

## LECTURE 2: DOUBLE AFFINE HECKE ALGEBRAS

JOSÉ SIMENTAL

ABSTRACT. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

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### 1. GOALS AND STRUCTURE OF THE TALK

This talk introduces one of the main objects of study in our seminar: the double affine Hecke algebra (DAHA). We will make the definitions in great (but not complete) generality, and we will emphasize the  $\mathfrak{gl}_n$  case. In the first part of the talk we will briefly recall from Seth's talk the main ingredients for the construction of DAHA: the affine Hecke algebras and Cherednik's basic representation. After this, we will give the definition of DAHA and exhibit an explicit basis of it. We will then spend some time looking at two explicit cases: the DAHA for  $A_1$  and that for  $\mathfrak{gl}_n$ . In particular, we will give explicit presentations by generators and relations, present their trigonometric and rational degenerations, and exhibit a large group of automorphisms of the DAHA for  $\mathfrak{gl}_n$ . After that, we will study certain operators on the polynomial representation of DAHA. The importance of these operators is that, first, they can be used to form a big commuting family of difference operators on the group algebra of the weight lattice and, second, they are connected to Macdonald polynomials, which is the topic of a subsequent talk in this seminar. The study of these difference operators naturally leads to the definition of spherical DAHA. We finish the notes with a discussion of trigonometric and rational degenerations in the general setting, the description of (trigonometric, difference-rational and rational) Dunkl operators, and applications to the theory of quantum integrable systems.

### 2. DOUBLE AFFINE HECKE ALGEBRAS

#### 2.1. Reminders.

2.1.1. *Root systems and Weyl groups.* We will work with affine root systems that are of the form  $R^a$ , where  $R$  is an irreducible finite root system (so, for example, we will ignore the affine root systems of the form  $(C_n^\vee, C_n)$ , etc.). Throughout these notes, we will use the following notation.

- $\{\alpha_1, \dots, \alpha_n\}$  denotes the set of simple roots and  $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$  the set of simple co-roots of  $R$ .
- $Q, Q^\vee$  denote the root and coroot lattice of  $R$ , respectively. Similarly,  $P, P^\vee$  denote the weight and coweight lattice of  $R$ , respectively.
- $W := \langle s_1, \dots, s_n \rangle$  is the Weyl group of  $R$ , where  $s_i$  denotes the reflection  $s_{\alpha_i}$ .
- $\alpha_0 := -\theta + \delta$ , so that  $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$  forms a set of simple roots for  $R^a$ .
- $W^a := \langle s_0, \dots, s_n \rangle$  is the Weyl group of  $R^a$ , aka the affine Weyl group. Recall that we have an isomorphism

$$W^a = W \ltimes t(Q^\vee)$$

- $W^{ae} := W \ltimes t(P^\vee)$  is the extended affine Weyl group.
- $\Omega \subseteq W^{ae}$  denotes the subgroup of all elements of length 0. This is a finite subgroup of  $W^{ae}$ , acting faithfully on the set of simple roots  $\{\alpha_0, \dots, \alpha_n\}$ , and it is actually isomorphic to  $P^\vee/Q^\vee$ .
- We have an isomorphism

$$W^{ae} = \Omega \ltimes W^a$$

where the action of  $\Omega'$  on  $W^a$  is given as follows: if  $\pi_r \in \Omega'$  is such that  $\pi_r(\alpha_i) = \alpha_j$ , then  $\pi_r s_i \pi_r^{-1} = s_j$ .

2.1.2. *Affine Hecke algebras.* Throughout this talk,  $\tau := \{\tau_0, \dots, \tau_n\}$  will denote a collection of *formal variables* such that  $\tau_i = \tau_j$  whenever the reflections  $s_i$  and  $s_j$  are conjugate in  $W^a$ , and let  $\mathbb{C}_\tau := \mathbb{C}(\tau_0, \dots, \tau_n)$  denote the field of rational functions in these variables. Recall that we have the *affine Hecke algebra*  $\mathcal{H}_\tau$  of  $W$ , which is a quotient of the group algebra  $\mathbb{C}_\tau B^{ae}$ , where  $B^{ae}$  is the extended affine braid group. We have two presentations of this algebra.

**The Coxeter presentation.**  $\mathcal{H}_\tau = \mathbb{C}_\tau \langle T_0, \dots, T_n, \Omega \rangle$  with the following relations.

- (a)  $T_i T_j \cdots = T_j T_i \cdots$ , where term has  $m_{ij}$  factors.
- (b)  $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$ .
- (c)  $\pi_r T_i \pi_r^{-1} = T_j$ , if  $\pi_r(\alpha_i) = \alpha_j$ .

**The Bernstein presentation.**  $\mathcal{H}_\tau = \mathbb{C}_\tau \langle T_1, \dots, T_n, Y^{P^\vee} \rangle$  with relations:

- (a') Relations (1) and (2) above for the  $T_i$ .
- (b')  $Y^\lambda Y^\mu = Y^{\lambda+\mu}$ .
- (c')  $T_i Y^\lambda = Y^\lambda T_i$  if  $\langle \lambda, \alpha_i \rangle = 0$ .
- (d')  $T_i Y^{s_i(\lambda)} T_i = Y^\lambda$  if  $\langle \lambda, \alpha_i \rangle = 1$ .

2.1.3. *Cherednik's basic representation.* Let us now recall Cherednik's basic representation of the affine Hecke algebra  $\mathcal{H}_\tau$ . We let  $q$  be a variable, and consider the affine Hecke algebra defined over the field  $\mathbb{C}_{q,\tau} := \mathbb{C}_\tau(q^{1/e})$ , where  $e$  is such that  $\langle P, P^\vee \rangle = \frac{1}{e}\mathbb{Z}$ . Now let  $\mathbb{C}_{q,\tau}[X]$  denote the group algebra of  $P$ . Note that this contains  $\mathbb{C}_\tau[\widehat{X}]$ , the group algebra of the affine weight lattice  $\widehat{P} := P \oplus \mathbb{Z}\delta$ , by setting  $X^{\lambda+r\delta} := q^r X^\lambda$ . So the extended affine Weyl group  $W^{ae}$  acts on  $\mathbb{C}_{q,\tau}[X]$  by setting, for  $w = t(\lambda)v$ ,  $\lambda \in P^\vee$ ,  $v \in W$  and  $\mu \in P$ ,

$$w(X^\mu) := X^{w(\mu)} = q^{-\langle \lambda, v(\mu) \rangle} X^{v(\mu)}$$

We have that  $\mathbb{C}_{q,\tau}[X]$  becomes a  $\mathcal{H}_\tau$ -module via the formulas:

$$\begin{aligned} \pi_r &\mapsto \pi_r, \pi_r \in \Omega \\ T_i &\mapsto \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{s_i - \text{id}}{X^{\alpha_i} - 1}, i = 0, \dots, n \end{aligned}$$

Let us be more explicit on the action of  $T_0$ . Recall that  $\alpha_0 = -\theta + \delta$ , and that we are identifying  $q = X^\delta$ . So it follows that  $s_0 X^\mu = X^{\mu - (-\theta + \delta)\langle \mu, -\theta^\vee \rangle} = X^\mu (q^{-1} X^\theta)^{\langle \mu, -\theta^\vee \rangle}$ . Thus,

$$T_0 : X^\mu \mapsto \left( \tau_0 (q^{-1} X^\theta)^{\langle \mu, -\theta^\vee \rangle} + (\tau_0 - \tau_0^{-1}) \frac{(q^{-1} X^\theta)^{\langle \mu, -\theta^\vee \rangle} - 1}{q X^{-\theta} - 1} \right) X^\mu$$

In particular, if  $\langle \mu, -\theta^\vee \rangle = 0$ , then  $T_0(X^\mu) = \tau_0 X^\mu$ , while if  $\langle \mu, -\theta^\vee \rangle = 1$ , then  $T_0(X^\mu) = \tau_0 q^{-1} X^{\theta + \mu} - (\tau_0 - \tau_0^{-1}) q^{-1} X^{\theta + \mu}$ .

**2.1.4. The induced representation.** Let us denote by  $\mathcal{H}_\tau^X$  the affine Hecke algebra for the root system  $(R^\vee)^a$ . In particular, we have the Bernstein presentation for this Hecke algebra, which is completely analogous to the Bernstein presentation above. We have the induced representation of  $\mathcal{H}_\tau^X$  on  $\mathbb{C}_{q,\tau}[X]$ , where the  $X^\mu$  act by multiplication and the  $T_i$  act by

$$T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{s_i - \text{id}}{X^{\alpha_i} - 1}, i = 1, \dots, n$$

Let us remark that the induced representation of  $\mathcal{H}_\tau^X$  on  $\mathbb{C}_{q,\tau}[X]$  is obtained by the eponymous representation on  $\mathbb{C}_\tau[X]$  by base-change to the field  $\mathbb{C}_{q,\tau}$ .

**2.2. Double affine Hecke algebras.** We are now ready to define the double affine Hecke algebra for  $R$ . The idea here is to glue together the affine Hecke algebras  $\mathcal{H}_\tau$  and  $\mathcal{H}_\tau^X$  along their common representation  $\mathbb{C}_{q,\tau}[X]$ .

**Definition 2.2.1.** The double affine Hecke algebra  $\mathbb{H} := \mathbb{H}(W)$  is the  $\mathbb{C}_{q,\tau}$ -algebra generated by elements  $T_0, \dots, T_n, \Omega, X^P$  with relations.

- (1) The relations (a)-(c) above for the affine Hecke algebra between  $T_0, \dots, T_n$  and  $\Omega$ .
- (2) Denote  $\alpha_0^\vee := -\theta^\vee$ . Then, for  $i = 0, \dots, n$ :

$$\begin{aligned} T_i X^\mu &= X^\mu T_i & \text{if } \langle \mu, \alpha_i^\vee \rangle &= 0 \\ T_i X^\mu &= X^{s_i(\mu)} T_i^{-1} & \text{if } \langle \mu, \alpha_i^\vee \rangle &= 1 \end{aligned}$$

- (3)  $\pi_r X^\mu \pi_r^{-1} = X^{\pi_r(\mu)}$ .

**Definition 2.2.2.** Note that, by its very definition, the DAHA  $\mathbb{H}$  admits a representation on the space  $\mathbb{C}_{q,t}[X]$ , where  $X^\mu$  acts by multiplication and both  $\pi_r$  and  $T_i$  act as in Cherednik's basic representation. We call this representation the polynomial representation of  $\mathbb{H}$ .

Note that, by Matsumoto's theorem, if  $w = \alpha_{i_1} \dots \alpha_{i_k}$  is a reduced decomposition of  $w \in W^a$ , then we have a well-defined element  $T_w \in \mathbb{H}$ .

**Theorem 2.2.3** (PBW theorem for DAHA). Every element  $h \in \mathbb{H}$  can be uniquely written in the form

$$h = \sum_{\substack{\mu \in P \\ \pi_r \in \Omega \\ w \in W^a}} a_{\mu,r,w} X^\mu \pi_r T_w, \quad a_{\mu,r,w} \in \mathbb{C}_{q,\tau}$$

The existence of such an expression for  $h$  is a standard exercise. The uniqueness is harder. We will use a standard trick that we have already seen in Seth's lecture: we will write down a representation of  $\mathbb{H}$  in a space in which the operators  $X^\mu \pi_r T_w$  are linearly independent. It turns out that we already know such a representation: the polynomial representation, cf. Definition 2.2.2.

