

INTRODUCTION TO TYPE A CATEGORICAL KAC-MOODY ACTIONS, I.

JOSÉ SIMENTAL

In these notes, we give a brief introduction to the theory of categorical actions of type A Kac-Moody algebras, as introduced in [CR] (in the case of \mathfrak{sl}_2) and [Rou] (a more general case). Intuitively speaking, these should be data of an exact category \mathcal{C} and a collection of exact endofunctors E_i, F_i of \mathcal{C} that descend to an action of a Kac-Moody algebra \mathfrak{g} on $[\mathcal{C}] := K_0(\mathcal{C}) \otimes \mathbb{C}$. This is, however, too general to give an interesting theory. The actual definition is much subtler and will be given next week.

In Section 1 we give a brief reminder of the definition of a Kac-Moody algebra $\mathfrak{g}(I)$ associated to a graph I . Before giving the definition of a categorical $\mathfrak{g}(I)$ -action this week we will provide, Section 2, an example of one via cyclotomic Hecke algebras (= Ariki-Koike algebras) that already appeared in Siddharth's talk. Next week we will give the definition of a categorical $\mathfrak{g}(I)$ -action, explain how to categorify divided powers of the Chevalley generators e_i, f_i , and how to categorify the action of a simple reflection in the Weyl group.

1. KAC-MOODY ALGEBRAS.

We (very briefly) recall the definition of a Kac-Moody algebra associated to a simply laced graph I . For a more detailed account see, for example, [Et, K].

So let I be a simply laced graph. Denote by $V(I)$ the set of vertices of I . The *Cartan matrix* of I , $C(I)$, is the $|V(I)| \times |V(I)|$ matrix $(a_{ij})_{i,j \in V(I)}$ defined by $a_{ii} = 2$ for all i ; $a_{ij} = -1$ if there is an edge $i - j$ in I ; and $a_{ij} = 0$ if there is no edge between i and j in I .

Definition 1.1. Let I be a simply laced graph. The Kac-Moody algebra $\mathfrak{g}(I)$ is the Lie algebra with generators e_i, f_i, h_i for $i \in V(I)$, known as the Chevalley generators of $\mathfrak{g}(I)$, and relations

$$(KM1) [h_i, e_j] = a_{ij}e_j. [h_i, f_j] = -a_{ij}f_j.$$

$$(KM2) [e_i, f_j] = \delta_{ij}h_j.$$

$$(KM3) \text{ If } i \neq j, \text{ ad}_{e_i}^{(1-a_{ij})} e_j = 0; \text{ ad}_{f_i}^{(1-a_{ij})} f_j = 0.$$

The relations (KM3) are known as the Serre relations.

In these notes, we will always assume that I is one of the following graphs.

I	$\mathfrak{g}(I)$
$\circ - \circ - \cdots - \circ$	\mathfrak{sl}_{n+1} (n vertices)
$\circ \swarrow \circ - \cdots - \searrow \circ$	$\widehat{\mathfrak{sl}}_n$ (n vertices)
$\cdots - \circ - \circ - \cdots$	\mathfrak{gl}_∞

2. CYCLOTOMIC HECKE ALGEBRAS.

2.1. Reminders. Our first example of a categorical action will be given by the representation theory of cyclotomic Hecke algebras. Here we recall their definition and a few facts from Siddharth's talk.

Definition 2.1. Let \mathbb{F} be a commutative domain, and $q \in \mathbb{F}^\times$. The affine Hecke algebra $\mathcal{H}_{\mathbb{F},q}^{\text{aff}}(n)$ is the unital associative \mathbb{F} -algebra generated by elements $T_1, \dots, T_{n-1}, X_1^\pm, \dots, X_n^\pm$ subject to the following relations:

- The subalgebra generated by T_1, \dots, T_{n-1} is isomorphic to the finite Hecke algebra of type A.
- The subalgebra generated by X_1^\pm, \dots, X_n^\pm is isomorphic to the algebra $\mathbb{F}[X_1^\pm, \dots, X_n^\pm]$ of Laurent polynomials in the variables X_i .
- $T_i X_j = X_j T_i$ if $i \neq j, j-1$; $T_i X_i T_i = q X_{i+1}$.

