CENTRAL ELEMENTS OF THE COMPLETED UNIVERSAL ENVELOPING ALGEBRA (PART 1)

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ABSTRACT. These notes were prepared for the Spring 2024 graduate learning seminar at MIT on representations of affine Kac–Moody algebras at the critical level. We closely follow Chapters 3.1 in [Fre07]. Our primary goal is to relate the vertex algebra associated with an affine Kac–Moody algebra with its completed universal enveloping algebra. In this first section, we first discuss commutation relations for the Sugawara operators on the vacuum module for an affine Kac–Moody algebra. We aim not to focus on the computations, but rather the general framework surrounding these computations, and more importantly, how to recover information about the the universal enveloping algebra from these computations.

1. SUGAWARA OPERATORS

We begin by establishing some notational conventions. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, and let $\hat{\mathfrak{g}}_\kappa := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t)) \oplus \mathbb{C}K$ be the corresponding affine Kac–Moody algebra, where κ is a choice of a nontrivial invariant bilinear form on $\mathfrak{g}$ (equivalently, this corresponds to choosing a nontrivial two-cocycle on $\mathfrak{g}((t))$). We will be particularly interested in the critical level $\kappa = \kappa_c = -\frac{1}{2}\kappa_{\mathfrak{g}}$, where $\kappa_{\mathfrak{g}}$ is the Killing form on $\mathfrak{g}$. Then, we let $U_\kappa(\hat{\mathfrak{g}}) := U(\hat{\mathfrak{g}}_\kappa)/(K-1)$ denote the quotient of the universal enveloping algebra of $\hat{\mathfrak{g}}_\kappa$ by the ideal generated by $K-1$. Write $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ for its completion with respect to the descending filtration by the left ideals

$$J_N := U_\kappa(\hat{\mathfrak{g}}) \cdot \hat{\mathfrak{g}}(N),$$

where $\hat{\mathfrak{g}}(N) := \operatorname{Span}_{\mathbb{C}}\{x \otimes t^m \mid x \in \mathfrak{g},\ m \geq N\}$. For any $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we write $x(n) := x \otimes t^n$. Let $V_\kappa(\mathfrak{g})$ denote the vacuum $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-module. Recall that this module was defined as the induced module

$$U_\kappa(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)} \mathbb{C}_\kappa,$$

where $\mathbb{C}_\kappa$ is the $U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)$-module where $\mathfrak{g}[[t]]$ acts trivially and K acts by one.

We fix a nontrivial invariant bilinear form κ_0 on $\mathfrak{g}$. Then, after fixing some basis $x_1, \ldots, x_{\dim \mathfrak{g}}$ for $\mathfrak{g}$, we let $x^1, \ldots, x^{\dim \mathfrak{g}}$ denote the dual basis with respect to κ_0. We will write $X_i(z) = \sum_{n \in \mathbb{Z}} x_i(n) z^{-n-1}$ for all $i = 1, \ldots, \dim \mathfrak{g}$.

Definition 1.1. For each $n \in \mathbb{Z}$, we define the **Segal–Sugawara elements** (or more concisely, *Sugawara elements*)

$$S_n := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{m_1+m_2=n} {:}\, x_i(m_1) x^i(m_2) \,{:} \in \widetilde{U}_\kappa(\hat{\mathfrak{g}}).$$

Equivalently, these are the Fourier coefficients of the formal power series

$$\frac{1}{2} : X_i(z)X^i(z) : = \sum_{n\in\mathbb{Z}} S_n z^{-n-2}.$$

We refer to the images of the Sugawara elements in End $V_\kappa(\mathfrak{g})$ as the *Sugawara operators*, and we will reuse the notation S_n for their images as well. Similarly, we write $x_i(m)$ for the image of $x_i(m) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ in End $V_\kappa(\mathfrak{g})$ under the action map. Recall that our overarching goal is to study the center of $\tilde{U}_{\kappa_c}(\hat{\mathfrak{g}})$, and to do so, we will show that the Sugawara elements are indeed central at the critical level. We should think about the Sugawara elements as the infinite-dimensional analogue to the quadratic Casimir elements for finite-dimensional semisimple Lie algebras. In fact, in the case $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2$, these Sugawara elements actually generate the center, analogous to how the quadratic Casimir generates the center for $U(\mathfrak{sl}_2)$.

