# Wakimoto modules

## §0. Setup

$$\begin{array}{c}\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h} \\ \cap \\ \mathfrak{g}\end{array} = \text{simple Lie alg.}, \quad \varkappa = \mathfrak{g}\text{-inv. bilinear form} \rightsquigarrow \hat{\mathfrak{g}}_\varkappa$$

$U_\varkappa(\mathfrak{g}) = U(\hat{\mathfrak{g}}_\varkappa)/(\mathbb{1} - 1)$

$\tilde{U}_\kappa(\mathfrak{g}) \curvearrowright V_\kappa(\mathfrak{g})$

$\kappa_{\mathfrak{g}}$ = Killing form, $\kappa_c = -\frac{1}{2}\kappa_{\mathfrak{g}}$

$\mathfrak{g} = \bigoplus_a \mathbb{C} \cdot J_a$, $\{J_a\}$ is a weighted basis of $\mathfrak{g}$

$$G \curvearrowright G/N_- \supset \underset{H}{B_+ \circlearrowleft} \rightsquigarrow \mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^H = \mathrm{Vect}(N_+) \oplus \mathbb{C}[N_+] \otimes \mathfrak{h}$$
$$\simeq \mathrm{Vect}(\mathfrak{n}_+) \oplus \mathbb{C}[\mathfrak{n}_+] \otimes \mathfrak{h}$$
$$\simeq \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{n}_+ \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{h}$$

differential operators

$$\rightsquigarrow U(\mathfrak{g}) \longrightarrow \mathcal{D}(B_+)^H = \mathcal{D}(N_+) \otimes U(\mathfrak{h})$$

$\big\downarrow$ affine analogue

(★) Thm 6.2.1 $\exists$ map of $\mathbb{Z}$-graded VA (satisfying some conditions)

$$w_\kappa : V_\kappa(\mathfrak{g}) \longrightarrow M_{\mathfrak{g}} \otimes V_{\kappa - \kappa_c}(\mathfrak{h})$$

universal $\rightsquigarrow$ enveloping alg

$$\tilde{U}_\kappa(\mathfrak{g}) \longrightarrow \tilde{\mathcal{A}}^{\mathfrak{g}} \hat{\otimes} \tilde{U}_{\kappa - \kappa_c}(\mathfrak{h})$$

**Def of $M_{\mathfrak{g}}$**

central ext'n

$$\hat{\Gamma} \twoheadrightarrow \Gamma = \mathfrak{n}_+((t)) \oplus \mathfrak{n}_+^*((t))\,dt$$

$[xf, yw] = \langle x, y \rangle \mathrm{Res}\, fw \cdot \mathbb{1}$

$$\begin{array}{ccl} \hat{\Gamma} & & \\ \cup & & \\ \Gamma_+ & = & \mathfrak{n}_+[[t]] \oplus \mathfrak{n}_+^*[[t]]\,dt \end{array}$$

$$\rightsquigarrow \tilde{\mathcal{A}}^{\mathfrak{g}} = \widetilde{U(\hat{\Gamma})/(\mathbb{1}-1)} \curvearrowright M_{\mathfrak{g}} = \mathrm{Ind}_{\Gamma_+ \oplus \mathbb{C}\mathbb{1}}^{\hat{\Gamma}} \mathbb{C}|0\rangle$$

$\Gamma_+ \cdot |0\rangle = 0$
$\mathbb{1} \cdot |0\rangle = |0\rangle$

**Def of $V_\nu(\mathfrak{h})$**

central ext'n given by sym bil. form $\nu$

$$\hat{\mathfrak{h}}_\nu \twoheadrightarrow \mathfrak{h}((t))$$

$[xf, yg] = -\nu(x, y)\, \mathrm{Res}\, fdg \cdot \mathbb{1}$

$$\begin{array}{c} \hat{\mathfrak{h}}_\nu \\ \cup \\ \mathfrak{h}[[t]] \end{array}$$

$$\rightsquigarrow \tilde{U}_\nu(\mathfrak{h}) = \widetilde{U(\hat{\mathfrak{h}}_\nu)/(\mathbb{1}-1)} \curvearrowright V_\nu(\mathfrak{h}) = \mathrm{Ind}_{\mathfrak{h}[[t]] \oplus \mathbb{C}\mathbb{1}}^{\hat{\mathfrak{h}}_\nu} \mathbb{C}|0\rangle$$

