4.3 Exercises

Exercise 4.1

The EOMs are eqns (4.2) and (4.3)

$$\bar{\sigma}^{n\alpha\beta}\partial_n\psi_\beta = iM\bar{\psi}^{\alpha}$$

$$\sigma^{n}{}_{\alpha\beta}\partial_n\bar{\psi}^{\beta} = iM\psi_\alpha$$

Then, using eqn (1.9),

$$-M^2\psi_\alpha = iM\,iM\psi_\alpha = iM\sigma^{n}{}_{\alpha\alpha}\partial_n\bar{\psi}^{\alpha} = \sigma^{n}{}_{\alpha\alpha}\partial_n\bar{\sigma}^{m\alpha\beta}\partial_m\psi_\beta = \sigma^{n}{}_{\alpha\alpha}\bar{\sigma}^{m\alpha\beta}\partial_n\partial_m\psi_\beta$$

$$= (\sigma^n\bar{\sigma}^m)_\alpha{}^{\beta}\partial_n\partial_m\psi_\beta = \frac{1}{2}(\sigma^{(n}\bar{\sigma}^{m)})_\alpha{}^{\beta}\partial_n\partial_m\psi_\beta = -\frac{1}{2}2\delta_\alpha^\beta\eta^{nm}\partial_n\partial_m\psi_\beta$$

$$= -\partial_n\partial^n\psi_\alpha = -\Box\psi_\alpha$$

We thus have the Klein-Gordon eqn

$$(\Box - M^2)\psi_\alpha = 0$$

Exercise 4.2

The terms proportional to M are

$$-i\int d^4x\Big(-\sqrt{2}M\,\varphi\bar{\xi}_\alpha\bar{\sigma}^{n\alpha\alpha}\partial_n\psi_\alpha - \bar{\psi}_\alpha\bar{\sigma}^{n\alpha\alpha}\sqrt{2}M\,\partial_n\bar{\varphi}\xi_\alpha$$

$$-\frac{iM}{2}\big(i\sqrt{2}\,\sigma^{m\alpha}{}_\alpha\bar{\xi}^\alpha\partial_m\varphi\,\psi_\alpha + i\sqrt{2}\,\sigma^{m}{}_{\alpha\alpha}\psi^\alpha\bar{\xi}^\alpha\partial_m\varphi$$

$$+ i\sqrt{2}\,\bar{\sigma}^{m}{}_{\alpha}{}^{\alpha}\xi_\alpha\partial_m\bar{\varphi}\,\bar{\psi}^\alpha + i\sqrt{2}\,\bar{\sigma}^{m\alpha\alpha}\bar{\psi}_\alpha\xi_\alpha\partial_m\bar{\varphi}\big)\Big)$$

$$= -i\int d^4x\Big(\cancel{\sqrt{2}M\partial_n\varphi\bar{\xi}_\alpha\bar{\sigma}^{n\alpha\alpha}\psi_\alpha - \bar{\psi}_\alpha\bar{\sigma}^{n\alpha\alpha}\sqrt{2}M\partial_n\bar{\varphi}\xi_\alpha}$$

$$- \frac{iM}{2}\Big(\cancel{-i\sqrt{2}\sigma^m{}_{\alpha\alpha}\bar{\xi}^\alpha\partial_m\varphi\psi^\alpha - i\sqrt{2}\sigma^m{}_{\alpha\alpha}\bar{\xi}^\alpha\psi^\alpha\partial_m\varphi}$$

$$\cancel{-i\sqrt{2}\bar{\sigma}^{m\alpha\alpha}\xi_\alpha\partial_m\bar{\varphi}\bar{\psi}_\alpha + i\sqrt{2}\bar{\sigma}^{m\alpha\alpha}\bar{\psi}_\alpha\xi_\alpha\partial_m\bar{\varphi}}\Big)\Big) = 0$$

where we have used the fact that

$$\psi^\alpha\xi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\varepsilon_{\alpha\gamma}\xi^\gamma = -\delta^\beta_\gamma\psi_\beta\xi^\gamma = \psi_\beta\xi^\beta$$

