I am having a little trouble proving wick's theorem. I'll start from the last step that I know is correct. We define

$$\left\langle \prod_{j=1}^{2m} x_{i_j} \right\rangle := \frac{\partial^{2m}}{\prod_{j=1}^{2m} \partial J_{i_j}} \frac{1}{2^m} \frac{1}{m!} \left(\vec{J} \cdot A^{-1} \vec{J} \right)^m \Bigg|_{\vec{J}=0},$$

where A is a symmetric $N \times N$ matrix and $m \in \mathbb{N}$. Notice that

$$\begin{aligned}\left(\vec{J} \cdot A^{-1} \vec{J} \right)^m &= \left(\sum_{k,l=1}^{N} J_k A^{-1}_{kl} J_l \right)^m = \sum_{k_1,\ldots,k_m,l_1,\ldots,l_m=1}^{N} \prod_{j=1}^{m} J_{k_j} A^{-1}_{k_j l_j} J_{l_j} \\ &= \sum_{k_1,\ldots,k_m,l_1,\ldots,l_m=1}^{N} \prod_{j=1}^{m} A^{-1}_{k_j l_j} \prod_{j=1}^{m} J_{k_j} J_{l_j}.\end{aligned}$$

Then

$$\left\langle \prod_{j=1}^{2m} x_{i_j} \right\rangle = \frac{1}{2^m} \frac{1}{m!} \sum_{k_1,\ldots,k_m,l_1,\ldots,l_m=1}^{N} \prod_{j=1}^{m} A^{-1}_{k_j l_j} \frac{\partial^{2m}}{\prod_{j=1}^{2m} \partial J_{i_j}} \prod_{j=1}^{m} J_{k_j} J_{l_j} \Bigg|_{\vec{J}=0}.$$

The differential operator is going to anniquilate the polynomial in $\vec{J}$ unless it contains exactly the components with respect to which the operator is differentiating the exact same amount of times. Any more are going to be killed when we put $\vec{J} = 0$ while any less are going to get killed by the derivatives. We can thus restrict the sum to $(k_1, \ldots k_m, l_1, \ldots l_m) = (i_{\sigma(1)}, \ldots, i_{\sigma(2m)})$ for some permutation $\sigma \in S_{2m}$. We thus have

$$\begin{aligned}\left\langle \prod_{j=1}^{2m} x_{i_j} \right\rangle &= \frac{1}{2^m} \frac{1}{m!} \sum_{\sigma \in S_{2m}} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}} \frac{\partial^{2m}}{\prod_{j=1}^{2m} \partial J_{i_j}} \prod_{j=1}^{m} J_{i_{\sigma(j)}} J_{i_{\sigma(j+m)}} \Bigg|_{\vec{J}=0} \\ &= \frac{1}{2^m} \frac{1}{m!} \sum_{\sigma \in S_{2m}} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}} \frac{\partial^{2m}}{\prod_{j=1}^{2m} \partial J_{i_j}} \prod_{j=1}^{2m} J_{i_{\sigma(j)}} \Bigg|_{\vec{J}=0} \\ &= \frac{1}{2^m} \frac{1}{m!} \sum_{\sigma \in S_{2m}} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}} \frac{\partial^{2m}}{\prod_{j=1}^{2m} \partial J_{i_j}} \prod_{j=1}^{2m} J_{i_j} \Bigg|_{\vec{J}=0}.\end{aligned}$$

Now, for every $r \in \{1, \ldots, N\}$ define $I_r = \{j \in \{1, \ldots, 2m\} | r = i_j\}$. Then $\{I_r | r \in \{1, \ldots, N\}\}$ is a partition of $\{1, \ldots, 2m\}$ and

$$\begin{aligned}\frac{\partial^{2m}}{\prod_{j=1}^{2m} \partial J_{i_j}} \prod_{j=1}^{2m} J_{i_j} &= \prod_{j=1}^{2m} \frac{\partial}{\partial J_{i_j}} \prod_{j=1}^{2m} J_{i_j} = \prod_{r=1}^{N} \prod_{j \in I_r} \frac{\partial}{\partial J_r} \prod_{r=1}^{N} \prod_{j \in I_r} J_r = \prod_{r=1}^{N} \frac{\partial^{|I_r|}}{\partial J_r^{|I_r|}} \prod_{r=1}^{N} J_r^{|I_r|} \\ &= \prod_{r=1}^{N} \frac{\partial^{|I_r|}}{\partial J_r^{|I_r|}} J_r^{|I_r|} = \prod_{r=1}^{N} |I_r|!.\end{aligned}$$

