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Teoria Quântica de Campos

2. Formalismo Lagrangiano para Campos Classicos

1. Ecuações de Euler Lagrange

Em correspondencia a mecánica clássica, para uma configuração de um multiplete de campos φ e uma região R do espaçotempo, definimos a ação

$$S_R(\varphi) = \int_R d^4x\, \mathcal{L}(\varphi(x), \partial\varphi(x)).$$

Observação: $\mathcal{L}$, a densidade Lagrangiana, introduce um campo $\mathcal{L}(\varphi, \partial\varphi)$ onde $\mathcal{L}(\varphi, \partial\varphi)(x) = \mathcal{L}(\varphi(x), \partial\varphi(x))$.

Agora considere uma variação infinitesimal do espaçotempo $x \longmapsto x' = x + \delta x$, onde δx é em geral dependente de x (e por o tanto, não é necessoramente uma translação). Sob esta, temos uma transformação da região $R \longmapsto R' = \{x' = x + \delta x \mid x \in R\}$. Simultaneamente considere uma variação infinitesimal $\varphi \longmapsto \varphi' = \varphi + \delta\varphi$ nos campos. A nova ação é

$$S_{R'}(\varphi') = \int_{R'} d^4x'\, \mathcal{L}(\varphi'(x'), \partial\varphi'(x')) = \int_{R'} d^4x\, \mathcal{L}(\varphi', \partial\varphi')(x')$$

$$= \int_R d^4x \det\left(\frac{\partial x^\mu + \delta x^\mu}{\partial x^\nu}\right) \mathcal{L}(\varphi', \partial\varphi')(x+\delta x).$$

Expandendo a primer orden em δx é lembrando que para todo espaço vetorial V de dim. finita $\det'(id_V) = tr$, temos

$$S_{R'}(\varphi') = \int_R d^4x \left(\det(\delta^\mu_\nu) + \partial_\mu \delta x^\mu\right)\left(\mathcal{L}(\varphi', \partial\varphi')(x) + \partial_\mu \mathcal{L}(\varphi', \partial\varphi')(x)\delta x^\mu\right)$$

$$= \int_R d^4x \left(1 + \partial_\mu \delta x^\mu\right)\Big(\mathcal{L}(\varphi+\delta\varphi, \partial\varphi + \partial\delta\varphi)(x)$$
$$+ \partial_\mu \mathcal{L}(\varphi+\delta\varphi, \partial\varphi+\partial\delta\varphi)(x)\delta x^\mu\Big)$$

$$= \int_R d^4x \left(1+\partial_\mu \delta x^\mu\right)\Big(\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi}(x)\delta\varphi(x) + \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}\partial_\mu\delta\varphi(x)$$
$$+ \partial_\mu \mathcal{L}(\varphi,\partial\varphi)(x)\delta x^\mu + \mathcal{L}(\varphi(x), \partial\varphi(x))\Big)$$

$$= S_R(\varphi) + \int_R d^4x \Big(\partial_\mu \delta x^\mu \mathcal{L}(\varphi(x), \partial\varphi(x)) + \partial_\mu \mathcal{L}(\varphi,\partial\varphi)(x)\delta x^\mu$$
$$+ \frac{\partial\mathcal{L}}{\partial\varphi}(\varphi,\partial\varphi)(x)\delta\varphi(x) - \partial_\mu \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}\delta\varphi(x)$$
$$+ \partial_\mu\left(\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}\delta\varphi\right)(x)\Big)$$

$$= S_R(\varphi) + \int_R d^4x \left(\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi}(x) - \partial_\mu \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}(x)\right)\delta\varphi(x)$$

$$+ \int_R d^4x\, \partial_\mu\left(\delta x^\mu \mathcal{L}(\varphi,\partial\varphi) + \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}\delta\varphi\right)(x).$$

Logo a variação da ação é

$$\delta S_R(\varphi) = S_{R'}(\varphi') - S_R(\varphi)$$

$$= \int_R d^4x \left(\left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x) \right) \delta\varphi(x) \right.$$

$$\left. + \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi} \delta\varphi + \mathcal{L}(\varphi,\partial\varphi)\delta x^\mu \right)(x) \right).$$

Principio de Ação Estacionaria: Os campos fisicos numa região R são t.q.

sob variações dos campos $\varphi \longmapsto \varphi' = \varphi + \delta\varphi$ infinitesimaes

onde $\delta\varphi|_{\partial R} = 0$, a ação é estacionaria.

Sob estas variações $\delta x^\mu = 0$. Logo, com o teorema de Stokes,

$$\delta S_R(\varphi) = \int d^4x \left(\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}(x) \right) \delta\varphi(x)$$

$$+ \cancelto{0}{\int_{\partial R} d\sigma^\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi} \delta\varphi},$$

onde $d\sigma^\mu(x)$ é o vetor de area normal a ∂R em x.

Então, devido a o teorema fundamental do cálculo de variações, os campos fisicos satisfacem as ecuaçoes de Euler-Lagrange

$$\partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi} - \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi} = 0.$$

2.1.1. Exemplos

Campo Escalar Real:

Temos a Lagrangiana

$$\mathcal{L}(\varphi, \partial\varphi) = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2.$$

Logo

$$\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi} = -m^2\varphi,$$

$$\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi} = \frac{1}{2}\frac{\partial}{\partial\partial_\mu\varphi}\left(\partial_\sigma\varphi\,\partial^\sigma\varphi\right)$$

$$= \frac{1}{2}\left(\frac{\partial\partial_\sigma\varphi}{\partial\partial_\mu\varphi}\partial^\sigma\varphi + \partial_\sigma\varphi\frac{\partial\partial^\sigma\varphi}{\partial\partial_\mu\varphi}\right)$$

$$= \frac{\partial\partial_\sigma\varphi}{\partial\partial_\mu\varphi}\partial^\sigma\varphi = \delta^\mu_\sigma\,\partial^\sigma\varphi = \partial^\mu\varphi.$$

Então as ecuações de Euler-Lagrange são a ecuação de Klein-Gordon

$$0 = \partial_\mu\partial^\mu\varphi + m^2\varphi = (\partial^2 + m^2)\varphi.$$

Campo Escalar Complexo:

Temos a Lagrangiana

$$\mathcal{L}(\varphi,\varphi^*,\partial\varphi,\partial\varphi^*) = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi.$$

Logo

$$\frac{\partial\mathcal{L}(\varphi,\varphi^*,\partial\varphi,\partial\varphi^*)}{\partial\varphi} = -m^2\varphi^*, \qquad \frac{\partial\mathcal{L}(\varphi,\varphi^*,\partial\varphi,\partial\varphi^*)}{\partial\varphi^*} = -m^2\varphi,$$

$$\frac{\partial\mathcal{L}(\varphi,\varphi^*,\partial\varphi,\partial\varphi^*)}{\partial\partial_\mu\varphi} = \partial^\mu\varphi^*, \qquad \frac{\partial\mathcal{L}(\varphi,\varphi^*,\partial\varphi,\partial\varphi^*)}{\partial\partial_\mu\varphi^*} = \partial^\mu\varphi.$$

Então, as ecuações de Euler-Lagrange são a equação de Klein-Gordon e seu conjugado

$$0 = \partial_\mu \partial^\mu \varphi + m^2 \varphi = (\partial^2 + m^2)\varphi,$$

$$0 = (\partial^2 + m^2)\varphi^*.$$

Campo de Dirac:

A Lagrangiana do campo de Dirac é

$$\mathcal{L}(\psi, \bar\psi, \partial\psi, \partial\bar\psi) = \frac{i}{2}\bar\psi \overleftrightarrow{\partial_\mu} \gamma^\mu \psi - M\bar\psi\psi,$$

onde $A\overleftrightarrow{\partial_\mu}B = A\partial_\mu B - \partial_\mu A B$. Logo

$$\frac{\partial\mathcal{L}(\psi, \bar\psi, \partial\psi, \partial\bar\psi)}{\partial\psi} = -\frac{i}{2}\partial_\mu\bar\psi\gamma^\mu - M\bar\psi,$$

$$\frac{\partial\mathcal{L}(\psi, \bar\psi, \partial\psi, \partial\bar\psi)}{\partial\bar\psi} = \frac{i}{2}\gamma^\mu\partial_\mu\psi - M\psi,$$

$$\frac{\partial\mathcal{L}(\psi, \bar\psi, \partial\psi, \partial\bar\psi)}{\partial\partial_\mu\psi} = \frac{i}{2}\bar\psi\gamma^\mu, \qquad \frac{\partial\mathcal{L}(\psi, \bar\psi, \partial\psi, \partial\bar\psi)}{\partial\partial_\mu\bar\psi} = -\frac{i}{2}\gamma^\mu\psi.$$

Então, as ecuações de Euler-Lagrange são a ecuação de Dirac e sua conjugada

$$0 = -\frac{i}{2}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\gamma^\mu\partial_\mu\psi + M\psi = -(i\not\partial - M)\psi,$$

$$0 = \bar\psi(i\overleftarrow{\not\partial} + M).$$

Campo Eletromagnético:

Temos a Lagrangiana

$$\mathcal{L}(A, F, \partial A, \partial F) = -\frac{1}{2}F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}(F^{\mu\nu} - \partial^\mu A^\nu - \partial^\nu A^\mu).$$

Observe que na segunda forma temos um camo F com Lagrangiana $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ sujeto a condição $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ com multiplicador de Lagrange $F_{\mu\nu}$. Em efeito

$$\frac{\partial \mathcal{L}(A,F,\partial A,\partial F)}{\partial F_{\mu\nu}} = \frac{1}{2} F^{\mu\nu} - \frac{1}{2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\frac{\partial \mathcal{L}(A,F,\partial A,\partial F)}{\partial A_\mu} = 0$$

$$\frac{\partial \mathcal{L}(A,F,\partial A,\partial F)}{\partial \partial_\mu F} = 0$$

$$\frac{\partial \mathcal{L}(A,F,\partial A,\partial F)}{\partial \partial_\mu A_\nu} = -\frac{1}{2} F^{\rho\sigma}\left(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu\right)$$

$$= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right).$$

Logo uma das ecuações de Euler-Lagrange é a ligadura

$$0 = -\frac{1}{2} F^{\mu\nu} + \frac{1}{2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \Longrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

e a outra são as ecuações homogeneas de Maxwell

$$0 = -\frac{1}{2}\left(\partial_\mu F^{\mu\nu} - \partial_\mu F^{\nu\mu}\right) = -\partial_\mu F^{\mu\nu} \Rightarrow 0 = \partial_\mu F^{\mu\nu}.$$

Usamos que pela primeira $F_{\mu\nu} = -F_{\nu\mu}$. Logo, lembrando que $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, podemos usar a Lagrangiana

$$\mathcal{L}(A,\partial A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Com esta

$$\frac{\partial \mathcal{L}(A,\partial A)}{\partial A_\mu} = 0,$$

$$\frac{\partial \mathcal{L}(A,\partial A)}{\partial \partial_\mu A_\nu} = -\frac{1}{2}\frac{\partial F_{\rho\sigma}}{\partial \partial_\mu A_\nu} F^{\rho\sigma} = -\frac{1}{2}\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma\right) F^{\rho\sigma}$$

$$= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) = -F^{\mu\nu},$$

e, as ecuações de Euler-Lagrange são

$$\partial_\mu F^{\mu\nu} = 0.$$

Electrodinámica:

Considera a interação do campo eletromagnético com um campo de Dirac descrita pela Lagrangiana

$$\mathcal{L}(\psi,\bar\psi,A,\partial\psi,\partial\bar\psi,\partial A) = \mathcal{L}_{\text{Dirac}}(\psi,\partial\psi) + \mathcal{L}_{\text{Maxwell}}(A,\partial A) + \underbrace{e\bar\psi\gamma_\mu\psi A^\mu}_{e\bar\psi \not{A}\psi}$$

Como

$$\frac{\partial}{\partial\psi}\left(e\bar\psi\gamma_\mu\psi A^\mu\right) = e\bar\psi\not{A},$$

$$\frac{\partial}{\partial\bar\psi}\left(e\bar\psi\gamma_\mu\psi A^\mu\right) = e\not{A}\psi,$$

$$\frac{\partial}{\partial A_\mu}\left(e\bar\psi\gamma^\sigma\psi A_\sigma\right) = e\bar\psi\gamma^\mu\psi$$

Logo as ecuações de Euler-Lagrange são

$$i(\not{\partial} - ie\not{A} - M)\psi = 0$$

$$\partial_\mu F^{\mu\nu} = -e\bar\psi\gamma^\nu\psi.$$

2.1.2. Derivação Funcional

Agora presentamos uma alternativa as ecuações de Euler-Lagrange. Considere um funcional F sobre o espacio de configuração do multiplete φ. Definimos a derivada funcional em x de F por

$$\frac{\delta F(\varphi)}{\delta \varphi(x)} = \lim_{\varepsilon \to 0} \frac{F(\varphi + \varepsilon \delta_x) - F(\varphi)}{\varepsilon}.$$

Em particular, para o funcional evaluación en y $ev_y(\varphi) = \varphi(y)$, temos

$$\frac{\delta \varphi(y)}{\delta \varphi(x)} = \lim_{\varepsilon \to 0} \frac{ev_y(\varphi + \varepsilon \delta_x) - ev_y(\varphi)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{\varphi(y)} + \varepsilon \delta(y-x) - \cancel{\varphi(y)}}{\varepsilon}$$

$$= \delta(y-x).$$

Logo, si x é um ponto interior de R

$$\frac{\delta S_R(\varphi)}{\delta \varphi(x)} = \int d^4y \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(y) \frac{\delta \varphi(y)}{\delta \varphi(x)} + \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(y) \frac{\partial}{\partial y^\mu} \frac{\delta \varphi(y)}{\delta \varphi(x)} \right)$$

$$= \int d^4y \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(y) - \frac{\partial}{\partial y^\mu} \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(y) \right) \frac{\delta \varphi(y)}{\delta \varphi(x)}$$

$$- \int d^4y \frac{\partial}{\partial y^\mu} \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(y) \frac{\delta \varphi(y)}{\delta \varphi(x)} \right)$$

$$= \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x)$$

$$- \cancelto{0}{\int d\sigma_\mu(y) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}}(y) \delta(y-x).$$

Então podemos reemplazar o principio de minima ação por $\frac{\delta S(\varphi)}{\delta \varphi} = 0$ para os configorações físicas.

Exercicio 2.1.

Suponga que a Lagrangiana depende dos primeros n derivados do campo. Então as ecuações de movimiento são

$$0 = \frac{\delta S(\varphi)}{\delta \varphi(x)} = \int d^4y \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi, \ldots, \partial^n\varphi)}{\partial \varphi}(y) \frac{\delta \varphi(y)}{\delta \varphi(x)} \right.$$

$$\left. + \sum_{k=1}^{n} \frac{\partial \mathcal{L}(\varphi, \partial\varphi, \ldots, \partial^n\varphi)}{\partial \partial_{\mu_1} \cdots \partial_{\mu_k} \varphi}(y) \prod_{i=1}^{k} \left(\frac{\partial}{\partial y^{\mu_i}} \right) \frac{\delta \varphi(y)}{\delta \varphi(x)} \right)$$

$$= \int d^4y \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi, \ldots, \partial^n\varphi)}{\partial \varphi}(y) + \sum_{k=1}^{n} (-1)^k \prod_{i=1}^{k} \left(\frac{\partial}{\partial y^{\mu_i}} \right) \frac{\partial \mathcal{L}(\varphi, \partial\varphi, \ldots, \partial^n\varphi)}{\partial \partial_{\mu_1} \cdots \partial_{\mu_k} \varphi}(y) \right) \frac{\delta \varphi(y)}{\delta \varphi(x)}$$

$$= \frac{\partial \mathcal{L}(\varphi, \partial\varphi, \ldots, \partial^n\varphi)}{\partial \varphi}(x) + \sum_{k=1}^{n} (-1)^k \prod_{i=1}^{k} \left(\frac{\partial}{\partial y^{\mu_i}} \right) \frac{\partial \mathcal{L}(\varphi, \partial\varphi, \ldots, \partial^n\varphi)}{\partial \partial_{\mu_1} \cdots \partial_{\mu_k} \varphi}.$$

2.2. Primer Teorema de Noether

Sea G um grupo de Lie. Suponga que (R, A) é uma representação afim de G sobre o espaço tempo $\mathbb{R}^4$.

Em particular, temos homomorphismos

$$R : G \longrightarrow GL(\mathbb{R}^4),$$

$$A : G \longrightarrow \mathbb{R}^4$$

t.q. para todo $g \in G$, e $x \in \mathbb{R}^4$

$$(R, A)(g)x = R(g)x + A(g).$$

Sea $\mathfrak{g}$ a álgebra de Lie de G. Se tem uma representação afim de $\mathfrak{g}$ inducida $(r,a):=(R',A')$. Intuitivamente, para $X\in\mathfrak{g}$ infinitesimal

$$R(e^X)\approx R(e_G+X)\approx R(e_G)+R'(X)=I+r(X),$$

$$A(e^X)\approx A(e_G+X)\approx A(e_G)+A'(X)=0+a(X)=a(X).$$

Logo, sob a ação do G por e_G+X

$$x'=(R,A)(e^X)x = x + r(X)x + a(X),$$

e decir

$$\delta x^\mu = r(X)^\mu{}_\nu x^\nu + a(X)^\mu.$$

Suponga ademais que o campo pertenece a uma representação D de G, i.e.

$$\varphi'(x)=D(g)\varphi((R,A)(g^{-1})x).$$

Então, sob X, si $d=D'$,

$$\varphi'(x)=(I+d(X))\varphi(x-r(X)x-a(X))$$
$$\approx \varphi(x)+d(X)\varphi(x)-\partial_\mu\varphi(x)r(X)^\mu{}_\nu x^\nu-\partial_\mu\varphi(x)a(X)^\mu.$$

Logo

$$\delta\varphi(x)=d(X)\varphi(x)-\partial_\mu\varphi(x)r(X)^\mu{}_\nu x^\nu-\partial_\mu\varphi(x)a(X)^\mu.$$

