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Teoria Quântica de Campos I

3. Quantização Canónica

3.1. Campo Escalar Complexo

3.1.1. Soluções da Ecuação de Klein-Gordon.

Seja φ um campo escalar complexo que satifaze a ecuação de Klein - Gordon

$$(\partial^2 + m^2)\varphi = 0.$$

Expresando em termos de sua transformada de Fourier, temos

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p \, e^{-ix\cdot p} \tilde{\varphi}(p).$$

Logo

$$0 = \frac{1}{(2\pi)^{3/2}} \int d^4p \, (-p^2 + m^2) e^{-ix\cdot p} \tilde{\varphi}(p).$$

Concluimos que só os modos que satisfazem a relação de disperção $p^2 = m^2$ contribuyen em φ. Logo existe χ t.q. $\tilde{\varphi}(p) = \delta(p^2 - m^2)\chi(p)$. Lembrando que $\theta(x) + \theta(-x) = 1$ casi em todas partes, e

$$\delta(p^2 - m^2) = \delta((p^0)^2 - E_{\vec{p}}^2) = \frac{1}{2E_{\vec{p}}}\left(\delta(p^0 - E_{\vec{p}}) + \delta(p^0 - E_{\vec{p}})\right),$$

onde $E_{\vec{p}} := m^2 + \vec{p}^2$, temos

$$\tilde{\varphi}(p) = \frac{1}{2E_{\vec{p}}}\left(\delta(p^0 - E_{\vec{p}}) + \delta(p^0 + E_{\vec{p}})\right)\left(\theta(p^0) + \theta(-p^0)\right)\chi(p)$$

$$= \frac{1}{2E_{\vec{p}}}\left(\delta(p^0 - E_{\vec{p}})\theta(p^0)\chi(p) + \delta(p^0 + E_{\vec{p}})\theta(-p^0)\chi(p)\right).$$

Logo, como a reverção temporal e a paridad são transformações de Lorentz,

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4p \frac{1}{2E_{\vec{p}}}\Big(\delta(p^0 - E_{\vec{p}})\theta(p^0)e^{-ix\cdot p}\chi(p) + \delta(p^0 + E_{\vec{p}})\theta(-p^0)e^{-ix\cdot p}\chi(p)\Big)$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^4p \frac{1}{2E_{\vec{p}}}\delta(p^0 - E_{\vec{p}})\theta(p^0)\left(e^{-ix\cdot p}\chi(p) + e^{ix\cdot p}\chi(-p)\right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^{3/2}2E_{\vec{p}}}\left(e^{-ix\cdot p}a(\vec{p}) + e^{ix\cdot p}b^*(\vec{p})\right)\Big|_{p^0 = E_{\vec{p}}},$$

onde $a(\vec{p}) := \chi(E_{\vec{p}}, \vec{p})$, $b^*(\vec{p}) := \chi(-E_{\vec{p}}, \vec{p})$. Aum mais

$$\varphi^*(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}2E_{\vec{p}}}\left(e^{-ix\cdot p}b(\vec{p}) + e^{ix\cdot p}a^*(\vec{p})\right)\Big|_{p^0 = E_{\vec{p}}}$$

Em termos de estos podemos expresar a e b. Para isso defina

$$f_{\vec{p}}(x) = \frac{e^{-ipx}}{(2\pi)^{3/2}}\Big|_{p^0 = E_{\vec{p}}}.$$

Eles satisfazem

$$i\int d^3\vec{x}\, f^*_{\vec{p}'}(t,\vec{x})\overleftrightarrow{\partial_0} f_{\vec{p}}(t,\vec{x}) = i\int \frac{d^3\vec{x}}{(2\pi)^3}\left(-iE_{\vec{p}}\, e^{i(p'-p)\cdot(t,\vec{x})} - iE_{\vec{p}'}\, e^{i(p'-p)\cdot(t,\vec{x})}\right)_{\substack{p^0=E_{\vec{p}}\\ p'^0=E_{\vec{p}'}}}$$

$$= \left(E_{\vec{p}} + E_{\vec{p}'}\right) e^{i(E_{\vec{p}'}-E_{\vec{p}})t}\, \delta^3(\vec{p}-\vec{p}')$$

$$= 2E_{\vec{p}}\,\delta(\vec{p}-\vec{p}'),$$

$$i\int d^3\vec{x}\, f_{\vec{p}'}(t,\vec{x})\overleftrightarrow{\partial_0} f_{\vec{p}}(t,\vec{x}) = i\int \frac{d^3\vec{x}}{(2\pi)^3}\left(-iE_{\vec{p}}\, e^{-i(p'+p)\cdot(t,\vec{x})} + iE_{\vec{p}'}\, e^{-i(p'+p)\cdot(t,\vec{x})}\right)_{\substack{p^0=E_{\vec{p}}\\ p'^0=E_{\vec{p}'}}}$$

$$= \left(E_{\vec{p}} - E_{\vec{p}'}\right) e^{-i(E_{\vec{p}}+E_{\vec{p}'})t}\, \delta^3(\vec{p}+\vec{p}')$$

$$= 0.$$

Então

$$i\int d^3\vec{x}\, f^*_{\vec{p}}(t,\vec{x})\overleftrightarrow{\partial_0}\varphi(t,\vec{x}) = \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}}\, i\int d^3\vec{x}\, f^*_{\vec{p}}(t,\vec{x})\overleftrightarrow{\partial_0} f_{\vec{p}'}(t,\vec{x})\, a(\vec{p}') = a(\vec{p}).$$

Similarmente

$$b(\vec{p}) = i\int d^3\vec{x}\, f^*_{\vec{p}}(t,\vec{x})\overleftrightarrow{\partial_0}\varphi^*(t,\vec{x}),$$

$$a^*(\vec{p}) = i\int d^3\vec{x}\, \varphi^*(t,\vec{x})\overleftrightarrow{\partial_0} f_{\vec{p}}(t,\vec{x}),$$

$$b^*(\vec{p}) = i\int d^3\vec{x}\, \varphi(t,\vec{x})\overleftrightarrow{\partial_0} f_{\vec{p}}(t,\vec{x}).$$

3.1.2. Relações de commutação

Agora, o proceso de quantização consiste em considerar a $\varphi(x)$ com um operador num espaço de Hilbert $\mathcal{H}$. Como veremos depois, isso tem problemas e, na verdade, é uma distribução com valor de operador em $\mathcal{H}$. Isso é, que para toda função f do espaçotempo e

$$\varphi(f) = \text{“} \int d^4x\, \varphi(x) f(x) \text{”}$$

o operador (geralmente não acotado) sobre $\mathcal{H}$. Obviaremos isso por agora. Simultaneamente remplazamos φ^* por um operador φ^+. Classicamente φ tem o campo

$$\pi := \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_0 \varphi} = \partial^0 \varphi^*$$

como campo conjuado. Así mesmo φ^* tem

$$\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_0 \varphi^*} = \partial^0 \varphi = \pi^*$$

como campo conjugado. Em particular, se tem os brackets de Poisson

$$\{\phi(t,\vec{x}), \pi(t,\vec{x}')\} = \{\phi^*(t,\vec{x}), \pi^*(t,\vec{x}')\} = \delta(\vec{x}-\vec{x}'),$$

mientras que todos os demas entre ϕ, π, ϕ^*, π^* se anulam.

O proceso de quantização consiste em considerar os operadores $\pi := \partial^0 \varphi^+$ e $\pi^+ = \partial^0 \varphi$, e substituir os parentesis de Poisson por $\frac{1}{i\hbar}$ pelos commutadores. Logo declaramos

$$[\varphi(t,\vec{x}), \pi(t,\vec{x}')] = [\varphi^+(t,\vec{x}), \pi^+(t,\vec{x}')] = i\,\delta(\vec{x}-\vec{x}')$$

mientras que os demais se anulam.
Para fazer sentido da descomposição de Fourier, tamben

fazemos que a, a^*, b, b^* sejam operadores a, a^+, b, b^+.

Com suas representações em termos dos campos, obtemos

as relações de commutação

$$[a(\vec{p}), a^+(\vec{p}\,')] = \Big[i\int d^3\vec{x}\big(f^*_{\vec{p}}(t,\vec{x})\,\pi^+(t,\vec{x}) - iE_{\vec{p}}\, f^*_{\vec{p}}(t,\vec{x})\,\varphi(t,\vec{x})\big),$$

$$i\int d^3\vec{x}\,'\big(-iE_{\vec{p}\,'}\,\varphi^+(t,\vec{x}\,')\,f_{\vec{p}\,'}(t,\vec{x}\,') - \pi(t,\vec{x}\,')\,f_{\vec{p}\,'}(t,\vec{x})\big)\Big]$$

$$= -\int d^3\vec{x}\,d^3\vec{x}\,'\Big(-iE_{\vec{p}\,'}\,f^*_{\vec{p}}(t,\vec{x})\,f_{\vec{p}\,'}(t,\vec{x}\,')\,[\pi^+(t,\vec{x}),\varphi^+(t,\vec{x}\,')]$$

$$+iE_{\vec{p}}\,f^*_{\vec{p}}(t,\vec{x})\,f_{\vec{p}\,'}(t,\vec{x}\,')\,[\varphi(t,\vec{x}),\pi(t,\vec{x}\,')]\Big)$$

$$= -\int d^3\vec{x}\,d^3\vec{x}\,'\Big(-E_{\vec{p}\,'}\,f^*_{\vec{p}}(t,\vec{x})\,f_{\vec{p}\,'}(t,\vec{x}\,')\,\delta(\vec{x}-\vec{x}\,')$$

$$-E_{\vec{p}}\,f^*_{\vec{p}}(t,\vec{x})\,f_{\vec{p}\,'}(t,\vec{x}\,')\,\delta(\vec{x}-\vec{x}\,')\Big)$$

$$= \int \frac{d^3\vec{x}}{(2\pi)^3}\,(E_{\vec{p}} + E_{\vec{p}\,'})\,e^{-i(p-p')(t,\vec{x})}\Big|_{p^0=E_{\vec{p}},\,p'^0=E_{\vec{p}\,'}}$$

$$= (E_{\vec{p}} + E_{\vec{p}\,'})\,e^{-i(E_{\vec{p}}-E_{\vec{p}\,'})t}\,\delta(\vec{p}-\vec{p}\,') = 2E_{\vec{p}}\,\delta(\vec{p}-\vec{p}\,').$$

De um jeito análogo, temos

$$[b(\vec{p}), b^+(\vec{p}\,')] = 2E_{\vec{p}}\,\delta(\vec{p}-\vec{p}\,')$$

e os demais nulos.

3.1.3. Representação do grupo de translações

Como vimos, a energia do campo é

$$H(t) = \int d^3\vec{x} \left(\pi(t,\vec{x}), \pi^+(t,\vec{x}) + \vec{\nabla}\varphi(t,\vec{x}) \cdot \vec{\nabla}\varphi^+(t,\vec{x}) + m^2 \varphi(t,\vec{x})\varphi^+(t,\vec{x}) \right)$$

Observe que não importa o ordenamento dos productos por as relações de commutação. Em particular se tem

$$[\partial_i \varphi(t,\vec{x}), \varphi(t,\vec{x}')] = \frac{\partial}{\partial x^i} [\varphi(t,\vec{x}), \varphi(t,\vec{x}')] = \frac{\partial}{\partial x^i} 0 = 0.$$

Do mesmo jeito

$$[\partial_i \varphi(t,\vec{x}), \partial_j \varphi^+(t,\vec{x}')] = 0.$$

O mesmo não vai a suceder sim H se escreve en termos de a e b. Sem embargo, podemos calcular

$$[H(t), \varphi(t,\vec{x})] = \int d^3\vec{x}' [\pi(t,\vec{x}'), \varphi(t,\vec{x})] \pi^+(t,\vec{x}')$$

$$= -i\pi^+(t,\vec{x}) = -i\partial^0 \varphi(t,\vec{x}).$$

Semelhantemente, usamdo a ecuação de Klein-Gordon

$$[H(t), \varphi^+(t,\vec{x})] = -i\partial^0 \varphi^+(t,\vec{x})$$

$$[H(t), \pi(t,\vec{x})] = \int d^3\vec{x}' \Big(\vec{\nabla}_{\vec{x}'} [\varphi(t,\vec{x}'), \pi(t,\vec{x})] \cdot \vec{\nabla}\varphi^+(t,\vec{x}') + m^2 \varphi^+(t,\vec{x}') [\varphi(t,\vec{x}'), \pi($$

$$= i\int d^3\vec{x}' \left(-\Delta\varphi^+(t,\vec{x}') + m^2 \varphi^+(t,\vec{x}') \right) \delta(\vec{x} - \vec{x}')$$

$$= -i\partial_0^2 \varphi^+(t,\vec{x}) = -i\partial^0 \pi(t,\vec{x})$$

$$[H(t), \pi^+(t,\vec{x})] = -i\partial^0 \pi^+(t,\vec{x}).$$

Issas são as ecuações de Heisenberg para nossos campos. Logo, si são um conjunto completo de operadores, vemos que

H é o gerador de translações temporais em $\mathcal{H}$. Em particular é constante no tempo. Isso vai a ser claro cuando o escrivamos em termos de a e b. Do mismo jeito, o momento do campo

$$P^k(t) = \int d^3\vec{x} \left(\pi(t,\vec{x})\, \partial^k \varphi(t,\vec{x}) + \pi^+(t,\vec{x})\, \partial^k \varphi^+(t,\vec{x}) \right)$$

Observe que agora si importa o ordenamento dos campos. Temos

$$[P^k(t), \varphi(t,\vec{x})] = \int d^3\vec{x}' \left([\pi(t,\vec{x}'), \varphi(t,\vec{x})]\, \partial^k \varphi(t,\vec{x}') \right)$$

$$= -i\, \partial^k \varphi(t,\vec{x}).$$

Em resumo

$$[P^\mu, \varphi] = -i\, \partial^\mu \varphi.$$

Logo os P^μ são os geradores do grupo de Translações em $\mathcal{H}$. Em particular, a translação por a^μ é

$$U(a) = e^{i P^\mu a_\mu}.$$

3.1.4. Renormalização da Energía

En termos de a e b, temos

$$\partial_\mu \varphi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\, 2E_{\vec{p}}}\, i p_\mu \left(-e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} b^+(\vec{p}) \right) \Big|_{p^0 = E_{\vec{p}}},$$

$$\partial_\mu \varphi^+(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\, 2E_{\vec{p}}}\, i p_\mu \left(-e^{-ip\cdot x} b(\vec{p}) + e^{ip\cdot x} a^+(\vec{p}) \right) \Big|_{p^0 = E_{\vec{p}}}.$$

Logo, sem convenção de somma

$$\partial_\mu \varphi(x) \partial_\mu \varphi^+(x) = -\int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3\, 2E_{\vec{p}}\, 2E_{\vec{p}\,'}}\, p_\mu p_\mu' \left(-e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} b^+(\vec{p}) \right) \times$$

$$\left(-e^{-ip'\cdot x} b(\vec{p}\,') + e^{ip'\cdot x} a^+(\vec{p}\,') \right) \Big|_{p^0 = E_{\vec{p}},\ p'^0 = E_{\vec{p}\,'}}$$

$$= -\int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3\, 4E_{\vec{p}} E_{\vec{p}\,'}}\, p_\mu p_\mu' \Big(e^{-i(p+p')\cdot x} a(\vec{p}) b(\vec{p}\,') + e^{i(p+p')\cdot x} b^+(\vec{p}) a^+(\vec{p}\,')$$

$$- e^{i(p'-p)\cdot x} a(\vec{p}) a^+(\vec{p}\,') - e^{i(p-p')\cdot x} b^+(\vec{p}) b(\vec{p}\,') \Big) \Big|_{\substack{p^0 = E_{\vec{p}} \\ p'^0 = E_{\vec{p}\,'}}}$$

Pelo tanto

$$\int d^3\vec{x}\, \partial_\mu \varphi(x) \partial_\mu \varphi^+(x) = -\int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{4E_{\vec{p}} E_{\vec{p}\,'}}\, p_\mu p_\mu' \Big(\delta(\vec{p}+\vec{p}\,') \Big(e^{-i(E_{\vec{p}} + E_{\vec{p}\,'}) x^0} a(\vec{p}) b(\vec{p}\,')$$

$$+ e^{i(E_{\vec{p}} + E_{\vec{p}\,'}) x^0} b^+(\vec{p}) a^+(\vec{p}\,') \Big)$$

$$- \delta(\vec{p}-\vec{p}\,') \Big(e^{i(E_{\vec{p}\,'} - E_{\vec{p}})\cdot x^0} a(\vec{p}) a^+(\vec{p})$$

$$+ e^{i(E_{\vec{p}} - E_{\vec{p}\,'})\cdot x^0} b^+(\vec{p}) b(\vec{p}) \Big) \Big) \Big|_{\substack{p^0 = E_{\vec{p}} \\ p'^0 = E_{\vec{p}\,'}}}$$

$$= -\int \frac{d^3\vec{p}}{4E_{\vec{p}}^2} \left\{ \begin{matrix} E_{\vec{p}}^2 \\ -(p_i)^2 \end{matrix} \right\} \left(e^{-2iE_{\vec{p}} x^0} a(\vec{p}) b(-\vec{p}) + e^{2iE_{\vec{p}} x^0} b^+(\vec{p}) a^+(\vec{p}) \right)$$

$$+ \int \frac{d^3\vec{p}}{4E_{\vec{p}}^2} \left\{ \begin{matrix} E_{\vec{p}}^2 \\ (p_i)^2 \end{matrix} \right\} \left(a(\vec{p}) a^+(\vec{p}) + b^+(\vec{p}) b(\vec{p}) \right).$$

Por outra parte

$$\varphi(\vec{x})\varphi^+(\vec{x}) = \int \frac{d^3\vec{p}\, d^3\vec{p}'}{(2\pi)^3 4E_{\vec{p}}E_{\vec{p}'}} \left(e^{-ix\cdot p} a(\vec{p}) + e^{ix\cdot p} b^+(\vec{p})\right)\left(e^{-ix\cdot p'} b(\vec{p}') + e^{ix\cdot p'} a^+(\vec{p}')\right)\Bigg|_{\substack{p^0=E_{\vec{p}}\\ p'^0=E_{\vec{p}'}}}$$

$$= \int \frac{d^3\vec{p}\, d^3\vec{p}'}{(2\pi)^3 4E_{\vec{p}}E_{\vec{p}'}} \Big(e^{-ix\cdot(p+p')} a(\vec{p})b(\vec{p}') + e^{ix\cdot(p+p')} b^+(\vec{p})a^+(\vec{p}')$$

$$+ e^{-i(p-p')\cdot x} a(\vec{p})a^+(\vec{p}') + e^{i(p-p')\cdot x} b^+(\vec{p})b(\vec{p}')\Big)\Bigg|_{\substack{p^0=E_{\vec{p}}\\ p'^0=E_{\vec{p}'}}}.$$

