

A general introduction to the Hilbert scheme of points on the plane

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1 The Hilbert scheme of points as a set

There are several objects in mathematics which have the magical property of appearing in various areas and which are, therefore, of the greatest interest for mathematicians. One of those, of which we shall discuss in these notes, is the *Hilbert scheme of points* on a surface. Hilbert schemes were first introduced, in their broadest generality, by A. Grothendieck in his EGA, although we will only discuss a special case in these notes. The setting is the following.

Let X be any quasi-projective surface. Let us remind that, given any zero-dimensional subscheme $Z \subset X$, the *length* of Z is $\ell(Z) := h^0(X, \mathcal{O}_Z)$. Then the Hilbert scheme of n points on X , denoted by $\text{Hilb}^n(X)$ or $X^{[n]}$ is, as a set:

$$\text{Hilb}^n(X) = \{Z \subset X \mid Z \text{ is a zero-dimensional subscheme with } \ell(Z) = n\}$$

Now, the question is whether it is possible to introduce a natural scheme structure on the set $\text{Hilb}^n(X)$, i.e. whether there exists a moduli space parametrizing such data. We will really only be interested in the case where X the complex plane, and in that case, it will turn out that the corresponding scheme will be smooth and symplectic. If we look carefully at the definition, we realize immediately that the support of any zero dimensional, length n subscheme consists of just n *points*, not necessarily distinct.

So there is a natural map:

$$\text{Hilb}^n(\mathbb{A}^2) \longrightarrow \text{Sym}^n(\mathbb{A}^2), \quad Z \mapsto \sum_{x \in \mathbb{A}^2} (\text{mult}_x Z) \cdot x,$$

where we write $\text{mult}_x Z$ for the multiplicity of Z at the point $x \in X$, the summation sign should be understood formally.

This map is called *Hilbert-Chow morphism*, we will see that it is indeed a morphism of algebraic varieties and, moreover, is a birational equivalence between the Hilbert scheme and the symmetric product of \mathbb{A}^2 .

Let us start with an easy example.

Example 1.1. Take $X = \mathbb{A}^2$ and $n = 2$.¹ Since everything is affine, the space we want to construct and study becomes:

$$\mathrm{Hilb}^n(\mathbb{A}^2) = \{I \subset \mathbb{C}[x, y] \mid I \text{ is an ideal with } \ell(I) = \dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n\}$$

We want to get a grasp on how the subschemes corresponding to such ideals look like. If $I = \mathfrak{m}_{x_1} \otimes \mathfrak{m}_{x_2}$ for some distinct points x_1 and x_2 , then its zero locus corresponds to the subscheme $\{x_1, x_2\} \subset \mathbb{A}^2$, which will be then in the Hilbert scheme. What happens if the two points collide? For each point $x \in X$ and a vector $v \neq 0 \in T_x \mathbb{A}^2$, we can define a length 2 ideal by setting $I = \{f \in \mathbb{C}[x, y] \mid f(x) = 0, df_x(v) = 0\}$. Hence any two points infinitesimally attached look like a point with the choice of a direction in the tangent space. Consider the product $\mathbb{A}^2 \times \mathbb{A}^2$. Moreover, we do not care about their order, therefore we take the quotient $\mathrm{Sym}^2(\mathbb{A}^2) := (\mathbb{A}^2 \times \mathbb{A}^2)/\mathfrak{S}_2$ by the symmetric group of order 2. What we get, though, will have singularities precisely along the locus which is stabilized by the action of \mathfrak{S}_2 , namely the diagonal $\Delta = \{(x, y) \mid x = y\}$. Therefore we blow up along the diagonal, and we obtain a nice, smooth variety

$$\mathrm{Hilb}^2(\mathbb{A}^2) := \mathrm{Bl}_{\Delta}((\mathbb{A}^2 \times \mathbb{A}^2)/\mathfrak{S}_2).$$

The variety $\mathrm{Hilb}^2(\mathbb{A}^2)$ is called *Hilbert scheme of two points on the plane*. Its points look like figures (1a) and (1b): if the two points were not coincident, they actually look like a pair of distinct points, while if they were, by blowing up we have attached a \mathbb{P}^1 to each point in the diagonal, therefore we want to add an arrow, to remind ourselves that in some sense, they have been infinitesimally attached in some direction.



