

# LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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## 12. CALOGERO-MOSER SYSTEMS AND QUANTUM MECHANICS

**12.1. Hamiltonian reduction, finished.** Let us recall the set-up of the end of the previous lecture. We have a symplectic affine variety  $X$  with form  $\omega$ . We equip this variety with a Hamiltonian action of a reductive algebraic group  $G$ , let  $\mu : X \rightarrow \mathfrak{g}^*$  be a moment map. Further, we choose a closed orbit  $Y \subset \mathfrak{g}^*$ . We assume that  $G$  acts freely on  $\mu^{-1}(Y)$ . Then, as we have seen,  $X//_Y G := \mu^{-1}(Y)//G$  is a smooth symplectic variety. A symplectic form  $\underline{\omega}$  on  $X//_Y G$  can be described as follows. We pick  $\alpha \in Y$ . Then  $\mu^{-1}(Y)//G$  is naturally identified with  $\mu^{-1}(\alpha)//G_\alpha$ . Then the symplectic form  $\underline{\omega}$  is defined as a unique form satisfying  $\pi^* \underline{\omega} = \iota^* \omega$ , where  $\pi : \mu^{-1}(\alpha) \rightarrow \mu^{-1}(\alpha)//G_\alpha$  is the quotient morphism, and  $\iota : \mu^{-1}(\alpha) \hookrightarrow X$  is the inclusion.

Recall that  $X//_Y G$  comes also with a Poisson structure, the bracket is defined directly on the algebra of functions of this variety. This description can be translated into a more geometric language as follows. It is enough to specify the Hamiltonian vector fields for the functions on  $X//_Y G$ .

Suppose  $F \in \mathbb{C}[X]$  is such that its restriction  $\underline{F}$  to  $\mu^{-1}(Y)$  is  $G$ -invariant. Then, tracking the definition of the bracket on  $\mathbb{C}[X//_Y G] = [\mathbb{C}[X]/\mathbb{C}[X]I]^G$ , where  $I \subset S(\mathfrak{g})$  is the ideal of  $Y$ , we see that  $v(F)$  preserves  $I$ , equivalently, is tangent to  $\mu^{-1}(Y)$ . The restriction of  $v(F)$  to  $\mu^{-1}(Y)$  is  $G$ -invariant and the induced (see (1) of Corollary 11.4 of the previous lecture) vector field on  $X//_Y G$  is  $v(\underline{F})$ .

Next, let  $x \in \mu^{-1}(\alpha)$ . We claim that  $v_x(F)$  is tangent to  $\mu^{-1}(\alpha)$ . This is equivalent to  $d_x \mu(v_x(F)) = 0$ . But  $\langle d_x \mu(v_x(F)), \xi \rangle = \omega_x(\xi_x, v_x(F)) = L_{\xi_x} F(x)$ . The latter is zero because the restriction of  $F$  to  $\mu^{-1}(Y)$  is  $G$ -invariant. Of course, the induced vector field on  $\mu^{-1}(\alpha)//G_\alpha$  is still  $v(\underline{F})$ .

Now we are in position to prove that the bracket induced by  $\underline{\omega}$  is the same as that of the reduction. This boils down to  $\iota_{v(\underline{F})} \underline{\omega} = d\underline{F}$ . Thanks to Step 4, the left hand side is the form on the reduction induced by  $\iota^*(\iota_{v(\underline{F})} \underline{\omega})$  as in (2) of Corollary 11.4. The right hand side is induced by  $\iota^*(dF)$ . Since  $\iota_{v(\underline{F})} \underline{\omega} = dF$ , we are done.

**12.2. CM system via reduction.** Let us return to the CM system. Let  $G = \mathrm{PGL}_n(\mathbb{C})$ . We have the Hamiltonian  $H = \frac{1}{2} \mathrm{tr}(Y^2)$  on  $R = T^* \mathrm{Mat}_n(\mathbb{C})$  and the induced Hamiltonian  $\underline{H}$  on  $C = \mu^{-1}(\alpha)//G_\alpha$ , where  $\alpha$  is the anti-unit matrix. The system we are interested in is obtained by restricting  $\underline{H}$  to the open subset  $C^{Reg} := T^*(\mathbb{C}^n)^{Reg}/\mathfrak{S}_n \hookrightarrow C$ . Thanks to the previous section, the trajectories for  $\underline{H}$  are obtained by projecting those for  $H$ .

