MAT 380, HOMEWORK 6, DUE DEC 6

There are 10 problems with 24 points total. Your score for this homework is the minimum of the total number of points and 20.

Problem 1, 2pts. This problem is about operations with complexes. Let $(M_\bullet, d_M), (N_\bullet, d_N)$ be complexes.

1, 1pt) Define the direct sum of these complexes.

2, 1pt) Define the tensor product (should be a complex).

Problem 2, 3pts. We consider the category $\mathsf{Comp}(A)$ of complexes of A-modules as well as its full subcategories $\mathsf{Comp}^{\leqslant 0}(A), \mathsf{Comp}^{\geqslant 0}(A)$ consisting of all complexes $(M_\bullet, d)$ with $M_i = \{0\}$ for $i > 0$ and $M_i = \{0\}$ for $i < 0$, respectively (*recall that "full" means that the sets of morphisms in the subcategory are the same as in the ambient category*). Recall that A-Mod is embedded into $\mathsf{Comp}(A)$ as the subcategory of all complexes $(M_\bullet, d)$ with $M_i = \{0\}$ for $i \neq 0$.

1, 1pt) Prove that $H_i(?)$ is a functor $\mathsf{Comp}(A) \to A$-Mod.

2, 1pt) Prove that $H_0(?) : \mathsf{Comp}^{\geqslant 0}(A) \to A$-Mod is left adjoint to the inclusion functor A-Mod $\hookrightarrow \mathsf{Comp}^{\geqslant 0}(A)$.

3, 1pt) Prove that $H_0(?) : \mathsf{Comp}^{\leqslant 0}(A) \to A$-Mod is right adjoint to the inclusion functor A-Mod $\hookrightarrow \mathsf{Comp}^{\leqslant 0}(A)$.

Problem 3, 2pts. Consider the following commutative diagram of A-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\varphi_1"] & M_2 \arrow[r] \arrow[d, "\varphi_2"] & M_3 \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
0 \arrow[r] & M_1' \arrow[r] & M_2' \arrow[r] & M_3' \arrow[r] & 0
\end{tikzcd}$$

Use the long exact sequence in homology to produce an exact sequence

$$0 \to \ker \varphi_1 \to \ker \varphi_2 \to \ker \varphi_3 \to \operatorname{coker} \varphi_1 \to \operatorname{coker} \varphi_2 \to \operatorname{coker} \varphi_3 \to 0.$$

Problem 4, 3pts. Let $P_k \to P_{k-1} \to \ldots \to P_0$ be a complex of projective modules. Prove that the following claims are equivalent:

a) The homology is zero (equivalently, the complex is exact if we complete it with zeroes).

b) The identity endomorphism of this complex is homotopic to zero.

c) The complex is isomorphic to the direct sum of complexes of the form $P \xrightarrow{\mathrm{id}} P$ with shifts (shifts of complexes were defined in Section 1 of Lecture 22).

Problem 5, 2pts. Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$ as a complex. Prove that the identity endomorphism of this complex is not homotopic to zero.

Problem 6, 2pts. Let $M_\bullet, M_\bullet', M_\bullet''$ be complexes of A-modules and $\varphi : M_\bullet \to M_\bullet', \varphi' : M_\bullet' \to M_\bullet''$ be homomorphisms of complexes. Prove that if $\varphi \sim 0$ or $\varphi' \sim 0$, then $\varphi'\varphi \sim 0$.

Problem 7, 2pts. Let A be an integral domain and $x, y \in A$ be elements such that the class of x is not a zero divisor in $A/(y)$. Produce a free resolution of $A/(x,y)$ with three terms.

Problem 8, 2pts. Extend the assignment that sends $M \in A$-Mod to its canonical free resolution

$$\ldots \to A^{\oplus \ker d_1} \xrightarrow{d_2} A^{\oplus \ker \pi} \xrightarrow{d_1} A^{\oplus M}$$

(here π is the natural surjection $A^{\oplus M} \twoheadrightarrow M$) to a functor A-Mod $\to \mathsf{Comp}^{\geqslant 0}(A)$.

Problem 9, 4pts. Let $(P_\bullet, d), (P'_\bullet, d')$ be two complexes of projective A-modules such that $P_i = P'_i = \{0\}$ for all $i < 0$. Let $\varphi : P_\bullet \to P'_\bullet$ be such that $H_i(\varphi) = 0$ for all i.

a, 2pts) Prove that $\varphi \sim 0$ if $H_i(P_\bullet) = H_i(P'_\bullet) = \{0\}$ for all $i > 0$.

b, 2pts) Prove $\varphi \sim 0$ in general.

Problem 10, 4pts. Let $A = \mathbb{F}[x, y]$, where $\mathbb{F}$ is a field, $I := (x, y), M := A/I$ and J be a maximal ideal in A different from I. Compute the following Ext^1-modules.

1) $\mathrm{Ext}^1_A(A/J, A/I)$.
2) $\mathrm{Ext}^1_A(I, A/I)$.
3) $\mathrm{Ext}^1_A(A/I, I)$.
4) $\mathrm{Ext}^1_A(A/I, A)$.