diag}}$. Next, let us compute the pull-back of $\mathcal{O}_{\tilde{\mathcal{N}}^{diag}}$ to $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} Z$. The intersection of $\tilde{\mathcal{N}}^{diag}$ and $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} Z$ is transversal and equals to Z^{diag} so the pull-back, $\mathrm{id} \boxtimes \iota^*(\mathcal{O}_{\tilde{\mathcal{N}}^{diag}})$ is $\mathcal{O}_{Z^{diag}}$. Finally, the restriction of $\tilde{\varpi}$ to Z^{diag} is the closed embedding $Z^{diag} \hookrightarrow \mathsf{St}_P$. So we see that $\psi_1(\mathcal{O}_{\tilde{\mathcal{N}}^{diag}}) = \mathcal{O}_{Z^{diag}}[\dim P/B]$. □

This finishes the proof of Theorem 6.1.

Proof of Corollary 6.3. Let us prove (1). We write $\tilde{\mathbb{O}}$ for the open orbit in $\tilde{\mathcal{N}}_P$, this is a G-equivariant cover of $\mathbb{O}$.

We first check that the equivalence

$$D^b(\mathrm{Coh}^{Z_P(e)} \mathcal{B}_e) \xrightarrow{\sim} D^b(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathbb{O}})) \tag{6.6}$$

is t-exact. The heart of the t-structure on the source is $\mathcal{A}_e$-mod$^{Z_P(e)}$ (in homological degree 0), while the heart in the target is naturally identified with $\mathcal{A}_e \otimes \mathcal{A}_{P,e}^{opp}$-mod$^{Z_P(e)}$, again in homological degree 0. Both triangulated categories are the derived categories of the hearts. But $\mathcal{A}_{P,e} = \mathrm{End}(\mathcal{T}_{P,e}(2\rho))$ and the restriction of (6.6) to the hearts is just tensoring with $\mathcal{T}_{P,e}(-2\rho)^*$. So it is exact.

What remains to check to prove (1) is that $D^b_{I^\circ}(\mathcal{F}l_P) \twoheadrightarrow D^b(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathbb{O}}))$ is t-exact. This follows directly from Theorem 6.2.

(2) is a direct corollary of the equivariance part from Theorem 6.1. To prove (3) note that, thanks to Theorem 6.1, the image of $\underline{\mathbb{C}}_{P^\vee/P^\vee}$ in $D^b(\mathrm{Coh}^{Z_G(e)} \mathcal{B}_e)$ is the restriction of $\mathcal{O}_{Z^{diag}}$ to $\mathcal{B}_e$. But the intersection of Z^{diag} with $\mathcal{B}_e$ is precisely $\mathcal{B}_{\mathfrak{m}}$. □

7. Duality

7.1. **Notation and content.** The notation $G, \mathfrak{g}, e, h, f, \nu, L, \mathfrak{g}^i, \underline{P}, P, \mathfrak{m}, \mathfrak{m}^-, T_0$ has the same meaning as in Section 2.1. Set $\underline{Q} := Z_{\underline{G}}(e, h, f)$.

Recall that we write W^a for the affine Weyl group of G. By ρ_L we denote half the sum of positive roots for L.

We continue to write p for a sufficiently large prime and $\chi \in \mathfrak{g}_{\mathbb{F}}^{(1)*}$ for the reduction of $(e\,\cdot)$ mod p. Recall the splitting bundle $\mathcal{V}^\chi_\mu$ on $\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)} \times_{\mathfrak{g}_{\mathbb{F}}^{(1)}} \mathfrak{g}_{\mathbb{F}}^{(1)\wedge_\chi}$, see (3.3). It carries a $\underline{Q}_{\mathbb{F}}$-equivariant structure as explained in Section 3.4.

In this section we introduce a duality functor, a contravariant t-exact self-equivalence $\mathbb{D}$ of $D^b(\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}})$ and study its properties. Section 7.2 defines this functor and states (and mostly proves) its basic properties. The most important property is that $\mathbb{D}$ fixes the

K_0-classes of χ-Weyl modules, Proposition 7.5. This proposition is proved in the next two sections: in Section 7.3 we treat the case when χ is distinguished and then in Section 7.4 we deal with the general case. Finally, in Section 7.5 we study an interplay between $\mathbb{D}$ and the derived localization equivalence.

7.2. **Duality functor: construction and basic properties.** Consider the standard anti-involution σ of $\mathfrak{g}$ defined by $\sigma_{\mathfrak{h}} = \mathrm{id}, \sigma(e_i) = f_i, \sigma(f_i) = e_i$. As was checked in [Lo, Section 2.6], one can replace (e,h,f) with a conjugate triple in such a way that $\sigma(e) = f, \sigma(f) = e, h$ is dominant. If n is the image in G of the matrix $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathrm{SL}_2$, then $\varsigma : x \mapsto \mathrm{Ad}(n)\sigma(x)$ is still an anti-involution that now fixes e,f (and maps h to $-h$). Also note that ς lifts to an involution of G. We can assume that T_0 is ς-stable. Clearly, ς maps $\mathfrak{g}^i$ to $\mathfrak{g}^{-i}$. It also fixes the parabolic subalgebra $\underline{\mathfrak{p}} \subset \underline{\mathfrak{g}}$. And, of course, we can reduce ς mod p. It is defined over $\mathbb{F}_p$ as long as $p-1$ is divisible by 4. Note also that ς fixes $\underline{Q}$.

For a module $M \in (\mathcal{U}_{\mathbb{F}}^{\chi})^{opp}\text{-mod}^{\underline{Q}}$, consider its twist with ς and denote it by ${}^{\varsigma}M$, it is an object of $\mathcal{U}_{\mathbb{F}}^{\chi}\text{-mod}^{\underline{Q}}$. Also consider the $\underline{Q}_{\mathbb{F}}$-module $\Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})$. We set ${}^{\mathsf{tw}}M := ({}^{\varsigma}M) \otimes \Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})$. The similar definition makes sense for $M' \in \mathcal{U}_{\mathbb{F}}^{\chi}\text{-mod}^{\underline{Q}}$.

Lemma 7.1. *We have* ${}^{\mathsf{tw}}({}^{\mathsf{tw}}M) \cong M$.

Proof. Note that $\underline{Q}_{\mathbb{F}} \cap Z(\underline{G}_{\mathbb{F}})$ (a normal subgroup of $\underline{Q}_{\mathbb{F}}$ containing $\underline{Q}_{\mathbb{F}}^{\circ}$) acts trivially on $\Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})$. Since $\varsigma^2 = \mathrm{id}$, the claim of the lemma reduces to checking that $\Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})^* \cong {}^{\varsigma}\Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})$ and it is enough to do this in the case when $\underline{\mathfrak{g}} = \mathfrak{g}$ and G is of adjoint type, here $\underline{Q} = A$. So we can assume that G is of adjoint type. Here A is either the sum of several copies of $\mathbb{Z}/2\mathbb{Z}$ or S_3, S_4, S_5. For all these groups the square of any one-dimensional representation is trivial. For S_3, S_4, S_5 there is a unique nontrivial one-dimensional representation and our claim follows. It remains to prove that if A is abelian, then $a \mapsto \varsigma(a)^{-1}$ is an inner automorphism of A.

For this consider the standard antiautomorphism $\sigma' : \mathfrak{g} \to \mathfrak{g}^{opp}, x \mapsto -x$. Assume that anti-automorphisms $\sigma^{-1}\sigma'$ is an inner automorphism of $\mathfrak{g}$. Then consider the element $n' \in G$, the image of $\mathrm{diag}(i,-i) \in \mathrm{SL}_2$ and set $\varsigma' := \mathrm{Ad}(n') \circ \sigma'$. Note that $\varsigma'(e) = \varsigma(e) = e, \varsigma'(f) = f, \varsigma'(h) = -h$. It follows that $\varsigma = \mathrm{Ad}(a') \circ \varsigma'$ for some $a' \in A$. It remains to observe that $\varsigma'(a) = a^{-1}$ for all $a \in A$.

Now we need to consider the situation when $\sigma^{-1}\sigma'$ is an outer automorphism. In type A, the group A is trivial, so we only need to consider type D_n (with odd n). Here $\sigma^{-1}\sigma'$ is induced by an element of O_{2n}. Now we can run the argument in the previous paragraph replacing G with O_{2n} and still arrive at the same conclusion. $\square$

Since $\mathcal{U}_{\mathbb{F}}^{\chi}$-mod consists of finite dimensional $\mathcal{U}_{\mathbb{F}}$-modules, we see that the functor

$$R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\bullet, \mathcal{U}_{\mathbb{F}})[\dim \mathfrak{g}]$$

is t-exact. Define $\mathbb{D} : \mathcal{U}_{\mathbb{F}}^{\chi}\text{-mod}^{\underline{Q}} \to \mathcal{U}_{\mathbb{F}}^{\chi}\text{-mod}^{\underline{Q},opp}$ by

$$\mathbb{D}(M) = {}^{\mathsf{tw}}R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(M, \mathcal{U}_{\mathbb{F}})[\dim \mathfrak{g}]. \tag{7.1}$$

The following lemmas establish basic properties of the functor $\mathbb{D}$.

Lemma 7.2. *We have* $\mathbb{D}^2 \cong \mathrm{id}$.

Proof. Since ${}^{\mathsf{tw}}\bullet$ is an involution, we have $R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}^{opp}}({}^{\mathsf{tw}}M, {}^{\varsigma}\mathcal{U}_{\mathbb{F}}) \cong {}^{\mathsf{tw}}R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(M,\mathcal{U}_{\mathbb{F}})$. But $\mathcal{U}_{\mathbb{F}} \cong {}^{\varsigma}\mathcal{U}_{\mathbb{F}}$ as a $\mathcal{U}_{\mathbb{F}}$-bimodule. Now the claim of this lemma follows from Lemma 7.1 combined with the claim that $R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\bullet,\mathcal{U}_{\mathbb{F}})$ is an involution. □

Lemma 7.3. *$\mathbb{D}$ maps $\mathcal{U}^{\chi}_{(\lambda),\mathbb{F}}$-$\mathrm{mod}^{Q}$ to $\mathcal{U}^{\chi}_{(\lambda),\mathbb{F}}$-$\mathrm{mod}^{Q}$ for every HC character λ.*

Proof. This follows from the classical fact that the principal anti-involution σ acts trivially on the HC center. □

Here is another useful property of $\mathbb{D}$ describing the interaction of this functor with the categorical braid group action.

Proposition 7.4. *We have $\mathbb{D}\circ T_w \cong T_{w^{-1}}^{-1}\circ\mathbb{D}$ for any $w\in W^a$.*

Proof. It is enough to consider the case $w = s$, in which case T_s is a classical wall-crossing functor. By what was recalled in Section 2.4, the functor T_s^{-1} is given by $\mathsf{T}^*\circ\mathsf{T}(\bullet)\to\bullet$, with the second term in the cohomological degree 1. So we need to show that $\mathbb{D}\circ\mathsf{T}\cong\mathsf{T}\circ\mathbb{D}$. The functor T has the form $\mathrm{pr}_{\lambda'}(V\otimes\bullet)$, for a suitable central character λ' (singular with the singularity corresponding to s) and a suitable finite dimensional irreducible G-module V. By Lemma 7.3, $\mathbb{D}$ intertwines the functors $\mathrm{pr}_{\lambda'}$ so it remains to show that $V\otimes\mathbb{D}(\bullet)\cong\mathbb{D}(V\otimes\bullet)$. Clearly, $\mathbb{D}(V\otimes\bullet)\cong{}^{\varsigma}(V^*)\otimes\mathbb{D}(\bullet)$, where in the right hand side V^* is viewed as right G-module and so the twist ${}^{\varsigma}(V^*)$ is a left G-module. And ${}^{\varsigma}(V^*)\cong{}^{\sigma}(V^*)\cong V$. This finishes the proof. □

Finally, let us state the main result of this section. In particular, it explains a reason why we twist with ς and not with some other anti-involution of $\mathfrak{g}$.

Proposition 7.5. *Let $\lambda = 0$. For all $x\in W^{a,P}$, we have $[\mathbb{D}W^{\chi}_{\mathbb{F}}(\mu_x)] = [W^{\chi}_{\mathbb{F}}(\mu_x)]$.*

This proposition will be proved in the next two sections.

7.3. **Behavior on K_0, distinguished case.** Throughout this section we assume that χ is distinguished.

We start with a series of lemmas.

Lemma 7.6. *Let $\lambda = 0$. The following claims are equivalent:*

(1) $\mathbb{D}W^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)\cong W^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)$.
(2) $[\mathbb{D}W^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)]\cong[W^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)]$.
(3) $[\mathbb{D}W^{\chi}_{\mathbb{F}}(\mu_x)]\cong[W^{\chi}_{\mathbb{F}}(\mu_x)]$ *for* $x\in W^{a,P}$.

Proof. We have $\dim W^{\chi}_{\mathbb{F}}(0) = p^{\dim\mathcal{P}}$. Since $\dim\mathcal{P} = \frac{1}{2}\dim G_{\mathbb{F}}\chi$, we use the main result of [P] to deduce that $W^{\chi}_{\mathbb{F}}(0)$ is irreducible. So (1) and (2) are equivalent.

Proposition 7.4 implies that $[\mathbb{D}]$ acts on $K_0(\mathcal{U}^{\chi}_{(0),\mathbb{F}}$-$\mathrm{mod}^{Q})$ by a W^a-linear automorphism. The equivalence (2)$\Leftrightarrow$(3) now follows from Lemma 2.5. □

Here is another technical result that we are going to need.

Lemma 7.7. *We have $W^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)\xrightarrow{\sim}W^{\chi}_{\mathbb{F}}(0)$.*

Proof. Note that $W^{\chi}_{\mathbb{F}}(\mu)$ is the specialization of the parabolic Verma module $\Delta^P_{\mathbb{F}}(\mu)$ to $\chi\in\mathfrak{g}^{(1)*}_{\mathbb{F}}$. Consider the parabolic Verma $\Delta^P_{\mathbb{C}}(2\rho_L-2\rho) := U(\mathfrak{g})\otimes_{U(\mathfrak{g}^{\mathfrak{p}})}\mathbb{C}_{2\rho_L-2\rho}$.

We claim that $\Delta^P_{\mathbb{C}}(2\rho_L-2\rho)\hookrightarrow\Delta^P_{\mathbb{C}}(0)$ and the quotient has proper support. Note that $\Delta^P_{\mathbb{C}}(2\rho_L-2\rho)$ is simple. The annihilators of $\Delta^P_{\mathbb{C}}(2\rho_L-2\rho),\Delta^P_{\mathbb{C}}(0)$ in $U(\mathfrak{g})$ coincide because both coincide with the kernel J of $U(\mathfrak{g})\twoheadrightarrow D(G/P)$, see [BB, Section 3.6]. The highest weight $2\rho_L - 2\rho$ corresponds to the longest element $\underline{w}_0$ of $W_{\underline{G}}$. This is a Duflo involution, so the socle of $\Delta_{\mathbb{C}}(0)/J\Delta_{\mathbb{C}}(0)$ is $\Delta^P_{\mathbb{C}}(2\rho_L - 2\rho)$ and the quotient of $\Delta_{\mathbb{C}}(0)/J\Delta_{\mathbb{C}}(0)$ by the socle has GK dimension smaller than that of $\Delta^P_{\mathbb{C}}(2\rho_L - 2\rho)$. It follows that the natural epimorphism $\Delta_{\mathbb{C}}(0)/J\Delta_{\mathbb{C}}(0) \twoheadrightarrow \Delta^P_{\mathbb{C}}(0)$ is actually an isomorphism. This yields the required embedding $\Delta^P_{\mathbb{C}}(2\rho_L - 2\rho) \hookrightarrow \Delta^P_{\mathbb{C}}(0)$.

The embedding $\Delta^P_{\mathbb{C}}(2\rho_L - 2\rho) \hookrightarrow \Delta^P_{\mathbb{C}}(0)$ is defined over a finite localization of $\mathbb{Z}$. Since p is large enough, we get $\Delta^P_{\mathbb{F}}(2\rho_L - 2\rho) \hookrightarrow \Delta^P_{\mathbb{F}}(0)$ and the quotient has proper support in $\mathfrak{m}_{\mathbb{F}}^{-,(1)}$. This support is closed and $P^{(1)}_{\mathbb{F}}$-stable. So the support of the quotient does not contain χ. We conclude that $W^\chi_{\mathbb{F}}(2\rho_L - 2\rho) \hookrightarrow W^\chi_{\mathbb{F}}(0)$. Since the dimensions coincide, this embedding is an iso. □

Proof of Proposition 7.5 for distinguished χ. The proof will be in several steps.