$$\lambda \in \mathfrak{h}^* \rightsquigarrow \pi_\nu^\lambda := \mathrm{Ind}_{\mathfrak{h}[[t]] \oplus \mathbb{C}\mathbb{1}}^{\hat{\mathfrak{h}}_\nu} \mathbb{C}|\lambda\rangle \in \mathrm{Mod}_{\tilde{U}_\nu(\mathfrak{h})}$$

$b \otimes t^n |\lambda\rangle = \delta_{n,0} \lambda(b) |\lambda\rangle \quad (b \in \mathfrak{h})$
$\mathbb{1}|\lambda\rangle = |\lambda\rangle$

$$\rightsquigarrow \tilde{U}_\kappa(\mathfrak{g}) \longrightarrow \tilde{\mathcal{A}}^{\mathfrak{g}} \hat{\otimes} \tilde{U}_{\kappa - \kappa_c}(\mathfrak{h}) \curvearrowright M_{\mathfrak{g}} \otimes \pi_{\kappa - \kappa_c}^\lambda =: W_{\lambda, \kappa} \in \mathrm{Mod}_{\tilde{U}_\kappa(\mathfrak{g})}$$

this is called Wakimoto module of level $\kappa$, highest wt $\lambda$.

They will play essential role in the proof of FF center thm.

### Example

$\mathfrak{g} = \mathfrak{sl}_2$, $\varkappa_0(X, Y) = \mathrm{tr}(XY)$

$\varkappa_{\mathfrak{g}} = 4\kappa_0$, $\varkappa_c = -2\kappa_0$, $\varkappa = k \cdot \kappa_0$

$\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f \qquad \kappa_0(e, f) = 1, \quad \kappa_0(h, h) = 2$

copy of e      copy of h

$\mathfrak{n}_+ = \mathbb{C} \cdot a$, $\mathfrak{n}_+^* = \mathbb{C}a^*$, $\mathfrak{h} = \mathbb{C} \cdot b$

$\rightsquigarrow N_+ = \mathrm{Spec}\, \mathbb{C}[a^*]$, $a = \frac{\partial}{\partial a^*} \in \mathrm{Vect}(N_+)$

$$\begin{array}{rcl} \mathfrak{sl}_2 & \longrightarrow & \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{n}_+ \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{h} \simeq \mathrm{Vect}(B_+)^H \\ e & \longmapsto & a \\ h & \longmapsto & -2a^*a + b \\ f & \longmapsto & -a^{*2}a + a^*b \end{array}$$

$([a, a^*] = 1)$

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$$\hat{\Gamma} \twoheadrightarrow \Gamma = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot a_n \bigoplus_{m \in \mathbb{Z}} \mathbb{C} a_m^* \qquad [a_n, a_m^*] = \delta_{n,-m} \mathbb{1}$$

$M_{\mathfrak{g}} = \mathcal{A}^{\mathfrak{g}} . |0\rangle$ where $a_n|0\rangle = 0 \quad n \geq 0$
$a_m^*|0\rangle = 0 \quad n \geq 1$

annihilating operators

$\deg a_n = \deg a_n^* = -n$

$[T, a_n] = -n a_{n-1} \quad [T, a_n^*] = -(n-1) a_{n-1}^*$

$Y(a_{-1}|0\rangle, z) = \sum a_n z^{-n-1} =: a(z)$

$Y(a_0^*|0\rangle, z) = \sum a_m^* z^{-m} =: a^*(z)$

$: \text{monomial in } a_n, a_m^* : \;=$ move annihilating operators to the right

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$$\hat{\mathfrak{h}}_{k+2} \twoheadrightarrow \mathfrak{h}((t)) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} b_n$$

$[b_n, b_m] = 2(k+2) n \delta_{n,-m} \cdot \mathbb{1}$

$V_{k+2}(\mathfrak{h}) = U_{k+2}(\mathfrak{h}) \cdot |0\rangle$ where $b_n|0\rangle = 0 \quad n \geq 0$

