Quantum Logic and the Orthocomplemented Lattice of Propositions

A logic based approach to Bell's inequalities[Burbano, 2017]

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Outline

Completeness of Quantum Mechanics

Einstein, Podolsy and Rosen, although weary of the success of quantum mechanics, wanted to probe its completeness[Einstein et al., 1935].

- In a complete physical theory every element of physical reality has a counterpart in the theory.
- If we can predict with certainty the value of a physical cuantity without disturbing the system, then there exists an element of physical reality corresponding to this physical quantity.

Let's Put It to the Test

Well, as we've learned from our mathematician friends, lets assume it is!

Heisenberg's Uncertainty Principle

If two observables are represented by operators which do not commute they cannot be measured simultaneously, i.e., they do not have a simultaneous physical reality[Hall, 2013].

Photon Polarization

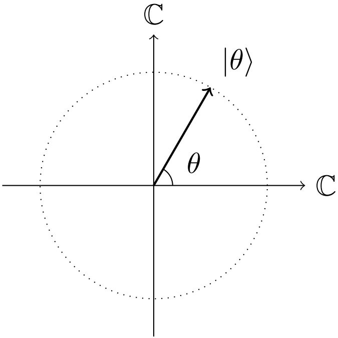
As an example consider the linear polarization of a photon.

- Hilbert space $\mathbb{C}^2$;
- Vector state describing polarization along the angle θ

$$|\theta\rangle = (\cos(\theta), \sin(\theta)); \quad (1)$$

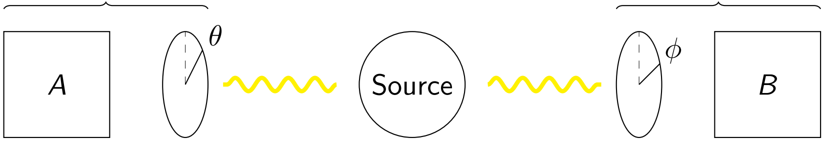
- Operator describing "The polarization of the photon is along θ"

$$P(\theta) = |\theta\rangle\langle\theta| \,. \quad (2)$$



Two Photons

$P_A(\theta) := P(\theta) \otimes \mathrm{id}_{\mathbb{C}^2}$ $\qquad$ $P_B(\phi) := \mathrm{id}_{\mathbb{C}^2} \otimes P(\phi)$



We emmit two photons in the state

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\pi/2\rangle - |\pi/2\rangle \otimes |0\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|\pi/4\rangle \otimes |3\pi/4\rangle - |3\pi/4\rangle \otimes |\pi/4\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2. \end{aligned} \tag{3}$$

Defeating Heisenberg's Uncertainty Principle

- Knowledge of $P_B(0) \Rightarrow$ Knowledge of $P_A(0)$.
- Knowledge of $P_B(\pi/4) \Rightarrow$ Knowledge of $P_A(\pi/4)$.

Since A and B don't interact measurements of B can't affect measurements of A!

Contradiction!

$P_A(0)$ and $P_A(\pi/4)$ have a simultaneous physical reality although $[P_A(0), P_A(\pi/4)] = 0$!

The Search for a Complete Theory

Under the definitions given by EPR quantum mechanics is not complete. Can we provide a complete theory of physical reality? Bell while studying this question arrived at his inequalities for a theory of hidden variables[Bell, 1964]. Our approach will be quite different.

Partially Ordered Sets

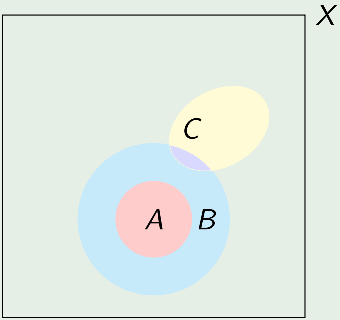
Definition

A partially ordered set (poset) $(P, \leq)$ is a set P along with a relation $\leq$ which is:

- reflexive: $p \leq p$ for all $p \in P$;
- anti-symmetric: $p \leq q$ and $q \leq p$ implies $p = q$ for all $p, q \in P$;
- transitive: $p \leq q$ and $q \leq r$ implies $p \leq r$ for all $p, q, r \in P$.