(a) A reduced, length 2 subscheme (b) A nonreduced, length 2 subscheme

The Hilbert-Chow morphism $\mathrm{Hilb}^2(\mathbb{A}^2) \rightarrow \mathrm{Sym}^2(\mathbb{A}^2)$, in this case, sends (x, y) with $x \neq y$ to the corresponding cycle $\{x, y\}$, and a point as in figure (1b) to the cycle corresponding to the point with multiplicity two, namely $(x, \text{direction}) \mapsto \{x, x\}$. So it is birational and, moreover, is a resolution of singularities.

Now, if we allow $n > 2$ the picture becomes a little more complicated, but the space $\mathrm{Hilb}^n(\mathbb{A}^2)$ can still be constructed, as we will see later. One also has a *universal (flat) family*

¹Let us remind, for those who have never seen this notation, that by $\mathbb{A}^2 = \mathbb{A}_{\mathbb{C}}^2$ we mean the complex plane \mathbb{C}^2 seen as an algebraic variety.

$$\begin{array}{ccc} F_n & \subset & \mathrm{Hilb}^n(\mathbb{A}^2) \times \mathbb{A}^2 \\ & & \downarrow \pi \\ & & \mathrm{Hilb}^n(\mathbb{A}^2) \end{array}$$

which has the following property: for each point $Z = V(I) \in \mathrm{Hilb}^n(\mathbb{A}^2)$, the corresponding fiber is precisely Z seen as a subscheme of \mathbb{A}^2 . It also satisfies a universal property: any family $Y \subset T \times \mathbb{A}^2$, finite, flat of degree n over a scheme T , is the pullback of F_n via a unique morphism $T \rightarrow \mathrm{Hilb}^n(\mathbb{A}^2)$. For $n > 2$, we still have that the generic point in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is a subscheme consisting on n distinct points. An opposite, in a sense, case is that of subschemes defined by monomial ideals, i.e. ideals I spanned by monomials $x^i y^j$. Each monomial $x^i y^j$ that does not belong to I corresponds to a box (i, j) in the diagram of a partition ν of n . We will index the monomial ideals $I = I_\nu$ with the corresponding partitions. Now, there is a natural torus action on $\mathbb{C}[x, y]$, given by

$$\begin{aligned} (\mathbb{C}^*)^2 &\ni \begin{pmatrix} t & 0 \\ 0 & q \end{pmatrix} \cdot x = tx, \\ (\mathbb{C}^*)^2 &\ni \begin{pmatrix} t & 0 \\ 0 & q \end{pmatrix} \cdot y = qy, \end{aligned}$$

which lift to the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ and to the universal family. We have the following

Proposition 1.2. *The fixed points of the torus action on $\mathrm{Hilb}^n(\mathbb{A}^2)$ are the monomial ideals I_ν .*

Proof. A monomial ideal $I \subset \mathbb{C}[x, y]$ is fixed if and only if it is doubly homogeneous, i.e, if and only if it is monomial. \square

If one wants to formally construct moduli spaces like the Hilbert scheme, one can use some GIT techniques, which is precisely what we are going to do next.

2 GIT and categorical quotients

We want to give a couple of reminders what it means to take a quotient of an algebraic variety (or, more in general, an algebraic scheme) by a group acting on it. First of all, let us remind what a reductive group is.

Definition 2.1. A *reductive group* over \mathbb{C} is an affine algebraic group G such that any rational representation of G is completely reducible.

Any semisimple algebraic group is reductive, as is any algebraic torus $(\mathbb{C}^*)^n$ and any general linear group. Of course, any finite group is reductive.

Now, let G be a reductive group acting on an *affine* algebraic variety X . We want to produce a well-behaved quotient, more precisely, to give an algebraic variety structure to the space of closed orbits of G . In other words, we want to find a variety Y whose closed points are

in one-to-one correspondence with the closed G -orbits in X , and such that there exists a G -invariant surjective morphism $\pi : X \longrightarrow Y$. The idea is the following: consider the ring $\mathbb{C}[X]$ of regular functions on X and its subring of G -invariant functions $\mathbb{C}[X]^G$. Clearly, there is an inclusion

$$i : \mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$$

which induces a morphism of the respective spectra:

$$\pi : \underbrace{\text{Spec} \mathbb{C}[X]}_{=X} \longrightarrow \text{Spec} \mathbb{C}[X]^G$$

A theorem by Hilbert tells us that the ring $\mathbb{C}[X]^G$ is finitely generated as an algebra over \mathbb{C} .

Definition 2.2. In the above setting, we define

$$X//G := \text{Spec} \mathbb{C}[X]^G$$

to be the *categorical quotient* or *affine algebro-geometric quotient* of X with respect to the G -action.

