QUANTIZED QUIVER VARIETIES

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1. Quantization: algebra level

1.1. **General formalism and basic examples.** Let $A = \bigoplus_{i\geqslant 0} A_i$ be a graded algebra equipped with a Poisson bracket $\{\cdot,\cdot\}$ that has degree -1 because $\{A_i, A_j\} \subset A_{i+j-1}$. By a quantization of A one means a pair $(\mathcal{A}, \iota)$, where

- $\mathcal{A}$ is a filtered associative unital algebra $\mathcal{A} = \bigcup_{i\geqslant 0} \mathcal{A}_{\leqslant i}$.
- $\iota : \operatorname{gr}\mathcal{A} \to A$ is an isomorphism of graded Poisson algebras.

Recall that $\operatorname{gr}\mathcal{A} := \bigoplus_{i\geqslant 0} \mathcal{A}_{\leqslant i}/\mathcal{A}_{\leqslant i-1}$ is a graded algebra. Thanks to the presence of ι, this algebra is commutative meaning that $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-1}$. We define the bracket on $\operatorname{gr}\mathcal{A}$ by $\{a + \mathcal{A}_{\leqslant i-1}, b + \mathcal{A}_{\leqslant j-1}\} := [a,b] + \mathcal{A}_{\leqslant i+j-2}$.

Let us consider some examples.

Example 1.1. Let R be a vector space. Consider the algebra $D(R)$ of linear differential operators on R. It is the quotient of the tensor algebra $T(R\oplus R^*)$ by the relations

$$[r_1, r_2] = 0 = [r_1^*, r_2^*], [r_1, r_1^*] = \langle r_1, r_1^*\rangle.$$

This algebra is filtered by the order of a differential operator so that $\deg R^* = 0, \deg R = 1$. The associated graded algebra is $\mathbb{C}[T^*R]$ and $D(R)$ is easily seen to be a quantization.

Example 1.2. The previous example can be generalized to any smooth affine variety X, we get the algebra $D(X)$ that quantizes $\mathbb{C}[T^*X]$.

Example 1.3. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then the universal enveloping algebra $U(\mathfrak{g})$ is a quantization of $S(\mathfrak{g})$.

1.2. **Quantum Hamiltonian reductions.** The quantum Hamiltonian reduction is a quantum counterpart of the classical Hamiltonian reduction from the previous lecture.

Let $\mathcal{A}$ be a quantization of A and let G be a reductive group acting rationally on $\mathcal{A}$ preserving the filtration. Suppose that there is a *quantum comoment map* $\Phi : \mathfrak{g} \to \mathcal{A}$, i.e., a G-equivariant linear map satisfying $[\Phi(\xi), \bullet] = \xi_{\mathcal{A}}$. Note that Φ is automatically a Lie algebra homomorphism.

As with the classical reduction, pick $\lambda \in \mathfrak{g}^{*G}$. Form the left ideal $\mathcal{I}_\lambda := \mathcal{A}\{\Phi(\xi) - \langle\lambda,\xi\rangle | \xi \in \mathfrak{g}\}$. Again, it is G-stable. Form the space $\mathcal{A}/\!/\!/_\lambda G := (\mathcal{A}/\mathcal{I}_\lambda)^G$. This is an algebra with product $(a+\mathcal{I}_\lambda)(b+\mathcal{I}_\lambda) = ab + \mathcal{I}_\lambda$. It is a useful exercise to check that the product is well-defined, you need to use that for $a + \mathcal{I}_\lambda \in \mathcal{A}/\!/\!/_\lambda G$ we have $[\Phi(\xi), a] \in \mathcal{I}_\lambda$ for any $\xi \in \mathfrak{g}$.

In the case when G is connected (the most interesting case for us), the algebra $\mathcal{A}/\!/\!/_\lambda G$ has an important alternative description.

Lemma 1.4. *Let $a + \mathcal{I}_\lambda \in \mathcal{A}/\!/\!/_\lambda G$. There is a unique endomorphism ζ_a of $\mathcal{A}/\mathcal{I}_\lambda$ that maps $1 + \mathcal{I}_\lambda$ to $a + \mathcal{I}_\lambda$. The map $a \mapsto \zeta_a$ is an isomorphism $\mathcal{A}/\!/\!/_\lambda G \xrightarrow{\sim} \operatorname{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{I}_\lambda)^{opp}$, where the superscript "opp" means the algebra with opposite product.*

Note that the algebra $\mathcal{A}///_\lambda G$ inherits a filtration from $\mathcal{A}$. We are interested in describing $\operatorname{gr}\mathcal{A}///_\lambda G$. Assume that $\operatorname{im}\Phi \subset \mathcal{A}_{\leqslant 1}$. Then we have the induced map $\varphi : \mathfrak{g} \to A_1$ that is a classical comoment map. Set $I := A\{\varphi(\xi), \xi \in \mathfrak{g}\}$. Note that $\operatorname{gr}\mathcal{I}_\lambda \supset I$ as $\varphi(\xi) \in \operatorname{gr}\mathcal{I}_\lambda$ for all $\xi \in \mathfrak{g}$. So we have an epimorphism $A///_0 G \twoheadrightarrow \operatorname{gr}\mathcal{A}_\lambda///_\lambda G$. We are interested in sufficient conditions for $\operatorname{gr}\mathcal{I}_\lambda = I$ for all λ.

