

Stable maps and quantum cohomology

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1 A brief discussion about the moduli space of marked curves

Let us begin our discussion by briefly recalling a few facts about the moduli space of marked genus zero curves. Historically, the first space which was taken into account was the moduli space \mathcal{M}_g , $g \geq 2$, of smooth Riemann surfaces C of genus g over \mathbb{C} . It can be constructed using various approaches (e.g. Teichmuller theory, Hodge theory, GIT) and it has several interesting properties:

- 1) It is a non-compact quasi-projective algebraic variety;
- 2) Locally, in a neighborhood of each point $[C]$, it looks like the quotient of an open ball in \mathbb{C}^{3g-3} by the automorphism group $\text{Aut}(C)$, therefore $\dim \mathcal{M}_g = 3g - 3$;
- 3) if we look at the space $\mathcal{C}_g = \{(C, p) \mid C \in \mathcal{M}_g, p \in C\}$, we may notice that it naturally maps to \mathcal{M}_g by forgetting the point p . In fact, \mathcal{C}_g may look at first glance like the universal curve over \mathcal{M}_g , but on a closer examination we see that this is true only over the open subset \mathcal{M}_g^0 consisting of automorphism-free curves, precisely because the set-theoretic fiber of \mathcal{C}_g over a point $[C]$ in \mathcal{M}_g is the quotient $C/\text{Aut}(C)$. Therefore the moduli space \mathcal{M}_g in general cannot be a fine moduli space.
- 4) From 2), it is evident that \mathcal{M}_g does not exist for $g = 0, 1$. This is also intuitive: in order to have a nice geometric object, even if we cannot hope for something smooth, we need each curve C to have a finite automorphism group so we can have only orbifold singularities. If C is a genus zero or a genus one curve, we know that its automorphism group will never be finite. However, if a curve C has genus $g \geq 2$, there is an upper bound of $84(g - 1)$ for the order of the automorphism group of C , which ensures the existence of the moduli space \mathcal{M}_g as an algebraic variety with orbifold singularities.

Since we are mainly interested in the genus zero case, we must find a way to fix the problem. The easiest way is to *rigidify* the problem by marking a certain number of distinct points on our curves and allowing only those automorphisms which leave them fixed. In this way, we get a new moduli space which we will call $\mathcal{M}_{g,n}$, whose elements are n -pointed curves $[(C, p_1, \dots, p_n)]$ modulo automorphisms. It has similar properties as \mathcal{M}_g :

- 1) It is a quasi-projective algebraic variety (again, it is not compact);

- 2) Each marked point will increase the dimension by one, therefore $\dim \mathcal{M}_{g,n} = 3g - 3 + n$;
- 3) We can now identify the universal curve \mathcal{C}_g over \mathcal{M}_g with the moduli space $\mathcal{M}_{g,1}$ and, analogously, we get that the universal curve over the moduli space $\mathcal{M}_{g,n}$ will be the space $\mathcal{M}_{g,n+1}$:

$$\begin{array}{ccc} & \mathcal{M}_{g,n+1} & \\ & \downarrow \pi & \\ & \mathcal{M}_{g,n}; & \end{array}$$

Notice that this is not a real universal family over the moduli space, since it is not fine: it will be a universal family only when we consider our moduli space as a stack, or for $g = 0$.

- 4) If we allow enough marked points, our moduli space $\mathcal{M}_{g,n}$ will now exist also for $g = 0, 1$. For dimensional reasons (and also intuitively) it appears clear that the minimum number of markings we need to allow in the genus zero case is three: with only one marking we can still spin our sphere around, and the same holds with two antipodal markings. Analogously, we need one marking in the genus one case. It is therefore clear that $\mathcal{M}_{g,n}$, if n is big enough, is a fine moduli space: no nontrivial automorphisms will fix n points on a curve if n is large enough.
- 5) The spaces $\mathcal{M}_{g,n}$ naturally come with forgetful morphisms: if $n_1 > n_2$, there is a natural *forgetful morphism*, which forgets the first $n_1 - n_2$ markings:

$$\begin{array}{ccc} \mathcal{M}_{0,n_1} & \xrightarrow{\phi_{n_1-n_2}} & \mathcal{M}_{0,n_2} \\ [(C, p_1, \dots, p_n)] & \mapsto & [(C, p_{n_1-n_2+1}, p_{n_1-n_2+2}, \dots, p_n)]. \end{array}$$

Clearly, this morphism only exists if the space on the right exists: for example, there is no forgetful morphism $\overline{\mathcal{M}}_{0,7} \rightarrow \overline{\mathcal{M}}_{0,0}$.

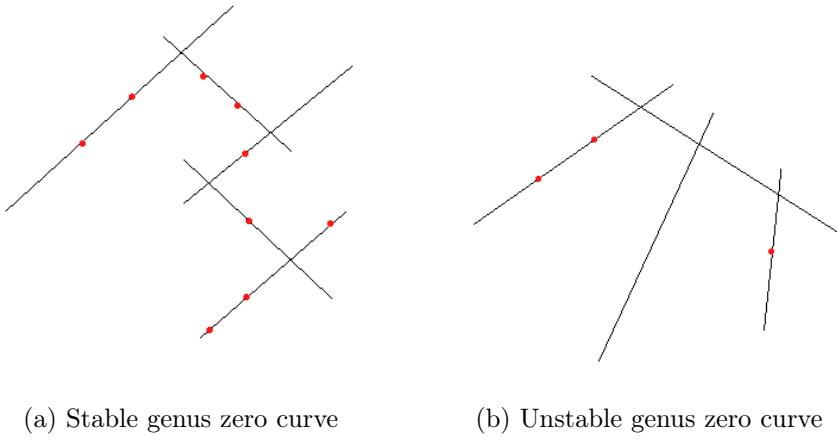
The only problem we now have to deal with is that our moduli space $\mathcal{M}_{g,n}$ is still not compact: intuitively, the limit of a one parameter family of smooth curves will be, in general, a singular curve. There are several compactifications of $\mathcal{M}_{g,n}$, but the one we want to take into account (and also the most popular and intuitive) is the *Deligne-Mumford compactification*, and in order to explain what it consists of we need to introduce the notion of *stable* curve.

Definition 1.1. A *stable n-pointed curve* is a complete connected curve with n marked points that has only nodes as singularities and has only finitely many automorphisms fixing each of the marked points.

