Applications of Tomita-Takesaki Theory to Quantum Physics I

Canonical Dynamics Induced by States in Thermal Equilibrium

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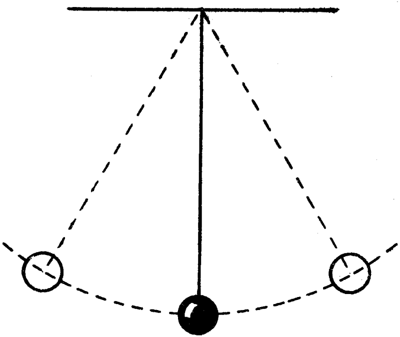
Motivation



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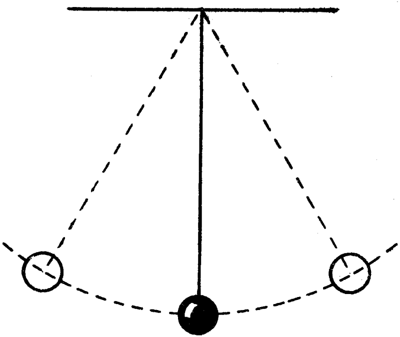


Can we obtain the equations of motion from the equilibrium state?

Maybe in quantum thermal systems.

$$e^{-\beta H} \circlearrowleft e^{-iHt}$$

$$\text{temperature} \iff i \times \text{time}$$

Classical and Quantum Theories

	Classical	Quantum
Auxiliary space	X Locally compact Hausdorff space	$\mathcal{H}$ Hilbert space
Observables	Real valued $f \in C(X)$	Selfadjoint operators O
States	Probability measures P	Density operators ρ
Expected Values	$\langle f \rangle_P := \int_X \mathrm{d}P\, f$	$\langle O \rangle_\rho = \mathrm{tr}(\rho O)$

C^*-algebras and States

Definition

A C^***-algebra** $\mathcal{A}$ is a Banach *-algebra for which $\|aa^*\| = \|a\|^2$ for all $a \in \mathcal{A}$.

Definition

A **state** $\omega : \mathcal{A} \to \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is a nonnegative linear functional such that $\|\omega\| = 1$.

Commutative Example

Example

If X is a locally compact Hausdorff space, the set of continuous functions vanishing at infinity $C_0(X)$ is a C^*-algebra. The norm is given by

$$\|f\| = \sup\{|f(x)||x \in X\} \tag{1}$$

and the algebraic operations are defined pointwise. By Riesz's Representation Theorem states ω on $C_0(X)$ can be identified with probability measures P where

$$\omega(f) = \int_X \mathrm{d}P\, f. \tag{2}$$

Noncommutative Example

Example

Let $\mathcal{H}$ be a Hilbert space. The set $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ is a C^*-algebra. The norm is given by

$$\|O\| = \sup\{O\psi|\psi \in \mathcal{H}, \|\psi\| = 1\}. \tag{3}$$

The involution is given by the adjoint operation. The rest of the operations are defined pointwise. By Gleason's theorem normal states ω can be identified with density operators ρ such that

$$\omega(O) = \mathrm{tr}(\rho O). \tag{4}$$

Moreover, every norm closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Structure Theorems

These are all the examples!

Theorem

Let $\mathcal{A}$ be a C^*-algebra. Then, it is isomorphic to a norm closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Moreover, if $\mathcal{A}$ is commutative, then it is isomorphic to $C_0(X)$ for a locally compact Hausdorff space X. This space is compact if and only if $\mathcal{A}$ is unital[Bratteli and Robinson, 1987].

Digression to Noncommutative Geometry

Remark

If in the commutative case we recover the topological space, what kind of object do we obtain the noncommutative setting? Can we do geometry there?

Algebraic Formulations of Physics

	Classical	Quantum	Algebraic
Auxiliary space	X	$\mathcal{H}$	
Observables	$f \in C(X)$	$O \in \mathcal{B}(\mathcal{H})$	$a \in \mathcal{A}$ selfadjoint
States	P	ρ	ω
Expected Values	$\langle f \rangle_P := \int_X \mathrm{d}P\, f$	$\langle O \rangle_\rho = \mathrm{tr}(\rho O)$	$\langle a \rangle_\omega = \omega(a)$

GNS Representation

Theorem

Let $\mathcal{A}$ be a C^*-algebra and ω a state on it. There exists a unique cyclic representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ with cyclic vector Ω_ω such that for all $a \in \mathcal{A}$ we have

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle. \tag{5}$$

[Bratteli and Robinson, 1987]

Cyclic means that $\overline{\pi_\omega(\mathcal{A})\Omega_\omega} = \mathcal{H}_\omega$.

