MAT 380, F19 FINAL, DUE DEC 15, 2019

The take home final is due Dec 15, 2019, by the end of day. Either e-mail the instructor or upload it to Canvas (preferred). There are four problems worth 10 points each. The maximum is 40 points.

All rings below are commutative and have 1.

Problem 1. Let M be an A-module. Prove that the functor $M \otimes_A \bullet : A\text{-Mod} \to A\text{-Mod}$ is left adjoint of $\mathrm{Hom}_A(M, \bullet) : A\text{-Mod} \to A\text{-Mod}$.

Problem 2. Let M be a finitely generated A-module. Recall that an A-module is called cyclic if it is isomorphic to the quotient of A by an ideal.

1, 5pts) Prove that there is a chain of submodules $\{0\} = M_0 \subset M_1 \subset \ldots \subset M_k = M$ such that M_i/M_{i-1} is cyclic.

2, 5pts) Assume now that A is a domain. Prove that the number of indexes i such that $M_i/M_{i-1} \cong A$ is independent of the choice of $M_1, \ldots, M_{k-1}$ as in part 1).

Problem 3. Let A_1, A_2 be $\mathbb{C}$-algebras.

1, 5pts) Let $\mathfrak{m} \subset A_1 \otimes_{\mathbb{C}} A_2$ be a maximal ideal such that $(A_1 \otimes_{\mathbb{C}} A_2)/\mathfrak{m} \cong \mathbb{C}$. Prove that there are maximal ideals $\mathfrak{m}_i \subset A_i, i = 1, 2$, such that $\mathfrak{m} = \mathfrak{m}_1 \otimes_{\mathbb{C}} A_2 + A_1 \otimes_{\mathbb{C}} \mathfrak{m}_2$.

2, 5pts) Let $\mathfrak{p} \subset A_1 \otimes_{\mathbb{C}} A_2$ be a prime ideal. Is it true that there are always prime ideals $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p} = \mathfrak{p}_1 \otimes_{\mathbb{C}} A_2 + A_1 \otimes_{\mathbb{C}} \mathfrak{p}_2$?

Problem 4. Consider the ring $A = \mathbb{C}[x]/(x^3)$ and its modules $M = \mathbb{C}[x]/(x)$ and $N = \mathbb{C}[x]/(x^2)$.

1, 3pts) Compute $\mathrm{Ext}^i_A(M, N)$ as an A-module for all i.

2, 3pts) Compute $\mathrm{Tor}_i^A(M, N)$ as an A-module for all i.

3, 4pts) Prove that neither M nor N have finite projective resolutions.