Macdonald positivity conjecture

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Abstract

This is the writeup of my talk at MIT-Northeastern graduate seminar "Quantum cohomology and representation theory". This notes follow papers [3] for an introductory part about Macdonald polynomials and [1] for the proof of Macdonald positivity conjecture.

1 Statement of the positivity conjecture

1.1 Transformed Macdonald polynomials

Let Λ_R be the ring of symmetric functions of inifinetely many variables over the ring R.

For the geometric formulation that will follow it will be convenient to work with *transformed* Macdonald polynomials (see [3]). See Appendix for the relation between transformed and standard Macdonald and Kostka-Macdonald polynomials.

Let $S_n - \mathrm{mod}$ be the category of finite-dimensional S_n-modules, let $S_n - \mathrm{mod}_{bg}$ be the category of finite dimensional bigraged S_n-modules. Let $F : K_0(S_n - \mathrm{mod}) \to \Lambda_{\mathbb{Q}}$ be the Frobenius character map, which is defined by $F(V_\lambda) = s_\lambda$ for an irreducible representation V_λ. Define, abusing the notation, the Frobenius series map $F : K_0(S_n - \mathrm{mod}_{bg}) \to \Lambda_{\mathbb{Q}(q,t)}$ by $FM(q,t) = \sum_{i,j} q^i t^j FM_{i,j}$. Let $f \in \Lambda_{\mathbb{Q}(q,t)}$ be a Frobenius series of $A \in K_0(S_n - \mathrm{mod}_{bg})$. Define

$$P_x f = \sum_k (-x)^k F(A \otimes \Lambda^k V)$$

where $V = \mathbb{C}^n$ is a standard representation of S_n and $x \in \mathbb{Q}(q,t)$.

Definition. Transformed Macdonald polynomials $H_\lambda(q,t)$ are defined by the following properties:

1. $P_q H_\lambda \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda\}$
2. $P_t H_\lambda \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda^*\}$
3. Coefficient of $s_{(n)}$ in H_λ is 1.

Existence of transformed Macdonald polynomials is non-trivial and is proved, for example, in [5].

1.2 Transformed Kostka-Macdonald polynomials and positivity

Transformed Kostka-Macdonald polynomials $\tilde{K}_{\mu,\lambda} \in \mathbb{Q}(q,t)$ are defined as the coefficients of decomposition

$$H_\lambda = \sum_\mu \tilde{K}_{\mu,\lambda} s_\mu.$$

The goal of this notes is to prove

Theorem 1 (Macdonald positivity conjecture). $\tilde{K}_{\mu,\lambda}$ *are polynomials with non-negative integer coefficients.*

In the following subsection we give the geometric formulation of this conjecture, due to Haiman.

1.3 Macdonald polynomials and Procesi bundles

Let Y be the Hilbert scheme of n points on $\mathbb{C}^2 = Spec\mathbb{C}[x,y]$. Let $\pi : Y \to \mathbb{C}^{2n}/S_n$ be the standard resolution. Recall that a *Procesi bundle* $\mathcal{P}$ was constructed in Gufang's talk and is a $(\mathbb{C}^\times)^2$-equivariant vector bundle on Y satisfying the following properties:

1. $End(\mathcal{P}) = \mathbb{C}[\mathbf{x},\mathbf{y}]\#S_n$.
2. $Ext^i(\mathcal{P},\mathcal{P}) = 0$ for $i > 0$.
3. $\mathcal{P}^{S_n} = \mathcal{O}$.

Haiman's idea of a proof of the positivity conjecture was to realize transformed Macdonald polynomials as Frobenius characters of bigraded S_n-modules $\mathcal{P}_\lambda$, where $\mathcal{P}_\lambda$ stands for the fiber of $\mathcal{P}$ in a fixed point of the $(\mathbb{C}^\times)^2$-action labeled by λ.

Now property (3) from the definition of transformed Macdonald polynomials is satisfied for $F\mathcal{P}_\lambda$ automatically, as $\mathcal{P}_\lambda \simeq \mathbb{C}[S_n]$ as an S_n-module.

We now reformulate the desired properties of $\mathcal{P}_\lambda$ in a more convenient way. Let $L \subset \mathbb{C}^{2n}$ be the Lagrangian spanned by x's. Note that

$$P_q F\mathcal{P}_\lambda = F[\mathcal{P}_\lambda \leftarrow \mathcal{P}_\lambda \otimes \Lambda^1 L \leftarrow \mathcal{P}_\lambda \otimes \Lambda^2 L \leftarrow ...]$$

where on the right we have a Frobenius series of a class of a fiber of a Koszul complex of $\mathcal{P}$ as of a $\mathbb{C}[\mathbf{x}]$-module. Now we need prove, in particular, that if $\mathcal{P}_\lambda \leftarrow \mathcal{P}_\lambda \otimes \Lambda^1 L \leftarrow \mathcal{P}_\lambda \otimes \Lambda^2 L \leftarrow ...$ contains representation V_μ, then $\mu \geq \lambda$. We need the following

Proposition 1. $\mathcal{P}$ *is flat over* $\mathbb{C}[\boldsymbol{x}]$.