Idea of Proof

Proof.

Recall $\mathcal{A}$ is a vector space. We can try to give it a Hilbert space structure. Note that $\omega(a^* b)$ is a reasonable attempt at an inner product. It fails though because in general

$$\mathcal{N}_\omega := \{a \in \mathcal{A} | \omega(a^* a) = 0\} \neq \{0\}. \tag{6}$$

However, on $\mathcal{A}/\mathcal{N}_\omega$ this is a well defined inner product. This can be completed into the Hilbert space $\mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{N}_\omega}$. The representation is now defined by PLoCS

$$\pi_\omega(a)[b] = [ab]. \tag{7}$$

The cyclic vector is then given by $\Omega_\omega := [1]$ if $\mathcal{A}$ is unital. □

GNS Space in Finite Dimensions

Example

Let $\mathcal{A} := M_n(\mathbb{C})$ and $\omega(a) := \mathrm{tr}(\rho a)$ for some density matrix ρ of dimension n. Note that

$$\begin{aligned} \omega(a^*a) &= \mathrm{tr}(\rho a^* a) = \mathrm{tr}(\sqrt{\rho}\sqrt{\rho}a^*a) = \mathrm{tr}(\sqrt{\rho}a^*a\sqrt{\rho}) = \mathrm{tr}(\sqrt{\rho}^* a^* a\sqrt{\rho}) \\ &= \mathrm{tr}((a\sqrt{\rho})^* a\sqrt{\rho}) = \|a\sqrt{\rho}\|_{HS}^2. \end{aligned} \tag{8}$$

Therefore $a \in \mathcal{N}_\omega$ if and only if $a\sqrt{\rho} = 0$. Assuming ρ is invertible we obtain $\mathcal{N}_\omega = \{0\}$. Thus

$$\mathcal{H}_\omega = M_n(\mathbb{C})/\{0\} \cong M_n(\mathbb{C}) \tag{9}$$

GNS inner product

Example

Since ρ is self-adjoint we may assume it has the form $\rho = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1, \ldots, \lambda_n > 0$ and $\sum_{i=1}^n \lambda_i = 1$. Consider the matrix units $E_{ij} := (\delta_{ij})_{ij} \in \mathcal{A}$. We have

$$\begin{aligned} \langle [E_{ij}], [E_{kl}] \rangle =& \omega(E_{ij}^* E_{kl}) = \omega(E_{ji} E_{kl}) = \omega(\delta_{ik} E_{jl}) = \delta_{ik} \operatorname{tr}(\rho E_{jl}) \\ =& \delta_{ik} \delta_{jl} \lambda_j. \end{aligned} \tag{10}$$

Therefore,

$$\beta := \{ e_i^{(\alpha)} := [E_{i\alpha}] / \sqrt{\lambda_\alpha} | i, \alpha \in \{1, \ldots, n\} \}. \tag{11}$$

is an orthonormal basis for $\mathcal{H}_\omega$.

GNS Representatives

Example

One has

$$\pi_\omega(a)e_i^{(\alpha)} = \frac{1}{\sqrt{\lambda_\alpha}}[aE_{i\alpha}] = \frac{1}{\sqrt{\lambda_\alpha}}\sum_{k=1}^{n} a_{ki}[E_{k\alpha}] = \sum_{k=1}^{n} a_{ki}e_k^{(\alpha)}. \quad (12)$$

Therefore ordering the basis appropriately we obtain

$$[\pi_\omega(a)]_\beta = \underbrace{\operatorname{diag}(a,\ldots,a)}_{n \text{ times}}. \quad (13)$$

It is then obvious that the representation decomposes as an n-fold sum of irreducible representations. Explicitely if $\mathcal{H}_\omega^{(\alpha)} := \operatorname{span}\{e_i^{(\alpha)} | i \in \{1,\ldots,n\}\}$ we obtain the decomposition into equivalent irreducible representations $\mathcal{H}_\omega = \oplus_{\alpha=1}^{n}\mathcal{H}_\omega^{(\alpha)}$.

