LECTURE 1: REMINDER ON AFFINE HECKE ALGEBRAS

SETH SHELLEY-ABRAHAMSON

Abstract. These are notes for a seminar talk at the MIT-Northeastern Spring 2017 Double Affine Hecke Algebras and Elliptic Hall Algebras (DAHAEHA) Seminar.

Contents

1. Goals

The purpose of this talk is to introduce affine and double affine Hecke algebras and certain structural results regarding these algebras. When possible, I will make all definitions and statements for general Cartan types. However, I will emphasize the type A case and use this case as an example throughout, as in the later parts of the seminar we will be primarily concerned with type A.

The main references are Macdonald's book [M] and Kirillov Jr.'s lecture notes [K]. Essentially everything in these notes can be found in those references in a more complete form.

2. Review of Coxeter Groups and Their Hecke Algebras

In this section we will quickly review Coxeter groups and their associated Hecke algebras. All statements and their proofs can be found in (one of) the references [GP] or [H]. We will largely omit proofs in this section, as these results and definitions are standard.

2.1. **Coxeter Groups.** Let I be a finite set, and let $m : I \times I \to \mathbb{Z}^{\geq 1} \cup \{\infty\}$ be a function satisfying $m(i,j) = m(j,i) \geq 2$ for all $i \neq j \in I$ and $m(i,i) = 1$ for all $i \in I$. One may view this data equivalently as a finite $(\mathbb{Z} \cup \{\infty\})$-labeled undirected graph Γ with vertex set I and edge labels at least 3 (by convention, a missing edge between i and j indicates $m(i,j) = 2$);

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we refer to Γ as the *Coxeter graph*. Given this data, let W be the group generated by the set

$$\{s_i : i \in I\}$$

with the relations

$$(s_i s_j)^{m(i,j)} = 1$$

(whenever $m(i,j) < \infty$). The groups W appearing in this manner are called *Coxeter groups*, and such a pair (W, I) is called a *Coxeter system*. Note that the generators s_i satisfy $s_i^2 = 1$. This allows the defining relations to be replaced with the *braid relations*

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

for all $i \neq j \in I$ whenever $m(i,j) < \infty$ (with $m(i,j)$ factors on each side) and the *quadratic relations*

$$s_i^2 = 1$$

for all $i \in I$. This presentation will be particularly convenient for us when we consider associated braid groups and Hecke algebras shortly.

Any Coxeter such group W admits a faithful real representation in which the generators s_i act by reflections; using this representation, one can show that for any $i \neq j \in I$ the order of the product $s_i s_j$ in W is precisely $m(i,j)$. In particular, the function $m(i,j)$, and hence the Coxeter graph Γ, describing the relations among the generators, is uniquely recovered from the Coxeter system (W, I). Note, however, that this data is not uniquely recovered from the Coxeter group W as an abstract group, and an abstract group may be a Coxeter group in many different ways. For example, when $n > 4$ is an odd integer, the Coxeter group B_n is isomorphic as an abstract group to the product $A_1 \times D_n$ (take the nontrivial element in A_1 to be $-1 \in B_n$).

Recall that a *finite real reflection group* is a finite subgroup $W \subset GL(V)$ of the general linear group $GL(V)$ of a finite-dimensional real vector space V that is generated by reflections, i.e. by elements $s \in W$ satisfying $\text{rank}(s-1) = 1$ and $s^2 = 1$. The following characterizes the finite Coxeter groups as the finite (real) reflection groups (with additional structure):

Theorem 2.1.1. *A finite group W is a real reflection group if and only if there is a generating subset $I \subset W$ such that (W, I) is a finite Coxeter system.*

Proof Sketch. This theorem is proved, for example, in Humphrey's book [H]. The idea of proof is as follows. Any Coxeter group admits a faithful representation generated by reflections in a finite-dimensional real vector space, and in particular finite Coxeter groups are finite real reflection groups. Conversely, given a finite real reflection group W with reflection representation V, choose a component $\mathcal{C}$ of the disconnected space

$$V^{reg} := V \setminus \cup_{s \in \text{Ref}(W)} \ker(s-1),$$

where $\text{Ref}(W) \subset W$ denotes the set of reflections in W with respect to its action on V. We refer to $\mathcal{C}$ as an (open) fundamental chamber for W. Then $\mathcal{C}$ is a simplicial cone with boundary defined by hyperplanes $\ker(s)$ for a certain subset $I \subset \text{Ref}(W)$ of the reflections (the *simple reflections*). Then I generates W, and (W, I) is a Coxeter system. The numbers $m(s, s')$ are obtained as the order of the products ss' for $s, s' \in I$ with $s \neq s'$. These orders $m(s, s')$ themselves are easily read off by the angle $2\pi/m(s, s')$ between the hyperplanes $\ker(s)$ and $\ker(s')$. $\square$

In particular, it follows from the classification of finite reflection groups that the irreducible finite Coxeter groups coincide with the finite Weyl groups (which come in types A through G) along with the dihedral groups $I_2(m)$ for $m \geq 3$ and the exceptional non-crystallographic Coxeter groups H_3 and H_4.

