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Supersymmetry

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Exercise 7.1.

Consider an R-symmetry transformation of the form (6.3) and (6.4)

$$\Phi(y,\theta) \longmapsto e^{2inK}\,\Phi(y, e^{-iK}\theta),$$

$$\bar{\Phi}(\bar{y},\bar{\theta}) \longmapsto e^{-2inK}\,\bar{\Phi}(\bar{y}, e^{iK}\bar{\theta}).$$

Then

$$\int d^4x\, d^2\theta\, g\, \Phi(x,\theta)^3 \longmapsto \int d^4x\, d^2\theta\, g\, e^{6inK}\,\Phi(x, e^{-iK}\theta)^3$$

$$= \int d^4x\, d^2\theta\, g\, e^{i(6n-2)K}\,\Phi(x,\theta)^3.$$

Thus the cubic term is invariant if and only if $n=\frac{1}{3}$.

Similarly,

$$\int d^4x\, d^2\bar{\theta}\, \bar{\Phi}(x,\bar{\theta})^3 \longmapsto \int d^4x\, d^2\bar{\theta}\, e^{-2inK}\,\bar{\Phi}(x, e^{iK}\bar{\theta})^3$$

$$= \int d^4x\, d^2\bar{\theta}\, \bar{\Phi}(x,\bar{\theta})^3.$$

We assume similar R-symmetry transformations for Ξ

$$\Xi(y,\theta) \longmapsto e^{2imK}\,\Xi(y, e^{-iK}\theta)$$

$$\bar{\Xi}(\bar{y},\bar{\theta}) \longmapsto e^{-2imK}\,\bar{\Xi}(\bar{y}, e^{iK}\bar{\theta}).$$

Then

$$\int d^4x\, d^2\theta\, \Phi(x,\theta)\Sigma(x,\theta) \longmapsto \int d^4x\, d^2\theta\, e^{2inK} e^{2imK}\, \Phi(x, e^{-iK}\theta)\Sigma(x, e^{-iK}\theta)$$

$$= \int d^4x\, d^2\theta\, e^{i(2n+2m-2)K}\, \Phi(x,\theta)\Sigma(x,\theta)$$

This term is invariant if and only if

$$m = 1 - n = \frac{2}{3}.$$

Then we automatically have

$$\int d^4x\, d^2\bar\theta\, \bar\Phi(x,\bar\theta)\bar\Sigma(x,\bar\theta) \longmapsto \int d^4x\, d^2\bar\theta\, e^{-2inK} e^{-2imK}\, \bar\Phi(x, e^{iK}\bar\theta)\bar\Sigma(x, e^{iK}\bar\theta)$$

$$= \int d^4x\, d^2\bar\theta\, e^{i(-2n-2m+2)K}\, \bar\Phi(x,\bar\theta)\bar\Sigma(x,\bar\theta)$$

$$= \int d^4x\, d^2\bar\theta\, \bar\Phi(x,\bar\theta)\bar\Sigma(x,\bar\theta).$$

For the last chiral superfield $W_\alpha = -\frac{1}{4}\bar D \bar D D_\alpha$, we note that since it is also chiral, it makes sense that

$$W_\alpha(y,\theta) \longmapsto e^{2ilK} W_\alpha(y, e^{-iK}\theta),$$

$$\bar W_\alpha(y,\theta) \longmapsto e^{-2ilK} \bar W_\alpha(\bar y, e^{iK}\bar\theta).$$

Then

$$\int d^4x\, d^2\theta\, W^\alpha W_\alpha \longmapsto \int d^4x\, d^2\theta\, e^{4ilK}\, W^\alpha(x, e^{-iK}\theta) W_\alpha(x, e^{-iK}\theta)$$

$$= \int d^4x\, d^2\theta\, e^{i(4l-2)K}\, W^\alpha(x,\theta) W_\alpha(x,\theta).$$

This is invariant if and only if

$$l = 1/2 .$$

Then

$$\int d^4x\, d^2\bar\theta\ \bar W(x,\bar\theta)^2 \longmapsto \int d^4x\, d^2\bar\theta\, e^{-2iK}\ \bar W(x, e^{iK}\bar\theta)^2$$

$$= \int d^4x\, d^2\bar\theta\ \bar W(x,\bar\theta)^2.$$

These transformations are achieved if we let V be a scalar under R-symmetry

$$V(x,\theta,\bar\theta) \longmapsto V(x, e^{-iK}\theta, e^{iK}\bar\theta) =: V'(x,\theta,\bar\theta).$$

To see this we note that for a superfield

$$f(x,\theta,\bar\theta) = g(x, e^{-iK}\theta, e^{iK}\bar\theta)$$

we have

$$\frac{\partial f}{\partial\theta^\alpha}(x,\theta,\bar\theta) = \frac{\partial g}{\partial\theta^\alpha}(x, e^{-iK}\theta, e^{iK}\bar\theta)\, e^{-iK}.$$

Then, given than in R-symmetry $\bar\theta \to e^{-iK}\bar\theta$, then

$$D_\alpha f(x,\theta,\bar\theta) = \frac{\partial}{\partial\theta^\alpha} g(x, e^{-iK}\theta, e^{iK}\bar\theta) + i(\sigma^m e^{-iK}\bar\theta)_\alpha \partial_m g(x, e^{-iK}\theta, e^{iK}\bar\theta)$$

$$= e^{-iK} D_\alpha g(x, e^{-iK}\theta, e^{iK}\bar\theta).$$

Similarly

$$\bar D^{\dot\alpha} f(x,\theta,\bar\theta) = e^{iK}\,\bar D^{\dot\alpha} g(x, e^{-iK}\theta, e^{iK}\bar\theta).$$

Thus

$$W_\alpha(x,\theta,\bar\theta) \longmapsto e^{ik}\, W_\alpha(x, e^{-ik}\theta, e^{ik}\theta).$$

Exercise 7.2

In the WZ gauge we have

$$-(\theta\sigma^m\bar\theta)\delta_Q A_m(x) + i\theta^2\bar\theta\delta_Q\bar\lambda(x) - i\bar\theta^2\theta\delta_Q\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2\delta_Q d(x)$$

$$= \delta_Q V(x) = (\xi Q + \bar\xi\bar Q)V(x,\theta,\bar\theta) + \Lambda(x+i\theta\sigma\bar\theta,\theta) + \bar\Lambda(x-i\theta\sigma\bar\theta,\bar\theta)$$

where

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^m},$$

$$\bar Q^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha\frac{\partial}{\partial x^m}.$$

