LECTURE 1: NAKAJIMA QUIVER VARIETIES

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1. Geometric invariant theory

Recall that an algebraic group G is called (linearly) *reductive* if any its rational (i.e., algebraic) representation is completely reducible. The finite groups, the group GL_n and the products $\mathrm{GL}_{n_1} \times \ldots \mathrm{GL}_{n_k}$ are reductive. Below G denotes a reductive algebraic group and X is an affine algebraic variety equipped with an (algebraic) action of G.

Results explained below in this section can be found in [PV].

1.1. **Categorical quotients and Hilbert-Mumford theorem.** The following results are essentially due to Hilbert.

Theorem 1.1. *The following is true.*

(1) *The algebra of invariants* $\mathbb{C}[X]^G := \{f \in \mathbb{C}[X] | f(g^{-1}x) = f(x), \forall x \in X, g \in G\}$ *is finitely generated.*
(2) *Set* $X//G := \mathrm{Spec}(\mathbb{C}[X]^G)$ *so that we have the quotient morphism* $\pi : X \to X//G$ *induced by the inclusion* $\mathbb{C}[X]^G \subset \mathbb{C}[X]$*. This morphism is surjective and every fiber contains exactly one closed orbit.*
(3) *If* $Y \subset X$ *is a closed* G*-stable subvariety, then a natural morphism* $Y//G \to X//G$ *is a closed embedding.*

The variety $X//G$ is called the categorical quotient of X (by the action of G). The name is justified by the observation that if $X \to Z$ is a G-invariant morphism, then it uniquely factorizes via $X//G$.

The Hilbert-Mumford theorem often allows to identify a unique closed orbit in the closure $\overline{Gx}$ of some orbit Gx.

Theorem 1.2. *Let* Gy *be a unique closed orbit in* $\overline{Gx}$*. Then there is an algebraic group homomorphism* $\gamma : \mathbb{C}^\times \to G$ *(a.k.a. one-parameter subgroup) such that* $\lim_{t\to 0} \gamma(t)x \in Gy$.

1.2. **GIT quotients.** The categorical quotient parameterizes closed orbits in X. Often, there are very few such. For example, consider the action of $G := \mathbb{C}^\times$ on $X := \mathbb{C}^n$ by dilations: $t.v := t^{-1}v$. Then the only closed orbit is zero and, indeed, $\mathbb{C}[X]^G = \mathbb{C}$. To remedy this situation one uses GIT quotients.

An additional parameter needed to form such a quotient is a character $\theta : G \to \mathbb{C}^\times$. Using it we can form the *semi-stable locus*

$$X^{\theta-ss} := \{x \in X | \exists f \in \mathbb{C}[X]^{G,n\theta} \text{ s.t. } n > 0 \text{ and } f(x) \neq 0\}.$$

Here, we write $\mathbb{C}[X]^{G,n\theta}$ for the space of semi-invariants,

$$\mathbb{C}[X]^{G,n\theta} := \{f \in \mathbb{C}[X] | f(g^{-1}x) = \theta(g)^n f(x)\}.$$

The subset $X^{\theta-ss} \subset X$ is open, it is the union of the principal open subsets X_f with $f \in \mathbb{C}[X]^{G,n\theta}, n > 0$. One can give an alternative characterization of $X^{\theta-ss}$ using the following lemma.

Lemma 1.3. *Consider the action of G on $X \times \mathbb{C}$ given by $g(x,z) := (gx, \theta(g)z)$. Then $X^{\theta-ss}$ consists precisely of points $x \in X$ such that $\overline{G(x,1)}$ does not intersect $X \times \{0\}$ or, equivalently, $x \in X$ such that there is no one-parameter subgroup $\gamma : \mathbb{C}^\times \to G$ such that $\lim_{t\to 0} \gamma(t)x$ exists and $\theta(\gamma(t)) = t^m$ with $m > 0$.*

The GIT quotient $X//^\theta G$ to be constructed will parameterize closed G-orbits in $X^{\theta-ss}$. Namely, consider the subspace $\bigoplus_{n\geqslant 0} \mathbb{C}[X]^{G,n\theta} \subset \mathbb{C}[X]$. This is a graded subalgebra (the grading is by n) in $\mathbb{C}[X]$. We set $X//^\theta G := \mathrm{Proj}(\bigoplus_{n\geqslant 0} \mathbb{C}[X]^{G,n\theta})$. Note that $X//^\theta G$ is glued from the affine charts $X_f//G$, where $f \in \mathbb{C}[X]^{G,n\theta}$ with $n > 0$. For $f \in \mathbb{C}[X]^{G,n\theta}, f' \in \mathbb{C}[X]^{G,n'\theta}$, we have open inclusions $X_{ff'}//G \subset X_f//G, X_{f'}//G$ and, moreover, $X_{ff'}//G = (X_f//G) \cap (X_{f'}//G)$ (inside $X//^\theta G$).