Again let (W, I) be a Coxeter system, finite or infinite. Any $w \in W$ equals some product $s_1 \cdots s_q$ of simple reflections. Let the *length* $l(w)$ of w be the minimal length of such an expression, and refer to any such minimal expression $w = s_{i_1} \cdots s_{i_{l(w)}}$ as a *reduced expression.* A typical element w of W admits many distinct reduced expressions, and it will be important to us to understand the relationship between these expressions. This question is answered by Matsumoto's Theorem. In particular, let RE denote the set of reduced expressions $\mathbf{s} = (s_{i_1}, ..., s_{i_l})$. Let $\sim_=$ denote the equivalence relation on RE given by $\mathbf{s} \sim_= \mathbf{s}'$ if and only if $s_{i_1} \cdots s_{i_l} = s_{i'_1} \cdots s_{i'_l}$. Let $\sim_{br}$ denote the equivalence relation on RE generated by the braid relations, i.e. by replacing a sequence $(s_{i_1}, s_{i_2}, ...)$ of length $m(i_1, i_2)$ with the sequence $(s_{i_2}, s_{i_1}, ...)$ of length $m(i_1, i_2)$. We then have:

Theorem 2.1.2 (Matsumoto)**.** $\sim_= \, = \, \sim_{br}$.

In other words, any two reduced expressions for the same element $w \in W$ are connected by a sequence of braid relations. The proof can be found in [L].

2.2. **Braid Groups.** Let (W, I) be a Coxeter system. The *braid group* B_W associated to (W, I) is the group generated by the set $\{T_i : i \in I\}$ subject to the *braid relations*

$$T_i T_j \cdots = T_j T_i \cdots$$

($m(i, j)$ factors on each side) for $i \neq j \in I$ with $m(i, j) \neq \infty$. In other words, the braid group B_W has a description by generators and relations identical to that of W except that the quadratic relations $s_i^2 = 1$ are omitted.

For any $w \in W$ and reduced expression $w = s_{i_1} \cdots s_{l(w)}$, it follows from the definition of the braid group and Matsumoto's theorem that the product $T_{i_1} \cdots T_{i_{l(w)}}$ is independent of the choice of reduced expression for w. We denote any such product by the symbol T_w. As $T_{i_1} = T_{s_{i_1}}$, it follows that the set $\{T_w : w \in W\}$ generates B_W, and it is immediate that the relation

$$T_w T_{w'} = T_{ww'} \quad \text{whenever} \quad l(ww') = l(w) + l(w')$$

holds in B_W. It is easy to see that this gives another presentation for B_W. Similarly, we may give a presentation for B_W by specifying generators $\{T_w : w \in W\}$ with relations

$$T_{s_i} T_w = T_{s_i w} \quad \text{whenever} \quad l(s_i w) > l(w)$$

(or the analogous "right handed" relations, or both types of relations simultaneously).

Clearly, there is a surjection $B_W \to W$ sending T_i to s_i, with kernel generated by the elements T_i^2. The kernel P_W is the *pure braid group.*

Remark 2.2.1. *When W is finite, so W is a finite real reflection group with reflection representation V, the braid group has a standard topological interpretation. In particular, let $V_\mathbb{C}$ denote the complexification of V, and let $V_\mathbb{C}^{reg}$ denote the set of points in $V_\mathbb{C}$ with trivial stabilizer in W (i.e., the complement of the reflection hyperplanes). Then there are identifications $\pi_1(V_\mathbb{C}^{reg}/W) = B_W$ and $\pi_1(V_\mathbb{C}^{reg}) = P_W$ compatible with the obvious short exact sequences.*

Example : **Type A** The symmetric group S_n on n letters is a real reflection group with respect to its standard representation by coordinate permutations in $\mathbb{R}^n$ - the transposition

(i, j) is given by reflection through the hyperplane $x_i = x_j$. This representation is faithful but not irreducible - the space $\{x : \sum_i x_i = 0\}$ is the *irreducible reflection representation* for S_n. A set of simple reflections can be given by the adjacent transpositions $(i, i+1)$ for $1 \leq i < n$, and the corresponding Coxeter graph is a line of $n-1$ connected dots (by convention an unlabeled connection indicates $m = 3$), and this type of reflection group is said to be of type A_{n-1}. When $W = S_n$, the braid group B_{S_n} is the familiar standard braid group B_n on n strands, and the pure braid group P_{S_n} is the standard pure braid group on n strands.

2.3. **Hecke Algebras.** In this section we will recall certain deformations of group algebras of Coxeter groups, the *Hecke algebras*. Let (W, I) be a Coxeter system. Let $\tau : I \to \mathbb{C}^\times$ be a function such that $\tau(i) = \tau(j)$ whenever s_i and s_j are conjugate in W, and write $\tau_i = \tau_{s_i} = \tau(i)$. The *Hecke algebra* $H_\tau(W, I)$ attached to the Coxeter system (W, I) and *parameter* τ is the $\mathbb{C}$-algebra with generators $\{T_i : i \in I\}$ and relations consisting of the braid relations seen above and the *Hecke relations* (or *quadratic relations*)

$$(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$$

for all $i \in I$. We will write $H_\tau(W)$ rather than $H_\tau(W, I)$ when the meaning is clear.