We then have

$$\left\langle \prod_{j=1}^{2m} x_{i_j} \right\rangle = \prod_{r=1}^{N} |I_r|! \frac{1}{2^m} \frac{1}{m!} \sum_{\sigma \in S_{2m}} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}}.$$

If this is correct so far, the rest should be combinatorics. Let $\sigma_1, \sigma_2 \in S_{2m}$. We say $\sigma_1 \sim \sigma_2$ if there exists a permutation $\mu \in S_m$ such that $\sigma_2(j) = \sigma_1(\mu(j))$ and $\sigma_2(j+m) = \sigma_1(\mu(j)+m)$ for all $j \in \{1, \ldots, m\}$. This is an equivalence relation, for every $[\sigma] \in S_{2m}/\sim$ we have $|[\sigma]| = |s_m| = m!$, and for every $\tilde{\sigma} \in [\sigma]$ we have

$$\prod_{j=1}^{m} A^{-1}_{i_{\tilde{\sigma}(j)} i_{\tilde{\sigma}(j+m)}} = \prod_{j=1}^{m} A^{-1}_{i_{\sigma(\mu(j))} i_{\sigma(\mu(j)+m)}} = \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}}$$

for some $\mu \in S_m$. Then

$$\begin{aligned}
\left\langle \prod_{j=1}^{2m} x_{i_j} \right\rangle &= \prod_{r=1}^{N} |I_r|! \frac{1}{2^m} \frac{1}{m!} \sum_{[\sigma] \in S_{2m}/\sim} \sum_{\tilde{\sigma} \in [\sigma]} \prod_{j=1}^{m} A^{-1}_{i_{\tilde{\sigma}(j)} i_{\tilde{\sigma}(j+m)}} \\
&= \prod_{r=1}^{N} |I_r|! \frac{1}{2^m} \frac{1}{m!} \sum_{[\sigma] \in S_{2m}/\sim} \sum_{\tilde{\sigma} \in [\sigma]} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}} \\
&= \prod_{r=1}^{N} |I_r|! \frac{1}{2^m} \sum_{[\sigma] \in S_{2m}/\sim} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}}
\end{aligned}$$

Now let $[\sigma_1], [\sigma_2] \in S_{2m}/\sim$. We say $[\sigma_1] \sim' [\sigma_2]$ if there exists a set $I \subset \{1, \ldots, m\}$ such that $\sigma_1(j) = \sigma_2(j)$ and $\sigma_1(j+m) = \sigma_2(j+m)$ for all $j \in I$, while $\sigma_1(j) = \sigma_2(j+m)$ and $\sigma_1(j+m) = \sigma_2(j)$ for all $j \in A^c$. It is easily checked that this is a well defined equivalence relation and for every $[[\sigma]] \in S_{2m}/\sim/\sim'$ we have both that $|[[\sigma]]| = |2^{\{1,\ldots,m\}}| = 2^m$ and for every $\tilde{\sigma} \in S_{2m}/\sim/\sim'$

$$A^{-1}_{i_{\tilde{\sigma}(j)} i_{\tilde{\sigma}(j+m)}} = A^{-1}_{i_{\tilde{\sigma}(j+m)} i_{\tilde{\sigma}(j)}} = A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}}.$$

We then have

$$\begin{aligned}
\left\langle \prod_{j=1}^{2m} x_{i_j} \right\rangle &= \prod_{r=1}^{N} |I_r|! \frac{1}{2^m} \sum_{[[\sigma]] \in S_{2m}/\sim/\sim'} \sum_{\tilde{\sigma} \in [[\sigma]]} \prod_{j=1}^{m} A^{-1}_{i_{\tilde{\sigma}(j)} i_{\tilde{\sigma}(j+m)}} \\
&= \prod_{r=1}^{N} |I_r|! \frac{1}{2^m} \sum_{[[\sigma]] \in S_{2m}/\sim/\sim'} \sum_{\tilde{\sigma} \in [[\sigma]]} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}} \\
&= \prod_{r=1}^{N} |I_r|! \sum_{[[\sigma]] \in S_{2m}/\sim/\sim'} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}}
\end{aligned}$$

The final result should be a summation over all pairings of the indices $(i_1, \ldots, i_{2m})$ of the product of the elements of A^{-1} corresponding to every pair in the pairing. However, the way that I understand it this is already $\sum_{[[\sigma]] \in S_{2m}/\sim/\sim'} \prod_{j=1}^{m} A^{-1}_{i_{\sigma(j)} i_{\sigma(j+m)}}$. Unless there is any more symmetry which I haven't seen and cancels the factor $\prod_{r=1}^{N} |I_r|!$, I must have done a mistake above.