Seam $\{T_A \mid A\in\{1,\dots,N:=\dim G\}\}$ geradores de $\mathfrak{g}$. Então, si $X=\delta\omega^A T_A$, temos

$$\delta x^\mu = \frac{\delta x^\mu}{\delta \omega^A}\delta\omega^A \qquad \text{e} \qquad \delta\varphi(x) = \frac{\delta\varphi(x)}{\delta\omega^A}\delta\omega^A$$

com

$$\frac{\delta x^\mu}{\delta\omega^A} := r(T_A)^\mu{}_\nu x^\nu + a(T_A)^\mu$$

$$\frac{\delta\varphi(x)}{\delta\omega^A} := d(T_A)\varphi(x) - \partial_\mu\varphi(x) r(T_A)^\mu{}_\nu x^\nu - \partial_\mu\varphi(x) a(T_A)^\mu.$$

Logo, si φ satisface as ecuações de movimento,

$$\delta S_R(\varphi) = \int_R d^4x\, \partial_\mu j^\mu{}_A\, \delta\omega^A,$$

onde

$$j^\mu{}_A := \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}\frac{\delta\varphi}{\delta\omega^A} + \mathcal{L}(\varphi,\partial\varphi)\frac{\delta x^\mu}{\delta\omega^A}.$$

Por o teorema fundamental de o calculo de variações, si para todo $\delta\omega^A$ temos uma simetria de a ação, $\delta S_R(\varphi)=0$, então $\partial_\mu j^\mu{}_A = 0$. Agora, si $X = \omega^A T_A$, tomando $j^\mu = \omega^A j^\mu{}_A$, obtemos tamben $\partial_\mu j^\mu = 0$. A esto se conheçe como o primer teorema de Noether. Em particular, si $R \subseteq \mathbb{R}^3$ é uma região e

$$Q_R(t) := \int_R d^3\vec{x}\; j^0(t,\vec{x}),$$

temos a ecuação de continuidade

$$Q'_R(t) = -\int_R d^3\vec{x}\; \vec{\nabla}\cdot\vec{j}(t,\vec{x}) = -\int_{\partial R} d\vec{S}\cdot\vec{j}(t,\vec{x}).$$

Si $\vec{j}$ decae o suficentemente rápido, $Q(t) = Q_{\mathbb{R}^3}$ é conservado

2.2.1. Simetrias

Translações

Pege o grupo $G = \mathbb{R}^4$ de translações. Então

$$R(v) = I \, , \qquad r(X) = 0 \, , \qquad D(v) = I \, ,$$

$$A(v) = v \, , \qquad a(X) = X_\varphi \qquad d(v) = 0 .$$

Sea δ_μ uma base de $\mathbb{R}^4$, digamos a cononica. Logo temos a corrente conservada

$$j^\mu{}_\nu = - \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} \partial_\sigma \varphi \, \delta^\sigma_\nu + \mathcal{L}(\varphi, \partial\varphi) \, \delta^\mu_\nu \, ,$$

i.e., temos o tensor

$$\Theta^{\mu\nu} = -j^{\mu\nu} = \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}(\varphi, \partial\varphi) ,$$

conhecido como o tensor de energia-momento canônico. Si a ação é invoriante sob translações, esta é uma corrente conservada e define o cuadrimomento do campo mediante suas cargas

$$P^\mu(t) = \int d^3\vec{x} \; \Theta^{0\mu}(t, \vec{x}) .$$

Campo Escalar (Real)

Temos

$$\Theta^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}(\varphi, \partial\varphi).$$

Em particular,

$$\Theta^{00} = \frac{1}{2}\left((\partial^0 \varphi)^2 + |\vec{\nabla}\varphi|^2 + m^2 \varphi^2 \right) > 0$$

$$\Theta^{0K} = \partial^0 \varphi \partial^K \varphi .$$

Campo Escalar Complexo

$$\Theta^{\mu\nu} = \partial^\mu \varphi^* \partial^\nu \varphi + \partial^\mu \varphi \partial^\nu \varphi^* - g^{\mu\nu} \mathcal{L}$$

Em particular

$$\Theta^{00} = |\partial^0 \varphi|^2 + |\vec{\nabla}\varphi|^2 + m^2 \varphi^2 > 0$$

$$\Theta^{0K} = \partial^0 \varphi^* \partial^K \varphi + \partial^0 \varphi \partial^K \varphi^* .$$

Campo de Dirac

$$\Theta^{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma^\mu \partial^\nu \psi - \frac{i}{2} \partial^\nu \bar{\psi} \gamma^\mu \psi - g^{\mu\nu} \mathcal{L}$$

$$= \frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial^\nu} \psi - g^{\mu\nu} \mathcal{L}$$

Em particular

$$\Theta^{00} = \cancel{\frac{i}{2} \bar{\psi} \gamma^0 \overleftarrow{\partial^0}} \psi - \cancel{\frac{i}{2} \bar{\psi} \gamma^0 \overleftrightarrow{\partial^0}} \psi + \frac{i}{2} \bar{\psi} \overleftrightarrow{\vec{\partial}} \psi + M \psi \bar{\psi},$$

$$\Theta^{0K} = \frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial^K} \psi .$$

Campo Eletromagnético

$$\Theta^{\mu\nu} = - F^{\mu\sigma} \partial^\nu A_\sigma - g^{\mu\nu} \mathcal{L}(A, \partial A).$$

Em

Considere o grupo de Lorentz $O(1,3)=G$. Sua algebra de Lie $o(1,3)=g$ é o grupo de matrices $i\omega^{\mu}{}_{\nu}$ onde

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

são reales. Uma base de g são as matrices $\frac{i}{2}M_{\mu\nu}$ com

$$\frac{i}{2}(M^{\mu\nu})_{\rho\sigma} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}, \qquad \mu<\nu.$$

Temos

$$R(\Lambda)=\Lambda, \qquad r(\Omega)=\Omega,$$
$$A(\Lambda)=0, \qquad a(\Omega)=0.$$

Definimos $\frac{i}{2}I_{\mu\nu} = d(M_{\mu\nu})$. Logo

$$J^{\mu}{}_{\nu\rho} = \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_{\mu}\varphi}\left(\frac{i}{2}I_{\nu\rho}\varphi(x) - \partial_{\mu}\varphi(x)\frac{1}{2}(M_{\nu\rho})^{\mu}{}_{\lambda}x^{\lambda}\right) + \mathcal{L}(\varphi,\partial\varphi)\frac{1}{2}(M_{\nu\rho})^{\mu}{}_{\sigma}x$$

$$= \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_{\mu}\varphi}\left(\frac{i}{2}I_{\nu\rho}\varphi(x) - \frac{1}{2}x_{\rho}\partial_{\nu}\varphi(x) + \frac{1}{2}x_{\nu}\partial_{\rho}\varphi(x)\right)$$

$$+\frac{1}{2}\mathcal{L}(\varphi,\partial\varphi)\left(x_{\rho}\delta^{\mu}_{\nu} - x_{\nu}\delta^{\mu}_{\rho}\right)$$

$$= -\frac{1}{2}\left(x_{\rho}\Theta^{\mu}{}_{\nu} - x_{\nu}\Theta^{\mu}{}_{\rho}\right) + i\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_{\mu}\varphi}I_{\nu\rho}\varphi(x)\Big).$$

Definimos o tensor de momento angular

$$M^{\mu\nu\lambda} = -x^{\lambda}\Theta^{\mu\nu} + x^{\nu}\Theta^{\mu\lambda} + i\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}I^{\nu\lambda}\varphi.$$

2.2.2. Tensor de Belintante.

Queremos um tensor de energia-momento conservado com o mesmo contenido que $\Theta^{\mu\nu}$. Suponga que temos $\Omega^{\rho\mu\nu} = -\Omega^{\mu\rho\nu}$ desvanece no infinito. Logo

$$T^{\mu\nu} = \Theta^{\mu\nu} - \partial_\rho \Omega^{\rho\mu\nu}$$

é conservado

$$\partial_\mu T^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} - \cancel{\partial_\mu \partial_\rho \Omega^{\rho\mu\nu}}^{0} = \partial_\mu \Theta^{\mu\nu}$$

si Θ o é, e tem o mesmo contenido de energia-momento

$$\begin{aligned}
\int_{\mathbb{R}^4} d^3\vec{x}\, T^{0\mu} &= \int_{\mathbb{R}^4} d^3\vec{x}\, T^{0\nu} - \int_{\mathbb{R}^4} d^3\vec{x}\, \partial_\rho \Omega^{\rho 0\nu} \\
&= \int_{\mathbb{R}^4} d^3\vec{x}\, T^{0\nu} - \cancel{\int_{\mathbb{R}^4} d^3\vec{x}\, \partial_0 \Omega^{00\nu}}^{0} + \int_{\mathbb{R}^4} d^3x\, \vec{\nabla}\cdot\vec{\Omega}^{\cdot 0\nu} \\
&= \int_{\mathbb{R}^4} d^3\vec{x}\, T^{0\nu} + \cancel{\int_{\partial\mathbb{R}^4} d\vec{S}\cdot\vec{\Omega}^{\cdot 0\nu}}^{0} \\
&= \int_{\mathbb{R}^4} d^3\vec{x}\, T^{0\nu}.
\end{aligned}$$

