Canonical Dynamics of von Neumann Algebras

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Abstract

To be done

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Chapter 1

Prerequisites

1.1 Differential Geometry

1.1.1 Differential Structure of Vector Spaces

We will at first be interested in studying the usual concepts of differential calculus on finite dimensional vector spaces over a field $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$. As we will see, the theory is completely equivalent to that of $\mathbb{R}^n$. Although this might at first seem to make the theory not worth studying, this is not the case. In fact, the study of differential structures on vector spaces makes the theory of multivariable differential calculus more transparent and shows that the "coordinate driven" example of $\mathbb{R}^n$ is not special amongst vector spaces. Only after this exercise is performed does the theory of multivariable calculus deserves the, usually misplaced name, *vector calculus.*

The usual definition of derivative in multivariable calculus requires the use of the standard norm on $\mathbb{R}^n$. Our first task will be to show that, as far as differential calculus is concerned, there is nothing special with this norm. We begin by defining an equivalence of norms through the same method equivalence of metrics is defined.

Definition 1.1.1. *Let V be a vector space. A norm $\|\cdot\|$ on V is said to be equivalent to a norm $\|\cdot\|'$ on V if there exists constants $a, b \in \mathbb{R}^+$ such that*

$$a\|v\| \leq \|v\|' \leq b\|v\| \tag{1.1}$$

for all $v \in V$.

This is in fact an equivalence relation.

Theorem 1.1.1. *The equivalence defined on definition 1.1.1 is an equivalence relation on the set of norms of V.*

Proof. Let $\|\cdot\|$ be a norm on V. Then for all $v \in V$

$$1\|v\| \leq \|v\| \leq 1\|v\| \tag{1.2}$$

showing that equivalence of norms is reflexive. If $\|\cdot\|'$ is a norm on V to which $\|\cdot\|$ is equivalent there exists $a, b \in \mathbb{R}^+$ such that for all $v \in V$

$$a\|v\| \leq \|v\|' \leq b\|v\|. \tag{1.3}$$

Therefore, for all $v \in V$

$$\frac{1}{b}\|v\|' \leq \|v\| \leq \frac{1}{a}\|v\|' \tag{1.4}$$

showing that $\|\cdot\|'$ is equivalent to $\|\cdot\|$. Thus, the equivalence of norms is reflexive. Finally assume $\|\cdot\|''$ is a norm on V equivalent to $\|\cdot\|'$. Then there exists $c, d \in \mathbb{R}^+$ such that for all $v \in V$

$$c\|v\|' \leq \|v\|'' \leq d\|v\|'. \tag{1.5}$$

Therefore for all $v \in V$

$$ca\|v\| \leq c\|v\|' \leq \|v\|'' \leq d\|v\|' \leq db\|v\|. \tag{1.6}$$

We conclude that $\|\cdot\|''$ is equivalent to $\|\cdot\|$ and the equivalence of norms in transitive. $\square$

Now we are ready to show that all norms are equivalent on finite dimensional vector spaces.

Theorem 1.1.2. *Let V be a finite dimensional vector space. Then all norms on V are equivalent.*

Proof. Given that equivalence of norms is an equivalence relation, we can fix a norm on V and show that every other norm is equivalent to it. To construct this norm choose a basis $\{v_1, \ldots, v_n\}$ of V. Define

$$\begin{aligned} \|\cdot\| : V &\to \mathbb{R}_0^+ \\ v &\mapsto \sum_{i=1}^n |k_i| \end{aligned} \tag{1.7}$$

where $k_1, \ldots, k_n \in \mathbb{F}$, the field over which V is defined, are the unique scalars such that

$$v = \sum_{i=1}^n k_i v_i. \tag{1.8}$$

From the fact that the codomain of $|\cdot|$ is $\mathbb{R}_0^+$ we have that the codomain of $\|\cdot\|$ is $\mathbb{R}_0^+$ as well. Since $0 = \sum_{i=1}^n 0v_i$ we have $\|0\| = 0$. Now let $v = \sum_{i=1}^n k_i v_i \in V$. If $0 = \|v\| = \sum_{i=1}^n |k_i|$, we have that $k_1 = \cdots = k_n = 0$ since the elements of the sum are non-negative. Thus $v = 0$. If $k \in \mathbb{F}$ then

$$\|kv\| = \|\sum_{i=1}^n kk_i v_i\| = \sum_{i=1}^n |kk_i| = |k| \sum_{i=1}^n |k_i| = |k|\|v\|. \tag{1.9}$$