**Theorem 2.2.4.** *Consider the polynomial representation  $\mathbb{C}_{q,\tau}[X]$  of  $\mathbb{H}$ . Then, the operators  $\{X^\mu \pi_r T_w : \mu \in P, \pi_r \in \Omega, w \in W^a\}$  are linearly independent over the field  $\mathbb{C}_{q,\tau}$ . In particular, the polynomial representation is faithful.*

*Proof.* Note that, even though the operators  $\pi_r T_w$  are not  $\mathbb{C}_{q,\tau}[X]$ -linear, we still have an action of  $\mathbb{C}_{q,\tau}[X]$  on  $\text{End}_{\mathbb{C}}(\mathbb{C}_{q,\tau}[X])$ ,  $f : \varphi \mapsto (x \mapsto f\varphi(x))$ . It clearly suffices to show that the operators  $\{\pi_r T_w\}_{\mu \in P, w \in W^a}$  are linearly independent over  $\mathbb{C}_{q,\tau}[X]$ . In order to do this, we will relate this action to the action of the extended affine Weyl group  $W^{ae}$  on  $\mathbb{C}_{q,\tau}[X]$ , which we know from Seth's talk it is faithful.

Recall that for  $i = 0, \dots, n$ ,  $T_i$  acts via the operator:

$$T_i := \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{s_i - \text{id}}{X^{\alpha_i} - 1} = \left( \tau_i + \frac{\tau_i - \tau_i^{-1}}{X^{\alpha_i} - 1} \right) s_i + \left( \frac{\tau_i^{-1} - \tau_i}{X^{\alpha_i} - 1} \right) \text{id}$$

It follows that, for  $\pi_r w \in W^{ae}$ ,  $w \in W^a$ , we can write

$$\pi_r T_w = \sum_{w' \leq w} f_{w',w} \pi_r w'$$

where  $f_{w',w} \in \mathbb{C}_{q,\tau}(X)$  are rational functions on  $X$  and the order on  $W^a$  is the usual Bruhat order. Note that  $f_{w,w} \neq 0$ .

Now assume that we have a linear combination of the form

$$\sum_{\substack{\pi_r \in \Omega \\ w \in W^a}} g_{r,w}(X) \pi_r T_w = 0$$

where  $g_{r,w} \in \mathbb{C}_{q,\tau}[X]$  are not all 0. It follows from the above that we get

$$\sum_{\substack{w \in W^a, \pi_r \in \Omega \\ w' \leq w}} g_{r,w}(X) f_{r,w',w}(X) \pi_r w' = 0$$

The operators  $\pi_r w$  are all distinct, since the representation of  $W^a$  on  $\mathbb{C}_{q,\tau}[X]$  is faithful, and can be extended to automorphisms of the field  $\mathbb{C}_{q,\tau}(X)$  (= the field of quotients of  $\mathbb{C}_{q,\tau}[X]$ ). It follows that the operators  $\pi_r w$  are linearly independent over the field  $\mathbb{C}_{q,\tau}(X)$ . So for every  $\pi_r \in \Omega$ ,  $w \in W^a$  we have

$$\sum_{w' \geq w} g_{r,w'} f_{r,w,w'} = 0$$

If we pick  $w_0 \in W^a$  such that  $w_0$  is maximal w.r.t. the Bruhat order in the set  $\{w \in W^a : g_{r,w} \neq 0 \text{ for some } \pi_r \in \Omega\}$  then we get  $g_{r,w_0} f_{r,w_0,w_0} = 0$ . But since  $f_{r,w_0,w_0} \neq 0$ , this is a contradiction. We are done.  $\square$

**Corollary 2.2.5.** *We define the following subalgebras of  $\mathbb{H}$ :*

- (1)  $\mathcal{H}^X := \langle T_1, \dots, T_n, X^\mu (\mu \in P) \rangle$ .
- (2)  $\mathcal{H}^Y := \langle T_0, \dots, T_n, \Omega \rangle$ .
- (3)  $H := \langle T_1, \dots, T_n \rangle$ .

*Then,  $\mathcal{H}^X$  is (isomorphic to) the affine Hecke algebra for the root system  $R^\vee$ ;  $\mathcal{H}^Y$  is (isomorphic to) the affine Hecke algebra for the root system  $R$ ; and  $H$  is (isomorphic to) the finite Hecke algebra of  $W$ .*

Recall from Seth's talk the definition of the elements  $Y^\lambda \in \mathcal{H}^Y$ ,  $\lambda \in P^\vee$ . Namely,  $Y^\lambda := T_{t(\lambda)}$  if  $\lambda \in P_+^\vee$ , while  $Y^\lambda := Y^\mu (Y^\nu)^{-1}$  if  $\lambda = \mu - \nu$  with  $\mu, \nu \in P_+^\vee$ . Since  $\mathcal{H}^Y$  is an affine Hecke algebra, the elements  $Y^\lambda$  are well-defined. The following result follows immediately from Seth's talk.

**Theorem 2.2.6** (PBW theorem for DAHA, v2). *Every element  $h \in \mathbb{H}$  can be uniquely written in the form*

$$h = \sum_{\substack{\lambda \in P^\vee \\ \mu \in P \\ w \in W}} a_{\lambda, \mu, w} X^\mu Y^\lambda T_w, \quad a_{\lambda, \mu, w} \in \mathbb{C}_{q, \tau}$$

Let us remark that the weight and co-weight lattice play a symmetric role in the definition of DAHA. In order to state this precisely, let  $\omega_1, \dots, \omega_n$  be the fundamental weights of  $R$ , so  $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ , and denote by  $\omega_1^\vee, \dots, \omega_n^\vee$  the fundamental coweights. We will denote  $X_i := X^{\omega_i}$ ,  $Y_i := Y^{\omega_i^\vee}$ .

**Theorem 2.2.7.** *The following assignment can be extended to a  $\mathbb{C}$ -automorphism of  $\mathbb{H}$ :*

$$X_i \mapsto Y_i, \quad Y_i \mapsto X_i, \quad T_j \mapsto T_j^{-1}, \quad \tau_j \mapsto \tau_j^{-1}, \quad q \mapsto q^{-1}$$

We will not prove Theorem 2.2.7 in full generality. We will show it for specific types of root systems below. Let us remark that a consequence of Theorem 2.2.7 is the following.

**Corollary 2.2.8** (PBW theorem for DAHA, v3). *Every element  $h \in \mathbb{H}$  can be uniquely written in the form*

$$h = \sum_{\substack{\lambda \in P^\vee \\ \mu \in P \\ w \in W}} a_{\lambda, \mu, w} Y^\lambda X^\mu T_w, \quad a_{\lambda, \mu, w} \in \mathbb{C}_{q, \tau}$$

Note that we could have also defined  $\mathbb{H}$  to be an algebra generated by  $T_1, \dots, T_{n-1}, X^P, Y^{P^\vee}$  with certain relations. Of course, the relations among  $(T_i, X^\mu)$  or among  $(T_i, Y^\lambda)$  can be explicitly written - they are just the relations of the affine Hecke algebra. But it is not easy to write the relations among  $(X^\mu, Y^\lambda)$ . We will give a couple of examples where these relations can actually be written. As we will see, they are topological in nature.

### 2.3. Example: DAHA for $A_1$ .

**2.3.1. Generators and relations.** We give explicit generators and relations for the DAHA of  $A_1$ . So we have that the (co-)root lattice is  $Q = Q^\vee = \mathbb{Z}\alpha$  and the (co-)weight lattice is  $P = P^\vee = \mathbb{Z}\rho$ , with  $\rho = \alpha/2$ . Let us denote  $s = s_\alpha$ . We have that  $\Omega = \{1, \pi_\rho\}$ , we denote  $\pi := \pi_\rho = t(\rho)s$ . Setting now  $X := X^\rho$ , we have that the DAHA  $\mathbb{H}$  is generated by  $T_0, T_1, X^{\pm 1}$  and  $\pi$ . Note, however, that  $T_0 = \pi T_1 \pi$ , so we may ignore  $T_0$  from our list of generators. Thus, we have

$$\mathbb{H} = \mathbb{C}_{q, \tau} \langle X, T, \pi \rangle / \left\{ \begin{array}{l} TXT = X^{-1}, \quad \pi X \pi^{-1} = qX^{-1}, \\ \pi^2 = 1, \quad (T - \tau)(T + \tau^{-1}) = 0 \end{array} \right\}$$

Setting  $Y := \pi T$ , we have the following alternative presentation of  $\mathbb{H}$ :

$$\mathbb{H} = \mathbb{C}_{q, \tau} \langle X, T, Y \rangle / \left\{ \begin{array}{l} TXT = X^{-1}, \quad Y^{-1} X^{-1} Y X = q^{-1} T^{-2}, \\ TY^{-1} T = Y, \quad (T - \tau)(T + \tau^{-1}) = 0 \end{array} \right\}$$

Note that this presentation reveals a symmetry between  $X$  and  $Y$ . The following proposition is obvious, note that its second part is a special case of Theorem 2.2.7.

**Lemma 2.3.1.** *We have a  $\mathbb{C}_{q, \tau}$  anti-involution  $\phi : \mathbb{H} \rightarrow \mathbb{H}^{\text{opp}}$ , defined on generators by the following formulas*

$$\phi(X) = Y^{-1}, \quad \phi(Y) = X^{-1}, \quad \phi(T) = T$$

and a  $\mathbb{C}$ -involution  $\varepsilon : \mathbb{H} \rightarrow \mathbb{H}$ , defined by

$$\varepsilon(X) = Y, \quad \varepsilon(Y) = X, \quad \varepsilon(T) = T^{-1}, \quad \varepsilon(\tau) = \tau^{-1}, \quad \varepsilon(q) = q^{-1}$$

**2.3.2. The polynomial representation.** Let us give formulas for the action of the elements  $X, Y, T$  on the polynomial representation  $\mathbb{C}_{q,\tau}[X]$ . First of all, we have that the action of  $T$  is given by

$$\tau s + (\tau - \tau^{-1}) \frac{s - \text{id}}{X^2 - 1}$$

while the action of  $t(\rho)$  is given by  $t(\rho)(X) = qX$ . Moreover, for a Laurent polynomial  $f(X) \in \mathbb{C}_{q,\tau}[X]$ , we have that  $t(\rho)f(X) = f(qX)$ , so that  $\pi(X) = t(\rho)s(X) = q^{-1}X^{-1}$ . Thus, we have

$$\begin{aligned} Y &= \pi \left( \tau s + (\tau - \tau^{-1}) \frac{s - \text{id}}{X^2 - 1} \right) \\ &= \tau t(\rho) + (\tau - \tau^{-1}) \pi \frac{s - \text{id}}{X^2 - 1} \\ &= \tau t(\rho) + (\tau - \tau^{-1}) \frac{t(\rho) - \pi}{q^{-2}X^{-2} - 1} \\ &= \tau t(\rho) + (\tau - \tau^{-1}) t(\rho) \frac{\text{id} - s}{X^{-2} - 1} \\ &= t(\rho) \left( \tau + (\tau - \tau^{-1}) \frac{\text{id} - s}{X^{-2} - 1} \right) \end{aligned}$$

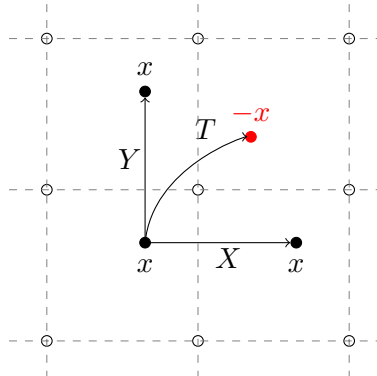
So, for example, we get  $Y(X^n) = \tau^{-1}(q^n X^n + q^{n-2} X^{n-2} + \dots + q^{2-n} X^{2-n})$ . The operator  $Y$  is known as the *difference-trigonometric Dunkl operator*.

