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Supersymmetry

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Exercise 7.1.

Consider an R-symmetry transformation of the form (6.3) and (6.4)

$$\Phi(y, \theta) \mapsto e^{2inK} \Phi(y, e^{-iK} \theta),$$

$$\bar{\Phi}(\bar{y}, \bar{\theta}) \mapsto e^{-2inK} \bar{\Phi}(\bar{y}, e^{iK} \bar{\theta}).$$

Then

$$\begin{aligned} \int d^4x d^2\theta g \Phi(x, \theta)^3 &\mapsto \int d^4x d^2\theta g e^{6inK} \Phi(x, e^{-iK} \theta)^3 \\ &= \int d^4x d^2\theta g e^{i(6n-2)K} \Phi(x, \theta)^3. \end{aligned}$$

Thus the cubic term is invariant if and only if  $n = \frac{1}{3}$ .

Similarly,

$$\begin{aligned} \int d^4x d^2\bar{\theta} g \bar{\Phi}(x, \bar{\theta})^3 &\mapsto \int d^4x d^2\bar{\theta} g e^{-2inK} \bar{\Phi}(x, e^{iK} \bar{\theta})^3 \\ &= \int d^4x d^2\bar{\theta} g \bar{\Phi}(x, \bar{\theta})^3. \end{aligned}$$

We assume similar R-symmetry transformations for  $\Sigma$

$$\Sigma(y, \theta) \mapsto e^{2imK} \Sigma(y, e^{-iK} \theta)$$

$$\bar{\Sigma}(\bar{y}, \bar{\theta}) \mapsto e^{-2imK} \bar{\Sigma}(\bar{y}, e^{iK} \bar{\theta}).$$

Then

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$$\begin{aligned} \int d^4x d^2\theta \bar{\Phi}(x, \theta) \bar{\Sigma}(x, \theta) &\longmapsto \int d^4x d^2\theta e^{2inK} e^{2imK} \bar{\Phi}(x, e^{-iK}\theta) \bar{\Sigma}(x, e^{-iK}\theta) \\ &= \int d^4x d^2\theta e^{i(2n+2m-2)K} \bar{\Phi}(x, \theta) \bar{\Sigma}(x, \theta) \end{aligned}$$

This term is invariant if and only if

$$m = 1 - n = \frac{2}{3}.$$

Then we automatically have

$$\begin{aligned} \int d^4x d^2\bar{\theta} \bar{\Phi}(x, \bar{\theta}) \bar{\Sigma}(x, \bar{\theta}) &\longmapsto \int d^4x d^2\bar{\theta} e^{-2inK} e^{-2imK} \bar{\Phi}(x, e^{iK}\bar{\theta}) \bar{\Sigma}(x, e^{iK}\bar{\theta}) \\ &= \int d^4x d^2\bar{\theta} e^{i(-2n-2m+2)K} \bar{\Phi}(x, \bar{\theta}) \bar{\Sigma}(x, \bar{\theta}) \\ &= \int d^4x d^2\bar{\theta} \bar{\Phi}(x, \bar{\theta}) \bar{\Sigma}(x, \bar{\theta}). \end{aligned}$$

For the last chiral superfield  $W_\alpha = -\frac{1}{4} \bar{D}\bar{D} D_\alpha$ , we note that

since it is also chiral, it makes sense that

$$\begin{aligned} W_\alpha(y, \theta) &\longmapsto e^{2ilK} W_\alpha(y, e^{-iK}\theta), \\ \bar{W}_\alpha(\bar{y}, \bar{\theta}) &\longmapsto e^{-2ilK} \bar{W}_\alpha(\bar{y}, e^{iK}\bar{\theta}). \end{aligned}$$

Then

$$\begin{aligned} \int d^4x d^2\theta W^\alpha W_\alpha &\longmapsto \int d^4x d^2\theta e^{4ilK} W^\alpha(x, e^{-iK}\theta) W_\alpha(x, e^{-iK}\theta) \\ &= \int d^4x d^2\theta e^{i(4l-2)K} W^\alpha(x, \theta) W_\alpha(x, \theta). \end{aligned}$$

This is invariant if and only if

$$l = \frac{1}{2}.$$

Then

$$\begin{aligned} \int d^4x d^2\bar{\theta} \bar{W}(x, \bar{\theta})^2 &\longrightarrow \int d^4x d^2\bar{\theta} e^{-2ik} \bar{W}(x, e^{ik}\bar{\theta})^2 \\ &= \int d^4x d^2\bar{\theta} \bar{W}(x, \bar{\theta})^2. \end{aligned}$$

These transformations are achieved if we let  $V$  be a scalar under  $R$ -symmetry

$$V(x, \theta, \bar{\theta}) \longrightarrow V(x, e^{-ik}\theta, e^{ik}\bar{\theta}) =: V'(x, \theta, \bar{\theta}).$$

To see this we note that for a superfield  $f(x, \theta, \bar{\theta}) = g(x, e^{-ik}\theta, e^{ik}\bar{\theta})$

$$f(x, \theta, \bar{\theta}) = g(x, e^{-ik}\theta, e^{ik}\bar{\theta})$$

we have

$$\frac{\partial f}{\partial \theta^\alpha}(x, \theta, \bar{\theta}) = \frac{\partial g}{\partial \theta^\alpha}(x, e^{-ik}\theta, e^{ik}\bar{\theta}) e^{-ik}.$$

