Homework

Take the superfield

$$\Phi(x,\theta) = \varphi(x) + \theta^\alpha \psi_\alpha(x) + \theta^\alpha \theta_\alpha F(x),$$

along with

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\theta^\beta \sigma^m{}_{\alpha\beta} \frac{\partial}{\partial x^m}$$

The following computation is useful

$$\frac{\partial}{\partial \theta^\alpha}(\theta^\beta \theta^\gamma) = \delta^\beta_\alpha \theta^\gamma - \delta^\gamma_\alpha \theta^\beta,$$

and in particular

$$\frac{\partial}{\partial \theta^\alpha}(\theta^\beta \theta_\beta) = \varepsilon_{\beta\gamma} \frac{\partial}{\partial \theta^\alpha}(\theta^\beta \theta^\gamma) = \varepsilon_{\beta\gamma}(\delta^\beta_\alpha \theta^\gamma - \delta^\gamma_\alpha \theta^\beta)$$

$$= \varepsilon_{\alpha\gamma}\theta^\gamma - \varepsilon_{\beta\alpha}\theta^\beta = 2\varepsilon_{\alpha\gamma}\theta^\gamma = 2\theta_\alpha$$

Therefore

$$D_\alpha \Phi(x,\theta) = \psi_\alpha(x) + 2\theta_\alpha F(x) + i\theta^\beta \sigma^m_{\alpha\beta}\left(\partial_m \varphi(x) + \theta^\gamma \partial_m \psi_\gamma(x) + \theta^\gamma \theta_\gamma \partial_m F(x)\right)$$

Note that

$$\theta^\beta \theta^\gamma \theta_\gamma = \varepsilon_{\gamma\delta}\theta^\beta \theta^\gamma \theta^\delta = \varepsilon_{\gamma\delta}\theta^\gamma \theta^\delta \theta^\beta$$

In $\theta^\gamma \theta^\delta \theta^\beta$ at least one of the indices has to be repeated since there are only two to choose from Thus $\theta^\gamma \theta^\delta \theta^\beta = 0$

and

$$D_\alpha \Phi(x,\theta) = \psi_\alpha(x) + 2\theta_\alpha F(x) + i\theta^\beta \sigma^m{}_{\alpha\beta} \partial_m \varphi(x) + i\theta^\beta \sigma^m{}_{\alpha\beta} \theta^\gamma \partial_m \psi_\gamma(x)$$

We now consider the action

$$S(\Phi) = \int d^3x \int d^2\theta\, D^\alpha \Phi(x) D_\alpha \Phi(x)$$

Only the terms quadratic in θ survive the θ integration

$$D_\beta \Phi(x) D_\alpha \Phi(x) = 4\theta_\beta \theta_\alpha F(x)^2 - \theta^\gamma \theta^\delta \sigma^m{}_{\beta\gamma} \sigma^n{}_{\alpha\delta} \partial_m \varphi(x) \partial_n \varphi(x)$$

$$+ 2i\theta_\beta \theta^\gamma \sigma^m{}_{\alpha\gamma} \partial_m \varphi(x) F(x) + 2i\theta^\gamma \theta_\alpha \sigma^m{}_{\beta\gamma} \partial_m \varphi(x) F(x)$$

$$+ i\theta^\gamma \theta^\delta \sigma^m{}_{\alpha\gamma} \psi_\beta(x) \partial_m \psi_\delta(x) + i\theta^\gamma \theta^\delta \sigma^m{}_{\beta\gamma} \partial_m \psi_\delta(x) \psi_\alpha(x)$$

$$+$$

The Berezin integrals are of the form

$$\int d^2\theta\, \theta^\alpha \theta^\beta = \frac{\partial}{\partial\theta^1} \frac{\partial}{\partial\theta^2} (\theta^\alpha \theta^\beta)\Big|_{\theta=0} = \frac{\partial}{\partial\theta^1}\left(\delta^\alpha_2 \theta^\beta - \delta^\beta_2 \theta^\alpha\right)\Big|_{\theta=0}$$

$$= \delta^\alpha_2 \delta^\beta_1 - \delta^\beta_2 \delta^\alpha_1 = \varepsilon^{\beta\alpha}$$

