

2. Formalismo Lagrangiano para Campos Clássicos

1. Equações de Euler Lagrange

Em correspondência a mecânica clássica, para uma configuração de um multiplete de campos φ e uma região R do espacotempo, definimos a ação

$$S_R(\varphi) = \int_R d^4x \mathcal{L}(\varphi(x), \partial\varphi(x)).$$

Observação: \mathcal{L} , a densidade Lagrangiana, introduz um

campo $\mathcal{L}(\varphi, \partial\varphi)$ onde $\mathcal{L}(\varphi, \partial\varphi)(x) = \mathcal{L}(\varphi(x), \partial\varphi(x))$.

Agora considere uma variação ^{infinitesimal} \uparrow do espacotempo $x \mapsto x' = x + \delta x$,

onde δx é em geral dependente de x (e por o

tanto, não é necessariamente uma translação). Sob esta,

temos uma transformação da região $R \mapsto R' = \{x' = x + \delta x \mid x \in R\}$.

Simultaneamente considere uma variação infinitesimal

$\varphi \mapsto \varphi' = \varphi + \delta\varphi$ nos campos. A nova ação é

$$S_{R'}(\varphi') = \int_{R'} d^4x' \mathcal{L}(\varphi'(x'), \partial\varphi'(x')) = \int_{R'} d^4x \mathcal{L}(\varphi', \partial\varphi')(x').$$

$$= \int_R d^4x \det \left(\frac{\partial x'^\mu + \delta x^\mu}{\partial x^\nu} \right) \left(\mathcal{L}(\varphi', \partial \varphi')(x + \delta x) \right). \quad (2)$$

Expandendo a primer orden em δx e lembrando que para todo espaço vetorial V de dim. finita $\det'(\text{id}_V) = \text{tr}$, temos

$$\begin{aligned} S_{R'}(\varphi') &= \int_R d^4x \left(\det(\delta^\mu_\nu) + \partial_\mu \delta x^\mu \right) \left(\mathcal{L}(\varphi', \partial \varphi')(x) + \partial_\mu \mathcal{L}(\varphi', \partial \varphi')(x) \delta x^\mu \right) \\ &= \int_R d^4x \left(1 + \partial_\mu \delta x^\mu \right) \left(\mathcal{L}(\varphi + \delta \varphi, \partial \varphi + \partial \delta \varphi)(x) \right. \\ &\quad \left. + \partial_\mu \mathcal{L}(\varphi + \delta \varphi, \partial \varphi + \partial \delta \varphi)(x) \delta x^\mu \right) \end{aligned}$$

$$\begin{aligned} &= \int_R d^4x \left(1 + \partial_\mu \delta x^\mu \right) \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(x) \delta \varphi(x) + \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \partial_\mu \delta \varphi(x) \right. \\ &\quad \left. + \partial_\mu \mathcal{L}(\varphi, \partial \varphi)(x) \delta x^\mu + \mathcal{L}(\varphi(x), \partial \varphi(x)) \right) \end{aligned}$$

$$\begin{aligned} &= S_R(\varphi) + \int_R d^4x \left(\partial_\mu \delta x^\mu \mathcal{L}(\varphi(x), \partial \varphi(x)) + \partial_\mu \mathcal{L}(\varphi, \partial \varphi)(x) \delta x^\mu \right. \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial \varphi}(\varphi, \partial \varphi)(x) \delta \varphi(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \delta \varphi(x) \right. \\ &\quad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \delta \varphi \right)(x) \right) \end{aligned}$$

$$= S_R(\varphi) + \int_R d^4x \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(x) \right) \delta \varphi(x)$$

$$+ \int_R d^4x \partial_\mu \left(\delta x^\mu \mathcal{L}(\varphi, \partial \varphi) + \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \delta \varphi \right)(x).$$

Logo a variação da ação é

(3)

$$\delta S_R(\varphi) = S_{R'}(\varphi') - S_R(\varphi)$$

$$= \int_R d^4x \left(\left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) \right) \delta\varphi(x) + \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} \delta\varphi + \mathcal{L}(\varphi, \partial\varphi) \delta x^\mu \right)(x) \right).$$

Princípio de Ação Estacionária: Os campos físicos numa região R são φ sob variações dos campos $\varphi \mapsto \varphi' = \varphi + \delta\varphi$ infinitesimais onde $\delta\varphi|_{\partial R} = 0$, a ação é estacionária.

