Iván Mauricio Burbano Aldana

Prof: Nathan Berkovits

Supersymmetry

Instituto de Física Teorica - UNESP

Exercise 7.1.

Consider an R-symmetry transformation of the form (6.3) and (6.4)

$$\Phi(y,\theta) \longmapsto e^{2inK}\,\Phi(y, e^{-iK}\theta),$$

$$\bar{\Phi}(\bar{y},\bar{\theta}) \longmapsto e^{-2inK}\,\bar{\Phi}(\bar{y}, e^{iK}\bar{\theta}).$$

Then

$$\int d^4x\, d^2\theta\, g\, \Phi(x,\theta)^3 \longmapsto \int d^4x\, d^2\theta\, g\, e^{6inK}\, \Phi(x, e^{-iK}\theta)^3$$

$$= \int d^4x\, d^2\theta\, g\, e^{i(6n-2)K}\, \Phi(x,\theta)^3.$$

Thus the cubic term is invariant if and only if $n=\frac{1}{3}$.

Similarly,

$$\int d^4x\, d^2\bar{\theta}\, \bar{\Phi}(x,\bar{\theta})^3 \longmapsto \int d^4x\, d^2\bar{\theta}\, e^{-2inK}\, \bar{\Phi}(x, e^{iK}\bar{\theta})^3$$

$$= \int d^4x\, d^2\bar{\theta}\, \bar{\Phi}(x,\bar{\theta})^3.$$

We assume similar R-symmetry transformations for Ξ

$$\Xi(y,\theta) \longmapsto e^{2imK}\,\Xi(y, e^{-iK}\theta)$$

$$\bar{\Xi}(\bar{y},\bar{\theta}) \longmapsto e^{-2imK}\,\bar{\Xi}(\bar{y}, e^{iK}\bar{\theta}).$$

Then

$$\int d^4x\, d^2\theta\, \Phi(x,\theta)\Sigma(x,\theta) \longmapsto \int d^4x\, d^2\theta\, e^{2inK} e^{2imK}\, \Phi(x, e^{-iK}\theta)\Sigma(x, e^{-iK}\theta)$$

$$= \int d^4x\, d^2\theta\, e^{i(2n+2m-2)K}\, \Phi(x,\theta)\Sigma(x,\theta)$$

This term is invariant if and only if

$$m = 1-n = \frac{2}{3}.$$

Then we automatically have

$$\int d^4x\, d^2\bar\theta\, \bar\Phi(x,\bar\theta)\bar\Sigma(x,\bar\theta) \longmapsto \int d^4x\, d^2\bar\theta\, e^{-2inK} e^{-2imK}\, \bar\Phi(x, e^{iK}\bar\theta)\bar\Sigma(x, e^{iK}\bar\theta)$$

$$= \int d^4x\, d^2\bar\theta\, e^{i(-2n-2m+2)K}\, \bar\Phi(x,\bar\theta)\bar\Sigma(x,\bar\theta)$$

$$= \int d^4x\, d^2\bar\theta\, \bar\Phi(x,\bar\theta)\bar\Sigma(x,\bar\theta).$$

For the last chiral superfield $W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha$, we note that since it is also chiral, it makes sense that

$$W_\alpha(y,\theta) \longmapsto e^{2ilK} W_\alpha(y, e^{-iK}\theta),$$

$$\bar W_\alpha(y,\theta) \longmapsto e^{-2ilK}\, \bar W_\alpha(\bar y, e^{iK}\bar\theta).$$

Then

$$\int d^4x\, d^2\theta\, W^\alpha W_\alpha \longmapsto \int d^4x\, d^2\theta\, e^{4ilK}\, W^\alpha(x, e^{-iK}\theta) W_\alpha(x, e^{-iK}\theta)$$

$$= \int d^4x\, d^2\theta\, e^{i(4l-2)K}\, W^\alpha(x,\theta) W_\alpha(x,\theta).$$

This is invariant if and only if

$$l = 1/2 .$$

Then

$$\int d^4x\, d^2\bar{\theta}\; \bar{W}(x,\bar{\theta})^2 \longmapsto \int d^4x\, d^2\bar{\theta}\; e^{-2iK}\; \bar{W}(x, e^{iK}\bar{\theta})^2$$

$$= \int d^4x\, d^2\bar{\theta}\; \bar{W}(x,\bar{\theta})^2 .$$

These transformations are achieved if we let V be a scalar under R-symmetry

$$V(x,\theta,\bar{\theta}) \longmapsto V(x, e^{-iK}\theta, e^{iK}\bar{\theta}) =: V'(x,\theta,\bar{\theta}).$$

To see this we note that for a superfield

$$f(x,\theta,\bar{\theta}) = g(x, e^{-iK}\theta, e^{iK}\bar{\theta})$$

we have

$$\frac{\partial f}{\partial \theta^\alpha}(x,\theta,\bar{\theta}) = \frac{\partial g}{\partial \theta^\alpha}(x, e^{-iK}\theta, e^{iK}\bar{\theta})\, e^{-iK}.$$