Since the action is

$$S = \int d^3x \int d^2\theta\, D^\alpha \Phi(x) D_\alpha \Phi(x) = \int d^3x \int d^2\theta\, \varepsilon^{\alpha\beta} D_\beta \Phi(x) D_\alpha \Phi(x),$$

we have the Berezin integrals

$$\varepsilon^{\alpha\beta} \int d^2\theta\, 4\theta_\beta \theta_\alpha F(x)^2 = 4\varepsilon^{\alpha\beta} \varepsilon_{\beta\gamma} \varepsilon_{\alpha\delta} \left(\delta^\gamma_2 \delta^\delta_1 - \delta^\delta_2 \delta^\gamma_1\right) F(x)^2$$

$$= 4\delta^{\alpha}_{\gamma}\left(\varepsilon_{\alpha 1}\delta^{\gamma}_{2} - \varepsilon_{\alpha 2}\delta^{\gamma}_{1}\right)F(x)^2 = 4\left(\varepsilon_{21} - \varepsilon_{12}\right)F(x)^2 = -8F(x)^2,$$

$$\varepsilon^{\alpha\beta}\int d^2\theta\left(-\,\theta^{\gamma}\theta^{\delta}\sigma^{m}{}_{\beta\gamma}\sigma^{n}{}_{\alpha\delta}\,\partial_m\varphi(x)\partial_n\varphi(x)\right) =$$

$$-\,\varepsilon^{\alpha\beta}\left(\delta^{\gamma}_{2}\delta^{\delta}_{1} - \delta^{\delta}_{2}\delta^{\gamma}_{1}\right)\sigma^{m}{}_{\beta\gamma}\sigma^{n}{}_{\alpha\delta}\,\partial_m\varphi(x)\partial_n\varphi(x) =$$

$$-\,\varepsilon^{\alpha\beta}\left(\sigma^{m}{}_{\beta 2}\sigma^{n}{}_{\alpha 1} - \sigma^{m}{}_{\beta 1}\sigma^{n}{}_{\alpha 2}\right)\partial_m\varphi(x)\partial_n\varphi(x) =$$

$$-\left(\sigma^{m}{}_{22}\sigma^{n}{}_{11} - \sigma^{m}{}_{12}\sigma^{n}{}_{21} - \sigma^{m}{}_{21}\sigma^{n}{}_{12} + \sigma^{m}{}_{11}\sigma^{n}{}_{22}\right)\partial_m\varphi(x)\partial_n\varphi(x) =$$

$$-(1-0-0+1)\partial_0\varphi(x)^2 - (0-1-1+0)\partial_1\varphi(x)^2 - (-1-0-0-1)\partial_2\varphi(x)^2$$

$$= -2\,\partial_m\varphi(x)\partial^m\varphi(x),$$

$$\int d^2\theta\,\varepsilon^{\alpha\beta}\left(2i\,\theta_{\beta}\theta^{\gamma}\sigma^{m}{}_{\alpha\gamma}\,\partial_m\varphi(x)F(x) + 2i\,\theta^{\gamma}\theta_{\alpha}\sigma^{m}{}_{\beta\gamma}\,\partial_m\varphi(x)F(x)\right) =$$

$$\int d^2\theta\left(-\,\varepsilon^{\alpha\beta}2i\,\theta^{\gamma}\theta_{\beta}\sigma^{m}{}_{\alpha\gamma}\,\partial_m\varphi(x)F(x) + \varepsilon^{\alpha\beta}2i\,\theta^{\gamma}\theta_{\alpha}\sigma^{m}{}_{\beta\gamma}\,\partial_m\varphi(x)F(x)\right) =$$

$$\int d^2\theta\; 4i\,\varepsilon^{\alpha\beta}\theta^{\gamma}\theta_{\alpha}\sigma^{m}{}_{\beta\gamma}\,\partial_m\varphi(x)F(x) =$$

$$4i\,\varepsilon^{\alpha\beta}\varepsilon_{\alpha\delta}\,\varepsilon^{\delta\gamma}\sigma^{m}{}_{\beta\gamma}\,\partial_m\varphi(x)F(x) = -4i\,\delta^{\beta}_{\delta}\,\varepsilon^{\delta\gamma}\sigma^{m}{}_{\beta\gamma}\,\partial_m\varphi(x)F(x) =$$

$$= -4i\,\cancelto{0}{\varepsilon^{\beta\gamma}\sigma^{m}{}_{\beta\gamma}}\,\partial_m\varphi(x)F(x) = 0,$$