$Y(b_{-1}|0\rangle, z) = \sum b_n z^{-n-1} =: b(z)$

$[T, b_n] = -n b_{n-1}$

When $\varkappa = \varkappa_c$ , $\widetilde{U}_0(\mathfrak{h}) = \mathrm{Fun}(\mathfrak{h}^*((t)))$

$$w_{\kappa_c} : \widetilde{U}_{\kappa_c}(\mathfrak{g}) \longrightarrow \widetilde{\mathcal{A}}^{\mathfrak{g}} \hat{\otimes} \mathrm{Fun}(\mathfrak{h}^*((t)))$$

$$\chi(t) \in \mathfrak{h}^*((t)) \rightsquigarrow \widetilde{U}_{\kappa_c}(\mathfrak{g}) \longrightarrow \widetilde{\mathcal{A}}^{\mathfrak{g}} \circlearrowright M_{\mathfrak{g}} =: W_{\chi(t)} \in \mathrm{Mod}_{\widetilde{U}_{\kappa_c}(\mathfrak{g})}$$

called <u>Wakimoto module of critical level</u>

<u>Rmk</u> · $W_{\chi(t)}, W_{\lambda,\kappa} \in \mathcal{O}$

$$\mathcal{O} = \{ V \in \underset{\substack{\text{affine Kac-Moody alg.}\\ \| \\ (\mathbb{C}\mathbb{1}\oplus\mathfrak{g}((t)))\rtimes\mathfrak{g}_{m,rot}}}{\hat{\mathfrak{g}}}\text{-mod} \mid \cdot\, V = \bigoplus_{\lambda\in\underset{\sim\ \text{Cartan of } \hat{\mathfrak{g}}}{\hat{\mathfrak{h}}^*}} V_\lambda \ \text{ weight space decomp}, \ \dim V_\lambda < \infty$$

$$\cdot\ \exists\ \lambda_1,\dots,\lambda_n \in \hat{\mathfrak{h}}^* \text{ s.t. all weights } \in \bigcup_{i=1}^{n} (\lambda_i - \mathbb{Z}_{\geq 0} \underbrace{\hat{\Phi}_+}_{\text{positive affine roots}}) \quad \}$$

· $W_{\chi(t)}$ are simple for some $\chi(t)$

## §1. <u>How to construct $V_\kappa(\mathfrak{g}) \longrightarrow V$ ?</u>

<u>Lem 6.1.1</u>

Let $V = \mathbb{Z}$-graded vertex algebra , the following data are in bijection

- A $\mathbb{Z}$-graded vertex alg. hom. $V_\kappa(\mathfrak{g}) \longrightarrow V$
- $x_a \in V$, $a = 1,\dots,\dim\mathfrak{g}$ , $\deg x_a = 1$ s.t.

$$\begin{tikzcd} w \arrow[d, mapsto] \\ x_a = w(J_{a,-1}|0\rangle) \end{tikzcd}$$

$$\begin{aligned} \hat{\mathfrak{g}}_\kappa &\longrightarrow \mathrm{End}(V) \quad \text{defines a Lie algebra homomorphism} \\ J_{a,n} &\longmapsto x_{a(n)} \\ \mathbb{1} &\longmapsto \mathrm{id} \end{aligned}$$

<u>Proof</u> only "$\Leftarrow$" needs a proof

$$Y(x_a, z)|0\rangle \in V[[z]] \Rightarrow x_{a(n)}|0\rangle = 0 \text{ for } n \geq 0$$

$$\xrightarrow{\substack{\text{universal}\\ \text{property}\\ \text{of induced}\\ \text{module}}} V_\kappa(\mathfrak{g}) \longrightarrow V \quad \text{linear map}$$

$$J_{a_1,n_1}\cdots J_{a_m,n_m}|0\rangle \longmapsto x_{a_1(n_1)}\cdots x_{a_m(n_m)}|0\rangle$$

