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Supersymmetry

Homework 11

Exercise 9.2.

Exercise 1.

The potential in the model described by (8.4) is of the form $V = F_K^* F_K$ with

$$F_K^* = -(\lambda_K + m_{iK} A_i + g_{ijK} A_i A_j),$$

with

$$\lambda_K = \lambda \delta_{0K}, \quad m_{ij} = m(\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j}), \quad g_{ijK} = g(\delta_{0i}\delta_{1j}\delta_{1K} + \delta_{1i}\delta_{0j}\delta_{1K} + \delta_{1i}\delta_{1j}\delta_{0K}).$$

Thus, the auxiliary fields are

$$F_0^* = -(\lambda + g A_1^2),$$

$$F_1^* = -(m A_2 + 2g A_0 A_1),$$

$$F_2^* = -m A_1.$$

Following O'Raifeartaigh's original paper, we will assume that the parameters λ, m and g are real. Then, expanding in F_0 and F_2,

$$V = \lambda^2 + \lambda g (A_1^2 + (A_1^*)^2) + m^2 A_1 A_1^* + F_1^* F_1 + g^2 (A_1 A_1^*)^2$$

Let $a_1 = \mathrm{Re}\, A_1$ and $b_1 = \mathrm{Im}\, A_1$. Then

$$V = \lambda^2 + \lambda g\left(a_1^2 + \cancel{2ia_1b_1} - b_1^2 + a_1^2 - \cancel{2ia_1b_1} - b_1^2\right)$$

$$+ m^2(a_1^2 + b_1^2) + g^2(a_1^2+b_1^2)^2 + F_1^* F_1$$

$$= \lambda^2 + (m^2 + 2\lambda g)a_1^2 + (m^2 - 2\lambda g)b_1^2 + g^2(a_1^2+b_1^2)^2 + F_1^*F_1$$

$$\geq \lambda^2 + (m^2 - 2\lambda g)b_1^2 .$$

In particular, if $m^2 > 2\lambda g$, $V \geq \lambda^2$. This minimum value is in fact achieved when $A_1 = A_2 = 0$. Indeed, in that case $F_1 = 0$ and $V = \lambda^2$ independently of the value of A_0.

Exercise 2

To obtain the mass spectrum, we expand the superpotential of the model

$$\int d^2\theta \left(\lambda \Phi_0 + m\Phi_1\Phi_2 + g\Phi_0\Phi_1^2\right) + \text{h.c.}$$

We clearly see

$$\int d^2\theta\, \Phi_0 = F_0$$

$$\int d^2\theta\, \Phi_1\Phi_2 = -\psi_1\psi_2 + A_1F_2 + F_1A_2.$$

$$\int d^2\theta\, \Phi_0\Phi_1^2 = A_0A_1F_1 - A_0\psi_1\psi_1 + A_0F_1A_1 - \psi_0A_1\psi_1 - \psi_0\psi_1A_1$$
$$F_0A_1A_1.$$

where we have used (1.30) of the Notes

Thus the superpotential is

$$\lambda F_0 + m A_1 F_2 + m F_1 A_2 + 2g A_0 A_1 F_1 + g (A_1)^2 F_0$$
$$- m \psi_1 \psi_2 - g A_0 \psi_1 \psi_1 - 2g A_1 \psi_0 \psi_1 + h.c.$$

By expanding the auxiliary fields, the quadratic terms of the superpotential are

$$- \lambda g (A_1^*)^2 - m^2 (A_1 A_1^*) - m^2 (A_2 A_2^*) - g \lambda (A_1)^2 - m \psi_1 \psi_2 + h.c.$$

$$= -2 \left(2\lambda g (a_1^2 - b_1^2) + m^2 (a_1^2 + b_1^2 + a_2^2 + b_2^2) \right) - (m \psi_1 \psi_2 + h.c.)$$

$$= -2 \left((m^2 + 2\lambda g) a_1^2 + (m^2 - 2\lambda g) b_1^2 + m^2 a_2^2 + m^2 b_2^2 \right) - (m \psi_1 \psi_2 + h.c)$$

We thus obtain 6 real bosons $(a_0, b_0, a_1, b_1, a_2, b_2)$ with respective masses $(0, 0, \sqrt{m^2 + 2\lambda g}, \sqrt{m^2 - 2\lambda g}, m, m)$.