In view of the connectedness of C , its automorphism group can fail to be finite only if C contains irreducible components of genus zero or one. Thus, the finiteness condition can be equivalently reformulated as:

Definition 1.2. A *stable curve* is a complete connected curve with n marked points that has only nodes as singularities and such that every smooth rational component contains at least three *special* points (and by special points we mean either marked or nodal points).

This reformulation allows us to depict what a stable curve is: in the genus zero case, for instance, each element of the compactified moduli space will be a *tree of \mathbb{P}^1 's*, such that each irreducible component has at least three points among the markings and the points it shares with the other components.



With this notion of stability, we can now set up our moduli functor:

$$\begin{aligned} \overline{\mathcal{M}}_{0,n} : \quad & \mathbf{Sch}_{\mathbb{C}} \longrightarrow \mathbf{Sets} \\ & S \quad \mapsto \quad \{\text{families of stable genus zero} \\ & \quad \text{curves over } S\} / \text{isomorphism}. \end{aligned}$$

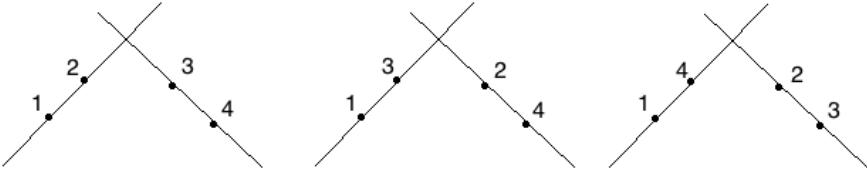
When $n \geq 3$, this functor admits a fine moduli space, $\overline{\mathcal{M}}_{0,n}$.

Notice that we have a universal curve also on the compactified space, now with n well-defined sections giving the markings in each fiber:

$$\begin{array}{c} \overline{\mathcal{M}}_{g,n+1} \\ \pi_1, \dots, \pi_n \curvearrowleft \downarrow \pi \\ \overline{\mathcal{M}}_{g,n}. \end{array}$$

Example 1.3. When $n = 3$, every curve in the moduli space must be smooth, since there are not enough markings to allow multiple components. Therefore, there is only one genus zero curve modulo (no nontrivial) automorphisms, and hence $\overline{\mathcal{M}}_{0,3} \cong pt$.

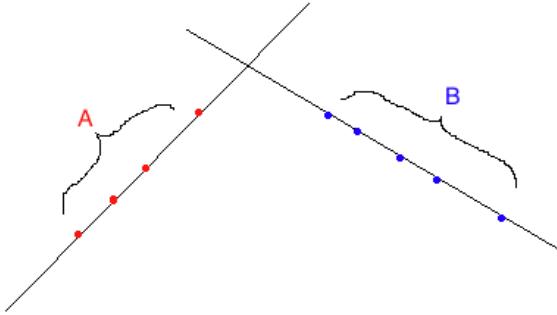
Example 1.4. When $n = 4$, each curve in the moduli space is smooth, or it has two components. If it is smooth, three out of the four markings can be taken by some automorphism to $\{0, 1, \infty\}$, therefore the curve is entirely determined, up to automorphisms, by the fourth marking, which is free to vary on the entire curve minus those three points, since the markings



must be distinct. Hence, $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$. It is not hard to guess what the compactified space must be, since there are only three distinct (up to isomorphism) curves with two components:

hence, obviously, $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$.

The boundary structure of $\overline{\mathcal{M}}_{0,n}$ is worth mentioning. It consists of divisors of the form $D(A|B)$, where A and B form a partition of $\{1, \dots, n\}$, and contain each at least two elements. A generic point of $D(A|B)$ is represented by two lines meeting at a node, with marked points labelled by A on one side and by B on the other:

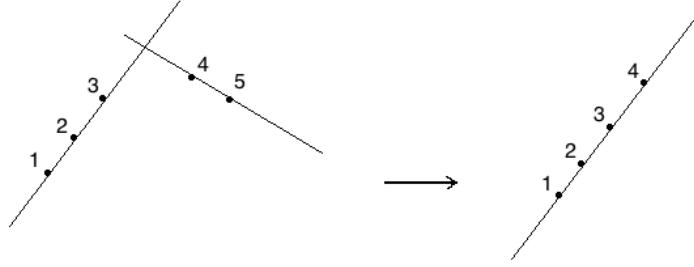


Notice that the forgetful morphisms defined in 5) for the noncompactified moduli space extend to its compactified version:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n_1} & \xrightarrow{\phi_{n_1-n_2}} & \overline{\mathcal{M}}_{0,n_2} \\ [(C, p_1, \dots, p_n)] & \mapsto & [(C, p_{n_1-n_2+1}, p_{n_1-n_2+2}, \dots, p_n)] . \end{array}$$

On these boundary points, the forgetful maps behaves subtly. Even though it is obvious what it does on the smooth locus, on the boundary it may happen that it sends a stable component in an unstable one: in this case, the unstable component gets contracted as shown in the picture (here the forgetful morphism is the one that forgets the fifth marked point):

In particular, for any subset $\{i, j, k, l\} \subset \{1, \dots, n\}$, the inverse image of a point $P(i, j|k, l)$ consisting of two components with markings i, j on one side and k, l on the other via the forgetful morphism $\overline{\mathcal{M}}_{0,n} \rightarrow \overline{\mathcal{M}}_{0,\{i,j,k,l\}}$ is a divisor on $\overline{\mathcal{M}}_{0,n}$. More precisely, it is a sum of



divisors of the form $D(A|B)$ with $i, j \in A$ and $k, l \in B$. Since the three boundary points in $\overline{\mathcal{M}}_{0,\{i,j,k,l\}} \cong \mathbb{P}^1$ are all linearly equivalent, it implies that their inverse images in $\overline{\mathcal{M}}_{0,n}$ are linearly equivalent, as well. Hence:

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A|B) = \sum_{\substack{i,k \in A \\ j,l \in B}} D(A|B) = \sum_{\substack{i,l \in A \\ j,k \in B}} D(A|B).$$

These relations will be important later on, as they will admit a generalization in the setting of stable maps.

