

Lec 2: Differential operators in char p

1. Weyl algebra

2. Differential operators.

1) Setting: p is prime (for applications $p \gg 0$), $\mathbb{F} = \bar{\mathbb{F}}$, $\text{char } \mathbb{F} = p$.

1.1) The center:

$\mathcal{A} = \mathbb{F}\langle x, \partial \rangle / (\partial x - x\partial = 1) \cap \mathbb{F}[x]$ by diff. op. generators.

If $\text{char } \mathbb{F} = 0$: \mathcal{A} is simple, $\mathbb{F}[x]$ is simple module (& faithful)

- exercise.

For $\text{char } \mathbb{F} = p$: $\mathbb{F}[x]$ is neither

- faithful: $\partial^p x^n = n(n-1)\dots(n-p+1)x^{n-p} = 0$

- simple: $(x^p) \subset \mathbb{F}[x]$ is a submodule.

\mathcal{A} has "big center": $x^p, \partial^p \in \mathcal{A}$ are central.

Central reduction: $\alpha, \beta \in \mathbb{F} \rightsquigarrow \mathcal{A}_{\alpha, \beta} := \mathcal{A} / (x^p - \alpha, \partial^p - \beta)$.

Proposition: i) x^p, ∂^p generate a subalgebra isomorphic to $\mathbb{F}[x^p, \partial^p]$

ii) $\mathbb{F}[x^p, \partial^p] = \text{center of } \mathcal{A}$

iii) \mathcal{A} is a free \mathbb{F} module over $\mathbb{F}[x^p, \partial^p]$

iv) $\mathcal{A}_{\alpha, \beta} \cong \text{Mat}_p(\mathbb{F}) \ncong \mathbb{F}$.

Proof:

\mathcal{A} has basis of $x^i \partial^j = [i = i_0 + pi_1, j = j_0 + pj_1, i_0, j_0 \in \{0, \dots, p-1\}]$
 $= x^{i_0} \partial^{j_0} (x^p)^{i_1} (\partial^p)^{j_1} \Rightarrow \text{iii) \& iv)}$.

Proof of iv):

Special case: $\alpha = \beta = 0$: $\mathcal{A}_{0,0} = \mathcal{A} / (x^p, \partial^p)$. Consider \mathcal{A} -module

$\mathbb{F}[x]/(x^p) \cap \mathcal{A}$ factors through $\mathcal{A}_{\alpha, \beta}$.

Exercise: $\mathbb{F}[x]/(x^p)$ is a simple $\mathcal{A}_{\alpha, \beta}$ -module.

$$\Rightarrow \mathcal{A}_{\alpha, \beta} \rightarrow \text{End}_{\mathbb{F}}(\mathbb{F}[x]/(x^p))$$

have $\dim = p^2$

so get an isomorphism.

General case - α, β are arbitrary: for $a, b \in \mathbb{F}$, then $x \mapsto x+a$,

$\partial \mapsto \partial + b$ extends to an autom'm of \mathcal{A} ; $x^p \mapsto (x+a)^p = x^p + a^p$

$\partial^p \mapsto \partial^p + b^p$: take $a = \sqrt[p]{\alpha}$, $b = \sqrt[p]{\beta} \rightsquigarrow \mathcal{A}_{\alpha, \beta} \xrightarrow{\sim} \mathcal{A}_{\alpha, \beta}$. \square of iv).

Proof of ii): if $D \in \text{center of } \mathcal{A}$, then it projects to (center of $\mathcal{A}_{\alpha, \beta}$) = \mathbb{F} $\nabla \alpha, \beta \in \mathbb{F}$.

Exercise: use that $\mathbb{F}[x^p, \partial^p]$ is normal to deduce that $\mathbb{F}[x^p, \partial^p] = \text{center of } \mathcal{A}$ \square

Rem: Direct analog of Proposition holds for

$$D(A^n) = \mathbb{F}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / ([\partial_i, x_j] = \delta_{ij})$$

1.2) Azumaya algebras.

Let X be scheme, \mathcal{R} is a coherent sheaf of \mathcal{O}_X -algebras.

Left & right actions of \mathcal{R} on itself \rightsquigarrow alg. homom'm

$$(1) \quad \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{R}^{\text{opp}} \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{R}).$$

Definition: \mathcal{R} is **Azumaya** if

(i) \mathcal{R} is a vector bundle

(ii) (1) is isomorphism.

Remark: If X is finite type scheme over $\mathbb{F} (= \bar{\mathbb{F}})$, then (ii)
 \Leftrightarrow (ii'): $\forall x \in X \Rightarrow$ the fiber R_x is a matrix algebra.

Example 0) V is a vector bundle on $X \rightsquigarrow R = \text{End}_{\mathcal{O}_X}(V)$

Such Azumaya algebras are called **split**.

1) $\text{char } \mathbb{F} = p$ then R is an Azumaya algebra over $\mathbb{F}[x, x^{-1}]$.

Problem: In example 1), Azumaya algebra is not split.

Let X be of finite type over $\mathbb{F} = \bar{\mathbb{F}}$, $x \in X \rightsquigarrow$ completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$, complete local ring $\hat{R}_x = \hat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_X} R$.

