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Supersymmetry

The Haag-Łopuszański-Sohnius Theorem

References:

Original papers: * Haag, R., Łopuszański, J.T., Sohnius, M. (1975). All Possible Generators of Supersymmetries of the S-matrix. Nuclear Physics B 88(2), 257-274

* Coleman, S., & Mandula, J. (1967). All Possible Symmetries of the S Matrix. Physical Review, 159 (5).

* Weinberg, S. (2000). The Quantum Theory of Fields: Supersymmetry (Vol. 3). Cambridge University Press.

In 1967, Coleman and Mandula came up with a theorem which, under quite general assumptions, constrained the possible symmetries of the S-matrix of a relativistic quantum theory. More specifically, they constrained its infinitesimal structured by describing the most general Lie algebra of infinitesimal symmetries of the S-matrix. Their investigation found that it had to be a direct sum of the Poincaré algebra with a finite dimensional algebra of internal symmetries. In particular, it prohibited the existence of symmetry generators with non-trivial commutation relations with the generators of Lorentz symmetries. This is a requirement in order to have symmetries mixing bosonic and fermionic degrees of freedom. However, as shown by all of the models we studied during the semester, this kind of symmetries can be achieved if they are parametrized by non-commuting parameters, in particular, Grassmann numbers. As shown by Exercise 4.3., this implies that the infinitesimal structure is not only characterized by commutation relations, but also by anticommutation relations. This leads to the concept of super Lie-algebras. Haag, Łopuszański and Sohnius characterized the most general super Lie algebra of symmetries of the S-matrix. As it turns out, this structure is quite rigid, leading to the SUSY algebra.

1. Coleman-Mandula Theorem

Let us start by stating the Coleman-Mandula Theorem, since it will be important in the proof of our main theorem. It is however, a theorem of relativistic quantum theory and we will not dwelve on its proof (for now at least). In order to state it, we need to first define what we mean by an infinitesimal symmetry of the S-matrix.

Definition: A symmetry of the S-matrix S is a selfadjoint operator A on our Hilbert space

$$\mathcal{H} \leq \bigoplus_{n=0}^{\infty} \bigotimes_{m=1}^{n} \mathcal{H} \quad \text{s.t.}$$

i) $\mathcal{H}$ is A-invariant (i.e. $A\psi \in \mathcal{H}$ for all $\psi \in \mathcal{H}$)

Should there be a spin-statistics factor?

ii) $$A(\psi_1 \otimes \cdots \otimes \psi_n) = \sum_{i=1}^{n} \psi_1 \otimes \cdots \otimes \psi_{i-1} \otimes A\psi_i \otimes \psi_{i+1} \otimes \cdots \otimes \psi_n$$

for all $\psi_1, \dots, \psi_n \in \mathcal{H}$;

iii) $[S, A] = 0$.

Remarks: i) is simply stating that a symmetry of the S-matrix transforms one-particle states into one-particle states. Along with ii), this means that such a symmetry does not change the number of particles of a state. Let A be a selfadjoint operator on $\mathcal{H}$, $\psi_1, \psi_2 \in \mathcal{H}$ and ε be some infinitesimal real parameter. Then

$$\begin{aligned}(e^{i\varepsilon A} \otimes e^{i\varepsilon A})(\psi_1 \otimes \psi_2) &= (\mathrm{id}_{\mathcal{H}\otimes\mathcal{H}} + i\varepsilon A \otimes \mathrm{id}_{\mathcal{H}} + i\varepsilon\, \mathrm{id}_{\mathcal{H}} \otimes A)(\psi_1 \otimes \psi_2) \\ &= \psi_1 \otimes \psi_2 + i\varepsilon (A\psi_1 \otimes \psi_2 + \psi_1 \otimes A\psi_2).\end{aligned}$$

We thus conclude that i) and ii) are just stating that symmetries of the S-matrix are second quantizations of one-particle operators.

Coleman - Mandula Theorem: Let g be the Lie algebra of symmetries of the S matrix in a relativistic quantum theory where:

i) for any mass M there are a finite number of particle types with mass less than M.

ii) all two particle states undergo some reaction at almost all energies, i.e., in terms of the connected S-matrix T, defined by

Is this almost in the sense of measures?

$$S = id_{\mathcal{F}} - i(2\pi)^4 \delta(P_\mu - P'_\mu)\, T,$$

$T|p,p'\rangle \neq 0$ for almost all energies.

iii) the amplitude of scattering of two bodies into two bodies are analytic functions of the scattering angle at almost all energies and angles.