Now choose $q_1, \dots, q_m \in \mathbb{F}^\times$. The cyclotomic Hecke algebra (or Ariki-Koike algebra) $H_{\mathbb{F},q_1, \dots, q_m}(n)$ is the quotient of $\mathcal{H}_{\mathbb{F},q}^{\text{aff}}(n)$ by the extra relation:

$$\prod_{j=1}^m (X_1 - q_j) = 0.$$

We remark that to pass from our definition of the cyclotomic Hecke algebra $H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ to the one given in [Ven, Section 3], we just set $T_0 := \pi(X_1)$, where $\pi : \mathcal{H}_{\mathbb{F},q}(n) \rightarrow H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ is the canonical projection. We define the *Jucys-Murphy* elements $L_1, \dots, L_n \in H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ by

$$L_i := \pi(X_i).$$

Note that an explicit formula for L_i is $L_i = q^{1-i} T_{i-1} \cdots T_1 T_0 T_1 \cdots T_{i-1}$. It follows from [Ven, Section 9], that every symmetric polynomial in the variables L_1, \dots, L_n belongs to the center of $H_{\mathbb{F},q,q_1,\dots,q_m}(n)$.

Siddharth has also constructed all the irreducible representations of $H_{\mathbb{F},q,q_1,\dots,q_m}(n)$ in the case where \mathbb{F} is a field and q, q_1, \dots, q_m are generic. Let us remark that, here, 'generic' means that q is not a root of unity and $q_i/q_j \notin \{q^k : k \in \mathbb{Z}\}$ for every $i, j = 1, \dots, m$. We do not need an explicit construction of these representations. We just remark that they are indexed by m -multipartitions of n , say V_λ is the irreducible representation corresponding to $\lambda \vdash_m n$. Each V_λ has as a basis $\{v_t\}$, where t runs over the set of all standard Young tableaux of shape λ . The action of the JM elements L_i on V_λ is given by

$$(1) \quad L_i v_t = q^{a-b} q_j v_t,$$

where the box with the number i appears in t in column a and row b of $\lambda^{(j)}$.

2.2. Induction and Restriction functors. Note that we have a natural embedding $\iota : H(n-1) \rightarrow H(n)$, where we denote $H(k) := H_{\mathbb{F},q,q_1,\dots,q_m}(k)$. Recall from [Ven, Section 4] that $H(n)$ is free of rank $m^n n!$ over \mathbb{F} , with basis $X_n := \{L_1^{c_1} \cdots L_n^{c_n} T_w : w \in \mathfrak{S}_n, 0 \leq c_i \leq m-1\}$. This has the following easy consequence:

Proposition 2.2. *The algebra $H(n)$ is free as a left $H(n-1)$ -module.*

Proof. By the definition of L_n and the relations on the affine Hecke algebra, it is clear that L_n commutes with $T_w, w \in \mathfrak{S}_{n-1}$. Thus, we have that

$$H(n) = \bigoplus_{0 \leq c_m < m} \bigoplus_{w \in \mathfrak{S}_n / \mathfrak{S}_{n-1}} H(n-1) L_m^{c_m} T_w,$$

where $w \in \mathfrak{S}_n / \mathfrak{S}_{n-1}$ runs over the coset representatives of \mathfrak{S}_{n-1} in \mathfrak{S}_n of minimal length. \square

The inclusion $\iota : H(n-1) \rightarrow H(n)$ also allows us to define induction and restriction functors for cyclotomic Hecke algebras. Namely, we define:

$$\begin{aligned} \text{Res}_n^{n+1} : H(n+1)\text{-mod} &\rightarrow H(n)\text{-mod} & \text{Ind}_n^{n+1} : H(n)\text{-mod} &\rightarrow H(n+1)\text{-mod}. \\ M &\mapsto \iota_n^*(M) & M &\mapsto H(n+1) \otimes_{H(n)} M \end{aligned}$$

By Proposition 2.2, both Res_n^{n+1} and Ind_n^{n+1} are exact functors. Moreover, it is clear that $(\text{Ind}_n^{n+1}, \text{Res}_n^{n+1})$ is an adjoint pair of functors. We also have the coinduction functor:

$$\begin{aligned} \text{CoInd}_n^{n+1} : H(n)\text{-mod} &\rightarrow H(n+1)\text{-mod} \\ M &\mapsto \text{Hom}_{H(n)}(H(n+1), M). \end{aligned}$$

This functor is right adjoint to Res_n^{n+1} .