Step 1. First, we prove that $\mathbb{D}M = {}^{\mathsf{tw}}\mathrm{Hom}_{\mathcal{U}^\chi_{\mathbb{F}}}(M, \mathcal{U}^\chi_{\mathbb{F}})$. Note that

$$\mathbb{D}M = {}^{\mathsf{tw}}\mathrm{Hom}(M, R\,\mathrm{Hom}_{\mathcal{U}_{\mathbb{F}}}(\mathcal{U}^\chi_{\mathbb{F}}, \mathcal{U}_{\mathbb{F}})[\dim \mathfrak{g}]).$$

So we just need to prove that the $\mathcal{U}^\chi_{\mathbb{F}}$-bimodules $\mathcal{U}^\chi_{\mathbb{F}}$ and $R\,\mathrm{Hom}_{\mathcal{U}_{\mathbb{F}}}(\mathcal{U}^\chi_{\mathbb{F}}, \mathcal{U}_{\mathbb{F}})[\dim \mathfrak{g}]$ are A-equivariantly isomorphic. Let $n := \dim \mathfrak{g}$ and let V denote the subspace in $S(\mathfrak{g}^{(1)}_{\mathbb{F}})$ generated by the elements $x - \langle \chi, x\rangle$ for $x \in \mathfrak{g}^{(1)}_{\mathbb{F}}$. This subspace is A-stable. Then the regular bimodule $\mathcal{U}^\chi_{\mathbb{F}}$ is quasi-isomorphic to the Koszul complex for V acting on $\mathcal{U}_{\mathbb{F}}$, denote it by $K_\bullet(V, \mathcal{U}_{\mathbb{F}})$. This isomorphism is $\mathcal{U}_{\mathbb{F}}$-bilinear and A-equivariant. Then we have an A-equivariant $\mathcal{U}_{\mathbb{F}}$-bilinear isomorphism

$$R\,\mathrm{Hom}_{\mathcal{U}_{\mathbb{F}}}(\mathcal{U}^\chi_{\mathbb{F}}, \mathcal{U}_{\mathbb{F}})[\dim \mathfrak{g}] \cong K_\bullet(V, \mathcal{U}_{\mathbb{F}} \otimes \Lambda^{top} V^*)$$

Now we observe that the action of A on V is via a homomorphism $A \to \mathrm{SO}(V)$. This is because the action is isomorphic to the action on A on $\mathfrak{g}^{(1)}_{\mathbb{F}}$, which factors through $G^{(1)}_{\mathbb{F}}$. It follows that $\Lambda^{top} V^*$ is the trivial A-module and hence indeed $\mathcal{U}^\chi_{\mathbb{F}} \cong R\,\mathrm{Hom}_{\mathcal{U}_{\mathbb{F}}}(\mathcal{U}^\chi_{\mathbb{F}}, \mathcal{U}_{\mathbb{F}})[\dim \mathfrak{g}]$.

Step 2. Both $W^\chi_{\mathbb{F}}(0), \mathbb{D}W^\chi_{\mathbb{F}}(0)$ are simple modules. By Lemma 7.7, $W^\chi_{\mathbb{F}}(0) = W^\chi_{\mathbb{F}}(2\rho_L - 2\rho)$. To prove the proposition we need to show that

$$\mathrm{Hom}_{\mathcal{U}^\chi_{\mathbb{F}}}(W^\chi_{\mathbb{F}}(2\rho_L - 2\rho), \mathbb{D}W^\chi_{\mathbb{F}}(0))^A \neq 0.$$

We write $\mathbb{F}_0$ for the one-dimensional trivial $\mathfrak{p}_{\mathbb{F}}$-module. By Step 1,

$$\begin{aligned}&\mathbb{D}W^\chi_{\mathbb{F}}(0) = {}^{\mathsf{tw}}\mathrm{Hom}_{\mathcal{U}^\chi_{\mathbb{F}}}(\mathcal{U}^\chi_{\mathbb{F}} \otimes_{U^0(\mathfrak{p}_{\mathbb{F}})} \mathbb{F}_0, \mathcal{U}^\chi_{\mathbb{F}}) = \\ &{}^{\varsigma}\mathrm{Hom}_{U^0(\mathfrak{p}_{\mathbb{F}})}(\mathbb{F}_0, \mathcal{U}^\chi_{\mathbb{F}}) \otimes \Lambda^{top}(\mathfrak{g}_F/\mathfrak{p}_{\mathbb{F}}) = \mathrm{Hom}_{U^0(\mathfrak{p}_{\mathbb{F}})^{opp}}(\mathbb{F}_0, \mathcal{U}^\chi_{\mathbb{F}}) \otimes \Lambda^{top}(\mathfrak{g}_{\mathbb{F}}/\mathfrak{p}_{\mathbb{F}}).\end{aligned}$$

The last equality holds because the ς-twist of the regular bimodule is again the regular bimodule (because the unit element in the twist is central) and ${}^{\varsigma}\mathbb{F}_0 \cong \mathbb{F}_0$. So

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^A}(W^\chi_{\mathbb{F}}(2\rho_L - 2\rho), \mathbb{D}W^\chi_{\mathbb{F}}(0)) = \\ &\left[\mathrm{Hom}_{U^0(\mathfrak{p}_{\mathbb{F}})}\left(\mathbb{F}_{2\rho_L - 2\rho}, \mathrm{Hom}_{U^0(\mathfrak{p}_{\mathbb{F}})^{opp}}(\mathbb{F}_0, \mathcal{U}^\chi_{\mathbb{F}})\right) \otimes \Lambda^{top}(\mathfrak{g}_{\mathbb{F}}, \mathfrak{p}_{\mathbb{F}})\right]^A = \\ &\left[\mathrm{Hom}_{U^0(\mathfrak{p}_{\mathbb{F}}) \otimes U^0(\mathfrak{p}_{\mathbb{F}})^{opp}}\left(\mathbb{F}_{2\rho_L - 2\rho} \otimes \mathbb{F}_0, \mathcal{U}^\chi_{\mathbb{F}}\right) \otimes \Lambda^{top}(\mathfrak{g}_{\mathbb{F}}/\mathfrak{p}_{\mathbb{F}})\right]^A.\end{aligned}$$

Note that $\Lambda^{top}(\mathfrak{g}_{\mathbb{F}}/\mathfrak{p}_{\mathbb{F}}) \cong \Lambda^{top}(\mathfrak{p}_{\mathbb{F}})$. Since $U^0(\mathfrak{p}_{\mathbb{F}})$ is an A-stable subalgebra of $\mathcal{U}^\chi_{\mathbb{F}}$, the space $\mathrm{Hom}(W^\chi_{\mathbb{F}}(2\rho_L - 2\rho), \mathbb{D}W^\chi_{\mathbb{F}}(0))$ contains

$$\left[\mathrm{Hom}_{U^0(\mathfrak{p}_{\mathbb{F}}) \otimes U^0(\mathfrak{p}_{\mathbb{F}})^{opp}}(\mathbb{F}_{2\rho_L - 2\rho} \otimes \mathbb{F}_0, U^0(\mathfrak{p}_{\mathbb{F}})) \otimes \Lambda^{top}(\mathfrak{p}_{\mathbb{F}})\right]^A.$$

Below in this proof we will see that the latter space is nonzero.

Step 3. Let $\mathfrak{u}$ be the Lie algebra of a unipotent algebraic group U over $\mathbb{F}$. We claim that there is a unique (up to rescaling) element $x(\mathfrak{u}) \in U^0(\mathfrak{u})$ annihilated by $\mathfrak{u}$ on the left and on the right. The proof is by induction on $\dim \mathfrak{u}$. Namely, set $\mathfrak{u}_1 := \mathfrak{u}/\mathfrak{z}(\mathfrak{u})$. Then $U^0(\mathfrak{u}) \twoheadrightarrow U^0(\mathfrak{u}_1)$. Let x' be a lift of $x(\mathfrak{u}_1)$ to $U^0(\mathfrak{u})$ and let $y_1, \ldots, y_k$ be a basis in $\mathfrak{z}(\mathfrak{u})$. The element $x(\mathfrak{u}) := x' \prod_{i=1}^k y_i^{p-1}$ is independent of the choice of x'. It is annihilated by $\mathfrak{u}$ on the left and on the right. On the other hand, if x is annihilated by $\mathfrak{u}$ on the left and on the right it must have the form $x'' \prod_{i=1}^k y_i^{p-1}$, where $x'' \in U^0(\mathfrak{u})$ is such that the projection of x'' to $U^0(\mathfrak{u}_1)$ is annihilated by $\mathfrak{u}_1$ on the left and on the right. So it must be proportional to $x(\mathfrak{u}_1)$. This implies the claim in the beginning of the paragraph.

Step 4. We still assume that $\mathfrak{u}$ is the Lie algebra of a unipotent algebraic group. Let $y_1, \ldots, y_N$ be a basis of $\mathfrak{u}$. Then $x(\mathfrak{u})$ is proportional to $\prod_{i=1}^N y_i^{p-1}$. Indeed, this follows from Step 3 for a special choice of basis and it is easy to see that the right hand side is independent of the choice of a basis. It follows that if S is an algebraic group acting on $\mathfrak{u}$ be algebraic Lie algebra automorphisms, then

$$sx(\mathfrak{u}) = \chi_{\Lambda^{top}(\mathfrak{u})}(s)^{p-1} x(\mathfrak{u}),$$

where $\chi_{\Lambda^{top}(\mathfrak{u})}$ is the character of the S-action in $\Lambda^{top}(\mathfrak{u})$.

Step 5. Now consider an algebraic group $F = S \ltimes U$ over $\mathbb{F}$, where S is connected reductive and U is unipotent. Assume that there is $x(\mathfrak{s}) \in U^0(\mathfrak{s})$ that is annihilated by $\mathfrak{s}$ on the left and on the right. It follows from Step 4, that the element $x(\mathfrak{f}) := x(\mathfrak{u})x(\mathfrak{s}) \in U^0(\mathfrak{f})$ is annihilated by $\mathfrak{f}$ on the right, while for any $y \in \mathfrak{f}$ we have

$$yx(\mathfrak{f}) = -\langle \chi_{\Lambda^{top}(\mathfrak{u})}, y \rangle x(\mathfrak{f}).$$

So to prove the claim in the end of Step 2, it remains to check that $x(\mathfrak{s})$ indeed exists.

Step 6. Let $\mathfrak{s} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition for $\mathfrak{s}$. Let $z_1, \ldots, z_r$ be an integral basis in $\mathfrak{h}$. For $k \in \mathbb{F}_p$, set $F_k(z) := (\prod_{i=1}^p (z-i))/(z-k) \in \mathbb{F}[z]$. We claim that

$$x(\mathfrak{s}) := x(\mathfrak{n})(\prod_{i=1}^r F_{\langle 2\rho, z_i \rangle}(z_i)) x(\mathfrak{n}^-)$$

satisfies the required properties. Note that $\prod_{i=1}^r F_{\langle 2\rho, z_i\rangle}(z_i)$ does not depend on the choice of $z_1, \ldots, z_r$ (up to rescaling): it is the unique element in $U^0(\mathfrak{h})$ annihilated by $z - \langle 2\rho, z\rangle$ for all $z \in \mathfrak{h}$.

It is enough to show that $x(\mathfrak{s})$ is annihilated by the Cartan generators e_i, f_i and also by $\mathfrak{h}$ on the left and on the right. For $\mathfrak{h}$, this is clear. Let $\mathfrak{n}_0$ be the H-stable complement of $\mathbb{F}e_i$ in $\mathfrak{n}$ and $\mathfrak{n}_0^-$ have the similar meaning. By Step 4, we have

$$x(\mathfrak{s}) = x(\mathfrak{n}_0) \left[e_i^{p-1} (\prod_{i=1}^r F_{\langle 2\rho, z_i\rangle}(z_i)) f_i^{p-1} \right] x(\mathfrak{n}_0^-)$$

The elements $x(\mathfrak{n}_0), x(\mathfrak{n}_0^-)$ commute with both e_i, f_i by Step 4. So we need to check that e_i, f_i annihilate the middle bracket. This reduces the computation to the case of $\mathfrak{s} = \mathfrak{sl}_2$: we choose $z_1 = \alpha_i^\vee$ and all other z_i vanishing on α_i.

In the case of $\mathfrak{sl}_2$ what we need to check is that $fe^{p-1}(\prod_{i\neq 2}(h-i))f^{p-1} = 0$ and $e^{p-1}(\prod_{i\neq 2}(h-i))f^{p-1}e = 0$ in $U^0(\mathfrak{sl}_2)$. The first equality easily follows from $fe^{p-1} = e^{p-2}(h-2) + e^{p-1}f$. The second is analogous. $\square$

7.4. **Behavior on K_0, general case.** Now let us discuss a compatibility between $\mathbb{D}$ and $\underline{\mathbb{D}}$, the similarly defined functor for $\underline{\mathcal{U}}^\chi_{\mathbb{F}}\text{-mod}^Q$. Let $\underline{\Delta}_\nu$ and $\underline{\nabla}_\nu$ denote the baby Verma and *dual baby Verma* functors $\underline{\mathcal{U}}_{\mathbb{F}}\text{-mod}^Q \to \mathcal{U}_{\mathbb{F}}\text{-mod}^Q$, the latter is defined by

$$\underline{\nabla}_\nu(M) := \operatorname{Hom}_{U^\chi(\mathfrak{g}^{\leqslant 0}_{\mathbb{F}})}(\mathcal{U}^\chi_{\mathbb{F}}, M).$$

Proposition 7.8. *We have $\mathbb{D}\circ\underline{\Delta}_\nu \cong \underline{\nabla}_\nu\circ\underline{\mathbb{D}}$, an isomorphism of functors $\underline{\mathcal{U}}^\chi_{\mathbb{F}}\text{-mod}^Q \to \mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{Q,opp}$.*

Proof. First, we compute $R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\underline{\Delta}_\nu(\underline{\mathcal{U}}_{\mathbb{F}}), \mathcal{U}_{\mathbb{F}})$. Let $\underline{\Delta}^r_\nu$ denote the analog of the functor $\underline{\Delta}$ for the categories of right modules (with the same parabolic). We claim that

$$R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\underline{\Delta}_\nu(\underline{\mathcal{U}}_{\mathbb{F}}), \mathcal{U}_{\mathbb{F}})[2\dim\mathfrak{g}^{>0}] \cong \underline{\Delta}^r_\nu\left(\underline{\mathcal{U}}_{\mathbb{F}}\otimes\left[\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right]^*\right). \tag{7.2}$$

This is an isomorphism of right $(\mathcal{U}_{\mathbb{F}}, \underline{G}_{\mathbb{F}})$-modules (in particular, we view $\left[\Lambda(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right]^*$ as a right $\underline{G}_{\mathbb{F}}$-module). To prove (7.2) we first note that $\underline{\Delta}_\nu(\underline{\mathcal{U}}_{\mathbb{F}})$ is the quotient of $\mathcal{U}_{\mathbb{F}}$ by the ideal generated by $\mathfrak{g}^{>0}_{\mathbb{F}}$ and $\mathfrak{g}^{<0,(1)}_{\mathbb{F}}$. So $\underline{\Delta}_\nu(\underline{\mathcal{U}}_{\mathbb{F}})$ is quasi-isomorphic to the Chevalley-Eilenberg complex of left modules. We denote this complex by $\mathsf{CE}_\ell(\mathfrak{g}^{>0}_{\mathbb{F}}\oplus\mathfrak{g}^{<0,(1)}_{\mathbb{F}}, \mathcal{U}_{\mathbb{F}})$. It follows that we have a quasi-isomorphism

$$R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\underline{\Delta}_\nu(\underline{\mathcal{U}}_{\mathbb{F}}), \mathcal{U}_{\mathbb{F}})[2\dim\mathfrak{g}^{>0}] \cong \mathsf{CE}_r(\mathfrak{g}^{>0}_{\mathbb{F}}\oplus\mathfrak{g}^{<0,(1)}_{\mathbb{F}}, \mathcal{U}_{\mathbb{F}}\otimes\left[\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right]^*),$$

where on the right we have the Chevalley-Eilenberg complex of right modules. This complex is quasi-isomorphic to (7.2). This proves (7.2).

In particular, we see that

$$R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\underline{\Delta}_\nu(M), \mathcal{U}_{\mathbb{F}}) = \underline{\Delta}^r_\nu\left(R\operatorname{Hom}_{\underline{\mathcal{U}}_{\mathbb{F}}}(M, \underline{\mathcal{U}}_{\mathbb{F}}\otimes\left[\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right]^*)\right)[2\dim\mathfrak{g}^{>0}]. \tag{7.3}$$

Now let us see what twisting by ς does. First of all, ${}^\varsigma\underline{\Delta}^r_\nu(\bullet) \cong \underline{\Delta}_{-\nu}({}^\varsigma\bullet)$, where $\underline{\Delta}_{-\nu}$ stands for the baby Verma module functor for the opposite parabolic $\mathfrak{g}^{\leqslant 0}$. Also

$${}^\varsigma\left[\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right]^* = \Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}.$$

Here the right hand side is a left $\underline{G}_{\mathbb{F}}$-module and the equality follows because ς is the identity on the character lattice of $\underline{G}_{\mathbb{F}}$.

So (7.3) yields

$$\mathbb{D}(\underline{\Delta}_\nu(M)) = \underline{\Delta}_{-\nu}(\underline{\mathbb{D}}(M)\otimes\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}). \tag{7.4}$$

Note that $\underline{\mathbb{D}}(M)$ is the maximal ν-weight subspace in $\underline{\Delta}_{-\nu}\left(\underline{\mathbb{D}}(M)\otimes\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right)$. This gives a homomorphism

$$\underline{\Delta}_{-\nu}(\underline{\mathbb{D}}(M)\otimes\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}) \to \underline{\nabla}_\nu(\underline{\mathbb{D}}(M)) \tag{7.5}$$

that is the identity on $\underline{\mathbb{D}}(M)$.