The gauge transformation

$$\Lambda(y,\theta) = a(y) + \sqrt{2}\,\theta b(y) + \theta\theta f(y),$$

will later be choosen to ensure that the result is indeed in the WZ gauge. We have

$$Q_\alpha V(x,\theta,\bar\theta) = -(\sigma^m\bar\theta)_\alpha A_m(x) + 2i\theta_\alpha\bar\theta\bar\lambda(x) - i\bar\theta^2\lambda_\alpha(x)$$

$$+\theta_\alpha\bar\theta^2 d(x) + i\underbrace{(\theta\sigma^m\bar\theta)(\sigma^n\bar\theta)_\alpha}\partial_n A_m(x)$$

$$\theta^\beta\sigma^m{}_{\beta\dot\beta}\bar\theta^{\dot\beta}\sigma^n{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$$
$$\shortparallel$$
$$=\frac{1}{2}\varepsilon^{\dot\beta\dot\alpha}\bar\theta^2\theta^\beta\sigma^m{}_{\beta\dot\beta}\sigma^n{}_{\alpha\dot\alpha}$$
$$\shortparallel$$
$$\frac{1}{2}\bar\theta^2\bar\sigma^{m\dot\alpha\beta}\theta_\beta\sigma^n{}_{\alpha\dot\alpha}$$
$$\shortparallel$$
$$\underbrace{\frac{1}{2}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta}$$

$$= \frac{1}{2}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta\partial_n A_m(x)$$

$$= \frac{1}{4}\bar\theta^2(-2\eta^{nm}\delta_\alpha{}^\beta)\theta_\beta\partial_n A_m(x)$$

$$+\frac{1}{4}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta F_{nm}(x)$$

$$= -\frac{1}{2}\bar\theta^2\theta_\alpha\partial^m A_m(x) + \frac{1}{4}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta F_{nm}(x)$$

$$+\underbrace{(\sigma^m\bar\theta)_\alpha\,\theta^2\,\bar\theta\,\partial_m\bar\lambda(x)}_{}$$

$$\theta^2\sigma^m{}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta}\,\partial_m\bar\lambda^{\dot\beta}(x) = -\theta^2\sigma^m{}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\,\partial_m\bar\lambda_{\dot\beta}(x)$$

$$= -\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\,\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}\,\partial_m\bar\lambda_{\dot\beta}(x)\theta^2$$

$$= -\frac{1}{2}\theta^2\bar\theta^2\left(\sigma^m\partial_m\bar\lambda(x)\right)_\alpha .$$

$$= -(\sigma^m\bar\theta)_\alpha A_m(x) + 2i\theta_\alpha\,\bar\theta\bar\lambda(x) - i\bar\theta^2\lambda_\alpha(x) + \theta_\alpha\bar\theta^2 d(x)$$

$$-\frac{i}{2}\bar\theta^2\theta_\alpha\,\partial^m A_m(x) + \frac{1}{4}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta\,F_{nm}(x) - \frac{1}{2}\theta^2\bar\theta^2\left(\sigma^m\partial_m\bar\lambda(x)\right)_\alpha$$

Similarly

$$\bar Q^{\dot\alpha}V(x,\theta,\bar\theta) = \underbrace{\theta^\alpha\sigma^m{}_{\alpha\dot\beta}\,\varepsilon^{\dot\beta\dot\alpha}A_m(x)}_{\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha A_m(x)} + i\theta^2\bar\lambda^{\dot\alpha}(x) - 2i\bar\theta^{\dot\alpha}\theta\lambda(x)$$

$$+\theta^2\bar\theta^{\dot\alpha}d(x) + \underbrace{i(\bar\sigma^n\theta)^{\dot\alpha}(\theta\sigma^m\bar\theta)\partial_n A_m(x)}_{}$$

$$= \bar\sigma^{n\dot\alpha\alpha}\theta_\alpha\,\theta^\beta\sigma^m{}_{\beta\dot\beta}\,\bar\theta^{\dot\beta}\partial_n A_m(x) = -\frac{1}{2}\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta}\theta^2\,\bar\sigma^{n\dot\alpha\alpha}\sigma^m{}_{\beta\dot\beta}\,\bar\theta^{\dot\beta}\partial_n A_m(x)$$

$$= -\frac{1}{2}\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta}\partial_n A_m(x) = -\frac{1}{4}\theta^2(-2\eta^{nm})\delta_{\dot\beta}{}^{\dot\alpha}\,\bar\theta^{\dot\beta}\partial_n A_m(x)$$

$$-\frac{1}{4}\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta}F_{nm}(x)$$

$$= \frac{1}{2}\theta^2\bar\theta^{\dot\alpha}\partial^m A_m(x) - \frac{1}{4}\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta}F_{nm}(x)$$

$$-\underbrace{\bar\theta^2(\bar\sigma^m\theta)^{\dot\alpha}\,\theta\partial_m\lambda(x)}_{}$$

$$\bar\theta^2\,\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha\theta^\beta\partial_m\lambda_\beta(x) = -\frac{1}{2}\varepsilon_{\alpha\beta}\,\theta^2\bar\theta^2\,\bar\sigma^{m\dot\alpha\alpha}\,\partial_m\lambda^\beta(x) = -\frac{1}{2}\theta^2\bar\theta^2\left(\bar\sigma^m\partial_m\lambda(x)\right)^{\dot\alpha}$$

$$= i\,(\bar\sigma^m\theta)^{\dot\alpha} A_m(x) + i\theta^2\bar\lambda^{\dot\alpha}(x) - 2i\,\bar\theta^{\dot\alpha}\theta\lambda(x) + \theta^2\bar\theta^{\dot\alpha} d(x)$$

$$+ \frac{i}{2}\,\theta^2\bar\theta^{\dot\alpha}\partial^m A_m(x) - \frac{i}{4}\,\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta}F_{nm}(x)$$

$$+ \frac{1}{2}\,\theta^2\bar\theta^2\left(\bar\sigma^m\partial_m\lambda(x)\right)^{\dot\alpha}.$$