Construiremos Ω t.q. $T^{\mu\nu} = T^{\nu\mu}$. Note que

$$H^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu} - \partial_\rho A^{\rho\mu\lambda\nu} + \Omega^{\lambda\mu\nu} - \Omega^{\nu\mu\lambda} + H^{\nu\mu\lambda},$$

onde

$$A^{\rho\mu\lambda\nu} = x^\lambda \Omega^{\rho\mu\nu} - x^\nu \Omega^{\rho\mu\lambda} = -A^{\rho\mu\nu\lambda} = -A^{\mu\rho\lambda\nu}$$

$$H^{\nu\mu\lambda} = i \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} I^{\nu\lambda} \varphi = -H^{\lambda\mu\nu}.$$

Logo $T^{\mu\nu} = T^{\nu\mu}$, se tem si

$$H^{\nu\mu\lambda} = \Omega^{\nu\mu\lambda} - \Omega^{\lambda\mu\nu}.$$

Em efecto

$$0 = \partial_\mu M^{\mu\nu\lambda} = T^{\nu\lambda} - T^{\lambda\nu} + \cancel{\partial_\mu \partial_\rho \Lambda^{\rho\mu\lambda\nu}} \xrightarrow{} 0 \, .$$

Esto se logra com

$$\frac{1}{2}\left(H^{\nu\mu\lambda} - H^{\mu\lambda\nu} + H^{\lambda\nu\mu}\right) =: \Omega^{\nu\mu\lambda},$$

pois

$$\Omega^{\mu\nu\lambda} = \frac{1}{2}\left(H^{\mu\nu\lambda} - H^{\nu\lambda\mu} + H^{\lambda\mu\nu}\right)$$

$$= \frac{1}{2}\left(-H^{\lambda\nu\mu} + H^{\mu\lambda\nu} - H^{\nu\mu\lambda}\right) = -\Omega^{\nu\mu\lambda}$$

e

$$\Omega^{\nu\mu\lambda} - \Omega^{\lambda\mu\nu} = \frac{1}{2}\left(H^{\nu\mu\lambda} - \cancel{H^{\mu\lambda\nu}} + \cancel{H^{\lambda\nu\mu}} - H^{\lambda\mu\nu} + \cancel{H^{\mu\nu\lambda}} - \cancel{H^{\nu\lambda\mu}}\right)$$

$$= H^{\nu\mu\lambda}.$$

A $T^{\mu\nu}$ se conhece como tensor de Belifante.

Exercicio 2.2.

Campo de Dirac

Temos para o campo de Dirac $D = S$. Logo

$$\frac{i}{2} I^{\nu\lambda} = \frac{1}{8}[\gamma_\mu, \gamma_\nu].$$

Logo

$$I^{\nu\lambda} = -\frac{i}{4}[\gamma^\nu, \gamma^\lambda].$$

Pelo tanto para o conjugado, como

$$(S(\Lambda)\psi)^\dagger \gamma^0 = \psi^\dagger \gamma^0 \gamma^0 S(\Lambda)\gamma^0 = \bar{\psi} S^{-1},$$

se tem

$$H^{\nu\mu\lambda} = i\,\frac{\partial\mathcal{L}(\psi,\bar\psi,\partial\psi,\partial\bar\psi)}{\partial\partial_\mu\psi}\left(-\frac{i}{4}\right)[\gamma^\nu,\gamma^\lambda]\psi + i\,\bar\psi\,\frac{i}{4}[\gamma^\nu,\gamma^\lambda]\frac{\partial\mathcal{L}(\psi,\bar\psi,\partial\psi,\partial\bar\psi)}{\partial\partial_\mu\bar\psi}$$

$$= \frac{i}{8}\bar\psi\gamma^\mu[\gamma^\nu,\gamma^\lambda]\psi + \frac{i}{8}\bar\psi[\gamma^\nu,\gamma^\lambda]\gamma^\mu\psi$$

$$= \frac{i}{8}\bar\psi\{\gamma^\mu,[\gamma^\nu,\gamma^\lambda]\}\psi.$$

Temos então

$$\Omega^{\nu\mu\lambda} = \frac{i}{16}\bar\psi\left(\{\gamma^\mu,[\gamma^\nu,\gamma^\lambda]\} - \{\gamma^\lambda,[\gamma^\mu,\gamma^\nu]\} + \{\gamma^\nu,[\gamma^\lambda,\gamma^\mu]\}\right)\psi.$$

Observe que

$$\{A,[B,C]\} - \{B,[C,A]\} = ABC - \cancel{ACB} + \cancel{BCA} - CBA$$
$$- \cancel{BCA} + BAC - CAB + \cancel{ACB}$$
$$= [\{A,B\},C] = 0$$

si $A,B,C \in \{\gamma^\mu \mid \mu\in\{0,\dots,3\}\}$. Logo

$$\Omega^{\nu\mu\lambda} = \frac{i}{16}\bar\psi\,\{\gamma^\nu,[\gamma^\lambda,\gamma^\mu]\}\psi.$$

Agora bem, lembre que

$$[\gamma^\nu,[\gamma^\lambda,\gamma^\mu]] = \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda - \gamma^\lambda\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\lambda\gamma^\nu$$
$$= 2\gamma^\nu\gamma^\lambda\gamma^\mu - \cancel{2g^{\mu\lambda}\gamma^\nu} - 2\gamma^\lambda\gamma^\mu\gamma^\nu + \cancel{2g^{\mu\lambda}\gamma^\nu}$$
$$= 2\gamma^\nu\gamma^\lambda\gamma^\mu + 4\gamma^\lambda\gamma^\nu\gamma^\mu - 4g^{\mu\nu}\gamma^\lambda$$
$$= 4g^{\lambda\nu}\gamma^\mu - 4g^{\mu\nu}\gamma^\lambda.$$

Logo

$$\Omega^{\nu\mu\lambda} = \frac{i}{4}\bar{\psi}(g^{\lambda\nu}\gamma^{\mu} - g^{\mu\nu}\gamma^{\lambda}) + \frac{i}{8}\bar{\psi}[\gamma^{\lambda},\gamma^{\mu}]\gamma^{\nu}\psi.$$

Logo

$$\partial_{\nu}\Omega^{\nu\mu\lambda} = \frac{i}{4}\bar{\psi}(\gamma^{\mu}\vec{\partial}^{\lambda} - \gamma^{\lambda}\vec{\partial}^{\mu} + \gamma^{\mu}\overleftarrow{\partial}^{\lambda} - \gamma^{\lambda}\overleftarrow{\partial}^{\mu})\psi + \frac{i}{8}\bar{\psi}[\gamma^{\lambda},\gamma^{\mu}](\vec{\partial}+\overleftarrow{\partial})$$

e

$$T^{\mu\nu} = \frac{i}{4}\bar{\psi}(2\gamma^{\mu}\vec{\partial}^{\nu} - 2\gamma^{\mu}\overleftarrow{\partial}^{\nu} - \cancel{\gamma^{\mu}\vec{\partial}^{\nu}} + \gamma^{\nu}\vec{\partial}^{\mu} + \cancel{\gamma^{\mu}\overleftarrow{\partial}^{\nu}} - \gamma^{\nu}\overleftarrow{\partial}^{\mu})\psi$$

$$- g^{\mu\nu}\mathcal{L} + \frac{i}{8}\bar{\psi}[\gamma^{\nu},\gamma^{\mu}](\vec{\partial}+\overleftarrow{\partial})\psi.$$

Como $T^{\mu\nu} = \frac{1}{2}T^{(\mu\nu)}$ e $[\gamma^{(\nu},\gamma^{\mu)}] = 0$, temos

$$T^{\mu\nu} = \frac{i}{4}\bar{\psi}(\gamma^{\mu}\overleftrightarrow{\partial}^{\nu} + \gamma^{\nu}\overleftrightarrow{\partial}^{\mu})\psi - g^{\mu\nu}\mathcal{L}$$

$$= \frac{i}{8}\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi - g^{\mu\nu}\mathcal{L}.$$

Campo de Maxwell.