Then, given that in  $R$ -symmetry  $\bar{\theta} \rightarrow e^{-ik}\bar{\theta}$ , then

$$\begin{aligned} D_\alpha f(x, \theta, \bar{\theta}) &= \frac{\partial}{\partial \theta^\alpha} g(x, e^{-ik}\theta, e^{ik}\bar{\theta}) + i(\sigma^m e^{-ik}\bar{\theta})_\alpha \partial_m g(x, e^{-ik}\theta, e^{ik}\bar{\theta}) \\ &= e^{-ik} D_\alpha g(x, e^{-ik}\theta, e^{ik}\bar{\theta}). \end{aligned}$$

Similarly

$$\bar{D}^{\dot{\alpha}} f(x, \theta, \bar{\theta}) = e^{ik} \bar{D}^{\dot{\alpha}} g(x, e^{-ik}\theta, e^{ik}\bar{\theta}).$$

Thus

(4)

$$W_{\alpha}^{\vee}(x, \theta, \bar{\theta}) \longmapsto e^{ik} W_{\alpha}(x, e^{-ik} \theta, e^{ik} \bar{\theta}).$$

In the WZ gauge we have

$$\begin{aligned}
 & -(\theta \sigma^m \bar{\theta}) \delta_Q A_m(x) + i \theta^2 \bar{\theta} \delta_Q \bar{\lambda}(x) - i \bar{\theta}^2 \theta \delta_Q \lambda(x) + \frac{1}{2} \theta^2 \bar{\theta}^2 \delta_Q d(x) \\
 & = \delta_Q V(x) = (\varepsilon Q + \bar{\varepsilon} \bar{Q}) V(x, \theta, \bar{\theta}) + \Lambda(x + i \theta \sigma \bar{\theta}, \theta) + \bar{\Lambda}(x - i \theta \sigma \bar{\theta}, \bar{\theta})
 \end{aligned}$$

where

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma^m_{\alpha \dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^m},$$

$$\bar{Q}^{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i \bar{\sigma}^{m \dot{\alpha} \alpha} \theta_\alpha \frac{\partial}{\partial x^m}.$$

The gauge transformation

$$\Lambda(y, \theta) = a(y) + i \bar{\theta} \theta b(y) + \theta \theta f(y),$$

will later be chosen to ensure that the result is indeed in the WZ gauge. We have

$$\begin{aligned}
 Q_\alpha V(x, \theta, \bar{\theta}) &= -(\sigma^m \bar{\theta})_\alpha A_m(x) + i \theta_\alpha \bar{\theta} \bar{\lambda}(x) - i \bar{\theta}^2 \lambda_\alpha(x) \\
 &\quad + \theta_\alpha \bar{\theta}^2 d(x) + i (\theta \sigma^m \bar{\theta}) (\sigma^n \bar{\theta})_\alpha \partial_n A_m(x) \\
 &= \frac{1}{2} \bar{\theta}^2 (\sigma^n \bar{\sigma}^m)_\alpha{}^\beta \theta_\beta \partial_n A_m(x) \\
 &\quad + \frac{1}{4} \bar{\theta}^2 (-2 \eta^{nm} \delta_\alpha{}^\beta) \theta_\beta \partial_n A_m(x) \\
 &\quad + \frac{1}{4} \bar{\theta}^2 (\sigma^n \bar{\sigma}^m)_\alpha{}^\beta \theta_\beta F_{nm}(x) \\
 &= -\frac{1}{2} \bar{\theta}^2 \theta_\alpha \partial^m A_m(x) + \frac{1}{4} \bar{\theta}^2 (\sigma^n \bar{\sigma}^m)_\alpha{}^\beta \theta_\beta F_{nm}(x) \\
 &\quad \leftarrow \begin{aligned} & \theta^\beta \sigma^m_{\beta \dot{\beta}} \bar{\theta}^{\dot{\beta}} \sigma^n_{\alpha \dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \\ & \quad \quad \quad \parallel \\ & \frac{1}{2} \varepsilon^{\dot{\beta} \dot{\alpha}} \bar{\theta}^2 \theta^\beta \sigma^m_{\beta \dot{\beta}} \sigma^n_{\alpha \dot{\alpha}} \\ & \quad \quad \quad \parallel \\ & \frac{1}{2} \bar{\theta}^2 \bar{\sigma}^{m \dot{\alpha} \beta} \theta_\beta \sigma^n_{\alpha \dot{\alpha}} \\ & \quad \quad \quad \parallel \\ & \frac{1}{2} \bar{\theta}^2 (\sigma^n \bar{\sigma}^m)_\alpha{}^\beta \theta_\beta \end{aligned}
 \end{aligned}$$

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$$+ (\sigma^m \bar{\theta})_\alpha \theta^2 \bar{\theta}^2 \partial_m \bar{\lambda}(x)$$

$$\begin{aligned} \theta^2 \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \partial_m \bar{\lambda}^{\dot{\beta}}(x) &= - \theta^2 \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} \partial_m \bar{\lambda}_{\dot{\beta}}(x) \\ &= - \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}^2 \sigma^m_{\alpha\dot{\alpha}} \partial_m \bar{\lambda}_{\dot{\beta}}(x) \bar{\theta}^2 \\ &= - \frac{1}{2} \theta^2 \bar{\theta}^2 (\sigma^m \partial_m \bar{\lambda}(x))_\alpha. \end{aligned}$$

$$\begin{aligned} &= - (\sigma^m \bar{\theta})_\alpha A_m(x) + i \theta_\alpha \bar{\theta}^2 \bar{\lambda}(x) - i \bar{\theta}^2 \lambda_\alpha(x) + \theta_\alpha \bar{\theta}^2 d(x) \\ &\quad - \frac{i}{2} \bar{\theta}^2 \theta_\alpha \partial^m A_m(x) + \frac{1}{4} \bar{\theta}^2 (\sigma^n \bar{\sigma}^m)_\alpha{}^\beta \theta_\beta F_{nm}(x) - \frac{1}{2} \theta^2 \bar{\theta}^2 (\sigma^m \partial_m \bar{\lambda}(x))_\alpha \end{aligned}$$

