Field Theory:

A Modern Primer

Chapter 1

Section 1.1

A. i) The action is invariant under the infinitesimal transformation $\delta x = \varepsilon$. Given that to $\mathcal{O}(\varepsilon^2)$

$$S[x+\delta x] = \int_{t_0}^{t_f} dt \, \frac{1}{2} m \left(\frac{d(x+\delta x)}{dt} \right)^2 = \int_{t_0}^{t_f} dt \, \frac{1}{2} m \left(\dot{x}^2 + 2\dot{x} \cdot \frac{d\delta x}{dt} \right)$$

$$= \int_{t_0}^{t_f} dt \, \frac{1}{2} m \left(\dot{x}^2 - 2\ddot{x} \cdot \delta x + 2 \frac{d}{dt} \left(\dot{x} \cdot \delta x \right) \right)$$

$$= S[x] - \int dt \, m\ddot{x} \cdot \delta x + \Big[m\dot{x} \cdot \delta x \Big]_{t_0}^{t_f},$$

if x satisfies Newton's second law, i.e. $\ddot{x} = 0$, we have

$$0 = \delta S = S[x+\delta x] - S[x] = \Big[m\dot{x} \cdot \delta x \Big]_{t_0}^{t_f} = \left(m\dot{x}(t_f) - m\dot{x}(t_0) \right) \cdot \varepsilon .$$

Since this is true for any ε, t_0, t_f we conclude that momentum is conserved.

ii) From equation (1.1.8) we have

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x} = -\nabla V = \frac{v}{a} \sin\left(\frac{r}{a} \right) \frac{x}{r}$$

B) We have from equation (1.1.8)

$$\frac{d}{dt}(m\dot{x}) = -\nabla V$$

c) Consider an infinitesimal rotation $\delta x^i = \omega^i{}_j x^j$. Given that

$$r^2 = x^2 = (x+\delta x)\cdot(x+\delta x) = x^2 + 2x^i \omega_{ij} x^j + \mathcal{O}(\omega_{ij}^2)$$

and $x^i x^j$ is symmetric, we have $\omega_{ij} = -\omega_{ji}$. Since r is left invariant by this transformation, S is too. Therefore by (1.1.6) we have

$$m\dot{x}\cdot\delta x = m\dot{x}^i \omega_{ij} x^j$$

is conserved by choosing a basis for the antisymmetric matrices $(\Omega^{ab})_{ij} = \delta^a_i \delta^b_j - \delta^b_i \delta^a_j$ we obtain that

$$m(\dot{x}^a x^b - \dot{x}^b x^a)$$

is conserved. This corresponds to the conservation of angular momentum. From now on let $L^{ab} = \dot{x}^a x^b - x^b \dot{x}^a$

Now consider the infinitesimal change $\delta x^i = \varepsilon L^{ik}$ for some fixed index k. The term $\frac{1}{2}m\dot{x}^2$ is left invariant since L^{ik} is constant as we just showed. On the other hand, to order ε^2

$$r+\delta r = \left((x^i+\delta x^i)(x_i+\delta x_i)\right)^{1/2} = \left(r^2 + 2\varepsilon x_i L^{ik}\right)^{1/2} = r\left(1+\varepsilon \frac{x_i L^{ik}}{r^2}\right) = r + \varepsilon \frac{x_i L^{ik}}{r}$$

and therefore the potential changes to

$$\frac{a}{r+\delta r} = \frac{a}{r}\left(1 - \frac{\delta r}{r}\right) = \frac{a}{r} - \frac{\varepsilon a x_i L^{ik}}{r^3}$$

$$= \frac{a}{r} - \frac{\varepsilon a}{r^3}\left(\dot{x}^i x^k x_i - \dot{x}^k r^2\right) = \frac{a}{r} - \frac{\varepsilon a}{r^3}\dot{x}^i x^k x_i + \frac{\varepsilon a \dot{x}^k}{r}$$

$$= \frac{a}{r} + \varepsilon \frac{d}{dt}\left(\frac{a x^k}{r}\right).$$

We conclude

$$S[x^i+\delta x^i] = S[x] + \varepsilon a\left[\frac{x^k}{r}\right]_{t_1}^{t_2}$$

However, since we have by equation (1.1.6)

$$S[x^i+\delta x^i] = S[x^i] + m\left[\delta x_i \dot{x}^i\right]_{t_1}^{t_2},$$

we conclude that $m x_i \delta x^i - \varepsilon a \frac{x^k}{r}$ is a constant. We thus obtain that the vector

$$m x_i L^{ik} - \frac{a x^k}{r} = m(x_i \dot{x}^i x^k - x_i x^i \dot{x}^k) - \frac{a x^k}{r}$$

is constant.