Lemma: \hat{R}_x is a matrix algebra over $\hat{\mathcal{O}}_{X,x}$.

Proof: R_x is matrix algebra $\Leftrightarrow \exists$ idempotent $e \in R_x \mid$

$R_x \xrightarrow{\sim} \text{End}_{\mathbb{F}}(R_x e)$. By lifting of idempotents, \exists idempot

$\hat{e} \in \hat{R}_x$ specializing to $e \Rightarrow \hat{R}_x \xrightarrow{\sim} \text{End}_{\hat{\mathcal{O}}_{X,x}}(\hat{R}_x \hat{e})$;

\hat{R}_x is projective $\Rightarrow \hat{R}_x \hat{e}$ is $\Rightarrow \hat{R}_x \hat{e}$ is free $\hat{\mathcal{O}}_{X,x}$ -module \square

2) Differential operators.

Setting: $\mathbb{F} = \bar{\mathbb{F}}$, $\text{char } \mathbb{F} = p$, X_0 is smooth affine variety over \mathbb{F}

$\rightsquigarrow A_0 = \mathbb{F}[X_0]$, $V = \text{Vect}(X_0) \rightsquigarrow \mathcal{D}(X_0)$ generated by A_0, V

w. suitable relations.

Goal: generalize results of Sec 1 to $\mathcal{D}(X_0)$

2.1) Central elements.

Exercise: $\forall f \in A_0 \Rightarrow f^p$ commutes with V inside $\mathcal{D}(X_0)$
 $\Rightarrow f^p \in Z(\mathcal{D}(X_0))$ (the center)

How about "contributions" of V to $Z(\mathcal{D}(X_0))$

Exercise: Let $\partial \in \text{Der}(A_0) (\hookrightarrow \text{End}_F(A_0))$. Then $\partial^p \in \text{End}_F(A_0)$ is a derivation.

Notation: "restricted" p th power from Exercise will be denoted by $\partial^{[p]}$.

$$\partial \in V \rightsquigarrow \partial^p \in \mathcal{D}(X_0)$$

Proposition: $\partial^p - \partial^{[p]} \in Z(\mathcal{D}(X_0))$.

Exercise: $X_0 = A'$. Show
 $(x\partial)^{[p]} = x\partial$.

Remark: If $X = A''$, $\partial = \partial_i \Rightarrow \partial_i^{[p]} = 0$

Exercise: Let B be an assoc'v \mathbb{F} -algebra, $b_1, b_2 \in B$. Then
 $[b_1^p, b_2] = \text{ad}(b_1)^p \cdot b_2$.

Proof of Prop'n: $\partial^p - \partial^{[p]}$ is central $\Leftrightarrow \begin{cases} [\partial^p f] = [\partial^{[p]} f] & \forall f \in A_0 \\ [\partial^p \partial'] = [\partial^{[p]} \partial'] & \forall \partial' \in V \end{cases}$

$$\cdot [\partial^p f] = [\text{apply Excr. to } B = \mathcal{D}(X_0)] = \text{ad}(\partial)^p f = \partial^p f$$

$$\cdot [\partial^{[p]} f] = \partial^{[p]} f = \partial^p f$$

$$[\partial^P, \partial'] = [\text{apply Exer. to } B = \text{End}_F(A_0)] = \text{ad}(\partial)^P \partial' \\ = [\partial^P, \partial']. \quad \square$$

2.2) Structure of the center

Q: What algebra do $f^P, \partial^P, \partial^{[P]} \in \mathcal{D}(X_0)$ generate?

Def'n: Let U be an \mathbb{F} -vector space. Define the **Frobenius twist** $U^{(1)}$ as follows:

- $U^{(1)} :=$ the same abelian group as U .
- $a \cdot^{(1)} u := a^{1/p} \cdot u \quad \forall a \in \mathbb{F}, u \in U$.

Similarly can twist associative algs, Lie algebras etc.

Proposition: The maps $f \mapsto f^P: A_0 \rightarrow Z(\mathcal{D}(X_0))$ & $\partial \mapsto \partial^P, \partial^{[P]}: V \rightarrow Z(\mathcal{D}(X_0))$ extend to \mathbb{F} -algebra homomorphism $\mathbb{F}[T^* X_0]^{(1)} \rightarrow Z(\mathcal{D}(X_0))$, which is injective.

Lemma: $\mathcal{D}(X_0)_{\leq p-1}$ act faithfully on $\mathbb{F}[X_0]$.

Sketch of proof (of Lemma):

The claim is local. So we can assume \exists etale morphism $\psi: X_0 \rightarrow \mathbb{A}^n \cong X_0, x_1, \dots, x_n \in \mathbb{F}[X_0], \partial_1, \dots, \partial_n \in V$ -pullbacks of eponymous elements under ψ .

$\mathcal{D}(X_0)_{\leq p-1}$ is a free left $\mathbb{F}[X_0]$ -module w. basis $\partial_1^{i_1}, \dots, \partial_n^{i_n}$ w. $i_1 + \dots + i_n \leq p-1$.

