Iván Mauricio Burbano Aldana

Prof. Nathan Berkovits

Supersymmetry

Instituto de Física Teórica, UNESP

Exercise 8.1.

In the Wess-Zumino gauge we have (8.23)

$$V = -(\theta\sigma^m\bar{\theta})A_m + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 d.$$

Thus

$$V^2 = (\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta})A_m A_n = -\frac{1}{2}\theta^2\bar{\theta}^2 A^m A_m,$$

and $V^3=0$. We conclude that

$$e^{\pm V} = 1 \pm V + \frac{1}{2}V^2,$$

and

$$\begin{aligned}
\mathcal{D}_\alpha = e^{-V}D_\alpha e^V &= \cancel{e^{-V}e^V} D_\alpha + \overbrace{e^{-V}D_\alpha(e^V)}^{A_\alpha} \\
&= D_\alpha + \left(1 - V + \frac{1}{2}V^2\right) D_\alpha\left(\cancelto{0}{1} + V + \frac{1}{2}V^2\right) \\
&= D_\alpha + D_\alpha(V) - V D_\alpha(V) + \frac{1}{2}D_\alpha(V^2) \\
&= D_\alpha + D_\alpha(V) - V D_\alpha(V) + \frac{1}{2}D_\alpha(V)V + \frac{1}{2}V D_\alpha(V) \\
&= D_\alpha + D_\alpha(V) - \frac{1}{2}[V, D_\alpha(V)],
\end{aligned}$$

which is the result of exercise (2) of chapter VII in Wess & Bagger. In here we used that

$$V D_\alpha V^2 = V^2 D_\alpha V = V^2 D_\alpha V^2 = 0,$$

which can be seen without need for computation by counting powers of θ and $\bar\theta$. To express this in terms of component fields, we start by computing $D_\alpha(V)$. We have

$$\frac{\partial}{\partial\theta^\alpha}(V) = -(\sigma^m\bar\theta)_\alpha A_m + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta^2\lambda_\alpha + \theta_\alpha\bar\theta^2 d,$$

and

$$i(\sigma^m\bar\theta)_\alpha\partial_m(V) = \underbrace{-i(\sigma^m\bar\theta)_\alpha(\theta\sigma^n\bar\theta)\partial_m A_n}$$

$$= +i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\theta^\beta\sigma^n{}_{\beta\dot\beta}\partial_m A_n$$

$$= \frac{i}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}\theta^\beta\sigma^n{}_{\beta\dot\beta}\partial_m A_n$$

$$= -\frac{i}{2}\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\sigma^{n\dot\alpha\beta}\theta_\beta\partial_m A_n$$

$$= -\frac{i}{2}\bar\theta^2(\sigma^m\bar\sigma^n)_\alpha{}^\beta\theta_\beta\partial_m A_n$$

$$= -\frac{i}{4}\bar\theta^2(-2\eta^{mn}\delta_\alpha^\beta)\theta_\beta\partial_m A_n - \frac{i}{4}\bar\theta^2(\sigma^m\bar\sigma^n)_\alpha{}^\beta\theta_\beta\partial_{[m}A_{n]}$$

$$= \frac{i}{2}\bar\theta^2\theta_\alpha\partial^m A_m - \frac{i}{4}\bar\theta^2(\sigma^m\bar\sigma^n)_\alpha{}^\beta\theta_\beta\partial_{[m}A_{n]}.$$

$$\underbrace{-(\sigma^m\bar\theta)_\alpha\theta^2\bar\theta\partial_m\bar\lambda}$$

$$= -\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\bar\lambda_{\dot\beta}\bar\theta^{\dot\beta} = \theta^2\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar\lambda_{\dot\beta}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}$$

$$= \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\theta^2\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar\lambda_{\dot\beta} = \frac{1}{2}\theta^2\bar\theta^2(\sigma^m\partial_m\bar\lambda)_\alpha$$

