3.2. Exercises

Exercise 3.1.

We have

$$\delta_1^Q \delta_2^Q F^m = \delta_1^Q \left(i\alpha_2^- \partial_- \psi_+^m - i\alpha_2^+ \partial_+ \psi_-^m \right)$$

$$= i\alpha_2^- \partial_- \left(\alpha_1^+ \partial_+ x^m + \alpha_1^- F^m \right) - i\alpha_2^+ \partial_+ \left(\alpha_1^- \partial_- x^m - \alpha_1^+ F^m \right)$$

$$= i\alpha_2^- \alpha_1^+ \partial_- \partial_+ x^m + i\alpha_2^- \alpha_1^- \partial_- F^m$$

$$- i\alpha_2^+ \alpha_1^- \partial_+ \partial_- x^m + i\alpha_2^+ \alpha_1^+ \partial_+ F^m$$

$$= i\left(\alpha_2^- \alpha_1^+ - \alpha_2^+ \alpha_1^- \right) \partial_+ \partial_- x^m + i\left(\alpha_2^- \alpha_1^- \partial_- + \alpha_2^+ \alpha_1^+ \partial_+ \right) F^m.$$

Then

$$[\delta_1^Q \delta_2^Q] = i\left(\cancel{\alpha_2^- \alpha_1^+} - \cancel{\alpha_2^+ \alpha_1^-} - \cancel{\alpha_1^- \alpha_2^+} + \cancel{\alpha_1^+ \alpha_2^-} \right) \partial_+ \partial_- x^m$$

$$+ i\left(\alpha_2^- \alpha_1^- \partial_- + \alpha_2^+ \alpha_1^+ \partial_+ - \alpha_1^- \alpha_2^- \partial_- - \alpha_1^+ \alpha_2^+ \partial_+ \right) F^m$$

$$= 2i\left(\alpha_2^+ \alpha_1^+ \partial_+ + \alpha_2^- \alpha_1^- \partial_- \right) F^m$$

$$= -2i\left(\alpha_1^+ \alpha_2^+ \partial_+ + \alpha_1^- \alpha_2^- \partial_- \right) F^m.$$

Exercise 3.2.

We have

$$\delta^Q x^m + i\kappa^+ \delta^Q \psi_+^m + i\kappa^- \delta^Q \psi_-^m + i\kappa^+\kappa^- \delta^Q F^m = \delta^Q \mathbb{X}^m$$

$$= \left(\alpha^+ Q^+ + \alpha^- Q^- \right) \mathbb{X}^m$$

$$=\left(\alpha^+\left(\frac{\partial}{\partial K^+}-iK^+\partial_+\right)+\alpha^-\left(\frac{\partial}{\partial K^-}-iK^-\partial_-\right)\right)\left(x^m+iK^+\psi_+^m+iK^-\psi_-^m+iK^+K^-F^m\right)$$

$$=\alpha^+\left(i\psi_+^m+iK^-F^m-iK^+\partial_+x^m+K^+K^-\partial_+\psi_-^m\right)$$

$$+\alpha^-\left(i\psi_-^m-iK^+F^m-iK^-\partial_-x^m+K^-K^+\partial_-\psi_+^m\right)$$

$$=\left(i\alpha^+\psi_+^m+i\alpha^-\psi_-^m\right)+iK^+\left(\alpha^+\partial_+x^m+\alpha^-F^m\right)+iK^-\left(\alpha^-\partial_-x^m-\alpha^+F^m\right)$$

$$+iK^+K^-\left(-i\alpha^+\partial_+\psi_-^m+i\alpha^-\partial_-\psi_+^m\right).$$

Comparing the coefficients of 1, iK^+, iK^- and iK^+K^- we obtain

$$\delta^Q x^m=i\alpha^+\psi_+^m+i\alpha^-\psi_-^m,$$

$$\delta^Q\psi_+^m=\alpha^+\partial_+x^m+\alpha^-F^m,$$

$$\delta^Q\psi_-^m=\alpha^-\partial_-x^m-\alpha^+F^m,$$

$$\delta^Q F=i\alpha^-\partial_-\psi_+^m-i\alpha^+\partial_+\psi_-^m,$$

which are precisely the transformations of x^m, ψ_+^m, ψ_-^m and F^m.