Lemma 1.5. *Let $\xi_1, \ldots, \xi_n$ be a basis in $\mathfrak{g}$. Assume that $\varphi(\xi_1), \ldots, \varphi(\xi_n)$ form a regular sequence in A. Then $\operatorname{gr}\mathcal{I}_\lambda = I$.*

Proof. The regularity condition is equivalent to the vanishing of the higher homology of the Koszul complex associated to $\varphi(\xi_1), \ldots, \varphi(\xi_n)$. We will need the vanishing of the first homology group. This means the following. Suppose that we have elements $a_1, \ldots, a_n \in A$ such that $\sum_{i=1}^n a_i\varphi(\xi_i) = 0$. Then there are elements $a_{ij} \in A, i, j = 1, \ldots, n$, with $a_{ij} = -a_{ji}$ such that $a_i = \sum_{j=1}^n a_{ij}\varphi(\xi_j)$.

Replacing $\Phi(\xi)$ with $\Phi(\xi) - \langle\lambda, \xi\rangle$, we may assume that $\lambda = 0$. Now suppose that there are elements $\alpha_1, \ldots, \alpha_n \in \mathcal{A}_{\leqslant d}$ such that the top degree component of $\sum_{i=1}^n \alpha_i\Phi(\xi_i)$ does not lie in I. It follows that $\sum_{i=1}^n \alpha_i\Phi(\xi_i) \in \mathcal{A}_{\leqslant d}$ and $\sum_{i=1}^n a_i\varphi(\xi_i) = 0$, where we write a_i for the degree d component of α_i. So we can find elements a_{ij} as above and of degree $d-1$. Lift a_{ij} to $\alpha_{ij} \in \mathcal{A}_{\leqslant d-1}$ with $\alpha_{ij} = -\alpha_{ij}$. So $\beta_i := \alpha_i - \sum_{j=1}^n \alpha_{ij}\Phi(\xi_j)$ lies in $\mathcal{A}_{\leqslant d-1}$. Moreover,

$$\sum_{i=1}^n \alpha_i\Phi(\xi_i) = \sum_{i=1}^n \beta_i\Phi(\xi_i) + \sum_{i,j=1}^n \alpha_{ij}\Phi(\xi_j)\Phi(\xi_i) =$$
$$\sum_{i=1}^n \beta_i\Phi(\xi_i) + \sum_{i<j} \alpha_{ji}[\Phi(\xi_i), \Phi(\xi_j)] = \sum_{i=1}^n \beta_i\Phi(\xi_i) + \sum_{i<j} \alpha_{ji}\Phi([\xi_i, \xi_j]).$$

Both summands in the final expression lie in $\mathcal{I}_\lambda$ but now β_i and α_{ij} are in $\mathcal{A}_{\leqslant d-1}$. So the same sum $\sum_{i=1}^n \alpha_i\Phi(\xi_i)$ can be expressed as $\sum_{i=1}^n \alpha_i'\Phi(\xi_i)$ with $\alpha_i' \in \mathcal{A}_{\leqslant d-1}$. By repeating this procedure we arrive at a contradiction with the choice of the α_i's. □

2. Quantization: sheaf level

2.1. **Quantizations of sheaves.** Now we want to develop the theory of filtered quantizations for more or less arbitrary Poisson varieties (or schemes) X. The variety needs to come with a $\mathbb{C}^\times$-action that rescales the Poisson bracket: $t.\{\cdot,\cdot\} = t^{-1}\{\cdot,\cdot\}$. When X is affine and the grading on $\mathbb{C}[X]$ is non-negative, for a quantization of X we take that of the graded Poisson algebra $\mathbb{C}[X]$.

Let us consider the situation when X is affine but $A := \mathbb{C}[X]$ is only $\mathbb{Z}$-graded. Let $\mathcal{A}$ be a filtered algebra with $\operatorname{gr}\mathcal{A} = A$. Note that the filtration induces a topology on $\mathcal{A}$: we say that a sequence (a_i) of elements of $\mathcal{A}$ converges to zero if there is a sequence of integers d_i with $\lim_{i\to\infty} d_i = -\infty$ and $a_i \in \mathcal{A}_{\leqslant d_i}$ (note that when $\mathcal{A}$ is $\mathbb{Z}_{\geqslant 0}$-graded the resulting topology is discrete). So we can consider the completion $\mathcal{A}' := \varprojlim_{i\to-\infty} \mathcal{A}/\mathcal{A}_{\leqslant i}$ with respect to this topology, its elements should be thought as Cauchy sequences of elements of $\mathcal{A}$. This is an algebra that comes with a natural filtration that is now complete and separated (meaning that the corresponding topology is). Also $\operatorname{gr}\mathcal{A}'$ is naturally identified with $\operatorname{gr}\mathcal{A}$. So there is basically no harm to restrict to the case when a quantization (in the sense of our previous definition) is complete and separated. This brings some advantages, e.g., one can now do "descending induction on degrees". If you want to know what this means, do the following exercise.

Exercise 2.1. Let $\mathcal{A}$ be a (complete and separated) quantization of A. Show that if A is Noetherian, then so is $\mathcal{A}$.