<u>Ex</u> Check this is a map of VA

$\square$

## §2. (☆) for $\mathfrak{sl}_2$

**Thm 6.2.1 for $\widehat{\mathfrak{sl}_2}$**

$$\exists \quad w_k : V_k(\mathfrak{sl}_2) \longrightarrow M_{\mathfrak{sl}_2} \otimes V_{k+2}(\mathfrak{h}) \quad \text{map of VA s.t.}$$

$$e_{-1}|0\rangle \longmapsto a_{-1}|0\rangle \quad =: \tilde{e}_{-1}|0\rangle \qquad \text{deg 1, } \underbrace{\text{wt } 2}_{\text{w.r.t. } h}$$

$$h_{-1}|0\rangle \longmapsto (-2a_0^* a_{-1} + b_{-1})|0\rangle \quad =: \tilde{h}_{-1}|0\rangle \qquad \text{deg 1, wt 0}$$

$$f_{-1}|0\rangle \longmapsto (-a_0^{*2} a_{-1} + a_0^* b_{-1} + k a_{-1}^*)|0\rangle \quad =: \tilde{f}_{-1}|0\rangle \qquad \text{deg 1, wt } -2$$

Compare

$$\mathfrak{sl}_2 \longrightarrow \overset{\overset{a^*}{\cap}}{\mathrm{Sym}\,\mathfrak{n}_+^*} \otimes \overset{\overset{a}{\cap}}{\mathfrak{n}_+} \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \overset{\overset{b}{\cap}}{\mathfrak{h}}$$

$$e \longmapsto a$$

$$h \longmapsto -2a^*a + b$$

$$f \longmapsto -a^{*2}a + a^*b$$

**Rmk** finite dim'l formulas + deg + wt pin down RHS

**Proof** Denote $Y(e_{-1}|0\rangle, z) = e(z)$, $Y(\tilde{e}_{-1}|0\rangle, z) = \tilde{e}(z)$

Use thm 6.1.1, sufficies to check commutator relations of $\tilde{e}_n, \tilde{h}_n, \tilde{f}_n$, hence sufficies to check

$w_k$ preserves OPE for $e(z)\cdot f(w)$, $h(z)\cdot f(w)$, $h(z)\cdot e(w)$

**Proof for "e · f"**

$$\tilde{e}(z)\cdot\tilde{f}(w) \sim \sum_{n\geq 0} \frac{Y(\tilde{e}_{(n)}\tilde{f}_{-1}|0\rangle, w)}{(z-w)^{n+1}}$$

$$= \sum_{n\geq 0} \frac{Y(a_n\cdot(-a_0^{*2}a_{-1} + a_0^* b_{-1} + k a_{-1}^*)|0\rangle, w)}{(z-w)^{n+1}} \quad \in \mathbb{C}[z,w]((\tfrac{z}{w}))$$

only $n = 0, 1$ get non-zero terms

$$= \frac{Y((-2a_0^* a_{-1} + b_{-1})|0\rangle, w)}{z-w} + \frac{k}{(z-w)^2}$$

$$= \frac{\tilde{h}(w)}{z-w} + \frac{k}{(z-w)^2}$$

$$e(z)\cdot f(w) \sim \frac{h(w)}{z-w} + \frac{k}{(z-w)^2}$$

**Ex** Do the same for $\underset{\text{more work}}{h\cdot f}$, $\underset{\text{easy}}{h\cdot e}$ □

## §3. Conformal structures in $sl_2$-case

Assume $k \neq -2$

normalized Sugawara element

Recall $S_k = \frac{1}{2(k+2)}(e_{-1}f_{-1} + f_{-1}e_{-1} + \frac{1}{2}h^2_{-1})|0\rangle \in V_k(sl_2)$ is a conformal vector, $S_k(z) := Y(S_k, z)$

w/ central charge $c_k = \frac{3k}{k+2}$ i.e. $S_k(z)S_k(w) = \frac{c_k/2}{(z-w)^4} + O(\frac{1}{(z-w)^3})$

**Prop 6.2.2** $w_k : V_k(sl_2) \longrightarrow M_{sl_2} \otimes V_{k+2}(\mathfrak{h})$ satisfies

$$w_k(S_k) = (a_{-1}a^*_{-1} + \frac{1}{4(k+2)}b^2_{-1} - \frac{1}{2(k+2)}b_{-2}) \cdot |0\rangle$$