In this section, we will apply the vertex algebra formalism to compute commutation relations between the Sugawara *operators* and the endomorphisms $x_i(m)$ on $V_\kappa(\mathfrak{g})$. We want to emphasize that the vertex algebra formalism is not strictly necessary for performing these computations – for instance, in [KR87, Lecture 10], these relations are computed using completely elementary means. However, we should view the following computations as a basic blueprint to illustrate the general techniques underlying more difficult computations in the subsequent talks, when the vertex algebra formalism will become indispensable. Essentially, the advantage of working with the vertex algebra is due to its simpler structure – the completed universal enveloping algebra admits a filtration by degree, but the filtered pieces are infinite-dimensional and difficult to work with. On the other hand, the vertex algebra has a bona fide grading where each graded piece is finite-dimensional and thus, easier to understand. We will exploit this graded structure repeatedly in the following computations. Moreover, the completed universal enveloping algebra acts by endomorphisms on the vacuum module, so we may hope to translate our vertex algebra computations to results on $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

In what follows, we show that, at the critical level, the Sugawara operators are central in the image of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ under the action map. Moreover, we will see that away from the critical level, the Sugawara operators form a Lie subalgebra isomorphic to a quotient of the Virasoro algebra. This latter result will lead us to a brief digression on conformal vertex algebras. Unfortunately, the computations in this section will apply only in the endomorphism algebra of the vacuum module $V_\kappa(\mathfrak{g})$, and it is not obvious that they should also hold in the completed universal enveloping algebra. This will be the aim of the next section.

The following relation (see page 9 in Ilya's second set of notes) forms the cornerstone for all computations (read: exercises!) in this section. For any vertex algebra V and $A, B \in V$, we have

$$[A_m, B_k] = \sum_{n\geq 0} \binom{m}{n} (A_n B)_{m+k-n}, \tag{1}$$

where we use the notation $Y(C, z) = \sum_{\ell\in\mathbb{Z}} C_\ell z^{-\ell-1}$ for any $C \in V$.

1.1. **Commutation Relations for Sugawara Operators.**

Proposition 1.2. [Fre07, Chapter 3.1.1] *For any $n, m \in \mathbb{Z}$ and $i = 1, \ldots, \dim \mathfrak{g}$, the following relation holds in* $\operatorname{End} V_\kappa(\mathfrak{g})$*:*

$$[S_n, x_i(m)] = \frac{\kappa_c - \kappa}{\kappa_0} \cdot m x_i(m+n),$$

where κ_0 is a fixed (nontrivial) invariant bilinear form on $\mathfrak{g}$ with respect to which $\{x_i\}$ and $\{x^i\}$ form a dual basis. In particular, at the critical level $\kappa = \kappa_c$, the operators S_k are central in the image of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ in $\operatorname{End} V_\kappa(\mathfrak{g})$*.*

Proof. Consider the vector

$$\sigma := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1) x^i(-1) |0\rangle \in V_\kappa(\mathfrak{g}),$$

so the kth Sugawara operator S_n is the $(n+1)$st Fourier coefficient σ_{n+1} in the vertex operator

$$Y(\sigma, z) = \sum_{n \in \mathbb{Z}} \sigma_n z^{-n-1} = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} : X_i(z) X^i(z) : .$$