**Exercise 12.1.** *The trajectories for  $H$  are of the form  $(X - tY, Y)$ .*

So what remains to prove is that the map  $C^{Reg} \rightarrow C$  induced by  $\iota : T^*(\mathbb{C}^n)^{Reg} \rightarrow \mu^{-1}(O)^{Reg}$  is an open inclusion of algebraic varieties that preserves the symplectic forms. First let us check the “open inclusion part”. The map  $\iota$  is a morphism and therefore so is the composition  $\pi_{G_\alpha} \circ \iota : T^*(\mathbb{C}^n)^{Reg} \rightarrow \mu^{-1}(O)^{Reg}//G$ , where  $\pi_{G_\alpha} : \mu^{-1}(\alpha) \rightarrow \mu^{-1}(\alpha)//G_\alpha$ . The latter is a

principal open subvariety of  $C$ . The composition is  $\mathfrak{S}_n$ -invariant by Exercise 11.3 and so descends to  $\underline{\iota} : C^{\text{Reg}} \rightarrow \mu^{-1}(O)^{\text{Reg}} // G$ . Again, by Exercise 11.3, this morphism is bijective. Now we can use a general fact that a bijective morphism into a smooth (or even normal) variety is an isomorphism. We will also check that our morphism is iso below.

Now let us show that our morphism is compatible with the symplectic forms. Let  $\omega_R$  be the symplectic form on  $R$ , it is given by  $\sum_{i,j=1}^n dx_{ij} \wedge dy_{ji}$ , where  $x_{ij}, y_{ij}$  are the matrix entries for  $X, Y$ . Let  $\underline{\omega}_R$  be the form on the reduction. Let  $\omega = \sum_{i=1}^n dx^i \wedge dy_i$  be the symplectic form on  $T^*(\mathbb{C}^n)^{\text{Reg}}$  and  $\underline{\omega}$  be the induced formula on  $C^{\text{Reg}}$  so that  $\omega = \pi_{\mathfrak{S}_n}^* \underline{\omega}$ . We need  $\iota^* \underline{\omega}_R = \underline{\omega}$ . First, we claim that  $\iota^* \omega_R = \omega$ . This is a direct computation that uses the explicit form of  $\iota$ :  $\iota^* \omega = \sum_{i=1}^n d\iota^*(x_{ij}) \wedge d\iota^*(y_{ij}) = \sum_{i=1}^n dx^i \wedge dy_i + \sum_{i \neq j} 0 \wedge d\frac{1}{x^j - x^i}$ . We remark that  $\iota^* \omega_R = \iota^* \pi_{G_\alpha}^* \underline{\omega}_R$  because the image of  $\iota$  lies in  $\mu^{-1}(\alpha)$ . So

$$\pi_{\mathfrak{S}_n}^* \underline{\omega} = \omega = \iota^* \omega_R = \iota^* \pi_{G_\alpha}^* \underline{\omega}_R = \pi_{\mathfrak{S}_n}^* \underline{\iota}^* \underline{\omega}_R,$$

the last equality follows from  $\pi_{G_\alpha} \circ \iota = \underline{\iota} \circ \pi_{\mathfrak{S}_n}$ . So we see that  $\pi_{\mathfrak{S}_n}^* \underline{\omega} = \pi_{\mathfrak{S}_n}^* (\underline{\iota}^* \underline{\omega}_R)$  and therefore  $\underline{\omega} = \underline{\iota}^* \underline{\omega}_R$ .

Actually the last equality shows that  $\underline{\iota}$  is étale: the kernel of  $d_p \underline{\iota}$  is forced to lie in the kernel of  $\underline{\iota}^* \underline{\omega}_R$  but the latter is non-degenerate. An étale bijective morphism of (smooth) varieties has to be an isomorphism.