Similarly

$$\bar{Q}^{\dot{\alpha}} V(x, \theta, \bar{\theta}) = \underbrace{\theta^\alpha \sigma^m_{\alpha\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\alpha}} A_m(x)}_{\bar{\sigma}^{m\dot{\alpha}\alpha} \theta_\alpha A_m(x)} + i \theta^2 \bar{\lambda}^{\dot{\alpha}}(x) - 2i \bar{\theta}^{\dot{\alpha}} \theta \lambda(x)$$

$$+ \theta^2 \bar{\theta}^{\dot{\alpha}} d(x) + i (\bar{\sigma}^n \theta)^{\dot{\alpha}} (\theta \sigma^m \bar{\theta}) \partial_n A_m(x)$$

$$\hookrightarrow = \bar{\sigma}^{n\dot{\alpha}\alpha} \theta_\alpha \theta^\beta \sigma^m_{\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_n A_m(x) = - \frac{1}{2} \varepsilon_{\alpha\gamma} \varepsilon^{\gamma\beta} \theta^2 \bar{\sigma}^{n\dot{\alpha}\alpha} \sigma^m_{\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_n A_m(x)$$

$$\begin{aligned} &= - \frac{1}{2} \theta^2 (\bar{\sigma}^n \sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_n A_m(x) = - \frac{1}{4} \theta^2 (-2\eta^{nm}) \delta_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} \partial_n A_m(x) \\ &\quad - \frac{1}{4} \theta^2 (\bar{\sigma}^n \sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} F_{nm}(x) \end{aligned}$$

$$= \frac{1}{2} \theta^2 \bar{\theta}^{\dot{\alpha}} \partial^m A_m(x) - \frac{1}{4} \theta^2 (\bar{\sigma}^n \sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} F_{nm}(x)$$

$$- \underbrace{\bar{\theta}^2 (\bar{\sigma}^m \theta)^{\dot{\alpha}} \theta \partial_m \lambda(x)}$$

$$\bar{\theta}^2 \bar{\sigma}^{m\dot{\alpha}\alpha} \theta_\alpha \theta^\beta \partial_m \lambda_\beta(x) = - \frac{1}{2} \varepsilon_{\alpha\beta} \theta^2 \bar{\theta}^2 \bar{\sigma}^{m\dot{\alpha}\alpha} \partial_m \lambda^\beta(x) = - \frac{1}{2} \theta^2 \bar{\theta}^2 (\bar{\sigma}^m \partial_m \lambda(x))^{\dot{\alpha}}$$

(7)

$$\begin{aligned}
&= i (\bar{\sigma}^m \theta)^{\dot{\alpha}} A_m(x) + i \theta^2 \bar{\lambda}^{\dot{\alpha}}(x) - 2i \bar{\theta}^{\dot{\alpha}} \theta \lambda(x) + \theta^2 \bar{\theta}^{\dot{\alpha}} d(x) \\
&\quad + \frac{1}{2} \theta^2 \bar{\theta}^{\dot{\alpha}} \partial^m A_m(x) - \frac{i}{4} \theta^2 (\bar{\sigma}^n \sigma^m)^{\dot{\alpha}\beta} \bar{\theta}^{\dot{\beta}} F_{nm}(x) \\
&\quad + \frac{1}{2} \theta^2 \bar{\theta}^2 (\bar{\sigma}^m \partial_m \lambda(x))^{\dot{\alpha}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\delta_Q V = & - (\xi \sigma^m \bar{\theta}) A_m(x) + 2i (\xi \theta) (\bar{\theta} \bar{\lambda}(x)) - i \bar{\theta}^2 (\xi \lambda(x)) + (\xi \theta) \bar{\theta}^2 d(x) \\
& - \frac{i}{2} \bar{\theta}^2 (\xi \theta) \partial^m A_m(x) + \frac{i}{4} \bar{\theta}^2 (\xi \sigma^n \bar{\sigma}^m \theta) F_{nm}(x) - \frac{1}{2} \theta^2 \bar{\theta}^2 (\xi \sigma^m \partial_m \bar{\lambda}(x)) \\
& + (\bar{\xi} \bar{\sigma}^m \theta) A_m(x) + i \theta^2 (\bar{\xi} \bar{\lambda}(x)) - 2i (\bar{\xi} \bar{\theta}) (\theta \lambda(x)) + \theta^2 (\bar{\xi} \bar{\theta}) d(x) \\
& + \frac{i}{2} \theta^2 (\bar{\xi} \bar{\theta}) \partial^m A_m(x) - \frac{i}{4} \theta^2 (\bar{\xi} \bar{\sigma}^n \sigma^m \bar{\theta}) F_{nm}(x) + \frac{1}{2} \theta^2 \bar{\theta}^2 (\bar{\xi} \bar{\sigma}^m \partial_m \lambda(x)) \\
& + a(x) + i (\theta \sigma^m \bar{\theta}) \partial_m a(x) + \frac{1}{4} (\theta \theta) (\bar{\theta} \bar{\theta}) \square a(x) \\
& + \sqrt{2} \theta b(x) + i \sqrt{2} (\theta \sigma^m \bar{\theta}) (\theta \partial_m b(x)) + \theta \theta f(x)
\end{aligned}$$

