# Braid group actions on categories of coherent sheaves

MIT-Northeastern Rep Theory Seminar

In this talk we will construct, following the recent paper of Bezrukavnikov and Riche, actions of certain groups on certain categories. Both the groups and the categories are relevant in representation theory, but all currently known constructions of the action are given by explicitly constructing action by generators and checking relations between them — this is what we will do. There are conjectures giving more "direct" meanings of these actions in terms of mirror symmetry, but these remain largely mysterious at present.

**Definition 1.** For a category $\mathcal{C}$, write $\mathrm{AutEquiv}(\mathcal{C})$ for the category of "autoequivalences", i.e. invertible functors, from $\mathcal{C}$ to itself. Write

$$\mathrm{Ob}\left(\mathrm{AutEquiv}(\mathcal{C})\right)$$

for the set of autoequivalences considered up to automorphism (of functors). This is a group.

**Definition 2.** An action of a group $G$ on a category $\mathcal{C}$ is a map of groups $\alpha : G \to \mathrm{AutEquiv}(\mathcal{C})$. Note that this is a *weak* action. The actions we will construct are conjectured, but not proven, to be *strict*, i.e. admit a choice of compatible isomorphisms between $\alpha(g_1 g_2)$ and $\alpha(g_1) \circ \alpha(g_2)$.

The categories we will be dealing with will be triangulated, and the functors between them will be triangulated functors. Associated to functor $\alpha(g)$ will be an endomorphism of the Grothendieck group $K^0(\mathcal{C})$.

For each group action, we will study the span of its image $H$ in $\mathrm{End}(K^0\mathcal{C})$, and on endomorphisms of smaller invariant subcategories. These images will be important algebras in their own right, and the "canonical" representations as endomorphisms of a Grothendieck group will be important in later talks.

Thus we will produce quadruples of the form "category $\mathcal{C}$, group $G$, algebra $H$, representations $V_e$" (classified by nilpotent orbits).

We will go through four "variations" of this construction, giving the talk a flavor of the musical movement called "theme and variations". The theme, which is self-contained, will contain the essential construction that will later be elaborated in the variations. In the first setting of the "theme" the group $G$ will be the braid group, the category — that of (derived) coherent sheaves on $\tilde{N}$, possibly with support conditions, and the algebra will be the group algebra of the Weyl group, $\mathbb{C}[W]$.

# 1 Theme: A classical braid group action on $\mathrm{coh}(\tilde{N})$.

Fix a root system corresponding to a Lie algebra $\mathfrak{g}$ (in this talk, either classical or affine).

**Definition 3.** The *braid group* Br of a root system is defined as $\pi_1\big((\mathfrak{h}\setminus\mathfrak{h}_\alpha)/W\big)$, where $\mathfrak{h}$ is the Cartan Lie subalgebra, $W$ is the Weyl group and $\mathfrak{h}_\alpha$ are the root hyperplanes corresponding to all roots (without any positivity conditions).

We will be using several different presentations of the braid group. First, we recall a presentation for the Weyl group $W$: Take for generators the $w_\alpha$ corresponding to simple reflections $\alpha$, for $\alpha$ running over simple roots.

**Lemma 4.** *The $w_\alpha$ generate the Weyl group $W$. For $g, h$ group elements, define the word $\{g,h\}^m = ghghg\dots$, where there are $m$ letters (e.g. if $m$ is odd, the last letter is an $m$). For a pair of roots $\alpha, \beta$, let $m(\alpha,\beta)$ be the denominator of the angle $\frac{\angle(\alpha,\beta)}{\pi}$. The following is a complete set of relations on the $w_\alpha$:*

$$\forall \alpha, \beta : \tag{1}$$

$$w_\alpha^2 = 1 \tag{2}$$

$$\{w_\alpha, w_\beta\}^{m(\alpha,\beta)} = \{w_\beta, w_\alpha\}^{m(\alpha,\beta)}. \tag{3}$$

*Here $w_\alpha$ run over the simple roots* (though if we imposed the same relation on reflections $w_\alpha$ corresponding to *all* roots we would get the same group.)

Now we can get a presentation of the braid group, by just getting rid of one of the relations.

