

# Quantum Logic

A logic based approach to Bell's inequalities[Burbano, 2017]

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# Outline

- 1 EPR Paradox
- 2 Bell's Inequalities
- 3 Lattice of Propositions in Quantum Mechanics

# Completeness of Quantum Mechanics

Einstein, Podolsky and Rosen, although weary of the success of quantum mechanics, wanted to probe its completeness[Einstein et al., 1935].

- In a complete physical theory every element of physical reality has a counterpart in the theory.
- If we can predict with certainty the value of a physical quantity without disturbing the system, then there exists an element of physical reality corresponding to this physical quantity.

# Let's Put It to the Test

Well, as we've learned from our mathematician friends, let's assume it is!

## Heisenberg's Uncertainty Principle

If two observables are represented by operators which do not commute they cannot be measured simultaneously, i.e., they do not have a simultaneous physical reality[Hall, 2013].

# Photon Polarization

As an example consider the linear polarization of a photon.

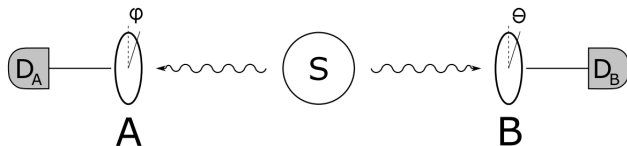
- Hilbert space  $\mathbb{C}^2$ ;
- Vector state describing polarization along the angle  $\theta$

$$|\theta\rangle = (\cos(\theta), \sin(\theta)); \quad (1)$$

- Operator describing “The polarization of the photon is along  $\theta$ ”

$$P(\theta) = |\theta\rangle\langle\theta|. \quad (2)$$

# Two Photons



We emit two photons in the state

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |\pi/2\rangle - |\pi/2\rangle \otimes |0\rangle) \\ &= \frac{1}{\sqrt{2}}(|\pi/4\rangle \otimes |3\pi/4\rangle - |3\pi/4\rangle \otimes |\pi/4\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2. \end{aligned} \quad (3)$$

Alice measures the first component  $P_A(\varphi) = P(\varphi) \otimes \text{id}_{\mathbb{C}^2}$  and Bob the second  $P_B(\theta) = \text{id}_{\mathbb{C}^2} \otimes P(\theta)$ . Such a state can be prepared through the decay of a Calcium atom [Reyes-Lega, 2013].

# Contradiction!

Through Bob's measurements we may acquire information of Alice's system due to the process known as collapse of the wave function. Indeed, if Bob measures  $P_B(0)$  then we can predict what the result of Alice's  $P_A(0)$  measurement will be. Since Bob's measurements cannot affect Alice's system  $P_A(0)$  becomes an element of physical reality. The same is true for  $P_B(\pi/4)$  and  $P_A(\pi/4)$ . Thus  $P_A(0)$  and  $P_A(\pi/4)$  both have a simultaneous reality! Since  $[P_A(0), P_A(\pi/4)] \neq 0$  we've arrived to a contradiction.

# The Search for a Complete Theory

Under the definitions given by EPR quantum mechanics is not complete. Can we provide a complete theory of physical reality? Bell while studying this question arrived at his inequalities for a theory of hidden variables[Bell, 1964]. Our approach will be quite different.



# Partially Ordered Sets

## Definition

A partially ordered set (poset)  $(P, \leq)$  is a set  $P$  along with a relation  $\leq$  which is:

- reflexive:  $p \leq p$  for all  $p \in P$ ;
- anti-symmetric:  $p \leq q$  and  $q \leq p$  implies  $p = q$  for all  $p, q \in P$ ;
- transitive:  $p \leq q$  and  $q \leq r$  implies  $p \leq r$  for all  $p, q, r \in P$ .

## Example

- $(\mathbb{R}, \leq)$
- $(P(X), \subseteq)$
- (Propositions,  $\Rightarrow$ )!

# Meet and Join

## Definition

Let  $(P, \leq)$  be a poset and  $p, q \in P$ . We define  $p \wedge q$  to be the greatest least bound of  $\{p, q\}$  if it exists. Similarly  $p \vee q$  is the lowest upper bound of  $\{p, q\}$  if it exists. If for every pair  $p, q \in P$  both  $p \wedge q$  and  $p \vee q$  exist the poset is said to be a lattice.

Notice that this definition can easily be extended to subsets with more than two elements.

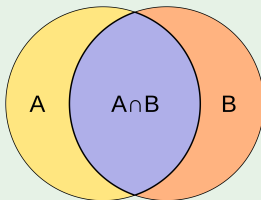
## Definition

A poset  $(P, \leq)$  is said to be bounded if there is a greatest lower bound  $0$  and a least upper bound  $1$ . A complement of  $p \in P$  is an element  $q \in P$  such that  $p \wedge q = 0$  and  $p \vee q = 1$

# Examples of Meets and Joins

## Example

- $(\mathbb{R}, \leq)$  is an unbounded lattice where  $x \wedge y = \min\{x, y\}$  and  $x \vee y = \max\{x, y\}$ .
- $(P(X), \subseteq)$  is a bounded lattice where  $A \wedge B = A \cap B$ ,  $A \vee B = A \cup B$ ,  $0 = \emptyset = A \cap A^c$ , and  $1 = X = A \cup A^c$ .