It is immediate from the Hecke relations that the generators T_i are invertible and that the Hecke relations can be equivalently written

$$T_i - T_i^{-1} = \tau_i - \tau_i^{-1}.$$

In particular, there is a natural surjection

$$\mathbb{C}B_W \to H_\tau(W), \quad T_i \mapsto T_i.$$

It follows that the Hecke algebra $H_\tau(W)$ has an alternative description as the quotient of the complex group algebra $\mathbb{C}B_W$ by the Hecke relations. We use the notation T_w for $w \in W$ to denote both elements of the braid group and their images in $H_\tau(W)$ when the meaning is clear.

When τ is the constant function 1, the Hecke relations read

$$T_i^2 = 1,$$

and in particular the Hecke algebra $H_1(W)$ is identified with the group algebra $\mathbb{C}W$ of W. In this way, the family of algebras $H_\tau(W)$ form a deformation of $\mathbb{C}W$. As it happens, this deformation is flat:

Theorem 2.3.1. *The set $\{T_w : w \in W\}$ forms a $\mathbb{C}$-basis of $H_\tau(W)$.*

The proof is standard. In particular, it is clear that the elements T_w span $H_\tau(W)$ because the span of the T_w contains 1, is stable under multiplication by the T_i, and the T_i generate $H_\tau(W)$. So, what one needs to do is to prove linear independence. This is achieved by the standard trick of writing down a representation of $H_\tau(W)$ in a space in which the linear operators by which the T_w act are manifestly linearly independent. In this case, one uses the regular representation as a model. In particular, one considers the $\mathbb{C}$-vector space H' with basis $\{e_w : w \in W\}$ and tries to define a representation of $H_\tau(W)$ in this space by letting the generator T_i act by

$$T_i(e_w) = \begin{cases} e_{s_i w} & \text{if } l(s_i w) > l(w) \\ e_{s_i w} + (\tau_i - \tau_i^{-1})e_w & \text{if } l(s_i w) < l(w). \end{cases}$$

It is obvious that the T_i satisfy the Hecke relations, so to see that this defines a representation of $H_\tau(W)$ one need only check the braid relations. For this, one considers the operators on H' that should correspond to right multiplication by T_i, and checks that the "left multiplication" operators commute with the "right multiplication" operators, reducing the check of the braid relations to the check that the braid relations hold when applied to the element e_1, which is obviously true. Details can be found, for example, in Humphrey's book [H].

Remark 2.3.2. *The same proof shows that when the parameter τ is viewed as a formal invertible variable(s), the Hecke algebra $H_\tau(W)$ is a free $\mathbb{C}[\tau_i^{\pm 1}]$-module with basis $\{T_w : w \in W\}$. The case of numeric τ is then obtained by specialization to $\mathbb{C}$.*

Remark 2.3.3. *In some other contexts, the presentation/definition of Hecke algebras attached to (W, I) looks slightly different in that the Hecke relation seen above. Specifically, the Hecke relation may be of the form $(T+1)(T-q) = 0$ (as one sees, for example, in the context of Hecke algebras attached to BN pairs) or $(T-1)(T+q)$ (as one sees, for example, in the context of the KZ functor appearing for rational Cherednik algebras). These two forms involving q are easily reconciled by a rescaling of the generators T_i (notice that the braid relations are homogenous). The version seen above with τ (often the letter v is used instead) amounts to choosing a square root of q and rescaling the generators, which is important from some representation theoretic perspective that we won't discuss here.*

3. Affine Hecke Algebras

3.1. **Affine Root Systems.** We will assume the reader is familiar with finite root systems. For the rest of this talk, any finite Coxeter group appearing will be a Weyl group, i.e. a crystallographic real reflection group (i.e., types H and I are excluded). Similarly, all finite root systems appearing will also be crystallographic. All definitions and results in this section can be found in [M, Chapter 1]. Also, the notion of "affine root system" I will use here is in the sense appearing in [M]; the roots appearing in these affine root systems are the real roots of the affine root systems discussed in the context of Kac-Moody Lie algebras.

Affine root systems are related and similar in spirit to finite root systems. The essential differences are that in the affine case there are infinitely many affine roots and an affine root determines an affine reflection, i.e. a reflection through an affine hyperplane, while in the finite case there are finitely many roots and a root determines an honest reflection through a linear hyperplane. The standard theory of finite root systems has an analogue for affine root systems. Specifically, one has an axiomatic definition and concise classification involving diagrams closely related to Dynkin diagrams, notions of affine Weyl groups with length functions, convenient fundamental domains (now called *alcoves* rather than *Weyl chambers*), etc. For the sake of concreteness, simplifying the notation, and with the goals of this seminar in mind, we will not consider arbitrary affine root systems, but rather only certain affine root systems R^a that are easily associated to finite irreducible reduced root systems R. This is not a significant restriction, and in fact the classification of arbitrary affine root systems is easily stated in terms of these affine root systems R^a and some mild additional constructions (see [M, Chapter 1, Section 3]). For the full-fledged general case of all the material to follow in this talk, see Macdonald's book [M].