**Problem 12.2.** *Prove (2) of the main theorem of the previous lecture.*

**12.3. Alternative realization of CM space.** We will need to realize  $C$  as a different Hamiltonian reduction. Namely, consider the quiver  $\underline{Q}$  with two vertices  $\infty$  and  $0$  and two arrows,  $a : 0 \rightarrow 0, e : \infty \rightarrow 0$ . Let  $Q$  be the double quiver. We consider the representation space  $\tilde{R}$  for  $Q$  with dimension vector  $n\epsilon_0 + \epsilon_\infty$ . Of course,  $\text{Rep}(\underline{Q}, v) = \text{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n$ , and  $\tilde{R} = T^* \text{Rep}(\underline{Q}, v) = \text{Mat}_n(\mathbb{C})^{\oplus 2} \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$ . We write an element of  $\tilde{R}$  as  $(X, Y, i, j)$  with  $i \in \mathbb{C}^n, j \in \mathbb{C}^{n*}$ . On  $\tilde{R}$  we have a natural action of  $\tilde{G} := \text{GL}_n(\mathbb{C})$  on  $\tilde{R}$  (while before we considered the action of  $\text{PGL}_n(\mathbb{C})$  on  $\text{Mat}_n(\mathbb{C})^{\oplus 2}$ ). The moment map  $\tilde{\mu} : \tilde{R} \rightarrow \text{Mat}_n(\mathbb{C})$  for the  $\tilde{G}$ -action on  $\tilde{R}$  is given by  $\tilde{\mu}(X, Y, i, j) = [X, Y] + ij$ .

**Proposition 12.1.** *The reduction  $\tilde{R} //_{-\tilde{E}} \tilde{G}$  is naturally identified with  $C = R // oG$ .*

*Proof.* We can consider the natural projection  $\rho : \tilde{R} \rightarrow R, \rho(X, Y, i, j) = (X, Y)$ . We have  $\tilde{\mu}(X, Y, i, j) = \mu \circ \rho(X, Y, i, j) + ij$ . So for  $(X, Y, i, j) \in \tilde{\mu}^{-1}(-E)$  we have  $\mu(\rho(X, Y, i, j)) + E = -ij$ . The trace of the left hand side is  $n$  and hence  $ij \neq 0$ . Clearly, any operator of rank 1 has the form  $-ij$ . It follows that  $\rho(\tilde{\mu}^{-1}(E)) \subset \mu^{-1}(O)$ . Moreover,  $ij = i'j' (\neq 0)$  if and only if  $i' = ti, j' = t^{-1}j$ , where  $t$  is a (uniquely determined) element of  $\mathbb{C}^\times$ . So the restriction of  $\rho$  to  $\tilde{\mu}^{-1}(-E)$  is a principal bundle over  $\mu^{-1}(O)$  for the group  $\mathbb{C}^\times$  (that acts as the center of  $\tilde{G}$ ). It follows that  $\mu^{-1}(O) = \tilde{\mu}^{-1}(-E) // \mathbb{C}^\times$ , this identification is  $G$ -equivariant. This leads to an identification of  $\tilde{\mu}^{-1}(-E) // \tilde{G}$  and  $\mu^{-1}(O) // G$ .  $\square$

**Problem 12.3.** *Check that the symplectic forms on  $\tilde{\mu}^{-1}(-E) // \tilde{G}$  and  $\mu^{-1}(O) // G$  are the same.*

A reason why one wants to consider  $\tilde{\mu} : R \rightarrow \text{Mat}_n(\mathbb{C})$  instead of  $\mu : R \rightarrow \mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C})$  is that the former map is flat and has reduced fibers. The latter is definitely not flat (fibers have different dimensions) and it is a big problem to determine whether the fibers are reduced (it is enough to check the reducedness of the zero fiber).

**12.4. Quantum Mechanics.** In Classical Mechanics, observables (i.e., physical quantities that can be measured on trajectories of our system) form a Poisson algebra. The equation of motion is the Hamilton equation,  $\dot{f} = \{H, f\}$ .

In the traditional formalism of Quantum Mechanics, observables are self-adjoint operators on Hilbert spaces, and the equation of motion is given by the Heisenberg equation  $\dot{F} = \frac{1}{\hbar}[H, F]$ , where the Hamiltonian  $H$  is also such an operator. Here  $\hbar$  is a normalized Plank constant, a purely imaginary number with very small absolute value.

Classical and quantum systems should correspond to each other: relatively large objects like insects or planets should obey classical laws, while quantum effects only appear for small objects, such as electrons. So one should be able to pass from a quantum system to classical (by taking the “quasi-classical” limit  $\hbar \rightarrow 0$ ) and vice versa (quantization). One of the problems with the traditional formalism of Quantum Mechanics is that it is very different from the classical set-up that makes even taking the quasi-classical limit a non-trivial procedure.