Logo

$$\int d^3\vec{x}\, \varphi(\vec{x})\varphi^+(\vec{x}) = \int \frac{d^3\vec{p}\, d^3\vec{p}'}{(4E_{\vec{p}}E_{\vec{p}'}} \Big(\delta(p+p')\Big(e^{-i(E_{\vec{p}}+E_{\vec{p}'})x^0} a(\vec{p})b(\vec{p}')$$

$$+ e^{i(E_{\vec{p}}+E_{\vec{p}'})x^0} b^+(\vec{p})a^+(\vec{p}')\Big)$$

$$+ \delta(\vec{p}-\vec{p}')\Big(e^{i(E_{\vec{p}}-E_{\vec{p}'})x^0} a(\vec{p})a^+(\vec{p}')$$

$$+ e^{i(E_{\vec{p}}-E_{\vec{p}'})x^0} b^+(\vec{p})b(\vec{p}')\Big)\Big)\Bigg|_{\substack{p^0=E_{\vec{p}}\\ p'^0=E_{\vec{p}'}}}$$

$$= \int \frac{d^3\vec{p}}{4E_{\vec{p}}^2} \Big(e^{-2iE_{\vec{p}}x^0} a(\vec{p})b(-\vec{p}) + e^{2iE_{\vec{p}}x^0} b^+(\vec{p})a^+(-\vec{p})$$

$$+ a(\vec{p})a^+(\vec{p}) + b^+(\vec{p})b(\vec{p})\Big).$$

Concluimos que

$$H = \int \frac{d^3\vec{p}}{4E_{\vec{p}}^2} \Big(\big(\cancelto{0}{-E_{\vec{p}}^2+\vec{p}^2+m^2}\big)\big(e^{-2iE_{\vec{p}}x^0} a(\vec{p})b(-\vec{p}) + e^{2iE_{\vec{p}}x^0} b^+(\vec{p})a^+(-\vec{p})\big) +$$

$$\big(E_{\vec{p}}^2 + \vec{p}^2 + m^2\big)\big(a(\vec{p})a^+(\vec{p}) + b^+(\vec{p})b(\vec{p})\big)\Big)$$

$$= \frac{1}{2}\int d^3\vec{p}\, \big(a(\vec{p})a^+(\vec{p}) + b^+(\vec{p})b(\vec{p})\big).$$

Observe que o Hamiltoniano é positiva pois para todo $|\psi\rangle \in \mathcal{H}$

$$\langle\psi|H|\psi\rangle = \frac{1}{2}\int d^3\vec{p}\left(\|a^+(\vec{p})|\psi\rangle\|^2 + \|b(\vec{p})|\psi\rangle\|^2\right) \geq 0.$$

Agora, observe que

$$[a(\vec{p}), H] = \frac{1}{2}\int d^3\vec{p}\,' a(\vec{p}\,')[a(\vec{p}), a^+(\vec{p}\,')] = E_{\vec{p}}\, a(\vec{p}).$$

Logo, si $|E\rangle$ é um autoestado de H com energía E

$$\begin{aligned} Ha(\vec{p})|E\rangle &= \vec{a}(p)H|E\rangle + [H, a(\vec{p})]|E\rangle \\ &= (E - E_{\vec{p}})a(\vec{p})|E\rangle. \end{aligned}$$

Logo $a(\vec{p})|E\rangle$ tem energía $E - E_{\vec{p}}$ definida a menos de que $a(\vec{p})|E\rangle = 0$. Do mismo jeito

$$[b(\vec{p}), H] = E_{\vec{p}}\, b(\vec{p}), \quad [a^+(\vec{p}), H] = -E_{\vec{p}}\, a^+(\vec{p}),$$

$$[b^+(\vec{p}), H] = -E_{\vec{p}}\, b^+(\vec{p}).$$

Logo

$$\begin{aligned} Hb(\vec{p})|E\rangle &= (E - E_{\vec{p}})\, b(\vec{p})|E\rangle, \\ Ha^+(\vec{p})|E\rangle &= (E + E_{\vec{p}})\, a^+(\vec{p})|E\rangle, \\ Hb^+(\vec{p})|E\rangle &= (E + E_{\vec{p}})\, b^+(\vec{p})|E\rangle. \end{aligned}$$

Agora, como H é positivo, temos dois opções. A primeira é que $E < \infty$ y pelo tanto existe $n \in \mathbb{N}$ t.q. $a(\vec{p})^n|E\rangle \neq 0$ mais $a(\vec{p})^{n+1}|E\rangle = 0$. Agora, $a(\vec{p})^n|E\rangle$ é um autovetor de H com autovalor $E - nE_{\vec{p}}$. Logo existe $m \in \mathbb{N}$ t.q.

$b(\vec{p})^m a(\vec{p})^n|E\rangle \neq 0$ mais $b(\vec{p})^{m+1} a(\vec{p})^n|E\rangle = 0$ e

$a(\vec{p})\, b(\vec{p})^m a(\vec{p})^n|E\rangle = b(\vec{p})^m a(\vec{p})^{n+1}|E\rangle = 0$. Agora temos um autovetor $b(\vec{p})^m a(\vec{p})^n|E\rangle$ de H com autovalor $E-(n+m)E_{\vec{p}}$. Sup. que existe outro $\vec{p}\,' \in \mathbb{R}^3$ t.q. $a(\vec{p}\,')b(\vec{p})^m a(\vec{p})^n|E\rangle \neq 0$ ou $b(\vec{p}\,')b(\vec{p})^m a(\vec{p})^n|E\rangle \neq 0$. Logo, do mesmo modo anterior, existen $n', m' \in \mathbb{N}$ (com $n'+m'>0$) t.q. o vetor $b(\vec{p}\,')^{m'} a(\vec{p}\,')^{n'} b(\vec{p})^m a(\vec{p})^n|E\rangle$ e anulado por $a(\vec{p})$, $b(\vec{p})$, $a(\vec{p}\,')$, $b(\vec{p}\,')$. Em particolar ese vetor e um autovetor de H com autovalor $E-(n+m)E_{\vec{p}}-(n'+m')E_{\vec{p}\,'} < E-(n+m)E_{\vec{p}}$. Escolhendo este tipo de momentos uma obtiene uma sucesión de autovetores com autovalores descentes

$$E \geq E-(n+m)E_{\vec{p}} > E-(n+m)E_{\vec{p}}-(n'+m')E_{\vec{p}\,'} > E-(n+m)E_{\vec{p}}-(n'+m')E_{\vec{p}\,'} - (n''+m'')E_{\vec{p}\,''} > \cdots$$

onde a diferencia entre cada autovalor é pelo menos umo. Como 0 é uma cota inferior, issa sucesión e finita. Logo só precisamos un numero finito de momentos $\vec{p}_1, \dots, \vec{p}_N$. De este modo construimos o vacuo $|0\rangle \in \mathcal{H}$. Ele tem a propiedad de que $a(\vec{p})|0\rangle = b(\vec{p})|0\rangle = 0$ para todo $\vec{p} \in \mathbb{R}^3$. Mais então, observando

$$H = \frac{1}{2}\int d^3\vec{p}\,\left(a^{+}(\vec{p})a(\vec{p}) + b^{+}(\vec{p})b(\vec{p})\right) + \int d^3\vec{p}\; E_{\vec{p}}\, \delta^3(\vec{0}),$$

chegamos a contradição de que a energia do vacuo é infinita

$$H|0\rangle = \int d^3\vec{p}\, E_{\vec{p}}\, \delta^3(\vec{0})\,|0\rangle.$$

Pelo tanto, a primera premisa é falsa e se tem $E=\infty$. Em particular, o argumento para a existencia do vacuo $|0\rangle$ não esta certo. Sim embargo, podemos repetir toda discução redefinendo

$$:H: = \frac{1}{2}\int d^3\vec{p}\ \left(a^+(\vec{p})a(\vec{p}) + b^+(\vec{p})b(\vec{p})\right),$$

pois as relações de commutação e a positividade de H ficam sem mudar. Com o vacuo $|0\rangle$ construido de este jeito, temos que a diferencia entre o velho Hamiltoniano e o novo, é a energia do vacuo (o cual asumiremos normalizado)

$$H - :H: = \langle 0|H|0\rangle = \int d^3\vec{p}\ E_{\vec{p}}\ \delta^3(\vec{0}).$$

Em particular, é so um número y pelo tanto H e $:H:$ se podem considerar como equivalentes, al diferir por uma redefinição do 0 de Energía. Asumiremos que $|0\rangle$ é unico.

3.1.5. Produto Normal.

Produtos de campos no mesmo ponto, como os de o Hamiltoniano não são bem definidos. Para ver isso, separe em creação e destrução o campo φ

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x).$$

Logo

$$\varphi(x)\varphi^*(x) = \varphi^{(+)}(x)\varphi^{*(+)}(x) + \varphi^{(+)}(x)\varphi^{*(-)}(x) + \varphi^{(-)}(x)\varphi^{*(+)}(x) + \varphi^{(-)}(x)\varphi^{*(-)}(x).$$

Em particular

$$\langle 0|\varphi(x)\varphi^*(x)|0\rangle = \langle 0|\varphi^{(-)}(x)\varphi^{*(+)}(x)|0\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{4E_{\vec{p}} E_{\vec{p}\,'}} e^{-ix\cdot(p-p')} \langle 0|a(\vec{p})a^{\dagger}(\vec{p}\,')|0\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_{\vec{p}}} = \infty.$$

Pelo tanto, dado um operador polinomial em a, b, a^{+} e b^{+}, definimos o orden normal como o operador com todos os a's e b's a dereita dos a^{+} e b^{+}. Observe que como os commutadores são números, a diferencia entre o orden normal e o operador original é um número. O orden normal é denotado por $: \; :$.

3.1.6. Espaço de Fock

Agora, podemos construir explicitamente um espaço de Hilbert $\mathcal{H}$ onde ϕ atua. Para poder interpretar nossos resultados em termos de partículas, calculamos o operador de momento

$$P^k = \int d^3\vec{x} \left(\pi(t,\vec{x})\,\partial^k\varphi(t,\vec{x}) + \pi^\dagger(t,\vec{x})\,\partial^k\varphi^\dagger(t,\vec{x})\right)$$

$$= -\int d^3\vec{x} \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3 4E_{\vec{p}}E_{\vec{p}\,'}} E_{\vec{p}}\, p'^k$$

$$\Big[\left(-e^{-ip\cdot x} b(\vec{p}) + e^{ip\cdot x} a^\dagger(\vec{p})\right)\left(-e^{-ip'\cdot x} a(\vec{p}\,') + e^{ip'\cdot x} b^\dagger(\vec{p}\,')\right) +$$

$$\left(-e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} b^\dagger(\vec{p})\right)\left(-e^{-ip'\cdot x} b(\vec{p}\,') + e^{ip'\cdot x} a^\dagger(\vec{p}\,')\right)\Big]\Big|_{\substack{p^0=E_{\vec{p}},\\ p'^0=E_{\vec{p}\,'}\\ x^0=t}}$$

$$= -\int d^3\vec{x} \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3 4\cancel{E_{\vec{p}}}E_{\vec{p}\,'}} \cancel{E_{\vec{p}}}\, p'^k$$

$$\Big[e^{-i(p+p')\cdot x}\left(b(\vec{p})a(\vec{p}\,') + a(\vec{p})b(\vec{p}\,')\right) + e^{i(p+p')\cdot x}\left(a^\dagger(\vec{p})b^\dagger(\vec{p}\,') + b^\dagger(\vec{p})a^\dagger(\vec{p}\,')\right)$$

$$- e^{-i(p-p')\cdot x}\left(b(\vec{p})b^\dagger(\vec{p}\,') + a(\vec{p})a^\dagger(\vec{p}\,')\right) - e^{i(p-p')\cdot x}\left(a^\dagger(\vec{p})a(\vec{p}\,') + b^\dagger(\vec{p})b(\vec{p}\,')\right)\Big]$$

$$\Big|_{p^0=E_{\vec{p}},\ p'^0=E_{\vec{p}\,'},\ x^0=t}$$

$$= -\int \frac{d^3\vec{p}}{4E_{\vec{p}}} p^k\Big(-e^{-2iE_{\vec{p}}t}\left(b(\vec{p})a(-\vec{p}) + a(\vec{p})b(-\vec{p})\right)$$

$$- e^{2iE_{\vec{p}}t}\left(a^\dagger(\vec{p})b^\dagger(-\vec{p}) + b^\dagger(\vec{p})a^\dagger(-\vec{p})\right)$$

$$- b(\vec{p})b^\dagger(\vec{p}) - b^\dagger(\vec{p})b(\vec{p}) - a(\vec{p})a^\dagger(\vec{p}) - a^\dagger(\vec{p})a(\vec{p})\Big)$$

$$= \int \frac{d^3\vec{p}}{2E_{\vec{p}}}\, p^K \left(a^+(\vec{p})a(\vec{p}) + b^+(\vec{p})\, b(\vec{p}) \right)$$

$$+ \int \frac{d^3\vec{p}}{4E_{\vec{p}}}\, p^K \Big(e^{-2iE_{\vec{p}}t} \left(b(\vec{p})a(-\vec{p}) + a(\vec{p})b(-\vec{p}) \right)$$

$$+ e^{2iE_{\vec{p}}t} \left(a^+(\vec{p})b^+(-\vec{p}) + b^+(\vec{p})a^+(-\vec{p}) \right) + 4E_{\vec{p}}\,\delta(\vec{0}) \Big).$$

Como a segunda integral é impar baixo a transformação $\vec{p} \longmapsto -\vec{p}$, se anula. Logo

$$P^K = \int \frac{d^3\vec{p}}{2E_{\vec{p}}}\, p^K \left(a^+(\vec{p})a(\vec{p}) + b^+(\vec{p})\, b(\vec{p}) \right).$$

Assim mesmo, calculamos a carga

$$Q = \int d^3\vec{x}\; j^0 = i \int d^3\vec{x} \left(\varphi^+(t,\vec{x})\,\pi^+(t,\vec{x}) - \varphi(t,\vec{x})\,\pi(t,\vec{x}) \right)$$

$$= - \int d^3\vec{x} \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3\, 4E_{\vec{p}}\, \cancel{E_{\vec{p}\,'}}}\, \cancel{E_{\vec{p}\,'}}$$

$$\Big[\left(e^{-ix\cdot p}\, b(\vec{p}) + e^{ix\cdot p}\, a^+(\vec{p}) \right)\left(- e^{-ip'\cdot x}\, a(\vec{p}\,') + e^{ip'\cdot x}\, b^+(\vec{p}\,') \right) -$$

$$\left(e^{-ip\cdot x}\, a(\vec{p}) + e^{ip\cdot x}\, b^+(\vec{p}) \right)\left(- e^{-ip'\cdot x}\, b(\vec{p}\,') + e^{ip'\cdot x}\, a^+(\vec{p}\,') \right) \Big] \Big|_{\substack{p^0 = E_{\vec{p}},\\ p'^0 = E_{\vec{p}\,'},\ x^0 = t}}$$

$$= - \int d^3\vec{x} \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3\, 4E_{\vec{p}}} \Big[- e^{-i(p+p')\cdot x} \left(b(\vec{p})a(\vec{p}\,') - a(\vec{p})b(\vec{p}\,') \right) +$$

$$e^{i(p+p')\cdot x} \left(a^+(\vec{p})b^+(\vec{p}\,') - b^+(\vec{p})a^+(\vec{p}\,') \right) +$$

$$e^{-i(p-p')\cdot x}\left(b(\vec{p})b^+(\vec{p}\,') + a(\vec{p})a^+(\vec{p}\,')\right) +$$

$$\left.- e^{i(p-p')\cdot x}\left(a^+(\vec{p})a(\vec{p}\,') - b^+(\vec{p})b(\vec{p}\,')\right)\Big]\Big]\right|_{p^0=E_{\vec{p}},\ p'^0=E_{\vec{p}'},\ x^0=t}$$

$$= -\int \frac{d^3\vec{p}}{4E_{\vec{p}}}\Big[-e^{-2iE_{\vec{p}}t}\left(b(\vec{p})a(-\vec{p}) - a(\vec{p})b(-\vec{p})\right) +$$

$$e^{2iE_{\vec{p}}t}\left(a^+(\vec{p})b^+(-\vec{p}) - b^+(\vec{p})a^+(-\vec{p})\right) +$$

$$b(\vec{p})b^+(\vec{p}) - a(\vec{p})a^+(\vec{p}) - a^+(\vec{p})a(\vec{p}) + b^+(\vec{p})b(\vec{p})\Big]$$

$$= \int \frac{d^3\vec{p}}{2E_{\vec{p}}}\left(a^+(\vec{p})a(\vec{p}) - b^+(\vec{p})b(\vec{p})\right)$$

$$+ \int \frac{d^3\vec{p}}{4E_{\vec{p}}}\Big[e^{-2iE_{\vec{p}}t}\left(b(\vec{p})a(-\vec{p}) - a(\vec{p})b(-\vec{p})\right) - e^{2iE_{\vec{p}}t}\left(a^+(\vec{p})b^+(-\vec{p}) - b^+(\vec{p})a^+(-\vec{p})\right)$$

$$+ \cancel{[a(\vec{p}), a^+(\vec{p})]} - \cancel{[b(\vec{p}), b^+(\vec{p})]}\Big].$$

Uma vez mais, a segunda integral se anula por paridad. Logo

$$Q = \int \frac{d^3\vec{p}}{2E_{\vec{p}}}\left(a^+(\vec{p})a(\vec{p}) - b^+(\vec{p})b(\vec{p})\right).$$

Vao ser util definir os operadores $n_a(\vec{p}) = \frac{1}{2E_{\vec{p}}}a^+(\vec{p})a(\vec{p})$, $n_b(\vec{p}) = \frac{1}{2E_{\vec{p}}}b^+(\vec{p})b(\vec{p})$, $N_a = \int d^3\vec{p}\, n_a(\vec{p})$, $N_b = \int d^3\vec{p}\, n_b(\vec{p})$ e $N = N_a + N_b$

Em termos de estos

$$H = \int d^3\vec{p}\, E_{\vec{p}}\left(n_a(\vec{p}) + n_b(\vec{p})\right),$$

$$P^k = \int d^3\vec{p}\, p^k\left(n_a(\vec{p}) + n_b(\vec{p})\right),$$

$$Q = N_a - N_b.$$

Se tem as relações de commutação

$$[n_a(\vec{p}), a^+(\vec{p}')] = a^+(\vec{p})\,\delta(\vec{p}-\vec{p}'), \quad [n_a(\vec{p}), b^+(\vec{p}')] = 0,$$

$$[n_b(\vec{p}), b^+(\vec{p}')] = b^+(\vec{p})\,\delta(\vec{p}-\vec{p}'), \quad [n_b(\vec{p}), a^+(\vec{p}')] = 0.$$