Therefore

$$\delta_Q V = -\,(\xi\sigma^m\bar\theta)A_m(x) + 2i(\xi\theta)(\bar\theta\bar\lambda(x)) - i\bar\theta^2(\xi\lambda(x)) + (\xi\theta)\bar\theta^2 d(x)$$

$$- \frac{i}{2}\,\bar\theta^2(\xi\theta)\partial^m A_m(x) + \frac{i}{4}\,\bar\theta^2(\xi\sigma^n\bar\sigma^m\theta)F_{nm}(x) - \frac{1}{2}\,\theta^2\bar\theta^2(\xi\sigma^m\partial_m\bar\lambda(x))$$

$$+ (\bar\xi\bar\sigma^m\theta)A_m(x) + i\theta^2(\bar\xi\bar\lambda(x)) - 2i(\bar\xi\bar\theta)(\theta\lambda(x)) + \theta^2(\bar\xi\bar\theta)d(x)$$

$$+ \frac{i}{2}\,\theta^2(\bar\xi\bar\theta)\partial^m A_m(x) - \frac{i}{4}\,\theta^2(\bar\xi\bar\sigma^n\sigma^m\bar\theta)F_{nm}(x) + \frac{1}{2}\,\theta^2\bar\theta^2(\bar\xi\bar\sigma^m\partial_m\lambda(x))$$

$$+ a(x) + i(\theta\sigma^m\bar\theta)\partial_m a(x) + \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\Box a(x)$$

$$+ \sqrt{2}\,\theta b(x) + i\sqrt{2}\,\underbrace{(\theta\sigma^m\bar\theta)(\theta\partial_m b(x))} + \theta\theta f(x)$$

$$\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\partial_m b_\beta(x) = +\frac{1}{2}\,\varepsilon^{\alpha\beta}\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m b_\beta(x)$$

$$= -\frac{1}{2}\,\theta^2\left(\partial_m b\,\sigma^m\bar\theta\right)$$

$$+ \bar a(x) - i(\theta\sigma^m\bar\theta)\partial_m\bar a(x) + \frac{1}{4}\,\theta^2\bar\theta^2\,\Box\bar a(x)$$

$$+ \sqrt{2}\,\bar\theta\bar b(x) - i\sqrt{2}\,\underbrace{(\theta\sigma^m\bar\theta)(\bar\theta\partial_m\bar b(x))} + \bar\theta\bar\theta\bar f(x)$$

$$\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta}\partial_m\bar b^{\dot\beta}(x) = -\frac{1}{2}\,\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar b_{\dot\beta}(x)$$

$$= -\frac{1}{2}\,\bar\theta^2\left(\theta\sigma^m\partial_m\bar b(x)\right).$$

By comparison on the $1, \theta, \bar\theta, \theta^2$ and $\bar\theta^2$ terms, we see that we need to choose our gauge transformation such that

$$a(x) = -\bar{a}(x)$$

$$-\sqrt{2}\, b^{\alpha}_{\alpha}(x) = \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{m\dot{\alpha}\alpha} A_m(x)$$

$$\sqrt{2}\, \bar{b}_{\dot{\alpha}}(x) = \xi^{\alpha} \sigma^{m}{}_{\alpha\dot{\alpha}} A_m(x)$$

$$f(x) = -i(\bar{\xi}\bar{\lambda}(x))$$

$$\bar{f}(x) = i(\xi\lambda(x)).$$

By comparing the $\theta\sigma^m\bar{\theta}$

$$(\theta\sigma^m\bar{\theta})\delta_Q A_m(x) = 2i(\xi\theta)(\bar{\theta}\bar{\lambda}(x)) - 2i(\bar{\xi}\bar{\theta})(\theta\lambda(x)) + i(\theta\sigma^m\bar{\theta})\partial_m a(x) - i(\theta\sigma^m\bar{\theta})\partial_m\bar{a}(x)$$

We thus have

$$\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\delta_Q A^n(x) = -(\theta\sigma^n\bar{\theta})(\theta\sigma^m\bar{\theta})\delta_Q A_m(x)$$

$$= 2i\underbrace{(\theta\sigma^n\bar{\theta})(\xi\theta)(\bar{\theta}\bar{\lambda}(x))}$$

$$\theta^{\alpha}\sigma^{n}{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\xi^{\beta}\theta_{\beta}\bar{\theta}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}}(x)$$

$$\shortparallel$$

$$\frac{1}{4}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\theta^2\bar{\theta}^2\sigma^{n}{}_{\alpha\dot{\alpha}}\xi_{\beta}\bar{\lambda}_{\dot{\beta}}(x)$$

$$\shortparallel$$

$$\frac{1}{4}\theta^2\bar{\theta}^2\bar{\sigma}^{n\dot{\beta}\beta}\xi_{\beta}\bar{\lambda}_{\dot{\beta}}(x) = -\frac{1}{4}\theta^2\bar{\theta}^2(\bar{\lambda}(x)\bar{\sigma}^n\xi)$$

$$-2i\underbrace{(\theta\sigma^n\bar{\theta})(\bar{\xi}\bar{\theta})(\theta\lambda(x))} + \frac{i}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^n\underbrace{(a(x)-\bar{a}(x))}_{2a(x)}$$

$$-\frac{1}{4}\theta^2\bar{\theta}^2(\bar{\xi}\bar{\sigma}^n\lambda(x))$$

$$= \frac{i}{2}\left(\bar{\xi}\bar{\sigma}^n\lambda(x) - \bar{\lambda}(x)\bar{\sigma}^n\xi\right)$$

$$= \frac{i}{2}\left(\bar{\xi}\bar{\sigma}^n\lambda(x) + \xi\sigma^n\bar{\lambda}(x)\right)$$

We conclude that

$$\boxed{\delta_Q A^n(x) = i\bar{\xi}\bar{\sigma}^n\lambda(x) + i\xi\sigma^n\bar{\lambda}(x) - 2i\partial^n a(x)}$$

Similarly, by comparing the $\bar\theta^2\theta$ terms we have

$$i\bar\theta^2(\theta\delta_Q\lambda(x)) = (\xi\theta)\bar\theta^2 d(x) - \cancel{\frac{i}{2}\bar\theta^2(\xi\theta)\partial^m A_m(x)} + \frac{i}{4}\bar\theta^2(\xi\sigma^n\bar\sigma^m\theta)F_{nm}(x)$$

$$+ \underbrace{\frac{i}{\sqrt{2}}\bar\theta^2\left(\theta\sigma^m\partial_m\bar b(x)\right)}$$

$$\frac{i}{\sqrt{2}}\bar\theta^2\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\frac{1}{\sqrt{2}}\xi^\beta\sigma^n{}_{\beta\dot\beta}\varepsilon^{\dot\alpha\dot\beta}\partial_m A_n(x)$$

$$= -\frac{i}{2}\bar\theta^2\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\xi_\beta\bar\sigma^{n\dot\alpha\beta}\partial_m A_n(x)$$

$$= -\frac{i}{2}\bar\theta^2\theta^\alpha(\sigma^m\bar\sigma^n)_\alpha{}^\beta\xi_\beta\partial_m A_n(x)$$

$$= -\frac{i}{4}\bar\theta^2\theta^\alpha(-2\eta^{mn}\delta_\alpha{}^\beta)\xi_\beta\partial_m A_n(x) - \frac{i}{4}\bar\theta^2\theta^\alpha(\sigma^m\bar\sigma^n)_\alpha{}^\beta\xi_\beta F_{mn}(x)$$

$$= \cancel{\frac{i}{2}\bar\theta^2(\theta\xi)\partial^m A_m(x)} - \frac{i}{4}\bar\theta^2(\theta\sigma^m\bar\sigma^n\xi)F_{mn}(x)$$

