LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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2. Algebras of Crawley-Boevey and Holland

This lecture consists of three different pieces. The first two are related to the deformation theory of Kleinian singularities and can be characterized as a general one and a specific one.

In the first lecture we have seen that a Kleinian singularity can be presented as the quotient $\mathbb{C}^2/\Gamma$, where Γ is a non-trivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. This allows to equip the algebra $\mathbb{C}[\mathbb{C}^2/\Gamma] = \mathbb{C}[x,y]^\Gamma$ with a positive grading.

Also we have given a general definition of a deformation. This definition is too general to be convenient. For graded algebras there is another notion – of a filtered deformations. This is explored in 2.1. Then we relate filtered deformations to the usual ones, via a *Rees algebra* construction, 2.2.

Then we proceed to describing some deformations of Kleinian singularities discovered by Crawley-Boevey and Holland in [CBH]. Their idea was to deform, first, not the algebras $\mathbb{C}[x,y]^\Gamma$ but closely related algebras $\mathbb{C}[x,y]\#\Gamma$ (known as "orbifolds" or "smash-products") and then produce deformations of $\mathbb{C}[x,y]^\Gamma$ from those of $\mathbb{C}[x,y]\#\Gamma$. We treat the algebras $\mathbb{C}[x,y]\#\Gamma$ in 2.3 and their filtered deformations (to be called CBH algebras) in 2.4.

After that we switch to an entirely different topic providing some generalities for the next lecture. Namely, we recall basic definitions regarding quivers and their representations.

2.1. **Filtered deformations.** Let us proceed to the definition of a filtered algebra. Let $\mathcal{A}$ be an associative algebra with unit. An increasing $\mathbb{Z}_{\geqslant 0}$-filtration (to be simply called a filtration below) on $\mathcal{A}$ is a collection of subspaces $\mathcal{A}^{\leqslant n}, n \in \mathbb{Z}_{\geqslant 0}$ with the following properties:

- $\mathcal{A}^{\leqslant n} \subset \mathcal{A}^{\leqslant m}$ whenever $n < m$.
- $\bigcup_n \mathcal{A}^n = \mathcal{A}$.
- $\mathcal{A}^{\leqslant n}\mathcal{A}^{\leqslant m} \subset \mathcal{A}^{\leqslant n+m}$ for any $n, m \in \mathbb{Z}_{\geqslant 0}$.
- $1 \in \mathcal{A}_{\leqslant 0}$.

Again, we say that a filtration is positive if $\mathcal{A}^{\leqslant 0}$ is spanned by 1. Roughly speaking, in a filtered algebra we have a notion of an "element of degree n" but cannot speak about homogeneous elements.

Every graded algebra is automatically filtered: just set $A^{\leqslant n} := \bigoplus_{i=0}^n A^i$. On the other hand, any quotient of a filtered (for example, graded) algebra inherits a natural filtration. More precisely, let $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{A}'$ be an epimorphism of algebras. Then we can set $\mathcal{A}'^{\leqslant n} := \varphi(\mathcal{A}^{\leqslant n})$.

Exercise 2.1. *Check that $\mathcal{A}'^{\leqslant n}$ is an algebra filtration.*

For example, consider the Weyl algebra $\mathrm{W}_2 := \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$. This algebra is spanned by monomials in x and y and, the filtration subspace $\mathrm{W}_2^{\leqslant n}$ is spanned by all monomials of degree $\leqslant n$.

The following problem presents a basis of W_2.

Problem 2.2. *Show that the monomials* $x^iy^j, i, j \geqslant 0$, *form a basis of* W_2. *For this, construct a representation of* W_2 *on* $\mathbb{C}[x]$.