Now we show that every ν-graded $\mathfrak{g}^{>0}_{\mathbb{F}}$-submodule of $\underline{\Delta}_{-\nu}\left(\underline{\mathbb{D}}(M)\otimes\Lambda^{top}(\mathfrak{g}^{>0}_{\mathbb{F}})^{\otimes 1-p}\right)$ intersects the maximal ν-weight subspace $\underline{\mathbb{D}}(M)$. This will follow if we check that every ν-graded submodule of $U^0(\mathfrak{g}^{>0}_{\mathbb{F}})$ has nonzero eigenspace with eigenvalue $(p-1)\sum_{\alpha|\langle\alpha,\nu\rangle>0}\alpha$. This claim, in its turn, follows from the analogous claim for $S(\mathfrak{g}^{>0}_{\mathbb{F}})/(\mathfrak{g}^{(1),>0}_{\mathbb{F}})$, where it is obvious.

It follows that (7.5) is an injection. Since the dimensions of the source and the target are the same, we see that (7.5) is an isomorphism. It follows that

$${}^\varsigma R\operatorname{Hom}_{\mathcal{U}_{\mathbb{F}}}(\underline{\Delta}_\nu(M), \mathcal{U}_{\mathbb{F}}) \cong \underline{\nabla}_\nu({}^\varsigma R\operatorname{Hom}_{\underline{\mathcal{U}}_{\mathbb{F}}}(M, \underline{\mathcal{U}}_{\mathbb{F}})).$$

This implies the claim of the proposition. $\square$

Proof of Proposition 7.5 in the general case. Thanks to the distinguished case of this proposition that we have established already in Section 7.3, we need to prove that if $[\underline{\mathbb{D}}M]=[M]$, then $[\mathbb{D}\underline{\Delta}_\nu(M)]=[\underline{\Delta}_\nu(M)]$. Using Proposition 7.8, we reduce to proving $[\underline{\Delta}_\nu(M)]=[\underline{\nabla}_\nu(M)]$. We will prove this for an arbitrary module $M\in\underline{\mathcal{U}}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}$.

Note that every module in $\mathcal{U}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}$ is also a module in $\underline{\mathcal{U}}^\chi_{\mathbb{F}}\text{-mod}^{\underline{Q}}$ by restriction. The corresponding map between the K_0-groups is injective because the images of simples are linearly independent thanks to the upper triangularity. To finish the proof observe that in $K_0(\mathcal{U}^\chi(\underline{\mathfrak{g}}_{\mathbb{F}})\text{-mod}^{\underline{Q}})$ we have

$$[\underline{\Delta}_\nu(M)]=[U^0(\mathfrak{g}_{\mathbb{F}}^{<0})\otimes M]=[\underline{\nabla}_\nu(M)].$$

$\square$

7.5. **Duality and localization.** The goal of this section is describe the autoequivalence

$$D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))\xrightarrow{\sim}D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))^{opp},$$

induced by $\mathbb{D}:\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}\xrightarrow{\sim}\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},opp}$ under the derived localization equivalence $R\Gamma(\mathcal{V}^\chi_0(\rho)\otimes\bullet)$. Here we assume that $\rho\in\mathfrak{X}(T)$ (we will explain what to do in the general case in the end of the section) and we use the $\underline{Q}_{\mathbb{F}}$-equivariant structure on $\mathcal{V}^\chi_0(\rho)$ introduced in Section 3.4.

Consider the Serre duality functor $R\operatorname{Hom}(\bullet,K_{(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}}})[\dim\mathfrak{g}]$, where $K_\bullet$ stands for the canonical bundle of $(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}}$, this bundle is trivial. This functor gives rise to an equivalence

$$D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))\xrightarrow{\sim}D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))^{opp},$$

where we view $\mathcal{B}_\chi$ as a derived subscheme of $\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}$ (equivalently, of $(\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}})_{\mathfrak{h}}$).

Note that ς gives rise to an automorphism of the $\mathfrak{h}^{(1)*}_{\mathbb{F}}$-scheme $\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}$ such that $\mathcal{B}_\chi$ is stable (as a derived subscheme). We twist the Serre duality functor by ς and then tensor with the $\underline{Q}/\underline{Q}^\circ$-module $\Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})$. The resulting contravariant autoequivalence of $D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))$ will be denoted by $\mathbb{D}_{coh}$.

Proposition 7.9. *Recall that we assume that $\rho\in\mathfrak{X}(T)$. Then we have the following commutative diagram*

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[r, "T_{w_0}^{-1}\circ\mathbb{D}_{coh}"] \arrow[d, "R\operatorname{Hom}(\mathcal{V}^\chi_0(\rho)\otimes\bullet)"] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))^{opp} \arrow[d, "R\operatorname{Hom}(\mathcal{V}^\chi_0(\rho)\otimes\bullet)"] \\
D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "\mathbb{D}"] & D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})^{opp}
\end{tikzcd}$$

Proof. We note that the claim of this proposition reduces to the case when G is semisimple and simply connected. We are going to assume this until the end of the proof.

Step 1. We write $\tilde{D}$ for $\tilde{D}_{\mathcal{B}_{\mathbb{F}}}$ to simplify the notation. We have $R\Gamma(\tilde{D})=\mathcal{U}_{\mathfrak{h},\mathbb{F}}$. The functors $R\operatorname{Hom}(\bullet,\mathcal{U}_{\mathbb{F}})$ and $R\operatorname{Hom}(\bullet,\mathcal{U}_{\mathfrak{h},\mathbb{F}})$ are isomorphic on $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}$. So we have the following commutative diagram

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\tilde{D}|_{\mathcal{B}_\chi})) \arrow[r, "R\,\mathrm{Hom}(\bullet,\tilde{D})"] \arrow[d, "R\Gamma"] & D^b(\mathrm{Coh}^{\underline{Q}}(\tilde{D}^{opp}|_{\mathcal{B}_\chi}))^{opp} \arrow[d, "R\Gamma"] \\
D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "R\,\mathrm{Hom}(\bullet,\mathcal{U}_{\mathbb{F}})"] & D^b(\mathcal{U}^{\chi,opp}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})^{opp}
\end{tikzcd}$$

Also we have the following commutative diagram, where in the top horizontal row we take Hom over $\mathcal{O}_{\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}}$ and in the bottom arrow we take Hom over $\tilde{D}_{\mathbb{F}}$.

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[r, "R\,\mathrm{Hom}(\bullet,\mathcal{O})"] \arrow[d, "\mathcal{V}^\chi_0(\rho)\otimes\bullet"] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))^{opp} \arrow[d, "(\mathcal{V}^\chi_0(\rho))^*\otimes\bullet"] \\
D^b(\mathrm{Coh}^{\underline{Q}}(\tilde{D}|_{\mathcal{B}_\chi})) \arrow[r, "R\,\mathrm{Hom}(\bullet,\tilde{D})"] & D^b(\mathrm{Coh}^{\underline{Q}}(\tilde{D}^{opp}|_{\mathcal{B}_\chi}))^{opp}
\end{tikzcd} \tag{7.6}$$

Combining the previous two diagram, twisting with ς and tensoring with $\Lambda^{top}(\underline{\mathfrak{g}}_{\mathbb{F}}/\underline{\mathfrak{p}}_{\mathbb{F}})$, we get the following commutative diagram.

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[r, "\mathbb{D}_{coh}"] \arrow[d, "R\Gamma(\mathcal{V}^\chi_0(\rho)\otimes\bullet)"] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))^{opp} \arrow[d, "R\Gamma({}^{\varsigma}(\mathcal{V}^\chi_0(\rho))^*\otimes\bullet)"] \\
D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "\mathbb{D}"] & D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})^{opp}
\end{tikzcd}$$

Step 2. The following commutative diagram is a consequence of Proposition 3.2.

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[r, "T_{w_0}"] \arrow[d, "R\Gamma(\mathcal{V}^\chi_0(\rho)\otimes\bullet)"] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[d, "R\Gamma(\mathcal{V}^\chi_0(\rho)\otimes\bullet)"] \\
D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "T_{w_0}"] & D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})
\end{tikzcd}$$

On the other hand, by [BMR1, Theorem 2.1.4], we have

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\tilde{D}|_{\mathcal{B}_\chi})) \arrow[r, "\mathcal{O}(2\rho)\otimes\bullet"] \arrow[d, "R\Gamma"] & D^b(\mathrm{Coh}^{\underline{Q}}(\tilde{D}|_{\mathcal{B}_\chi})) \arrow[d, "R\Gamma"] \\
D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "T_{w_0}"] & D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})
\end{tikzcd}$$

This commutative diagram implies the following one.

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[r, "\mathrm{id}"] \arrow[d, "R\Gamma(\mathcal{V}^\chi_{-2\rho}(\rho)\otimes\bullet)"] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[d, "R\Gamma(\mathcal{V}^\chi_0(\rho)\otimes\bullet)"] \\
D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "T_{w_0}"] & D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})
\end{tikzcd}$$

So we conclude that

$$R\Gamma(\mathcal{V}^{\chi}_{-2\rho}\otimes\bullet)\cong R\Gamma(\mathcal{V}^{\chi}_{0}\otimes T^{-1}_{w_0}(\bullet)). \tag{7.7}$$

Step 3. Suppose, for a moment, that we know that $\mathcal{V}^{\chi}_{-2\rho}(\rho)$ and ${}^{\varsigma}(\mathcal{V}^{\chi}_0(\rho))^*$ are $\underline{Q}_{\mathbb{F}}$-equivariantly isomorphic. Using (7.6), we get

$$\mathbb{D}\circ R\Gamma(\mathcal{V}^{\chi}_0(\rho)\otimes\bullet)\cong R\Gamma(\mathcal{V}^{\chi}_{-2\rho}(\rho)\otimes\mathbb{D}_{coh}(\bullet)).$$

By (7.7), we get

$$R\Gamma(\mathcal{V}^{\chi}_{-2\rho}(\rho)\otimes\mathbb{D}_{coh}(\bullet))\cong R\Gamma(\mathcal{V}^{\chi}_0(\rho)\otimes T^{-1}_{w_0}\mathbb{D}_{coh}(\bullet)).$$

The last two isomorphisms imply the commutative diagram in the statement of the proposition.

Step 4. It remains to show that we have a $\underline{Q}_{\mathbb{F}}$-equivariant isomorphism

$$\mathcal{V}^{\chi}_{-2\rho}(\rho)\cong {}^{\varsigma}(\mathcal{V}^{\chi}_0(\rho))^*.$$

We will prove a stronger statement: there is a $\underline{Q}_{\mathbb{F}}\times\mathbb{F}^{\times}$-equivariant isomorphism

$$\mathcal{V}^{\mathbb{F}\chi}_{-2\rho}(\rho)\cong {}^{\varsigma}(\mathcal{V}^{\mathbb{F}\chi}_0(\rho))^*, \tag{7.8}$$

where the bundles $\mathcal{V}^{\mathbb{F}\chi}_{?}$ were introduced in Section 3.4. Note that the action of $\mathbb{F}^{\times}$ is contracting (to $(\mathcal{B}_\chi,0)$). Therefore it is enough to prove that there is a $G_{\mathbb{F}}$-equivariant isomorphism

$$\mathcal{V}^{0}_{-2\rho}(\rho)\cong {}^{\varsigma}(\mathcal{V}^{0}_0(\rho))^*. \tag{7.9}$$

Indeed, we restrict (7.9) to the formal neighborhood of $\mathcal{B}_\chi$ and get (7.8) thanks to the $\mathbb{F}^{\times}$-equivariance.

The left hand side of (7.9) is a splitting bundle for $\tilde{D}^{-2\rho}_{\mathcal{B},\mathbb{F}}$ restricted to

$$\tilde{\mathcal{N}}^{(1)\wedge_0}_{\mathbb{F}}:=\tilde{\mathcal{N}}^{(1)}_{\mathbb{F}}\times_{\mathfrak{g}^{(1)}_{\mathbb{F}}}\mathfrak{g}^{(1)\wedge_0}_{\mathbb{F}}.$$

The right hand side is a splitting bundle for ${}^{\varsigma}(\tilde{D}_{\mathcal{B},\mathbb{F}})^{opp}$ restricted to $\tilde{\mathcal{N}}^{(1)\wedge}_{\mathbb{F}}$. Consider the antiautomorphism σ' of $\mathfrak{g}_{\mathbb{F}}$ given by $x\mapsto -x$. It lies in the same $\mathrm{Ad}(\mathfrak{g}_{\mathbb{F}})$-coset as ς so we get a $G_{\mathbb{F}}$-equivariant isomorphism ${}^{\varsigma}(\tilde{D}_{\mathcal{B}})^{opp}\cong {}^{\sigma'}(\tilde{D}_{\mathcal{B}})^{opp}$. Note that ${}^{\sigma'}(\tilde{D}_{\mathcal{B}})^{opp}\cong\tilde{D}^{-2\rho}_{\mathcal{B}}$, a $G_{\mathbb{F}}$-equivariant isomorphism of Azumaya algebras.So (7.9) is true up to a twist with a line bundle. We need to show that this line bundle is trivial and it is sufficient to prove the restriction of the line bundle to $\mathcal{B}^{(1)}_{\mathbb{F}}$ is trivial. Similarly to the proof of (1) of Lemma 3.5, it is enough to show (7.9) on the level of K_0-classes in the non-equivariant K-theory. Thanks to (1) of Lemma 3.5 this equality reduces to

$$[(\mathrm{Fr}_{\mathcal{B}*}\mathcal{O})(\rho)]^*=[(\mathrm{Fr}_{\mathcal{B}*}\mathcal{O}(-2\rho))(\rho)]. \tag{7.10}$$

Recall that we have an isomorphism $\mathrm{Fr}_*\mathcal{O}(-\rho)\cong\mathcal{O}^{(1)}(-\rho)^{\oplus p^{\dim G/B}}$. So

$$[(\mathrm{Fr}_{\mathcal{B}*}\mathcal{O})(\rho)]^*=p^{\dim G/B}[\mathcal{O}(\rho/p)]^*,[(\mathrm{Fr}_{\mathcal{B}*}\mathcal{O}(-2\rho))(\rho)]=p^{\dim G/B}[\mathcal{O}(-\rho/p)].$$

(7.10) follows. □

Remark 7.10. Now we no longer assume that $\rho\in\mathfrak{X}(T)$ (compare to Remark 6.4). Instead, let ρ' be a character of T that pairs by 1 with all simple coroots. Then the argument of the proof of Proposition 7.9 shows that the following diagram is commutative:

$$\begin{tikzcd} D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi)) \arrow[r, "T_{w_0}^{-1}\circ \mathbb{D}_{coh}"] \arrow[d, "R\,\mathrm{Hom}(\mathcal{V}_0^\chi(\rho')\otimes \bullet)"] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_\chi))^{opp} \arrow[d, "R\,\mathrm{Hom}(\mathcal{V}_0^\chi(2\rho-\rho')\otimes \bullet)"] \\ D^b(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "\mathbb{D}"] & D^b(\mathcal{U}^\chi_{(0),\tilde{\mathbb{F}}}\text{-mod}^{\underline{Q}})^{opp} \end{tikzcd}$$

Note that $\mathcal{V}_0^\chi(\rho'), \mathcal{V}_0^\chi(2\rho-\rho')$ differ by a twist with a character of T.

8. K_0-classes of equivariantly simple objects

8.1. **Notation and content.** Assume that $\lambda+\rho$ is regular and pick a representative μ° of $W\cdot\lambda$ in the anti-dominant p-alcove. The meaning of $G, \underline{G}, L, P, \nu, \chi, W^a, W_P, W^{a,P}, \mu_x, \alpha_0, \ldots, \alpha_r$ is as in Section 2.1 and the meaning of $\tilde{\mathcal{N}}, \tilde{\mathfrak{g}}$ is as in Section 3.1. We write $\underline{\rho}, \rho_L$ for the elements ρ for the Levi subalgebras $\underline{\mathfrak{g}}, \mathfrak{l}$.

In Section 8.3 and some subsequent sections we will introduce some additional notation related to (affine) Hecke algebras.

The goal of this section is to finish the proofs of Theorems 1.1 and 1.2 (see Theorems 8.6 and 8.21 below). In Section 8.2 contains two technical results that describe an interplay between the parabolic induction and the derived localization equivalences. Then in Section 8.4 we prove Theorem 8.6, a stronger version of Theorem 1.1. Then in Section 8.5 we start explaining our approach to proving Theorem 1.2: it is based on the study of the graded lift of the contravariant duality functor $\mathbb{D}$ from Section 7. In this respect it is similar to what was done in [BM], and, in fact, a Koszulity result, Theorem 4.4, from that paper is a crucial part of our approach. Unlike in [BM], we end up with explicit character formulas, which requires a substantial additional work. Sections 8.6-8.9 contain a proof of Theorem 1.2. In Section 8.10 we discuss a relation between the equivariantly irreducible modules and usual irreducible modules. And then in Section 8.11 we speculate on a categorification of Theorem 8.21.

8.2. **Parabolic induction and splitting bundles.** The goal of this section is to establish two results on an interplay of splitting bundles with different instances of parabolic induction functors. The first one plays an important role in the part of the section, where we determine K_0-classes of A-equivariantly irreducible modules in the case when χ is distinguished. Our second result is in the case of general χ, it will be used to extend the computation of irreducible K_0-classes from the distinguished case to the general one.