The quotient morphism $\pi : X \longrightarrow X//G$ has some good properties proved by Hilbert:

Theorem 2.3. *Let X be an affine algebraic variety, G be a reductive algebraic group acting on X and $\pi : X \longrightarrow X//G$ denote the quotient morphism. Then the following is true:*

- (i) π is surjective,
- (ii) Every fiber of π contains a single closed orbit.
- (iii) Let $X_1 \subset X$ be a closed subvariety (or, more generally, a closed subscheme). Then $\pi(X_1) \subset X//G$ is closed and a natural morphism $X_1//G \rightarrow \pi(X_1)$ is an isomorphism.

We also have that the closure of any G -orbit in X contains a single closed orbit, since the closure $\overline{G \cdot x}$ obviously contains a closed orbit, and \overline{Gx} is contained in a single fiber.

Let us give a trivial example of categorical quotient:

Example 2.4. Let $X = \mathbb{C}^n$ and $G = \mathbb{C}^*$ be the one-dimensional torus acting diagonally, i.e. $(t, (x_1, \dots, x_n)) \mapsto (tx_1, \dots, tx_n)$. Then $\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]$ and $\mathbb{C}[X]^G = \mathbb{C}$, since there are no non-constant invariant polynomials. Therefore $\mathbb{C}^n // \mathbb{C}^* = \text{Spec} \mathbb{C}[X]^G = \text{Spec} \mathbb{C} \cong pt$.

This example, as well as many others, show that categorical quotients are not always the best that can be done: sometimes parametrizing closed orbits is useless, and we may want to parametrize other kinds of orbits. A more refined and, usually, effective construction is a slight modification of the one we just saw.

The first, intuitive idea would be to take some affine, G -stable open affine subsets X_1, \dots, X_n of X , take their categorical quotients with respect to the G -action and then glue the quotients together. However, the space we obtain is not always a nice space, but it turns out that if we

carefully choose *which* affine open subsets to consider, then the object we get is a well-behaved geometrical object.

First, we choose a character $\chi : G \longrightarrow \mathbb{C}^*$. Then, a function $f \in \mathbb{C}[X]$ is called χ^n -semiinvariant if it verifies $f(g^{-1} \cdot x) = \chi(g)^n \cdot f(x)$ for some non-negative power n (recall that a natural G -action on $\mathbb{C}[X]$ is given by $(g \cdot f)(x) := f(g^{-1} \cdot x)$). The χ^n -semiinvariant functions span a ring,

$$\bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n},$$

which is a finitely generated graded algebra.

Definition 2.5. A point $x \in X$ is called χ -semistable if there exists a χ^n -semiinvariant f such that $f(x) \neq 0$. A point which is not semistable is called *unstable*.

Call X^{ss} the locus of semistable points in X (we write $X^{\chi-ss}$ when we want to indicate the dependence on χ). By definition, X^{ss} can be covered by open sets of the form $X_f := \{x \in X \mid f(x) \neq 0\}$ with $f \in \mathbb{C}[X]^{G,\chi^n}$ for some n , and each of this open sets is acted on by G .

Definition 2.6. The *GIT quotient*² of X with respect to G and χ is

$$X//^\chi G := \text{Proj} \left(\bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n} \right).$$

Notice that if $\chi = 0$, then the functions we consider are just the invariant functions. Therefore the GIT quotient is a generalization of the categorical quotient.

We remark that, for $f \in \mathbb{C}[X]^{G,\chi^n}$, the subvariety X_f is affine and G -stable so we can take the categorical quotient $X_f//G$. For $f_1 \in \mathbb{C}[X]^{G,\chi^{n_1}}$, $f_2 \in \mathbb{C}[X]^{G,\chi^{n_2}}$, we have G -equivariant inclusions $X_{f_i} \subset X_{f_1 f_2} = X_{f_1} \cap X_{f_2}$ that give rise to morphisms $X_{f_i} // G \rightarrow X_{f_1 f_2} // G$. By the very definition of Proj, the variety $X//^\chi G$ is glued from the open subsets $X_f // G$ along the gluing morphisms of the form $X_{f_1} // G \rightarrow X_{f_1 f_2} // G$. The Proj construction guarantees that the result of the gluing is an algebraic variety. Also we would like to point out that there is a natural quotient morphism $X^{\chi-ss} \rightarrow X//^\chi G$.

