Field Theory:

A Modern Primer

Chapter 1

Section 1.1

A. i) The action is invariant under the infinitesimal transformation $\delta x = \varepsilon$. Given that to $\mathcal{O}(\varepsilon^2)$

$$S[x+\delta x] = \int_{t_0}^{t_f} dt \, \frac{1}{2} m \left(\frac{d(x+\delta x)}{dt} \right)^2 = \int_{t_0}^{t_f} dt \, \frac{1}{2} m \left(\dot{x}^2 + 2\dot{x} \cdot \frac{d\delta x}{dt} \right)$$

$$= \int_{t_0}^{t_f} dt \, \frac{1}{2} m \left(\dot{x}^2 - 2\ddot{x} \cdot \delta x + 2 \frac{d}{dt} \left(\dot{x} \cdot \delta x \right) \right)$$

$$= S[x] - \int dt \, m\ddot{x} \cdot \delta x + \Big[m\dot{x} \cdot \delta x \Big]_{t_0}^{t_f},$$

if x satisfies Newton's second law, i.e. $\ddot{x} = 0$, we have

$$0 = \delta S = S[x+\delta x] - S[x] = \Big[m\dot{x} \cdot \delta x \Big]_{t_0}^{t_f} = \left(m\dot{x}(t_f) - m\dot{x}(t_0) \right) \cdot \varepsilon .$$

Since this is true for any ε, t_0, t_f we conclude that momentum is conserved.

ii) From equation (1.1.8) we have

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x} = -\nabla V = \frac{v}{a} \sin\left(\frac{r}{a}\right) \frac{x}{r}$$

B) We have from equation (1.1.8)

$$\frac{d}{dt}(m\dot{x}) = -\nabla V$$

c) Consider an infinitesimal rotation $\delta x^i = \omega^i{}_j x^j$. Given that

$$r^2 = x^2 = (x+\delta x)\cdot(x+\delta x) = x^2 + 2\, x^i \omega_{ij} x^j + \mathcal{O}(\omega_{ij}^2)$$

and $x^i x^j$ is symmetric, we have $\omega_{ij} = -\omega_{ji}$. Since r is left invariant by this transformation, S is too. Therefore by (1.1.6) we have

$$m\dot{x}\cdot\delta x = m\dot{x}^i \omega_{ij} x^j$$

is conserved by choosing a basis for the antisymmetric matrices $(\Omega^{ab})_{ij} = \delta^a_i \delta^b_j - \delta^b_i \delta^a_j$ we obtain that

$$m(\dot{x}^a x^b - \dot{x}^b x^a)$$

is conserved. This corresponds to the conservation of angular momentum. From now on let $L^{ab} = \dot{x}^a x^b - x^b \dot{x}^a$

Now consider the infinitesimal change $\delta x^i = \varepsilon L^{ik}$ for some fixed index k. The term $\frac{1}{2} m\dot{x}^2$ is left invariant since L^{ik} is constant as we just showed. On the other hand, to order ε^2

$$r+\delta r = \left((x^i+\delta x^i)(x_i+\delta x_i)\right)^{1/2} = \left(r^2 + 2\varepsilon x_i L^{ik}\right)^{1/2} = r\left(1+\varepsilon \frac{x_i L^{ik}}{r^2}\right) = r + \varepsilon \frac{x_i L^{ik}}{r}$$

and therefore the potential changes to

$$\frac{a}{r+\delta r} = \frac{a}{r}\left(1 - \frac{\delta r}{r}\right) = \frac{a}{r} - \frac{\varepsilon a x_i L^{ik}}{r^3}$$

$$= \frac{a}{r} - \frac{\varepsilon a}{r^3}\left(\dot{x}^i x^k x_i - \dot{x}^k r^2\right) = \frac{a}{r} - \frac{\varepsilon a}{r^3}\dot{x}^i x^k x_i + \frac{\varepsilon a \dot{x}^k}{r}$$

$$= \frac{a}{r} + \varepsilon \frac{d}{dt}\left(\frac{a x^k}{r}\right).$$

We conclude

$$S[x^i+\delta x^i] = S[x] + \varepsilon a \left[\frac{x^k}{r}\right]_{t_1}^{t_2}$$

However, since we have by equation (1.1.6)

$$S[x^i+\delta x^i] = S[x^i] + m\left[\delta x_i \dot{x}^i\right]_{t_1}^{t_2},$$

we conclude that $m x_i \delta x^i - \varepsilon a \frac{x^k}{r}$ is a constant. We thus obtain that the vector

$$m x_i L^{ik} - \frac{a x^k}{r} = m\left(x_i \dot{x}^i x^k - x_i x^i \dot{x}^k\right) - \frac{a x^k}{r}$$

is constant.

