# Lecture 26.

## 1) Projective modules vs locally free modules.

BONUS: What's next (in the study of Commutative Algebra)?

### 1.1) Main result.

$A$ is a commutative ring. The goal of this lecture is to prove the following result.

**Theorem:** For a finitely generated $A$-module $M$ TFAE:

(a) $M$ is projective.

(b) $\forall$ max. ideal $\mathfrak{m} \subset A$, the local'n $M_{\mathfrak{m}}$ is free $A_{\mathfrak{m}}$-module

If $A$ is Noetherian, then (a) & (b) are equivalent to (c)

(c) $M$ is locally free, i.e. $\exists f_1,\dots,f_k \in A$ s.t. $(f_1,\dots,f_k)=A$ & $M_{f_i}$ is free $A_{f_i}$-module, $\forall i=1,\dots k$.

### 1.2) Projective modules over local ring.

**Thm:** Every finitely generated projective module $P$ over a local ring $A$ is free.

Proof: Let $\mathfrak{m} \subset A$ denote the maximal ideal. The quotient $P/\mathfrak{m}P$ is a finite dimensional vector space, let $\overline{m}_1,\dots \overline{m}_\ell$ be a basis, and let $m_1,\dots m_\ell$ be preimages of these elements in $P$ (under $P \twoheadrightarrow P/\mathfrak{m}P$). By Corollary in Section 2.1 of Lec 25, $m_1,\dots m_\ell$ span $P$. So the homomorphism

$$\pi: A^{\oplus \ell} \longrightarrow P, \quad (a_1,\dots,a_\ell) \mapsto \sum_{i=1}^{\ell} a_i m_i$$

is surjective & the induced homomorphism $(A/\mathfrak{m})^{\oplus \ell} \longrightarrow P/\mathfrak{m}P$ is an isomorphism. Since $P$ is projective, Thm from Sec. 3.1 of Lec 22 (and its proof) shows that $A^{\oplus \ell} \simeq P \oplus P'$ w. $P' := \ker \pi$. It follows that $(A/\mathfrak{m})^{\oplus \ell} \simeq P/\mathfrak{m}P \oplus P'/\mathfrak{m}P'$. But $(A/\mathfrak{m})^{\oplus \ell}$ & $P/\mathfrak{m}P$ are isomorphic dim $\ell$ vector spaces over $A/\mathfrak{m}$. So $P'/\mathfrak{m}P' = \{0\}$. The $A$-module $P'$ admits a surjective homomorphism from $A^{\oplus \ell}$. So it's finitely generated. Applying Nakayama lemma we see that $P' = \{0\}$. So $\pi$ is an isomorphism. $\square$

## 1.3) Proof of (a) $\Rightarrow$ (b)

$M$ is finitely generated $\Leftrightarrow \exists\, n > 0$ w. $\pi: A^{\oplus n} \twoheadrightarrow M$; $M$ is projective $\Rightarrow A^{\oplus n} \simeq M \oplus M'$ for $M' = \ker \pi$.
Localize at $A \setminus \mathfrak{m}$: by Prob. 7 in HW3 $A_{\mathfrak{m}}^{\oplus n} \simeq M_{\mathfrak{m}} \oplus M'_{\mathfrak{m}} \Rightarrow$ $M_{\mathfrak{m}}$ is a finitely generated projective $A_{\mathfrak{m}}$-module. The ring $A_{\mathfrak{m}}$ is local. By Thm in Sect. 1.2, $M_{\mathfrak{m}}$ is free $\square$

Note that in this proof we only need that $M$ is finitely generated. However the proof of (b) $\Rightarrow$ (a) requires the stronger condition that $M$ is finitely presented.

## 1.2) Proofs of (b) $\Rightarrow$ (a):

Let $P$ be a finitely presented $A$-module and $M, N$ be $A$-modules w. surjective $A$-linear map $M \twoheadrightarrow N$. Consider the induced $A$-linear map $\varphi: \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$. Let $C$ denote its cokernel, i.e. the quotient of the target by the image. Of course, $P$ is projective iff any such $C$ is $\{0\}$.