Recall, Section 6.2, that for a distinguished χ, we have an irreducible component $\mathcal{B}_{\mathfrak{m}} \cong \mathcal{B}_\chi$ that is naturally identified with $P^{(1)}_{\mathbb{F}}/B^{(1)}_{\mathbb{F}}$.

Proposition 8.1. *We have* $R\Gamma(\mathcal{V}_0^\chi|_{\mathcal{B}_{\mathfrak{m}}}) = W^\chi_{\mathbb{F}}(2\rho_L - 2\rho)$.

Our second result gives a geometric interpretation of the baby Verma functor

$$\underline{\Delta}_\nu : \underline{\mathcal{U}}^\chi_{(-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}} \to \mathcal{U}^\chi_{(-2\rho)}\text{-mod}^{\underline{Q}}.$$

We have a natural embedding $\xi : \underline{\tilde{\mathfrak{g}}}^{(1)}_{\mathbb{F}} \hookrightarrow \tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}$ that sends an arbitrary Borel subalgebra $\underline{\mathfrak{b}}_{\mathbb{F}} \subset \underline{\mathfrak{g}}^{(1)}_{\mathbb{F}}$ to $\underline{\mathfrak{b}}_{\mathbb{F}} \oplus \mathfrak{g}^{>0}_{\mathbb{F}}$. This map gives rise to the corresponding embedding of (derived) Springer fibers at χ to be also denoted by ξ. For $\mu \in \mathfrak{X}(T)$, we write $\underline{\mathcal{V}}^\chi_\mu$ for the splitting bundle of $\tilde{D}^\mu_{\underline{\mathcal{B}}_{\mathbb{F}}}$ defined by the formula analogous to (3.3).

Proposition 8.2. *We have the following commutative diagram.*

$$\begin{tikzcd} D^b(\mathrm{Coh}^{\underline{Q}}(\underline{\mathcal{B}}_{\chi})) \arrow[r, "\xi_*"] \arrow[d, "R\Gamma(\underline{\mathcal{V}}^{\chi}_{2\underline{\rho}-2\rho}\otimes\bullet)"'] & D^b(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_{\chi})) \arrow[d, "R\Gamma(\mathcal{V}^{\chi}_0\otimes\bullet)"] \\ D^b(\underline{\mathcal{U}}^{\chi}_{(-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}}) \arrow[r, "\Delta_\nu"] & D^b(\mathcal{U}^{\chi}_{(-2\rho)}\text{-mod}^{\underline{Q}}) \end{tikzcd}$$

Let us explain a general construction that goes into the proofs of these two propositions. Let P' be a parabolic subgroup of G containing B and let L' denote the standard Levi subgroup of P'. Below we take $P'=P$ (for Proposition 8.1) and $P'=G^{\geqslant 0}$ (for Proposition 8.2). Let M' denote the unipotent radical of P'. This, in particular, gives rise to the Verma functor $\Delta^{P'}_{\mathbb{F}}: U(\mathfrak{l}'_{\mathbb{F}})\text{-mod}\to\mathcal{U}_{\mathbb{F}}\text{-mod}$.

We consider the fiberwise lagrangian subvariety $\tilde{Y}^{(1)}_{\mathbb{F}}\subset\tilde{\mathfrak{g}}^{(1)}_{\mathbb{F}}\times\tilde{\mathfrak{l}}'^{(1)}_{\mathbb{F}}$, given by $\{(\mathfrak{b}^{(1)}_{\mathbb{F}},x)|\mathfrak{b}^{(1)}_{\mathbb{F}}\supset\mathfrak{m}'^{(1)}_{\mathbb{F}}\}$ under the natural embedding.

The embedding $\xi_0: P'_{\mathbb{F}}/B_{\mathbb{F}}\hookrightarrow G_{\mathbb{F}}/B_{\mathbb{F}}$ gives rise to the D-module pushforward functor $\xi_{0,*}:\mathrm{Coh}(\tilde{D}_{P'/B,\mathbb{F}})\to\mathrm{Coh}(\tilde{D}_{G/B,\mathbb{F}})$. This functor can be viewed as tensoring with the $\tilde{D}_{G/B,\mathbb{F}}$-$\tilde{D}_{P'/B,\mathbb{F}}$-bimodule $\xi_{0,*}(\tilde{D}_{P'/B,\mathbb{F}})$. We can view $\xi_{0,*}(\tilde{D}_{P'/B,\mathbb{F}})$ as a vector bundle on $\tilde{Y}^{(1)}_{\mathbb{F}}$. This is a splitting bundle for the Azumaya algebra $\left(\tilde{D}_{G/B,\mathbb{F}}\otimes\tilde{D}^{opp}_{P'/B,\mathbb{F}}\right)|_{\tilde{Y}^{(1)}_{\mathbb{F}}}$.

Lemma 8.3. *Let $\chi\in\mathfrak{g}^{(1)*}_{\mathbb{F}}$ be a nilpotent element vanishing on $\mathfrak{m}'^{(1)}_{\mathbb{F}}$ and let $\underline{\chi}$ be the induced element in $\mathfrak{l}'^{(1)*}_{\mathbb{F}}$. We write $Q'_{\mathbb{F}}$ for the reductive part of the centralizer of χ in $P'_{\mathbb{F}}$. Then the restriction of $\xi_{0,*}(\tilde{D}_{P/B,\mathbb{F}})$ to the preimage of $(\mathfrak{g}^{(1)*}_{\mathbb{F}}\times\mathfrak{l}'^{(1)*}_{\mathbb{F}})^{\wedge_{\chi,\underline{\chi}}}$ in $\tilde{Y}^{(1)}_{\mathbb{F}}$ is $\underline{Q}_{\mathbb{F}}$-equivariantly isomorphic to the restriction of*

$$\mathcal{V}^{\chi}_0\otimes(\underline{\mathcal{V}}^{\underline{\chi}}_{2\rho_{L'}-2\rho})^*.$$

Proof. Similarly to the proof of Step 4 of Proposition 7.9 the isomorphism we need to prove will follow once we replace $(\chi,\underline{\chi})$ with the line $\mathbb{F}(\chi,\underline{\chi})$. The latter isomorphism, in its turn, will follow once we know that there is an $L'_{\mathbb{F}}$-equivariant isomorphism between

(i) the restriction of $\xi_{0,*}(\tilde{D}_{P/B,\mathbb{F}})$ to the preimage of $(\mathfrak{g}^{(1)*}_{\mathbb{F}}\times\mathfrak{l}'^{(1)*}_{\mathbb{F}})^{\wedge_{0,0}}$ in $\tilde{Y}^{(1)}_{\mathbb{F}}$,
(ii) and the restriction of $\mathcal{V}^0_0\otimes(\underline{\mathcal{V}}^0_{2\rho_{L'}-2\rho})^*$.

The bundles in (i) and (ii) are still splitting bundles for the same Azumaya algebra, the restriction of $\tilde{D}_{G/B,\mathbb{F}}\otimes\tilde{D}^{opp}_{P/B,\mathbb{F}}$ to the preimage of $(\mathfrak{g}^{(1)*}_{\mathbb{F}}\times\mathfrak{l}'^{(1)*}_{\mathbb{F}})^{\wedge_{0,0}}$ in $\tilde{Y}^{(1)}_{\mathbb{F}}$. So (i) and (ii) differ by a twist with an $L_{\mathbb{F}}$-equivariant line bundle. Such a line bundle is given by a character of the group scheme $B_{\mathbb{F}}P'^{(1)}_{\mathbb{F}}$. The character lattice of this group scheme embeds into the character lattice of $B_{\mathbb{F}}$. So the line bundle of interest is given by a suitable character of $B_{\mathbb{F}}$. Since $\mathfrak{X}(B_{\mathbb{F}})$ is torsion free, to prove that this character is trivial, we need to verify that the K_0-classes of the top exterior powers of the splitting bundles are the same.

Let us start with computing the class for the restriction $\xi_{0*}(\tilde{D}_{P'/B,\mathbb{F}})$. We have

$$[\xi_{0*}(\tilde{D}_{P'/B,\mathbb{F}})]=[\mathrm{Fr}_{\tilde{Y},*}K_{\tilde{Y}_{\mathbb{F}}}].$$

Note that the class of $K_{\tilde{Y}_{\mathbb{F}}}$ in $K_0^{L'_{\mathbb{F}}}(\tilde{Y}_{\mathbb{F}})=\mathfrak{X}(T_{\mathbb{F}})$ is $2(\rho_{L'}-\rho)$. It follows that

$$[\Lambda^{top}\mathrm{Fr}_{\tilde{Y},*}K_{\tilde{Y}_{\mathbb{F}}}]=p^{\dim\tilde{Y}-1}(p-1)(\rho_{L'}-\rho). \tag{8.1}$$

Now we compute the class for the restriction of $\mathcal{V}_0^0 \otimes (\underline{\mathcal{V}}^0_{2\rho_{L'}-2\rho})^*$. Its the same as that of the restriction to P'/B. Part (1) of Lemma 3.5 implies that this restriction is

$$\mathrm{Fr}_{\mathcal{B},\mathbb{F}}(\mathcal{O}_{\mathcal{B},\mathbb{F}}) \otimes \left(\mathrm{Fr}_{P'/B,*}(\mathcal{O}_{P'/B}(2\rho_{L'} - 2\rho))\right)^* . \tag{8.2}$$

Since the canonical bundle of $\mathcal{B}$ is $\mathcal{O}(-2\rho)$ and the equivariant canonical bundle of P'/B is $\mathcal{O}(-2\rho_{L'})$, an easy computation shows that the top degree exterior power of (8.2) coincides with (8.1). This finishes the proof. □

We will also need the following standard lemma.

Lemma 8.4. *We have* $\Gamma(\xi_{0*}(\tilde{D}_{P'/B,\mathbb{F}})) = \mathcal{U}_{\mathfrak{h},\mathbb{F}}/\mathcal{U}_{\mathfrak{h},\mathbb{F}}\mathfrak{m}'_{\mathbb{F}}$.

Now we proceed to proving the two propositions in the beginning of the section. Proposition 8.2 is a straightforward corollary of Lemmas 8.3,8.4.

Proof of Proposition 8.1. Thanks to Lemma 8.3, $\mathcal{V}_0^\chi|_{\mathcal{B}_\mathfrak{m}}$ is naturally identified with the fiber at χ of

$$\xi_{0*}(\tilde{D}_{P/B,\mathbb{F}}) \otimes_{\tilde{D}_{P/B,\mathbb{F}}} \underline{\mathcal{V}}^0_{2\rho_L - 2\rho}|_{P^{(1)}_{\mathbb{F}}/B^{(1)}_{\mathbb{F}}}.$$

Note that $\xi_{0*}(\tilde{D}_{P/B,\mathbb{F}})$ is a free right $\tilde{D}_{P/B,\mathbb{F}}$-module. It follows that

$$R\Gamma\left(\xi_{0*}(\tilde{D}_{P/B,\mathbb{F}}) \otimes_{\tilde{D}_{P/B,\mathbb{F}}} \underline{\mathcal{V}}^0_{2\rho_L - 2\rho}|_{P^{(1)}_{\mathbb{F}}/B^{(1)}_{\mathbb{F}}}\right) \xrightarrow{\sim} \Gamma\left(\xi_{0*}(\tilde{D}_{P/B,\mathbb{F}})\right) \otimes_{\Gamma(\tilde{D}_{P/B,\mathbb{F}})} R\Gamma\left(\underline{\mathcal{V}}^0_{2\rho_L - 2\rho}|_{P^{(1)}_{\mathbb{F}}/B^{(1)}_{\mathbb{F}}}\right).$$

Now we use Lemma 8.4 to show that

$$R\Gamma(\mathcal{V}_0^\chi|_{\mathcal{B}_{0,\chi}}) = \underline{\Delta}^\chi(R\Gamma(\mathcal{V}^0_{2\rho_L - 2\rho}|_{P^{(1)}_{\mathbb{F}}/B^{(1)}_{\mathbb{F}}})).$$

Then we use (1) of Lemma 3.5 that says, in particular, that

$$\mathcal{V}^0_{2\rho_L - 2\rho}|_{P^{(1)}_{\mathbb{F}}/B^{(1)}_{\mathbb{F}}} = \mathrm{Fr}_{P/B,*}\,\mathcal{O}_{P/B}(2\rho_L - 2\rho).$$

So we see that $R\Gamma(\mathcal{V}_0^\chi|_{\mathcal{B}_\mathfrak{m}}) = W^\chi_{\mathbb{F}}(2\rho_L - 2\rho)$. □

8.3. **Reminder on affine Hecke algebras.** Consider the affine Hecke algebra $\mathcal{H}^a_G$ for G over $\mathbb{Z}[v^{\pm1}]$, where v is an indeterminate. For $x \in W^a$, let H_x denote the standard basis element of $\mathcal{H}^a_G$. Recall that the product on $\mathcal{H}^a_G$ is determined by

$$H_x H_y = H_{xy} \text{ if } \ell(xy) = \ell(x) + \ell(y),$$
$$(H_s + v)(H_s - v^{-1}) = 0,$$

where s is a simple affine reflection.

The Hecke algebra $\mathcal{H}^a_G$ comes with a $\mathbb{Z}[v^{\pm1}]$-linear ring involution, called the bar-involution and denoted by $\bar{\bullet}$, it is given by $\bar{H}_x := H^{-1}_{x^{-1}}$. As Kazhdan and Lusztig checked in [KL], there is a unique basis $C_x, x \in W^a$, of $\mathcal{H}^a_G$ with the following two properties:

- $\bar{C}_x = C_x$ for all $x \in W^a$,
- and $C_x - H_x \in v^{-1}\operatorname{Span}_{\mathbb{Z}[v^{-1}]}(H_y | y \in W^a)$ for all $x \in W^a$.

Then $C_x = \sum_{y \preceq x} c_{xy}(v) H_y$, where $\prec$ stands for the Bruhat order and $c_{xy} \in \mathbb{Z}[v^{-1}]$ is a Kazhdan-Lusztig polynomial.

We will need a parabolic version of this construction. Let P be a parabolic subgroup of G. The Hecke algebra $\mathcal{H}_{W_P}$ of the parabolic subgroup $W_P \subset W^a$ embeds into $\mathcal{H}^a_G$. Consider the sign representation $\mathrm{sgn}_P \cong \mathbb{Z}[v^{\pm1}]$ of $\mathcal{H}_{W_P}$, where H_w acts via $(-v)^{\ell(w)}$, and the induced module $\mathcal{H}^{a,P}_G := \mathcal{H}^a_G \otimes_{\mathcal{H}_{W_P}} \mathrm{sgn}_P$. For $x \in W^{a,P}$, define H^P_x as $H_x \otimes 1$. We note that sgn_P embeds into $\mathcal{H}^a_{W_L}$ via $1 \mapsto C_{w_{0,P}} = \sum_{w \in W_P} (-v)^{-\ell(ww_{0,P})} H_w$. This gives rise to an embedding

$\mathcal{H}_G^{a,P} \hookrightarrow \mathcal{H}_G^a$ so that C_x for $x \in W^{a,P}$ lies in the image. So, for $x \in W^{a,P}$ we can expand C_x as $\sum_{y\in W^{a,P}} c^P_{xy}(v)H^P_y$. The coefficient c^P_{xy} is known as a parabolic Kazhdan-Lusztig polynomial.

Let us now recall the representation theoretic meaning of the values $c^P_{xy}(1)$. For $x \in W^{a,P}$, let Δ^P_x, L^P_x denote the standard and simple objects in $\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P)$. Then the classes in K_0 are related as follows:

$$[L^P_x] = \sum_{y\in W^{a,P}} c^P_{xy}(1)[\Delta^P_y]. \tag{8.3}$$

Now recall that, for $\theta \in \mathfrak{X}(T)$, we have the corresponding element $X_\theta \in \mathcal{H}^a_G$. We have

$$\bar{X}_\theta = H_{w_0} X_{w_0(\theta)} H^{-1}_{w_0}. \tag{8.4}$$

Also we will need the coherent geometric realization of $\mathcal{H}^a_G$ and the corresponding formula for the bar-involution. Namely, consider the $\mathbb{C}^\times$-action on $\mathfrak{g}$ via $(t, x) := t^{-2}x$. This gives rise to the action of $\mathbb{C}^\times$ on $\mathsf{St}_\mathfrak{h}$. Consider the equivariant K_0-group $K_0^{G\times\mathbb{C}^\times}(\mathsf{St}_\mathfrak{h})$, which is an algebra with respect to convolution. It is also a module over $K_0^{\mathbb{C}^\times}(\mathrm{pt})$ and hence a $\mathbb{Z}[v^{\pm1}]$-algebra. By a theorem of Kazhdan-Lusztig and Ginzburg, see [CG, Section 7], we have a $\mathbb{Z}[v^{\pm1}]$-algebra isomorphism $\mathcal{H}^a_G \cong K_0^{G\times\mathbb{C}^\times}(\mathsf{St}_\mathfrak{h})$ that sends X_θ to the class of the line bundle $\mathcal{O}(\theta)$ on the diagonal.