Then, given than in R-symmetry $\bar{\theta} \to e^{-iK}\bar{\theta}$, then

$$D_\alpha f(x,\theta,\bar{\theta}) = \frac{\partial}{\partial \theta^\alpha} g(x, e^{-iK}\theta, e^{iK}\bar{\theta}) + i(\sigma^m e^{-iK}\bar{\theta})_\alpha \partial_m g(x, e^{-iK}\theta, e^{iK}\bar{\theta})$$

$$= e^{-iK} D_\alpha g(x, e^{-iK}\theta, e^{iK}\bar{\theta}).$$

Similarly

$$\bar{D}^{\dot{\alpha}} f(x,\theta,\bar{\theta}) = e^{iK}\, \bar{D}^{\dot{\alpha}} g(x, e^{-iK}\theta, e^{iK}\bar{\theta}).$$

Thus

$$W_\alpha^{*}(x,\theta,\bar{\theta}) \longmapsto e^{iK}\, W_\alpha(x, e^{-ik}\theta, e^{ik}\theta).$$

Exercise 7.2

In the WZ gauge we have

$$-(\theta\sigma^m\bar\theta)\delta_Q A_m(x) + i\theta^2\bar\theta\delta_Q\bar\lambda(x) - i\bar\theta^2\theta\delta_Q\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2\delta_Q d(x)$$

$$= \delta_Q V(x) = (\xi Q + \bar\xi\bar Q)V(x,\theta,\bar\theta) + \Lambda(x+i\theta\sigma\bar\theta,\theta) + \bar\Lambda(x-i\theta\sigma\bar\theta,\bar\theta)$$

where

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^m},$$

$$\bar Q^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha\frac{\partial}{\partial x^m}.$$

The gauge transformation

$$\Lambda(y,\theta) = a(y) + \sqrt{2}\,\theta b(y) + \theta\theta f(y),$$

will later be choosen to ensure that the result is indeed in the WZ gauge. We have

$$Q_\alpha V(x,\theta,\bar\theta) = -(\sigma^m\bar\theta)_\alpha A_m(x) + 2i\theta_\alpha\bar\theta\bar\lambda(x) - i\bar\theta^2\lambda_\alpha(x)$$

$$+ \theta_\alpha\bar\theta^2 d(x) + \underbrace{i(\theta\sigma^m\bar\theta)(\sigma^n\bar\theta)_\alpha}\partial_n A_m(x)$$

$$\theta^\beta\sigma^m{}_{\beta\dot\beta}\bar\theta^{\dot\beta}\sigma^n{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$$

$$= \frac{1}{2}\varepsilon^{\dot\beta\dot\alpha}\bar\theta^2\theta^\beta\sigma^m{}_{\beta\dot\beta}\sigma^n{}_{\alpha\dot\alpha}$$

$$= \frac{1}{2}\bar\theta^2\bar\sigma^{m\dot\alpha\beta}\theta_\beta\sigma^n{}_{\alpha\dot\alpha}$$

$$= \underbrace{\frac{1}{2}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta}$$

$$= \frac{1}{2}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta\partial_n A_m(x)$$

$$= \frac{1}{4}\bar\theta^2(-2\eta^{nm}\delta_\alpha{}^\beta)\theta_\beta\partial_n A_m(x)$$

$$+ \frac{1}{4}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta F_{nm}(x)$$

$$= -\frac{1}{2}\bar\theta^2\theta_\alpha\partial^m A_m(x) + \frac{1}{4}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta F_{nm}(x)$$

$$\underbrace{+(\sigma^m\bar\theta)_\alpha\,\theta^2\bar\theta\partial_m\bar\lambda(x)}$$

$$\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta}\partial_m\bar\lambda^{\dot\beta}(x) = -\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\partial_m\bar\lambda_{\dot\beta}(x)$$

$$= -\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar\lambda_{\dot\beta}(x)\theta^2$$

$$= -\frac{1}{2}\theta^2\bar\theta^2(\sigma^m\partial_m\bar\lambda(x))_\alpha .$$

$$= -(\sigma^m\bar\theta)_\alpha A_m(x) + 2i\theta_\alpha\bar\theta\bar\lambda(x) - i\bar\theta^2\lambda_\alpha(x) + \theta_\alpha\bar\theta^2 d(x)$$

$$-\frac{i}{2}\bar\theta^2\theta_\alpha\partial^m A_m(x) + \frac{1}{4}\bar\theta^2(\sigma^n\bar\sigma^m)_\alpha{}^\beta\theta_\beta F_{nm}(x) - \frac{1}{2}\theta^2\bar\theta^2(\sigma^m\partial_m\bar\lambda(x))_\alpha$$