Finally, of $u = \sum_{i=1}^n l_i v_i \in V$ then

$$\|v+u\| = \|\sum_{i=1}^{n}(k_i + l_i)v_i\| = \sum_{i=1}^{n}|k_i + l_i| \leq \sum_{i=1}^{n}(|k_i| + |l_i|) = \|v\| + \|u\|. \tag{1.10}$$

We have thus show that $\|\cdot\|$ is a norm on V.

Now let $\|\cdot\|'$ be any other norm on V. Let $M := \max\{\|v_i\|' | i \in \{1, \ldots, n\}\}$. Then for any vector $v = \sum_{i=1}^n k_i v_i \in V$ we have

$$\|v\|' = \|sum_{i=1}^n k_i v_i\|' \leq \sum_{i=1}^{n}|k_i|\|v_i\|' \leq M \sum_{i=1}^{n}|k_i| = M\|v\|. \tag{1.11}$$

In particular, for any $u, v \in V$ we have

$$|\|u\|' - \|v\|'| \leq \|u - v\|' \leq M\|u - v\|. \tag{1.12}$$

Let $\epsilon \in \mathbb{R}^+$ and $v \in V$. Then for every $u \in V$ such that $\|u - v\| < \epsilon/M$ we have

$$|\|u\| - \|v\|| \leq M\|u - v\| < M\epsilon/M = \epsilon. \tag{1.13}$$

Therefore, $\|\cdot\|'$ is continuous with respect to the topology on V induced by the metric induced by $\|\cdot\|$. Consider now the unit sphere $S := \{v \in V | \|v\| = 1\}$ centered at 0 according to the metric induced by $\|\cdot\|$. It is compact. Therefore, the continuity of $\|\cdot\|'$ implies $a := \inf \|S\|', b := \sup \|S\|' \in \mathbb{R}$. Moreover, since a and b are both limit points of $\|S\|' \subseteq \mathbb{R}_0^+$, the fact that $\mathbb{R}_0^+$ is closed implies $a, b \in \mathbb{R}_0^+$. Finally, for all $v \in V$

$$a\|v\| \leq \|\frac{1}{\|v\|}v\|'\|v\| = \|v\|' = \|\frac{1}{\|v\|}v\|'\|v\| \leq b\|v\|. \tag{1.14}$$

We conclude $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. □

1.1.2 Manifolds and their Philosophy

In differential geometry we are interested in the study of spaces that locally resemble finite dimensional vector spaces. Of course, the implementation of locality is done through a topological structure.

Definition 1.1.2. *A locally euclidean space is a topological space X for which there exists an $n \in \mathbb{N}$ such that every point $p \in M$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. The number $\dim M := n$ is called the dimension of M.*

Some remarks regarding the restrictions done in our definition are in order. You could be worried since $\mathbb{R}^n$ is a very special n-dimensional vector space. However, all finite dimensional vector spaces are isomorphic to $\mathbb{R}^n$ for some n. This remains true if we consider the natural topological and differential structures on these vector spaces. Although these isomorphisms are not canonical, the physically meaningful object will be the locally Euclidean space. The role of the homeomorphisms will be to give mathematical structure to these spaces and will not have any actual physical interpretation.

You may also wonder if we could extend this definition to allow for the dimension to change from point to point. Suppose M is a topological space. Further, assume $p, q \in M$ have open neighborhoods U and V respectively such that $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ and $\psi : V \to \psi(V) \subseteq \mathbb{R}^m$ are homeomorphisms. If $U \cap V \neq \varnothing$ we have a homeomorphism between open subsets

$$\psi \circ \phi^{-1} : \phi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^m. \tag{1.15}$$

Invariance of domain then guarantees that $n = m$. If we then demand $n \neq m$ we must have $U \cap V = \varnothing$. If every point in M has an open neighborhood homeomorphic to an open neighborhood of $\mathbb{R}^n$ then n must be a local invariant. Indeed, for every $\tilde{p} \in U$ the previous argument shows that every open neighborhood of $\tilde{p}$ homeomorphic to an open subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$ must satisfy $k = n$. Thus, the connected component containing p is a locally Euclidean space. Similarly, the connected component containing q is a locally Euclidean space. If we require $n \neq m$ we must then conclude that these connected components are disjoint. We conclude that the connected components of M are always locally Euclidean spaces (with a fixed dimension) and M can be studied by considering each connected component separately.