Consider the ς-twisted and $\dim\mathfrak{g}$-shifted Serre duality functor

$$\tilde{\mathbb{D}}_{coh} := {}^{\varsigma}R\operatorname{Hom}_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}(\bullet, K_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}})[\dim\mathfrak{g}] : D^b(\mathrm{Coh}^{G\times\mathbb{C}^\times}(\mathsf{St}_\mathfrak{h})) \xrightarrow{\sim} D^b(\mathrm{Coh}^{G\times\mathbb{C}^\times}(\mathsf{St}_\mathfrak{h}))^{opp},$$

where we consider $K_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}$ with its natural $G\times\mathbb{C}^\times$-equivariant structure. The following result is [Lu3, Proposition 9.12] (note that our H_x is $\tilde{T}_x^{-1}$).

Lemma 8.5. *Under the identification $\mathcal{H}^a_G \cong K_0^{G\times\mathbb{C}^\times}(\mathsf{St}_\mathfrak{h})$, we have*

$$\bar{a} = v^{-2\ell(w_0)} H_{w_0}[\tilde{\mathbb{D}}_{coh}](a)H^{-1}_{w_0}.$$

Here and below we write $[\tilde{\mathbb{D}}_{coh}]$ for the operator on K_0 induced by $\tilde{\mathbb{D}}_{coh}$.

8.4. **Character formulas in the distinguished case.** We assume that χ is distinguished. Let $w_{0,P}$ be the longest element in W_P. Consider the left cell $\mathfrak{c}_P \subset W^a$ containing $w_{0,P}$. It is contained in $W^{a,P}$.

Let $V_L(\mu_x)$ denote the irreducible representation of L (over $\mathbb{C}$) with highest weight μ_x. Let $V_L(\mu_x)$ be the irreducible L-module with highest weight μ_x and $\hat{d}_L(x)$ be its $\underline{Q}$-character. We write ch_{M^-} for the $\underline{Q}$-character of $U^0(\mathfrak{m}^-_\mathbb{F})$.

The following is one of the main results of the paper. In particular, it implies Theorem 1.1.

Theorem 8.6. *Let $\lambda+\rho$ be regular and χ be distinguished. The following claims are true:*

(1) *There is a bijection between $\mathfrak{c}_P$ and $\mathrm{Irr}(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}})$.*
(2) *For the simple object $L^\chi_{x,\mathbb{F}} \in \mathrm{Irr}(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}})$ corresponding to $x \in \mathfrak{c}_P$, the multiplicity of $L^\chi_{x,\mathbb{F}}$ in $W^\chi_\mathbb{F}(\mu_y)$ coincides with the multiplicity of L^P_x in Δ^P_y.*
(3) *The classes $[W^\chi_\mathbb{F}(\mu_y)], y \in W^{a,P}$, span $K_0(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}})$.*
(4) *If $x \in \mathfrak{c}_P$, then $\sum_{y\in W^{a,P}} c^P_{xy}(1)\mathsf{ch}_{M^-}[W^\chi_\mathbb{F}(\mu_y)] = [L^\chi_{x,\mathbb{F}}]$.*
(5) *If $x \notin \mathfrak{c}_P$, then $\sum_{y\in W^{a,P}} c^P_{xy}(1)[W^\chi_\mathbb{F}(\mu_y)] = 0$.*

Proof. In the proof we can assume that $\lambda = 0$ thanks to the translation functors. Recall that the restriction of $\mathcal{T}_{\mathbb{F}}(-\rho)$ to $\tilde{\mathfrak{g}}_{\mathbb{F}}^{(1)\wedge_\chi}$ has the same indecomposable summands as $\mathcal{V}_0^\chi$, Lemma 4.6. It follows that $K_0(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}})$ is identified with $K_0(\mathcal{A}_e\text{-mod}^{\underline{Q}})$. Both are identified with $K_0(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_e))$ and these identifications are W^a-equivariant. Thanks to Proposition 8.1, the class of $R\Gamma(\mathcal{T}(-\rho)|_{\mathcal{B}_{\mathfrak{m}}})$ in $K_0(\mathcal{A}_e\text{-mod}^{\underline{Q}})$ is mapped to $[W^\chi_{\mathbb{F}}(2\rho_L - 2\rho)] \in K_0(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}})$. Then we use Corollary 6.3 to get a W^a-equivariant surjective map

$$K_0(\mathrm{Perv}_{I^\circ}(\mathcal{F}l_P)) \twoheadrightarrow K_0(\mathcal{U}^\chi_{\lambda,\mathbb{F}}\text{-mod}^{\underline{Q}}).$$

It maps $[L^P_x]$ to a class of a simple object if $x \in \mathfrak{c}_P$ or zero else. This proves (1). Thanks to Lemma 2.5, the image of $[\Delta^P_y]$ is $[W^\chi_{\mathbb{F}}(\mu_y)]$ for all $y \in W^{a,P}$. This observation together with the rest of the proof now implies (2)-(5). □

8.5. **Canonical basis.** Now we assume that χ is a general nilpotent element. The goal of this section is to explain a general approach to computing the multiplicities of the simple objects in $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}$ in the χ-Weyl modules $W^\chi_{\mathbb{F}}(\mu_w), w \in W^{a,P}$. This general approach is a ramification of what was used in [BM].

Let $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}$ denote the Serre subcategory of $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}$ consisting of all modules M such that $\mathfrak{t}_{0,\mathbb{F}}$ acts on the graded component M_v by v mod p for all characters v of $\underline{Q}^\circ = T_0$. Consider the completion $\hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0})$ of $K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0})$ consisting of all infinite sums of the form

$$\sum_{\mu|\langle\mu,\nu\rangle\leqslant N} a_\mu[W^\chi_{\mathbb{F}}(\mu)]$$

for some $N \in \mathbb{Z}_{>0}$.

Lemma 8.7. *The following claims are true.*

1. *The classes of simples form a topological basis in $\hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0})$.*
2. *The classes $[\underline{\Delta}_\nu(\underline{\mathcal{L}})]$ where $\underline{\mathcal{L}}$ runs over the set of simples in the categories of the form $\underline{\mathcal{U}}^\chi_{w\cdot 0}\text{-mod}^{\underline{Q},0}$, form a topological basis in $\hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0})$.*
3. *The classes $[W^\chi_{\mathbb{F}}(x\cdot(-2\rho))]$ for $x \in W^{a,P}$ form a topological generating set for $\hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})$.*

Proof. Part (1) is obvious, part (2) is standard, and part (3) follows from (3) of Theorem 8.6. □

Note that the topological bases in (1) and (2) are both labelled by $\mathrm{Irr}(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})$: we write $\underline{\Delta}_{\mathcal{L}}$ for the unique object of the form $\underline{\Delta}_\nu(\underline{\mathcal{L}})$ with an epimorphism onto $\mathcal{L}$.

Recall, Theorem 4.4, that the algebra $\mathcal{A}_{\mathfrak{h}}|_S$ is Koszul. This yields a positive grading on the specialization $\mathcal{A}_{\mathfrak{h},e}$. The grading is Q-stable, see Remark 4.5, which gives a graded lift of $\mathcal{A}_{\mathfrak{h},e}\text{-mod}^{\underline{Q}}$ to be denoted by $\mathcal{A}_{\mathfrak{h},e}\text{-mod}^{\underline{Q},gr}$. Since the grading on $\mathcal{A}_{\mathfrak{h}}|_S$ comes from a $\mathbb{C}^\times$-equivariant structure on $\mathcal{T}_{\mathfrak{h}}|_S$, we get a derived equivalence

$$D^b(\mathcal{A}_{\mathfrak{h},e}\text{-mod}^{\underline{Q},gr}) \xrightarrow{\sim} D^b(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)). \tag{8.5}$$

Thanks to Lemma 4.6, the algebras $(\mathcal{A}_{\mathfrak{h},e})_{\mathbb{F}}$ and $\mathcal{U}^\chi_{(0),\mathbb{F}}$ are Morita equivalent. Under this Morita equivalence, the shift of $\mathfrak{X}(T_0)$-grading by v in $(\mathcal{A}_{\mathfrak{h},e})_{\mathbb{F}}\text{-mod}^{\underline{Q}}$ corresponds to a shift by pv in $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}$. So the Morita equivalence gives rise to $(\mathcal{A}_{\mathfrak{h},e})_{\mathbb{F}}\text{-mod}^{\underline{Q}} \xrightarrow{\sim} \mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}$. From this equivalence we get a graded lift $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr}$ of $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}$. We can also assume that the grading on $\mathcal{U}^{\chi}_{(0),\mathbb{F}}$ corresponding to the positive grading on $(\mathcal{A}_{\mathfrak{h},e})_{\mathbb{F}}$ is also positive. We get an equivalence

$$D^b(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr}) \xrightarrow{\sim} D^b(\mathrm{Coh}^{\underline{Q}^{(1)}\times\mathbb{F}^\times}(\mathcal{B}_\chi)) \tag{8.6}$$

intertwining the grading shift functors (to be denoted by $\langle 1\rangle$).

Recall the contravaiant equivalence $\mathbb{D}:\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}\to\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},opp}$.

Lemma 8.8. *We have $\mathbb{D}\mathcal{L}\cong\mathcal{L}$ for all simple objects in $\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}$.*

Proof. This follows from (3) of Lemma 8.7 combined with Proposition 7.5. □

Now we proceed to discussing graded lifts for the objects $\mathcal{L},\underline{\Delta}_{\mathcal{L}}$, and an equivalence $\mathbb{D}$. First of all, the equivalences $\mathbb{D}_{coh}$ and $T_{w_0}^{-1}$ both admit natural graded lifts to $D^b(\mathrm{Coh}^{\underline{Q}\times\mathbb{F}^\times}(\mathcal{B}_\chi))$ to be denoted by $\tilde{\mathbb{D}}_{coh}$ and $\tilde{T}_{w_0}^{-1}$. Note that the latter intertwines the grading shift functors, while the former intertwines $\langle 1\rangle$ with $\langle -1\rangle$. Transferring $(\tilde{T}_{w_0}^{-1}\circ\tilde{\mathbb{D}}_{coh})\langle -\ell(w_0)\rangle$ to $D^b(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ using (8.6) we get a self-equivalence

$$\tilde{\mathbb{D}}: D^b(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr}) \xrightarrow{\sim} D^b(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})^{opp}$$

that intertwines $\langle 1\rangle$ with $\langle -1\rangle$. By Proposition 7.9, $\tilde{\mathbb{D}}$ is a graded lift of $\mathbb{D}$.

Rescaling the grading, if necessary, we achieve, thanks to Lemma 8.8, that each $\mathcal{L}\in\mathrm{Irr}(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}})$ admits a unique graded lift $\tilde{\mathcal{L}}\in\mathrm{Irr}(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ such that $\tilde{\mathbb{D}}(\tilde{\mathcal{L}})=\tilde{\mathcal{L}}$.

Lemma 8.9. *We have a unique graded lift $\tilde{\underline{\Delta}}_{\mathcal{L}}$ of $\underline{\Delta}_{\mathcal{L}}$ with $\tilde{\underline{\Delta}}_{\mathcal{L}}\twoheadrightarrow\tilde{\mathcal{L}}$. The simple constituents of the kernel $\tilde{\underline{\Delta}}_{\mathcal{L}}\twoheadrightarrow\tilde{\mathcal{L}}$ are of the form $\tilde{\mathcal{L}}'\langle -i\rangle$ for $i>0$.*

Proof. Let $\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}_{\leqslant i}$ denote the Serre subcategory of $\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}$ spanned by all simples $\mathcal{L}$ with ν-highest weights $\leqslant i$. Let $\pi_{\leqslant i}:\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}_{\leqslant i}\twoheadrightarrow \mathrm{gr}_i\,\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0}$ be the quotient functor and let $\pi^!_{\leqslant i}$ be its left adjoint. Then $\underline{\Delta}_{\mathcal{L}}$ is nothing else but $\pi^!_{\leqslant i}\pi_{\leqslant i}(\mathcal{L})$. The existence of $\tilde{\underline{\Delta}}_{\mathcal{L}}$ follows from here. And the claim on the simple constituents follows from $\mathcal{U}^{\chi}_{(0),\mathbb{F}}$ being positively graded. □

Consider the completed K_0-group $\hat{K}_0(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ defined similarly to $\hat{K}_0(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},0})$. This is a topological $\mathbb{Z}[v^{\pm 1}]$-module, where $v=[\langle 1\rangle]$. Both families $[\tilde{\underline{\Delta}}_{\mathcal{L}}]$ and $[\tilde{\mathcal{L}}]$ are topological bases of $\hat{K}_0(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$.

Our goal is to express the basis $[\tilde{\mathcal{L}}]$ via the classes $H_x[\tilde{W}^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)]$ (for $x\in W^{a,P}$): note that $W^{\chi}_{\mathbb{F}}(2\rho_L-2\rho)$ is of the form $\underline{\Delta}_{\mathcal{L}}$ so it makes sense to speak about the graded lift. This will be done as follows. First, we will study an action of the affine Hecke algebra of G on $\hat{K}_0(\mathcal{U}^{\chi}_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ and explicitly describe the module structure and the classes $[\tilde{\underline{\Delta}}_{\mathcal{L}}]$, Section 8.6. Then we describe the action of $[\tilde{\mathbb{D}}]$ on this module, Section 8.7, this is the main part. In Section 8.8 we introduce semi-periodic affine Kazhdan-Lusztig polynomials. Finally, in Section 8.9 we fully compute the basis of simples.

8.6. **Module structure.** The affine Hecke algebra $\mathcal{H}^a_G$ contains $\mathcal{H}^a_{\underline{G}}$ as a subalgebra and is a free right $\mathcal{H}^a_{\underline{G}}$-module with basis H_u, where u runs over the elements of W with u shortest in $uW_{\underline{G}}$. This subset of W will be denoted by $W^{\underline{G},-}$.

Recall that $\mathcal{H}^a_G$ gets identified with $K_0^{G\times\mathbb{C}^\times}(\mathsf{St}_{\mathfrak{h}})$. The action of $D^b(\mathrm{Coh}^{G\times\mathbb{C}^\times}(\mathsf{St}_{\mathfrak{h}}))$ on $D^b(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e))$ gives rise to a left $\mathcal{H}^a_G$-action on $K_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e) = K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$. The class $[\tilde{T}_x^{-1}]$ acts by H_x.

First, we need a result on a compatibility of dualities. Let $a\mapsto \bar{a}$ denote the standard bar-involution on $\mathcal{H}^a_G$.

Proposition 8.10. *For $a\in\mathcal{H}^a_G$ and $m\in K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$. Then $[\tilde{\mathbb{D}}](am) = \bar{a}([\tilde{\mathbb{D}}]m)$.*

Proof. Recall, Lemma 8.5, that

$$\bar{a} = v^{-2\ell(w_0)}H_{w_0}[\tilde{\mathbb{D}}_{coh}](a)H_{w_0}^{-1}.$$

So our claim boils down to

$$[\mathbb{D}_{coh}](am) = v^{-2\ell(w_0)}\left([\tilde{\mathbb{D}}_{coh}]a\right)\left([\tilde{\mathbb{D}}_{coh}]m\right).$$

This is [Lu3, Lemma 9.5]. $\square$

Our next goal is to describe the abelian group $K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ as an $\mathcal{H}^a_G$-module and also compute the classes $[\tilde{\Delta}_{\mathcal{L}}]$.

Let $\mathfrak{C}_{\underline{P}}$ be the left cell module over $\mathcal{H}^a_{\underline{G}}$ corresponding to the left cell $\mathfrak{c}_{\underline{P}}$. This left cell labels the simples in $\underline{\mathcal{U}}^\chi_{(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}}$, thanks to (1) of Theorem 8.6. For $x\in\mathfrak{c}_{\underline{P}}$ we write $C^{\mathfrak{C}}_x$ labeled by x for the Kazhdan-Lusztig basis element in the natural $\mathbb{Z}[v^{-1}]$-lattice of $\mathfrak{C}_{\underline{P}}$. We write $\hat{\mathfrak{C}}_{\underline{P}}$ for the completion of $\mathfrak{C}_{\underline{P}}$, it consists of all sums

$$\sum_{\langle\nu,x\rangle\leqslant N} a_x C^{\mathfrak{C}}_x,$$

where we abuse the notation and write $\langle\nu,x\rangle$ for the pairing of ν with the projection of x to $\mathfrak{X}(\underline{G})$.

Consider the $\mathcal{H}^a_G$-modules

$${}^G\mathfrak{C}_{\underline{P}} := \mathcal{H}^a_G\otimes_{\mathcal{H}^a_{\underline{G}}}\mathfrak{C}_{\underline{P}},\ {}^G\hat{\mathfrak{C}}_{\underline{P}} := \mathcal{H}^a_G\otimes_{\mathcal{H}^a_{\underline{G}}}\hat{\mathfrak{C}}_{\underline{P}}$$

and the elements $M_{u,x} := H_u C^{\mathfrak{C}}_x \in {}^G\hat{\mathfrak{C}}_{\underline{P}}$, where $u\in W^{\underline{G},-}$, the set of shortest coset representatives for the action of $W_{\underline{G}}$ on the right, $x\in\mathfrak{c}_{\underline{P}}$. Note that the elements $M_{u,x}$ form a topological basis in ${}^G\hat{\mathfrak{C}}_{\underline{P}}$.