$$\begin{aligned}
\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \theta^\beta \partial_m b_\beta(x) &= + \frac{1}{2} \varepsilon^{\alpha\beta} \theta^2 \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m b_\beta(x) \\
&= - \frac{1}{2} \theta^2 (\partial_m b \sigma^m \bar{\theta})
\end{aligned}$$

$$\begin{aligned}
&+ \bar{a}(x) - i (\theta \sigma^m \bar{\theta}) \partial_m \bar{a}(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 \square \bar{a}(x) \\
&+ \sqrt{2} \bar{\theta} \bar{b}(x) - i \sqrt{2} (\theta \sigma^m \bar{\theta}) (\bar{\theta} \partial_m \bar{b}(x)) + \bar{\theta} \bar{\theta} \bar{f}(x)
\end{aligned}$$

$$\begin{aligned}
\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} \partial_m \bar{b}_\beta(x) &= - \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}^2 \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \partial_m \bar{b}_\beta(x) \\
&= - \frac{1}{2} \bar{\theta}^2 (\theta \sigma^m \partial_m \bar{b}(x)).
\end{aligned}$$

By comparison on the  $1, \theta, \bar{\theta}, \theta^2$  and  $\bar{\theta}^2$  terms, we see that we need to choose our gauge transformation such that

$$a(x) = -\bar{a}(x)$$

$$-\sqrt{2} b_{\alpha}^{\alpha}(x) = \bar{\xi}_{\alpha}^{\alpha} \bar{\sigma}^{m\alpha\omega} A_m(x)$$

$$-\sqrt{2} \bar{b}_{\dot{\alpha}}(x) = \xi^{\alpha} \sigma^m_{\alpha\dot{\alpha}} A_m(x)$$

$$t(x) = -i(\bar{\xi} \bar{\lambda}(x))$$

$$\bar{t}(x) = i(\xi \lambda(x)).$$

By comparing the  $\theta \sigma^m \bar{\theta}$

$$(-(\theta \sigma^m \bar{\theta}) \delta_Q A_m(x) = 2i(\xi \theta)(\bar{\theta} \bar{\lambda}(x)) - 2i(\bar{\xi} \bar{\theta})(\theta \lambda(x)) + i(\theta \sigma^m \bar{\theta}) \partial_m a(x) - i(\theta \sigma^m \bar{\theta}) \partial_m \bar{a}(x)$$

We thus have

$$-\frac{1}{2} (\theta \theta)(\bar{\theta} \bar{\theta}) \delta_Q A^n(x) = -(\theta \sigma^n \bar{\theta})(\theta \sigma^m \bar{\theta}) \delta_Q A_m(x)$$

$$= 2i \underbrace{(\theta \sigma^n \bar{\theta})(\xi \theta)(\bar{\theta} \bar{\lambda}(x))}_{\text{II}}$$

$$\theta^{\alpha} \sigma^n_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \xi^{\beta} \theta_{\beta} \bar{\theta}_{\dot{\beta}} \bar{\lambda}^{\dot{\beta}}(x)$$

$$\stackrel{\text{II}}{=} \frac{1}{4} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \theta^2 \bar{\theta}^2 \sigma^n_{\alpha\dot{\alpha}} \xi_{\beta} \bar{\lambda}_{\dot{\beta}}(x)$$

$$\stackrel{\text{II}}{=} \frac{1}{4} \theta^2 \bar{\theta}^2 \bar{\sigma}^{n\dot{\beta}\beta} \xi_{\beta} \bar{\lambda}_{\dot{\beta}}(x) = -\frac{1}{4} \theta^2 \bar{\theta}^2 (\bar{\lambda}(x) \bar{\sigma}^n \xi)$$

$$- 2i \underbrace{(\theta \sigma^n \bar{\theta})(\bar{\xi} \bar{\theta})(\theta \lambda(x))}_{\text{III}} + \frac{i}{2} (\theta \theta)(\bar{\theta} \bar{\theta}) \partial^n \underbrace{(a(x) - \bar{a}(x))}_{2a(x)} - \frac{1}{4} \theta^2 \bar{\theta}^2 (\bar{\xi} \bar{\sigma}^n \lambda(x))$$

$$= \frac{i}{2} (\bar{\xi} \bar{\sigma}^n \lambda(x) - \bar{\lambda}(x) \bar{\sigma}^n \xi)$$

$$= \frac{i}{2} (\bar{\xi} \bar{\sigma}^n \lambda(x) + \xi \sigma^n \bar{\lambda}(x))$$

We conclude that

$$\boxed{\delta_Q A^n(x) = i \bar{\xi} \bar{\sigma}^n \lambda(x) + i \xi \sigma^n \bar{\lambda}(x) - 2i \partial^n a(x)}$$



