KMS states and Tomita-Takesaki Theory

Iván Mauricio Burbano Aldana

Advised by: Prof. Andrés Fernando Reyes Lega

Universidad de los Andes

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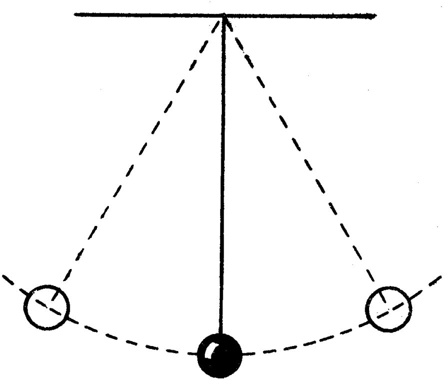
Motivation



Can we obtain the equations of motion from the equilibrium state?

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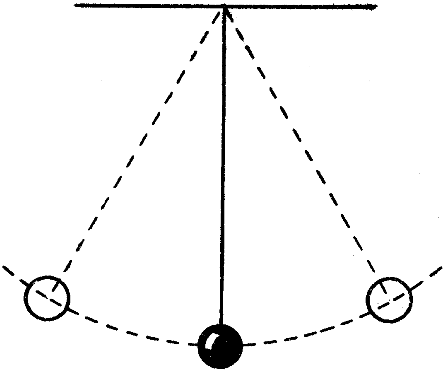

Can we obtain the equations of motion from the equilibrium state?

Maybe in quantum thermal systems.

$$e^{-\beta H} \circlearrowright e^{-iHt}$$

$$\text{temperature} \iff i \times \text{time}$$

Outline

Elements of Classical and Quantum Theories

Classical theories

- Auxiliary space: locally compact Hausdorff space X;

Quantum theories

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- States: probability measures P on X;

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- Auxiliary space: separable Hilbert space $\mathcal{H}$
- Observables: self-adjoint operators on $\mathcal{H}$
- States: positive, self-adjoint, normalized and trace-class operators ρ on $\mathcal{H}$;

Elements of Classical and Quantum Theories

Classical theories

- Auxiliary space: locally compact Hausdorff space X;
- Observables: continuous functions $C(X)$ on X;
- States: probability measures P on X;
- Expected values: $\int f dP$.

Quantum theories

- Auxiliary space: separable Hilbert space $\mathcal{H}$
- Observables: self-adjoint operators on $\mathcal{H}$
- States: positive, self-adjoint, normalized and trace-class operators ρ on $\mathcal{H}$;
- Expected values: $\mathrm{tr}(A\rho)$.

Algebraic Quantum Mechanics

- Observables: A C^*-algebra $\mathcal{A}$:
  - Complete normed vector space with product and involution;
  - C^* property: $\|A^*A\| = \|A\|^2$;
  - A C^*-algebra can always be realized as a uniformly closed subset of the bounded operators on a Hilbert space[Bratteli and Robinson, 1987]. It is called a von Neumann algebra or W^*-algebra if $\mathcal{A}'' = \mathcal{A}$ where the commutant $\mathfrak{A}'$ of a set $\mathfrak{A}$ of bounded operators on a Hilbert space is defined as the set of all bounded operatos which commute with every element of $\mathfrak{A}$.

- States: Positive normalized linear functionals $\omega : \mathcal{A} \to \mathbb{C}$.

GNS Construction

Start with a C^*-algebra $\mathcal{A}$ and a state ω.

- $\mathcal{N}_\omega := \{A \in \mathcal{A} | \omega(AA^*) = 0\}$
- Hilbert space $\mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{N}_\omega}$ with $\langle [A], [B] \rangle := \omega(A^* B)$
- Define the representation extending

$$\begin{aligned} \pi_\omega : \mathcal{A} \to \quad & \mathcal{B}(\mathcal{H}_\omega) \\ A \mapsto & \pi_\omega(A) : \mathcal{H}_\omega \to \mathcal{H}_\omega \\ & \qquad\qquad [B] \mapsto [AB] \end{aligned}$$

- Cyclic vector $\Omega_\omega := [1]$, that is, $\overline{\mathcal{A}\Omega_\omega} = \mathcal{H}_\omega$
- This is the unique $*$-representation of $\mathcal{A}$ with a cyclic vector Ω_ω such that $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$.