Let $\mathcal{L}$ be a simple object in $\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q}}$ and $\underline{\mathcal{L}}$ be the corresponding simple in

$$\bigoplus_{w\in W^{\underline{G},-}}\underline{\mathcal{U}}^\chi_{w\cdot(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q},0}.$$

Suppose $\underline{\mathcal{L}}\in\mathrm{Irr}(\underline{\mathcal{U}}^\chi_{2\underline{\rho}-2\rho,\mathbb{F}}\text{-mod}^{\underline{Q},0})$. Let $x\in\mathfrak{c}_{\underline{P}}$ be the Weyl group element labelling $\underline{\mathcal{L}}$.

Lemma 8.11. *There is an $\mathcal{H}^a_G$-linear map $\upsilon: {}^G\mathfrak{C}_{\underline{P}}\to K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ such that, in the notation of the previous paragraph, $\upsilon(M_{1,x}) = [\underline{\tilde{\Delta}}_{\mathcal{L}}]$.*

Proof. The proof is in two steps.

Step 1. Recall that we identify $K_0(\underline{\mathcal{U}}^\chi_{(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ with $K_0^{\underline{Q}\times\mathbb{F}^\times}(\underline{\mathcal{B}}_\chi)$ (using the splitting bundle $\underline{\mathcal{V}}^\chi_{2\underline{\rho}-2\rho}$). We claim that we have an isomorphism of based $\mathcal{H}^a_{\underline{G}}$-modules

$$\underline{\upsilon}:\mathfrak{C}_{\underline{P}}\xrightarrow{\sim} K_0(\underline{\mathcal{U}}^\chi_{(2\rho_L-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q},gr}).$$

First of all, note that $C^{\underline{P}}_x$ generates the $\mathcal{H}^a_{\underline{G}}$-module $\mathfrak{C}_{\underline{P}}$ so there can be at most one $\mathcal{H}^a_{\underline{G}}$-linear map $\underline{v}$ with $\underline{v}(C^{\underline{P}}_{w_{0,P}}) = [\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)]$. Let us show that such a module homomorphism exists.

First, let us show that we have a homomorphism

$$\mathcal{H}^a_{\underline{G}} \otimes_{\mathcal{H}_{W_P}} \mathrm{sgn}_P \to K_0(\underline{\mathcal{U}}^\chi_{2\rho_L-2\rho,\mathbb{F}}\text{-mod}^{\underline{Q},gr}) \tag{8.7}$$

sending $C^{\underline{P}}_{w_{0,P}}$ to $[\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)]$. Note that for a simple reflection s in W_P, the functor T_s homologically shifts $\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)$ (this is already the case with the corresponding parabolic Verma module). So $H_s[\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)]$ is of the form $-v^?[\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)]$. We conclude $(H_s + v)[\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)] = 0$. This gives (8.7).

Now we prove that (8.7) factors through the projection $\mathcal{H}^a_{\underline{G}} \otimes_{\mathcal{H}_{W_P}} \mathrm{sgn}_P \twoheadrightarrow \mathfrak{C}_{\underline{P}}$. It follows from the $\mathbb{C}^\times$-equivariant version of [B, Theorem 55] (proved in the same way as the version there) that the $\mathcal{H}^a_{\underline{G}}$-action on $K_0^{\underline{Q}\times\mathbb{F}^\times}(\underline{\mathcal{B}}_\chi) \cong K_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e)$ belongs to the two-sided cell corresponding to $\underline{G}e$. And $\mathfrak{C}_{\underline{P}}$ is the largest quotient of $\mathcal{H}^a_{\underline{G}} \otimes_{\mathcal{H}_{W_P}} \mathrm{sgn}_P$ that belongs to that two-sided cell.

Step 2. We map

$$K_0(\mathcal{U}^\chi_{(2\rho_L-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q},gr}) \xrightarrow{\sim} K_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e)$$

to

$$K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr}) \xrightarrow{\sim} K_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$$

via ξ_*. The map ξ_* is easily seen to be a $\mathcal{H}^a_{\underline{G}}$-linear. So ξ_* induces a $\mathcal{H}^a_G$-linear map ${}^G\mathfrak{C}_{\underline{P}} \to K_0(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e))$. By Proposition 8.2, it indeed sends $M_{1,x}$ to $[\underline{\tilde{\Delta}}_{\mathcal{L}}]$. □

Let $\mathcal{L}$ be such as in the lemma and let $u \in W^{\underline{G},-}$. Note that the element $u \cdot (2\underline{\rho} - 2\rho)$ is dominant for $\underline{G}$. The categories $\underline{\mathcal{U}}^\chi_{(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}}$ and $\underline{\mathcal{U}}^\chi_{u^{-1}\cdot(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}}$ are identified by means of the translation functor from $2\underline{\rho}-2\rho$ to $u^{-1}\cdot(2\underline{\rho}-2\rho)$. For $\underline{\mathcal{L}} \in \mathrm{Irr}(\underline{\mathcal{U}}^\chi_{(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}})$ we write $\underline{\mathcal{L}}^u$ for the corresponding simple object in $\underline{\mathcal{U}}^\chi_{u\cdot(2\underline{\rho}-2\rho),\mathbb{F}}\text{-mod}^{\underline{Q}}$.

Our next result is as follows.

Proposition 8.12. *Let $u, \mathcal{L}, \mathcal{L}^u$ have the same meaning as above. Then*

$$\tilde{T}_u^{-1}\underline{\tilde{\Delta}}_{\mathcal{L}} \cong \underline{\tilde{\Delta}}_{\mathcal{L}^u}. \tag{8.8}$$

Proof. First, we prove the ungraded analog of (8.8). We prove that by induction with respect to the Bruhat order on $W^{\underline{G},-}$. For $u = 1$, there is nothing to prove. Now we assume that we know that claim for all u' such that $u' \prec u$. Note that there is a simple reflection s such that $su \prec u, su \in W^{\underline{G},-}$. So we need to check that $\tilde{T}_s^{-1}\underline{\Delta}_{\mathcal{L}^{su}} \cong \underline{\Delta}_{\mathcal{L}^u}$.

Step 1. Set $\lambda_1 := (su)^{-1}\cdot(2\underline{\rho} - 2\rho)$ and $\lambda_2 := u^{-1}\cdot(2\underline{\rho} - 2\rho)$. Let us write $\underline{\mathcal{U}}_{\lambda_1\to\lambda_2}$ for the translation $\underline{\mathcal{U}}_{\lambda_2}$-$\underline{\mathcal{U}}_{\lambda_1}$-bimodule (over $\mathbb{C}$), this is a Morita equivalence bimodule. We claim that

$$T_s^{-1}\Delta^P(\underline{\mathcal{U}}_{\lambda_1}) \cong \Delta^P(\underline{\mathcal{U}}_{\lambda_1\to\lambda_2}). \tag{8.9}$$

Recall that $T_s\Delta^P(\underline{\mathcal{U}}_{\lambda_1})$ is given by the complex

$$\mathsf{T}^*\mathsf{T}\Delta^P(\underline{\mathcal{U}}_{\lambda_1}) \to \Delta^P(\underline{\mathcal{U}}_{\lambda^1}),$$

where T is a translation to the wall given by s and the second term is in homological degree 0. Since $su \prec u$, we see that $\mathsf{T}^*\mathsf{T}\Delta^P(\underline{\mathcal{U}}_{\lambda_1}) = \Delta^P(M)$, where M is a $\underline{\mathcal{U}}$-module that fits to the exact sequence

$$0 \to \underline{\mathcal{U}}_{\lambda_1 \to \lambda_2} \to M \to \underline{\mathcal{U}}_{\lambda_1} \to 0.$$

The homomorphism $\Delta^P(M) \to \Delta^P(\mathcal{U}_{\lambda_1})$ induced by the second arrow in the exact sequence above is surjective, so we see that (8.9) indeed holds.

Step 2. Note that (8.9) is defined over $\mathbb{Q}$ and hence can be reduced mod p for $p \gg 0$. Note that

$$T_s\underline{\Delta}_\nu(\mathcal{L}^{su}) \cong T_s\Delta^P(\underline{\mathcal{U}}_{\lambda_1,\mathbb{F}}) \otimes^L_{\underline{\mathcal{U}}_{\lambda_1,\mathbb{F}}} L^{su} \cong \Delta^P(\underline{\mathcal{U}}_{\lambda_1 \to \lambda^2,\mathbb{F}}) \otimes^L_{\underline{\mathcal{U}}_{\lambda_1,\mathbb{F}}} \cong \underline{\Delta}_\nu(\mathcal{L}^u).$$

This is precisely the ungraded version of (8.8).

Step 3. By Step 2, both $T_u^{-1}\tilde{\underline{\Delta}}_{\mathcal{L}}, \tilde{\underline{\Delta}}_{\mathcal{L}^u}$ are graded lifts of $\underline{\Delta}_{\mathcal{L}^u}$. The module $\underline{\Delta}_{\mathcal{L}^u}$ is indecomposable so its graded lifts differ by a shift of grading. Note that

(i) the difference of the ν-highest weights of L^u and L is small comparing to p,

(ii) and if $u_1 \prec u_2$, then the difference of the ν-highest weights of $\mathcal{L}^{u_2}$ and $\mathcal{L}^{u_1}$ is positive.

The element $H_u - \overline{H}_u \in \mathcal{H}_W$ is a linear combination of $H_{u'}$ for $u' \prec u$. Using Proposition 8.10, we see that $[\tilde{\mathbb{D}}]H_u[\tilde{\underline{\Delta}}_{\mathcal{L}}] - H_u[\tilde{\mathbb{D}}][\tilde{\underline{\Delta}}_{\mathcal{L}}]$ is a linear combination of $H_{u'}[\tilde{\mathbb{D}}][\tilde{\underline{\Delta}}_{\mathcal{L}}]$ with $u' \prec u$. Combining this observation with (i) and (ii), we see that $[H_u\tilde{\underline{\Delta}}_{\mathcal{L}}]$ is the sum of $[\tilde{\mathcal{L}}]$ and the classes with smaller ν-highest weights. We conclude that the shift of grading from $\tilde{T}_u^{-1}\tilde{\underline{\Delta}}_{\mathcal{L}}$ to $\tilde{\underline{\Delta}}_{L^u}$ is trivial and hence $\tilde{T}_u^{-1}\tilde{\underline{\Delta}}_{\mathcal{L}} \cong \tilde{\underline{\Delta}}_{L^u}$. □

Corollary 8.13. *Under the linear map ${}^G\mathfrak{C}_{\underline{P}} \to K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$ from Lemma 8.11, the element $M_{u,x}$ is mapped to $[\tilde{\underline{\Delta}}_{\mathcal{L}^u}]$, where x labels $\mathcal{L}$. In particular, this map induces an isomorphism ${}^G\hat{\mathfrak{C}}_{\underline{P}} \to \hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$.*

8.7. **Computation of $[\tilde{\mathbb{D}}]$.** In this section we explain how to compute the involution $[\tilde{\mathbb{D}}]$ of $K_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$. Note that $[\tilde{\mathbb{D}}]$ is continuous so it extends a semi-linear automorphism of $\hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr})$. Corollary 8.13 provides an identification

$$\hat{K}_0(\mathcal{U}^\chi_{(0),\mathbb{F}}\text{-mod}^{\underline{Q},gr}) \cong {}^G\hat{\mathfrak{C}}_{\underline{P}}.$$

By the definition, as a topological $\mathcal{H}^a_G$-module, the right hand side is generated by $\mathfrak{C}_{\underline{P}}$. Thanks to Proposition 8.10, it is enough to compute $[\tilde{\mathbb{D}}]$ on $\mathfrak{C}_{\underline{P}}$.

Let us state the answer. Let $\underline{w}_0$ denote the longest element in $W_{\underline{G}}$. Set $u_{\underline{G}} := \underline{w}_0^{-1}w_0$. For $\theta \in \mathfrak{X}(P)$, the notation $\theta \to +\infty$ means that $\langle \theta, \alpha_i^\vee \rangle \to +\infty$ for all simple roots α_i of G that are not roots of $\underline{G}$. For $\theta \in \mathfrak{X}(T)$, set $\theta^* := -w_0(\theta)$.

Proposition 8.14. *For $m \in \Sigma_{P^0}$, the limit*

$$\lim_{\theta \to +\infty} H_{u_{\underline{G}}} X_{-\theta^*} H^{-1}_{u_{\underline{G}}} X_{-\theta}[\tilde{\mathbb{D}}]m$$

exists in $\hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$ and is equal to $[\tilde{\mathbb{D}}]m$.

The proof will be given after a series of lemmas. Recall the inclusion $\xi : \underline{\mathcal{B}}_e \hookrightarrow \mathcal{B}_e$ that is induced by the inclusion $\underline{\mathcal{B}} \hookrightarrow \mathcal{B}$ via $\underline{\mathfrak{b}} \mapsto \underline{\mathfrak{b}} \oplus \mathfrak{g}^{>0}$. We also have another inclusion, $\xi^- : \underline{\mathcal{B}}_e \hookrightarrow \mathcal{B}_e$ induced from $\underline{\mathfrak{b}} \mapsto \underline{\mathfrak{b}} \oplus \mathfrak{g}^{<0}$.

Recall also, Section 8.5, that $\tilde{\mathbb{D}} : D^b(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)) \xrightarrow{\sim} D^b(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e))$ is given by $\tilde{T}_{w_0}^{-1}\tilde{\mathbb{D}}_{coh}\langle -\ell(w_0)\rangle$.

Lemma 8.15. *We have* $\tilde{\mathbb{D}} \circ \xi_* \cong \tilde{T}_{u_{\underline{G}}}^{-1} \circ \xi_*^- \circ \underline{\tilde{\mathbb{D}}}[2\ell(u_{\underline{G}})]\langle -\ell(u_{\underline{G}})\rangle$.

Proof. Recall that $\tilde{\mathbb{D}}_{coh}$ is the ς-twisted and $\dim \mathfrak{g}$-shifted Serre duality functor. Note that $\varsigma \circ \xi \cong \xi^- \circ \underline{\varsigma}$, where $\underline{\varsigma} = \varsigma|_{\underline{\mathfrak{g}}}$. It follows that $\tilde{\mathbb{D}}_{coh} \circ \xi_* \cong \xi_*^- \circ \underline{\tilde{\mathbb{D}}}_{coh}[2(\ell(u_{\underline{G}})]\langle 2\ell(u_{\underline{G}})\rangle$. It remains to show that $\tilde{T}_{w_0}^{-1} \circ \xi^- \cong \tilde{T}_{u_{\underline{G}}}^{-1} \circ \xi^- \circ \tilde{T}_{\underline{w}_0}^{-1}$, equivalently $\tilde{T}_{\underline{w}_0}^{-1} \circ \xi^- \cong \xi^- \circ \tilde{T}_{\underline{w}_0}^{-1}$. This is standard from the construction of the elements $\tilde{T}_i$ for Dynkin roots α_i, see [BR]. □

To prove Proposition 8.14, it remains to establish the following formula

$$[\xi_*^-] = v^{-\ell(u_{\underline{G}})} \lim_{\theta \to +\infty} X_{-\theta^*} H_{u_{\underline{G}}}^{-1} X_{-\theta}[\xi_*]. \tag{8.10}$$

Note that $\mathfrak{X}(P)$ naturally acts on the right on ${}^G\hat{\mathfrak{C}}_{\underline{P}}$ by $\mathcal{H}_G^a$-linear automorphisms. Let t_θ denote the image of θ under this action. We have

$$X_{-\theta}[\xi_*] = [\xi_*]t_{-\theta}. \tag{8.11}$$

In order to prove this we need to find an alternative presentation of $\hat{K}_0(\mathrm{Coh}^{\underline{Q}}(\mathcal{B}_e))$ and get some information on $H_u^{-1}[\xi_*]$ in terms of this presentation. Let $u \in W$ be such that $u^{-1} \in W^{\underline{G},-}$. Let us write $\mathfrak{m}^u$ for the maximal nilpotent subalgebra of the parabolic subalgebra $\underline{\mathfrak{g}} + u^{-1}(\mathfrak{b})$. Then we have the embedding $\xi^u : \underline{\mathcal{B}}_e \hookrightarrow \mathcal{B}_e$ induced from $\underline{\mathfrak{b}} \mapsto \underline{\mathfrak{b}} \oplus \mathfrak{m}^u$. Note that $\xi = \xi^1$ and $\xi^- = \xi^{u_{\underline{G}}}$. The corresponding embedding $\underline{\tilde{\mathfrak{g}}} \hookrightarrow \tilde{\mathfrak{g}}$ will also be denoted by ξ^u.

Lemma 8.16. *The map* $\bigoplus_u [\xi_*^u] : \hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e)^{\oplus |W|/|W_{\underline{G}}|} \to \hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$ *is an isomorphism.*

Proof. The localization theorem in the equivariant K-theory applied to the action of $\underline{Q}^\circ$ on $\tilde{\mathfrak{g}}$ says that the maps above are mutually inverse isomorphisms

$$K_0(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e))_{loc} \leftrightarrow K_0(\mathrm{Coh}^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e))_{loc}, \tag{8.12}$$

where the subscript "loc" means that we localize the classes

$$[\Lambda^\bullet N_{\tilde{\mathfrak{g}}}(\xi_u(\underline{\tilde{\mathfrak{g}}}))], \tag{8.13}$$

where the notation $N_Y X$ means a normal bundle to a smooth subvariety Y in a smooth variety X. The classes (8.13) in $K_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e)$ have inverses in $\hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e)$. It follows that the isomorphism (8.12) extends to an isomorphism between $\hat{K}_0$'s. □

For $m \in \hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$ we set

$$\mathsf{pr}_u(m) := [\xi_*^u]\left(\bigoplus_u [\xi_*^u]\right)^{-1} m,$$

so that $m = \sum_u \mathsf{pr}_u(m)$.