Similarly, by comparing the  $\bar{\theta}^2 \theta$  terms we have (9)

$$\begin{aligned}
 i \bar{\theta}^2 (\theta \delta_Q \lambda(x)) &= (\xi \theta) \bar{\theta}^2 d(x) - \frac{i}{2} \bar{\theta}^2 (\xi \theta) \cancel{\partial^m A_m(x)} + \frac{i}{4} \bar{\theta}^2 (\xi \sigma^n \bar{\sigma}^m \theta) F_{nm}(x) \\
 &\quad + \underbrace{\frac{i}{\sqrt{2}} \bar{\theta}^2 (\theta \sigma^m \partial_m \bar{b}(x))}_{=0} \\
 &= \frac{i}{\sqrt{2}} \bar{\theta}^2 \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \frac{1}{\sqrt{2}} \xi^\beta \sigma^n_{\beta\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \partial_m A_n(x) \\
 &= -\frac{i}{2} \bar{\theta}^2 \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \xi_\beta \bar{\sigma}^n{}^{\dot{\alpha}\beta} \partial_m A_n(x) \\
 &= -\frac{i}{2} \bar{\theta}^2 \theta^\alpha (\sigma^m \bar{\sigma}^n)_{\alpha}{}^{\beta} \xi_\beta \partial_m A_n(x) \\
 &= -\frac{i}{4} \bar{\theta}^2 \theta^\alpha (-2\eta^{mn} \delta_{\alpha}{}^{\beta}) \xi_\beta \partial_m A_n(x) - \frac{i}{4} \bar{\theta}^2 \theta^\alpha (\sigma^m \bar{\sigma}^n)_{\alpha}{}^{\beta} \xi_\beta F_{mn}(x) \\
 &= \cancel{\frac{i}{2} \bar{\theta}^2 (\theta \xi) \partial^m A_m(x)} - \frac{i}{4} \bar{\theta}^2 (\theta \sigma^m \bar{\sigma}^n \xi) F_{mn}(x)
 \end{aligned}$$

Notice that

$$\begin{aligned}
 \frac{i}{4} \bar{\theta}^2 (\xi \sigma^n \bar{\sigma}^m \theta) F_{nm}(x) &= \frac{i}{8} \bar{\theta}^2 (\xi \sigma^{[n} \bar{\sigma}^{m]} \theta) F_{nm}(x) \\
 &= \frac{i}{2} \bar{\theta}^2 (\xi \sigma^{nm} \theta) F_{nm}(x).
 \end{aligned}$$

Moreover, using the hint of Exercise 1.2

$$\begin{aligned}
 (\xi \sigma^{nm} \theta) &= \xi^\alpha \sigma^{nm}_{\alpha}{}^{\beta} \theta_\beta = -\xi^\alpha \sigma^{nm}_{\alpha\beta} \theta^\beta \\
 &= -\xi^\alpha \sigma^{nm}_{\beta\alpha} \theta^\beta = \theta^\beta \sigma^{nm}_{\beta\alpha} \xi^\alpha \\
 &= -(\theta \sigma^{nm} \xi).
 \end{aligned}$$

Thus

$$-i \bar{\theta}^2 (\theta \delta_Q \lambda(x)) = (\xi \theta) \bar{\theta}^2 d(x) + i \bar{\theta}^2 (\xi \sigma^{nm} \theta) F_{nm}(x),$$

i.e.

$$\delta_Q \lambda^\alpha(x) = -\bar{\xi}^{\dot{\beta}} \sigma^{nm}{}_{\beta}{}^{\alpha} F_{nm}(x) + i \bar{\xi}^{\dot{\alpha}} d(x),$$

or

$$\boxed{\delta_Q \lambda_\alpha(x) = \sigma^{nm}{}_{\alpha}{}^{\beta} \bar{\xi}_{\dot{\beta}} F_{nm}(x) + i \bar{\xi}_{\dot{\alpha}} d(x).}$$

Repeating with  $\theta^2 \bar{\theta}$ , we have

$$i \theta^2 \bar{\theta} \delta_Q \bar{\lambda}(x) = \theta^2 (\bar{\xi} \bar{\theta}) d(x) + \cancel{\frac{i}{2} \theta^2 (\bar{\xi} \bar{\theta}) \partial^m A_m(x)} - \frac{i}{4} \theta^2 (\bar{\xi} \bar{\sigma}^n \sigma^m \bar{\theta}) F_{nm}(x) \\ - \cancel{\frac{i}{\sqrt{2}} \theta^2 (\partial_m b \sigma^m \bar{\theta})}$$

$$\frac{i}{\sqrt{2}} \theta^2 \frac{1}{\sqrt{2}} \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{nm\dot{\alpha}}{}_{\beta} \partial_m A_n(x) \sigma^m{}_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \\ = \frac{i}{2} \theta^2 \bar{\xi}_{\dot{\alpha}} (\bar{\sigma}^n \sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_m A_n(x) \\ = \cancel{\frac{i}{\sqrt{2}} \theta^2 \bar{\xi}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial^m A_m(x)} + \frac{i}{4} \theta^2 (\bar{\xi} \bar{\sigma}^n \sigma^m \bar{\theta}) F_{mn}(x) \\ = \theta^2 (\bar{\xi} \bar{\theta}) d(x) - i \theta^2 \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{nm\dot{\alpha}}{}_{\beta} \bar{\theta}^{\dot{\beta}} F_{nm}(x).$$

Thus

$$\boxed{\delta_Q \bar{\lambda}^{\dot{\alpha}}(x) = \bar{\sigma}^{nm\dot{\alpha}}{}_{\beta} \bar{\xi}^{\dot{\beta}} F_{nm}(x) - i \bar{\xi}^{\dot{\alpha}} d(x).}$$

Finally, the terms  $\theta^2 \bar{\theta}^2$  give

$$\frac{1}{2} \theta^2 \bar{\theta}^2 \delta_Q d(x) = -\frac{1}{2} \theta^2 \bar{\theta}^2 (\bar{\xi} \sigma^m \partial_m \bar{\lambda}(x)) + \frac{1}{2} \theta^2 \bar{\theta}^2 (\bar{\xi} \bar{\sigma}^m \partial_m \lambda(x)) \\ + \cancel{\frac{1}{4} \theta^2 \bar{\theta}^2 \square (a + \bar{a})(x)},$$

Therefore,

$$\left| \delta_Q d(x) = \bar{\xi} \bar{\sigma}^m \partial_m \lambda(x) - \bar{\xi} \sigma^m \partial_m \bar{\lambda}(x) \right|$$

By fixing the gauge transformation to  $a=0$ , we obtain (6.20).