The inclusion $j : \mathbb{C}[X]^G \hookrightarrow \bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n}$ induces a natural projective morphism

$$\begin{array}{ccc} \Psi : & X//_\chi G & \longrightarrow X//G \\ & \parallel & \parallel \\ & \text{Proj} \left(\bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n} \right) & \longrightarrow \text{Spec } \mathbb{C}[X]^G \end{array}$$

This morphism makes the following diagram commutative

²Here GIT is an acronym for “Geometric Invariant Theory”. Please do not consult an English dictionary for this word

$$\begin{array}{ccc}
X^{\chi-ss} & \longrightarrow & X \\
\downarrow & & \downarrow \\
X//{}^\chi G & \longrightarrow & X//G
\end{array}$$

The top horizontal map is the inclusion and the vertical maps are quotient morphisms. This morphism will be very important to us later on.

Example 2.7 (\mathbb{P}^1). We can consider the diagonal action of \mathbb{C}^* on \mathbb{A}^2 by $t \cdot (x_1, x_2) = (tx_1, tx_2)$. So for $f \in \mathbb{C}[x_1, x_2]$, we have $(t \cdot f)(x_1, x_2) = f(t^{-1}x_1, t^{-1}x_2)$. If $\chi(t) = t^{-1}$, then $\mathbb{C}[X]^{G, \chi^n} = \mathbb{C}[x_1, x_2]^n$ and $X//{}^\chi G = \mathbb{P}^1$. If $\chi(t) = t$, then $\mathbb{C}[X]^{G, \chi^n} = 0$, and $X^{\chi-ss} = \emptyset$.

3 The Hilbert scheme of points on \mathbb{A}^2 as a GIT quotient

3.1 $\text{Hilb}^n(\mathbb{A}^2)$ as a GIT quotient

Let us now focus on the case where $X = \mathbb{A}^2$. It turns out that the Hilbert scheme of points $\mathbb{A}^{2[n]}$ has a natural and handy description.

Theorem 3.1. *Let*

$$\begin{aligned}
\bar{H} &:= \{(B_1, B_2, i) \in \text{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathbb{C}^n \mid [B_1, B_2] = 0\}, \\
\tilde{H} &:= \{(B_1, B_2, i) \in \bar{H} \mid \mathbb{C}[B_1, B_2]i = \mathbb{C}^n\}.
\end{aligned}$$

Consider the natural action of $\text{GL}_n(\mathbb{C})$ over \tilde{H} given by $g \cdot (B_1, B_2, i) = (gB_1g^{-1}, gB_2g^{-1}, gi)$. Then $\tilde{H} = \bar{H}^{\det^{-1}-ss}$, the $\text{GL}_n(\mathbb{C})$ -action on \tilde{H} is free and the GIT quotient

$$H := \tilde{H}/\text{GL}_n(\mathbb{C})^{\det^{-1}}$$

is in a natural bijection with $\text{Hilb}^n(\mathbb{A}^2)$.

It is easy to see the action of $\text{GL}_n(\mathbb{C})$ is free: if $gB_1g^{-1} = B_1$, $gB_2g^{-1} = B_2$ and $gi = i$, then g acts by the identity on $\mathbb{C}[B_1, B_2]i$ that coincides with the whole space \mathbb{C}^n . On the other hand, the action of $\text{GL}_n(\mathbb{C})$ on \bar{H} is no longer free, because $(0, 0, 0) \in \bar{H}$.

Let us make a couple of remarks.

- 1) First, let us try to understand what is \tilde{H} . As a subset,

$$\begin{aligned}
\tilde{H} &\subset \text{End}(\mathbb{C}^n) \times \text{End}(\mathbb{C}^n) \times \mathbb{C}^n \\
&\cong \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \times \mathbb{C}^n \\
&\cong \mathbb{C}^{2n^2+n}.
\end{aligned}$$

Now, the condition $[B_1, B_2] = 0$ is clearly a closed condition of rank at most n^2 , which makes the set \tilde{H} into an algebraic subvariety of \mathbb{C}^{2n^2+n} of dimension at least $n^2 + n$.