- (Propositions,  $\Rightarrow$ ) form a bounded lattice where  $p \wedge q$  is the conjunction of the propositions,  $p \vee q$  is the disjunction,  $0$  is always true, and  $1$  is always false. The complement of  $p$  is its negation  $\neg p$ .

# Distributivity

## Definition

A lattice  $(L, \leq)$  is said to be distributive if  $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$  and  $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$  for all  $p, q, r \in L$ .

$p$	$q$	$r$	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

# Boolean Algebras

## Definition

A Boolean algebra is a distributive bounded lattice in which every element has a complement.

One can prove that in these algebras the complement is unique. Therefore, the complement of  $p$  in a Boolean algebra will be denoted by  $p'$ .

## Interpretation

We will interpret EPR's requirements of a complete physical theory to be that the set of propositions one may ask of the theory be a Boolean algebra.

# Bell's Inequalities I

Let  $(B, \leq)$  be a Boolean algebra. Define

$$\begin{aligned} f : B \times B &\rightarrow B \\ (p, q) &\mapsto f(p, q) := (p \wedge q) \vee (p' \wedge q'). \end{aligned} \tag{4}$$

Note that for all  $p_1, q_1, p_2, q_2 \in B$

$$\begin{aligned} (p_1 \wedge q_1) \wedge ((p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1)) &= \\ (p_1 \wedge q_1 \wedge q_2) \vee (p_1 \wedge q_1 \wedge (p_2 \vee q_2)') \vee (p_1 \wedge q_1 \wedge p_2) &= \\ (p_1 \wedge q_1) \wedge (q_2 \vee (p_2 \vee q_2)' \vee p_2) = (p_1 \wedge q_1) \wedge 1 = p_1 \wedge q_1. \end{aligned} \tag{5}$$

# Bell's Inequalities II

We conclude

$$p_1 \wedge q_1 \leq (p_1 \wedge q_2) \vee (p'_2 \wedge q'_2) \vee (p_2 \wedge q_1). \quad (6)$$

Similarly

$$p'_1 \wedge q'_1 \leq (p'_1 \wedge q'_2) \vee (p_2 \wedge q_2) \vee (p'_2 \wedge q'_1). \quad (7)$$

Therefore

$$f(p_1, q_1) \leq f(p_1, q_2) \vee f(p_2, q_2) \vee f(p_2, q_1). \quad (8)$$

# Bell's Inequalities III: Degrees of Plausibility

In quantum mechanics we are more comfortable with the assignment of probabilities to propositions [Jaynes, 2003]. Any reasonable assignment  $P : B \rightarrow \mathbb{R}$  of degree of plausibility to physical propositions must be such that  $P(p) \leq P(q)$  if  $p \leq q$ . Moreover,  $P(p \vee q) \leq P(p) + P(q)$ . We thus arrive at

## Theorem (Bell's Inequalities)

*Let  $(B, \leq)$  be a Boolean algebra with an assignment of degrees of plausibility  $P$ . Then*

$$P(f(p_1, q_1)) \leq P(f(p_1, q_2)) + P(f(p_2, q_2)) + P(f(p_2, q_1)). \quad (9)$$



# What are Propositions in Quantum Mechanics?

Propositions in quantum mechanics should be observables with only two possible values when measured: True or False[Wilce, 2012].

- Observable  $\rightarrow$  Self-adjoint operator
- Spectrum  $\{\text{False}, \text{True}\} \rightarrow \{0, 1\}$

## Definition

We define propositions in quantum mechanics to be the orthogonal projections  $L(\mathcal{H}) := \{P \in \mathcal{B}(\mathcal{H}) | P^2 = P = P^*\}$ .

# Geometry on Hilbert Spaces

Once again, much like mathematicians, given that it is not clear how to define a poset structure on  $L(\mathcal{H})$  we have to proceed by duality.

## Theorem

*Every closed subspace of  $\mathcal{H}$  is the image of an orthogonal projection. Conversely, the image of every orthogonal projection is a closed subspace of  $\mathcal{H}$ .*

We may thus understand  $L(\mathcal{H})$  as the set of closed subspaces of  $\mathcal{H}$ .

## Partial Order on $L(\mathcal{H})$

We inherit the Poset structure from  $P(\mathcal{H})$ .

### Definition

The poset of propositions in quantum mechanics is  $(L(\mathcal{H}), \subseteq)$ .

This forms a bounded lattice:

- $P \wedge Q = P \cap Q$ ;
- $P \vee Q = \overline{\text{span}(P \cup Q)}$ ;
- $0 = \{0\}$ ;
- $1 = \mathcal{H}$ .

### Theorem

If  $P, Q \in L(\mathcal{H})$  commute  $P \wedge Q = PQ$  and  $P \vee Q = P + Q - PQ$ .