The final constant of motion is obtained by letting $\delta x = \varepsilon \dot{x}$.

Then the Kinetic energy changes by

$$\frac{1}{2}m(\dot{x}^i+\varepsilon\ddot{x}^i)(\dot{x}_i+\varepsilon\ddot{x}^i) = \frac{1}{2}m\dot{x}^i\dot{x}_i + m\varepsilon\ddot{x}^i x_i = \frac{1}{2}m\dot{x}^i\dot{x}_i + \varepsilon\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^i\dot{x}_i\right)$$

The potential changes by

$$\frac{a}{r+\delta r} = \frac{a}{((x^i+\delta x^i)(x_i+\delta x_i))^{1/2}} = \frac{a}{(r^2+2x^i\delta x_i)^{1/2}} = \frac{a}{r}\left(1-\frac{x^i\delta x_i}{r^2}\right)$$

$$= \frac{a}{r} - \varepsilon\frac{ax^i\dot{x}_i}{r^3} = \frac{a}{r}+\varepsilon\frac{d}{dt}\left(\frac{a}{r}\right)$$

Much like before we have

$$S[x]+\varepsilon\Big[m\dot{x}^i\dot{x}_i\Big]_{t_1}^{t_2} = S[x]+m\Big[\dot{x}^i\delta x_i\Big]_{t_1}^{t_2} = S[x+\delta x] = S[x]+\varepsilon\left[\frac{1}{2}m\dot{x}^i\dot{x}_i+\frac{a}{r}\right]_{t_1}^{t_2}.$$

We thus conclude that

$$\frac{1}{2}m\dot{x}^i\dot{x}_i - \frac{a}{r}$$

is a constant of motion.

Observation: The solution given for the Runge-Lenz vector was inspired on Gorni, G. & Zampieri, G. "Revisiting Nother's Theorem on constants of motion".

D) Consider an infinitesimal variation $\delta x = \varepsilon \dot{x}$. Then as we saw in the previous problem

$$\frac{1}{2} m \left(\frac{d(x+\delta x)}{dt} \right)^2 = \frac{1}{2} m \dot{x}^2 + \varepsilon \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 \right).$$

On the other hand

$$V(t, x+\delta x) = V(t,x) + \partial_i V(t,x) \delta x^i = V(t,x) + \varepsilon \partial_i V(t,x) \dot{x}^i$$

$$= V(t,x) + \varepsilon \left(\frac{dV(t,x)}{dt} - \frac{\partial V(t,x)}{\partial t} \right).$$

Therefore the change in the action is

$$\varepsilon \left[m \dot{x}^2 \right]_{t_1}^{t_2} = m \left[\dot{x} \delta x \right]_{t_1}^{t_2} = \delta S = \varepsilon \left[\frac{1}{2} m \dot{x}^2 \right]_{t_1}^{t_2} - \varepsilon \left[V \right]_{t_1}^{t_2} + \varepsilon \int_{t_1}^{t_2} dt \, \frac{\partial V}{\partial t}.$$

We conclude that if $E = \frac{1}{2} m \dot{x}^2 + V$

$$\frac{dE}{dt} = \frac{\partial V}{\partial t}.$$

In particular, if V is time independent the E is conserved.

Section 1.2.

A) Let L_1 and L_2 be Lorentz transformations. Then

$$g = (L_1 L_1^{-1})^T g\,(L_1 L_1^{-1}) = (L_1^{-1})^T L_1^T g L_1 L_1^{-1} = (L_1^{-1})^T g L_1^{-1}$$

and

$$(L_1 L_2)^T g\,(L_1 L_2) = L_2^T L_1^T g L_1 L_2 = L_2^T g L_2 = g,$$

showing that the set of Lorentz transformatios is closed under multiplication and taking inverses. It is therefore a sobgroup of the 4×4 matrices.

