LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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8. Spherical SRA

Let V be a finite dimensional vector space equipped with a symplectic form ω and Γ be a finite subgroup in $\mathrm{Sp}(V)$. Let S denote the subset of all symplectic reflections $s\in\Gamma$, i.e., all elements with $\mathrm{rk}(s-1_V)=2$ and let $S=S_1\sqcup S_2\sqcup\ldots\sqcup S_m$ be the decomposition of S into conjugacy classes. To each $s\in S$ we assign the form $\omega_s\in\bigwedge^2 V^*$ by

$$\omega_s(u,v)=\begin{cases}\omega(u,v), u,v\in \mathrm{im}(s-1_V),\\ 0, u\in\ker(s-1_V).\end{cases}$$

In Lecture 6 we have introduced the universal SRA. We pick independent variables $t,c_1,\ldots,c_m$ and consider the vector space P with basis $t,c_1,\ldots,c_m$. Then we define the algebra H by

$$H=S(P)\otimes T(V)\#\Gamma/(u\otimes v-v\otimes u-t\omega(u,v)-\sum_{i=1}^m c_i\sum_{s\in S_i}\omega_s(u,v)s|\quad u,v\in V).$$

Our goal is to prove that H is a graded deformation of $S(V)\#\Gamma$, meaning H is free as an $S(P)$-module.

In Lecture 7 we have computed some Hochschild cohomology of $S(V)\#\Gamma$. Namely, we have seen that $\mathrm{HH}^2(S(V)\#\Gamma)^i=0$ for $i\leqslant -3$, and $\mathrm{HH}^3(S(V)\#\Gamma)^i=0$ for $i\leqslant -4$. This implies that there is a universal graded deformation H_{un} of $S(V)\#\Gamma$ over $S(P_{un})$, where $P_{un}:=\left(\mathrm{HH}^2(S(V)\#\Gamma)^{-2}\right)^*$. We have computed P_{un} in the case when V is symplectically irreducible as Γ-module, we have found that $\dim P_{un}=m+1$. In fact, in general, $\dim P_{un}=m+\dim(\bigwedge^2 V)^\Gamma$, this is proved completely analogously to the last exercise in the previous lecture. The universality means that for any other graded deformation $\tilde{H}$ of $S(V)\#\Gamma$ over $S(\tilde{P})$ there is a unique linear map $P_{un}\to\tilde{P}$ such that there is an $S(\tilde{P})$-linear isomorphism $S(\tilde{P})\otimes_{S(P_{un})}H_{un}\xrightarrow{\sim}\tilde{H}$ that is the identity modulo $\tilde{P}$.

The first thing we will do in this lecture: we will identify H_{un} and H in the case when V is symplectically irreducible. This will easily imply that H is a deformation, in general. We will also see that the isomorphism in the end of the previous paragraph is unique that makes H_{un} a universal object in the categorical sense.

Our original goal was to study the deformations of the invariant subalgebra $S(V)^\Gamma$. We get a deformation eHe over $S(P)$ (it is almost for sure is not universal in the categorical sense; in general it is unknown whether this exhausts all reasonable deformations). We will study an interplay between H and eHe. This interplay is provided by the bimodules He and eH. We will see that eHe and H are mutual centralizers of each other in these bimodules (the double centralizer theorem).

Finally, we will discuss what is known about Morita equivalence between the specializations $eH_{t,c}e$ and $H_{t,c}$ to numerical parameters.

8.1. **Universal SRA as a universal deformation.**

Theorem 8.1. *Suppose that V is symplectically irreducible. Then H is a deformation of $S(V)\#\Gamma$, and there is an isomorphism $P_{un} \cong P$ making H and H_{un} equivalent in the sense explained above.*

Proof. We will check that H_{un} is given by the same generators and relations as H.

Let us deal with generators first. Let π denote the natural projection $H_{un} \twoheadrightarrow SV\#\Gamma$. Since P_{un} has degree 2, π identifies the degree 0 component of H_{un} with $(S(V)\#\Gamma)^0 = \mathbb{C}\Gamma$ and the degree 1 component with $(S(V)\#\Gamma)^1 = V \otimes \mathbb{C}\Gamma$. In particular, there is a natural inclusion of V into H_{un}. The $S(P_{un})$-subalgebra generated by V and $\mathbb{C}\Gamma$ is graded and surjects onto $S(V)\#\Gamma$. It follows from the next exercise that this subalgebra coincides with H_{un}.