Lemma: Suppose $S$ is a multiplicative subset of $A$ s.t. $P[S^{-1}]$ is a projective $A[S^{-1}]$-module. Then $C[S^{-1}]=\{0\}$.

Proof: It's here that we use that $P$ is finitely presented. Recall (Problem 8 in HW3): for multiplicative $S\subset A$ have natural isomorphism

$$\operatorname{Hom}_A(P,?)[S^{-1}] \longrightarrow \operatorname{Hom}_{A[S^{-1}]}(P[S^{-1}],?[S^{-1}])$$

It is an isomorphism of functors $A\text{-Mod}\to A[S^{-1}]\text{-Mod}$ so we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(P,M)[S^{-1}] \arrow[r] \arrow[d, "\wr"] & \operatorname{Hom}_A(P,N)[S^{-1}] \arrow[d, "\wr"] \\
\operatorname{Hom}_{A[S^{-1}]}(P[S^{-1}],M[S^{-1}]) \arrow[r] & \operatorname{Hom}_{A[S^{-1}]}(P[S^{-1}],N[S^{-1}])
\end{tikzcd}$$

The bottom arrow is surjective b/c $P[S^{-1}]$ is projective. So $\varphi[S^{-1}]$ is surjective. The functor $\bullet[S^{-1}]$ is exact (Sect. 3.3 of Lec 20). So it sends exact sequence

$$\operatorname{Hom}_A(P,M)\xrightarrow{\varphi}\operatorname{Hom}_A(P,N)\longrightarrow C\longrightarrow 0 \quad \text{to}$$

exact sequence

$$\operatorname{Hom}_A(P,M)[S^{-1}]\xrightarrow{\varphi[S^{-1}]}\operatorname{Hom}_A(P,N)[S^{-1}]\longrightarrow C[S^{-1}]\longrightarrow 0$$

Since $\varphi[S^{-1}]$ is surjective, $C[S^{-1}]=\{0\}$. $\square$

Proof of (b) $\Rightarrow$ (a): $P_{\mathfrak{m}}$ is free, hence projective for all max. ideals $\mathfrak{m}$. So $C_{\mathfrak{m}} = \{0\}$ $\forall$ $\mathfrak{m}$, thx to the previous lemma. Suppose $C \neq \{0\}$. Take $c \in C \setminus \{0\}$ and consider the nonzero submodule $Ac \simeq A/I$, where $I = \{a \in A \mid ac = 0\}$. Let $\mathfrak{m}$ be a maximal ideal containing $I$. We claim $(A/I)_{\mathfrak{m}} \neq \{0\}$. Indeed, $(A/I)_{\mathfrak{m}} = A_{\mathfrak{m}}/I_{\mathfrak{m}}$ & $I_{\mathfrak{m}} \subset \mathfrak{m}_{\mathfrak{m}}$, the maximal ideal of $A_{\mathfrak{m}}$. Since the functor $\bullet_{\mathfrak{m}}$ sends embeddings to embeddings we have $(A/I)_{\mathfrak{m}} \hookrightarrow C_{\mathfrak{m}}$ so the target is nonzero. Contradiction. $\square$

1.3) Proof of c) $\Rightarrow$ a): is similar to b) $\Rightarrow$ a). In the same notation, we have $C[f_i^{-1}] = \{0\}$ for all $i$. Let $c \in C$. Since $\frac{c}{1} = 0$ in $C[f_i^{-1}]$, $\exists$ $m_i > 0$ s.t. $f_i^{m_i} c = 0$. Consider the ideal $(f_1^{m_1}, \dots, f_k^{m_k}) \subset A$. It contains $(f_1, \dots, f_k)^m$ w. $m = m_1 + \dots + m_k$. The latter ideal is $A$, so $\exists$ $a_1, \dots, a_k \in A$ s.t. $a_1 f_1^{m_1} + \dots + a_k f_k^{m_k} = 1$. From $f_i^{m_i} c = 0$ we conclude $c = (\sum a_i f_i^{m_i}) c = 0$. So $c = 0$ and we are done.

