Lecture 24

1) Algebra homomorphisms vs polynomial maps
2) Geometric singnificance of localization

Refs: [E], Section 1.6, Intro to Section 2.

BONUS: 1) What is an algebraic variety?
2) Projective varieties.

1) Algebra homomorphisms vs polynomial maps.

Let $\mathbb{F}$ be algebraically closed field. Let X be an algebraic subset of $\mathbb{F}^n$ (the set of solutions to a system of polynomial equations). Recall that to X we assign the ideal $I(X)= =\{f\in \mathbb{F}[x_1,\dots x_n] \mid f|_X=0\}$ & the algebra $\mathbb{F}[X]=\mathbb{F}[x_1,\dots x_n]/I(X)$ whose elements can be interpreted as polynomial functions on X, and the natural projection $\mathbb{F}[x_1,\dots x_n]\twoheadrightarrow \mathbb{F}[X]$ is the restriction to X. Set $\bar{x}_i := x_i|_X$, note that $\bar{x}_1,\dots \bar{x}_n$ generate the algebra $\mathbb{F}[X]$.

1.1) Polynomial maps.

Definition: Let $X\subset \mathbb{F}^n$, $Y\subset \mathbb{F}^m$ be algebraic subsets. A map $\varphi: X\to Y$ is called polynomial if $\exists f_1,\dots f_m\in \mathbb{F}[X]$ s.t. $\varphi(\alpha)=(f_1(\alpha),\dots f_m(\alpha))$ $\forall \alpha\in X$ (in particular, $(f_1(\alpha),\dots, f_m(\alpha))\in Y$).

Rem: polynomial map $X\to \mathbb{F}$ = polynomial function on X.

Exercise: the composition of polynomial maps is polynomial.

In particular, let $\varphi : X \to Y$ be a polynomial map & $g \in \mathbb{F}[Y]$, i.e. $g: Y \to \mathbb{F}$. Consider the composed polynomial map $g \circ \varphi : X \to \mathbb{F}$. When viewed as an element of $\mathbb{F}[X]$, $g\circ\varphi$ will be denoted by $\varphi^*(g)$ and called the pullback from g (under φ).

Lemma: 1) $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X]$ is algebra homomorphism.

2) On the generators $\bar{y}_j (:= y_j|_Y)$, $\varphi^*(\bar{y}_j) = f_j$

Proof:

1) is exercise, compare to Prob 1 in HW4.

2): $\varphi^*(\bar{y}_j)(\alpha) = \bar{y}_j(\varphi(\alpha)) = f_j(\alpha) \Rightarrow \varphi^*(\bar{y}_j) = f_j$. $\square$

Examples: 1) Inclusion map $\iota : X \hookrightarrow \mathbb{F}^n$ is polynomial, $\iota^* : \mathbb{F}[x_1, \dots x_n] \twoheadrightarrow \mathbb{F}[X]$ is the restriction map. More generally, if $X \subset Y \subset \mathbb{F}^n$ are algebraic subsets, then the inclusion map $\iota : X \hookrightarrow Y$ is polynomial & $\iota^* : \mathbb{F}[Y] \to \mathbb{F}[X]$ is $g \mapsto g|_X$.

2) $X = \mathbb{F}$, $Y = V(y_1^2 - y_2^3) \subset \mathbb{F}^2$, $\varphi : X \to Y$, $x \mapsto (x^3, x^2)$ is a polynomial map. Since $y_1^2 - y_2^3$ is irreducible, the ideal $(y_1^2 - y_2^3)$ is prime, hence radical, so $\mathbb{F}[Y] = \mathbb{F}[y_1, y_2]/(y_1^2 - y_2^3)$. By 2) of Lemma, $\varphi^*(\bar{y}_1) = x^3$, $\varphi^*(\bar{y}_2) = x^2$, this determines φ^* uniquely because $\bar{y}_1, \bar{y}_2$ generate the algebra $\mathbb{F}[Y]$.

1.2) Main result.

Theorem: $\varphi \mapsto \varphi^*$ defines a bijection between:

(I) $\{$polynomial maps $\varphi: X \to Y\}$

(II) $\{$algebra homomorphisms $\mathbb{F}[Y] \to \mathbb{F}[X]\}$.

Proof: Given $\varphi = (f_1, \dots, f_m)$, φ^* is the unique algebra homomorphism $\mathbb{F}[Y] \to \mathbb{F}[X]$ s.t. $\varphi^*(\bar{y}_j) = f_j$. We'll use this observation to construct the inverse map (II) $\to$ (I).