### Proof.

[Wilce, 2012]



# Degrees of Plausibility in Quantum Mechanics

In quantum mechanics every state determines a degree of plausibility on  $L(\mathcal{H}), \subseteq$ . In particular, vector states  $|\psi\rangle \in \mathcal{H}$  determine the degree of plausibility

$$P_\psi(P) = \langle \psi | P | \psi \rangle. \quad (10)$$

One verifies that if  $P \leq Q$  then

$$\begin{aligned} \|P|\psi\rangle\|^2 &= \langle \psi | P^2 | \psi \rangle = \langle \psi | P | \psi \rangle = P_\psi(P) = P(PQ) \\ &= \langle \psi | PQ | \psi \rangle \leq \|P|\psi\rangle\| \|Q|\psi\rangle\| \leq \|Q|\psi\rangle\|^2 = P_\psi(Q). \end{aligned} \quad (11)$$

Moreover, one can check that

$$P_\psi(P \vee Q) \leq P_\psi(P) + P_\psi(Q). \quad (12)$$

## Back to EPR

Notice that  $[P_A(\theta), P_B(\phi)] = 0$ . Thus the proposition we assign to their conjunction “Alice measures polarization at an angle  $\theta$  while Bob at an angle  $\phi$ ” is  $P_A(\theta) \wedge P_B(\phi) = P_A(\theta)P_B(\phi) = P(\theta) \otimes P(\phi)$ . The expected values are

$$\begin{aligned} P(P_A(\theta) \wedge P_B(\phi)) &= \langle \psi | P_A(\theta) P_B(\phi) | \psi \rangle = \\ \frac{1}{\sqrt{2}} \langle \psi | (P(\theta) | 0 \rangle \otimes P(\phi) | \pi/2 \rangle - P(\theta) | \pi/2 \rangle \otimes P(\phi) | 0 \rangle) &= \\ \frac{1}{\sqrt{2}} \langle \psi | (\cos(\theta) | \theta \rangle \otimes \sin(\phi) | \phi \rangle - \sin(\theta) | \theta \rangle \otimes \cos(\phi) | \phi \rangle) &= \quad (13) \\ \frac{1}{2} (\cos(\theta) \sin(\phi) - \sin(\theta) \cos(\phi))^2 &= \\ \frac{1}{2} \sin(\theta - \phi)^2. \end{aligned}$$

## Violation of Bell's Inequalities

It makes sense to choose as complements  $P_A(\theta)' = P_A(\theta + \pi/2)$  and  $P_B(\theta)' = P_B(\theta + \pi/2)$ . It is clear then that

$$P(P_A(\theta)' \wedge P_B(\phi)') = \frac{1}{2} \sin(\theta - \phi)^2. \quad (14)$$

On the other hand, notice that

$$\begin{aligned} (P_A(\theta)' \wedge P_B(\phi)')(P_A(\theta) \wedge P_B(\phi)) &= \\ P_A(\theta + \pi/2)P_B(\phi + \pi/2)P_A(\theta)P_B(\phi) &= \\ (P(\theta + \pi/2) \otimes P(\phi + \pi/2))(P(\theta) \otimes P(\phi)) &= \\ P(\theta + \pi/2)P(\theta) \otimes P(\phi + \pi/2)P(\phi) &= \\ 0 = (P_A(\theta) \wedge P_B(\phi))(P_A(\theta)' \wedge P_B(\phi)'). \end{aligned} \quad (15)$$

Thus, we obtain

$$\begin{aligned} f(P_A(\theta), P_B(\phi)) &= (P_A(\theta) \wedge P_B(\phi)) \vee (P_A(\theta)' \wedge P_B(\phi)') \\ &= P_A(\theta) \wedge P_B(\phi) + P_A(\theta)' \wedge P_B(\phi)'. \end{aligned} \quad (16)$$

Since expected values are linear,  $P(f(P_A(\theta), P_B(\phi))) = \sin(\theta - \phi)^2$ . Thus Bell's inequalities dictate

$$\begin{aligned} 1 &= \sin(0 - \pi/2)^2 = P(f(P_A(0), P_B(\pi/2))) \\ &\leq \sin(0 - \pi/6)^2 + \sin(\pi/3 - \pi/6)^2 + \sin(\pi/3 - \pi/2)^2 = 3/4. \end{aligned} \quad (17)$$

There is no correct physical theory whose propositions satisfy a Boolean algebra.



# What failed?

## Theorem

*In a distributive bounded lattice elements have at most one complement.*

## Proof.

Suppose  $q$  and  $r$  are complements of  $p$  in a distributive bounded lattice. Then

$$q = q \wedge 1 = q \wedge (p \vee r) = (q \wedge p) \vee (q \wedge r) = 0 \vee (q \wedge r) = q \wedge r. \quad (18)$$

This  $q \leq r$ . Similarly one can show  $r \leq q$ . By anti-symmetry  $q = r$ . □

It is clear that in the bounded lattice  $L(\mathcal{H})$  complements are not unique. Thus the lattice is not distributive.

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