Similarly

$$\bar Q^{\dot\alpha}V(x,\theta,\bar\theta) = \underbrace{\theta^\alpha\sigma^m{}_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\alpha}A_m(x)}_{\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha A_m(x)} + i\theta^2\bar\lambda^{\dot\alpha}(x) - 2i\bar\theta^{\dot\alpha}\theta\lambda(x)$$

$$+\theta^2\bar\theta^{\dot\alpha}d(x) + \underbrace{i(\bar\sigma^n\theta)^{\dot\alpha}(\theta\sigma^m\bar\theta)\partial_n A_m(x)}$$

$$= \bar\sigma^{n\dot\alpha\alpha}\theta_\alpha\theta^\beta\sigma^m{}_{\beta\dot\beta}\bar\theta^{\dot\beta}\partial_n A_m(x) = -\frac{1}{2}\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta}\theta^2\bar\sigma^{n\dot\alpha\alpha}\sigma^m{}_{\beta\dot\beta}\bar\theta^{\dot\beta}\partial_n A_m(x)$$

$$= -\frac{1}{2}\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\partial_n A_m(x) = -\frac{1}{4}\theta^2(-2\eta^{nm})\delta_{\dot\beta}{}^{\dot\alpha}\bar\theta^{\dot\beta}\partial_n A_m(x)$$

$$-\frac{1}{4}\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}F_{nm}(x)$$

$$= \frac{1}{2}\theta^2\bar\theta^{\dot\alpha}\partial^m A_m(x) - \frac{1}{4}\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}F_{nm}(x)$$

$$\underbrace{-\bar\theta^2(\bar\sigma^m\theta)^{\dot\alpha}\,\theta\partial_m\lambda(x)}$$

$$\bar\theta^2\bar\sigma^{m\dot\alpha\alpha}\theta_\alpha\theta^\beta\partial_m\lambda_\beta(x) = -\frac{1}{2}\varepsilon_{\alpha\beta}\theta^2\bar\theta^2\bar\sigma^{m\dot\alpha\alpha}\partial_m\lambda^\beta(x) = -\frac{1}{2}\theta^2\bar\theta^2(\bar\sigma^m\partial_m\lambda(x))^{\dot\alpha}$$

$$= i\,(\bar\sigma^m\theta)^{\dot\alpha} A_m(x) + i\theta^2\bar\lambda^{\dot\alpha}(x) - 2i\,\bar\theta^{\dot\alpha}\theta\lambda(x) + \theta^2\bar\theta^{\dot\alpha} d(x)$$

$$+ \frac{i}{2}\,\theta^2\bar\theta^{\dot\alpha}\partial^m A_m(x) - \frac{i}{4}\,\theta^2(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta} F_{nm}(x)$$

$$+ \frac{1}{2}\,\theta^2\bar\theta^2(\bar\sigma^m\partial_m\lambda(x))^{\dot\alpha}.$$

Therefore

$$\delta_Q V = -(\xi\sigma^m\bar\theta)A_m(x) + 2i(\xi\theta)(\bar\theta\bar\lambda(x)) - i\bar\theta^2(\xi\lambda(x)) + (\xi\theta)\bar\theta^2 d(x)$$

$$- \frac{i}{2}\,\bar\theta^2(\xi\theta)\partial^m A_m(x) + \frac{i}{4}\,\bar\theta^2(\xi\sigma^n\bar\sigma^m\theta)F_{nm}(x) - \frac{1}{2}\,\theta^2\bar\theta^2(\xi\sigma^m\partial_m\bar\lambda(x))$$

$$+ (\bar\xi\bar\sigma^m\theta)A_m(x) + i\theta^2(\bar\xi\bar\lambda(x)) - 2i(\bar\xi\bar\theta)(\theta\lambda(x)) + \theta^2(\bar\xi\bar\theta)d(x)$$

$$+ \frac{i}{2}\,\theta^2(\bar\xi\bar\theta)\partial^m A_m(x) - \frac{i}{4}\,\theta^2(\bar\xi\bar\sigma^n\sigma^m\bar\theta)F_{nm}(x) + \frac{1}{2}\theta^2\bar\theta^2(\bar\xi\bar\sigma^m\partial_m\lambda(x))$$

$$+ a(x) + i(\theta\sigma^m\bar\theta)\partial_m a(x) + \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\Box a(x)$$

$$+ \sqrt{2}\,\theta b(x) + i\sqrt{2}\,\underbrace{(\theta\sigma^m\bar\theta)(\theta\partial_m b(x))} + \theta\theta f(x)$$

$$\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\partial_m b_\beta(x) = +\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m b_\beta(x)$$

$$= -\frac{1}{2}\,\theta^2(\partial_m b\sigma^m\bar\theta)$$

$$+ \bar a(x) - i(\theta\sigma^m\bar\theta)\partial_m\bar a(x) + \frac{1}{4}\theta^2\bar\theta^2\Box\bar a(x)$$

$$+ \sqrt{2}\,\bar\theta\bar b(x) - i\sqrt{2}\,\underbrace{(\theta\sigma^m\bar\theta)(\bar\theta\partial_m\bar b(x))} + \bar\theta\bar\theta\bar f(x)$$

$$\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta}\partial_m\bar b^{\dot\beta}(x) = -\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar b_{\dot\beta}(x)$$

$$= -\frac{1}{2}\,\bar\theta^2(\theta\sigma^m\partial_m\bar b(x)).$$

By comparison on the $1, \theta, \bar\theta, \theta^2$ and $\bar\theta^2$ terms, we see that we need to choose our gauge transformation such that

$$a(x) = -\bar{a}(x)$$

$$-\sqrt{2}\, b_{\alpha}^{\alpha}(x) = \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{m\dot{\alpha}\alpha} A_m(x)$$