Note also that we have natural morphisms $X^{\theta-ss} \to X//^\theta G$ (the quotient morphism, it is affine and surjective, every fiber contains a single closed orbit) and $X//^\theta G \to X//G$, this morphism is projective. The following diagram is commutative.

$$\begin{tikzcd} X^{\theta-ss} \arrow[r] \arrow[d, "\subseteq"] & X//^\theta G \arrow[d] \\ X \arrow[r] & X//G \end{tikzcd}$$

Finally, note that for the trivial character θ (we will write $\theta = 0$) we just have $X^{\theta-ss} = X$ and $X//^0 G = X//G$.

To finish our discussion of GIT quotients, let us revisit the example of $G = \mathbb{C}^\times$ and $X = \mathbb{C}^n$. We have $\theta(t) = t^m$ for some $m \in \mathbb{Z}$. If $m > 0$, then $X^{\theta-ss} = \mathbb{C}^n \setminus \{0\}$ and $X//^\theta G = \mathbb{P}^{n-1}$. If $m < 0$, then $X^{\theta-ss} = \varnothing$.

1.3. **Representations of quivers.** Here we are going to present an example of reductive group actions we will care about.

By a quiver we mean an oriented graph. Formally, it can be presented as a quadruple (Q_0, Q_1, t, h), where Q_0, Q_1 are finite sets (vertices and arrows) and $t, h : Q_1 \to Q_0$ are tail and head maps. We will need to consider co-framed representations of Q. The data of such a representation consists of vector spaces $V_k, W_k, k \in Q_0$, and linear maps $x_a : V_{t(a)} \to V_{h(a)}, a \in Q_1$, and $i_k : V_k \to W_k, k \in Q_0$. When $V_k, W_k, k \in Q_0$, are fixed, the set of co-framed representations of Q naturally forms a vector space

$$\bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{k\in Q_0} \mathrm{Hom}_{\mathbb{C}}(V_k, W_k).$$

Let us introduce the dimension vector $v := (v_k)_{k\in Q_0}$, where $v_k := \dim V_k$, and the framing vector $w := (w_k)_{k\in Q_0}$. The representation space above will be denoted by $R(Q, v, w)$. It has a natural action by the group $G = \mathrm{GL}(v) := \prod_{k\in Q_0} \mathrm{GL}(V_k)$. We are interested in categorical and GIT quotients for the action of G on $R := R(Q, v, w)$.

Note that we can speak about sub- and quotient (co-framed) representations, direct sums, extensions and so on.

The following two lemmas characterize closed and stable orbits for a suitable choice of the character θ. The proofs are based on the Hilbert-Mumford theorem.

Lemma 1.4. *The G-orbit of the collection $(x_a, i_k) \in R$ is closed if and only if $i_k = 0$ for all k and the representation $(x_a)_{a\in Q_1} \in R(Q, v, 0)$ is semisimple.*

The character group of G is identified with $\mathbb{Z}^{Q_0}$, an element $(\theta_k)_{k\in Q_0}$ corresponds to the character θ given by $(g_k)_{k\in Q_0} \mapsto \prod_{k\in Q_0} \det(g_k)^{\theta_k}$.

Lemma 1.5. *If $\theta_k > 0$ for all $k \in Q_0$, then the subset $R^{\theta-ss}$ consists of all representations (x_a, i_k) such that the only x_a-stable collection of subspaces in $(\ker i_k)_{k\in Q_0}$ is the zero one.*

In particular, the action of G on $R^{\theta-ss}$ is free (note, however, that $R^{\theta-ss}$ may be empty).

For example, when the quiver Q has one vertex and no arrows, the space R is just $\mathrm{Hom}(V, W)$. The subset $R^{\theta-ss}$ for $\theta = \det$ consists of all injective maps. It follows that $R//^\theta G = \mathrm{Gr}(v, w)$.

2. Moment maps and Hamiltonian reduction

2.1. **Hamiltonian actions and moment maps.** Let X be a smooth algebraic variety. The definition of a symplectic form on X mirrors the classical definition in the C^∞-setting, but now we require the form to be algebraic. By a symplectic algebraic variety we mean X together with a symplectic form ω. Here is a classical example: let X_0 be a smooth algebraic variety and $X := T^*X_0$ be its cotangent bundle. On X, we have a canonical 1-form, say α defined as follows. Let $\pi : X \to X_0$ denote the projection. Pick a point $(x, \beta) \in X$, where $x \in X_0, \beta \in T^*_x X_0$. Let $\xi \in T_{(x,\beta)}X$. Then set $\alpha_{(x,\beta)}(\xi) = \langle \beta, d_{(x,\beta)}\pi(\xi)\rangle$. Define ω by $\omega := -d\alpha$. In particular, if X_0 is a vector space U, then $X = U \oplus U^*$ and ω is defined by $\omega(u, u') = \omega(a, a') = 0$ and $\omega(u, a) = \langle a, u\rangle$, where $u, u' \in U, a, a' \in U^*$.