Exercise 8.1. *Let $M = \bigoplus_{i=0}^{\infty} M$ be a graded module over $S(P)$, where P is a vector space. Let M_0 be a graded $S(P)$-submodule such that $M_0 \twoheadrightarrow M/PM$. Show that $M_0 = M$.*

So we get a natural epimorphism $S(P_{un}) \otimes T(V)\#\Gamma \twoheadrightarrow H_{un}$.

Let us proceed to relations, i.e., to describing the kernel of the epimorphism above. For $u, v \in V \subset H_{un}$, the element $[u, v]$ has degree 2 and lies in $\ker \pi$. But the degree 2 component of $\ker \pi$ is $P_{un} \otimes \mathbb{C}\Gamma$. So there is a map $\kappa : \bigwedge^2 V \to P_{un} \otimes \mathbb{C}\Gamma$ such that $[u, v] = \kappa(u, v)$. Let $\tilde{H}_{un} := S(P_{un}) \otimes T(V)\#\Gamma/(u \otimes v - v \otimes u - \kappa(u, v))$. Then $\tilde{H}_{un}/P_{un}\tilde{H}_{un} = S(V)\#\Gamma$, while there is an epimorphism $\tilde{H}_{un} \twoheadrightarrow H_{un}$. Since H_{un} is a free graded $S(P_{un})$-module, the following exercise implies $\tilde{H}_{un} = H_{un}$.

Exercise 8.2. *Let M_1, M_2 be two non-negatively graded $S(P)$-modules, where P is a vector space. Suppose that M_2 is a graded free module. Consider an epimorphism $M_1 \twoheadrightarrow M_2$ that induces an isomorphism $M_1/PM_1 \xrightarrow{\sim} M_2/PM_2$. Show that this epimorphism is an isomorphism.*

We claim that there are $t', c_1', \ldots, c_m' \in P_{un}$ such that $\kappa(u, v) = t'\omega(u, v) + \sum_{i=1}^m c_i' \sum_{s \in S_i} \omega_s(u, v)s$. This follows from our computations in Lecture 6 (for example, using passing to a numerical specialization of H_{un}). Also we remark that $t', c_1', \ldots, c_m'$ is a basis in P_{un} – here finally we will use that H_{un} is a universal deformations, the arguments above worked for any deformation. It is enough to show that $t', c_1' \ldots, c_m'$ span P_{un} because $\dim P_{un} = m + 1$. Let P_{un}' be the subspace spanned by $t', c_1', \ldots, c_m'$. Deformations $S(P_{un}) \otimes_{S(P_{un})} H_{un}$ and H_{un} are equivalent for any linear map $P_{un} \to P_{un}$ that is the identity on P_{un}' (they are just algebras given by exactly the same relations). But a linear map $P_{un} \to P_{un}$ with $S(P_{un}) \otimes_{S(P_{un})} \sim H_{un}$ and H_{un} has to be unique and hence $P_{un}' = P_{un}$. $\square$

Exercise 8.3. *Use the theorem to deduce that H is a graded deformation of $S(V)\#\Gamma$ even if V is not symplectically irreducible.*

We remark that any graded deformation H' of $S(V)\#\Gamma$ over $S(P')$ has no nontrivial self-equivalences. This is because any such self-equivalence is forced to be the identity on $\mathbb{C}\Gamma$ and V (thanks to $\deg P' = 2$). But the $S(P')$-algebra H' is generated by $\mathbb{C}\Gamma$ and V. So the equivalence is the identity and H_{un} is a universal object in the categorical sense.

8.2. **Algebra eHe and bimodule eHe.**

Let e be the idempotent $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}\Gamma$. We can view e as an element of H or of its numerical specialization $H_{t,c}$. Then we get spherical subalgebras $eHe \subset H, eH_{t,c}e \subset H_{t,c}$.

Exercise 8.4. *Prove that eHe is a graded deformation of $S(V)^\Gamma$ over $S(P)$. Also prove that the specialization of eHe at $t, c_1, \ldots, c_m \in \mathbb{C}$ coincides with $eH_{t,c}e$ so that $\operatorname{gr} eH_{t,c}e = S(V)^\Gamma$.*

We can consider the $S(P)$-module He and also its specializations $H_{t,c}e$. The space He has commuting actions of H on the left and eHe on the right and so becomes an H-eHe-bimodule.