Sob estas variações $\delta x^\mu = 0$. Logo, com o teorema de Stokes,

$$\delta S_R(\varphi) = \int d^4x \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) \right) \delta\varphi(x) + \int_{\partial R} d\sigma^\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} \delta\varphi,$$

onde $d\sigma^\mu(x)$ é o vetor de área normal a ∂R em x .

Então, devido a o teorema fundamental do cálculo de variações, os campos físicos satisfazem as equações de Euler-Lagrange

$$\partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} - \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi} = 0.$$

2.1.1. Exemplos

Campo Escalar Real:

Temos a Lagrangiana

$$\mathcal{L}(\varphi, \partial\varphi) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2.$$

Logo

$$\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi} = -m^2 \varphi,$$

$$\begin{aligned} \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} &= \frac{1}{2} \frac{\partial}{\partial \partial_\mu \varphi} (\partial_\sigma \varphi \partial^\sigma \varphi) \\ &= \frac{1}{2} \left(\frac{\partial \partial_\sigma \varphi}{\partial \partial_\mu \varphi} \partial^\sigma \varphi + \partial_\sigma \varphi \frac{\partial \partial^\sigma \varphi}{\partial \partial_\mu \varphi} \right) \\ &= \frac{\partial \partial_\sigma \varphi}{\partial \partial_\mu \varphi} \partial^\sigma \varphi = \delta_\sigma^\mu \partial^\sigma \varphi = \partial^\mu \varphi. \end{aligned}$$

Então as equações de Euler-Lagrange são a equação de

Klein-Gordon

$$0 = \partial_\mu \partial^\mu \varphi + m^2 \varphi = (\partial^2 + m^2) \varphi.$$

Campo Escalar Complexo:

Temos a Lagrangiana

$$\mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*) = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi.$$

Logo

$$\frac{\partial \mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*)}{\partial \varphi} = -m^2 \varphi^*,$$

$$\frac{\partial \mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*)}{\partial \varphi^*} = -m^2 \varphi,$$

$$\frac{\partial \mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*)}{\partial \partial_\mu \varphi} = \partial^\mu \varphi^*,$$

$$\frac{\partial \mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*)}{\partial \partial_\mu \varphi^*} = \partial^\mu \varphi.$$

Então, as equações de Euler-Lagrange são

a equação de Klein-Gordon e seu conjugado

⑤

$$0 = \partial_\mu \partial^\mu \psi + m^2 \psi = (\partial^2 + m^2) \psi,$$

$$0 = (\partial^2 + m^2) \psi^*.$$

Campo de Dirac:

A Lagrangiana do campo de Dirac é

$$\mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi}) = \frac{i}{2} \bar{\psi} \overleftrightarrow{\partial}_\mu \gamma^\mu \psi - M \bar{\psi} \psi,$$

onde $A \overleftrightarrow{\partial}_\mu B = A \partial_\mu B - \partial_\mu A B$. Logo

$$\frac{\partial \mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi})}{\partial \psi} = -\frac{i}{2} \bar{\psi} \overleftrightarrow{\partial}_\mu \gamma^\mu - M \bar{\psi},$$

$$\frac{\partial \mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi})}{\partial \bar{\psi}} = \frac{i}{2} \gamma^\mu \partial_\mu \psi - M \psi,$$

$$\frac{\partial \mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi})}{\partial \partial_\mu \psi} = \frac{i}{2} \bar{\psi} \gamma^\mu, \quad \frac{\partial \mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi})}{\partial \partial_\mu \bar{\psi}} = -\frac{i}{2} \gamma^\mu \psi.$$

Então, as equações de Euler-Lagrange são a equação de

Dirac e sua conjugada

$$0 = -\frac{i}{2} \gamma^\mu \partial_\mu \psi - \frac{i}{2} \gamma^\mu \partial_\mu \psi + M \psi = -(i \not{\partial} - M) \psi,$$

$$0 = \bar{\psi} (i \overleftrightarrow{\not{\partial}} + M).$$

Campo Eletromagnético:

Temos a Lagrangiana

$$\begin{aligned} \mathcal{L}(A, F, \partial A, \partial F) &= -\frac{1}{2} F_{\mu\nu} (\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} F_{\mu\nu} (\partial^\mu A^\nu - \partial^\nu A^\mu). \end{aligned}$$