3.1.1. *Affine functions, affine reflections, and translations.* Fix a Euclidean vector space V with inner product $(\cdot, \cdot)$. Identify V with its dual V^* via the inner product. Let F be the set of affine-linear functions on V, i.e. the functions $f : V \to \mathbb{R}$ that are sums of linear functionals and constant functions. Then $F = V \oplus \mathbb{R}\delta$, where $\delta \in F$ is the constant function with value 1. For any $f \in F$, let $Df \in V$ denote the projection of f to V under the splitting $F = V \oplus \mathbb{R}\delta$ (then Df is the gradient of f in the usual sense of calculus, and $f = Df + f(0)$). Extend the inner product $(\cdot,\cdot)$ to F by defining

$$(v + c\delta, w + d\delta) = (v, w)$$

for all $v, w \in V$ and $c, d \in \mathbb{R}$. On F, the form $(\cdot,\cdot)$ is a degenerate symmetric bilinear form with kernel $\mathbb{R}\delta$.

Let $f \in F$ be non-constant. Then $(f, f) > 0$, and we define

$$f^\vee := \frac{2f}{(f, f)}.$$

The subset $f^{-1}(0) \subset V$ is an affine hyperplane, and we denote by s_f the orthogonal reflection in V through this affine hyperplane. The affine reflection s_f is given by a familiar formula:

$$s_f(x) = x - f^\vee(x)Df = x - f(x)Df^\vee.$$

Then s_f also acts on functions g on V by the usual formula $s_f.g = g \circ s_f^{-1} = g \circ s_f$, and clearly this action preserves the space F. This action of s_f is given by the familiar formula

$$s_f(g) = g - (g, f^\vee)f = f - (g, f)f^\vee.$$

Naturally, a translation $t : V \to V$ is an affine linear transformation of V of the form $t(x) = x + v$, for some $v \in V$; we denote this translation by $t(v)$. For a subset $X \subset V$, we define $t(X) := \{t(v) : v \in X\}$. When $L \subset V$ is a lattice, $t(L)$ is a lattice isomorphic to L. Any translation $t(v)$ also acts on the space F of affine-linear functions on V:

$$t(v)(f) = f - (v, f)\delta.$$

3.1.2. *The affine root systems R^a.* Fix a finite irreducible reduced root system $R \subset V$ spanning V (so R has rank $\dim V$). As usual, let $Q := \sum_{\alpha \in R} \mathbb{Z}\alpha$ denote the root lattice, let $Q^\vee := \sum_{\alpha \in R} \mathbb{Z}\alpha^\vee$ denote the coroot lattice, let $P \subset V$ be the weight lattice (i.e. those $\lambda \in V$ with integral pairing with all coroots), and let $P^\vee$ be the coroot lattice (i.e. those $\lambda \in V$ with integral pairing with all roots.

Define the associated *affine root system* R^a to be the subset of F given by:

$$R^a := \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\}.$$

We call elements $a \in R^a$ *affine roots.* Note that R is a subset of R^a. Let W denote the Weyl group attached to R, i.e. the subgroup of $GL(V)$ generated by the reflections $\{s_\alpha : \alpha \in R\}$. Similarly, let W^a, the *affine Weyl group*, be the group of invertible affine transformations of V generated by the s_a for $a \in R^a$. Clearly, $W \subset W^a$.

Proposition 3.1.1. *The lattice $t(Q^\vee)$ is a normal subgroup of W^a and $W^a = W \ltimes t(Q^\vee)$.*

Proof. Let $a = \alpha + n\delta$ be an affine root. Then we have

$$s_\alpha s_a(x) = s_\alpha(x - ((x, \alpha) + n)\alpha^\vee) = x - (x, \alpha)\alpha^\vee + ((x, \alpha) + n)\alpha^\vee = x + n\alpha^\vee$$

so $s_a s_\alpha = t(n\alpha^\vee)$. It follows that $t(Q^\vee)$ is a subgroup of W^a. We also see that $s_a = t(n\alpha^\vee)s_\alpha$, so W^a is generated by W and $t(Q^\vee)$. It's also clear that for any $w \in W$ and $\lambda \in Q^\vee$, we have $wt(\lambda)w^{-1} = t(w\lambda)$, so $t(Q^\vee)$ is normal in W^a and $W^a = W.t(Q^\vee)$. As W fixes $0 \in V$, it follows that $W \cap t(Q^\vee) = 1$ and the claim follows. $\square$

It's now easy to see the following:

Proposition 3.1.2.
(1) R^a spans F.
(2) $s_a(b) \in R^a$ for all $a, b \in R^a$.
(3) $(a^\vee, b) \in \mathbb{Z}$ for all $a, b \in R^a$.
(4) the action of W^a on V is proper, i.e. for any compact subset $K \subset V$ the set of $w \in W^a$ such that $wK \cap K \neq \emptyset$ is finite.