2 The moduli space of stable maps

The main object of our discussion, following what we briefly said in the introduction, will be a powerful generalization of the moduli space of pointed curves. The objects of our interest will from now on live on this space, therefore it will be important for us to understand its geometric structure and its main features. Let us take X to be a smooth projective variety (although later on we will want to impose stronger conditions). We want to consider all the maps $\mu : (C, p_1, \dots, p_n) \rightarrow X$, where (C, p_1, \dots, p_n) is a smooth genus zero, n -pointed curve and the map μ is represented by a certain fixed cycle in X , i.e. $\mu_*[C] = \beta \in H_2(X, \mathbb{Z})$. We also need to specify what an isomorphism between two such objects is:

Definition 2.1. An isomorphism between two n -pointed maps $(C, p_1, \dots, p_n, \mu)$ and $(C', p'_1, \dots, p'_n, \mu')$ is an isomorphism between the underlying n -pointed curves $\tau : (C, p_1, \dots, p_n) \rightarrow (C', p'_1, \dots, p'_n)$ such that the diagram

$$\begin{array}{ccc} (C, p_1, \dots, p_n) & \xrightarrow{\tau} & (C', p'_1, \dots, p'_n) \\ & \searrow \mu & \swarrow \mu' \\ & X & \end{array}$$

commutes.

In analogy to the case of curves, we have the notion of a *family* of maps.

Definition 2.2. A *family* of genus zero, n-pointed morphisms to X of class β parametrized by a scheme S is a flat, projective map $\pi : \mathcal{C} \rightarrow S$ together with n sections p_1, \dots, p_n such that each geometric fiber $(\mathcal{C}_s, p_1(s), \dots, p_n(s)) \rightarrow X$ is a genus zero, n-pointed map:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mu} & X \\ p_1, \dots, p_n \downarrow \pi & \swarrow & \\ S. & & \end{array}$$

We can now set up our moduli functor:

$$\begin{aligned} \mathfrak{M}_{0,n}(X, \beta) : \mathfrak{Sch}_{\mathbb{C}} &\longrightarrow \mathfrak{Sets} \\ S &\mapsto \{ \text{stable families of maps over } S \\ &\quad \text{from genus zero, n-pointed curves to } X \\ &\quad \text{representing the class } \beta \} / \text{isomorphism} \end{aligned}$$

Theorem 2.3. The moduli functor described above admits a coarse moduli space $\mathcal{M}_{0,n}(X, \beta)$, which is actually a fine moduli space when $n \geq 3$.

As before, it is quite easy to see that this moduli space is not compact for the same reason as the moduli space of curves. In order to compactify it, we need to introduce a suitable notion of stability that should possibly match the stability we have in the case of curves. Therefore, we want to allow domain curves with nodal singularities plus some restrictions.

Definition 2.4. A genus zero, n-pointed map $\mu : (C, p_1, \dots, p_n) \rightarrow X$ is stable if every component of C that gets contracted by μ is stable in the sense of pointed curves, i.e., if it contains at least three special points.

Notice that this condition is equivalent, as before, to the automorphism groups of the map being finite, which is exactly what we need to have a well-behaved moduli space. Indeed, for dimensional reasons, on each irreducible component of C our map μ is either a branched cover of its image, and then the group $\text{Aut}(\mu)$ consists precisely of the automorphisms of the branched cover (whose number is finite), or it contracts everything to a point, and in this case its automorphisms are precisely the automorphisms of the connected component itself. If we modify the moduli functor accordingly, we will find out that it admits a projective, coarse moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ (these spaces are not generally fine, not even if $n \geq 3$).

Example 2.5. If $\beta = 0$, then the fibers of the map μ have strictly positive dimension, therefore the map is constant. This implies that $\overline{\mathcal{M}}_{0,n}(X, 0) \cong \overline{\mathcal{M}}_{0,n} \times X$ and when X is a point we actually recover the moduli space $\overline{\mathcal{M}}_{0,n}$.

Example 2.6. When $X = \mathbb{P}^r$, the class β must be a multiple of the class of a line, since $H_2(\mathbb{P}^r, \mathbb{Z}) \cong \mathbb{Z}$. If $\beta = d[\text{line}]$, we will just write $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. A basic example is $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 1)$, which is just the moduli space of lines in \mathbb{P}^r , i.e. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 1) \cong \text{Gr}(2, m+1)$.

Let us now list and discuss the main properties of the newly constructed moduli space.

- 1) The moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is compact (this is a consequence of the semistable reduction theorem).

- 2) We can identify the non-compactified moduli space $\mathcal{M}_{0,n}(X, \beta) \subset \overline{\mathcal{M}}_{0,n}(X, \beta)$ with the open locus corresponding to maps from non-singular curves.
- 3) There are n natural *evaluation maps*:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n}(X, \beta) & \xrightarrow{\text{ev}_i} & X \\ [(C, p_1, \dots, p_n, \mu)] & \mapsto & \mu(p_i) . \end{array}$$

- 4) If $n_1 > n_2$, there is a natural *forgetful morphism*, which forgets the first $n_1 - n_2$ markings:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n_1}(X, \beta) & \xrightarrow{\phi_{n_1 - n_2}} & \overline{\mathcal{M}}_{0,n_2}(X, \beta) \\ [(C, p_1, \dots, p_n, \mu)] & \mapsto & [(C, p_{n_1 - n_2 + 1}, p_{n_1 - n_2 + 2}, \dots, p_n, \mu)] . \end{array}$$

Notice that this morphism only exists if the space on the right exists: for example, there is no forgetful morphism $\overline{\mathcal{M}}_{0,7}(X, 0) \rightarrow \overline{\mathcal{M}}_{0,0}(X, 0)$.

- 5) In analogy to the case of curves, we have a notion of *universal map* (here $\tilde{\mathcal{C}}$ will be a stack):

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\tilde{\mu}} & X \\ p_{\tilde{1}}, \dots, p_{\tilde{n}} \swarrow \downarrow \pi & & \\ \overline{\mathcal{M}}_{0,n}(X, \beta). & & \end{array}$$

We can identify our universal curve with the space $\overline{\mathcal{M}}_{0,n+1}(X, \beta)$: the map π will be therefore identified with the forgetful morphism ϕ_1 and the map $\tilde{\mu}$ with the evaluation morphism ev_1 .