**2.3.3. Topological interpretation.** Let  $E = \mathbb{C}/\Lambda$  be an elliptic curve, where we take the lattice  $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\iota$ . Let  $0 \in E$  be the zero point, and consider the automorphism  $-1 : x \mapsto -x$  of  $E$ . Note that  $\pi_1((E \setminus \{0\})/\mathbb{Z}_2)$  is trivial, as  $(E \setminus \{0\})/\mathbb{Z}_2$  being a disk is contractible. We will consider the orbifold fundamental group  $\pi_1^{\text{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$ , where  $x \in E \setminus \{0\}$  is a generic point (i.e., not one of the three branching points of  $E \setminus \{0\} \rightarrow (E \setminus \{0\})/\mathbb{Z}_2$ ).

Let us recall that the orbifold fundamental group is generated by homotopy classes of paths in  $E \setminus \{0\}$  from  $x$  to  $\pm x$ , with multiplication defined by  $\gamma_1 \circ \gamma_2$  is  $\gamma_2$  followed by  $-\gamma_1$ , if  $\gamma_2$  connects  $x$  to  $-x$ . So we have an exact sequence

$$1 \rightarrow \pi_1(E \setminus \{0\}, x) \rightarrow \pi_1^{\text{orb}}((E \setminus \{0\})/\mathbb{Z}_2, x) \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

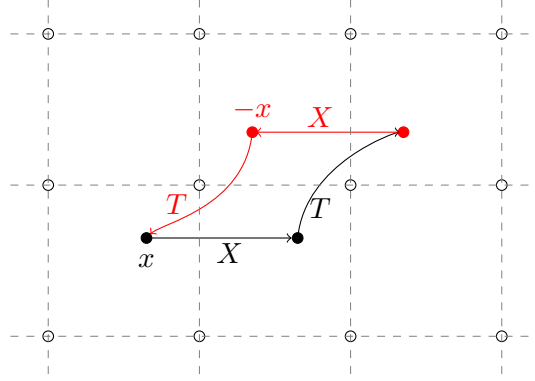
i.e.,  $\pi_1^{\text{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$  is an extension by  $\mathbb{Z}_2$  of the group  $\pi_1(E \setminus \{0\}, x)$ , the fundamental group of the punctured torus. The latter group has three generators,  $X$  (the “horizontal” cycle of the torus),  $Y$  (the “vertical” cycle of the torus) and  $C$  (a loop around the missing point 0). The orbifold fundamental group  $\pi_1^{\text{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$  is then generated by  $X, Y$  and an element  $T$  (a half-loop around 0) connecting  $x$  to  $-x$  such that  $T^2 = C$ .



The elements  $X, Y, T$  satisfy the relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad Y^{-1}X^{-1}YXT^2 = 1$$

Let us show, for example, the first relation, which says that  $TXTX = 1$ . So we have to first go through the loop  $X$ , then through the path  $T$ , and then through the path  $-X$ , since the endpoint of  $T$  is  $-x$ . Finally, we go through the path  $-T$ . It follows from the following picture that this path is null-homotopic.



So we see that  $\mathbb{H}$  can be seen as a quotient of the group algebra  $\mathbb{C}_{q,\tau}\pi_1^{\text{orb}}((E \setminus \{0\})/\mathbb{Z}_2, x)$ , as follows. First of all, recall that our base field  $\mathbb{C}_{q,\tau}$  includes  $q^{\pm 1/2}$ , since  $\langle \rho, \rho \rangle = 1/2$ . Now set  $\tilde{T} := q^{-1/2}T$ ,  $\tilde{X} := q^{1/2}X$ ,  $\tilde{Y} := q^{-1/2}Y$ , so that  $\tilde{X}, \tilde{Y}, \tilde{T}$  satisfy the relations of  $\mathbb{H}$  with the exception of the quadratic relation for  $\tilde{T}$ . Thus:

$$\mathbb{H} = \mathbb{C}_{q,\tau}\pi_1^{\text{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x) / ((T - q^{1/2}\tau)(T + q^{1/2}\tau^{-1}))$$

**2.3.4. Trigonometric degeneration.** Now let  $\hbar, c$  be variables. Set  $Y := \exp(\hbar\hat{y})$ ,  $q := \exp(\hbar)$ ,  $\tau := q^c = \exp(\hbar c)$  and  $T := s \exp(\hbar cs)$ , where  $s \in S_2$  is the non-trivial element. We can consider  $\mathbb{H}$  as a  $\mathbb{C}[[\hbar, c]]$ -algebra, with the same generators and relations as above. Then,  $\mathbb{H}/\hbar\mathbb{H}$  is generated by  $s, \hat{y}$  and  $X$ , with relations

$$s^2 = 1, \quad sXs = X^{-1}, \quad s\hat{y} + \hat{y}s = 2c, \quad X^{-1}\hat{y}X - \hat{y} = 1 - 2cs$$

We call  $\mathbb{H}^{\text{trig}} := \mathbb{C}[c]\langle s, X, \hat{y} \rangle$  with the relations above the *trigonometric* DAHA of  $A_1$ .

**Lemma 2.3.2.** *Every element  $h \in \mathbb{H}^{\text{trig}}$  can be uniquely written as*

$$h = \sum_{\substack{m \in \mathbb{Z} \\ n \in \mathbb{Z}_{\geq 0} \\ i=0,1}} a_{m,i,n} X^m s^i \hat{y}^n, \quad a_{m,i,n} \in \mathbb{C}[c]$$

The lemma again can be proved using a faithful representation of  $\mathbb{H}^{\text{trig}}$ . Here the space is  $\mathbb{C}[c][X^{\pm 1}]$ . The element  $s$  acts by  $X \mapsto X^{-1}$  and  $X$  acts by multiplication. To give the action of  $\hat{y}$ , first define the *trigonometric derivative* by  $\partial(X) = X$ . Then,  $\hat{y}$  acts by the *trigonometric Dunkl operator*

$$D^{\text{trig}} := \partial - 2c \frac{1}{1 - X^{-2}} (\text{id} - s) + c$$

This is known as the *differential polynomial representation* of  $\mathbb{H}^{\text{trig}}$ . We also have a different polynomial representation of  $\mathbb{H}^{\text{trig}}$ , which stems from the fact that the variables  $X$  and  $\hat{y}$  are not symmetric. This is an action of  $\mathbb{H}^{\text{trig}}$  on  $\mathbb{C}[c][\hat{y}]$ . Note that we have an action of  $S_2$  on  $\mathbb{C}[c][\hat{y}]$ , where  $s$  acts by  $y \mapsto -y$ . Then,  $s \in \mathbb{H}^{\text{trig}}$  acts on  $\mathbb{C}[c][\hat{y}]$  via the operator

$$S := s - \frac{c}{\hat{y}}(s - \text{id})$$

To define an action of  $X$ , let  $\pi : \mathbb{C}[c][\hat{y}] \rightarrow \mathbb{C}[c][\hat{y}]$  be defined by  $f(\hat{y}) \mapsto f(-\hat{y} + 1)$ . Then,  $X$  acts via the operator  $\pi S$ . This is known as the *difference-rational polynomial representation* of  $\mathbb{H}^{\text{trig}}$ . The operator  $\pi S$  is known as the *difference-rational Dunkl operator*.

**Corollary 2.3.3.** *The following are subalgebras of  $\mathbb{H}^{\text{trig}}$ :*

- (1) *The group algebra of the extended affine Weyl group for  $A_1$ : it is isomorphic to the subalgebra of  $\mathbb{H}^{\text{trig}}$  generated by  $s, X$ .*
- (2) *The degenerate affine Hecke algebra for  $A_1$ : it is isomorphic to the subalgebra of  $\mathbb{H}^{\text{trig}}$  generated by  $s, \hat{y}$ .*

**2.3.5. Rational degeneration.** Now in  $\mathbb{H}^{\text{trig}}$  set  $X = \exp(\hbar x)$  and  $y = \hbar \hat{y}$ . Then, modulo  $\hbar$ , the elements  $s, x, y$ , satisfy the following relations

$$s^2 = 1, \quad sx = -xs \quad sy = -ys \quad yx - xy = 1 - 2cs$$

Define the algebra  $\mathbb{H}^{\text{rat}} := \mathbb{C}[c]\langle s, x, y \rangle$  with the relations above. This is known as the *rational DAHA* of  $A_1$ .

**Lemma 2.3.4.** *Every element  $h \in \mathbb{H}^{\text{rat}}$  can be uniquely written in the form*

$$h = \sum_{\substack{m, n \in \mathbb{Z} \\ i=0,1}} a_{m,i,n} x^m s^i y^n, \quad a_{m,i,n} \in \mathbb{C}[c]$$

Lemma 2.3.4 may be proven using the *polynomial representation* of  $\mathbb{H}^{\text{rat}}$ . This is the representation on  $\mathbb{C}[c][x]$ , where  $s$  acts by  $x \mapsto -x$ ,  $x$  acts by multiplication and  $y$  acts by the *rational Dunkl operator*

$$D^{\text{rat}} := \frac{d}{dx} + c \frac{1}{x}(s - \text{id})$$

## 2.4. Example: DAHA for $\mathfrak{gl}_n$ .

**2.4.1. The affine Hecke algebra for  $\mathfrak{gl}_n$ , revisited.** We will now define the DAHA for  $\mathfrak{gl}_n$ , which is different (but closely related to) from the DAHA of type  $A_n$ . So the first step is to study the affine Hecke algebra for  $\mathfrak{gl}_n$ , which has already appeared at the end of Seth's talk. Recall that we denote  $\mathbb{C}_\tau := \mathbb{C}(\tau)$ , the field of rational functions on the variable  $\tau$ .