It turns out that from a filtered algebra we can cook a graded one by taking the *associated graded algebra*. Namely, set $A^n := \mathcal{A}^{\leqslant n}/\mathcal{A}^{\leqslant n-1}$ (where we assume that $\mathcal{A}^{\leqslant -1} = 0$). Then $A := \bigoplus_{n=0}^{+\infty} A^n$ becomes a graded associative algebra. The product is defined as follows. Pick $a \in \mathcal{A}^{\leqslant n}/\mathcal{A}^{\leqslant n-1}, b \in \mathcal{A}^{\leqslant m}/\mathcal{A}^{\leqslant m-1}$ and lift them to elements $\bar{a} \in \mathcal{A}^{\leqslant n}, \bar{b} \in \mathcal{A}^{\leqslant m}$. Then $\bar{a}\bar{b}$ is an element in $\mathcal{A}^{\leqslant n+m}$. Moreover, since $\mathcal{A}^{\leqslant n-1}\mathcal{A}^{\leqslant m}, \mathcal{A}^{\leqslant n}\mathcal{A}^{\leqslant m-1} \subset \mathcal{A}^{\leqslant n+m-1}$, the class of $\bar{a}\bar{b}$ in $A^{n+m} = \mathcal{A}^{\leqslant n+m}/\mathcal{A}^{\leqslant n+m-1}$ does not depend on the choice of $\bar{a}, \bar{b}$. By definition, ab is that class.

Exercise 2.3. *Check that this product is associative and has a unit.*

The associated graded algebra of $\mathcal{A}$ is denoted by $\operatorname{gr}\mathcal{A}$.

Now let us give a definition of a filtered deformation.

Definition 2.1. Let A be a $\mathbb{Z}_{\geqslant 0}$-graded algebra. We say that a filtered algebra $\mathcal{A}$ is a *filtered deformation* of A if $\operatorname{gr}\mathcal{A} \cong A$.

For example, let A be a graded algebra and let I be a two-sided ideal. For an element $a \in A$ we write $\operatorname{gr} a$ for the top degree summand of a. We write $\operatorname{gr} I$ for the set $\{\operatorname{gr} a | a \in I\}$. Recall that A/I is equipped with a natural filtration.

Exercise 2.4. *Show that* $\operatorname{gr} I$ *is a two-sided ideal of* A *and identify* $\operatorname{gr} A/I$ *with* $A/\operatorname{gr} I$.

For example, if I is generated by elements $a_1, \ldots, a_k$, then $\operatorname{gr} a_1, \ldots, \operatorname{gr} a_k \in \operatorname{gr} I$. But it is not necessary that $\operatorname{gr} I$ is generated by $\operatorname{gr} a_1, \ldots, \operatorname{gr} a_k$. However, with a wise choice of generators, this is so in many example.

For example, take $A = \mathbb{C}\langle x, y\rangle$ and let I be the ideal generated by $xy - yx - 1$ so that $A/I = \mathrm{W}_2$. Then of course $\operatorname{gr} I$ contains the element $xy - yx$ and so we have a natural epimorphism $\mathbb{C}[x, y] \twoheadrightarrow \operatorname{gr}\mathrm{W}_2$. This epimorphism maps x^iy^j to x^iy^j. Recall however that the elements x^iy^j form a basis of W_2. It follows that the epimorphism is an isomorphism, equivalently, that $\operatorname{gr}\mathrm{W}_2 = \mathbb{C}[x, y]$.

2.2. **Rees algebras.** The two notions of deformations are related as follows: given a filtered deformation $\mathcal{A}$ of A one can produce a deformation of A over $\mathbb{C}[h]$ in the sense of the general definition of the previous lecture, where h is an independent variable, whose fiber at $h = 0$ is A, while the fiber at any nonzero h is $\mathcal{A}$. This is achieved by using an object called the *Rees algebra* of $\mathcal{A}$.

By definition, the Rees algebra $R_h(\mathcal{A})$ is the subspace $\bigoplus_{i=0}^{+\infty} \mathcal{A}^{\leqslant i}h^i \subset \mathcal{A}[h]$. This subspace contains $1, h$ and is closed under multiplication (of polynomials with coefficients in $\mathcal{A}$). Also $R_h(\mathcal{A})$ is graded as a subalgebra of $\mathcal{A}[h]$ so that $R_h(\mathcal{A})^i := \mathcal{A}^{\leqslant i}h^i$.

Problem 2.5. *Establish natural isomorphisms* $R_h(\mathcal{A})/hR_h(\mathcal{A}) \cong \operatorname{gr}\mathcal{A}, R_h(\mathcal{A})/(h-\alpha)R_h(\mathcal{A}) \cong \mathcal{A}$, *where* $\alpha \in \mathbb{C} \setminus \{0\}$. *Also check that* $R_h(\mathcal{A})$ *is flat over* $\mathbb{C}[h]$.