We will compute the commutation relations for the Fourier coefficients of $Y(\sigma, z)$ and the Fourier coefficients of

$$X_i(z) = Y(x_i(-1) |0\rangle, z) = \sum_{m \in \mathbb{Z}} x_i(m) z^{-m-1}.$$

Consider the following special case of (1):

$$[x_i(m), \sigma_n] = \sum_{\ell \geq 0} \binom{m}{\ell} (x_i(\ell)\sigma)_{m+n-\ell}$$

for any $m, n \in \mathbb{Z}$. Hence, it suffices to compute the Fourier coefficients of $Y(x_i(\ell)\sigma, z)$ for each $\ell \geq 0$. By definition,

$$x_i(\ell)\sigma = \frac{1}{2} \sum_{j=1} x_i(n) x_j(-1) x^j(-1) |0\rangle .$$

Observe that this vector is homogeneous of degree $n-2$, and since all nonzero homogeneous components of $V_\kappa(\mathfrak{g})$ have nonnegative degree, we have $x_i(\ell)\sigma = 0$ for all $\ell > 2$. Hence, it suffices to consider only three cases $\ell = 0, 1, 2$. (Here, we see a slight advantage of the vertex algebra formalism!) In the interest of time, the rest of the proof will be left as an exercise during the talk. For a more direct approach without the vertex algebra theory, refer to [KR87, Lecture 10].

For completeness, here are the results of the cases.

$$x_i(0)\sigma = 0, \quad x_i(1)\sigma = \frac{\kappa - \kappa_c}{\kappa_0} x_i(-1) |0\rangle , \quad x_i(2)\sigma = 0.$$

Let us adopt Einstein summation notation for the remainder of this proof (so indices that appear as an upper and lower index in the same term are summed over).

Case 0. Suppose $\ell = 0$. Using the Lie bracket on $\hat{\mathfrak{g}}_\kappa$ and $x_i(0)\,|0\rangle = 0$, we have

$$x_i(0)\sigma = \frac{1}{2}([x_i, x_j](-1)x^j(-1) + x_j(-1)[x_i, x^j](-1))\,|0\rangle\,.$$

If we write $[x_i, x_j] = d_j^k x_k$, then we see that

$$d_j^k = \kappa_0([x_i, x_j], x^k) = -\kappa_0(x_j, [x_i, x^k]),$$

so that $[x_i, x^j] = -d_k^j x^k$. Hence,

$$x_i(0)\sigma = \frac{1}{2}(d_j^k x_k(-1)x^j(-1) - d_k^j x_j(-1)x^k(-1))\,|0\rangle = 0.$$

Case 1. Now, take $\ell = 1$. In this case, the relations in $\hat{\mathfrak{g}}_\kappa$ give us

$$\begin{aligned}
x_i(1)\sigma &= \frac{1}{2}([x_i, x_j](0)x^j(-1)\,|0\rangle + x_j(-1)[x_i, x^j](0)\,|0\rangle) \\
&\quad + \frac{1}{2}(\kappa(x_i, x_j)x^j(-1)\,|0\rangle + \kappa(x_i, x^j)x_j(-1)\,|0\rangle) \\
&= \frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle + \frac{1}{2}(\kappa(x_i, x_j)x^j(-1)\,|0\rangle) + \frac{1}{2}\frac{\kappa}{\kappa_0}x_i(-1)\,|0\rangle\,, \\
&= \frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle + \frac{\kappa}{\kappa_0}x_i(-1)\,|0\rangle\,,
\end{aligned}$$

where we used the fact that $\kappa(x_i, x^j) = \delta_{ij}\frac{\kappa}{\kappa_0}$ for the second equality and $\kappa(x_i, x_j)x^j = \frac{\kappa}{\kappa_0}x_i$ for the last equality. Now, using $[x_i, x_j](0)\,|0\rangle = 0$, let us rewrite

$$\frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle = \frac{1}{2}[[x_i, x_j], x^j](-1)\,|0\rangle = \frac{1}{2}[x^j, [x_j, x_i]](-1)\,|0\rangle\,.$$

