Iván Mauricio Burbano Aldana

Prof. Nathan Berkovitz

Instituto de Física Teórica, UNESP

Homework 12

Exercise 9.3.

We are considering the action

$$S = S_{FI} + \int d^4x \left[\int d^4\theta \left(\bar{\Phi}_1 e^{-eV} \Phi_1 + \bar{\Phi}_2 e^{2eV} \Phi_2 \right) + \lambda \left(\int d^2\theta \, \Phi_1 \Phi_1 \Phi_2 + c.c. \right) \right],$$

with

$$S_{FI} = \int d^4x \left(\int d^2\theta \, \frac{1}{2} W_\alpha W^\alpha + 2K \int d^4\theta \, V \right).$$

In here Φ_1 and Φ_2 are chiral superfields, V is a vector superfield, $W_\alpha = -\frac{1}{4} \bar{D}\bar{D} D_\alpha V$, and, in order to have a real action, we take $e, \lambda, K \in \mathbb{R}$.

To study this model, we begin by expanding the action in component fields. Using the result of

Exercise 6.4, and eqn (6.22), we obtain

$$S_{FI} = \int d^4x \left(\frac{1}{2} d^2 - \frac{1}{4} F^{mn} F_{mn} - i\lambda\sigma^m \partial_m \bar{\lambda} + Kd \right),$$

where we've chosen to work in the Wess-Zumino gauge. In this we also have the kinetic terms of (7.12) (or (7.7) in Wess & Bagger)

$$\int d^4x \int d^4\theta \left(\bar{\Phi}_1 e^{-eV} \Phi_1 + \bar{\Phi}_2 e^{2eV} \Phi_2 \right)$$

$$= \int d^4x \Big(\bar{\varphi}_1 D^m D_m \varphi_1 + \bar{\varphi}_2 D^m D_m \varphi_2$$

$$- i \bar{\psi}_1 \bar{\sigma}^m D_m \psi_1 - i \bar{\psi}_2 \bar{\sigma}^m D_m \psi_2 + \bar{F}_1 F_1 + \bar{F}_2 F_2$$

$$+ \frac{ie}{\sqrt{2}} \left(\varphi_1 \bar{\psi}_1 \bar{\lambda} - \bar{\varphi}_1 \psi_1 \lambda \right) - ie\sqrt{2} \left(\varphi_2 \bar{\psi}_2 \bar{\lambda} - \bar{\varphi}_2 \psi_2 \lambda \right)$$

$$- \frac{1}{2} e d \bar{\varphi}_1 \varphi_1 + e d \bar{\varphi}_2 \varphi_2 \Big),$$

where

$$D_m \varphi_1 = \left(\partial_m - \frac{i}{2} e A_m \right) \varphi_1, \qquad D_m \psi_1 = \left(\partial_m - \frac{i}{2} e A_m \right) \psi_1,$$

$$D_m \varphi_2 = \left(\partial_m + i e A_m \right) \varphi_2, \qquad D_m \psi_2 = \left(\partial_m + i e A_m \right) \psi_2.$$

Finally, our last term is of the form we computed in Exercise 2 of homework 11

$$\lambda \int d^4x \int d^2\theta \, \Phi_1 \Phi_1 \Phi_2 = \lambda \int d^4x \Big(\varphi_1 \varphi_1 F_2 - \varphi_1 \psi_1 \psi_2 + \varphi_1 F_1 \varphi_2$$

$$- \psi_1 \varphi_1 \psi_2 - \psi_1 \psi_1 \varphi_2 + F_1 \varphi_1 \varphi_2 \Big)$$

$$= \lambda \int d^4x \Big(\varphi_1 \varphi_1 F_2 + 2 \varphi_1 \varphi_2 F_1 - \varphi_2 \psi_1 \psi_1$$

$$- 2 \varphi_1 \psi_1 \psi_2 \Big)$$

We now study the equations of motion for the auxiliary fields. By varying F_1, F_2 and d we obtain

$$\bar{F}_1 = -2\lambda \varphi_1 \varphi_2$$

$$\bar{F}_2 = -\lambda \varphi_1 \varphi_1 \,,$$

$$d = -K + \frac{1}{2} e \bar{\varphi}_1 \varphi_1 - e \bar{\varphi}_2 \varphi_2 = -K + \frac{1}{2} e \left(\bar{\varphi}_1 \varphi_1 - 2 \bar{\varphi}_2 \varphi_2 \right)$$