Proof of Proposition 8.14. We need to prove (8.10). The proof is in several steps.

Step 1. Consider the direct sum decomposition of $\hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$ from Lemma 8.16. We claim that for $m \in \hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$ the limit

$$\lim_{\theta \to +\infty} X_{-\theta^*} m t_{-\theta}$$

exists and equals to $\mathsf{pr}_{u_{\underline{G}}}(m)$. This is equivalent to showing that for $m \in \mathrm{im}[\xi_*^u]$ we have

$$\lim_{\theta \to +\infty} X_{-\theta^*} m t_{-\theta} = \begin{cases} m, & \text{if } u = u_{\underline{G}}, \\ 0, & \text{else} \end{cases}$$

Recall that X_{θ^*} is the class of the line bundle $\mathcal{O}(\theta^*)$ on the diagonal in the Steinberg variety $\mathsf{St}_{\mathfrak{h}}$. We have $(\xi^u)^*\mathcal{O}(-\theta^*) = \underline{\mathcal{O}}(-u\theta^*)$. Note that for $u = u_{\underline{G}}$, we have $-u\theta^* = \theta$ and hence $\underline{\mathcal{O}}(-u\theta^*)$ is a trivial bundle with K-equivariant structure via θ. It follows that for $m \in \operatorname{im}[\xi_*^{u_{\underline{G}}}]$ we have $X_{-\theta^*} m t_{-\theta} = m$.

Let $u \neq u_{\underline{G}}$. Set $\underline{m} := (\bigoplus_u [\xi_*^u])^{-1} m$. We need to show that

$$X_{-u(\theta^*)}\underline{m}t_{-\theta} \to 0 \tag{8.14}$$

in $\hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e)$. Consider the image of $X_{-u(\theta^*)}\underline{m}t_{-\theta}$ in $\hat{K}_0(\underline{\mathcal{U}}^\chi_{(2\underline{\rho}-2\rho,\mathbb{F})}\text{-mod}^{\underline{Q},gr})$. We can assume that the image of $\underline{m}$ there is the class of a simple object, say $\tilde{L}$. The ν-highest weight of simples that appear in the cohomology of $(X_{-u(\chi^*)}\tilde{L}) \otimes \mathbb{F}_{-\chi}$ will be of the form $k + \langle -u\chi^* - \chi, \nu\rangle$, where k is the ν-highest weight of L. We reduce (8.14) to checking that $\lim_{\chi\to+\infty}\langle -u\chi^* - \chi, \nu\rangle = -\infty$. Set $u' = u\underline{w}_0^{-1}w_0$ so that $-u\chi^* = u'\chi$ and $u' \in W^{\underline{G},-}$. Then $u'\chi - \chi$ is a nonnegative linear combination of roots of $\mathfrak{g}^{<0}$. As $\chi \to +\infty$ the minimum of the coefficients in this linear combination also goes to $-\infty$. It follows that $\lim_{\chi\to+\infty}\langle -u\chi^* - \chi, \nu\rangle = -\infty$. This finishes the proof of this step.

Step 2. It remains to prove that

$$v^{-\ell(u_{\underline{G}})}\mathsf{pr}_{u_{\underline{G}}}\left(H^{-1}_{u_{\underline{G}}}[\xi_*]m\right) = [\xi_*^-]m. \tag{8.15}$$

For this we will prove similar statements for related K_0-groups, where it is easier to do computations.

Consider the partial Steinberg variety $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \underline{\tilde{\mathfrak{g}}}$ (in fact, it is a derived scheme) and its $\underline{G} \times \mathbb{C}^\times$-equivariant K-theory. The convolution map

$$K_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \underline{\tilde{\mathfrak{g}}}) \times K_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e) \to K_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e)$$

is continuous so gives rise to

$$\hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \underline{\tilde{\mathfrak{g}}}) \times \hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\underline{\mathcal{B}}_e) \to \hat{K}_0^{\underline{Q}\times\mathbb{C}^\times}(\mathcal{B}_e).$$

This map is $\mathcal{H}^a_G$-linear in the first argument due to associativity of convolution. Similarly to Lemma 8.16, we have the decomposition

$$\bigoplus_u [\tilde{\xi}_*^u] : \hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\underline{\mathsf{St}}_{\mathfrak{h}})^{\oplus|W|/|W_{\underline{G}}|} \xrightarrow{\sim} \hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \underline{\tilde{\mathfrak{g}}})),$$

where $\underline{\mathsf{St}}_{\mathfrak{h}}$ is the analog of $\mathsf{St}_{\mathfrak{h}}$ for $\underline{\mathfrak{g}}$ and the meaning of $\tilde{\xi}^u$ is similar to that of ξ^u.

Let δ denote the class of diagonal in $\hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\underline{\mathsf{St}}_{\mathfrak{h}})$, the unit in this algebra. Then we have $[\xi_*^u](\bullet) = [\tilde{\xi}_*^u](\delta) * \bullet$. So it is enough to prove that

$$v^{-\ell(u_{\underline{G}})}\mathsf{pr}_{u_{\underline{G}}}H^{-1}_{u_{\underline{G}}}[\tilde{\xi}_*](\delta) = [\tilde{\xi}_*^{u_{\underline{G}}}](\delta). \tag{8.16}$$

Step 3. Now consider the convolution map

$$\hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \underline{\tilde{\mathfrak{g}}}) \times \hat{K}_0^{T\times\mathbb{C}^\times}(T^*\underline{\mathcal{B}}) \to \hat{K}_0^{T\times\mathbb{C}^\times}(T^*\mathcal{B}),$$

where the second and the third completions are again with respect to ν. Let us write $\mathbb{C}_{1\underline{B}}$ for the skyscraper sheaf at the point $1\underline{B} \in \underline{\mathcal{B}}$. We claim that the map

$$a \mapsto a * [\mathbb{C}_{1\underline{B}}] : \hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \underline{\tilde{\mathfrak{g}}})) \to \hat{K}_0^{T\times\mathbb{C}^\times}(T^*\mathcal{B}).$$

is injective. This claim reduces to the case when $G = \underline{G}$ is semisimple using the localization theorem. In this case $K_0^{T\times\mathbb{C}^\times}(T^*\mathcal{B})$ is the periodic $\mathcal{H}^a_G$-module up to localization, see [Lu3, Section 10]. Our claim translates to the claim that the standard basis element labelled by 1. It is now immediate.

Step 4. We write $\mathbb{C}_{wB}, w \in W$, for the skyscraper sheaf at $wB \in T^*\mathcal{B}$ with the trivial $T \times \mathbb{C}^\times$-action. The classes $[\mathbb{C}_{wB}]$ form a basis in the $K_0(\mathrm{Rep}(T \times \mathbb{C}^\times))_{loc}$-module $K_0^{T\times\mathbb{C}^\times}(T^*\mathcal{B})_{loc}$, where "loc" stands for localization functor, where we invert all nonzero elements in $K_0^{T\times\mathbb{C}^\times}(\mathrm{pt})$. Hence what remains to prove is that $H_{u_{\underline{G}}}^{-1}[\mathbb{C}_{1B}] = v^{\ell(u_{\underline{G}})}[\mathbb{C}_{(u_{\underline{G}})^{-1}B}]+?$, where ? stands for the sum of the basis elements 1_{wB} with $w \prec u_{\underline{G}}$ with some coefficients in $K_0^{T\times\mathbb{C}^\times}(pt)_{loc}$. Note that this statement immediately reduces to the following claim: $H_s^{-1}[\mathbb{C}_{wB}] = v[\mathbb{C}_{swB}]+?[\mathbb{C}_{wB}]$ if $sw > w$. And it is sufficient to verify that statement in the case when $G = \mathrm{SL}_2$ and $w = 1$.

Step 5. The character of $\mathbb{C}^\times$ on the cotangent fibers is v^{-2}. The action of $G\times\mathbb{C}^\times$ on $T^*\mathbb{P}^1$ factors through that of GL_2 on the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$. Let $\tilde{T}$ stand for the maximal torus $\{\mathrm{diag}(t_1, t_2)\} \subset \mathrm{GL}_2$. We also write t_1, t_2 for the corresponding equivariant parameters. Our convention is that (t_1, t_2) acts on the standard homogeneous coordinate functions x, y on $\mathbb{P}^1$ by t_1, t_2, respectively. The point $1B \in \mathbb{P}^1$ is $[1 : 0]$ (i.e., $x = 1, y = 0$) and the point sB is $[0 : 1]$. Then $v^{-2} = t_1t_2$.

Consider the element $c = v^{-1}H_s + 1$. According to [CG, Section 7.5], the element c acts on $K_0(\mathrm{Coh}^{T\times\mathbb{C}^\times}(T^*\mathbb{P}^1))$ as the convolution with

$$[\mathcal{O}_{\mathbb{P}^1}] \boxtimes ([\mathcal{O}_{T^*\mathbb{P}^1}] - t_1t_2[\pi^*\Omega_{\mathbb{P}^1}])\,.$$

The class of $\Omega_{\mathbb{P}^1}|_{[1:0]}$ is t_1^{-2}. It follows that $c[\mathbb{C}_{[1:0]}] = [\mathcal{O}_{\mathbb{P}^1}](1 - t_2t_1^{-1})$. On the other hand, in the coordinate chart $(y \neq 0)$, the class $[\mathbb{C}_{[0:1]}]$ is that of the complex $\mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$, where the map is the multiplication by x/y, of weight $t_1t_2^{-1}$. It follows that the class $[\mathbb{C}_{[0:1]}]$ in the coordinate chart is also $[\mathcal{O}_{\mathbb{P}^1}](1-t_2t_1^{-1})$. It follows that $(v^{-1}H_s+1)[\mathbb{C}_{[1:0]}] = [\mathbb{C}_{[0:1]}]+?[\mathbb{C}_{[1:0]}]$. Therefore $H_s^{-1}[\mathbb{C}_{[1:0]}] = v[\mathbb{C}_{[0:1]}]+?[\mathbb{C}_{[1:0]}]$. This finishes the proof. □

8.8. **Semi-periodic Kazhdan-Lusztig polynomials.** The goal of this section is to define the completed semi-periodic module for $\mathcal{H}^a_G$ with its standard basis, a bar-involution on this module, and the canonical basis. Then we relate this canonical basis to affine Kazhdan-Lusztig polynomials.

Fix a standard Levi subgroup $\underline{G} \subset G$. Let $\mathfrak{X}^+(\underline{G})$ denote the intersection of the positive Weyl chamber with $\mathfrak{X}(\underline{G})$. We consider the completion $\hat{\mathcal{H}}^a_{\underline{G}}$ defined as follows: it consists of all infinite sums $\sum_{x\in W^a_{\underline{G}}} a_xH_x$, where for each $N \in \mathbb{Z}$, the set $\{x \in W^a_L | \langle\nu, x\rangle > N, a_x \neq 0\}$ is finite. Here we abuse the notation and write $\langle\nu, x\rangle$ for the pairing of ν and the projection of x to $\mathfrak{X}(\underline{G})$.

Then set $\hat{\mathcal{H}}^a_G := \mathcal{H}^a_G \otimes_{\mathcal{H}^a_{\underline{G}}} \hat{\mathcal{H}}^a_{\underline{G}}$. This space carries a natural structure of a $\mathcal{H}^a_G$-$\mathcal{H}^a_{\underline{G}}$-bimodule. It is also a complete topological $\mathbb{Z}[v^{\pm1}]$-module and the topology is compatible with the bimodule structure.

Now we define a standard (topological) basis of $\hat{\mathcal{H}}^a_G$. Let $x \in W^a$. We can uniquely decompose x as $w\underline{x}$, where $\underline{x} \in W^a_{\underline{G}}$ and $w \in W^{\underline{G},-}$. Set $H^\infty_x := H_wH_{\underline{x}}$. It is clear that these elements form a topological basis in $\hat{\mathcal{H}}^a_G$.

Now we proceed to defining a bar-involution on $\hat{\mathcal{H}}^a_G$. For $\theta \in \mathfrak{X}(\underline{G})$ and $a \in \mathcal{H}^a_G$ we set

$$\overline{a}^\theta := \overline{aX_\theta}X_{-\theta}, \tag{8.17}$$

where in the right hand side $\overline{\bullet}$ denotes the usual bar-involution on $\mathcal{H}^a_G$. The following lemma describes elementary properties of $\overline{\bullet}^\theta$.

Lemma 8.17. *The following claims are true:*

(1) $\overline{\bullet}^\theta$ *is an involution.*
(2) *We have* $\overline{abc}^\theta = \overline{a}\cdot\overline{b}^\theta\overline{c}$ *for all* $a, b \in \mathcal{H}^a_G$ *and* $c \in \mathcal{H}_{W_{\underline{G}}}$.
(3) *We have* $\overline{1}^\theta = H_{u_{\underline{G}}} X_{-\theta^*} H^{-1}_{u_{\underline{G}}} X_{-\theta}$.

Proof. (1) and (2) are straightforward from the definition. Let us prove (3). We have $\overline{1}^\theta = \overline{X_\theta} X_{-\theta}$. As we recalled in Section 8.3, $\overline{X_\theta} = H_{w_0} X_{w_0(\theta)} H^{-1}_{w_0}$. But $w_0(\theta) = -\theta^*$, hence $H_{w_0} X_{-\theta^*} H^{-1}_{w_0} = H_{u_{\underline{G}}} X_{-\theta^*} H^{-1}_{u_{\underline{G}}}$. This proves (3). ☐

Here is a stabilization property for the involution $\overline{\bullet}^\theta$.

Proposition 8.18. *For all* $a \in \hat{\mathcal{H}}^a_G$*, the limit* $\lim_{\theta\to\infty} \overline{a}^\theta$ *exists in* $\hat{\mathcal{H}}^a_G$*. Moreover,* $\lim_{\theta\to\infty} \overline{\bullet}^\theta$ *extends to a continuous involution on* $\hat{\mathcal{H}}^a_G$.

Proof. Thanks to (2) of Lemma 8.17 and the continuity of the product, the only thing we need to check is the existence of $\lim_{\theta\to+\infty} \overline{1}^\theta$. For this, we use the proof of Proposition 8.14. Namely, consider the completed K_0-group $\hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\underline{\mathfrak{g}}})$. We have an inclusion $\mathcal{H}^a_G \hookrightarrow \hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\underline{\mathfrak{g}}}), a \mapsto a * [\tilde{\xi}_*]\delta$. This map extends to a topological homomorphism of $\mathcal{H}^a_G$-modules $\hat{\mathcal{H}}^a_G \xrightarrow{\sim} \hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\underline{\mathfrak{g}}})$. The argument of Step 2 of the proof of Proposition 8.14 shows that this homomorphism is a topological isomorphism. The proof of the subsequent steps of Proposition 8.14 shows that $\lim_{\theta\to+\infty} H_{u_{\underline{G}}} X_{-\theta^*} H^{-1}_{u_{\underline{G}}} X_{-\theta}[\tilde{\xi}_*]\delta$ exists in $\hat{K}_0^{\underline{G}\times\mathbb{C}^\times}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\underline{\mathfrak{g}}})$. This finishes the proof of the proposition. ☐

We denote the limiting involution by $\overline{\bullet}^\infty$. Now we discuss the canonical basis for this involution.

Proposition 8.19. *The following claims are true:*

(1) *There is a unique collection of elements* $C^\infty_x \in H^\infty_x + v^{-1}\operatorname{Span}^{top}_{\mathbb{Z}[v^{-1}]}(H^\infty_y | y \in W^a)$ *(where the superscript "top" means that we take all converging sums) such that* $\overline{C^\infty_x}^\infty = C^\infty_x$.
(2) *The coefficient of* H^∞_y *in* C^∞_x *coincides with* c_{xt_θ, yt_θ}*, where* $\theta \in \mathfrak{X}^+(\underline{G})$ *is large enough (depending on* x, y*).*

Proof. The proof is in several steps. We write $\Delta, \Delta_{\underline{G}}$ for the root systems of $\mathfrak{g}, \underline{\mathfrak{g}}$. The superscript "+"means the system of positive roots.