**Lemma 5.** *The braid group* Br *can be presented with generators $w_\alpha$, where $\alpha$ runs over* simple *roots and*

$$\forall \alpha, \beta : \tag{4}$$

$$\{T_\alpha, T_\beta\}^{m(\alpha,\beta)} = \{T_\beta, T_\alpha\}^{m(\alpha,\beta)}. \tag{5}$$

Here the generators $T_\alpha$ correspond to a half-turn around one of the hyperplanes in $\check{\mathfrak{h}}$. Note that if $\alpha, \beta$ are orthogonal then $m(\alpha,\beta) = 2$ and the relation says $T_\alpha, T_\beta$ commute. For type $A_n$ the roots are indexed by $1, \dots, n$ and relations are $T_iT_j = T_jT_i$ for $i \neq j \pm 1$ and $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, the usual relation for the ordinary braid group.

We see from the presentations that Br surjects onto $W$. This map comes from the topological fact that $\pi_1(X/W)$ maps to $W$ for any free action of a group $W$ on a topological space $X$ (this map is surjective so long as $X$ is connected).

For $\mathcal{C}$ an abelian category, let $K^0(\mathcal{C})$, the $K$-group, or Grothendieck group of $\mathcal{C}$ denote the quotient of the set $\mathrm{Ob}(\mathcal{C})$ by the relation of "additivity": $[A]+[C] = [B]$ for $0 \to A \to B \to C \to 0$ exact. We will use the following simple lemma.

**Lemma 6.** *1. The map* $\mathrm{Ob}(\mathcal{C}) \to K^0(\mathcal{C})$ *extends to the bounded derived category, with* $A^*$ *a complex in* $D^b(\mathcal{C})$ *mapping to*

$$A^*] = \sum_i (-1)^i [A^i].$$

2. *We can rewrite* $[A^*] = \sum_i (-1)^i [H^i(A^*)]$ *by inductively applying the additivity relation to short exact sequences.*

3. *For* $A^* \to B^* \to C^* \to$ *an exact triangle,* $[B^*] = [A^*] + [C^*]$*.*

For $X$ a smooth algebraic variety there is a map called the (topological) Chern character, $\chi : \mathrm{Ob}(\mathrm{coh}(X)) \to H^*(X)$, which we extend to $\chi : \mathrm{Ob}(D^b \,\mathrm{coh}(X)) \to H^*(X)$ using $\chi(A^*) = \sum_i \chi(A^i)$ for bounded complexes. Indeed, the best way to define it for non-locally-free sheaves is to first extend to the derived category of locally free sheaves, then, using that $X$ is smooth, take finite projective resolutions.

The map $\chi$ satisfies the following properties:

1. additivity: $\chi(A) + \chi(C) = \chi(B)$ for $0 \to A \to B \to C \to 0$ a short exact sequence. Equivalently, $\chi$ factors through the Grothendieck group $K^0(\mathcal{C})$.

2. multiplicativity: $\chi(A \overset{L}{\otimes} B) = \chi(A) \cdot \chi(B)$.

3. functoriality with respect to pullback: $\chi(Rf^*(V)) = f^*(\chi(V))$ is an equality in $H^*(X)$ for $f : X \to Y$ a map of varieties and $V$ a sheaf on $Y$.

4. the Grothendieck-Riemann-Roch theorem, a.k.a. "twisted" functoriality with respect to pushforward:

$$\mathrm{Td}_Y \cdot f_* \chi(F) = f_*(\mathrm{Td}_X \cdot \chi_*(F))$$

   (here we assume $X, Y$ are smooth.)

For the last property, we define the *Todd class* $\mathrm{Td}_X = 1 + O(c_i(T_X))$ of a smooth variety $X$ to be a certain linear combination of the Chern classes of its tangent bundle. (See Appendix).

Because of property (1) above, $\chi$ factors through the Grothendieck group, and we write (abusing notation) $\chi : K^0(\mathrm{coh}(X)) \to H^*(X)$. When $X$ is not smooth then the Chern character can be generalized to land in a different homology theory, the Borel-Moore homology defined by Yi in the first lecture. Namely, we have

**Lemma 7.** *There is a chern character for arbitrary separable schemes* $X$ *over* $\mathbb{C}$ *of finite type* $\chi : \mathrm{Ob}(\mathrm{coh}\, X) \to H_*^{BM}(X)$*, additive in short exact sequences (item (1) above). When* $X$ *is in fact smooth, this character coincides with the Chern character* $\chi$ *defined above, using the isomorphism* $H_{n-i}^{BM}(X) \cong H^i(X)$ *for smooth* $X$*.*

These properties characterize at most uniquely $\chi$ in the singular setting in terms of $\chi$ for the non-singular setting. That there is indeed such a character is proved in [CG] chapter 5.9, using differential-geometric techniques.

