2.1. Exercises

Exercise 2.1.

$$\{D, Q\} = \left\{ \frac{\partial}{\partial \kappa} + i\kappa \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \kappa} - i\kappa \frac{\partial}{\partial \tau} \right\}$$

$$= \cancelto{0}{\left\{ \frac{\partial}{\partial \kappa}, \frac{\partial}{\partial \kappa} \right\}} + i \cancelto{0}{\left\{ \kappa \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \kappa} \right\} - i \left\{ \frac{\partial}{\partial \kappa}, \kappa \frac{\partial}{\partial \tau} \right\}} + \left\{ \kappa \frac{\partial}{\partial \tau}, \kappa \frac{\partial}{\partial \tau} \right\}$$

$$= 2\kappa \frac{\partial}{\partial \tau} \kappa \frac{\partial}{\partial \tau} = 2\cancelto{0}{\kappa^2} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} = 0$$

Exercise 2.2.

Let us begin by noticing that for any quantity F, whose change under an infinitesimal supersymmetry transformation is $\delta F = \alpha Q F$, we have

$$\delta \int d\tau \int d\kappa \, F = \int d\tau \int d\kappa \, \delta F = \int d\tau \int d\kappa \, \alpha Q F = \int d\tau \int d\kappa \, \alpha \left(\frac{\partial}{\partial \kappa} - i\kappa \frac{\partial}{\partial \tau} \right) F$$

$$= \int d\tau \, \frac{\partial}{\partial \kappa} \left(\alpha \left(\frac{\partial}{\partial \kappa} - i\kappa \frac{\partial}{\partial \tau} \right) F \right) \Big|_{\kappa = 0}$$

$$= \int d\tau \left(-\alpha \cancelto{0}{\frac{\partial}{\partial \kappa} \frac{\partial}{\partial \kappa}} F + \alpha i \frac{\partial}{\partial \tau} F + i\alpha\kappa \frac{\partial}{\partial \kappa} \frac{\partial}{\partial \tau} F \right)_{\kappa = 0}$$

$$= \alpha i \int d\tau \, \frac{\partial}{\partial \tau} F .$$

If we further assume that F vanishes at the boundaries of integration, we have that $\delta \int d\tau \int dK F = 0$.

Now assum that both F and G satisfy the conditions above.

It is clear that FG also vanishes at the boundary of integration.

On the other hand, since αQ is a first order bosonic derivation,

$$\delta(FG) = \delta F G + F \delta G = \alpha Q(F) G + F \alpha Q(G) = \alpha Q(FG).$$

We conclude that FG is also covariant under supersymmetry and $\int d\tau \int dK\, FG$ is a supersymmetric invariant. We thus only need to show that f and Dg are covariant under supersymmetry if they have the correct vanishing properties.

Since $f = f(\mathbb{X})$, we have

$$\alpha Q f = \alpha \left(\frac{\partial}{\partial K} - iK \frac{\partial}{\partial \tau} \right) f = \alpha \left(\frac{\partial}{\partial K} f - iK \frac{\partial}{\partial \tau} f \right) = \alpha \left(\frac{df}{d\mathbb{X}} \frac{\partial}{\partial K} \mathbb{X} - iK \frac{df}{d\mathbb{X}} \frac{\partial}{\partial \tau} \mathbb{X} \right)$$

$$= \frac{df}{d\mathbb{X}} \alpha \left(\frac{\partial}{\partial K} \mathbb{X} - iK \frac{\partial}{\partial \tau} \mathbb{X} \right) = \frac{df}{d\mathbb{X}} \alpha Q \mathbb{X} = \delta f.$$

Notice that we used the fact that $\mathbb{X}$ is bosonic and, therefore, that f is too. On the other hand, since $g = g(\mathbb{X})$ as well,

$$\delta D_g = D\alpha Q g = -\alpha DQg = \alpha Q D_g ,$$

where we used that D is fermionic and $\{D, Q\} = 0$. Thus f and D_g are both covariant under supersymmetry and $\int d\tau \int d\kappa \, f D_g$ is a supersymmetric invariant.

Exercise 2.3.

We have that

$$-\frac{ie}{c} D\mathbb{X}^m \mathbb{A}(\mathbb{X}) = -\frac{ie}{c}\left(i\psi^m + i\kappa(\dot{x}^m + i\kappa\dot{\psi}^m)\right)\left(A_m(x) + i\kappa\psi^n \partial_n A_m(x)\right)$$

$$= \left(\frac{e}{c}\psi^m + \frac{e\kappa}{c}\dot{x}^m\right)\left(A_m(x) + i\kappa\psi^n \partial_n A_m(x)\right)$$

$$= \frac{e}{c}\psi^m A_m(x) + \frac{e\kappa}{c}\dot{x}^m A_m(x) + \frac{ie}{c}\kappa\psi^n\psi^m \partial_n A_m(x)$$

$$= \frac{e}{c}\psi^m A_m(x) + \kappa\left(\frac{e}{c}\dot{x}^m A_m(x) + i\frac{e}{c}\psi^n\psi^m \partial_n A_m(x)\right)$$

$$= \frac{e}{c}\psi^m A_m(x) + \kappa\left(\frac{e}{c}\dot{x}^m A_m(x) + i\frac{e}{2c}\psi^n\psi^m F_{nm}(x)\right)$$

since $\kappa^2 = 0$, $\psi^m \kappa\psi^n = -\kappa\psi^m\psi^n = \kappa\psi^n\psi^m$, and

$$\psi^n\psi^m F_{nm}(x) = \psi^n\psi^m\left(\partial_n A_m(x) - \partial_m A_n(x)\right) = \psi^n\psi^m\partial_n A_m(x) - \psi^n\psi^m\partial_m A_n(x)$$

$$= \psi^n\psi^m\partial_n A_m(x) + \psi^m\psi^n\partial_m A_n(x) = 2\psi^n\psi^m\partial_n A_m(x).$$

