Mecánica Cuántica Avanzada: Examen 3

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Problema 2

(a) Se tiene que

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\dagger} =&\frac{\partial}{\partial\partial_\mu\phi^\dagger}\big((\partial_\nu\phi^\dagger)(\partial^\nu\phi)\big)=\partial^\nu\phi\frac{\partial}{\partial\partial_\mu\phi^\dagger}\big(\partial_\nu\phi^\dagger\big)=\partial^\nu\phi\delta^\mu_\nu=\partial^\mu\phi,\\ \frac{\partial\mathcal{L}}{\partial\phi^\dagger}=&-m^2\phi.\end{aligned} \tag{1}$$

Por lo tanto, las ecuaciones de Euler-Lagrange entregan la de Klein-Gordon

$$0=\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\dagger}-\frac{\partial\mathcal{L}}{\partial\phi^\dagger}=\partial_\mu\partial^\mu\phi+m^2\phi=(\Box+m^2)\phi. \tag{2}$$

(b) Bajo esta transformación $\phi\mapsto(e^{-iq\theta}\phi)^\dagger=e^{iq\theta}\phi^\dagger$ y el Lagrangiano es

$$\begin{aligned}\mathcal{L}(e^{-iq\theta}\phi,e^{iq\theta}\phi^\dagger)=&(\partial_\mu(e^{iq\theta}\phi^\dagger))(\partial^\mu(e^{-iq\theta}\phi))-m^2e^{iq\theta}\phi^\dagger e^{-iq\theta}\phi\\ =&(e^{iq\theta}\partial_\mu\phi^\dagger)(e^{-iq\theta}\partial^\mu\phi)-m^2\phi^\dagger\phi\\ =&(\partial_\mu\phi^\dagger)(\partial^\mu\phi)-m^2\phi^\dagger\phi=\mathcal{L}(\phi,\phi^\dagger).\end{aligned} \tag{3}$$

Se concluye que en efecto el Lagrangiano es invariante.

(c) Asumiendo $q\theta$ como pequeño se tiene $\phi\mapsto\phi-iq\theta\phi$ y $\phi^\dagger\mapsto\phi^\dagger+iq\theta\phi^\dagger$. Ya que el Lagrangiano es invariante, la corriente de Noether obtenida es particularmente sencilla

$$j^\mu=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(-iq\theta\phi)+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\dagger}(iq\theta\phi^\dagger)=iq\theta\big((\partial^\mu\phi)\phi^\dagger-\big(\partial^\mu\phi^\dagger\big)\phi\big). \tag{4}$$

Esto nos va a ser util una vez cuanticemos el campo. Ahora bien, a partir de la ecuación del enunciado

$$
\begin{aligned}
\phi^\dagger(x) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_\mathbf{p}}} \big(a^\dagger(\mathbf{p})e^{ip\cdot x} + \hat{a}(\mathbf{p})e^{-ip\cdot x}\big), \\
\partial^\mu\phi(x) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_\mathbf{p}}} ip^\mu(-a(p)e^{-ip\cdot x} + \hat{a}^\dagger(p)e^{ip\cdot x}), \\
\partial^\mu\phi^\dagger(x) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_\mathbf{p}}} ip^\mu\big(a^\dagger(\mathbf{p})e^{ip\cdot x} - \hat{a}(\mathbf{p})e^{-ip\cdot x}\big).
\end{aligned} \tag{5}
$$

Entonces, en terminos de los operadores de creación y destrucción se obtiene

$$
\begin{aligned}
(\partial^\mu\phi(x))\phi^\dagger(x) = &\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} ip^\mu \\
&\Big(-a(\mathbf{p})a^\dagger(\mathbf{q})e^{-i(p-q)\cdot x} + \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{q})e^{i(p-q)\cdot x} \\
&-a(\mathbf{p})\hat{a}(\mathbf{q})e^{-i(p+q)\cdot x} + \hat{a}^\dagger(\mathbf{p})a^\dagger(\mathbf{q})e^{i(p+q)\cdot x}\Big), \\
(\partial^\mu\phi^\dagger(x))\phi(x) = &\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} ip^\mu \\
&\Big(a^\dagger(\mathbf{p})a(\mathbf{q})e^{i(p-q)\cdot x} - \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})e^{-i(p-q)\cdot x} \\
&-\hat{a}(\mathbf{p})a(\mathbf{q})e^{-i(p+q)\cdot x} + a^\dagger(\mathbf{p})\hat{a}^\dagger(\mathbf{q})e^{i(p+q)\cdot x}\Big).
\end{aligned} \tag{6}
$$