B) Let L' be a Lorentz transformation. In the frame to which L' transforms, L becomes $(L')^{-1} L L'$. We have

$$\det((L')^{-1} L L') = (\cancel{\det L'})^{-1} \det L\, \cancel{\det L'} = \det L$$

showing that $\det L$ is Lorentz invariant.

For any Lorentz transformation L we have

$$L^T = g L^{-1} g.$$

Since g is a Lorentz transformation (as a matrix) the

the previous exercise shows that L^T is also one. Therefore (1.2.12) is also valid for L^T

$$(L^0{}_0)^2 - \sum_{i=1}^{3} (L^i{}_0)^2 = 1 = (L^0{}_0)^2 - \sum_{i=1}^{3} (L^0{}_i)^2.$$

Let L' be another Lorentz transformation. Then by the Cauchy-Schwartz inequalities and letting $L'' = LL'$ we obtain

$$L''^0{}_0 = L^0{}_\mu L'^\mu{}_0 = L^0{}_0 L'^0{}_0 + \sum_{i=1}^{3} L^0{}_i L'^i{}_0$$

and

$$|L''^0{}_0 - L^0{}_0 L'^0{}_0| = \left| \sum_{i=1}^{3} L^0{}_i L'^i{}_0 \right| \le \left(\sum_{i=1}^{3} (L^0{}_i)^2 \right)^{1/2} \left(\sum_{i=1}^{3} (L'^i{}_0)^2 \right)^{1/2}$$

$$= \sqrt{(L^0{}_0)^2 - 1}\sqrt{(L'^0{}_0)^2 - 1} \le |L^0{}_0 L'^0{}_0|.$$

We conclude that

$$L^0{}_0 L'^0{}_0 - |L^0{}_0 L'^0{}_0| \le L''^0{}_0 \le L^0{}_0 L'^0{}_0 + |L^0{}_0 L'^0{}_0|,$$

$$L''^0{}_0 - |L^0{}_0 L'^0{}_0| \le L^0{}_0 L'^0{}_0 \le L''^0{}_0 + |L^0{}_0 L'^0{}_0|$$

Therefore $L^0{}_0 L'^0{}_0 > 0$ if and only if $L''^0{}_0 > 0$ and $L^0{}_0 L'^0{}_0 < 0$ if und only if $L''^0{}_0 < 0$. From that we conclude that

$((L')^{-1})^0{}_0 L^0{}_0 \geq 0$ and therefore $((L')^{-1} L L')^0{}_0$ has the same sign as $L^0{}_0$.

C) Considere $L \in L_-^{\uparrow}$ y sea $P = g$ como matriz. Luego $P^2 = 1$, $\det(LP) = \det L \det P = (-1)(-1) = 1$ y

$$(LP)^0{}_0 = L^0{}_\mu P^\mu{}_0 = L^0{}_0 P^0{}_0 = L^0{}_0 \geq 1.$$

En vista de que P es una transformación de Lorentz se tiene que $LP \in L_+^{\uparrow}$ y $L = LPP$.

Ahora tome $L \in L_-^{\downarrow}$ y $T = -P$. Es claro que T es una transformación de Lorentz, $T^2 = 1$, $\det(LT) = (-1)(-1) = 1$ y

$$(LT)^0{}_0 = L^0{}_\mu T^\mu{}_0 = L^0{}_0 T^0{}_0 = -L^0{}_0 \geq 1.$$

Luego $LT \in L_+^{\uparrow}$ y $L = LTT$.

Finalmente, sea $L \in L_+^{\downarrow}$. Entonces $\det(LTP) = \det(TP) = 1$ y $(LTP)^0{}_0 = L^0{}_\mu T^\mu{}_\nu P^\nu{}_0 = L^0{}_\mu T^\mu{}_0 = -L^0{}_0 \geq 1$. Luego $LTP \in L_+^{\uparrow}$ y $L = LTPPT$.