Inverting the symplectic form ω, we get a bivector ω^{-1} that defines a Poisson bracket $\{\cdot, \cdot\}$ on the sheaf $\mathcal{O}_X$, i.e., a skew-symmetric operation $\{\cdot, \cdot\} : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$ satisfying the Jacobi and the Leibnitz identities. Using the bracket, for a local function f on X, we can define the local vector field $v(f) := \{f, \cdot\}$ called the Hamiltonian vector field on X.

Let us discuss group actions on X. Let G be an algebraic group acting on X algebraically. The action gives rise to a G-equivariant map (and Lie algebra homomorphism) from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra $\mathrm{Vect}(X)$ of vector fields on X. This map will be denoted by $\xi \mapsto \xi_X$. If X is affine, then the corresponding action of $\mathfrak{g}$ on $\mathbb{C}[X]$ is just obtained by differentiating the G-action. On the other hand, we have the Lie algebra homomorphism $f \mapsto v(f), \mathbb{C}[X] \to \mathrm{Vect}(X)$. This homomorphism is G-equivariant provided G preserves the form ω (we will say in this case that the G-action is symplectic).

Now we are ready to give definitions of a Hamiltonian G-action and of the corresponding moment map. We say that a symplectic G-action on X is *Hamiltonian* if it comes equipped with a G-equivariant map $\mathfrak{g} \to \mathbb{C}[X], \xi \mapsto H_\xi$, such that $\xi_X = v(H_\xi)$. Note that the map $\xi \mapsto H_\xi$ is defined uniquely up to adding an element of $\mathfrak{g}^{*G}$: if χ is such an element, then we can take the map $\xi \mapsto H_\xi + \langle \chi, \xi\rangle$. By the *moment map*, we mean $\mu : X \to \mathfrak{g}^*$ given by $\langle \mu(x), \xi\rangle := H_\xi(x)$.

Let us give an example of a Hamiltonian action. In the notation above, let G act on X_0. Then this action canonically lifts to an action on $X = T^*X_0$ and the latter is symplectic. We claim that we can take $H_\xi = \xi_{X_0}$ (we can view a vector field on X as a function on X that is linear on the fibers of $\pi : X \to X_0$). This is left as an exercise (that will require some understanding of the brackets between functions and vector fields on X_0 viewed as functions on X).

Let us provide some properties of μ (left as exercises).

Lemma 2.1. *The kernel of $d_x\mu$ coincides with the ω-orthogonal complement of $T_x(Gx)$. The image of $d_x\mu$ is the annihilator of $\mathfrak{g}_x := \{\xi \in \mathfrak{g} | \xi_{X,x} = 0\}$ in $\mathfrak{g}^*$. In particular, μ is a submersion in x provided the stabilizer G_x is finite.*

2.2. **Hamiltonian reductions.** One can form a Poisson variety from a symplectic variety with Hamiltonian G-action by the procedure known as Hamiltonian reduction.

First, let us consider an algebraic situation. Let A be a Poisson algebra, $\mathfrak{g}$ be a Lie algebra equipped with a Lie algebra homomorphism $\varphi : \mathfrak{g} \to A$. Fix a character $\lambda : \mathfrak{g} \to \mathbb{C}$. Then we can define the Hamiltonian reduction $A///_\lambda \mathfrak{g}$ as follows. Set $I_\lambda := A\{\varphi(\xi) - \langle \lambda, \xi\rangle, \xi \in \mathfrak{g}\}$. This is a two-sided ideal stable under the adjoint action of $\mathfrak{g}$. Then define $A///_\lambda \mathfrak{g}$ as $(A/I_\lambda)^{\mathfrak{g}}$ (the invariants are taken with respect to the adjoint action of $\mathfrak{g}$). Then $A///_\lambda \mathfrak{g}$ is a commutative associative algebra that has a natural Poisson bracket: $\{a + I_\lambda, b + I_\lambda\} := \{a, b\} + I_\lambda$. Note that bracket is only well-defined on $A///_\lambda \mathfrak{g}$, not on the whole algebra A/I_λ.

For example, let X be an affine symplectic variety equipped with a Hamitonian action of the reductive group G. Then we can take $A := \mathbb{C}[X], \varphi := \mu^*$. The algebra $\mathbb{C}[X]///_\lambda \mathfrak{g}$ is the algebra of regular functions on the scheme $X///_\lambda G := \mu^{-1}(\lambda)//G$.

We want some sufficient conditions for $X///_\lambda G$ to be a symplectic variety. The proof of the next lemma is based on on the following fact: if Y is a smooth affine variety with a free G-action, then $Y//G$ is also smooth and $\pi : Y \to Y//G$ is a locally trivial bundle in étale topology. This fact is a consequence of the Luna slice theorem. The remaining steps in the proof of the lemma are left as an exercise.