Example

- $(\mathbb{R}, \leq)$
- $(P(X), \subseteq)$
- (Propositions, $\Rightarrow$)!



Meet and Join

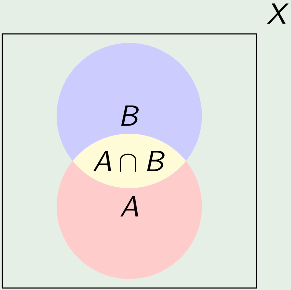
Definition

Let $(P, \leq)$ be a poset and $p, q \in P$. We define $p \wedge q$ to be the greatest lower bound of $\{p, q\}$ if it exists. Similarly $p \vee q$ is the least upper bound of $\{p, q\}$ if it exists. If for every pair $p, q \in P$ both $p \wedge q$ and $p \vee q$ exist the poset is said to be a lattice.

Examples of Meets and Joins

Example

- $(\mathbb{R}, \leq)$ is a lattice where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.
- $(P(X), \subseteq)$ is a lattice where $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.



Meet and Join in (Propositions, $\Rightarrow$)

Example

(Propositions, $\Rightarrow$) form a lattice where $p \wedge q$ is the conjunction of the propositions and $p \vee q$ is the disjunction. Indeed,

- $p \wedge q \rightarrow p$ and $p \wedge q \Rightarrow q$ showing that $p \wedge q$ is a lower bound;
- if $r \Rightarrow p$ and $r \Rightarrow q$ then $r \Rightarrow p \wedge q$.

Some Properties of Meets and Joins

Theorem

Let $(P, \leq)$ *be a poset. For all* $p, q, r \in P$*:*

- $p \leq q$ *if and only if* $p \wedge q = p$ *if and only if* $p \vee q = q$*;*
- *(idempotency)* $p \wedge p = p = p \vee p$*.*

Boundedness

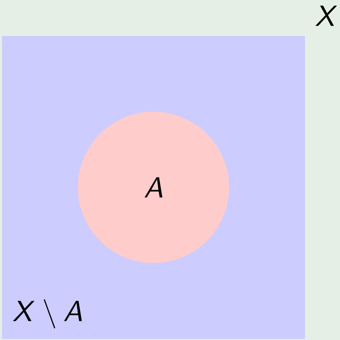
Definition

A poset $(P, \leq)$ is said to be bounded if there is a greatest lower bound 0 and a least upper bound 1. A complement of $p \in P$ is an element $q \in P$ such that $p \wedge q = 0$ and $p \vee q = 1$

Bounded and Unbounded Lattices

Example

- $(\mathbb{R}, \leq)$ is an unbounded lattice.
- $(P(X), \subseteq)$ is a bounded lattice in which $0 = \emptyset$, $1 = X$, and the complement of A is $X \setminus A$.
- (Propositions, $\Rightarrow$) is a bounded lattice in which 0 is always false, 1 is always true, and the complement of a proposition p is its negation $\neg p$.



Distributivity

Definition

A lattice $(L, \leq)$ is said to be distributive if
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
for all $p, q, r \in L$.

Distributivity in (Propositions, $\Rightarrow$)

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Boolean Algebras

Definition

A Boolean algebra is a distributive bounded lattice in which every element has a complement.

The complement of p in a Boolean algebra will be denoted by p'.

Interpretation

We will interpret EPR's requirements of a complete physical theory to be that the set of propositions one may ask of the theory be a Boolean algebra.

Coincidences

Let $(B, \leq)$ be a Boolean algebra. Define the coincidence function

$$
\begin{aligned}
f : B \times B &\to B \\
(p, q) &\mapsto f(p, q) := (p \wedge q) \vee (p' \wedge q').
\end{aligned} \tag{4}
$$

Distributivity in Coincidences

Note that for all $p_1, q_1, p_2, q_2 \in B$

$$\begin{aligned} (p_1 \wedge q_1) \wedge ((p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1)) = \\ (p_1 \wedge q_1) \wedge ((p_1 \wedge q_2) \vee (p_2 \vee q_2)' \vee (p_2 \wedge q_1)) = \\ (p_1 \wedge q_1 \wedge q_2) \vee (p_1 \wedge q_1 \wedge (p_2 \vee q_2)') \vee (p_1 \wedge q_1 \wedge p_2) = \\ (p_1 \wedge q_1) \wedge (q_2 \vee (p_2 \vee q_2)' \vee p_2) = (p_1 \wedge q_1) \wedge 1 = p_1 \wedge q_1. \end{aligned} \tag{5}$$