The final constant of motion is obtained by letting $\delta x = \varepsilon \dot{x}$.

Then the kinetic energy changes by

$$\frac{1}{2} m (\dot{x}^i + \varepsilon \ddot{x}^i)(\dot{x}_i + \varepsilon \ddot{x}^i) = \frac{1}{2} m \dot{x}^i \dot{x}_i + m \varepsilon \ddot{x}^i \dot{x}_i = \frac{1}{2} m \dot{x}^i \dot{x}_i + \varepsilon \frac{d}{dt}\left(\frac{1}{2} m \dot{x}^i \dot{x}_i\right)$$

The potential changes by

$$\frac{a}{r + \delta r} = \frac{a}{((x^i + \delta x^i)(x_i + \delta x_i))^{1/2}} = \frac{a}{(r^2 + 2x^i \delta x_i)^{1/2}} = \frac{a}{r}\left(1 - \frac{x^i \delta x_i}{r^2}\right)$$

$$= \frac{a}{r} - \varepsilon \frac{a x^i \dot{x}_i}{r^3} = \frac{a}{r} + \varepsilon \frac{d}{dt}\left(\frac{a}{r}\right)$$

Much like before we have

$$S[x] + \varepsilon \left[m \dot{x}^i \dot{x}_i \right]_{t_1}^{t_2} = S[x] + m \left[\dot{x}^i \delta x_i \right]_{t_1}^{t_2} = S[x + \delta x] = S[x] + \varepsilon \left[\frac{1}{2} m \dot{x}^i \dot{x}_i + \frac{a}{r} \right]_{t_1}^{t_2}.$$

We thus conclude that

$$\frac{1}{2} m \dot{x}^i \dot{x}_i - \frac{a}{r}$$

is a constant of motion.

Observation: The solution given for the Runge-Lenz vector was inspired on Gorni, G. & Zampieri, G. "Revisiting Nother's Theorem on constants of motion".

D) Consider an infinitesimal variation $\delta x = \varepsilon \dot{x}$. Then as we saw in the previous problem

$$\frac{1}{2} m \left(\frac{d(x+\delta x)}{dt} \right)^2 = \frac{1}{2} m \dot{x}^2 + \varepsilon \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 \right).$$

On the other hand

$$V(t, x+\delta x) = V(t,x) + \partial_i V(t,x) \delta x^i = V(t,x) + \varepsilon \partial_i V(t,x) \dot{x}^i$$

$$= V(t,x) + \varepsilon \left(\frac{dV(t,x)}{dt} - \frac{\partial V(t,x)}{\partial t} \right).$$

Therefore the change in the action is

$$\varepsilon \left[m \dot{x}^2 \right]_{t_1}^{t_2} = m \left[\dot{x} \delta x \right]_{t_1}^{t_2} = \delta S = \varepsilon \left[\frac{1}{2} m \dot{x}^2 \right]_{t_1}^{t_2} - \varepsilon \left[V \right]_{t_1}^{t_2} + \varepsilon \int_{t_1}^{t_2} dt \frac{\partial V}{\partial t}.$$

We conclude that if $E = \frac{1}{2} m \dot{x}^2 + V$

$$\frac{dE}{dt} = \frac{\partial V}{\partial t}.$$

In particular, if V is time independent the E is conserved.

Section 1.2.

A) Let L_1 and L_2 be Lorentz transformations. Then

$$g = (L_1 L_1^{-1})^T g\, (L_1 L_1^{-1}) = (L_1^{-1})^T L_1^T g L_1 L_1^{-1} = (L_1^{-1})^T g L_1^{-1}$$

and

$$(L_1 L_2)^T g\, (L_1 L_2) = L_2^T L_1^T g L_1 L_2 = L_2^T g L_2 = g,$$

showing that the set of Lorentz transformatios is closed under multiplication and taking inverses. It is therefore a sobgroup of the 4×4 matrices.

B) Let L' be a Lorentz transformation. In the frame to which L' transforms, L becomes $(L')^{-1} L L'$. We have

$$\det((L')^{-1} L L') = (\det \cancel{L'})^{-1} \det L \det \cancel{L'} = \det L$$

showing that $\det L$ is Lorentz invariant. Moreover,

$$((L')^{-1} L L')^0{}_0 = ((L')^{-1})^0{}_\mu \Lambda^\mu{}_\nu L'^\nu{}_0$$

where $((L')^{-1})^\mu{}_\alpha L'^\alpha{}_\nu = \delta^\mu{}_\nu$