Thus, in component fields

$$D_\alpha V = -(\sigma^m\bar\theta)_\alpha A_m + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta^2\lambda_\alpha + \theta_\alpha\bar\theta^2\left(d + \frac{i}{2}\partial^m A_m\right)$$
$$- \frac{i}{4}\bar\theta^2(\sigma^m\bar\sigma^n)_\alpha{}^\beta\theta_\beta\partial_{[m}A_{n]} + \frac{1}{2}\theta^2\bar\theta^2(\sigma^m\partial_m\bar\lambda)_\alpha.$$

We now calculate $[V, D_\alpha V]$. By eliminating the terms that clearly cancel due to their powers in θ and $\bar\theta$,

$$[V, D_\alpha V] = \underbrace{\left[-(\theta\sigma^m\bar\theta)A_m\,,\, -(\sigma^n\bar\theta)_\alpha A_n\right]}$$

$$= +(\theta\sigma^m\bar\theta)(\sigma^n\bar\theta)_\alpha [A_m, A_n]$$

$$= \theta^\beta \sigma^m{}_{\beta\dot\beta}\, \sigma^n{}_{\alpha\dot\alpha}\, \bar\theta^{\dot\beta}\bar\theta^{\dot\alpha}[A_m, A_n]$$

$$= \frac{1}{2}\varepsilon^{\dot\beta\dot\alpha}\,\bar\theta^2\theta^\beta\sigma^m{}_{\beta\dot\beta}\,\sigma^n{}_{\alpha\dot\alpha}[A_m, A_n]$$

$$= \frac{1}{2}\bar\theta^2\,\bar\sigma^{m\dot\alpha\beta}\theta_\beta\,\sigma^n{}_{\alpha\dot\alpha}[A_m, A_n]$$

$$= \frac{1}{2}\bar\theta^2\theta_\beta\,(\sigma^n\bar\sigma^m)_\alpha{}^\beta[A_m, A_n]$$

$$= -\frac{1}{2}\bar\theta^2\theta_\beta\,(\sigma^m\bar\sigma^n)_\alpha{}^\beta[A_m, A_n]$$

$$= \frac{i}{2}\bar\theta^2(\sigma^m\bar\sigma^n)_\alpha{}^\beta\theta_\beta\;\; i[A_m, A_n]$$

$$+ \underbrace{\left[-(\theta\sigma^m\bar\theta)A_m\,,\, 2i\theta_\alpha\bar\theta\bar\lambda\right]}$$

$$= +2i\,\theta_\alpha\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\,\bar\sigma^{m\dot\beta\beta}\theta_\beta\,[A_m, \bar\lambda^{\dot\alpha}]$$

$$= -\frac{i}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\,\theta^2\bar\theta^2\,\bar\sigma^{m\dot\beta\beta}\,[A_m, \bar\lambda^{\dot\alpha}]$$

$$= -\frac{i}{2}\theta^2\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}[A_m, \bar\lambda^{\dot\alpha}]$$

$$= -\frac{i}{2}\theta^2\bar\theta^2[A_m, (\sigma^m\bar\lambda)_\alpha]$$

$$+ \underbrace{\left[i\theta^2\bar\theta\bar\lambda\,,\, -(\sigma^m\bar\theta)_\alpha A_m\right]}$$

$$= -i\theta^2\bar\theta_{\dot\beta}\bar\lambda^{\dot\beta}\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}A_m + i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}A_m\theta^2\bar\theta_{\dot\beta}\bar\lambda^{\dot\beta}$$

$$= +i\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta_{\dot\beta}\bar\theta^{\dot\alpha}\bar\lambda^{\dot\beta}A_m - i\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta_{\dot\beta}\bar\theta^{\dot\alpha}A_m\bar\lambda^{\dot\beta}$$

$$= i\theta^2\sigma^m{}_{\alpha\dot\alpha}\,\varepsilon_{\dot\beta\dot\gamma}\frac{1}{2}\varepsilon^{\dot\gamma\dot\alpha}\bar\theta^2\,[\bar\lambda^{\dot\beta}, A_m]$$

$$= \frac{i}{2}\theta^2\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}[\bar\lambda^{\dot\alpha}, A_m] = -\frac{i}{2}\theta^2\bar\theta^2[A_m, (\sigma^m\bar\lambda)_\alpha]$$