Remark 3.1.3. *The preceding proposition says that R^a is indeed an affine root system in the axiomatic sense.*

3.1.3. *Alcoves and positive and simple roots.* Let $\alpha_1, ..., \alpha_n$ be a choice of simple positive roots for the finite root system R, determining a Weyl chamber

$$\mathcal{C} := \{x \in V : \alpha_i(x) \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Recall that the Weyl group W of R is then a Coxeter group with respect to the corresponding simple reflections through the walls of $\mathcal{C}$. Recall also that the set of *positive roots* $R_+ \subset R$ is given by

$$R_+ := \{\alpha \in R : \alpha(x) \geq 0 \text{ for all } x \in \mathcal{C}\},$$

the negative roots are $R_- := -R_+$, and that every positive root is a linear combination of positive simple roots with nonnegative integer coefficients. There is a very similar story for the affine Weyl group W^a that we now explain.

The set of affine hyperplanes $\{a^{-1}(0) : a \in R^a\}$ is a locally finite arrangement of real hyperplanes in V, and it follows that the complement is open and has a natural W^a-action. A connected component of this complement is called an *alcove*, and we denote the set of alcoves by $\mathcal{A}$. Let A denote the closure of the unique alcove A° contained in $\mathcal{C}$ and such that $0 \in \overline{A^\circ}$. We call A an *affine Weyl chamber* for R^a. Clearly, A is a n-dimensional simplex

$$A = \{x \in V : a_i(x) \geq 0 \text{ for } 0 \leq i \leq n\}$$

with $n+1$ walls given by affine hyperplanes $\{a_i^{-1}(0) : 0 \leq i \leq n\}$ for some uniquely determined affine roots $a_0, a_1, ..., a_n$. We call the a_i the *simple affine roots*, or just *simple roots* when the meaning is clear. Up to reordering, we have $a_i = \alpha_i$ for $1 \leq i \leq n$ (corresponding to the walls that A shares with $\mathcal{C}$) and a root a_0 with nonzero constant term that defines the remaining wall of A. Let I be the set $\{0, ..., n\}$ and let I_0 be the set $I \backslash \{0\}$. For $0 \leq i \leq n$, define the i^{th} *simple reflection* by $s_i := s_{a_i}$. Sometimes by abuse of notation I'll confuse $i \in I$ with s_i.

Let's fix some terminology and notation. Define the *positive (affine) roots* $R^a_+ \subset R^a$ by

$$R^a_+ := \{a \in S : a(x) \geq 0 \text{ for all } x \in A\}$$

and define the *negative (affine) roots* R^a_- by $R^a_- := -R^a_+$.

Let $\theta \in R^+$ denote the highest root of the finite root system R. For $i \in I_0$, let $m_i \in \mathbb{Z}^{>0}$ denote the unique positive integers such that

$$\theta = \sum_{i \in I_0} m_i \alpha_i.$$

Proposition 3.1.4. $a_0 = -\theta + \delta$.

Proof. Certainly $a_0 = \alpha + n\delta$ for some $\alpha \in R$ and $n \in \mathbb{Z}$. As $0 \in A \backslash a_0^{-1}(0)$, we have $a_0(0) > 0$, so $n > 0$. Certainly for any $n > 0$ the simplex A' defined by any $\alpha + n\delta$ along with $\alpha_1, ..., \alpha_n$ contains A, so we have $n = 1$. Any root α can be written $\alpha = \sum_i k_i \alpha_i$, and we have $k_i \geq -m_i$ for all i. As the 1-dimensional faces of $\mathcal{C}$ are given by $\mathbb{R}^{\geq 0}\lambda_i$, where the λ_i are the fundamental weights, it is clear that simplex determined by $-\theta + \delta$ and the α_i is contained in the simplex determined by any $\alpha + \delta$ and the α_i, and the claim follows. □

Corollary 3.1.5. *A has the alternative, slightly more concrete description:*

$$A = \{x \in V : (\alpha_i, x) \geq 0 \text{ for } i \in I_0 \text{ and } (x, \theta) \leq 1\}.$$

We can now give a convenient description of the positive affine roots and see that the simple positive affine roots give a basis for R^a in the same familiar way that the simple positive roots give a basis for R:

Corollary 3.1.6.
(1) R^a_+ has the following description:

$$R^a_+ = \{\alpha + r\delta : \alpha \in R, r \geq \chi(\alpha)\}$$

where χ is the indicator function on R of the subset $R^- \subset R$ of negative roots.
(2) $R^a = R^a_+ \coprod R^a_-$ and every positive affine root $a \in R^a_+$ is of the form

$$a = \sum_{i \in I} n_i a_i$$

for some non-negative integers n_i.

Proof. (1) follows easily from the previous corollary (in particular, note that any root $\alpha \in R$ is positive if and only if $\alpha(x) \in [0, 1]$ for all $x \in A$). (2) follows from (1) and the fact that for any root $\alpha \in R$ the difference $\theta - \alpha$ is a sum of positive simple roots α_i with nonnegative coefficients. □

3.1.4. *W^a as a Coxeter group, and its length function.* Proofs of the following two propositions can be found in [H, Chapter 4].