There is a different and significantly simplified approach to Quantum Mechanics based on the deformation theory: we need to choose the simplest formalism, where the Heisenberg equation still makes sense and where the quasi-classical limit is easy. Namely, for an algebra of observables we take a (flat) deformation  $A_\hbar$  of a commutative algebra  $A$  over  $\mathbb{C}[\hbar]$ , where we view  $\hbar$  as an independent variable. We also require  $A_\hbar$  to be separated with respect to the  $\hbar$ -adic topology (this definitely holds when  $A_\hbar$  is a free  $\mathbb{C}[\hbar]$ -module). Since  $A_\hbar$  is flat and commutative modulo  $\hbar$ , the expression  $\frac{1}{\hbar}[\tilde{a}, \tilde{b}]$  makes sense for all  $\tilde{a}, \tilde{b} \in A_\hbar$ . So we can consider the Heisenberg equation. Also, modulo  $\hbar$ , the expression  $\frac{1}{\hbar}[\tilde{a}, \tilde{b}]$  depends only on the classes of  $\tilde{a}, \tilde{b}$  modulo  $\hbar$ . This defines a bracket on  $A$  and the bracket is Poisson. So when we set  $\hbar = 0$ , the Heisenberg equation becomes the Hamilton equation.

A drawback of this approach is that it is unclear what a trajectory of a point should be (it’s also somewhat tricky in the original formalism). Still, the notion of a first integral makes sense and one can define a completely integrable system. More precisely, let  $A$  be an algebra of functions on an affine symplectic variety of dimension  $2n$  and let  $A_\hbar$  be a deformation of  $A$  as above that induces the bracket on  $A$  coming from the symplectic form. Pick a Hamiltonian  $H$ . By a completely integrable system we mean a collection  $H_1, \dots, H_n \in A_\hbar$  of pairwise commuting elements of  $A_\hbar$  including  $H$  such that the classes of  $H_1, \dots, H_n$  modulo  $\hbar$  are algebraically independent. The same definition works when the Poisson bracket on  $\text{Spec}(A)$  is non-degenerate generically.

Let us introduce a basic class of deformations that we will need in the sequel. First of all, we have seen that the Weyl algebra  $W(V)$  is a filtered deformation of  $S(V)$  (compatible with Poisson brackets). So its homogenized version  $W_\hbar(V) = T(V)[\hbar]/(u \otimes v - v \otimes u - \hbar\omega(u, v))$  is a deformation of  $S(V)$  in the above sense.

This example can be globalized as follows. Let  $X_0$  be a smooth affine algebraic variety. We consider the algebra  $D_\hbar(X_0)$  of homogenized differential operators that is the quotient of  $T(\mathbb{C}[X_0] \oplus \text{Vect}(X_0))[\hbar]$  by the relations

$$f \otimes g = fg, f \otimes \xi = f\xi, \xi \otimes f = f\xi + \hbar L_\xi f, \xi \otimes \eta - \eta \otimes \xi = \hbar[\xi, \eta], f, g \in \mathbb{C}[X_0], \xi, \eta \in \text{Vect}(X_0).$$

When we specialize  $\hbar = 1$  we get the usual algebra of differential operators  $D(X_0)$ . Also if we specialize  $\hbar = 0$ , we recover  $\mathbb{C}[T^*X_0]$ .

**Exercise 12.4.** Show that the algebra  $D_{\hbar}(X_0)$  is a deformation of  $\mathbb{C}[T^*X_0]$  compatible with the usual bracket there. Hint: how does the sheaf  $D_{\hbar}(X_0)$  on  $X_0$  behave under étale base changes?

In fact, one can filter the algebra  $D(X_0)$  by the order of a differential operator and then  $D_{\hbar}(X_0)$  becomes the Rees algebra of  $D(X_0)$ .