and

$$\bar{\psi}_\alpha\bar{\sigma}^{n\alpha\alpha}\psi_\alpha = \varepsilon_{\alpha\beta}\bar{\psi}^\beta\bar{\sigma}^{n\alpha\alpha}\varepsilon_{\alpha\beta}\psi^\beta = \varepsilon_{\beta\alpha}\varepsilon_{\beta\alpha}\bar{\psi}^\beta\bar{\sigma}^{n\alpha\alpha}\psi^\beta$$

$$= \bar{\psi}^\beta\sigma^n{}_{\beta\beta}\psi^\beta.$$

Exercise 4.3.

Indeed we have

$$[\xi^1\kappa^1, \xi^2\kappa^2] = \xi^1\kappa^1\xi^2\kappa^2 - \xi^2\kappa^2\xi^1\kappa^1 = -\xi^1\xi^2\kappa^1\kappa^2 - \xi^1\xi^2\kappa^2\kappa^1 = -\xi^1\xi^2\{\kappa^1,\kappa^2\}$$

for fermionic symbols $\xi^1, \xi^2, \kappa^1,$ and κ^2. Thus

$$[\xi_1Q + \bar{\xi}_1\bar{Q}, \xi_2Q + \bar{\xi}_2\bar{Q}] = [\xi_1{}^\alpha Q_\alpha + \bar{\xi}_{1\alpha}\bar{Q}^\alpha, \xi_2{}^\beta Q_\beta + \bar{\xi}_{2\beta}\bar{Q}^\beta]$$

$$= [\xi_1^\alpha Q_\alpha, \xi_2^\beta Q_\beta] + [\xi_1^\alpha Q_\alpha, \bar{\xi}_{2\beta}\bar{Q}^\beta]$$

$$+ [\bar{\xi}_{1\alpha}\bar{Q}^\alpha, \xi_2{}^\beta Q_\beta] + [\bar{\xi}_{1\alpha}\bar{Q}^\alpha, \bar{\xi}_{2\beta}\bar{Q}^\beta]$$

$$= -\xi_1{}^\alpha\xi_2{}^\beta\{Q_\alpha, Q_\beta\} + \xi_1{}^\alpha\bar{\xi}_2{}^\beta\{Q_\alpha, \bar{Q}_\beta\}$$

$$+ \bar{\xi}_1{}^\alpha\xi_2{}^\beta\{\bar{Q}_\alpha, Q_\beta\} - \bar{\xi}_1{}^\alpha\bar{\xi}_2{}^\beta\{\bar{Q}_\alpha, \bar{Q}_\beta\}$$

This is however equal to

$$2(\xi_1 \sigma^m \bar{\xi}_2 + \bar{\xi}_1 \bar{\sigma}^m \xi_2) P_m = 2(\xi_1{}^\alpha \sigma^m{}_{\alpha\dot{\alpha}} \bar{\xi}_2{}^{\dot{\alpha}} + \bar{\xi}_{1\dot{\alpha}} \bar{\sigma}^{m\dot{\alpha}\alpha} \xi_{2\alpha}) P_m$$

We immediately see that there are no $\xi_1^\alpha \xi_2^\beta$ or $\bar{\xi}_1{}^{\dot{\alpha}} \bar{\xi}_2{}^{\dot{\beta}}$ terms.

Thus

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$$

The $\xi_1^\alpha \bar{\xi}_2^{\dot{\alpha}}$ term shows

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^m{}_{\alpha\dot{\alpha}} P_m$$