2.3. **Orbifolds.** Deformations of the algebra $\mathbb{C}[x, y]^\Gamma$ are not so easy to define and study. One approach could be to take the defining equation, say $x_1^{r+1} - x_2x_3 = 0$ and modify it adding terms of lower degree: $P(x_1) - x_2x_3 = 0$, where $P(x_1) = x_1^{r+1} + a_rx_1^r + \ldots + a_0$ (we can assume that $a_r = 0$). We do get commutative deformations in this way and actually can extend this approach to non-commutative deformations as well, see [H],[Sm]. In the other types the commutative deformations are also not hard to write down, but non-commutative ones are considerably harder. We will make some remarks about that in the end of the lecture.

The problem in studying deformations of $\mathbb{C}[x,y]^\Gamma$ ideologically comes from the fact that the variety $\mathbb{C}^2/\Gamma$ is not smooth. We are not going to explain here why smoothness is related to a nice deformation theory, however, we will see several manifestations of this principle later. Fortunately, one can replace $\mathbb{C}[x,y]^\Gamma$ with a closely related algebra, the smash-product $\mathbb{C}[x,y]\#\Gamma$, that is smooth in some precise sense although is no longer commutative. Then one can deform $\mathbb{C}[x,y]\#\Gamma$ in a way "compatible" with Γ and cook a deformation of $\mathbb{C}[x,y]^\Gamma$ out of it. This was proposed by Crawley-Boevey and Holland in [CBH] and this is an approach we are going to take.

Let A be an associative algebra with unit equipped with an action of a finite group Γ by automorphisms. Let us define the algebra $A\#\Gamma$. As a vector space, it coincides with the tensor product $A\otimes\mathbb{C}\Gamma$, where $\mathbb{C}\Gamma$ stands for the group algebra of Γ. It is enough to define the product on the elements of the form $f\otimes\gamma$, $f\in A, \gamma\in\Gamma$, then we can extend it by linearity. We set

$$f_1\otimes\gamma_1\cdot f_2\otimes\gamma_2 := f_1\gamma_1(f_2)\otimes\gamma_1\gamma_2,$$

where $\gamma_1(f_2)$ stands for the image of f_2 under the action of γ_1.

The algebra $\mathbb{C}[x,y]\#\Gamma$ carries a natural grading: with Γ in degree 0 and x,y in degree 1.

One can easily recover the algebra A^Γ from $A\#\Gamma$. Namely, consider the idempotent e in $\mathbb{C}\Gamma$ corresponding to the trivial representation, $e=\frac{1}{\Gamma}\sum_{\gamma\in\Gamma}\gamma$ (recall that being an idempotent means that $e^2=e$). The algebra $\mathbb{C}\Gamma$ is embedded into $A\#\Gamma$ via $\gamma\mapsto 1\otimes\gamma$. So we can view e as an element of $A\#\Gamma$. The subspace $e(A\#\Gamma)e$ is closed under the multiplication. It is not a subalgebra in the sense that it does not contain 1. But it has its own unit, e. It is called a *spherical subalgebra* of $A\#\Gamma$.

Lemma 2.2. *The map $a\mapsto ae$ identifies A^Γ with the spherical subalgebra $e(A\#\Gamma)e$ and is an isomorphism of unital algebras.*

Proof. First of all, we remark that A^Γ embedded into $A\#\Gamma$ via $a\mapsto a\otimes 1$ commutes with $\mathbb{C}\Gamma$. It follows that the map $a\mapsto ae=ea$ is an algebra homomorphism. Then we have the equalities $(A\#\Gamma)e=A\otimes e, e(A\otimes e)=A^\Gamma\otimes e$ in $A\#\Gamma=A\otimes\mathbb{C}\Gamma$ (check them; use that $\mathbb{C}\Gamma e=\mathbb{C}e$ and $\gamma ae=\gamma(a)e$). This implies the claim. □

Exercise 2.6. *In the notation of the previous proof, check that A^Γ coincides with the center of $A\#\Gamma$.*