$$\sqrt{2}\, \bar{b}_{\dot{\alpha}}(x) = \xi^{\alpha} \sigma^{m}{}_{\alpha\dot{\alpha}} A_m(x)$$

$$f(x) = -i\left(\bar{\xi}\bar{\lambda}(x)\right)$$

$$\bar{f}(x) = i\left(\xi\lambda(x)\right).$$

By comparing the $\theta\sigma^m\bar{\theta}$

$$-(\theta\sigma^m\bar{\theta})\delta_Q A_m(x) = 2i(\xi\theta)(\bar{\theta}\bar{\lambda}(x)) - 2i(\bar{\xi}\bar{\theta})(\theta\lambda(x)) + i(\theta\sigma^m\bar{\theta})\partial_m a(x) - i(\theta\sigma^m\bar{\theta})\partial_m\bar{a}(x)$$

We thus have

$$\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\delta_Q A^n(x) = -(\theta\sigma^n\bar{\theta})(\theta\sigma^m\bar{\theta})\delta_Q A_m(x)$$

$$= 2i\,\underbrace{(\theta\sigma^n\bar{\theta})(\xi\theta)(\bar{\theta}\bar{\lambda}(x))}$$

$$\theta^{\alpha}\sigma^{n}{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\xi^{\beta}\theta_{\beta}\bar{\theta}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}}(x)$$

$$\parallel$$

$$\frac{1}{4}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\theta^2\bar{\theta}^2\sigma^{n}{}_{\alpha\dot{\alpha}}\xi_{\beta}\bar{\lambda}_{\dot{\beta}}(x)$$

$$\parallel$$

$$\frac{1}{4}\theta^2\bar{\theta}^2\bar{\sigma}^{n\dot{\beta}\beta}\xi_{\beta}\bar{\lambda}_{\dot{\beta}}(x) = -\frac{1}{4}\theta^2\bar{\theta}^2(\bar{\lambda}(x)\bar{\sigma}^n\xi)$$

$$-2i\,\underbrace{(\theta\sigma^n\bar{\theta})(\bar{\xi}\bar{\theta})(\theta\lambda(x))} \qquad -\frac{i}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^n\underbrace{(a(x)-\bar{a}(x))}_{2a(x)}$$

$$-\frac{1}{4}\theta^2\bar{\theta}^2(\bar{\xi}\bar{\sigma}^n\lambda(x))$$

$$= \frac{i}{2}\left(\bar{\xi}\bar{\sigma}^n\lambda(x) - \bar{\lambda}(x)\bar{\sigma}^n\xi\right)$$

$$= \frac{i}{2}\left(\bar{\xi}\bar{\sigma}^n\lambda(x) + \xi\sigma^n\bar{\lambda}(x)\right)$$

We conclude that

$$\boxed{\delta_Q A^n(x) = i\bar{\xi}\bar{\sigma}^n\lambda(x) + i\xi\sigma^n\bar{\lambda}(x) - 2i\partial^n a(x)}$$

Similarly, by comparing the $\bar{\theta}^2\theta$ terms we have (9)

$$i\bar{\theta}^2(\theta\delta_Q\lambda(x)) = (\xi\theta)\bar{\theta}^2 d(x) - \cancel{\frac{i}{2}\bar{\theta}^2(\xi\theta)\partial^m A_m(x)} + \frac{i}{4}\bar{\theta}^2(\xi\sigma^n\bar{\sigma}^m\theta)F_{nm}(x)$$

$$+ \underbrace{\frac{i}{\sqrt{2}}\bar{\theta}^2(\theta\sigma^m\partial_m\bar{b}(x))}$$

$$\frac{i}{\sqrt{2}}\bar{\theta}^2\theta^\alpha\sigma^m{}_{\alpha\dot{\alpha}}\frac{1}{\sqrt{2}}\xi^\beta\sigma^n{}_{\beta\dot{\beta}}\varepsilon^{\dot{\alpha}\dot{\beta}}\partial_m A_n(x)$$

$$= -\frac{i}{2}\bar{\theta}^2\theta^\alpha\sigma^m{}_{\alpha\dot{\alpha}}\xi_\beta\bar{\sigma}^{n\dot{\alpha}\beta}\partial_m A_n(x)$$

$$= -\frac{i}{2}\bar{\theta}^2\theta^\alpha(\sigma^m\bar{\sigma}^n)_\alpha{}^\beta\xi_\beta\partial_m A_n(x)$$

$$= -\frac{i}{4}\bar{\theta}^2\theta^\alpha(-2\eta^{mn}\delta_\alpha{}^\beta)\xi_\beta\partial_m A_n(x) - \frac{i}{4}\bar{\theta}^2\theta^\alpha(\sigma^m\bar{\sigma}^n)_\alpha{}^\beta\xi_\beta F_{mn}(x)$$

$$= \cancel{\frac{i}{2}\bar{\theta}^2(\theta\xi)\partial^m A_m(x)} - \frac{i}{4}\bar{\theta}^2(\theta\sigma^m\bar{\sigma}^n\xi)F_{mn}(x)$$