Notice that

$$\frac{1}{4}\bar\theta^2(\xi\sigma^n\bar\sigma^m\theta)F_{nm}(x) = \frac{1}{8}\bar\theta^2(\xi\sigma^{[n}\bar\sigma^{m]}\theta)F_{nm}(x)$$

$$= \frac{1}{2}\bar\theta^2(\xi\sigma^{nm}\theta)F_{nm}(x).$$

Moreover, using the hint of Exercise 1.2

$$(\xi\sigma^{nm}\theta) = \xi^\alpha\sigma^{nm}{}_\alpha{}^\beta\theta_\beta = -\xi^\alpha\sigma^{nm}{}_{\alpha\beta}\theta^\beta$$

$$= -\xi^\alpha\sigma^{nm}{}_{\beta\alpha}\theta^\beta = \theta^\beta\sigma^{nm}{}_{\beta\alpha}\xi^\alpha$$

$$= -(\theta\sigma^{nm}\xi).$$

Thus

$$-i\bar\theta^2(\theta\delta_Q\lambda(x)) = (\xi\theta)\bar\theta^2 d(x) + i\bar\theta^2(\xi\sigma^{nm}\theta)F_{nm}(x),$$

i.e.

$$\delta_Q \lambda^\alpha(x) = -\xi^\beta \sigma^{nm}{}_\beta{}^\alpha F_{nm}(x) + i\xi^\alpha d(x),$$

or

$$\boxed{\delta_Q \lambda_\alpha(x) = \sigma^{nm}{}_\alpha{}^\beta \xi_\beta F_{nm}(x) + i\xi_\alpha d(x).}$$

Repeating with $\theta^2\bar\theta$, we have

$$i\theta^2\bar\theta\delta_Q\bar\lambda(x) = \theta^2(\bar\xi\bar\theta)d(x) + \cancel{\frac{i}{2}\theta^2(\bar\xi\bar\theta)\partial^m A_m(x)} - \frac{i}{4}\theta^2(\bar\xi\bar\sigma^n\sigma^m\bar\theta)F_{nm}(x)$$

$$\underline{-\frac{i}{\sqrt2}\theta^2(\partial_m b\sigma^m\bar\theta)}$$

$$\frac{i}{\sqrt2}\theta^2\frac{1}{\sqrt2}\bar\xi_{\dot\alpha}\bar\sigma^{n\dot\alpha\alpha}\partial_m A_n(x)\sigma^m{}_{\alpha\dot\beta}\bar\theta^{\dot\beta}$$

$$= \frac{i}{2}\theta^2\bar\xi_{\dot\alpha}(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\partial_m A_n(x)$$

$$= \cancel{\frac{i}{2}\theta^2\bar\xi_{\dot\alpha}\bar\theta^{\dot\alpha}\partial^m A_m(x)} + \frac{i}{4}\theta^2(\bar\xi\bar\sigma^n\sigma^m\bar\theta)F_{mn}(x)$$

$$= \theta^2(\bar\xi\bar\theta)d(x) - i\theta^2\bar\xi_{\dot\alpha}\bar\sigma^{nm\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}F_{nm}(x).$$

Thus

$$\boxed{\delta_Q\bar\lambda^{\dot\alpha}(x) = \bar\sigma^{nm\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}F_{nm}(x) - i\bar\xi^{\dot\alpha}d(x).}$$

Finally, the terms $\theta^2\bar\theta^2$ give

$$\frac12\theta^2\bar\theta^2\delta_Q d(x) = -\frac12\theta^2\bar\theta^2(\xi\sigma^m\partial_m\bar\lambda(x)) + \frac12\theta^2\bar\theta^2(\bar\xi\bar\sigma^m\partial_m\lambda(x))$$

$$+ \frac14\theta^2\bar\theta^2\cancelto{0}{\Box(a+\bar a)(x)}.$$

Therefore,

$$\boxed{\delta_Q d(x) = \bar{\xi}\bar{\sigma}^m \partial_m \lambda(x) - \xi \sigma^m \partial_m \bar{\lambda}(x).}$$

By fixing the gauge transformation to $a=0$, we obtain (6.20).

Exercise 7.3.

We begin from the transformation rule (7.22)

$$A_m \longmapsto M A_m M^\dagger + i M \partial_m M^\dagger.$$

To lighten the notation we will work without Matrix indices. We then have

$$F_{mn} \mapsto \partial_m (M A_n M^\dagger + i M \partial_n M^\dagger) - m \longleftrightarrow n$$

$$- i \left[M A_m M^\dagger + i M \partial_m M^\dagger, M A_n M^\dagger + i M \partial_n M^\dagger \right]$$

$$= M \partial_m A_n M^\dagger + \partial_m M A_n M^\dagger + M A\cancel{{}_n \partial_m} M^\dagger + i \partial_m M \partial_n M^\dagger + i M \cancel{\partial_m \partial_n} M^\dagger$$

$$- m \cancel{\longleftrightarrow} n$$

$$- i \left(M A_m \cancel{M^\dagger M} A_n M^\dagger - M A_n \cancel{M^\dagger M} A_m M^\dagger \right.$$

$$+ i M A_m \cancel{M^\dagger M} \partial_n M^\dagger - i M \partial_n M^\dagger M A_m M^\dagger$$

$$+ i M \partial_m M^\dagger M A_n M^\dagger - i M A_n \cancel{M^\dagger M} \partial_m M^\dagger$$

$$\left. - M \partial_m M^\dagger M \partial_n M^\dagger + M \partial_n M^\dagger M \partial_m M^\dagger \right).$$

To further proceed we observe that the restriction $MM^\dagger = I$ restricts

$$0 = \partial_m I = \partial_m M M^\dagger + M \partial_m M^\dagger.$$

Thus $\partial_m M^\dagger = - M^\dagger \partial_m M M^\dagger$ and

$$F_{mn} \longmapsto M(\partial_m A_n - \partial_n A_m)M^\dagger + \partial_m \cancel{M A_n} M^\dagger + i\cancel{\partial_m M \partial_n} M^\dagger$$

$$- \partial_n \cancel{M A_m} M^\dagger - i\,\partial_n \cancel{M \partial_m} M^\dagger$$

$$- i M[A_m, A_n]M^\dagger + \cancel{MM^\dagger}\partial_n M \cancel{M^\dagger M} A_m M^\dagger$$

$$- \cancel{MM^\dagger}\partial_m M \cancel{M^\dagger M} A_n M^\dagger + \cancel{MM^\dagger}\partial_m M \cancel{M^\dagger M} M^\dagger \partial_n M M^\dagger$$

$$- i\cancel{\partial_m M \partial_n} M^\dagger + i\partial_n \cancel{M \partial_m} M^\dagger$$

$$= M F_{mn} M^\dagger.$$

Nathan Exercise:

In the Wess-Zumino gauge we have

$$V(x,\theta,\bar\theta) = -(\theta\sigma^m\bar\theta)A_m(x) + i\theta^2(\bar\theta\bar\lambda(x)) - i\bar\theta^2(\theta\lambda(x)) + \frac{1}{2}\theta^2\bar\theta^2 d(x)$$

$$V(x,\theta,\bar\theta)^2 = \underbrace{(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta)A_m(x)A_n(x)}$$

$$-\frac{1}{2}\theta^2\bar\theta^2 A^m(x)A_m(x).$$

Thus

$$e^{-V(x,\theta,\bar\theta)} = 1 + (\theta\sigma^m\bar\theta)A_m(x) - i\theta^2(\bar\theta\bar\lambda(x)) + i\bar\theta^2(\theta\lambda(x)) - \frac{1}{2}\theta^2\bar\theta^2 d(x)$$

$$-\frac{1}{4}\theta^2\bar\theta^2 A^m(x)A_m(x).$$

On the other hand the matter chiral field is

$$\Phi(x + i\theta\sigma\bar\theta, \theta) = \varphi(x) + i(\theta\sigma^m\bar\theta)\partial_m\varphi(x) + \frac{1}{4}\theta^2\bar\theta^2\Box\varphi(x)$$

$$+ \sqrt{2}\,\theta\psi(x) + \frac{i}{\sqrt{2}}\theta^2(\bar\theta\bar\sigma^m\partial_m\psi(x)) + \theta^2 F(x)$$

and

$$\bar\Phi(x - i\theta\sigma\bar\theta, \bar\theta) = \bar\varphi(x) - i(\theta\sigma^m\bar\theta)\partial_m\bar\varphi(x) + \frac{1}{4}\theta^2\bar\theta^2\Box\bar\varphi(x)$$

$$+ \sqrt{2}\,\bar\theta\bar\psi(x) - \frac{i}{\sqrt{2}}\bar\theta^2(\partial_m\bar\psi(x)\bar\sigma^m\theta) + \bar\theta^2\bar F(x).$$

Thus

$$\int d^4\theta\, \Phi(x+i\theta\sigma\bar\theta,\theta)\, e^{-V(x,\theta,\bar\theta)}\, \bar\Phi(x-i\theta\sigma\bar\theta,\bar\theta)$$

$$= \frac{1}{4}\varphi(x)\Box\bar\varphi(x) + \frac{i}{2}\varphi(x)A^m(x)\partial_m\bar\varphi(x)$$

$$+ \underbrace{\int d^4\theta\, \varphi(x)\left(-i\theta^2(\bar\theta\bar\lambda(x))\right)\sqrt{2}(\bar\theta\bar\psi(x))} \quad + \underbrace{\int d^4\theta\, \varphi(x)\, i\bar\theta^2(\theta\lambda(x))\frac{i}{\sqrt{2}}(\partial_m\bar\psi(x)\bar\sigma^m\theta)}$$

$$= -\int d^4\theta\, \frac{i}{\sqrt{2}}\varepsilon_{\dot\alpha\dot\beta}\,\theta^2\bar\theta^2\varphi(x)\bar\lambda^{\dot\alpha}(x)\bar\psi^{\dot\beta}(x) \qquad = \underbrace{-\frac{1}{\sqrt{2}}\int d^4\theta\, \varphi(x)\bar\theta^2\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\lambda_\alpha(x)\sigma^m{}_{\beta\dot\beta}\partial_m\bar\psi^{\dot\beta}(x)}$$

$$= \frac{i}{\sqrt{2}}\varphi(x)\left(\bar\lambda(x)\bar\psi(x)\right) \qquad -\frac{1}{2\sqrt{2}}\varphi(x)\left(\lambda\sigma^m\partial_m\bar\psi(x)\right)$$

$$\qquad\qquad = \frac{1}{2\sqrt{2}}\varphi(x)\left(\partial_m\bar\psi(x)\bar\sigma^m\lambda(x)\right)$$

$$-\frac{1}{2}\varphi(x)d(x)\bar\varphi(x) - \frac{1}{4}\varphi(x)A^m(x)A_m(x)\bar\varphi(x)$$

$$-\frac{1}{2}\partial^m\varphi(x)\partial_m\bar\varphi(x) - \frac{i}{2}\bar\varphi(x)A^m(x)\partial_m\varphi(x) + \frac{1}{4}\bar\varphi(x)\Box\varphi(x)$$

$$+ \underbrace{\int d^4\theta\, \sqrt{2}(\theta\psi(x))(\theta\sigma^m\bar\theta)A_m(x)\sqrt{2}(\bar\theta\bar\psi(x))}$$

$$= \int d^4\theta\, 2\,\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\psi_\alpha(x)\sigma^m{}_{\beta\dot\beta}\bar\theta^{\dot\beta}A_m(x)\bar\psi_{\dot\alpha}(x)\bar\theta^{\dot\alpha}$$

$$= -\int d^4\theta\, \varepsilon^{\alpha\beta}\frac{1}{2}\varepsilon^{\dot\beta\dot\alpha}\theta^2\bar\theta^2\psi_\alpha(x)\sigma^m{}_{\beta\dot\beta}A_m(x)\bar\psi_{\dot\alpha}(x)$$

$$= \frac{1}{2}\left(\psi(x)\sigma^m\bar\psi(x)\right)A_m(x)$$

$$+ \underbrace{\int d^4\theta\, \sqrt{2}(\theta\psi(x))\, i\bar\theta^2(\theta\lambda(x))\bar\varphi(x)}$$

$$= -\frac{i}{\sqrt{2}}\varepsilon^{\alpha\beta}\psi_\alpha(x)\lambda_\beta(x)\bar\varphi(x) = \frac{i}{\sqrt{2}}\left(\psi(x)\lambda(x)\right)\bar\varphi(x)$$