Proposition 2.3. *There is an isomorphism of functors $\text{Ind}_n^{n+1} \cong \text{CoInd}_n^{n+1}$.*

Proof. Let $M \in H(n)\text{-mod}$. Since $H(n+1)$ is a free left $H(n)$ -module, there is a natural isomorphism $\text{Ind}_n^{n+1}(M) = \text{Hom}_{H(n)}(H(n+1), M) \xrightarrow{\cong} \text{Hom}_{H(n)}(H(n+1), H(n)) \otimes_{H(n)} M$. Thus, we only need to show that the $H(n+1)$ - $H(n)$ -bimodule $\text{Hom}_{H(n)}(H(n+1), H(n))$ is isomorphic (as a bimodule) to $H(n+1)$. This follows because $H(n+1), H(n)$ are symmetric algebras. Indeed, we have:

$$\begin{aligned} \text{Hom}_{H(n)}(H(n+1), H(n)) &\xrightarrow{\cong} \text{Hom}_{H(n)}(H(n+1), H(n)^*) \\ &\xrightarrow{\cong} \text{Hom}_{\mathbb{F}}(H(n) \otimes_{H(n)} H(n+1), \mathbb{F}) \\ &\xrightarrow{\cong} \text{Hom}_{\mathbb{F}}(H(n+1), \mathbb{F}) \\ &\xrightarrow{\cong} H(n+1). \end{aligned}$$

\square

2.3. Action on K_0 . Since Res and Ind are exact functors, they descend to maps

$$[\text{Res}_n^{n+1}] : [H(n+1)\text{-mod}] \longleftrightarrow [H(n)\text{-mod}] : [\text{Ind}_n^{n+1}].$$

In the generic case, an explicit formula for the maps $[\text{Res}]$ and $[\text{Ind}]$ is easy to find. Recall that $[H(n)\text{-mod}]$ has basis $\{[V_\lambda] : \lambda \vdash_m n\}$. By the construction of the representations V_λ , we have that:

$$(2) \quad [\text{Res}_{n-1}^n][V_\lambda] = \sum_{x \in \text{rem}(\lambda)} [V_{\lambda-\{x\}}],$$

where $\text{rem}(\lambda)$ denotes the set of removable boxes of λ . By adjunction and Frobenius reciprocity, we have that

$$(3) \quad [\text{Ind}_{n-1}^n][V_\mu] = \sum_{x \in \text{add}(\mu)} [V_{\mu \cup \{x\}}],$$

where $\text{add}(\mu)$ denotes the set of addable boxes of μ . We would like to have similar formulas for $[\text{Ind}]$ and $[\text{Res}]$ in the general case, not just the generic one.

To do this, we introduce a specialization map. Consider $\mathbb{F}[\mathbf{t}]_{(\mathbf{t}-1)}$, the localization of $\mathbb{F}[\mathbf{t}]$ at the ideal $(\mathbf{t}-1)$, and its completion, $\mathbb{S} := \mathbb{F}[\mathbf{t}]_{(\mathbf{t}-1)}^\wedge$. Let \mathbb{K} be the fraction field of \mathbb{S} . Note that \mathbb{S} is a complete discrete valuation ring, with residue field \mathbb{F} . Let $\mathbf{q} := q\mathbf{t}^m$, and, for $i = 1, \dots, n$, $\mathbf{q}_i := q_i \mathbf{t}^{i-1}$. We will consider the cyclotomic Hecke algebras $H_{\mathbb{S}} := H_{\mathbb{S}, \mathbf{q}, \dots, \mathbf{q}_m}$ and $H_{\mathbb{K}} := H_{\mathbb{K}, \mathbf{q}, \dots, \mathbf{q}_m}$. We have that $H_{\mathbb{F}} = H_{\mathbb{S}} \otimes_{\mathbb{S}} \mathbb{F}$, and $H_{\mathbb{K}} = H_{\mathbb{S}} \otimes_{\mathbb{S}} \mathbb{K}$.