**Definition 2.4.1.** *Let  $n > 0$ . The affine Hecke algebra of  $\mathfrak{gl}_n$ ,  $\mathcal{H}_n$ , is the  $\mathbb{C}_\tau$ -algebra with generators  $T_1, \dots, T_{n-1}, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$  and relations*

$$\begin{aligned} (\text{Quadratic relations}) \quad & (T_i - \tau)(T_i + \tau^{-1}) = 0, \text{ for } i = 1, \dots, n-1. \\ (\text{Braid relations}) \quad & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } i = 1, \dots, n-2; T_i T_j = T_j T_i \text{ if } |i-j| > 1. \\ (\text{Action relations}) \quad & T_i^{-1} Y_i T_i^{-1} = Y_{i+1}, i = 1, \dots, n-1; T_j Y_i = Y_i T_j \text{ if } j \neq i, i-1. \\ (\text{Laurent relations}) \quad & Y_i Y_j = Y_j Y_i \text{ for } i, j = 1, \dots, n; Y_i Y_i^{-1} = 1. \end{aligned}$$

In order to define DAHA, we will need Cherednik's basic representation for  $\mathcal{H}_n$ . This is more easily given in the Coxeter presentation, so we will need an analogue of the Coxeter presentation of  $\mathcal{H}_n$ . Let us introduce the following element of  $\mathcal{H}_n$ :

$$\pi := T_1^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_i \cdots T_{n-1}$$

Note that, thanks to the action relations, the element  $\pi$  is well-defined, i.e., it does not depend on  $i = 1, \dots, n$ . So, for example,  $\pi = Y_1^{-1} T_1 \cdots T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1} Y_n$ .



**Lemma 2.4.2.** *The element  $\pi^n$  is central in  $\mathcal{H}_n$ .*

*Proof.* We have

$$\begin{aligned}\pi^n &= (Y_1^{-1}T_1 \cdots T_{n-1})(Y_1^{-1}T_1 \cdots T_{n-1}) \cdots (Y_1^{-1}T_1 \cdots T_{n-1}) \\ &= Y_1^{-1}Y_2^{-1} \cdots Y_n^{-1}A_1A_2 \cdots A_n\end{aligned}$$

where  $A_i := T_1^{-1}T_2^{-1} \cdots T_{n-i}^{-1}T_{n-i+1} \cdots T_{n-1}$ , so, for example,  $A_1 = T_1^{-1}T_2^{-1} \cdots T_{n-1}^{-1}$  and  $A_n = T_1T_2 \cdots T_{n-1}$ . We claim that  $A_1A_2 \cdots A_n = 1$ , note that this will finish the proof of the lemma. Indeed, this can already be seen in the braid group  $B_n$ : first of all, the associated permutation in  $S_n$  of every  $A_i$ 's is the cycle  $1 \mapsto n \mapsto n-1 \mapsto \cdots \mapsto 2 \mapsto 1$ , and the  $n$ -th power of this cycle is the identity. So  $A_1 \cdots A_n$  is, at least, an element of the pure braid group.

Now note that, in  $A_i$ , the strand starting at 1 passes *below* the strands starting at  $2, \dots, n-i$  and *above* the strands starting at  $n-i+1, \dots, n-1$ . So, in the product  $A_1 \cdots A_n$ , the strand connecting 1 to 1 passes below all other strands; the strand connecting 2 to 2 passes above the strand connecting 1 to 1 and below all other strands and, in general, the strand connecting  $i$  to  $i$  passes above the strand connecting  $j$  to  $j$  if  $j < i$ , and below the strand connecting  $j$  to  $j$  if  $j > i$ . So  $A_1 \cdots A_n = 1$ .  $\square$

**Lemma 2.4.3.** *We have  $\pi T_i \pi^{-1} = T_{i+1}$ ,  $i = 1, \dots, n-2$ .*

*Proof.* Here we use  $\pi = T_1^{-1} \cdots T_{n-1}^{-1} Y_n$ . So

$$\begin{aligned}\pi T_i \pi^{-1} &= (T_1^{-1} \cdots T_{n-1}^{-1} Y_n) T_i (Y_n^{-1} T_{n-1} \cdots T_1) \\ &= (T_1^{-1} \cdots T_i^{-1}) (T_{i+1}^{-1} T_i T_{i+1}) (T_i \cdots T_1)\end{aligned}$$

Now we use the identity  $T_{i+1}^{-1} T_i T_{i+1} = T_i T_{i+1} T_i^{-1}$ , which follows immediately from the braid relation involving  $i, i+1$ . From here, the result follows easily.  $\square$

**Theorem 2.4.4.** *The affine Hecke algebra  $\mathcal{H}_n$  is generated by  $T_1, \dots, T_{n-1}, \pi^{\pm 1}$  with relations:*

- (1) *The braid and quadratic relations involving the  $T_i$ .*
- (2)  *$\pi T_i \pi^{-1} = T_{i+1}$ ,  $i = 1, \dots, n-2$ .*
- (3)  *$\pi^n$  is central.*

*Proof.* Let  $\mathcal{H}'_n$  denote the algebra defined in the statement of the theorem. Define

$$Y_i := T_i \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1}.$$

We have to check that the  $Y_i$ 's satisfy the action and commutativity relations. Let us check the action relations. First of all, it is clear that  $T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$ , for  $i = 1, \dots, n$ . Now, if  $j > i$  we have

$$\begin{aligned}T_j Y_i &= T_j T_i T_{i+1} \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1} \\ &= T_i \cdots T_j T_{j-1} T_j \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1} \\ &= T_i \cdots T_{j-1} T_j \cdots T_{n-1} T_{j-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1} \\ &= T_i \cdots T_{n-1} \pi^{-1} T_j T_1^{-1} \cdots T_{i-1}^{-1} \\ &= Y_i T_j\end{aligned}$$

and if  $j < i-1$  we have

$$\begin{aligned}T_j Y_i &= T_i \cdots T_{n-1} T_j \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1} \\ &= T_i \cdots T_{n-1} \pi^{-1} T_{j+1} T_1^{-1} \cdots T_{i-1}^{-1} \\ &= T_i \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_{j+1} T_j^{-1} T_{j+1} \cdots T_{i-1}^{-1} \\ &= T_i \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_j^{-1} T_{j+1}^{-1} T_j T_{j+2} \cdots T_{i-1}^{-1} \\ &= Y_i T_j\end{aligned}$$

Let us proceed to the commutation relations. We prove them in several steps.

*Step 1:* If  $Y_1 Y_j = Y_j Y_1$  for every  $j = 1, \dots, n$ , then  $Y_i Y_j = Y_j Y_i$  for every  $i, j = 1, \dots, n$ . Indeed, assume that  $i < j$ . Then, using the action relations that we have already shown:

$$\begin{aligned} Y_i Y_j &= T_{i-1}^{-1} \dots T_1^{-1} Y_1 T_1^{-1} \dots T_{i-1}^{-1} Y_j \\ &= T_{i-1}^{-1} \dots T_1^{-1} Y_1 Y_j T_1^{-1} \dots T_{i-1}^{-1} \\ &= T_{i-1}^{-1} \dots T_1^{-1} Y_j Y_1 T_1^{-1} \dots T_{i-1}^{-1} \\ &= Y_j T_{i-1}^{-1} \dots T_1^{-1} Y_1 T_1^{-1} \dots T_{i-1}^{-1} \\ &= Y_j Y_i \end{aligned}$$

*Step 2:* If  $Y_1 Y_2 = Y_2 Y_1$ , then  $Y_1 Y_j = Y_j Y_1$  for every  $j = 1, \dots, n$ . This is done similarly to Step 1.

*Step 3:*  $Y_1 Y_2 = Y_2 Y_1$ . We need to show that  $Y_1 T_1^{-1} Y_1 T_1^{-1} = T_1^{-1} Y_1 T_1^{-1} Y_1$ . The left-hand side of this equation becomes

$$\begin{aligned} (2.4.1) \quad Y_1 T_1^{-1} Y_1 T_1^{-1} &= T_1 \dots T_{n-1} \pi^{-1} T_1^{-1} T_1 \dots T_{n-1} \pi^{-1} T_1^{-1} \\ &= T_1 \dots T_{n-1} T_1 \dots T_{n-2} \pi^{-2} T_1^{-1} \end{aligned}$$

And the right-hand side becomes

$$\begin{aligned} (2.4.2) \quad T_1^{-1} Y_1 T_1^{-1} Y_1 &= T_1^{-1} T_1 \dots T_{n-1} \pi^{-1} T_1^{-1} T_1 \dots T_n \pi^{-1} \\ &= T_2 \dots T_{n-1} T_1 \dots T_{n-2} \pi^{-2} \end{aligned}$$

Now we use that  $\pi^n$  is central in  $\mathcal{H}'_n$ . Indeed, we have  $\pi^{-n} T_1 \pi^n = T_1$ , which implies that  $\pi^{-2} T_1 \pi^2 = T_{n-1}$ , or  $T_1 \pi^2 = \pi^2 T_{n-1}$ , so  $\pi^{-2} T_1^{-1} = T_{n-1}^{-1} \pi^{-2}$ . We use this on the right-hand side of Equation (2.4.1). Now inductively use the identity  $T_{i-1} T_i^{-1} = T_i^{-1} T_{i-1}^{-1} T_i T_{i-1}$ , together with the braid relations, to get an equality with (2.4.2).  $\square$

We also need an analog of Cherednik's basic representation. This is given by the following.

**Theorem 2.4.5.** *The following assignment defines a representation of  $\mathcal{H}_n$  on the space  $\mathbb{C}_{q,\tau}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ :*

$$\begin{aligned} T_i &\mapsto \tau s_i + (\tau - \tau^{-1}) \frac{s_i - \text{id}}{1 - X_i X_{i+1}^{-1}} \\ \pi(X_1^{a_1} \dots X_n^{a_n}) &= q^{-a_n} X_1^{a_n} X_2^{a_1} \dots X_n^{a_{n-1}} \end{aligned}$$

*Proof.* We need to check that these operators satisfy the relations of  $\mathcal{H}_n$ . That the  $T_i$  satisfy the braid and quadratic relations is very similar to what Seth has already done. Note also that  $\pi^n(X_1^{a_1} \dots X_n^{a_n}) = q^{-\sum a_i} X_1^{a_1} \dots X_n^{a_n}$ . Since the operators  $T_i$  preserve the grading, it follows that they commute with  $\pi^n$ . The only relation we need to check now is that  $\pi T_i = T_{i+1} \pi$  for  $i = 1, \dots, n-2$ . This is clear.  $\square$

Let us examine the relations between the operators  $X_i$  (multiplication) and  $\pi$ . First of all, it is clear that for  $i = 1, \dots, n-1$ , we have that  $\pi X_i = X_{i+1} \pi$ . For  $i = n$ , we get  $\pi X_n = q^{-1} X_1 \pi$ . And since  $\pi^n$  is a grading operator, we get that  $\pi^n X_i = q^{-1} X_i \pi^n$ .

**2.4.2. Generators and relations.** The DAHA for  $\mathfrak{gl}_n$  is the  $\mathbb{C}_{q,\tau}$ -algebra generated by the operators  $T_j, j = 1, \dots, n-1$ ,  $X_i, i = 1, \dots, n$  and  $\pi$ . Let us give a precise definition by generators and relations.