Lemma 2.2. *Suppose that the G-action on $\mu^{-1}(\lambda)$ is free. Then $X///_\lambda G$ is a symplectic variety of dimension* $\dim X - 2\dim G$. *The symplectic form $\underline{\omega}$ on $X///_\lambda G$ is a unique form satisfying $\pi^*\underline{\omega} = \iota^*\omega$. Here we write π for the quotient morphism $\mu^{-1}(\lambda) \to X///_\lambda G$ and ι for the inclusion $\mu^{-1}(\lambda) \hookrightarrow X$.*

Let us discuss GIT reductions. Let θ be a character of G such that the G-action on $\mu^{-1}(\lambda)^{\theta-ss}$ is free. The uniqueness of the form on the reduction in the previous lemma shows that the symplectic forms on the reductions $\mu^{-1}(0)_f///_\lambda G$ glue to a global symplectic form $\underline{\omega}$ on $X///^\theta_\lambda G := \mu^{-1}(\lambda)//^\theta G$. For similar reasons, even if the G-action is not free, the reduction $X///^\theta_\lambda G$ is a Poisson variety. Examples of Poisson varieties obtained in such a way (Nakajima quiver varieties) will be provided in the next section.

3. Nakajima Quiver varieties

3.1. **Definition.** Let $Q = (Q_0, Q_1, t, h)$ be a quiver. Fix dimension and (co)framing vectors v, w and consider the space

$$R = R(Q, v, w) := \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{k\in Q_0} \mathrm{Hom}_{\mathbb{C}}(V_k, W_k).$$

This space comes equipped with a natural action of the group $G = \mathrm{GL}(v) := \prod_{k\in Q_0} \mathrm{GL}(V_k)$. Nakajima quiver varieties are GIT Hamiltonian reductions of the space T^*R by the action of G.

We want to interpret the space T^*R and the moment map $\mu : T^*R \to \mathfrak{g}^*$ more linear algebraically. First of all, for two finite dimensional vector spaces, U, U', we identify $\mathrm{Hom}_{\mathbb{C}}(U, U')^*$ with $\mathrm{Hom}_{\mathbb{C}}(U', U)$ via the trace form: $(A, B) := \mathrm{tr}(AB)$. Then we get

$$T^*R = \bigoplus_{a\in Q_1} (\mathrm{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{h(a)}, V_{t(a)})) \oplus \bigoplus_{k\in Q_0} (\mathrm{Hom}(V_k, W_k) \oplus \mathrm{Hom}(W_k, V_k)).$$

We will write (x_a, x_{a^*}, i_k, j_k) for a typical element in T^*R meaning that $x_a \in \mathrm{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)})$, $x_{a^*} \in \mathrm{Hom}_{\mathbb{C}}(V_{h(a)}, V_{t(a)}), i_k \in \mathrm{Hom}_{\mathbb{C}}(V_k, W_k), j_k \in \mathrm{Hom}_{\mathbb{C}}(W_k, V_k)$.

Lemma 3.1. *We have* $\mu(x_a, x_{a^*}, i_k, j_k) = \sum_{a\in Q_1}(x_a x_{a^*} - x_{a^*}x_a) - \sum_{k\in Q_0} j_k i_k$.

Proof. First, let us consider an easy case: the action of $G = \mathrm{GL}(V)$ on $T^*\operatorname{Hom}_{\mathbb{C}}(V,W) = \operatorname{Hom}_{\mathbb{C}}(V,W)\oplus\operatorname{Hom}_{\mathbb{C}}(W,V)$. We claim that the moment map $\mu : T^*\operatorname{Hom}_{\mathbb{C}}(V,W)\to\mathfrak{gl}(V)$ sends $(i,j)\in\operatorname{Hom}_{\mathbb{C}}(V,W)\oplus\operatorname{Hom}_{\mathbb{C}}(W,V)$ to $-ji$. The moment map is specified by $\operatorname{tr}(\mu(i,j)\xi) = \operatorname{tr}(\xi_{\operatorname{Hom}(V,W),i}j)$. We have $\xi_{\operatorname{Hom}(V,W),i} = -i\xi$. So $\operatorname{tr}(\xi_{\operatorname{Hom}(V,W),i}j) = \operatorname{tr}(-i\xi j) = \operatorname{tr}(-ji\xi)$ that implies $\mu(i,j) = -ji$.

Similarly, one checks that the moment map for the $\mathrm{GL}(V)$-action on $T^*\operatorname{End}(V) = \operatorname{End}(V)^{\oplus 2}$ sends $(x,x^*)\in T^*\operatorname{End}(V)$ to $[x,x^*]$.

The proof of the lemma follows from these two computations and the following two observations:

- Let U_1, U_2 be two vector spaces and G be an algebraic group acting on U_1, U_2. Let μ_1,μ_2 denote the moment maps for the G-action on T^*U_1, T^*U_2. Then the moment map for the G-action on $T^*U_1\oplus T^*U_2$ equals $\mu(u_1,u_2) = \mu_1(u_1)+\mu_2(u_2)$.
- Let U be a symplectic vector space equipped with an action of $G_1\times G_2$. Let $\mu_{G_1\times G_2}, \mu_{G_1}, \mu_{G_2}$ be the moment maps for the actions of G, G_1, G_2 on T^*U. Then $\mu_{G_1\times G_2} = (\mu_{G_1}, \mu_{G_2})$.

