Lecture 4

1) Submodules & quotient modules.
2) Finitely generated and free modules.
References: [AM], Chapter 2, Sections 2,3,5.

1.1) Submodules: Let A be a comm've ring.
Definition: let M be an A-module; a submodule in M is an abelian subgroup $N \subset M$ s.t. $a \in A, n \in N \Rightarrow an \in N$.
Rem: N has a natural A-module str're.
Examples: 0) $\{0\}, M \subset M$ are submodules.
1) A is a field (so module = vector space): Submodule = subspace.
2) $A = \mathbb{Z}$ (so module = abelian group): Submodule = subgroup.
3) $A = \mathbb{F}[x]$ ($\mathbb{F}$ is a field). A module M = $\mathbb{F}$-vector space w. operator $X: M \to M$. A submodule $N \subset M$ - subspace s.t. $X(N) \subseteq N$. Conversely, every X-stable subspace is submodule b/c $f(x)m = f(X)m$ & $X(N) \subseteq N \Rightarrow f(X)(N) \subset N$.
4) A is any ring, $M = A$: submodule = ideal.

1.2) Constructions w. submodules.
1) $\varphi: M \to N$ A-module homom'm: $\ker\varphi \subset M$ & $\operatorname{im}\varphi \subset N$ are submodules, left as exercise.
2) $m_1, \ldots m_k \in M \rightsquigarrow \operatorname{Span}_A(m_1, \ldots m_k) := \{\sum_{i=1}^k a_i m_i \mid a_i \in A\}$ - this is special case of image: $\underline{m} = (m_1, \ldots m_k) \rightsquigarrow \varphi_{\underline{m}}: A^{\oplus k} \to M$
$\varphi_{\underline{m}}(a_1, \ldots a_k) := \sum_{i=1}^k a_i m_i$ (see the last example in 2.3 of

Lecture 3). Then $\mathrm{Span}_A(m_1, \dots m_k) = \mathrm{im}\, \varphi_{\underline{m}}$. Note also that this generalizes the ideal generated by a given collection of elements (3.1 of Lecture 1). More generally, for an index set I, $(m_i)_{i \in I} \rightsquigarrow \mathrm{Span}_A(m_i \mid i \in I) := \{$finite A-linear combinations of m_i's$\}$.

3) Sums & intersections: $M_1, M_2 \subset M$ submodules $M_1 \cap M_2$, $M_1 + M_2 = \{m_1 + m_2 \mid m_i \in M_i\}$ - submodules.

4) Product w. ideal: $N \subset M$ submodule, $I \subset A$ ideal $IN := \{\sum_{i=1}^{\ell} a_i n_i \mid a_i \in I, n_i \in N\}$ - submodule, exercise. (compare to product of ideals in 1.1 of Lecture 2).

1.3) Quotient modules: M is A-module, $N \subset M$ submodule $\rightsquigarrow$ abelian group $M/N = \{m + N \mid m \in M\}$ & abelian group homom'm $\pi: M \twoheadrightarrow M/N$, $\pi(m) := m + N$. Then M/N has a natural A-module str're. The following is analogous to Proposition in 3.2 of Lecture 1.

Proposition: 1) The map $A \times (M/N) \to M/N$, $(a, m+N) \mapsto am + N$ is well-defined ($am + N$ only depends on $m + N$ & not on m itself) and equips M/N w. A-module structure.

2) This module str're is unique s.t. $\pi: M \to M/N$ is a module homomorphism.

3) (Universal property of M/N & $\pi: M \twoheadrightarrow M/N$.) Let $\varphi: M \to M'$ be A-module homom'm s.t. $N \subset \ker \varphi$. Then

$\exists!$ module homom'm $\underline{\varphi}: M/N \to M'$ s.t. the following diagram is commutative

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ M/N \arrow[r, dashed, "\underline{\varphi}"'] & M' \end{tikzcd}$$

$\underline{\varphi}$ is given by: $\underline{\varphi}(m+N) := \varphi(m)$

Proof: exercise.

Remarks:

1) Let $I \subset A$ be ideal $\rightsquigarrow$ submodule $IM \subset M \rightsquigarrow$ quotient M/IM, an A-module; $a \in I \Rightarrow a(m+IM) = am + IM = 0$. Apply Observation II in 2.2 of Lec 3 to $A \twoheadrightarrow A/I$. We get that M/IM is an A/I-module. For example, if $I = \mathfrak{m}$ is maximal $\Rightarrow A/\mathfrak{m}$ is a field so $M/\mathfrak{m}M$ is vector space over $A/\mathfrak{m}$. This gives one way to reduce the study of modules over rings to study of vector spaces over fields.

2) We have standard "isomorphism theorems":

- for $\varphi: M \to N$, A-module homom'm, then $M/\ker\varphi \xrightarrow{\sim} \operatorname{im}\varphi$ (A-module isomorphism).
- for submodules $K \subset N \subset M$, have $(M/K)/(N/K) \xrightarrow{\sim} M/N$.
- for submodules $N_1, N_2 \subset M$, have $N_1/N_1 \cap N_2 \xrightarrow{\sim} (N_1+N_2)/N_2$

Have natural abelian group isomorphisms, they are module homomorphisms.