⑥

Observe que na segunda forma temos um campo F com Lagrangiana $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ sujeito a condição $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ com multiplicador de Lagrange $F_{\mu\nu}$. Em efeito

$$\frac{\partial \mathcal{L}(A, F, \partial A, \partial F)}{\partial F_{\mu\nu}} = \frac{1}{2} F^{\mu\nu} - \frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\partial \mathcal{L}(A, F, \partial A, \partial F)}{\partial A_\mu} = 0$$

$$\frac{\partial \mathcal{L}(A, F, \partial A, \partial F)}{\partial \partial_\mu F} = 0$$

$$\begin{aligned} \frac{\partial \mathcal{L}(A, F, \partial A, \partial F)}{\partial \partial_\mu A_\nu} &= -\frac{1}{2} F^{\sigma\rho} (\delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu) \\ &= -\frac{1}{2} (F^{\mu\nu} - F^{\nu\mu}). \end{aligned}$$

Logo uma das equações de Euler-Lagrange é a ligação

$$0 = -\frac{1}{2} F^{\mu\nu} + \frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu) \Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

e a outra são as equações homogêneas de Maxwell

$$0 = -\frac{1}{2} (\partial_\mu F^{\mu\nu} - \partial_\mu F^{\nu\mu}) = -\partial_\mu F^{\mu\nu} \Rightarrow 0 = \partial_\mu F^{\mu\nu}.$$

Usamos que pela primeira $F_{\mu\nu} = -F_{\nu\mu}$. Logo, lembrando

que $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, podemos usar a Lagrangiana

$$\mathcal{L}(A, \partial A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}(A, \partial A)}{\partial A_\mu} = 0 \quad \frac{\partial \mathcal{L}}{\partial A_\mu} F^{\mu\nu} = -\frac{j^\nu}{c}$$

$$\begin{aligned} \frac{\partial \mathcal{L}(A, \partial A)}{\partial \partial_\mu A_\nu} &= -\frac{1}{2} \frac{\partial F_{\rho\sigma}}{\partial \partial_\mu A_\nu} F^{\rho\sigma} = -\frac{1}{2} (\delta_\rho^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\rho^\nu) F^{\rho\sigma} \\ &= -\frac{1}{2} (F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu}, \end{aligned}$$

e, as equações de Euler-Lagrange são

$$\partial_\mu F^{\mu\nu} = 0.$$

Electrodinâmica:

Considere a interação do campo eletromagnético com um campo de Dirac descrita pela Lagrangiana

$$\mathcal{L}(\psi, \bar{\psi}, A, \partial\psi, \partial\bar{\psi}, \partial A) = \mathcal{L}_{\text{Dirac}}(\psi, \partial\psi) + \mathcal{L}_{\text{Maxwell}}(A, \partial A) + \underbrace{e \bar{\psi} \gamma_\mu \psi A^\mu}_{e \bar{\psi} \not{A} \psi}$$

Como

$$\frac{\partial}{\partial \psi} (e \bar{\psi} \gamma_\mu \psi A^\mu) = e \bar{\psi} \not{A},$$

$$\frac{\partial}{\partial A_\mu} (e \bar{\psi} \gamma_\mu \psi A^\mu) = e \bar{\psi} \gamma^\mu \psi$$

$$\frac{\partial}{\partial \bar{\psi}} (e \bar{\psi} \gamma_\mu \psi A^\mu) = e \not{A} \psi,$$

Logo as equações de Euler-Lagrange são

$$i(\not{\partial} - ie \not{A} - M)\psi = 0$$

$$\partial_\mu F^{\mu\nu} = e \bar{\psi} \gamma^\nu \psi.$$

2.1.2. Derivação Funcional

Agora apresentamos uma alternativa as equações de

Euler-Lagrange. Considere um funcional F sobre o

espacio de configuração do multiplete φ . Definimos a derivada funcional em x de F por

(8)

$$\frac{\delta F(\varphi)}{\delta \varphi(x)} = \lim_{\varepsilon \rightarrow 0} \frac{F(\varphi + \varepsilon \delta_x) - F(\varphi)}{\varepsilon}.$$