Logo

$$[N_a, a^+(\vec{p})] = a^+(\vec{p}), \qquad [N_a, b^+(\vec{p})] = 0$$

$$[N_b, b^+(\vec{p})] = b^+(\vec{p}), \qquad [N_b, a^+(\vec{p})] = 0$$

$$[H, a^+(\vec{p})] = E_{\vec{p}}\, a^+(\vec{p}), \qquad [H, b^+(\vec{p})] = E_p\, b^+(\vec{p})$$

$$[P^K, a^+(\vec{p})] = p^K a^+(\vec{p}), \qquad [P^K, b^+(\vec{p})] = p^K b^+(\vec{p})$$

$$[Q, a^+(\vec{p})] = a^+(\vec{p}) \qquad [Q, b^+(\vec{p})] = -b^+(\vec{p}).$$

Logo, si $|N_a\rangle$, $|N_b\rangle$, $|E\rangle$, $|p^K\rangle$, $|q\rangle$ são autoestados de N_a, N_b, H, P^K, Q resp. com autovalores N_a, N_b, E, p^K, q, resp. se tem que

$$N_a\, a^+(\vec{p})|N_a\rangle = (N_a+1)\, a^+(\vec{p})|N_a\rangle, \qquad N_a\, b^+(\vec{p})|N_a\rangle = N_a\, b^+(\vec{p})|N_a\rangle,$$

$$N_b\, b^+(\vec{p})|N_b\rangle = (N_b+1)\, b^+(\vec{p})|N_b\rangle, \qquad N_b\, b^+(\vec{p})|N_b\rangle = N_b\, b^+(\vec{p})|N_b\rangle,$$

$$H\, a^+(\vec{p})|E\rangle = (E+E_{\vec{p}})\, a^+(\vec{p})|E\rangle, \qquad H\, b^+(\vec{p})|E\rangle = (E+E_{\vec{p}})\, b^+(\vec{p})|E\rangle$$

$$P^K a^+(\vec{p})|p'^K\rangle = (p'^K+p^K)\, a^+(\vec{p})|p'^K\rangle, \qquad P^K b^+(\vec{p})|p'^K\rangle = (p'^K+p^K)\, b^+(\vec{p})|p'^K\rangle$$

$$Q\, a^+(\vec{p})|q\rangle = (q+1)\, a^+(\vec{p})|q\rangle, \qquad Q\, b^+(\vec{p})|q\rangle = (q-1)\, b^+(\vec{p})|q\rangle.$$

Temos então a siguente interpretação. $a^+(\vec{p})$ cria uma exitação com energía $E_{\vec{p}}$, momento $\vec{p}$ e carga $+1$. Esta é conhecida como partícula. $b^+(\vec{p})$ cria uma exitação com

$E_{\vec{p}}$, momento $\vec{p}$ e carga -1. Esta é conhecida como antipartícula. Para as dois se tem $E_{\vec{p}}^2 - \vec{p}^2 = m^2$. Logo as dois tem masa m. De fato, para todo ψ normalizado

$$\| a^+(\vec{p})|\psi\rangle\|^2 = \langle\psi|a(\vec{p})a^+(\vec{p})|\psi\rangle = \langle\psi|a^+(\vec{p})a(\vec{p})|\psi\rangle + 2E_{\vec{p}}\,\delta(0)$$

$$= \|a(\vec{p})|\psi\rangle\|^2 + 2E_{\vec{p}}\,\delta(0) \geq 2E_{\vec{p}}\,\delta(0) = \infty.$$

Com esta interpretação agora podemos construir um espaço de Hilbert $\mathfrak{F}$ onde estos operadores actuam. Seja $\mathcal{H}_a$ o espaço gerado por a base impropia $\{|\vec{p}_i\rangle := a(\vec{p})|0\rangle,\ \vec{p}\in\mathbb{R}^3\}$. Então $|\psi\rangle\in\mathcal{H}_a$ sim e só sim

$$|\psi\rangle = \int d^3\vec{p}\,|\vec{p}_i\rangle\langle\vec{p}_i|\psi\rangle$$

com $\langle\vec{p}_i|\psi\rangle\in L^2(\mathbb{R}^3)$. Em particular $\mathcal{H}_a \cong L^2(\mathbb{R}^3)$. O espaço gerado por $\{|\vec{p}_1,\vec{p}_2;\rangle := a^+(\vec{p}_1)a^+(\vec{p}_2)|0\rangle = a^+(\vec{p}_2)a^+(\vec{p}_1)|0\rangle = |\vec{p}_2,\vec{p}_1;\rangle \mid \vec{p}_1,\vec{p}_2\in\mathbb{R}^3\}$ é do mismo jeito, isomorfo as funções simetrias em $L^2(\mathbb{R}^3\times\mathbb{R}^3)\cong L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)\cong \bigotimes_{i=1}^{2}\mathcal{H}_a$, é dizer, a $\mathrm{Sym}\left(\bigotimes_{i=1}^{2}\mathcal{H}_a\right)$. Continuando de esse jeito, construimos o espaço de Fock $\mathfrak{F}_a = \bigoplus_{n=0}^{\infty}\mathrm{Sym}\left(\bigotimes_{i=1}^{n}\mathcal{H}_a\right)$, onde se toma $\mathrm{Sym}\left(\bigotimes_{i=1}^{0}\mathcal{H}_a\right) = \mathrm{Sym}(\mathbb{C}) = \mathbb{C}\cong\mathrm{span}\{|0\rangle\}$. Um espaço $\mathfrak{F}_b$ se construye do mesmo jeito com os operadores $b^+(\vec{p})$. Tomamos $\mathfrak{F} = \mathfrak{F}_a\otimes\mathfrak{F}_b$.

Para ver que em efeito, issos vetores são uma base impropia, considere os vetores da forma

$$\prod_{i=1}^{N_b} b^+(\vec{p}_i)\prod_{j=1}^{N_a} a^+(\vec{q}_j)|0\rangle .$$

Estudemos

$$\star = \langle 0|\prod_{k=1}^{N_a'} a(\vec{q}\,'_k)\prod_{l=1}^{N_b'} b(\vec{p}\,'_l)\prod_{i=1}^{N_b} b^+(\vec{p}_i)\prod_{j=1}^{N_a} a^+(\vec{q}_j)|0\rangle$$

$$= \langle 0|\prod_{l=1}^{N_b'} b(\vec{p}\,'_l)\prod_{i=1}^{N_b} b^+(\vec{p}_i)\prod_{k=1}^{N_a'} a(\vec{q}\,'_k)\prod_{j=1}^{N_a} a^+(\vec{q}_j)|0\rangle .$$

$$a(\vec{q}\,'_{N_a'})\prod_{j=1}^{N_a} a^+(\vec{q}_j)|0\rangle = a^+(\vec{q}_1)\,a(\vec{q}\,'_{N_a'})\prod_{j=2}^{N_a} a^+(\vec{q}_j)|0\rangle$$

$$+ 2E_{\vec{q}\,'_{N_a'}}\,\delta(\vec{q}_1-\vec{q}\,'_{N_a'})\prod_{j=2}^{N_a} a^+(\vec{q}_j)|0\rangle$$

$$= 2E_{\vec{q}\,'_{N_a'}}\sum_{i=1}^{N_a}\delta(\vec{q}_i-\vec{q}\,'_{N_a'})\prod_{\substack{j=1\\ j\neq i}}^{N_a} a^+(\vec{q}_j)|0\rangle . \quad \star\star$$

Então, para $N_a' < N_a$, se tem

$$\prod_{k=1}^{N_a'} a(\vec{q}\,'_k)\prod_{j=1}^{N_a} a^+(\vec{q}_j)|0\rangle = \left(\prod_{s=1}^{N_a'} 2E_{\vec{q}\,'_s}\right)\sum_{\substack{m_1,\dots,m_{N_a'}=1\\ m_1\neq\dots\neq m_{N_a'}}}^{N_a}\ \prod_{n=1}^{N_a'}\delta(\vec{q}_{m_n}-\vec{q}\,'_n)\prod_{\substack{r=1\\ r\neq m_1,\dots,m_{N_a'}}}^{N} a^+(\vec{q}_r)|0\rangle .$$

Em particular

$$\star = \left(\prod_{s=1}^{N_a'} 2E_{\vec{q}\,'_s}\right) \sum_{\substack{m_1,\dots,m_{N_a'}=1 \\ m_1\neq\dots\neq m_{N_a'}}}^{N_a} \prod_{n=1}^{N_a'} \delta(\vec{q}_{m_n}-\vec{q}\,'_n)\langle 0|\prod_{l=1}^{N_b'} b(\vec{p}\,'_l)\prod_{i=1}^{N_b} b^+(\vec{p}_i)\prod_{\substack{r=1 \\ r\neq m_1,\dots,m_{N_a'}}}^{N_a} a^+(\vec{q}_r)|0\rangle$$

$$=0,$$

pois $\langle 0|a^+(\vec{p})=0$. Si $N_a > N_a'$, se tem que

$$\langle 0|\prod_{k=1}^{N_a'} a(\vec{q}\,'_k)\prod_{j=1}^{N_a} a^+(\vec{q}_j)$$

é o conjugado de

$$\prod_{j=1}^{N_a} a(\vec{q}_j)\prod_{k=1}^{N_a'} a(\vec{q}\,'_k)|0\rangle.$$

Logo, com o resultado anterior sob a troca $N_a' \longleftrightarrow N_a$, $\vec{q} \leftrightarrow \vec{q}\,'$, se tem

$$\star = \left(\prod_{s=1}^{N_a} 2E_{\vec{q}_s}\right) \sum_{\substack{m_1,\dots,m_{N_a}=1 \\ m_1\neq\dots\neq m_{N_a}}}^{N_a'} \prod_{n=1}^{N_a} \delta(\vec{q}\,'_{m_n}-\vec{q}_n)\langle 0|\prod_{\substack{r=1 \\ r\neq m_1,\dots,m_{N_a}}}^{N_a'} a(\vec{q}\,'_r)\prod_{l=1}^{N_b'} b(\vec{p}\,'_l)\prod_{i=1}^{N_b} b^+(\vec{p}_i)|0\rangle$$

$$=0,$$

pois $a^+(\vec{p})|0\rangle=0$. O mesmo é certo se $N_b > N_b'$ o $N_b' > N_b$.

Si $N_a = N_a'$, se tem

$$\prod_{k=1}^{N_a} a(\vec{q}\,'_k)\prod_{j=1}^{N_a} a^+(\vec{q}_j)|0\rangle = \left(\prod_{s=1}^{N_a} 2E_{\vec{q}\,'_s}\right) \sum_{\substack{m_1,\dots,m_{N_a}=1 \\ m_1\neq\dots\neq m_{N_a}}}^{N_a} \prod_{n=1}^{N_a} \delta(\vec{q}_{m_n}-\vec{q}\,'_n)|0\rangle.$$

Pelo tanto, si $N_a = N_a'$ e $N_b = N_b'$

$$\star = \prod_{s=1}^{N_a} (2E_{\vec{q}_s'}) \prod_{r=1}^{N_b} (2E_{\vec{p}_r'}) \sum_{\substack{m_1,\dots,m_{N_a}=1 \\ m_1\neq\cdots\neq m_{N_a}}}^{N_a}{}' \sum_{\substack{n_1,\dots,n_{N_b}=1 \\ n_1\neq\cdots\neq n_{N_b}}}^{N_b}{}' \prod_{i=1}^{N_a} \delta(\vec{q}_{m_i} - \vec{q}_i') \prod_{j=1}^{N_b} \delta(\vec{p}_{n_j} - \vec{p}_j').$$

$$= \sum_{\substack{m_1,\dots,m_{N_a}=1 \\ m_1\neq\cdots\neq m_{N_a}}}^{N_a}{}' \sum_{\substack{n_1,\dots,n_{N_b}=1 \\ n_1\neq\cdots\neq n_{N_b}}}^{N_b}{}' \left(\prod_{i=1}^{N_a} 2E_{\vec{q}_i'} \delta(\vec{q}_{m_i} - \vec{q}_i') \right) \left(\prod_{j=1}^{N_b} 2E_{\vec{p}_j'} \delta(\vec{p}_{n_j} - \vec{p}_j') \right)$$

$$= \sum_{\sigma_a \in S_{N_a}}{}' \sum_{\sigma_b \in S_{N_b}}{}' \left(\prod_{i=1}^{N_a} 2E_{\vec{q}_i} \delta(\vec{q}_i - \vec{q}\,'_{\sigma_a(i)}) \right) \left(\prod_{j=1}^{N_b} 2E_{\vec{p}_j} \delta(\vec{p}_j - \vec{p}\,'_{\sigma_b(j)}) \right),$$

onde $S_N := \{ \sigma : \{1,\dots,N\} \to \{1,\dots,N\} \mid \sigma \text{ é uma bijeção} \}$ e o grupo de permutações de N letras e tem cardinalidade $|S_N| = N!$. Logo definimos

$$|\vec{q}_1,\dots,\vec{q}_{N_a}; \vec{p}_1,\dots,\vec{p}_{N_b}\rangle = \frac{1}{\sqrt{N_a!N_b!}} \prod_{i=1}^{N_a} a^+(\vec{q}_i) \prod_{j=1}^{N_b} b^+(\vec{p}_j)|0\rangle,$$

os cuales tem a normalização

$$\langle \vec{q}_1',\dots,\vec{q}\,'_{N_a'}; \vec{p}_1',\dots,\vec{p}\,'_{N_b'} | \vec{q}_1,\dots,\vec{q}_{N_a}; \vec{p}_1,\dots,\vec{p}_{N_b}\rangle =$$

$$\frac{\delta_{N_a N_a'} \delta_{N_b N_b'}}{N_a! N_b!} \sum_{\sigma_a \in S_{N_a}}{}' \sum_{\sigma_b \in S_{N_b}}{}' \left(\prod_{i=1}^{N_a} 2E_{\vec{q}_i} \delta(\vec{q}_i - \vec{q}\,'_{\sigma_a(i)}) \right) \left(\prod_{j=1}^{N_b} 2E_{\vec{p}_j} \delta(\vec{p}_j - \vec{p}\,'_{\sigma_b(j)}) \right).$$

Sobre eles é claro que

$$a^{+}(\vec{q}_{N_a+1})|\vec{q}_1,\dots,\vec{q}_{N_a};\vec{p}_1,\dots,\vec{p}_{N_b}\rangle = \sqrt{N_a+1}\,|\vec{q}_1,\dots,\vec{q}_{N_a+1};\vec{p}_1,\dots,\vec{p}_{N_b}\rangle$$

e o resultado análogo para os b^{+}'s. Com o cálculo ** é claro que

$$a(\vec{q}_{N_a+1})|\vec{q}_1,\dots,\vec{q}_{N_a};\vec{p}_1,\dots,p_{N_b}\rangle = \frac{2E_{\vec{q}_{N_a+1}}}{\sqrt{N_a}}\sum_{i=1}^{N_a}\delta(\vec{q}_{N_a+1}-\vec{q}_i)\times$$

$$|\vec{q}_1,\dots,\hat{\vec{q}}_i,\dots,\vec{q}_{N_a}|\vec{p}_1,\dots,\vec{p}_{N_b}\rangle.$$

3.1.7. Funções de Green

Por agora vamos a voltar a o problema de Klein-Gordon classico. Em particular, queremos encontras as funções de de Green do operador de Klein-Gordon ∂^2+m^2, isso é as funções G

$$(\partial^2+m^2)G(x) = -\delta(x)$$

Asumendo que tal G tem uma transformada de Fourier inversa

$$G(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\,\tilde{G}(p),$$

se tem

$$\int \frac{d^4p}{(2\pi)^4}\left(-p^2+m^2\right)e^{ip\cdot x}\,\tilde{G}(p) = -\int \frac{d^4p}{(2\pi)^4}\,e^{ip\cdot x}.$$

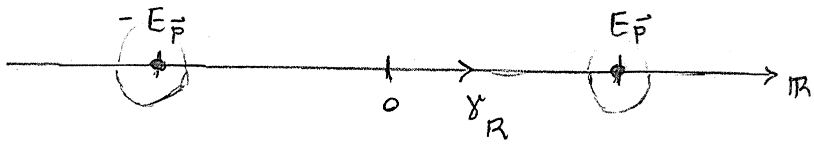
Pelo tanto

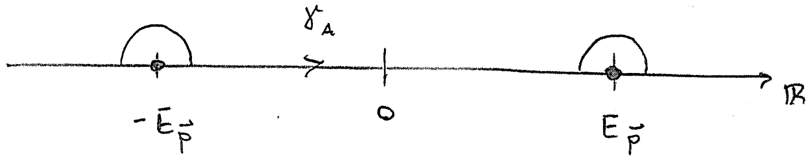
$$\tilde{G}(p) = \frac{1}{p^2 - m^2} = \frac{1}{p^{0\,2} - \vec{p}^{\,2} - m^2} = \frac{1}{(p^0)^2 - E_{\vec{p}}^2} = \frac{1}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})} \, ,$$

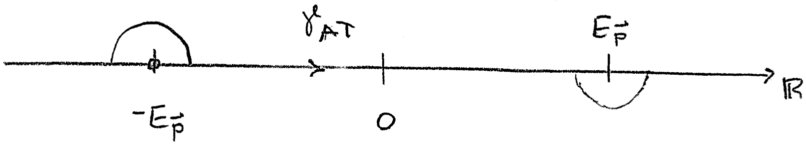
e

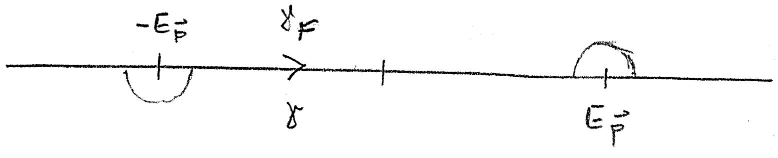
$$G(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \, e^{-i\vec{p}\cdot\vec{x}} \int \frac{dp^0}{2\pi} \, \frac{e^{ip^0 x^0}}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})} \, ,$$

which clearly does not converge. Então nossa premisa estava errada. Sem embargo, podemos deformar o contorno de integração para evadir os polos. Temos 4 opções









Para explorarlas, definimos

$$f : \mathbb{C} \setminus \{-E_{\vec{p}}, E_{\vec{p}}\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{e^{izx^0}}{(z-E_{\vec{p}})(z+E_{\vec{p}})},$$

$$I_\gamma := \int_\gamma \frac{dz}{2\pi} f(z).$$

Observamos que os resíduos são

$$\operatorname{Res}_{-E_{\vec{p}}} f = -\frac{e^{-iE_{\vec{p}}x^0}}{2E_{\vec{p}}},$$

$$\operatorname{Res}_{E_{\vec{p}}} f = \frac{e^{iE_{\vec{p}}x^0}}{2E_{\vec{p}}}.$$

Estos são útiles para calcular I_γ com γ fechado.

Agora bem $\gamma_R, \gamma_A, \gamma_F, \gamma_{AT}$ não são fechados. Mais, podemos calcular as integrais correspondentes considerando $\underbrace{(\gamma_i \cap [-R,R]) \cup RC_j}_{\gamma_{ij}}$ com $i \in \{R, A, F, AT\}$, e $j \in \{+,-\}$,

$C_+ = S^1 \cap \mathbb{C}^+$, $C_- = S^1 \cap \mathbb{C}^-$ de manera que $\lim_{R\to\infty} I_{RC_j} = 0$.