Lemma 8.2. *The right eHe-module He is finitely generated.*

Proof. We know that $S(V) = He/PHe$ is finitely generated over $S(V)^\Gamma = eHe/PeHe$. We can choose finitely many homogeneous generators $m_1, \ldots, m_n$. Then lift them to homogeneous elements $\tilde{m}_1, \ldots, \tilde{m}_n$ of He. Exercise 8.1 implies that $\tilde{m}_1, \ldots, \tilde{m}_n$ generate the right eHe-module He. $\square$

Similarly, eH is a finitely generated left eHe-module.

8.3. **Double centralizer property.** We are going to prove that the algebras $H_{t,c}, eH_{t,c}e$ are mutual centralizers in the bimodule $H_{t,c}e$. One statement here is easy.

Exercise 8.5. *The homomorphism $eH_{t,c}e^{opp} \to \mathrm{End}_{H_{t,c}}(H_{t,c}e)$ is an isomorphism.*

The following theorem is due to Etingof and Ginzburg, [EG].

Theorem 8.3. *The homomorphism $H_{t,c} \to \mathrm{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ is an isomorphism.*

Proof. The proof is organized as follows. We start with the case $t = 0, c = 0$. We first prove the injectivity, which is easier, and then the surjectivity, which is harder. After that the general case is done basically by passing to associated graded.

Step 1. We claim that the natural map $S(V)\#\Gamma \to \mathrm{End}_{S(V)^\Gamma}(S(V))$ is injective. In what follows we will identify $S(V)$ with $\mathbb{C}[V]$ using the identification of V and V^* coming from the symplectic form.

Let V^0 denote the subset in V consisting of all points with trivial stabilizers. This subset is open and, since Γ acts faithfully – only the unit element acts as 1_V, we have $V^0 \neq \varnothing$. Let $\sum_\gamma f_\gamma \gamma$ lie in the kernel of $\mathbb{C}[V]\#\Gamma \to \mathrm{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$. This means that $\sum_{\gamma\in\Gamma} f_\gamma \gamma(g) = 0$ for any $g \in \mathbb{C}[V]$. Pick $v \in V^0$. For any complex numbers z_γ there is $g \in \mathbb{C}[V]$ such that $g(\gamma^{-1}v) = z_\gamma$. It follows that $\sum_\gamma f_\gamma(v) z_\gamma = 0$ and so $f_\gamma(v) = 0$. Since V^0 is open and non-empty, we deduce that $f_\gamma = 0$.

Step 2. To prove that the homomorphism $\mathbb{C}[V]\#\Gamma \to \mathrm{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$ is surjective we will need the following lemma.

Lemma 8.4. *Let X be a smooth affine variety equipped with a free action of a finite group Γ. Then the homomorphism $\mathbb{C}[X]\#\Gamma \to \mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ is an isomorphism.*

Proof of Lemma. We remark that both $\mathbb{C}[X]\#\Gamma$ and $\mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ are locally free $\mathbb{C}[X]^G$-modules of rank $|\Gamma|^2$. To prove this we only need to check that $\mathbb{C}[X]$ is a locally free $\mathbb{C}[X]^G$-module of rank $|\Gamma|$. Pick a point $x \in X$ and let $\pi : X \to X/\Gamma$ denote the quotient morphism. Then there are $g_\gamma \in \mathbb{C}[X], \gamma \in \Gamma$, such that the matrix $(g_\gamma(\gamma' x))_{\gamma,\gamma'\in\Gamma}$ is non-degenerate. These elements form a basis of the $\mathbb{C}[X]^\Gamma$-module $\mathbb{C}[X]$ after an appropriate localization. This implies that $\mathbb{C}[X]$ is locally free of rank $\mathbb{C}[X]^\Gamma$.

Now to show that the homomorphism $\mathbb{C}[X]\#\Gamma \to \mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ is bijective it is enough to show that it is injective fiberwise, i.e., the induced homomorphism $A_y\#\Gamma \to \mathrm{End}(A_y)$ is injective for any $y \in X/\Gamma$, where $A_y = \mathbb{C}[X]/\mathbb{C}[X]\mathfrak{m}_y$, $\mathfrak{m}_y$ being the maximal ideal of y in $\mathbb{C}[X]^\Gamma$. But the algebra A_y is just the algebra of functions on $\pi^{-1}(y)$, a free Γ-orbit. The injectivity is checked as in the proof of step 1. $\square$