When X is not affine, its structure of a scheme is not determined by a single algebra but rather by the structure sheaf $\mathcal{O}_X$. However, this is not a sheaf of graded algebras in the Zariski topology: for an open subset U one only expects $\mathbb{C}[U]$ to be naturally graded if U is $\mathbb{C}^\times$-stable. This motivates the following definition. The conical topology on X is the topology where "open" means Zariski open and $\mathbb{C}^\times$-stable. So $\mathcal{O}_X$ becomes a sheaf of graded algebras when viewed in the conical topology. It turns out that when X is normal (e.g., smooth) every point has an open affine neighborhood in the conical topology, this is Sumihiro's theorem.

By a quantization of $\mathcal{O}_X$ we mean a pair $(\mathcal{D}, \iota)$, where

- $\mathcal{D}$ is a sheaf of filtered algebras in the conical topology. We assume that the filtration on $\mathcal{D}$ is complete and separated, meaning that $\mathcal{D} \xrightarrow{\sim} \varprojlim_{i\to-\infty} \mathcal{D}/\mathcal{D}_{\leqslant i}$.
- $\iota : \operatorname{gr}\mathcal{D} \xrightarrow{\sim} \mathcal{O}_X$ is an isomorphism of graded sheaves of Poisson algebras.

Recall that the inverse limit of sheaves is just taken section-wise: for any open U, we have $[\varprojlim_{i\to-\infty} \mathcal{D}/\mathcal{D}_{\leqslant i}](U) = \varprojlim_{i\to-\infty}[\mathcal{D}/\mathcal{D}_{\leqslant i}(U)]$.

We will give an example a bit later. First, let us point out that in the case of affine varieties we now have two formalisms: quantizations as algebras and as sheaves. These two formalisms are equivalent. Given a sheaf quantization $\mathcal{D}$, we can take its global sections, $\mathcal{A} := \Gamma(\mathcal{D})$. Conversely, given an algebra quantization we can *microlocalize* $\mathcal{A}$ to get a sheaf in conical topology. Overall, this situation is very similar to the classical one, when we have the algebra of functions and the structure sheaf of an affine algebraic variety.

Here is an example of a sheaf quantization (and also of microlocalization, the general formalism here will be presented as an exercise in the appendix). We will explain how $D(R)$ sheafifies to T^*R. For this we will need a new realization of $D(R)$.

Consider the space $\mathbb{C}[T^*R]$ with a new product. Pick a basis $x_1, \ldots, x_n$ in R and let $y_1, \ldots, y_n \in R^*$ be the dual basis. Set $\nu := \sum_{i=1}^n \partial_{x_i} \otimes \partial_{y_i}$, a differential endomorphism of $\mathbb{C}[T^*R] \otimes \mathbb{C}[T^*R]$ that is independent of the choice of $x_1, \ldots, x_n$. Further, let $\mu : \mathbb{C}[T^*R] \otimes \mathbb{C}[T^*R] \to \mathbb{C}[T^*R]$ denote the usual commutative product. Define a new product (a "star-product") on $\mathbb{C}[T^*R]$ by the formula

$$f * g = \mu \circ \exp(\nu)(f \otimes g).$$

For example, $x_i * x_j = x_i x_j, y_i * y_j = y_i y_j, y_i * x_j = y_i x_j, x_i * y_j = x_i y_j + \delta_{ij}$. From here one produces a filtration preserving isomorphism of $D(R)$ and $(\mathbb{C}[T^*R], *)$, where y_i goes to y_i, and x_i goes to ∂_{y_i}.

We grade $\mathbb{C}[T^*R]$ with respect to x's. Let $U \subset T^*R$ be an open subspace stable with respect to the corresponding $\mathbb{C}^\times$-action. Pick $F, G \in \mathbb{C}[U]$. We want to compute the product $F * G$. It is enough to do so when F, G are homogeneous (let $i = \deg F + \deg G$). Note that the sum $F*G$ is infinite, in general (F, G will have some denominators and so all non-vanishing derivatives), but has the form $\sum_{j\leqslant i} a_j$, where $a_j \in \mathbb{C}[U]$ is a homogeneous element of degree j. But $*$ is well defined on the completion $\mathbb{C}[U]^\wedge$ whose elements are of the form $\sum_{j\leqslant i} a_j$. So now we can define the sheaf of algebras (in conical topology) D_R by $D_R(U) := (\mathbb{C}[U]^\wedge, *)$. This is the microlocalization of the algebra $D(R)$. Global sections of D_R obviously coincide with $D(R)$.

2.2. **GIT quantum Hamiltonian reduction.** Now suppose that G is a reductive group acting on R and let $\theta : G \to \mathbb{C}^\times$ be a character. Assume that G acts freely on $\mu^{-1}(0)^{\theta-ss}$. Our goal is to produce quantizations $D_R///^\theta_\lambda G$ of $X := T^*R///^\theta_0 G$. We will do this by using a suitable version of the quantum Hamiltonian reduction.