Given an algebra homomorphism $\tau: \mathbb{F}[Y] \to \mathbb{F}[X]$ define $\psi_\tau: X \to \mathbb{F}^m$ by $\psi_\tau := (\tau(\bar{y}_1), \dots, \tau(\bar{y}_m))$. We claim $\operatorname{im} \psi_\tau \subset Y$ (so that ψ_τ can be viewed as a polynomial map $X \to Y$). We have $\operatorname{im} \psi_\tau \subset Y \Leftrightarrow G(\operatorname{im} \psi_\tau) = 0 \ \forall G \in I(Y) \Leftrightarrow$

$$G(\tau(\bar{y}_1), \dots \tau(\bar{y}_m)) = 0 \qquad (*)$$

Note that $G(\bar{y}_1, \dots, \bar{y}_m) = 0$ b/c $G \in I(Y)$ & τ preserves polynomial relations b/c it's an algebra homomorphism. $(*)$ follows.

So we have maps $\varphi \mapsto \varphi^*: (I) \leftrightarrows (II): \tau \mapsto \psi_\tau$. We have $\psi_\tau^*(\bar{y}_i) = [i\text{th coordinate in } \tau] = \tau(\bar{y}_i) \Rightarrow \psi_\tau^* = \tau$ b/c $\bar{y}_i$'s generate. On the other hand, $\psi_{\varphi^*} = (\varphi^*(\bar{y}_1), \dots \varphi^*(\bar{y}_m)) = (\varphi_1, \dots, \varphi_m) = \varphi$. So these maps are mutually inverse, finishing the proof. $\square$

1.3) Affine varieties:

Similarly to Prob 1 in HW4, we see that:

- for algebraic subsets $X \subset \mathbb{F}^n, Y \subset \mathbb{F}^m, Z \subset \mathbb{F}^k$ & polynomial maps $\varphi: X \to Y, \psi: Y \to Z$, we have $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
- $(id_X)^* = id_{\mathbb{F}[X]}$.

Consider categories:

1) $\mathcal{C}$: $Ob(\mathcal{C}) = \{$finitely generated $\mathbb{F}$-algebras w/o nonzero nilpotents$\}$

Morphisms: homomorphisms of algebras.

2) $\tilde{\mathcal{C}}$: $Ob(\tilde{\mathcal{C}}) = \{A \in Ob(\mathcal{C})$ w. finite collection of algebra generators$\}$

Morphisms: same as in $\mathcal{C}$.

3) $\tilde{\mathcal{D}}$: $Ob(\tilde{\mathcal{D}}) = \{$algebraic subsets in some $\mathbb{F}^n\}$

Morphisms: polynomial maps.

We have functors: • $\tilde{\mathcal{D}} \to \tilde{\mathcal{C}}^{opp}$: $X \mapsto \mathbb{F}[X]$ w. generators given by coordinate functions, $\bar{x}_i$, $\varphi \mapsto \varphi^*$.

• $\tilde{\mathcal{C}}^{opp} \to \tilde{\mathcal{D}}$: $A = \mathbb{F}[x_1, \dots x_n]/I$ (identification coming from generators) $\mapsto V(I)$, $\tau \mapsto \psi_\tau$.

Crucially important exercise: these two functors are inverse to each other.

Definition: The category of affine varieties (over $\mathbb{F}$) is $\mathcal{D} := \mathcal{C}^{opp}$.

The objects in $\mathcal{D}$ can be thought of as algebraic subsets "irrespective" of embedding into $\mathbb{F}^n$. Morphisms are still polynomial maps.

Example: Let $X_1 = \mathbb{F}$ & $X_2 = V(x_2 - x_1^2) \subset \mathbb{F}^2$. They have isomorphic algebras of functions: polynomials in one variable but different embeddings into $\mathbb{F}^n$'s. From the point of view of algebraic geometry they behave in the same way so can be viewed as the same variety.

While this definition of an affine variety looks like cheating, we can talk about

- Algebra of polynomial functions $\mathbb{F}[X]$ of an affine variety X (X viewed as an object of $\mathcal{C}$)
- Points of X: algebra homomorphisms $\mathbb{F}[X] \to \mathbb{F}$.

And so on.

2) Geometric significance of localization.