Exercise: Look at how $\sum_i f_i \partial^i$ acts on monomials x^i to conclude that $\mathcal{D}(X_0)_{\leq p-1}$ acts faithfully on $\mathbb{F}[X_0]$.

□ of Lemma.

Proof of Proposition: • extension to homomorphism:

$\mathbb{F}[T^*X_0] = S_{A_0}(V)$. So we need to show that

$$\gamma: V \rightarrow Z(\mathcal{D}(X_0)) \text{ satisfies: } \begin{cases} \gamma(f\partial) = f^p \gamma(\partial) & (i) \\ \gamma(\partial_1 + \partial_2) = \gamma(\partial_1) + \gamma(\partial_2) & (ii) \end{cases}$$

$\forall f \in A_0, \partial, \partial_1, \partial_2 \in V$.

Proof of (i): Note $\forall \partial \in V \Rightarrow \partial^p f = \partial^{[p]} f \quad \forall f \in A_0$

So $\gamma(f\partial)$ & $f^p \gamma(\partial)$ act by 0 on $\mathbb{F}[X_0]$.

$$\gamma(f\partial) - f^p \gamma(\partial) = (f\partial)^p - f^p \partial^p + \text{lower order terms}$$

$\in \mathcal{D}(X_0)_{\leq p-1}$ & acts by 0 on $\mathbb{F}[X_0]$. By Lemma,

$$\gamma(f\partial) = f^p \gamma(\partial)$$

Proof of (ii): exercise

Proof of injectivity of $\mathbb{F}[T^*X_0]^{(1)} \xrightarrow{\text{is graded}} Z(\mathcal{D}(X_0))$

Rescale the grading on $\mathbb{F}[T^*X_0]^{(1)}$ so that V lives in $\deg p$

Then $\mathbb{F}[T^*X_0]^{(1)} \rightarrow \mathcal{D}(X_0)$ is a homomorphism of filt.

algebras. Its associated graded is

$$\mathbb{F}[T^*X_0]^{(1)} \longrightarrow \mathbb{F}[T^*X_0], F \mapsto F^p$$

is injective b/c $\mathbb{F}[T^*X_0]^{(1)}$ has no nilpotent elements □

Definition: Let X be a scheme over \mathbb{F} . Then its Frobenius twist $X^{(1)}$ is the same scheme over \mathbb{F} but with \mathbb{F} -linear structure twisted by $a \mapsto a^{1/p}$

If $X = \text{Spec}(B)$, then $X^{(1)} = \text{Spec}(B^{(1)})$

Previous prop'n: $\mathcal{D}(X_0)$ is a q -coherent sheaf of algebras on $(T^*X_0)^{(1)}$.

Thm: $\mathbb{F}[[T^*X_0]]^{(1)} \xrightarrow{\sim} \mathcal{Z}(\mathcal{D}(X_0))$ & $\mathcal{D}(X_0)$ is Azumaya algebra over $(T^*X_0)^{(1)}$

Sketch of proof: The question is local in X_0 , so can assume

\exists etale $X_0 \rightarrow \mathbb{A}^n \rightsquigarrow$ etale $T^*X_0 \rightarrow T^*\mathbb{A}^n \rightsquigarrow$
 $\psi^{(1)}: (T^*X_0)^{(1)} \rightarrow (T^*\mathbb{A}^n)^{(1)}$

Exercise: $\mathcal{D}(X_0) \xrightarrow[\mathcal{O}_{(T^*X_0)^{(1)}}]{} (\psi^{(1)})^* \mathcal{D}(\mathbb{A}^n)$.

This reduces the proof to case of \mathbb{A}^n - Sec 1. \square

2.3) Concluding remarks:

o) All constrns in Sec 2.1 & 2.2 glue nicely. In particular, for any smooth X_0 , \mathcal{D}_{X_0} can be viewed as Azumaya algebra on $(T^*X_0)^{(1)}$.

1) On the Poisson level have smth similar:

Exercise: • If A is a Poisson \mathbb{F} -algebra, then $\{a^P, \cdot\} = 0$
 so $A^P \subset$ Poisson center & we can view A as a Poisson

$A^{(1)}$ -algebra (via $A^{(1)} \rightarrow A^P$)

• For smooth affine X_0 , Poisson center of $\mathbb{F}[T^*X_0]$ coincides w. $\mathbb{F}[T^*X_0]^P \hookleftarrow \mathbb{F}[T^*X_0]^{(1)}$

2) Take algebraic group $G =: X_0 \curvearrowright G \cap \text{Vect}(X_0)$ & $\mathfrak{o}_g = \text{Vect}(X_0)^G$, $G \cap \mathcal{D}(X_0)$ & $\mathcal{D}(X_0)^G = U(g)$.
 $(\mathbb{F}[T^*X_0]^{(1)})^G = S(g)^{(1)}$. So have an algebra embedding
 $S(g)^{(1)} \hookrightarrow \text{center of } U(g)$, $f \mapsto \underbrace{f^P}_{\in U(g)} - \underbrace{f^{[P]}}_{\in \mathfrak{o}_g}$.

The image is called the P -center.