We thus conclude that

$$\mathcal{D}_\alpha = D_\alpha + A_\alpha ,$$

with

$$A_\alpha = -(\sigma^m \bar\theta)_\alpha A_m + 2i\theta_\alpha \bar\theta \bar\lambda - i\bar\theta^2 \lambda_\alpha + \theta_\alpha \bar\theta^2 \left(d + \frac{i}{2}\partial^m A_m\right)$$

$$- \frac{i}{4}\bar\theta^2 (\sigma^m \bar\sigma^n)_\alpha{}^\beta \theta_\beta F_{mn} + \frac{1}{2}\theta^2\bar\theta^2 \left((\sigma^m \partial_m \bar\lambda)_\alpha + i[A_m, (\sigma^m \bar\lambda)_\alpha]\right),$$

where

$$F_{mn} = \partial_{[m} A_{n]} + i[A_m, A_n].$$

We now proceed to the calculation of

$$\{\mathcal{D}_\alpha, \bar D_{\dot\alpha}\} = \{D_\alpha + A_\alpha, \bar D_{\dot\alpha}\} = \{D_\alpha, \bar D_{\dot\alpha}\} + \{A_\alpha, \bar D_{\dot\alpha}\}$$

$$= -2i\sigma^m{}_{\alpha\dot\alpha}\partial_m + A_\alpha \bar D_{\dot\alpha} + \bar D_{\dot\alpha} A_\alpha$$

$$= -2i\sigma^m{}_{\alpha\dot\alpha}\partial_m + \cancel{A_\alpha \bar D_{\dot\alpha}} + \bar D_{\dot\alpha}(A_\alpha) - \cancel{A_\alpha \bar D_{\dot\alpha}}$$

$$= -2i\sigma^m{}_{\alpha\dot\alpha}\partial_m + \bar D_{\dot\alpha}(A_\alpha).$$

The minus sign in the product rule is clear from the expression we found for A_α. Indeed, both $\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$ and $i(\theta\sigma^m)_{\dot\alpha}\partial_m$ are grassmann odd terms which will have to go through the odd number of grassmann odd terms in each component of A_α. We can now calculate

$$-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} A_\alpha = +\sigma^m{}_{\alpha\dot\alpha} A_m + 2i\theta_\alpha \bar\lambda_{\dot\alpha} + 2i\bar\theta_{\dot\alpha}\lambda_\alpha - 2\theta_\alpha\bar\theta_{\dot\alpha}\left(d + \frac{i}{2}\partial^m A_m\right)$$

$$-\frac{i}{2}\bar\theta_{\dot\alpha}(\sigma^m\bar\sigma^n)_\alpha{}^\beta\theta_\beta F_{mn}+\theta^2\bar\theta_{\dot\alpha}\left((\sigma^m\partial_m\bar\lambda)_\alpha+i[A_m,(\sigma^m\bar\lambda)_\alpha]\right),$$

and

$$-i(\theta\sigma^m)_{\dot\alpha}\partial_m A_\alpha=\underbrace{i(\theta\sigma^m)_{\dot\alpha}(\sigma^n\bar\theta)_\alpha\partial_m A_n}$$

$$=i\theta^\beta\sigma^m{}_{\beta\dot\alpha}\sigma^n{}_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m A_n$$

$$=i\theta^\beta\sigma^m{}_{\beta\dot\gamma}\delta_{\dot\alpha}^{\dot\gamma}\delta_\alpha^\gamma\sigma^n{}_{\gamma\dot\beta}\bar\theta^{\dot\beta}\partial_m A_n$$

$$=-\frac{i}{2}\theta^\beta\sigma^m{}_{\beta\dot\gamma}\sigma_{p\alpha\dot\alpha}\bar\sigma^{p\dot\gamma\gamma}\sigma^n{}_{\gamma\dot\beta}\bar\theta^{\dot\beta}\partial_m A_n$$

$$=-\frac{i}{2}\sigma_{p\alpha\dot\alpha}(\theta\sigma^m\bar\sigma^p\sigma^n\bar\theta)\partial_m A_n$$

$$=-\frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\sigma^{(m|}\bar\sigma^p\sigma^{|n)}\bar\theta)\partial_m A_n$$