By our choice of parameters, the algebra $H_{\mathbb{K}}$ is semisimple. Indeed, it is clear that \mathbf{q} is not a root of unity and that $\mathbf{q}_i/\mathbf{q}_j$ is not a power of \mathbf{q} . So we have the simple modules $V_\lambda^{\mathbb{K}}$. We introduce the specialization map $d : K_0(H_{\mathbb{K}}\text{-mod}) \rightarrow K_0(H_{\mathbb{F}}\text{-mod})$ as follows. Let $M \in H_{\mathbb{K}}\text{-mod}$. Pick an \mathbb{S} -lattice in M of maximal rank, say L . So L is a $H_{\mathbb{S}}$ -submodule of M with $M = L \otimes_{\mathbb{S}} \mathbb{K}$. The class $[L \otimes_{\mathbb{S}} \mathbb{F}] \in K_0(H_{\mathbb{F}})$ depends only on $[M]$, not on the choice of representative of this class, nor on the choice of a lattice L . This gives us our specialization map.

We claim that $d : [H_{\mathbb{K}}\text{-mod}] \rightarrow [H_{\mathbb{F}}\text{-mod}]$ is surjective. To see this, we check that its dual map d^* is injective. This is where we use that we are working with Grothendieck groups with coefficients in a field, rather than just a domain. Recall that, if A is a finite dimensional \mathbb{F} -algebra, then the dual to $[A\text{-mod}]$ is $[A\text{-proj}]$: a pairing is given by $(M, P) \mapsto \dim_{\mathbb{F}} \text{Hom}_A(P, M)$. So we have to check that $d^* : [H_{\mathbb{F}}\text{-proj}] \rightarrow [H_{\mathbb{K}}\text{-proj}]$ is injective. In other words, we have to check that any projective $H_{\mathbb{F}}$ -module has a unique, up to isomorphism, deformation to a projective $H_{\mathbb{K}}$ -module. Existence of this deformation is an easy consequence of Hensel's lemma. Uniqueness follows from the fact that $\text{Ext}_{H_{\mathbb{F}}}^1(P, P) = 0$ for any projective $H_{\mathbb{F}}$ -module. So d is surjective. We define

$$[V_\lambda] := d[V_\lambda^{\mathbb{K}}].$$

Then, $\{[V_\lambda] : \lambda \vdash_m n\}$ generates $[H_{\mathbb{F}}\text{-mod}]$. We remark that this is not, in general, a basis. But we can give the action of $[\text{Res}]$, $[\text{Ind}]$ on $[V_\lambda]$. Indeed, it is an easy consequence of the definitions that the diagrams

$$\begin{array}{ccc} [H_{\mathbb{K}}(n)\text{-mod}] & \xrightarrow{d} & [H_{\mathbb{F}}(n)\text{-mod}] \\ \downarrow [\text{Res}_{n-1}^n] & & \downarrow [\text{Res}_{n-1}^n] \\ [H_{\mathbb{K}}(n-1)\text{-mod}] & \xrightarrow{d} & [H_{\mathbb{F}}(n-1)\text{-mod}] \end{array} \quad \begin{array}{ccc} [H_{\mathbb{K}}(n)\text{-mod}] & \xrightarrow{d} & [H_{\mathbb{F}}(n)\text{-mod}] \\ \uparrow [\text{Ind}_{n-1}^n] & & \uparrow [\text{Ind}_{n-1}^n] \\ [H_{\mathbb{K}}(n-1)\text{-mod}] & \xrightarrow{d} & [H_{\mathbb{F}}(n-1)\text{-mod}] \end{array}$$

commute. It follows that formulas (2), (3) are valid in general, not just in the generic case.