2.1) Localizing one element.

Let $X \subset \mathbb{F}^n$ be an algebraic subset & $f \in \mathbb{F}[X]$. We want to find a geometric interpretation of the localization $\mathbb{F}[X][f^{-1}]$.

Let $f_1, \dots, f_m$ be generators of $I(X)$. Then Lemma in Sec 1.1 of Lecture 11 tells us that

$$\mathbb{F}[X][f^{-1}] = \mathbb{F}[X][t]/(tf-1) \simeq \mathbb{F}[x_1, \dots x_n, t]/(f_1, \dots, f_m, tf-1).$$

The corresponding algebraic subset of $\mathbb{F}^{n+1}$ is
$\{(\alpha_1, \dots, \alpha_n, z) \in \mathbb{F}^{n+1} \mid f_i(\alpha_1, \dots \alpha_n) = 0 \ \forall i = 1, \dots m;\ z f(\alpha_1, \dots \alpha_n) = 1\}$

The projection $\mathbb{F}^{n+1} \to \mathbb{F}^n$ forgetting the z coordinate identifies this algebraic subset w. $\{\alpha \in X \mid f(\alpha) \neq 0\}$. Denote this subset by X_f. We note that it's <u>not</u> an algebraic subset of $\mathbb{F}^n$ in our terminology. This subset of X is called a principal open subset.

Here's an explanation of the terminology.

Definition: · a subset $Y \subset X$ is called Zariski closed if it's an algebraic subset of $\mathbb{F}^n$.

• A subset $U \subset X$ is Zariski open if $X \setminus U$ is Zariski closed.

Example: $X_f \subset X$ is Zariski open.
Exercise: Any Zariski open subset of X is the union of, in fact, finitely many, principal open subsets.

Zariski open/closed subsets are open/closed subsets in a topology (called Zariski topology). Principal open subsets form a "base of topology."

2.2) Localization at the complement of a maximal ideal.
Let $X \subset \mathbb{F}^n$ be algebraic subset, $A := \mathbb{F}[X]$, $\mathfrak{m} \subset A$ a maximal ideal. Recall, Sec 2 of Lec 12, that for any prime ideal $\mathfrak{p} \subset A$, subset $S := A \setminus \mathfrak{p} \subset A$ is multiplicative. We write $A_{\mathfrak{p}} := A[S^{-1}]$. Take $\mathfrak{p} := \mathfrak{m}$.

Note that $A_{\mathfrak{m}}$ is not finitely generated (in general) so is not the algebra of functions of an algebraic subset. It still has a geom'c meaning that we are going to discuss now.

For simplicity, assume X is irreducible $\Leftrightarrow A = \mathbb{F}[X]$ is domain $\rightsquigarrow$ fraction field $\mathrm{Frac}(A) = \{\frac{f}{g} \mid g \neq 0\}$, every localization of A is contained in $\mathrm{Frac}(A)$ as a subring, Corollary in Sec 1.2 of Lec 11.

By Corollary in Sec 1.4 of Lec 22, the maximal ideals of A are in bijection w. X: $\mathfrak{m} \leftrightarrow \alpha$ w. $\mathfrak{m} = \{f \in A \mid f(\alpha) = 0\}$. Then

$A_{\mathfrak{m}} = \{\frac{f}{g} \mid g(\alpha) \neq 0\} = \bigcup_{g \mid g(\alpha) \neq 0} A_g = \bigcup_g \mathbb{F}[X_g]$ (recall that X_g

is an algebraic subset (in $\mathbb{F}^{n+1}$) so it makes sense to speak about its algebra of functions.

Conclusion:

Every element of $A_{\mathfrak{m}}$ is a function on a Zariski open subset containing α, but which subset we choose depends on this element.

Remark: When X is reducible, conclusion still holds but $A_{\mathfrak{m}} = \bigcup_g \mathbb{F}[X_g]$ makes no sense b/c $\mathbb{F}[X_g] = A_g$ are, a priori, not subrings in a fixed ring (in general). To fix this one replaces the union w. the "direct limit."

Remark (on terminology): Recall (Sec 2 of Lec 12) that a commutative ring B is called local if it has unique maximal ideal.

For example, $A_{\mathfrak{p}}$ is local (w. maximal ideal $\mathfrak{p}_{\mathfrak{p}}$), in particular $A_{\mathfrak{m}}$ is local. The discussion above gives a geometric justification to this terminology: this algebra controls what happens locally (in Zariski topology) near $\alpha \in X$.

