Lecture 16.

1) Products & coproducts
2) Tensor products

Ref: [HS], II.5, [AM], Section 2.7.

1) Products & coproducts.

1.1) Definition of product.

Product of functors: $F_1, F_2 : \mathcal{C} \longrightarrow Sets$. Can take direct products of sets $\rightsquigarrow F_1 \times F_2 : \mathcal{C} \longrightarrow Sets$

for $Y \in Ob(\mathcal{C}) \rightsquigarrow F_1 \times F_2(Y) = F_1(Y) \times F_2(Y)$

for $Y \xrightarrow{f} Y' \rightsquigarrow F_1 \times F_2(f) = F_1(f) \times F_2(f) : F_1(Y) \times F_2(Y) \to F_1(Y') \times F_2(Y')$.

Definition: Let $X_1, X_2 \in Ob(\mathcal{C})$. By their product we mean an object, denoted $X_1 \times X_2$, representing the functor $\mathcal{F}_{X_1}^{opp} \times \mathcal{F}_{X_2}^{opp}$ (i.e. $Y \mapsto Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2)$ on objects).

Rem: $X_1 \times X_2$ may fail to exist, we'll see an example below.

Exercise: 1) For any $F_1, F_2 : \mathcal{C} \to Sets$, have $F_1 \times F_2 \xRightarrow{\sim} F_2 \times F_1$.
2) For $X_1, X_2 \in Ob(\mathcal{C})$, $X_2 \times X_1$ exists iff $X_1 \times X_2$ exists, and $X_2 \times X_1 \xrightarrow{\sim} X_1 \times X_2$.

Alternative definition (via universal property): Let $X_1, X_2 \in Ob(\mathcal{C})$. By the product of X_1 & X_2 we mean (X, π_1, π_2) w. $X \in Ob(\mathcal{C})$, $X \xrightarrow[\pi_i]{} X_i$ $(i = 1, 2)$ s.t. $\forall\ Y \in Ob(\mathcal{C})$, $Y \xrightarrow[\psi_i]{} X_i$

$\exists!\ Y \xrightarrow[\psi]{} X$ s.t. the following diagram is commutative

$$\begin{tikzcd}
 & Y \arrow[dl, bend right, "\psi_1"'] \arrow[d, dashed, "\psi"'] \arrow[dr, bend left, "\psi_2"] & \\
X_1 & X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & X_2
\end{tikzcd}$$

i.e. $\psi \mapsto (\pi_1 \circ \psi, \pi_2 \circ \psi) : \mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X_1) \times \mathrm{Hom}_{\mathcal{C}}(Y, X_2)$.

Lemma: The two definitions are equivalent.

Proof: An object constructed in Altern. def'n satisfies Definition:

Let (X, π_1, π_2) be as above. We need a functor isomorphism

$\eta : \mathrm{Hom}_{\mathcal{C}}(\cdot, X) \xRightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(\cdot, X_1) \times \mathrm{Hom}_{\mathcal{C}}(\cdot, X_2)$

• Construct $\eta_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X_1) \times \mathrm{Hom}_{\mathcal{C}}(Y, X_2)$

$$\psi \longmapsto (\pi_1 \circ \psi, \pi_2 \circ \psi)$$

η_Y is a bijection b/c of the universal property.

• Check η is a functor morphism, i.e. $\forall\ Y' \xrightarrow{f} Y$

the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y, X) \arrow[r, "\eta_Y : \psi \mapsto (\pi_1 \circ \psi, \pi_2 \circ \psi)"] \arrow[d, "\psi \mapsto \psi \circ f"] & \mathrm{Hom}_{\mathcal{C}}(Y, X_1) \times \mathrm{Hom}_{\mathcal{C}}(Y, X_2) \arrow[d, "(\psi_1, \psi_2) \mapsto (\psi_1 \circ f, \psi_2 \circ f)"] \\
\mathrm{Hom}_{\mathcal{C}}(Y', X) \arrow[r, "\psi' \mapsto (\pi_1 \circ \psi', \pi_2 \circ \psi')"] & \mathrm{Hom}_{\mathcal{C}}(Y', X_1) \times \mathrm{Hom}_{\mathcal{C}}(Y', X_2)
\end{tikzcd}$$

– exercise.

Now: Definition $\rightsquigarrow$ Alternative definition:

Note $F_1 \times F_2$ admits a functor morphism $\pi^i : F_1 \times F_2 \Rightarrow F_i$

π^i_Y is the projection $F_1(Y) \times F_2(Y) \to F_i(Y)$ (exercise). Now we have

$\eta : Hom_{\mathcal{C}}(\cdot, X_1 \times X_2) \xRightarrow{\sim} Hom_{\mathcal{C}}(\cdot, X_1) \times Hom_{\mathcal{C}}(\cdot, X_2) \rightsquigarrow$

$\pi^i \circ \eta : Hom_{\mathcal{C}}(\cdot, X_1 \times X_2) \Rightarrow Hom_{\mathcal{C}}(\cdot, X_i)$. By Yoneda (for $\mathcal{C}^{opp}$)

$\pi^i \circ \eta = \eta^{\pi^i}$ for the unique $\pi_i \in Hom_{\mathcal{C}}(X_1 \times X_2, X_i)$ $(i=1,2)$ so that $(\pi^i \circ \eta)_Y(\psi) = \pi_i \circ \psi$, $\forall \psi \in Hom_{\mathcal{C}}(Y, X_1 \times X_2)$. Since η is a functor isomorphism $\Rightarrow$

