

Williamson

Lecture 1

\mathfrak{g} -complex sl/simple Lie alg

\mathfrak{B}_B -Borel

\mathfrak{H} -Cartan

\mathcal{O} -category $\mathcal{O} \subset \mathfrak{g}\text{-mod}$

$\mathcal{O}_{\text{prime block}}$

$\mathfrak{h} \cap W$ -Weyl gr-p

S -simple refl-ns

L_x -simple module w.h.wt. $x \circ \alpha$

Δ_x -Verma

P_x -proj-ve cover

Kazhdan-Lusztig conj $[L_x : L_y] = p_{xy}(1)$, p_{xy} -KL polyn-l

(Kostul-dual formuln)

\mathbb{D}_x -dual Verma $\Rightarrow [\mathbb{D}_x : L_y] = [\Delta_x : L_y]$

$\dim \text{Hom}(P_y, \mathbb{D}_x) = (P_y : \Delta_x)$ - mult. of Δ_x in Verma filt-n of P_y

Under $\mathbb{Z}W \xrightarrow{\sim} [\mathcal{O}]$, $w \mapsto [\Delta_w]$

KL conj: $C'_y(1) \leftrightarrow [P_y]$, where C'_y is KL polynomial

KL conj. has 2 proofs: • via Beilinson-Bernstein localization:

$U(g)/(Z^+)$ -mod $\xleftarrow[\Gamma(\cdot)]{\sim} D_{G/B}$ -mod

$D_{G/B}$ -mod_{reg, loc} $\xleftrightarrow{\sim} \text{Rep}(G/B)$

↑ Riemann Hilbert

↓ KL polyn-s

- Soergel's approach

Soergel functor: $\mathcal{O} \xrightarrow{\mathbb{V}} \text{Mod-} C$

$\mathbb{V} = \text{Hom}(P_{w_0}, \cdot)$, $\text{End}(P_{w_0}) = S(\mathfrak{g}) / (S(\mathfrak{g})_+^W) = H^*(G^V/B^V) =: C$

"coinvariant algebra"

$\text{Proj} \subset \mathcal{O}$ - full additive subcat. of proj-ve objects

\mathbb{V} is fully faithful on Proj.
We can describe combinatorially image of Proj. combinatorially using Soergel modules

\mathcal{O}_\circ or translation functors $\mathcal{U}_S, S \in S$ (aka wall-crossing functors)

Basic facts: 1) \mathcal{U}_S are exact $[\Delta_x \mathcal{U}_S] = [\Delta_{x_S}] + [\Delta_x]$

so under $\mathbb{Z}W \xrightarrow{\sim} [\mathcal{O}_\circ]$

$$\begin{matrix} \mathcal{O}_\circ \\ \hookrightarrow \\ 1+S \end{matrix} \quad \begin{matrix} \mathcal{O}_\circ \\ \hookrightarrow \\ \mathcal{U}_S \end{matrix}$$

2) If $x = st\dots u$ is reduced expression, then

$$\Delta_{st\dots u} \mathcal{U}_x = P_x \bigoplus_x P_y^{\oplus m_y}$$

$$3) \mathbb{V}(M \mathcal{U}_S) \xrightarrow{\sim} \mathbb{V}(M) \otimes_{C^S} C, \quad \mathbb{V}(\Delta_x) = \mathbb{I} \text{ (b/c } \text{soc } \Delta_x = L_{w_0})$$

$$\text{Gnd-n: } \mathbb{V}(\Delta_{st\dots u} \mathcal{U}_x) = \mathbb{I} \otimes_{C^S} C \otimes_{C^t} C \otimes_{C^u} C = \mathbb{V}(P_x) \bigoplus_y \mathbb{V}(P_y)^{\oplus m_y} \quad y < x$$

$$\Rightarrow \text{can characterize } \mathbb{V}(P_x) = \mathbb{D}_x$$

as a unique up to iso summand in $\mathbb{I} \otimes_{C^S} C \otimes_{C^t} C \otimes_{C^u} C$

which ~~isn't~~ doesn't appear in smaller expressions

Def: full additive subcategory in $\text{Mod-}C$ generated by summands of $\mathbb{I} \otimes_{C^S} C \otimes_{C^t} C \otimes_{C^u} C$ are Soergel modules