The $\bar{\xi}_1{}^{\dot{\alpha}} \xi_2{}^\beta$ term on the other hand shows

$$\{\bar{Q}_{\dot{\alpha}} Q_\beta\} = 2\sigma^m{}_{\alpha\dot{\alpha}} P_m$$

as well

Exercise 44

Indeed

$$(\sigma^m \bar{\xi}_2)_\alpha \partial_m (\xi_1 \psi) = \sigma^m{}_{\alpha\dot{\alpha}} \bar{\xi}_2^{\dot{\alpha}} \xi_1^\beta \partial_m \psi_\beta = \sigma^m{}_{\alpha\dot{\alpha}} \bar{\xi}_2{}^{\dot{\alpha}} \xi_1{}^\beta \partial_n \psi_\gamma \delta^\gamma_\beta \eta^{mn}$$

$$= -\frac{1}{2} \sigma^m{}_{\alpha\dot{\alpha}} \bar{\xi}_2{}^{\dot{\alpha}} \xi_1{}^\beta \partial_n \psi_\gamma (\sigma^m{}_{\beta\dot{\beta}} \bar{\sigma}^{n\dot{\beta}\gamma} + \sigma^n{}_{\beta\dot{\beta}} \bar{\sigma}^{m\dot{\beta}\gamma})$$

$$= -\frac{1}{2} \varepsilon_{\alpha\delta} \varepsilon_{\dot{\alpha}\dot{\delta}} \bar{\sigma}^{m\dot{\delta}\delta} \bar{\xi}_2{}^{\dot{\alpha}} \xi_1{}^\beta \partial_n \psi_\gamma \sigma^m{}_{\beta\dot{\beta}} \bar{\sigma}^{n\dot{\beta}\gamma}$$

$$- \frac{1}{2} \sigma^m{}_{\alpha\dot{\alpha}} \bar{\xi}_2{}^{\dot{\alpha}} \xi_1{}^\beta \partial_n \psi_\gamma \sigma^n{}_{\beta\dot{\beta}} \bar{\sigma}^{m\dot{\beta}\gamma}$$

$$= \delta^\delta_\beta \delta^{\dot{\delta}}_{\dot{\beta}} \varepsilon_{\alpha\delta} \varepsilon_{\dot{\alpha}\dot{\delta}} \bar{\xi}_2^{\dot{\alpha}} \xi_1{}^\beta \partial_n \psi_\gamma \bar{\sigma}^{n\dot{\beta}\gamma}$$

$$+\delta_\alpha^\gamma \delta_\alpha^\beta \bar\xi_2{}^{\dot\alpha} \xi_1{}^\beta \partial_n \psi_\gamma \sigma^n{}_{\beta\dot\beta}$$

$$= -\bar\xi_{2\dot\beta} \xi_{1\alpha} \partial_n \psi_\gamma \bar\sigma^{n\dot\beta\gamma} + \bar\xi_2{}^{\dot\beta} \xi_1{}^\beta \partial_n \psi_\alpha \sigma^n{}_{\beta\dot\beta}$$

$$= -\bar\xi_2{}^{\dot\beta} \xi_{1\alpha} \partial_n \psi^\gamma \sigma^n{}_{\gamma\dot\beta} - \xi_1{}^\beta \bar\xi_2{}^{\dot\beta} \partial_n \psi_\alpha \sigma^n{}_{\beta\dot\beta}$$

$$= -(\sigma^n \bar\xi_2)_\gamma \xi_{1\alpha} \partial_n \psi^\gamma - (\xi_1 \sigma^n \bar\xi_2) \partial_n \psi_\alpha$$

We used the relations (Wess & Bagger)

$$(\sigma^m \bar\sigma^n + \sigma^n \bar\sigma^m)_\alpha{}^\alpha = -2\delta_\alpha^\alpha \eta^{mn},$$

$$\sigma^m{}_{\alpha\dot\alpha} \bar\sigma^{m\dot\beta\beta} = -2\delta_\alpha^\beta \delta_{\dot\beta}^{\dot\beta}$$