Note that $\mathfrak{g}$ acts on $\mathfrak{g}(-1)\,|0\rangle$ by the adjoint action, so the latter sum above is simply the action of the quadratic Casimir $\frac{1}{2}x^j x_j$ on $x_i(-1)\,|0\rangle$ via the adjoint action. The adjoint representation of $\mathfrak{g}$ is irreducible since $\mathfrak{g}$ is simple, so it follows that the action of the quadratic Casimir is by scalar multiplication by some $\lambda \in \mathbb{C}$, that is,

$$\frac{1}{2}[x_i, x_j](0)x^j(-1)\,|0\rangle = \lambda x_i(-1)\,|0\rangle\,.$$

Let $\kappa_\mathfrak{g}$ denote the Killing form on $\mathfrak{g}$. Then, the scalar by which $\frac{1}{2}x_j x^j$ acts on the adjoint representation is precisely

$$\lambda = \frac{1}{2\dim\mathfrak{g}}\operatorname{Tr}(\operatorname{ad}(x^j)\circ\operatorname{ad}(x_j)) = \frac{1}{2\dim\mathfrak{g}}\kappa_\mathfrak{g}(x^j, x_j) = \frac{1}{2}\frac{\kappa_g}{\kappa_0}.$$

Thus, we conclude that

$$x_i(1)\sigma = \left(\frac{(1/2)\kappa_\mathfrak{g} + \kappa}{\kappa_0}\right)x_i(-1)\,|0\rangle = \left(\frac{\kappa - \kappa_c}{\kappa_0}\right)x_i(-1)\,|0\rangle\,.$$

Case 2. Finally, take $\ell = 2$. In this case, the relations in $\hat{\mathfrak{g}}_\kappa$ give us

$$x_i(2)\sigma = \frac{1}{2}([x_i, x_j](1)x^j(-1)\,|0\rangle + x_j(-1)[x_i, x^j](1)\,|0\rangle)$$
$$= \frac{1}{2}[x_i, x_j](1)x^j(-1)\,|0\rangle = -\frac{1}{2}\kappa(x_i, [x_j, x^j])\,|0\rangle\,.$$

If we assume without loss of generality that $\{x_j\}$ is orthonormal with respect to κ_0, then $x^j = x_j$ for all $j = 1, \ldots, \dim \mathfrak{g}$ and hence the right-hand side is equal to zero. Note that we can make such an assumption since the Sugawara elements S_k do not depend on the choice of basis, and hence, the left-hand side of the equation above is also independent of the choice of basis (we can extend x_i to an orthonormal basis). Thus, the result is zero.

Finishing Touches. To conclude, the cases above give us

$$[x_i(m), \sigma_n] = \sum_{\ell \geq 0} \binom{m}{\ell} (x_i(\ell)\sigma)_{m+n-\ell} = \left(\frac{\kappa - \kappa_c}{\kappa_0}\right) m x_i(m + n - 1).$$

Using the fact that $\sigma_{n+1} = S_n$, we conclude

$$[S_n, x_i(m)] = \left(\frac{\kappa_c - \kappa}{\kappa_0}\right) m x_i(m + n).$$ □

Away from the critical level, let us define the *normalized Sugawara elements*

$$L_n := \frac{\kappa_0}{\kappa - \kappa_c} S_n.$$

Observe that Proposition 1.2 can be rephrased as

$$[L_n, x_i(m)] = -m x_i(m + n). \tag{2}$$

that is, the adjoint action of L_n on $V_\kappa(\mathfrak{g})$ is the action of the derivation $-t^{n+1}\partial_t$. Recall that these derivations form a topological basis for the Witt algebra $\mathbb{C}((t))\partial_t$, so a natural question is whether the Sugawara operators form a Lie subalgebra of $\operatorname{End} V_\kappa(\mathfrak{g})$ isomorphic to the Witt algebra. As it turns out, we will show that the commutation relations between the normalized Sugawara operators are the same as the commutation relations not of the Witt algebra, but rather a quotient of its central extension, the Virasoro algebra.

Proposition 1.3. [Fre07, Chapter 3.1.2] *For any $n, m \in \mathbb{Z}$, the following relation holds in* $\operatorname{End} V_\kappa(\mathfrak{g})$*:*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c_\kappa,$$

where $c_\kappa = \dim \mathfrak{g} \cdot \frac{\kappa}{\kappa - \kappa_c}$.