1.4) Proof of b) $\Rightarrow$ c).

Now assume $A$ is Noetherian.

Lemma: Let $P$ be a finitely generated $A$-module & $\mathfrak{m} \subset A$ be a maximal ideal s.t. $P_{\mathfrak{m}}$ is free. Then $\exists$ $f \in A \setminus \mathfrak{m}$ s.t. $P[f^{-1}]$ is a free $A[f^{-1}]$-module.

Proof: Let $\frac{p_1}{f_1}, \dots \frac{p_k}{f_k}$ form a basis in the $A_{\mathfrak{m}}$-module $P_{\mathfrak{m}}$. Consider the $A$-linear map $\varphi: A^{\oplus k} \longrightarrow P$ given by $(a_1,\dots,a_k) \mapsto \sum_{i=1}^{k} a_i p_i$.

Set $S = A \setminus \mathfrak{m}$. The map $\varphi[S^{-1}]$ is an isomorphism so $(\ker\varphi)[S^{-1}] = \ker(\varphi[S^{-1}]) = 0$, $(\operatorname{coker}\varphi)[S^{-1}] = \operatorname{coker}(\varphi[S^{-1}]) = 0$.

We claim that $\exists\, s' \in S \mid s'(\ker\varphi) = 0$. Indeed, $\ker\varphi$ is finitely generated over $A$, say, by elements $m_1, \dots m_e$. Since $\frac{m_i}{1} = 0$ we see that $\exists\, s_i \in S$ s.t. $s_i m_i = 0$. Set $s' = s_1 \dots s_e$ so that $s' m_i = 0$ for all $i$. Since $m_1, \dots, m_e$ span $\ker\varphi$, we see that $s'(\ker\varphi) = 0$.

Similarly, $\exists\, s'' \in S$ w. $s''(\operatorname{coker}\varphi) = 0$. Set $f = s's''$. We claim $(\operatorname{coker}\varphi)[f^{-1}] = (\ker\varphi)[f^{-1}] = 0$ indeed in, say $(\ker\varphi)[f^{-1}]$ we have $\frac{m}{f^k} = \frac{fm}{f^{k+1}} = \frac{0}{1}$. It follows that the kernels & cokernels of $\varphi[f^{-1}]$ are zero. So $A[f^{-1}]^{\oplus k} \xrightarrow{\sim} P[f^{-1}]^{\oplus k}$ $\square$

Proof of $b) \Rightarrow c)$: Using the lemma, we see that for any maximal ideal $\mathfrak{m} \subset A$ $\exists\, f_{\mathfrak{m}} \in A \setminus \mathfrak{m}$ s.t. $P[f_{\mathfrak{m}}^{-1}]$ is free. Note that $\operatorname{Span}_A(f_{\mathfrak{m}})$ is not contained in any maximal ideal so it equals $A$ hence $1 = \sum_{\mathfrak{m}} a_{\mathfrak{m}} \cdot f_{\mathfrak{m}}$ for $a_{\mathfrak{m}} \in A$ w. only finitely many $a_{\mathfrak{m}} \neq 0$. This gives (c). $\square$

BONUS: What's next (in studying Commutative Algebra)?

A short answer: a whole lot, see Eisenbud's book

- we've discussed finite ring extensions a bit. There's more to it, like going up & going down theorems. These are best understood geometrically.
- we've briefly touched upon completions. There's more to it, incl. the Artin-Rees lemma, Hensel lemma etc.
- we haven't discussed the dimension theory at all, but it's very important. Neither we talked about regular rings, algebraic counterparts of smooth (a.k.a. nonsingular) affine varieties.
- various homological algebra considerations starting with Hilbert's Syzygy Theorem.
- and so on.