Exercise 4.5

Indeed, up to surface terms

$$\delta_Q S = \int d^4x \Big(-\sqrt{2}\,\xi^\alpha \partial_m \psi_\alpha \partial^m \bar\varphi - \sqrt{2}\,\partial_m \varphi\, \bar\xi_{\dot\alpha} \partial^m \bar\psi^{\dot\alpha}$$

$$- i\left(i\sqrt{2}\, \bar\sigma^m{}_{\alpha}{}^{\alpha} \xi_\alpha \partial_m \bar\varphi + \cancel{\sqrt{2}\,\bar\xi_{\dot\alpha}} \bar F \right) \bar\sigma^{n\dot\alpha\beta} \partial_n \psi_\beta$$

$$- i \bar\psi_{\dot\alpha} \bar\sigma^{n\dot\alpha\beta} \partial_n \left(i\sqrt{2}\, \sigma^m{}_{\beta\dot\beta} \bar\xi^{\dot\beta} \partial_m \varphi + \sqrt{2}\, \xi_\beta F \right)$$

$$+ i\sqrt{2}\, \cancel{\bar\xi_{\dot\alpha} \bar\sigma^{m\dot\alpha\alpha}} \partial_m \psi_\alpha \bar F + F\, i\sqrt{2}\, \xi^\alpha \sigma^m{}_{\alpha\dot\alpha} \partial_m \bar\psi^{\dot\alpha}$$

$$+ M\Big(\cancel{\sqrt{2}\,\xi^\alpha \psi_\alpha F} + \varphi\, i\sqrt{2}\, \bar\xi_{\dot\alpha} \bar\sigma^{m\dot\alpha\alpha} \partial_m \psi_\alpha$$

$$+ \cancel{\sqrt{2}\,\bar\xi_{\dot\alpha} \bar\psi^{\dot\alpha}} \bar F + \bar\varphi\, i\sqrt{2}\, \xi^\alpha \sigma^m_{\alpha\dot\alpha} \partial_m \bar\psi^{\dot\alpha}$$

$$- \frac{1}{2}\left(i\sqrt{2}\, \sigma^{m\alpha}{}_{\dot\alpha} \bar\xi^{\dot\alpha} \partial_m \varphi + \cancel{\sqrt{2}\,\xi^\alpha F} \right) \psi_\alpha$$

$$- \frac{1}{2} \psi^{\alpha} \left(i\sqrt{2}\, \sigma^{m}{}_{\alpha\dot\alpha} \bar\xi^{\dot\alpha} \partial_m \varphi + \cancel{\sqrt{2}\, \xi_\alpha F} \right)$$

$$- \frac{1}{2} \left(i\sqrt{2}\, \bar\sigma^{m}{}_{\dot\alpha}{}^{\alpha} \xi_\alpha \partial_m \bar\varphi + \cancel{\sqrt{2}\, \bar\xi_{\dot\alpha} \bar F} \right) \bar\psi^{\dot\alpha}$$

$$- \frac{1}{2} \bar\psi_{\dot\alpha} \left(i\sqrt{2}\, \bar\sigma^{m\dot\alpha\alpha} \xi_\alpha \partial_m \bar\varphi + \cancel{\sqrt{2}\, \bar\xi^{\dot\alpha} \bar F} \right) \Big) \Big)$$

$$= \int d^4x \Big(- \sqrt{2}\, \xi^{\alpha} \partial_m \psi_\alpha \partial^m \bar\varphi - \sqrt{2}\, \partial_m \varphi \bar\xi_{\dot\alpha} \partial^m \bar\psi^{\dot\alpha}$$

$$+ \sqrt{2}\, \bar\sigma^{m\dot\alpha\alpha} \sigma^{n}{}_{\beta\dot\alpha} \xi_\alpha \partial_m \bar\varphi \partial_n \psi^{\beta} + \sqrt{2}\, \bar\psi_{\dot\alpha} \bar\sigma^{n\dot\alpha\beta} \sigma^{m}{}_{\beta\dot\beta} \bar\xi^{\dot\beta} \partial_n \partial_m \varphi$$

$$- i\sqrt{2}\, \bar\psi_{\dot\alpha} \bar\sigma^{n\dot\alpha\beta} \xi_\beta \partial_n F + i\sqrt{2}\, \xi_\alpha \bar\sigma^{m\dot\alpha\alpha} \partial_m \bar\psi_{\dot\alpha} F$$