Agora bem

$$|I_{RC_j}| = \left| \pm \int_0^{\pm\pi} d\theta \, iRe^{i\theta} \frac{e^{iR\cos\theta x^0} e^{-R\sin\theta x^0}}{(Re^{i\theta} - E_{\vec{p}})(Re^{i\theta} + E_{\vec{p}})} \right|$$

$$\leq \int_0^{\pm\pi} d\theta \; R \; \frac{e^{-R\sin\theta x^0}}{|Re^{i\theta} - E_{\vec{p}}||Re^{i\theta} + E_{\vec{p}}|} \longrightarrow 0$$

sem $\sin(\theta)x^0 > 0$ no domínio de integração. Logo, temos que escolher C_+ para $x^0 > 0$ e C_- para $x^0 < 0$. Então

$$I_{\gamma_R} = i\,\theta(x^0)\,\frac{1}{2E_{\vec{p}}}\left(e^{iE_{\vec{p}}x^0} - e^{-iE_{\vec{p}}x^0}\right),$$

$$I_{\gamma_A} = -i\,\theta(-x^0)\,\frac{1}{2E_{\vec{p}}}\left(e^{iE_{\vec{p}}x^0} - e^{-iE_{\vec{p}}x^0}\right),$$

$$I_{\gamma_{AT}} = i \begin{cases} \dfrac{e^{iE_{\vec{p}}x^0}}{2E_{\vec{p}}} & x^0 > 0 \\[2ex] \dfrac{e^{-iE_{\vec{p}}x^0}}{2E_{\vec{p}}} & x^0 < 0 \end{cases} = \frac{e^{iE_{\vec{p}}|x^0|}}{2E_{\vec{p}}},$$

$$I_{\gamma_F} = -\frac{e^{-iE_{\vec{p}}|x^0|}}{2E_{\vec{p}}}.$$

Definimos

$$\Delta_i(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}}\, I_{\gamma_i},$$

e

$$\Delta^+(x) = i\int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\, e^{ip\cdot x}\Big|_{p^0=E_{\vec{p}}},$$

$$\Delta(x) = \Delta^+(x) - \Delta^+(-x).$$

Logo

$$\Delta_R(x) = \Theta(x^0)\,\Delta(x),$$

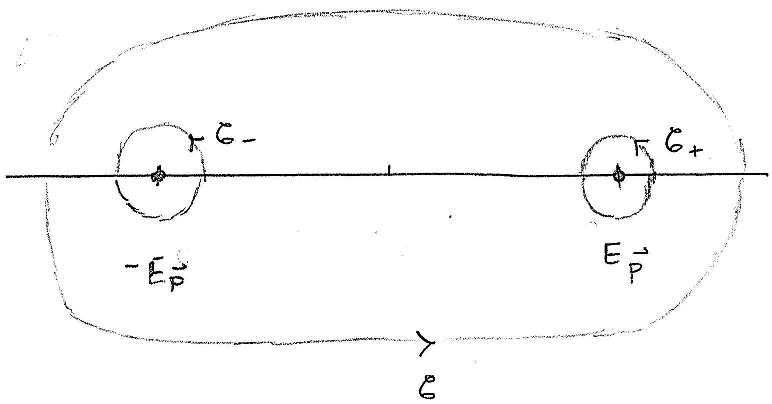
$$\Delta_A(x) = -\Theta(-x^0)\,\Delta(x)$$

$$\Delta_{AT}(x) = \Theta(x^0)\,\Delta^+(x) + \Theta(-x^0)\,\Delta^+(-x)$$

$$\Delta_F(x) = -\Theta(x^0)\,\Delta^+(-x) - \Theta(-x^0)\,\Delta^+(x),$$

são funções de Green do operador do Klein-Gordon.

Observamos que as funções auxiliares Δ^+, $-\Delta^+(-\cdot) =: \Delta^-$ e Δ são as integrales tomadas por o contorno $\mathcal{C}_+$, $\mathcal{C}_-$ e $\mathcal{C}$ resp.



3.1.8. Commutatividade Local

Numa teoría relativistica medições separadas espacialmente não debem interferir. Em particular, si dois observavels estam separados espacialmente, devem ser compativels, i.e. commutar. Se tem por exemplo

$$[\varphi(x), \varphi^+(y)] = \int \frac{d^3\vec{p}\; d^3\vec{p}\,'}{(2\pi)^3\, 2E_{\vec{p}}\, 2E_{\vec{p}'}} \times \Big(e^{-ix\cdot p} e^{-iy\cdot p'} \cancelto{0}{[a(\vec{p}), b(\vec{p}\,')]}$$

$$+ e^{-ix\cdot p} e^{iy\cdot p'} [a(\vec{p}), a^+(\vec{p}\,')] + e^{ix\cdot p} e^{-iy\cdot p'} [b^+(\vec{p}), b(\vec{p}\,')]$$

$$+ e^{ix\cdot p} e^{iy\cdot p'} \cancelto{0}{[b^+(\vec{p}), a^+(\vec{p}\,')]} \Big)\Big|_{p^0 = E_{\vec{p}},\; p'^0 = E_{\vec{p}'}}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}} \left(e^{-i(x-y)\cdot p} - e^{i(x-y)\cdot p} \right)$$

$$= i\Delta(x-y).$$

Para ver que para ver que esto se anulara para $(x-y)^2 < 0$, note que Δ e invariante de Lorentz pois

$$\frac{d^3\vec{p}}{2E_{\vec{p}}} = d^4p\; \delta(p^2 - m^2)\,\theta(p^0)$$

o é. Então, se $(x-y)^2$, mediante uma transformação de Lorentz se pode fazer $x^0 = y^0$. Logo

$$[\varphi(x), \varphi^+(y)] = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}} \left(e^{i(\vec{x}-\vec{y})\cdot\vec{p}} - e^{-i(\vec{x}-\vec{y})\cdot\vec{p}} \right) = 0$$

pois é impar sob $\vec{p} \to -\vec{p}$. A função

$$i\Delta(x-y) = \langle 0|[\varphi(x),\varphi^+(y)]|0\rangle$$

é conhecida como função de Pauli-Jordan. Podemos notar que Δ^+ tambem aparece deste jeito

$$\langle 0|\varphi(x)\varphi^+(y)|0\rangle = \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3 2E_{\vec{p}} 2E_{\vec{p}\,'}} e^{-ix\cdot p} e^{iy\cdot p'} \langle 0|a(\vec{p})a^+(\vec{p}\,')|0\rangle \Big|_{\substack{p^0=E_{\vec{p}},\\ p'^0=E_{\vec{p}\,'}}}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-i(x-y)\cdot p}\Big|_{p^0=E_{\vec{p}}} = -i\Delta^+(y-x) = i\Delta^-(x-y),$$

$$\langle 0|\varphi^+(x)\varphi(y)|0\rangle = \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3 2E_{\vec{p}} 2E_{\vec{p}\,'}} e^{-ix\cdot p} e^{+iy\cdot p'} \langle 0|b(\vec{p})b^+(\vec{p}\,')|0\rangle \Big|_{\substack{p^0=E_{\vec{p}},\\ p'^0=E_{\vec{p}\,'}}}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-i(x-y)\cdot p}\Big|_{p^0=E_{\vec{p}}} = -i\Delta^+(y-x) = i\Delta^-(x-y).$$

Outra função importante é

$$i\Delta^1(x-y) = \langle 0|\{\varphi(x),\varphi^+(y)\}|0\rangle = i(\Delta^-(x-y) - \Delta^+(x-y)).$$

Observe que

$$(\Box + m^2)\Delta^+(x) = i\int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} (-p^2+m^2)\Delta^+(x)\Big|_{p^0=E_{\vec{p}}} = 0.$$

Logo Δ, Δ^+, Δ^- e Δ_1 são sol. da ecuação de Klein-Gordon. Finalmente, dados dos campos A e B, definimos

$$T(A(x)B(y)) = \theta(x^0-y^0)A(x)B(y) + \theta(y^0-x^0)B(y)A(x).$$

Logo

$$\langle 0| T\varphi(x)\varphi^{+}(y)|0\rangle = i\theta(x^0-y^0)\Delta^{-}(x-y) - i\theta(y^0-x^0)\,\Delta^{+}(x-y)$$
$$= i\Delta_F(x).$$

É por isso que Δ_F se conhece como a função de Green de Feynman.

3.1.9. Simetrias Discretas

Como a^+ e b^+ geram partículas de cargas distintas, definimos o operador de conjugação de carga extendendo (por segunda quantização)

$$C a^{+}(\vec{p})|0\rangle = b^{+}(\vec{p})|0\rangle,$$
$$C b^{+}(\vec{p})|0\rangle = a^{+}(\vec{p})|0\rangle.$$

Como leva uma base a outra, C é unitario.

Equivalentemente

$$\varphi^{*} = C\varphi C^{-1} \iff \begin{cases} b^{+}(\vec{p}) = C a^{+}(\vec{p}) C^{-1} \iff b(\vec{p}) = C\, a(\vec{p})\, C^{-1}, \\ a^{+}(\vec{p}) = C b^{+}(\vec{p}) C^{-1}. \end{cases}$$

Observe que $C^2 = 1$. Logo $C = C^{-1} = C^{+}$.

Exercício 3.3.

Pege

$$C = \exp\left(i\frac{\pi}{2} \underbrace{\int \frac{d^3\vec{p}}{2E_{\vec{p}}} \left(b^+(\vec{p}) - a^+(\vec{p})\right)\left(b(\vec{p}) - a(\vec{p})\right)}_{\hat{C}.} \right)$$

Se tem

$$[\hat{C}, a(\vec{q})] = \int \frac{d^3\vec{p}}{2E_{\vec{p}}} \left(-[a^+(\vec{p}), a(\vec{q})]\,(b(\vec{p}) - a(\vec{p}))\right)$$

$$= b(\vec{q}) - a(\vec{q})$$

$$[\hat{C}, b(\vec{q})] = a(\vec{q}) - b(\vec{q})$$

$$[\hat{C}, [\hat{C}, a(\vec{q})]] = a(\vec{q}) - b(\vec{q}) - \left(b(\vec{q}) - a(\vec{q})\right) = 2(a(\vec{q}) - b(\vec{q}))$$

$$[\hat{C}, [\hat{C}, b(\vec{q})]] = b(\vec{q}) - a(\vec{q}) - (a(\vec{q}) - b(\vec{q})) = 2(b(\vec{q}) - a(\vec{q}))$$

$$\vdots$$

$$\underbrace{[\hat{C}, \ldots, [\hat{C}}_{n}, a(\vec{q})]] = (-2)^{n-1}\left(b(\vec{q}) - a(\vec{q})\right) = -[\hat{C}, \ldots, [\hat{C}, b(\vec{q})]].$$

Logo, usando a formula de Baker-Campbell-Hausdorff

$$C a(\vec{q}) C^{-1} = a(\vec{q}) + \sum_{n=1}^{\infty} \frac{(i\frac{\pi}{2})^n}{n!} [\hat{C}, \ldots, [\hat{C}, a(\vec{q})]]$$

$$= a(\vec{q}) + \sum_{n=1}^{\infty} \frac{(i\pi/2)^n}{n!} (-2)^{n-1} \left(b(\vec{q}) - a(\vec{q})\right)$$

$$= a(\vec{q}) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-i\pi)^n}{n!} \left(b(\vec{q}) - a(\vec{q})\right) =$$

$$= a(\vec{q}) - \frac{1}{2}(e^{-i\pi} - 1)\,(b(\vec{q}) - a(\vec{q}))$$

$$= a(\vec{q}) + (b(\vec{q}) - a(\vec{q})) = b(\vec{q}),$$

e, devido a simetria

$$C b(\vec{q}) C^{-1} = a(\vec{q}).$$

Logo C é de fato o operador de conjugação de carga.

Além, se tem claramente

$$CHC^{-1} = H$$
$$CP^{k}C^{-1} = P^{k}$$
$$C j_{\mu}(t,\vec{x}) C^{-1} = -j_{\mu}(t,\vec{x}).$$

Definimos o operador de paridade $P|\vec{p};\rangle = |-\vec{p};\rangle$, $P|\,;\vec{p}\rangle = |\,;-\vec{p}\rangle$. Equivalentemente,

$$P\varphi(\vec{x},t)P^{-1} = \varphi(-\vec{x},t) \Leftrightarrow P a(\vec{p})P^{-1} = a(-\vec{p}) \quad \text{e} \quad P b(\vec{p}) P^{-1} b(-\vec{p}).$$

É claro que $P = P^{-1} = P^{+}$, $PHP^{-1} = H$, $PP^{K}P^{-1} = -P^{K}$, e $P\vec{j}(t,\vec{x})P^{-1} = -\vec{j}(t,-\vec{x})$, $PQP^{-1} = Q$. Do mesmo jeito anterior

$$P = \exp\left(-\frac{i\pi}{2}\int \frac{d^3\vec{p}}{2E_{\vec{p}}}\left(a^{+}(\vec{p})a(\vec{p}) + b^{+}(\vec{p})b(\vec{p}) - a^{+}(\vec{p})a(-\vec{p}) - b^{+}(\vec{p})b(-\vec{p})\right)\right).$$

Finalmente, queremos um operador de inverção temporal T. Suponga que T é unitario e $T\varphi(t,\vec{x})T = \varphi(-t,\vec{x})$. Então se tem

$$[\varphi(-t,\vec{x}), THT^{-1}] = [T\varphi(t,\vec{x})T^{-1}, THT^{-1}] = T[\varphi(t,\vec{x}), H]T^{-1}$$

$$= T\, i\partial_0 \varphi(t,\vec{x}) T^{-1} = i\frac{\partial}{\partial t}\left(T\varphi(t,\vec{x})T^{-1}\right) = i\frac{\partial}{\partial t}\left(\varphi(-t,\vec{x})\right)$$

$$= -i\partial_0 \varphi(-t,\vec{x}) = -[\varphi(-t,\vec{x}), H].$$

Logo $THT^{-1} = -H$. Se $|E\rangle$ tem energía E,

$$HT|E\rangle = -TH|E\rangle = -ET|E\rangle.$$

Isso é uma contradição pois H é positivo. Concluimos, segum o teorema de Wigner, que T tem que ser antiunitario. Seja $\bar{\mathfrak{F}}$ o espaço conjugado de $\mathfrak{F}$. Então se tem um operador lineal $V = TK$ onde K e a identidad $\mathfrak{F} \longrightarrow \bar{\mathfrak{F}}$. É lineal pois

$$V(\alpha|\psi\rangle) = TK(\alpha|\psi\rangle) = T\alpha^*|\psi\rangle = \alpha T|\psi\rangle$$

Se tem

$$Tj_0(\vec{x},t)T^{-1} = T(i\varphi^* \overleftrightarrow{\partial_0}\varphi)(\vec{x},t)T^{-1} = +i\varphi^*(-t,\vec{x})\,\partial_0\varphi(-t,\vec{x})$$

$$= j_0(\vec{x},-t),$$

$$T\vec{j}(t,\vec{x})T^{-1} = -\vec{j}(-t,\vec{x}),$$
$$T\vec{P}T^{-1} = -\vec{P}.$$

3.2. Campo de Dirac

Agora vamos a voltar a fazer a discução anterior para o caso de um campo que satisfaz a ecuação de Dirac

$$(i\partial\!\!\!/ - M)\psi = 0.$$

3.2.1. Soluções da ecuação de Dirac

Comencemos observando que

$$\partial\!\!\!/^2 = \partial^\mu \gamma_\mu \partial^\nu \gamma_\nu = \frac{1}{2}\partial^{(\mu}\partial^{\nu)}\gamma_\mu\gamma_\nu = \frac{1}{2}\partial^\mu\partial^\nu \gamma_{(\mu}\gamma_{\nu)}$$
$$= \frac{1}{2}\partial^\mu\partial^\nu \{\gamma_\mu,\gamma_\nu\} = \frac{1}{\cancel{2}}\partial^\mu\partial^\nu \cancel{2} g_{\mu\nu} = \partial^\mu\partial_\mu = \partial^2.$$

Logo

$$0 = (i\partial\!\!\!/ + M)(i\partial\!\!\!/ - M)\psi = (-\partial^2 - M^2)\psi = -(\partial^2 + M^2)\psi.$$

Então ψ satisfaz a ecuação de Klein-Gordon com massa M. Logo,

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\, 2E_{\vec{p}}} \left(u(\vec{p})e^{-ip\cdot x} + v(\vec{p})e^{ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}}.$$

Então, para que ψ satisfaga a ecuação de Dirac

$$0=(i\partial\!\!\!/ - M)\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\, 2E_{\vec{p}}} \Big(\big(i\gamma^\mu(-ip_\mu) - M\big) u(\vec{p})\, e^{-ip\cdot x} + \big(i\gamma^\mu(ip_\mu) - M\big) v(\vec{p})\, e^{ip\cdot x}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\, 2E_{\vec{p}}} \Big((p\!\!\!/ - M)\, u(\vec{p})\, e^{-ip\cdot x} - (p\!\!\!/ + M)\, v(\vec{p})\, e^{ip\cdot x} \Big)\Big|_{p^0 = E_{\vec{p}}},$$

o equivalentemente

$$(p\!\!\!/ - M)\, u(\vec{p}) = 0 = (p\!\!\!/ + M)\, v(\vec{p})$$

onde $p\!\!\!/ := \cancel{(E_{\vec{p}}, \vec{p})}$. Logo $u(\vec{p})$ é um autovetor de $p\!\!\!/$ com autovalor M e $v(\vec{p})$ também com autovalor $-M$. Issos são os unicos autovalores de $p\!\!\!/$ pois

$$p\!\!\!/^2 = p^\mu \gamma_\mu p^\nu \gamma_\nu = \frac{1}{2} p^{(\mu} p^{\nu)} \gamma_\mu \gamma_\nu = \frac{1}{2} p^\mu p^\nu \{\gamma_\mu, \gamma_\nu\} = p^2.$$

Para hallar uma base de $\mathbb{C}^4 = \mathrm{Ker}(p\!\!\!/ - M) \oplus \mathrm{Ker}(p\!\!\!/ + M)$, empezamos com $\vec{p} = \vec{0}$. Então $p\!\!\!/ = M\gamma_0$. Logo $\mathrm{Ker}(0\!\!\!/ - M) = \mathrm{span}\{u^{(1)}(\vec{0}), u^{(2)}(\vec{0})\}$ onde $u^{(r)}(\vec{0}) := (\eta^{(r)}, 0, 0)$ e $\{\eta^{(1)}, \eta^{(2)}\}$ é a base canónica de $\mathbb{C}^2$. Similarmente, como

$$i\sigma^2 \eta^{(1)} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$i\sigma^2 \eta^{(2)} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

se tem $\mathrm{Ker}(p\!\!\!/ + M) = \mathrm{span}\{v^{(1)}(\vec{0}), v^{(2)}(\vec{0})\}$, com $v^{(r)}(\vec{0}) = (0, 0, i\sigma^2 \eta^{(r)})$.

Agora bem

$$(\not{p} \pm M)(\not{p} \mp M) = \not{p}^2 - M^2 = (p^2 - M^2)\Big|_{p^0 = E_{\vec{p}}} = 0.$$

Logo, definendo

$$u^{(r)}(\vec{p}) = C(\vec{p})\,(\not{p} + M)\,u^{(r)}(\vec{0}),$$

$$v^{(r)}(\vec{p}) = \tilde{C}(\vec{p})\,(\not{p} - M)\,v^{(r)}(\vec{0}),$$

com $r \in \{1,2\}$, se tem $u^{(r)}(\vec{p}) \in \mathrm{Ker}(\not{p} - M)$ e $v^{(r)}(\vec{p}) \in \mathrm{Ker}(\not{p} + M)$.