Exercise 3.3

We have for $\alpha,\beta\in\{\pm\}$ (no summation convention)

$$Q_\alpha Q_\beta=\left(\frac{\partial}{\partial K^\alpha}-iK^\alpha\partial_\alpha\right)\left(\frac{\partial}{\partial K^\beta}-iK^\beta\partial_\beta\right)$$

$$=\frac{\partial}{\partial K^\alpha}\frac{\partial}{\partial K^\beta}-iK^\alpha\partial_\alpha\frac{\partial}{\partial K^\beta}-i\delta_\alpha^\beta\partial_\beta+iK^\beta\frac{\partial}{\partial K^\alpha}\partial_\beta-iK^\alpha K^\beta\partial_\alpha\partial_\beta.$$

Then

$$\{Q_\alpha, Q_\beta\} = \cancel{\frac{\partial}{\partial K^\alpha}\frac{\partial}{\partial K^\beta}} - \cancel{i K^\alpha \partial_\alpha \frac{\partial}{\partial K^\beta}} - i\delta^\beta_\alpha \partial_\beta + \cancel{i K^\beta \frac{\partial}{\partial K^\alpha}\partial_\beta} - \cancel{i K^\alpha K^\beta \partial_\alpha \partial_\beta}$$

$$+ \cancel{\frac{\partial}{\partial K^\beta}\frac{\partial}{\partial K^\alpha}} - \cancel{i K^\beta \partial_\beta \frac{\partial}{\partial K^\alpha}} - i\delta^\alpha_\beta \partial_\alpha + \cancel{i K^\alpha \frac{\partial}{\partial K^\beta}\partial_\alpha} - \cancel{i K^\beta K^\alpha \partial_\beta \partial_\alpha}.$$

$$= -2i\,\delta_{\alpha\beta}\partial_\alpha.$$

On the other hand $\sigma^a_{\alpha\beta} = 0$ if $\alpha \neq \beta$. Therefore

$$\{Q_\alpha, Q_\beta\} = 0 = 2\sigma^a_{\alpha\beta} P_a$$

if $\alpha \neq \beta$. Finally,

$$2\sigma^a_{11} P_a = 2(P_0 + P_1) = P_+ + \cancel{P_-} + P_+ - \cancel{P_-} = 2P_+ = -2i\partial_+ = \{Q_+, Q_+\} = \{Q_1, Q_1\},$$

$$2\sigma^a_{22} P_a = 2(P_0 - P_1) = \cancel{P_+} + P_- - \cancel{P_+} + P_- = 2P_- = -2i\partial_- = \{Q_-, Q_-\} = \{Q_2, Q_2\}.$$

Exercise 3.4.

There is a problem in the definition of $\bar\sigma$ since $\bar\sigma^{a\beta\alpha} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\gamma}\sigma^a_{\gamma\delta}$ has to many γ indices. We will assume that the correct formula is $\bar\sigma^{a\beta\alpha} := \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\sigma^a_{\gamma\delta}$. Moreover, we will take $\partial_0 = \partial_\tau$ and $\partial_1 = \partial_\sigma$.

With this convention, in matrix notation we have $\bar\sigma = \varepsilon(\sigma^a)^T\varepsilon^T$.

By comparison with Exercise 1.1., in which σ^0 and σ^3 where equal to our current σ^0 and σ^1, we conclude that

$\bar{\sigma}^0 = \sigma^0$ and $\bar{\sigma}^1 = -\sigma^1$. Then

$$\psi_\alpha^m \bar{\sigma}^{a\alpha\beta} \partial_a \psi_{\beta m} = \sum_{\alpha=1}^{2} \psi_\alpha^m \bar{\sigma}^{a\alpha\alpha} \partial_a \psi_{\alpha m} = \psi_1^m \partial_0 \psi_{1m} + \psi_2^m \partial_0 \psi_{2m} - \psi_1^m \partial_1 \psi_{1m} + \psi_2^m \partial_1 \psi_{2m}$$

$$= \psi_1^m (\partial_0 - \partial_1) \psi_{1m} + \psi_2^m (\partial_0 + \partial_1) \psi_{2m}$$

$$= \psi_+^m (\partial_\tau - \partial_\sigma) \psi_{+m} + \psi_-^m (\partial_\tau + \partial_\sigma) \psi_{-m} = \psi_+^m \partial_- \psi_{+m} + \psi_-^m \partial_+ \psi_{-m}.$$