Proof. Write $\bar{\sigma} = \frac{\kappa_0}{\kappa - \kappa_c}\sigma$, so that $L_n = \bar{\sigma}_{n+1}$. In this case, we need to compute the OPE

$$Y(\bar{\sigma}, z)Y(\bar{\sigma}, w) \sim \sum_{n \geq 0} \frac{Y(\bar{\sigma}_n \cdot \bar{\sigma}, w)}{(z - w)^{n+1}}.$$

Note $\deg \bar{\sigma}_n = \deg S_{n-1} = 1-n$, so $\deg \bar{\sigma}_n \cdot \bar{\sigma} = 3-n$. Hence, $\bar{\sigma}_n \cdot \bar{\sigma} = 0$ for all $n \geq 4$. We consider the following four cases.

Case 0. First consider $n = 0$. Observe that $\bar{\sigma}_0 \left|0\right\rangle = 0$, so it follows that

$$\bar{\sigma}_0 \cdot \bar{\sigma} = \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([\bar{\sigma}_0, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_0, x^i(-1)]\right)\left|0\right\rangle$$

From (2), we see that the adjoint action of $\bar{\sigma}_0$ is the same as the action of the derivation $-\partial_t = T$, so it follows that

$$\bar{\sigma}_0 \cdot \bar{\sigma} = \frac{\kappa_0}{2(\kappa - \kappa_c)} - \partial_t(x_i(-1)x^i(-1))\left|0\right\rangle = T\bar{\sigma}.$$

Case 1. Now suppose $n = 1$. In this case, (2) tells us that $[\bar{\sigma}_1, -]$ acts as the grading operator, so

$$\begin{aligned}\bar{\sigma}_1 \cdot \bar{\sigma} &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([\bar{\sigma}_1, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_1, x^i(-1)]\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(-1)x^i(-1) + x_i(-1)x^i(-1)\right)\left|0\right\rangle = 2\bar{\sigma}.\end{aligned}$$

Case 2. Now suppose $n = 2$. In this case, Equation (2) tells us that $[\bar{\sigma}_2, x(m)] = -mx(m+1)$ for any $x \in \mathfrak{g}$, so it follows that

$$\begin{aligned}\bar{\sigma}_2 \cdot \bar{\sigma} &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([\bar{\sigma}_2, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_2, x^i(-1)]\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(0)x^i(-1) + x_i(-1)x^i(0)\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(0)x^i(-1)\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}[x_i, x^i](-1)\left|0\right\rangle.\end{aligned}$$

If we choose $\{x_i\}$ orthonormal with respect to κ_0 (note that this choice does not affect $\bar{\sigma}$), then we see that the right-hand side vanishes, so $\bar{\sigma}_2 \cdot \bar{\sigma} = 0$.

Case 3. Finally, suppose $n = 3$. In this case, Equation (2) tells us that $[\bar{\sigma}_3, x(m)] = -mx(m+2)$ for any $x \in \mathfrak{g}$, so it follows that

$$\begin{aligned}\bar{\sigma}_3 \cdot \bar{\sigma} &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([\bar{\sigma}_3, x_i(-1)]x^i(-1) + x_i(-1)[\bar{\sigma}_3, x^i(-1)]\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(1)x^i(-1) + x_i(-1)x^i(1)\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left(x_i(1)x^i(-1)\right)\left|0\right\rangle \\ &= \frac{\kappa_0}{2(\kappa - \kappa_c)}\left([x_i, x^i](-1) + \kappa(x_i, x^i)\right)\left|0\right\rangle.\end{aligned}$$

Once again picking $\{x_i\}$ orthonormal with respect to κ_0, we deduce that

$$\bar{\sigma}_3 \cdot \bar{\sigma} = \frac{\kappa_0}{2(\kappa - \kappa_c)} \kappa(x_i, x^i) \left|0\right\rangle = \frac{\kappa \cdot \dim \mathfrak{g}}{2(\kappa - \kappa_c)} \left|0\right\rangle .$$

Hence, our OPE is

$$Y(\sigma, z) Y(\sigma, w) \sim \frac{\partial_w Y(\sigma, w)}{(z-w)} + \frac{2Y(\sigma, w)}{(z-w)^2} + \frac{c_\kappa/2}{(z-w)^4},$$

where $c_\kappa = \frac{\kappa \cdot \dim \mathfrak{g}}{\kappa - \kappa_c}$. The desired commutation relations follow from this OPE. □

It follows that the Sugawara operators span a Lie subalgebra of End $V_\kappa(\mathfrak{g})$ isomorphic to a quotient of the Virasoro algebra Vir with central charge given by c_κ. The fact that the elements $\{L_n\}_{n \in \mathbb{Z}}$ are indeed linearly independent in End $V_\kappa(\mathfrak{g})$ is an exercise.