Agora bem,

$$0 = \overline{(\not{p} - M)u^{(r)}(\vec{p})} = u^{(r)}(\vec{p})^{+}(p^{\mu}\gamma_{\mu}^{+} - M)\,\gamma^{0}\Big|_{p^0 = E_{\vec{p}}}$$

$$= \bar{u}^{(r)}(\vec{p})\,(p^{\mu}\gamma_0\gamma_{\mu}^{+}\gamma_0 - M)\Big|_{p^0 = E_{\vec{p}}} = \bar{u}^{(r)}(\vec{p})(\not{p} - M)$$

$$0 = \bar{v}^{(r)}(\vec{p})(\not{p} + M),$$

Além, como $u^{(r)}(\vec{0}) = \gamma^0 u^{(r)}(\vec{0})$ se tem

$$\bar{u}^{(r)}(\vec{0})\gamma^K u^{(s)}(\vec{0}) = \bar{u}^{(r)}(\vec{0})\gamma^K\gamma^0 u^{(s)}(\vec{0}) = -\bar{u}^{(r)}(\vec{0})\gamma^0\gamma^K u^{(s)}(\vec{0})$$

$$= -\bar{u}^{(r)}(\vec{0})\gamma^K u^{(s)}(\vec{0}) = 0.$$

Logo

$$\bar{u}^{(r)}(\vec{p})u^{(s)}(\vec{p}) = C(\vec{p})\,\bar{u}^{(r)}(\vec{p})(\not{p} + M)\,u^{(s)}(\vec{0}) = 2MC(\vec{p})\,\bar{u}^{(r)}(\vec{p})\,u^{(s)}(\vec{0})$$

$$= 2M|C(\vec{p})|^2\,\bar{u}^{(r)}(\vec{0})\,(\not{p} + M)\,u^{(s)}(\vec{0})$$

$$= 2M|C(\vec{p})|^2\,\bar{u}^{(r)}(\vec{0})\,(E_{\vec{p}} - \vec{p}\cdot\vec{\gamma} + M)\,u^{(s)}(\vec{0})$$

$$= 2M(M + E_{\vec{p}})|C(\vec{p})|^2\,\bar{u}^{(r)}(\vec{0})u^{(s)}(\vec{0}).$$

Pelo tanto, escolhendo $C(\vec{p}) = \dfrac{1}{\sqrt{2M(M+E_{\vec{p}})}}$ se tem

$$\bar{u}^{(r)}(\vec{p})\, u^{(s)}(\vec{p}) = \delta_{rs}.$$

Escolhendo $\tilde{C}(\vec{p}) = -C(\vec{p})$ se tem $\bar{v}^{(r)}(\vec{p})v^{(s)}(\vec{p}) = -\delta_{rs}$ do mesmo modo. Aún mais,

$$0 = \bar{u}^{(r)}(\vec{p})\{-\not{p} + M,\, \gamma^0\} u^{(s)}(\vec{p}) = \bar{u}^{(r)}(\vec{p})\left(-p_\mu 2g^{\mu 0} + 2M\gamma^0\right) u^{(s)}(\vec{p})\Big|_{p^0 = E_{\vec{p}}}$$

$$= -2E_{\vec{p}}\, \bar{u}^{(r)}(\vec{p})\, u^{(s)}(\vec{p}) + 2M\, u^{+(r)}(\vec{p})\, u^{(s)}(\vec{p})$$

$$= -2M\left(u^{(r)}(\vec{p})^{+} u^{(s)}(\vec{p}) - \frac{E_{\vec{p}}}{M}\delta_{rs}\right),$$

i.e.

$$u^{(r)}(\vec{p})^{+} u^{(s)}(\vec{p}) = \frac{E_{\vec{p}}}{M}\delta_{rs}.$$

Do mesmo modo (Exercício 4.2)

$$0 = \bar{v}^{(r)}(\vec{p})\{\not{p} + M,\gamma^0\} v^{(s)}(\vec{p}) = \bar{v}^{(r)}(\vec{p})\left(2E_{\vec{p}} + 2M\gamma^0\right) v^{(s)}(\vec{p})$$

$$= -2E_{\vec{p}}\,\delta_{rs} + 2M\, v^{(r)}(\vec{p})^{+} v^{(s)}(\vec{p}),$$

i.e.

$$v^{(r)}(\vec{p})^{+} v^{(s)}(\vec{p}) = +\frac{E_{\vec{p}}}{M}\delta_{rs}$$

Em particular $\{u^{(1)}(\vec{p}), u^{(2)}(\vec{p}), v^{(1)}(\vec{p}), v^{(2)}(\vec{p})\}$ é linealmente independente e, de fato, uma base de $\mathbb{C}^4$.

Agora bem

$$\not{p} = E_{\vec{p}}\gamma^0 - \vec{p}\cdot\vec{\gamma} = \begin{pmatrix} E_{\vec{p}}\sigma^0 & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E_{\vec{p}}\sigma^0 \end{pmatrix}.$$

Logo

$$u^{(r)}(\vec{p}) = \frac{1}{\sqrt{2M(M+E_{\vec{p}})}} \begin{pmatrix} E_{\vec{p}}\sigma^0 + M & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E_{\vec{p}}\sigma^0 + M \end{pmatrix} \begin{pmatrix} \eta^{(r)} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2M(M+E_{\vec{p}})}} \begin{pmatrix} (M+E_{\vec{p}})\eta^{(r)} \\ (\vec{p}\cdot\vec{\sigma})\eta^{(r)} \end{pmatrix},$$

$$v^{(r)}(\vec{p}) = \frac{-1}{\sqrt{2M(M+E_{\vec{p}})}} \begin{pmatrix} E_{\vec{p}}\sigma^0 - M & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E_{\vec{p}}\sigma^0 - M \end{pmatrix} \begin{pmatrix} 0 \\ i\sigma^2\eta^{(r)} \end{pmatrix} = \frac{1}{\sqrt{2M(M+E_{\vec{p}})}} \begin{pmatrix} (\vec{p}\cdot\vec{\sigma})i\sigma^2\eta^{(r)} \\ (M+E_{\vec{p}})i\sigma^2\eta^{(r)} \end{pmatrix}.$$

Exercício 4.1.

Considere $\vec{p} = p^3\hat{k}$. A velocidade asoçada é $\vec{v} = v^3\hat{k}$ com

$$p^3 = \frac{m v^3}{\sqrt{1-(v^3)^2}}.$$

Logo

$$(p^3)^2\left(1-(v^3)^2\right) = m^2(v^3)^2$$

$$v^3 = \frac{p_3}{\sqrt{m^2+p_3^2}} = \frac{p_3}{E_{\vec{p}}},$$

Então considere o boost de Lorentz do momento $\vec{0}$ a momento $\vec{p}$

$$\Lambda = \exp\left(\frac{i\theta}{2}\,\sigma_{03}\right) \quad \text{com} \quad \sigma_{03} = -i\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \quad \text{e} \quad \tanh\theta = v_3$$

Como

$$(i\sigma_{03})^2 = +\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} = +\begin{pmatrix} (\sigma^3)^2 & 0 \\ 0 & (\sigma^3)^2 \end{pmatrix} = I_4,$$

se tem

$$\Lambda = \sum_{n=0}^{\infty} \frac{(\theta/2)^n}{n!}(i\sigma_{03})^n = \sum_{n=0}^{\infty}\frac{(\theta/2)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(\theta/2)^{n+1}}{(2n+1)!}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$

$$= \cosh(\theta/2) + \sinh(\theta/2)\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$

Agora Lembremos umas fórmolas hipergeometricas,

$$\cosh(x)\cosh(y) = \frac{e^{x+y} + e^{x-y} + e^{y-x} + e^{-(x+y)}}{4},$$

$$\sinh(x)\sinh(y) = \frac{e^{x+y} - e^{x-y} - e^{y-x} + e^{-(x+y)}}{4},$$

Logo

$$\cosh(x)\cosh(y) + \sinh(x)\sinh(y) = \frac{\cancel{2}e^{x+y} + \cancel{2}e^{-(x+y)}}{\cancel{4}_2} = \cosh(x+y),$$

e,

$$\cosh(x)^2 - \sinh(x)^2 = \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2}{4} = 1.$$

Em particular

$$\cosh(2x) = \cosh(x)^2 + \sinh(x)^2 = 2\cosh(x)^2 - 1$$

e

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x) + 1}{2}},$$

lembrando que $\cosh(\mathbb{R}) \subseteq (0, \infty)$. Similarmente

$$\cosh(2x) = 1 + 2\sinh(x)^2$$

e

$$\sinh\left(\frac{x}{2}\right) = \operatorname{sgn}(x)\sqrt{\frac{\cosh(x) - 1}{2}}.$$

Agora bem

$$1 - (v_3)^2 = 1 - \tanh(\theta)^2 = \frac{1}{\cosh(\theta)^2},$$

pelo que $\cosh(\theta) = \dfrac{\operatorname{sgn}\theta}{\sqrt{1 - v_3^2}} = \sqrt{\dfrac{E_{\vec{p}}^2}{E_{\vec{p}}^2 - \vec{p}^{\,2}}} = \dfrac{E_{\vec{p}}}{M}$ e

$$\cosh(\theta/2) = \sqrt{\frac{E_{\vec{p}} + M}{2M}} = \frac{M + E_{\vec{p}}}{\sqrt{2M(M + E_{\vec{p}})}},$$

$$\sinh(\theta/2) = \sqrt{\frac{E_{\vec{p}} - M}{2M}} = \sqrt{\frac{E_{\vec{p}}^2 - M^2}{2M(E_{\vec{p}} + M)}} = \frac{p^3}{\sqrt{2M(M + E_{\vec{p}})}}.$$

Logo

$$\Lambda = \frac{1}{\sqrt{2M(M+E_{\vec{p}})}} \begin{pmatrix} (M+E_{\vec{p}})\sigma^0 & p^3\sigma^3 \\ p^3\sigma^3 & (M+E_{\vec{p}})\sigma^0 \end{pmatrix}.$$

Logo $\quad \Lambda u^{(r)}(\vec{0}) = u^{(r)}(\vec{p}) \quad$ e $\quad \Lambda v^{(r)}(\vec{0}) = v^{(r)}(\vec{p})$.

3.2.2. Operadores de Projeção

Pegue

$$\Lambda_+(\vec{p}) = \sum_{r=1}^{2} u^{(r)}(\vec{p})\bar{u}^{(r)}(\vec{p}) = \frac{1}{2M(M+E_{\vec{p}})} \sum_{r=1}^{2} (\not{p}+M)u^{(r)}(\vec{0})\bar{u}^{(r)}(\vec{0})(\not{p}+M).$$

Se tem

$$\sum_{r=1}^{2} u^{(r)}(\vec{0})\bar{u}^{(r)}(\vec{0}) = \sum_{r=1}^{2} \begin{pmatrix} \eta^{(r)} \\ 0 \end{pmatrix} \begin{pmatrix} \eta^{(r)+} & 0 \end{pmatrix} = \sum_{r=1}^{2} \begin{pmatrix} \eta^{(r)}\eta^{(r)+} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{I+\gamma^0}{2}.$$

Logo

$$\Lambda_+(\vec{p}) = \frac{(\not{p}+M)(I+\gamma^0)(\not{p}+M)}{4M(M+E_{\vec{p}})} = \frac{\not{p}^2 + M^2 + 2\not{p}M + \not{p}\gamma^0\not{p} + M\{\not{p},\gamma^0\} + \gamma^0 M^2}{4M(M+E_{\vec{p}})}$$

$$= \frac{2M^2 + 2\not{p}M - \cancel{\not{p}^2\gamma^0} + \not{p}\{\gamma^0,\not{p}\} + M\{\not{p},\gamma^0\} + \gamma^0\cancel{M^2}}{4M(M+E_{\vec{p}})}$$

$$= \left.\frac{2M(M+\not{p}) + (M+\not{p})p_\mu 2g^{0\mu}}{4M(M+E_{\vec{p}})}\right|_{p^0=E_{\vec{p}}} = \frac{\cancel{2}\cancel{(M+E_{\vec{p}})}(\not{p}+M)}{\cancel{4}_2 M\cancel{(M+E_{\vec{p}})}}$$

$$= \frac{\not{p}+M}{2M}.$$

Do mesmo modo,

$$\sum_{r=1}^{2} v^{(r)}(\vec{0})\,\bar{v}^{(r)}(\vec{0}) = -\sum_{r=1}^{2} \begin{pmatrix} 0 \\ i\sigma^2\eta^{(r)} \end{pmatrix} \begin{pmatrix} 0 & -i\,\eta^{(r)+}\sigma^2 \end{pmatrix}$$

$$= -\sum_{r=1}^{2} \begin{pmatrix} 0 & 0 \\ 0 & +\sigma^2\eta^{(r)}\eta^{(r)+}\sigma^2 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & 0 \\ 0 & +(\sigma^2)^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} = -\frac{I_4 - \gamma^0}{2},$$

e definimos

$$\Lambda_-(\vec{p}) = -\sum_{r=1}^{2} v^{(r)}(\vec{p})\,\bar{v}^{(r)}(\vec{p}) = -\frac{1}{2M(M+E_{\vec{p}})}\sum_{r=1}^{2} (\not{p}-M)\,v^{(r)}(\vec{0})\,\bar{v}^{(r)}(\vec{0})\,(\not{p}-M)$$

$$= +\frac{(\not{p}-M)(I_4-\gamma^0)(\not{p}-M)}{4M(M+E_{\vec{p}})} = +\frac{\not{p}^2 - 2\not{p}M + M^2 - \not{p}\gamma^0\not{p} + M\{\not{p},\gamma^0\} - \gamma^0 M^2}{4M(M+E_{\vec{p}})}$$

$$= +\frac{2M(M-\not{p}) + \cancel{\not{p}^2\gamma^0} + (M-\not{p})\{\not{p},\gamma^0\} - \cancel{\gamma^0 M^2}}{4M(M+E_{\vec{p}})}$$

$$= +\frac{\cancel{2}(M-\not{p})\cancel{(M+E_{\vec{p}})}}{{}_{2}\cancel{4}M\cancel{(M+E_{\vec{p}})}} = -\frac{\not{p}-M}{2M}.$$

Os dois são projeções pois

$$\Lambda_+(\vec{p})^2 = \frac{(\not{p}+M)^2}{4M^2} = \frac{\not{p}^2 + 2\not{p}M + M^2}{4M^2} = \frac{2M(M+\not{p})}{4M^2} = \frac{\not{p}+M}{2M} = \Lambda_+(\vec{p}),$$

$$\Lambda_-(\vec{p})^2 = \frac{(\not{p}-M)^2}{4M^2} = \frac{\not{p}^2 - 2\not{p}M + M^2}{4M^2} = \frac{2M(M-\not{p})}{4M^2} = -\frac{\not{p}-M}{2M} = \Lambda_-(\vec{p}).$$

É claro que $\Lambda_+(\vec{p})$ projecta sobre $\mathrm{Ker}(\not{p}-M)$ ao largo de

$Ker(\not{p}+M)$ e o opuesto para $\Lambda_-(\vec{p})$. De fato,

$$\Lambda_+(\vec{p})\Lambda_-(\vec{p}) = -\frac{(\not{p}+M)(\not{p}-M)}{4M^2} = 0 = \Lambda_-(\vec{p})\Lambda_+(\vec{p}) \quad \text{e}$$

$$\Lambda_+(\vec{p}) + \Lambda_-(\vec{p}) = \frac{\cancel{\not{p}}+M-(\cancel{\not{p}}-M)}{2M} = 1.$$

$Ker(\not{p}-M)$ e conhecido como as soluções de frecuençia positiva e momento $\vec{p}$. $Ker(\not{p}+M)$ é o de frecuençã negativa.

Aparte dos operadores de frecuençia também temos operadores de spín. Lembre que tanto na representação $(1/2,0)$ como $(0,1/2)$ o gerador de rotações pelo eje z é $\frac{1}{2}\sigma_z$. Logo na representação de Dirac $(1/2,0)\oplus(0,1/2)$, os gerador destas rotações é

$$S_z = \left(\frac{1}{2}\sigma^3\right)\oplus\left(\frac{1}{2}\sigma^3\right) = \frac{1}{2}(\sigma^3\oplus\sigma^3) = \frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix} = \frac{1}{2}\Sigma_z.$$

Os vetores $u^{(1)}(\vec{0})$, $u^{(2)}(\vec{0})$, $v^{(1)}(\vec{0})$ e $v^{(2)}(\vec{0})$ são autovetores de S_z com autovalores $1/2$, $-1/2$, $-1/2$, $1/2$ resp. Logo

$$\frac{I+\Sigma_z}{2} = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

é o projetor sobre autovetores correspodentes aos autovalores $1/2$. A estos os chamoremos de spin $1/2$ no eje $+Oz$.

Observe que

$$\gamma_5\gamma^0\gamma^3 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}\begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\sigma^0 \\ \sigma^0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \Sigma_z .$$

Logo $\quad \Sigma_z = -\gamma_5\gamma_0\gamma_3 = \gamma_5\gamma_3\gamma_0 = \gamma_5 \not{\eta}\, \gamma_0 ,$

onde $\eta^\mu = (0,0,0,1)^\mu$. Definimos $\pi(\eta) = \dfrac{I + \gamma_5\not{\eta}}{2}$. Então

$$\pi(\eta)\, u^{(1)}(\vec{0}) = u^{(1)}(\vec{0}), \qquad \pi(\eta)\, u^{(2)}(\vec{0}) = 0,$$

$$\pi(\eta)\, v^{(1)}(\vec{0}) = v^{(1)}(\vec{0}), \qquad \pi(\eta)\, v^{(2)}(\vec{0}) = 0,$$

$$\pi(-\eta)\, u^{(1)}(\vec{0}) = 0, \qquad \pi(-\eta)\, u^{(2)}(\vec{0}) = u^{(2)}(\vec{0}),$$

$$\pi(-\eta)\, v^{(1)}(\vec{0}) = 0, \qquad \pi(-\eta)\, v^{(2)}(\vec{0}) = v^{(2)}(\vec{0}).$$

Em geral, dado $\eta^2 = -1$, $\pi(\eta)$ projecta sobre os estados de frecuença positiva, momento $\vec{p}$ com $p^\mu \eta_\mu \big|_{p^0 = E_{\vec{p}}} = 0$ e spin $1/2$ na direção $\vec{\eta}$ de frecuença positiva o direção opuesto para frecuença negativa. Em geral pegamos

$$\pi(\eta)\, u(\vec{p},\eta) = u(\vec{p},\eta)$$

$$\pi(\eta)\, v(\vec{p},\eta) = v(\vec{p},\eta).$$

3.2.3. Teoria de Fourier

A solução mais geral é então

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^{3/2} E_{\vec{p}}/M} \left(a^{(r)}(\vec{p}) u^{(r)}(\vec{p}) e^{-ip\cdot x} + b^{(r)}(\vec{p})^* v^{(r)}(\vec{p}) e^{ip\cdot x} \right)\Big|_{p^0=E}$$

O espinor adjunto fica então

$$\bar{\psi}(x) = \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^{3/2} E_{\vec{p}}/M} \left(b^{(r)}(\vec{p}) \bar{v}^{(r)}(\vec{p}) e^{-ip\cdot x} + a^{(r)}(\vec{p})^* \bar{u}^{(r)}(\vec{p}) e^{ip\cdot x} \right)\Big|_{p^0=E}$$