$$+ M \Big(\cancel{- i\sqrt{2}\, \partial_m \varphi \bar\xi_{\dot\alpha} \bar\sigma^{m\dot\alpha\alpha} \psi_\alpha} \;\; \cancel{- i\sqrt{2}\, \partial_m \bar\varphi \xi^{\alpha} \sigma^{m}{}_{\alpha\dot\alpha} \bar\psi^{\dot\alpha}}$$

$$\cancel{- i\sqrt{2}\, \psi_\alpha \bar\sigma^{m\dot\alpha\alpha} \bar\xi_{\dot\alpha} \partial_m \varphi} \;\; \cancel{- i\sqrt{2}\, \bar\psi^{\dot\alpha} \sigma^{m}{}_{\alpha\dot\alpha} \xi^{\alpha} \partial_m \bar\varphi} \Big)$$

$$= \int d^4x \Big(+ \sqrt{2}\, \xi^{\alpha} \psi_\alpha \partial_m \partial^m \bar\varphi + \sqrt{2}\, \partial^m \partial_m \varphi \bar\xi_{\dot\alpha} \bar\psi^{\dot\alpha}$$

$$- \sqrt{2} \left(\sigma^n \bar\sigma^m \right)_{\beta}{}^{\alpha} \xi_\alpha \partial_m \partial_n \bar\varphi \psi^{\beta} + \sqrt{2}\, \bar\psi_{\dot\alpha} \left(\bar\sigma^n \sigma^m \right)^{\dot\alpha}{}_{\dot\beta} \bar\xi^{\dot\beta} \partial_n \partial_m \varphi$$

$$\cancel{- i\sqrt{2}\, \bar\psi_{\dot\alpha} \bar\sigma^{n\dot\alpha\beta} \xi_\beta \partial_n F - i\sqrt{2}\, \xi_\alpha \bar\sigma^{m\dot\alpha\alpha} \bar\psi_{\dot\alpha} \partial_m F} \Big)$$

$$= \int d^4x \left(\cancel{\sqrt{2}\, \xi^{\alpha} \psi_\alpha \Box \bar\varphi} + \cancel{\sqrt{2}\, \bar\xi_{\dot\alpha} \bar\psi^{\dot\alpha} \Box \varphi} + \cancel{\sqrt{2}\, \eta^{nm} \delta_{\beta}^{\alpha} \xi_\alpha \partial_m \partial_n \bar\varphi \psi^{\beta}} \cancel{- \sqrt{2}\, \bar\psi_{\dot\alpha} \eta^{nm} \delta_{\dot\beta}^{\dot\alpha} \bar\xi^{\dot\beta} \partial_n \partial_m \varphi} \right)$$

$$= 0$$

Exercise 4.6

We define

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^{m}_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \frac{\partial}{\partial x^m} , \qquad \bar Q^{\dot\alpha} = \frac{\partial}{\partial \bar\theta_{\dot\alpha}} - i\bar\sigma^{m\dot\alpha\beta} \theta_\beta \frac{\partial}{\partial x^m}$$

Since Φ is expressed in terms of y^m, let us consider the change of coordinates $(x^m, \theta, \bar{\theta}) \longmapsto (y^m, \theta, \bar{\theta})$ Since

$$\frac{\partial y^m}{\partial x^n} = \delta^m_n, \qquad \frac{\partial y^m}{\partial \theta^\alpha} = i\sigma^m{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}},$$

$$\frac{\partial y^m}{\partial \bar{\theta}_{\dot{\alpha}}} = -i\theta^\beta \sigma^m{}_{\beta\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} = -i\varepsilon^{\beta\gamma}\theta_\gamma \sigma^m{}_{\beta\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} = -i\bar{\sigma}^{m\dot{\alpha}\alpha}\theta_\alpha$$