Al restar estos términos se pueden factorizar las exponenciales comunes de manera que se obtiene

$$
\begin{aligned}
j^\mu(x) =&iq\theta \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} ip^\mu \\
&\Big(\big(\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q}) - a(\mathbf{p})a^\dagger(\mathbf{q})\big)e^{-i(p-q)\cdot x} + \\
&\big(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{q}) - a^\dagger(\mathbf{p})a(\mathbf{q})\big)e^{i(p-q)\cdot x} \\
&(\hat{a}(\mathbf{p})a(\mathbf{q}) - a(\mathbf{p})\hat{a}(\mathbf{q}))e^{-i(p+q)\cdot x} + \\
&\big(\hat{a}^\dagger(\mathbf{p})a^\dagger(\mathbf{q}) - a^\dagger(\mathbf{p})\hat{a}^\dagger(\mathbf{q})\big)e^{i(p+q)\cdot x}\Big).
\end{aligned} \tag{7}
$$

Para el cálculo de la carga conservada note que

$$
\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\, e^{\pm i(p\pm q)\cdot x} =& \int \mathrm{d}^3\mathbf{x}\, e^{\pm i(E_\mathbf{p}\pm E_\mathbf{q})t} e^{\mp i(\mathbf{p}\pm\mathbf{q})\cdot\mathbf{x}} \\
=&(2\pi)^3 e^{\pm i(E_\mathbf{p}\pm E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}\pm\mathbf{q}).
\end{aligned} \tag{8}
$$

Por lo tanto

$$
\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\, j^\mu(x) =& iq\theta \int \mathrm{d}^3\mathbf{p}\, \mathrm{d}^3\mathbf{q}\, \frac{1}{2\sqrt{E_\mathbf{p} E_\mathbf{q}}} ip^\mu \\
& \Big(\big(\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q}) - a(\mathbf{p})a^\dagger(\mathbf{q})\big) e^{-i(E_\mathbf{p}-E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + \\
& \big(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{q}) - a^\dagger(\mathbf{p})a(\mathbf{q})\big) e^{i(E_\mathbf{p}-E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
& (\hat{a}(\mathbf{p})a(\mathbf{q}) - a(\mathbf{p})\hat{a}(\mathbf{q})) e^{-i(E_\mathbf{p}+E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) + \\
& \big(\hat{a}^\dagger(\mathbf{p})a^\dagger(\mathbf{q}) - a^\dagger(\mathbf{p})\hat{a}^\dagger(\mathbf{q})\big) e^{i(E_\mathbf{p}+E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \Big) .
\end{aligned}
\tag{9}
$$

Más aún, ya que $E_\mathbf{p} = \sqrt{m^2 + \mathbf{p}^2} = \sqrt{m^2 + (-\mathbf{p})^2} = E_{-\mathbf{p}}$ el coeficiente de cada termino va a ser igual para $\mathbf{q} = \pm\mathbf{p}$ y podemos realizar la integración sobre $\mathbf{q}$

$$
\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\, j^\mu(x) =& iq\theta \int \mathrm{d}^3\mathbf{p}\, \frac{1}{2E_\mathbf{p}} ip^\mu \\
& \big(\big(\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p}) - a(\mathbf{p})a^\dagger(\mathbf{p})\big) + \big(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) - a^\dagger(\mathbf{p})a(\mathbf{p})\big) \\
& (\hat{a}(\mathbf{p})a(-\mathbf{p}) - a(\mathbf{p})\hat{a}(-\mathbf{p})) e^{-2iE_\mathbf{p}t} + \\
& \big(\hat{a}^\dagger(\mathbf{p})a^\dagger(-\mathbf{p}) - a^\dagger(\mathbf{p})\hat{a}^\dagger(-\mathbf{p})\big) e^{2iE_\mathbf{p}t} \big) .
\end{aligned}
\tag{10}
$$

En particular, la carga conservada se obtiene al poner $\mu = 0$

$$
\begin{aligned}
Q =& iq\theta \int \mathrm{d}^3\mathbf{p}\, \frac{i}{2} \\
& \big(\big(\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p}) - a(\mathbf{p})a^\dagger(\mathbf{p})\big) + \big(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) - a^\dagger(\mathbf{p})a(\mathbf{p})\big) \\
& (\hat{a}(\mathbf{p})a(-\mathbf{p}) - a(\mathbf{p})\hat{a}(-\mathbf{p})) e^{-2iE_\mathbf{p}t} + \\
& \big(\hat{a}^\dagger(\mathbf{p})a^\dagger(-\mathbf{p}) - a^\dagger(\mathbf{p})\hat{a}^\dagger(-\mathbf{p})\big) e^{2iE_\mathbf{p}t} \big) .
\end{aligned}
\tag{11}
$$