In here $a_0 = \operatorname{Re} A_0$, $b_0 = \operatorname{Im} A_0$, $a_1 = \operatorname{Re} A_1$ and $b_1 = \operatorname{Im} A_1$.

The factor of two appears because

$$\partial_\mu A^* \partial^\mu A = 2 \left(\frac{1}{2} \partial_\mu a \partial^\mu a + \frac{1}{2} \partial_\mu b \partial^\mu b \right)$$

for some general scalar field $A = a + i b$. For the fermionic part, we have

$$m\psi_1\psi_2 = \begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix}0 & m/2 \\ m/2 & 0\end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}.$$

The eigenvalues are those M satisfying

$$M^2 - \frac{m^2}{4} = 0,$$

i.e. $M = \pm \frac{m}{2}$. The corresponding orthonormalized eigenbasis is $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$. We thus obtain two fermions with mass m (see (4.1) of the notes) given by $\frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ and $\frac{i}{\sqrt{2}}(\psi_1 - \psi_2)$ and ψ_0 with 0 mass. We further have

$$\frac{\partial F_0^*}{\partial A_0} = 0, \quad \frac{\partial F_0^*}{\partial A_1} = -2g, \quad \frac{\partial F_0^*}{\partial A_2} = 0,$$

$$\frac{\partial F_1^*}{\partial A_0} = -2gA_1, \quad \frac{\partial F_1^*}{\partial A_1} = -2gA_0, \quad \frac{\partial F_1^*}{\partial A_2} = -m,$$

$$\frac{\partial F_2^*}{\partial A_0} = 0, \quad \frac{\partial F_2^*}{\partial A_1} = -m, \quad \frac{\partial F_2^*}{\partial A_2} = 0.$$

Thus

$$\det\left(\frac{\partial F_k^*}{\partial A_i}\right) = 0.$$

3. Recall that the most general supersymmetric renormalizable

Lagrangian made out of chiral fields is of the form

$$S = \int d^4x \left\{ \int d^4\theta\, \bar{\Phi}_K \Phi_K + \left[\int d^2\theta \left(\lambda_K \Phi_K + \frac{1}{2} M_{K\ell} \Phi_K \Phi_\ell + \frac{1}{3} g_{jK\ell} \Phi_j \Phi_K \Phi_\ell + h.c. \right) \right] \right\}$$

with $M_{K\ell}$ and $g_{jK\ell}$ symmetric on all of their indices.

Under R-symmetry the kinetic term of the action transforms is invariant

$$\int d^4x \int d^4\theta\, \bar{\Phi}_K \Phi_K \longmapsto \int d^4x \int d^4\theta\, \cancel{e^{-2in_K K}}\, \bar{\Phi}_K(\bar{y}, e^{iK}\bar{\theta})\, \cancel{e^{2in_K K}}\, \Phi_K(y, e^{-iK}\theta)$$

$$= \int d^4x \int d^2\theta\, \cancel{e^{-2iK}}\, d^2\bar{\theta}\, \cancel{e^{2iK}}\, \bar{\Phi}_K(\bar{y}, \bar{\theta})\, \Phi_K(y, \theta)$$

$$= \int d^4x \int d^4\theta\, \bar{\Phi}_K \Phi_K.$$

However, invariance under R-symmetry restricts the coefficients of the superpotential. Under R-symmetry, the superpotential transforms into

$$\int d^2\theta\, e^{-2iK} \Big(\lambda_K e^{2in_K K} \Phi_K + \frac{1}{2} M_{K\ell}\, e^{2i(n_K + n_\ell)K} \Phi_K \Phi_\ell$$

$$+ \frac{1}{3} g_{jK\ell}\, e^{2i(n_j + n_K + n_\ell)K} \Phi_j \Phi_K \Phi_\ell \Big) + h.c.$$