D) First we need to find an expresion for a boost in an arbitrary direction $\vec{v}$. We have

$$x'^{\mu} = L(\vec{v})^{\mu}{}_{\nu} x^{\nu} = \left(\gamma (x^0 - \vec{v}\cdot\vec{x}) \ , \left(\vec{x} - \frac{\vec{v}\cdot\vec{x}\,\vec{v}}{\|\vec{v}\|^2} \right) + \gamma \left(\frac{\vec{v}\cdot\vec{x}}{\|\vec{v}\|^2} \vec{v} - \vec{v} x^0 \right) \right)$$

$$= \left(\gamma (x^0 - \vec{v}\cdot\vec{x}) , \ \vec{x} + (\gamma - 1) \frac{\vec{v}\cdot\vec{x}}{\|\vec{v}\|^2} \vec{v} - \gamma \vec{v} x^0 \right)$$

from the fact that we know how x^0, $\frac{\vec{x}\cdot\vec{v}\,\vec{v}}{\|\vec{v}\|^2}$ and its complement changes. We conclude that

$$L(\vec{v}) = \begin{bmatrix} \gamma & -\gamma v^1 & -\gamma v^2 & -\gamma v^3 \\ -\gamma v^1 & 1 + (\gamma-1) \frac{{v^1}^2}{\|\vec{v}\|^2} & (\gamma - 1) \frac{v^2 v^1}{\|\vec{v}\|^2} & (\gamma-1) \frac{v^3 v^1}{\|\vec{v}\|^2} \\ -\gamma v^2 & (\gamma - 1) \frac{v^1 v^2}{\|\vec{v}\|^2} & 1 + (\gamma - 1) \frac{{v^2}^2}{\|\vec{v}\|^2} & (\gamma - 1) \frac{v^3 v^2}{\|\vec{v}\|^2} \\ -\gamma v^3 & (\gamma - 1) \frac{v^1 v^3}{\|\vec{v}\|^2} & (\gamma - 1) \frac{v^2 v^3}{\|\vec{v}\|^2} & 1 + (\gamma - 1) \frac{{v^3}^2}{\|\vec{v}\|^2} \end{bmatrix}.$$

Dado $R \in SO(3)$ definimos

$$L(R) = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}.$$

Sea $L \in \mathcal{L}^{\uparrow}_{+}$. Defina $v^i = L^i{}_0 / L^0{}_0$. Note que

$$\|\vec{v}\|^2 = 1 - \frac{1}{(L^0{}_0)^2} \leq 1,$$

$$\gamma = (1 - \|\vec{v}\|^2)^{-1/2} = L^0{}_0$$

y

$$L(\vec{v})^i{}_k = \delta^i{}_k + (\gamma - 1)\frac{v^i v^k}{\|\vec{v}\|^2} = \delta^i{}_k + \frac{L^0{}_0 - 1}{1 - (L^0{}_0)^{-2}} \frac{L^i{}_0}{L^0{}_0}\frac{L^k{}_0}{L^0{}_0}$$

$$= \delta^i{}_k + \frac{L^0{}_0 - 1}{(L^0{}_0)^2 - 1} L^i{}_0 L^k{}_0 = \delta^i{}_k + \frac{L^i{}_0 L^k{}_0}{1 + L^0{}_0}.$$

Defina $\mathcal{R} = L(\vec{v})^{-1} L = L(-\vec{v}) L$. Luego $\det \mathcal{R} = 1$,

$$\mathcal{R}^0{}_0 = L(-\vec{v})^0{}_\mu L^\mu{}_0 = (L^0{}_0)^2 - \sum_{i=1}^{3} (L^i{}_0)^2 = 1$$

y

$$\mathcal{R}^0{}_i = L(-\vec{v})^0{}_\mu L^\mu{}_i = L^0{}_0 L^0{}_i - \sum_{j=1}^{3} L^j{}_0 L^j{}_i = g_{0i} = 0.$$

Entonces existe $R \in SO(3)$ t.q. $\mathcal{R} = L(R)$. Concluimos que

$$L = L(\vec{v}) L(R)$$

Suponga que existe $\tilde{\vec{v}} \in \mathbb{R}^3$ y $\tilde{R} \in SO(3)$ t.q.