**Proof** $w_k(S_k)$ has deg $-2$, wt $0$ ( $\deg a_i = i$, $\deg a^*_i = i$, $\deg b_i = i$

$\Rightarrow w_k(S_k) \in \mathrm{Span}(b^2_{-1}, b_{-2}, a_{-1}a^*_{-1}$ wt $a_i = 2$, wt $a^*_i = -2$, wt $b_i = 0$ )

$a_{-2}a^*_0$, $a^2_{-1}a^{*2}_0$, $a_{-1}a^*_0 b_{-1}$ )

$\leftarrow$ only possible monomials s.t. deg $= -2$, wt $= 0$

$Y(w_k(S_k), z) =: \sum L_n z^{-n-2}$, $\deg L_n = -n$

**Observation 1** $L_n \cdot P(a^*_0)|0\rangle = 0$ for $n \geq 0$, $P(a^*_0) \in \mathbb{C}[a^*_0]$

**Proof** $n > 0$ true for deg reason

$n = 0$ $L_0 \cdot P(a^*_0)|0\rangle = \frac{1}{2(k+2)}(:e_0f_0: + :f_0e_0: + \frac{1}{2}:h_0^2: + \underbrace{\text{other terms}}\ ) \cdot P(a^*_0)|0\rangle$

: $\deg 0$ monomial : $\cdot P(a^*_0)|0\rangle \neq 0 \Rightarrow$ monomial $\in \mathbb{C}[a^*_0, a_0] \Rightarrow \overset{''}{0}$

$= \frac{1}{2(k+2)}(:a_0 \cdot (-a^{*2}_0 a_0): + :(-a^{*2}_0 a_0) \cdot a_0: + :\frac{1}{2}(-2a^*_0 a_0)^2:) P(a^*_0)|0\rangle$

$= 0$

abuse of notation
means putting annihilating to the right $\square$

However, the ( )$_{(1)}$ part of above monomials acts on $\mathbb{C}[a^*_0] \cdot |0\rangle$ by

$(b^2_{-1})_{(1)}|_{\mathbb{C}[a^*_0]|0\rangle}$ similar for other terms

$\overset{''}{0}$, $0$, $0$

$-a^*_0 a_0$, $a^{*2}_0 a^2_0$, $0$ viewed as differential operators on $\mathbb{C}[a^*_0]$

$\Rightarrow w_k(S_k) \in \mathrm{Span}(b^2_{-1}, b_{-2}, a_{-1}a^*_{-1}, a_{-1}a^*_0 b_{-1}) \cdot |0\rangle$

wt 2, deg $\geq 0$

**Observation 2** $L_n \cdot a_{-1}|0\rangle = 0$ $n > 0$

$L_0 \cdot a_{-1}|0\rangle = a_{-1}|0\rangle$

$\overset{''}{w_k(L_0 e_{-1}|0\rangle)}$ ''

On the other hand, $(b_{-1}^2)_{(1)} \cdot a_{-1}|0\rangle = 0$

$$(b_{-2})_{(1)} \cdot a_{-1}|0\rangle = 0$$

$$(a_{-1}a^*_{-1})_{(1)} \cdot a_{-1}|0\rangle = a_{-1}|0\rangle$$

$$(a_{-1}a^*_0 b_{-1})_{(1)} \cdot a_{-1}|0\rangle = 0$$

$$\Rightarrow w_k(S_k) \in (a_{-1}a^*_{-1} + \mathrm{Span}(b_{-1}^2 \,,\, b_{-2} \,,\, a_{-1}a^*_0 b_{-1})) \cdot |0\rangle$$

<u>Observation 3</u> $L_n\, w_k(h_{-1}|0\rangle) = \overset{\overset{= w_k(L_n h_{-1}|0\rangle)}{\|}}{0} \qquad n > 0$

$$L_0\, w_k(h_{-1}|0\rangle) \underset{w_k(L_0 h_{-1}|0\rangle)}{=} w_k(h_{-1}|0\rangle)$$