- There are bijections between:

$$\begin{tikzcd} \{\text{submodules } L \subset M \mid N \subset L\} \arrow[d, bend right, "L \mapsto \pi(L) = L/N"'] \\ \{\text{submodules } \underline{L} \subset M/N\} \arrow[u, bend right, "\underline{L} \mapsto \pi^{-1}(\underline{L})."'] \end{tikzcd}$$

We have seen a similar claim for ideals in 3.2 of Lecture 1.

2.1) Finitely generated modules

Definition: • Elements $m_i \in M$ ($i \in I$) are generators (a.k.a. spanning set) of M if $M = \mathrm{Span}_A(m_i \mid i \in I)$, i.e. $\forall m \in M$ is A-linear combination of finite number of m_i's.

• M is finitely generated if it has a finite spanning set.

Remarks: • $A^{\oplus I}$ is finitely generated $\Leftrightarrow$ I is finite.

• If M is fin. generated, then so is M/N $\forall N \subset M$:

$M = \mathrm{Span}_A(m_1, \dots m_k) \Rightarrow M/N = \mathrm{Span}_A(\pi(m_1), \dots, \pi(m_k))$.

• $\underline{m} = (m_1, \dots m_k) \rightsquigarrow \varphi_{\underline{m}}: A^{\oplus k} \to M$, $\varphi_{\underline{m}}(a_1, \dots a_k) = \sum_{i=1}^{k} a_i m_i$.

$M = \mathrm{Span}_A(m_1, \dots m_k) \Leftrightarrow \varphi_{\underline{m}}$ is surj've $\Rightarrow M \xrightarrow{\sim} A^{\oplus k}/\ker \varphi_{\underline{m}}$

So: fin. gen'd modules = quotients of $A^{\oplus k}$ for some $k \in \mathbb{Z}_{\geq 0}$.

2.2) Free modules.

Definition: • Elements m_i, $i \in I$, form a basis in M if $\forall m \in M$ is uniquely written as A-linear combination of m_i, $i \in I$.

• M is free if it has a basis.

Examples: 1) For any set I, $A^{\oplus I}$ is free, for a basis can take coordinate vectors $e_i, i \in I$: $e_i = (0,..,0,1,0...)$
ith position

2) If A is field, then every module (a.k.a. vector space) is free. If A is not a field, there are non-free modules: let $J \subset A$ be ideal, $J \neq \{0\}, A \Rightarrow A/J$ is not free (over A). Indeed, for any vector e in a basis we must have $ae \neq 0$ $\forall a \in A$. But for any $e \in A/J$ we have $ae = 0$ $\forall a \in J$.

Remark: Every free module is isomorphic to $A^{\oplus I}$ for some set I: choose basis $m_i \in M$ $(i \in I)$: $\varphi_{\underline{m}}: A^{\oplus I} \xrightarrow{\sim} M$.

Lemma: Every basis in $M = A^{\oplus k}$ has exactly k elements.
Proof: Assume the contrary: $\exists\, \ell \neq k$ w. $A^{\oplus k} \xrightarrow{\sim} A^{\oplus \ell}$. As in the case of fields, any A-linear map $A^{\oplus k} \to A^{\oplus \ell}$ is given by multiplication w. uniquely determined $\ell \times k$-matrix w. coeff's in A. Also for $T \in Mat_k(A)$, the map $\vec{v} \mapsto T \cdot \vec{v}: A^{\oplus k} \to A^{\oplus k}$ is invertible $\Leftrightarrow \det(T) \in A$ is invertible. WLOG, assume $k > \ell$ and let isomorphisms $A^{\oplus k} \xrightarrow{\sim} A^{\oplus \ell}$, $A^{\oplus \ell} \to A^{\oplus k}$ be given by $T_1 \in Mat_{\ell,k}(A)$, $Mat_{k,\ell}(A)$. So $\det(T_2 T_1) = 0$ b/c rows of $T_2 T_1$ are A-linear combinations of the ℓ rows of T_1 & $\ell < k$ $\square$

Remark: Why to care about modules:

– they generalize various interesting objects (vector spaces w. n pairwise commuting operators, abelian groups).

– every ideal in A & every A-algebra can be viewed as modules (e.g. ideal = A-module embedded into A). Some properties we care about depend just on a module str're, e.g. whether an ideal can be generated by a single element.

– certain modules, e.g. "projective" $\Leftrightarrow$ "locally free" are important from the geometric perspective. More on this in the end of the class.

Up next: Modules generalize vector spaces. When we study vector spaces in Linear Algebra, we concentrate on finite dimensional ones. What's an analog of finite dimensional vector spaces for modules? The first guess would be that we need to consider finitely generated modules. However, there's an issue: while a subspace in a finite dimensional vector space is finite dimensional, a submodule in a finitely generated module may fail to be finitely generated.

Example: I infinite set, $A := \mathbb{F}[x_i]_{i\in I}$

Module $M := A$ (gen'd by 1), $N := (x_i)_{i\in I}$

Claim: N is not finitely generated.

Recall every $f\in A$ is polynomial in fin. many variables. If $f_1, \ldots f_k$

$\in N \rightsquigarrow \sum_{i=1}^{k} g_i f_i \ (g_i \in A)$: deg 1 terms in $\sum_{i=1}^{k} g_i f_i \in$
$\mathrm{Span}_{\mathbb{F}}($ deg 1 terms of $f_1, \dots f_k)$ - only involves fin. many X_i's.
so $f_1, \dots f_k$ cannot span N.

A module, where every submodule is finitely generated, is called Noetherian. We will study this property in the next lecture.