Temos $d(M_{\mu\nu}) = M_{\mu\nu}$ e pelo tanto $I_{\mu\nu} = -iM_{\mu\nu}$. Logo

$$H^{\nu\mu\lambda} = i(-F^{\mu\rho})(-iM^{\nu\lambda})_{\rho\sigma}A^{\sigma}$$

$$= -F^{\mu\nu}A^{\lambda} + F^{\mu\lambda}A^{\nu}.$$

Então

$$\Omega^{\nu\mu\lambda} = \frac{1}{2}(\cancel{F^{\mu\lambda}A^{\nu}} - F^{\mu\nu}A^{\lambda} - \cancel{F^{\lambda\nu}A^{\mu}} + \cancel{F^{\lambda\mu}A^{\nu}} + F^{\nu\mu}A^{\lambda} - \cancel{F^{\nu\lambda}A^{\mu}})$$

$$= F^{\nu\mu}A^{\lambda},$$

e

$$\partial_\rho \Omega^{\rho\mu\nu} = \partial_\rho(F^{\rho\mu}A^\nu) = \cancel{\partial_\rho F^{\rho\mu}}A^\nu + F^{\rho\mu}\partial_\rho A^\nu.$$

O tensor de Belifante então é

$$\begin{aligned} T^{\mu\nu} &= -F^{\mu\sigma}\partial^\nu A_\sigma - F^{\rho\mu}\partial_\rho A^\nu - g^{\mu\nu}\mathcal{L}(A,\partial A) \\ &= F^{\rho\mu}(\partial^\nu A_\rho - \partial_\rho A^\nu) - g^{\mu\nu}\mathcal{L}(A,\partial A) \\ &= F_\rho{}^\mu F^{\nu\rho} - g^{\mu\nu}\mathcal{L}(A,\partial A). \end{aligned}$$

2.2.3. Transformações Conformes

O grupo multiplicativo $(0,\infty)$ actua sobe o espaçotempo por dilatações

$$R(\lambda) = \lambda \qquad r(\lambda) = \lambda$$

$$A(\lambda) = 0 \qquad a(\lambda) = 0.$$

Si $b \in \mathbb{R}^4$, e $x \in \mathbb{R}^4$ é t.q

$$1 - 2b\cdot x + b^2x^2 \neq 0,$$

então, a transformação conforme espécial de x é

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b.x + b^2x^2}.$$

Note que esta não é uma transformação afim e não é global, no sentido que existe $x \in \mathbb{R}^4$ t.q.

$$1 - 2b\cdot x + b^2x^2 = 0.$$

É claro que o grupo de Poincaré junto com as dilatações claramente preservam o cone de Luz. As transformações conformes especiais também. Para ver isso, observe

$$x'^2 = \frac{x^2 - 2b\cdot x\, x^2 + b^2 x^4}{\sigma(x)^2} = \frac{x^2 \cancel{\sigma(x)}}{\sigma(x)^{\cancel{2}}} = \frac{x^2}{\sigma(x)},$$

onde $\sigma(x) = 1 - 2b\cdot x + b^2 x^2$. Além mais,

$$(x'-y')^2 = x'^2 - 2x'\cdot y' + y'^2 = \frac{x^2}{\sigma(x)} + \frac{y^2}{\sigma(y)} - 2\frac{(x - bx^2)\cdot(y-by^2)}{\sigma(x)\sigma(y)}$$

$$= \frac{1}{\sigma(x)\sigma(y)}\left(x^2\sigma(y) + y^2\sigma(x) - 2(x-bx^2)\cdot(y-by^2)\right)$$

$$= \frac{1}{\sigma(x)\sigma(y)}\left(x^2 - 2b\cdot\cancel{y}\,x^2 + b^2\cancel{y^2}x^2 + y^2 - 2b\cdot\cancel{x\,y^2} + b^2\cancel{x^2}y^2 \right.$$
$$\left. - 2x\cdot y + 2\cancel{x}\,b\,y^2 + 2b\cancel{y}\,x^2 - 2b^2\cancel{x^2}y^2\right)$$

$$= \frac{1}{\sigma(x)\sigma(y)}(x-y)^2.$$

Exercício 2.3.

Asuma que $D(\lambda) = \lambda^{-d}$. Então para $\lambda = 1+\varepsilon$ com ε infinitesimal $d(\varepsilon) = -d\varepsilon$ e

$$\delta x = \varepsilon x,$$

$$\delta\varphi(x) = -\varepsilon d\varphi(x) - \partial_\mu\varphi(x)\,\varepsilon x^\mu.$$

Logo, sob esta transformação, a variação da ação é

$$\delta S_R(\varphi) = \int d^4x \left(\left(\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x) \right) \left(-\varepsilon d\varphi(x) - \varepsilon x^\mu \partial_\mu \varphi(x)\right) \right.$$

$$\left. + \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi} (-\varepsilon d\varphi - \varepsilon x^\nu \partial_\nu \varphi) + \mathcal{L}(\varphi,\partial\varphi)\varepsilon x^\mu \right)(x) \right)$$

$$= \int d^4x \left(-\varepsilon d\varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}(x) + \varepsilon d\varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}}(x) \right.$$

$$- \varepsilon x^\mu \partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}(x) + \varepsilon x^\nu \partial_\nu \varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}}(x)$$

$$- \varepsilon d\varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}}(x) - \varepsilon d \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x) \partial_\mu \varphi(x)$$

$$- \varepsilon x^\nu \partial_\nu \varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}}(x) - \varepsilon \partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x)$$

$$- \varepsilon x^\nu \partial_\mu \partial_\nu \varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x) + \varepsilon x^\mu \partial_\mu \mathcal{L}(\varphi,\partial\varphi)(x)$$

$$\left. + 4\varepsilon \mathcal{L}(\varphi(x), \partial\varphi(x)) \right)$$

$$= -\varepsilon \int d^4x \left(-d\varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}(x) + (d+1)\partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x) \right.$$

$$- 4\mathcal{L}(\varphi(x), \partial\varphi(x)) + x^\mu \partial_\mu \varphi(x) \cancel{\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}}(x)$$

$$+ x^\nu \cancel{\partial_\mu \partial_\nu \varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}}(x) - x^\mu \cancel{\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}}(x) \partial_\mu \varphi(x)$$

$$\left. - x^\mu \cancel{\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\nu \varphi}(x) \partial_\mu \partial_\nu \varphi(x)} \right)$$

$$= -\varepsilon \int d^4x \left(d\varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \varphi}(x) + (d+1)\partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial \partial_\mu \varphi}(x) - 4\mathcal{L}(\varphi,\partial\varphi)(x) \right)$$

Logo as dilatações som uma simetria da ação si o Lagrangiano satisface

$$d\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi}(x)\varphi(x) + (d+1)\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}(x)\partial_\mu\varphi(x) = 4\mathcal{L}(\varphi(x),\partial\varphi(x)).$$

Considere o Lagrangiano

$$\mathcal{L}(\varphi,\partial\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi.$$

Se tem

$$d\cancel{\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi}}^{\,0}(x)\varphi(x) + (d+1)\frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}(x)\partial_\mu\varphi(x) = (d+1)\partial^\mu\varphi(x)\partial_\mu\varphi(x).$$

Logo, si $d = D-1 = 4-1 = 3$, as dilatações são uma simetria. Logo

$$j^\mu = \frac{\partial \mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi}(-d\varphi(x) - x^\nu\partial_\nu\varphi(x)) + \mathcal{L}(\varphi,\partial\varphi)x^\mu$$

$$= -d\pi^\mu\varphi - x^\nu\partial^\mu\varphi\partial_\nu\varphi + \mathcal{L}(\varphi,\partial\varphi)x^\mu$$

$$= -d\pi^\mu\varphi - x_\nu(\partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}(\varphi,\partial\varphi))$$

$$= -d\pi^\mu\varphi - x_\nu \Theta^{\mu\nu}.$$

Agora procuremos a dimenção de massa de φ.

Se tem

$$1 = [\hbar] = [S] = [d^D x][\mathcal{L}] = L^D [\partial_\mu \varphi \partial^\mu \varphi] = L^{\cancel{D}\, D-2} \cancel{L^{-2}} [\varphi]^2$$

$$= L^{D-2}[\varphi]^2 = M^{2-D}[\varphi]^2$$

onde se uso $L = M^{-1}$ como é claro de

$$1 = [\hbar] = [\vec{r} \times \vec{p}] = [\vec{r}][m][\cancel{\vec{v}}]^{1} = LM.$$

Logo $[\varphi] = M^{\frac{D-2}{2}}$. Para o campo de Dirac temos analogamente

$$1 = [S] = [d^D x][M \bar{\psi} \psi] = L^D M [\psi]^2 = M^{1-D}[\psi]^2,$$

é dizer

$$[\psi] = M^{\frac{D-1}{2}}.$$

2.2.4. Transformações de Calibre Globais

Além das simetrias geometricas estudiadas, uma teoria pode ter simetrias internas. Estas estam caracterizadas por $R(g) = I$, $A(g) = 0$ para todo $g \in G$. Pelo tanto $r(X) = a(X) = 0$ para todo $X \in \mathfrak{g}$. Por outra parte, D é não trivial e muitas veces é a representação fondamental. Asoçada temos a corrente conservada

$$j^\mu = \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} d(X)\varphi$$

para todo $X \in \mathfrak{g}$.