Proposition 3.1.7.
(1) The simple affine reflections $s_0, ..., s_n$ generate W^a, and in fact the pairs (W^a, I) and (W, I_0) are Coxeter systems. As usual, the entries $m(i, j)$ of the Coxeter matrix are read off from the relative angles of the affine hyperplanes $a_i^{-1}(0)$ (equivalently, from the pairings $(a_i^\vee, a_j)$).
(2) W^a acts simply transitively on the set $\mathcal{A}$ of alcoves, and A is a fundamental domain for the action of W^a on V.

Let $l : W^a \to \mathbb{Z}^{\geq 0}$ be the length function on W^a as a Coxeter group with generating simple reflections $s_0, ..., s_n$.

Proposition 3.1.8. *For any $w \in W^a$, the length $l(w)$ can be equivalently described as:*
(a) The length of a reduced expression for w.
(b) $l(w) = |R^a_+ \cap w^{-1}R^a_-|$
(c) The number of affine hyperplanes $a^{-1}(0)$ with $a \in R^a_+$ separating A and wA.

3.2. **Extended Affine Weyl Groups.** In light of the decomposition $W^a = W \ltimes t(Q^\vee)$, we can enlarge the group W^a by replacing the coroot lattice $Q^\vee$ with a larger lattice L' on which W acts.

Definition 3.2.1. *The* extended affine Weyl group W^{ae} *attached to R^a is the semidirect product*

$$W^{ae} := W \ltimes t(P^\vee).$$

Clearly, W^{ae} admits a natural action on V extending that of W^a.

Proposition 3.2.2. *The extended affine Weyl group W^{ae} acts on the affine roots R^a.*

Proof. For a coweight $\lambda \in P^\vee$, the translation $t(\lambda)$ acts on F by

$$t(\lambda)(a) = a - (\lambda, a)\delta.$$

Any $\lambda \in P^\vee$ has integral pairing with any $a \in R^a$, and the claim follows. □

So it follows as well that W^{ae} acts on the set of affine hyperplanes and the set of alcoves $\mathcal{A}$ etc., but the action on the alcoves is no longer faithful.

Now we want to relate the extended affine Weyl group W^{ae} to the affine Weyl group W^a:

Definition 3.2.3. *Extend the definition of the length function l from the affine Weyl group W^a to the extended affine Weyl group W^{ae} by either of the two equivalent definitions (b) or (c) appearing in Proposition 3.1.8*

Definition 3.2.4. *Let Ω be the finite group*

$$\Omega := \{w \in W^{ae} : l(w) = 0\} = \{w \in W^{ae} : wA = A\}.$$

Definition 3.2.5. *Recall that a weight $\lambda \in P$ is called* miniscule *if $0 \leq (\lambda, \alpha^\vee) \leq 1$ for* every *positive root $\alpha \in R^+$. Similarly, recall that a coweight $\lambda' \in P^\vee$ is called* miniscule *if $0 \leq (\lambda', \alpha) \leq 1$ for every positive root $\alpha \in R^+$.*

Recall that the miniscule weights form a system of representatives for P/Q, just as the miniscule coweights do for $P^\vee/Q^\vee$.

Proposition 3.2.6.
(1) $W^{ae} = \Omega \ltimes W^a$.
(2) $\Omega \cong P^\vee/Q^\vee$. In particular, every $\pi_r \in \Omega$ is of the form

$$\pi_r = t(b_r)w_r$$

for some miniscule coweight $b_r \in P^\vee$ and $w_r \in W^a$.

Proof. (1) follows immediately from the facts that W^a acts simply transitively on the set of alcoves and that Ω is the set-wise stabilizer of the alcove A. (2) follows from the semidirect product decomposition/definition of W^a and W^{ae} and the fact that the minscule coweights form a system of representatives for $P^\vee/Q^\vee$. □

Remark 3.2.7. *In fact, in (2) above one has $w_r \in W$ (see* [M, Chapter 2, Section 5]*).*

Remark 3.2.8. *The faithful action of Ω on the alcove A gives rise to a faithful action of Ω on the set of walls of A, and therefore on the set of simple roots. If $\pi_r(a_i) = a_j$, then $\pi_r s_i \pi_r^{-1} = s_j$. This describes the semidirect product appearing in (1) concretely, and we see that the action of Ω on W^a is by diagram automorphisms.*

Let P_+ denote the dominant weights (i.e. those $\lambda \in P$ with $(\lambda, \alpha_i^\vee) \geq 0$ for all $i \in I_0$) and let $P_+^\vee$ denote the dominant coweights (i.e. those $\lambda' \in P^\vee$ with $(\lambda', \alpha_i) \geq 0$ for all $i \in I_0$). As usual, let $\rho \in P_+$ denote the half sum of the positive roots.

We have the following facts about the length function l on the extended affine Weyl group W^{ae}:

Proposition 3.2.9.

(1) The restriction of the length function on W^{ae} to W^a coincides with the usual length function, and $l(\pi w) = l(w\pi) = l(w)$ for all $\pi \in \Omega$, $w \in W^{ae}$.