Notice that

$$\frac{i}{4}\bar{\theta}^2(\xi\sigma^n\bar{\sigma}^m\theta)F_{nm}(x) = \frac{i}{8}\bar{\theta}^2(\xi\sigma^{[n}\bar{\sigma}^{m]}\theta)F_{nm}(x)$$

$$= \frac{i}{2}\bar{\theta}^2(\xi\sigma^{nm}\theta)F_{nm}(x).$$

Moreover, using the hint of Exercise 1.2

$$(\xi\sigma^{nm}\theta) = \xi^\alpha\sigma^{nm}{}_\alpha{}^\beta\theta_\beta = -\xi^\alpha\sigma^{nm}{}_{\alpha\beta}\theta^\beta$$

$$= -\xi^\alpha\sigma^{nm}{}_{\beta\alpha}\theta^\beta = \theta^\beta\sigma^{nm}{}_{\beta\alpha}\xi^\alpha$$

$$= -(\theta\sigma^{nm}\xi).$$

Thus

$$-i\bar{\theta}^2(\theta\delta_Q\lambda(x)) = (\xi\theta)\bar{\theta}^2 d(x) + i\bar{\theta}^2(\xi\sigma^{nm}\theta)F_{nm}(x),$$

i.e.

$$\delta_Q \lambda^\alpha(x) = -\xi^\beta \sigma^{nm}{}_\beta{}^\alpha F_{nm}(x) + i\xi^\alpha d(x),$$

or

$$\boxed{\delta_Q \lambda_\alpha(x) = \sigma^{nm}{}_\alpha{}^\beta \xi_\beta F_{nm}(x) + i\xi_\alpha d(x).}$$

Repeating with $\theta^2\bar\theta$, we have

$$i\theta^2\bar\theta\delta_Q\bar\lambda(x) = \theta^2(\bar\xi\bar\theta)d(x) + \cancel{\tfrac{i}{2}\theta^2(\bar\xi\bar\theta)\partial^m A_m(x)} - \tfrac{i}{4}\theta^2(\bar\xi\bar\sigma^n\sigma^m\bar\theta)F_{nm}(x)$$

$$\underline{-\tfrac{i}{\sqrt2}\theta^2(\partial_m b\sigma^m\bar\theta)}$$

$$\frac{i}{\sqrt2}\theta^2\frac{1}{\sqrt2}\bar\xi_{\dot\alpha}\bar\sigma^{n\dot\alpha\alpha}\partial_m A_n(x)\sigma^m{}_{\alpha\dot\beta}\bar\theta^{\dot\beta}$$

$$= \frac{i}{2}\theta^2\bar\xi_{\dot\alpha}(\bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\partial_m A_n(x)$$

$$= \cancel{-\frac{i}{2}\theta^2\bar\xi_{\dot\alpha}\bar\theta^{\dot\alpha}\partial^m A_m(x)} + \frac{i}{4}\theta^2(\bar\xi\bar\sigma^n\sigma^m\bar\theta)F_{mn}(x)$$

$$= \theta^2(\bar\xi\bar\theta)d(x) - i\theta^2\bar\xi_{\dot\alpha}\bar\sigma^{nm\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}F_{nm}(x).$$

Thus

$$\boxed{\delta_Q\bar\lambda^{\dot\alpha}(x) = \bar\sigma^{nm\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}F_{nm}(x) - i\bar\xi^{\dot\alpha}d(x).}$$

Finally, the terms $\theta^2\bar\theta^2$ give

$$\frac12\theta^2\bar\theta^2\delta_Q d(x) = -\frac12\theta^2\bar\theta^2(\xi\sigma^m\partial_m\bar\lambda(x)) + \frac12\theta^2\bar\theta^2(\bar\xi\bar\sigma^m\partial_m\lambda(x))$$

$$+ \cancel{\frac14\theta^2\bar\theta^2\Box(a+\bar a)(x)} \to 0.$$

Therefore,

$$\boxed{\delta_Q d(x) = \bar{\xi}\bar{\sigma}^m \partial_m \lambda(x) - \xi\sigma^m \partial_m \bar{\lambda}(x).}$$

By fixing the gauge transformation to $a=0$, we obtain (6.20).

Exercise 7.3.

We begin from the transformation rule (7.22)

$$A_m \longmapsto M A_m M^+ + i M \partial_m M^+.$$

To lighten the notation we will work without Matrix indices. We then have

$$\begin{aligned}
F_{mn} \mapsto\ & \partial_m (M A_n M^+ + i M \partial_n M^+) - m \longleftrightarrow n \\
& - i\left[M A_m M^+ + i M \partial_m M^+, M A_n M^+ + i M \partial_n M^+\right] \\
=\ & M \partial_m A_n M^+ + \partial_m M A_n M^+ + M A_n \cancel{\partial_m} M^+ + i \partial_m M \partial_n M^+ + i M \cancel{\partial_m \partial_n} M^+ \\
& - m \cancel{\longleftrightarrow} n \\
& - i\big(M A_m \cancel{M^+ M} A_n M^+ - M A_n \cancel{M^+ M} A_m M^+ \\
& + i M A_m \cancel{M^+ M} \partial_n M^+ - i M \partial_n M^+ M A_m M^+ \\
& + i M \partial_m M^+ M A_n M^+ - i M A_n \cancel{M^+ M} \partial_m M^+ \\
& - M \partial_m M^+ M \partial_n M^+ + M \partial_n M^+ M \partial_m M^+\big).
\end{aligned}$$