It is enough to define the sections of a sheaf (and check the sheaf axioms) on a base of topology. Such a base in the case of X is formed by the $\mathbb{C}^\times$-stable open subsets $X_f := (T^*R)_f///_0 G$, where $f \in \mathbb{C}[T^*R]$ is a homogeneous $n\theta$-semiinvariant element. The subsets X_f, X_g are glued together by means of the natural inclusions $X_{fg} \hookrightarrow X_f, X_g$. We define our sheaf by setting $\mathcal{D}(X_f) := D_R((T^*R)_f)///_\lambda G$. The restriction map $\mathcal{D}(X_f) \to \mathcal{D}(X_{fg})$ is the map $D_R((T^*R)_f)///_\lambda G \to D_R((T^*R)_{fg})///_\lambda G$ induced from the inclusion $D_R((T^*R)_f) \hookrightarrow D_R((T^*R)_{fg})$. By the construction, we get a pre-sheaf with respect to our covering. This pre-sheaf carries a natural filtration. It is complete and separated with respect to the filtration, this follows from the next lemma.

Lemma 2.2. *The algebra $D_R((T^*R)_f)///_\lambda G$ is complete and separated with respect to its natural filtration. Moreover, it is a quantization of $\mathbb{C}[(T^*R)_f]///_0 G$.*

Proof. To prove the claim about the filtration, it is enough to prove that the filtration on $D_R((T^*R)_f)/\mathcal{I}_\lambda$ is complete and separated (our algebra is a closed subspace there). This will follow if we check that the ideal $\mathcal{I}_\lambda$ is closed. This is a consequence of the following exercise.

Exercise 2.3. Let $\mathcal{A}$ be a filtered quantization of A (and, as usual, A is finitely generated). Show that every left ideal in $\mathcal{A}$ is closed.

To prove the claim about the quantization, we need to check that $\operatorname{gr}\mathcal{I}_\lambda = I$. Since the action of G on $\mu^{-1}(0)_f$ is free, the conditions of Lemma 1.5 hold and we are done. $\square$

So now we have a filtered pre-sheaf $\mathcal{D}$ (complete and separated) with $\operatorname{gr}\mathcal{D} = \mathcal{O}_X$.

What remains to show is that the presheaf $\mathcal{D}$ is indeed a sheaf. This boils down to the following: let $f, f_1, \ldots, f_k \in \mathbb{C}[T^*R]$ be such that $X_f = X_{f_1} \cup X_{f_2} \cup \ldots \cup X_{f_k}$ and let $a_i \in \mathcal{D}(X_{f_i})$ be sections that agree on intersections. Then there is a unique section $a \in \mathcal{D}(X_f)$ that restricts to a_i for all i. We claim that this is a consequence of $\operatorname{gr}\mathcal{D} = \mathcal{O}_X$ and the observation that the presheaf $\mathcal{O}_X$ is a sheaf. Indeed, let d be the highest degree of the a_i's and let $a_i^d \in \mathbb{C}[X_{f_i}]$ denote the top degree component. Then the elements a_i^d agree on intersections and so glue to $a^d \in \mathbb{C}[X_f]$. Lift a^d to $\bar{a}^d \in \mathcal{D}(X_f)_{\leqslant d}$. Replacing a_i with $a_i - \bar{a}^d$ we reduce degree by 1. Then we use the "descending induction on degree" (+convergence) to show the existence of a. The uniqueness is similar in spirit.

2.3. **Classification of quantizations.** We finish this section by classifying filtered quantizations of X under some vanishing assumption. The following theorem is partly due to Bezrukavnikov and Kaledin, [BK], and partly due to the author, [L, Section 2].

Theorem 2.4. *The following is true.*

(1) *There is a natural* period *map* Per *from the set* $\mathsf{Quant}(X)$ *of isomorphism classes of filtered quantizations of X to $H^2_{DR}(X)$.*
(2) *If $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ (that we assume from now on), then* $\mathsf{Per} : \mathsf{Quant}(X) \xrightarrow{\sim} H^2_{DR}(X)$.
(3) *If* $\mathsf{Per}(\mathcal{D}) = -\mathsf{Per}(\mathcal{D}')$*, then $\mathcal{D}' \cong \mathcal{D}^{opp}$.*

Let us explain how to compute the period of $D_R///^\theta_\lambda G$. We will see that the period is given by an affine map $\mathfrak{g}^{*G} \to H^2_{DR}(X)$.

Recall that a quantum comoment map is defined up to adding a character. Define a new "symmetrized" quantum comoment map $\Phi^{sym}(\xi) = \xi_R + \frac{1}{2}\operatorname{tr}_R(\xi)$. Let us explain what makes this map symmetrized. Note that $D(R)$ and $D(R^*)$ are identified via $r \in D(R) \mapsto r \in D(R^*), r^* \in D(R) \mapsto -r^* \in D(R^*)$ (some version of the Fourier transform; note that the roles of vectors and coverctors is interchanged). Now $\xi \mapsto \xi_{R^*}$ is also a quantum comoment map for the G-action on $D(R)$, and one can show that $\Phi^{sym}(\xi) = \frac{1}{2}(\xi_R + \xi_{R^*})$ (start with the case of $G = \mathrm{GL}(R)$ in this case both ξ_R, ξ_{R^*} can be written explicitly).