Campo Escalar Complexo

Temos $G = U(1)$ e $D(e^{i\alpha}) = e^{i\alpha}$. Então $\mathfrak{g} = \mathbb{R}$ e $d(i\alpha) = i\alpha$. Logo

$$j^\mu = -i\left(\partial^\mu \varphi^* \varphi - \partial^\mu \varphi \varphi^*\right) = i\left(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*\right)$$

Campo de Dirac

Temos o mesmo que arriba

$$j^\mu = -i\left(\frac{i}{2}\bar{\psi}\gamma^\mu\psi + \frac{i}{2}\bar{\psi}\gamma^\mu\psi\right) = \bar{\psi}\gamma^\mu\psi.$$

Considere $\psi = (\psi^1, \ldots, \psi^N)$ con ψ^A espinores de Dirac. Suponga que ψ pertenece a representação fundamental de SU(n). Logo a ação de $U \in SU(n)$ sobre ψ é

$$\psi' = U\psi \qquad (\psi'^A = U^A{}_B \psi^B).$$

A algebra de Lie su(n) é definida por

$$I = (I+X)(I+X)^\dagger = I + X + X^\dagger$$

$$1 = \det(I+X) = \det(I) + \det'(I)X = 1 + tr(X).$$

para todo $X \in su(n)$. Logo

$$su(n) = \{ iX \in M_n(\mathbb{C}) \mid X = X^\dagger \text{ e } tr(X) = 1 \}.$$

Em particular, $\dim_{\mathbb{R}} su(n) = n^2 - 1$. Pege geradoes iT_A, $A \in \{1, \ldots, n^2-1\}$ de su(n). Então para ε^A infinitesimoes

$$\bar{\psi} T_A \psi \longmapsto \bar{\psi} e^{-i\varepsilon^B T_B} T_A e^{i\varepsilon^A T_A} \psi = \bar{\psi}(I - i\varepsilon^B T_B) T_A (I + i\varepsilon^C T_C)\psi$$

$$= \bar{\psi} T_A \psi - i\varepsilon^B \bar{\psi} T_B T_A \psi + i\varepsilon^B \bar{\psi} T_A T_B \psi$$

$$= \bar{\psi} T_A \psi + i\varepsilon^B \bar{\psi} [T_A, T_B] \psi$$

$$= \bar{\psi} T_A \psi + i\varepsilon^B i C^C{}_{AB} \bar{\psi} T_C \psi$$

$$= (\delta^C_A + \varepsilon^B C^C{}_{BA}) \bar{\psi} T_C \psi.$$

Concluimos que $\bar{\psi} T_A \psi$ transforma com a representação adjunta.

2.3. Teoría de Yang-Mills.

Considere uma teoría com uma ação $S(\varphi)=\int d^4x\, \mathcal{L}(\varphi,\partial\varphi)(x)$ invariante baixo transformações globais de calibre. Seja o grupo G e a representação D. Assuma além que G esta compacto e pelo tanto a exponencial é sobreyectiva. Sejam iT_A geradores de $\mathfrak{g}$ na representação d. Considere a ação inducida de

$$G_\infty = \{f:\mathbb{R}^4 \longrightarrow G \mid f \text{ é suave}\},$$

de dimenção infinita. Para todo $f\in G_\infty$ existe $\alpha_A:\mathbb{R}^4\longrightarrow\mathbb{R}$ suave t.q. $D(f(x))=e^{i\alpha^A(x)T_A}$. Considere agora a transformação $\partial_\mu\varphi$

$$\partial_\mu\varphi \longmapsto \partial_\mu\left(e^{i\alpha^A T_A}\varphi\right) = e^{i\alpha^A T_A}\partial_\mu\varphi + e^{i\alpha^A T_A}\, i\,\partial_\mu\alpha^A T_A \varphi$$
$$\neq e^{i\alpha^A T_A}\partial_\mu\varphi.$$

Logo em geral a teoria não é invariante baixo G_∞. Para corregir isso introducimos campos $A_\mu = A_\mu{}^A T_A:\mathbb{R}^4\longrightarrow d(\mathfrak{g})$. Considere

$$D_\mu = \partial_\mu - igA_\mu,$$

onde g é conhecida como a carga de φ. Logo

$$D_\mu\varphi \longmapsto (\partial_\mu - igA'_\mu)(U\varphi) = U\partial_\mu\varphi + \partial_\mu U\varphi - igA'_\mu U\varphi$$

Logo, si

$$A'_\mu = UA_\mu U^{-1} + \frac{1}{ig}(\partial_\mu U)U^{-1},$$

se tem

$$D_\mu\varphi \longmapsto UD_\mu\varphi.$$

Então a ação $S(\varphi, A) = \int d^4x\, \mathcal{L}(\varphi, D\varphi)$ é invariante baixo G_∞.

Considere a curvatura

$$[D_\mu, D_\nu]\varphi = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]\varphi = \big(\overbrace{[\partial_\mu,\partial_\nu]}^{0} - ig[\partial_\mu, A_\nu] - ig[A_\mu, \partial_\nu] - g^2[A_\mu, A_\nu]\big)\varphi$$

$$= \big(-ig\,\partial_\mu A_\nu - \cancel{igA_\nu\partial_\mu} + \cancel{igA_\nu\partial_\mu} - \cancel{igA_\mu\partial_\nu} + ig\,\partial_\nu A_\mu + \cancel{igA_\mu\partial_\nu}$$

$$- g^2[A_\mu, A_\nu]\big)\varphi$$

$$= -ig\,F_{\mu\nu}\,\varphi$$

onde

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

A curvatura transforma na representação adjunta

$$F'_{\mu\nu} = \partial_\mu\Big(UA_\nu U^{-1} + \frac{1}{ig}\partial_\nu U U^{-1}\Big) - \partial_\nu\Big(UA_\mu U^{-1} + \frac{1}{ig}\partial_\mu U U^{-1}\Big)$$

$$- ig\Big[UA_\mu U^{-1} + \frac{1}{ig}\partial_\mu U U^{-1},\, UA_\nu U^{-1} + \frac{1}{ig}\partial_\nu U U^{-1}\Big]$$

$$= \partial_\mu U \cancel{A_\nu U^{-1}} + U\partial_\mu A_\nu U^{-1} + U\cancel{A_\nu \partial_\mu U^{-1}} + \frac{1}{ig}\cancel{\partial_\mu \partial_\nu U} U^{-1}$$

$$+ \frac{1}{ig}\cancel{\partial_\nu U \partial_\mu U^{-1}} - \cancel{\partial_\nu U A_\mu U^{-1}} - U\partial_\nu A_\mu U^{-1} - U A_\mu \cancel{\partial_\nu U^{-1}}$$

$$- \frac{1}{ig}\cancel{\partial_\nu \partial_\mu U} U^{-1} - \frac{1}{ig}\cancel{\partial_\mu U}\partial_\nu U^{-1} - ig\left(UA_\mu \cancel{U^{-1}U} A_\nu U^{-1} - UA_\nu \cancel{U^{-1}U} A_\mu U^{-1}\right)$$

$$- UA_\mu U^{-1}\cancel{\partial_\nu U} U^{-1} + \partial_\nu U \cancel{U^{-1}U} A_\mu U^{-1} - \partial_\mu U \cancel{U^{-1}U} A_\nu U^{-1}$$

$$+ UA_\nu U^{-1}\cancel{\partial_\mu U} U^{-1} - \frac{1}{ig}\left(\partial_\mu U \cancel{U^{-1}\partial_\nu} U U^{-1} - \partial_\nu U \cancel{U^{-1}}\partial_\mu U U^{-1}\right)$$

$$= U F_{\mu\nu} U^{-1}. \qquad \left(0 = \partial_\mu (UU^{-1}) = \partial_\mu U U^{-1} + U\partial_\mu U^{-1}\right)$$

Logo o Lagrangiana de Yang-Mills

$$\mathcal{L}_{Y.M.}(A, \partial A) = -\frac{1}{2}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)$$

é invariante sob transformações de calibre. Logo

$$\frac{\partial \mathcal{L}_{Y.M.}(A,\partial A)}{\partial \partial_\mu A_\nu{}^A} = -\frac{1}{2}F^{\sigma\rho B}\frac{\partial F_{\sigma\rho}{}^B}{\partial \partial_\mu A_\nu{}^A} = -\frac{1}{2}F^{\sigma\rho B}\left(\delta^\mu_\sigma \delta^\nu_\rho \delta^B_A - \delta^\mu_\rho \delta^\nu_\sigma \delta^B_A\right)$$

$$= -\frac{1}{2}F^{\mu\nu A} + \frac{1}{2}F^{\nu\mu A} = -F^{\mu\nu A},$$

$$\frac{\partial \mathcal{L}_{Y.M.}(A,\partial A)}{\partial A_\nu{}^A} = -\frac{1}{2}F^{\sigma\rho B}\frac{\partial F_{\sigma\rho}{}^B}{\partial A_\nu{}^A} = -\frac{1}{2}F^{\sigma\rho B}\left(g\, C^B{}_{DE}\left(\delta^\nu_\sigma \delta^D_A A_\rho^E + A_\sigma^D \delta^\nu_\rho \delta^E_A\right.\right.$$

$$= -\frac{g}{2}F^{\nu\rho B}C^B{}_{AE}A_\rho{}^E - \frac{g}{2}F^{\sigma\nu B}C^B{}_{DA}A_\sigma{}^D$$

$$= -g\,F^{\nu\rho B}C^B{}_{AE}A_\rho{}^E.$$

onde $tr(T_a T_b) = \frac{1}{2}\delta_{ab}$, assumiendo que G es semisimple. Logo