Thus, the scalar potential is given by

$$\mathcal{V} = -\frac{1}{2} d^2 - K d - \bar{F}_1 F_1 - \bar{F}_2 F_2 + \frac{1}{2} de \left(\bar{\varphi}_1 \varphi_1 - 2 \bar{\varphi}_2 \varphi_2 \right) +$$

$$\left(-2\lambda \varphi_1 \varphi_2 F_1 - \lambda F_2 \varphi_1 \varphi_1 + c.c. \right)$$

$$= -\frac{1}{2} d^2 - \cancel{Kd} - \cancel{\bar{F}_1 F_1} - \cancel{\bar{F}_2 F_2} + d \left(d + \cancel{K} \right) + \left(\bar{F}_2 F_2 + \bar{F}_1 F_1 + \cancel{c.c.} \right)$$

$$= \frac{1}{2} d^2 + \bar{F}_1 F_1 + \bar{F}_2 F_2$$

$$= \frac{1}{2}K^2 - \frac{1}{2}Ke(\bar{\varphi}_1\varphi_1 - 2\bar{\varphi}_2\varphi_2) + \lambda^2(\bar{\varphi}_1\varphi_1)^2 + 4\lambda^2\,\bar{\varphi}_1\varphi_1\bar{\varphi}_2\varphi_2$$

$$+ \frac{1}{8}e^2(\bar{\varphi}_1\varphi_1 - 2\bar{\varphi}_2\varphi_2)^2.$$

To study the minima of this potential, let $y_1 = \bar{\varphi}_1\varphi_1$ and $y_2 = \bar{\varphi}_2\varphi_2$. Then, the interior critical points satisfy

$$0 = \frac{\partial \mathcal{V}}{\partial y_1} = -\frac{1}{2}Ke + 2\lambda^2 y_1 + 4\lambda^2 y_2 + \frac{1}{4}e^2(y_1 - 2y_2)$$

$$0 = \frac{\partial \mathcal{V}}{\partial y_2} = Ke + 4\lambda^2 y_1 - \frac{1}{2}e^2(y_1 - 2y_2)$$

By dividing the second eqn by 2 and adding both eqns one obtains

$$4\lambda^2 y_1 + 4\lambda^2 y_2 = 0$$

Thus, at interior critical points we have $y_2 = -y_1$. Since interior points satisfy $y_1, y_2 > 0$, this is impossible! (Unless $\lambda =$

We conclude that the minima is at the boundary $y_1 = 0$ or $y_2 = 0$. In the case $y_1 = 0$,

$$\mathcal{V} = \frac{1}{2}K^2 + Key_2 + \frac{1}{2}e^2 y_2^2$$

An interior critical point of this would satisfy $y_2 = -\frac{K}{e}$ which is, again, an imposibility. (unless $Ke < 0$)

A minimum is then found at $y_2 = 0$, where $\gamma = \frac{1}{2}K^2$. On the line $y_2 = 0$,

$$\gamma = \frac{1}{2}K^2 - \frac{1}{2}Key_1 + \lambda^2 y_1^2 + \frac{1}{8}e^2 y_1^2$$

$$= \frac{1}{2}K^2 - \frac{1}{2}Key_1 + \left(\frac{1}{8}e^2 + \lambda^2\right) y_1^2 .$$

An interior critical point would then have

$$y_1 = \frac{\frac{1}{2}Ke}{2\left(\frac{1}{8}e^2 + \lambda^2\right)} = \frac{Ke}{\frac{1}{2}e^2 + 4\lambda^2} = \frac{2Ke}{e^2 + 8\lambda^2},$$

where the potential is

$$\gamma = -\frac{\frac{1}{4}K^2e^2}{4\left(\frac{1}{8}e^2 + \lambda^2\right)} + \frac{1}{2}K^2$$

$$= -\frac{K^2e^2}{2e^2 + 16\lambda^2} + \frac{1}{2}K^2$$

$$= \frac{1}{2}K^2\left(1 - \frac{e^2}{e^2 + 8\lambda^2}\right) \leq \frac{1}{2}K^2.$$

This is plausible as long as $Ke \geq 0$. Finally, Let us go back to the case $\lambda = 0$. Then the two interior critical point equations become identical. They are solved by

$$y_1 = 2y_2 + 2K/e.$$

On this line the potential takes the value

$$\mathcal{V} = \frac{1}{2}K^2 - \frac{1}{8}K\cancel{e}\,8K/\cancel{e} + \frac{1}{\cancel{8}\,2}\cancel{e^2}\,\cancel{4}K^2/\cancel{e^2}$$

$$= \frac{1}{2}K^2 - \frac{1}{2}K^2 = 0.$$

We conclude that we have the following cases:

- $\lambda = 0$: Supersymmetry is not spontaneously broken but gauge symmetry is. The e.v. satisfy

$$y_1 = 2y_2 + 2K/e$$

- $\lambda \neq 0$ and $Ke > 0$: Supersymmetry is spontaneously broken to a potential minimum of

$$\mathcal{V} = \frac{1}{2}K^2\left(1 - \frac{e^2}{e^2 + 8\lambda^2}\right).$$

Gauge symmetry is also broken, the minimum being attained at $y_2 = 0$ and $y_1 = \frac{2\kappa e}{e^2 + 8\lambda^2}$.

- $\lambda \neq 0$ and $\kappa e < 0$: Supersymmetry is unbroken. Gauge symmetry is broken, with the minimum being attained at $y_1 = 0$ and $y_2 = -\frac{\kappa}{e}$.

We will not consider $\kappa e = 0$. In that case, either the Fayet-Iliopoulos term reduces to super-Maxwell, or the matter and radiation fields decouple. For simplicity, let us also take $\lambda \neq 0$.

In the case $\kappa e > 0$, take

$$\langle \varphi_1 \rangle := \sqrt{\frac{2\kappa e}{e^2 + 8\lambda^2}}, \qquad \langle \varphi_2 \rangle := 0,$$

and $\tau_i := \varphi_i - \langle \varphi_i \rangle$. The quadratic terms in the τ fields in the Lagrangian all come from the scalar potential

$$-\frac{1}{2}Ke\left(\bar{\tau}_1\tau_1 - 2\bar{\tau}_2\tau_2\right) + \lambda^2\left(\bar{\tau}_1\tau_1 + \bar{\tau}_1\langle\varphi_1\rangle + \langle\bar{\varphi}_1\rangle\tau_1 + \langle\varphi_1\rangle\langle\bar{\varphi}_1\rangle\right)^2$$

$$+ 4\lambda^2\left(\bar{\tau}_1\tau_1 + \bar{\tau}_1\langle\varphi_1\rangle + \langle\bar{\varphi}_1\rangle\tau_1 + \langle\bar{\varphi}_1\rangle\langle\varphi_1\rangle\right)\bar{\tau}_2\tau_2$$

$$+ \frac{1}{8}e^2\left(\bar{\tau}_1\tau_1 + \bar{\tau}_1\langle\varphi_1\rangle + \langle\bar{\varphi}_1\rangle\tau_1 + \langle\bar{\varphi}_1\rangle\langle\varphi_1\rangle - 2\bar{\tau}_2\tau_2\right)^2 + \cdots$$

$$= -\frac{1}{2}Ke\left(\bar{\tau}_1\tau_1 - 2\bar{\tau}_2\tau_2\right) + 2\lambda^2\langle\varphi_1\rangle^2\bar{\tau}_1\tau_1 + \lambda^2\langle\varphi_1\rangle^2\left(\tau_1+\bar{\tau}_1\right)^2$$

$$+ 4\lambda^2\langle\varphi_1\rangle^2\bar{\tau}_2\tau_2 + \frac{1}{4}e^2\langle\varphi_1\rangle^2\bar{\tau}_1\tau_1 + \frac{1}{8}e^2\langle\varphi_1\rangle^2\left(\tau_1+\bar{\tau}_1\right)^2$$

$$- \frac{1}{2}e^2\langle\varphi_1\rangle^2\bar{\tau}_2\tau_2 + \cdots$$

$$= \left(-\frac{1}{2}Ke + 2\lambda^2\langle\varphi_1\rangle^2 + 4\lambda^2\langle\varphi_1\rangle^2 + \frac{1}{4}e^2\langle\varphi_1\rangle^2 + \frac{1}{2}e^2\langle\varphi_1\rangle^2\right)(\mathrm{Re}\,\tau_1)^2$$

$$+ \left(-\frac{1}{2}Ke + 2\lambda^2\langle\varphi_1\rangle^2 + \frac{1}{4}e^2\langle\varphi_1\rangle^2\right)(\mathrm{Im}\,\tau_1)^2$$

$$+ \left(Ke + 4\lambda^2\langle\varphi_1\rangle^2 - \frac{1}{2}e^2\langle\varphi_1\rangle^2\right)\bar{\tau}_2\tau_2 .$$