$$L(\vec{v}) L(R) = L = L(\tilde{\vec{v}}) L(\tilde{R}).$$

Luego

$$1 = L(-\vec{v})L(\tilde{\vec{v}})L(\tilde{R})L(R)^{-1}.$$

Thefefore

$$1 = L(-\vec{v})^{0}{}_{\mu}L(\tilde{\vec{v}})^{\mu}{}_{\nu}L(\tilde{R})^{\nu}{}_{\sigma}L(R^{-1})^{\sigma}{}_{0} = L(-\vec{v})^{0}{}_{\mu}L(\tilde{\vec{v}})^{\mu}{}_{\nu}L(\tilde{R})^{\nu}{}_{0}$$

$$= L(-\vec{v})^{0}{}_{\mu}L(\tilde{\vec{v}})^{\mu}{}_{0} = \frac{1 - \vec{v}\cdot\tilde{\vec{v}}}{\sqrt{1-\vec{v}^2}\sqrt{1-\tilde{\vec{v}}^2}}$$

and we conclude $\vec{v} = \tilde{\vec{v}}$. Therefore $1 = L(\tilde{R})L(R)^{-1}$ and $\tilde{R} = R$.

This solution was adapted from Scheck, F. "Mechanics"

E) We have that under a Lorentz transformation $B_{\mu\nu}$ transforms by

$$B'_{\mu\nu} = \Lambda_{\mu}{}^{\sigma}\Lambda_{\nu}{}^{\rho}B_{\sigma\rho} = -\Lambda_{\mu}{}^{\sigma}\Lambda_{\nu}{}^{\rho}B_{\rho\sigma} = -B_{\nu\mu}.$$

In particular, we confirm $B'_{\mu\nu}$ is antisymmetric.

Note that

$$\varepsilon^{\alpha\beta\delta\varepsilon}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\delta}\Lambda^{\sigma}{}_{\varepsilon} = \det\Lambda\ \varepsilon^{\mu\nu\rho\sigma}$$

Therefore, since $\det\Lambda^2 = 1$

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}B'_{\rho\sigma} = \frac{1}{2}\det\Lambda\, \varepsilon^{\alpha\beta\delta\varepsilon}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\delta}\Lambda^{\sigma}{}_{\varepsilon}\Lambda_{\rho}{}^{\xi}\Lambda_{\sigma}{}^{\zeta}B_{\xi\zeta}$$

$$= \frac{1}{2}\det\Lambda\, \varepsilon^{\alpha\beta\delta\varepsilon}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\delta^{\xi}_{\delta}\delta^{\zeta}_{\varepsilon}B_{\xi\zeta}$$

$$= \frac{1}{2}\det\Lambda\, \varepsilon^{\alpha\beta\delta\varepsilon}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}B_{\delta\varepsilon}$$

$$= \det\Lambda\, \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}B^{\alpha\beta} = \det\Lambda\, B'^{\mu\nu}.$$

In particular, if $\Lambda \in L_+^{\uparrow}$ (which is the most we can hope to get a representation for since it is the connected component of the identity) we get that the antisymmetric self-dual second-rank tensors are closed under the action of $L_+^{\uparrow}$.

Section 1.3.

A) The product is defined by

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2).$$

It is associative:

$$((\Lambda_1, a_1)(\Lambda_2, a_2))(\Lambda_3, a_3) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)(\Lambda_3, a_3)$$

$$= (\Lambda_1\Lambda_2\Lambda_3, a_1 + \Lambda_1 a_2 + \Lambda_1\Lambda_2 a_3)$$

$$= (\Lambda_1, a_1)(\Lambda_2\Lambda_3, a_2 + \Lambda_2 a_3)$$

$$= (\Lambda_1, a_1)((\Lambda_2, a_2)(\Lambda_3, a_3)).$$

It has an identity:

$$(1,0)(\Lambda, a) = (\Lambda, a) = (\Lambda, a)(1,0).$$

Every element has an inverse:

$$(\Lambda, a)(\Lambda^{-1}, -\Lambda^{-1}a) = (\Lambda\Lambda^{-1}, a - \Lambda\Lambda^{-1}a) = (1, 0)$$

$$= (\Lambda^{-1}\Lambda, -\Lambda^{-1}a + \Lambda^{-1}a) = (\Lambda^{-1}, -\Lambda^{-1}a)(\Lambda, a).$$