Em particular, para o funcional avaliação em y

$ev_y(\varphi) = \varphi(y)$, temos

$$\begin{aligned} \frac{\delta \varphi(y)}{\delta \varphi(x)} &= \lim_{\varepsilon \rightarrow 0} \frac{ev_y(\varphi + \varepsilon \delta_x) - ev_y(\varphi)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varphi(y) + \varepsilon \delta(y-x) - \varphi(y)}{\varepsilon} \\ &= \delta(y-x). \end{aligned}$$

Logo, si x é um ponto interior de R

$$\begin{aligned} \frac{\delta S_R(\varphi)}{\delta \varphi(x)} &= \int d^4 y \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(y) \frac{\delta \varphi(y)}{\delta \varphi(x)} + \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(y) \frac{\partial}{\partial y^\mu} \frac{\delta \varphi(y)}{\delta \varphi(x)} \right) \\ &= \int d^4 y \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(y) - \frac{\partial}{\partial y^\mu} \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(y) \right) \frac{\delta \varphi(y)}{\delta \varphi(x)} \\ &\quad - \int d^4 y \frac{\partial}{\partial y^\mu} \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(y) \frac{\delta \varphi(y)}{\delta \varphi(x)} \right) \\ &= \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(x) \\ &\quad - \int d\sigma_\mu(y) \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(y) \delta(y-x). \end{aligned}$$

(9)

Então podemos reemplazar o princípio de mínima ação por $\frac{\delta S(\varphi)}{\delta \varphi} = 0$ para as configurações físicas.

Exercício 2.1.

Suponha que a Lagrangiana depende dos primeiros n derivados do campo. Então as equações de movimento são

$$\begin{aligned} 0 = \frac{\delta S(\varphi)}{\delta \varphi(x)} &= \int d^4 y \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi, \dots, \partial^n \varphi)}{\partial \varphi} (y) \frac{\delta \varphi(y)}{\delta \varphi(x)} \right. \\ &\quad \left. + \sum_{k=1}^n \frac{\partial \mathcal{L}(\varphi, \partial \varphi, \dots, \partial^n \varphi)}{\partial \partial_{\mu_1} \dots \partial_{\mu_k} \varphi} (y) \prod_{i=1}^k \left(\frac{\partial}{\partial y^{\mu_i}} \right) \frac{\delta \varphi(y)}{\delta \varphi(x)} \right) \\ &= \int d^4 y \left(\frac{\partial \mathcal{L}(\varphi, \partial \varphi, \dots, \partial^n \varphi)}{\partial \varphi} (y) + \sum_{k=1}^n (-1)^k \prod_{i=1}^k \left(\frac{\partial}{\partial y^{\mu_i}} \right) \frac{\partial \mathcal{L}(\varphi, \partial \varphi, \dots, \partial^n \varphi)}{\partial \partial_{\mu_1} \dots \partial_{\mu_k} \varphi} (y) \right) \frac{\delta \varphi(y)}{\delta \varphi(x)} \\ &= \frac{\partial \mathcal{L}(\varphi, \partial \varphi, \dots, \partial^n \varphi)}{\partial \varphi} (x) + \sum_{k=1}^n (-1)^k \prod_{i=1}^k \left(\frac{\partial}{\partial y^{\mu_i}} \right) \frac{\partial \mathcal{L}(\varphi, \partial \varphi, \dots, \partial^n \varphi)}{\partial \partial_{\mu_1} \dots \partial_{\mu_k} \varphi} \end{aligned}$$

2.2. Primer Teorema de Noether

Seja G um grupo de Lie. Suponha que (R, A) é uma representação afim de G sobre o espaço tempo \mathbb{R}^4 .

Em particular, temos homomorfismos

$$R: G \longrightarrow GL(\mathbb{R}^4),$$

$$A: G \longrightarrow \mathbb{R}^4$$

t.q. para todo $g \in G$, $e \in \mathbb{R}$, $x \in \mathbb{R}^4$

$$(R, A)(g)x = R(g)x + A(g).$$

Seja \mathfrak{g} a álgebra de Lie de G . Se tem uma representação afim de \mathfrak{g} induzida $(r, a) := (R', A')$. Intuitivamente, para $X \in \mathfrak{g}$ infinitesimal

$$R(e^X) \approx R(e_0 + X) \approx R(e_0) + R'(X) = I + r(X),$$

$$A(e^X) \approx A(e_0 + X) \approx A(e_0) + A'(X) = 0 + a(X) = a(X).$$

Logo, sob a ação de G por $e_0 + X$

$$x' = (R, A)(e^X)x = x + r(X)x + a(X),$$

e decir

$$\delta x^\mu = r(X)^\mu{}_\nu x^\nu + a(X)^\mu.$$

Suponha ademais que o campo pertence a uma representação D de G , i.e.