$\eta_Y = ((\pi^1 \circ \eta)_Y, (\pi^2 \circ \eta)_Y) : Hom_{\mathcal{C}}(Y, X_1 \times X_2) \to Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2)$

is a bijection

$$\begin{array}{ccc} Hom_{\mathcal{C}}(Y, X_1 \times X_2) & \xrightarrow{\sim} & Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2). \\ \psi & \longmapsto & (\pi_1 \circ \psi, \pi_2 \circ \psi) \ (= \eta_Y(\psi)) \end{array}$$

So we recover the universal property. $\square$

1.2) Coproduct := product in $\mathcal{C}^{opp}$. Notation: $X_1 * X_2$

Universal property: have $X_i \xrightarrow{\iota_i} X_1 * X_2$ s.t. $\forall Y \in Ob(\mathcal{C})$, $X_i \xrightarrow{\varphi_i} Y$, $i=1,2$ $\exists! \ X_1 * X_2 \xrightarrow{\varphi} Y$ s.t. the following is comm've:

$$\begin{tikzcd} & Y & \\ & X \arrow[u, dashed, "\varphi"] & \\ X_1 \arrow[ur, "\iota_1"'] \arrow[uur, bend left, "\varphi_1"] & & X_2 \arrow[ul, "\iota_2"] \arrow[uul, bend right, "\varphi_2"'] \end{tikzcd}$$

Equivalently, $X_1 * X_2$ represents $Hom_{\mathcal{C}}(X_1, \cdot) \times Hom_{\mathcal{C}}(X_2, \cdot)$.

1.3) Examples:

Products in Sets, Groups, Rings, A-Mod etc.:
product = direct product (exercise – on universal property).

Coproducts:
In Sets, coproduct = disjoint union.
In A-Mod, coproduct = direct sum (see Prob 6 in HW1)
In A-CommAlg: tensor product of algebras, to be covered later.

1.4) Remarks.

1) we can talk about (co)products indexed by an arbitrary set, I (rather than $\{1,2\}$), for example, direct sums & products in A-Mod.

2) products may fail to exist: Let $\mathcal{C}$ be the full subcat. in $\mathbb{F}$-Vect w. objects being odd-dimensional vector spaces. For any two objects in $\mathcal{C}$, there's no product (in $\mathcal{C}$).

2) Tensor products of modules

2.1) Bilinear maps:

Let A be a comm've ring, M_1, M_2, N be A-modules $\rightsquigarrow$ set
$Bilin_A(M_1 \times M_2, N) = \{A\text{-bilinear maps } M_1 \times M_2 \to N\}$.

Digression: why should we care about bilinear maps – b/c they are everywhere!

1) Linear algebra: for an $\mathbb{F}$-vector space V can talk about bilinear forms := bilinear maps $V \times V \to \mathbb{F}$, fundamentally important in linear algebra & beyond.

2) if M is an A-module $\Rightarrow$ mult'n map $A \times M \to M$ is A-bilinear.

3) the composition map $Hom_A(L,M) \times Hom_A(M,N) \to Hom_A(L,N)$, where L, M, N are A-modules is A-bilinear, Problem 7 in HW1.

4) If B is an A-algebra, then the product map $B \times B \to B$ is A-bilinear.

Observation: $F_{M_1,M_2} := Bilin_A(M_1 \times M_2, \bullet)$ is actually a functor $A\text{-Mod} \longrightarrow Sets$:

To $\psi \in Hom_A(N,N')$ we assign

$$\begin{array}{rcl} F_{M_1,M_2}(\psi): Bilin_A(M_1 \times M_2, N) & \longrightarrow & Bilin_A(M_1 \times M_2, N') \\ \beta & \longmapsto & \psi \circ \beta \end{array}$$

Exercise: Show $\psi \circ \beta$ is A-bilinear & F_{M_1,M_2} is indeed a functor $A\text{-Mod} \to Sets$.

2.2) Definition of tensor product:

Definition: By the tensor product $M_1 \otimes_A M_2$ we mean a representing object for $Bilin_A(M_1 \times M_2, \bullet)$ i.e. want a functor isomorphism $Hom_A(M_1 \otimes_A M_2, \bullet) \xRightarrow{\sim} Bilin_A(M_1 \times M_2, \bullet)$

Equivalently (compare to products, exercise) we can define tensor products via a universal property:

==tensor product== of M_1 & M_2 is an A-module $M_1 \otimes_A M_2$ w. a bilinear map $M_1 \times M_2 \longrightarrow M_1 \otimes_A M_2$, $(m_1, m_2) \longmapsto m_1 \otimes m_2$ w. the following universal property:

$\forall$ A-module N & A-bilinear map $\beta : M_1 \times M_2 \longrightarrow N$ $\exists!$ A-linear map $\tilde{\beta} : M_1 \otimes_A M_2 \longrightarrow N$ s.t. $\beta(m_1, m_2) = \tilde{\beta}(m_1 \otimes m_2)$

$$\begin{tikzcd}
M_1 \times M_2 \arrow[d, "{(m_1, m_2) \mapsto m_1 \otimes m_2}"'] \arrow[dr, "\beta"] & \\
M_1 \otimes_A M_2 \arrow[r, dashed, "\tilde{\beta}"] & N
\end{tikzcd}$$

Rem: Under isomorphism $\mathrm{Bilin}_A(M_1 \times M_2, M_1 \otimes_A M_2) \xrightarrow{\sim} \mathrm{Hom}_A(M_1 \otimes_A M_2, M_1 \otimes_A M_2)$, the map $(m_1, m_2) \mapsto m_1 \otimes m_2$ corresponds to the identity on the r.h.s.

In the next lecture we'll see that tensor product always exists.