*Remark* 8. Note that property (2) of $\chi$ above does not make sense for $X$ not smooth, as Borel-Moore homology does not in general have multiplication. The action map $H_*^{BM}(X)$ on $H_*^{BM}(Y)$ does however make sense when $Y \subset X$ is a closed (possibly singular) subvariety in a smooth $X$. In this case we still have (2) of Lemma 7, so long as $A$ is a sheaf of $X$ and $B$ is a sheaf of $Y$. (See appendix).

### 1.1 Statements of results

For $e \in N$ a nilpotent, let $\mathbb{B}_e$ be the Springer fiber over $e$, i.e. $\mathbb{B}_e = \tilde{N}_e$.

**Theorem 9.** *1. For any nilpotent $e$, there is a natural action of* Br *on $D^b(\tilde{N})$, taking generators $T_w$ to convolution with pushforwards of the varieties $\Lambda_0^w$ defined in Yi's talk.*

*2. Further, let $D^b \operatorname{coh}(\tilde{N})_{\mathbb{B}_e-supp}$ be the category of sheaves whose cohomology groups are supported on $\mathbb{B}_e$, the action of* Br *preserves these subcategories. On the level of their Grothendieck groups, its action factors through the quotient* $\mathrm{Br} \to W$.

**Definition 10.** For $X$ smooth, we have a filtration on $K^0$ called the *support* filtration. The group $F^i(K^0)$ is the subgroup of $K^0(\operatorname{coh}(X))$ spanned by complexes of sheaves with cohomology supported on a degree $\leq i$-subvariety.

With respect to this grading, the Chern character $\chi$ is a filtered map (where the filtration on cohomology is by $H^{\geq i}$).

**Lemma 11.** *The associated graded of the Chern character, $gr(\chi) : gr(K^0(\tilde{N})) \to H^*(\tilde{N})$ is compatible with pushforwards of smooth maps.*

The following lemma relates this result to results of the previous talk:

**Lemma 12.** *We have an isomorphism on the level of Grothendieck groups,*

$$K^0(\operatorname{coh} \tilde{N}_{\mathbb{B}_e-supp}) \cong K^0(\operatorname{coh}(\mathbb{B}_e)).$$

In terms of these identifications, the $W$-action on $\operatorname{coh} \tilde{N}_{\mathbb{B}_e-\mathrm{supp}}$ induces $W$-action on $K^0(\operatorname{coh} \mathbb{B}_e)$.

**Theorem 13.** *This action by $W$ on the Grothendieck group is compatible, via the Chern character, with the action Yi defined on $H_{top}^{BM}(\mathbb{B}_e)$, after taking certain associated gradeds.*

In fact, it will turn out that in this case the two groups $H_*^{BM}(\mathbb{B}_e)$ and $K^0(\mathbb{B}_e)$ are isomorphic. Hence many results of Yi port across. In particular, the action constucted gives a contains every representation of $W$ as an isotypic component under the component group $G_e^0$ of the stabilizer $G_e$ of $e \in N$.

## 1.2 Constructing the representation.

There is a standard way of constructing triangulated functors $D^b coh(X) \to D^b coh(Y)$: pick a coherent sheaf $K \in coh(X \times Y)$, such that it has a finite resolution by locally free sheaves.

**Definition 14.** Now we define the *Fourier-Mukai* transformation with *Kernel* $K$, denoted $FM(K)$, is the composition functor

$$FM : coh(X) \xrightarrow{\pi_1^*} coh(X \times Y) \xrightarrow{\otimes K} coh(X \times Y) \xrightarrow{R(\pi_2)_*} coh(Y),$$

with $F \mapsto (\pi_2)_*(K \otimes \pi_1^* F)$.

The "Fourier" in "Fourier-Mukai" comes from the following analogy: if we replaced "pushforward" by "integral" and the Kernel $K$ by $e^{x \cdot y}$, we would have the ordinary Fourier transform. Note that composition corresponds to convolution, $FM(K) \circ FM(K') = FM(K * K')$, which, again in analogy with integrals, corresponds to multiplication of matrices (with entries parametrized by $X \times Y$).