Locally Euclidean spaces are useful because they can inherit the local structure of finite dimensional vector spaces. For example,

1.2 Functional Analysis

1.2.1 Banach Spaces

A way to induce a topology compatible with the algebraic structure of a vector space is to consider the metric associated to a norm.

Theorem 1.2.1. *Let V be a normed vector space. Then the function*

$$\begin{aligned} d : V \times V &\to V \\ (x, y) &\mapsto d(x, y) := \|x - y\| \end{aligned} \tag{1.16}$$

is a metric on V. Moreover, the topology on V induced by this metric makes V a topologival vector space.

Proof. Given that $\|\cdot\| : V \to \mathbb{R}_0^+$ we have that for all $x, y \in V$

$$d(x, y) = \|x - y\| \geq 0. \tag{1.17}$$

Moreover, $\|x\| = 0$ if and only if $x = 0$ for all $x \in V$. This implies that $\|x - y\| = d(x, y) = 0$ if and only if $x - y = 0$, or in other words, $x = y$ for all

$x, y \in V$. We also have from the definition of a norm that $\|kx\| = |k|\|x\|$ for all $k \in \mathbb{F}$ and $x \in V$. Thus, for all $x, y \in V$

$$d(x,y) = \|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x). \quad (1.18)$$

Finally, we have the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$. Therefore, for all $x, y, z \in V$

$$\begin{aligned} d(x,z) =&\|x-z\| = \|(x-y)+(y-z)\| \\ \leq&\|x-y\| + \|y-z\| = d(x,y) + d(y,z). \end{aligned} \quad (1.19)$$

We conclude that d is a metric on V.

We now consider the topology induced by d on V. Let $a, b \in V$ and $\epsilon \in \mathbb{R}^+$. For all $x, y \in V$ we have that if $\|x-a\| < \epsilon/2$ and $\|y-b\| < \epsilon/2$, then the triangle inequality guarantees

$$\begin{aligned} \|(x+y)-(a+b)\| =&\|(x-a)+(y-b)\| \leq \|x-a\| + \|y-b\| && (1.20) \\ <&\epsilon/2 + \epsilon/2 = \epsilon. && (1.21) \end{aligned}$$

Thus, the sum is continuous. Now let $a \in V$, $k \in \mathbb{F}$ and $\epsilon \in \mathbb{R}^+$. For all $l \in \mathbb{F}$ and $x \in V$ if

$$\begin{aligned} \|x-a\| <&\left(\sqrt{1+\frac{\epsilon}{|k|\|a\|}}-1\right)\|a\|, \\ |l-k| <&\left(\sqrt{1+\frac{\epsilon}{|k|\|a\|}}-1\right)|k| \end{aligned} \quad (1.22)$$

then the triangle inequality once again guarantees

$$\begin{aligned} \|lx-ka\| =&\|(l-k)(x-a)+kx+la-ka-ka\| \\ =&\|(l-k)(x-a)+k(x-a)-(l-k)a\| \\ \leq&|l-k|\|x-a\| + |k|\|x-a\| + |l-k|\|a\| \\ <&\left(\sqrt{1+\frac{\epsilon}{|k|\|a\|}}-1\right)^2\|a\||k| + 2\left(\sqrt{1+\frac{\epsilon}{|k|\|a\|}}-1\right)\|a\||k| \\ =&\left(\sqrt{1+\frac{\epsilon}{|k|\|a\|}}+1\right)\left(\sqrt{1+\frac{\epsilon}{|k|\|a\|}}-1\right)\|a\||k| \\ =&\left(1+\frac{\epsilon}{|k|\|a\|}-1\right)\|a\||k| = \epsilon. \end{aligned} \quad (1.23)$$

This shows that scalar multiplication is also continuous and V is a topological vector space when equipped with the topology induced by d. □

Definition 1.2.1. *Let V be a normed space. Then the metric d defined by* (1.16) *is called the metric induced by the norm of V.*

Although metric spaces are usually provided with their metric topology, they are endowed with more structure than topological spaces. In particular, two metric spaces can be homeomorphic without both being complete. During this section we will focus on a particular type of topological vector spaces.