W^*-algebras

Definition

A C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called a von Neumann algebra or W^*-algebra if $\mathcal{A}'' = \mathcal{A}$ where

$$\mathcal{A}' = \{b \in \mathcal{B}(\mathcal{H}) | ab = ba \text{ for all } a \in \mathcal{A}\}. \tag{14}$$

Cyclic representations of W^*-algebras

Theorem (★)

If $\mathfrak{M}$ is a W^*-algebra and ω is a faithful ($\omega(a^*a) = 0 \to a = 0$) normal ($\omega(a) = \mathrm{tr}(\rho a)$) state then its cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ satisfies

- *π_ω is faithful (injective);*
- *$\pi_\omega(\mathfrak{M})$ is a von Neumann algebra;*
- *Ω_ω is separating for $\pi_\omega(\mathfrak{M})$ ($\pi_\omega(a)\Omega_\omega = 0 \to \pi_\omega(a) = 0$).*

Dynamical Systems

Time evolution is represented by a one-parameter group of automorphisms

$$\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$$
$$t \mapsto \tau_t.$$

Dynamical systems consist of an $C(W)^*$-algebra with a time evolution which satisfies certain continuity properties.

Example

Given a Hamiltonian H on a Hilbert space $\mathcal{H}$ the Schrödinger time evolution s is given by

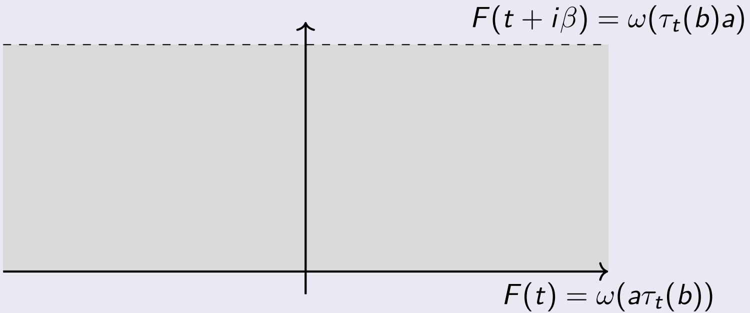
$$s_t(O) = e^{iHt} O e^{-iHt} \tag{15}$$

and $(\mathcal{B}(\mathcal{H}), s)$ is a dynamical system.

KMS States

Definition

Let $(\mathcal{A}, \tau)$ be a dynamical system. ω is said to be a (τ, β)-KMS state if for all $a, b \in \mathcal{A}$ there exists a bounded continuous F on the strip analytic on its interior such that for all for all $t \in \mathbb{R}$



KMS states as Equilibrium states

KMS states are a candidate for a general definition of thermodynamic equilibrium in quantum systems[Haag et al., 1967]:

- KMS states are invariant under the dynamics $\omega(\tau_t(a)) = \omega(a)$;
- In finite dimensional Hilbert spaces with Schrödinger's time evolution τ, the only possible (τ, β)-KMS states are the β-Gibbs states

$$\begin{aligned} \mathcal{B}(\mathcal{H}) &\to \mathbb{C} \\ a &\mapsto \frac{\operatorname{tr}\left(ae^{-\beta H}\right)}{\operatorname{tr}\left(e^{-\beta H}\right)}. \end{aligned}$$

- It is clear that the Gibbs prescription cannot be the characterization of equilibrium in the thermodynamic limit since coexistence of different phases demands that there cannot be a general unique correspondence between the Hamiltonian (evolution group) and states[Connes, 1994].

Tomita-Takesaki Theory

For a W^*-algebra $\mathfrak{M}$ equipped with a cyclic and separating vector Ω the polar decomposition $S = J\Delta^{1/2}$ of the closure of

$$\begin{aligned} S_0 : \mathfrak{M}\Omega &\to \mathcal{H} \\ A\Omega &\mapsto A^*\Omega \end{aligned} \tag{16}$$

yields:

- a one-parameter unitary group $t \mapsto \Delta^{it}$;
- a modular conjugation J.