$$\varphi'(x) = D(g) \varphi((R, A)(g^{-1})x).$$

Então, sob X , si $d = D'$,

$$\varphi'(x) = (I + d(X)) \varphi(x - r(X)x - a(X))$$

$$\approx \varphi(x) + d(X) \varphi(x) - \partial_\mu \varphi(x) r(X)^\mu{}_\nu x^\nu - \partial_\mu \varphi(x) a(X)^\mu.$$

Logo

$$\delta \varphi(x) = d(X) \varphi(x) - \partial_\mu \varphi(x) r(X)^\mu{}_\nu x^\nu - \partial_\mu \varphi(x) a(X)^\mu.$$

Sejam $\{T_A | A \in \{1, \dots, N := \dim \mathfrak{g}\}\}$ geradores de \mathfrak{g} . Então, si $X = \delta \omega^A T_A$, temos

$$\delta x^\mu = \frac{\delta x^\mu}{\delta \omega^A} \delta \omega^A \quad e \quad \delta \varphi(x) = \frac{\delta \varphi(x)}{\delta \omega^A} \delta \omega^A$$

(11)

com

$$\frac{\delta x^\mu}{\delta \omega^A} := r(T_A)^\mu{}_\nu x^\nu + a(T_A)^\mu$$

$$\frac{\delta \varphi(x)}{\delta \omega^A} := d(T_A)\varphi(x) - \partial_\mu \varphi(x) r(T_A)^\mu{}_\nu x^\nu - \partial_\mu \varphi(x) a(T_A)^\mu.$$

Logo, si φ satisfaz as equações de movimento,

$$\delta S_R(\varphi) = \int_R d^4x \partial_\mu j^\mu{}_A \delta \omega^A,$$

onde

$$j^\mu{}_A := \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \frac{\delta \varphi}{\delta \omega^A} + \mathcal{L}(\varphi, \partial \varphi) \frac{\delta x^\mu}{\delta \omega^A}.$$

Por o teorema fundamental de o cálculo de variações,

si para todo $\delta \omega^A$ temos uma simetria de a ação,

$\delta S_R(\varphi) = 0$, então $\partial_\mu j^\mu{}_A = 0$. Agora, si $X = \omega^A T_A$, tomando

$j^\mu = \omega^A j^\mu{}_A$, obtemos também $\partial_\mu j^\mu = 0$. A isto se conhece

como o primeiro teorema de Noether. Em particular,

si $R \subseteq \mathbb{R}^3$ é uma região, e

$$Q_R(t) := \int_R d^3\vec{x} j^0(t, \vec{x}),$$

temos a equação de continuidade

$$Q'_R(t) = - \int_R d^3\vec{x} \vec{\nabla} \cdot \vec{j}(t, \vec{x}) = - \int_{\partial R} d\vec{S} \cdot \vec{j}(t, \vec{x}).$$

Se \vec{j} decair o suficientemente rápido, $Q(t) = Q_{\mathbb{R}^3}$ é conservado (L2)

2.2.1. Simetrias

Translações

Pegue o grupo $G = \mathbb{R}^4$ de translações. Então

$$R(v) = \mathbb{I}, \quad r(X) = 0, \quad D(v) = \mathbb{I},$$

$$A(v) = v, \quad a(X) = X, \quad d(v) = 0.$$

Seja δ_μ uma base de \mathbb{R}^4 , digamos a canônica. Logo temos a corrente conservada

$$j^\mu{}_\nu = - \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \partial_\nu \varphi + \mathcal{L}(\varphi, \partial \varphi) \delta_\nu^\mu,$$

i.e., temos o tensor

$$\Theta^{\mu\nu} = -j^{\mu\nu} = \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}(\varphi, \partial \varphi),$$

conhecido como o tensor de energia-momento canônico. Se a ação é invariante sob translações, esta é uma corrente conservada e define o quadrimomento do campo mediante suas cargas

$$P^\mu(t) = \int d^3 \vec{x} \Theta^{0\mu}(t, \vec{x}).$$

Campo Escalar (Real ou complexo)