2.4. **Definition of CBH algebras.** Although the algebra $\mathbb{C}[x,y]\#\Gamma$ is no longer commutative, it has an advantage over $\mathbb{C}[x,y]^\Gamma$: a presentation of the former via generators and relations becomes kind of simpler. Namely,

$$\mathbb{C}[x,y]\#\Gamma := \mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx). \tag{1}$$

So to get a filtered deformation of $\mathbb{C}[x,y]\#\Gamma$ we can just correct a relation $xy-yx=0$ by a smaller degree (i.e., degree 0 or 1) term, and get something of the form $xy-yx=c$. The quotient comes equipped with a standard quotient filtration explained above. The degree 0 part of $\mathbb{C}\langle x,y\rangle\#\Gamma$ is $\mathbb{C}\Gamma$ and the degree 1 part is $\operatorname{Span}(x,y)\otimes\mathbb{C}\Gamma$. The following problem shows that we are forced to take c in the center of $\mathbb{C}\Gamma$ (that is the same as the subalgebra $(\mathbb{C}\Gamma)^\Gamma$ of Γ-invariants under the adjoint action).

Problem 2.7. *Show that if* $\operatorname{gr}\mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx-c)=\mathbb{C}[x,y]\#\Gamma$*, then* c *lies in the center of* $\mathbb{C}\Gamma$ *(that is equal to* $(\mathbb{C}\Gamma)^\Gamma$*, where the invariants are taken with respect to the adjoint action).*

For $c\in(\mathbb{C}\Gamma)^\Gamma$ we set $H_c:=\mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx-c)$. This is an algebra introduced by Crawley-Boevey and Holland. They checked that $\operatorname{gr}H_c=\mathbb{C}[x,y]\#\Gamma$. We are not going to show this right now, in fact, we will obtain a more general result later. We would like to remark that the claim that $\operatorname{gr}H_c=\mathbb{C}[x,y]\#\Gamma$ is equivalent to saying that the elements $x^iy^j\gamma$, where $i+j\leqslant n$ form a basis in $H_c^{\leqslant n}$.

Now we can take the spherical subalgebra $eH_ce\subset H_c$. This algebra is filtered, we just restrict the filtration from H_c, i.e., $(eH_ce)^{\leqslant n}:=eH_ce\cap H_c^{\leqslant n}$. Equivalently, $(eH_ce)^{\leqslant n}=eH_c^{\leqslant n}e$.

Exercise 2.8. *Deduce* $\operatorname{gr}eH_ce=\mathbb{C}[x,y]^\Gamma$ *from* $\operatorname{gr}H_c=\mathbb{C}[x,y]\#\Gamma$ *(i.e., show that taking the spherical subalgebra commutes with taking the associated graded).*

So we get a family of filtered deformations of $\mathbb{C}[x,y]^\Gamma$ indexed by the space $(\mathbb{C}\Gamma)^\Gamma$.

In fact, it is expected (and proved for Γ of types A,D) that all filtered deformations of $\mathbb{C}[x,y]^\Gamma$ are of this form. It is a general fact (to be proved later in this course) that the algebra eH_ce is commutative if and only if $c_1=0$. It is known (for this one uses results of Slodowy, [Sl], together with some other considerations that will be explained later) that all commutative filtered deformations of $\mathbb{C}^2/\Gamma$ are of the form eH_ce. To prove that all non-commutative deformations are of the form eH_ce in type A is relatively easy. Type D is due to Boddington, [B], this is much harder and more technical, see also [L].

The following problem describes the algebras eH_ce in type A more explicitly.

Problem 2.9. *Let* Γ *be the group* $\mathbb{Z}/(r+1)\mathbb{Z}$*. We write* $x,y\in H_c$ *for the images of* $x,y\in\mathbb{C}\langle x,y\rangle\#\Gamma$.

1) Show that the algebra H_c *is* $\mathbb{Z}$*-graded with* Γ *in degree* 0*,* x *in degree* 1 *and* y *in degree* -1.