**Definition 2.4.6.** *The DAHA  $\mathbb{H}_n$  is the  $\mathbb{C}_{q,\tau}$ -algebra generated by  $T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}, \pi^{\pm 1}$  with relations:*

- (1) *The quadratic relations for  $T_1, \dots, T_{n-1}$ :  $(T_i - \tau)(T_i + \tau^{-1}) = 0$ .*

- (2) The braid relations for  $T_1, \dots, T_{n-1}$ :  $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ ,  $T_i T_j = T_j T_i$  if  $|i - j| > 1$ .
- (3) The Laurent relations for  $X_1^{\pm}, \dots, X_n^{\pm}$ :  $X_i X_j = X_j X_i$ ,  $X_i X_i^{-1} = X_i^{-1} X_i = 1$ .
- (4) The action relations involving  $T_i, X_j$ :  $T_i X_i T_i = X_{i+1}$  if  $i = 1, \dots, n-1$ ;  $T_i X_j = X_j T_i$  if  $i \neq j, j-1$ .
- (5)  $\pi X_i = X_{i+1} \pi$ ,  $i = 1, \dots, n-1$ ;  $\pi^n X_i = q^{-1} X_i \pi^n$ .
- (6)  $\pi T_i = T_{i+1} \pi$ ,  $i = 1, \dots, n-2$ ;  $\pi^n T_i = T_i \pi^n$ ,  $i = 1, \dots, n-1$ .

**Remark 2.4.7.** Let us remark that the relations  $\pi X_n = q^{-1} X_1 \pi$  and  $\pi^2 T_{n-1} = T_1 \pi^2$  are formal corollaries of the relations (5), (6) in Definition 2.4.6.

Now set

$$Y_i := T_i \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1} \in \mathbb{H}_n$$

It is clear that the  $Y_i$ 's satisfy the Laurent relations, as well as the relations

$$T_i^{-1} Y_i T_i^{-1} = Y_{i+1}, \quad T_i Y_j = Y_j T_i \text{ if } i \neq j, j-1$$

Let us examine the relations of  $Y_i$  with  $X_j$ . First of all, since  $Y_1 \cdots Y_n = \pi^{-n}$ , we get

$$(2.4.3) \quad \tilde{Y} X_j = q X_j \tilde{Y}$$

where  $\tilde{Y} := Y_1 \cdots Y_n$ . Now, setting  $\tilde{X} := X_1 \cdots X_n$ , we have that  $\tilde{X}$  commutes with all the  $T_i$ 's while we have that  $\pi \tilde{X} = q^{-1} \tilde{X} \pi$ . This easily implies that

$$(2.4.4) \quad \tilde{X} Y_j = q^{-1} Y_j \tilde{X}.$$

Finally, we have the following relation.

$$\begin{aligned}
 Y_2^{-1} X_1 Y_2 X_1^{-1} &= (T_1 \pi T_{n-1}^{-1} \cdots T_2^{-1}) X_1 (T_2 \cdots T_{n-1} \pi^{-1} T_1^{-1}) X_1^{-1} \\
 &= T_1 \pi T_{n-1}^{-1} \cdots T_2^{-1} T_2 \cdots T_{n-1} X_1 \pi^{-1} T_1^{-1} X_1^{-1} \\
 &= T_1 (\pi X_1 \pi^{-1}) T_1^{-1} X_1^{-1} \\
 &= T_1 X_2 (T_1^{-1} X_1^{-1} T_1^{-1}) T_1 \\
 &= T_1 (X_2 X_2^{-1}) T_1 \\
 &= T_1^2.
 \end{aligned}
 \tag{2.4.5}$$

**Theorem 2.4.8.** The DAHA  $\mathbb{H}_n$  is isomorphic to the  $\mathbb{C}_{q,\tau}$ -algebra generated by  $T_1, \dots, T_{n-1}$ ,  $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ ,  $Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$  subject to the following relations.

- (1) The quadratic and braid relations for  $T_1, \dots, T_{n-1}$ .
- (2) The Laurent relations for  $\{X_1^{\pm 1}, \dots, X_n^{\pm 1}\}$  and for  $\{Y_1^{\pm 1}, \dots, Y_n^{\pm 1}\}$ .
- (3) The action relations for  $(T_i, X_j)$  and for  $(T_i, Y_j)$ .
- (4) Relations (2.4.3), (2.4.4) and (2.4.5).

*Proof.* Let  $\mathbb{H}'_n$  denote the algebra defined in the statement of the theorem. Define  $\pi$  by

$$\pi := T_1^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_i \cdots T_{n-1}$$

Thanks to the action relations involving  $T$  and  $Y$ ,  $\pi$  is independent of  $i$ . We need to check that  $T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}, \pi$  satisfy the relations of  $\mathbb{H}_n$ . Note that we only need to check the relations involving  $\pi$  and  $X_i$ . Moreover, since  $\pi^n = Y_1^{-1} \cdots Y_n^{-1}$ , we only need to check that  $\pi X_i = X_{i+1} \pi$  for  $i = 1, \dots, n-1$ . Furthermore, note that because of the action relations involving  $T, X$  and the relations  $\pi T_i = T_{i+1} \pi$ , we only need to check the relation  $\pi X_1 = X_2 \pi$ . Using the relation (2.4.5) we have

$$\begin{aligned}
\pi X_1 &= T_1^{-1} Y_2^{-1} T_2 \cdots T_{n-1} X_1 \\
&= T_1^{-1} (Y_2^{-1} X_1) T_2 \cdots T_{n-1} \\
&= T_1^{-1} (T_1^2 X_1 Y_2^{-1}) T_2 \cdots T_{n-1} \\
&= T_1 X_1 (T_1 T_1^{-1}) Y_2^{-1} T_2 \cdots T_{n-1} \\
&= (T_1 X_1 T_1) T_1^{-1} Y_2^{-1} T_2 \cdots T_{n-1} \\
&= X_2 \pi
\end{aligned}$$

and the result follows.  $\square$

Just as in the  $A_1$  case, the  $T, X, Y$  presentation of the DAHA  $\mathbb{H}_n$  has the advantage of revealing a symmetry between the  $X$  and  $Y$  parameters.

**Lemma 2.4.9.** *The following defines a  $\mathbb{C}_{q,\tau}$ -linear anti-involution of  $\mathbb{H}_n$*

$$\phi(X_i) = Y_i^{-1}, \quad \phi(Y_i) = X_i^{-1}, \quad \phi(T_j) = T_j, \quad 1 \leq i \leq n, 1 \leq j \leq n-1$$

and the following defines a  $\mathbb{C}$ -linear involution of  $\mathbb{H}_n$ :

$$\varepsilon(X_i) = Y_i, \quad \varepsilon(Y_i) = X_i, \quad \varepsilon(T_j) = T_j^{-1}, \quad \varepsilon(\tau) = \tau^{-1}, \quad \varepsilon(q) = q^{-1}, \quad 1 \leq i \leq n, 1 \leq j \leq n-1$$

*Proof.* For the first statement, we only need to check that the relation (2.4.5) is self-dual with respect to  $\phi$ . Note that we can write this relation as:

$$\begin{aligned}
(2.4.6) \quad 1 &= T_1^{-1} Y_2^{-1} X_1 Y_2 X_1^{-1} T_1^{-1} \\
&= (T_1^{-1} Y_2^{-1} T_1^{-1}) (T_1 X_1 T_1) (T_1^{-1} Y_2 T_1^{-1}) (T_1 X_1^{-1} T_1) T_1^{-2} \\
&= Y_1^{-1} X_2 (T_1^{-2} Y_1 T_1^{-2}) (T_1^2 X_2^{-1} T_1^2) T_1^{-2} \\
&= Y_1^{-1} X_2 T_1^{-2} Y_1 X_2^{-1}
\end{aligned}$$

which we can rewrite as  $T_1^2 = Y_1 X_2^{-1} Y_1^{-1} X_2$ , and so (2.4.5) is self-dual with respect to  $\phi$ . Note that this is also the equation required to prove that  $\varepsilon$  extends to a morphism  $\mathbb{H} \rightarrow \mathbb{H}$ . This finishes the proof.  $\square$

**Exercise 2.4.10.** *The following relations hold in  $\mathbb{H}_n$ :*

$$Y_{i+1}^{-1} X_i Y_{i+1} X_i^{-1} = T_i^2, \quad Y_{j+1}^{-1} X_i Y_{j+1} X_i^{-1} = T_j \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_j^{-1}, \quad j > i$$

**2.4.3. Topological interpretation.** Let  $E$  be a 2-torus. Consider the  $n$ -fold product  $E^n$ , and let  $(E^n)^{reg} := \{(x_1, \dots, x_n) \in E^n : x_i \neq x_j \text{ if } i \neq j\}$ ,  $C := (E^n)^{reg}/S_n$ . The fundamental group  $\pi_1(C)$  is known as the *elliptic braid group*.

**Lemma 2.4.11.** *We have  $\pi_1(C) = \langle T_1, \dots, T_{n-1}, X_1, \dots, X_n, Y_1, \dots, Y_n \rangle$  with relations*

$$\begin{aligned}
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & T_i T_j &= T_j T_i \text{ if } |i-j| > 1, & X_i X_j &= X_j X_i \\
Y_i Y_j &= Y_j Y_i & T_i X_i T_i &= X_{i+1} & T_i^{-1} Y_i T_i^{-1} &= Y_{i+1} \\
T_i X_j &= X_j T_i, i \neq j, j-1 & T_i Y_j &= Y_j T_i, i \neq j, j-1 & Y_2^{-1} X_1 Y_2 X_1^{-1} &= T_1^2 \\
(Y_1 \cdots Y_n) X_j &= X_j (Y_1 \cdots Y_n) & & & (X_1 \cdots X_n) Y_j &= Y_j (X_1 \cdots X_n)
\end{aligned}$$

In the previous lemma, the generator  $X_i$  corresponds to the  $i$ -th point going around a loop in the “horizontal” direction on  $E$ ;  $Y_i$  corresponds to the  $i$ -th point going around in the “vertical” direction on  $E$ ; while  $T_i$  corresponds to the transposition of the  $i$ -th and  $(i+1)$ -th points. Let us remark that, unlike the  $A_1$  case, it is *not* possible to renormalize the generators so that the DAHA  $\mathbb{H}_n$  becomes an honest quotient of the group algebra of  $\pi_1(C)$ . However, one may form a *twisted group algebra*, which is a deformation of the group algebra  $\pi_1(C)$  arising from a central extension of  $\pi_1(C)$  (so that the central element  $z$  becomes  $q$  in the twisted group algebra) and we indeed have

$$\mathbb{H}_n = \mathbb{C}_{q,\tau}^{tw} \pi_1(C) / ((T_i - \tau)(T_i + \tau^{-1}))_{i=1,\dots,n-1}$$

2.4.4. *From  $\mathfrak{gl}_n$  to  $\mathfrak{sl}_n$ .* Let us explain how to recover the DAHA for the root system  $A_{n-1}$  from  $\mathbb{H}_n$ . First of all, in the lattice generated by  $Y_i$  we must have  $Y_1 \cdots Y_n = 1$ . Thus, we pass to the algebra

$$\widetilde{\mathbb{H}}_n := \mathbb{H}_n / (\pi^n - 1)$$

In this algebra, we take the subalgebra generated by  $T_1, \dots, T_{n-1}$ , the elements  $\overline{Y}_i := Y_1 \cdots Y_i$ ,  $i = 1, \dots, n-1$  and their multiplicative inverses, and the elements  $\overline{X}_i := X_1 \cdots X_i (X_1 \cdots X_n)^{-i/n}$ . We remark that the element  $X_1 \cdots X_n$  does have an  $n$ -root in  $\mathbb{H}_n$  (this can be seen, for example, using the automorphism  $\varepsilon$  defined in Lemma 2.4.9 and using the fact that  $Y_1 \cdots Y_n = \pi^{-n}$ ) so this expression makes sense. We also take  $\overline{X}_i^{-1}$ . This subalgebra is isomorphic to  $\mathbb{H}(A_{n-1})$ .

