Lecture 1

1) Filtered quantizations

1.1) Basic definitions.

Let $\mathcal{A}$ be an associative algebra/$\mathbb{C}$ w. 1. By a grading (or more precisely an algebra $\mathbb{Z}$-grading on $\mathcal{A}$) we mean a vector space decomposition $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ w. $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$ & (automatically) $1\in\mathcal{A}_0$. If $\mathcal{A}_i = \{0\}$ $\forall\, i<0$, then we talk about $\mathbb{Z}_{\geqslant 0}$-gradings.

Example: If V is a vector space, then its symmetric algebra, $S(V)$, is $\mathbb{Z}_{\geqslant 0}$-graded w. $S(V)_i = S^i(V)$.
More generally, if $\mathcal{A}_0$ is a commutative algebra & V is its module, then $S_{\mathcal{A}_0}(V)$ is non-negatively graded.

By a filtration (or more precisely, an ascending exhaustive algebra $\mathbb{Z}$-filtration) on $\mathcal{A}$ we mean a collection of subspaces $\mathcal{A}_{\leq i} \subset \mathcal{A}$ $(i\in\mathbb{Z})$ s.t.

- $\mathcal{A}_{\le i} \subset \mathcal{A}_{\le i+1} \ \forall i$
- $\bigcup_{i\in\mathbb{Z}} \mathcal{A}_{\le i} = \mathcal{A}$
- $\mathcal{A}_{\le i}\mathcal{A}_{\le j} \subset \mathcal{A}_{\le i+j}$
- $1 \in \mathcal{A}_{\le 0}$.

We'll give examples a bit later, for now let's highlight connections between these two structures. First, if $\mathcal{A} = \bigoplus_i \mathcal{A}_i$ is a grading, then $\mathcal{A}_{\le i} := \bigoplus_{j\le i} \mathcal{A}_j$ defines a filtration. In the opposite direction, to a filtered algebra (= algebra w. filtration) $\mathcal{A} = \bigcup_i \mathcal{A}_{\le i}$ we can assign its associated graded algebra $\mathrm{gr}\,\mathcal{A} = \bigoplus_i \mathcal{A}_{\le i}/\mathcal{A}_{\le i-1}$ (& $(\mathrm{gr}\,\mathcal{A})_i = \mathcal{A}_{\le i}/\mathcal{A}_{\le i+1}$). The product is uniquely recovered from that of homogeneous elements, where it's given by:

$$(a+\mathcal{A}_{\le i-1})(b+\mathcal{A}_{\le j-1}) = ab + \mathcal{A}_{\le i+j-1},\ a\in\mathcal{A}_{\le i},\ b\in\mathcal{A}_{\le j}$$

Examples: 1) Let $\mathfrak{g}$ be a Lie algebra. Then to $\mathfrak{g}$ we can assign its universal enveloping algebra

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\overbrace{(x\otimes y - y\otimes x - [x,y] \mid x,y\in\mathfrak{g})} =: J$$

Note that $T(\mathfrak{g})$ is graded (by degree of tensor monomial) hence filtered. This induces the PBW filtration on $\mathcal{U}(\mathfrak{g})$:

$$\mathcal{U}(\mathfrak{g})_{\leq i} = T(\mathfrak{g})_{\leq i} + \mathcal{J} = \mathrm{Span}_{\mathbb{C}}(x_1 \dots x_i \in \mathcal{U}(\mathfrak{g}) \mid x_i \in \mathfrak{g})$$

We have $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathrm{gr}\,\mathcal{J}$, where $\mathrm{gr}\,\mathcal{J}$ is the graded ideal in $\mathcal{U}(\mathfrak{g})$ spanned by top deg. terms in $\mathcal{J}$. In particular $x \otimes y - y \otimes x \in \mathrm{gr}\,\mathcal{J}$ $\forall\, x, y \in \mathfrak{g}$, hence $S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}\,\mathcal{U}(\mathfrak{g})$. The PBW theorem states that this is an isomorphism.