$$-\frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\sigma^m\bar\sigma^p\sigma^n\bar\theta)\partial_{[m}A_{n]}$$

$$=-\frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\,2(g^{mp}\sigma^n+g^{pn}\sigma^m-g^{mn}\sigma^p)\bar\theta)\partial_m A_n$$

$$-\frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\sigma^m\bar\sigma^p\sigma^n\bar\theta)\partial_{[m}A_{n]}$$

$$=-\frac{i}{2}\sigma_{p\alpha\dot\alpha}\Big((\theta\sigma^m\bar\theta)(\partial^p A_m+\partial_m A^p)$$
$$-(\theta\sigma^p\bar\theta)\partial^m A_m\Big)$$

$$-\frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\sigma^m\bar\sigma^p\sigma^n\bar\theta)\partial_{[m}A_{n]}$$

$$=-\frac{i}{2}(\theta\sigma^m\bar\theta)\sigma^p{}_{\alpha\dot\alpha}\partial_{(p}A_{m)}$$

$$\underbrace{-\frac{i}{2}\bar\theta_{\dot\beta}\sigma_{p\alpha\dot\alpha}\bar\sigma^{p\dot\beta\beta}\theta_\beta\partial^m A_m}_{=i\bar\theta_{\dot\alpha}\theta_\alpha\partial^m A_m}$$

$$-\frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\sigma^m\bar\sigma^p\sigma^n\bar\theta)\partial_{[m}A_{n]}$$

$$\underbrace{+2(\theta\sigma^m)_{\dot\alpha}\,\theta_\alpha\,\bar\theta\,\partial_m\bar\lambda}$$

$$= 2\theta^\beta\sigma^m{}_{\beta\dot\alpha}\,\theta_\alpha\,\bar\theta\,\partial_m\bar\lambda$$

$$= \cancel{2}\,\varepsilon^{\beta\gamma}\frac{1}{\cancel{2}}\varepsilon_{\gamma\alpha}\,\theta^2\sigma^m{}_{\beta\dot\alpha}\,\bar\theta\,\partial_m\bar\lambda$$

$$= \sigma^m{}_{\alpha\dot\alpha}\,\theta^2\bar\theta\,\partial_m\bar\lambda$$

$$-(\theta\sigma^m)_{\dot\alpha}\,\bar\theta^2\partial_m\lambda_\alpha\ \underbrace{-\,i(\theta\sigma^n)_{\dot\alpha}\,\theta_\alpha\,\bar\theta^2\left(\partial_n d+\tfrac{i}{2}\partial_n\partial^m A_m\right)}$$

$$\underbrace{-\frac{i}{2}\,\theta^2\delta_\alpha^\beta\,\sigma^n{}_{\beta\dot\alpha}\,\bar\theta^2\left(\partial_n d+\tfrac{i}{2}\partial_n\partial^m A_m\right)}$$

$$-\frac{i}{2}\,\sigma^n{}_{\alpha\dot\alpha}\,\theta^2\bar\theta^2\left(\partial_n d+\tfrac{i}{2}\partial_n\partial^m A_m\right)$$

$$\underbrace{-\frac14(\theta\sigma^m)_{\dot\alpha}\,\bar\theta^2(\sigma^n\bar\sigma^p)_\alpha{}^\beta\,\theta_\beta\,\partial_m F_{np}}$$

$$= -\frac14\bar\theta^2\,\theta^\gamma\sigma^m{}_{\gamma\dot\alpha}\,\sigma^n{}_{\alpha\dot\beta}\,\bar\sigma^{p\dot\beta\beta}\,\theta_\beta\,\partial_m F_{np}$$

$$= -\frac14\bar\theta^2\,\theta^\gamma\sigma^m{}_{\gamma\dot\gamma}\,\delta^{\dot\gamma}_{\dot\alpha}\,\delta^\sigma_\alpha\,\sigma^n{}_{\sigma\dot\beta}\,\bar\sigma^{p\dot\beta\beta}\,\theta_\beta\,\partial_m F_{np}$$

$$= \frac12\bar\theta^2\,\theta^\gamma\sigma^m{}_{\gamma\dot\gamma}\,\sigma_{q\alpha\dot\alpha}\,\bar\sigma^{q\dot\gamma\sigma}\,\sigma^n{}_{\sigma\dot\beta}\,\bar\sigma^{p\dot\beta\beta}\,\theta_\beta\,\partial_m F_{np}$$

$$= \frac12\sigma_{q\alpha\dot\alpha}\,\bar\theta^2\left(\theta\sigma^m\bar\sigma^q\sigma^n\bar\sigma^p\theta\right)\partial_m F_{np}.$$