We follow the proof by Roman Bezrukavnikov. Note that as $\mathcal{O}$ is a direct summand of $\mathcal{P}$, $\mathcal{P}$ is a direct summand of $\mathcal{E}nd(\mathcal{P})$, so it is enough to prove the flatness of the latter. Now $\mathbb{C}[\mathbf{x}]$ if free over $\mathbb{C}[\mathbf{x}]^{S_n}$ so we will prove flatness over $\mathbb{C}[\mathbf{x}]^{S_n}$. Because of the equivariance it is enough to prove the flatness of the completion $\hat{\mathcal{P}}_0$ of $\mathcal{P}$ on the fiber $\pi^{-1}(0)$.

Now in Gufang's talk $\mathcal{P}$ was constructed by lifting from the positive characteristic, so we may assume that we are working over the filed $\Bbbk, char\Bbbk = p \gg 0$.

Let $Fr : Y \to Y^{(1)}$ be the Frobenius morphism. Recall that Gufang constructed an Azumaya algebra $\mathbb{O}$ on $Y^{(1)}$. It splits on the completion of $\pi^{-1}(0)$ with splitting bundle $\mathcal{S}$ having the same indecomposable summands as $\mathcal{P}$. Now recall that $\mathbb{O}$ is the deformation of $Fr_*\mathcal{O}$. So desired flatness follows from the flatness of $\hat{Y}$ over $\mathbb{C}^n/S_n$ after the base change and deformation. The latter flatness follows from Sasha's talk: projection $\hat{Y} \to \mathbb{C}^n/S_n$ is equidimensional and hence flat – equidimensional morphism between two smooth varieties is flat.

Corollary. *Theorem 1 is equivalent to the following: there exists a Procesi bundle $\mathcal{P}$ such that if $I_\lambda \in \operatorname{supp} e_\mu \mathcal{P}/\boldsymbol{x}\mathcal{P}$ then $\mu \geq \lambda$, if $I_\lambda \in \operatorname{supp} e_\mu \mathcal{P}/\boldsymbol{y}\mathcal{P}$ then $\mu \geq \lambda^*$.*

Indeed, from flatness it follows that the Koszul complex

$$0 \leftarrow \mathcal{P} \leftarrow \mathcal{P} \otimes \Lambda^1 L \leftarrow \mathcal{P} \otimes \Lambda^2 L \leftarrow ...$$

is the resolution of $\mathcal{P}/\mathbf{x}\mathcal{P}$.

This is the formulation of a Macdonald positivity conjecture we will now prove.

2 Appendix. Macdonald polynomials

Remember that Sasha introduced Jack polynomials P^α_λ as an orthoganilzation of the basis m_λ of the space $\Lambda_{\mathbb{Q}(\alpha)}$ equipped with the form given by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{l(\lambda)} z_\lambda \delta_{\lambda,\mu}$$

where $l(\lambda) = \lambda^*_1, z_\lambda = \prod l^{n_l} n_l!, \lambda = (1^{n_1}, 2^{n_2}, ...)$.

Now we define the following form on $\Lambda_{\mathbb{Q}(q,t)}$:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} z_\lambda \delta_{\lambda,\mu}.$$

Definition. Macdonald polynomials $P_\lambda(q,t) \in \Lambda_{\mathbb{Q}(q,t)}$ are characterized by the following two properties:

1. $P_\lambda = m_\lambda +$ lower terms.
2. $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$.

Proposition 2 ([5])**.** *Polynomials satisfying above properties exist.*

We list some straightforward properties of $P_\lambda(q,t)$.

1. $P_\lambda(q,q) = s_\lambda$.
2. $P_\lambda(0,t)$ are Hall-Littlewood polynomials.
3. $\lim_{t\to 1} P_\lambda(t^\alpha, t) = P^{(\alpha)}_\lambda$.
4. $P_\lambda(1,t) = e_{\lambda^*}$, $P_\lambda(q,1) = m_\lambda$.

2.1 Kostka-Macdonald polynomials

Let $f \in \Lambda_{\mathbb{Q}(q,t)}$ be a Frobenius series of $A \in K_0(S_n - \mathrm{mod}_{bg})$. Define

$$Q_x f = \sum_k x^k F(A \otimes S^k V)$$

where $V = \mathbb{C}^n$ is a standard representation of S_n and $x \in \mathbb{Q}(q,t)$.

Proposition 3 ([3]). *Polynomials $J_\lambda = t^{n(\lambda)} Q_{t^{-1}} H_\lambda(q, t^{-1})$ are scalar multiples of Macdonald polynomials P_λ. Here $n(\lambda) = \sum_i (i-1)\lambda_i$.*

Macdonald defined Kostka-Macdonald polynomials $K_{\mu,\lambda}$ as coefficients in the decomposition

$$J_\lambda = \sum_\mu K_{\mu,\lambda} Q_t s_\lambda.$$

Proposition 4. $\tilde{K}_{\mu,\lambda}(q,t) = t^{n(\lambda)} K_{\mu,\lambda}(q, t^{-1})$.

References

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