Estas expansões se podem invertir (sem trocar ordems de a e b em preparação para o caso quântico)

$$\int \frac{d^3\vec{x}}{(2\pi)^{3/2}} e^{ip\cdot x} \bar{u}^{(r)}(\vec{p}) \gamma^0 \psi(x)\Big|_{p^0=E_{\vec{p}}} =$$

$$\sum_{s=1}^{2} \int \frac{d^3\vec{x}\, d^3\vec{p}\,'}{(2\pi)^3 E_{\vec{p}\,'}/M} e^{i(p-p')\cdot x} a^{(s)}(\vec{p}\,') u^{+(r)}(\vec{p}) u^{(s)}(\vec{p}\,')\Big|_{p^0=E_{\vec{p}},\, p'^0=E_{\vec{p}\,'}} =$$

$$\sum_{s=1}^{2} \int \frac{d^3\vec{p}\,'}{E_{\vec{p}\,'}/M} \delta(\vec{p}-\vec{p}\,') a^{(s)}(\vec{p}) u^{+(r)}(\vec{p}) u^{(s)}(\vec{p}) = a^{(r)}(\vec{p}),$$

$$\int \frac{d^3\vec{x}}{(2\pi)^{3/2}} e^{ip\cdot x} \bar{\psi}(x) \gamma^0 v^{(r)}(\vec{p})\Big|_{p^0=E_{\vec{p}}} =$$

$$\sum_{s=1}^{2} \int \frac{d^3\vec{x}\, d^3\vec{p}\,'}{(2\pi)^3 E_{\vec{p}\,'}/M} e^{i(p-p')\cdot x} b^{(s)}(\vec{p}\,') v^{+(s)}(\vec{p}\,') v^{(r)}(\vec{p})\Big|_{p^0=E_{\vec{p}},\, p'^0=E_{\vec{p}\,'}} =$$

$$\sum_{s=1}^{2} \int \frac{d^3\vec{p}\,'}{E_{\vec{p}}/M} \delta(\vec{p}-\vec{p}\,') b^{(s)}(\vec{p}) v^{+(s)}(\vec{p}) v^{(r)}(\vec{p}) = b^{(r)}(\vec{p}).$$

Em termos da teoría de Fourier, podemos expresar as operadores do grupo de Poincaré. Para o Hamiltoniano temos

$$\Theta^{00} = \frac{i}{2}\bar{\psi}\gamma^0 \overleftrightarrow{\partial}_0 \psi - \mathcal{L},$$

Como ψ satisfaz a ecuaçã a de Dirac

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\overleftrightarrow{\not{\partial}}\psi - M\bar{\psi}\psi = \bar{\psi}\cancelto{0}{(i\not{\partial} - M)\psi} - \frac{i}{2}\partial_\mu\cancelto{0}{(\bar{\psi}\gamma^\mu\psi)} = 0,$$

pois a corrente $j^\mu = \bar{\psi}\gamma^\mu\psi$ é conservada. Logo

$$\Theta^{00} = \frac{i}{2}\bar{\psi}\gamma^0\overleftrightarrow{\partial}_0\psi.$$

Logo

$$H = \int d^3\vec{x}\,\Theta^{00} = \sum_{r,s=1}^{2}\int \frac{d^3\vec{x}\,d^3\vec{p}\,d^3\vec{p}\,'}{(2\pi)^3 E_{\vec{p}}E_{\vec{p}\,'}/M^2}\,\frac{i}{2}\times$$

$$\Big[\left(b^{(r)}(\vec{p})\bar{v}^{(r)}(\vec{p})e^{-ip\cdot x} + a^{(r)}(\vec{p})^*\bar{u}^{(r)}(\vec{p})e^{ip\cdot x}\right)\gamma^0\times$$

$$\left(-iE_{\vec{p}\,'}a^{(s)}(\vec{p}\,')u^{(s)}(\vec{p}\,')e^{-ip'\cdot x} + iE_{\vec{p}\,'}b^{(s)}(\vec{p}\,')^* v^{(s)}(\vec{p}\,')e^{ip'\cdot x}\right) -$$

$$\left(-iE_{\vec{p}}\,b^{(r)}(\vec{p})\bar{v}^{(r)}(\vec{p})e^{-ip\cdot x} + iE_{\vec{p}}\,a^{(r)}(\vec{p})^*\bar{u}^{(r)}(\vec{p})e^{ip\cdot x}\right)\gamma^0\times$$

$$\left(a^{(s)}(\vec{p}\,')u^{(s)}(\vec{p}\,')e^{-ip'\cdot x} + b^{(s)}(\vec{p}\,')^* v^{(s)}(\vec{p}\,')e^{ip'\cdot x}\right)\Big]\Big|_{p^0=E_{\vec{p}},\ p'^0=E_{\vec{p}\,'}}$$

$$= \sum_{r,s=1}^{2}\int \frac{d^3\vec{x}\,d^3\vec{p}\,d^3\vec{p}\,'}{(2\pi)^3 E_{\vec{p}}E_{\vec{p}\,'}/M^2}\,\frac{i}{2}\times\Big[-iE_{\vec{p}\,'}e^{-i(p+p')\cdot x}\,b^{(r)}(\vec{p})a^{(s)}(\vec{p}\,')v^{(r)}(\vec{p})^+u^{(s)}(\vec{p}\,')$$

$$-iE_{\vec{p}'}\, e^{i(p-p')\cdot x}\, a^{(r)}(\vec{p})^*\, a^{(s)}(\vec{p}')\, u^{(r)}(\vec{p})^\dagger\, u^{(s)}(\vec{p}')$$

$$+iE_{\vec{p}'}\, e^{i(p'-p)\cdot x}\, b^{(r)}(\vec{p})\, b^{(s)}(\vec{p}')^*\, v^{(r)}(\vec{p})^\dagger\, v^{(s)}(\vec{p}')$$

$$+iE_{\vec{p}'}\, e^{i(p+p')\cdot x}\, a^{(r)}(\vec{p})^*\, b^{(s)}(\vec{p}')^*\, u^{(r)}(\vec{p})^\dagger\, v^{(s)}(\vec{p}')$$

$$+iE_{\vec{p}}\, e^{-i(p+p')\cdot x}\, b^{(r)}(\vec{p})\, a^{(s)}(\vec{p}')\, v^{(r)}(\vec{p})^\dagger\, u^{(s)}(\vec{p}')$$

$$+iE_{\vec{p}}\, e^{i(p'-p)\cdot x}\, b^{(r)}(\vec{p})\, b^{(s)}(\vec{p}')^*\, v^{(r)}(\vec{p})^\dagger\, v^{(s)}(\vec{p}')$$

$$-iE_{\vec{p}}\, e^{i(p-p')\cdot x}\, a^{(r)}(\vec{p})^*\, a^{(s)}(\vec{p}')\, u^{(r)}(\vec{p})^\dagger\, u^{(s)}(\vec{p}')$$

$$-iE_{\vec{p}}\, e^{i(p+p')\cdot x}\, a^{(r)}(\vec{p})^*\, b^{(s)}(\vec{p}')^*\, u^{(r)}(\vec{p})^\dagger\, v^{(s)}(\vec{p}')\Big]_{p^0=E_{\vec{p}},\; p'^0=E_{\vec{p}'}}$$

$$= \sum_{r,s=1}^{2} \int \frac{d^3\vec{p}}{\cancel{E_{\vec{p}}}/M}\left(-\frac{\cancel{E_{\vec{p}}}}{2}\right)\Big[\cancel{-e^{-2iE_{\vec{p}}x^0}\, b^{(r)}(\vec{p})\, a^{(s)}(-\vec{p})\, v^{(r)}(\vec{p})^\dagger\, u^{(s)}(-\vec{p})}$$

$$- a^{(r)}(\vec{p})^*\, a^{(s)}(\vec{p})\, \frac{\cancel{E_{\vec{p}}}}{\cancel{M}}\,\delta_{rs}$$

$$+ b^{(r)}(\vec{p})\, b^{(s)}(\vec{p})^*\, \frac{\cancel{E_{\vec{p}}}}{\cancel{M}}\,\delta_{rs}$$

$$\cancel{+ e^{2iE_{\vec{p}}x^0}\, a^{(r)}(\vec{p})^*\, b^{(s)}(-\vec{p})^*\, u^{(r)}(\vec{p})^\dagger\, v^{(s)}(-\vec{p})}$$

$$\cancel{+ e^{-2iE_{\vec{p}}x^0}\, b^{(r)}(\vec{p})\, a^{(s)}(-\vec{p})\, v^{(r)}(\vec{p})^\dagger\, u^{(s)}(-\vec{p})}$$

$$+ b^{(r)}(\vec{p})\, b^{(s)}(\vec{p})^*\, \frac{\cancel{E_{\vec{p}}}}{\cancel{M}}\,\delta_{rs}$$

$$- a^{(r)}(\vec{p})^*\, a^{(s)}(\vec{p})\, \frac{\cancel{E_{\vec{p}}}}{\cancel{M}}\,\delta_{rs}$$

$$\cancel{- e^{2iE_{\vec{p}}x^0}\, a^{(r)}(\vec{p})^*\, b^{(s)}(-\vec{p})^*\, u^{(r)}(\vec{p})^\dagger\, v^{(s)}(-\vec{p})}$$

$$= \sum_{r=1}^{2} \int d^3\vec{p}\, M\left(a^{(r)}(\vec{p})^*\, a^{(s)}(\vec{p}) - b^{(r)}(\vec{p})\, b^{(s)}(\vec{p})^*\right).$$

Para o momento temos

$$\Theta^{0i} = \frac{i}{2}\left(\bar\psi\gamma^0\partial^i\psi - \partial^i\bar\psi\gamma^0\psi\right)$$

$$= i\bar\psi\gamma^0\partial^i\psi - \frac{i}{2}\partial^i(\bar\psi\gamma^0\psi).$$

Como

$$\vec P_R = \int_R d^3\vec x\,\vec\Theta^{0\cdot}(t,\vec x) = -\int_R d^3\vec x\; i\bar\psi(t,\vec x)\gamma^0\vec\nabla\psi(t,\vec x) + \frac{i}{2}\int_R d^3\vec x\;\vec\nabla(\bar\psi\gamma^0\psi)(t,\vec x)$$

$$= -\int_R d^3\vec x\; i\bar\psi(t,\vec x)\gamma^0\vec\nabla\psi(t,\vec x) + \frac{i}{2}\int_{\partial R} d\vec S\;\bar\psi(t,\vec x)\gamma^0\psi(t,\vec x),$$

podemos tomar $\vec P = \int d^3\vec x\;\vec{\mathcal P}(t,\vec x)$ com $\vec{\mathcal P}(t,\vec x) = -i\bar\psi(t,\vec x)\gamma^0\vec\nabla\psi(t,\vec x)$.

Em geral pegamos $\mathcal P^\mu = i\bar\psi\gamma^0\partial^\mu\psi$. Logo

$$\vec P(t,\vec x) = -\sum_{r,s=1}^{2}\int\frac{d^3\vec x\,d^3\vec p\,d^3\vec p'}{(2\pi)^3 E_{\vec p}E_{\vec p'}/M^2}\, i\left(b^{(r)}(\vec p)\bar v^{(r)}(\vec p)e^{-ip\cdot x} + a^{(r)}(\vec p)^*\bar u^{(r)}(\vec p)e^{ip\cdot x}\right)\times$$

$$\gamma^0\left(i\vec p' a^{(s)}(\vec p')u^{(s)}(\vec p')e^{-ip'\cdot x} - i\vec p' b^{(s)}(\vec p')^* v^{(s)}(\vec p')e^{ip'\cdot x}\right)\Big|_{p^0=E_{\vec p},\,p'^0=E_{\vec p'}}$$

$$= -\sum_{r,s=1}^{2}\int\frac{d^3\vec x\,d^3\vec p\,d^3\vec p'}{(2\pi)^3 E_{\vec p}E_{\vec p'}/M^2}\, i\Big(i\vec p' e^{-i(p+p')\cdot x} b^{(r)}(\vec p)a^{(s)}(\vec p')v^{(r)}(\vec p)^\dagger u^{(s)}(\vec p')$$

$$+ i\vec p' e^{i(p-p')\cdot x} a^{(r)}(\vec p)^* a^{(s)}(\vec p')u^{(r)}(\vec p)^\dagger u^{(s)}(\vec p')$$

$$- i\vec p' e^{i(p'-p)\cdot x} b^{(r)}(\vec p)b^{(s)}(\vec p')^* v^{(r)}(\vec p)^\dagger v^{(s)}(\vec p')$$

$$- i\vec p' e^{i(p+p')\cdot x} a^{(r)}(\vec p)^* b^{(s)}(\vec p')^* u^{(r)}(\vec p)^\dagger v^{(s)}(\vec p')\Big)$$

$$= -\sum_{r,s=1}^{2}\int\frac{d^3\vec p}{E_{\vec p}^{\cancel{2}}/M^{\cancel{2}}}(-\vec p)\Big(e^{-2iE_{\vec p}x^0} b^{(r)}(\vec p)a^{(s)}(-\vec p)\underbrace{\cancel{v^{(r)}(\vec p)^\dagger u^{(s)}(-\vec p)}}_{0}$$

$$+ a^{(r)}(\vec p)^* a^{(s)}(\vec p)\cancel{E_{\vec p}}/\cancel{M}\,\delta_{rs}$$

$$- b^{(r)}(\vec p)b^{(s)}(\vec p)^*\cancel{E_{\vec p}}/\cancel{M}\,\delta_{rs}$$

$$- e^{2iE_{\vec p}x^0} a^{(r)}(\vec p)^* b^{(s)}(-\vec p)^* \underbrace{\cancel{u^{(r)}(\vec p)^\dagger v^{(s)}(-\vec p)}}_{0}\Big)$$

$$=+\sum_{r=1}^{2}\int\frac{d^3\vec{p}}{E_{\vec{p}}/M}\,\vec{p}\left(a^{(r)}(\vec{p})^{*}a^{(r)}(\vec{p})-b^{(r)}(\vec{p})b^{(s)}(\vec{p})^{*}\right).$$

Fizimos uso de que $v^{(r)}(\vec{p})^{+}u^{(s)}(-\vec{p})=0=u^{(r)}(\vec{p})^{+}v^{(s)}(-\vec{p})$. De fato

$$0=v^{(r)}(\vec{p})^{+}(-\cancel{(-\not p)}+M)u^{(s)}(-\vec{p})=\bar{v}^{(r)}(\vec{p})\gamma^{0}\left(-E_{\vec{p}}\gamma^{0}-\vec{p}\cdot\vec{\gamma}+M\right)u^{(s)}(-\vec{p})$$

$$=\bar{v}^{(r)}(\vec{p})\left(-E_{\vec{p}}\gamma^{0}+\vec{p}\cdot\vec{\gamma}+M\right)\gamma^{0}u^{(s)}(-\vec{p})=\bar{v}^{(r)}(\vec{p})(-\not p+M)\gamma^{0}u^{(s)}(-\vec{p})$$

$$=2M\,\bar{v}^{(r)}(\vec{p})\gamma^{0}u^{(s)}(-\vec{p})=2M\,v^{(r)}(\vec{p})^{+}u^{(s)}(-\vec{p}).$$

Para as transformações de Lorentz, temos o tensor densidade de momento angular

$$M^{\mu\nu\lambda}=-x^{\lambda}\Theta^{\mu\nu}+x^{\nu}\Theta^{\mu\lambda}+i\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\psi}I^{\nu\lambda}\psi-i\,\overline{I^{\nu\lambda}\psi}\,\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\bar{\psi}}$$

com $I^{\nu\lambda}=-\frac{1}{2}\sigma^{\nu\lambda}$. Logo

$$M^{\mu\nu\lambda}=-x^{\lambda}\Theta^{\mu\nu}+x^{\nu}\Theta^{\mu\lambda}+i\frac{i}{2}\bar{\psi}\gamma^{\mu}\left(-\frac{1}{2}\sigma^{\nu\lambda}\right)\psi+i\,\overline{\psi}\gamma^{0}\left(-\frac{1}{2}\sigma^{\nu\lambda}\right)^{+}\gamma^{0}\left(-\frac{i}{2}\gamma^{\mu}\psi\right)$$

$$=-x^{\lambda}\Theta^{\mu\nu}+x^{\nu}\Theta^{\mu\lambda}+\frac{1}{4}\bar{\psi}\gamma^{\mu}\sigma^{\nu\lambda}\psi+\frac{1}{4}\bar{\psi}\gamma^{0}(\sigma^{\nu\lambda})^{+}\gamma^{0}\gamma^{\mu}\psi.$$

Agora bem, $\sigma^{\mu\nu}=\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$. Logo

$$\gamma^{0}(\sigma^{\mu\nu})^{+}\gamma^{0}=-\frac{i}{2}\gamma^{0}[(\gamma^{\nu})^{+},(\gamma^{\mu})^{+}]\gamma^{0}=-\frac{i}{2}\left[\gamma^{0}(\gamma^{\nu})^{+}\gamma^{0},\gamma^{0}(\gamma^{\mu})^{+}\gamma^{0}\right]$$

$$=-\frac{i}{2}[\gamma^{\nu},\gamma^{\mu}]=\sigma^{\mu\nu}.$$

Então

$$M^{\mu\nu\lambda}=-x^{\lambda}\Theta^{\mu\nu}+x^{\nu}\Theta^{\mu\lambda}+\frac{1}{4}\bar{\psi}\{\gamma^{\mu},\sigma^{\nu\lambda}\}\psi.$$

Em particular, para a grandeza conservada

$$M^{0\mu\nu} = -x^\nu \Theta^{0\mu} + x^\mu \Theta^{0\nu} + \frac{1}{4}\bar{\psi}\{\gamma^0, \sigma^{\mu\nu}\}\psi$$

$$= -x^\nu P^\mu + x^\mu P^\nu + \frac{1}{4}\bar{\psi}\{\gamma^0, \sigma^{\mu\nu}\}\psi$$

$$= +\frac{i}{2} x^\nu \partial^\mu(\bar{\psi}\gamma^0\psi) - \frac{i}{2} x^\mu \partial^\nu(\bar{\psi}\gamma^0\psi)$$

$$= -x^\nu P^\mu + x^\mu P^\nu + \frac{1}{4}\bar{\psi}\{\gamma^0, \sigma^{\mu\nu}\}\psi$$
$$+ \frac{i}{2}\partial_\rho\left(x^\nu g^{\mu\rho}\bar{\psi}\gamma^0\psi - x^\mu g^{\nu\rho}\bar{\psi}\gamma^0\psi\right),$$

e

$$M^{\mu\nu} = \int d^3\vec{x}\, M^{0\mu\nu} = \int d^3\vec{x}\left(-x^\nu P^\mu + x^\mu P^\nu + \frac{1}{4}\bar{\psi}\{\gamma^0, \sigma^{\mu\nu}\}\psi\right).$$

Observe agora que

$$\gamma^0 \sigma^{ij} = \frac{i}{2}\gamma^0\left(\gamma^i\gamma^j - \gamma^j\gamma^i\right) = \frac{i}{2}\left(\gamma^i\gamma^j - \gamma^j\gamma^i\right)\gamma^0 = \sigma^{ij}\gamma^0,$$

Logo (error em Livro $\{\gamma^0, \sigma^{\mu\nu}\} = 2\gamma^0\sigma^{\mu\nu}$. De fato $\{\gamma^0, [\gamma^0, \gamma^i]\} = 0 \neq 2\gamma^i = \gamma^0[\gamma^0, \gamma^i]$)

$$M^{ij} = \int d^3\vec{x}\left(x^i P^j - x^j P^i + \frac{1}{2}\bar{\psi}\gamma^0\sigma^{\mu\nu}\psi\right).$$