Let $D_R^{sym}///^\theta_\lambda G$ denote the reduction, where we use the quantum comoment map Φ^{sym}, so that

$$D_R^{sym}///^\theta_\lambda G = D_R///^\theta_{\lambda+\frac{1}{2}\operatorname{tr}_R} G.$$

Lemma 2.5 (Proposition 5.4.4 in [L]). *The period of $D_R^{sym}///^\theta_0 G$ equals 0.*

Now let us describe the differential of the affine map. Note that we have a natural map $\iota : \mathfrak{g}^{*G} \to H^2_{DR}(X)$. Namely, to a character χ of G we can assign a G-equivariant line bundle V_χ on T^*R, which is trivial as the line bundle and has the G-action given by χ^{-1}. We can restrict V_χ to $\mu^{-1}(0)^{\theta-ss}$ and consider its descent to the quotient $\mu^{-1}(0)^{\theta-ss}/G$ getting a line bundle $\mathcal{O}(\chi)$. By definition,

$$\Gamma(U, \mathcal{O}(\chi)) = \mathbb{C}[\pi^{-1}(U)]^{G,\chi},$$

where we write $\pi : \mu^{-1}(0)^{\theta-ss} \to \mu^{-1}(0)^{\theta-ss}/G$ for the quotient morphism. In particular, the bundle $\mathcal{O}(\theta)$ is ample.

Now to χ we can assign the first Chern class $c_1(\mathcal{O}(\chi))$. By $\mathbb{C}$-linearity, this extends to $\iota : \mathfrak{g}^{*G} \to H^2_{DR}(T^*R///^\theta_0 G)$.

Theorem 2.6. *The period of $D_R///^\theta_\lambda G$ is $\iota(\lambda - \frac{1}{2}\operatorname{tr}_R)$.*

This result follows from [L, Section 3, Proposition 5.4.4].

In the cases of interest (Nakajima quiver varieties for quivers of finite and affine types), the map ι is actually an isomorphism (and is conjectured to be an isomorphism for any quiver).

3. Quantized quiver varieties

3.1. **Algebras $\mathcal{A}_\lambda(v)$ and sheaves $\mathcal{A}^\theta_\lambda(v)$.** We will apply the constructions of the previous section to the case when $R = R(Q, v, w)$ and $G = \mathrm{GL}(v)$. Below θ stands for a generic stability condition. The corresponding sheaf $D_R///^\theta_\lambda G$ will be denoted by $\mathcal{A}^\theta_\lambda(v)$. We will also write $\mathcal{A}^0_\lambda(v)$ for $D(R)///_\lambda G$.

Lemma 3.1. *The algebra $\Gamma(\mathcal{A}^\theta_\lambda(v))$ is a quantization of $\mathbb{C}[\mathcal{M}(v)]$. Furthermore, $H^i(\mathcal{M}^\theta(v), \mathcal{A}^\theta_\lambda(v)) = 0$ for $i > 0$.*

Proof. For brevity, set $X := \mathcal{M}^\theta(v)$. Since X is a symplectic resolution, $H^i(X, \mathcal{O}_X) = 0$ for all i. Let us prove that $\operatorname{gr}\Gamma(\mathcal{D}) = \mathbb{C}[X]$. First of all, note that there is an injective homomorphism from the left hand side to the right hand side coming from $\operatorname{gr}\mathcal{D} = \mathcal{O}_X$. We need to show that any element $f \in \mathbb{C}[X]_d$ lifts to a global section of $\mathcal{D}$. We can find an open affine $\mathbb{C}^\times$-stable covering $X = \bigcup_i X_i$, where X_i is of the form $(T^*R)_{g_i}///_0 G$, where g_i is a $(G, n\theta)$-semiinvariant homogeneous element. Then $\operatorname{gr}\Gamma(X_i, \mathcal{D}) = \mathbb{C}[X_i]$, by the construction of $\mathcal{D}$. Choose lifts $\hat{f}_i \in \Gamma(X_i, \mathcal{D})_{\leqslant d}$ of $f|_{X_i}$. Note that $\hat{f}_i - \hat{f}_j \in$

$\Gamma(X_i \cap X_j, \mathcal{D})_{\leqslant d-1}$. Let f_{ij} denote the class of $\hat{f}_i - \hat{f}_j$ in $\mathbb{C}[X_i \cap X_j]$. Then the f_{ij}'s form a 1-cocycle, that is a coboundary thanks to $H^1(X, \mathcal{O}_X) = 0$. So let $f_{ij} = g_i - g_j$ and let $\hat{g}_i$ be a lift of g_i to $\Gamma(X_i, \mathcal{D})_{\leqslant d-1}$. Replacing $\hat{f}_i$ with $\hat{f}_i - \hat{g}_i$, we see that now $\hat{f}_i - \hat{g}_i \in \Gamma(X_i \cap X_j)_{\leqslant d-2}$. Now to complete the proof we apply the descending induction on the degree (and the convergence). The proof that the higher cohomology of $\mathcal{D}$ vanish is similar. □

Note also that we have a natural algebra homomorphism $\mathcal{A}^0_\lambda(v) \to \Gamma(\mathcal{A}^\theta_\lambda(v))$.

Lemma 3.2. *We have $\mathcal{A}^0_\lambda(v) \xrightarrow{\sim} \Gamma(\mathcal{A}^\theta_\lambda(v))$ for λ Zariski generic.*

This is a quantum analog of the equality $\mu^{-1}(\lambda) = \mu^{-1}(\lambda)^{\theta-ss}$ for generic λ. A somewhat stronger statement is [BL, Proposition 2.7].