Thus, R-invariance demands that

$$\lambda_k \neq 0 \Longrightarrow n_k = 1$$

$$M_{kl} \neq 0 \Longrightarrow n_k + n_l = 1$$

$$g_{jkl} \neq 0 \Longrightarrow n_j + n_k + n_l = 1.$$

Therefore, with the charges of the problem, our action is R-invariant if and only if, up to permutations, the only non-vanishing parameters are λ_0, λ_2, M_{10}, M_{12}, g_{110}, g_{112}. The discrete transformation however requires $\lambda_2 = M_{12} = g_{112} = 0$. By defining $\lambda := \lambda_0$, $m := M_{12}$, and $g := g_{110}$, we obtain

$$S = \int d^4x \left\{ \int d^4\theta\, \bar{\Phi}_K \Phi_K + \left[\int d^2\theta \left(\lambda \Phi_0 + m \Phi_1 \Phi_2 + g\, \Phi_0 \Phi_1^2 \right) \right] \right\},$$

which is precisely the O'Raifeartaigh model.

Exercise 9.1.

In exercise 8.3. we already wrote the action of the MSSM without mass. We will use the same notation as there. To introduce the Higgs, we introduce four chiral superfields H_{RA}, with $R, A \in \{1, 2\}$. Under the A index they form on SU(2) doublet. Then

$$(V_L H_R)_A := V_{LA}{}^B H_{RB}$$

These form a singlet of SU(3)

$$(V_S H_R)_A = 0$$

(this is an error in the previous homework. We should have

$$(V_S L_i)_A = 0 ,$$

$$(V_S E_i)_A = 0 ,$$

$$(V_L \bar{E}_i)_A = \bar{E}_{iA} ,$$

$$(V_L \bar{U}_i)_{AM} = 0 ,$$

$$(V_L \bar{D}_i)_{AM} = 0 .).$$

They however transform with opposite hypercharges

$$(V_Y H)_{RA} = (-1)^{R} V_Y H_{RA} .$$

To our previous action we add a kinetic term for these new fields

$$\sum_{R,A=1}^{2} \int d^4x \int d^4\theta \; \bar{H}^{RA} \left(e^{V} H \right)_{RA} .$$

The minimal coupling between the Higgs fields and the vector field will give the latter mass once the local symmetry is spontoneously broken. To introduce this breakage, we add the superpotential (respecting the conditions for renormalizability of our previous exercise) $\int d^4x \int d^2\theta \; W \; + \; h.c.$ where

$$\begin{aligned} W = \sum_{i,j=1}^{3} \Big(& h^u_{ij} \, Q_{iAM} \, U_j^{\;M} \, H_{2B} \, \varepsilon^{AB} \\ & + h^d_{ij} \, Q_{iAM} \, D_j^{\;M} \, H_{1B} \, \varepsilon^{AB} \\ & + h^e_{ij} \, L_{iA} \, E_j \, H_{1B} \, \varepsilon^{AB} \Big) + \mu H_{1A} H_{2B} \, \varepsilon^{AB} . \end{aligned}$$

This contains both the "Higgs potential responsible for the spontaneous symmetry breaking" and the Yokawa couplings which gives the matter fermions and their bosonic superpartners mass. The whole action is then given by

$$S = \int d^4x \left\{ \int d^4\theta \left[\sum_{i=1}^{3} \left(\bar{Q}_i^{\ AM} (e^V Q_i)_{AM} + \bar{L}_i^{\ A} (e^V L_i)_A \right.\right.\right.$$

$$+ \bar{U}_{iM} (e^V U_i)^M + \bar{D}_{iM} (e^V D_i)^M$$

$$\left.\left. + \bar{E}_i (e^V E_i) \right) + \bar{H}^{RA} (e^V H)_{RA} \right]$$

$$+ \left[\int d^2\theta \left(W_{S\alpha M}{}^{N} W_S{}^{\alpha}{}_{N}{}^{M} + W_{L\alpha A}{}^{B} W_L{}^{\alpha}{}_{B}{}^{A} \right.\right.$$