$$+\int d^4\theta \, \frac{i}{\sqrt{2}} \left(\bar\theta \bar\sigma^m \partial_m \psi(x)\right)\left(\theta\sigma^n\bar\theta\right) A_n(x) \frac{i}{\sqrt{2}} \left(\partial_m \bar\psi(x) \bar\sigma^n \theta\right)$$

$$= -\int d^4\theta \, \frac{1}{2} \partial_m \psi^\alpha(x) \sigma^m{}_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \theta^\beta \sigma^n{}_{\beta\dot\beta} \bar\theta^{\dot\beta} A_n(x) \theta^\gamma \sigma^\kappa{}_{\gamma\dot\gamma} \partial_\kappa \bar\psi^{\dot\gamma}(x)$$

$$= -\int d^4\theta \, \frac{1}{2} \, \frac{1}{2} \varepsilon^{\dot\alpha\dot\beta} \bar\theta^2 \, \frac{1}{2} \varepsilon^{\beta\gamma} \theta^2 \partial_m \psi^\alpha(x) \sigma^m{}_{\alpha\dot\alpha} \sigma^n{}_{\beta\dot\beta} A_n(x) \sigma^\kappa{}_{\gamma\dot\gamma} \partial_\kappa \bar\psi^{\dot\gamma}(x)$$

$$= \frac{1}{8} \partial_m \psi^\alpha(x) \sigma^m{}_{\alpha\dot\alpha} \bar\sigma^{n\dot\alpha\gamma} A_n(x) \sigma^\kappa{}_{\gamma\dot\gamma} \partial_\kappa \bar\psi^{\dot\gamma}(x)$$

$$= \frac{1}{8} \left(\partial_m \psi(x) (\sigma^m \bar\sigma^n \sigma^\kappa) \partial_\kappa \bar\psi(x)\right)$$

$$-\int d^4\theta \, \frac{i}{\sqrt{2}} \left(\bar\theta \bar\sigma^m \partial_m \psi(x)\right) i\theta^2 \left(\bar\theta \bar\lambda(x)\right) \bar\varphi(x)$$

$$= \frac{1}{2\sqrt{2}} \varepsilon_{\dot\alpha\dot\beta} \bar\sigma^{m\dot\alpha\alpha} \partial_m \psi_\alpha(x) \bar\lambda^{\dot\beta}(x) \bar\varphi(x)$$

$$= -\frac{1}{2\sqrt{2}} \left(\bar\lambda(x) \bar\sigma^m \partial_m \psi(x)\right) \bar\varphi(x) = \frac{1}{2\sqrt{2}} \left(\partial_m \psi(x) \sigma^m \bar\lambda(x)\right)$$

$$+ F(x) \bar F(x)$$

$$+ \int d^4\theta \, \frac{i}{\cancel{\sqrt{2}}} \theta^2 \left(\bar\theta \bar\sigma^m \partial_m \psi(x)\right) \cancel{\sqrt{2}} \left(\bar\theta \bar\psi(x)\right)$$

$$= \frac{i}{2\sqrt{\ }} \varepsilon_{\dot\alpha\dot\beta} \bar\sigma^{m\dot\alpha\alpha} \partial_m \psi_\alpha(x) \cdot \bar\psi^{\dot\beta}(x) = -\frac{i}{2} \left(\bar\psi(x) \bar\sigma^m \partial_m \psi(x)\right)$$

$$= \frac{i}{2} \left(\partial_m \psi(x) \sigma^m \bar\psi(x)\right)$$

$$- i \int d^4\theta \, \left(\theta\psi(x)\right) \bar\theta^2 \left(\partial_m \bar\psi(x) \bar\sigma^m \theta\right)$$

$$= +\frac{i}{2} \varepsilon^{\alpha\beta} \psi_\alpha(x) \sigma^m{}_{\beta\dot\beta} \partial_m \bar\psi^{\dot\beta}(x) = -\frac{i}{2} \left(\psi(x) \sigma^m \partial_m \bar\psi(x)\right)$$

$$= \frac{1}{4}\varphi(x)\Box\bar{\varphi}(x) + \frac{1}{4}\bar{\varphi}(x)\Box\varphi(x) - \frac{1}{2}\partial^m\varphi(x)\partial_m\bar{\varphi}(x)$$

$$+ \frac{i}{2}\varphi(x)A^m(x)\partial_m\bar{\varphi}(x) - \frac{i}{2}\bar{\varphi}(x)A^m(x)\partial_m\varphi(x)$$

$$- \frac{1}{4}\varphi(x)A^m(x)A_m(x)\bar{\varphi}(x) -$$

$$+ \frac{i}{2}\left(\partial_m\psi(x)\sigma^m\bar{\psi}(x)\right) - \frac{i}{2}\left(\psi(x)\sigma^m\partial_m\bar{\psi}(x)\right)$$

$$+ \frac{1}{2}\left(\psi(x)\sigma^m\bar{\psi}(x)\right)A_m(x)$$

$$+ \frac{i}{\sqrt{2}}\varphi(x)\left(\bar{\lambda}(x)\bar{\psi}(x)\right) + \frac{i}{\sqrt{2}}\bar{\varphi}(x)\left(\lambda(x)\psi(x)\right)$$

$$- \frac{1}{2}\varphi(x)d(x)\bar{\varphi}(x) + F(x)\bar{F}(x)$$

$$= -\partial^m\varphi(x)\partial_m\varphi(x) + \frac{i}{2}A^m(x)\varphi(x)\partial_m\bar{\varphi}(x) - \frac{i}{2}A_m(x)\bar{\varphi}(x)\partial^m\varphi(x)$$

$$- \frac{1}{4}A^m(x)\varphi(x)A_m(x)\bar{\varphi}(x)$$

$$- i\left(\psi(x)\sigma^m\partial_m\bar{\psi}(x)\right) - i\frac{i}{2}\left(\bar{\psi}(x)\sigma^m A_m(x)\bar{\psi}(x)\right)$$

$$+ F(x)\bar{F}(x) + \frac{i}{\sqrt{2}}\varphi(x)\left(\bar{\lambda}(x)\bar{\psi}(x)\right) + \frac{i}{\sqrt{2}}\bar{\varphi}(x)\left(\lambda(x)\psi(x)\right)$$

$$- \frac{1}{2}\varphi(x)\bar{\varphi}(x)d(x) + \text{surface terms}$$

$$= -\left(\partial^m - \frac{i}{2}A^m(x)\right)\varphi(x)\left(\partial_m + \frac{i}{2}A_m(x)\right)\bar{\varphi}(x) - i\left(\psi(x)\sigma^m\left(\partial_m + \frac{i}{2}A^m(x)\right)\bar{\psi}(x)\right)$$