$$0 = -\partial_\mu F^{\mu\nu A} + g F^{\nu\rho B} C^B{}_{AE} A_\rho{}^E$$

$$= -\partial_\mu F^{\mu\nu A} - g\, C^A{}_{EB} A_\mu{}^E F^{\mu\nu B}$$

$$= -\partial_\mu F^{\mu\nu A} + ig\, [T_E, T_B]^A A_\mu{}^E F^{\mu\nu B}$$

$$= -(\partial_\mu F^{\mu\nu A} + ig [A_\nu, F^{\mu\nu}]^A = -(D_\mu F^{\mu\nu})^A$$

onde

$$D_\mu \theta := \partial_\mu \theta - ig [A_\mu, \theta]$$

para θ na representação adjunta. Aca usamos

$$Tr(T_A[T_B, T_C]) = i C^D{}_{BC} Tr(T_A T_D) = \frac{i}{2} C^A{}_{BC}$$

$$\shortparallel$$

$$Tr(T_A T_B T_C) - Tr(T_A T_C T_B) = Tr([T_A, T_B] T_C) = i C^D{}_{AB} Tr(T_D T_C)$$

$$= \frac{i}{2} C^C{}_{AB} \Rightarrow C^A{}_{BC} = C^C{}_{AB}$$

2.3.1. Exemplos

Modelo de Forças Nucleares

Como o proton e o neutron tem spin $1/2$ e massas similares, nuestro modelo tem um doplete $\psi = (\psi_p, \psi_n)$ com ψ_p e ψ_n espinores de Dirac. Asumimos que ψ está na representação fundamental de $SU(2)$.

por outro lado, os piones $\vec{\varphi} = (\pi^+, \pi^0, \pi^-)$ estam na representação adjunta, que tem dimenção $\dim su(2) = 2^2 - 1 = 3$. A interação é descrita por o escalar

$$i \bar{\psi} \gamma^5 \vec{\tau} \psi \cdot \vec{\varphi}.$$

Modelo $CP(n-1)$

Temos

$$\mathcal{L}(Z, Z^*, \partial Z, \partial Z^*) = \partial_\mu Z^* \partial^\mu Z + \frac{f}{2n} (Z^* \overleftrightarrow{\partial_\mu} Z)^2$$

com Z_a, $a \in \{1, \ldots, n\}$ satisfazendo $Z^* Z = \frac{n}{2f}$. É claro que esta teoría e invariante de calibre $U(1)$ con Z_a na representação fundamental. Bajo transformações locais

$$\mathcal{L}(e^{i\alpha} Z, e^{-i\alpha} Z^*, \partial(e^{i\alpha} Z), \partial(e^{-i\alpha} Z^*)) =$$

$$(e^{-i\alpha} \partial_\mu Z^* - i \partial_\mu \alpha e^{-i\alpha} Z^*)(e^{i\alpha} \partial^\mu Z + i \partial^\mu \alpha e^{i\alpha} Z)$$

$$+ \frac{f}{2n} \left(e^{-i\alpha} Z^* (e^{i\alpha} \partial_\mu Z + i \partial_\mu \alpha e^{i\alpha} Z) - (e^{-i\alpha} \partial_\mu Z^* - i \partial_\mu \alpha e^{-i\alpha} Z^*) e^{i\alpha} Z \right)^2 =$$

$$\partial_\mu Z^* \partial^\mu Z + i Z \partial^\mu \alpha \partial_\mu Z^* - i Z^* \partial_\mu \alpha \partial^\mu Z + \partial_\mu \alpha \partial^\mu \alpha Z^* Z$$

$$+ \frac{f}{2n} \left(Z^* \partial_\mu Z + i \partial_\mu \alpha Z^* Z - Z \partial_\mu Z^* + i \partial_\mu \alpha Z^* Z \right)^2 =$$

$$\partial_\mu Z^* \partial^\mu Z + i Z \partial^\mu \alpha \partial_\mu Z^* - i Z^* \cancel{\partial_\mu \alpha \partial^\mu Z} + \frac{n}{2f} \cancel{\partial_\mu \alpha \partial^\mu \alpha}$$

$$+ \frac{f}{2n} (Z^* \overleftrightarrow{\partial_\mu} Z)^2 + i \frac{f}{n} \cancel{(Z^* \partial_\mu Z) \partial^\mu \alpha \frac{n}{f}} - \cancel{\frac{2f}{n} (\partial_\mu \alpha)^2 \left(\frac{n}{2f} \right)^2}$$

$$= \mathcal{L}(Z, Z^*, \partial Z, \partial Z^*).$$

Logo a Lagrangiana é invariante sim introducir o campo calibrante. Aum así,

$$\mathcal{L}(Z, \partial Z) = (D_\mu Z)^* D^\mu Z \,,$$

como $D_\mu Z = (\partial_\mu + i A_\mu) Z$ e $A_\mu = \frac{if}{n} Z^* \overleftrightarrow{\partial}_\mu Z$. Em efeito

$$D_\mu Z = \partial_\mu Z - \frac{f}{n} (Z^* \overleftrightarrow{\partial}_\mu Z) Z.$$

Logo

$$\begin{aligned}(D_\mu Z)^* D^\mu Z &= \partial_\mu Z^* \partial^\mu Z - \partial_\mu Z^* \frac{f}{n} (Z^* \overleftrightarrow{\partial}^\mu Z) Z \\ &\quad - \frac{f}{n} (Z \overleftrightarrow{\partial}_\mu Z^*) Z^* \partial^\mu Z + \overset{\frac{f}{2n}}{\cancel{\frac{f^2}{n^2}}} (Z^* \overleftrightarrow{\partial}_\mu Z)(Z \overleftrightarrow{\partial}^\mu Z^*) \cancel{Z Z^*} \\ &= \partial_\mu Z^* \partial^\mu Z - \frac{f}{2n} (Z^* \overleftrightarrow{\partial}_\mu Z)^2 + \frac{f}{n} (Z^* \overleftrightarrow{\partial}^\mu Z)(Z^* \partial^\mu Z - Z \partial^\mu Z^*) \\ &= \partial_\mu Z^* \partial^\mu Z + \frac{f}{2n} (Z^* \overleftrightarrow{\partial}_\mu Z)^2 = \mathcal{L}(Z, Z^*, \partial Z, \partial Z^*).\end{aligned}$$

Cromodinâmica Quântica

Esta é uma teoría com $G = SU(3)$. Se tem

$\psi = (\psi_{up}, \psi_{down}, \psi_{charm}, \psi_{strange}, \psi_{top}, \psi_{bottom})$ e cada entrada é um elemento de representação fundamental de $SU(3)$. A campo de calibre tem 8 componentes pois $\dim su(3) = 3^2 - 1 = 8$. Logo

$$\mathcal{L}(\psi, \partial\psi, \bar\psi, \partial\bar\psi, A, \partial A) = -\frac{1}{2} \operatorname{Tr}(F^{\mu\nu} F_{\mu\nu}) + i \bar\psi \gamma_\mu D^\mu \psi - \bar\psi M \psi$$

onde $M = \operatorname{diag}(M_{up}, M_{down}, M_{charm}, M_{strange}, M_{Top}, M_{Bottom})$. Logo a ecuação de movimento é

$$(D^\mu F_{\mu\nu})^A = -g\bar{\psi}\gamma_\nu T^A\psi.$$

⟶ Ver o final ☆

Modelo Sigma Linear

Considere a Lagrangiana

$$\mathcal{L} = \frac{1}{2}\sum_{A=1}^{4} \partial_\mu\phi_A\partial^\mu\phi_A$$

onde $\phi = (\varphi_1, \varphi_2, \varphi_3, \sigma)$. Considere o grupo de calibre $O(4)$. Então $o(4) = \{X \in M_4(\mathbb{C}) \mid X^t = -X\}$. Uma base são

$$(M_{AB})_{CD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}; \qquad A \leq B$$

Logo

$$J^{\mu AB} = -\frac{1}{\cancel{2}}\sum_{E=1}^{4} \cancel{2}\partial^\mu\phi_E (M^{AB})_{ED}\phi_D$$

$$= -(\partial^\mu\phi^A\phi^B - \partial^\mu\phi^B\phi^A) = \partial^\mu\phi^{[B}\phi^{A]}$$

Agora considere as transformações geradas pelo $\varepsilon_{A4} \propto M_{04}$ infinitesimal. Si pegamos $\vec{\varepsilon} = (\varepsilon_{14}, \varepsilon_{24}, \varepsilon_{34})$, temos

$$\delta\sigma = -\vec{\varepsilon}\cdot\vec{\varphi} \qquad \text{e} \qquad \delta\vec{\varphi} = \vec{\varepsilon}\cdot\sigma.$$

Si um espinor de Dirac transforma-se tal que

$$\delta\psi = -\frac{i}{2}\gamma_5(\vec{\varepsilon}\cdot\vec{\tau})\psi,$$

podemos ver que $(\bar{\psi}\psi, i\bar{\psi}\gamma_5\vec{\tau}\psi)$ é um vector.

