Quantum Logic

A logic based approach to Bell's inequalities[Burbano, 2017]

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Outline

1. EPR Paradox
2. Bell's Inequalities
3. Lattice of Propositions in Quantum Mechanics

Completeness of Quantum Mechanics

Einstein, Podolsy and Rosen, although weary of the success of quantum mechanics, wanted to probe its completeness[Einstein et al., 1935].

- In a complete physical theory every element of physical reality has a counterpart in the theory.
- If we can predict with certainty the value of a physical cuantity without disturbing the system, then there exists an element of physical reality corresponding to this physical quantity.

Let's Put It to the Test

Well, as we've learned from our mathematician friends, lets assume it is!

Heisenberg's Uncertainty Principle

If two observables are represented by operators which do not commute they cannot be measured simultaneously, i.e., they do not have a simultaneous physical reality[Hall, 2013].

Photon Polarization

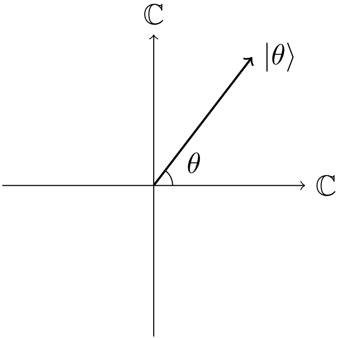
As an example consider the linear polarization of a photon.

- Hilbert space $\mathbb{C}^2$;
- Vector state describing polarization along the angle θ

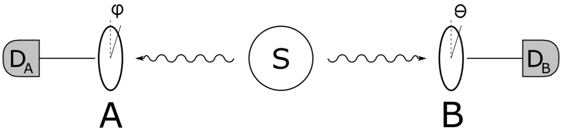
$$|\theta\rangle = (\cos(\theta), \sin(\theta)); \quad (1)$$

- Operator describing "The polarization of the photon is along θ"

$$P(\theta) = |\theta\rangle\langle\theta| . \quad (2)$$



Two Photons



We emmit two photons in the state

$$\begin{aligned} |\psi\rangle =& \frac{1}{\sqrt{2}}(|0\rangle \otimes |\pi/2\rangle - |\pi/2\rangle \otimes |0\rangle) \\ =& \frac{1}{\sqrt{2}}(|\pi/4\rangle \otimes |3\pi/4\rangle - |3\pi/4\rangle \otimes |\pi/4\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2. \end{aligned} \tag{3}$$

Alice measures the first component $P_A(\varphi) = P(\varphi) \otimes \mathrm{id}_{\mathbb{C}^2}$ and Bob the second $P_B(\theta) = \mathrm{id}_{\mathbb{C}^2} \otimes P(\theta)$. Such a state can be prepared through the decay of a Calcium atom[Reyes-Lega, 2013].

Contradiction!

Through Bob's measurements we may acquire information of Alice's system due to the process known as collapse of the wave function. Indeed, if Bob measures $P_B(0)$ then we can predict what the result of Alice's $P_A(0)$ measurement will be. Since Bob's measurements cannot affect Alice's system $P_A(0)$ becomes and element of physical reality. The same is true for $P_B(\pi/4)$ and $P_A(\pi/4)$. Thus $P_A(0)$ and $P_A(\pi/4)$ both have a simultaneous reality! Since $[P_A(0), P_A(\pi/4)] \neq 0$ we've arrived to a contradiction.

The Search for a Complete Theory

Under the definitions given by EPR quantum mechanics is not complete. Can we provide a complete theory of physical reality? Bell while studying this question arrived at his inequalities for a theory of hidden variables[Bell, 1964]. Our approach will be quite different.

Partially Ordered Sets

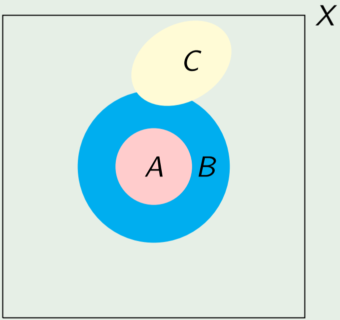
Definition

A partially ordered set (poset) $(P, \leq)$ is a set P along with a relation $\leq$ which is:

- reflexive: $p \leq p$ for all $p \in P$;
- anti-symmetric: $p \leq q$ and $q \leq p$ implies $p = q$ for all $p, q \in P$;
- transitive: $p \leq q$ and $q \leq r$ implies $p \leq r$ for all $p, q, r \in P$.

Example

- $(\mathbb{R}, \leq)$
- $(P(X), \subseteq)$
- (Propositions, $\Rightarrow$)!



Meet and Join

Definition

Let $(P, \leq)$ be a poset and $p, q \in P$. We define $p \wedge q$ to be the greatest least bound of $\{p, q\}$ if it exists. Similarly $p \vee q$ is the lowest upper bound of $\{p, q\}$ if it exists. If for every pair $p, q \in P$ both $p \wedge q$ and $p \vee q$ exist the poset is said to be a lattice.

Notice that this definition can easily be extended to subsets with more than two elements.