To further proceed we observe that the restriction $MM^+ = I$ restricts

$$0 = \partial_m I = \partial_m M M^+ + M \partial_m M^+.$$

Thus $\partial_m M^+ = -M^+ \partial_m M M^+$ and

$$F_{mn} \longmapsto M(\partial_m A_n - \partial_n A_m)M^\dagger + \cancel{\partial_m M A_n} M^\dagger + i\,\cancel{\partial_m M \partial_n} M^\dagger$$

$$- \cancel{\partial_n M A_m} M^\dagger - i\,\partial_n \cancel{M \partial_m} M^\dagger$$

$$- i M[A_m, A_n]M^\dagger + \cancel{MM^\dagger}\partial_n M \cancel{M^\dagger} M A_m M^\dagger$$

$$- \cancel{MM^\dagger}\partial_m M \cancel{M^\dagger} M A_n M^\dagger + \cancel{MM^\dagger}\partial_m M \cancel{M^\dagger M} M^\dagger \partial_n M M^\dagger$$

$$- i\,\cancel{\partial_m M \partial_n} M^\dagger + i\partial_n \cancel{M \partial_m} M^\dagger$$

$$= M F_{mn} M^\dagger.$$

Nathan Exercise:

In the Wess-Zumino gauge we have

$$V(x,\theta,\bar\theta) = -(\theta\sigma^m\bar\theta)A_m(x) + i\theta^2(\bar\theta\bar\lambda(x)) - i\bar\theta^2(\theta\lambda(x)) + \frac{1}{2}\theta^2\bar\theta^2 d(x)$$

$$V(x,\theta,\bar\theta)^2 = \underbrace{(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta)A_m(x)A_n(x)}$$

$$-\frac{1}{2}\theta^2\bar\theta^2 A^m(x)A_m(x).$$

Thus

$$e^{-V(x,\theta,\bar\theta)} = 1 + (\theta\sigma^m\bar\theta)A_m(x) - i\theta^2(\bar\theta\bar\lambda(x)) + i\bar\theta^2(\theta\lambda(x)) - \frac{1}{2}\theta^2\bar\theta^2 d(x)$$

$$- \frac{1}{4}\theta^2\bar\theta^2 A^m(x)A_m(x).$$

On the other hand the matter chiral field is

$$\Phi(x + i\theta\sigma\bar\theta, \theta) = \varphi(x) + i(\theta\sigma^m\bar\theta)\partial_m\varphi(x) + \frac{1}{4}\theta^2\bar\theta^2\Box\varphi(x)$$

$$+ \sqrt{2}\,\theta\psi(x) + \frac{i}{\sqrt{2}}\theta^2(\bar\theta\bar\sigma^m\partial_m\psi(x)) + \theta^2 F(x)$$

and

$$\bar\Phi(x - i\theta\sigma\bar\theta, \bar\theta) = \bar\varphi(x) - i(\theta\sigma^m\bar\theta)\partial_m\bar\varphi(x) + \frac{1}{4}\theta^2\bar\theta^2\Box\bar\varphi(x)$$

$$+ \sqrt{2}\,\bar\theta\bar\psi(x) + \frac{i}{\sqrt{2}}\bar\theta^2(\partial_m\bar\psi(x)\bar\sigma^m\theta) + \bar\theta^2\bar F(x).$$

Thus

$$\int d^4\theta\, \Phi(x+i\theta\sigma\bar\theta,\theta)\, e^{-V(x,\theta,\bar\theta)}\, \bar\Phi(x-i\theta\sigma\bar\theta,\bar\theta)$$

$$= \frac{1}{4}\varphi(x)\Box\bar\varphi(x) + \frac{i}{2}\varphi(x)A^m(x)\partial_m\bar\varphi(x)$$

$$+ \underbrace{\int d^4\theta\, \varphi(x)\left(-i\theta^2(\bar\theta\bar\lambda(x))\right)\sqrt{2}(\bar\theta\bar\psi(x))} \quad + \cancel{\underbrace{\int d^4\theta\, \varphi(x)\, i\bar\theta^2(\theta\lambda(x))\frac{i}{\sqrt{2}}(\partial_m\bar\psi(x)\bar\sigma^m\theta)}}$$

$$= -\int d^4\theta\, \frac{i}{\sqrt{2}}\varepsilon_{\dot\alpha\dot\beta}\,\theta^2\bar\theta^2\varphi(x)\bar\lambda^{\dot\alpha}(x)\bar\psi^{\dot\beta}(x) \qquad \cancel{= -\frac{1}{\sqrt{2}}\underbrace{\int d^4\theta\, \varphi(x)\bar\theta^2\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\lambda_\alpha(x)\sigma^m{}_{\beta\dot\beta}\,\partial_m\bar\psi^{\dot\beta}(x)}}$$