Under suitable conditions (say if $X, Y$ are smooth and proper), all exact functors $coh(X) \to coh(Y)$ are given by Fourier-Mukai transforms. We make the following further shorthand.

**Definition 15.**

1. If $Z \subset X \times Y$ is a closed subset, define $FM(Z) = FM(O(Z))$ (with Kernel the pushforward of the constant sheaf).
2. If $M$ is a sheaf over $X$, define $FM(M) = \Delta_*(M)$. Evidently,

$$FM(M) : N \mapsto M \otimes N$$

for $M$ another sheaf.

Note that if $\Gamma \subset X \times Y$ is a graph of a map $Y \to X$ then $FM(\Gamma) = Rf^*$.

We'll define the map by taking a limit of actions of $W$ on the part of Grothendieck's simultaneous resolution $\tilde{\mathfrak{g}}$ over the subset of regular elements $\mathfrak{h}^{\mathrm{reg}}$. Remember from Yi's talk that the space $\tilde{N}$ is the fiber over 0 of the map $\tilde{\mathfrak{g}} \to \mathfrak{h}$. Let $\mathfrak{h}^{reg}$ be the open of regular values in $\mathfrak{h}$ (if $\mathfrak{g} = sl_n$, this is $n$-element tuples of $\mathfrak{h}$ with different values). Let $\tilde{\mathfrak{g}}_{\mathfrak{h}\mathrm{reg}}$ be the preimage in $\mathfrak{g}$ of $\mathfrak{h}_{\mathrm{reg}}$. Then any element $w \in W$ of the Weyl group defines an action on $\mathfrak{h}$ lifting to an action $\alpha(w)$ on $\tilde{\mathfrak{g}}_{\mathfrak{h}\mathrm{reg}}$.

Consider in $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ the graph

$$\Lambda^w_{\mathfrak{h}\mathrm{reg}} = \Gamma(\alpha(w)) \subset \tilde{\mathfrak{g}}_{\mathfrak{h}\mathrm{reg}} \times \tilde{\mathfrak{g}}_{\mathfrak{h}\mathrm{reg}}.$$

Now take the limit in the sense of algebraic geometry, and make the following definition.

**Definition 16.** Let $\Lambda^0_w$ be the fiber over $0 \times 0$ of the closure $\bar{\Lambda}^w_{\mathfrak{h}\mathrm{reg}}$.

Now we are ready to define our action. Let the functor $\Phi(w) = FM(\Lambda_w^0)$ denote convolution with the cycle defined above.

Note that $K$ theory, like Borel-Moore homology, is compatible with specialization of a flat family, and the family $\bar{\Lambda}^w_{\mathfrak{h}\mathrm{reg}}$ is flat over $\mathfrak{h}$. Further, specialization commutes with convolution. Putting this together (and using that we have an honest $W$-action on $\Lambda^w_{\mathfrak{h}\mathrm{reg}}$), we see that, on $K^0$, the $\Phi(w)$ satisfy

$$K^0(\Phi(w_1))K^0(\Phi(w_1)) = K^0\Phi(w_1 w_2).$$

Hence we have constructed an action of $W$ on $K^0(\tilde{N})$, which is isomorphic to $H^{BM}_{\tilde{N}}$! The character $\chi$ doesn't quite take the action here to the one Yi defined: some twisting by Todd classes is required to make the identification precise. But since the Todd class is $1 + O(c_i(T_X))$, and $c_i(T_X)$ have higher cohomological degree, we can show that convolution by the $FM(\Lambda_0^w)$ preserve the support filtration on $K$ theory, and on associated gradeds they intertwine the action on Borel-Moore homology with that on $K$ theory.

But the $\Phi(w)$ on the level of the category itself do not multiply the same way the do in $K$ theory. The problem is that "geometric specialization" as above does not preserve convolution product (although its image in $K$ theory does). The simplest example of this is the (derived) tensor product of two skyscraper sheaves $O_x, O_y$ on some space $X$ jumps when $x$ and $y$ come together. In our case, the map $W \to \mathrm{AutEquiv}(coh(\tilde{N}))$ is not a map of groups. E.g. for $sl_2$, the functor $F_w^2$ (for $w$ the unique reflection) is not isomorphic to the identity functor. Can we salvage the action of anything but the free group on $w_i$?