- 6) (**Expected dimension**). We now want to compute the expected (or *virtual*) dimension of our moduli space. The expected dimension comes from counting the moduli naively: what can vary in our problem is the curve, or the map when the curve is fixed. We already know that the curve can vary in its moduli space $\overline{\mathcal{M}}_{0,n}$, whose dimension is $n - 3$. To understand how the map can vary, we will need to compute its *first order deformation* space. The deformation long exact sequence around a point $[(C, p_1, \dots, p_n, \mu)] \in \overline{\mathcal{M}}_{0,n}(X, \beta)$ is:

$$\begin{aligned} 0 &\longrightarrow \text{Aut}(C, p_1, \dots, p_n, \mu) \longrightarrow \text{Aut}(C, p_1, \dots, p_n) \longrightarrow \\ &\longrightarrow \text{Def}(\mu) \longrightarrow \text{Def}(C, p_1, \dots, p_n, \mu) \longrightarrow \text{Def}(C, p_1, \dots, p_n) \longrightarrow \\ &\longrightarrow \text{Ob}(\mu) \longrightarrow \text{Ob}(C, p_1, \dots, p_n, \mu) \longrightarrow 0 \end{aligned}$$

which we can roughly think of as the long exact sequence in cohomology associated to the short exact sequence:

$$0 \longrightarrow T_C(-p_1 - \dots - p_n) \longrightarrow \mu^* T_X \longrightarrow N_{C/X} \longrightarrow 0,$$

therefore we can identify the two spaces:

$$\begin{aligned}\text{Def}(\mu) &\cong H^0(C, \mu^* T_X), \\ \text{Ob}(\mu) &\cong H^1(C, \mu^* T_X).\end{aligned}$$

Notice that in the case when $\text{Ob}(\mu) = 0$ for each genus zero map μ , the deformations are unobstructed, therefore our moduli space is nonsingular when viewed as a stack, and the dimension of the space giving the variation of the map is just $h^0(C, \mu^* T_X)$. This justifies the following definition:

Definition 2.7. A smooth projective variety X is *convex* if for any genus zero map $\mu : C \longrightarrow X$ we have

$$H^1(C, \mu^* T_X) = 0.$$

We have, therefore, the following lemma.

Lemma 2.8. *Let X be a convex, projective variety. Then the virtual dimension of the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is*

$$\text{vdim } \overline{\mathcal{M}}_{0,n}(X, \beta) = \dim X + \int_{\beta} c_1(X) + n - 3.$$

Proof. By counting the moduli, following the discussion above, we can conclude that the virtual dimension of the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ at a given point $[(C, p_1, \dots, p_n, \mu)]$ is :

$$\text{vdim } \overline{\mathcal{M}}_{0,n}(X, \beta) = \underbrace{n - 3}_{\text{how the pointed curve varies}} + \underbrace{h^0(C, \mu^* T_X)}_{\text{how the map varies}}.$$

Now, since we have assumed our target variety X to be convex, we have that

$$h^0(C, \mu^* T_X) = \chi(C, \mu^* T_X)$$

and, by Riemann-Roch:

$$\begin{aligned}\chi(C, \mu^* T_X) &= \deg(\mu^* T_X) - \text{rk}(\mu^* T_X)(g(C) - 1) \\ &= \deg(\mu^* T_X) + \text{rk}(\mu^* T_X) \\ &= \int_C c_1(\mu^* T_X) + \text{rk}(T_X) \\ &= \int_C \mu^* c_1(T_X) + \dim X\end{aligned}$$

$$\begin{aligned}
&= \int_{\mu_*[C]} c_1(X) + \dim X \\
&= \int_{\beta} c_1(X) + \dim X,
\end{aligned}$$

hence the result. \square

The following lemma provides us an ample class of examples of convex varieties.

Lemma 2.9. *Suppose X is an n -dimensional variety such that the vector bundle μ^*T_X is generated by its global sections for any genus zero map μ . Then X is convex.*

Proof. For $\mu : C \rightarrow X$, consider the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow H^0(C, \mu^*T_X) \otimes \mathcal{O}_C \rightarrow \mu^*T_X \rightarrow 0$$

where \mathcal{K} is the kernel of the natural evaluation map. We have a long exact sequence in cohomology:

$$\begin{aligned}
0 \rightarrow H^0(C, \mathcal{K}) &\rightarrow H^0(C, \mathcal{O}_C^{\oplus n}) \rightarrow H^0(C, \mu^*X) \rightarrow \\
&\rightarrow H^1(C, \mathcal{K}) \rightarrow H^1(C, \mathcal{O}_C^{\oplus n}) \rightarrow H^1(C, \mu^*T_X) \rightarrow 0.
\end{aligned}$$

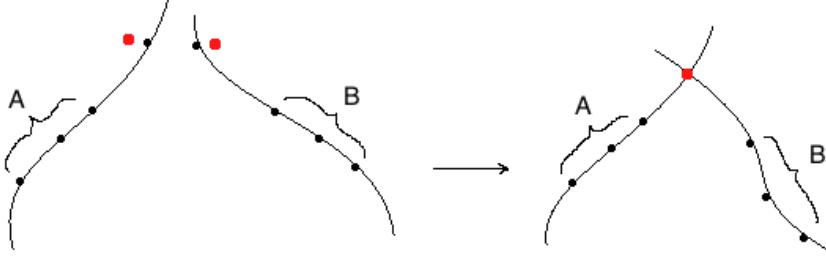
The sequence obviously ends after the first cohomology groups because we are on a curve. Moreover, since the genus of C is zero, we have that by Serre duality $H^1(C, \mathcal{O}_C) \cong H^0(C, \omega_C) = H^0(C, \mathcal{O}_C(-2)) = 0$, hence $H^1(C, \mu^*T_X) = 0$. \square

This shows, for instance, that all the homogeneous varieties are examples of convex varieties and hence the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ with X homogeneous is well-behaved.

- 7) **(Boundary divisors)** When X is convex, the spaces $\overline{\mathcal{M}}_{0,n}(X, \beta)$ have fundamental boundary divisors analogous to the divisors $D(A|B)$ on $\overline{\mathcal{M}}_{0,n}$. Let $n \geq 4$. Let $A \cup B$ be a partition of $\{0, \dots, n\}$. Let $\beta_1 + \beta_2 = \beta$ be a sum in $H_2(X, \mathbb{Z})$. There is a divisor on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ determined by

$$\begin{aligned}
D(A, B; \beta_1, \beta_2) &= \overline{\mathcal{M}}_{0, A \cup \{\bullet\}}(X, \beta_1) \times_X \overline{\mathcal{M}}_{0, B \cup \{\bullet\}}(X, \beta_2) \\
D(A, B; \beta_1, \beta_2) &\subset \overline{\mathcal{M}}_{0,n}(X, \beta).
\end{aligned}$$

A moduli point in $D(A, B; \beta_1, \beta_2)$ corresponds to a map with reducible domain $C = C_1 \cup C_2$ where $\mu_*[C_1] = \beta_1$ and $\mu_*[C_2] = \beta_2$. The points labeled by A lie on C_1 and



the points labeled by B lie on C_2 . The curves C_1 and C_2 are connected at the points labeled \bullet .