De fato,

$$\delta\bar{\psi} = \left(-\frac{i\gamma_5}{2}(\vec{\varepsilon}\cdot\vec{\tau})\psi\right)^{+}\gamma_0 = \frac{i}{2}\psi^{+}(\vec{\varepsilon}\cdot\vec{\tau})^{+}\gamma_5^{+}\gamma_0$$

$$= -\frac{i}{2}\psi^{+}\gamma_0(\vec{\varepsilon}\cdot\vec{\tau})\gamma_5 = -\frac{i}{2}\bar{\psi}\gamma_5(\vec{\varepsilon}\cdot\vec{\tau}),$$

lembrando que $\tau_i^{+} = \tau_i$, $\gamma_5^{+} = \gamma_5$, $\{\gamma_5, \gamma^{\mu}\} = 0$, e as matrices gamma e de Pauli actuam em espaços distintos. Então

$$\delta(\bar{\psi}\psi) = -\frac{i}{2}\bar{\psi}\gamma_5(\vec{\varepsilon}\cdot\vec{\tau})\psi - \frac{i}{2}\bar{\psi}\frac{\gamma_5}{2}(\vec{\varepsilon}\cdot\vec{\tau})\psi$$

$$= -\vec{\varepsilon}\cdot\left(i\bar{\psi}\gamma_5\vec{\tau}\psi\right),$$

$$\delta\left(i\bar{\psi}\gamma_5\vec{\tau}\psi\right)^{i} = \frac{1}{2}\bar{\psi}\cancel{\gamma_5}(\vec{\varepsilon}\cdot\vec{\tau})\cancel{\gamma_5}\tau^{i}\psi + \frac{1}{2}\bar{\psi}\cancel{\gamma_5}\tau^{i}\cancel{\gamma_5}(\vec{\varepsilon}\cdot\vec{\tau})\psi$$

$$= \frac{1}{2}\bar{\psi}\{\vec{\varepsilon}\cdot\vec{\tau}, \tau^{i}\}\psi = \varepsilon^{j}\bar{\psi}\delta^{ji}\psi = \varepsilon^{i}\bar{\psi}\psi.$$

O termo cinético da Lagrangiana de Dirac é invariante

$$\delta(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi) = -\bar{\psi}\frac{i\gamma_5}{2}\vec{\varepsilon}\cdot\vec{\tau}\gamma^{\mu}\partial_{\mu}\psi - \bar{\psi}\gamma^{\mu}\partial_{\mu}\frac{i\gamma_5}{2}(\vec{\varepsilon}\cdot\vec{\tau})\psi$$

$$= -\bar{\psi}\frac{i}{2}\vec{\varepsilon}\cdot\vec{\tau}\{\gamma_5, \gamma^{\mu}\}\partial_{\mu}\psi = 0.$$

Mais o termo de massa não é. Então a Lagrangiana do modelo sigma limed originalmente proposto por Gell-Mann e Levy é

$$\mathcal{L}(\phi = (\vec{\varphi}, \sigma), \psi, \partial\phi, \partial\psi) = i\bar{\psi}\not{\partial}\psi + \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \lambda\phi^4 + g\left(\sigma\bar{\psi}\psi + i\bar{\psi}\gamma_5\vec{\tau}\psi\cdot\vec{\varphi}\right).$$

Modelo sigma não linear

O modelo linear tem 4 campos escalares, mais só temos 3 píons. Logo campo σ não esta associado a nenhuma partícola física. Para eliminar o campo σ pegamos um vínculo invariante por $O(4)$

$$\vec{\varphi}^2 + \sigma^2 = f^2.$$

Então $\vec{\varphi} \cdot \partial_\mu \vec{\varphi} + \sigma \partial_\mu \sigma = 0$. Logo

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2} \sum_{i=1}^{3} \partial_\mu \vec{\varphi} \cdot \partial^\mu \vec{\varphi} + \frac{1}{2} \underbrace{\partial_0 \sigma \, \partial^0 \sigma}$$

$$\frac{\sigma \partial_0 \sigma \, \sigma \partial^0 \sigma}{\sigma^2} = \frac{(\vec{\varphi} \cdot \partial_\mu \vec{\varphi})(\vec{\varphi} \cdot \partial^\mu \vec{\varphi})}{F^2 - \varphi^2}.$$

Esta é invariante pela transformação não linear

$$\delta \vec{\varphi} = \vec{\varepsilon} \sigma = \vec{\varepsilon} \sqrt{f^2 - \varphi^2}.$$

★ A corrente $-g \bar{\Psi} \gamma_\nu T^A \psi$ é conservada em a Cromodinâmica Quântica pos $D^\nu D^\mu F_{\mu\nu} = 0$. De fato, pela identidade de Jacobi

$$0 = [A^\mu, [A^\nu, F_{\mu\nu}]] + [A^\nu, [F_{\mu\nu}, A^\mu]] + [F_{\mu\nu}, [A^\mu, A^\nu]]$$

$$= 2[A^\mu, [A^\nu, F_{\mu\nu}]] + [F_{\mu\nu}, [A^\mu, A^\nu]].$$

Logo $[A^\mu, [A^\nu, F_{\mu\nu}]] = -\frac{1}{2} [F_{\mu\nu}, [A^\mu, A^\nu]]$

$$D^\nu D^\mu F_{\mu\nu} = \partial^\nu \cancel{\partial^\mu F_{\mu\nu}}^{=0} - ig\,\partial^\nu [A^\mu, F_{\mu\nu}] - ig[A^\nu, \partial^\mu F_{\mu\nu}]$$

$$+ g^2 [A^\nu, [A^\mu, F_{\mu\nu}]]$$

$$= -ig\,[\partial^\nu A^\mu, F_{\mu\nu}] - ig\,[\cancel{A^\mu, \partial^\nu} F_{\mu\nu}] - ig\,[\cancel{A^\nu, \partial^\mu} F_{\mu\nu}]$$

$$+ \frac{g^2}{2} [F_{\nu\mu}, [A^\nu, A^\mu]]$$

$$= -\frac{ig}{2} [\partial^{[\nu} A^{\mu]} - ig[A^\nu, A^\mu], F_{\mu\nu}]$$

$$= \frac{ig}{2} [F^{\mu\nu}, F_{\mu\nu}] = 0$$

2.3.2. Segundo Teorema de Noether

O primer teorema de Noether não é certo para transformações de calibre locales. Isso é porque $\partial\omega$ não e constante.

Sim embargo, ha um analogo do teorema para estas.

Para isso considere uma configuração dos campos φ e A.

Dada uma ação S, considere o funcional

$$\tilde{S} : \alpha \longmapsto S\Big(e^{i\alpha^A(x)T_A}\varphi,\ e^{i\alpha^A(x)T_A}A_\mu e^{-i\alpha^A(x)T_A} + \frac{1}{ig}\left(\partial_\mu e^{i\alpha^A(x)T_A}\right)e^{-i\alpha^A(x)T_A}\Big).$$

Em particular, si temos uma simetria local, este funcional toma o valor constante $S(\varphi, A)$. Agora bem,

$$\left.\frac{\delta e^{i\alpha^B(y)T_A}\varphi}{\delta\alpha^A(x)}\right|_{\alpha=0} = i\,\delta(y-x)\,T_A\varphi.$$

$$\left.\frac{\delta\left(e^{i\alpha^B(y)T_B}A_\mu e^{-i\alpha^B(y)T_B}\right)}{\delta\alpha^A(x)}\right|_{\alpha=0} = i\,\delta(y-x)\,[T_A, A_\mu] = i\,\delta(y-x)\,A_\mu^B\, i\,C^c{}_{AB}T_c$$

$$\left.\frac{\delta\left(\partial_\mu e^{i\alpha^B(y)T_B}e^{-i\alpha^B(y)T_B}\right)}{\delta\alpha^A(x)}\right|_{\alpha=0} = i\,\partial_\mu\delta(y-x)\,T_A.$$

Logo, si deixamos

$$\frac{\delta\mathcal{L}(\varphi,\partial\varphi)}{\delta\varphi} = \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\partial_\mu\varphi},$$

temos

$$\frac{\delta \tilde{S}(\alpha)}{\delta \alpha^A(x)} = i\,\frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta \varphi}(x) T_A \varphi(x) + i\,\frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta A_\mu{}^B}(x)\, i C^B{}_{AC} A_\mu{}^C$$

$$- \frac{1}{g}\partial_\mu \frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta A_\mu{}^B}\, \delta^B_A .$$

Logo, si temos uma simetria local, Obtemos a relação

$$\frac{1}{g}\partial_\mu \frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta A_\mu{}^A} + \frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta A_\mu{}^B} C^B{}_{AC} A_\mu{}^C = - \frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta \varphi}\, i T_A \varphi .$$

Esta relação não depende das ecuações de movimento!

Definendo

$$j^\mu{}_A(x) := - \frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta A_\mu{}^A}(x),$$

temos

$$\partial_\mu j^\mu{}_A(x) + g\, C^B{}_{AC} A_\mu{}^C j^\mu{}_B = - \frac{\delta \mathcal{L}(\varphi, \partial\varphi, A, \partial A)}{\delta \varphi} T_A \varphi .$$