We remark that usually one imposes one more assumption on  $\mathcal{A}_{\hbar}$ : that it is complete in the  $\hbar$ -adic topology. This is to reflect the physical fact that  $\hbar$  is very small. We are not going to do this so far. If we have a deformation  $\mathcal{A}_{\hbar}$  satisfying our assumptions, then its  $\hbar$ -adic completion  $\mathcal{A}'_{\hbar} := \varprojlim_n \mathcal{A}_{\hbar}/\hbar^n \mathcal{A}_{\hbar}$  is still flat and separated and, in addition,  $\hbar$ -adically complete (the condition that  $\mathcal{A}_{\hbar}$  is separated precisely means that a natural homomorphism  $\mathcal{A}_{\hbar} \rightarrow \mathcal{A}'_{\hbar}$  is an embedding). In fact, in some cases one can recover  $\mathcal{A}_{\hbar}$  from  $\mathcal{A}'_{\hbar}$ .

**Exercise 12.5.** Let  $\mathcal{A}_{\hbar}$  be a  $\mathbb{Z}_{\geq 0}$ -graded  $\mathbb{C}[\hbar]$ -algebra with  $\hbar$  being of positive degree. Let  $\mathcal{A}'_{\hbar}$  be the  $\hbar$ -adic completion of  $\mathcal{A}_{\hbar}$ . Explain how to recover  $\mathcal{A}_{\hbar}$  back from  $\mathcal{A}'_{\hbar}$  using some natural action of  $\mathbb{C}^{\times}$  on  $\mathcal{A}'_{\hbar}$ .

**12.5. Quantum Calogero-Moser system.** We will consider a quantum CM system associated to an arbitrary real reflection group  $W$ . Let  $\mathfrak{h}$  be the complexification of the reflection representation of  $W$ , so  $\mathfrak{h}$  is an irreducible  $W$ -module that comes equipped with a  $W$ -invariant symmetric form  $(\cdot, \cdot)$ . Let  $S$  be the set of reflections in  $W$ . To each reflection  $s \in W$  we assign  $\alpha_s \in \mathfrak{h}^*$ , a nonzero vector with  $s\alpha_s = -\alpha_s$ . Further we pick a conjugation invariant function  $c_{\bullet} : S \rightarrow \mathbb{C}$ . To the function  $c$  we associate the following potential:

$$V = - \sum_{s \in S} \frac{c_s(c_s + \hbar)(\alpha_s, \alpha_s)}{\alpha_s^2}$$

This potential can be viewed as an element of  $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ , where  $\mathfrak{h}^{Reg} = \{x \in \mathfrak{h} | \langle \alpha_s, x \rangle \neq 0, \forall s \in S\}$ , this is precisely the locus, where  $W$  acts freely. If we consider  $W$  of type  $A$  and specialize  $\hbar = 0$ , then we get  $V = - \sum_{i \neq j} \frac{2c^2}{(x^i - x^j)^2}$ , because there is only one conjugacy class of reflections. So  $V$  is a usual CM potential.

A quantum mechanical system with potential  $V$  has Hamiltonian  $H = \Delta + V$  with  $\Delta := \sum_{i=1}^{\dim \mathfrak{h}} y_i^2$ , where vectors  $y_i$  form an orthonormal basis in  $\mathfrak{h}$ , is the Laplace operator. We view  $H$  as an element of  $D_{\hbar}(\mathfrak{h}^{Reg})$ , where  $\mathfrak{h}$  is viewed a subspace in  $\text{Vect}(\mathfrak{h}^{Reg})$  consisting of constant vector fields. This is a so called *Olshanetsky-Perelomov* Hamiltonian.

**Exercise 12.6.** Let  $X_0$  be a smooth affine variety acted freely by a finite group  $\Gamma$ . Equip  $D_{\hbar}(X_0)$  with a natural  $\Gamma$ -action by  $\mathbb{C}[\hbar]$ -algebra automorphisms and then identify  $D_{\hbar}(X_0)^{\Gamma}$  with  $D_{\hbar}(X_0/\Gamma)$ .

So we get a  $W$ -action on  $D_{\hbar}(\mathfrak{h}^{Reg})$  by algebra automorphisms. We remark that both  $\Delta$  and  $V$  are  $W$ -invariant and so  $H$  is  $W$ -invariant too. In the sequel, for  $\mathcal{A}_{\hbar}$  we will take  $D_{\hbar}(\mathfrak{h}^{Reg})^W$ , by the previous exercise, this is the same as  $D_{\hbar}(\mathfrak{h}^{Reg}/W)$ .