$$+ F(x)\bar{F}(x) + \frac{i}{\sqrt{2}}\varphi(x)\left(\bar{\lambda}(x)\bar{\psi}(x)\right) + \frac{i}{\sqrt{2}}\bar{\varphi}(x)\left(\lambda(x)\psi(x)\right)$$

$$- \frac{1}{2}\varphi(x)\bar{\varphi}(x)d(x) + \text{surface terms}$$

Thus, indeed

$$S := \int d^4x\, d^4\theta\; \Phi(x + i\theta\sigma\bar\theta, \theta)\, e^{-V(x,\theta,\bar\theta)}\, \bar\Phi(x - i\theta\sigma\bar\theta, \bar\theta)$$

$$= \int d^4x \Big(-\nabla_m \varphi \nabla^m \bar\varphi - i\, \psi \sigma^m \nabla_m \bar\psi + F\bar F + \frac{i}{\sqrt{2}} \varphi(x) \big(\bar\lambda(x) \bar\psi(x)\big)$$

$$+ \frac{i}{\sqrt{2}}\, \bar\varphi(x) \big(\lambda(x) \psi(x)\big) - \frac{1}{2} \varphi(x) \bar\varphi(x)\, d(x) \Big)$$

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Supersymmetry

Exercise 8.3.

Let us begin by describing the standard model (without the Higgs) in order to fix notation. Let us begin with the fermionic part. Due to the electroweak interaction, we have to divide our description into left and right weyl spinors. Similarly, due to the strong interaction, we have to further divide this into quarks and leptons. The left Leptons are grouped into the fields L_i, with $i\in\{1,2,3\}$ running over the three generations. Each of this is further divided into $SU(2)$ doublets

$$l_1=\begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix},\quad l_2=\begin{pmatrix}\nu_{\mu L}\\ \mu_L\end{pmatrix},\quad l_3=\begin{pmatrix}\nu_{\tau L}\\ \tau_L\end{pmatrix},$$

which we will denote by $l_i=\begin{pmatrix}\nu_{Li}\\ e_i\end{pmatrix}=\begin{pmatrix}l_{i1}\\ l_{i2}\end{pmatrix}$ to write the Lagrangian as explicitely as possible. For all $i\in\{1,2,3\}$ and $A\in\{1,2\}$, L_{iA} is a left Weyl spinor, i.e. transforms in the $(1/2,0)$ representation of the proper orthochronous Lorentz group $L_+^\uparrow$. At the risk of introducing cumbersome notation, we will denote by $l_{iA\alpha}$, with $\alpha\in\{1,2\}$, the α-th component of the spinor l_{iA}. There is no color index, indicating that these form a singlet under $SU(3)$. The story for quarks is different. We, however, start in a similar fashion, by taking the fields q_i with $i \in \{1,2,3\}$. Explicitely

$$q_1 = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad q_2 = \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad q_3 = \begin{pmatrix} t_L \\ b_L \end{pmatrix},$$

although we will retain the notation $q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix} = \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix}$.

For each $i \in \{1,2,3\}$, q_i is an $SU(2)$ doublet. However, now for each $A \in \{1,2\}$, q_{iA} is a $SU(3)$ triplet

$$q_{iA} = \begin{pmatrix} q_{iA1} \\ q_{iA2} \\ q_{iA3} \end{pmatrix}.$$

It is at this level where q_{iAM}, $M \in \{1,2,3\} \cong \{\text{red, blue, green}\}$ running over the colors, is a left Weyl spinor, with components $q_{iAM\alpha}$.

The matter with right chirality forms a singlet under $SU(2)$. We, however, are still forced to separate our fields into quarks and leptons due to the strong interactions. For the latter, we have three right weyl spinors $\bar{e}_{Ri}$, $i \in \{1,2,3\}$ running over generations, each a singlet under $SU(3)$ and with components $\bar{e}_{Ri\dot{\alpha}}$. To be explicit,

$$\bar{e}_{R1} = \bar{e}_R, \qquad \bar{e}_{R2} = \bar{\mu}_R, \qquad \bar{e}_{R3} = \bar{\tau}_R.$$

Although likely to change in the future, the current standard model does not contemplate right handed neutrinos. This is because they don't have charge or color. Thus, if they are massless, they wouldn't interact with anything, making them undetectable. For the quarks, we have the fields $\bar{u}_{Ri}$, $\bar{d}_{Ri}$, $i \in \{1,2,3\}$ running over generations. Explicitely

$$\bar{u}_{R1} = \bar{u}_R, \qquad \bar{u}_{R2} = \bar{c}_R, \qquad \bar{u}_{R3} = \bar{t}_R,$$

$$\bar{d}_{R1} = \bar{d}_R, \qquad \bar{d}_{R2} = \bar{s}_R, \qquad \bar{d}_{R3} = \bar{b}_R,$$

each being a singlet under $SU(2)$. Each forms a triplet $(\bar{u}_{RiM})$ $(\bar{d}_{RiM})$, with $M \in \{1,2,3\} \cong \{\text{red, blue, green}\}$, under $SU(3)$. Each $\bar{u}_{RiM}$, $\bar{d}_{RiM}$ is a right Weyl spinor with components $\bar{u}_{RiM\dot{\alpha}}$, $\bar{d}_{RiM\dot{\alpha}}$.

Although we have already hinted at the transformation properties of these groupings by using the word multiplet, let us be more explicit. The structure group of the Standard Model is

$$G = SU(3) \times SU(2) \times U(1).$$

It acts on the fields we have described by

$$((U,V,z)\, l_i)_A = z^{-3}\, V_A{}^B\, l_{iB}$$

$$((U,V,z)\, q_i)_{AM} = z\, U_M{}^N\, V_A{}^B\, q_{iBN}$$

$$(U,V,z)\, \bar{e}_{Ri} = z^{-6}\, \bar{e}_{Ri}$$

$$((U,V,z)\, \bar{u}_{Ri})_M = z^{4}\, U_M{}^N\, \bar{u}_{RiN}$$

$$((U,V,z)\, \bar{d}_{Ri})_M = z^{-2}\, U_M{}^N\, \bar{d}_{RiN}.$$

These actions induce through differentiation an action of the Lie algebra $\mathfrak{g} = su(3) \oplus su(2) \oplus u(1)$ of G. It is with this representations that we can make the $su(3)$ valued fields G_μ, the $su(2)$ valued fields W_μ, and the $u(1)$ valued fields B_μ act on our previous fields. In particular, we have our covariant derivatives

$$(D_\mu l_i)_A = \left(\partial_\mu - ig(W_\mu)_A{}^B + ig' B_\mu \delta_A{}^B\right) l_{iB}$$

$$(D_\mu q_i)_{AM} = \left(\partial_\mu - ig_s (G_\mu)_M{}^N \delta_A{}^B - ig (W_\mu)_A{}^B \delta_M{}^N - \tfrac{1}{3} ig' B_\mu \delta_A{}^B \delta_M{}^N\right) q_{iBN}$$

$$D_\mu \bar{e}_{Ri} = \left(\partial_\mu + 2ig' B_\mu\right) \bar{e}_{Ri}$$

$$(D_\mu \bar{u}_{Ri})_M = \left(\partial_\mu - ig_s (G_\mu)_M{}^N - \tfrac{4}{3} ig' B_\mu \delta_M{}^N\right) \bar{u}_{RiN}$$

$$(D_\mu \bar{d}_{Ri})_M = \left(\partial_\mu - ig_s (G_\mu)_M{}^N + \tfrac{2}{3} ig' B_\mu \delta_M{}^N\right) \bar{d}_{RiN}.$$

