Williamson
Lecture 2

Last time: why is KL conj. true? $\to$ BB loc. + calc-n of stalks of IC's
$\downarrow$
$\mathcal{O}_0 \curvearrowleft \mathcal{H}$ (Hecke cat.) + Hodge th. of $\mathcal{U}$

In char p: hope Rep $G \curvearrowleft \mathcal{H}$ — missing

G - reduct. alg. gr-p / $k = \bar{k}$ - field of char p
$\mathrm{Rep}(G)$ = abelian cat. of rational rep-s of G.
Goal: understand Rep G.
$G \supset B \supset T$, $X = X^*(T) \supset X^+$ (domin. wts), $R \subset X$ - roots
$\lambda \in X^+ \rightsquigarrow \nabla(\lambda) = \mathrm{Ind}_B^G \lambda$ has simple socle $L(\lambda)$
$\Delta(\lambda) \twoheadrightarrow L(\lambda)$
Weyl module
$\mathrm{Irr}\, G = \{L(\lambda) \mid \lambda \in X^+\}$
W_f - finite Weyl gr-p, $S_f \subset W_f$ - finite simple refl-ns
$W = W_p = p\mathbb{Z}R \rtimes W_f \curvearrowright X$
Warning: W is affine Weyl gr-p (of Langlands dual group)
$x \cdot \lambda = x(\lambda + \rho) - \rho$
Linkage principle $\mathrm{Rep}\, G = \bigoplus_{\Omega} \mathrm{Rep}_{\Omega} G$, $\Omega \in X/W\cdot$, $\mathrm{Rep}\,\Omega = \langle L(\lambda) \mid \lambda \in \Omega \cap X^+ \rangle$

Rem: • Block decomposition is finer
• can use red-n of center of $U(\mathfrak{g})$ + action of center of G to get this decomp-n (at least for $p \gg 0$)

Assume $p \geq h \iff \mathrm{Stab}_{W\cdot} 0 = \{1\}$
$\rightsquigarrow \mathrm{Rep}_0 = \mathrm{Rep}_{W\cdot 0} G$ (principal block)
(highest wts of simples in Rep_0 is $W \cdot 0 \cap X^+ \xrightarrow{\sim} W_f \backslash W \xrightarrow{\sim} {}^f W$ — min. coset reps

$\xrightarrow{\sim}$ dominant alcoves

Jantzen's translation principle: all blocks of $\operatorname{Rep} G$ are equivalent to or "simpler than" Rep_0.

Rem: situation is more difficult for $p<h$

Tilting modules

T is <u>tilting</u> if it has a Δ-flag & ∇-flag

Tilt = the additive cat-y of tilting modules

Tilt_0 - princ. block

Why tiltings?

1) indec. tilt $\xleftrightarrow{\sim} X^+$ (via taking highest wt)

$\{ch\, T(\lambda)\}$ gives positive basis of $(\mathbb{Z}X)^W$

2) $\operatorname{Ext}^i(\Delta(\lambda), \nabla(\mu))=0$ unless $i=0, \lambda=\mu \Rightarrow \operatorname{Ext}^i(T(\lambda), T(\mu))=0,\ \forall i>0$

$\rightsquigarrow K^b(\operatorname{Tilt}) \xrightarrow{\sim} D^b(\operatorname{Rep})$ (so Tilt is min. homol. skeleton)

3) $T, T' \Rightarrow T\otimes T' \in \operatorname{Tilt} \Rightarrow$ translation functors act on Tilt.

4) $G = GL_n$, V-nat-l module, $V^{\otimes m} = \bigoplus_\lambda V_\lambda \otimes T(\lambda)$

V_λ's are simple modules for $k S_m$

knowing $ch\, T(\lambda) \Leftrightarrow$ knowledge of all $\leq n$ row decomp. #'s for S_m's

$ch\, T(\lambda)$: known for GL_2, open for GL_3.

Lusztig character formula: $p \geq h$ & $y\cdot 0$ is restricted, then

$$ch\, L(y\cdot 0) = \sum (-1)^{\ell(y)+\ell(x)} P_{w_0x, w_0y}\, ch\, \Delta(x\cdot 0)$$

$\rightsquigarrow$ characters of all $L(y)$

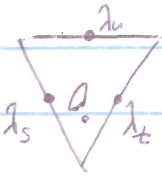
- true for $p \gg 0$ & false for $h \leq p \leq c^h$ (c is const.)

In cat. $\mathcal{O}$: $\operatorname{Ext}^\bullet(\bigoplus_x L_x) \underset{\text{Koszul}}{=} \operatorname{End}(\bigoplus P_x) \underset{\text{Ringel}}{=} \operatorname{End}(\bigoplus_x T_x)$

Back to $\operatorname{Rep} G$.

When LCF doesn't hold, then $\mathrm{Ext}^\bullet(\bigoplus L_x)$ is not formal

Translation functors



fix an S-singular λ_s, $s \in S$

$$\mathrm{Rep}_0 \underset{\Theta_s^{on}}{\overset{\Theta_s^{out}}{\rightleftarrows}} \mathrm{Rep}_{\lambda_s} \quad \text{- biadj-t functors}$$

Set $\mathcal{V}_s = \Theta_s^{out} \circ \Theta_s^{on}$ (from now on $L_x := L(x\cdot 0)$, $\Delta_x := \Delta(x \cdot 0)$)

Basic fact: $[\Delta_x \mathcal{V}_s] = \begin{cases} [\Delta_x] + [\Delta_{xs}] & \text{if } xs \in {}^f W \\ 0 & \text{, else} \end{cases}$

Hence $\mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W \xrightarrow{\sim} [\mathrm{Rep}_0] \circlearrowleft \mathcal{V}_s$