In fact, the algebra $\Gamma(\mathcal{A}^\theta_\lambda(v))$ is independent of θ, this is a general result about quantizations of symplectic resolutions due to Braden, Proudfoot and Webster, [BPW, Section 3.3]. We will write $\mathcal{A}_\lambda(v)$ for this algebra.

Let us discuss the dependence of the algebras $\mathcal{A}_\lambda(v), \mathcal{A}^0_\lambda(v)$ and of the sheaf $\mathcal{A}^\theta_\lambda(v)$ on the orientation of Q. Recall that the variety $\mathcal{M}^\theta(v)$ together with the $\mathbb{C}^\times$-action induced from the dilation action on T^*R is independent of the orientation.

The algebra $D(R)$ is independent of the choice of the orientation: when we change the orientation of an arrow a, we get an isomorphism $D(R) \cong D(R')$ that on the generating spaces $T^*R \to T^*R'$ is given as in Lecture 1 (a partial Fourier transform). Let us note that the quantum comoment map $\Phi = \Phi_Q$ depends on the orientation, while Φ^{sym} does not, this follows from $\Phi^{sym}(\xi) = \frac{1}{2}(\xi_R + \xi_{R^*})$. It follows that $\mathcal{A}^{sym}_\lambda(v), \mathcal{A}^{sym,\theta}_\lambda(v)$ are independent of the orientation. From here we conclude that, if Q' is obtained from Q by reversing some arrows, we get $\mathcal{A}_\lambda(Q, v) = \mathcal{A}_{\lambda+\chi}(Q', v)$, where χ is a suitable element in $\mathbb{Z}^{Q_0}$ (the trace of some representation of G).

Note that the action we currently consider, $t.(r, r^*) = (r, t^{-1}r^*)$ is <u>different</u>, in particular, it does depend on the orientation of Q. So the other thing that depends on the orientation is a filtration on $\mathcal{A}^{sym}_\lambda(v)$, etc. However, the difference between different filtrations is sort of negligible. Namely, let Q' be obtained by reversing one arrow, say a, in Q. Let α, α' be the corresponding actions of $\mathbb{C}^\times$ on T^*R, they commute. Note that the action $\alpha'(t)\alpha(t)^{-1}$ on T^*R is symplectic, it is induced by the $\mathbb{C}^\times$-action on R that rescales the summand $\mathrm{Hom}(V_{t(a)}, V_{h(a)})$ and leaves all other summands fixed. So this action induces a Hamiltonian $\mathbb{C}^\times$-action on $D(R)$ and hence an inner grading on that algebra: let h be the image of 1 under the quantum comoment map for the torus action, then $[h\,\cdot]$ acts by the multiplication by i on the ith graded component. If $F_{\leqslant i}D(R), F'_{\leqslant i}D(R)$ are the filtrations on $D(R)$ induced by the actions $t \mapsto \alpha(t), \alpha'(t)$, then they are related with the grading on $D(R)$ via $F_{\leqslant i}D(R)_j = F'_{\leqslant i+j}D(R)_j$. The similar equality holds for $\mathcal{A}^{sym}_\lambda(v), \mathcal{A}^{0,sym}_\lambda(v)$. It also holds for $\mathcal{A}^{\theta,sym}_\lambda(v)$ but we need to consider this as a sheaf in a refinement of the conical topology, we need to consider open subsets stable under both $\mathbb{C}^\times$-actions. A punchline is that the filtrations corresponding to different orientations are obtained from one another by twists with internal gradings.

Of course, one can ask, why we do not use the filtration coming from the dilation action that is independent of the choice of the orientation. The answer is that that action satisfies $t.\{\cdot,\cdot\} = t^{-2}\{\cdot,\cdot\}$, while all the orientation dependent actions we consider satisfy $t.\{\cdot,\cdot\} = t^{-1}\{\cdot,\cdot\}$. The latter condition is better for certain technical reasons.

3.2. **Example: Differential operators.** Let us present an easy example of the sheaf $\mathcal{A}^\theta_\lambda(v)$. Let Q be the quiver with a single vertex and no arrows. Further, assume that $\theta > 0$ so that $\mathcal{M}^\theta_0(v) = T^*\operatorname{Gr}(v,w)$. Recall that $(T^*R)^{\theta-ss} = T^*(R^{\theta-ss})$ in this case. So $D_R|_{(T^*R)^{\theta-ss}}$ in this case is just (the microlocalization of) the sheaf $D_{R^{\theta-ss}}$. We claim that $\mathcal{A}^\theta_\lambda(v)$ is the (microlocalization of) the sheaf of λ-twisted differential operators on $\operatorname{Gr}(v,w)$.

To explain why this is the case we start with the following lemma.