Definition 1.2.2. *A Banach space is a normed vector space which is complete when endowed with the metric induced by its norm.*

Recall that every metric space has a unique completion up to isomorphism of metric spaces. We will now see that the same is true for normed vector spaces. This implies that every normed vector space determines uniquely a Banach space which extends it.

Theorem 1.2.2. *Let V be a normed vector space. Then, there exists a Banach space $\tilde{V}$ and a linear isometry $\phi : V \to \tilde{V}$ for which overline$\phi(V) = \tilde{V}$. Moreover, for every Banach space W for which a linear isometry $\varphi : V \to W$ with $\overline{\varphi(V)} = W$ exists, there is a unique (canonical) isomorphism of normed spaces $\psi : \tilde{V} \to W$ for which $\psi \circ \phi = \varphi$, that is, the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\varphi"'] & \tilde{V} \arrow[d, dotted, "\psi"] \\ & W. \end{tikzcd}$$

Proof. Let $\tilde{V}$ be the completion of V when equipped with the metric induced by its norm. Recall that $\tilde{V} = C/{\sim}$ where C is the set of Cauhy sequences in V and $\sim$ is the equivalence relation defined by $(x_n) \sim (y_n)$ if and only if $\|x_n - y_n\| \to 0$ for all $(x_n), (y_n) \in C$. The metric on $\tilde{V}$ is given by

$$\begin{aligned} d : \tilde{V} \times \tilde{V} &\to \tilde{V} \\ ([(x_n)]_\sim, [(y_n)]_\sim) &\mapsto \lim_{n\to\infty} \|x_n - y_n\|. \end{aligned} \tag{1.24}$$

To define a vector space structure on $\tilde{V}$ we need to begin by considering the behavior of the vector space structure of V with respect to $\sim$.

Let $(x_n), (y_n) \in C$, $k \in \mathbb{F}$ and $\epsilon \in \mathbb{R}^+$. There exists an $N \in \mathbb{N}^+$ such that for all $n, m \in \mathbb{N}^{\geq N}$ we have

$$\begin{aligned} \|x_n - x_m\| &< \epsilon/2; \\ \|x_n - x_m\| &< \epsilon/|k|; \\ \|y_n - y_m\| &< \epsilon/2. \end{aligned} \tag{1.25}$$

Therefore, for all $n, m \in \mathbb{N}^{\geq N}$

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\| &= \|(x_n - x_m) + (y_n - y_m)\| \\ &\leq \|x_n - x_m\| + \|y_n - y_m\| < \epsilon/2 + \epsilon/2 = \epsilon; \\ \|kx_n - kx_m\| &= \|k(x_n - x_m)\| = |k|\|x_n - x_m\| \\ &< |k|\epsilon/|k| = \epsilon. \end{aligned} \tag{1.26}$$

This shows that the point-wise addition of the elements of Cauchy sequences and the multiplication of a Cauchy sequence by a scalar yield new Cauchy sequences.

Assume now that for $(a_n), (b_n) \in C$ we have that $(x_n) \sim (a_n)$ and $(y_n) \sim (b_n)$. Therefore, for all $n \in \mathbb{N}^{\geq N}$

$$\begin{aligned} \|(x_n + y_n) - (a_n + b_n)\| =& \|(x_n - a_n) + (y_n - b_n)\| \\ \leq& \|x_n - a_n\| + \|y_n - b_n\| < \epsilon/2 + \epsilon/2 = \epsilon; \\ \|kx_n - ka_n\| =& \|k(x_n - a_n)\| = |k|\|x_n - a_n\| < |k|\epsilon/|k| = \epsilon. \end{aligned} \tag{1.27}$$

We conclude that $(x_n + y_n) \sim (a_n + b_n)$ and $(kx_n) \sim (ka_n)$.