### Exercise 7.3.

We begin from the transformation rule (7.22)

$$A_m \mapsto M A_m M^\dagger + i M \partial_m M^\dagger$$

To lighten the notation we will work without Matrix indices. We then have

$$\begin{aligned} F_{mn} &\mapsto \partial_m (M A_n M^\dagger + i M \partial_n M^\dagger) - m \leftrightarrow n \\ &\quad - i [M A_m M^\dagger + i M \partial_m M^\dagger, M A_n M^\dagger + i M \partial_n M^\dagger] \\ &= M \partial_m A_n M^\dagger + \partial_m M A_n M^\dagger + \cancel{M A_n \partial_m M^\dagger} + i \partial_m M \partial_n M^\dagger + \cancel{i M \partial_m \partial_n M^\dagger} \\ &\quad - m \leftrightarrow n \\ &\quad - i (M A_m M^\dagger M A_n M^\dagger + \cancel{M A_n M^\dagger M A_m M^\dagger} \\ &\quad + i M A_m M^\dagger M \partial_n M^\dagger - i M \partial_n M^\dagger M A_m M^\dagger \\ &\quad + i M \partial_m M^\dagger M A_n M^\dagger - \cancel{i M A_n M^\dagger M \partial_m M^\dagger} \\ &\quad - M \partial_m M^\dagger M \partial_n M^\dagger + M \partial_n M^\dagger M \partial_m M^\dagger). \end{aligned}$$

To further proceed we observe that the restriction  $MM^\dagger = I$  restricts

$$0 = \partial_m I = \partial_m M M^\dagger + M \partial_m M^\dagger$$

Thus  $\partial_m M^\dagger = -M^\dagger \partial_m M M^\dagger$  and

$$\begin{aligned}
F_{mn} &\mapsto M(\partial_m A_n - \partial_n A_m)M^\dagger + \cancel{\partial_m M A_n M^\dagger} + \cancel{i\partial_m M \partial_n M^\dagger} \dots \\
&\quad - \cancel{\partial_n M A_m M^\dagger} - \cancel{i\partial_n M \partial_m M^\dagger} \dots \\
&\quad - iM[A_m, A_n]M^\dagger + \cancel{MM^\dagger \partial_n M M^\dagger} A_m M^\dagger \\
&\quad - \cancel{MM^\dagger \partial_m M M^\dagger} A_n M^\dagger + \cancel{MM^\dagger \partial_m M M^\dagger} M M^\dagger \partial_n M M^\dagger \\
&\quad - \cancel{i\partial_m M \partial_n M^\dagger} + \cancel{\partial_n M \partial_m M^\dagger} \\
&= MF_{mn}M^\dagger.
\end{aligned}$$

Nathan Exercise:

In the Wess-Zumino gauge we have

$$\begin{aligned}
V(x, \theta, \bar{\theta}) &= -(\theta \sigma^m \bar{\theta}) A_m(x) + i\theta^2 (\bar{\theta} \bar{\lambda}(x)) - i\bar{\theta}^2 (\theta \lambda(x)) + \frac{1}{2} \theta^2 \bar{\theta}^2 d(x) \\
V(x, \theta, \bar{\theta})^2 &= \underbrace{(\theta \sigma^m \bar{\theta})(\theta \sigma^n \bar{\theta}) A_m(x) A_n(x)} \\
&\quad - \frac{1}{2} \theta^2 \bar{\theta}^2 A^m(x) A_m(x).
\end{aligned}$$

Thus

$$\begin{aligned}
e^{-V(x, \theta, \bar{\theta})} &= 1 + (\theta \sigma^m \bar{\theta}) A_m(x) - i\theta^2 (\bar{\theta} \bar{\lambda}(x)) + i\bar{\theta}^2 (\theta \lambda(x)) - \frac{1}{2} \theta^2 \bar{\theta}^2 d(x) \\
&\quad - \frac{1}{4} \theta^2 \bar{\theta}^2 A^m(x) A_m(x).
\end{aligned}$$

On the other hand the matter chiral field is

$$\begin{aligned}
\Phi(x + i\theta \sigma \bar{\theta}, \theta) &= \varphi(x) + i(\theta \sigma^m \bar{\theta}) \partial_m \varphi(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 \square \varphi(x) \\
&\quad + \sqrt{2} \theta \psi(x) + \frac{i}{\sqrt{2}} \theta^2 (\bar{\theta} \bar{\sigma}^m \partial_m \psi(x)) + \theta^2 F(x)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\Phi}(x - i\theta \sigma \bar{\theta}, \bar{\theta}) &= \bar{\varphi}(x) - i(\theta \sigma^m \bar{\theta}) \partial_m \bar{\varphi}(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 \square \bar{\varphi}(x) \\
&\quad + \sqrt{2} \bar{\theta} \bar{\psi}(x) + \frac{i}{\sqrt{2}} \bar{\theta}^2 (\partial_m \bar{\psi}(x) \bar{\sigma}^m \theta) + \bar{\theta}^2 \bar{F}(x).
\end{aligned}$$

