1.4. Exercises

Exercise 1.1.

When translated to matrix form, eq (1.21) becomes

$$\bar{\sigma}^{m} = \varepsilon\, \sigma^{m\,T} \varepsilon^{T}, \quad \text{onde} \quad \varepsilon = \left[\varepsilon^{\alpha\beta}\right]^{2}_{\alpha,\beta=1} = \left[\varepsilon^{\dot{\alpha}\dot{\beta}}\right]^{2}_{\dot{\alpha},\dot{\beta}=1}$$

We can thus, by direct calculation, confirm

$$\bar{\sigma}^{0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{0}$$

$$\bar{\sigma}^{1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma^{1}$$

$$\bar{\sigma}^{2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma^{0}$$

$$\bar{\sigma}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma^3$$

Exercise 1.2.

Following our index contraction convention in (1.17)

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = (\psi_\alpha)^*(\chi^\alpha)^* = (\psi_\alpha)^\dagger(\chi^\alpha)^\dagger = (\chi^\alpha\psi_\alpha)^\dagger = -(\psi_\alpha\chi^\alpha)^\dagger$$

$$= -(\varepsilon_{\alpha\beta}\varepsilon^{\alpha\gamma}\psi^\beta\chi_\gamma)^\dagger$$

We notice that $\varepsilon_{\alpha\beta}\varepsilon^{\alpha\gamma} = -\delta^\gamma_\beta$ since $\varepsilon^2 = -I_2$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore

$$\bar{\psi}\bar{\chi} = -(-\delta^\gamma_\beta\psi^\beta\chi_\gamma)^\dagger = (\psi^\beta\chi_\beta)^\dagger = (\psi\chi)^\dagger.$$

On the other hand, to first order

$$\delta(\psi\xi) = \delta\psi\xi + \psi\delta\xi = \delta\psi^\alpha\xi_\alpha + \psi^\alpha\delta\xi_\alpha$$

$$= \varepsilon^{\alpha\beta}\delta\psi_\beta\xi_\alpha + \psi^\alpha\delta\xi_\alpha$$

$$= \varepsilon^{\alpha\beta} M_\beta{}^\gamma \psi_\gamma \xi_\alpha + \psi^\alpha M_\alpha{}^\beta \xi_\beta$$

$$= \varepsilon^{\alpha\beta} M^{mn} (\sigma_{mn})_\beta{}^\gamma \psi_\gamma \xi_\alpha + \psi^\alpha M^{mn} (\sigma_{mn})_\alpha{}^\beta \xi_\beta$$

$$= \varepsilon^{\alpha\beta} \varepsilon_{\gamma\delta} M^{mn} (\sigma_{mn})_\beta{}^\gamma \psi^\delta \xi_\alpha + \psi^\alpha M^{mn} (\sigma_{mn})_\alpha{}^\beta \xi_\beta$$

$$= M^{mn} \psi^\alpha \xi_\beta \left(\varepsilon^{\beta\delta} \varepsilon_{\gamma\alpha} (\sigma_{mn})_\delta{}^\gamma + (\sigma_{mn})_\alpha{}^\beta \right)$$

$$= M^{mn} \psi^\alpha \xi_\beta \left(\varepsilon^{\beta\delta} \varepsilon_{\gamma\delta} (\sigma_{mn})_\alpha{}^\gamma + (\sigma_{mn})_\alpha{}^\beta \right)$$

Used the hint

$$= M^{mn} \psi^\alpha \xi_\beta \left(-\delta^\beta_\gamma (\sigma_{mn})_\alpha{}^\gamma + (\sigma_{mn})_\alpha{}^\beta \right)$$

$$= M^{mn} \psi^\alpha \xi_\beta \left(-(\sigma_{mn})_\alpha{}^\beta + (\sigma_{mn})_\alpha{}^\beta \right) = 0.$$

Exercise 1.3.

By direct computation we have

$$c = i\gamma^0\gamma^2 = i \begin{pmatrix} 0 & \sigma^0 \\ \bar\sigma^0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ \bar\sigma^2 & 0 \end{pmatrix} = i \begin{pmatrix} \sigma^0 \bar\sigma^2 & 0 \\ 0 & \bar\sigma^0 \sigma^2 \end{pmatrix}$$

$$= i \begin{pmatrix} -\sigma^0\sigma^2 & 0 \\ 0 & \sigma^0\sigma^2 \end{pmatrix} = i \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}.$$

