MATH 380, HOMEWORK 6, DUE DEC 9

There are 8 problems worth 27 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-5. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are commutative.

Problem 1, 2pts total. *Exactness and Homs.* Let M_1, M_2, M_3 be A-modules and let $\varphi_1 : M_1 \to M_2$ and $\varphi_2 : M_2 \to M_3$ be A-linear maps with $\varphi_2\varphi_1 = 0$.

1, 1pt) Prove that $0 \to M_1 \to M_2 \to M_3$ is exact if and only if the corresponding sequence

$$0 \to \operatorname{Hom}_A(N, M_1) \to \operatorname{Hom}_A(N, M_2) \to \operatorname{Hom}_A(N, M_3)$$

is exact for every A-module N.

2, 1pt) Prove that $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if the corresponding sequence

$$0 \to \operatorname{Hom}_A(M_3, N) \to \operatorname{Hom}_A(M_2, N) \to \operatorname{Hom}_A(M_1, N)$$

is exact for every A-module N.

Problem 2, 3pts total. *Adjointness implies opposite exactness.* Let A, B be rings. Let an additive functor $F : A\text{-Mod} \to B\text{-Mod}$ be left adjoint to an additive functor $G : B\text{-Mod} \to A\text{-Mod}$. Suppose that the bijections $\eta_{X,Y}$ in the definition of adjoint functors (Section 3 of Lecture 18) are abelian group homomorphisms.

1, 1pts) Show that F is right exact and G is left exact.

2, 2pts) Show that the following claims are equivalent:

(i) F sends projective modules to projective modules.
(ii) $F(A)$ is a projective B-module.
(iii) G is exact.

Problem 3, 4pts total. *This problem indicates why tensor product functors are ubiquitous.* Let A be a Noetherian ring. We write $A\text{-mod}$ for the category of finitely generated A-modules. Let F be an additive functor $A\text{-mod} \to \mathbb{Z}\text{-Mod}$. Further, let For be the forgetful functor

$$A\text{-mod} \to \mathbb{Z}\text{-Mod}$$

1, 1pt) Equip $F(A)$ with an A-module structure so that the underlying abelian group structure is the default structure on $F(A)$, i.e., the structure coming from the fact that the target of F is $\mathbb{Z}$-Mod. *Hint: use that F is a functor.*

2, 1pt) Let M_1, M_2 be objects of $A\text{-mod}$. Use the inclusion and projection maps between $M_i, i = 1, 2$, and $M_1 \oplus M_2$ to establish an isomorphism $F(M_1 \oplus M_2) \cong F(M_1) \oplus F(M_2)$.

3, 2pts) Suppose that F is right exact. Establish an isomorphism of functors $\mathsf{For}(F(A) \otimes_A \bullet)$ and F.

There's a direct analog of 2) for left exact functors $A\text{-mod}^{opp} \to \mathbb{Z}\text{-Mod}$*, these are Homs to some A-module. And with some care you can replace* $\mathbb{Z}\text{-Mod}$ *with categories of modules over more general rings.*

Problem 4, 4pts. *That ideal in* $\mathbb{Z}[\sqrt{-5}]$ *is here one last time*! Let $A = \mathbb{Z}[\sqrt{-5}]$ and I be the ideal $(2, 1+\sqrt{-5})$.

1, 1pt) Prove that the surjective homomorphism $\pi : A^{\oplus 2} \twoheadrightarrow I, (a,b) \mapsto 2a + (1+\sqrt{-5})b$ splits, i.e., there $\iota : I \to A^{\oplus 2}$ with $\pi\iota = \mathrm{id}_I$. Deduce that I is projective.

2, 1pt) Show that I is not free (*and so we get an example of a finitely generated projective module that is not free*).

3, 1pt) Identify the kernel of π with I and conclude that $I \oplus I \cong A \oplus A$.

4, 1pt) Prove that $I \otimes I \cong A$ (*hint: this is a part of this problem not just because it deals with the ideal* I).

Problem 5, 2pts. Consider $A = \mathbb{C}[x,y]$ and the ideal $I = (x,y)$. Prove that the A-module I is not projective.

Problem 6, 4pts total. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ and I be an ideal in A. Set $J = \sqrt{I}$.

1, 2pts) Prove that there is $n > 0$ such that $J^n \subset I$.

2, 2pts) Use part (1) to conclude that the following two claims are equivalent:

(a) $V(I) \subset \mathbb{C}^n$ is finite.

(b) I has finite codimension in A, i.e., $\dim_{\mathbb{C}} A/I < \infty$.

Problem 7, 4pts. Let $\mathbb{F}$ be an infinite (for simplicity) field. Let M be an $\mathbb{F}[x_1, \ldots, x_n]$-module. Prove that the following two claims are equivalent:

(a) M is finite dimensional over $\mathbb{F}$.

(b) M has finite length as a module over $\mathbb{F}[x_1, \ldots, x_n]$.

Problem 8, 4pts. Let $\mathbb{F}$ be an algebraically closed field. Let $X \subset \mathbb{F}^n, Y \subset \mathbb{F}^m$ be algebraic subsets, $A := \mathbb{F}[X], B := \mathbb{F}[Y]$ be their algebras of polynomial functions. Finally, let $\tau : B \to A$ be an algebra homomorphism, and $\psi_\tau : X \to Y$ be the corresponding polynomial map. Recall the bijections between X (resp., Y) and algebra homomorphisms $A \to \mathbb{F}$ (resp., $B \to \mathbb{F}$), *see the 2nd corollary in Section 1.4 of Lecture 22.* Prove that, under this identification, ψ_τ sends a homomorphism $\eta : A \to \mathbb{F}$ to $\eta \circ \tau$.