Temos

$$\Theta^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}(\varphi, \partial\varphi).$$

Em particular,

$$\Theta^{00} = \frac{1}{2} \left((\partial^0 \varphi)^2 + |\vec{\nabla} \varphi|^2 + m^2 \varphi^2 \right) > 0$$

$$\Theta^{0K} = \partial^0 \varphi \partial^K \varphi.$$

Campo Escalar Complexo

$$\Theta^{\mu\nu} = \partial^\mu \varphi^* \partial^\nu \varphi + \partial^\mu \varphi \partial^\nu \varphi^* - g^{\mu\nu} \mathcal{L}.$$

Em particular

$$\Theta^{00} = |\partial^0 \varphi|^2 + |\vec{\nabla} \varphi|^2 + m^2 \varphi^2 > 0$$

$$\Theta^{0K} = \partial^0 \varphi^* \partial^K \varphi + \partial^0 \varphi \partial^K \varphi^*.$$

Campo de Dirac

$$\Theta^{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma^\mu \partial^\nu \psi - \frac{i}{2} \partial^\nu \bar{\psi} \gamma^\mu \psi - g^{\mu\nu} \mathcal{L}$$

$$= \frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \psi - g^{\mu\nu} \mathcal{L}.$$

Em particular

$$\Theta^{00} = \frac{i}{2} \bar{\psi} \gamma^0 \overleftrightarrow{\partial}^0 \psi - \frac{i}{2} \overleftrightarrow{\partial}^0 \bar{\psi} \gamma^0 \psi + \frac{i}{2} \bar{\psi} \overleftrightarrow{\partial}^0 \psi + m \bar{\psi} \psi, > 0$$

$$\Theta^{0K} = \frac{i}{2} \bar{\psi} \gamma^0 \overleftrightarrow{\partial}^K \psi.$$

Campo Eletromagnético

$$\Theta^{\mu\nu} = -F^{\mu\sigma} \partial^\nu A_\sigma - g^{\mu\nu} \mathcal{L}(A, \partial A).$$

Em

Considere o grupo de Lorentz $O(1,3)=G$. Sua álgebra de Lie $\mathfrak{o}(1,3)=\mathfrak{g}$ é o grupo de matrizes $i\omega^\mu{}_\nu$, onde

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

são reais. Uma base de \mathfrak{g} são as matrizes $\frac{i}{2}M_{\mu\nu}$ com

$$\left(\frac{i}{2}M^{\mu\nu}\right)_{\rho\sigma} = \delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu, \quad \mu < \nu.$$

Temos

$$R(\Lambda) = \Lambda, \quad r(\Omega) = \Omega,$$

$$A(\Lambda) = 0, \quad a(\Omega) = 0.$$

Definimos $\frac{i}{2}I_{\mu\nu} = d(M_{\mu\nu})$. Logo

$$\begin{aligned} J^\mu{}_{\nu\rho} &= \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} \left(\frac{i}{2} I_{\nu\rho} \varphi(x) - \partial_\mu \varphi(x) \frac{i}{2} (M_{\nu\rho})^\lambda{}_\sigma x^\sigma \right) + \mathcal{L}(\varphi, \partial\varphi) \frac{i}{2} (M_{\nu\rho})^\mu{}_\sigma x^\sigma \\ &= \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} \left(\frac{i}{2} I_{\nu\rho} \varphi(x) - \frac{1}{2} x_\rho \partial_\nu \varphi(x) + \frac{1}{2} x_\nu \partial_\rho \varphi(x) \right) \\ &\quad + \frac{i}{2} \mathcal{L}(\varphi, \partial\varphi) (x_\rho \delta_\nu^\mu - x_\nu \delta_\rho^\mu) \\ &= -\frac{1}{2} \left(x_\rho \Theta^{\mu\nu}{}_\nu - x_\nu \Theta^{\mu\rho}{}_\rho \right) + i \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} I_{\nu\rho} \varphi(x). \end{aligned}$$

Definimos o tensor de momento angular

$$M^{\mu\nu\lambda} = -x^\lambda \Theta^{\mu\nu} + x^\nu \Theta^{\mu\lambda} + i \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} I^{\nu\lambda} \varphi.$$