We can then write the Kinetic Lagrangian for fermions including the interaction with bosons

$$\mathcal{L}_f = \sum_{i=1}^{3} \Big(\bar{q}_i^{AM} \bar{\sigma}^\mu (\overleftrightarrow{D}_\mu q_i)_{AM} + \bar{l}_i^{A} \bar{\sigma}^\mu (\overleftrightarrow{D}_\mu l_i)_A$$

$$+ \bar{u}_{RiM} \bar{\sigma}^\mu (D_\mu u_{Ri})^M + \bar{d}_{RiM} \bar{\sigma}^\mu (D_\mu d_{Ri})^M$$

$$+ \bar{e}_{Ri} \bar{\sigma}^\mu (D_\mu e_{Ri}) \Big).$$

The gauge bosons propagate through the Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} \operatorname{Tr}(G^{\mu\nu} G_{\mu\nu}) - \frac{1}{4} \operatorname{Tr}(W^{\mu\nu} W_{\mu\nu}) - \frac{1}{4} B^{\mu\nu} B_{\mu\nu},$$

where we have the curvatures

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i g_s [G_\mu, G_\nu],$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - i g [W_\mu, W_\nu],$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

Thus the action of the standard model is

$$S = \int d^4x \, (\mathcal{L}_f + \mathcal{L}_{YM}).$$

To do a minimal extension of the standard model to a supersymmetric one, we introdue a chiral field for every left-handed spinor. We thus have the chiral superfield

$$L_{iA}(x, \theta, \bar{\theta}),$$

where $L_{iA}|_\theta = l_{iA}$, $i \in \{1,2,3\}$ runs through generations, and $A \in \{1,2\}$ runs through flavors. Similarly, we introduce the chiral superfields Q_{iAM}, where $Q_{iAM}|_\theta = q_{iAM}$, $i \in \{1,2,3\}$ runs through the generations, $A \in \{1,2\}$ runs through flavors, and $M \in \{1,2,3\} \cong \{\text{red, blue, green}\}$ runs through colors. On the other hand, for the right-handed spinors, we introduce antichiral superfields. We thus introduce the antichiral superfields $\bar{E}_i$, $\bar{U}_{iM}$, and $\bar{D}_{iM}$, where $\bar{E}_i|_\theta = \bar{e}_{Ri}$, $\bar{U}_{iM}|_\theta = \bar{u}_{RiM}$, $\bar{D}_{iM}|_\theta = \bar{d}_{iM}$, $i \in \{1,2,3\}$ runs through generations, and $M \in \{1,2,3\} \cong \{\text{red, blue, green}\}$ runs through colors.

For the gauge Bosons, we introduce the Lie algebra valued vector superfields V_s, V_L, and V_y, with values in $so(3)$, $su(2)$, and $u(1)$ respectively. In particular $V_s|_{\theta\sigma^\mu\bar\theta} = G_\mu$, $V_L|_{\theta\sigma^\mu\bar\theta} = W_\mu$, and $V_y|_{\theta\sigma^\mu\bar\theta} = B_\mu$. We define their actions on our matter fields in correspondance to the non-supersymmetric models. Thus

$$
\begin{array}{lll}
(V_s L_i)_A = L_{iA} & (V_L L_i)_A = V_{LA}{}^B L_{iB} & (V_y L_i)_A = -V_y L_{iA} \\
(V_s Q_i)_{AM} = (V_s)_M{}^N Q_{iAN} & (V_L Q_i)_{AM} = V_{LA}{}^B Q_{iBM} & (V_y Q_i)_{AM} = \frac{1}{3} V_y Q_{iAM} \\
(V_s \bar{E}_i)_A = \bar{E}_{iA} & (V_L \bar{E}_i)_A = \bar{E}_{iA} & (V_y \bar{E}_i)_A = -2 V_y \bar{E}_{iA} \\
(V_s \bar{U}_i)_{AM} = V_{sM}{}^N \bar{U}_{iAN} & (V_L \bar{U}_i)_{AM} = \bar{U}_{iAM} & (V_y \bar{U}_i)_{AM} = \frac{4}{3} V_y \bar{U}_{iAM} \\
(V_s \bar{D}_i)_{AM} = V_{sM}{}^N \bar{D}_{iAN} & (V_L \bar{D}_i)_{AM} = \bar{D}_{iAM} & (V_y \bar{D}_i)_{AM} = -\frac{2}{3} V_y \bar{D}_{iAM}.
\end{array}
$$

Then, defining $V = V_s \oplus V_L \oplus V_Y$ taking values in $su(3) \oplus su(2) \oplus u(1)$, we have the suppersymmetric standard model action

$$
\begin{aligned}
S = \int d^4x \Bigg(\sum_{i=1}^{3} \Bigg(\int d^4\theta \Big(& \bar{Q}_i{}^{AM} (e^V Q_i)_{AM} + \bar{L}_i{}^A (e^V L_i)_A \\
& + \bar{U}_{iM} (e^V U_i)^M + \bar{D}_{iM} (e^V D_i)^M \\
& + \bar{E}_i (e^V E_i) \Big) \Bigg) \\
+ \int d^2\theta \Big(& (W_{s\alpha})_M{}^N (W_s{}^\alpha)_M{}^N + (W_{L\alpha})_A{}^B (W_L{}^\alpha)_A{}^B \\
& + B_{Y\alpha} B_Y{}^\alpha \Big) + c.c. \Bigg),
\end{aligned}
$$

where

$$W_{s\alpha} = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-V_s} D_\alpha e^{V_s} \right)$$

$$W_{L\alpha} = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-V_L} D_\alpha e^{V_L} \right)$$

$$W_{Y\alpha} = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-V_Y} D_\alpha e^{V_Y} \right)$$