Let us now make a brief digression on the Virasoro algebra. The Witt algebra has a unique (up to scaling) nontrivial 2-cycle, and the corresponding 1-dimensional central extension is the Virasoro algebra

$$\mathrm{Vir} = W \oplus \mathbb{C}C,$$

with bracket relations such that C is central and

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} C$$

for any $m, n \in \mathbb{Z}$. Similar to the vacuum module in the affine Kac–Moody case, we can construct a vertex algebra associated to the Virasoro algebra. Fix $c \in \mathbb{C}$. Then, let $\mathbb{C}_c$ be the one-dimensional representation of $\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C \subset \mathrm{Vir}$ where $\mathbb{C}[[t]]\partial_t$ acts by zero and C acts by c. Define the Virasoro vacuum module as the induced representation

$$\mathrm{Vir}_c := U(\mathrm{Vir}) \otimes_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C} \mathbb{C}_c.$$

We can give Vir_c a natural vertex algebra structure in a way analogous to the vacuum module $V_\kappa(\mathfrak{g})$. Observe that, by the Poincaré–Birkhoff–Witt theorem, Vir_c has a basis given by

$$L_{i_1} L_{i_2} \cdots L_{i_n} \left|0\right\rangle,$$

where $\left|0\right\rangle = 1 \otimes 1$ and $i_1 \leq i_2 \leq \cdots \leq i_n < -1$ are integers. Let us define the field

$$L(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

Note that $L(z)$ satisfies essentially the same OPE as the renormalized Sugawara field $Y(\bar{\sigma}, z) \in \mathrm{End}\, V_\kappa(\mathfrak{g})((z))$ since the renormalized Sugawara operators L_n satisfy the same mutual commutation relations as the Fourier coefficients L_n. That is,

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}.$$

Hence, we see that $L(z)$ is mutually local with itself, and after setting $T = -\partial_t$, we can apply the reconstruction theorem (see page 12 in Ilya's first set of notes) to deduce the following result.

Proposition 1.4. [Fre07, Theorem 3.1.1] *There exists a unique vertex algebra structure on* Vir_c *such that the vacuum vector is given by* $|0\rangle$ *and* $Y(L_{-2}|0\rangle, z) = L(z)$. *More precisely, the state-field correspondence is given by*

$$Y(L_{-1-i_1}\cdots L_{-1-i_n}|0\rangle, z) = \frac{1}{(i_1-1)!\cdots(i_n-1)!} :\partial_z^{i_1}L(z)\cdots\partial_z^{i_n}L(z):$$

for any $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$.

1.2. **Conformal Vertex Algebras.** We will now introduce a structure on vertex algebras that both the Virasoro vertex algebra and the Kac–Moody vertex algebra possess. Namely, observe that the Kac–Moody vertex algebra (away from the critical level) contains a vector $\bar{\sigma}$ such that the Fourier coefficients of $Y(\bar{\sigma}, z)$ behave like the Fourier coefficients of the field $L(z)$. In other words, these vertex algebras admit an internal action of the Virasoro algebra, a symmetry that recurs in many interesting vertex algebras. The Virasoro algebra acts by infinitesimal generators of the conformal transformations of the Riemann sphere, so these vertex algebras are given the name "conformal." We axiomatize this structure with the following definition.

Definition 1.5. A **conformal vertex algebra** of central charge $c \in \mathbb{C}$ is a $\mathbb{Z}$-graded vertex algebra V containing a degree two vector $\nu \in V$, called a *conformal vector*, such that

(i) the Fourier coefficients L_n^ν in
$$Y(\nu, z) = \sum_{n\in\mathbb{Z}} L_n^\nu z^{-n-2}$$
satisfy the defining relations of the Virasoro algebra with central charge c,
(ii) the translation operator T must coincide with the coefficient L_{-1}^ν,
(iii) the coefficient L_0^ν is the grading operator, that is, it acts by scalar multiplication by n on the degree n graded component of V.