Details are left as exercises. □

Note that the space $\mathfrak{g}^{*G}$ is identified with $\mathbb{C}^{Q_0}$ via $(\lambda_k)_{k\in Q_0}\mapsto\sum_{k\in Q_0}\lambda_k\operatorname{tr}_{V_k}$. So, following Nakajima,[N1, N2], for $\lambda\in\mathbb{C}^{Q_0}, \theta\in\mathbb{Z}^{Q_0}$, we can define the Nakajima quiver variety $\mathcal{M}^\theta_\lambda(Q,v,w) := \mu^{-1}(\lambda)/\!/^\theta G$. The most important case for us is when $\lambda = 0$. Often, we will fix Q and w, then we will just write $\mathcal{M}^\theta_\lambda(v)$. We will also write $\mathcal{M}^\theta(v)$ instead of $\mathcal{M}^\theta_0(v)$.

Note that by the construction of the quiver varieties, we have a projective morphism $\phi_\lambda : \mathcal{M}^\theta_\lambda(v)\to\mathcal{M}^0_\lambda(v)$.

3.2. **Examples.**

Example 3.2. Consider the simplest possible quiver: one vertex and no arrows so that $R = \operatorname{Hom}(V,W), G = \mathrm{GL}(V), T^*R = \operatorname{Hom}(V,W)\oplus\operatorname{Hom}(W,V)$ and $\mu(i,j) = -ji$. So $\mu^{-1}(\lambda) = \{(i,j)|ji = -\lambda\}$. In particular, $\mu^{-1}(\lambda) = \varnothing$ if and only if $\lambda\neq 0$ and $v > w$. If $\lambda\neq 0$, then the action of G on $\mu^{-1}(\lambda)$ is free, and $\mu^{-1}(\lambda)^{\theta-ss} = \mu^{-1}(\lambda)$ no matter what θ is. The map $\mu^{-1}(\lambda)\to\mathfrak{gl}(W), (i,j)\mapsto ij$ descends to an isomorphism of $\mathcal{M}^\theta_\lambda(v)$ with the $\mathrm{GL}(W)$-orbit of the diagonal matrix with eigenvalues $-\lambda$ (v times) and 0 ($w-v$ times).

Let us now consider the case when $\lambda = 0$. Let us start with $\theta > 0$. In this case, similarly to Lemma 1.5, $(T^*R)^{\theta-ss}$ consists of all pairs (i,j) such that i is injective. The condition $(i,j)\in\mu^{-1}(0)^{\theta-ss}$ means that j vanishes on $\operatorname{im} i$. Note that $\operatorname{Hom}(W/V,V)$ can be interpreted as the cotangent space to $\operatorname{Gr}(v,W)$ at the point V. So $\mathcal{M}^\theta_0(v) = T^*\operatorname{Gr}(v,w)$ (here $\theta > 0$).

Let us consider the case when $\theta < 0$. Dually to Lemma 1.5, $(T^*R)^{\theta-ss}$ consists of all pairs (i,j) such that j is surjective. The condition that $(i,j)\in\mu^{-1}(0)^{\theta-ss}$ means that i is a map $V\to\ker j$. Up to the G-conjugacy, j defines a point in $\operatorname{Gr}(w-v,w)$ (via taking the kernel). Then the space of the maps i is again the cotangent fiber and $\mathcal{M}^\theta_0(v) = T^*\operatorname{Gr}(w-v,w)$.

Finally, let us consider the case when $\theta = 0$ (and any v). We still have an invariant map $\mu^{-1}(0)\to\mathfrak{gl}(W), (i,j)\mapsto ij$. Its image consists of all matrices with square 0 of rank not exceeding $\min(v,\lfloor w/2\rfloor)$. In fact, $\mathcal{M}^0_0(v)$ coincides with this image.

This example can be generalized to the case of a type A Dynkin quiver: $Q_0 = \{1, 2, \ldots, r\}$, $Q_1 = \{a_1, \ldots, a_{r-1}\}$ with $h(a_i) = i, t(a_i) = i + 1$. Take an arbitrary dimension vector v and w with $w_2 = \ldots = w_r = 0$. Take a stability condition θ with positive components. Then $\mathcal{M}_0^\theta(v) = T^* \operatorname{Fl}(v_r, \ldots, v_1; w_1)$ (in particular, it is empty if $v_i < v_{i+1}$ or $v_1 > w_1$).

Example 3.3. Now let Q be the quiver with one vertex and a single loop. We will consider the situation, when $w = 1, \lambda = 0$. In this case $R = \operatorname{End}(V) \oplus V^*, G = \operatorname{GL}(V), T^*R = \operatorname{End}(V)^{\oplus 2} \oplus V^* \oplus V$ and the moment map is $\mu(X, Y, i, j) = [X, Y] - ji$.