2.4.5. *Trigonometric degeneration.* Let us introduce the trigonometric degeneration of the DAHA  $\mathbb{H}_n$ . This is done completely analogously to the  $A_1$  case. So the first thing we need to do is to think of  $\mathbb{H}_n$  as a  $\mathbb{C}[c][[\hbar]]$ -algebra. Set

$$Y_i := e^{\hbar \hat{y}_i}, \quad q := e^{\hbar}, \quad t := e^{\hbar c}, \quad T_j := s_j e^{\hbar c s_j}, \quad i = 1, \dots, n, \quad j = 1, \dots, n-1$$

where  $s_i \in S_n$  is the transposition  $(i, i+1)$ . We have  $\mathbb{H}_n^{\text{trig}} := \mathbb{H}_n / \hbar \mathbb{H}_n$ . So  $\mathbb{H}^{\text{trig}}$  is generated by  $s_i, i = 1, \dots, n-1$ ,  $X_i^{\pm 1}, i = 1, \dots, n$ , and  $\hat{y}_i, i = 1, \dots, n$ . We have  $s_i^2 = 1$ . From the identity  $T_i X_i T_i = X_{i+1}$ , we get  $s_i X_i s_i = X_{i+1}$ . Let us now examine the identity  $T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$ , we have

$$(s_i - \hbar c s_i^2 + \hbar^2 \frac{c^2 s_i^3}{2!} + \dots)(1 + \hbar \hat{y}_i + \hbar^2 \frac{\hat{y}_i^2}{2!} + \dots)(s_i - \hbar c s_i^2 + \hbar^2 \frac{c^2 s_i^3}{2!} + \dots) = 1 + \hbar \hat{y}_{i+1} + \hbar^2 \frac{\hat{y}_{i+1}^2}{2!} + \dots$$

looking at the coefficient of  $\hbar$  we get the identity  $s_i \hat{y}_i s_i - 2c s_i = \hat{y}_{i+1}$  or, equivalently,

$$s_i \hat{y}_i - \hat{y}_{i+1} s_i = 2c.$$

Similarly, we have the following relations:

$$\begin{aligned} (\hat{y}_1 + \dots + \hat{y}_n) X_j &= X_j (1 + \hat{y}_1 + \dots + \hat{y}_n), \\ X_1 \cdots X_n \hat{y}_j &= (-1 + \hat{y}_j) X_1 \cdots X_n, \\ X_1 \hat{y}_2 X_1^{-1} - \hat{y}_2 &= 2c s_1 \end{aligned}$$

**Definition 2.4.12.** *The trigonometric double affine Hecke algebra for  $\mathfrak{gl}_n$ ,  $\mathbb{H}_n^{\text{trig}}$ , is the  $\mathbb{C}[c]$ -algebra with generators  $s_1, \dots, s_{n-1}, \hat{y}_1, \dots, \hat{y}_n, X_1^{\pm 1}, \dots, X_n^{\pm 1}$  subject to the relations*

- (1)  $s_i^2 = 1, i = 1, \dots, n-1$ ;  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ ;  $s_i s_j = s_j s_i$  if  $|i - j| > 1$ .
- (2)  $s_i X_i s_i = X_{i+1}$ ;  $s_i X_j = X_j s_i$  if  $i \neq j, j-1$ .
- (3)  $s_i \hat{y}_i - \hat{y}_{i+1} s_i = 2c$ ,  $s_i \hat{y}_j = \hat{y}_j s_i$  if  $i \neq j, j-1$ .
- (4)  $(\hat{y}_1 + \dots + \hat{y}_n) X_j = X_j (1 + \hat{y}_1 + \dots + \hat{y}_n)$ .
- (5)  $X_1 \cdots X_n \hat{y}_j = (\hat{y}_j - 1) X_1 \cdots X_n$ .
- (6)  $X_1 \hat{y}_2 X_1^{-1} - \hat{y}_2 = 2c s_1$

**Lemma 2.4.13.** *Every element  $h \in \mathbb{H}_n^{\text{trig}}$  can be uniquely written as a sum*

$$h = \sum_{\substack{P \in \mathbb{C}[X_i^{\pm 1}] \\ w \in S_n \\ f \in \mathbb{C}[\hat{y}_i]}} a_{P,w,f} P(X) w f(\hat{y}), \quad a_{P,w,f} \in \mathbb{C}[c]$$

Of course, the lemma is proven by means of an action of  $\mathbb{H}_n^{\text{trig}}$  on its *polynomial representation*  $\mathbb{C}[c][X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , where  $s_i$  acts by transposing the  $i$ -th and  $(i+1)$ -th variables,  $X_i$  acts by multiplication. Now, for  $i = 1, \dots, n$ , define the trigonometric derivative by  $\partial_i(X_j) = \delta_{ij}X_j$ . So  $\hat{y}_i$  acts by the *trigonometric Dunkl operator*

$$D_i^{\text{trig}} := \partial_i + 2c \sum_{i \neq j} \frac{1}{1 - X_i X_j^{-1}} (\text{id} - s_{ij}) - 2c s_i$$

**Corollary 2.4.14.** *The following are subalgebras of  $\mathbb{H}_n^{\text{trig}}$ :*

- (1) *The group algebra of the extended affine Weyl group of  $\mathfrak{gl}_n$ , which is isomorphic to the subalgebra of  $\mathbb{H}_n^{\text{trig}}$  generated by  $s_1, \dots, s_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}$ .*
- (2) *The degenerate affine Hecke algebra of  $\mathfrak{gl}_n$ , which is isomorphic to the subalgebra of  $\mathbb{H}_n^{\text{trig}}$  generated by  $s_1, \dots, s_{n-1}, \hat{y}_1, \dots, \hat{y}_n$ .*

Let us remark that we also have a *difference-rational polynomial representation* of  $\mathbb{H}_n^{\text{trig}}$ . This is the representation of  $\mathbb{H}_n^{\text{trig}}$  on the space  $\mathbb{C}[c][\hat{y}_1, \dots, \hat{y}_n]$  which is defined as follows. The element  $\hat{y}_i$  just acts by multiplication, the element  $s_i$  acts by the *Demazure-Lusztig operator*:

$$S_i := \tilde{s}_i - 2c \frac{1}{\hat{y}_i - \hat{y}_{i+1}} (\tilde{s}_i - \text{id})$$

where  $\tilde{s}_i$  is the operator on  $\mathbb{C}[c][\hat{y}_1, \dots, \hat{y}_n]$  that transposes the variables  $\hat{y}_i$  and  $\hat{y}_{i+1}$ . To state the action of  $X_i$ , first define the operator  $\pi : \mathbb{C}[c][\hat{y}_1, \dots, \hat{y}_n] \rightarrow \mathbb{C}[c][\hat{y}_1, \dots, \hat{y}_n]$  by

$$\pi(f(\hat{y}_1, \dots, \hat{y}_n)) = f(\hat{y}_2, \dots, \hat{y}_n, \hat{y}_1 - 1)$$

And now define the action of  $X_i$  to be by the operator  $S_{i-1} \cdots S_1 \pi S_{n-1} \cdots S_i$ . This is known as the *difference-rational Dunkl operator*.

**2.4.6. Rational degeneration.** Now we define the rational degeneration for the DAHA  $\mathbb{H}_n$ . Similarly to what was done for the case of  $A_1$ , let  $X_i = \exp(\hbar x_i)$ ,  $y_i := \hbar \hat{y}_i$ . Then  $s_1, \dots, s_{n-1}, y_1, \dots, y_n, x_1, \dots, x_n$  satisfy the following relations modulo  $\hbar$ .

$$\begin{aligned} s_i x_i s_i &= x_{i+1}, & s_i y_i s_i &= y_{i+1}, & s_i x_j &= x_j s_i (i \neq j, j-1), & s_i y_j &= y_j s_i (i \neq j, j-1), \\ y_i x_j - x_j y_i &= 2c s_{ij}, i \neq j & (y_1 + \cdots + y_n) x_j &= 1 + x_j (y_1 + \cdots + y_n) \end{aligned}$$

Note that, in view of all the other relations, the last relation is equivalent to  $y_i x_i - x_i y_i = 1 - 2c \sum_{j \neq i} s_{ij}$ . These are the defining relations for the *rational DAHA* of  $\mathfrak{gl}_n$ ,  $\mathbb{H}_n^{\text{rat}}$ .

**Lemma 2.4.15.** *Every element  $h \in \mathbb{H}_n^{\text{rat}}$  can be written as*

$$h = \sum_{\substack{f \in \mathbb{C}[x] \\ w \in S_n \\ g \in \mathbb{C}[y]}} a_{f,w,g} f(x) w g(y), \quad a_{f,w,g} \in \mathbb{C}[c]$$

We have the polynomial representation  $\mathbb{C}[x_1, \dots, x_n]$  of  $\mathbb{H}_n^{\text{rat}}$ . Here,  $S_n$  acts by permutation of the indices,  $x_i$  acts by multiplication, and  $y_i$  acts via the rational Dunkl operator

$$D_i^{\text{rat}} := \frac{d}{dx_i} - 2c \sum_{j \neq i} \frac{1}{x_i - x_j} (\text{id} - s_{ij})$$

Let us remark that similar degenerations  $\mathbb{H}^{\text{trig}}, \mathbb{H}^{\text{rat}}$  exist for a general root system  $R$ . This will be the subject of the talk next week.

**2.4.7. Braid group action.** The main goal of this section is to produce a braid group action on  $\mathbb{H}_n$  by algebra automorphisms.

**Lemma 2.4.16.** *The following assignment can be extended to an automorphism of  $\mathbb{H}_n$ :*

$$(2.4.7) \quad \rho_1(T_i) = T_i, i = 1, \dots, n-1, \quad \rho_1(X_j) = X_j, j = 1, \dots, n, \quad \rho_1(\pi) = X_1^{-1}\pi$$

*Proof.* The only relation that is not immediate to check that it is preserved is  $\pi^n T_i = T_i \pi^n$ . Using the relations  $\pi X_i^{-1} = X_{i+1}^{-1} \pi$  if  $i < n$

$$\begin{aligned} (X_1^{-1}\pi) \cdots (X_1^{-1}\pi) &= X_1^{-1}(\pi X_1^{-1}) \cdots (\pi X_1^{-1})\pi \\ &= X_1^{-1}X_2^{-1} \cdots X_n^{-1}\pi^n \end{aligned}$$

Which is the product of a symmetric polynomial in the  $X_i$ 's and  $\pi^n$ . Both terms commute with all  $T_i$ . From here, the result follows.  $\square$

For completeness, let us give a formula for  $\rho_1(Y_i)$ . Recall that we have  $Y_1 = T_1 \cdots T_{n-1} \pi^{-1}$ , so  $\rho_1(Y_1) = T_1 \cdots T_{n-1} \pi^{-1} X_1 = Y_1 X_1$ . Now, using the fact that  $T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$  we get:

$$(2.4.8) \quad \rho_1(Y_i) = Y_i X_i (T_{i-1}^{-1} \cdots T_1^{-1}) (T_1^{-1} \cdots T_{i-1}^{-1})$$

The following lemma can be checked similarly to Lemma 2.4.16.