Then $\mathfrak{g} = \mathfrak{o}(1,3) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of some compact Lie group.

Although we won't dwell on the proof, let us discuss a non-rigorous but insightful argument to why this theorem should be true.

"Kinematic" proof: Consider

$$\{P_\mu \mid \mu \in \{0,\dots,3\}\}' \equiv \{P_\mu\}' := \{A \in \mathfrak{g} \mid [A, P_\mu] = 0 \text{ for all } \mu \in \{0,\dots,3\}\}$$

This is clearly a Lie algebra due to the linearity of the Lie Bracket and the Jacobi identity. In particular, the Poincaré algebra implies $P_\nu \in \{P_\mu\}'$. Moreover, this is "stable" under Lorentz transformations, since for all $A \in \{P_\mu\}'$ and $\Lambda \in SO(1,3)^\uparrow$, we have

$$[U(\Lambda) A U(\Lambda)^{-1}, P^\mu] = \Lambda_\nu{}^\mu [U(\Lambda) A U(\Lambda)^{-1}, \Lambda^\nu{}_\sigma P^\sigma]$$

$$= \Lambda^\mu{}_\nu [U(\Lambda) A U(\Lambda)^{-1}, U(\Lambda) P^\nu U(\Lambda)^{-1}]$$

$$= \Lambda^\mu{}_\nu U(\Lambda) \underbrace{\cancel{[A, P^\nu]}}_{0} U(\Lambda)^{-1} = 0.$$

Thus $\{P_\mu\}'$ spans a representation D of $SO(1,3)^\uparrow$. $\{P_\mu\}$ is an invariant subspace

$$D(\Lambda)P_\mu := U(\Lambda)P_\mu U(\Lambda)^{-1} = \Lambda_\mu{}^\nu P_\nu .$$

Due to Weyl's theorem, there exists a subspace h of $\{P_\mu\}'$ s.t. D decomposes into

$$\{P_\mu\}' = \mathrm{span}\{P_\mu\} \oplus h.$$

By the very definition of $\{P_\mu\}'$, this decomposition is also valid at the level of Lie algebras. Of course, to apply Weyl's theorem, we've assumed $\{P_\mu\}'$ to be finite dimensional. We will now further assume h is compact. Due to the decomposition of our Lie algebra, its Killing form g decomposes into

$$g = 0 \oplus h.$$

Since h is compact, h is negative definite. Moreover, g is invariant under Lorentz transformations since D acts by automorphisms of $\{P_\mu\}'$, namely, for all $A, B \in \{P_\mu\}'$ we have

$$[D(\Lambda)A,\, D(\Lambda)B] = [U(\Lambda)AU(\Lambda)^{-1},\, U(\Lambda)BU(\Lambda)^{-1}]$$

$$= U(\Lambda)\,[A,B]\,U(\Lambda)^{-1}.$$

Thus, we may consider the representation

$$\tilde{D}(\Lambda) = (-h)^{+1/2}\, D(\Lambda)|_h\, (-h)^{-1/2}$$

on h. This representation is real and orthogonal

$$\tilde{D}(\Lambda)^T \tilde{D}(\Lambda) = (-h)^{-1/2}\, D(\Lambda)|_h^T\, (-h)^{1/2} (-h)^{1/2}\, D(\Lambda)|_h\, (-h)^{-1/2}$$

$$= (-h)^{-1/2}\, D(\Lambda)^T|_h\, (-h)\, D(\Lambda)|_h\, (-h)^{-1/2}$$

$$= (-h)^{-1/2} \left(\cancelto{-h}{-D(\Lambda)^T h\, D(\Lambda)}\right)\Big|_h (-h)^{-1/2}$$

$$= id_h.$$

Thus, since $SO(1,3)^\uparrow$ is not compact, $\tilde{D}$ has to be trivial. Then h is invariant under Lorentz transformations and it commutes with their generators.