We thus conclude

$$\{D_\alpha,\bar D_{\dot\alpha}\} = -2i\sigma^m{}_{\alpha\dot\alpha}(\partial_m+A_m) - 2i\theta_\alpha\bar\lambda_{\dot\alpha} - 2i\bar\theta_{\dot\alpha}\lambda_\alpha - 2\theta_\alpha\bar\theta_{\dot\alpha}\left(d+\tfrac{i}{2}\partial^m A_m\right)$$

$$-\frac{i}{2}\bar\theta_{\dot\alpha}(\sigma^m\bar\sigma^n\theta)_\alpha F_{mn} + \theta^2\bar\theta_{\dot\alpha}\left((\sigma^m\partial_m\bar\lambda)_\alpha + i[A_m,(\sigma^m\bar\lambda)_\alpha]\right)$$

$$-\frac{i}{2}(\theta\sigma^m\bar\theta)\,\sigma^n{}_{\alpha\dot\alpha}\,\partial_{(n}A_{m)} + i\bar\theta_{\dot\alpha}\theta_\alpha\partial^m A_m - \frac{i}{4}\sigma_{p\alpha\dot\alpha}(\theta\sigma^m\bar\sigma^p\sigma^n\bar\theta)\,\partial_{[m}A_{n]}$$

$$+\sigma^m{}_{\alpha\dot\alpha}\,\theta^2\bar\theta\,\partial_m\bar\lambda - \frac{i}{2}\sigma^n{}_{\alpha\dot\alpha}\,\theta^2\bar\theta^2\left(\partial_n d+\tfrac{i}{2}\partial_n\partial^m A_m\right)$$

$$+\frac12\sigma_{q\alpha\dot\alpha}\,\bar\theta^2\left(\theta\sigma^m\bar\sigma^q\sigma^n\bar\sigma^p\theta\right)\partial_m F_{np}$$

Exercise 8.2.

The Lagrangian of QCD is (for massless quarks)

$$\mathcal{L} = -\frac{1}{2g^2} \mathrm{Tr}\left(F^{mn} F_{mn}\right) + \frac{i}{2} \bar{\Psi} \gamma_m \overleftrightarrow{D}^m \Psi ,$$

where $F_{mn} = \partial_m A_n - \partial_n A_m - i g [A_m, A_n]$, A_m is a $SO(3)$ gauge field and $\Psi = (\Psi_1, \ldots, \Psi_6)$ is a multiplet of Dirac spinors. As we have learned, a supersymmetric extension of this $SO(3)$ theory is obtained by introducing the real superfield V in (8.1) which contains A_m as a component field. To include matter, for each $\Psi_a = (\phi_a, \bar{\chi}_a)$ we introduce two chiral superfields $\Phi_a{}^A$ and Σ'_{aA}. The first contains ϕ_a as a component field while the second χ_a. In these fashion we obtain the multiplets $\Phi = (\Phi_1, \ldots, \Phi_6)$ and $\Sigma' = (\Sigma'_1, \ldots, \Sigma'_6)$. Our action is then

$$S(\Phi, \Sigma', V) = \int d^4x\, d^4\theta \left(\bar{\Phi} e^{-V} \bar{\Phi} + \bar{\Sigma}' e^{V} \Sigma' \right)$$
$$+ \frac{1}{4g^2} \int d^4x \left(\int d^2\theta\, \mathrm{Tr}(W^2) + \int d^2\bar{\theta}\, \mathrm{Tr}(\bar{W}^2) \right),$$

where $W_\alpha = -\frac{1}{4} \bar{D}\bar{D}\left(e^{-V} D_\alpha e^{V}\right)$. Notice that in the matter part there is a summation over the flavor indices a.