B). We first calculate

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\alpha\beta\gamma} = {} & \cancelto{4}{\delta^{\mu}_{\mu}}\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\sigma}_{\gamma} - \cancelto{4}{\delta^{\mu}_{\mu}}\delta^{\nu}_{\alpha}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\beta} + \cancelto{4}{\delta^{\mu}_{\mu}}\delta^{\nu}_{\gamma}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} - \cancelto{4}{\delta^{\mu}_{\mu}}\delta^{\nu}_{\gamma}\delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha} \\
& + \cancelto{4}{\delta^{\mu}_{\mu}}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\alpha} - \cancelto{4}{\delta^{\mu}_{\mu}}\delta^{\nu}_{\beta}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\gamma} \\
& + \delta^{\mu}_{\beta}\delta^{\nu}_{\mu}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\gamma} - \delta^{\mu}_{\beta}\delta^{\nu}_{\mu}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\alpha} + \delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}\delta^{\rho}_{\mu}\delta^{\sigma}_{\alpha} - \delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\mu} \\
& + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\mu} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\delta^{\rho}_{\mu}\delta^{\sigma}_{\gamma} \\
& + \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\mu}\delta^{\sigma}_{\gamma} - \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\mu} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\gamma}\delta^{\rho}_{\beta}\delta^{\sigma}_{\mu} - \delta^{\mu}_{\alpha}\delta^{\nu}_{\gamma}\delta^{\rho}_{\mu}\delta^{\sigma}_{\beta} \\
& + \delta^{\mu}_{\alpha}\delta^{\nu}_{\mu}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\beta} - \delta^{\mu}_{\alpha}\delta^{\nu}_{\mu}\delta^{\rho}_{\beta}\delta^{\sigma}_{\gamma} \\
& + \delta^{\mu}_{\gamma}\delta^{\nu}_{\alpha}\delta^{\rho}_{\mu}\delta^{\sigma}_{\beta} - \delta^{\mu}_{\gamma}\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\sigma}_{\mu} + \delta^{\mu}_{\gamma}\delta^{\nu}_{\beta}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\mu} - \delta^{\mu}_{\gamma}\delta^{\nu}_{\beta}\delta^{\rho}_{\mu}\delta^{\sigma}_{\alpha} \\
& + \delta^{\mu}_{\gamma}\delta^{\nu}_{\mu}\delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha} - \delta^{\mu}_{\gamma}\delta^{\nu}_{\mu}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} \\
= {} & \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\sigma}_{\gamma} - \delta^{\nu}_{\alpha}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\beta} + \delta^{\nu}_{\gamma}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\nu}_{\gamma}\delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha} + \delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\alpha} - \delta^{\nu}_{\beta}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\gamma}.
\end{aligned}
$$

Por lo tanto, como $W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} S_{\rho\sigma} P_{\nu}$,

$$
\begin{aligned}
W^{\mu}W_{\mu} &= \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} P_{\nu} S_{\rho\sigma}\varepsilon_{\mu\alpha\beta\gamma} P^{\alpha} S^{\beta\gamma} \\
&= \frac{1}{4} P_{\nu} S_{\rho\sigma}\left(P^{\nu}S^{\rho\sigma} - P^{\nu}S^{\sigma\rho} + P^{\rho}S^{\sigma\nu} - P^{\sigma}S^{\rho\nu} + P^{\sigma}S^{\nu\rho} - P^{\rho}S^{\nu\sigma}\right) \\
&= \frac{1}{2} m^2 S_{\rho\sigma} S^{\rho\sigma} + \frac{1}{2} P_{\nu} S_{\rho\sigma}\left(P^{\rho}S^{\sigma\nu} - P^{\sigma}S^{\rho\nu}\right)
\end{aligned}
$$

Notando que $S_{\rho\sigma} P^{\sigma} S^{\rho\nu} = -S_{\sigma\rho} P^{\sigma} S^{\rho\nu} = -S_{\rho\sigma} P^{\rho} S^{\sigma\nu}$ se obtiene

$$W^{\mu} W_{\mu} = \frac{1}{2} m^2 S_{\rho\sigma} S^{\rho\sigma} + \frac{1}{2} P_{\nu} S_{\rho\sigma} P^{\rho} S^{\sigma\nu}$$