2.2.2. Tensor de Belinfante.

Queremos um tensor de energia-momento conservado com o mesmo conteúdo que $\Theta^{\mu\nu}$. Suponha que

temas $\Omega^{\rho\mu\nu} = -\Omega^{\mu\rho\nu}$ desvanece no infinito. Logo (15)

$$T^{\mu\nu} = \Theta^{\mu\nu} - \partial_\rho \Omega^{\rho\mu\nu}$$

é conservado

$$\partial_\mu T^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} - \cancel{\partial_\mu \partial_\rho \Omega^{\rho\mu\nu}} = \partial_\mu \Theta^{\mu\nu}$$

si Θ o é, e tem o mesmo conteúdo de energia-momento

$$\begin{aligned} \int_{\mathbb{R}^4} d^3 \vec{x} T^{0\mu} &= \int_{\mathbb{R}^4} d^3 \vec{x} T^{0\nu} - \int_{\mathbb{R}^4} d^3 \vec{x} \partial_\rho \Omega^{\rho 0 \nu} \\ &= \int_{\mathbb{R}^4} d^3 \vec{x} T^{0\nu} - \int_{\mathbb{R}^4} d^3 \vec{x} \cancel{\partial_0 \Omega^{00\nu}} + \int_{\mathbb{R}^4} d^3 \vec{x} \vec{\nabla} \cdot \vec{\Omega}^{0\nu} \\ &= \int_{\mathbb{R}^4} d^3 \vec{x} T^{0\nu} + \int_{\mathbb{R}^4} d\vec{S} \cdot \vec{\Omega}^{0\nu} \\ &= \int_{\mathbb{R}^4} d^3 \vec{x} T^{0\nu} \end{aligned}$$

Construiremos Ω t.q. $T^{\mu\nu} = T^{\nu\mu}$. Note que

$$H^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu} - \partial_\rho A^{\rho\mu\lambda\nu} + \Omega^{\lambda\mu\nu} - \Omega^{\nu\mu\lambda} + H^{\nu\mu\lambda},$$

onde

$$A^{\rho\mu\lambda\nu} = x^\lambda \Omega^{\rho\mu\nu} - x^\nu \Omega^{\rho\mu\lambda} = -A^{\rho\mu\nu\lambda} = -A^{\mu\rho\lambda\nu}$$

$$H^{\nu\mu\lambda} = i \frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial \partial_\mu \varphi} I^{\nu\lambda} \varphi = \pm H^{\lambda\mu\nu}.$$

Logo $T^{\mu\nu} = T^{\nu\mu}$, se tem si

$$H^{\nu\mu\lambda} = \Omega^{\nu\mu\lambda} - \Omega^{\lambda\mu\nu}.$$

Em efecto

$$0 = \partial_\mu M^{\mu\nu\lambda} = T^{\nu\lambda} - T^{\lambda\nu} + \partial_\mu \partial_\nu A^{\mu\lambda\nu} = 0$$

Esto se logra com

$$\frac{1}{2} (H^{\nu\mu\lambda} - H^{\mu\lambda\nu} + H^{\lambda\nu\mu}) =: \Omega^{\nu\mu\lambda}$$

pois

$$\begin{aligned} \Omega^{\mu\nu\lambda} &= \frac{1}{2} (H^{\mu\nu\lambda} - H^{\nu\lambda\mu} + H^{\lambda\mu\nu}) \\ &= \frac{1}{2} (-H^{\lambda\nu\mu} + H^{\mu\lambda\nu} - H^{\nu\mu\lambda}) = -\Omega^{\nu\mu\lambda} \end{aligned}$$

c

$$\begin{aligned} \Omega^{\nu\mu\lambda} - \Omega^{\lambda\mu\nu} &= \frac{1}{2} (H^{\nu\mu\lambda} - \cancel{H^{\mu\lambda\nu}} + \cancel{H^{\lambda\nu\mu}} - H^{\lambda\mu\nu} + \cancel{H^{\mu\nu\lambda}} - \cancel{H^{\nu\lambda\mu}}) \\ &= H^{\nu\mu\lambda} \end{aligned}$$

A $T^{\mu\nu}$ se conhece como tensor de Belitante. Note que

Exercício 2.2.