Bell's Inequalities in Boolean Algebras

We conclude

$$p_1 \wedge q_1 \leq (p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1). \tag{6}$$

Similarly

$$p_1' \wedge q_1' \leq (p_1' \wedge q_2') \vee (p_2 \wedge q_2) \vee (p_2' \wedge q_1'). \tag{7}$$

Therefore

Theorem (Bell's Inequalities for Boolean Algebras)

Let $(B, \leq)$ be a Boolean algebra. For all p_1, q_1, p_2, q_2 we have

$$P(f(p_1, q_1)) \leq P(f(p_1, q_2)) + P(f(p_2, q_2)) + P(f(p_2, q_1)). \tag{8}$$

Degrees of Plausibility

In quantum mechanics we are more comfortable with the assignment of probabilities to propositions[Jaynes, 2003]. Any reasonable assignment

$$P : B \to [0, 1] \tag{9}$$

of degree of plausibility to physical propositions must be such that

- $p \leq q \Rightarrow P(p) \leq P(q)$;
- (sum rule) $P(p \vee q) \leq P(p) + P(q)$.

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Bell's Inequalities

Theorem (Bell's Inequalities)

Let $(B, \leq)$ be a Boolean algebra with an assignation of degrees of plausibility P. Then

$$P(f(p_1, q_1)) \leq P(f(p_1, q_2)) + P(f(p_2, q_2)) + P(f(p_2, q_1)). \quad (10)$$

What are Propositions in Quantum Mechanics?

Propositions in quantum mechanics should be observables with only two possible values when measured: True or False[Wilce, 2012].

- Observable $\rightarrow$ Self-adjoint operator
- Spectrum $\{\text{False}, \text{True}\} \rightarrow \{0, 1\}$

Definition

We define propositions in quantum mechanics to be the orthogonal projections $L(\mathcal{H}) := \{P \in \mathcal{B}(\mathcal{H}) | P^2 = P = P^*\}$.

Geometry on Hilbert Spaces

Once again, much like mathematicians, given that it is not clear how to define a poset structure on $L(\mathcal{H})$ we have to proceed by duality.

Theorem

Every closed subspace of $\mathcal{H}$ is the image of an orthogonal projection. Conversely, the image of every orthogonal projection is a closed subspace of $\mathcal{H}$.

We may thus understand $L(\mathcal{H})$ as the set of closed subspaces of $\mathcal{H}$.

Partial Order on $L(\mathcal{H})$

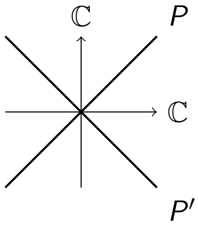
We inherit the Poset structure from $P(\mathcal{H})$.

Definition

The poset of propositions in quantum mechanics is $(L(\mathcal{H}), \subseteq)$.

This forms a bounded lattice in which every element has a complement:

- $P \wedge Q = P \cap Q$
- $P \vee Q = \overline{\text{span}(P \cup Q)}$
- $0 = \{0\}$
- $1 = \mathcal{H}$
- P' is the projection onto $P^{\perp}$



Useful Theorems for Calculations

Theorem

Let $P, Q \in L(\mathcal{H})$. Then,

- $P \leq Q \Leftrightarrow P = PQ = QP;$
- *if P and Q commute* $P \wedge Q = PQ$ *and* $P \vee Q = P + Q - PQ.$

Degrees of Plausibility in Quantum Mechanics

In quantum mechanics every state determines a degree of plausibility on $(L(\mathcal{H}), \subseteq)$. In particular, vector states $|\psi\rangle \in \mathcal{H}$ determine the degree of plausibility

$$P_\psi(P) = \langle\psi|P|\psi\rangle \,. \tag{11}$$

One verifies that if $P \leq Q$ then

$$\begin{aligned} \|P\,|\psi\rangle\,\|^2 &= \langle\psi|P^2|\psi\rangle = \langle\psi|P|\psi\rangle = P_\psi(P) = P_\psi(PQ) \\ &= \langle\psi|PQ|\psi\rangle \leq \|P\,|\psi\rangle\,\|\|Q\,|\psi\rangle\,\| \leq \|Q\,|\psi\rangle\,\|^2 \\ &= P_\psi(Q). \end{aligned} \tag{12}$$