Not-n: SMod

$$\text{So } \text{Proj}_\circ \xrightarrow{\sim} \text{SMod}$$

~~KL-conj~~ $[\text{Proj}_\circ] = [\text{SMod}]$

$$\mathbb{Z}W = [\mathcal{O}_\circ]$$

$$\text{KL-conj: } C_x(1) \leftrightarrow [\mathbb{D}_x] \quad (*)$$

To conclude the proof, there are 2 possibilities

1: Soergel: identify \mathbb{D}_x w. $IH^*(\overline{B^v \times B^v} / B^v)$ & apply decomposition theorem

2 (Elias-Williamson): an "algebraic Hodge theory" for \mathbb{D}_x to deduce equality $(*)$

Problem: in related situations it's not clear how to define \mathbb{V}

Example: Rep G , where G is reductive alg-c grp / $\bar{\mathbb{F}}_p$

Def: \mathcal{H} (= Soergel bimodules, a.k.a Hecke category) = $\langle B_s \mid s \in S \rangle$
 $[R = S(\mathbb{H})]$

$\int \otimes_{\mathbb{C}^{\text{mod}}}, \text{Kar}$
 $R\text{-6mod gr}$

$B_s := R \otimes_{R^S} R(1)$ (gen-r in deg-1) under taking tensor prod-s,
grading shifts, direct sums & summands

So indecomposable Soergel bimodules are indec. summands of

$B_{\underline{s}} := B_s B_{s_1} \dots B_{s_n}$ (where we omit " \otimes_R ")

Rmk: Krull-Schmidt theorem holds in $S\text{Mod}$ and \mathcal{H}

Trivial Lem: $M \in C\text{-mod} \Rightarrow M \otimes_{C^S} C = MB_S$

Since $\text{Proj}_0 \xrightarrow{\sim} S\text{Mod}$, we deduce Proj_0 is a right \mathcal{H} -module cat-y

w. B_s acting
by v_s .
Observation: Suppose \mathcal{O}_0 or Proj_0 has structure of a right \mathcal{H} -module
cat-y st. the morphisms $\text{id} \rightarrow B_s, B_s \rightarrow \text{id}, B_s \rightarrow B_s B_s, B_s B_s \rightarrow B_s$
are induced by adj-s (more in Lecture 2), then we have an equivalence
of \mathcal{H} -modules ~~$\mathcal{O}_0 \rightleftarrows \text{Proj}_0$~~ $\mathcal{O}_0 \rightleftarrows \text{Proj}_0 \rightleftarrows C \otimes_R \mathcal{H}$

Rmk: It seems hard to produce \mathcal{H} -action directly, but using KLR alg-s &
Brendan's results this can be done for $q\mathcal{O}_n^*(\mathbb{C})$

Part 2: \mathcal{H} by generators & relations (w. B Elias)

(W, S) is Coxeter gr-p, $W \backslash \mathbb{H}$ is refl-n rep-n
 $\alpha \in \mathbb{H}$ - coroot, $\alpha_s \in \mathbb{H}^*$ - root

\mathbb{H}/K , K is a field ($\cong \mathbb{Z}$)

Def: $(I \subset S)$ is finitary if $W_I = \langle I \rangle$ is finite

Recall: \mathcal{H} (Hecke algebra) / $\mathbb{Z}[v^{\pm 1}]$

have gen-ns $h_s, s \in S$, & rel-ns

$$h_s^2 = (v^{-1} - v) h_s + 1 \quad (\text{rk } 1 \text{ rel-n})$$

$$\underbrace{h_s h_{s_1} \dots}_{m_{st}} = \underbrace{h_{s_1} h_{s_2} \dots}_{m_{st}}, m_{st} = \text{ord}(st), \text{ for } t, s \in S \text{ is finitary}$$

H_{BS} = full subcat. of $R\text{-mod}_g$ w. objects $B_x = B_5 B_{\frac{x}{5}} \dots B_n$ for all expressions x . and morphism spaces graded by degree.

$\mathcal{H} \simeq \text{Kar}(H_{BS})$ "additive graded Karoubi envelope."