Remarks: Both Weinberg and Mandula, J.E. (2015), Scholarpedia, 10(2):7476, following Weinbergs simplified treatment, avoid mentioning the decomposition into subrepresentations. I think this is a mistake induced by the use of indices. I may be wrong. Other than this, I think that a perfectly rigorous proof can be achieved by proving that h is indeed compact, and showing that the only possible generators in g that don't commute with translations are the generators of Lorentz transformations.

2. Introduction to Supermathematics.

The theory of supermathematics deals with super structures. These are important for stating our main theorem.

Definition: A super vector space is a $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_0 \oplus V_1$, i.e. a pair of vector spaces (V_0, V_1). The elements of $V_0 \cup V_1$ are called homogeneous, and we define the degree map

$$p : V_0 \cup V_1 \longrightarrow \mathbb{Z}_2$$

by $v \in V_{p(v)}$ for all $v \in V_0 \cup V_1$.

Definition: A super Lie algebra is a super vector space $\mathfrak{g}$ with a bilinear map

$$[\cdot , \cdot\} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

s.t.

i) $[X, Y\} \in \mathfrak{g}_{d(X)+d(Y)}$,

ii) $[X, Y\} = -(-1)^{d(X)d(Y)} [Y, X\}$, and

iii) (Jacobi identity)

$$0 = (-1)^{d(X)d(Z)} [X, [Y, Z\}\} + (-1)^{d(Y)d(X)} [Y, [Z, X\}\}$$
$$+ (-1)^{d(Z)d(Y)} [Z, [X, Y\}\},$$

for all $X, Y, Z \in \mathfrak{g}_0 \cup \mathfrak{g}_1$

3. Haag - Łopuszańanski - Sohnius Theorem

We are now ready for our main theorem,

Haag-Lopuszanski-Sohnius Theorem: Let $\mathfrak{g}$ be a super Lie algebra of symmetries of the S-matrix on a massive theory with the assumptions of the Coleman-Mandula theorem. Then $\mathfrak{g}_1$ is generated by a basis of the form $\{ Q^L_\alpha, \bar{Q}^L_{\dot\alpha} := (Q^L_\alpha)^\dagger \mid \alpha, \dot\alpha \in \{1,2\},$ $L \in \{1,\dots,N\}\}$, where, for each $L \in \{1,\dots,K\}$, $\{Q^L_1, Q^L_2\}$ span the $(1/2, 0)$ representation of $SL(2,\mathbb{C})$. In fact,

$$[Q^L_\alpha, \bar{Q}^M_{\dot\beta}\} = 2\delta^{LM} \sigma^m{}_{\alpha\dot\beta} P_m,$$

$$[Q^L_\alpha, Q^M_\beta\} = \varepsilon_{\alpha\beta} Z^{LM},$$

$$[Q^L_\alpha, P_m\} = 0,$$

$$[Q^L_\alpha, M_{mn}\} = \frac{1}{2} (\sigma_{mn})_\alpha{}^\beta Q^L_\beta,$$

where the Z^{LM}'s are antisymmetric and central elements of $\mathfrak{g}$, i.e. $[Z^{LM}, X\} = 0$ for all $X \in \mathfrak{g}$.

Remarks: $\mathfrak{g}$ has, by definition, a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

as a super vector space. This is however, not a decomposition as a super Lie algebra, permiting that the fermionic generators be in a non-trivial representation of $SL(2,\mathbb{C})$. By fermionic generators we mean the elements of g_1. On the other hand, elements of g_0 are called bosonic. The latter is the Lie algebra of symmetries of the S-matrix. It is thus already characterized by the Coleman-Mandula theorem.

"Proof": Consider $Q \in g_1$. Then, for all $A \in SL(2,\mathbb{C})$, we have $U(A)QU(A)^{-1}$ is also a symmetry of the S-matrix

$$[U(A)QU(A)^{-1}, S] = [U(A)QU(A)^{-1}, U(A)SU(A)^{-1}]$$
$$= U(A)\,\cancelto{0}{[Q,S]}\,U(A)^{-1} = 0$$

(we should've done this calculation for proving that $\{P_\mu\}'$ was stable under Lorentz transformations in the proof of the Coleman-Mandula theorem). We will further assume that for all $Q \in g_1$ and $A \in SL(2,\mathbb{C})$

$$U(A) Q U(A)^{-1} \in g_1 ,$$

(see remarks at the end) as well as that g_1 is finite dimensional (this requires a lot of functional analysis coming from unspoken assumptions in the Coleman – Mandula Theorem, see Haag). Then, g_1 spans a finite-dimensional representation of $SL(2,\mathbb{C})$. Thus,