$$= \frac{i}{\sqrt{2}}\varphi(x)(\bar\lambda(x)\bar\psi(x)) \qquad \cancel{-\frac{1}{2\sqrt{2}}\varphi(x)(\lambda\sigma^m\partial_m\bar\psi(x))}$$

$$\cancel{\| \quad \frac{1}{2\sqrt{2}}\varphi(x)(\partial_m\bar\psi(x)\bar\sigma^m\lambda(x))}$$

$$-\frac{1}{2}\varphi(x)d(x)\bar\varphi(x) - \frac{1}{4}\varphi(x)A^m(x)A_m(x)\bar\varphi(x)$$

$$-\frac{1}{2}\partial^m\varphi(x)\partial_m\bar\varphi(x) - \frac{i}{2}\bar\varphi(x)A^m(x)\partial_m\varphi(x) + \frac{1}{4}\bar\varphi(x)\Box\varphi(x)$$

$$+ \underbrace{\int d^4\theta\, \sqrt{2}(\theta\psi(x))(\theta\sigma^m\bar\theta)A_m(x)\sqrt{2}(\bar\theta\bar\psi(x))}$$

$$= \int d^4\theta\, \cancel{2}\frac{1}{\cancel{2}}\varepsilon^{\alpha\beta}\theta^2\psi_\alpha(x)\sigma^m{}_{\beta\dot\beta}\bar\theta^{\dot\beta}A_m(x)\bar\psi_{\dot\alpha}(x)\bar\theta^{\dot\alpha}$$

$$= -\int d^4\theta\, \varepsilon^{\alpha\beta}\frac{1}{2}\varepsilon^{\dot\beta\dot\alpha}\theta^2\bar\theta^2\psi_\alpha(x)\sigma^m{}_{\beta\dot\beta}A_m(x)\bar\psi_{\dot\alpha}(x)$$

$$= \frac{1}{2}(\psi(x)\sigma^m\bar\psi(x))A_m(x)$$

$$+ \underbrace{\int d^4\theta\, \sqrt{2}(\theta\psi(x))\, i\bar\theta^2(\theta\lambda(x))\bar\varphi(x)}$$

$$= -\frac{i}{\sqrt{2}}\varepsilon^{\alpha\beta}\psi_\alpha(x)\lambda_\beta(x)\bar\varphi(x) = \frac{i}{\sqrt{2}}(\psi(x)\lambda(x))\bar\varphi(x)$$

$$+\int d^4\theta\, \frac{i}{\sqrt{2}}\left(\bar\theta\bar\sigma^m\partial_m\psi(x)\right)\left(\theta\sigma^n\bar\theta\right)A_n(x)\frac{i}{\sqrt{2}}\left(\partial_m\bar\psi(x)\bar\sigma^n\theta\right)$$

$$=-\int d^4\theta\, \frac{1}{2}\,\partial_m\psi^\alpha(x)\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\theta^\beta\sigma^n{}_{\beta\dot\beta}\bar\theta^{\dot\beta}A_n(x)\,\theta^\gamma\sigma^\kappa{}_{\gamma\dot\gamma}\partial_\kappa\bar\psi^{\dot\gamma}(x)$$

$$=-\int d^4\theta\, \frac{1}{2}\,\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\,\frac{1}{2}\varepsilon^{\beta\gamma}\theta^2\,\partial_m\psi^\alpha(x)\sigma^m{}_{\alpha\dot\alpha}\sigma^n{}_{\beta\dot\beta}A_n(x)\sigma^\kappa{}_{\gamma\dot\gamma}\partial_\kappa\bar\psi^{\dot\gamma}(x)$$

$$=\frac{1}{8}\,\partial_m\psi^\alpha(x)\sigma^m{}_{\alpha\dot\alpha}\,\bar\sigma^{n\dot\alpha\gamma}A_n(x)\sigma^\kappa{}_{\gamma\dot\gamma}\partial_\kappa\bar\psi^{\dot\gamma}(x)$$

$$=\frac{1}{8}\left(\partial_m\psi(x)(\sigma^m\bar\sigma^n\sigma^\kappa)\partial_\kappa\bar\psi(x)\right)$$

$$-\int d^4\theta\, \frac{i}{\sqrt{2}}\left(\bar\theta\bar\sigma^m\partial_m\psi(x)\right) i\theta^2\left(\bar\theta\bar\lambda(x)\right)\bar\varphi(x)$$

$$=\frac{1}{2\sqrt{2}}\,\varepsilon_{\dot\alpha\dot\beta}\,\bar\sigma^{m\dot\alpha\alpha}\partial_m\psi_\alpha(x)\bar\lambda^{\dot\beta}(x)\bar\varphi(x)$$

$$=-\frac{1}{2\sqrt{2}}\left(\bar\lambda(x)\bar\sigma^m\partial_m\psi(x)\right)\bar\varphi(x)=\frac{1}{2\sqrt{2}}\left(\partial_m\psi(x)\sigma^m\bar\lambda(x)\right)$$

$$+F(x)\bar F(x)$$

$$+\int d^4\theta\, \frac{i}{\cancel{\sqrt{2}}}\theta^2\left(\bar\theta\bar\sigma^m\partial_m\psi(x)\right)\cancel{\sqrt{2}}\left(\bar\theta\bar\psi(x)\right)$$