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Exercise 8.3.

Let us begin by describing the standard model (without the Higgs) in order to fix notation. Let us begin with the fermionic part. Due to the electroweak interaction, we have to divide our description into left and right weyl spinors. Similarly, due to the strong interaction, we have to further divide this into quarks and leptons. The left Leptons are grouped into the fields L_i, with $i\in\{1,2,3\}$ running over the three generations. Each of this is further divided into $SU(2)$ doublets

$$l_1=\begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix},\qquad l_2=\begin{pmatrix}\nu_{\mu L}\\ \mu_L\end{pmatrix},\qquad l_3=\begin{pmatrix}\nu_{\tau L}\\ \tau_L\end{pmatrix},$$

which we will denote by $l_i=\begin{pmatrix}\nu_{Li}\\ e_i\end{pmatrix}=\begin{pmatrix}l_{i1}\\ l_{i2}\end{pmatrix}$ to write the Lagrangian as explicitely as possible. For all $i\in\{1,2,3\}$ and $A\in\{1,2\}$, L_{iA} is a left Weyl spinor, i.e. transforms in the $(1/2,0)$ representation of the proper orthochronous Lorentz group $L_+^\uparrow$. At the risk of introducing cumbersome notation, we will denote by $l_{iA\alpha}$, with $\alpha\in\{1,2\}$, the α-th component of the spinor l_{iA}. There is no color index, indicating that these form a singlet under $SU(3)$. The story for quarks is different. We, however, start in a similar fashion, by taking the fields q_i with $i \in \{1,2,3\}$. Explicitely

$$q_1 = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad q_2 = \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad q_3 = \begin{pmatrix} t_L \\ b_L \end{pmatrix},$$

although we will retain the notation $q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix} = \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix}$.

For each $i \in \{1,2,3\}$, q_i is an $SU(2)$ doublet. However, now for each $A \in \{1,2\}$, q_{iA} is a $SU(3)$ triplet

$$q_{iA} = \begin{pmatrix} q_{iA1} \\ q_{iA2} \\ q_{iA3} \end{pmatrix}.$$

It is at this level where q_{iAM}, $M \in \{1,2,3\} \cong \{\text{red, blue, green}\}$ running over the colors, is a left Weyl spinor, with components $q_{iAM\alpha}$.

The matter with right chirality forms a singlet under $SU(2)$. We, however, are still forced to separate our fields into quarks and leptons due to the strong interactions. For the latter, we have three right weyl spinors $\bar{e}_{Ri}$, $i \in \{1,2,3\}$ running over generations, each a singlet under $SU(3)$ and with components $\bar{e}_{Ri\dot{\alpha}}$. To be explicit,

$$\bar{e}_{R1} = \bar{e}_R, \qquad \bar{e}_{R2} = \bar{\mu}_R, \qquad \bar{e}_{R3} = \bar{\tau}_R.$$

Although likely to change in the future, the current standard model does not contemplate right handed neutrinos. This is because they don't have charge or color. Thus, if they are massless, they wouldn't interact with anything, making them undetectable. For the quarks, we have the fields $\bar{u}_{Ri}$, $\bar{d}_{Ri}$, $i \in \{1,2,3\}$ running over generations. Explicitely

$$\bar{u}_{R1} = \bar{u}_R, \qquad \bar{u}_{R2} = \bar{c}_R, \qquad \bar{u}_{R3} = \bar{t}_R,$$

$$\bar{d}_{R1} = \bar{d}_R, \qquad \bar{d}_{R2} = \bar{s}_R, \qquad \bar{d}_{R3} = \bar{b}_R,$$

each being a singlet under $SU(2)$. Each forms a triplet $(\bar{u}_{RiM})$ $(\bar{d}_{RiM})$, with $M \in \{1,2,3\} \cong \{\text{red, blue, green}\}$, under $SU(3)$.

Each $\bar{u}_{RiM}$, $\bar{d}_{RiM}$ is a right Weyl spinor with components $\bar{u}_{RiM\dot{\alpha}}$, $\bar{d}_{RiM\dot{\alpha}}$.