2) We can write c *as* $\sum_{\gamma\in\Gamma}c_\gamma\gamma$*. Produce element* $h\in(H_c)_{\leqslant 2}$ *that commutes with* Γ *and satisfies* $[h,x]=c_1x,[h,y]=-c_1y$ *(such an element is defined uniquely up to adding a constant provided* $c_1\neq 0$*).*

3) Set $x_1:=eh,x_2:=ex^n,x_3:=ey^n$*. Check that there are polynomials* P,Q *in one variable of degree* $r+1$ *such that* $x_2x_3=P(x_1),x_3x_2=Q(x_1)$ *in* eH_ce*. How are these polynomials related? Express their coefficients via the coefficients* c_γ.

4) Use $\operatorname{gr}eH_ce=\mathbb{C}[x,y]^\Gamma$ *to show that* $eH_ce=\mathbb{C}\langle x_1,x_2,x_3\rangle/([x_1,x_2]=(r+1)c_1x_2,[x_1,x_3]=-(r+1)c_1x_3,x_2x_3=P(x_1),x_3x_2=Q(x_1))$.

2.5. **Quivers and their representations.** Formally, a quiver Q is a collection of the following data: two sets, Q_0 (vertices) and Q_1 (arrows) and two maps $Q_1\to Q_0$, head h and tail t that to each arrow assign its target and source vertices. Mostly, people consider the case when both Q_0 and Q_1 are finite. A representation of a quiver Q is a collection of vector spaces $V_i,i\in Q_0$, and of linear maps $x_a:V_{t(a)}\to V_{h(a)},a\in Q_1$.

For example, we can consider a quiver with a single vertex, say 1, and a single arrow, a, with $t(a)=h(a)=1$. We can draw this quiver as an oriented loop. This is a so called Jordan quiver, the reason is that its representation is a vector space together with its linear endomorphism.

The dimension of a representation (V_i,x_a) is the vector $(\dim V_i)_{i\in Q_0}$. In these lectures, we will only consider finite dimensional representations. Usually, for V_i we take the coordinate vector space $\mathbb{C}^{v_i}$. Then the set of representations of given dimension naturally becomes a vector space: for example, the sum of the representations $(x_a)_{a\in Q_1}, (x'_a)_{a\in Q_1}$ is, by definition, $(x_a + x'_a)_{a\in Q_1}$. The space of representations of Q of dimension v (where recall v is a vector $(v_i)_{i\in Q_0}$) is denoted by $\mathrm{Rep}(Q, v)$. As a vector space, it is naturally identified with $\bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_{t(a)}}, \mathbb{C}^{v_{h(a)}})$.

On $\mathrm{Rep}(Q, v)$ we have a natural action of the group $\mathrm{GL}(v) := \prod_{i\in Q_0} \mathrm{GL}(\mathbb{C}^{v_i})$ induced by its natural action on $\bigoplus_i \mathbb{C}^{v_i}$ by the change of bases. In more detail, an element $g = (g_i)_{i\in Q_0}, g_i \in \mathrm{GL}(\mathbb{C}^{v_i})$ maps on an element $(x_a)_{a\in Q_1}$ to $(x'_a)_{a\in Q_1}$ with $x'_a = g_{h(a)} x_a g^{-1}_{t(a)}$. The elements of the same $\mathrm{GL}(v)$-orbit can be thought as a single representation but written in different basis. A basic problem in the study of quiver representations is therefore to describe the orbits.

References

[B] P. Boddington, *Deformations of type D Kleinian singularities.* arXiv:0612853.

[CBH] W. Crawley-Boevey, M. Holland. *Noncommutative deformations of Kleinian singularities.* Duke Math. J. 92(1998), 605-635.

[L] P. Levy. *Isomorphism Problems of Noncommutative Deformations of Type D Kleinian Singularities.* arXiv:0610490.

[H] T.J. Hodges. *Noncommutative deformations of type A Kleinian singularities.* J. Algebra, 161(1993), 271-290.

[Sl] P. Slodowy. *Simple singularities and simple algebraic groups.* Lect. Notes Math., v.815. Springer, Berlin/Heidelberg/New York, 1980.

[Sm] S.P. Smith, *A class of algebras similar to the enveloping algebra of* $\mathfrak{sl}_2$, Trans. Amer. Math. Soc. 322 (1990), 285-314.