$$\int d^2\theta\,\varepsilon^{\alpha\beta}\left(i\,\theta^{\gamma}\theta^{\delta}\sigma^{m}{}_{\alpha\gamma}\,\psi_{\beta}(x)\partial_m\psi_{\delta}(x) + i\,\theta^{\gamma}\theta^{\delta}\sigma^{m}{}_{\beta\gamma}\,\partial_m\psi_{\delta}(x)\psi_{\alpha}(x)\right) =$$

$$\int d^2\theta\left(\varepsilon^{\alpha\beta}i\,\theta^{\gamma}\theta^{\delta}\sigma^{m}{}_{\alpha\gamma}\psi_{\beta}(x)\partial_m\psi_{\delta}(x) - \varepsilon^{\alpha\beta}i\,\theta^{\gamma}\theta^{\delta}\sigma^{m}{}_{\beta\gamma}\psi_{\alpha}(x)\partial_m\psi_{\delta}(x)\right) =$$

$$\int d^2\theta\; 2i\,\varepsilon^{\alpha\beta}\theta^{\gamma}\theta^{\delta}\sigma^{m}{}_{\alpha\gamma}\psi_{\beta}(x)\partial_m\psi_{\delta}(x) = 2i\,\varepsilon^{\alpha\beta}\varepsilon^{\delta\gamma}\sigma^{m}{}_{\alpha\gamma}\psi_{\beta}\partial_m\psi_{\delta}(x)$$

$$= -2i\,\psi^{\alpha}(x)\sigma^{m}{}_{\alpha\gamma}\partial_m\psi^{\gamma}(x)$$

Then

$$S(\Phi) = -2 \int d^3x \left(\partial_m \varphi(x) \partial^m \varphi(x) + i \psi^\alpha(x) \sigma^m{}_{\alpha\gamma} \partial_m \psi^\gamma(x) + 4F(x)^2 \right)$$

If we add to the action the term

$$\int d^3x \int d^2\theta\, m \Phi(x)^2 = \int d^3x \int d^2\theta\, m \left(-\theta^\alpha \theta^\beta \psi_\alpha(x) \psi_\beta(x) + 2\varepsilon_{\alpha\beta} \theta^\alpha \theta^\beta \varphi(x) F(x) \right)$$

$$= \int d^3x\, m\, \varepsilon^{\beta\alpha} \left(-\psi_\alpha(x) \psi_\beta(x) + 2\varepsilon_{\alpha\beta} \varphi(x) F(x) \right)$$

$$= \int d^3x\, m \left(\psi_\alpha(x) \psi^\alpha(x) + 2\delta^\beta_\beta \varphi(x) F(x) \right)$$

$$= \int d^3x \left(m \psi_\alpha(x) \psi^\alpha(x) + 4m \varphi(x) F(x) \right),$$

we obtain

$$S(\Phi) = -2 \int d^3x \left(\partial_m \varphi(x) \partial^m \varphi(x) - 2m \varphi(x) F(x) + i \psi^\alpha(x) \sigma^m_{\alpha\beta} \partial_m \psi^\beta(x) - \frac{1}{2} m \psi_\alpha(x) \psi^\alpha(x) \right.$$
$$\left. + 4F(x)^2 \right)$$

Varying F we find its equation of motion

$$8F(x) - 2m\varphi(x) = 0,$$

ie

$$F(x) = \frac{1}{4} m \varphi(x)$$

Then an equivalent action for φ and ψ is

$$S(\varphi, \psi) = -2 \int d^3x \left(\partial_m \varphi(x) \partial^m \varphi(x) - \frac{1}{2} m^2 \varphi(x)^2 + \frac{1}{4} m^2 \varphi(x)^2 \right.$$
$$\left. + i \psi^\alpha(x) \sigma^m_{\alpha\beta} \partial_m \psi^\beta(x) - \frac{1}{2} m \psi_\alpha(x) \psi^\alpha(x) \right)$$

$$= -2\int d^3x \; \Big(\partial_m \varphi(x)\partial^m \varphi(x) - \frac{1}{4} m^2 \, \varphi(x)^2$$

$$+ \, i\psi^\alpha(x)\, \sigma^m_{\alpha\beta}\, \partial_m \psi^\beta(x) \; - \frac{1}{2} m \psi_\alpha(x)\psi^\alpha(x) \Big)$$