$$=\frac{i}{2\sqrt{\ }}\,\varepsilon_{\dot\alpha\dot\beta}\,\bar\sigma^{m\dot\alpha\alpha}\partial_m\psi_\alpha(x)\cdot\bar\psi^{\dot\beta}(x)=-\frac{i}{2}\left(\bar\psi(x)\bar\sigma^m\partial_m\psi(x)\right)$$

$$=\frac{i}{2}\left(\partial_m\psi(x)\sigma^m\bar\psi(x)\right)$$

$$-\,i\int d^4\theta\,\left(\theta\psi(x)\right)\bar\theta^2\left(\partial_m\bar\psi(x)\bar\sigma^m\theta\right)$$

$$=+\frac{i}{2}\,\varepsilon^{\alpha\beta}\,\psi_\alpha(x)\,\sigma^m{}_{\beta\dot\beta}\partial_m\bar\psi^{\dot\beta}(x)=-\frac{i}{2}\left(\psi(x)\sigma^m\partial_m\bar\psi(x)\right)$$

$$= \frac{1}{4}\varphi(x)\Box\bar{\varphi}(x) + \frac{1}{4}\bar{\varphi}(x)\Box\varphi(x) - \frac{1}{2}\partial^m\varphi(x)\partial_m\bar{\varphi}(x)$$

$$+ \frac{i}{2}\varphi(x)A^m(x)\partial_m\bar{\varphi}(x) - \frac{i}{2}\bar{\varphi}(x)A^m(x)\partial_m\varphi(x)$$

$$- \frac{1}{4}\varphi(x)A^m(x)A_m(x)\bar{\varphi}(x) -$$

$$+ \frac{i}{2}\left(\partial_m\psi(x)\sigma^m\bar{\psi}(x)\right) - \frac{i}{2}\left(\psi(x)\sigma^m\partial_m\bar{\psi}(x)\right)$$

$$+ \frac{1}{2}\left(\psi(x)\sigma^m\bar{\psi}(x)\right)A_m(x)$$

$$+ \frac{i}{\sqrt{2}}\varphi(x)\left(\bar{\lambda}(x)\bar{\psi}(x)\right) + \frac{i}{\sqrt{2}}\bar{\varphi}(x)\left(\lambda(x)\psi(x)\right)$$

$$- \frac{1}{2}\varphi(x)d(x)\bar{\varphi}(x) + F(x)\bar{F}(x)$$

$$= -\partial^m\varphi(x)\partial_m\varphi(x) + \frac{i}{2}A^m(x)\varphi(x)\partial_m\bar{\varphi}(x) - \frac{i}{2}A_m(x)\bar{\varphi}(x)\partial^m\varphi(x)$$

$$- \frac{1}{4}A^m(x)\varphi(x)A_m(x)\bar{\varphi}(x)$$

$$- i\left(\psi(x)\sigma^m\partial_m\bar{\psi}(x)\right) - i\frac{i}{2}\left(\bar{\psi}(x)\sigma^m A_m(x)\bar{\psi}(x)\right)$$

$$+ F(x)\bar{F}(x) + \frac{i}{\sqrt{2}}\varphi(x)\left(\bar{\lambda}(x)\bar{\psi}(x)\right) + \frac{i}{\sqrt{2}}\bar{\varphi}(x)\left(\lambda(x)\psi(x)\right)$$

$$- \frac{1}{2}\varphi(x)\bar{\varphi}(x)d(x) + \text{surface terms}$$

$$= -\left(\partial^m - \frac{i}{2}A^m(x)\right)\varphi(x)\left(\partial_m + \frac{i}{2}A_m(x)\right)\bar{\varphi}(x) - i\left(\psi(x)\sigma^m\left(\partial_m + \frac{i}{2}A^m(x)\right)\bar{\psi}(x)\right)$$

$$+ F(x)\bar{F}(x) + \frac{i}{\sqrt{2}}\varphi(x)\left(\bar{\lambda}(x)\bar{\psi}(x)\right) + \frac{i}{\sqrt{2}}\bar{\varphi}(x)\left(\lambda(x)\psi(x)\right)$$

$$- \frac{1}{2}\varphi(x)\bar{\varphi}(x)d(x) + \text{surface terms}$$

Thus, indeed

$$S := \int d^4x\, d^4\theta\ \Phi(x + i\theta\sigma\bar{\theta}, \theta)\, e^{-V(x,\theta,\bar{\theta})}\, \bar{\Phi}(x - i\theta\sigma\bar{\theta}, \bar{\theta})$$

$$= \int d^4x \Big(-\nabla_m \varphi \nabla^m \bar{\varphi} - i \psi \sigma^m \nabla_m \bar{\psi} + F\bar{F} + \frac{i}{\sqrt{2}} \varphi(x) \big(\bar{\lambda}(x) \bar{\psi}(x)\big)$$

$$+ \frac{i}{\sqrt{2}} \bar{\varphi}(x) \big(\lambda(x) \psi(x)\big) - \frac{1}{2} \varphi(x) \bar{\varphi}(x) d(x) \Big)$$