**Lemma 2.4.17.** *The following assignment can be extended to an automorphism of  $\mathbb{H}_n$ :*

$$(2.4.9) \quad \rho_2(T_i) = T_i, i = 1, \dots, n-1; \quad \rho_2(Y_j) = Y_j, \quad \rho_2(X_j) = X_j Y_j (T_{j-1} \cdots T_1) (T_1 \cdots T_{j-1}).$$

**Remark 2.4.18.** *Note that  $\rho_2 = \varepsilon \rho_1 \varepsilon$ , where  $\varepsilon : \mathbb{H} \rightarrow \mathbb{H}$  is the  $\mathbb{C}$ -linear involution defined in Lemma 2.4.9.*

Let us give a formula for  $\rho_2^{-1}$ . We have that  $\rho_2^{-1}(T_i) = T_i, \rho_2^{-1}(Y_j) = Y_j$ , while

$$\rho_2^{-1}(X_j) = X_j (T_{j-1}^{-1} \cdots T_1^{-1}) (T_1^{-1} \cdots T_{j-1}^{-1}) Y_j^{-1}$$

**Lemma 2.4.19.** *Consider the braid group on three strands,  $B_3 := \langle \sigma_1, \sigma_2 : \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$ . The assignment*

$$\sigma_1 \mapsto \rho_1, \quad \sigma_2 \mapsto \rho_2^{-1}$$

*gives an action of  $B_3$  on  $\mathbb{H}_n$ .*

*Proof.* We need to check that  $\rho_1, \rho_2^{-1}$  satisfy the relation:

$$(2.4.10) \quad \rho_1 \rho_2^{-1} \rho_1 = \rho_2^{-1} \rho_1 \rho_2^{-1}$$

It is obvious that when we evaluate both sides on  $T_i$  we just get  $T_i$ . Then, thanks to the action relations and the fact that both sides of (2.4.10) are automorphisms, we just need to check that  $\rho_1 \rho_2^{-1} \rho_1(X_1) = \rho_2^{-1} \rho_1 \rho_2^{-1}(X_1)$ , and a similar equation for  $Y_1$ . We have:

$$\begin{aligned} \rho_1 \rho_2^{-1} \rho_1(X_1) &= \rho_1 \rho_2^{-1}(X_1) = \rho_1(X_1 Y_1^{-1}) = X_1 X_1^{-1} Y_1^{-1} = Y_1^{-1} \\ \rho_2^{-1} \rho_1 \rho_2^{-1}(X_1) &= \rho_2^{-1}(Y_1^{-1}) = Y_1^{-1} \end{aligned}$$

It is similarly easy to check that  $\rho_1 \rho_2^{-1} \rho_1(Y_1) = \rho_2^{-1} \rho_1 \rho_2^{-1}(Y_1) = Y_1 X_1 Y_1^{-1}$ . The lemma follows.  $\square$

Let us remark that the automorphisms  $\rho_1, \rho_2$  descend to the rational degeneration of  $\mathbb{H}_n$ .

**Lemma 2.4.20.** *The following define automorphisms of  $\mathbb{H}_n^{\text{rat}}$ :*

$$\begin{aligned} \rho_1(s_i) &= s_i & \rho_1(x_j) &= x_j & \rho_1(y_j) &= y_j + x_j \\ \rho_2(s_i) &= s_i & \rho_2(x_j) &= x_j + y_j & \rho_2(y_j) &= y_j \end{aligned}$$

*We still have the relations  $\rho_1\rho_2^{-1}\rho_1 = \rho_2^{-1}\rho_1\rho_2^{-1}$ . Moreover,  $(\rho_1\rho_2^{-1}\rho_1)^4 = \text{id}$ , so that we have an action of  $\text{SL}_2(\mathbb{Z})$  on  $\mathbb{H}_n^{\text{rat}}$ , cf. Lemma 3.3.5.*

It is interesting to note that, according to [C2, 2.12.4], the automorphisms  $\rho_1, \rho_2$  have no trigonometric analogue.

### 3. THE POLYNOMIAL REPRESENTATION

**3.1. Upper triangularity of  $Y^\lambda$ .** We study the polynomial representation more carefully. Our first goal is to see that the operators  $Y^\lambda$  are upper triangular with respect to a certain partial order on  $P$ . First of all, recall that we have the partial order  $<$  on  $P^+$ , which is defined by  $\nu > \mu$  if  $\nu - \mu \in Q^+$ . We extend this order to  $P$ .

**Definition 3.1.1.** *For  $\mu \in P$ , let  $\mu^+ \in P^+$  be the dominant weight lying in the orbit  $W\mu$ . Define a partial order on the weight lattice  $P$  as follows:  $\nu \prec \mu$  if  $\nu^+ < \mu^+$ , or  $\nu^+ = \mu^+$  and  $\nu > \mu$  (note the change of signs!)*

Let us give some properties of the order  $\prec$  that will be useful later.

**Lemma 3.1.2.** *Let  $\mu \in P$  and let  $\alpha \in R_+$*

- (1) *If  $\langle \mu, \alpha^\vee \rangle = r > 0$ , then  $s_\alpha(\mu) \succ \mu$ , while  $\mu - \alpha, \dots, \mu - (r-1)\alpha \prec \mu$ .*
- (2) *If  $\langle \mu, \alpha^\vee \rangle = -r < 0$ , then  $\mu + \alpha, \dots, \mu + (r-1)\alpha, s_\alpha(\mu) \prec \mu$ .*

*Proof.* See e.g. [M, Section 2.6]. □

Recall now from Seth's talk that, if  $\lambda \in P_+^\vee$  is such that  $t(\lambda) = \pi_r s_{i_\ell} \cdots s_{i_1}$  is a reduced expression, then  $Y^\lambda = \pi_r T_{i_\ell} \cdots T_{i_1}$ . We would like to use this to obtain some information about the operator  $Y^\lambda$ . First of all, if  $a \in R^a$  is a root, define the operator

$$G(a) := \tau_a + (\tau_a - \tau_a^{-1}) \frac{\text{id} - s_a}{X^{-a} - 1}$$

where  $\tau_a = \tau_i$  if  $w(a) = a_i$  for some element  $w \in W^a$ . Note that  $T_i = s_i G(a_i)$  and, if  $w \in W^a$ ,  $G(w(a)) = wG(a)w^{-1}$ . In particular, if  $s$  is a reflection we have  $G(a)s = sG(s(a))$ . Thus, for  $\lambda \in P_+^\vee$  we have

$$Y^\lambda = \pi_r s_{i_\ell} G(a_{i_\ell}) s_{i_{\ell-1}} G(a_{i_{\ell-1}}) \cdots s_{i_1} G(a_{i_1}) = t(\lambda) G(a^{(\ell)}) \cdots G(a^{(1)}),$$

where  $a^{(j)} = s_{i_1} \cdots s_{i_{j-1}}(a_{i_j})$ .

**Theorem 3.1.3.** *For  $\lambda \in P_+^\vee, \mu \in P$  we have*

$$Y^\lambda(X^\mu) = \sum_{\nu \preceq \mu} c_{\mu, \nu} X^\nu$$

with  $c_{\mu, \nu} \in \mathbb{C}_{q, \tau}$ .

*Proof.* Assume first that  $\lambda \in P_+^\vee$ , so that in particular  $Y_\lambda = t(\lambda) G(a^{(\ell)}) \cdots G(a^{(1)})$  with  $a^{(i)} = \alpha_i + k_i \delta, \alpha_i \in R^+$ . So let  $a = \alpha + k\delta$  with  $\alpha \in R^+$ . Then we have:

$$G(a)X^\mu = \tau_a X^\mu + (\tau_a - \tau_a^{-1}) \frac{X^{s_a(\mu)} - X^\mu}{1 - X^{-a}}$$

Now assume that  $\langle \alpha^\vee, \mu \rangle = r > 0$ , so  $s_a(\mu) = \mu - ra$ . Thus, we have



$$G(a)X^\mu = \tau_a X^\mu - (\tau_a - \tau_a^{-1})(X^\mu + X^{\mu-a} + \dots + X^{\mu-(r-1)a}) = \tau_a^{-1} X^\mu + \dots$$

where the ellipsis stands for lower order terms, see Lemma 3.1.2. The case  $\langle \alpha^\vee, \mu \rangle < 0$  is similar, for  $\langle \alpha^\vee, \mu \rangle = 0$  we just have  $G(a)X^\mu = \tau_a X^\mu$ . Since  $t(\lambda)$  is diagonal, we have that  $Y^\lambda$  is a composition of upper triangular operators and the result follows.

Now if  $\lambda = \lambda' - \lambda''$  with  $\lambda', \lambda'' \in P_+^\vee$ , then  $Y^\lambda = Y^{\lambda'}(Y^{\lambda''})^{-1}$ . Since the inverse of an upper triangular operator is again upper triangular, the result follows.  $\square$

**3.2. Difference operators.** The goal of this section is to produce some difference operators on the space  $\mathbb{C}_{q,\tau}[X]$  using the representation theory of DAHA. Recall from the proof of Theorem 2.2.4 that for every  $w \in W^{ae}$ , the extended affine Weyl group, the action of  $T_w$  on  $\mathbb{C}_{q,\tau}[X]$  may be written as

$$(3.2.1) \quad T_w = \sum_{\substack{\lambda \in P^\vee \\ w \in W}} g_{\lambda,w} t(\lambda) w, \quad g_{\lambda,w} \in \mathbb{C}_{q,\tau}(X)$$

so in particular the same is true for  $Y^\lambda \in \mathcal{H}^Y \subseteq \mathbb{H}$ . Recall that the center of  $\mathcal{H}^Y$  is precisely  $\mathbb{C}_{q,\tau}[Y]^W$ .

**Lemma 3.2.1.** *Let  $f(Y) \in \mathbb{C}_{q,\tau}[Y]^W$ . Then, the action of  $f(Y)$  on  $\mathbb{C}_{q,\tau}[X]$  preserves the space of  $W$ -invariants  $\mathbb{C}_{q,\tau}[X]^W$ .*

*Proof.* Note that, from the formula for the action of  $T_i$ ,  $i = 1, \dots, n$ , on  $\mathbb{C}_{q,\tau}[X]$  it follows that  $p(X) \in \mathbb{C}_{q,\tau}[X]$  is  $W$ -invariant if and only if

$$T_i p(X) = \tau_i p(X), i = 1, \dots, n$$

From here, the result follows easily using the fact that  $f(Y)$  commutes with  $T_i$ .  $\square$

Now let  $f$  be an operator on  $\mathbb{C}_{q,\tau}[X]$  of the form (3.2.1). We define its restriction by

$$\text{Res}(f) := \sum_{\substack{\lambda \in P^\vee \\ w \in W}} g_{\lambda,w} \tau(\lambda)$$

In particular,  $\text{Res}(f)$  is a difference operator on  $\mathbb{C}_{q,\tau}[X]$  and  $f|_{\mathbb{C}_{q,\tau}[X]^W} = \text{Res}(f)|_{\mathbb{C}_{q,\tau}[X]^W}$ . For  $f \in \mathbb{C}_{q,\tau}[Y]^W$ , we denote  $L_f := \text{Res}(f)$ . This is a difference operator on  $\mathbb{C}_{q,\tau}[X]$  preserving the space of invariants  $\mathbb{C}_{q,\tau}[X]^W$ .