(2) $l(ws_i) = \begin{cases} l(w)+1 & \text{if } w(a_i) \in R^a_+ \\ l(w)-1 & \text{if } w(a_i) \in R^a_- \end{cases}$

(3) If $w \in W^{ae}$ an $\lambda \in P^\vee$, then

$$l(wt(\lambda)) = \sum_{\alpha \in R^+} |(\lambda, \alpha) + \chi(w\alpha)|$$

where χ is the indicator function of the negative roots $R_- \subset R$.

Proof. (1) is immediate from the definition and (2) follows from (1), the definition of l, and the fact that s_i permutes the set $R^a_+ \backslash \{a_i\}$. A proof of (3) can be found in [M, Chapter 2] - it is not difficult and it makes use of the description

$$R^a_+ = \{\alpha + r\delta : \alpha \in R, r \geq \chi(\alpha)\}.$$

□

The translation elements $t(\lambda) \in W^{ae}$ will be of particular relevance in what follows, so we record some facts about these elements and the length function in the following corollary:

Corollary 3.2.10.

(1) If $\lambda \in P^\vee$, then $l(t(\lambda)) = 2(\lambda^+, \rho)$, where λ^+ is the dominant coweight lying in the W-orbit of λ.

(2) If $\lambda \in P^\vee_+$, then $l(wt(\lambda)) = l(w) + l(t(\lambda))$.

(3) If $(\lambda, \alpha_i) = 0$ for any $i \in I_0$, then $l(t(\lambda)s_i) = l(s_i t(\lambda)) = l(t(\lambda)) + 1$

(4) If $(\lambda, \alpha_i) = -1$, then $l(s_i t(\lambda)) = l(t(\lambda)) - 1$.

Proof. Follows immediately from Proposition 3.2.9(3). □

3.3. **Affine and Extended Affine Braid Groups.**

Definition 3.3.1. *The* affine braid group B^a *attached to the affine Weyl group W^a is the braid group attached to the Coxeter system (W^a, I). The* extended affine braid group B^{ae} *attached to the extended affine Weyl group W^{ae} has exactly the same description as B^a except with W^{ae} and its length function in place of W^a. In particular, B^{ae} has generators T_w for $w \in W^{ae}$ and relations $T_w T'_w = T_{ww'}$ whenever $l(ww') = l(w) + l(w')$.*

Remark 3.3.2. *Like for finite-type braid groups, the affine braid group B^a has a topological interpretation as the fundamental group $\pi_1(V^{reg}_{\mathbb{C}}/W^a)$ where*

$$V^{reg}_{\mathbb{C}} = (\mathbb{C} \otimes_{\mathbb{R}} V) \backslash \cup_{a \in R^a} (1 \otimes a)^{-1}(0)$$

(this quotient is sensible because the action of W^a on V is proper). In light of the decomposition $W^a = W \ltimes t(Q^\vee)$, it is easy to see that $V^{reg}_{\mathbb{C}}/W^a$ can also be described as T^{reg}/W, where $T = V^{reg}_{\mathbb{C}}/t(Q^\vee)$ is the complement of the corresponding hypersurfaces in the complex torus $V_{\mathbb{C}}/t(Q^\vee)$.

Theorem 3.3.3. *The elements T_π with $\pi \in \Omega$ form a subgroup of B^{ae} isomorphic to Ω, and we have*

$$B^{ae} = \Omega \ltimes B^a$$

where the action of Ω *on* B^a *is by the same diagram automorphisms discussed earlier, i.e. if* $\pi_r(a_i) = a_j$ *then* $\pi_r T_i \pi_r^{-1} = T_j$*, where* T_i *denotes* T_{s_i} *and* π_r *denotes* T_{π_r}*.*

Proof. This is easy to see from our previous discussion of braid groups and the fact that $\pi_r s_i = s_j \pi_r$ and that $l(\pi_r s_i) = 1 = l(s_i)$ etc. □

Along with the description of B^a by generators and relations from our discussion of Coxeter groups, this gives a more tractable description of B^{ae} by generators and relations.

Recall that the affine and extended affine Weyl groups had the semidirect product decompositions

$$W^a = W \ltimes t(Q^\vee) \qquad W^{ae} = W \ltimes t(P^\vee).$$

We now will upgrade this decomposition to the affine and extended affine braid groups, which will allow us to do the same for the affine Hecke algebras to come.

Definition 3.3.4. *For* $\lambda \in P^\vee$*, define elements* $Y^\lambda \in B^{ae}$ *by*

(1) $Y^\lambda = T_{t(\lambda)}$ *if* $\lambda \in P^\vee_+$

(2) $Y^\lambda = Y^\mu (Y^\nu)^{-1}$ *if* $\lambda = \mu - \nu$ *with* $\mu, \nu \in P^\vee_+$*.*

Theorem 3.3.5. Y^λ *is well-defined for all* $\lambda \in P^\vee$*, and we have* $Y^\lambda Y^\mu = Y^{\lambda+\mu}$ *for all* $\lambda, \mu \in P^\vee$*. The mapping* $\lambda \mapsto Y^\lambda$ *determines a lattice isomorphism between* $P^\vee$ *and the subgroup* $\{Y^\lambda : \lambda \in P^\vee\}$ *of* B^{ae}*.*

Proof. Clearly the Y^λ are well-defined for $\lambda \in P^\vee_+$ and satisfy $Y^\lambda Y^\mu = Y^{\lambda+\mu}$. It follows immediately that the Y^λ are well-defined for all $\lambda \in P^\vee$ and satisfy $Y^\lambda Y^\mu = Y^{\lambda+\mu}$. The restriction of the natural surjection $B^{ae} \to W^{ae}$ gives the inverse lattice isomorphism, completing the proof.