A generator in an irreducible component (j,j') of this representation has the index structure $Q_{\alpha_1 \cdots \alpha_{2j} \dot{\alpha}_1 \cdots \dot{\alpha}_{2j'}}$. Furthermore, by the spin-statistics theorem (this looks to me that we are using more hypothesis than I claimed, namely, from local field theory), $2(j+j') \in 2\mathbb{N}^+ + 1$. Now consider

$$R_{\alpha_1 \cdots \alpha_{2j} \beta_1 \cdots \beta_{2j'} \dot{\alpha}_1 \cdots \dot{\alpha}_{2j'} \dot{\beta}_1 \cdots \dot{\beta}_{2j}}$$

$$:= [Q_{\alpha_1 \cdots \alpha_{2j} \dot{\alpha}_1 \cdots \dot{\alpha}_{2j'}} , \bar{Q}_{\beta_1 \cdots \beta_{2j'} \dot{\beta}_1 \cdots \dot{\beta}_{2j}} \} .$$

By the Jacobi identity, this is also a bosonic symmetry of the S-matrix. It further belongs to a representation

$$(j,j')\otimes(j',j)\oplus(j',j)\otimes(j,j')\ni(j+j',j+j').$$

In particular, by making use of raising and lowering operators, one may construct an element of $(j+j',j+j')$ out of $R_{1\cdots 1\dot{1}\cdots\dot{1}}$. It will also be a symmetry of the S-matrix, since the bosonic symmetries are themselves a representation. But, by the Coleman-Mandula Theorem, the bosonic symmetries are $P_m\in(\frac{1}{2},\frac{1}{2})$, $M_{mn}\in(1,0)\oplus(0,1)$ and the internal symmetries in $(0,0)$. Thus $j+j'\leq\frac{1}{2}$. Since $2(j+j')\in 2\mathbb{N}+1$, we conclude $j+j'=\frac{1}{2}$ to obtain a non-vanishing commutator. But, since $\{Q,Q^+\}=0$ implies $Q=0$, we have that $Q\in(\frac{1}{2},0)\cup(0,\frac{1}{2})$. We take $Q^L_\alpha\in(\frac{1}{2},0)$.

Notice that

$$[Q^L_\alpha, \bar{Q}^M_{\dot\alpha}\} \in [(1/2,1/2)\oplus(1/2,1/2)]_S = (1/2,1/2),$$

and thus, by the Coleman - Mandula theorem,

$$[Q^L_\alpha, Q^M_{\dot\alpha}\} = 2N^{LM}\sigma^m{}_{\alpha\dot\alpha}P_m.$$

We will now show N^{LM} is positive-definite.

Taking $\alpha=\dot\alpha$ we have

$$\sigma^m{}_{00}P_m = P_0 + P_3$$

$$\sigma^m{}_{11}P_m = P_0 - P_3.$$

Since our theory is massless, $P^\mu P_\mu > 0$. Thus

$$P_0^2 > P_1^2 + P_2^2 + P_3^2 \geq P_3^2.$$

Moreover, in relativistic field theories $P_0 > 0$.

Thus

$$P_0 > |P_3|.$$

We conclude that $\sigma^m{}_{\alpha\dot\alpha}P_m$ is positive when $\alpha=\dot\alpha$. Taking $c_1,\dots,c_N\in\mathbb{C}$, we have with $\alpha=\dot\alpha$

$$c_L^* [Q^L_\alpha, \bar{Q}^M_{\dot\alpha}\} c_M^* = 2c_L N^{LM} c_M^* \sigma^m{}_{\alpha\dot\alpha}P_M$$

$$\shortparallel$$

$$[Q_\alpha, \bar{Q}_{\dot\alpha}\}_0 \geq 0,$$

where $Q_\alpha := c_L Q^L_\alpha$. Since the Q^L_α are linearly independent, it is enough to take $\vec{c} \neq 0$ to ensure that $Q_\alpha \neq 0$. There is thus some ψ in its domain s.t. $Q_\alpha \psi \neq 0$. Then

$$\langle \psi, \{Q_\alpha, \bar{Q}_{\dot\alpha}\} \psi \rangle = \|\bar{Q}_{\dot\alpha}\psi\|^2 + \|Q_\alpha \psi\|^2$$
$$\geq \|Q_\alpha \psi\|^2 > 0.$$