We thus have a real boson $\mathrm{Re}\,\tau_1$ of mass

$$\left(\left(6\lambda^2 + \frac{3}{4}e^2\right)\frac{2Ke}{e^2+8\lambda^2} - \frac{1}{2}Ke\right)^{1/2}$$

$$= \left(\frac{3}{2}Ke - \frac{1}{2}Ke\right)^{1/2} = \sqrt{\frac{Ke}{2}} ,$$

another real boson $\mathrm{Im}\,\tau_1$ of mass

$$\left(-\frac{1}{2}\kappa e + \frac{1}{4}\left(8\lambda^2 + e^2\right) \langle \varphi_1 \rangle^2 \right)^{1/2} =$$

$$\left(-\frac{1}{2}\kappa e + \frac{1}{2}\kappa e \right)^{1/2} = 0,$$

i.e., which is actually massless, and a complex scalar $\bar{z}_2 z_2$ of mass

$$\left(\kappa e - \frac{1}{\cancel{2}}\left(e^2 - 8\lambda^2\right) \frac{\cancel{2}\kappa e}{e^2 + 8\lambda^2} \right)^{1/2} =$$

$$\left(\kappa e \left(\frac{\cancel{e^2} + 8\lambda^2 - \cancel{e^2} + 8\lambda^2}{e^2 + 8\lambda^2} \right)\right)^{1/2} =$$

$$4|\lambda| \left(\frac{\kappa e}{e^2 + 8\lambda^2} \right)^{1/2}.$$

Similarly, the quadratic terms in the spinors are

$$\frac{ie\langle \varphi_1 \rangle}{\sqrt{2}} \left(\bar{\psi}_1 \bar{\lambda} - \psi_1 \lambda \right) - 2\lambda \langle \varphi_1 \rangle \left(\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2 \right)$$

$$= -\frac{e\langle \varphi_1 \rangle}{\sqrt{2}} i\psi_1 \lambda - 2\lambda \langle \varphi_1 \rangle \psi_1 \psi_2 + \text{c.c.}$$

Sorry for the double notation

We thus have a fermionic mass matrix

$$M = \langle \varphi_1 \rangle \begin{bmatrix} 0 & -\lambda & e/2\sqrt{2} \\ \lambda & 0 & 0 \\ e/2\sqrt{2} & 0 & 0 \end{bmatrix}.$$

Matrices of the form

$$\begin{bmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{bmatrix}$$

have eigenvalues μ satisfying

$$0 = -\mu^3 + a^2\mu + b^2\mu = -\mu\left(\mu^2 - (a^2+b^2)\right),$$

i.e. $0, \pm\sqrt{a^2+b^2}$. The 0 eigenvalue corresponds by inspection to

$$\frac{1}{\sqrt{a^2+b^2}}(0, b, -a).$$

The $\pm\sqrt{a^2+b^2}$ is obtained with

$$\frac{1}{\sqrt{2a^2+2b^2}}\left(\pm\sqrt{a^2+b^2}, a, b\right)$$

$$\| $$

$$\frac{1}{\sqrt{2}}\left(\pm 1, \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

With our matrix, we have

$$\sqrt{a^2+b^2} = \langle \varphi_1 \rangle \sqrt{\lambda^2 + \frac{e^2}{8}}$$

$$= \frac{1}{2}\sqrt{Ke}.$$

We thus have that the fermionic mass terms are

$$\frac{\sqrt{Ke}}{2}\left[\left(\frac{1}{\sqrt{2}}\left(\psi_1 + \frac{2\sqrt{2}\lambda}{\sqrt{e^2+8\lambda^2}}\psi_2 + i\frac{e}{\sqrt{e^2+8\lambda^2}}\lambda\right)\right)^2 + \left(\frac{i}{\sqrt{2}}\left(-\psi_1 + \frac{2\sqrt{2}\lambda}{\sqrt{e^2+8\lambda^2}}\psi_2 + i\frac{e}{\sqrt{e^2+8\lambda^2}}\lambda\right)\right)^2\right].$$