(in fact, it has dimension $n^2 + 2n$). The condition specifying \tilde{H} inside of \bar{H} (to be referred to as the *cyclicity condition*), on the other hand, is clearly open: it can be easily translated into a condition of linear independence, which is open because it is given by the non-vanishing of some determinant of the form

$$\det(B_1^{a_1} B_2^{b_1} i, B_1^{a_2} B_2^{b_2} i, \dots, B_1^{a_n} B_2^{b_n} i). \quad (1)$$

- 2) Now let us establish a bijection between the set of the $\mathrm{GL}_n(\mathbb{C})$ -orbits in \tilde{H} and $\mathrm{Hilb}^n(\mathbb{A}^2)$. As we noticed above, a point in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is given by an ideal $I \subset \mathbb{C}[x, y]$ such that $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n$. On $\mathbb{C}[x, y]/I \cong_{Vect_{\mathbb{C}}} \mathbb{C}^n$ we have two natural endomorphisms given by the multiplication by each of the two variables modulo I :

$$\begin{aligned} B_1 : \mathbb{C}[x, y]/I &\longrightarrow \mathbb{C}[x, y]/I \quad ; \quad f(x, y) \mapsto m_x(f(x, y)) := x \cdot f(x, y) \bmod I \\ B_2 : \mathbb{C}[x, y]/I &\longrightarrow \mathbb{C}[x, y]/I \quad ; \quad f(x, y) \mapsto m_y(f(x, y)) := y \cdot f(x, y) \bmod I \end{aligned}$$

The condition $[B_1, B_2] = 0$ obviously holds. Finally, define $i : \mathbb{C} \longrightarrow \mathbb{C}[x, y]/I$ by $i(1) = 1 \bmod I$. It is then clear that the cyclicity condition holds.

Vice versa, consider a triple $(B_1, B_2, i) \in \tilde{H}$. We need to produce an ideal $I \subset \mathbb{C}[x, y]$. Consider the homomorphism

$$\phi : \mathbb{C}[x, y] \longrightarrow \mathbb{C}^n$$

given by $f \mapsto f(B_1, B_2)i$. It is clear that ϕ is surjective, by the cyclicity condition. The ideal we are looking for will be now given by $\mathrm{Ker}\phi$, which satisfies all the required conditions. In fact, due to the cyclicity condition, $\mathrm{Ker}\phi = \{f | f(B_1, B_2) = 0\}$, it is an exercise to check this.

We now have a correspondence:

$$\{I \subset \mathbb{C}[x, y] \mid \ell(I) = n\} \longrightarrow \left\{ (B_1, B_2, i) \left| \begin{array}{l} B_1 = m_x \bmod I \\ B_2 = m_y \bmod I \\ i \text{ such that } i = 1 \bmod I \end{array} \right. \right\}$$

$$\{\mathrm{Ker}\phi \text{ where } \phi(f) = f(B_1, B_2)i\} \longleftarrow \{(B_1, B_2, i) \mid \text{as above}\}.$$

It is an exercise to check that these two maps are mutually inverse bijections between the set of ideals in $\mathbb{C}[x, y]$ of codimension n and the set of G -orbits in \tilde{H} .

What remains to do to prove the theorem above is to establish the equality $\bar{H}^{\det^{-1}-ss} = \tilde{H}$. The inclusion $\tilde{H} \subset \bar{H}^{\det^{-1}-ss}$ follows from the observation that the functions in (1) actually belong to $\mathbb{C}[\bar{H}]^{G,\det^{-1}}$. The other inclusion is more subtle. One can show that $\bigoplus_{n \geq 0} \mathbb{C}[\bar{H}]^{G,\det^{-n}}$ is generated by $\mathbb{C}[\bar{H}]^{G,\det^{-1}}$ as a $\mathbb{C}[\bar{H}]^G$ -algebra, this follows from the 1st and 2nd fundamental theorems of Invariant theory, see [PV, Section 9]. This claim easily implies the required inclusion.

We are going to use a different approach based on the Hilbert-Mumford theorem (no, this wasn't a joint work).

Theorem 3.2. *Let a reductive group G act on an affine algebraic variety X . Pick $x \in X$. Then there is a one-parametric subgroup $\gamma : \mathbb{C}^* \rightarrow G$ such that $\lim_{t \rightarrow 0} \gamma(t) \cdot x$ exists and lies in a unique closed orbit in \overline{Gx} .*

This theorem is very useful because actions of one-dimensional tori are much easier than of general reductive groups.

There is also the following useful but easy observation that gives an alternative description of $X^{\chi-ss}$. Namely, let \mathbb{C}_χ be the 1-dimensional representation of G with action given by χ .

Lemma 3.3. *$X^{\chi-ss}$ coincides with the set of all points $x \in X$ such that the closure of the G -orbit of $(x, 1) \in X \times \mathbb{C}_\chi$ (with diagonal G -action) does not intersect $X \times \{0\} \subset X \times \mathbb{C}_\chi$.*

The proof is pretty basic and is left as an exercise.