Then

$$\frac{ie}{c}\int d\tau \int d\kappa \, D\mathbb{X}^m \mathbb{A}_m(\mathbb{X}) =$$

$$= \frac{e}{c}\int d\tau \frac{\partial}{\partial \kappa}\left(\psi^m A_m(x) + \kappa\left(\dot{x}^m A_m(x) + \frac{i}{2}\psi^n\psi^m F_{nm}(x)\right)\right)\Big|_{\kappa=0}$$

$$= \frac{e}{c}\int d\tau \left(\dot{x}^m A_m(x) + \frac{i}{2}\psi^n\psi^m F_{nm}(x)\right)$$

Exercise 2.4.

We have

$$\frac{\partial \hat{\mathcal{L}}}{\partial \mathbb{X}^m} = -\frac{ie}{c} D\mathbb{X}^n \frac{\partial \mathbb{A}_n}{\partial \mathbb{X}^m} = -\frac{ie}{c} D\mathbb{X}^n \partial_m \mathbb{A}_n(\mathbb{X})$$

$$\frac{\partial \hat{\mathcal{L}}}{\partial D\mathbb{X}^m} = -\frac{iM}{2}\dot{\mathbb{X}}_m - \frac{ie}{c}\mathbb{A}_m(\mathbb{X}),$$

$$\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\mathbb{X}}^m} = -\frac{iM}{2} D\mathbb{X}_m .$$

Notice that $\frac{\partial}{\partial \tau}$ commutes with $\frac{\partial}{\partial t}$, κ and $\frac{\partial}{\partial \kappa}$. Therefore, it also commutes with D. We conclude that the EOMs are

$$-\frac{i}{2}MD\dot{\mathbb{X}}_m - \frac{iM}{2}D\dot{\mathbb{X}}_m - \frac{ie}{c}D\mathbb{A}_m(\mathbb{X}) = -\frac{ie}{c}D\mathbb{X}^n\partial_m \mathbb{A}_n(\mathbb{X})$$

i.e.

$$MD\dot{\mathbb{X}}_m = \frac{e}{c}\left(D\mathbb{X}^n\partial_m \mathbb{A}_n(\mathbb{X}) - D\mathbb{A}_m(\mathbb{X})\right)$$

Since

$$D\mathbb{A}_m(\mathbb{X}) = \left(\frac{\partial}{\partial \kappa} + i\kappa\frac{\partial}{\partial \tau}\right)\left(A_m(x) + i\kappa\psi^n\partial_n A_m(x)\right)$$

$$= i\psi^n \partial_n A_m(x) + i\kappa \frac{\partial}{\partial\tau} A_m(x) = i\psi^n \partial_n A_m(x) + i\kappa \partial_n A_m(x)\dot{x}^n$$

$$= (i\psi^n + i\kappa \dot{x}^n)\partial_n A_m(x) = (i\psi^n + i\kappa(\dot{x}^n + i\kappa\dot{\psi}^n))\partial_n A_m(x)$$

$$= D\mathbb{X}^n \partial_n A_m(x),$$

we have

$$M D\dot{\mathbb{X}}_m = \frac{e}{c} D\mathbb{X}^n F_{mn}(\mathbb{X})$$

Since by (2.36)

$$D\dot{\mathbb{X}}_m = i\dot{\psi}_m + i\kappa(\ddot{x}_m + i\kappa\ddot{\psi}_m) = i\dot{\psi}_m + i\kappa\ddot{x}_m,$$

and

$$F_{mn}(\mathbb{X}) = F_{mn}(x + i\kappa\psi) = F_{mn}(x) + i\kappa\psi^p \partial_p F_{mn}(x),$$

we have

$$iM\dot{\psi}_m + i\kappa M\ddot{x}_m = \frac{e}{c}(i\psi^n + i\kappa\dot{x}^n)(F_{mn}(x) + i\kappa\psi^p\partial_p F_{mn}(x))$$

$$= i\frac{e}{c}\psi^n F_{mn} + i\kappa\frac{e}{c}\left(\dot{x}^n F_{mn}(x) - i\psi^n\psi^p\partial_p F_{mn}(x)\right)$$

Multiplying the Jacobi identity by $\psi^n\psi^p$ we get

$$0 = \psi^n\psi^p(\partial_p F_{mn}(x) + \partial_m F_{np}(x) + \partial_n F_{pm}(x))$$

$$= \psi^n\psi^p\partial_p F_{mn}(x) + \psi^n\psi^p\partial_m F_{np}(x) + \psi^n\psi^p\partial_n F_{pm}(x)$$

$$= \psi^n \psi^p \partial_p F_{mn} + \psi^n \psi^p \partial_m F_{np}(x) + \psi^p \psi^n \partial_n F_{mp}(x)$$

$$= 2 \psi^n \psi^p \partial_p F_{mn}(x) + \psi^n \psi^p \partial_m F_{np}(x).$$

Therefore

$$i M \dot{\psi}_m + i K M \ddot{x}_m = i \frac{e}{c} \psi^n F_{mn} + i K \frac{e}{c} \left(\dot{x}^n F_{mn}(x) + \frac{i}{2} \psi^n \psi^p \partial_m F_{np}(x) \right).$$

We thus have

$$M \ddot{x}_m = \frac{e}{c} \left(\dot{x}^n F_{mn}(x) + \frac{i}{2} \psi^n \psi^p \partial_m F_{np}(x) \right),$$

$$M \dot{\psi}_m = \frac{e}{c} \psi^n F_{mn}(x).$$