Lecture 9 (minor update 10/2)

1) Integral extensions of rings.
2) Integral closure.

Ref: [AM], Section 5.1.

1) Integral extensions of rings.

As usual, A is a commutative ring. The notions of finitely generated and finite A-algebras were recalled in Sec 2 of Lec 8.

1.1) Definition & examples.

Definition: Let B be a commutative A-algebra.

- $b\in B$ is **integral** over A if $\exists$ monic (i.e. leading coeff $=1$) $f\in A[x] \mid f(b)=0$.
- B is **integral** over A if $\forall\, b\in B$ is integral (over A).

Exercise: If B is integral over A & C is a quotient of B, then C is integral over A.

Example: 1) Let $f(x)\in A[x]$ be a monic polynomial. Then $\bar{x}:=x+(f)\in B:=A[x]/(f)$ is integral. Also note that B is finite over A (generated by $1,\bar{x},\ldots \bar{x}^{d-1}$ for $d:=\deg f$). Below we'll see that B is integral over A.

2) Let $A=K\subset B=L$ be a field extension. Then B is integral over $A \Leftrightarrow B$ is algebraic over A (we can divide by the leading coefficient).

Rem: In this example the two meanings of being finite are equivalent. But being finitely generated as an algebra is much stronger than being finitely generated as a field (for the latter we allow taking inverses among operations we use, for the former we only use multiplication & addition/subtraction. We'll later see (under some restrictions although this result holds in general) that if L is a finitely generated K-algebra, then $\dim_K L < \infty$.

1.2) Finite vs integral.

Reminder: for field extensions: finite $\Leftrightarrow$ [algebraic & finitely generated (as a field extension)].

Thm: Let B be an A-algebra. TFAE

(a) B is integral and finitely generated over A.

(b) B is finite over A.

The proof of (a) $\Rightarrow$ (b) is based on the following lemma. Note that if A_1 is an A-algebra & A_2 is an A_1-algebra, then A_2 is also an A-algebra: the homomorphism $A \to A_2$ is the composition $A \to A_1 \to A_2$.

Lemma 1: Suppose A_1 is finite over A & A_2 is finite over A_1. Then A_2 is finite over A.

Proof: Have $a_1, \ldots a_k \in A_1$ & $b_1, \ldots b_\ell \in A_2$ s.t. $A_1 = \mathrm{Span}_A(a_1, \ldots a_k)$, $A_2 = \mathrm{Span}_{A_1}(b_1, \ldots, b_\ell)$.

Exercise: $A_2 = \mathrm{Span}_A(b_i a_j \mid i = 1, \ldots, \ell, j = 1, \ldots k)$ $\square$

Proof of (a) $\Rightarrow$ (b) of Thm: B is generated by finitely many elements $b_1, \dots b_k \in B$. Each of them is integral over A. Let B_i, $i=1,\dots k$, be the A-subalgebra of B generated by $b_1, \dots b_i$, i.e. $B_i = \mathrm{Span}_A(b_1^{d_1} \dots b_i^{d_i} \mid d_1, \dots, d_i \in \mathbb{Z}_{\geq 0})$. By the construction, we have $B_k = B$ and $B_i \subset B_{i+1}$. Set $B_0 := A$. Since all b_i are integral over A, we also have that b_i is integral over B_{i-1}. By the construction, b_i generates B_i as an algebra over B_{i-1}. We claim that B_i is finite over B_{i-1} $\forall i$. This claim together w. Lemma 1 yields (a) $\Rightarrow$ (b). So let's prove the claim.

Let $f_i \in B_{i-1}[x]$ be a monic polynomial s.t. $f_i(b_i)=0$. Then the unique B_{i-1}-algebra homomorphism $B_{i-1}[x] \to B_i$ w. $x \mapsto b_i$ factors as $B_{i-1}[x] \twoheadrightarrow B_{i-1}[x]/(f_i) \to B_i$. But b_i generates B_i over B_{i-1}. So $B_{i-1}[x]/(f_i) \twoheadrightarrow B_i$. The source is a finite over B_{i-1} by Example 1. So is the target. $\square$

To prove (b) $\Rightarrow$ (a) we will need the following lemma: a module version of the Cayley-Hamilton theorem from Linear algebra. We will prove a more general form that will be used later.

Lemma 2: Let M be an finitely generated A-module, $I \subset A$ an ideal, $\varphi: M \to M$ A-linear map s.t $\varphi(M) \subset IM$. Then there is a polynomial $f(x) \in A[x]$ of the form

(*) $f(x) = x^n + a_1 x^{n-1} + \dots + a_n$ with $a_k \in I^k$ $\forall k$

s.t $f(\varphi)=0$.