Remark 2.4. We would like to make some comments about the elements $[V_\lambda] \in [H(n)\text{-mod}]$. The algebra $H(n)$ has the structure of a cellular algebra, cf. [GL]. This means that there is a basis of $H(n)$ that satisfies some upper triangularity conditions. This basis is indexed by pairs of standard tableaux of the same shape λ , where λ is an m -multipartition of n . It follows from the theory of cellular algebras that $H(n)$ comes equipped with a set of representations C_λ , $\lambda \vdash_m n$, called cell modules. For each λ , the representation C_λ has a natural bilinear form, φ^λ , whose radical is an $H(n)$ -submodule. This bilinear form may be zero. The set $\{D_\lambda := C_\lambda / \text{rad } \varphi^\lambda : \varphi^\lambda \neq 0\}$ forms a complete list of irreducible $H(n)$ -modules. Moreover, D_λ is the unique irreducible quotient of C_λ and, if D_μ appears as a composition factor of C_λ , then $\mu \trianglerighteq \lambda$ under the dominance ordering. We have that $[V_\lambda] = [C_\lambda]$ and the transition matrix from the basis $\{C_\lambda\}$ to the basis $\{D_\lambda\}$ is upper unitriangular. This is another way to see that the specialization map is surjective. We can also look at cell modules from the point of view of rational Cherednik algebras. Let \mathbb{H} be a cyclotomic rational Cherednik algebra whose category \mathcal{O} maps to $H(n)\text{-mod}$ under the KZ functor. Then, we have that $C_\lambda = \text{KZ}(\Delta(\lambda))$,

where $\Delta(\lambda)$ is the Verma module for \mathbb{H} , see [CGG]. The upper unitriangularity of the transition matrix also follows from here.

2.4. i -Induction and i -Restriction. We will need refined versions of the induction and restriction functors. From now on, we assume that \mathbb{F} is an algebraically closed field. Let $L_n \in H_{\mathbb{F}}(n)$ be the n -th JM element. It is a direct consequence of the definition that L_n centralizes the subalgebra $H_{\mathbb{F}}(n-1)$. So, for every $H_{\mathbb{F}}(n)$ -module M , we can think of L_n as a $H_{\mathbb{F}}(n-1)$ -endomorphism of $\text{Res}_{n-1}^n(M)$. For $a \in \mathbb{F}$, let $(\text{Res}_{n-1}^n)_a(M)$ be the a -generalized eigenspace for the action of L_n on M . This construction is functorial and we have $\text{Res}_{n-1}^n = \bigoplus_{a \in \mathbb{F}} (\text{Res}_{n-1}^n)_a$. Clearly, $(\text{Res}_{n-1}^n)_a$ is an exact functor, so it induces a map on K_0 . By adjointness, we have a decomposition $\text{Ind}_{n-1}^n = \bigoplus_{a \in \mathbb{F}} (\text{Ind}_{n-1}^n)_a$ such that, for each $a \in \mathbb{F}$, the functor $(\text{Res}_{n-1}^n)_a$ is left adjoint to $(\text{Ind}_{n-1}^n)_a$.

Proposition 2.5. *For every $a \in \mathbb{F}$, the functors $(\text{Res}_{n-1}^n)_a, (\text{Ind}_{n-1}^n)_a$ are biadjoint.*

Proof. For every k , denote $\mathcal{L}_k = L_1 + \cdots + L_k \in H(k)$. Since this is a symmetric polynomial in the JM elements, it actually belongs to the center of $H(k)$. So we have a decomposition

$$H(k)\text{-mod} = \bigoplus_{b \in \mathbb{F}} (H(k)\text{-mod})_b,$$

where $(H(k)\text{-mod})_b$ consists of those modules on which \mathcal{L}_k acts with generalized eigenvalue b . Now, let $M \in (H(n)\text{-mod})_b$. Then, since $\mathcal{L}_n = \mathcal{L}_{n-1} + L_n$, we have that $(\text{Res}_{n-1}^n)_a(M)$ is the projection of the $H(n-1)$ -module $\text{Res}_{n-1}^n(M)$ to $(H(n-1)\text{-mod})_{b-a}$. Since $((\text{Res}_{n-1}^n)_a, (\text{Ind}_{n-1}^n)_a)$ is an adjoint pair, it follows that for $N \in (H(n-1)\text{-mod})_b$, $(\text{Ind}_{n-1}^n)_a(N)$ is the projection of $\text{Ind}_{n-1}^n(M)$ to $(H(n)\text{-mod})_{b+a}$. The result now follows since $\text{Res}_{n-1}^n, \text{Ind}_{n-1}^n$ are biadjoint. \square