$(1+s) \circlearrowright$

$$\begin{array}{ccc} \mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W & \xrightarrow{\sim} & [\mathrm{Rep}_0] \\ \Psi & & \Psi \\ 1 \otimes x & \longmapsto & [\Delta_x] \\ x \in {}^f W & & \end{array}$$

So Rep_0 is a categorification of anti-spherical module

Moral: translation functors acting on Rep_0 provide this categorification

p-canonical basic

$(W,S) \curvearrowright \mathfrak{h} \rightsquigarrow$ Hecke cat-y $\mathcal{H}$ (by generators & rel-ns), $\mathcal{H} = \mathrm{Kar}(\mathcal{H}_{BS})$

1) $\mathcal{H}$ is a graded monoidal additive cat-y, shift functor (1)

2) $\mathcal{H}$ is Krull-Schmidt

3) $\mathcal{H}$ is gen-d by $\{B_s\}_{s \in S}$

4) $\exists!$ isom. $H \longrightarrow [\mathcal{H}]$ of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$\underline{h}_s := h_s + v \longmapsto [B_s] \quad (v[B] = [B(1)])$$

$[\mathcal{H}] \xrightarrow{ch} H$ (inverse)

5) $\forall x \in W$ $\exists!$ (up to iso) B_x s.t. B_x is indec. &

$$ch(B_x) = h_x + \sum_{y<x} {}^p h_{y,x} h_y, \quad {}^p h_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$$

The map $(x,n) \mapsto B_x(n)$ is $W \times \mathbb{Z} \xrightarrow{\sim} \mathrm{Indec}(\mathcal{H})$

Def: ${}^p\underline{h}_x := ch(B_x)$ is p-canon basis of $\mathcal{H}$

Rem: • doesn't change if we extend $\Bbbk$

• basis depends on $\mathfrak{h}$ (2-canon. basis for $C_3 \neq$ for B_3)

• basis may be alger. computed (slowly...)

Categorifying anti-spherical module

From now on, W is affine Weyl group

$AS := sgn \otimes_{H_f} H$, where sgn is given by $1 \cdot h_s = -v$

Lem: $AS \xleftrightarrow{\sim} H / \bigoplus_{s \in S_f} \underline{h}_s H = H / \bigoplus_{x \notin {}^fW} \mathbb{Z}[v^{\pm 1}] \underline{h}_x$

KL basis

Define: $\mathcal{AS} := \mathcal{H} / \langle B_x \mid x \notin {}^fW \rangle_{\oplus, [m]}$ - quot-t of additive categories

1) $\mathcal{AS}$ is graded, additive, Krull-Schmidt right $\mathcal{H}$-module

$\Rightarrow [\mathcal{AS}] \curvearrowleft [\mathcal{H}] = H$

2) $[\mathcal{AS}] \xleftrightarrow{\sim} AS$ (right H-modules)

3) ${}^fW \times \mathbb{Z} \xrightarrow{\sim} Indec(\mathcal{AS})$

Get a p-canonical basis in anti-spherical module: $n_x := 1 \otimes h_x$

${}^p\underline{n}_x := \sum {}^pn_{y,x}\, n_y$, ${}^p\underline{n}_x = 1 \otimes {}^p\underline{h}_x$

Let $G \supset B \supset T$ be as above

Rem: $W \to W_f \curvearrowright \mathfrak{h} := (\cancel{\quad} \otimes_{\mathbb{Z}} \Bbbk)$ $\mathbb{Z}R$

$\rightsquigarrow W \curvearrowright R(\mathfrak{h}) \rightsquigarrow \mathcal{H}$

Conj (GW-Riche): can choose adjunctions for $(\Theta_s^{on}, \Theta_s^{out})$, $(\Theta_s^{out}, \Theta_s^{on})$ s.t. $\mathbb{1} \to \mho_s$, $\mho_s \to \mathbb{1}$, $\mho_s^2 \to \mho_s$, $\mho_s \to \mho_s^2$ s.t. assignment $B_s \to \mho_s$ & $2m_{st}$-valent vertex from lecture 1 s.t. these data defines a right $\mathcal{H}$-module structure on $\mathcal{H}_0$

Rem: Conj is true for GL_n (via KLR theory)

Rmk: In Lec 1 need rel-n $= \boxed{\alpha_s}$

Since $\mathfrak{h}$ = Span (root lattice), don't need to specify action of R

Thm (GW+SR) Suppose conj. holds. Then $\exists$ essent. surj. functor
$\phi: \mathcal{AS} \longrightarrow \mathrm{Tilt}_0$ of right $\mathcal{H}$-modules s.t. ϕ induces equivalence after forgetting grading on the l.h.s, i.e. $\mathcal{AS}$ is graded lift of Tilt_0
($\phi(B_x) = T_x$)

Cor: $(T_x : \Delta_y) = {}^p n_{y,x}(1)$

Rem: there's also conj. for small p (checked against all known decomp. #'s for S_n)

Thm: $\mathcal{H} \xrightarrow{\sim} \mathrm{Parity}_B(G/B)$, $p \neq 2$ ($\forall$ Kac-Moody G)

Can also define a t-structure on $K^b(\mathcal{AS})$ w heart = graded lift of Rep_0

Rmk: $p \geq 2h-2$ (shouldn't be necessary — should $p \geq h$) one can deduce a formula for simple characters (not replacing $h_{w_0x, w_0y}(1)$ by ${}^p h_{w_0x, w_0y}(1)$)

$$ch\, L(y \cdot 0) = \sum (-1)^{\ell(x)+\ell(y)} \, {}^p g_{xy}(1) \; ch\, \Delta(x \cdot 0)$$

inverse matrix to ${}^p h_{xy}$