Lecture 2

0) Introduction

1) Formal quantizations

2) Microlocalization.

0) Introduction

Let A be $\mathbb{Z}$-graded Poisson algebra (where, for simplicity, $\deg\{\cdot,\cdot\}=-1$), $A=\bigoplus_{i\in\mathbb{Z}} A_i$.

We have defined the notion of a filtered quantization, $\mathcal{A}$ of A. This is a filtered associative algebra $\mathcal{A}=\bigcup_{i\in\mathbb{Z}}\mathcal{A}_{\leqslant i}$ w. $[\mathcal{A}_{\leqslant i},\mathcal{A}_{\leqslant j}]\subset \mathcal{A}_{\leqslant i+j-1}$ ($\rightsquigarrow \operatorname{gr}\mathcal{A}$ comes w. $\{\cdot,\cdot\}$ of deg -1) & a fixed graded Poisson algebra isomorphism $\operatorname{gr}\mathcal{A}\xrightarrow{\sim} A$. If $\mathcal{A}_{\leqslant -1}\neq\{0\}$ we also need to impose the technical condition: $\mathcal{A}\xrightarrow{\sim}\varprojlim_{i\to-\infty}\mathcal{A}/\mathcal{A}_{\leqslant i}$.

In this lecture we'll define some kind of a localization of $\mathcal{A}$ by an element of A. And this requires a connection with a related class of quantizations: formal ones.

1) Formal quantizations

Let A be a commutative algebra. By its formal deformation (over $\mathbb{C}[[\hbar]]$) we mean an associative $\mathbb{C}[[\hbar]]$-algebra $\mathcal{A}_\hbar$ together w. an algebra isomorphism $\iota: \mathcal{A}_\hbar/(\hbar) \xrightarrow{\sim} A$ s.t.

(i) $\hbar$ is not a zero divisor in $\mathcal{A}_\hbar$

(ii) $\mathcal{A}_\hbar$ is complete & separated in the $\hbar$-adic topology i.e. $\mathcal{A}_\hbar \xrightarrow{\sim} \varprojlim_n \mathcal{A}_\hbar/(\hbar^n)$.

Note that $\mathcal{A}_\hbar/(\hbar)$ carries a Poisson bracket. Indeed, since $\mathcal{A}_\hbar/(\hbar)$ is commutative, $[a,b]\in\hbar\mathcal{A}_\hbar$ $\forall a,b\in\mathcal{A}_\hbar$. Thx to (i), $\frac{1}{\hbar}[a,b]\in\mathcal{A}_\hbar$ is uniquely defined. We set

$$\{a+(\hbar), b+(\hbar)\} = \frac{1}{\hbar}[a,b]+(\hbar)$$

getting a Poisson bracket on $\mathcal{A}_\hbar/(\hbar)$ (exercise). If A was equipped with a Poisson bracket & ι is a Poisson isomorphism, then we say that $\mathcal{A}_\hbar$ is a formal quantization of A.

1.1) From filtered quantizations to graded formal.

By a grading on $\mathcal{A}_\hbar$ we mean a collection of gradings on each $\mathcal{A}_\hbar/(\hbar^n)$ s.t.

- $\deg \hbar = 1$
- $\mathcal{A}_\hbar/(\hbar^{n+1}) \twoheadrightarrow \mathcal{A}_\hbar/(\hbar^n)$ is graded $\forall n$.

($\mathcal{A}_\hbar$ itself cannot be graded b/c of completeness)

Let's explain how to get from a filtered quantization to a graded formal quantization. We'll do this in two steps. Let $\mathcal{A}$ be a filtered quantization of graded A.

1) From $\mathcal{A} = \bigcup_{i\in\mathbb{Z}} \mathcal{A}_{\leq i}$ we form the Rees algebra

$$R_\hbar(\mathcal{A}) := \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{\leq i}\hbar^i \subset \mathcal{A}[\hbar^{\pm 1}], \text{ a graded subalgebra}$$

Note that $R_\hbar(\mathcal{A})/(\hbar) \xrightarrow{\sim} \operatorname{gr}\mathcal{A}$, $R_\hbar(\mathcal{A})/(\hbar-1) \xrightarrow{\sim} \mathcal{A}$ (exercise)

2) Set $\mathcal{A}_\hbar := \varprojlim_{n\to\infty} R_\hbar(\mathcal{A})/(\hbar^n)$

Exercise: Prove that $\mathcal{A}_\hbar$ is a graded formal quantization of A. Hint: prove first that $\mathcal{A}_\hbar/(\hbar^n) \xrightarrow{\sim} R_\hbar(\mathcal{A})/(\hbar^n)$.

1.2) From graded formal to filtered

Now let $\mathcal{A}_\hbar$ be a graded formal quantization of A. We want to produce a filtered quantization. Recall that $\mathcal{A}_\hbar(n) := \mathcal{A}_\hbar/(\hbar^n)$ is graded: $\mathcal{A}_\hbar(n) = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_\hbar(n)_i$. The projection

$\mathcal{A}_\hbar(n+1) \twoheadrightarrow \mathcal{A}_\hbar(n)$ is a graded algebra homomorphism. Set

$$\mathcal{A}_{\hbar,i} := \varprojlim_{n\to\infty} \mathcal{A}_\hbar(n)_i, \quad \mathcal{A}_\hbar^{fin} := \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{\hbar,i}.$$

Exercise: 1) Show that $\mathcal{A}_\hbar^{fin}$ is a $\mathbb{C}[\hbar]$-subalgebra of $\mathcal{A}_\hbar$ & a graded $\mathbb{C}[\hbar]$-algebra.