**Theorem 17** (Bezrukavnikov-Riche)**.** *1. The $\Phi(w)$ are invertible;*

*2. We have a canonical isomorphism of functors*

$$\Phi_{w_1}\Phi_{w_2} \cong \Phi_{w_1 w_2} \textit{ if } l(w_1) + l(w_2) = l(w_1 w_2).$$

The fact that $l(w_1)l(w_2) = l(w_1 w_2)$ is necessary can be seen by projecting $\tilde{N} \to \mathbb{B}$. Then the set-theoretic images of $\Lambda_0^w$ are the Bruhat double coset $O_w$, and on the level of sets, we have $O_{w_1} * O_{w_2} = O_{w_1 w_2}$ iff $l(w_1 w_2) = l(w_1) + l(w_2)$. The proof that this is much more mysterious, and proceeds by reduction to characteristic $p$.

## 1.3 Standard modules

The $\Lambda^w_{\mathfrak{h}\mathrm{reg}}$ are, set-theoretically, contained in the fibered product $\tilde{\mathfrak{g}} \times_{\mathfrak{h}/W} \tilde{\mathfrak{g}}$. This means that $\Lambda_0^w$ is, set-theoretically, contained in $Z = \tilde{N} \times_N \tilde{N}$ (in fact Yi constructed these as classes in $Z$ in his talk). This means that the functors $FM(\Lambda_0^w)$ preserve the category. Hence they act on $K^0(\mathrm{coh}\,\tilde{N}_{\mathbb{B}_e-\mathrm{supp}}) \cong K^0(\mathrm{coh}(\mathbb{B}_e))$. These actions are again isomorphic to Yi's on the level of associated gradeds.

**Corollary 18.** *In particular, the proof that $(G_e)^0$-isotypic components classify representations of $W$ still goes through.*

## 2 Equivariant coherent sheaves: $G$-equivariance

We're done with our theme. Now on to variations. These will require a good understanding of the category of equivariant modules over a scheme with a $\mathcal{G}$ action for an algebraic group $\mathcal{G}$.

Suppose $R$ is a commutative ring with action by an algebraic group $\mathcal{G}$ (so denoted to distinguish it from our original Lie group $G$). Let $\alpha(g) : R \to R$ (map of rings) be this map. Say $M$ is a module over $R$.

**Definition 19.** A $\mathcal{G}$-equivariant structure on $M$ is a $\mathcal{G}$-action on $M$ such that

$$(gr)(gm) = g(rm) \tag{6}$$

For an automorphism $\alpha(g)$ of $R$, the pushforward $\alpha(g)_*(M)$ twists the action of $R$ by $M$ by $\alpha(g) : R \to R$. Formula (6) is equivalent to the $g$-action on $M$ defining a map $M \to \rho(g)_*(M)$. Now suppose $X$ is a scheme with action by a group $\mathcal{G}$. We modify the above definition of an equivariant module as follows.

**Definition 20.** Given a scheme $X$ with action $\alpha : \mathcal{G} \to \mathrm{Aut}(X)$, an $\mathcal{G}$-*equivariang sheaf over* $X$ is a coherent sheaf $M$ together with a collection of isomorphisms $\rho(g) : M \to \alpha(g)_* M$ with the following compatibility condition:

$$\big(\alpha(g_1)_*[\rho(g_2)]\big)\rho(g_1) = \rho(g_2 g_1) : M \to \alpha(g_2 g_2)_*(M). \tag{7}$$

It goes without saying that both the action $\alpha(g)$ on $X$ and the mapping $g \mapsto \rho(g)$ must be algebraic in $g$ in the obvious sense.

The category of $\mathcal{G}$-equivariant sheaves as above is abelian and monoidal and will be denoted $\mathrm{coh}^{\mathcal{G}}(X)$. When $X$ is affine and $M$ is a module, the above simply encodes the associativity condition on the action $\rho$ of $\mathcal{G}$ on $M$.

*Remark* 21. Note also that pullbacks could be substituted for pushforwards in the above using the canonical isomorphism $\alpha(g)_* M \cong \alpha(g^{-1})^* M$.