Proposition 3.4. *We have $\bar{H}^{\det^{-1}-ss} \subset \tilde{H}$.*

Proof. We just need to show that $\overline{G(x, 1)} \cap (\bar{H} \times \{0\}) = \emptyset$ for any $x = (B_1, B_2, i) \in \bar{H}$. Assume the converse. The Hilbert-Mumford theorem implies that there is a one-parametric subgroup $\gamma : \mathbb{C}^* \rightarrow G$ such that $\lim_{t \rightarrow 0} \gamma(t) \cdot (B_1, B_2, i, 1)$ exists and lies in $\bar{H} \times \{0\}$. This means that the limits as $t \rightarrow 0$ of $\gamma(t)B_1\gamma(t)^{-1}, \gamma(t)B_2\gamma(t)^{-1}, \gamma(t)i$ exist while $\det(\gamma(t)) = t^m$ with $m < 0$. One can diagonalize $\gamma(t)$: in some basis $\gamma(t) = \text{diag}(t^{m_1}, \dots, t^{m_n})$. Let $V_{\geq 0}$ denote the span of eigenvectors corresponding to non-negative degrees m_i . Then it is an exercise to check that B_1, B_2 map $V_{\geq 0}$ to $V_{\geq 0}$, while $i \in V_{\geq 0}$. The cyclicity condition then implies that $V_{\geq 0} = \mathbb{C}^n$, which contradicts $\det(\gamma(t)) = t^m$ with $m < 0$. \square

This completes the proof of Theorem 3.1.

3.2 $\text{Sym}^n(\mathbb{A}^2)$ as a categorical quotient

We also can realize the symmetric product $\text{Sym}^n(\mathbb{A}^2)$ as a categorical quotient.

Theorem 3.5. *There exists a bijection*

$$\text{Sym}^n(\mathbb{A}^2) \cong \bar{H} // \text{GL}_n(\mathbb{C}),$$

Proof. The proof follows from the fact that an orbit is closed if and only if it contains an element of the form $(B_1, B_2, 0)$ such that both B_1 and B_2 are simultaneously diagonalizable. Knowing this, by associating the orbit with the set of simultaneous eigenvalues of (B_1, B_2) we will have the claimed bijection. To prove this fact, we first observe that for closed orbits, we must have $i = 0$ because of the actions of constant matrices. It is a well-known fact from

linear algebra that commuting matrices can be simultaneously conjugated to the triangular form. Then we can take a suitable one-parametric subgroup of diagonal matrices and act on B_1 and B_2 such that the limit is a pair of diagonal matrices. Since the orbit of (B_1, B_2) is closed, the limit lies in the orbit, and hence B_1, B_2 are simultaneously diagonalizable. Now let us show that the orbit $G(B_1, B_2, 0)$ is closed provided B_1, B_2 are diagonal. As we have seen above, $\overline{G(B_1, B_2)}$ contains a pair (B'_1, B'_2) of diagonal matrices. But the simultaneous eigenvalues of (B_1, B_2) and (B'_1, B'_2) have to coincide and hence (B_1, B_2) is conjugate to (B'_1, B'_2) . \square

One can actually show that the map above is an isomorphism of algebraic varieties. A key ingredient is the fact that $\mathbb{C}[\bar{H}]^G$ is generated by polynomials of the form $\text{tr}(B_1^a B_2^b)$. To deduce the claim about an isomorphism from here is an exercise.

Now we can interpret the Hilbert-Chow map.

Proposition 3.6. *Under the identifications $\bar{H}/\det^{-1} G \cong \text{Hilb}^n(\mathbb{A}^2)$, $\bar{H}/G \cong \text{Sym}^n(\mathbb{A}^2)$ as above, the Hilbert-Chow morphism is identified with the natural projective morphism $\bar{H}/\det^{-1} G \rightarrow \bar{H}/G$.*

Proof. As above, we can simultaneously conjugate B_1, B_2 to triangular matrices, let $(\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n)$ be the eigenvalues. By the commutative diagram describing the morphism $\bar{H}/\det G \rightarrow \bar{H}/G$, it maps the orbit of (B_1, B_2, i) to the closed orbit in its closure, meaning to that of $(\text{diag}(\lambda_1, \dots, \lambda_n), \text{diag}(\mu_1, \dots, \mu_n), 0)$.