Theorem (Tomita-Takesaki)

- $J\mathfrak{M}J = \mathfrak{M}'$;
- $\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}$ *for all* $t \in \mathbb{R}$.

Modular Operators in Finite Dimensions

Example

In our previous example we have

$$S[a] = S\pi_\omega(a)[1] = \pi_\omega(a^*)[1] = [a^*]. \tag{17}$$

From the action on the basis that diagonalizes the density operator

$$Se_i^{(\alpha)} = \frac{1}{\sqrt{\lambda_\alpha}} S[E_{i\alpha}] = \frac{1}{\sqrt{\lambda_\alpha}}[E_{\alpha i}] = \sqrt{\frac{\lambda_i}{\lambda_\alpha}} e_\alpha^{(i)} \tag{18}$$

We can obtain the polar decomposition by

$$\begin{aligned} Je_i^{(\alpha)} &= e_\alpha^{(i)} \\ \Delta e_i^{(\alpha)} &= \frac{\lambda_i}{\lambda_\alpha} e_i^{(\alpha)} \end{aligned} \tag{19}$$

Modular Automorphism Group

Definition

Let $\mathfrak{M}$ be a von Neumann algebra and ω be a faithful normal state. Due to ★ we can perform the modular constructions on the cyclic representation $(\pi_\omega(\mathfrak{M}), \pi_\omega, \Omega_\omega)$. We define the modular automorphism group of $(\mathfrak{M}, \omega)$ by

$$\alpha_t(a) = \pi_\omega^{-1}(\Delta^{it}\pi_\omega(a)\Delta^{-it}). \tag{20}$$

Theorem (★★)

$(\mathfrak{M}, \alpha)$ *is a* W^**-dynamical system*

Proof.

[Duvenhage, 1999] □

Modular Automorphism Group in Finite Dimensions

Example

$$\begin{aligned}
\Delta^{it}\pi_\omega(a)\Delta^{-it}e_i^{(\alpha)} &= \left(\frac{\lambda_i}{\lambda_\alpha}\right)^{-it}\sum_{k=1}^{n} a_{ki}\left(\frac{\lambda_k}{\lambda_\alpha}\right)^{it} e_k^{(\alpha)} \\
&= \sum_{k=1}^{n} a_{ki}\left(\frac{\lambda_k}{\lambda_i}\right)^{it} e_k^{(\alpha)} \\
&= \pi_\omega\left(\sum_{i,k=1}^{n} a_{ki}\left(\frac{\lambda_k}{\lambda_i}\right)^{it} E_{ki}\right) e_i^{(\alpha)}.
\end{aligned} \tag{21}$$

Therefore the modular automorphism group is

$$\alpha_t(a) = \sum_{i,j=1}^{n} \lambda_i^{it} a_{ij} \lambda_j^{-it} E_{ij} = \rho^{it} a \rho^{-it} \tag{22}$$

The Canonical Time Evolution

Theorem (★★★)

Let $\mathfrak{M}$ be a von Neumann algebra and ω be a faithful normal state. Then $(\mathfrak{M}, \tau)$ with $\tau_t(a) = \alpha_{-t/\beta}(a)$ and α the modular group of $(\mathfrak{M}, \omega)$ is the unique W^*-dynamical system such that ω is a (τ, β)-KMS state.

Proof.

[Duvenhage, 1999] □

Modular Hamiltonian in Finite Dimensions

Example

As we saw before

$$\tau_t(a) = \alpha_{-t/\beta}(a) = e^{iHt} a e^{-iHt} \tag{23}$$

where the modular Hamiltonian is given by

$$e^{iHt} = \rho^{-it/\beta}. \tag{24}$$

We conclude that indeed ρ is a β-Gibbs state for this Hamiltonian!

$$\rho = e^{-\beta H}. \tag{25}$$

On von Neumann Algebras as Dynamical Objects

- Through the modular group, states induce dynamics on the algebra of operators.
- The physical relevance of such prescription for evolution is guaranteed by the fact that it is the unique dynamical law which makes the state an equilibrium state.
- One can use an analog of the Radon-Nikodym theorem to connect the modular groups induced by different states. Such a connection brings forward a canonical homomorphism from $\mathbb{R}$ into the automorphism group of $\mathfrak{M}$ modulus inner automorphisms. This suggests that the emergence of the dynamical law might have a deeper origin.

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