$$\left.\left. + B_{Y\alpha} B_Y{}^{\alpha} \right) + c.c. \right]$$

$$+ \left[\int d^2\theta \left(\sum_{i,j=1}^{3} \left(h^u_{ij} Q_{iAM} U_j^{\ M} H_{2B} \varepsilon^{AB} + h^d_{ij} Q_{iAM} D_j^{\ M} H_{2B} \varepsilon^{AB} \right.\right.\right.$$

$$\left. + h^e_{ij} L_{iA} E_j H_{1B} \varepsilon^{AB} \right)$$

$$\left.\left. + \mu H_{1A} H_{2B} \varepsilon^{AB} \right) \right] .$$

This Lagrangian has

$$\underbrace{3 \times 3 \times 2}_{Q} + \underbrace{3 \times 2}_{L} + \underbrace{3 \times 3}_{U} + \underbrace{3 \times 3}_{D} + \underbrace{3}_{E} + \underbrace{2 \times 2}_{H} = 49$$

chiral fields. It thus have 49 fermions (distinguishing right and left) and 49 bosons. On the other hand, it has $\dim (SU(3) \times SU(2) \times U(1)) = 8 + 3 + 1 = 12$ vector superfields.

This yields 12 gauge bosons and 12 fermionic superpartners. This action, although supersymmetric, does not include symmetry breaking. Indeed, contrary to what we previewed, the minimum of $\mu H_{1A} H_{2B} \varepsilon^{AB}$ is clearly $H_{1A} = H_{1B} = 0$, which is invariant under the internal symmetry group. Thus, no particles have mass except for the Higgs particles. Indeed, if

$$H_{RA} = h_{RA} + \sqrt{2}\,\theta \psi_{RA} + \theta\theta F_{RA},$$

in the Lagrangian we have

$$\int d^2\theta\, \mu H_{1A} H_{2B} \varepsilon^{AB} + h.c = \int d^2\theta\, \mu \left(H_{11} H_{22} - H_{12} H_{21} \right) + h.c.$$

$$= \mu \left(h_{11} F_{22} + F_{11} h_{22} - \psi_{11} \psi_{22} - h_{12} F_{21} - F_{12} h_{21} + \psi_{12} \psi_{21} \right)$$

$$+ h.c.$$

Therefore, the EOMs for the auxiliary fields are

$$F_{11}^* = -\mu h_{22}, \qquad F_{12}^* = +\mu h_{21}, \qquad F_{21}^* = \mu h_{12}, \qquad F_{22}^* = -\mu h_{11},$$

so that the potential becomes

$$-\mu^2 \sum_{R,A=1}^{2} |h_{RA}|^2 - \mu \left(\psi_{11} \psi_{22} + \psi_{12} \psi_{21} \right) + h.c.$$

We can thus identify

Four scalar complex fields $\{h_{RA} \mid R, A \in \{1,2\}\}$ of mass $\sqrt{\mu}$. For the spinor content we diagonalize in the basis $(\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})$,

$$\begin{bmatrix} 0 & 0 & 0 & \mu/2 \\ 0 & 0 & \mu/2 & 0 \\ 0 & \mu/2 & 0 & 0 \\ \mu/2 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection, this matrix has eigenvectors $(1,0,0,1)$ and $(0,1,1,0)$ associated to the eigenvalue $\mu/2$ and $(1,0,0,-1)$ and $(0,1,-1,0)$ associated to $-\mu/2$. Thus

$$\mu\left(\psi_{11}\psi_{22} + \psi_{12}\psi_{21}\right) = \frac{\mu}{2}\left(\left(\frac{(\psi_{11}+\psi_{22})}{\sqrt{2}}\right)^2 + \left(i\,\frac{(\psi_{11}-\psi_{22})}{\sqrt{2}}\right)^2 + \left(\frac{(\psi_{12}+\psi_{21})}{\sqrt{2}}\right)^2 + \left(i\,\frac{(\psi_{12}-\psi_{21})}{\sqrt{2}}\right)^2\right),$$

giving four spinors of mass μ.