Thus

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$$\int d^4\theta \Phi(x+i\theta\sigma\bar{\theta}) e^{-V(x,\theta,\bar{\theta})} \bar{\Phi}(x-i\theta\sigma\bar{\theta},\bar{\theta})$$

$$= \frac{1}{4} \varphi(x) \square \bar{\varphi}(x) + \frac{i}{2} \varphi(x) A^m(x) \partial_m \bar{\varphi}(x)$$

$$+ \int d^4\theta \varphi(x) (-i\theta^2 (\bar{\theta} \lambda(x))) \sqrt{2} (\bar{\theta} \bar{\psi}(x)) + \int d^4\theta \varphi(x) i \bar{\theta}^2 (\theta \lambda(x)) \frac{i}{\sqrt{2}} (\partial_m \bar{\psi}(x) \bar{\sigma}^m \theta)$$

$$= - \int d^4\theta \frac{i}{\sqrt{2}} \varepsilon_{\dot{\alpha}\dot{\beta}} \theta^2 \bar{\theta}^2 \varphi(x) \bar{\lambda}^{\dot{\alpha}}(x) \bar{\psi}^{\dot{\beta}}(x) = - \frac{i}{\sqrt{2}} \int d^4\theta \varphi(x) \bar{\theta}^2 \frac{1}{2} \varepsilon^{\alpha\beta} \theta^2 \lambda_{\alpha}(x) \sigma^m_{\beta\dot{\beta}} \partial_m \bar{\psi}^{\dot{\beta}}(x)$$

$$= \frac{i}{\sqrt{2}} \varphi(x) (\bar{\lambda}(x) \bar{\psi}(x))$$

$$- \frac{1}{2\sqrt{2}} \varphi(x) (\lambda \sigma^m \partial_m \bar{\psi}(x))$$

$$= \frac{1}{2\sqrt{2}} \varphi(x) (\partial_m \bar{\psi}(x) \bar{\sigma}^m \lambda(x))$$

$$- \frac{1}{2} \varphi(x) \square \bar{\varphi}(x) - \frac{i}{4} \varphi(x) A^m(x) \partial_m \bar{\varphi}(x)$$

$$- \frac{1}{2} \partial^m \varphi(x) \partial_m \bar{\varphi}(x) - \frac{i}{2} \bar{\varphi}(x) A^m(x) \partial_m \varphi(x) + \frac{1}{4} \bar{\varphi}(x) \square \varphi(x)$$

$$+ \int d^4\theta \sqrt{2} (\theta \psi(x)) (\theta \sigma^m \bar{\theta}) A_m(x) \sqrt{2} (\bar{\theta} \bar{\psi}(x))$$

$$= \int d^4\theta \frac{1}{2} \varepsilon^{\alpha\beta} \theta^2 \psi_{\alpha}(x) \sigma^m_{\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}} A_m(x) \bar{\psi}_{\dot{\alpha}}(x) \bar{\theta}^{\dot{\alpha}}$$

$$= - \int d^4\theta \varepsilon^{\alpha\beta} \frac{1}{2} \varepsilon^{\dot{\beta}\dot{\alpha}} \theta^2 \bar{\theta}^2 \psi_{\alpha}(x) \sigma^m_{\beta\dot{\beta}} A_m(x) \bar{\psi}_{\dot{\alpha}}(x)$$

$$= \frac{1}{2} (\psi(x) \sigma^m \bar{\psi}(x)) A_m(x)$$

$$+ \int d^4\theta \sqrt{2} (\theta \psi(x)) i \bar{\theta}^2 (\theta \lambda(x)) \bar{\varphi}(x)$$

$$= - \frac{i}{\sqrt{2}} \varepsilon^{\alpha\beta} \psi_{\alpha}(x) \lambda_{\beta}(x) \bar{\varphi}(x) = \frac{i}{\sqrt{2}} (\psi(x) \lambda(x)) \bar{\varphi}(x)$$

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$$+ \int d^4\theta \frac{i}{\sqrt{2}} (\bar{\theta} \bar{\sigma}^m \partial_m \psi(x)) (\theta \sigma^n \bar{\theta}) A_n(x) \frac{i}{\sqrt{2}} (\partial_m \bar{\psi}(x) \bar{\sigma}^m \theta)$$

$$= - \int d^4\theta \frac{1}{2} \partial_m \psi^\alpha(x) \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \theta^\beta \sigma^n_{\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}} A_n(x) \theta^\gamma \sigma^k_{\gamma\dot{\gamma}} \partial_k \bar{\psi}^{\dot{\gamma}}(x)$$

$$= - \int d^4\theta \frac{1}{2} \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}^2 \frac{1}{2} \varepsilon^{\beta\gamma} \theta^2 \partial_m \psi^\alpha(x) \sigma^m_{\alpha\dot{\alpha}} \sigma^n_{\beta\dot{\beta}} A_n(x) \sigma^k_{\gamma\dot{\gamma}} \partial_k \bar{\psi}^{\dot{\gamma}}(x)$$

$$= - \frac{1}{8} \partial_m \psi^\alpha(x) \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \theta^\beta \sigma^n_{\beta\dot{\beta}} A_n(x) \sigma^k_{\gamma\dot{\gamma}} \partial_k \bar{\psi}^{\dot{\gamma}}(x)$$

$$= - \frac{1}{8} (\partial_m \psi(x) (\sigma^m \bar{\sigma}^n \sigma^k) \partial_k \bar{\psi}(x))$$

$$- \int d^4\theta \frac{i}{\sqrt{2}} (\bar{\theta} \bar{\sigma}^m \partial_m \psi(x)) i \theta^2 (\bar{\theta} \bar{\lambda}(x)) \bar{\varphi}(x)$$