Then

$$2 c_L N^{LM} c^*_M \langle \psi, \sigma^m{}_{\alpha\dot\alpha} P_m \psi \rangle > 0.$$

Since $\langle \psi, \sigma^m{}_{\alpha\dot\alpha} P_m \psi \rangle > 0$, we conclude that

$$c_L N^{LM} c^*_M > 0$$

for all $\vec{c} \neq 0$. Thus N is positive definite. By replacing $Q^L_\alpha \longrightarrow (N^{-1/2})^{LM} Q^M_\alpha$, we obtain

$$\{Q^L_\alpha, \bar{Q}^M_{\dot\alpha}\} = 2\delta^{LM} \sigma^m{}_{\alpha\dot\alpha} P_m.$$

Let us now study the commutation relations of Q^L_α and P_m. We have

$$\{Q^L_\alpha, P_m] \in (1/2, 0) \otimes (1/2, 1/2) = (1, 1/2) \oplus (0, 1/2)$$

is a fermionic symmetry generator. Since there are no such generators in $(1, 1/2)$, we have

$$[\sigma^m{}_{\beta\dot{\gamma}} P_m, Q^L_\alpha] = \varepsilon_{\beta\alpha} K^L{}_M \bar{Q}^M_{\dot{\gamma}}.$$

By taking the adjoint, we have

$$[\bar{Q}^L_{\dot{\alpha}}, \sigma^m{}_{\beta\dot{\beta}} P_m] = \varepsilon_{\dot{\beta}\dot{\alpha}} (K^+)_M{}^L Q^M_\beta.$$

To determine the K-matrix there is a rather nifty trick. We begin by doing the seemingly arbitrary calculation

$$[\sigma^m{}_{11} P_m, \{Q^L_2, Q^E_1\}]$$

$$= -(\{Q^L_2, \cancel{[Q^E_1, \sigma^m{}_{11} P_m]}\} - \{Q^E_1, [\sigma^m{}_{11} P_m, Q^L_2]\})$$

$$= \varepsilon_{12} K^L{}_F \{Q^E_1, \bar{Q}^F_{\dot{1}}\} = -2K^L{}_F \delta^{EF} \sigma^m{}_{1\dot{1}} P_m$$

$$= -2K^L{}_E \sigma^m{}_{11} P_m,$$

which is doable thanks to the Jacobi identity. Similarly, we have

$$[\sigma^m{}_{11}P_m, \{\bar{Q}^M_2, \bar{Q}^E_1\}] =$$

$$= -\left(\{\bar{Q}^M_2, \cancel{[\bar{Q}^E_1, \sigma^m{}_{11}P_m]}^{\,0}\} - \{\bar{Q}^E_1, [\sigma^m{}_{11}P_m, \bar{Q}^M_2]\}\right)$$

$$= -\varepsilon_{12}(K^+)_F{}^M \{\bar{Q}^E_1, Q^F_1\},$$

$$= 2(K^+)_F{}^M \delta^{EF} \sigma^m{}_{11}P_m = 2(K^+)_E{}^M \sigma^m{}_{00}P_m$$

We then have

$$0 = [\sigma^m{}_{11}P_m, [\sigma^m{}_{11}P_m, \{Q^L_2, \bar{Q}^M_2\}]]$$

$$= [\sigma^m{}_{11}P_m, -(\{Q^L_2, [\bar{Q}^M_2, \sigma^m{}_{11}P_m]\}$$

$$- \{\bar{Q}^M_2, [\sigma^m{}_{00}P_m, Q^L_2]\}]$$

$$= [\sigma^m{}_{11}P_m, (K^+)_E{}^M \{Q^L_2, Q^E_1\} - K^L{}_E \{\bar{Q}^M_2, \bar{Q}^E_1\}]$$

$$= -4(KK^+)^{LM} \sigma^m{}_{00}P_m.$$

By tracing over L and M,

$$0 = -4\,\mathrm{tr}(KK^+)\sigma^m{}_{00}P_m$$

$$= -4\,\|KK^+\|_1 \sigma^m{}_{00}P_m.$$

Thus $KK^+ = 0$ and we conclude that $K=0$ due to its polar decomposition. We conclude that

$$[Q^L_\alpha, P_m] = 0.$$

We are left with only one more commutation relation. We have the Bosonic generators

$$\{Q^L_\alpha, Q^M_\beta\} \in (1/2, 0) \otimes (1/2, 0) = (1,0) \oplus (0,0).$$