Ya que $[a(\mathbf{p}), \hat{a}(\mathbf{q})] = [a^\dagger(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})] = 0$ esto se puede reescribir de manera más sugestiva

$$
\begin{aligned}
Q =& iq\theta \int \mathrm{d}^3\mathbf{p}\, \frac{i}{2} \\
& \big(\big(\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p}) - a(\mathbf{p})a^\dagger(\mathbf{p})\big) + \big(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) - a^\dagger(\mathbf{p})a(\mathbf{p})\big) \\
& (\hat{a}(\mathbf{p})a(-\mathbf{p}) - \hat{a}(-\mathbf{p})a(\mathbf{p})) e^{-2iE_\mathbf{p}t} + \\
& \big(\hat{a}^\dagger(\mathbf{p})a^\dagger(-\mathbf{p}) - \hat{a}^\dagger(-\mathbf{p})a^\dagger(\mathbf{p})\big) e^{2iE_\mathbf{p}t} \big) .
\end{aligned}
\tag{12}
$$

Esta forma sugiere considerar el cambio de variable $\mathbf{p} \mapsto -\mathbf{p}$ bajo el cual la integral se mantiene invariante debido a que la energía lo hace y el cambio en los diferenciales se compensa con el de la orientación de la región de integración

$$
\begin{aligned}
\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \hat{a}(-\mathbf{p})a(\mathbf{p})e^{-2iE_{\mathbf{p}}t} =& \int_{-\mathbb{R}^3} \mathrm{d}^3(-\mathbf{p})\, \hat{a}(\mathbf{p})a(-\mathbf{p})e^{-2iE_{-\mathbf{p}}t} \\
=&(-1)^3 \int_{-\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \hat{a}(\mathbf{p})a(-\mathbf{p})e^{-2iE_{\mathbf{p}}t} \\
=& \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \hat{a}(\mathbf{p})a(-\mathbf{p})e^{-2iE_{\mathbf{p}}t}, \\
\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \hat{a}^\dagger(-\mathbf{p})a^\dagger(\mathbf{p})e^{-2iE_{\mathbf{p}}t} =& \int_{-\mathbb{R}^3} \mathrm{d}^3(-\mathbf{p})\, \hat{a}^\dagger(\mathbf{p})a^\dagger(-\mathbf{p})e^{-2iE_{-\mathbf{p}}t} \\
=&(-1)^3 \int_{-\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \hat{a}^\dagger(\mathbf{p})a^\dagger(-\mathbf{p})e^{-2iE_{\mathbf{p}}t} \\
=& \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \hat{a}^\dagger(\mathbf{p})a^\dagger(-\mathbf{p})e^{-2iE_{\mathbf{p}}t}.
\end{aligned}
\tag{13}
$$

Entonces los coeficientes de las exponenciales se anulan de manera que al sacar el factor $i/2$ de la integral obtenemos

$$
\begin{aligned}
Q =&\frac{q}{2}\theta \int \mathrm{d}^3\mathbf{p} \\
&\big(\big(a(\mathbf{p})a^\dagger(\mathbf{p}) - \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\big) + \big(a^\dagger(\mathbf{p})a(\mathbf{p}) - \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})\big)\big)
\end{aligned}
\tag{14}
$$

Ya que $[a(\mathbf{p}), a^\dagger(\mathbf{q})] = \delta^{(3)}(\mathbf{p}-\mathbf{q}) = [\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})]$ se obtiene que

$$
\begin{aligned}
&a(\mathbf{p})a^\dagger(\mathbf{q}) - \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q}) \\
=&a^\dagger(\mathbf{q})a(\mathbf{p}) + [a(\mathbf{p}), a^\dagger(\mathbf{q})] - \hat{a}^\dagger(\mathbf{q})\hat{a}(\mathbf{p}) - [\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})] \\
=&a^\dagger(\mathbf{q})a(\mathbf{p}) - \hat{a}^\dagger(\mathbf{q})\hat{a}(\mathbf{p}).
\end{aligned}
\tag{15}
$$

Poniendo $\mathbf{q} = \mathbf{p}$ vemos que los dos términos de la suma son iguales y se obtiene una carga conservada

$$
Q = q\theta \int \mathrm{d}^3\mathbf{p}\, \big(a^\dagger(\mathbf{p})a(\mathbf{p}) - \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})\big). \tag{16}
$$

Si una cantidad se conserva es claro que cualquier múltiplo de ella también. Ya que el parámetro de la transformación $q\theta$ no tiene una interpretación clara por el momento, podemos redefinir la carga de Noether a

$$
Q = \int \mathrm{d}^3\mathbf{p}\, \big(a^\dagger(\mathbf{p})a(\mathbf{p}) - \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})\big). \tag{17}
$$

Podemos definir los operadores

$$
\begin{aligned}
\mathcal{N}_a :=&a^\dagger(\mathbf{p})a(\mathbf{p}), \\
\mathcal{N}_{\hat{a}} :=&\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}).
\end{aligned}
\tag{18}
$$

de manera que

$$Q = \int \mathrm{d}^3\mathbf{p}\,(\mathcal{N}_a - \mathcal{N}_{\hat{a}}). \tag{19}$$

En la interpretación de partículas esto corresponde a que estas se crean de a pares.

Problema 3