**Corollary 3.2.2.** *The operators  $L_f, f \in \mathbb{C}_{q,\tau}[Y]^W$ , are pairwise commutative and  $W$ -invariant.*

*Proof.* Let  $f, g \in \mathbb{C}_{q,\tau}[Y]^W$ . Then  $L_f L_g = \text{Res}(f) \text{Res}(g)$ . Since  $g$  is  $W$ -invariant,  $\text{Res}(f) \text{Res}(g) = \text{Res}(fg) = \text{Res}(gf)$ . Since now  $f$  is  $W$ -invariant, we get  $\text{Res}(gf) = \text{Res}(g) \text{Res}(f) = L_g L_f$ .  $\square$

**Remark 3.2.3.** *It follows from Theorem 3.1.3 that the operators  $L_f : \mathbb{C}_{q,\tau}[X]^W \rightarrow \mathbb{C}_{q,\tau}[X]^W$  are upper triangular with respect to the basis formed by  $\{x_\lambda := \sum_{\lambda' \in W_\lambda} X^{\lambda'}\}_{\lambda \in P^+}$  and the dominance ordering on  $P^+$ .*

The operators  $L_f$  are intimately related to the theory of Macdonald's polynomials. This will be the subject of a subsequent talk.

3.2.1. *Example:  $A_1$ .* Let us consider the example of a root system of type  $A_1$ . We keep the notation of Section 2.3.2, with one small caveat. Now we set  $X := X^\alpha$ , so that  $\mathbb{C}_{q,\tau}[X]$  is the algebra of polynomials in  $X^{\pm 1/2}$ . With this convention, the action of  $T$  is given by

$$\tau s + (\tau - \tau^{-1}) \frac{s - \text{id}}{X - 1}$$

While the action of  $t(\rho)$  is  $t(\rho)(X) = q^2 X$ , so  $t(\rho)f(X) = f(q^2 X)$  and, in particular,  $\pi_\rho(X) = t(\rho)s(X) = q^{-2}X^{-1}$ . Note that we have

$$Y^\rho = t(\rho) \left( \tau + (\tau - \tau^{-1}) \frac{\text{id} - s}{X^{-1} - 1} \right)$$

Let us now deal with  $Y^{-\rho} = T^{-1}\pi_\rho$ . Here we will use the reflection  $s_0$  on the affine Weyl group: it is easy to check that we have a relation  $\pi_\rho T^{-1}\pi_\rho = T_0^{-1}$ , so that  $Y^{-\rho} = \pi_\rho T_0^{-1} = \pi_\rho(T_0 + (\tau^{-1} - \tau))$ . Thus,

$$\begin{aligned} Y^{-\rho} &= \pi_\rho \left( \tau_{s_0} + (\tau - \tau^{-1}) \frac{s_0 - \text{id}}{q^{-2}X^{-1} - 1} + (\tau^{-1} - \tau) \right) \\ &= t(\rho)s \left( \tau t(\alpha)s + (\tau - \tau^{-1}) \frac{t(\alpha)s - q^{-2}X^{-1}}{q^{-2}X^{-1} - 1} \right) \\ &= \tau t(-\rho) + (\tau - \tau^{-1}) t(\rho) \frac{t(-\alpha) - q^{-2}Xs}{q^{-2}X - 1} \\ &= \tau t(-\rho) + (\tau - \tau^{-1}) \frac{t(-\rho) - Xt(\rho)s}{X - 1} \end{aligned}$$

Thus,  $\text{Res } Y^\rho = \tau t(\rho)$ ,  $\text{Res } Y^{-\rho} = \frac{\tau X - \tau^{-1}}{X - 1} t(-\rho) + (\tau - \tau^{-1}) \frac{1}{X^{-1} - 1} t(\rho)$ . So

$$(3.2.2) \quad \text{Res}(Y^\rho + Y^{-\rho}) = \frac{\tau X^{-1} - \tau^{-1}}{X^{-1} - 1} t(\rho) + \frac{\tau X - \tau^{-1}}{X - 1} t(-\rho).$$

This is (a scalar multiple of) Macdonald's difference operator for  $A_1$ . The symmetric polynomials here are spanned by binomials of the form  $X^{i/2} + X^{-i/2}$ ,  $i \geq 0$ . It is an easy exercise to check that the operator (3.2.2) indeed preserves the space of symmetric polynomials, and that it is upper triangular with respect to the basis  $x_i := X^{i/2} + X^{-i/2}$ .

**3.3. Spherical DAHA.** We have seen that the operators  $L_f, f \in \mathbb{C}_{q,\tau}[X]^W$  define difference operators on the space of  $W$ -invariant polynomials on  $X$ . We can actually define a smaller algebra than the DAHA  $\mathbb{H}$  which includes all the operators  $L_f$  and which acts on  $\mathbb{C}_{q,\tau}[X]^W$ . This is known as the *spherical* DAHA and it is constructed as follows.

Let  $\mathbb{C}_\tau$  be the 1-dimensional (over  $\mathbb{C}_\tau$ ) representation of the finite Hecke algebra  $H$  where  $T_i$  acts by  $\tau_i$ ,  $i = 1, \dots, n$ . We can realize this representation as  $\mathbb{C}_\tau = H\mathbf{e}$ , where  $\mathbf{e} \in H$  is an idempotent which is constructed as follows. For  $w \in W$  let  $\tau_w := \tau_{i_1} \cdots \tau_{i_k}$ , where  $w = s_{i_1} \cdots s_{i_k}$  is a reduced decomposition. Note that  $\tau_w$  is well-defined, since it is the scalar by which  $w$  acts on the representation  $\mathbb{C}_\tau$ . Now define  $\tilde{\mathbf{e}} := \sum_{w \in W} \tau_w T_w$ .

**Lemma 3.3.1.** *For  $i = 1, \dots, n$ , we have  $T_i \tilde{\mathbf{e}} = \tau_i \tilde{\mathbf{e}}$ .*

*Proof.* We will do a direct calculation. We will need the following equation that we have already seen in Seth's talk. In the finite Hecke algebra  $H$ :

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ T_{s_i w} + (\tau_i - \tau_i^{-1}) T_w & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

where the length  $\ell$  is the usual one in  $W$ , i.e., the length of a reduced expression of  $w$ . Thus, we have:

$$T_i \tilde{\mathbf{e}} = \sum_{\substack{w \in W \\ \ell(s_i w) > \ell(w)}} \tau_w T_{s_i w} + \sum_{\substack{w \in W \\ \ell(s_i w) < \ell(w)}} \tau_w (T_{s_i w} + (\tau_i - \tau_i^{-1}) T_w)$$

Now we find the coefficient of  $T_w$  in the previous expression. We have two cases. If  $\ell(s_i w) < \ell(w)$ , then we have that the coefficient of  $T_w$  is  $\tau_{s_i w} + \tau_w(\tau_i - \tau_i^{-1}) = \tau_i \tau_w$ , since  $\tau_w = \tau_i \tau_{s_i w}$ . If  $\ell(s_i w) > \ell(w)$ , then the coefficient of  $T_w$  is simply  $\tau_{s_i w} = \tau_i \tau_w$ . We are done.  $\square$

**Remark 3.3.2.** Similarly, we can see that  $\tilde{\mathbf{e}} T_i = \tau_i \tilde{\mathbf{e}}$  for  $i = 1, \dots, n$ .

Thanks to the previous lemma,  $\tilde{\mathbf{e}}^2 = \sum_{w \in W} \tau_w T_w \tilde{\mathbf{e}} = \sum_{w \in W} \tau_w^2 \tilde{\mathbf{e}}$ . Thus

$$\mathbf{e} := \left( \sum_{w \in W} \tau_w^2 \right)^{-1} \tilde{\mathbf{e}}$$

is an idempotent.

**Definition 3.3.3.** Define the spherical DAHA as  $\mathbb{SH} := \mathbf{e} \mathbb{H} \mathbf{e}$ . This is a non-unital subalgebra of  $\mathbb{H}$ , with unit  $\mathbf{e}$ .

**Remark 3.3.4.** In the  $\mathfrak{gl}_n$  case, note that the automorphisms  $\rho_1, \rho_2$  of  $\mathbb{H}_n$  preserve the idempotent  $\mathbf{e}$ , hence they also preserve the spherical subalgebra. So we have an action of  $B_3$  on  $\mathbb{SH}_n$ .

The following result will be important to connect DAHA's to EHA's, which is one of the objectives of the course. First, we recall a well-known result. For a proof, see e.g. [KT, Appendix A].

**Lemma 3.3.5.** The group  $\mathrm{SL}_2(\mathbb{Z})$  is a quotient of the braid group on three strands  $B_3 = \langle \sigma_1, \sigma_2 : \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$ . The quotient map  $B_3 \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z})$  is given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

the kernel of this map is generated by  $(\sigma_1 \sigma_2 \sigma_1)^4$ .

**Theorem 3.3.6.** The braid group action on  $\mathbb{SH}_n$  factors through  $\mathrm{SL}_2(\mathbb{Z})$ , that is,  $(\rho_1 \rho_2^{-1} \rho_1)|_{\mathbb{SH}_n} = \mathrm{id}_{\mathbb{SH}_n}$ .

*Proof.* According to [C2, 3.2.2],  $(\rho_1 \rho_2^{-1} \rho_1)^4$  is conjugation by  $T_{w_0}^{-2}$ , where  $w_0$  is the longest element of  $S_n$ . Since  $T_{w_0} \mathbf{e} = \tau_{w_0} \mathbf{e}$ , the result follows easily.  $\square$

Note that, if  $M$  is a  $\mathbb{H}$ -module, then  $\mathbf{e}M$  becomes a  $\mathbb{SH}$ -module. For the polynomial representation we have:

$$\mathbf{e} \mathbb{C}_{q,\tau}[X] = \{f \in \mathbb{C}_{q,\tau}[X] : T_i f = \tau_i f, i = 1, \dots, n\} = \mathbb{C}_{q,\tau}[X]^W$$

Now note that, for  $f \in \mathbb{C}_{q,\tau}[Y]^W$ , we have that  $\mathbf{e} L_f \mathbf{e}|_{\mathbb{C}_{q,\tau}[X]^W} = L_f|_{\mathbb{C}_{q,\tau}[X]^W}$ . Thus, the action of the spherical DAHA  $\mathbb{SH}$  on  $\mathbb{C}_{q,\tau}[X]^W$  already includes the operators  $L_f$  defined above.

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DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY. BOSTON, MA 02115. USA.

*E-mail address:* `simentalrodriguez.j@husky.neu.edu`