In the following we will give several ways of constructing classes of equivariant sheaves, which we will use for our construction of new functors from $\mathrm{coh}^{\mathcal{G}}(\tilde{N})$ to itself.

**Definition 22.** Suppose $V$ is a representation of $G$. Consider the sheaf $O(V) = O_X \otimes V$. Then for any map $f : X \to X$ — in particular, for action maps $f = \alpha(g^{-1})_*$, we have canonically $f^* O(V) \cong O(V)$. Using this identification, define the equivariance maps $\rho(g) : O(V) \to O(V)$ using the representation action of $g$ on $V$.

As a special case of this when $\mathcal{G} = \mathbb{C}^*$, the character sheaf $O(\chi)$ given by the one-dimensional representation $\chi : \mathbb{C}^* \to \mathbb{C}$. Another case of interest for us is the character $\chi_\lambda : B \to \mathbb{C}^*$ given by pulling back to $B$ a weight $\lambda : T \to \mathbb{C}^*$.

**Theorem 23.** *Suppose $G$ acts freely on a space $X$ and $M$ is an equivariant sheaf over $X$. Then*

1. *The sheaf $M$ is locally free along $G$-orbits $O_x$ of $X$.*

2. *The restriction of $M$ to $O_x$ has a unique $G$-equivariant section taking a given value at the specific point $x \in O_x$.*

3. *The above spaces glue into a sheaf $M/G$ on the quotient $X/G$. Namely, an open subset $U_0$ of $X/G$ is equivalent to a $G$-equivariant subset $U \subset X$, and we define $\Gamma(U_0, M/\mathcal{G}) = \Gamma(U, M)^{\mathcal{G}}$.*

4. *Further, this construction commutes with tensor product of equivariant bundles on $X$.*

Applying this to the space $X = G$ and $\mathcal{G} = B$, we have (as for any $B$-equivariant space) a character sheaf $O_\lambda$ on $X = G$. This induces a bundle, which we will denote similarly by $O(\lambda)$, on $X/\mathcal{G} = G/B$. By item 2 above, this is a line bundle. The bundle is, by construction, equivariant with respect to left multiplication by $G$.

Define functors $\Theta_\lambda : \mathrm{coh}(\tilde{N}) \to \mathrm{coh}(\tilde{N})$ to be twists by the line bundles $O(\lambda)$. By part 4 of the lemma above, these satisfy $\Theta(\lambda) \circ \Theta(\lambda') = \Theta(\lambda\lambda')$, and in particular the functors $\Theta(\lambda)$ span a lattice isomorphic to the root lattice $R$.

**Theorem 24.** *Suppose $T_\alpha$ is a generator of the braid group* $\mathrm{Br}$ *corresponding to the simple reflection $s_\alpha$. Then the functors $\Theta_\lambda$ and $\Phi_\lambda$ commute if $(\lambda, \check{\alpha}) = 0$ and satisfy the relation*

$$\Phi_\alpha \Theta_{s_\alpha(\lambda)} \Phi_\alpha = \Theta_\lambda \tag{8}$$

*when $(\lambda, \check{\alpha}) = 1$.*

We are now set up for the first variation on the braid group action from the "theme". Namely, consider the group spanned by the group Br (generated by $T_\alpha$) and the weight lattice $R$, with elements $v_\lambda$ for $\lambda \in R$. Impose the relation 8,

$$T_\alpha v_{s_\alpha(\lambda)} T_\alpha = v_\lambda.$$

**Lemma 25.** *The free product on the two groups* Br *and $R$ modulo the relation above gives a presentation for the affine braid group $Br_{A\!f\!f}$, the braid group corresponding to the affine root system $\hat{\mathfrak{g}}$.*

The proof is in the appendix of [BR]. Now we have a new group acting in the first variation: namely, the affine braid group $\mathrm{Br}_{\mathrm{Aff}}$. We want to understand its image in End of the Grothendieck group of an appropriate category. In this variation, we take the category $\mathrm{coh}^G(\tilde{N})$ of $G$-equivariant sheaves on $\tilde{N}$. The $O(\lambda)$ are naturally $G$-equivariant by construction, hence the functors $\Theta_\lambda$ can be taken to be $G$-equivariant. Further, the $\Lambda_t^w$ are clearly $G$-equivariant for regular $g$, and hence so are the $\Lambda_0^w$.