2) Show that $\mathcal{A}_\hbar^{fin}/(\hbar) \xrightarrow{\sim} A$ & $\mathcal{A}_\hbar^{fin}/(\hbar-1)$ is a filtered quantization of A (w.r.t. the filtration induced by the grading on $\mathcal{A}_\hbar^{fin}$).

3) Show that the assignments $\mathcal{A} \mapsto \varprojlim_{n\to\infty} R_\hbar(\mathcal{A})/(\hbar^n)$ & $\mathcal{A}_\hbar \mapsto \mathcal{A}_\hbar^{fin}/(\hbar-1)$ are mutually inverse bijections between filtered & graded formal quantizations of A.

Rem: $\mathcal{A}_\hbar^{fin}/(\hbar-1)$ is complete w.r.t. the filtration b/c $\mathcal{A}_\hbar$ is complete in the $\hbar$-adic topology. This kind of explains that condition in the general definition of a filtered quantization.

2) Microlocalization.

Setting: Let A be a Poisson algebra & $\mathcal{A}_\hbar$ be its formal quantization. For $f \in A$ we want to construct from $\mathcal{A}_\hbar$ a formal quantization of $A[f^{-1}]$, a kind of localization, to be denoted by $\mathcal{A}_\hbar[f^{-1}]$.

Then we will explain a modification for filtered quantizations.

2.1) Localization in noncommutative rings

Let R be an associative (unital) ring & $S \subset R$ be a subset closed under products. We want to have the localization R_S i.e. a ring with a homomorphism $\iota: R \to R_S$ s.t. $\forall$ ring R':

$\iota^*: \mathrm{Hom}_{Rings}(R_S, R') \xrightarrow{\sim} \{\varphi: R \to R' \mid \varphi(s) \text{ is invertible } \forall s \in S\}$

Moreover, we want it to consist of "fractions" rs^{-1}. Sufficient conditions for the latter were found by Ore:

(O1) $\forall a \in R, s \in S \Rightarrow \exists b \in R, t \in S \mid at = sb$ "$\Leftrightarrow s^{-1}a = bt^{-1}$"

(a.k.a. every left fraction is also a right action)

(O2) $\forall a \in R, s \in S$ s.t. $sa = 0 \Rightarrow \exists t \in S \mid at = 0$

Note that these conditions are vacuous if R is commutative.

Example/exercise: Suppose $s \in R$ satisfies $\mathrm{ad}(s)^n = 0$ for some $n \in \mathbb{Z}_{>0}$, where $\mathrm{ad}(s): R \to R$ is given by $r \mapsto [s,r]$. Then $\{s^m \mid m \in \mathbb{Z}_{\geq 0}\}$ satisfies (O1) & (O2).

Assume (O1) & (O2). Consider the following relation on $R \times S$: $(a,s) \sim (a',s')$ if $\exists\, t,t' \in S$ w. $at = a't'$ & $st = s't'$. It turns out that:

- this is an equivalence relation, let a/s denote the equivalence class of (a,s)
- and the set of equivalence classes has a ring structure making it the localization R_S.

2.2) Construction of $\mathcal{A}_\hbar[f^{-1}]$

Recall that any formal quantization is the inverse limit of its quotients by $(\hbar^n)$. Set $\mathcal{A}_\hbar(n) = \mathcal{A}_\hbar/(\hbar^n)$. We are going to define an inverse system $\mathcal{A}_\hbar(n)[f^{-1}]$ of flat $\mathbb{C}[\hbar]/(\hbar^n)$-algebras.

Let $\hat{f}$ denote a lift of f to $\mathcal{A}_\hbar(n)$. Since $[\mathcal{A}_\hbar(n), \mathcal{A}_\hbar(n)] \subset \hbar\mathcal{A}_\hbar(n)$, we see that $S := \{\hat{f}^m\}$ satisfies (O1) & (O2). Then we set $\mathcal{A}_\hbar(n)[f^{-1}] := \mathcal{A}_\hbar(n)_S$. Here are the properties:

1) $\mathcal{A}_\hbar(n)[f^{-1}]$ is independent of the choice of $\hat{f}$: for a different lift $\hat{f}'$ $\exists$ $n>0$ s.t.

$$\hat{f}'\hat{f}^{n-1} = \hat{f}^n(1+\hbar a) \text{ for some } a \in \mathcal{A}_\hbar(n)$$

2) $\mathcal{A}_\hbar(n)[f^{-1}]$ is flat over $\mathbb{C}[\hbar]/(\hbar^n)$

3) We have naturel epimorphism

$$\mathcal{A}_\hbar(n+1)[f^{-1}] \twoheadrightarrow \mathcal{A}_\hbar(n)[f^{-1}]$$

The proofs are exeruses. Now thx to these properties we can set $\mathcal{A}_\hbar[f^{-1}] := \varprojlim_n \mathcal{A}_\hbar(n)[f^{-1}]$.

Exeruse: $\mathcal{A}_\hbar[f^{-1}]$ is a formal quantization of $A[f^{-1}]$.

2.3) Filtered setting

One can ask when $\mathcal{A}_\hbar[f^{-1}]$ has a grading. The answer: when f is homogeneous: each $\mathcal{A}_\hbar(n)[f^{-1}]$ is graded & the homomorphism in 3) is graded. So using the correspondence between filtered & graded formal quantizations in Sec 1 we define $\mathcal{A}[f^{-1}]$. Note that if $\mathcal{A}$ is $\mathbb{Z}_{\geqslant 0}$-filtered and $f \in \mathcal{A}_i$ for $i > 0$, then $\mathcal{A}[f^{-1}]$ is only $\mathbb{Z}$-filtered (in essentially all cases).