Note that the first condition is equivalent to $Y(\nu, z)$ satisfying the OPE

$$Y(\nu,z)Y(\nu,w) \sim \frac{c/2}{(z-w)^4} + \frac{2Y(\nu,w)}{(z-w)^2} + \frac{\partial_w Y(\nu,w)}{(z-w)}.$$

Observe that every conformal vertex algebra of central charge c can is a representation of Vir with central charge c. The vacuum module $V_\kappa(\mathfrak{g})$ is conformal of central charge c_κ when κ is non-critical, and the conformal vector is given by $\bar{\sigma}$. Similarly, the Virasoro vacuum module Vir_c is tautologically conformal of central charge c.

Perhaps unsurprisingly, Vir_c plays an important role among conformal vertex algebras of central charge c. More precisely, we will soon show that any conformal vertex algebra of central charge c admits a unique homomorphism from Vir_c. This homomorphism is not only a Vir-module homomorphism but also a *vertex algebra homomorphism*.

Definition 1.6. A **homomorphism of vertex algebras** is a linear map $\varphi : V_1 \to V_2$ with

(i) $\varphi(|0\rangle_1) = |0\rangle_2$,
(ii) $\varphi \circ T_1 = T_2 \circ \varphi$, and
(iii) for any $A, B \in V_1$, we have $\varphi(Y_1(A,z)B) = Y_2(\varphi(A), z)\varphi(B)$.

Before we proceed to the following proposition, we make a few simple but crucial observations regarding any conformal vertex algebra V. From $Y(\nu,z)\left|0\right\rangle = \nu$, we can deduce that $\nu = L^{\nu}_{-2}\left|0\right\rangle$. In fact, applying the strong reconstruction theorem to the subspace spanned by the vectors of the form $L^{\nu}_{-1-i_1}\cdots L^{\nu}_{-1-i_n}\left|0\right\rangle$ (for $i_1 \geq \cdots \geq i_n \geq 0$) with the field $Y(\nu,z)$ allows us to deduce that

$$Y(L^{\nu}_{-1-i_1}\cdots L^{\nu}_{-1-i_n}\left|0\right\rangle,z) = \frac{1}{(i_1-1)!\cdots(i_n-1)!} :\partial_z^{i_1}Y(\nu,z)\cdots\partial_z^{i_n}Y(\nu,z): \tag{3}$$

for any $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$.

Proposition 1.7. [Fre07, Lemma 3.1.2] *If V is a conformal vertex algebra of central charge c with conformal vector ν, then there exists a unique homomorphism of vertex algebras $\mathrm{Vir}_c \to V$ sending $L_{-2}\left|0\right\rangle$ to ν. This linear map is also a homomorphism of* Vir*-modules.*

Proof. Observe that such a vertex algebra homomorphism must be a homomorphism of Vir-modules. Indeed, given any $A \in \mathrm{Vir}_c$, the relation $\varphi(L_jA) = L^{\nu}_j\varphi(A)$ (for any $j \in \mathbb{Z}$) immediately follows from comparing coefficients on both sides of

$$\varphi(Y(L_{-2}\left|0\right\rangle,z)A) = Y(\nu,z)\varphi(A).$$

Since $\left|0\right\rangle$ generates Vir_c as a Vir-module, it follows that such a vertex algebra homomorphism is uniquely determined after we know the corresponding action of Vir on V. This action is determined by the image of $L_{-2}\left|0\right\rangle$, so it follows that φ is uniquely determined by the image of $L_{-2}\left|0\right\rangle$. It remains to prove that such a homomorphism exists. We simply consider the map $\mathrm{Vir}_c \to V$ given by

$$L_{i_1}L_{i_2}\cdots L_{i_n}\left|0\right\rangle \mapsto L^{\nu}_{i_1}L^{\nu}_{i_2}\cdots L^{\nu}_{i_n}\left|0\right\rangle,$$

which is easily seen to be a vertex algebra homomorphism thanks to (3). □

We conclude this section with a sufficient condition for a vector in a $\mathbb{Z}_{\geq 0}$-graded vertex algebra to be conformal.