Isso sugere a descomposição do momento angular numa parte orbital

$$L^{ij} = \int d^3\vec{x}\left(x^i P^j - x^j P^i\right)$$

e uma de spin

$$S^{ij} := \frac{1}{2}\int d^3\vec{x}\,(\bar{\psi}\,\gamma^0\sigma^{ij}\,\psi).$$

Agora estudemos M_{ij} no espaço de momentos. Observe

$x^i P^j = i\bar{\psi}\,\gamma^0\, x^i\partial^j\psi$. Logo

$$M^{ij} = \int d^3\vec{x}\;\bar{\psi}\,\gamma^0\left(ix^i\partial^j - ix^j\partial^i + \frac{1}{2}\sigma^{ij}\right)\psi.$$

Agora bem

$$\left(ix^i\partial^j - ix^j\partial^i + \frac{1}{2}\sigma^{ij}\right)e^{\pm ip\cdot x}$$

$$= \left(\mp x^i p^j \pm x^j p^i + \frac{1}{2}\sigma^{ij}\right)e^{\pm ip\cdot x}$$

$$= \Big(\underbrace{-ip^j\frac{\partial}{\partial p^i} + ip^i\frac{\partial}{\partial p^j}}_{:=\,n_p^{ij}} + \frac{1}{2}\sigma^{ij}\Big)e^{\pm ip\cdot x}$$

Logo

$$M^{ij} = \sum_{r,s=1}^{2}\int\frac{d^3\vec{x}\,d^3\vec{p}\,d^3\vec{p}\,'}{(2\pi)^3 E_{\vec{p}}E_{\vec{p}\,'}/M^2}\left(b^{(r)}(\vec{p})\,\bar{v}^{(r)}(\vec{p})\,e^{-ip\cdot x} + a^{(r)}(\vec{p})^*\,\bar{u}^{(r)}(\vec{p})\,e^{ip\cdot x}\right)\gamma^0$$

$$\Big[\Big(n_{p'}^{ij}(e^{-ip'\cdot x}) + \frac{1}{2}\sigma^{ij}e^{-ip'\cdot x}\Big)a^{(s)}(\vec{p}\,')\,u^{(s)}(\vec{p}\,') +$$

$$\Big(n_{p'}^{ij}(e^{ip'\cdot x}) + \frac{1}{2}\sigma^{ij}e^{ip'\cdot x}\Big)b^{(s)}(\vec{p}\,')^*\,v^{(s)}(\vec{p}\,')\Big]$$

$$= \sum_{r,s=1}^{2} \int \frac{d^3\vec{x}\, d^3\vec{p}\, d^3\vec{p}'}{(2\pi)^3\, E_{\vec{p}} E_{\vec{p}'}/M^2}$$

$$\Big[\, b^{(r)}(\vec{p})\, v^{(r)}(\vec{p})^{+} \Big(n^{ij}_{p'} \big(e^{-i(p+p')\cdot x} \big) + \tfrac{1}{2}\sigma^{ij} e^{-i(p+p')\cdot x} \Big) a^{(s)}(\vec{p}')\, u^{(s)}(\vec{p}')$$

$$+ a^{(r)}(\vec{p})^{*}\, u^{(r)}(\vec{p})^{+} \Big(n^{ij}_{p'} \big(e^{i(p-p')\cdot x} \big) + \tfrac{1}{2}\sigma^{ij} e^{i(p-p')\cdot x} \Big) a^{(s)}(\vec{p}')\, u^{(s)}(\vec{p}')$$

$$+ b^{(r)}(\vec{p})\, v^{(r)}(\vec{p})^{+} \Big(n^{ij}_{p'} \big(e^{i(p'-p)\cdot x} \big) + \tfrac{1}{2}\sigma^{ij} e^{i(p'-p)\cdot x} \Big) b^{(s)}(\vec{p}')^{*}\, v^{(s)}(\vec{p}')$$

$$+ a^{(r)}(\vec{p})^{*}\, u^{(r)}(\vec{p})^{+} \Big(n^{ij}_{p'}\, e^{i(p+p')\cdot x} + \tfrac{1}{2}\sigma^{ij} e^{i(p+p')\cdot x} \Big) b^{(s)}(\vec{p}')^{*}\, v^{(s)}(\vec{p}') \Big].$$

Mediante integração por partes

$$n^{ij}_{p'} \big(e^{\pm i(p \pm p')\cdot x} \big) \to -\big(e^{\pm i(p \pm p')\cdot x} \big)\, n^{ij}_{p'}.$$

Logo

$$M^{ij} = \sum_{r,s=1}^{2} \int \frac{d^3\vec{p}}{E^2_{\vec{p}}/M^2} \Big(b^{(r)}(-\vec{p})\, v^{(r)}(-\vec{p})^{+}\, m^{ij}_{p} \big(a^{(s)}(\vec{p})\, u^{(s)}(\vec{p}) \big)\, e^{-2iE_{\vec{p}} x^0}$$

$$+ a^{(r)}(\vec{p})^{*}\, u^{(r)}(\vec{p})^{+}\, m^{ij}_{p} \big(a^{(s)}(\vec{p})\, u^{(s)}(\vec{p}) \big)$$

$$+ b^{(r)}(\vec{p})\, v^{(r)}(\vec{p})^{+}\, m^{ij}_{p} \big(b^{(s)}(\vec{p})^{*}\, v^{(s)}(\vec{p}) \big)$$

$$+ a^{(r)}(-\vec{p})^{*}\, u^{(r)}(-\vec{p})^{+}\, m^{ij}_{p} \big(b^{(s)}(\vec{p})^{*}\, v^{(s)}(\vec{p}) \big)\, e^{2iE_{\vec{p}} x^0} \Big),$$

onde

$$m^{ij}_{\vec{p}} = i p^j \frac{\partial}{\partial p^i} - i p^i \frac{\partial}{\partial p^j} + \frac{1}{2}\sigma^{ij}$$

Agora bem

$$0 = v^{(r)}(-\vec{p})^{+}\, m^{ij}_{\vec{p}}\, (\not{p} - M)\, u^{(s)}(\vec{p}).$$

$$= v^{(r)}(-\vec{p})^{+}\, m^{ij}_{\vec{p}}\, (p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} - M)\, u^{(s)}(\vec{p})$$

$$= v^{(r)}(-\vec{p})^{+}\, (p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} - M)\, m^{ij}_{\vec{p}}\, u^{(s)}(\vec{p})$$

$$= \bar{v}^{(r)}(-\vec{p})\, (p^0\gamma^0 + \vec{p}\cdot\vec{\gamma} - M)\, \gamma^0 m^{ij}_{\vec{p}}\, u^{(s)}(\vec{p})$$

$$= \bar{v}^{(r)}(-\vec{p})\, (\cancel{(-\not{p})} - M)\, \gamma^0 m^{ij}_{\vec{p}}\, u^{(s)}(\vec{p})$$

$$= -2M\, \bar{v}^{(r)}(-\vec{p})\, \gamma^0 m^{ij}_{\vec{p}}\, u^{(s)}(\vec{p}) = -2M\, v^{(r)}(-\vec{p})^{+}\, m^{ij}_{\vec{p}}\, u^{(s)}(\vec{p})$$

pois

$$n^{ij}_{\vec{p}}\, p^0 = 0, \qquad \sigma^{ij}\gamma^0 = \gamma^0\sigma^{ij},$$

$$[\sigma^{ij}, \vec{p}\cdot\vec{\gamma}] = \frac{i}{2}\left[[\gamma^i, \gamma^j], \gamma^k\right] p^k$$

$$= -2i\,(g^{ik}\gamma^j - g^{jk}\gamma^i)\, p^k$$

$$= +2i\,(p^i\gamma^j - p^j\gamma^i),$$

$$n^{ij}_{\vec{p}}\, \vec{p}\cdot\vec{\gamma} = -i p^j\gamma^i + i p^i\gamma^k, \quad \text{isso é}$$

$$m^{ij}_{\vec{p}}\, \vec{p}\cdot\vec{\gamma} = \cancel{ip^j}\gamma^i - \cancel{ip^i\gamma^j} + i\,(\cancel{p^i\gamma^j} - p^j\gamma^i)$$

$$= \vec{p}\cdot\vec{\gamma}\, n^{ij}_{p} + \frac{1}{2}\vec{p}\cdot\vec{\gamma}\, \sigma^{ij} = \vec{p}\cdot\vec{\gamma}\, m^{ij}_{p}.$$

Pelo tanto

$$M^{ij} = \sum_{r,s=1}^{2} \int \frac{d^3\vec{p}}{(E_{\vec{p}}/M)^2} \Big(a^{(r)}(\vec{p})^{*}\, u^{(r)}(\vec{p})^{+}\, m^{ij}_{\vec{p}}\, \big(a^{(s)}(\vec{p})\, u^{(s)}(\vec{p})\big)$$

$$+\, b^{(r)}(\vec{p})\, v^{(r)}(\vec{p})^{+}\, m^{ij}_{p}\, \big(b^{(s)}(\vec{p})^{*}\, v^{(s)}(\vec{p})\big)\Big)$$

3.2.4. Relações de Anticommutação

Tenemos o Lagrangiano equivalente

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial} - M)\psi = \frac{i}{2}\,\bar{\psi}\,\overleftrightarrow{\not{\partial}}\,\psi - M\bar{\psi}\psi + \frac{i}{2}\,\partial_\mu(\bar{\psi}\gamma^\mu\psi).$$

O momento conjugado é

$$\Pi = \frac{\partial\mathcal{L}}{\partial\partial_0\psi} = i\bar{\psi}\gamma^0 = i\psi^+.$$

Em vez de relações de commutação, pegomos relações de anticommutação

$$\{\psi_\alpha(\vec{x},t),\, \psi_\beta^+(\vec{y},t)\} = -i\,\{\psi_\alpha(\vec{x},t), \Pi_\beta(\vec{y},t)\} = \delta(\vec{x}-\vec{y})\delta_{\alpha\beta}.$$

Por outro lado, para os operadores de Fourier,

$$\{a^{(r)}(\vec{p}),\, a^{(s)}(\vec{p}\,')^+\} = \int \frac{d^3\vec{x}\,d^3\vec{x}\,'}{(2\pi)^3}\, e^{ip\cdot x}\, \{\bar{u}^{(r)}(\vec{p})^*_\alpha\,\psi_\alpha(\vec{x},t),\, \psi_\beta(\vec{x}\,',t)^*\, u^{(s)}_\beta(\vec{p}\,')\}$$
$$e^{-ip'\cdot x'}$$

$$= \int \frac{d^3\vec{x}\,d^3\vec{x}\,'}{(2\pi)^3}\, e^{ip\cdot x} e^{-ip'\cdot x'}\, \frac{E_{\vec{p}}}{M}\,\delta_{rs}\,\delta(\vec{x}-\vec{x}\,')$$

$$= \frac{E_{\vec{p}}}{M}\,\delta_{rs}\,\delta(\vec{p}-\vec{p}\,') = \{b^{(r)}(\vec{p}),\, b^{(s)}(\vec{p}\,')^+\}.$$

com os demais nulos.

3.2.5. Representação do grupo de Poincaré.

Se tem

$$[H, \psi_\alpha(t,\vec{x})] = \int d^3\vec{y}\ \frac{i}{2}\ [(\psi^+)^\beta(t,\vec{y})\overleftrightarrow{\partial_0}\psi_\beta(t,\vec{y}),\ \psi_\alpha(t,\vec{x})].$$

Como $\partial_0\psi$ não aparece nos momentos conjugados para calcular issos commutadores temos que fazer uso da ecuação de Dirac. se tem

$$0 = (i\not\partial - M)\psi = i\gamma^0\partial_0\psi + i\vec{\gamma}\cdot\vec{\nabla}\psi - M\psi.$$

Logo

$$\partial_0\psi = -i\gamma^0(-i\vec{\gamma}\cdot\vec{\nabla}\psi + M\psi) = -\gamma^0\vec{\gamma}\cdot\vec{\nabla}\psi - i\gamma^0 M\psi.$$

Então

$$\{\partial_0\psi_\beta(t,\vec{y}), \psi_\alpha(t,\vec{x})\} = -(\gamma^0)_\beta{}^\gamma(\gamma^i)_\gamma{}^\delta\frac{\partial}{\partial y^i}\cancelto{0}{\{\psi_\delta(t,\vec{y}), \psi_\alpha(t,\vec{x})\}}$$

$$- i(\gamma^0)_\beta{}^\gamma M\cancelto{0}{\{\psi_\gamma(t,\vec{y}), \psi_\alpha(t,\vec{x})\}}$$

$$= 0.$$

Agora bem, se tem

$$[AB, C] = ABC - CAB = -ACB - CAB = -\{A, C\}B,$$

se $\{B, C\} = 0$. Logo

$$[H, \psi_\alpha(t,\vec{x})] = \frac{i}{2}\int d^3\vec{y}\left(-\{(\psi^+)^\beta(t,\vec{y}), \psi_\alpha(t,\vec{x})\}\,\partial_0\psi_\beta(t,\vec{y})\right.$$

$$\left. + \{\partial_0(\psi^+)^\beta(t,\vec{y}), \psi_\alpha(t,\vec{x})\}\,\psi_\beta(t,\vec{y})\right)$$

Para o segundo commutador, a gente tem

$$0 = \bar{\psi}(i\overleftarrow{\not{\partial}} + M) = i\partial_0\bar{\psi}\gamma^0 + i\vec{\nabla}\bar{\psi}\cdot\vec{\gamma} + M\bar{\psi}.$$

Logo

$$\partial_0\psi^+ = -i\left(-i\vec{\nabla}\psi^+\gamma^0\cdot\vec{\gamma} - M\psi^+\gamma^0\right) = -\vec{\nabla}\psi^+\gamma^0\cdot\vec{\gamma} + iM\psi^+\gamma^0,$$

e

$$\{\partial_0(\psi^+)^\beta(t,\vec{y}), \psi_\alpha(t,\vec{x})\} = -\frac{\partial}{\partial y^i}\{(\psi^+)^\delta(t,\vec{y}), \psi_\alpha(t,\vec{x})\}(\gamma^0)_\delta{}^\gamma(\gamma^i)_\gamma{}^\beta$$

$$+ iM\{(\psi^+)^\gamma(t,\vec{y}), \psi_\alpha(t,\vec{x})\}(\gamma^0)_\gamma{}^\beta$$

$$= -\frac{\partial}{\partial y^i}\delta(\vec{y}-\vec{x})(\gamma^0\gamma^i)_\alpha{}^\beta + iM\delta(\vec{y}-\vec{x})(\gamma^0)_\alpha{}^\beta.$$

Então

$$[H, \psi_\alpha(t,\vec{x})] = \frac{i}{2}\int d^3\vec{y}\left(-\partial_0\psi_\alpha(t,\vec{y}) + (\gamma^0\gamma^i)_\alpha{}^\beta\partial_i\psi_\beta(t,\vec{y}) + iM(\gamma^0)_\alpha{}^\beta\psi_\beta(t,\vec{y})\right)$$

$$\delta(\vec{y}-\vec{x})$$

$$= \frac{i}{2}\left(-\partial_0\psi_\alpha(t,\vec{x}) + \left(\gamma^0\vec{\gamma}\cdot\vec{\nabla}\psi(t,\vec{x}) + iM\gamma^0\psi(t,\vec{x})\right)_\alpha\right)$$

$$= -i\partial_0\psi_\alpha(t,\vec{x}).$$

Logo a ecuação de Heisenberg é satisfeita e H é o Hamiltoniano da nossa teoría. De modo mais facil se tem

$$[\vec{P}, \psi_\alpha(t,\vec{x})] = -i\int d^3\vec{y}\,[(\psi^+)^\beta(t,\vec{y})\,\vec{\nabla}\psi_\beta(t,\vec{y}), \psi_\alpha(t,\vec{x})]$$

$$= i\int d^3\vec{y}\,\{(\psi^+)^\beta(t,\vec{y}), \psi_\alpha(t,\vec{x})\}\,\vec{\nabla}\psi_\beta(t,\vec{y})$$

$$= i\,\vec{\nabla}\psi_\alpha(t,\vec{x}),$$

indicando que $\vec{P}$ é o operador de momento.

3.2.3. Adendo de carga

Esquecimos fazer este calculo na teoría de Fourier

$$Q = \int d^3\vec{x}\,(\bar{\psi}(\vec{x})\,\gamma^0\psi(\vec{x})) = \sum_{r,s=1}^{2}\int \frac{d^3\vec{x}\,d^3\vec{p}\,d^3\vec{p}\,'}{(2\pi)^3 E_{\vec{p}}E_{\vec{p}\,'}/M^2} \times$$

$$\left(b^{(r)}(\vec{p})\,\bar{v}^{(r)}(\vec{p})e^{-ip\cdot x} + a^{(r)}(\vec{p})^*\,\bar{u}^{(r)}(\vec{p})e^{ip\cdot x}\right)\times\gamma^0\times$$

$$\left(a^{(s)}(\vec{p}\,')\,u^{(s)}(\vec{p}\,')e^{-ip'\cdot x} + b^{(s)}(\vec{p}\,')^*\,v^{(s)}(\vec{p}\,')e^{ip'\cdot x}\right)\Big|_{p^0=E_{\vec{p}},\ p'^0=E_{\vec{p}\,'}}$$

$$= \sum_{r,s=1}^{2}\int\frac{d^3\vec{p}}{E_{\vec{p}}^2/M^2}\Big(e^{-2iE_{\vec{p}}x^0}\,b^{(r)}(\vec{p})\,a^{(r)}(-\vec{p})\,\cancel{\bar{v}^{(r)}(\vec{p})^+ u^{(s)}(-\vec{p})}^{\,0} +$$

$$a^{(r)}(\vec{p})^*\,a^{(s)}(\vec{p})\,\frac{E_{\vec{p}}}{M}\delta_{rs} + b^{(r)}(\vec{p})\,b^{(s)}(\vec{p})^*\,\delta_{rs}\frac{E_{\vec{p}}}{M} +$$

$$a^{(r)}(\vec{p})^*\,b^{(s)}(\vec{p})\,\cancel{\bar{u}^{(r)}(\vec{p})\,v^{(s)}(-\vec{p})}^{\,0}\,e^{2iE_{\vec{p}}x^0}\Big)$$

$$= \sum_{r=1}^{2}\int\frac{d^3\vec{p}}{E_{\vec{p}}/M}\left(a^{(r)}(\vec{p})^*\,a^{(r)}(\vec{p}) + b^{(r)}(\vec{p})\,b^{(r)}(\vec{p})^*\right).$$

3.2.6. Espaço de Fock

Como no caso de Klein-Gordon, temos

$$H = \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{E_{\vec{p}}/M} \, E_{\vec{p}} \left(a^{(r)}(\vec{p})^{+} a^{(r)}(\vec{p}) + b^{(r)}(\vec{p})^{+} b^{(r)}(\vec{p}) \right) - 2\int d^3\vec{p} \, E_{\vec{p}} \, \delta(\vec{0}).$$

Este Hamiltoniano não e definido positivo! O termo negativo $-\infty$ parece estar de acordo com a interpretação do mar de Dirac. Definendo o ordem normal deixondo os operadores de creação a izquerda (mais montendo os signos por coda troca), temos o novo Hamiltoniano

$$H = \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{E_{\vec{p}}/M} \, E_{\vec{p}} \left(a^{(r)}(\vec{p})^{+} a^{(r)}(\vec{p}) + b^{(r)}(\vec{p})^{+} b^{(r)}(\vec{p}) \right)$$

$$= \sum_{r=1}^{2} \int d^3\vec{p} \, E_{\vec{p}} \left(n_a^{(r)}(\vec{p}) + n_b^{(r)}(\vec{p}) \right),$$

com $n_a^{(r)}(\vec{p}) = \dfrac{1}{E_{\vec{p}}/M} \, a^{(r)}(\vec{p})^{+} a^{(r)}(\vec{p})$ e $n_b^{(r)}(\vec{p}) = \dfrac{1}{E_{\vec{p}}/M} \, b^{(r)}(\vec{p})^{+} b^{(r)}(\vec{p})$.