Although we have already hinted at the transformation properties of these groupings by using the word multiplet, let us be more explicit. The structure group of the Standard Model is

$$G = SU(3) \times SU(2) \times U(1).$$

It acts on the fields we have described by

$$((U,V,z)\, l_i)_A = z^{-3}\, V_A{}^B\, l_{iB}$$

$$((U,V,z)\, q_i)_{AM} = z\, U_M{}^N\, V_A{}^B\, q_{iBN}$$

$$(U,V,z)\,\bar{e}_{Ri} = z^{-6}\,\bar{e}_{Ri}$$

$$((U,V,z)\,\bar{u}_{Ri})_M = z^{4}\, U_M{}^N\, \bar{u}_{RiN}$$

$$((U,V,z)\,\bar{d}_{Ri})_M = z^{-2}\, U_M{}^N\, \bar{d}_{RiN}.$$

These actions induce through differentiation an action of the Lie algebra $\mathfrak{g} = su(3) \oplus su(2) \oplus u(1)$ of G. It is with this representations that we can make the $su(3)$ valued fields G_μ, the $su(2)$ valued fields W_μ, and the $u(1)$ valued fields B_μ act on our previous fields. In particular, we have our covariant derivatives

$$(D_\mu l_i)_A = \left(\partial_\mu - ig(W_\mu)_A{}^B + ig' B_\mu \delta_A{}^B\right) l_{iB}$$

$$(D_\mu q_i)_{AM} = \left(\partial_\mu - ig_s (G_\mu)_M{}^N \delta_A{}^B - ig(W_\mu)_A{}^B \delta_M{}^N - \tfrac{1}{3} ig' B_\mu \delta_A{}^B \delta_M{}^N\right) q_{iBN}$$

$$D_\mu \bar{e}_{Ri} = \left(\partial_\mu + 2ig' B_\mu\right)\bar{e}_{Ri}$$

$$(D_\mu \bar{u}_{Ri})_M = \left(\partial_\mu - ig_s (G_\mu)_M{}^N - \tfrac{4}{3} ig' B_\mu \delta_M{}^N\right)\bar{u}_{RiN}$$

$$(D_\mu \bar{d}_{Ri})_M = \left(\partial_\mu - ig_s (G_\mu)_M{}^N + \tfrac{2}{3} ig' B_\mu \delta_M{}^N\right)\bar{d}_{RiN}.$$

We can then write the kinetic Lagrangian for fermions including the interaction with bosons

$$\mathcal{L}_f = \sum_{i=1}^{3} \Big(\bar{q}_i^{\,AM}\, \bar{\sigma}^\mu (\overleftrightarrow{D}_\mu q_i)_{AM} + \bar{l}_i^{\,A}\, \bar{\sigma}^\mu (\overleftrightarrow{D}_\mu l_i)_A$$

$$+ \bar{u}_{RiM}\, \bar{\sigma}^\mu (D_\mu u_{Ri})^M + \bar{d}_{RiM}\, \bar{\sigma}^\mu (D_\mu d_{Ri})^M$$

$$+ \bar{e}_{Ri}\, \bar{\sigma}^\mu (D_\mu e_{Ri}) \Big).$$

The gauge bosons propagate through the Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4}\,\mathrm{Tr}(G^{\mu\nu} G_{\mu\nu}) - \frac{1}{4}\,\mathrm{Tr}(W^{\mu\nu} W_{\mu\nu}) - \frac{1}{4} B^{\mu\nu} B_{\mu\nu},$$

where we have the curvatures

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i g_s [G_\mu, G_\nu],$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - i g [W_\mu, W_\nu],$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu .$$