Back to EPR

Notice that $[P_A(\theta), P_B(\phi)] = 0$. Thus the proposition we assign to their conjunction "Alice measures polarization at an angle θ while Bob at an angle ϕ" is

$$P_A(\theta) \wedge P_B(\phi) = P_A(\theta)P_B(\phi) = P(\theta) \otimes P(\phi) \tag{13}$$

with an expected value

$$P_\psi(P_A(\theta) \wedge P_B(\phi)) = \langle\psi|P_A(\theta)P_B(\phi)|\psi\rangle = \frac{1}{2}\sin(\theta - \phi)^2. \tag{14}$$

Complements of Polarization

It makes sense to choose as complements $P_A(\theta)' = P_A(\theta + \pi/2)$ and $P_B(\theta)' = P_B(\theta + \pi/2)$. It is clear then that

$$P(P_A(\theta)' \wedge P_B(\phi)') = \frac{1}{2}\sin(\theta - \phi)^2. \tag{15}$$

To calculate de disjunction, notice that

$$\begin{aligned}(P_A(\theta)' \wedge P_B(\phi)')(P_A(\theta) \wedge P_B(\phi)) =\\ P_A(\theta + \pi/2)P_B(\phi + \pi/2)P_A(\theta)P_B(\phi) =\\ (P(\theta + \pi/2) \otimes P(\phi + \pi/2))(P(\theta) \otimes P(\phi))) =\\ P(\theta + \pi/2)P(\theta) \otimes P(\phi + \pi/2)P(\phi) =\\ 0 = (P_A(\theta) \wedge P_B(\phi))(P_A(\theta)' \wedge P_B(\phi)').\end{aligned} \tag{16}$$

Expectation of Coincidences

We obtain

$$\begin{aligned} f(P_A(\theta), P_B(\phi)) =& (P_A(\theta) \wedge P_B(\phi)) \vee (P_A(\theta)' \wedge P_B(\phi)') \\ =& P_A(\theta) \wedge P_B(\phi) + P_A(\theta)' \wedge P_B(\phi)'. \end{aligned} \tag{17}$$

Since expectation values are linear,

$$P(f(P_A(\theta), P_B(\phi))) = \sin(\theta - \phi)^2. \tag{18}$$

Violation of Bell's Inequalities

Thus Bell's inequalities dictate

$$\begin{aligned}1 =& \sin(0-\pi/2)^2 = P(f(P_A(0), P_B(\pi/2))) \\ \leq& P(f(P_A(0), P_B(\pi/6))) + P(f(P_A(\pi/3), P_B(\pi/6))) \\ &+ P(f(P_A(\pi/3), P_B(\pi/2))) \\ =& \sin(0-\pi/6)^2 + \sin(\pi/3-\pi/6)^2 + \sin(\pi/3-\pi/2)^2 \\ =& 3/4.\end{aligned} \tag{19}$$

Conclusion

There is no correct physical theory whose propositions satisfy a Boolean algebra.

What failed?

Theorem

In a distributive bounded lattice elements have at most one complement.

Proof.

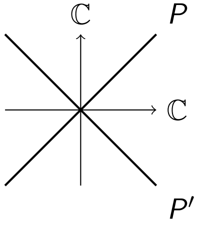
Suppose q and r are complements of p in a distributive bounded lattice. Then

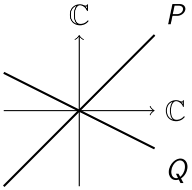
$$q = q \wedge 1 = q \wedge (p \vee r) = (q \wedge p) \vee (q \wedge r) = 0 \vee (q \wedge r) = q \wedge r. \quad (20)$$

This $q \leq r$. Similarly one can show $r \leq q$. By anti-symmetry $q = r$. $\square$

Many Complements!

It is clear that in the bounded lattice $L(\mathcal{H})$ complements are not unique. Thus the lattice is not distributive.





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