2) Let X be a smooth affine variety over $\mathbb{C}$, and let $\mathrm{Vect}(X)$ denote the Lie algebra of vector fields on X = derivations of $\mathbb{C}[X]$. By definition, the algebra of differential operators, $\mathcal{D}(X)$ is generated by spaces $\mathbb{C}[X]$ & $\mathrm{Vect}(X)$ subject to the following relations, where we write $*$ for the product in $\mathcal{D}(X)$: $\forall\, f, g \in \mathbb{C}[X]$, $\xi, \eta \in \mathrm{Vect}(X)$

$f * g = fg$ (product in $\mathbb{C}[X]$)

$f * \xi = f\xi$ (action of $\mathbb{C}[X]$ on $\mathrm{Vect}(X)$ by multiplications)

$\xi * f = f\xi + \xi . f$ (the 2nd summand is the action of $\mathrm{Vect}(X)$ on $\mathbb{C}[X]$)

$\xi * \eta - \eta * \xi = [\xi, \eta]$

The algebra $\mathcal{D}(X)$ acts on $\mathbb{C}[X]$ (w. $\mathbb{C}[X]$ acting by multiplications & $\mathrm{Vect}(X)$ acting by derivations) justifying the name.

We introduce the filtration on $\mathcal{D}(X)$ by degree in $\mathrm{Vect}(X)$ (it is usually referred to as the filtration by order of differential operator). Then $\mathrm{gr}\,\mathcal{D}(X)$ is the quotient of the algebra generated by $\mathbb{C}[X], \mathrm{Vect}(X)$ modulo the relations obtained by taking top degree parts of the filtrations defining $\mathcal{D}(X)$: i.e. $f*g = fg$, $f*\xi = \xi * f = f\xi$, $\xi * \eta = \eta * \xi$. This algebra is nothing else but $S_{\mathbb{C}[X]}(\mathrm{Vect}(X)) = \mathbb{C}[T^*X]$. So we get a graded algebra epimorphism $\mathbb{C}[T^*X] \twoheadrightarrow \mathrm{gr}\,\mathcal{D}(X)$. It's an isomorphism, the proof will be sketched later.

Subexample/exercise: In the case when $X = \mathbb{A}^n$, we have:

$$\mathcal{D}(\mathbb{A}^n) = \underbrace{\mathbb{C}\langle x_1, \dots x_n, \partial_1, \dots \partial_n \rangle}_{\text{free algebra}} / (x_i x_j = x_j x_i,\ \partial_i \partial_j = \partial_j \partial_i,\ \partial_i x_j - x_j \partial_i = \delta_{ij}).$$

1.2) Poisson brackets & quantizations

Let A be a commutative algebra. By a Poisson bracket on A we mean a skew-symmetric bilinear operation $\{\cdot,\cdot\}: A \times A \to A$

satisfying Leibniz & Jacobi identities:

$$\{a,bc\}=\{a,b\}c+b\{a,c\}$$

$$\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0 \quad \forall a,b,c\in A.$$

Examples/constructions:

1) $A=S(\mathfrak{g})$ w. unique bracket satisfying $\{x,y\}:=[x,y]$ (and extended to A using the Leibniz identity).

2) Let $\tilde{X}$ be a smooth affine variety. We can talk about regular (a.k.a. algebraic) i-forms on X. By a symplectic form on $\tilde{X}$ we mean a closed non-degenerate 2-form. Such a form ω gives an identification $T\tilde{X}\xrightarrow{\sim}T^*\tilde{X}$ & hence a Poisson bracket on $\mathbb{C}[\tilde{X}]$. A special case is provided by $\tilde{X}=T^*X$. Here $\mathbb{C}[\tilde{X}]$ $=S_{\mathbb{C}[X]}(\mathrm{Vect}(X))$ & on the generators the bracket is given by:

$$\{f,g\}=0,\ \{\xi,f\}=\xi.f,\ \{\xi,\eta\}=[\xi,\eta]$$

3) Let $\mathcal{A}$ be a filtered algebra s.t. $\mathrm{gr}\mathcal{A}$ is commutative i.e. $[\mathcal{A}_{\leq i},\mathcal{A}_{\leq j}]\subset\mathcal{A}_{\leq i+j-1}$. Then $\mathrm{gr}\mathcal{A}$ comes w. a unique Poisson

bracket that on the homogeneous elements is given by:

$$\{a+\mathcal{A}_{\leq i-1}, b+\mathcal{A}_{\leq j-1}\} = [a,b] + \mathcal{A}_{\leq i+j-2}.$$