Finally, let us turn to the case of $\theta = 0$. We have a map $(\mathbb{C}^v)^2 \to \mu^{-1}(0)$ given by $(x_1, \ldots, x_v, y_1, \ldots, y_v) \mapsto (\operatorname{diag}(x_1, \ldots, x_v), \operatorname{diag}(y_1, \ldots, y_v), 0, 0)$. This map gives rise to the morphism $\mathbb{C}^{2v}/\mathfrak{S}_v \to \mathcal{M}_0^0(v)$ that makes the following diagram commutative.

$$\begin{array}{ccc} \mathbb{C}^{2v} & \longrightarrow & \mathbb{C}^{2v}/\mathfrak{S}_v \\ \downarrow & & \downarrow \\ \mu^{-1}(0) & \longrightarrow & \mathcal{M}_0^0(v) \end{array}$$

It turns out that this morphism is an isomorphism. It is surjective because any two matrices X, Y whose commutator has rank 1 are upper triangular in some basis. It is a closed embedding because the algebra $\mathbb{C}[x_1, \ldots, x_v, y_1, \ldots, y_v]^{\mathfrak{S}_v}$ is spanned by elements of the form $f(x_1 + \alpha y_1, \ldots, x_v + \alpha y_v)$, where f is a symmetric polynomial. See [GG] for details.

Now consider $\theta < 0$. Then $(T^*R)^{\theta-ss}$ consists of all quadruples (X, Y, i, j) such that $\mathbb{C}\langle X, Y\rangle i = V$. If, in addition, $[X, Y] = ji$, then one can show that $j = 0$ (an exercise). So X and Y commute. Using this one can identify $\mathcal{M}_0^{-1}(v)$ with the Hilbert scheme $\operatorname{Hilb}_v(\mathbb{C}^2)$. Recall that the latter parameterizes codimension v ideals in $\mathbb{C}[x, y]$ and is a smooth irreducible variety of dimension $2v$. Namely, to the G-orbit of (X, Y, i) we assign the ideal of all polynomials $f \in \mathbb{C}[x, y]$ such that $f(X, Y)i = 0$. Going in the opposite direction, given a codimension v ideal $I \subset \mathbb{C}[x, y]$ we set $V := \mathbb{C}[x, y]/I$. For i, we take the map $\mathbb{C} \to V, 1 \mapsto 1 + I$, while for X, Y we take the multiplications by x, y, respectively. These maps are mutually inverse bijections between $\operatorname{Hilb}_n(\mathbb{C}^2)$ and the set of G-orbits in $\mu^{-1}(0)^{\theta-ss}$. But the G-action on $\mu^{-1}(0)^{\theta-ss}$ is free, so the orbit space is $\mathcal{M}_0^{-1}(v)$.

Let us consider the case $\theta > 0$. In this case we still get $\mathcal{M}_0^1(v) = \operatorname{Hilb}_v(\mathbb{C}^2)$. To an ideal I we now assign the space $V = (\mathbb{C}[x, y]/I)^*$, operators X and Y that are the dual operators to the multiplication by x, y, and j is $1 + I$ viewed as a linear map $V \to \mathbb{C}$.

One also has an algebro-geometric interpretation of $\mathcal{M}_0^\theta(v, w)$, where the quiver is still the same. This is a so called *Gieseker moduli space* of rank w degree v torsion free sheaves on $\mathbb{P}^2$ that are trivialized at the line at infinity. This is a smooth irreducible variety of dimension $2vw$.

3.3. **Properties.** To start with, let us note that $\mathcal{M}_\lambda^\theta(Q, v, w)$ does not depend on the orientation of Q. Indeed, let Q' be the quiver obtained from Q by reversing a single arrow, say a. Then the isomorphism $T^*R \to T^*R'$ that maps x_a to x'_{a^*}, x_{a^*} to $-x_a$ and fixing all other components is a G-equivariant symplectomorphisms that intertwines the moment maps. It induces an isomorphism $\mathcal{M}_\lambda^\theta(Q, v, w) \xrightarrow{\sim} \mathcal{M}_\lambda^\theta(Q', v, w)$.

Let us note that the varieties $\mathcal{M}^\theta_0(v)$ come with a $\mathbb{C}^\times$-action induced from the dilation action on T^*R, $t.v := t^{-1}v$. The action is compatible with the Poisson structure, the action of t on the bracket multiplies it by t^{-2}.