Therefore

$$\int d\tau d\sigma \left(\psi_+{}^m \partial_- \psi_{+m} + \psi_-{}^m \partial_+ \psi_{-m} \right) = \int d\tau d\sigma = \psi_\alpha{}^m \bar{\sigma}^{a\alpha\beta} \psi_{\beta m}.$$

Exercise 3.5.

We have

$$D_+ X^m = \left(\frac{\partial}{\partial \kappa^+} + i\kappa^+ \partial_+ \right) \left(x^m + i\kappa^+ \psi_+^m + i\kappa^- \psi_-^m + i\kappa^+ \kappa^- F^m \right)$$

$$= i\psi_+{}^m + i\kappa^- F^m + i\kappa^+ \partial_+ x^m - \kappa^+ \kappa^- \partial_+ \psi_-{}^m,$$

and

$$D_- X^m = \left(\frac{\partial}{\partial \kappa^-} + i\kappa^- \partial_- \right) \left(x^m + i\kappa^+ \psi_+^m + i\kappa^- \psi_-^m + i\kappa^+ \kappa^- F^m \right)$$

$$= i\psi_-{}^m - i\kappa^+ F^m + i\kappa^- \partial_- x^m - \kappa^- \kappa^+ \partial_- \psi_+{}^m.$$

Therefore

$$D_+ \mathbb{X}^m D_- \mathbb{X}_m = -\psi_+^m \psi_{-m} + \psi_+{}^m \kappa^+ F_m - \psi_+{}^m \kappa^- \partial_- x^m - i\psi_+{}^m \kappa^- \kappa^+ \partial_- \psi_+{}^m$$

$$- \kappa^- F^m \psi_{-m} + \kappa^- F^m \kappa^+ F_m$$

$$- \kappa^+ \partial_+ x^m \psi_{-m} - \kappa^+ \partial_+ x^m \kappa^- \partial_- x_m$$

$$- i\kappa^+ \kappa^- \partial_+ \psi_-^m \psi_{-m}.$$

Through Berezin integration we obtain

$$\int d\kappa^+ d\kappa^- \, D_+ \mathbb{X}^m D_- \mathbb{X}_m = \frac{\partial}{\partial \kappa^+} \frac{\partial}{\partial \kappa^-} D_+ \mathbb{X}^m D_- \mathbb{X}_m \Big|_{\kappa^\pm = 0}$$

$$= \frac{\partial}{\partial \kappa^+} \Big(\psi_+{}^m \partial_- x^m + i\psi_+{}^m \kappa^+ \partial_- \psi_+{}^m$$

$$- F^m \psi_{-m} + F^m \kappa^+ F_m$$

$$+ \kappa^+ \partial_+ x^m \partial_- x_m$$

$$+ i\kappa^+ \partial_+ \psi_-{}^m \psi_{-m} \Big) \Big|_{\kappa^\pm = 0}$$

$$= -i\psi_+{}^m \partial_- \psi_+{}^m + F^m F_m + \partial_+ x^m \partial_- x_m + i\partial_+ \psi_-{}^m \psi_{-m}$$

We thus have

$$\frac{T}{2} \int d\tau d\sigma \int d\kappa^+ d\kappa^- D_+ \mathbb{X}^m D_- \mathbb{X}^m$$

$$= \frac{T}{2} \int d\tau d\sigma \left(\partial_+ x^m \partial_- x_m - i\psi_+{}^m \partial_- \psi_{+m} - i\psi_{-m} \partial_+ \psi_{-m} + F^m F_m \right).$$