Definition: Suppose A is a $\mathbb{Z}_{\geq 0}$-graded commutative algebra equipped w. Poisson bracket of degree -1: $\{A_i, A_j\} \subset A_{i+j-1}$ $\forall i,j$. By a filtered quantization of A we mean a filtered algebra $\mathcal{A}$ s.t. $\operatorname{gr}\mathcal{A}$ is commutative together w. an isomorphism of graded Poisson algebras $\operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$ (strictly speaking, this isomorphism is a part of data of a quantization)

Examples: 1) $S(\mathfrak{g}) \xrightarrow{\pi} \operatorname{gr} U(\mathfrak{g})$ is a homomorphism of Poisson algebras (and an isomorphism by the PBW theorem) Indeed, it's enough that $\pi(\{x,y\}) = \{\pi(x), \pi(y)\}$ for $x,y \in \mathfrak{g}$ because these elements generate $S(\mathfrak{g})$ as an algebra; both sides are equal to $[x,y] \in \operatorname{gr} U(\mathfrak{g})$. So $U(\mathfrak{g})$ quantizes $S(\mathfrak{g})$.

2) For similar reasons, $\mathbb{C}[T^*X] \twoheadrightarrow \operatorname{gr} D(X)$ is a Poisson homomorphism (exercise).

Premium exercise: Prove that $\mathbb{C}[T^*X] \xrightarrow{\pi} \operatorname{gr} \mathcal{D}(X)$ is an isomorphism as follows:

1) $\mathcal{D}(X) \neq \{0\}$ - use the representation of $\mathcal{D}(X)$ in $\mathbb{C}[X]$.

2*) $\ker \pi = \{0\}$ or $\mathbb{C}[T^*X]$. Deduce this from a more general observation: if $\tilde{X}$ is a symplectic smooth affine variety, then the only Poisson ideals in $\mathbb{C}[\tilde{X}]$ are $\{0\}$ & $\mathbb{C}[\tilde{X}]$.
Hint: If A is a commutative algebra, $I \subset A$ an ideal, ∂ is a derivation of A preserving I, then ∂ preserves $\sqrt{I}$.

1.3) Remarks

1) Why is this deformation quantization? Choosing sections $\mathcal{A}_{\leq i}/\mathcal{A}_{\leq i-1} \hookrightarrow \mathcal{A}_{\leq i}$ of the projections $\mathcal{A}_{\leq i} \twoheadrightarrow \mathcal{A}_{\leq i}/\mathcal{A}_{\leq i-1}$ we identify $\mathcal{A}$ w. $\operatorname{gr}\mathcal{A} =: A$. The product on $\mathcal{A}$ is then of the form: $a * b = ab + \sum_{\ell \geq 1} D_\ell(a,b)$, where D_ℓ has degree $-\ell$, here a, b are arbitrary homogeneous elements of $\operatorname{gr}\mathcal{A}$. In other words, we deform the initial commutative product on A (of degree 0) by adding lower degree terms.

2) There are two ways how the definition of filtered quantizations can be generalized:

2.1) The case when $\deg\{\cdot,\cdot\}=-d$ (for $d\in\mathbb{Z}_{\geq 1}$). Here we require $[\mathcal{A}_{\leq i},\mathcal{A}_{\leq j}]\subset\mathcal{A}_{\leq i+j-d}$, which yields a deg $-d$ bracket on $\operatorname{gr}\mathcal{A}$ - again by taking the top degree part of the commutator. An example is provided by $D(\mathbb{A}^n)$, where we introduce a new filtration (called Bernstein filtration) by placing both x_i,∂_i in deg 1. The resulting filtered algebra is a quantization of $\mathbb{C}[x_1,\dots x_n,\partial_1,\dots\partial_n]$ with the standard Poisson bracket (coming from the standard symplectic form on $\mathbb{C}^{2n}$) known as the Weyl algebra.

2.2) $\mathbb{Z}$-graded Poisson algebras: here we require that $\mathcal{A}$ is complete & separated w.r.t. the filtration, i.e.:

$$\mathcal{A}\xrightarrow{\sim}\varprojlim_{i\to-\infty}\mathcal{A}/\mathcal{A}_{\leq i}$$

We'll revisit this condition next time.