$$= - \frac{1}{2\sqrt{2}} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\alpha}} \partial_m \psi_\alpha(x) \bar{\lambda}^{\dot{\beta}}(x) \bar{\varphi}(x)$$

$$= - \frac{1}{2\sqrt{2}} (\bar{\lambda}(x) \bar{\sigma}^m \partial_m \psi(x)) \bar{\varphi}(x) = \frac{1}{2\sqrt{2}} (\partial_m \psi(x) \sigma^m \bar{\lambda}(x))$$

$$+ F(x) \bar{F}(x)$$

$$+ \int d^4\theta \frac{i}{\sqrt{2}} \theta^2 (\bar{\theta} \bar{\sigma}^m \partial_m \psi(x)) \sqrt{2} (\bar{\theta} \bar{\psi}(x))$$

$$= \frac{i}{2\sqrt{2}} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\alpha}} \partial_m \psi_\alpha(x) \bar{\psi}^{\dot{\beta}}(x) = - \frac{i}{2} (\bar{\psi}(x) \bar{\sigma}^m \partial_m \psi(x))$$

$$= \frac{i}{2} (\partial_m \psi(x) \sigma^m \bar{\psi}(x))$$

$$= i \int d^4\theta (\theta \psi(x)) \bar{\theta}^2 (\partial_m \bar{\psi}(x) \bar{\sigma}^m \theta)$$

$$= + \frac{i}{2} \varepsilon^{\alpha\beta} \psi_\alpha(x) \bar{\sigma}^m_{\beta\dot{\beta}} \partial_m \bar{\psi}^{\dot{\beta}}(x) = - \frac{i}{2} (\psi(x) \sigma^m \partial_m \bar{\psi}(x))$$

$$\begin{aligned}
&= \frac{1}{4} \psi(x) \square \bar{\psi}(x) + \frac{1}{4} \bar{\psi}(x) \square \psi(x) - \frac{1}{2} \partial^m \psi(x) \partial_m \bar{\psi}(x) \\
&\quad + \frac{i}{2} \psi(x) A^m(x) \partial_m \bar{\psi}(x) - \frac{i}{2} \bar{\psi}(x) A^m(x) \partial_m \psi(x) \\
&\quad - \frac{1}{4} \psi(x) A^m(x) A_m(x) \bar{\psi}(x) - \\
&\quad + \frac{i}{2} (\partial_m \psi(x) \sigma^m \bar{\psi}(x)) + \frac{i}{2} (\psi(x) \sigma^m \partial_m \bar{\psi}(x)) \\
&\quad + \frac{1}{2} (\psi(x) \sigma^m \bar{\psi}(x)) A_m(x) \\
&\quad + \frac{i}{\sqrt{2}} \psi(x) (\bar{\lambda}(x) \bar{\psi}(x)) + \frac{i}{\sqrt{2}} \bar{\psi}(x) (\lambda(x) \psi(x)) \\
&\quad - \frac{1}{2} \psi(x) d(x) \bar{\psi}(x) + F(x) \bar{F}(x) \\
&= -\partial^m \psi(x) \partial_m \psi(x) + \frac{i}{2} A^m(x) \psi(x) \partial_m \bar{\psi}(x) - \frac{i}{2} A_m(x) \bar{\psi}(x) \partial^m \psi(x) \\
&\quad - \frac{1}{4} A^m(x) \psi(x) A_m(x) \bar{\psi}(x) \\
&\quad - i (\psi(x) \sigma^m \partial_m \bar{\psi}(x)) - i \frac{i}{2} (\bar{\psi}(x) \sigma^m A_m(x) \bar{\psi}(x)) \\
&\quad + F(x) \bar{F}(x) + \frac{i}{\sqrt{2}} \psi(x) (\bar{\lambda}(x) \bar{\psi}(x)) + \frac{i}{\sqrt{2}} \bar{\psi}(x) (\lambda(x) \psi(x)) \\
&\quad - \frac{1}{2} \psi(x) \bar{\psi}(x) d(x) + \text{surface terms} \\
&= -(\partial^m - \frac{i}{2} A^m(x)) \psi(x) (\partial_m + \frac{i}{2} A_m(x)) \bar{\psi}(x) - i (\psi(x) \sigma^m (\partial_m + \frac{i}{2} A_m(x)) \bar{\psi}(x)) \\
&\quad + F(x) \bar{F}(x) + \frac{i}{\sqrt{2}} \psi(x) (\bar{\lambda}(x) \bar{\psi}(x)) + \frac{i}{\sqrt{2}} \bar{\psi}(x) (\lambda(x) \psi(x)) \\
&\quad - \frac{1}{2} \psi(x) \bar{\psi}(x) d(x) + \text{surface terms}
\end{aligned}$$

Thus, indeed

$$S := \int d^4x d^4\theta \, \Phi(x + i\theta\sigma\bar{\theta}, \theta) e^{-V(x, \theta, \bar{\theta})} \bar{\Phi}(x - i\theta\sigma\bar{\theta}, \bar{\theta})$$

$$= \int d^4x \left( -\nabla_m \psi \nabla^m \bar{\psi} - i \psi \sigma^m \nabla_m \bar{\psi} + F \bar{F} + \frac{i}{\sqrt{2}} \varphi(x) (\lambda(x) \bar{\psi}(x)) \right. \\ \left. + \frac{i}{\sqrt{2}} \bar{\psi}(x) (\lambda(x) \psi(x)) - \frac{1}{2} \varphi(x) \bar{\varphi}(x) d(x) \right)$$