Thus the action of the standard model is

$$S = \int d^4x \,(\mathcal{L}_f + \mathcal{L}_{YM}).$$

To do a minimal extension of the standard model to a supersymmetric one, we introdue a chiral field for every left-handed spinor. We thus have the chiral superfield

$$L_{iA}(x, \theta, \bar{\theta}),$$

where $L_{iA}|_\theta = l_{iA}$, $i \in \{1,2,3\}$ runs through generations, and $A \in \{1,2\}$ runs through flavors. Similarly, we introduce the chiral superfields Q_{iAM}, where $Q_{iAM}|_\theta = q_{iAM}$, $i \in \{1,2,3\}$ runs through the generations, $A \in \{1,2\}$ runs through flavors, and $M \in \{1,2,3\} \cong \{red, blue, green\}$ runs through colors. On the other hand, for the right-handed spinors, we introduce antichiral superfields. We thus introduce the antichiral superfields $\bar{E}_i$, $\bar{U}_{iM}$, and $\bar{D}_{iM}$, where $\bar{E}_i|_\theta = \bar{e}_{Ri}$, $\bar{U}_{iM}|_\theta = \bar{u}_{RiM}$, $\bar{D}_{iM}|_\theta = \bar{d}_{iM}$, $i \in \{1,2,3\}$ runs through generations, and $M \in \{1,2,3\} \cong \{red, blue, green\}$ runs through colors.

For the gauge Bosons, we introduce the Lie algebra valued vector superfields V_s, V_L, and V_Y, with values in $so(3)$, $su(2)$, and $u(1)$ respectively. In particular $V_s|_{\theta\sigma^\mu\bar{\theta}} = G_\mu$, $V_L|_{\theta\sigma^\mu\bar{\theta}} = W_\mu$, and $V_Y|_{\theta\sigma^\mu\bar{\theta}} = B_\mu$. We define their actions on our matter fields in correspondance to the non-supersymmetric models. Thus

$$(V_s L_i)_A = L_{iA} \qquad (V_L L_i)_A = V_{LA}{}^B L_{iB} \qquad (V_Y L_i)_A = -V_Y L_{iA}$$

$$(V_s Q_i)_{AM} = (V_s)_M{}^N Q_{iAN} \qquad (V_L Q_i)_{AM} = V_{LA}{}^B Q_{iBM} \qquad (V_Y Q_i)_{AM} = \tfrac{1}{3} V_Y Q_{iAM}$$

$$(V_s \bar{E}_i)_A = \bar{E}_{iA} \qquad (V_L \bar{E}_i)_A = \bar{E}_{iA} \qquad (V_Y \bar{E}_i)_A = -2V_Y \bar{E}_{iA}$$

$$(V_s \bar{U}_i)_{AM} = V_{sM}{}^N \bar{U}_{iAN} \qquad (V_L \bar{U}_i)_{AM} = \bar{U}_{iAM} \qquad (V_Y \bar{U}_i)_{AM} = \tfrac{4}{3} V_Y \bar{U}_{iAM}$$

$$(V_s \bar{D}_i)_{AM} = V_{sM}{}^N \bar{D}_{iAN} \qquad (V_L \bar{D}_i)_{AM} = \bar{D}_{iAM} \qquad (V_Y \bar{D}_i)_{AM} = -\tfrac{2}{3} V_Y \bar{D}_{iAM}.$$

Then, defining $V = V_S \oplus V_L \oplus V_Y$ taking values in $su(3) \oplus su(2) \oplus u(1)$, we have the suppersymmetric standard model action

$$
\begin{aligned}
S = \int d^4x \Bigg(\sum_{i=1}^{3} \Bigg(\int d^4\theta \Big(& \bar{Q}_i{}^{AM} \left(e^V Q_i\right)_{AM} + \bar{L}_i{}^{A} \left(e^V L_i\right)_A \\
& + \bar{U}_{iM} \left(e^V U_i\right)^M + \bar{D}_{iM} \left(e^V D_i\right)^M \\
& + \bar{E}_i \left(e^V E_i\right) \Big) \\
+ \int d^2\theta \Big(& (W_{S\alpha})_M{}^N (W_S{}^\alpha)_M{}^N + (W_{L\alpha})_A{}^B (W_L{}^\alpha)_A{}^B \\
& + B_{Y\alpha} B_Y{}^\alpha \Big) + c.c. \Bigg),
\end{aligned}
$$

where

$$W_{S\alpha} = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-V_S} D_\alpha e^{V_S} \right)$$

$$W_{L\alpha} = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-V_L} D_\alpha e^{V_L} \right)$$

$$W_{Y\alpha} = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-V_Y} D_\alpha e^{V_Y} \right)$$