BONUS: 1) What is an algebraic variety?

We've discussed affine (algebraic) varieties. Now we are going to address the question in the title.

A common approach to constructing geometric objects is to "glue" them from simpler objects. For example, C^∞-manifolds are glued from balls in Euclidian spaces: $M = \bigcup_\alpha D_\alpha$, where $D_\alpha \xrightarrow[\sim]{\varphi_\alpha} \{v \in \mathbb{R}^n \mid \|v\| < 1\}$. The condition is, roughly, that for all α, β in the index set, the images of $D_\alpha \cap D_\beta$ under $\varphi_\alpha, \varphi_\beta$ are open subsets in $\{v \in \mathbb{R}^n \mid \|v\| < 1\}$ and the resulting composition

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(D_\alpha \cap D_\beta) \xrightarrow[\sim]{\varphi_\alpha^{-1}} D_\alpha \cap D_\beta \xrightarrow[\sim]{\varphi_\beta} \varphi_\beta(D_\alpha \cap D_\beta)$$

is C^∞ (which makes sense b/c this is a map between open subsets in $\mathbb{R}^n$). Thank to this definition it makes to speak about various C^∞-objects, e.g. C^∞-maps $M \to N$.

Similarly, it makes sense to speak about complex analytic manifolds: we use balls in $\mathbb{C}^n$ and require that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is complex analytic (you might have studied that for $n=1$ — in which case the resulting objects appear when you study analytic continuation of holomorphic functions).

Something like that happens for algebraic varieties. The building blocks are affine algebraic varieties and they are glued together using polynomial isomorphisms: if the variety of interest is reasonable ("separated" in a suitable sense) the intersection of two open affine subvarieties is again affine so we can just use what we have in this lecture.

We can define the notion of a polynomial map (a.k.a. morphism): $\varphi: X \to Y$ is a morphism if we can cover $X = \bigcup U_i$, $Y = \bigcup V_j$ w. open affine varieties s.t. $\forall i\ \exists j \mid \varphi(U_i) \subset V_j$ & $\varphi: U_i \to V_j$ is a polynomial map of affine varieties.

B2) Projective varieties and graded algebras.

Here comes the most important example of the construction sketched above.

We start with $\mathbb{F}^{n+1}$ (viewed as a vector space). The projective space $\mathbb{P}^n (= \mathbb{P}(\mathbb{F}^{n+1}))$ as a set consists of 1-dimensional subspaces in $\mathbb{F}^{n+1}$. In other words, it consists of equivalence classes $[x_0:\ldots:x_n]$ w. $(x_0,\ldots x_n)\in \mathbb{F}^{n+1}\setminus\{0\}$, where equivalent means proportional. Let us explain how gluing works.

Let $U_i=\{[x_0:\ldots:x_n] \mid x_i\neq 0\}$, $i=0,\ldots n$. Then the map $U_i \xrightarrow[\sim]{\varphi_i} \mathbb{F}^n$: $[x_0:\ldots:x_n]\mapsto \left(\frac{x_0}{x_i},\ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i},\ldots \frac{x_n}{x_i}\right)$ is a bijection that will be used to identify U_i w. $\mathbb{F}^n$. Note that $\varphi_i(U_i\cap U_j)$ is given by non-vanishing of a single coordinate so is an affine variety. And one can show that

$$\varphi_j\circ\varphi_i^{-1}: \quad \varphi_i(U_i\cap U_j)\xrightarrow{\sim}\varphi_j(U_i\cap U_j)$$

is a polynomial isomorphism.

Example: let $n=1$. Let $y_0=\frac{x_1}{x_0}$ & $y_1=\frac{x_0}{x_1}$ be coordinates on $\varphi_0(U_0)\simeq\mathbb{F}$ & $\varphi_1(U_1)\simeq\mathbb{F}$. Then $\varphi_i(U_0\cap U_1)$ is given by $y_i\neq 0$ & $\varphi_1\circ\varphi_0^{-1}$ sends y_0 to y_0^{-1}, which is a polynomial isomorphism as we have inverted y_0.

So $\mathbb{P}^n$ is an algebraic variety in the sense of the 1st part.