Recall that $\Gamma \subset \mathrm{Sp}(V)$. In particular, for any $\gamma \in \Gamma$ the fixed point subspace V^γ has codimension at least 2. So $\operatorname{codim}_V V \setminus V^0 \geqslant 2$. For every point $v \in V^0$ we can find $f_v \in \mathbb{C}[V]^\Gamma$ such that $f_v(v) \neq 0, f_v(V \setminus V^0) = 0$. Let $V_v^0 := \{u \in V | f_v(u) \neq 0\}$, this is a Γ-stable affine open subset of V^0. For convenience, we can choose a finite covering $V^0 = \bigcup_i V_i$ by subsets of the form V_v^0, let f_i denote the corresponding polynomial, so that $\mathbb{C}[V_i] = \mathbb{C}[V]_f$ and $\mathbb{C}[V_i]^\Gamma = \mathbb{C}[V]_f^\Gamma$. By general Commutative algebra, $\operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i])$ is just the localization of $\operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$ by f_i. In particular, we have a homomorphism $\iota_i : \operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V]) \to \operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i])$. It is injective: if we have $\iota_i(\varphi)(f/f_i^k) = 0$, then $\iota_i(\varphi)(f) = 0$ for any $f \in \mathbb{C}[V]$.

Pick $\varphi \in \operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$. Now, by Lemma applied to $X = V_i$, we have $\iota_i(\varphi) = \sum_{\gamma\in\Gamma} f_\gamma^i \gamma$ for some (uniquely determined) elements $f_\gamma^i \in \mathbb{C}[V_i]$.

Set $V_{ij} = V_i \cap V_j$. We claim that $f_\gamma^i|_{V_{ij}} = f_\gamma^j|_{V_{ij}}$. We have, again injective, homomorphisms $\operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V]), \operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i]), \operatorname{End}_{\mathbb{C}[V_j]^\Gamma}(\mathbb{C}[V_j]) \to \operatorname{End}_{\mathbb{C}[V_{ij}]^\Gamma}(\mathbb{C}[V_{ij}])$, denote them by $\iota_{ij}, \iota'_j, \iota'_i$, respectively. Of course, $\iota_{ij} = \iota'_i \circ \iota_j = \iota'_j \circ \iota_i$. Clearly, ι'_i sends $\sum_{\gamma\in\Gamma} f_\gamma^j \gamma$ to the same element (where we now view the f_γ^j's as elements of $\mathbb{C}[V_{ij}]$ not of $\mathbb{C}[V_j]$) and the similar claim holds for ι'_j. But, by the lemma above, the natural homomorphism $\mathbb{C}[V_{ij}]\#\Gamma \to \operatorname{End}_{\mathbb{C}[V_{ij}]^\Gamma}(\mathbb{C}[V_{ij}])$ is injective. It follows that $f_\gamma^i = f_\gamma^j$ in $\mathbb{C}[V_{ij}]$.

So the functions f_γ^i glue to a regular function f_γ on V^0. But recall that $\operatorname{codim} V \setminus V^0 \geqslant 2$. It follows that f_γ is regular on the whole V. The element $\sum_\gamma f_\gamma \gamma \in \mathbb{C}[V]\#\Gamma$ produces the endomorphism φ. This follows, for example, from the injectivity of $\operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V]) \to \operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i])$ and the construction of f_γ.

Step 3. Let us equip the algebra $\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ with a filtration.

Exercise 8.6. *Let $\mathcal{A}$ be a $\mathbb{Z}_{\geqslant 0}$-filtered algebra and M be its module. Equip M with a filtration compatible with that on $\mathcal{A}$ in such a way that $\operatorname{gr} M$ is finitely generated $\operatorname{gr}\mathcal{A}$-module. We set $\operatorname{End}_{\mathcal{A}}(M)^{\leqslant n} := \{\psi \in \operatorname{End}_{\mathcal{A}}(M) | \psi(M^{\leqslant m}) \subset M^{\leqslant n+m}, \quad \forall m\}$.*

(1) *Show that this is a $\mathbb{Z}$-filtration and that $\operatorname{End}_{\mathcal{A}}(M)^{\leqslant n} = 0$ for $n \ll 0$.*
(2) *Construct a natural homomorphism $\operatorname{gr}\operatorname{End}_{\mathcal{A}}(M) \to \operatorname{End}_{\operatorname{gr}\mathcal{A}}(\operatorname{gr} M)$ of graded algebras.*
(3) *Show that this homomorphism is injective.*