We obtain a massless spinor

$$\frac{i}{\sqrt{e^2+8\lambda^2}}\left(e\psi_2 - 2\sqrt{2}\underset{\text{parameter}}{\lambda}\overset{\text{spinor}}{\lambda}\right)$$

and two spinors of mass $\sqrt{Ke}$

$$\frac{1}{\sqrt{2}}\left(\psi_1 + \frac{2\sqrt{2}\lambda}{\sqrt{e^2+8\lambda^2}}\psi_2 + \frac{e}{\sqrt{e^2+8\lambda^2}}\lambda\right),$$

$$\frac{i}{\sqrt{2}}\left(-\psi_1 + \frac{2\sqrt{2}\lambda}{\sqrt{e^2+8\lambda^2}}\psi_2 + \frac{e}{\sqrt{e^2+8\lambda^2}}\lambda\right).$$

Finally, out of the coupling between radiation and the scalar matter, we obtain a mass term for A_m

$$-\frac{1}{4}e^2 \langle \varphi_1 \rangle^2 A^m A_m .$$

Thus, the vector field adquires a mass of

$$\frac{e}{\sqrt{2}} \sqrt{\frac{k e}{c^2 + 8\lambda^2}} .$$

We can repeat all of this analysis for the case $ke < 0$. In that case

$$\langle \varphi_1 \rangle = 0, \qquad \langle \varphi_2 \rangle = \sqrt{-\frac{k}{e}}$$

The quadratic terms in the scalars are

$$-\frac{1}{2} ke \left(\bar{\tau}_1 \tau_1 - 2 \bar{\tau}_2 \tau_2 \right) + 4\lambda^2 \langle \varphi_2 \rangle^2 \bar{\tau}_1 \tau_1$$

$$+ \frac{1}{8} e^2 \left(\bar{\tau}_1 \tau_1 - 2 \bar{\tau}_2 \tau_2 - 2 \langle \varphi_2 \rangle \left(\bar{\tau}_2 + \tau_2 \right) - 2 \langle \varphi_2 \rangle^2 \right)^2$$

$$= -\frac{1}{2} ke \left(\bar{\tau}_1 \tau_1 - 2 \bar{\tau}_2 \tau_2 \right) + 4\lambda^2 \langle \varphi_2 \rangle^2 \bar{\tau}_1 \tau_1$$

$$-\frac{1}{2} e^2 \langle\varphi_2\rangle^2 \bar\tau_1\tau_1 + e^2\langle\varphi_2\rangle^2 \bar\tau_2\tau_2 + \frac{1}{2} e^2\langle\varphi_2\rangle^2(\bar\tau_2+\tau_2)^2$$

$$= \left(\cancel{-\frac{1}{2}\kappa e} - \frac{4\lambda^2\kappa}{e} + \cancel{\frac{1}{2}\kappa e}\right)\bar\tau_1\tau_1$$

$$(+\kappa e - \kappa e - 2\kappa e)(\mathrm{Re}\,\tau_2)^2 + (+\kappa e - \kappa e)(\mathrm{Im}\,\tau_2)^2.$$

We thus get a complex boson τ_1 with mass

$$\sqrt{-\frac{4\lambda^2\kappa}{e}} = +\sqrt{\frac{4\lambda^2|\kappa e|}{e^2}}$$

$$= \sqrt{4\lambda^2\left|\frac{\kappa}{e}\right|} = 2|\lambda|\sqrt{\left|\frac{\kappa}{e}\right|}$$

and real scalar $\mathrm{Re}\,\tau_2$ with mass $\sqrt{2|\kappa e|}$ and another real scalar $\mathrm{Im}(\tau_2)$ with mass 0. The Fermionic mass terms are

$$-e\sqrt{2}\langle\varphi_2\rangle(\bar\psi_2 i\bar\lambda - \psi_2 i\lambda) + (-\lambda\langle\varphi_2\rangle\psi_1\psi_1 + c.c.)$$

$$= -e\sqrt{2}\langle\varphi_2\rangle(-\psi_2 i\lambda) - \lambda\langle\varphi_2\rangle\psi_1\psi_1 + c.c.$$

We thus have the fermion $\mathrm{sign}(\lambda)\,\psi_1$ with mass $2|\lambda|\sqrt{-\frac{\kappa}{e}}$. Using (9.19), we identify another fermions, namely

$$\frac{1}{\sqrt{2}}\left(\psi_2 - i\,\mathrm{sgn}(e)\lambda\right)$$

and

$$\frac{1}{\sqrt{2}}\, i\left(\psi_2 + i\,\mathrm{sgn}(e)\lambda\right),$$

both with mass $\sqrt{-2ke}$. Finally, the mass for the gauge field comes from

$$-\,e^2 \langle \varphi_2 \rangle^2 A^m A_m\,,$$

i.e., A_m adquires the mass

$$\sqrt{ke}\,.$$