On the other hand,

$$w_k(h_{-1}|0\rangle) = (-2a^*_0 a_{-1} + b_{-1})|0\rangle$$

$L_0$-part $\qquad$ $L_1$-part

all non-zero terms

$$\underbrace{Y(b_{-1}^2|0\rangle, z)}_{2b_{-1}b_1 z^{-2}} \cdot (-2a^*_0 a_{-1} + b_{-1})|0\rangle = 4(k+2)\, b_{-1}|0\rangle \cdot z^{-2} + 0 \cdot z^{-3} + \cdots$$

$$\underbrace{Y(b_{-2}|0\rangle, z)}_{-2b_1 z^{-3}} \cdot (-2a^*_0 a_{-1} + b_{-1})|0\rangle = \qquad -4(k+2)|0\rangle \cdot z^{-3}$$

$$\underbrace{Y(a_{-1}a^*_{-1}|0\rangle, z)}_{-a_{-1}a^*_1 z^{-2} - a_0 a^*_1 z^{-3}} \cdot (-2a^*_0 a_{-1} + b_{-1})|0\rangle = -2a^*_0 a_{-1}|0\rangle \cdot z^{-2} - 2 \cdot |0\rangle z^{-3}$$

$$\underbrace{Y(a_{-1}a^*_0 b_{-1}|0\rangle, z)}_{(a_{-1}a^*_0 b_1 + a_0 a^*_1 b_{-1}) z^{-2}} \cdot (-2a^*_0 a_{-1} + b_{-1})|0\rangle = (2(k+2)a^*_0 a_{-1} + 2b_{-1})|0\rangle z^{-2} + 0 \cdot z^{3} + \cdots$$

$$\Rightarrow w_k(S_k) = (a_{-1}a^*_{-1} + \frac{1}{4(k+2)} b_{-1}^2 - \frac{1}{2(k+2)} b_{-2})|0\rangle \qquad \square$$

<u>Rmk</u> $a^*_{-1}a_{-1}|0\rangle \in M_{sl_2}$ is a conformal vector of $M_{sl_2}$ w/ central charge $2$

$\frac{1}{4(k+2)} b_{-1}^2 - \frac{1}{2(k+2)} b_{-2}$ is a conformal vector of $V_{k+2}(\mathfrak{h})$ w/ central charge $\frac{k-4}{k+2}$

## §4. (☆) for general $\mathfrak{g}$

$\{\alpha_i\}$ = simple roots

$(a_\alpha = \text{copy of } e_\alpha)$ $\quad (b_i = \text{copy of } h_i)$

Recall
$$\mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^H = \overset{\mathbb{C}[a_\alpha^* | \alpha\in\Phi_+]}{\overset{\shortparallel}{\mathrm{Sym}\,\mathfrak{n}_+^*}} \otimes \overset{\bigoplus_{\alpha\in\Phi_+}\mathbb{C}\cdot a_\alpha}{\overset{\shortparallel}{\mathfrak{n}_+}} \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \overset{\bigoplus_i \mathbb{C} b_i}{\overset{\shortparallel}{\mathfrak{h}}}$$
$$e_i \longmapsto a_{\alpha_i} + \sum_{\beta\in\Delta_+} P^i_\beta(\underline{a}^*)\, a_\beta$$
$$h_i \longmapsto -\sum_{\beta\in\Delta_+} \beta(h_i)\, a_\beta^* a_\beta + b_i$$
$$f_i \longmapsto \sum_{\beta\in\Delta_+} Q^i_\beta(\underline{a}^*)\cdot a_\beta + a^*_{\alpha_i} b_i$$

Affine $\rightsquigarrow$ analogue

Thm 6.2.1 $\exists$ map of VA

$$w_\kappa : V_\kappa(\mathfrak{g}) \longrightarrow M_{\mathfrak{g}} \otimes V_{\kappa-\kappa_c}(\mathfrak{h})$$
$$e_{i,-1}|0\rangle \longmapsto \Big(a_{\alpha_i,-1} + \sum_{\beta\in\Delta_+} P^i_\beta(\underline{a}^*_0)\, a_{\beta,-1}\Big)|0\rangle \qquad \text{deg } 1,\ \text{wt } \alpha_i$$
$$h_{i,-1}|0\rangle \longmapsto \Big(-\sum_{\beta\in\Delta_+} \beta(h_i)\, a^*_{\beta,0} a_{\beta,-1} + b_{i,-1}\Big)|0\rangle \qquad \text{deg } 1,\ \text{wt } 0$$
$$f_{i,-1}|0\rangle \longmapsto \Big(\sum_{\beta\in\Delta_+} Q^i_\beta(\underline{a}^*_0)\, a_{\beta,-1} + a^*_{\alpha_i,0} b_{i,-1} + \underbrace{(c_i + (\kappa-\kappa_c)(e_i, f_i))}_{} a^*_{\alpha_i,-1}\Big)|0\rangle \qquad \text{deg } 1,\ \text{wt } -\alpha_i$$