Finally, the fiber product in the definition of $D(A, B; \beta_1, \beta_2)$ corresponds to the condition that the maps must take the same value in X on the marked point \bullet in order to be glued. For i, l, k, l distinct in $\{1, \dots, n\}$, set

$$D(i, j|k, l) = \sum_{\substack{A \cup B = \{1, \dots, n\}, i, j \in A, k, l \in B \\ \beta_1, \beta_2 \in H_2(X, \mathbb{Z}), \beta_1 + \beta_2 = \beta}} D(A, B; \beta_1, \beta_2).$$

Using the projection $\overline{\mathcal{M}}_{0,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0, \{i, j, k, l\}} \cong \mathbb{P}^1$, the fundamental linear equivalences

$$D(i, j|k, l) = D(i, k|j, l) = D(i, l|j, k) \quad (1)$$

on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ are obtained via pullback of the 4-point linear equivalences on $\overline{\mathcal{M}}_{0, \{i, j, k, l\}}$ as in the case of pointed curves. These relations among the boundary divisors are fundamental to prove the associativity of the quantum product that we will introduce later.

Moreover, we have the following:

Theorem 2.10. *Let X be a nonsingular, projective, convex variety. The boundary of $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is a divisor with normal crossing (up to a finite group quotient).*

3 Gromov-Witten invariants

From now on, we will always work in the setting when the genus of our pointed maps is zero and the target variety X is convex, unless otherwise stated. As we saw in the previous sections, the moduli space of stable maps $\mathcal{M}_{0,n}(X, \beta)$ has a natural compactification, namely the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}(X, \beta)$. Now that we have a compact space, we can integrate top cohomology classes. We have also seen that the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ comes equipped with n natural evaluation morphisms:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n}(X, \beta) & \xrightarrow{\text{ev}_i} & X \\ [(C, p_1, \dots, p_n, \mu)] & \mapsto & \mu(p_i) . \end{array}$$

Given an arbitrary ordered n-tuple $\gamma_1, \dots, \gamma_n \in H^*(X)$, we can pullback each γ_i by the i-th evaluation morphisms, obtaining a well defined class $\text{ev}_i^*(\gamma_i) \in H^*(\overline{\mathcal{M}}_{0,n}(X, \beta))$, and then integrate the class given by the cup product of them all. The number

$$\langle \gamma_1 \cdots \gamma_n \rangle_{0,\beta}^X = \int_{[\overline{\mathcal{M}}_{0,n}(X, \beta)]} \text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_n^*(\gamma_n)$$

is called *genus g, n-point Gromov-Witten invariant*. Before proceeding further, we can draw a few first glance properties:

- The hypotheses we set at the beginning of the section, i.e. genus zero and convex target space, ensure us that we are integrating over a physical fundamental class rather than on the virtual fundamental class of our moduli space: as we saw before, in this case the first order deformations are unobstructed, therefore our moduli space (now seen as a stack) is smooth and therefore the virtual and the actual fundamental class coincide.
- Even though we began by choosing an ordered n-tuple of cohomology classes, it follows from the definition that the number $\langle \gamma_1 \cdots \gamma_n \rangle_{0,\beta}^X$ is invariant up to re-ordering the classes γ_i (and up to Koszul signs): each element σ in the group of permutations \mathfrak{S}_n induces an automorphism of the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ by simply permuting the marked points.
- Since a homogeneous class $\gamma \in H^{2i}(X)$ is Poincaré dual to some cycle of (complex) codimension i and pullback in cohomology preserves the degree, the form $\text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_n^*(\gamma_n)$ is a top form iff

$$\sum_{i=1}^n 2|\gamma_i| = \dim \overline{\mathcal{M}}_{0,n}(X, \beta) = \dim X + \int_{\beta} c_1(X) + n - 3. \quad (2)$$

- If $n = 0$, the only invariant we have is the 0-point invariant: it occurs when $\dim \overline{\mathcal{M}}_{0,0}(X, \beta) = 0$, i.e. when

$$\dim X + \int_{\beta} c_1(X) = 3. \quad (3)$$

Now, while if $\dim X = 0$ then it must be the case that $\int_{\beta} c_1(X) = 0$ while if $\dim X = 3$, then β must be nonzero, otherwise the moduli space would be empty: since the map is constant, every irreducible component gets contracted and there are no marked points to stabilize them, therefore every map becomes unstable. Now, suppose that $\dim X > 0$ and $\beta \neq 0$. Then we have the following

Lemma 3.1. *Let $\mu : \mathbb{P}^1 \longrightarrow X$ be a nonconstant morphism to a nonsingular, convex projective space X . Then $\int_{\mu_*[\mathbb{P}^1]} c_1(X) \geq 2$.*

Proof. Consider the differential

$$d\mu : T_{\mathbb{P}^1} \longrightarrow \mu^* T_X.$$

Since $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, each section will be a homogeneous degree 2 polynomial and we can pick some generic $s \in T_{\mathbb{P}^1}$ which vanishes at two distinct points p_1 and p_2 . Now, since the differential is nonzero because of the non constancy of μ and s is generic, we can further assume that $d\mu(s) \neq 0$. Moreover, every vector bundle on \mathbb{P}^1 splits into direct sum of line bundles, i.e. $\mu^*(T_X) \cong \bigoplus \mathcal{O}(d_i)$ with $d_i \geq 0$ for each i , since T_X is globally generated: this implies that $d\mu(s)$, a section which vanishes at two points at least, must be a homogeneous polynomial of degree at least two, i.e., that $d_i \geq 2$ for some i . \square

By the Lemma, we get that (3) holds only when $\dim X = 1$ and $\int_{\beta} c_1(X) = 2$, hence it only occurs when $X \cong \mathbb{P}^1$ and, in that case, $I_1 = 1$ is the unique 0-point invariant.

Warning: the proof of the Lemma above only holds when X is convex. There are some cases of great interest, in which X is not convex, where $n = 0$ and either $\dim X = 2$ and $\int_{\beta} c_1(X) = 1$ or $\dim X = 3$ and $\int_{\beta} c_1(X) = 0$.