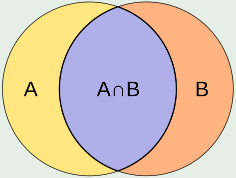
Definition

A poset $(P, \leq)$ is said to be bounded if there is a greatest lower bound 0 and a least upper bound 1. A complement of $p \in P$ is an element $q \in P$ such that $p \wedge q = 0$ and $p \vee q = 1$

Examples of Meets and Joins

Example

- $(\mathbb{R}, \leq)$ is an unbounded lattice where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.
- $(P(X), \subseteq)$ is a bounded lattice where $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $0 = \emptyset = A \cap A^c$, and $1 = X = A \cup A^c$.



- (Propositions, $\Rightarrow$) form a bounded lattice where $p \wedge q$ is the conjunction of the propositions, $p \vee q$ is the disjunction, 0 is always true, and 1 is always false. The complement of p is its negation $\neg p$.

Distributivity

Definition

A lattice $(L, \leq)$ is said to be distributive if
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
for all $p, q, r \in L$.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Boolean Algebras

Definition

A Boolean algebra is a distributive bounded lattice in which every element has a complement.

One can prove that in these algebras the complement is unique. Therefore, the complement of p in a Boolean algebra will be denoted by p'.

Interpretation

We will interpret EPR's requirements of a complete physical theory to be that the set of propositions one may ask of the theory be a Boolean algebra.

Bell's Inequalities I

Let $(B, \leq)$ be a Boolean algebra. Define

$$\begin{aligned} f : B \times B &\to B \\ (p, q) &\mapsto f(p, q) := (p \wedge q) \vee (p' \wedge q'). \end{aligned} \tag{4}$$

Note that for all $p_1, q_1, p_2, q_2 \in B$

$$\begin{aligned} (p_1 \wedge q_1) \wedge ((p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1)) = \\ (p_1 \wedge q_1 \wedge q_2) \vee (p_1 \wedge q_1 \wedge (p_2 \vee q_2)') \vee (p_1 \wedge q_1 \wedge p_2) = \\ (p_1 \wedge q_1) \wedge (q_2 \vee (p_2 \vee q_2)' \vee p_2) = (p_1 \wedge q_1) \wedge 1 = p_1 \wedge q_1. \end{aligned} \tag{5}$$

Bell's Inequalities II

We conclude

$$p_1 \wedge q_1 \leq (p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1). \tag{6}$$

Similarly

$$p_1' \wedge q_1' \leq (p_1' \wedge q_2') \vee (p_2 \wedge q_2) \vee (p_2' \wedge q_1'). \tag{7}$$

Therefore

$$f(p_1, q_1) \leq f(p_1, q_2) \vee f(p_2, q_2) \vee f(p_2, q_1). \tag{8}$$

Bell's Inequalities III: Degrees of Plausibility

In quantum mechanics we are more comfortable with the assignment of probabilities to propositions[Jaynes, 2003]. Any reasonable assignment $P : B \to \mathbb{R}$ of degree of plausibility to physical propositions must be such that $P(p) \leq P(q)$ if $p \leq q$. Moreover, $P(p \vee q) \leq P(p) + P(q)$. We thus arrive at

Theorem (Bell's Inequalities)

Let $(B, \leq)$ be a Boolean algebra with an assignation of degrees of plausibility P. Then

$$P(f(p_1, q_1)) \leq P(f(p_1, q_2)) + P(f(p_2, q_2)) + P(f(p_2, q_1)). \quad (9)$$

What are Propositions in Quantum Mechanics?

Propositions in quantum mechanics should be observables with only two possible values when measured: True or False[Wilce, 2012].

- Observable $\rightarrow$ Self-adjoint operator
- Spectrum $\{\text{False}, \text{True}\} \rightarrow \{0, 1\}$

Definition

We define propositions in quantum mechanics to be the orthogonal projections $L(\mathcal{H}) := \{P \in \mathcal{B}(\mathcal{H}) | P^2 = P = P^*\}$.

Geometry on Hilbert Spaces

Once again, much like mathematicians, given that it is not clear how to define a poset structure on $L(\mathcal{H})$ we have to proceed by duality.

Theorem

Every closed subspace of $\mathcal{H}$ is the image of an orthogonal projection. Conversely, the image of every orthogonal projection is a closed subspace of $\mathcal{H}$.

We may thus understand $L(\mathcal{H})$ as the set of closed subspaces of $\mathcal{H}$.

Partial Order on $L(\mathcal{H})$

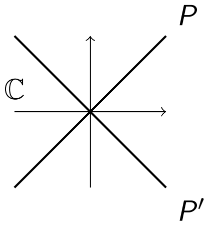
We inherit the Poset structure from $P(\mathcal{H})$.

Definition

The poset of propositions in quantum mechanics is $(L(\mathcal{H}), \subseteq)$.

This forms a bounded lattice in which every element has a complement:

- $P \wedge Q = P \cap Q$
- $P \vee Q = \overline{\text{span}(P \cup Q)}$
- $0 = \{0\}$
- $1 = \mathcal{H}$
- P' is the projection onto $P^{\perp}$



Theorem

If $P, Q \in L(\mathcal{H})$ commute $P \wedge Q = PQ$ and $P \vee Q = P + Q - PQ$.