Rmk finite dim'l case + deg + wt pin down RHS except

## §5. Conformal structures in general case

Recall $S_\kappa = \frac{1}{2}\sum \overset{\text{dual under } \kappa-\kappa_c}{\overbrace{J_{a,-1} J^a_{-1}}} |0\rangle$ is a conformal vector of $V_\kappa(\mathfrak{g})$ $(\kappa \neq \kappa_c)$

$Y(S_\kappa|0\rangle, z) =: \sum L_\kappa z^{-\kappa-2} \rightsquigarrow [L_\kappa, J_{a,n}] = -n J_{a,n+m}$

Prop 6.2.2

For $\kappa \neq \kappa_c$

$w_\kappa : V_\kappa(\mathfrak{g}) \longrightarrow M_{\mathfrak{g}} \otimes V_{\kappa-\kappa_c}(\mathfrak{h})$ satisfies

$$w_\kappa(S_\kappa) = \Big(\underbrace{\sum_{\alpha\in\Delta_+} a_{\alpha,-1} a^*_{\alpha,-1}}_{\in\, M_{\mathfrak{g}}\otimes 1} + \underbrace{\frac{1}{2}\sum_{i=1}^{\ell} \overset{\text{dual under } \kappa-\kappa_c}{\overbrace{b_{i,-1} b^i_{-1}}} - \overset{\text{dual under } \kappa-\kappa_c \text{ to } \rho\in\mathfrak{h}^*}{\rho_{-2}}}_{\in\, 1\otimes V_{\kappa-\kappa_c}(\mathfrak{h})}\Big)|0\rangle$$

Proof Similar to $\mathfrak{sl}_2$-case $\square$

## §6. Quasi-conformal structures

$$\mathrm{Der}_+\mathcal{O} = \mathbb{C}\cdot\{L_1, L_2, \cdots\}$$

**Recall** $\mathrm{Der}\,\mathcal{O} = \mathbb{C}\cdot\{L_{-1}, L_0, L_1, \cdots\}$ , $L_k = -t^{k+1}\partial_t$

$$\cap$$

$$\mathrm{Vir} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}L_i \oplus \mathbb{C}\cdot C$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{m,-n}\cdot C$$

**Def** A quasi-conformal structure on a $\mathbb{Z}$-graded VA is

$\mathrm{Der}\,\mathcal{O} \circlearrowright V$ s.t.

- $[L_m, A_{(k)}] = \sum_{n\geq -1}\binom{m+1}{n+1}(L_n\cdot A)_{(m+k-n)}$ for all $A\in V$
- $L_{-1} = T$
- $L_0 =$ grading
- $\mathrm{Der}_+\mathcal{O}$ acts nilpotently

e.g. A conformal vector $\omega\in V \rightsquigarrow Y(\omega, z) = \sum L_n z^{-n-2}$

$\rightsquigarrow L_n \in \mathrm{End}_{\mathbb{C}}(V)$ $n = -1, 0, 1, \cdots$

$\rightsquigarrow$ quasi-conformal structure on $V$

e.g. For $V = V_\kappa(\mathfrak{g})$ $(\kappa \neq \kappa_c)$

$\omega = S_\kappa \rightsquigarrow L_n\cdot J_{a,m}|0\rangle = -mJ_{a,m+n}|0\rangle$ (*)

When $\kappa = \kappa_c$ , $S_{\kappa_c}$ doesn't exist , but (*) still makes sense

and defining $\mathrm{Der}\,\mathcal{O} \circlearrowright V_{\kappa_c}(\mathfrak{g})$ , which is a quasi-conformal structure