On the other hand

$$\gamma^0 \psi_M^* = \begin{pmatrix} 0 & \sigma^0 \\ \bar\sigma^0 & 0 \end{pmatrix} \begin{pmatrix} (\psi_\alpha)^* \\ (\bar\psi^{\dot\alpha})^* \end{pmatrix} = \begin{pmatrix} 0 & \sigma^0 \\ \bar\sigma^0 & 0 \end{pmatrix} \begin{pmatrix} \bar\psi_{\dot\alpha} \\ \psi^\alpha \end{pmatrix} =$$

$$= -\begin{pmatrix} \psi^{\alpha} \\ \bar{\psi}_{\alpha} \end{pmatrix}$$

We can thus apply it noticing that $i\bar{\sigma}^2 = -\varepsilon$ and $i\sigma^2 = \varepsilon$

$$c\gamma^0 (\Psi_M)^* = i \begin{pmatrix} 0 & \bar{\sigma}^2 \\ \sigma^2 & 0 \end{pmatrix} \begin{pmatrix} (\psi_{\alpha})^* \\ (\bar{\psi}^{\dot{\alpha}})^* \end{pmatrix} = \begin{pmatrix} \varepsilon_{\alpha\beta} \psi^{\beta} \\ \varepsilon^{\dot{\alpha}\dot{\beta}} \psi_{\dot{\beta}} \end{pmatrix} = \begin{pmatrix} \psi_{\alpha} \\ \psi^{\dot{\alpha}} \end{pmatrix} = \Psi_M.$$

Exercise 1.4.

$\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}$ is 1 if $\alpha = \delta$ and $\beta = \gamma$ but $\alpha \neq \beta$. On the other hand, it is -1 if $\alpha = \gamma$, $\beta = \delta$, and $\alpha \neq \beta$. In the other cases it is null. This can be expressed synthetically in the equation

$$\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta} = \delta^{\alpha}_{\delta}\,\delta^{\beta}_{\gamma} - \delta^{\alpha}_{\gamma}\,\delta^{\beta}_{\delta}.$$

Thus

$$-\frac{1}{2}\varepsilon^{\alpha\beta}\theta^{\gamma}\theta_{\gamma} = -\frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}\,\theta^{\gamma}\theta^{\delta} = -\frac{1}{2}\left(\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} - \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\right)\theta^{\gamma}\theta^{\delta}$$

$$= -\frac{1}{2}\left(\theta^{\beta}\theta^{\alpha} - \theta^{\alpha}\theta^{\beta}\right) = -\frac{1}{2}\left(-\theta^{\alpha}\theta^{\beta} - \theta^{\alpha}\theta^{\beta}\right) = \theta^{\alpha}\theta^{\beta},$$

proving the first equation. The second is then a simple consequence

$$\theta_\alpha\theta_\beta = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\theta^\gamma\theta^\delta = -\frac{1}{2}\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\varepsilon^{\gamma\delta}\,\theta\theta = -\frac{1}{2}\delta_\alpha^\delta\varepsilon_{\beta\delta}\,\theta\theta$$

$$= -\frac{1}{2}\varepsilon_{\beta\alpha}\,\theta\theta = \frac{1}{2}\varepsilon_{\alpha\beta}\,\theta\theta.$$

Finally, for the third we have an application of the last two

$$(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta) = \theta^\alpha\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\theta^\gamma\sigma^n_{\gamma\dot\delta}\bar\theta^{\dot\delta} = -\sigma^m_{\alpha\dot\beta}\sigma^n_{\gamma\dot\delta}\,\theta^\alpha\theta^\gamma\bar\theta^{\dot\beta}\bar\theta^{\dot\delta}$$

$$= \frac{1}{4}\sigma^m_{\alpha\dot\beta}\sigma^n_{\gamma\dot\delta}\,\varepsilon^{\alpha\gamma}\varepsilon^{\dot\beta\dot\delta}(\theta\theta)(\bar\theta\bar\theta)$$

$$= \frac{1}{4}\sigma^m_{\alpha\dot\beta}\bar\sigma^{n\dot\beta\alpha}(\theta\theta)(\bar\theta\bar\theta) = \frac{1}{4}(\sigma^m\bar\sigma^n)_\alpha{}^\alpha(\theta\theta)(\bar\theta\bar\theta)$$

(1.9)

$$= -\frac{1}{2}\eta^{mn}(\theta\theta)(\bar\theta\bar\theta).$$

Exercise 1.5.

We have, as an application of the previous exercise,

$$(\theta\varphi)(\theta\psi) = \theta^\alpha\varphi_\alpha\theta^\beta\psi_\beta = -\varphi_\alpha\psi_\beta\theta^\alpha\theta^\beta = \frac{1}{2}\varepsilon^{\alpha\beta}\varphi_\alpha\psi_\beta(\theta\theta) = -\frac{1}{2}\varphi^\beta\psi_\beta(\theta\theta)$$

$$= -\frac{1}{2}(\varphi\psi)(\theta\theta).$$

References I found useful:

H.J.W. Müller-Kirsten, A. Wiedemann, Introduction to Supersymmetry, 2nd ed., World Scientific.