Now, let us try to give a geometrical interpretation of Gromow-Witten invariants. We can intuitively think of the class $\text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_n^*(\gamma_n)$ as being the Poincaré dual of the locus of stable maps sending each marking to the support of the corresponding class γ_i : the invariant $\langle \gamma_1 \cdots \gamma_n \rangle_{0,n}^X$ will then count how many such maps occur. We thus understand that Gromov-Witten theory is somehow connected to enumerative geometry. To further clarify this idea, let us examine a very specific case.

Example 3.2. Suppose now that $X \cong \mathbb{P}^2$. Then, since $H^2(\mathbb{P}^2) \cong \mathbb{Z}[\text{line}]$, it must be that $\beta = d[\text{line}]$ for some d , and equation (2) becomes:

$$\begin{aligned} \sum_{i=1}^n \text{codim}(\gamma_i) &= \dim \mathbb{P}^2 + d \int_{[\text{line}]} c_1(T_{\mathbb{P}^2}) + n - 3 \\ &= 2 - d \int_{[\text{line}]} c_1(\mathcal{O}_{\mathbb{P}^2}(-3)) + n - 3 \\ &= 2 + 3d + n - 3 \\ &= n + 3d - 1 \end{aligned}$$

We can now choose $\gamma_i = [pt]$ for each i , so the equation above becomes $2n = n + 3d - 1$ and hence the integral does not vanish only if $n = 3d - 1$. Now, the number

$$N_d = \int_{[\overline{\mathcal{M}}_{0,3d-1}(X,d)]} \text{ev}_1^*(pt) \cup \dots \cup \text{ev}_n^*(pt)$$

counts exactly the number of degree d maps from \mathbb{P}^1 to \mathbb{P}^2 which send each of the $3d - 1$ markings to the corresponding number of general points in \mathbb{P}^2 . Since the class of the image of each such map is a degree d curve in \mathbb{P}^2 , what we are actually counting is *the number of degree d curves passing through $3d - 1$ general points in \mathbb{P}^2* . The low degree cases were already well known several decades ago: $N_1 = N_2 = 1$, since there is just one line passing through two general points and only one conic passing through five general points in the plane; the degree three case can be easily solved by hand to find out that there are $N_3 = 12$ (possibly nodal) cubics passing through eight general points in the plane, and the degree four case was first done in 1870's by Zeuthen, who showed that there were 620 possibly nodal quadrics passing through eleven general points in the plane. The general case has been unknown until 1993, when Ruan and Tian proved Kontsevich's formula using Gromov-Witten theory: by using fundamental relations among boundary components of the moduli space $\overline{\mathcal{M}}_{0,3d-1}(X, d)$, it can be fairly easily proved that:

$$N_d = \sum_{d_1+d_2=d, d_1, d_2 > 0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

This amazing recursive algorithm was a complete surprise, and led Gromov-Witten theory to a period of wide popularity.

Let us now examine three basic properties of Gromov-Witten invariants.

- (1) $\beta = 0$. As we saw in the previous sections, in this case the map μ is constant, the moduli space $\overline{\mathcal{M}}_{0,n}(X, 0)$ is therefore isomorphic to $\overline{\mathcal{M}}_{0,n} \times X$ and the canonical evaluation maps are all equal to projection $p : \overline{\mathcal{M}}_{0,n} \times X \rightarrow X$ onto the second factor. The corresponding the Gromov-Witten invariant becomes:

$$\begin{aligned} \langle \gamma_1 \cdots \gamma_n \rangle_{0,\beta}^X &= \int_{[\overline{\mathcal{M}}_{0,n} \times X]} \text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_n^*(\gamma_n) \\ &= \int_{[\overline{\mathcal{M}}_{0,n} \times X]} p^*(\gamma_1 \cup \dots \cup \gamma_n) \\ &= \int_{p_*[\overline{\mathcal{M}}_{0,n} \times X]} \gamma_1 \cup \dots \cup \gamma_n. \end{aligned}$$

Now, if $n < 3$, the moduli space is empty, therefore the integral is zero. If $n \geq 4$, then the space $\overline{\mathcal{M}}_{0,n}$ has positive dimension, therefore the fibers of the projection morphism p have positive dimension and the push forward of the fundamental class is thus zero. The only case in which the integral is possibly nonzero is when $n = 3$ and in that case

$$\langle \gamma_1 \cup \gamma_2 \cup \gamma_3 \rangle_{0,0}^X = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$$

is the 3-point invariant, containing all the triple intersections on X .

- (2) One of the classes, say γ_1 , is the unit: $\gamma_1 = 1 \in H^0(X)$. In this case, for the Gromov-Witten invariant to be non vanishing *it must be* $\beta = 0$, and we are back to the first case. Indeed, if $\beta \neq 0$, then the cohomology class $\text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_n^*(\gamma_n) = \text{ev}_1^*(\gamma_2) \cup \dots \cup \text{ev}_n^*(\gamma_n)$ "does not see" the first marking, and it is therefore the pullback of some class ω in $\overline{\mathcal{M}}_{0,n-1}(X, \beta)$ via the forgetful map we described previously:

$$\begin{aligned} \phi_1: \quad & \overline{\mathcal{M}}_{0,n}(X, \beta) & \longrightarrow & \overline{\mathcal{M}}_{0,n-1}(X, \beta) \\ & [(C, p_1, \dots, p_n, \mu)] & \mapsto & [(C, p_2, \dots, p_n, \mu)]. \end{aligned}$$

Therefore:

$$\begin{aligned} \langle 1 \cdot \gamma_2 \cdots \gamma_n \rangle_{0,\beta}^X &= \int_{[\overline{\mathcal{M}}_{0,n}(X, \beta)]} \phi_1^* \omega \\ &= \int_{\phi_{1,*}[\overline{\mathcal{M}}_{0,n}(X, \beta)]} \omega \\ &= 0 \end{aligned}$$

because the fibers of ϕ_1 have positive dimension. If $\beta = 0$, on the other hand, we are back to case (a), and hence it must also be $n = 3$. The only surviving invariant is, therefore, $\langle 1 \cdot \gamma_2 \cdot \gamma_3 \rangle_{0,0}^X = \int_X \gamma_2 \cup \gamma_3$.