Now let I be the ideal corresponding to (B_1, B_2, i) , i.e., consisting of all polynomials with f with $f(B_1, B_2) = 0$. Take a composition series of $\mathbb{C}[x, y]/I$, with constituents S_1, \dots, S_n . By Nullstellensatz, simple modules over $\mathbb{C}[x, y]$ are one-dimensional and correspond to points in \mathbb{A}^2 , and the image of $\mathbb{C}[x, y]/I$ under the Hilbert-Chow morphism are precisely the points appearing in the composition series. By putting the matrices B_1, B_2 into upper triangular form, it is easy to see that these constituents correspond to the (λ_i, μ_i) 's. Now we are done by the definition of the Hilbert-Chow map. \square

4 Hamiltonian reductions

Now we are going to show that $\text{Hilb}^n(\mathbb{A}^2)$ is a smooth and symplectic variety. For this, we will realize $\text{Hilb}^n(\mathbb{A}^2)$ as a *GIT Hamiltonian reduction*.

4.1 Moment maps

Before realizing the Hilbert scheme as a Hamiltonian reduction, we want to remind the basics of Symplectic geometry and Hamiltonian reductions. It only makes sense to speak about Hamiltonian reductions for symplectic varieties equipped with *Hamiltonian G -actions*.

Let M be an algebraic variety. Recall that a *symplectic form* on M will be any closed, non-degenerate 2-form ω . The existence of such a form surprisingly makes the geometry of M extremely rigid but, on the other hand, it implies a certain number of good properties.

Here is our main starting example of symplectic varieties.

Example 4.1 (Cotangent bundles). Take any n -dimensional variety Z and consider its cotangent bundle $M = T^*Z$. We claim that M has always a symplectic structure. First, consider a

canonical 1-form α defined as follows. Any point in M can be thought of as a pair $p = (x, y)$ where $x \in Z$ and y is a vector in T_x^*Z . Consider the projection

$$\pi : M \longrightarrow Z, p = (x, y) \mapsto x$$

and define α point-wise via

$$\alpha_p = (d\pi_p)^*y$$

or, equivalently,

$$\alpha_p(v) = y((d\pi_p)(v)).$$

Now define

$$\omega = -d\alpha.$$

Now we want to compute the form in coordinates. If we worked in the \mathbb{C}^∞ or analytic setting, we could use the usual coordinates, while in our present setting we want everything to be algebraic, so we need to use étale coordinates, instead. Namely, we can introduce étale coordinates in a neighborhood of each point $x \in Z$. It is an easy exercise to show that, if $\{x_1, \dots, x_n\}$ are étale coordinates and $\{y_1, \dots, y_n\}$ are their corresponding fiberwise coordinates, then $\alpha = \sum_{i=1}^n y_i dx_i$ and $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.

Suppose now that a symplectic variety M is acted on by a reductive group G . Then we have that there is a natural map that takes an element in the Lie algebra \mathfrak{g} of G and to it associates a complete vector field on M , namely $\mathfrak{g} \longrightarrow \text{Vect}(M), \xi \mapsto \xi^M$, where ξ^M is the derivation of the structure sheaf \mathcal{O}_M corresponding to ξ . An important special case: if V is a vector space and the G -action is linear, then $\xi_v^V = \xi \cdot v$, the image of v under the operator corresponding to ξ . Also, by the nondegeneracy of ω , we get an isomorphism

$$TM \longrightarrow T^*M, X \mapsto i_X \omega$$

where $i_X \omega$ denotes the contraction of the 2-form ω with the vector field X . This implies that for every one form β , there exists a unique vector field X such that $i_X \omega = \beta$. In particular, for any $f \in \mathbb{C}[M]$, there exists a unique vector field, which we call X_f , such that $i_{X_f} \omega = df$. It is an easy computation to show that, in Darboux coordinates,

$$X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i} \right).$$

We thus get, for free, a Poisson bracket on the algebra $C^\infty(M)$ by setting

$$\{f, g\} := \omega(X_f, X_g).$$

4.2 Hamiltonian reductions

Let us give the following

Definition 4.2. Let G be a reductive group and (M, ω) a symplectic G -variety. Then the action of G is called *Hamiltonian* (and the variety (M, ω, G, μ) is called *Hamiltonian G -space*) if there exists a map $\mu : M \rightarrow \mathfrak{g}^*$ such that

1. For every $\xi \in \mathfrak{g}$, if $\mu^\xi : M \rightarrow \mathbb{C}$ is defined by $\mu^\xi(p) := \langle \mu(p), \xi \rangle$, then

$$i_{\xi^M} \omega = d\mu^\xi.$$

2. μ is equivariant with respect to the action of G on M and the co-adjoint action of G on \mathfrak{g}^* .