**Lemma 26.** *On the level of Grothendieck groups, the action of $Br_{A\!f\!f}$ factors through the Weyl group $W_{A\!f\!f}$ of the affine root system.*

*Remark* 27. The bundles $O(\lambda)$ are nontrivial already without $G$ equivariance, hence we could study the action of $\mathrm{Br}_{\mathrm{Aff}}$ (with generators corresponding to $\Phi_w, \Theta_\lambda$) already on $\mathrm{coh}(\tilde{N})$. The image in endomorphisms of the Grothendieck group, however, would be less interesting and the standard modules (constructed below) would be less useful.

*Remark* 28. The affine Weyl group $W_{\mathrm{Aff}}$ is isomorphic to $W \ltimes R$.

### 2.1 Standard modules

For the standard modules in this setting we again want to take equivariant $K$ theory of the $\mathbb{B}_e$ as $e$ runs over different nilpotent orbits. These fibers only have action by the stabilizer $G_e$, and not full equivariance. Still, both the $\Phi_w$ and the $O(\lambda)$ act on $K^{G_e}(\mathrm{coh}(\tilde{N})_{\mathbb{B}-\mathrm{sup}})$ with affine braid group relations, and these actions factor through the Weyl group $W_{\mathrm{Aff}}$ on the level of $K^0$.

## 3 Appendix: the Chern character and Riemann-Roch.

### 3.1 The Chern character

**Definition 29.** Suppose $P$ is a monic polynomial of degree $n$ over $\mathbb{C}$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $P$. Define

$$\Psi^n(P) = \prod_{i=1}^{n}(x - \alpha_i).$$

*Remark* 30. Note that the coefficients of $\Psi^n(P)$ are polynomial expressions of the coefficients of $P$, with coefficients in $\mathbb{Z}$. In particular $\Psi^n(P)$ is defined for $P$ with coefficients in an arbitrary (commutative) rings.

Suppose $X$ is a vector space and $V/X$ is a vector bundle. Recall that associated to $V$ is a "total Chern class", $c(V)(t) = 1 + c_1(V)t + c_2(V)t^2 + \ldots$, homogeneous if $t$ has cohomological degree $-2$.

**Definition 31.** The topological Chern character

$$\chi(V) = \sum \frac{\Psi^n(c(V)(t))}{n!}.$$

In general this is in $\prod H^i(X)$ (with potentially infinitely many nonzero terms), although in the cases of interest to us $X$ is finite-dimensional, so a direct sum is sufficient. By default the parameter $t$ is set to 1, as giving $\chi(V)(1)$ is equivalent to giving $\chi(V)(t)$ by considerations of equivariance.

**Lemma 32.** *1. $\chi(V') + \chi(V'') = \chi(V)$ if $0 \to V'' \to V \to V' \to 0$ is a short exact sequence, and $\chi(V')\chi(V'') = \chi(V') \otimes \chi(V'')$.*

2. *$\chi$ induces a map (abusing notation) $\chi : K^0(X, \mathbb{Q}) \to \prod H^i(X, \mathbb{Q})$, which is a map of rings.*

We can extend the notion of Chern character to singular spaces at the cost of giving up multiplicativity. Namely, recall that Poincaré duality gives an isomorphism for $X$ smooth of dimension $n$ (but not necessarily closed!) $H^i(X) \cong H^{BM}_{n-i}(X)$. We have the following lemma.

**Lemma 33.** *There is a notion of Chern class with values in Borel-Moore homology, $\chi : \mathrm{Ob}(\mathrm{coh}(X)) \to H^{BM}_*(X)$ satisfying the following properties.*

1. *$\chi([A]) + \chi([C]) = \chi([B])$ for $0 \to A \to B \to C \to 0$ exact*

2. *$\chi$ is functorial with respect to pullback of spaces.*

*These two properties can be shown to characterize this generalization. In the cases of interest to us, life will be much simpler, as for Springer fibers both $H^{BM}_*(\mathbb{B}_e)$ and $K^0(\mathbb{B}_e)$ are spanned by classes of subvarieties.*

### 3.2 Todd Classes

### 3.3 Grothendieck-Riemann-Roch

## References

[BR] Bezrukavnikov-Riche, `arXiv:1101.3702`

[CG] Representation Theory and Complex Geometry by Neil Chriss and Victor Ginzburg