□

We will now describe the commutation relations between elements Y^λ and T_i. Note that as the lattice $\{Y^\lambda : \lambda \in P^\vee\}$ is generated by the elements Y^{ω_i}, where the $\omega_i \in P^\vee_+$ are the fundamental dominant coweights, it suffices to explain the commutation relations between Y^λ and T_i in the case that $(\alpha_i, \lambda) \in \{0, 1\}$.

Theorem 3.3.6.

(1) The elements Y^λ*,* $\lambda \in P^\vee$*, and the elements* $T_1, ..., T_n$ *together generate* B^{ae} *as a group.*

(2) If $(\lambda, \alpha_i) = 0$ *for some* $i \in I_0$*, then* $T_i Y^\lambda = Y^\lambda T_i$*.*

(3) If $(\lambda, \alpha_i) = 1$ *for some* $i \in I_0$*, then* $Y^\lambda = T_i Y^{s_i \lambda} T_i$*.*

Proof. For statement (1), in view of the decomposition $B^{ae} = \Omega \ltimes B^a$, it follows that the elements Y^λ together with the $T_0, ..., T_n$ generate B^{ae}, so we need only understand why generator T_0 is redundant in this collection. But this follows from the equality $s_0 s_\theta = t(\theta^\vee)$ in W^a. Indeed, note that

$$s_\theta(a_0) = s_\theta(-\theta + \delta) = \theta + \delta \in R^a_+$$

so the expression $s_0 \cdot s_\theta$ is reduced (i.e. $l(s_0 s_\theta) = l(s_0) + l(s_\theta)$). It follows that we have an equality

$$T_0 T_{s_\theta} = T_{t(\theta^\vee)} = Y^{\theta^\vee}$$

in the affine braid group.

In the case $\lambda \in P^\vee_+$, (2) follows from Corollary 3.2.10 which says that $l(t(\lambda)s_i) = l(s_i t(\lambda)) = l(t(\lambda)) + 1$. This then gives

$$T_i Y^\lambda = T_i T_{t(\lambda)} = T_{s_i t(\lambda)} = T_{t(\lambda) s_i} = Y^\lambda T_i$$

as needed. The general case then follows from the fact that any $\lambda \in P^\vee$ with $(\lambda, \alpha_i) = 0$ can be written as $\lambda = \mu - \nu$ with $\mu, \nu \in P_+^\vee$ and $(\mu, \alpha_i) = (\nu, \alpha_i) = 0$.

For case (3), we can again reduce to the case $\lambda \in P_+^\vee$ by noticing that any $\lambda \in P^\vee$ satisfying $(\lambda, \alpha_i) = 1$ can be written as $\lambda = \mu - \nu$ for some $\mu, \nu \in P_+^\vee$ satisfying $(\mu, \alpha_i) = 1$ and $(\nu, \alpha_i) = 0$. So take any $\lambda \in P_+^\vee$ satisfying $(\lambda, \alpha_i) = 1$. Define another element

$$\pi := \lambda + s_i\lambda = 2\lambda - \alpha_i^\vee.$$

Note that $\pi \in P_+^\vee$. From Corollary 3.2.10, we know that if $l(t(\lambda)) = 2(\lambda, \rho) = p$ then $l(t(\pi)) = 2p - 2$. From the same corollary (statement (3)) we also know that $l(s_i t(\pi)) = l(t(\pi)) + 1 = 2p - 1$, and from statement (4) we know that $l(t(\lambda)s_i) = l(t(\lambda)) - 1 = p - 1$. It follows that each side of the equality

$$s_i \cdot t(\pi) = (t(\lambda)s_i) \cdot t(\lambda)$$

is a reduced expression, and therefore that

$$T_i Y^\pi = T_{t(\lambda)s_i} Y^\lambda.$$

But $Y^\pi = Y^{s_i\lambda}Y^\lambda$ and $T_{t(\lambda)s_i} = T_{t(\lambda)}T_i^{-1} = Y^\lambda T_i^{-1}$. Rearranging the above equality, we get

$$T_i Y^{s_i\lambda} T_i = Y^\lambda$$

as needed. □

We can now state a presentation for B^{ae} analogous to the decomposition $W^{ae} = W \ltimes t(P^\vee)$ of the extended affine Weyl group W^{ae} seen earlier.

Theorem 3.3.7. *The extended affine braid group B^{ae} is generated by the finite-type braid group B (generated by $T_1, ..., T_n$) and the lattice $Y^{P^\vee}$ subject only to the relations appearing in (2) and (3) of the previous Theorem 3.3.6.*

The proof can be found in [M, Section 3.3]. The idea is to define elements T_0 and U_i in the group described by generators and relations in the theorem, and then to show by calculation that the relations of the extended affine braid group hold.

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