By the Coleman - Mandula theorem the only bosonic generators in $(1,0)$ are linear combinations of M_{mn}. However, $\{Q^L_\alpha, Q^M_\beta\}$ commutes with the momenta. Therefore

$$\{Q^L_\alpha, Q^M_\beta\} = \varepsilon_{\alpha\beta} Z^{LM}$$

with Z^{LM} scalar. We immediately see that

$$[Z^{LM}, P_m] = 0.$$

Moreover, the Jacobi identity dictates

$$0 = [\{Q^L_\alpha, Q^M_\beta\}, \bar{Q}^K_{\dot\alpha}] + [\underbrace{\cancelto{0}{\{Q^M_\beta, \bar{Q}^K_{\dot\alpha}\}}}_{\sim P_m}, Q^L_\alpha] + [\underbrace{\cancelto{0}{\{\bar{Q}^K_{\dot\alpha}, Q^M_\beta\}}}_{\sim P_m}, Q^L_\alpha]$$

$$= \varepsilon_{\alpha\beta} [Z^{LM}, \bar{Q}^K_{\dot\alpha}].$$

Similarly

$$0 = + [\cancelto{0}{Z^{LM}, \overbrace{\{Q^K_\alpha , \bar{Q}^R_{\dot\beta}\}}^{P_m}}] + \{Q^K_\alpha , [\cancelto{0}{\bar{Q}^R_{\dot\beta}, Z^{LM}}]\}$$

$$= - \{\bar{Q}^R_{\dot\beta}, \underbrace{[Z^{LM}, Q^K_\alpha]}_{\in (0,0) \otimes (1/2,0) = (1/2,0)}\}$$

$$= N^{LMK}{}_E \{\bar{Q}^R_{\dot\beta}, Q^E_\alpha\}$$

$$= 2N^{LMK}{}_E \, \delta^{RE} \sigma^m{}_{\alpha\dot\beta} P_m = 2N^{LMK}{}_R \, \sigma^m{}_{\alpha\dot\beta} P_m .$$

We conclude $N^{LMKR} = 0$ and thus

$$[Z^{LM}, Q^K_\alpha] = 0.$$

In a similar fashion we can proof

$$[Z^{LM}, Z^{EF}] = [Z^{LM}, (Z^{EF})^\dagger] = 0.$$

Thus the Z^{LM} are indeed central charges.

We end by noting that the Z^{LM} are antisymmetric in the indices L and M since

$$\{Q^L_\alpha , Q^M_\beta\} = \varepsilon_{\alpha\beta} Z^{LM}$$

is symmetric under the simultaneous exchange of α, L with β, M while ε is antisymmetric

$$\varepsilon_{\alpha\beta} Z^{LM} = \{Q^L_\alpha, Q^M_\beta\} = \{Q^M_\beta, Q^L_\alpha\} = \varepsilon_{\beta\alpha} Z^{ML} = -\varepsilon_{\alpha\beta} Z^{ML}.$$

▨

For our final remarks, let us first define a last piece of supermathematics.

Definition: A superalgebra is a super vector space $\mathcal{A}$ which is an associative algebra whose multiplication satisfies

$$d(ab) = d(a) + d(b)$$

for all $a, b \in \mathcal{A}_0 \cup \mathcal{A}_1$.

Remark: Every superalgebra is a super Lie algebra when equipped with

$$[a, b\} = ab - (-1)^{d(a)d(b)}\, ba.$$

In the previous theorem, we left an open end, namely, that $U(\Lambda) Q U(\Lambda)^{-1}$ was bosonic. This can be understood if we assume $\mathfrak{g}$ comes from a superalgebra. Then, since the Lorentz generators are Bosonic, it is reasonable to assume Lorentz transformations are as well.

A rigorous version of this should require some really nice analysis- to close the homogeneous components of a superalgebra under the functional calculus.