We can give the action of $(\text{Res}_{n-1}^n)_a, (\text{Ind}_{n-1}^n)_a$ in the generic case. This follows from (1). To express this, we introduce some notation. Let $\lambda \vdash_m n$ be an m -multipartition. Assume that the box \square is column a , row b of $\lambda^{(i)}$. We define the *content* of \square to be:

$$\text{cont}(\square) := q^{a-b}q_i$$

Then, we have the following identity:

$$(4) \quad (\text{Res}_{n-1}^n)_a(V_{\lambda}) = \bigoplus_{\substack{x \in \text{rem}(\lambda) \\ \text{cont}(x)=a}} V_{\lambda-\{x\}}.$$

And, by Frobenius reciprocity, we get

$$(\text{Ind}_{n-1}^n)_a(V_{\mu}) = \bigoplus_{\substack{x \in \text{add}(\mu) \\ \text{cont}(x)=a}} V_{\mu \cup \{x\}}.$$

In the general case, we can only get similar formulas at the level of the Grothendieck group. We again use specialization maps. Recall the notation $\mathbb{S}, \mathbb{K}, \mathbf{q}, \mathbf{q}_1, \dots, \mathbf{q}_m$ from Subsection 2.3. We remark that all eigenvalues of L_n on an $H_{\mathbb{K}}(n)$ -module are, actually, in \mathbb{S} . This follows from semisimplicity of $H_{\mathbb{K}}(n)$ and (1). For an element $\mathbf{a} \in \mathbb{S}$, we denote by a its projection to \mathbb{F} .

Proposition 2.6. *The following diagram commutes:*

$$(5) \quad \begin{array}{ccc} [H_{\mathbb{K}}(n)\text{-mod}] & \xrightarrow{d} & [H_{\mathbb{F}}(n)]\text{-mod} \\ \sum_{\mathbf{a} \mapsto a} [\text{Res}_{n-1}^n]_{\mathbf{a}} \downarrow & & \downarrow [\text{Res}_{n-1}^n]_a \\ [H_{\mathbb{K}}(n-1)\text{-mod}] & \xrightarrow{d} & [H_{\mathbb{F}}(n-1)\text{-mod}] \end{array}$$

Proof. It is enough to show that, for a multipartition $\lambda \vdash_m n$, $d(\sum_{\mathbf{a} \mapsto a} [\text{Res}_{n-1}^n]_{\mathbf{a}} [V_{\lambda}^{\mathbb{K}}]) = [\text{Res}_{n-1}^n]_a [V_{\lambda}]$. Recall that $V_{\lambda}^{\mathbb{K}}$ has a \mathbb{K} -basis $\{v_t : t \text{ is a standard tableau of shape } \lambda\}$. Then, as an \mathbb{S} -lattice L we can take the $H_{\mathbb{S}}(n)$ -module generated by $\{v_t\}$. From here, the result follows. \square

For the rest of this section, we will make the following assumption on parameters:

- (†) *There exists $\ell \in \mathbb{Z}_{>0}$ such that $q = \sqrt[\ell]{1}$ is a primitive ℓ -root of 1, and for every $i = 1, \dots, r$, $q_i = q^{k_i}$ for some $k_i \in \mathbb{Z}/\ell\mathbb{Z}$.*