The previous argument shows that the maps

$$\begin{aligned} \tilde{V} \times \tilde{V} &\to \tilde{V} \\ ([(x_n)]_\sim, [(y_n)]_\sim) &\mapsto [(x_n)]_\sim + [(y_n)]_\sim := [(x_n + y_n)]_\sim \\ \mathbb{F} \times \tilde{V} &\to \tilde{V} \\ (k, [(x_n)]_\sim) &\mapsto k[(x_n)]_\sim := [(kx_n)]_\sim \end{aligned} \tag{1.28}$$

are well defined. We now need to show this is a vector space structure over $\tilde{V}$. Let $k, l \in \mathbb{F}$ and $[(x_n)]_\sim, [(y_n)]_\sim, [(z_n)]_\sim \in \tilde{V}$. Then

$$\begin{aligned} ([(x_n)]_\sim + [(y_n)]_\sim) + [(z_n)]_\sim =& [(x_n + y_n)]_\sim + [(z_n)]_\sim \\ =& [((x_n + y_n) + z_n)]_\sim \\ =& [(x_n + (y_n + z_n)]_\sim \\ =& [(x_n)]_\sim + [(y_n + z_n)]_\sim \\ =& [(x_n)]_\sim + ([(y_n)]_\sim + [(z_n)]_\sim); \\ [(x_n)]_\sim + [(0)]_\sim =& [(x_n) + 0]_\sim = [(x_n)]_\sim; \\ [(x_n)]_\sim + [(-x_n)]_\sim =& [(x_n - x + n)]_\sim = [(0)]_\sim; \\ [(x_n)]_\sim + [(y_n)]_\sim =& [(x_n + y_n)]_\sim = [(y_n + x_n)]_\sim \\ =& [(y_n)]_\sim + [(x_n)]_\sim; \\ 1[(x_n)]_\sim =& [(1x_n)]_\sim = [(x_n)]_\sim; \\ k([(x_n)]_\sim + [(y_n)]_\sim) =& k[(x_n + y_n)]_\sim = [(k(x_n + y_n))]_\sim \\ =& [(kx_n + ky_n)]_\sim = [(kx_n)]_\sim + [(ky_n)]_\sim \\ =& k[(x_n)]_\sim + k[(y_n)]_\sim; \\ (k + l)[(x_n)]_\sim =& [((k + l)x_n)]_\sim = [(kx_n + lx_n)]_\sim \\ =& [(kx_n)]_\sim + [(lx_n)]_\sim \\ =& k[(x_n)]_\sim + l[(x_n)]_\sim. \end{aligned} \tag{1.29}$$

We see then that $\tilde{V}$ is a vector space over $\mathbb{F}$ where the additive neutral element is $0 := [(0)]_\sim$ and the additive inverse of a sequence $[(x_n)]_\sim \in V$ is $-[(x_n)]_\sim := [(-x_n)]_\sim$.

We would now like to give $\tilde{V}$ a norm which induces the metric (1.24). If such a norm exists, then the normed space $\tilde{V}$ is a Banach space. Recall that for two elements x and y in a normed space we have

$$\begin{aligned} \|x\| =& \|x - y + y\| \leq \|x - y\| + \|y\|; \\ \|y\| =& \|x - y - x\| \leq \|x - y\| + \|-x\| = \|x - y\| + \|x\|. \end{aligned} \tag{1.30}$$

Therefore, $|\|x\| - \|y\|| \leq \|x - y|$. Let $(x_n) \in C$ and $\epsilon \in \mathbb{R}^+$. Then there exists $N \in \mathbb{N}^+$ such that for all $n, m \in \mathbb{N}^{\geq N}$ we have that $|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\| < \epsilon$. This shows that $(\|x_n\|)$ is Cauchy for all $(x_n) \in C$ and, due to the completeness of $\mathbb{R}$,

$$\begin{aligned} &\| \cdot \| : \tilde{V} \to \mathbb{R}^+ \\ &[(x_n)]_\sim \mapsto \|[(x_n)]_\sim\| := \lim_{n\to\infty} \|x_n\| \end{aligned} \tag{1.31}$$

is well defined. □