Proposition 1.8. [Fre07, Lemma 3.1.2] *A $\mathbb{Z}_{\geq 0}$-graded vertex algebra V is conformal with central charge c if and only if there exists a (nonzero) degree two vector $\nu \in V$ such that the coefficients L^{ν}_n in $Y(\nu,z) = \sum_{n\in\mathbb{Z}} L^{\nu}_n z^{-n-2}$ satisfy the following conditions:*

(1) $L^{\nu}_{-1} = T$,
(2) L^{ν}_0 *acts as the grading operator,*
(3) $L^{\nu}_2 \cdot \nu = (c/2)\left|0\right\rangle$.

Proof. For the forward direction, conditions (1) and (2) follow by definition. For condition (3), we recall that $\nu = L^{\nu}_{-2}\left|0\right\rangle$. It follows that

$$L^{\nu}_2\nu = [L^{\nu}_2, L^{\nu}_{-2}]\left|0\right\rangle = (L^{\nu}_0 + \frac{c}{2})\left|0\right\rangle = \frac{c}{2}\left|0\right\rangle,$$

as needed.

Conversely, suppose V contains a nonzero $\nu \in V_2$ such that the Fourier coefficients of $Y(\nu,z)$ satisfy the conditions above. We need to show that the coefficients L^{ν}_n satisfy the defining relations of the Virasoro algebra. By the OPE formalism from before, it suffices to show that $Y(v,z)$ satisfies the following OPE:

$$Y(v,z)Y(v,w) \sim \frac{c/2}{(z-w)^4} + \frac{2Y(v,w)}{(z-w)^2} + \frac{\partial_w Y(v,w)}{(z-w)}.$$

By definition,

$$Y(v,z)Y(v,w) \sim \sum_{n\geq 0} \frac{Y(L^v_{n-1}v,w)}{(z-w)^{n+1}}.$$

Note that $L^v_{n-1}v$ has degree $(1-n)+2 = 3-n$. Since the degrees of V are nonnegative, it suffices to consider only the cases $0 \leq n \leq 3$. In the case $n=0$, we have $L^v_{-1}v = Tv$, it follows that $Y(L^v_{-1}v,w) = \partial_w Y(v,w)$. Similarly, we are given $L^v_0 v = \deg v \cdot v = 2v$, and $L^v_2 v = (c/2)\,|0\rangle$. Hence, it follows that

$$Y(v,z)Y(v,w) \sim \frac{c/2}{(z-w)^4} + \frac{\alpha(w)}{(z-w)^3} + \frac{2Y(v,w)}{(z-w)^2} + \frac{\partial_w Y(v,w)}{(z-w)}, \tag{4}$$

where $\alpha(w) := Y(L_1 v, w)$. It remains to show that $\alpha(w) = 0$. By swapping the roles of z and w, we obtain

$$Y(v,w)Y(v,z) \sim \frac{c/2}{(z-w)^4} - \frac{\alpha(z)}{(w-z)^3} + \frac{2Y(v,z)}{(z-w)^2} - \frac{\partial_w Y(v,w)}{(z-w)}. \tag{5}$$

By locality, the right-hand sides of (4) and (5) must be equal to each other. If we perform a Taylor series expansion on the right-hand side of (5) in terms of w and take only the singular terms (at $z=w$), we have

$$\frac{c/2}{(z-w)^4} - \frac{\alpha(w) + \partial_w\alpha(w)(z-w) + \partial_w^2\alpha(w)(z-w)^2}{(z-w)^3} + O((z-w)^{-2}).$$

In particular, the coefficient of $(z-w)^{-3}$ when the right-hand side of (5) is expanded in terms of w is $-\alpha(w)$. It follows that $\alpha(w) = 0$, completing the proof. □

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