One can generalize this construction. Let $F_1,\ldots F_k\in\mathbb{F}[x_0,\ldots x_n]$ be <u>homogeneous</u> polynomials of degree >0. If F_i vanishes at a nonzero point in $\mathbb{F}^{n+1}$, then it also vanishes on the line between this point & 0. So it makes sense to speak about the zero locus of F_i in $\mathbb{P}^n$ (note that F_i is NOT a function $\mathbb{P}^n \to \mathbb{F}$). This gives rise to the zero locus $V(F_1, \ldots F_k)$ and hence to the notion of an algebraic subset of $\mathbb{P}^n$.

Exercise: $V(F_1, \ldots F_k) \cap U_i$ is an algebraic subset in $U_i \xrightarrow{\sim} \mathbb{F}^n$.

So $V(F_1, \ldots F_k)$ is an algebraic variety, varieties of that kind are called <u>projective</u>.

Here's a reason why we care about them. Let $\mathbb{F} = \mathbb{C}$. So $\mathbb{C}^n$ has the usual topology. And so does $\mathbb{P}^n$ with U_i's being open subsets.

Important exercise: $\mathbb{P}^n$ is compact – in the usual topology.

And so, every $V(F_1, \ldots F_k)$ is compact. In Geometry & Topology we like compact spaces more than noncompact as they behave better in many ways. And while not all compact (in the usual topology) algebraic varieties are projective, the projective ones are nicest.

Now we discuss a connection between projective varieties & graded algebras. The vanishing locus of $V(F_1, \ldots F_k)$ depends only on $(F_1, \ldots F_k)$, a homogeneous ideal.

Exercise: If $I \subset \mathbb{F}[x_0, \ldots x_n]$ is a homogeneous ideal, then so is its radical.

In fact, $V(F_1, \ldots F_k)$ only depends on $\sqrt{(F_1, \ldots F_k)}$, similarly to the affine case. This gives rise to a bijection between

- Algebraic subsets of $\mathbb{P}^n$
- and radical homogeneous ideals in $\mathbb{F}[x_0, \ldots x_n]$ not containing 1.

Exercise: What ideal corresponds to $\emptyset$?

So starting from an algebraic subset in $\mathbb{P}^n$ we get a finitely generated reduced <u>graded</u> algebra, the quotient of $\mathbb{F}[x_0, \dots x_n]$ by the corresponding ideal. Note that the elements of this algebra are not functions on the initial algebraic subset of $\mathbb{P}^n$.

Conversely, let $A = \bigoplus_{i=0}^{\infty} A_i$ be a fin. gen'd graded reduced $\mathbb{F}$-algebra w. $A_0 = \mathbb{F}$. From this algebra we can construct a projective variety. Namely, if A is generated by A_1 ($\Leftrightarrow A$ is a graded quotient of $\mathbb{F}[x_0, \dots x_n]$), then we consider the algebraic subset of $\mathbb{P}^n$ defined by the kernel of $\mathbb{F}[x_0, \dots x_n] \twoheadrightarrow A$, which is a homogeneous ideal.

In general - if A isn't generated by A_1 - we have the following:

Exercise: $\exists\, d > 0$ s.t. $A_{(d)} := \bigoplus_{i=0}^{\infty} A_{di}$ is generated by A_d.

A fun fact: the projective variety we get is independent of the choice of d up to an isomorphism.

Example: Take $A = \mathbb{F}[x_0, x_1]$ (w. usual grading). It gives rise to the projective line $\mathbb{P}^1$. Now consider $A_{(2)}$. It's generated by $y_0 := x_0^2$, $y_1 := x_0 x_1$, $y_2 := x_1^2$. All relations between these elements are generated by $y_0 y_2 - y_1^2$. The corresponding algebraic subset is $\{[y_0 : y_1 : y_2] \mid y_0 y_2 = y_1^2\}$. Denote it by X.

We are going to construct two mutually inverse polynomial maps between $\mathbb{P}^1$ & X. Let $\varphi: \mathbb{P}^1 \to X$ be given by $[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_1^2]$. Now we define $\psi: X \to \mathbb{P}^1$:

$$\psi([y_0:y_1:y_2]) = \begin{cases} [y_0:y_1], \text{ if } y_2 \neq 0 \\ [y_1:y_2], \text{ if } y_0 \neq 0. \end{cases}$$

Exercise: Check φ, ψ are well-defined & mutually inverse maps. Furthermore, check that φ, ψ are morphisms (in the sense explained in the end of B1).

A connection with projective varieties is one of the reasons to care about graded algebras.