Lemma 3.3. *Let X_0 be a smooth affine variety equipped with a free action of a reductive group G. Then $D(X_0)///_0G$ is naturally isomorphic to $D(X_0/G)$.*

Proof. We will construct a natural homomorphism $D(X_0/G) \to D(X_0)///_0G$ and check that it is an isomorphism. The algebra $D(X_0/G)$ is generated by $\mathbb{C}[X_0/G] = \mathbb{C}[X_0]^G$ and $\operatorname{Vect}(X_0/G)$. The latter is nothing else but $[\operatorname{Vect}(X_0)/\mathbb{C}[X_0]\Phi(\mathfrak{g})]^G$ (note that $\mathbb{C}[X_0]\Phi(\mathfrak{g})$ are precisely the vector fields tangent to the fibers of the quotient morphism $X_0 \to X_0/G$). So we get natural homomorphisms $\mathbb{C}[X_0/G], \operatorname{Vect}(X_0/G) \to D(X_0)///_0G$. One can show that they extend to a homomorphism $D(X_0/G) \to D(X_0)///_0G$.

On the other hand, the algebra $D(X_0)///_0G$ naturally acts on $\mathbb{C}[X_0]^G$ (namely, $C[X_0]^G$ is annihilated by $D(X_0)\Phi(\mathfrak{g})$ hence we get the required action). The action is by differential operators, this becomes clear if one uses Grothendieck's inductive definition. This gives rise to a homomorphism $D(X_0)///_0G \to D(X_0/G)$.

The two homomorphisms we have constructed are mutually inverse to one another. To check this one can use étale base change with respect to X/G achieving that $X = G\times X/G$. In the latter case the statement should be clear. $\square$

One can generalize this lemma to the reduction with arbitrary λ: one gets twisted differential operators on X_0/G. We will only need the case when λ comes from a character of G. As above, λ defines a line bundle $\mathcal{O}(\lambda)$ on X/G. The following lemma generalizes the previous one.

Lemma 3.4. *Let X_0 be a smooth affine variety, θ be a character of G such that the G-action on $X_0^{\theta-ss}$ is free. Let λ be a character of $\mathfrak{g}$. Then $D_{X_0}///^\theta_\lambda G$ is (the microlocalization of) the sheaf $D^\lambda_{X_0//^\theta G}$ of the differential operators on $\mathcal{O}(\lambda)$ (a.k.a. $\mathcal{O}(\lambda)$-twisted differential operators).*

This shows that in the case of interest $\mathcal{A}^\theta_\lambda(v)$ is the sheaf of λ-twisted differential operators on $\operatorname{Gr}(v,w)$. Similarly, for a Dynkin quiver of type A (as in Example 3.2 of the previous lecture), we get twisted differential operators on the flag variety.

3.3. **Example: spherical RCA.** Let us proceed to the case of a quiver with a single vertex and one loop. In this case, ν is dominant. We will give an alternative description of the algebra $\mathcal{A}_\lambda(v) = \mathcal{A}^0_\lambda(v)$. By Lemma 1.5, the algebra $\mathcal{A}^0_\lambda(v)$ quantizes $\mathcal{M}^0_0(v) = \mathbb{C}^{2v}/\mathfrak{S}_v$ because $\mu^{-1}(0)$ has codimension $\dim G$ in this case, that was checked in [GG]. In order to give the alternative description, we will produce an a priori different family of quantizations of $\mathbb{C}^{2v}/\mathfrak{S}_v$.

We will start with filtered deformations of a related algebra, the smash product. Let $\mathfrak{h} := \mathbb{C}^v, W := \mathfrak{S}_v$ and consider the algebra $A := S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$ that equals $S(\mathfrak{h}\oplus\mathfrak{h}^*)\otimes\mathbb{C}W$ as a vector space with product $f_1\otimes\gamma_1\cdot f_2\otimes\gamma_2 := f_1\gamma_1(f_2)\otimes\gamma_1\gamma_2$. Note that one can recover $S(\mathfrak{h}\oplus\mathfrak{h}^*)^W$ inside A as follows. Let $e \in \mathbb{C}W$ be the averaging idempotent. Consider the subset $eAe \subset A$. This subset is a non-unital subalgebra, where e is a unit. We have an isomorphism of $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ with eAe given by $f \mapsto fe = ef$.

Now let us produce a filtered deformation of A. Note that A can be presented as the quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$ by the relations $[v, v'] = 0$ for $v, v' \in \mathfrak{h}\oplus\mathfrak{h}^*$. We will deform these relations by adding elements of $\mathbb{C}W$. Namely, let t, c be complex numbers and consider the quotient $H_{t,c}$ of $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ by the relations

$$[x, x'] = [y, y'] = 0, [y, x] = t\langle y, x\rangle - c\sum_{i<j}(x_i - x_j)(y_i - y_j)(ij),$$

where $\langle y, x\rangle$ stands for a natural pairing and we write $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ for the coordinates of x, y. The algebra $H_{t,c}$ (known as a Rational Cherednik algebra) is filtered: the filtration is induced from the grading on $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$, where $\deg \mathfrak{h}^* = \deg W = 0$ and $\deg \mathfrak{h} = 1$. The PBW theorem holds for $H_{t,c}$: $\operatorname{gr} H_{t,c} = S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$. It follows that $\operatorname{gr} eH_{t,c}e = S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$. Furthermore, the bracket on $S(\mathfrak{h} \oplus h^*)^W$ induced by the deformation as above equals $t\{\cdot, \cdot\}$. So the algebra $eH_{1,c}e$ is a quantization of $S(\mathfrak{h}\oplus\mathfrak{h}^*)^W$.