Pela positividade do novo Hamiltoniano, podemos do mesmo jeito construir um vacío $|0\rangle \in \mathcal{H}$ t.q.

$$0 = a^{(r)}(\vec{p})|0\rangle = b^{(r)}(\vec{p})|0\rangle$$

para todo $r \in \{1,2\}$ e $\vec{p} \in \mathbb{R}^3$. Temos

$$[A, BC] = ABC - BCA = -BAC - BCA = -B\{A, C\}$$

se $\{A, B\} = 0$. Logo

$$[a^{(r)}(\vec{p})^{+}, n_a^{(s)}(\vec{q})] = -a^{(s)}(\vec{q})^{+}\delta_{rs}\delta(\vec{p}-\vec{q}) = -a^{(r)}(\vec{p})^{+}\delta_{rs}\delta(\vec{p}-\vec{q})$$

$$[b^{(r)}(\vec{p})^{+}, n_b^{(s)}(\vec{q})] = -b^{(r)}(\vec{p})^{+}\delta_{rs}\delta(\vec{p}-\vec{q})$$

$$[a^{(r)}(\vec{p})^{+}, n_b^{(s)}(\vec{q})] = [b^{(r)}(\vec{p})^{+}, n_a^{(s)}(\vec{q})] = 0.$$

Em particular, se tem

$$[a^{(r)}(\vec{p})^{+}, H] = -E_{\vec{p}}a^{(r)}(\vec{p})^{+} \qquad [b^{(r)}(\vec{p})^{+}, H] = -E_{\vec{p}}b^{(r)}(\vec{p})^{+}.$$

Para o momento e para a carga se tem sim precizar ordem normal

$$P^{k} = \sum_{r=1}^{2}\int \frac{d^3\vec{p}}{E_{\vec{p}}/M}\, p^{k}\left(a^{(r)}(\vec{p})^{+}a^{(r)}(\vec{p}) + b^{(r)}(\vec{p})^{+}b^{(r)}(\vec{p})\right) - \cancelto{0}{2\int d^3\vec{p}\,\vec{p}\,\delta(\vec{0})}$$

$$= \sum_{r=1}^{2}\int d^3\vec{p}\, p^{k}\left(n_a^{(r)}(\vec{p}) + n_b^{(r)}(\vec{p})\right).$$ Mais

$$Q = \sum_{r=1}^{2}\int \frac{d^3\vec{p}}{E_{\vec{p}}/M}\left(a^{(r)}(\vec{p})^{+}a^{(r)}(\vec{p}) - b^{(r)}(\vec{p})^{+}b^{(r)}(\vec{p})\right) + 2\int d^3\vec{p}\,\delta(\vec{0}),$$

de acordo com o mar de Dirac. O orden normal da o novo operador

$$Q = \sum_{r=1}^{2}\int d^3\vec{p}\left(n_a^{(r)}(\vec{p}) - n_b^{(r)}(\vec{p})\right).$$

Se tem

$$[a^{(r)}(\vec{p})^+, P^k] = -p^k a^{(r)}(\vec{p})^+, \qquad [b^{(r)}(\vec{p})^+, P^k] = -p^k b^{(r)}(\vec{p})^+,$$

$$[a^{(r)}(\vec{p})^+, Q] = -a^{(r)}(\vec{p})^+, \qquad [b^{(r)}(\vec{p})^+, Q] = b^{(r)}(\vec{p})^+.$$

Pelo tanto $a^{(r)}(\vec{p})^+$ gera uma partícula de energía $E_{\vec{p}}$, momento $\vec{p}$ e carga $+1$. O operador $b^{(r)}(\vec{p})^+$ gera partículas da misma energía e momento, mas com carga opuesta. Para o espín, temos que estudar a partícula em reposo. Se tem

$$[a^{(r)}(\vec{0})^+, M^{ij}] = -\sum_{s,t=1}^{2} \int \frac{d^3\vec{p}}{(E_{\vec{p}}/M)^2} a^{(s)}(\vec{p})^+ u^{(s)}(\vec{p})^+ m_p^{ij}\left(u^{(t)}(\vec{p})\{a^{(r)}(\vec{0})^+, a^{(t)}(\vec{p})\}\right)$$

$$= -\sum_{s=1}^{2} \int \frac{d^3\vec{p}}{E_{\vec{p}}/M} a^{(s)}(\vec{p})^+ u^{(s)}(\vec{p})^+ m_p^{ij}\left(u^{(r)}(\vec{p})\delta(\vec{p}-\vec{0})\right)$$

$$= -\frac{1}{2}\sum_{s=1}^{2} \int \frac{d^3\vec{p}}{E_{\vec{p}}/M} a^{(s)}(\vec{p})^+ u^{(s)}(\vec{p})^+ \sigma^{ij} u^{(r)}(\vec{0})\, \delta(\vec{p}-\vec{0})$$

$$= -\frac{1}{2}\sum_{s=1}^{2} u^{(s)}(\vec{0})^+ \sigma^{ij} u^{(r)}(\vec{0})\, a^{(s)}(\vec{0})^+.$$

Agora bem, si $i=j$ se tem $\sigma^{ij}=0$. Se $i\neq j$ se tem

$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = i\gamma^i\gamma^j$$

$$= i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = i\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

Então

$$u^{(s)}(\vec{0})^{\dagger}\sigma^{ij}u^{(r)}(\vec{0}) = -\varepsilon^{ijk}\begin{pmatrix}\eta^{(s)\dagger} & 0\end{pmatrix}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\begin{pmatrix}\eta^{(r)}\\ 0\end{pmatrix}$$

$$= -\varepsilon^{ijk}\begin{pmatrix}\eta^{(s)\dagger}\sigma^k & 0\end{pmatrix}\begin{pmatrix}\eta^{(r)}\\ 0\end{pmatrix}$$

$$= -\varepsilon^{ijk}\eta^{(s)\dagger}\sigma^k\eta^{(r)} = -\varepsilon^{ijk}\sigma^k{}_{sr}.$$

Logo

$$[a^{(r)}(\vec{0})^{\dagger}, M^{ij}] = \frac{1}{2}\varepsilon^{ijk}\sum_{s=1}^{2}\sigma^k{}_{sr}\,a^{(s)}(\vec{0})^{\dagger}$$

Definindo o vetor de momento angular $J^k = \frac{\varepsilon^{ijk}}{2}M^{ij}$,

temos

$$[a^{(r)}(\vec{0})^{\dagger}, J^k] = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{ijl}\sum_{s=1}^{2}\sigma^l{}_{sr}\,a^{(s)}(\vec{0})^{\dagger}$$

$$= \frac{1}{4}(2\delta_{kl})\sum_{s=1}^{2}\sigma^l{}_{sr}\,a^{(s)}(\vec{0})^{\dagger}$$

$$= \frac{1}{2}\sum_{s=1}^{2}\sigma^k{}_{sr}\,a^{(s)}(\vec{0})^{\dagger}.$$

Agora bem $M^{ij}|0\rangle = 0$. Logo $a^{(1)}(\vec{0})^{\dagger}$ gera uma partícula de spin $1/2$ e $a^{(2)}(\vec{0})^{\dagger}$ uma de spin $-\frac{1}{2}$. Do mesmo jeito, $b^{(1)}(\vec{0})^{\dagger}$ gera uma partícula de spin $1/2$ e $b^{(2)}(\vec{0})^{\dagger}$ de spin $-1/2$. Um estado geral é da forma

$$|\vec{q}_1 r_1, \dots, \vec{q}_n r_n; \vec{p}_1 s_1, \dots, \vec{p}_m s_m\rangle = \prod_{i=1}^{n} a^{(r_i)}(\vec{q}_i)^{\dagger}\prod_{j=1}^{m} b^{(s_j)}(\vec{p}_j)^{\dagger}|0\rangle.$$

Observe que é antisimétrico nos qr e ps. Logo, nossas partículas são fermiones. Em particular todos os qr's tem que ser distintos. O mesmo se tem para os ps's.

Observe

$$\langle \vec{q}\,'_1 r'_1, \ldots, \vec{q}\,'_{n'} r'_{n'}; \vec{p}\,'_1 s'_1, \ldots, \vec{p}\,'_{m'} s'_{m'} | \vec{q}_1 r_1, \ldots, \vec{q}_n r_n; \vec{p}_1 s_1, \ldots, \vec{p}_m s_m \rangle$$

$$= \delta_{n'n} \delta_{m'm}$$

3.2.7. Funções de Green

Agora consideramos o problema de hachar as funções de Green S da equação de Dirac

$$(i \not\partial - M) \underbrace{S(x)}_{\text{matriz}} = i \delta(x).$$

Como no caso escalar, podemos estudar a teoria de Fourier de S se existe. Se tem

$$S(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot x} \tilde{S}(p),$$

e

$$\int \frac{d^4 p}{(2\pi)^4} (+\not p - M) e^{-ip\cdot x} \tilde{S}(p) = i \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot x}.$$

Logo $\tilde{S}(p) = \dfrac{i}{\not p - M} = \dfrac{i(\not p + M)}{p^2 - M^2}$. Como no caso escalar, a integral não existe. Além

$$\tilde{S}(p) = i(\not{p} + M)\,\tilde{G}(p)$$

onde $\tilde{G}(p)$ é a tranformada de fourier das funções de Green da ecuação de Dirac. Logo para toda função S existe G t.q.

$$S(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, i(\not{p} + M)\tilde{G}(p)$$

$$= i \int \frac{d^4p}{(2\pi)^4}\, (i\not{\partial} + M)\, e^{-ip\cdot x}\, \tilde{G}(p)$$

$$= i\,(i\not{\partial} + M)\,G(x).$$

Em particular, temos

$$S_F(x) := i\,(i\not{\partial} + M)\,\Delta_F(x)$$

$$= \lim_{\varepsilon \to 0} \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{i}{p^2 - M^2 + i\varepsilon}$$

segum a escolhencia da p.g. 23.

3.2.8. Commotatividade Local

Considere agora

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^3\, E_{\vec{p}} E_{\vec{p}\,'}/M^2} \sum_{r,s=1}^{2} \Big(\{a^{(r)}(\vec{p}), a^{(s)\dagger}(\vec{p}\,')\}\, u_\alpha^{(r)}(\vec{p}) \bar{u}_\beta^{(s)}(\vec{p}\,') \times$$

$$e^{-ip\cdot x + ip'\cdot y} + \{b^{(r)}(\vec{p})^\dagger, b^{(s)}(\vec{p})\}\, v_\alpha^{(r)}(\vec{p}) \bar{v}_\beta^{(s)}(\vec{p}\,')$$

$$e^{ip\cdot x - ip'\cdot y}\Big)$$

$$= \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3\, E_{\vec{p}}/M} \Big(u_\alpha^{(r)}(\vec{p}) \bar{u}_\beta^{(r)}(\vec{p})\, e^{-ip\cdot(x-y)}$$

$$+ v_\alpha^{(r)}(\vec{p})\, \bar{v}_\beta^{(r)}(\vec{p})\, e^{ip\cdot(x-y)}\Big)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}} \Big((\not{p} + M)_{\alpha\beta}\, e^{-ip\cdot(x-y)}$$

$$+ (\not{p} - M)_{\alpha\beta}\, e^{ip\cdot(x-y)}\Big)$$

$$= (i\not{\partial} + M)_{\alpha\beta}\, \Delta(x-y).$$

Em particular $\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = 0$ para $(x-y)^2 < 0$.

Como $[A, BC] = \{A, B\}C - B\{A, C\}$, se tem que os bilineales em ψ hermiticos são candidatos a observavels.

A função de Green de Feynman também é interpretável como propagador. Defina para dos campos A e B

$$T(A,B)(x,y) \equiv \text{"}T(A(x)\,B(y))\text{"}$$
$$:= \Theta(x^0-y^0)\,A(x)\,B(y) + \Theta(y^0-x^0)\,B(y)\,A(x)\,(-1)^{\deg A \deg B}$$

onde deg é o grado Grassmann.

Então

$$\langle 0|T(\psi_\alpha,\bar{\psi}_\beta)(x,y)|0\rangle = \Theta(x^0-y^0)(i\not{\partial}+M)\int\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\,e^{-ip\cdot(x-y)}$$
$$+\Theta(y^0-x^0)(i\not{\partial}+M)\int\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\,e^{ip\cdot(x-y)}$$
$$=(i\not{\partial}+M)\,\Delta_F(x-y) = S_F(x-y)\,.$$

3.2.9. Simetrias discretas

Seja U o operador unitario de conjugação de carga. Este tem que satisfacer

$$U\psi U^{-1} = (\bar{\psi}C)^T$$

para uma matriz C. Para que esta seja uma simetría observe que para toda solução ψ da ec. de Dirac se tem

$$0 = \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + M)C = i\partial_\mu \bar{\psi} C C^{-1}\gamma^\mu C + M\bar{\psi}C.$$

$$= \left(i\,(C^{-1}\gamma^\mu C)^t\, \partial_\mu (\bar{\psi}C)^t + M(\bar{\psi}C)^t \right)^t$$

Logo $(\bar{\psi}C)^t$ satisfaz a ecuação de Dirac se e so se $C^{-1}\gamma^\mu C = -(\gamma^\mu)^t$. Uma escolha possivel é $C = i\gamma^2\gamma^0$ pois commuta com γ^1 e γ^3 mais anticomuta com γ^0 e γ^2. Se tem $C = -C^{-1} = -C^t = -C^+$. Logo

$$\overline{U\psi U^{-1}} = U\psi^+ U^{-1}\gamma^0 = U\bar{\psi}U^{-1}$$

$$\shortparallel$$

$$\overline{(\bar{\psi}\, C)^t}$$

$$\shortparallel$$

$$(\bar{\psi}C)^* \gamma^0 = (\psi^+\gamma^0 C)^*\gamma^0 = \psi^T \gamma^0 (C^+)^T\gamma^0$$

$$= -\psi^T(\gamma^0 C\gamma^0)^T = \psi^T C^T = (C\psi)^T.$$

Do jeito geral, os bilineales então transforman

$$:\bar\psi\Gamma\psi: \mapsto :(C\psi)^T\Gamma(\bar\psi C)^T: = -:(\bar\psi C\Gamma^T C\psi)^T: = -:\bar\psi C\Gamma^T C\psi:$$

Como (Exercício 4.4)

$$-\ C^2 = 1 \Longrightarrow :\bar\psi\psi: \longmapsto :\bar\psi\psi:,$$

$$-\ C(\gamma^5)^T C = -C^2\gamma^5 = \gamma^5 \Longrightarrow :\bar\psi\gamma^5\psi: \longmapsto :\bar\psi\gamma^5\psi:,$$

$$-\ C(\gamma^\mu)^T C = C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \Longrightarrow :\bar\psi\gamma^\mu\psi: \longmapsto -:\bar\psi\gamma^\mu\psi:,$$

$$-\ C(\gamma^\mu\gamma^5)^T C = C\gamma^5 C^{-1} C(\gamma^\mu)^T C^{-1} = -\gamma^5(-\gamma^\mu) = -\gamma^5\gamma^\mu = \gamma^\mu\gamma^5$$

$$\Longrightarrow :\bar\psi\gamma^\mu\gamma^5\psi: \longmapsto :\bar\psi\gamma^\mu\gamma^5\psi.$$

Para as transformações de paridade, considere o operador unitario

$$U\psi(x)U^{-1} = A\psi(t,-\vec{x}).$$

Para uma solução da ecuação de Dirac ψ, se tem (Exercício 4.5)

$$(i\partial\!\!\!/ - M)U\psi U^{-1} = t^\nu_\mu i\gamma^\mu A\partial_\nu\psi(t,-\vec{x}) - MA\psi(t,-\vec{x}), \quad t^\nu_\mu = \begin{cases} 1 & \nu=0=\mu \\ -1 & \mu=\nu\neq 0 \\ 0 & \mu\neq\nu \end{cases}$$

Logo, a paridade é uma simetria se A anticommuta com γ^1, γ^2 e γ^3 e commuta com γ^0. Então escolhemos $A=\gamma^0$.

Tem $\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}$. Então

$$\gamma^0 u^{(r)}(\vec{p}) = u^{(r)}(-\vec{p}),$$

$$\gamma^0 v^{(r)}(\vec{p}) = -v^{(r)}(-\vec{p}).$$

Logo

$$U a^{(r)}(\vec{p}) U^{-1} = \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} e^{ip\cdot x} \bar{u}^{(r)}(\vec{p}) \gamma^0 \gamma^0 \psi(t, -\vec{x})$$

$$= \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} e^{ip\cdot x} \bar{u}^{(r)}(-\vec{p}) \gamma^0 \psi(t, -\vec{x})$$

$$= \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} e^{i(E_{\vec{p}} t + (-\vec{p})\cdot\vec{x})} \bar{u}^{(r)}(-\vec{p}) \gamma^0 \psi(x)$$

$$= a^{(r)}(-\vec{p}) \qquad \text{e}$$

$$U b^{(r)}(\vec{p}) U^{-1} = -b^{(r)}(-\vec{p}),$$

A inversão temporal T é antiunitaria. Demandamos

$$T \psi(x) T^{-1} = B \psi(-t, \vec{x}).$$

Para as soluções da ec. de Dirac se tem

$$(i\not\partial - M)\, T\psi(x)T^{-1} = -i t^{\nu}_{\mu} \gamma^{\mu *} A\, \partial_{\mu} \psi(-t, \vec{x}) - M A\, \psi(-t, \vec{x}).$$

Logo, para que a reverção temporal seja uma simetria se precisa que

$$A^{-1}\gamma^{0*}A = \gamma^{0}, \qquad A^{-1}\gamma^{k*}A = -\gamma^{k},$$

o que se tem se $A = i\gamma^{1}\gamma^{3}$. Então $A^{\dagger} = A = A^{-1}$. Os bilineais então transformam como

$$T\bar{\psi}\gamma^{\mu}\psi T^{-1} = T\psi^{+}(t,\vec{x})T^{-1}\, T\gamma^{0}\gamma^{\mu}T^{-1}\, T\psi(t,\vec{x})T^{-1}$$

$$= \psi^{+}(-t,\vec{x})\, A^{-1}\gamma^{0*}\gamma^{\mu*}A\, \psi(t,\vec{x})$$

$$= \bar{\psi}(-t,\vec{x})\, B^{-1}\gamma^{\mu *} B\, \psi(t,\vec{x}).$$

Em particular,

$$\bar{\psi}\gamma^{0}\psi \longmapsto \bar{\psi}(-t,\vec{x})\gamma^{0}\psi(-t,\vec{x})$$

$$\bar{\psi}\gamma^{k}\psi \longmapsto -\bar{\psi}(-t,\vec{x})\gamma^{k}\psi(-t,\vec{x}).$$