Step 1. Let $x = wt_\zeta$ for $w \in W, \zeta \in \mathfrak{X}(T)$. Suppose that $\langle\zeta, \alpha^\vee\rangle < 0$ and $\alpha > 0 \Rightarrow \alpha \in \Delta_{\underline{G}}$. We claim that $H^\infty_x = H_x$. Indeed, what we need to prove is that if $w = w'w''$, where $w'' \in W_{\underline{G}}$ and $w' \in W^{\underline{G},-}$, then $\ell(x) = \ell(w') + \ell(w''t_\zeta)$. Indeed, we have

$$\ell(x) = \sum_{\alpha, w(\alpha)>0} |\langle\zeta, \alpha^\vee\rangle| + \sum_{\alpha>0, w(\alpha)<0} |1 + \langle\zeta, \alpha^\vee\rangle| = \sum_{\alpha\in\Delta^+\setminus\Delta^+_{\underline{G}}, w(\alpha)>0} \langle\zeta, \alpha^\vee\rangle +$$

$$\sum_{\alpha\in\Delta^+\setminus\Delta^+_{\underline{G}}, w(\alpha)<0} (1 + \langle\zeta, \alpha^\vee\rangle) + \sum_{\alpha\in\Delta^+_{\underline{G}}, w(\alpha)>0} |\langle\zeta, \alpha^\vee\rangle| + \sum_{\alpha\in\Delta^+_{\underline{G}}, w(\alpha)<0} |1 + \langle\zeta, \alpha^\vee\rangle|$$

$$= \ell(w') + \sum_{\alpha\in\Delta^+\setminus\Delta^+_{\underline{G}}} \langle\zeta, \alpha^\vee\rangle \sum_{\alpha\in\Delta^+_{\underline{G}}, w''(\alpha)>0} |\langle\zeta, \alpha^\vee\rangle| + \sum_{\alpha\in\Delta^+_{\underline{G}}, w'(\alpha)<0} |1 + \langle\zeta, \alpha^\vee\rangle| =$$

$$\ell(w') + \ell(w''t_\zeta).$$

Step 2. Recall that $\preceq$ denotes the Bruhat order on W^a. Then we have the following well-defined order $\preceq^\infty$ on W^a: $x \preceq^\infty y$ if and only if $xt_\theta \preceq yt_\theta$ for all large enough $\theta \in \mathfrak{X}^+(\underline{G})$. Note that $\hat{\mathcal{H}}^a_G$ is the completion with respect to $\preceq^\infty$. Also note that $\overline{H^\infty_x}^\infty \in H^\infty_x + \mathrm{Span}_{\mathbb{Z}[v^{\pm 1}]}(H^\infty_y | y \preceq^\infty x)$. From here it follows that the elements C^∞_x are unique if they exist.

Step 3. The existence of the limit of $\overline{\bullet}^\theta$ can be interpreted as follows. Take $x = wt_\zeta \in W^a$ be as in Step 1 and $\theta \in \mathfrak{X}^+(\underline{G})$. Then $H_{t_\theta} = X_\theta$ and $\ell(x) + \ell(t_\theta) = \ell(xt_\theta)$, hence

$$H_x X_\theta = H_{xt_\theta}. \tag{8.18}$$

Proposition 8.18 means that for all $x, y \in W^a$, and all large enough $\chi \in \mathfrak{X}^+(\underline{G})$ the coefficient of H_{yt_θ} in $\overline{H_{xt_\theta}}$ is independent of θ.

Let us show c_{xt_θ, yt_θ} is independent of θ as long as θ is large enough. Consider the $\mathbb{Z}[v^{\pm 1}]$-module M_θ with basis M_z with $yt_\theta \preceq zt_\theta \preceq xt_\theta$. Note that the labelling set for the basis elements is independent of θ as long as θ is large enough. Consider the $\mathbb{Z}[v^{\pm 1}]$-linear embedding $M_\theta \hookrightarrow \mathcal{H}^a_G$ given by $M_z \mapsto H^\infty_x$. We get the involution on M_θ by pulling back $\overline{\bullet}^\theta$ under that embedding. Note that both the $\mathbb{Z}[v^{-1}]$-lattice spanned by the basis elements M_z and the involution are independent of θ as long as θ is large enough. The latter follows from the previous paragraph. It follows that the canonical basis for these data is independent of the choice of an involution. This implies the claim in the beginning of the paragraph. This establishes (2). Let us denote this stable element by $c^\infty_{x,y}$.

Note that, by the construction, we have $c^\infty_{xt_\theta, yt_\theta} = c^\infty_{x,y}$ for all $\theta \in \mathfrak{X}(\underline{G})$ and $c^\infty_{x,y} \neq 0 \Rightarrow y \preceq^\infty x$.

Step 4. Set

$$C^\infty_x = \sum_{y \preceq^\infty x} c^\infty_{x,y} H^\infty_x.$$

This sum makes sense in $\hat{\mathcal{H}}^a_G$. Combining the discussion of Steps 2 and 3, we see that the element C^∞_x satisfies the conditions of (1) of the proposition. This finishes the proof. □

8.9. **The basis of simples.** For $x \in W^{a,P}$, set $H^{\infty,P}_x = \sum_{u \in W_P} (-v)^{-\ell(u)} H^\infty_{xu}$. These elements form a topological basis in $\hat{\mathcal{H}}^{a,P}_G$ (embedded into $\hat{\mathcal{H}}^a_G$).

Proposition 8.20. *The following statements hold:*

1. *The elements C^∞_x with $x \in W^{a,P}$ form a topological basis in $\hat{\mathcal{H}}^{a,P}_G$.*
2. *The kernel of $\hat{\mathcal{H}}^a_G \twoheadrightarrow {}^G\hat{\mathfrak{C}}_P$ is topologically spanned by the elements C^∞_x with x of the form $u\underline{x}$ with $u \in W^{\underline{G},-}$ and $\underline{x} \in W^{a,P}_{\underline{G}} \setminus \mathfrak{c}_{P^0}$.*
3. *Let $x = u\underline{x}$ be such that $u \in W^{\underline{G},-}$ and $\underline{x} \in \mathfrak{c}_{\underline{P}}$. Let $\mathcal{L}$ be the irreducible module in $\mathcal{U}^\chi_{(0),\mathbb{F}}$-mod$^{\underline{Q}}$ labelled by x. The image of $H_u C_{\underline{x}}$ in ${}^G\hat{\Sigma}_P$ coincides with $[\tilde{\Delta}_{\mathcal{L}}]$, while the image of C^∞_x coincides with $[\tilde{\mathcal{L}}]$.*

Proof. By (2) of Lemma 8.17, we have

$$\overline{bc}^\infty = \overline{b}^\infty \overline{c}, \forall b \in \hat{\mathcal{H}}^a_G, c \in \mathcal{H}_{W_{\underline{G}}}. \tag{8.19}$$

Thanks to (2) of Proposition 8.19, (1) of the present proposition follows from the analogous property for the usual Kazhdan-Lusztign basis.

Let us prove (2) and (3). Let π denote natural projection $\hat{\mathcal{H}}^{a,P}_G \twoheadrightarrow {}^G\hat{\mathfrak{C}}_{\underline{P}}$. Recall the isomorphism ${}^G\hat{\mathfrak{C}}_{\underline{P}} \cong \hat{K}^{Q \times \mathbb{C}^\times}_0(\mathcal{B}_e)$ from Corollary 8.13. Thanks to Proposition 8.14, we see

that
$$[\tilde{\mathbb{D}}]\pi(C_{w_{0,P}}) = \lim_{\theta\to+\infty} H_{u_{\underline{G}}} X_{-\theta^*} H_{u_{\underline{G}}}^{-1} X_{-\theta}[\tilde{\underline{\mathbb{D}}}]\pi(C_{w_{0,P}}).$$
In the case when $G = \underline{G}$, the element $\pi(C_{w_{0,P}})$ corresponds to the class $[\tilde{W}^\chi_{\mathbb{F}}(2\rho_L - 2\rho)]$ hence is fixed by $[\tilde{\mathbb{D}}]$. In general, we get that $\pi(C_{w_{0,P}})$ is fixed by $[\tilde{\underline{D}}]$.

Applying (3) of Lemma 8.17, we see that
$$\pi(\overline{C_{w_{0,P}}}^\infty) = [\tilde{\mathbb{D}}]\pi(C_{w_{0,P}}).$$
Now we can combine Proposition 8.10 with (2) of Lemma 8.17 to see that π intertwines $\bar{\bullet}^\infty$ and $[\tilde{\mathbb{D}}]$.

For $x = u\underline{x} \in W^{a,P}$, the image of $H_u C_{\underline{x}}$ in ${}^G\hat{\mathfrak{C}}_{\underline{P}}$ is zero if $\underline{x} \notin \mathfrak{c}_{\underline{P}}$ and coincides with $[\tilde{\Delta}_{\mathcal{L}}]$ if $x \in \mathfrak{c}_{\underline{P}}$. This is a consequence of Corollary 8.13.

Note that the topological $\mathbb{Z}[v^{-1}]$-spans of $H_u C^\infty_{\underline{x}}$ and of C_x in $\hat{\mathcal{H}}^{a,P}_G$ coincide. The image of this $\mathbb{Z}[v^{-1}]$-lattice in $\hat{\Sigma}^P$ coincides with the $\mathbb{Z}[v^{-1}]$-lattice topologically spanned by $[\tilde{\Delta}_{\mathcal{L}}]$'s. The elements $\pi(C^\infty_{u\underline{x}})$, where $\underline{x} \in \mathfrak{c}_{\underline{P}}$ in ${}^G\hat{\mathfrak{C}}_{\underline{P}}$ as well as the classes $[\tilde{\mathcal{L}}]$ satisfies the canonical basis conditions analogous to those of (1) of Proposition 8.19. For the same reason as in Step 2 of the proof of Proposition 8.19, $[\tilde{\mathcal{L}}] = \pi(C^\infty_x)$. This shows (3). And if $\underline{x} \notin \mathfrak{c}_{\underline{P}}$, then
$$C^\infty_x \in v^{-1}\operatorname{Span}^{top}_{\mathbb{Z}[v^{-1}]}([\tilde{\mathcal{L}}]).$$
Such an element can only be self-dual if it is equal to zero. This completes the proof. □

Our next result implies Theorem 1.2.

Theorem 8.21. *Let $x = u\underline{x} \in W^{a,P}$ be such that $\underline{x} \in \mathfrak{c}_{\underline{P}}$ and $u \in W^{\underline{G},-}$. Let $L^\chi_{x,\mathbb{F}}$ be the corresponding simple object in $\mathcal{U}^\chi_{(0),\mathbb{F}}$-mod$^{\underline{Q}}$. We have the following identity in $\hat{K}_0$:*

$$[L^\chi_{u\underline{x},\mathbb{F}},] = \sum_{y\in W^{a,P}} c^\infty_{x,y}(1)[W^\chi_{\mathbb{F}}(\mu_y)]. \tag{8.20}$$

And if $\underline{x} \notin \mathfrak{c}_{\underline{P}}$, we get

$$0 = \sum_{y\in W^{a,P}} c^\infty_{x,y}(1)[W^\chi_{\mathbb{F}}(\mu_y)]. \tag{8.21}$$

Proof. Theorem 8.6 expresses the classes of $[\underline{\Delta}_{\mathcal{L}}]$ via the classes $[W^\chi_{\mathbb{F}}(\mu_y)]$. Now (8.20), (8.21) follow from (3) and (2) of Proposition 8.20. □

8.10. **From equivariantly irreducible to usual irreducible.** The goal of this section is to explain how to compute the dimensions of irreducible $\mathcal{U}^\chi_{(0),\mathbb{F}}$-modules. The group $\underline{Q}_{\mathbb{F}}$ acts on $\operatorname{Irr}(\mathcal{U}^\chi_{(0),\mathbb{F}})$. Let V be an equivariantly irreducible module. By Lemma 2.1, it is completely reducible and all of its irreducible summands have the same dimension. Every irreducible $\mathcal{U}^\chi_{(0),\mathbb{F}}$-module U occurs in some V so to compute the dimension of U one needs to divide the dimension of V by the number of irreducible summands.

The computation of this number easily reduces to the case when χ is distinguished: by taking highest weight spaces for ν. We will produce a recipe to compute the number of irreducible summand of V based on the representation theory of affine Hecke algebras. This method works best when G is of adjoint type and A is commutative, which is always the case for classical Lie algebras. From now on we assume that χ is distinguished and A is abelian.

First, let us recall a definition of a centrally extended set with a group action. Let Y be a finite set together with an action of a finite group Γ and $\mathbb{K}$ be an algebraically closed field. By a centrally extended Γ-set structure on Y we mean a Γ-invariant assignment $y \mapsto s_y \in H^2(\Gamma_y, \mathbb{K}^\times)$. To a centrally extended set Y one can assign a category $\mathsf{Sh}^\Gamma(Y)$ of Γ-equivariant sheaves of finite dimensional vector spaces on Y: for such a sheaf its fiber at y is a projective representation of Γ with Schur multiplier s_y.

An example we need is as follows. Take $\mathbb{K} := \mathbb{F}, Y := \operatorname{Irr}(\mathcal{U}^\chi_{(0),\mathbb{F}}), \Gamma := A$. The structure of a centrally extended A-set on Y comes from the action of A on $\mathcal{U}^\chi_{(0),\mathbb{F}}$ by algebra automorphisms.

Each A-equivariantly irreducible $\mathcal{U}^\chi_{(0),\mathbb{F}}$-module V gives rise to an irreducible object in $\mathsf{Sh}^A(Y)$. Such an object is supported on a single orbit, say Ay. Its fiber at y is an irreducible projective representation V_y of A_y with Schur multiplier s_y. Then the number of irreducible constituents of V coincides with $|Ay| \dim V_y$. We remark that this number is a power of 2. To compute it we need to know $|Ay|$ and also $\dim K_0(\mathsf{Sh}^A(Ay))$, which coincides with the number of the irreducible projective A_y-modules with Schur multiplier s_y.

The A-orbits in Y are identified with the right cells inside the two-sided cell corresponding to $\mathbb{O}$.Recall that $\mathfrak{c}_P$ denotes the left cell containing $w_{0,P}$. For $y \in Y$, we write $\mathfrak{c}_y$ for the right cell corresponding to Ay. Note that the irreducible objects in $\mathsf{Sh}^A(Y)$ are in bijection with $\mathfrak{c}_P$. And the irreducible objects in $\mathsf{Sh}^A(Ay)$ are in bijection with $\mathfrak{c}_y \cap \mathfrak{c}_P$.

Now we interpret $|Ay|$. Let J^a_G denote Lusztig's asymptotic Hecke algebra for W^a_G. Lusztig has constructed a $\mathbb{Z}[v]$-algebra homomorphism $\mathcal{H}^a_G \to J^a_G[v^{\pm 1}]$.Let $J^a_{G,\mathbb{O}}$ denote the direct summand of J^a_G corresponding to $\mathbb{O}$. It follows from that $J^a_{G,\mathbb{O}}$ is identified with $K_0(\mathsf{Sh}^A(Y \times Y))$. For a generic number $\alpha \in \mathbb{C}^\times$, we have $\mathcal{H}^a_G|_{v=\alpha} \twoheadrightarrow J^a_{G,\mathbb{O}}$. The representation of $\mathcal{H}^a_G|_{v=\alpha}$ at $K^{\mathbb{C}^\times}(\mathcal{B}_e)|_{v=\alpha}$ is pulled back from the $J^a_{G,\mathbb{O}}$-module $K_0(\mathsf{Sh}(Y))$ (here we consider complexified K_0-groups). Let e_y denote the idempotent in $J^a_{G,\mathbb{O}}$ corresponding to Ay. Let $\tilde{e}_y$ be its preimage in $\mathcal{H}^a_G$. Then $|Ay|$ coincides with the trace of $\tilde{e}_y$ in $K^{\mathbb{C}^\times}(\mathcal{B}_e)|_{v=\alpha}$.

8.11. **Towards categorification of Theorem 8.21.** Theorem 8.6 is essentially a K_0-manifestation of results of Section 6.2 but Theorem 1.2 is a purely K-theoretic statement. One could try to categorify it by producing a constructible realization of $D^b(\operatorname{Coh}^{\underline{G}}(\tilde{\mathfrak{g}} \times^L_{\mathfrak{g}} \underline{\tilde{\mathcal{N}}}))$ or the heart of one of its t-structures, e.g. $\operatorname{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \underline{\mathcal{A}}^{opp}\text{-mod}^{\underline{G}})$, where $\underline{\mathcal{A}}$ is an analog of $\mathcal{A}$ for $\underline{G}$.

We expect that $\operatorname{Perv}(\mathcal{A}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{g}]} \underline{\mathcal{A}}_{\mathfrak{h}}^{opp}\text{-mod}^{\underline{G}})$ should be equivalent to the category of I°-equivariant perverse sheaves on a "quarter-infinite" affine flag variety. Morally, this should be a category of perverse sheaves on $G^\vee((t))/J'$, where J' is a mixed parabolic subgroup constructed as follows. Let $\underline{J}$ denote the parahoric subgroup of $\underline{G}^\vee((t))$ corresponding to $\underline{P} \subset \underline{G}$. We set $J' := \underline{J} \ltimes G^{\vee,>0}((t))$. An issue, however, that the space $G^\vee((t))/J'$ behaves pretty badly. In the case when $\underline{G} = T$ the issue was circumvented in [ABBGM], where the right version of of $\operatorname{Perv}_{I^\circ}(G^\vee((t))/J')$ was constructed. Moreover, it was proved that the multiplicities in this category are given by the periodic affine Kazhdan-Lusztig polynomials. And finally, they have established an equivalence between $\operatorname{Perv}_{I^\circ}(G^\vee((t))/J')$ and $\mathcal{A}_{\mathfrak{h},0}\text{-mod}^T$, which is what the category of perverse bimodules becomes in this particular case. It is an interesting question of whether these constructions and results generalize to the case of an arbitrary Levi subgroup $\underline{G}$.

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