Now let us state a sufficient condition (due to Nakajima, [N1, Theorem 2.8]) for the G-action on $\mu^{-1}(\lambda)^{\theta-ss}$ to be free (so that $\mathcal{M}^\theta_\lambda(v)$ is smooth and symplectic). For this we will need the notion of a root for Q. Let us define the symmetric form $(\cdot,\cdot)$ on $\mathbb{C}^{Q_0}$ by

$$(x,y) := 2\sum_{k\in Q_0} x_k y_k - \sum_{a\in Q_1}(x_{t(a)}y_{h(a)} + x_{h(a)}y_{t(a)}).$$

We also define the "usual scalar product" on $\mathbb{C}^{Q_0}$

$$x\cdot y := \sum_{k\in Q_0} x_k y_k.$$

We say that a nonzero element $\alpha\in\mathbb{Z}^{Q_0}_{\geqslant 0}$ is a root if $(\alpha,\alpha)\leqslant 2$ and the support of α (the set of all i such that $\alpha_i\neq 0$) is connected. We say that (θ,λ) is *generic* if $\theta\cdot\alpha\neq 0$ or $\lambda\cdot\alpha\neq 0$ for every root α subject to $\alpha\leqslant v$ (component-wise).

Proposition 3.4. *If (θ,λ) is generic, then the G-action on $\mu^{-1}(\lambda)^{\theta-ss}$ is free. In particular, all components of $\mathcal{M}^\theta_\lambda(v)$ have dimension $2\dim R - 2\dim G = 2w\cdot v - (v,v)$.*

Note that there are generic λ (meaning that $(0,\lambda)$ is generic) as well as generic θ. For example, θ with all strictly positive components is generic. Consider the union of hyperplanes $\{\theta|\alpha\cdot\theta = 0\}$ for roots $\alpha\leqslant v$ in $\mathbb{R}^{Q_0}$. The cones cut by this arrangement will be called *chambers*. It is a standard fact from GIT that for θ,θ' lying in the interior of the same chamber, we have $\mu^{-1}(0)^{\theta-ss} = \mu^{-1}(0)^{\theta'-ss}$ and hence $\mathcal{M}^\theta_0(v) = \mathcal{M}^{\theta'}_0(v)$. There is a generalization of this to arbitrary λ: we just need to consider hyperplanes defined by α with $\lambda\cdot\alpha = 0$. In particular, if λ is generic, then $\mathcal{M}^\theta_\lambda(v)$ is independent of θ, we have already seen this in an example.

The property of $\mathcal{M}^\theta(v)$ with generic θ which is the most important for us is that this variety is a symplectic resolution of singularities. Let give the general definition.

Definition 3.5. We say that a smooth symplectic variety X is a symplectic resolution if $\mathbb{C}[X]$ is a finitely generated algebra and the morphism $X\to\operatorname{Spec}(\mathbb{C}[X])$ is birational and projective (and so is a resolution of singularities).

Proposition 3.6. *The variety $\mathcal{M}^\theta(v)$ is a symplectic resolution.*

This is something we have already seen in Example 3.3, $\operatorname{Hilb}_v(\mathbb{C}^2)$ is a resolution of singularities of $(\mathbb{C}^2)^v/\mathfrak{S}_v$, the resolution morphism takes an ideal I to its support counted with multiplicities (so we do get an unordered v-tuple of complex numbers).

Proof. Fix a generic λ and consider the varieties $\mathcal{M}^\theta_{\mathbb{C}\lambda}(v) := \mu^{-1}(\mathbb{C}\lambda)^{\theta-ss}/G$ and $\mathcal{M}^0_{\mathbb{C}\lambda}(v) := \mu^{-1}(\mathbb{C}\lambda)//G$. Both are schemes over $\mathbb{C}\lambda$. We have a natural morphism $\phi_{\mathbb{C}\lambda}:\mathcal{M}^\theta_{\mathbb{C}\lambda}(v)\to\mathcal{M}^0_{\mathbb{C}\lambda}(v)$ that is an isomorphism over $\mathbb{C}^\times\lambda$. Note that all components of $\mathcal{M}^\theta_{\mathbb{C}\lambda}(v)$ have dimension $\dim T^*R - 2\dim G + 1$.

Let $\bar{\mathcal{M}}_{\mathbb{C}\lambda}(v)$ be the image of $\phi_{\mathbb{C}\lambda}$, this is a closed subvariety in $\mathcal{M}^0_{\mathbb{C}\lambda}(v)$ because $\phi_{\mathbb{C}\lambda}$ is projective. So it coincides with the closure of the preimage of $\mathbb{C}^\times\lambda$ and has dimension $\dim\mathcal{M}^\theta(v)+1$. Hence the fiber $\bar{\mathcal{M}}^0(v)$ of $\bar{\mathcal{M}}^0_{\mathbb{C}\lambda}(v)$ over 0 has dimension $\dim\mathcal{M}^\theta(v)$ and admits a surjective projective morphism from $\mathcal{M}^\theta(v)$. Applying the Stein decomposition to this morphism we decompose it to the composition of $\varphi:\mathcal{M}^\theta(v)\to\mathcal{M}(v)$, where

$\mathcal{M}(v)$ is a normal variety and φ is a resolution of singularieties, and a finite morphism $\tau : \mathcal{M}(v) \to \bar{\mathcal{M}}^0(v)$. Since τ is finite, $\mathcal{M}(v)$ is affine and hence $\mathcal{M}(v) = \operatorname{Spec}(\mathbb{C}[\mathcal{M}^\theta(v)])$. $\square$