Exercise 8.7. *Let us retain the conventions of the previous exercise. Let $\mathcal{B}$ be another $\mathbb{Z}_{\geqslant 0}$-filtered algebra such that M becomes a filtered $\mathcal{A}\otimes\mathcal{B}$-module. Show that there is a homomorphism $\mathcal{B} \to \operatorname{End}_{\mathcal{A}}(M)$ of filtered algebras. Moreover, show that the composite homomorphism $\operatorname{gr}\mathcal{B} \to \operatorname{gr}\operatorname{End}_{\mathcal{A}}(M) \to \operatorname{End}_{\operatorname{gr} A}(\operatorname{gr} M)$ coincides with the homomorphism induced by the action of $\operatorname{gr}\mathcal{A}\otimes\operatorname{gr}\mathcal{B}$ on $\operatorname{gr} M$.*

The right $eH_{t,c}e$-module $H_{t,c}e$ satisfies the conditions of the exercise. So we get a monomorphism $\operatorname{gr}\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e) \to \operatorname{End}_{\operatorname{gr} eH_{t,c}e^{opp}}(\operatorname{gr} H_{t,c}e) = \operatorname{End}_{S(V)^\Gamma}(S(V))$ of graded algebras. Clearly, $H_{t,c}e$ is filtered as an $H_{t,c}\otimes eH_{t,c}e^{opp}$-module. So we get the induced homomorphism $S(V)\#\Gamma = \operatorname{gr} H_{t,c} \to \operatorname{gr}\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$. The composite homomorphism $S(V)\#\Gamma \to \operatorname{End}_{S(V)^\Gamma}(S(V))$ is the same as one from Steps 1,2 and so is an isomorphism. We deduce that $\operatorname{gr} H_{t,c} \to \operatorname{gr}\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ is an isomorphism. According to the following exercise, the homomorphism $H_{t,c} \to \operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ is an isomorphism.

Exercise 8.8. *Let M, N be $\mathbb{Z}$-filtered vector spaces such that $M^{\leqslant n} = 0$ for $n \ll 0$. Let $\varphi : M \to N$ be a filtration preserving linear map. Show that if $\operatorname{gr}\varphi : \operatorname{gr} M \to \operatorname{gr} N$ is an isomorphism, then φ is an isomorphism.*

$\square$

Problem 8.9. *Show that H, eHe also satisfy the double centralizer property.*

8.4. **Spherical parameters.** The double centralizer property for $eH_{t,c}e$ and $H_{t,c}$ is not to be confused with the Morita equivalence condition: $H_{t,c}eH_{t,c} = H_{t,c}$, the latter is far more restrictive. For example, the irreducible modules for $S(V)\#\Gamma$ with zero action of V are precisely the Γ-irreducibles. All such modules but the trivial one are annihilated by e.

The parameter (t, c) such that $H_{t,c}eH_{t,c} = H_{t,c}$ is called *spherical*. Let us explain what is known about spherical parameters when $t = 1$. The case when $t = 0$ will be mentioned in the next lecture. This dichotomy is justified by the next exercise.

Exercise 8.10. *Let $a \in \mathbb{C}^\times$. Establish a natural isomorphism between $H_{t,c}$ and $H_{at,ac}$.*

In the case when $\dim V = 2$ the description of spherical parameters was obtained by Crawley-Boevey and Holland in [CBH]. Namely, recall that (in the notation of lectures 1-4) to t, c we can assign the $r + 1$-tuple $(\lambda_i)_{i=0}^r$ by

$$\lambda_i = \operatorname{tr}_{N_i}(t\omega(u, v) + \sum_{i=1}^{m} c_i \sum_{s\in S_i} \omega_s(u, v)s)$$

Then the parameter (t, c) is spherical (no matter whether $t = 0$ or not) if and only if $\sum_{i=1}^r \lambda_i\alpha_i \neq 0$ for any root $\alpha = \sum_{i=1}^n \alpha_i\epsilon_i$ of the corresponding finite root system (we use the convention that $\epsilon_1, \ldots, \epsilon_r$ correspond to simple roots).

The answer is known (and easy to state) also for the Rational Cherednik algebra of type A – corresponding to the group $\mathfrak{S}_n$ and the double of its reflection representation. In this case, c is a single complex number. It is known, see [BEG], that $(1, c)$ is spherical if and only if $c \neq \frac{r}{d}$, where $d = 2, 3, \ldots, n$ and r is an integer with $-d < r < 0$.

Dunkl and Griffeth, [DG], obtained the description of the spherical parameters for the complex reflection groups $G(\ell, 1, n)$. The answer is too complicated to be reproduced here.

Finally, let us mention that there is a conjecture of Etingof on the structure of the spherical parameters for all groups of the form $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma_1^n$, [E], that generalizes results of Dunkl and Griffeth. At the moment it is unclear how to prove that conjecture.

References

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