It is easy to see that the form in the l.h.s. in (1) is closed. Then (1) says that it is exact. In our case, this is the only additional condition: since we require that G is reductive, we can always achieve (2) (by averaging μ with respect to the G -action).

The morphism μ above is called a *moment map*.

Example 4.3. Consider the example above, namely $M = T^*Z$ for some smooth variety Z . We have a chain of maps:

$$\mathfrak{g} \longrightarrow \text{Vect}(Z) \hookrightarrow \mathbb{C}[M], \quad \xi \mapsto \xi^Z \mapsto H^{\xi^M}$$

where H^{ξ^Z} is the function $H^{\xi^Z}(x, y) = \langle y, \xi_x^Z \rangle$. Then we can construct a moment map by setting:

$$\langle \mu(p), \xi \rangle = H^{\xi^Z}(p). \tag{2}$$

It is a useful exercise to check that this map indeed satisfies required axioms.

The map $\mu^* : \mathfrak{g} \rightarrow \mathbb{C}[M]$, $\xi \mapsto H^{\xi^Z}$, dual to μ in the sense of (2), is called a *comoment map*. The following is the main result of this subsection.

Theorem 4.4. Let M be a smooth affine algebraic variety equipped with an action of a reductive group G such that the action on $\mu^{-1}(0)$ is free. Then the following is true:

1. $\mu^{-1}(0)$ is a smooth subvariety of codimension $\dim G$.
2. Every fiber of the quotient morphism $\pi : \mu^{-1}(0) \rightarrow M_{\text{red}} := \mu^{-1}(0) // G$ is a single orbit.
3. M_{red} is a smooth variety of dimension $\dim M - 2 \dim G$.
4. Let $\iota : \mu^{-1}(0) \hookrightarrow M$ denote the inclusion. Then there is a unique 2-form ω_{red} on M_{red} such that $\pi^* \omega_{\text{red}} = \iota^* \omega$. The form ω_{red} is symplectic.

To prove the theorem we need the following lemma describing the image and the kernel of $d_x \mu : T_x M \rightarrow \mathfrak{g}^*$. We write \mathfrak{g}_x for $\{\xi \in \mathfrak{g} \mid \xi_x^M = 0\}$, this is the tangent algebra of the stabilizer G_x .

Lemma 4.5. *We have $\text{im } d_x\mu = \mathfrak{g}_x^\perp$ and $\ker d_x\mu = (T_xGx)^\perp$ (the first \perp means the annihilator in the dual space, the second is a skew-orthogonal complement w.r.t. the symplectic form).*

The proof is an exercise.

Proof of the theorem. Since the G -action on $\mu^{-1}(0)$ is free, we see that $\mathfrak{g}_x = \{0\}$ for all $x \in \mu^{-1}(0)$. It follows that μ is a submersion at all points of $\mu^{-1}(0)$. (1) follows.

To prove (2) recall that every fiber of π contains a unique closed orbit. But since the action is free, all orbits have the same dimension (equal $\dim G$) and hence are closed.

To show (3) we will show that M_{red} is smooth of required dimension as a complex analytic manifold. For this we take a small transversal slice to an orbit. Shrinking it, we can assume that π embeds it into M_{red} as an open complex submanifold. This slice will be a coordinate chart in M_{red} . Our claim follows.

Let us prove (4). The form $\iota^*\omega$ vanishes on T_xGx for any $x \in \mu^{-1}(0)$, this is an exercise, and is G -invariant. The existence (and uniqueness) of ω_{red} follows. The claim that this form is symplectic is an exercise. \square

The variety M_{red} is called *Hamiltonian reduction* of M . Its ring of function is given by $(\mathbb{C}[M]/\mu^*(\mathfrak{g})\mathbb{C}[M])^G$: this also allows us to give a Poisson bracket to the quotient M_{red} , as well. Indeed, we define $\{f + \mu^*(\mathfrak{g})\mathbb{C}[M], g + \mu^*(\mathfrak{g})\mathbb{C}[M]\} := \{f, g\} + \mu^*(\mathfrak{g})\mathbb{C}[M]$. It can be checked that the bracket is well defined, and that this gives a Poisson structure to the quotient. We remark that in order to define this bracket we do not need to assume that the G -action on $\mu^{-1}(0)$ is free (and that M is symplectic, it is enough to assume that M is Poisson).

References

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