For similar reasons, $\mathcal{M}^\theta_\lambda(v)$ is a symplectic resolution, we write $\mathcal{M}_\lambda(v)$ for $\operatorname{Spec}(\mathcal{M}^\theta_\lambda(v))$ and φ_λ for the resolution morphism $\mathcal{M}^\theta_\lambda(v) \to \mathcal{M}_\lambda(v)$ (we put subscript "v" if we want to indicate the dependence on v). Then we get a natural (finite by the construction) morphism $\zeta_\lambda : \mathcal{M}_\lambda(v) \to \mathcal{M}^0_\lambda(v)$ and so $\phi_\lambda = \zeta_\lambda \circ \varphi_\lambda$.

Being a symplectic resolution has important implications for Čech cohomology.

Lemma 3.7. *Let X be a symplectic resolution. Then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. By the Grauert-Riemenschneider theorem, $H^i(X, K_X) = 0$ for $i > 0$, where K_X stands for the canonical bundle. But X is symplectic, so $K_X = \mathcal{O}_X$. $\square$

3.4. **Lagrangian subvariety and cohomology.** Consider the natural morphism $\phi : \mathcal{M}^\theta(v) \to \mathcal{M}^0(v)$ (we will write ϕ_v when we need to specify the dimension vector).

Recall the $\mathbb{C}^\times$-action on T^*R by dilations. The induced grading on $\mathbb{C}[T^*R]$ is the standard one. So $\mathbb{C}[\mathcal{M}^0(v)]$ is positively graded, equivalently, the induced $\mathbb{C}^\times$-action contracts $\mathcal{M}^0(v)$ to a single point to be denoted by 0. We are interested in the structure of $\phi^{-1}(0)$.

The following result is due to Nakajima.

Proposition 3.8 (Theorem 5.8 in [N1])**.** *Suppose Q has no loops. Let θ be generic. The subvariety $\phi^{-1}(0) \subset \mathcal{M}^\theta(v)$ is lagrangian, meaning that all its irreducible components have dimension $\frac{1}{2}\dim \mathcal{M}^\theta(v)$ and the restriction of the symplectic form to the smooth points in $\phi^{-1}(0)$ is zero.*

Since $\mathbb{C}^\times$ contracts $\mathcal{M}^0(v)$ to 0 and ϕ is $\mathbb{C}^\times$-equivariant by the construction, we also have the following claim.

Proposition 3.9 (Corollary 5.5 in[N1])**.** *The variety $\mathcal{M}^\theta(v)$ is homotopy equivalent to $\varphi^{-1}(0)$ so $H_*(\mathcal{M}^\theta(v))$ is identified with $H_*(\varphi^{-1}(0))$. In particular, $H_{mid}(\mathcal{M}^\theta(v))$ (where "mid" means $\dim_{\mathbb{C}} \mathcal{M}^\theta(v)$) has a natural basis indexed by the irreducible components of $\phi^{-1}(0)$.*

The reason why we are interested in the homology (and the middle homology in particular) lies in Geometric Representation theory and will be explained later.

Finally, let us quote one more result of Nakajima.

Proposition 3.10 (Corollary 4.2,Section 9 in [N1])**.** *Let θ be generic. Then the varieties $\mathcal{M}^\theta_\lambda(v)$ corresponding to different (λ, θ) are diffeomorphic (as C^∞-manifolds). The spaces $H_*(\mathcal{M}^\theta_\lambda(v))$ are canonically identified for all generic (λ, θ).*

This has an important corollary.

Proposition 3.11. *The variety $\mathcal{M}^\theta_\lambda(v)$ is irreducible provided (λ, θ) is generic.*

Proof. By results of Crawley-Boevey, [CB], the variety $\mathcal{M}^\theta_\lambda(v)$ is connected when λ is generic. Now we can apply the first statement in Proposition 3.10. $\square$

Corollary 3.12. *The fiber of $0 \in \mathcal{M}^0_0(v)$ under the morphism $\mathcal{M}_0(v) \to \mathcal{M}^0_0(v)$ consists of a single point (to be denoted by 0). In particular, $\varphi^{-1}(0) = \phi^{-1}(0)$.*

Proof. The algebra $\mathbb{C}[\mathcal{M}_0^\theta(v)]$ has no zero divisors so $\mathcal{M}_0(v)$ is irreducible. The morphism $\eta : \mathcal{M}_0(v) \to \mathcal{M}_0^0(v)$ is finite and $\mathbb{C}^\times$-equivariant. So $\eta^{-1}(0)$ is a finite set, and $\mathcal{M}_0(v)$ get contracted to this set. It follows that $\mathcal{M}_0(v)//\mathbb{C}^\times = \eta^{-1}(0)$. But $\mathcal{M}_0(v)//\mathbb{C}^\times$ is irreducible. This establishes all claims of this corollary. $\square$

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