- (3) One of the classes, say γ_1 , is in $H^2(X)$. Then:

$$\langle \gamma_1 \cdots \gamma_n \rangle_{0,\beta}^X = \left(\int_\beta \gamma_1 \right) \cdot \langle \gamma_2 \cdots \gamma_n \rangle_{0,\beta}^X$$

since the number $\left(\int_\beta \gamma_1 \right)$ counts precisely how many choices we have for the first marking to lie in the support of γ_1 , since γ_1 is Poincaré dual to a hypersurface.

Historically, properties (2) and (3) together are called "divisor axiom".

Example 3.3. Take $X = \mathbb{P}^3$, $d = 1$. Then the Gromov-Witten invariant $\langle \text{line}, \text{line}, \text{line}, \text{line} \rangle_{0,4}^{\mathbb{P}^3}$ counts the number of lines that intersect four generic lines in the space. This number can be computed via intersection theory, and it is known to be equal to 2. Therefore

$$\langle \text{line}, \text{line}, \text{line}, \text{line} \rangle_{0,4}^{\mathbb{P}^3} = 2.$$

4 Quantum Cohomology

We now want to define a new "cohomology theory", whose multiplicative structure will be a deformation of the usual cup product in cohomology. How to deform the intersection product in a meaningful way? The answer to this question is suggested by Gromov-Witten

theory. Remember that when we considered Gromov-Witten invariants on the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ with $\beta = 0$ we found that the only surviving invariant was the three point invariant, namely

$$\langle \gamma_1 \gamma_2 \gamma_3 \rangle_{0,0}^X = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

Actually, the three-point invariant allows us to express the cup product of any two cycles. To see this, let us fix a basis $\{T_0, \dots, T_m\}$ for the cohomology ring $H^*(X, \mathbb{Z})$ such that $T_0 = 1 \in H^0(X, \mathbb{Z})$, $\{T_1, \dots, T_p\}$ is a basis for the Kähler part $H^{1,1}(X, \mathbb{Z})$ ($= H^{1,1}(X) \cap H^2(X, \mathbb{Z})$) and $\{T_{p+1}, \dots, T_m\}$ is a homogeneous basis for the other cohomology groups. Let us call (g_{ij}) the intersection matrix, namely

$$g_{ij} = \int_X T_i \cup T_j$$

and let (g^{ij}) denote its inverse matrix. Now, notice that for each $i, j \in \{0, \dots, m\}$ we can express the cup product $T_i \cup T_j$ as

$$T_i \cup T_j = \sum_{k,h} \left(\int_X T_i \cup T_j \cup T_k \right) g^{hk} T_h,$$

that is

$$T_i \cup T_j = \sum_{k,h} \langle T_i T_j T_k \rangle_{0,0}^X g^{hk} T_h.$$

Since the three-point invariant occurs when the class β is equal to zero, we can deform the cup product by allowing nonzero classes: we will though need to balance somehow the growth of the invariant $\langle \gamma_1 \gamma_2 \gamma_3 \rangle_{0,\beta}^X$ when the class β varies. Fix a Kähler class $\omega \in H^{1,1}(X)$, and define a new product, the *quantum product* as:

$$T_i * T_j = \sum_{k,h} \sum_{\beta} e^{-\int_{\beta} \omega} \langle T_i T_j T_k \rangle_{0,\beta}^X g^{hk} T_h = T_i \cup T_j + \sum_{k,h} \sum_{\beta \neq 0} e^{-\int_{\beta} \omega} \langle T_i T_j T_k \rangle_{0,\beta}^X g^{hk} T_h \quad (4)$$

Notice that when $\omega \rightarrow \infty$ in the Kähler cone, we have that

$$\sum_{k,h} \sum_{\beta \neq 0} e^{-\int_{\beta} \omega} \langle T_i T_j T_k \rangle_{0,\beta}^X g^{hk} T_h \longrightarrow 0,$$

therefore $T_i * T_j \longrightarrow T_i \cup T_j$. There are several convergence assumptions we can make about (4).

A) *There exist only finitely many effective β satisfying (2).*

This is the case when X is Fano.

B) *The previous condition is not satisfied, but the sum in (4) converges, at least for large ω .*

Conjecturally, this is the case for Calabi-Yau manifolds.

Notice that with this definition we have a family of products, hence a family of rings $(H^*(X, \mathbb{C}), *_\omega)$ parametrized by the Kähler cone. If we modify the module structure of our ring, though, we can incorporate the dependence of the product on the Kähler class ω in it. Let $\omega = y_1 T_1 + \dots + y_p T_p$. Then our product becomes

$$T_i * T_j = T_i \cup T_j + \sum_{k,h} \sum_{\beta \neq 0} e^{-y_1 \int_\beta T_1} \cdots e^{-y_p \int_\beta T_p} \langle T_i T_j T_k \rangle_{0,\beta}^X g^{hk} T_h$$

and, by setting $q_i = e^{-y_i}$ for each $i = 1, \dots, p$:

$$T_i * T_j = T_i \cup T_j + \sum_{k,h} \sum_{\beta \neq 0} q_1^{\int_\beta T_1} \cdots q_p^{\int_\beta T_p} \langle T_i T_j T_k \rangle_{0,\beta}^X g^{hk} T_h.$$

We now have q_1, \dots, q_p as formal variables.

Definition 4.1. The ring

$$QH_s^*(X) = (H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \widehat{\mathbb{C}[q_1^{\pm 1}, \dots, q_p^{\pm 1}]}, *)$$

where the hat stands for a suitable completion, is the *small quantum cohomology ring* of X .

Notice that when the variety X is Fano, as we said before, it actually becomes a polynomial ring since the sum in (4) is finite.

We can now observe three basic facts:

- 1) The quantum product is obviously graded commutative, since the usual cup product is.
- 2) The class $T_0 = 1$ is a unit for the quantum product: indeed, we have that

$$T_0 * T_j = T_0 \cup T_j + \sum_{k,h} \sum_{\beta \neq 0} q_1^{\int_\beta T_1} \cdots q_p^{\int_\beta T_p} \langle T_0 T_j T_k \rangle_{0,\beta}^X g^{hk} T_h,$$

but by property 2) of Gromov-Witten invariants, if $T_0 = 1$ then β must be equal to zero, hence the big sum on the right disappears and we are left with

$$T_0 * T_j = T_0 \cup T_j = T_j.$$

- 3) The quantum product is associative: we are not going to prove this in details, but the proof strongly relies on the relations (1) among the boundary divisors of the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$.

We now want to examine two examples of interest.