Will describe H_{BS} by generators & relations

Consider monoidal cat. D_{BS} gen'd as \otimes -cat. by $s \in S$ meaning that objects are sequences x :

1111 tstus

Morphisms: linear combns of colored decorated planar graphs w. following local generators

$$\text{rank 1} \quad \left\{ \begin{array}{c} \bullet \\ s \end{array}, \begin{array}{c} ! \\ t \end{array}, \begin{array}{c} \lambda \\ t \end{array}, \begin{array}{c} Y \\ t \end{array} \in \text{Hom}(t, tt) \right.$$

$$\text{rank 0} \quad \boxed{\square} \quad f \in R = S(\mathbb{Y}^*)$$

$$\text{rank 2} \quad \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \diagup \diagdown \diagup \diagdown \end{array} \quad \text{"2m}_{st} \text{ valent vertex for } \{s, t\} \subset S \text{ binary}$$

Monoidal cat.: w.r.t. horizontal (\otimes) and vertical (composition)

Properties: concatenations.

Relations: $\boxed{1} = \phi$, $\boxed{f} \boxed{g} = \boxed{fg}$: rk 0

rk 1: $\begin{array}{c} | \\ t \end{array}$ is Frobenius object $\begin{array}{c} \lambda \\ \circ \end{array} = 1$, $\lambda = \lambda$ etc

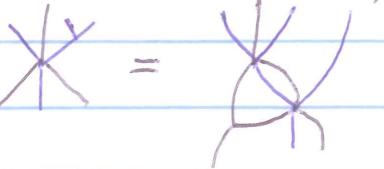
any diagram w. 1 color only depends on thickening up to iso
and

$$Q = 0 \text{ (where } \cap = \lambda \text{)}$$

and: Polynomial sliding $\boxed{f} = \boxed{sf} + \boxed{\partial sf}$

$\mathfrak{J}_s f$ -divided difference operator, $\mathfrak{J}_s f = \frac{f - sf}{ds}$

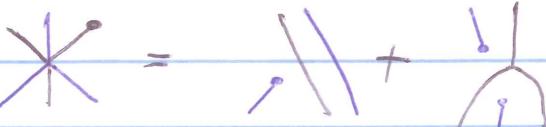
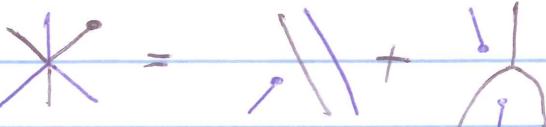
Rank 2: "2 color associativity"

e.g.  $=$  $(m_{st}=3)$

 $=$  $(m_{st}=2)$

Jones-Wenzl

 $=$ 

 $=$  $+ \text{dot}$ $m_{st}=3$

Rank 3: Zamolodchikov relns

In Soergel bimodules:

$D_{BS} \rightarrow \mathcal{H}_{BS}$

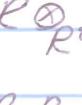
$\mathbb{P} \mapsto [B_t \rightarrow R(1), f \otimes g \mapsto fg \quad (\deg 1)]$

$!$ $\mapsto [R(-1) \rightarrow B_t : 1 \mapsto \frac{1}{2}(g \otimes 1 + 1 \otimes g)]$

$\mathbb{E} \mapsto [R \xrightarrow{f} R \quad (\deg = \deg f)]$

$\lambda \mapsto [B_s^2 \rightarrow B_s(-1) \quad (f \otimes g \otimes h \mapsto f \circ_s g \otimes h)]$

$Y \mapsto [B_s \rightarrow B_s^2(-1) \quad (f \otimes g \mapsto f \otimes 1 \otimes g)]$

 $[$  $\xrightarrow{B_{st}s} \mathcal{S} = R \otimes_{R \otimes B_s B_t} R(3)$ 

Thm (Elias-Khovanov W = S_n , Libedinsky - Elias for dihedral groups,
Elias-Williamson, in gen'l)

$$D_{BS} \xrightarrow{\sim} H_{BS}, \text{ hence } \text{Ker}(D_{BS}) \xrightarrow{\sim} H \quad (\text{in char 0})$$

or if $\tilde{\gamma}$ is refl-n faithful repn
& char $\neq 2$ | $\tilde{\gamma}$ faithful & refl-n in $\tilde{\gamma}$ =
abstr. refl-n.)