Degrees of Plausibility in Quantum Mechanics

In quantum mechanics every state determines a degree of plausibility on $(L(\mathcal{H}), \subseteq)$. In particular, vector states $|\psi\rangle \in \mathcal{H}$ determine the degree of plausibility

$$P_\psi(P) = \langle\psi|P|\psi\rangle\,. \tag{10}$$

One verifies that if $P \leq Q$ then

$$\begin{aligned}\|P\,|\psi\rangle\,\|^2 =& \langle\psi|P^2|\psi\rangle = \langle\psi|P|\psi\rangle = P_\psi(P) = P(PQ)\\ =& \langle\psi|PQ|\psi\rangle \leq \|P\,|\psi\rangle\,\|\|Q\,|\psi\rangle\,\| \leq \|Q\,|\psi\rangle\,\|^2\\ =& P_\psi(Q).\end{aligned} \tag{11}$$

Moreover, one can check that

$$P_\psi(P \vee Q) \leq P_\psi(P) + P_\psi(Q). \tag{12}$$

Back to EPR

Notice that $[P_A(\theta), P_B(\phi)] = 0$. Thus the proposition we assign to their conjunction "Alice measures polarization at an angle θ while Bob at an angle ϕ" is $P_A(\theta) \wedge P_B(\phi) = P_A(\theta)P_B(\phi) = P(\theta) \otimes P(\phi)$. The expected values are

$$\begin{aligned} P(P_A(\theta) \wedge P_B(\phi)) = \langle\psi|P_A(\theta)P_B(\phi)|\psi\rangle = \\ \frac{1}{\sqrt{2}} \langle\psi| \left(P(\theta) |0\rangle \otimes P(\phi) |\pi/2\rangle - P(\theta) |\pi/2\rangle \otimes P(\phi) |0\rangle\right) = \\ \frac{1}{\sqrt{2}} \langle\psi| \left(\cos(\theta) |\theta\rangle \otimes \sin(\phi) |\phi\rangle - \sin(\theta) |\theta\rangle \otimes \cos(\phi) |\phi\rangle\right) = \\ \frac{1}{2}(\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi))^2 = \\ \frac{1}{2}\sin(\theta - \phi)^2. \end{aligned} \tag{13}$$

Coincidences in Quantum Mechanics

It makes sense to choose as complements $P_A(\theta)' = P_A(\theta + \pi/2)$ and $P_B(\theta)' = P_B(\theta + \pi/2)$. It is clear then that

$$P(P_A(\theta)' \wedge P_B(\phi)') = \frac{1}{2}\sin(\theta - \phi)^2. \tag{14}$$

On the other hand, notice that

$$\begin{aligned}(P_A(\theta)' \wedge P_B(\phi)')(P_A(\theta) \wedge P_B(\phi)) &= \\ P_A(\theta + \pi/2)P_B(\phi + \pi/2)P_A(\theta)P_B(\phi) &= \\ (P(\theta + \pi/2) \otimes P(\phi + \pi/2))(P(\theta) \otimes P(\phi)) &= \\ P(\theta + \pi/2)P(\theta) \otimes P(\phi + \pi/2)P(\phi) &= \\ 0 = (P_A(\theta) \wedge P_B(\phi))(P_A(\theta)' \wedge P_B(\phi)'). &\end{aligned} \tag{15}$$

Thus, we obtain

$$\begin{aligned} f(P_A(\theta), P_B(\phi)) =& (P_A(\theta) \wedge P_B(\phi)) \vee (P_A(\theta)' \wedge P_B(\phi)') \\ =& P_A(\theta) \wedge P_B(\phi) + P_A(\theta)' \wedge P_B(\phi)'. \end{aligned} \tag{16}$$

Violations of Bell's Inequalities

Since expected values are linear,

$$P(f(P_A(\theta), P_B(\phi))) = \sin(\theta - \phi)^2. \tag{17}$$

Thus Bell's inequalities dictate

$$\begin{aligned} 1 =& \sin(0 - \pi/2)^2 = P(f(P_A(0), P_B(\pi/2))) \\ \leq& \sin(0 - \pi/6)^2 + \sin(\pi/3 - \pi/6)^2 + \sin(\pi/3 - \pi/2)^2 \\ =& 3/4. \end{aligned} \tag{18}$$

Conclusion

There is no correct physical theory whose propositions satisfy a Boolean algebra.

What failed?

Theorem

In a distributive bounded lattice elements have at most one complement.

Proof.

Suppose q and r are complements of p in a distributive bounded lattice. Then

$$q = q \wedge 1 = q \wedge (p \vee r) = (q \wedge p) \vee (q \wedge r) = 0 \vee (q \wedge r) = q \wedge r. \quad (19)$$

This $q \leq r$. Similarly one can show $r \leq q$. By anti-symmetry $q = r$. □

It is clear that in the bounded lattice $L(\mathcal{H})$ complements are not unique. Thus the lattice is not distributive.

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