Example 4.2. (Quantum cohomology of the projective space)

Let $X = \mathbb{P}^r$. Then, since $H^{1,1}(X) \cong \mathbb{C} \cdot [T]$ where $T = T_1$ is the hyperplane class, we have only one variable, namely $q = q_1$. Moreover, since \mathbb{P}^r is Fano, the small quantum cohomology ring is actually a polynomial ring. As a basis, we will take $\{T_i\}_{i=0, \dots, r}$ such that T_i = Poincaré dual of a linear subspace of codimension i . With this basis, the intersection matrix (g_{ij}) becomes:

$$\begin{pmatrix} 0 & & & 1 \\ & \dots & & 1 \\ & 1 & & \\ 1 & & & 0 \end{pmatrix},$$

which coincides with its own inverse, therefore $g_{ij} = g^{ij}$ for each i, j . Remember that in general the number $\langle T_1^{n_1} \cdots T_m^{n_m} \rangle_{0,\beta}^X$ is nonzero only if

$$\sum_i n_i (\text{codim } T_i - 1) = \dim X + \int_{\beta} c_1(X) - 3,$$

hence we have that $\langle T_i T_j T_k \rangle_{0,d}^X \neq 0$ only if $i+j+k-3 = r+d(r+1)-3$, i.e. $i+j+k = r+d(r+1)$. Since i, j, k vary from zero to r , this can only happen when $d = 0$ or when $d = 1$ and in both cases $\langle T_i T_j T_k \rangle_{0,d}^X = 1$. It follows that:

- (i) If $i + j \leq r$, then it must be $d = 0$ because $k = 0, \dots, r$ and if $d = 1$ we would have $i + j + k = 2r + 1$. This implies that the sum over $\beta \neq 0$ in the definition of quantum product disappears, and we have $T_i * T_j = T_i \cup T_j = T_{i+j}$;
- (ii) If $r + 1 \leq i + j \leq 2r$, then it must be $d = 1$, therefore $k = 2r + 1 - i - j$. Now, the only terms surviving in the definition of quantum product are:

$$T_i * T_j = \underbrace{T_i \cup T_j}_{=0} + \underbrace{g^{k,r-k}}_{=1} \underbrace{q^{\int_{[line]} T}}_{=q} T_{r-k} = q T_{r-(2r+1-i-j)} = q T_{i+j-r-1}.$$

Now, the quantum cohomology ring has additively the same structure of the ordinary cohomology, therefore it is linearly generated by the T_i 's (i.e., by the powers of T with respect to the ordinary cup product). Multiplicatively, we need to understand what the relations are. From (i), we get that all the powers of T up to T^r are in the subring generated by T . From (ii), we get that $T^{r+1} = q$. Therefore the small quantum cohomology ring, as a $\mathbb{C}[q]$ -module, is a quotient of $\mathbb{C}[q][T]$. The kernel is exactly $(T^{r+1} - q)$: indeed, it is enough to consider a polynomial of minimum degree in the quantum ring. Its degree cannot be less than $r + 1$ since the T_i 's are linearly independent, and if its degree is greater than $r + 1$, then it is q from (ii). Now, even though $\mathbb{C}[T, q]$ is not a PID, it is enough to use the Euclidean division (since the polynomial $T^{r+1} - q$ is monic) to show that it actually generates the entire kernel. This proves that the quantum cohomology ring of \mathbb{P}^r is:

$$QH_s \mathbb{P}^r = \mathbb{C}[T, q]/(T^{r+1} - q).$$

Example 4.3. (Quantum cohomology of the flag manifold)

Consider the complex flag manifold $F = Fl(\mathbb{C}^{n+1})$. As we all know, it has a natural complex structure and it comes with distinguished line bundles $L_i \rightarrow F$, such that the fiber of L_i over a point $[E] = \{0 \subset E_1 \subset \dots \subset E_n \subset \mathbb{C}^{n+1}\} \in F$ is the vector space E_{i+1}/E_i for each

$i = 0, \dots, n$. Also in this case we can chose the Schubert cycles as a basis and, for simplicity, we want to call $\bar{q}_i = q_i^{\int_{\beta} T_i}$. Set

$$u_i = c_1(L_i)$$

for each i . We obviously have that

$$c(L_0 \oplus \dots \oplus L_n) = (1 + u_0) \cdots (1 + u_n) = \sigma_0(u) + \dots + \sigma_n(u)$$

where $\sigma_i(u)$ is the elementary i -th symmetric polynomial in the u_i 's. It is known that the cohomology ring of the flag manifold is:

$$H^*(F, \mathbb{C}) = \mathbb{C}[u_0, \dots, u_n]/\langle \sigma_0(u), \dots, \sigma_n(u) \rangle.$$

Now, Givental and Kim proved that:

$$QH_s^*(F) \cong \mathbb{C}[u_0, \dots, u_n, \bar{q}_1, \dots, \bar{q}_n]/I$$

where I is the ideal generated by the coefficients of the characteristic polynomial of the matrix

$$\begin{pmatrix} u_0 & \bar{q}_1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \bar{q}_n \\ & & & -1 & u_n \end{pmatrix}.$$

Notice that if $\bar{q}_i = 0$, the coefficients of the characteristic polynomial become the elementary symmetric polynomials, and therefore we recover the classical cohomology. We are not going to prove this, but we can say a few words about the quadratic relation. In the standard cohomology ring, we have that $u_0^2 + \dots + u_n^2 = 0$. Here, we claim than $u_0^{*2} + \dots + u_n^{*2} = 2(\bar{q}_1, \dots, \bar{q}_n)$. Indeed, let us introduce some auxiliary variables p_i such that

$$p_i = -u_0 - \dots - u_i.$$

Notice that the sum $2(p_1 + \dots + p_n)$ represents the anti-canonical class of the flag manifold. Now, it is clear that $p_i * p_j$ must be equal to $p_i \cup p_j$ plus a linear combination of the \bar{q}_i 's. We can interpret the coefficients of this linear combination as being the number of holomorphic curves passing through 0, 1 and $+\infty$ to the divisors p_i, p_j and the generic point of F . Therefore, It must be

$$p_i * p_j = p_i \cup p_j + \delta_{ij} \bar{q}_i$$

which yields:

$$\sum_i \frac{u_i^{*2}}{2} = \sum_i \frac{(p_{i-1} - p_i)^{*2}}{2} = \sum_i p_i^{*2} + \sum_i p_{i-1} * p_i = \sum_i \bar{q}_i.$$