It follows from (4) and commutativity of the diagram (5) that, under the assumption (\dagger), the functors $(\text{Res}_{n-1}^n)_a$ vanish unless $a = q^i$ for some $i = 0, \dots, \ell - 1$. In this setting, we define the i -restriction and i -induction functors:

$$i\text{-}\text{Res}_{n-1}^n := (\text{Res}_{n-1}^n)_{q^i}, \quad i\text{-}\text{Ind}_{n-1}^n := (\text{Ind}_{n-1}^n)_{q^i},$$

so that $\text{Res}_{n-1}^n = \bigoplus_{i=0}^{\ell-1} i\text{-}\text{Res}_{n-1}^n$, $\text{Ind}_{n-1}^n = \bigoplus_{i=0}^{\ell-1} i\text{-}\text{Ind}_{n-1}^n$. Using (4) and (5) again, it is easy to see the action of i -Ind and i -Res at the level of the (complexified) Grothendieck groups. For a box \square of the Young diagram of a multipartition $\lambda \vdash_m n$, we define the ℓ -content of \square to be:

$$\text{cont}_\ell(\square) = a - b + k_i \pmod{\ell},$$

where the box \square is on column a and row b of $\lambda^{(i)}$. Then, we have

$$(6) \quad [i\text{-}\text{Res}_{n-1}^n][V_\lambda] = \sum_{\substack{x \in \text{rem}(\lambda) \\ \text{cont}_\ell(x)=i}} [V_{\lambda-\{x\}}], \quad [i\text{-}\text{Ind}_{n-1}^n][V_\mu] = \sum_{\substack{x \in \text{add}(\mu) \\ \text{cont}_\ell(x)=i}} [V_{\mu \cup \{x\}}].$$

2.5. Categorification functors. We denote $H(n) := H_{\mathbb{F}}(n)$, and throughout this Subsection we assume (\dagger).

Now let $\mathcal{C} := \bigoplus_{n \geq 0} H(n)\text{-mod}$, where, by definition, $H(0)\text{-mod}$ is just the category of \mathbb{F} -vector spaces. Let $E := \bigoplus \text{Res}_n^{n+1}$ (we define Res_{-1}^0 to be just the zero functor) and $F := \bigoplus \text{Ind}_n^{n+1}$ be endofunctors of the category \mathcal{C} . We have seen that:

- The endofunctors E and F are biadjoint.
- There exists an endomorphism $L := \bigoplus L_n$ of E that yields a decomposition $E = \bigoplus_{i \in \mathbb{Z}/\ell\mathbb{Z}} E_i$ into generalized eigenfunctors. By adjointness, this induces a decomposition $F = \bigoplus_{i \in \mathbb{Z}/\ell\mathbb{Z}} F_i$ such that each pair E_i, F_i consists of biadjoint endofunctors, cf. Proposition 2.5.
- Let $f_i := [F_i] : [\mathcal{C}] \rightarrow [\mathcal{C}], e_i := [E_i] : [\mathcal{C}] \rightarrow [\mathcal{C}]$ be the induced maps in the complexified Grothendieck group of \mathcal{C} . Using (6) we can see that $e_i, f_i, i = 1, \dots, \ell$ induce an action of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_\ell$ on $[\mathcal{C}]$. Indeed, $[\mathcal{C}]$ is a quotient of the level m Fock space of $\widehat{\mathfrak{sl}}_\ell$ that has as a basis the set of all m -multipartitions.

Note that we do not have categorical analogues of the Chevalley generators $h_i, i \in I$. However, there is a decomposition of \mathcal{C} lifting the decomposition of $[\mathcal{C}]$ into weight spaces. This is induced by the action of the center of the algebra $\mathcal{H}_q^{\text{aff}}(n)$ on $H(n)$ -modules. Recall that the center of $\mathcal{H}_q^{\text{aff}}(n)$ is Z_n , the space of symmetric Laurent polynomials on x_1, \dots, x_n . Since $H(n)$ is a quotient of $\mathcal{H}_q^{\text{aff}}(n)$, this induces a decomposition by central characters:

$$H(n)\text{-mod} = \bigoplus_{\chi} (H(n)\text{-mod})_{\chi}.$$

Identifying central characters of $\mathcal{H}_q^{\text{aff}}(n)$ with points in $(\mathbb{F}^\times)^n/\mathfrak{S}_n$ we have, by (\dagger), that $(H(n)\text{-mod})_{\chi} = 0$ unless $\chi \in \mathcal{I}^n/\mathfrak{S}_n$, where $\mathcal{I} = \{q, q^2, \dots, q^{\ell-1}\}$. For such χ , let

$$\text{wt}(\chi) = \sum_{i=0}^{\ell-1} m_i \alpha_i,$$

where α_i is the simple root of $\widehat{\mathfrak{sl}}_\ell$ corresponding to the Chevalley generators e_i, f_i, h_i ; and m_i is the multiplicity of q^i on χ . Note that χ is uniquely determined by $\text{wt}(\chi)$. Let $\varpi := \sum_{i=1}^m \omega_{k_i}$, where $\omega_0, \dots, \omega_{\ell-1}$ are the fundamental weights of $\widehat{\mathfrak{sl}}_\ell$.

Proposition 2.7. *For $\chi \in \mathcal{I}^n/\mathfrak{S}_n$, $\widehat{\mathfrak{sl}}_\ell$ acts on $[H(n)\text{-mod}]_{\chi}$ with weight $\varpi + \text{wt}(\chi)$.*

There is an extra piece of structure we have not seen yet. Namely, consider the endofunctor $E^2 : \mathcal{C} \rightarrow \mathcal{C}$, that sends $M \in H(n)\text{-mod}$ to $\text{Res}_{n-2}^{n-1} \text{Res}_{n-1}^n(M) = \text{Res}_{n-2}^n(M)$. Note that the element $T_{n-1} \in H(n)$ centralizes the subalgebra $H(n-2)$, so we can consider it as an endomorphism of $\text{Res}_{n-2}^n(M)$. Then, similarly to above, we can consider an endomorphism T of E^2 , given by $T = \bigoplus T_{n-1}$. We remark that the endomorphism T satisfies the relations

$$(7) \quad (1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E) \in \text{End}(E^3),$$

$$(8) \quad (T + 1_E)(T - q 1_E) = 0 \in \text{End}(E^2),$$

Let us explain the notation. For $M \in \mathcal{C}$, $(1_E T) \circ (T 1_E) \circ (1_E T)_M : E^3 M \rightarrow E^3 M$ is the endomorphism given by $E(T_M) \circ T_{EM} \circ E(T_M)$, where $T_M : E^2 M \rightarrow E^2 M$ and $T_{EM} : E^2 EM \rightarrow E^2 EM$. The other notation is similar.

Then, (7) is nothing more than the braid relation $T_{n-1}T_{n-2}T_{n-1} = T_{n-2}T_{n-1}T_{n-2}$, while (8) is nothing else but the Hecke relation $(T_{n-1} + 1)(T_{n-1} - q) = 0$. We also have a relation between the endomorphisms $L \in \text{End}(E)$, $T \in \text{End}(E^2)$,

$$(9) \quad T \circ (L1_E) \circ T = q1_E L \in \text{End}(E^2),$$

which is just different way to say that $T_{n-1}L_{n-1}T_{n-1} = qL_n$.

Remark 2.8. *Throughout this subsection we assumed the condition (\dagger) on the parameters. Similar results are obtained if we assume that $q_i = q^{k_i}$ for some $k_i \in \mathbb{Z}$, and $q \in \mathbb{F}^\times$ is not a root of unity. In this case, we obtain an action of \mathfrak{gl}_∞ on $[\mathcal{C}]$. More generally, let $S = \{q_1, \dots, q_m\}$. We have an equivalence relation on S : $q_i \sim q_j$ if $q_i/q_j \in q^\mathbb{Z}$. This yields a partition $S = \bigsqcup_{k=1}^j S_k$. Then, we obtain an action of $\widehat{\mathfrak{sl}_\ell^j}$ (if q is a primitive ℓ -root of unity) or of \mathfrak{gl}_∞^j (if q is not a root of unity) on \mathcal{C} . This follows from, for example, [Ari, Theorem 13.30].*

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E-mail address: simentalrodriguez.j@husky.neu.edu