The following result is essentially due to Gan and Ginzburg.

Proposition 3.5 ([GG]). *We have an isomorphism of quantizations $eH_{1,c}e \cong \mathcal{A}_c(v)$.*

This proposition is useful because $H_{1,c}$ is easier to study than $\mathcal{A}_c(v)$.

4. Appendix: exercises on microlocalization

4.1. **Completed Rees algebra.** Let $\mathcal{A}$ be a $\mathbb{Z}$-filtered algebra. Set $R_\hbar(\mathcal{A}) := \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{\leqslant i}\hbar^i \subset \mathcal{A}[\hbar^{\pm1}]$.

a) Show that $R_\hbar(\mathcal{A})$ is a graded $\mathbb{C}[\hbar]$-subalgebra in $\mathcal{A}[\hbar^{\pm1}]$ (the degree is with respect to $\hbar$). Identify $R_\hbar(\mathcal{A})/(\hbar)$ with $\operatorname{gr}\mathcal{A}$ and $R_\hbar(\mathcal{A})/(\hbar - a)$ with $\mathcal{A}$ for $a \neq 0$.

We also consider the completed Rees algebra $R^\wedge_\hbar(\mathcal{A})$, the $\hbar$-adic completion of $R_\hbar(\mathcal{A})$, so that $R^\wedge_\hbar(\mathcal{A})$ is complete and separated in the $\hbar$-adic topology and carries a $\mathbb{C}^\times$-action by $\mathbb{C}$-algebra automorphisms with $t.\hbar = t\hbar$. This action is rational on all quotients mod $\hbar^k$.

Now let $\mathcal{A}_\hbar$ be a $\mathbb{C}[[\hbar]]$-algebra that is complete and separated in the $\hbar$-adic topology that comes equipped with a $\mathbb{C}^\times$-action by $\mathbb{C}$-algebra automorphisms that is rational on all quotients $\mathcal{A}_\hbar/(\hbar^k)$ and satisfying $t.\hbar = t\hbar$. If $A := \mathcal{A}_\hbar/(\hbar)$ is commutative and finitely generated, we will call $\mathcal{A}_\hbar$ a *graded formal quantization* of A. We define $\mathcal{A}_{\hbar,fin}$ as the span of all elements $a \in \mathcal{A}_\hbar$ with $t.a = t^i a$ for some $i \in \mathbb{Z}$.

b) Prove that $\mathcal{A}_{\hbar,fin}$ is a graded subalgebra of $\mathcal{A}_\hbar$ that is dense in the $\hbar$-adic topology and satisfies $\mathcal{A}_{\hbar,fin}/(\hbar) = A$.

c) Prove that $\mathcal{A}_{\hbar,fin}/(\hbar - 1)$ is a filtered quantization of A.

d) Prove that the maps $\mathcal{A} \mapsto R^\wedge_\hbar(\mathcal{A})$ and $\mathcal{A}_\hbar \mapsto \mathcal{A}_{\hbar,fin}/(\hbar - 1)$ are mutually inverse bijections between filtered quantizations and graded formal quantizations.

4.2. **(Micro)localization for formal quantizations.** Let $\mathcal{A}_\hbar$ be a formal quantization of A (we do not require the presence of $\mathbb{C}^\times$-actions/gradings, A is just required to be a finitely generated commutative algebra). We are going to sheafify $\mathcal{A}_\hbar$ in the Zariski topology on $\operatorname{Spec}(A)$.

a) Let $f \in A$ be a nonzero divisor and let $\hat{f} \in \mathcal{A}_k := \mathcal{A}_\hbar/(\hbar^k)$ be a lift of f. Show that $[\hat{f}, \cdot]^k = 0$ and deduce from here that every left fraction by $\hat{f}$ is also a right fraction.

Show that the localization $\mathcal{A}_k[\hat{f}^{-1}]$ (defined by the same universality property as in the commutative case) makes sense and is independent of the choice of the lift. We will denote this localization by $\mathcal{A}_k[f^{-1}]$

b) Show that the algebras $\mathcal{A}_k[f^{-1}]$ form an inverse system. Further show that $\mathcal{A}_\hbar[f^{-1}] := \varprojlim_{k\to\infty} \mathcal{A}_k[f^{-1}]$ is a formal quantization of $A[f^{-1}]$.

c) Establish a natural homomorphism $\mathcal{A}_\hbar[f^{-1}] \to \mathcal{A}_\hbar[(fg)^{-1}]$.

d) Show that $\mathcal{A}_\hbar$ naturally sheafifies to a sheaf $\mathcal{D}_\hbar$ on $\operatorname{Spec}(A)$. Show that $\Gamma(\mathcal{D}_\hbar) = \mathcal{A}_\hbar$.

Note that if $\mathcal{A}_\hbar$ is graded, then $\mathcal{A}_\hbar[f^{-1}]$ is graded provided f is $\mathbb{C}^\times$-semiinvariant. So we can get the microlocalization of $\mathcal{A}_{\hbar,fin}/(\hbar-1)$ by taking the sheaf $\mathcal{D}_{\hbar,fin}/(\hbar-1)$ that makes sense in the conical topology.

e) Work out the details.

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