Cyclic representations of W^*-algebras

Theorem (★)

If $\mathfrak{M}$ is a W^*-algebra and ω is a faithful ($\omega(A^*A) = 0 \to A = 0$) normal ($\omega(A) = \mathrm{tr}(\rho A)$) state then its cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ satisfies

- π_ω *is faithful (injective);*
- $\pi_\omega(\mathfrak{M})$ *is a von Neumann algebra;*
- Ω_ω *is separating for* $\pi_\omega(\mathfrak{M})$ ($\pi_\omega(A)\Omega_\omega = 0 \to \pi_\omega(A) = 0$).

Dynamical Systems

Time evolution is represented by a one-parameter group of automorphisms

$$\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$$
$$t \mapsto \tau_t.$$

Dynamical systems consist of an $C^*(W^*)$-algebra with a time evolutions which satisfy certain continuity properties.

KMS States

Definition

Let $(\mathcal{A}, \tau)$ be a dynamical system. We say that a state ω is a (τ, β)-KMS state if for all $A, B \in \mathcal{A}$ there exists a continuous bounded function $F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$ analytic on $\mathfrak{D}_\beta$ (the strip of the complex plain bounded by $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \beta$) such that

$$F_{A,B}(t) = \omega(A\tau_t(B))$$
$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A)$$

for all $t \in \mathbb{R}$.

KMS states as Equilibrium states

KMS states are a candidate for a general definition of thermodynamic equilibrium in quantum systems[Haag, 1992][Duvenhage, 1999]:

- KMS states are invariant under the dynamics $\omega(\tau_t(A)) = \omega(A)$;
- In finite dimensional Hilbert spaces with Schrödinger's time evolution τ, the only possible (τ, β)-KMS states are the β-Gibbs states

$$\begin{aligned} \mathcal{B}(\mathcal{H}) &\to \mathbb{C} \\ A &\mapsto \frac{\operatorname{tr}\left(Ae^{-\beta H}\right)}{\operatorname{tr}\left(e^{-\beta H}\right)}. \end{aligned}$$

Tomita-Takesaki Theory

For a W^*-algebra $\mathfrak{M}$ equipped with a cyclic and separating vector Ω Tomita-Takesaki theory yields:

- a one-parameter unitary group $t \mapsto \Delta^{it}$;
- a modular conjugation J.

Theorem (Tomita-Takesaki)

- $J\mathfrak{M}J = \mathfrak{M}'$;
- $\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}$ *for all* $t \in \mathbb{R}$.

Proof.

[Duvenhage, 1999] $\square$

Tomita-Takesaki, Time Evolution and KMS States

Theorem (★)

$t \mapsto \Delta^{it}$ is the unique strongly continuous one-parameter unitary group on $\mathcal{H}$ that satisfies the KMS condition with respect to $\mathcal{K}$ such that $\Delta^{it}\mathcal{K} \subseteq \mathcal{K}$ for all $t \in \mathbb{R}$.

Theorem (★)

Let $\mathfrak{M}$ be a von Neuman algebra and ω a faithful normal state. Consider the unitary group $t \mapsto \Delta^{it}$ associated to the pair $(\pi_\omega(\mathfrak{M}), \Omega_\omega)$. Then the one-parameter group of automorphisms given by $\alpha_t = \pi_\omega^{-1}(\Delta^{it}\pi_\omega(A)\Delta^{-it})$ makes $(\mathfrak{M}, \alpha)$ a W^*-dynamical system.

Proof.

[Duvenhage, 1999] □

The Canonical Time Evolution

Theorem (★★★)

Let $\mathfrak{M}$ be a von Neumann algebra and ω be a faithful normal state. Then $(\mathfrak{M}, \tau)$ with $\tau_t(A) = \alpha_{-t/\beta}(A)$ and α the modular group of $(\mathfrak{M}, \omega)$ is the unique W^*-dynamical system such that ω is a (τ, β)-KMS state.

Proof.

[Duvenhage, 1999] □

Further work

- Classical KMS states and Tomita-Takesaki theory.
- Understanding KMS states from "first principles":
  - stability;
  - passivity.
- Relativistic generalization of KMS states.
- Entropy ambiguities.

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