Proof: Note that M becomes an $A[x]$-module w. x acting by φ. Pick generators $m_1, \dots m_n \in M$. We have elements $a_{ij} \in I$, $i = 1, \dots n$ s.t.

$$(1) \qquad x m_i = \sum_{j=1}^{n} a_{ij} m_j$$

Form the matrix $X = xI - (a_{ij})$. Then $\det(X) \in A[x]$. Note that $\det(X)$ is a polynomial $f(x)$ as in condition (*) (use the initial definition of det, left as exercise) Also note that $\det(X)$ acts by $f(\varphi)$ on M. So it's enough to show that $\det(X)$ acts by 0.

Let $\vec{m} = (m_1, \dots m_n)$ viewed as a column vector. Then $X\vec{m} = \vec{0}$ by (1). Consider the "adjoint" matrix $X' = (x'_{ij})$ w. $x'_{ij} = (-1)^{i+j} \cdot \det$ (the matrix obtained from X by removing row $\# i$ & column $\# j$) so that $X'X = \det(X) I$. Then $X\vec{m} = \vec{0} \Rightarrow \det(X)\vec{m} = X'X\vec{m} = \vec{0}$. So

$$(2) \qquad \det(X) m_i = 0 \ \forall i.$$

Since $m_1, \dots m_n$ span the A- (and hence $A[x]$-) module M, (2) $\Rightarrow f(\varphi) m = \det(X) m = 0 \ \forall m \in M$. This finishes the proof □

Proof of (b) $\Rightarrow$ (a): Let B be a finite A-algebra. It's finitely generated as module generators are also algebra generators. We need to show that $\forall b \in B$ is integral over A. In Lemma 2 we take $M := B$, $I = A$ and $\varphi: M \to M$, $m \mapsto bm$. We conclude: $\exists$ monic polynomial $f(x) \in A[x]$ s.t. $f(\varphi) = 0$. But applying $f(\varphi)$ to $1 \in B$ we get $f(\varphi)1 = f(b) = 0$. So b is integral over A. □

Exercise: Under the assumptions of Thm, if A is Noetherian, then B is Noetherian.

1.3) Consequences of Thm.

Corollary 1: i) If $f(x)\in A[x]$ is monic, then $A[x]/(f(x))$ is integral over A.

ii) If $\beta\in B$ is integral over A, then $A[\alpha]$, the A-subalgebra of B generated by α, is integral over A.

Proof: Using Example 1 & (b) $\Rightarrow$ (a) of Thm we get (i). In ii) if $f(x)\in A[x]$ is a monic polynomial w $f(\beta)=0$, then $A[x]/(f(x))\twoheadrightarrow A[\beta]$. Since $A[x]/(f(x))$ is integral (by (i)), $A[\beta]$ is also integral $\square$

Corollary 2 (transitivity of integral extensions): If B is an A-algebra integral over A, and C is a B-algebra integral over B, then C is integral over A (as an A-algebra).

Proof: Take $\gamma\in C$; it's integral over $B \rightsquigarrow \exists\, b_0,\dots b_{k-1}\in B$ s.t. $\gamma^k - b_{k-1}\gamma^{k-1}-\dots-b_0=0$. We write $A[b_0,\dots b_{k-1}]$ for the subalgebra of B generated by $b_0,\dots b_{k-1}$. So γ is integral over $A[b_0,\dots b_{k-1}]\subset B$. But $b_0,\dots b_{k-1}$ are integral over A. We use (a) $\Rightarrow$ (b) of Thm to show that $A[b_0,\dots b_{k-1}]$ is finite over A, while $A[b_0,\dots b_{k-1},\gamma]$, the subalgebra of C generated by $b_0,\dots,\gamma$, is finite over $A[b_0,\dots b_{k-1}]$. Using Lemma 1, we see that $A[b_0,\dots b_{k-1},\gamma]$ is finite over A. By (b) $\Rightarrow$ (a) of Thm, γ is integral over A and we are done. $\square$

2) Integral closure.

Proposition 1: Let B be an A-algebra. If $\alpha, \beta \in B$ are integral over A, then so are $\alpha+\beta, \alpha\beta, a\alpha$ ($\forall a \in A$).

Proof: Consider subalgebras $A[\alpha] \subset A[\alpha,\beta] \subset B$, $A[\alpha]$ is integral over A, $A[\alpha,\beta]$ is integral over $A[\alpha]$. By Corollary 2, $A[\alpha,\beta]$ is integral over A. Since $\alpha\beta, \alpha+\beta, a\alpha \in A[\alpha,\beta]$, they are integral over A. $\square$

Corollary/definition: The elements in B integral over A form an A-subalgebra of A called the integral closure of A in B.

We'll denote the integral closure $\bar{A}^B$. Note that this is a direct generalization of algebraic closures of fields.

Example: If $A = K \subset B = L$ are fields, then $\bar{K}^L$ is the algebraic closure of K in L.

Proposition 2: The integral closure of $\bar{A}^B$ in B is $\bar{A}^B$.

Proof: apply Corollary 2, left as exercise.