we have

$$\left(\frac{\partial}{\partial \theta^\alpha}\right)_{(x,\theta,\bar{\theta})} = \left(\frac{\partial y^m}{\partial \theta^\alpha}\right)_{(x,\theta,\bar{\theta})}\left(\frac{\partial}{\partial y^m}\right)_{(y,\theta,\bar{\theta})} + \left(\frac{\partial \theta^\beta}{\partial \theta^\alpha}\right)_{(x,\theta,\bar{\theta})}\left(\frac{\partial}{\partial \theta^\beta}\right)_{(x,\theta,\bar{\theta})}$$

$$+ \cancelto{0}{\left(\frac{\partial \bar{\theta}_{\dot{\alpha}}}{\partial \theta^\alpha}\right)_{(x,\theta,\bar{\theta})}}\left(\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}\right)_{(x,\theta,\bar{\theta})}$$

$$= i\sigma^m{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m} + \frac{\partial}{\partial \theta^\alpha}$$

Similarly

$$\left(\frac{\partial}{\partial x^m}\right)_{(x,\theta,\bar{\theta})} = \frac{\partial}{\partial y^m}$$

Then

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \cancel{i\sigma^m{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}} - \cancel{i\sigma^m{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}} = \frac{\partial}{\partial \theta^\alpha},$$

$$Q_\alpha \Phi(y,\theta) = \sqrt{2}\,\psi_\alpha(y) + 2\theta_\alpha F(y)$$

On the other hand

$$\left(\frac{\partial}{\partial \bar\theta_\alpha}\right)_{(x,\theta,\bar\theta)} = -i\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha \frac{\partial}{\partial y^m} + \frac{\partial}{\partial \bar\theta_\alpha}$$

Thus

$$\bar\theta_{\dot\alpha} = \frac{\partial}{\partial \bar\theta_\alpha} - 2i\sigma^{m\dot\alpha\alpha}\theta_\alpha \frac{\partial}{\partial y^m},$$

and

$$\bar Q^\alpha \Phi(y,\theta) = -2i\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha \left(\partial_m \varphi(y) + \sqrt{2}\theta^\beta \partial_m \psi_\beta(y) + \theta^\beta\theta_\beta \partial_m F(y)\right)$$

We are left with the supersymmetry transformation

$$\delta_Q \varphi(y) + \sqrt{2}\theta^\alpha \delta_Q \psi_\alpha(y) + \theta^\alpha\theta_\alpha \delta_Q F(y) = \delta_Q \Phi(y,\theta) =$$

$$\sqrt{2}\xi^\alpha \psi_\alpha(y) + 2\xi^\alpha\theta_\alpha F(y) - 2i\bar\xi_\alpha \bar\sigma^{m\alpha\alpha}\theta_\alpha \partial_m \varphi(y) - 2i\bar\xi_\alpha\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha \sqrt{2}\theta^\beta \partial_m \psi_\beta(y) =$$

$$\sqrt{2}\xi\psi(y) + \sqrt{2}\theta^\alpha\left(\sqrt{2}\xi_\alpha F(y) + \sqrt{2}i\sigma^m{}_{\alpha\alpha}\bar\xi^{\dot\alpha}\partial_m\varphi(y)\right) - 2\sqrt{2}i\left(\bar\xi\bar\sigma^m\theta\right)\left(\theta\partial_m\psi(y)\right) =$$

$$\sqrt{2}\xi\psi(y) + \sqrt{2}\theta^\alpha\left(\sqrt{2}\xi_\alpha F(y) + \sqrt{2}i(\sigma^m\bar\xi)_\alpha \partial_m\varphi(y)\right) + \theta^\alpha\theta_\alpha\sqrt{2}i\left(\bar\xi\bar\sigma^m\partial_m\psi(y)\right)$$

Then

$$\delta_Q \varphi = \sqrt{2}\xi\psi,$$

$$\delta_Q \psi_\alpha = \sqrt{2}i(\sigma^m\bar\xi)_\alpha \partial_m\varphi + \sqrt{2}\xi_\alpha F,$$

$$\delta_Q F = \sqrt{2}i(\bar\xi\bar\sigma^m\partial_m\psi)$$