Campo de Dirac

Temos para o campo de Dirac $D = S$. Logo

$$\frac{i}{2} \bar{\psi} \gamma^{\nu\lambda} \psi = -\frac{i}{8} [\gamma_\mu, \gamma_\nu]$$

Logo

$$\bar{\psi} \gamma^{\nu\lambda} \psi = -\frac{i}{4} [\gamma_\mu^\nu, \gamma_\mu^\lambda]$$

Pelo tanto para o conjugado, como

$$(S(\Lambda)\psi)^\dagger \gamma^0 = \psi^\dagger \gamma^0 \gamma^0 S(\Lambda) \gamma^0 = \bar{\psi} S^{-1}$$

$$\begin{aligned}
 H^{\nu\mu\lambda} &= i \frac{\partial \mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi})}{\partial \partial_\mu \psi} \left(-\frac{i}{4} \right) [\gamma^\nu, \gamma^\lambda] \psi + i \bar{\psi} \frac{i}{4} [\gamma^\nu, \gamma^\lambda] \frac{\partial \mathcal{L}(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi})}{\partial \partial_\mu \bar{\psi}} \\
 &= \frac{i}{8} \bar{\psi} \gamma^\mu [\gamma^\nu, \gamma^\lambda] \psi + \frac{i}{8} \bar{\psi} [\gamma^\nu, \gamma^\lambda] \gamma^\mu \psi \\
 &= \frac{i}{8} \bar{\psi} \{ \gamma^\mu, [\gamma^\nu, \gamma^\lambda] \} \psi.
 \end{aligned}$$

Temos então

$$\begin{aligned}
 \Omega^{\nu\mu\lambda} &= \frac{i}{16} \bar{\psi} (\{ \gamma^\mu, [\gamma^\nu, \gamma^\lambda] \} - \{ \gamma^\lambda, [\gamma^\mu, \gamma^\nu] \} + \{ \gamma^\nu, [\gamma^\lambda, \gamma^\mu] \}) \psi \\
 &= 0
 \end{aligned}$$

Observe que

$$\begin{aligned}
 \{A, [B, C]\} - \{B, [C, A]\} &= ABC - \cancel{ACB} + \cancel{BCA} - CBA \\
 &\quad - \cancel{BCA} + BAC - \cancel{CAB} + \cancel{ACB} \\
 &= [\{A, B\}, C] = 0
 \end{aligned}$$

Se $A, B, C \in \{\gamma^\mu \mid \mu \in \{0, \dots, 3\}\}$. Logo

$$\Omega^{\nu\mu\lambda} = \frac{i}{16} \bar{\psi} (\{ \gamma^\nu, [\gamma^\lambda, \gamma^\mu] \}) \psi.$$

Agora bem, lembre que

$$[\gamma^\nu, [\gamma^\lambda, \gamma^\mu]] = \gamma^\nu \gamma^\lambda \gamma^\mu - \gamma^\nu \gamma^\mu \gamma^\lambda - \gamma^\lambda \gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\lambda \gamma^\nu$$

$$= 2\gamma^\nu \gamma^\lambda \gamma^\mu - \cancel{2g^{\mu\lambda} \gamma^\nu} - 2\gamma^\lambda \gamma^\mu \gamma^\nu + \cancel{2g^{\mu\lambda} \gamma^\nu}$$

$$[\gamma^\lambda, [\gamma^\mu, \gamma^\nu]] = 2\gamma^\lambda \gamma^\mu \gamma^\nu + 4\gamma^\lambda \gamma^\nu \gamma^\mu - 4g^{\mu\nu} \gamma^\lambda$$

$$= 4g^{\lambda\nu} \gamma^\mu - 4g^{\mu\nu} \gamma^\lambda - \cancel{2g^{\mu\lambda} \gamma^\nu} + \cancel{2g^{\mu\lambda} \gamma^\nu}$$

Logo $= 2\gamma^\lambda \gamma^\mu \gamma^\nu + 2\gamma^\mu \gamma^\lambda \gamma^\nu - 4\gamma^{\mu\nu} \gamma^\lambda$

$$\Omega^{\nu\mu\lambda} = \frac{i}{4} \bar{\psi} (g^{\lambda\nu} \gamma^\mu - g^{\mu\nu} \gamma^\lambda) + \frac{i}{8} \bar{\psi} [\gamma^\lambda, \gamma^\mu] \gamma^\nu \psi.$$

Logo Logo

$$\Omega^{\nu\mu\lambda} = \frac{i}{4} \bar{\psi} (\gamma^\mu \partial^\lambda - \gamma^\lambda \partial^\mu + \gamma^\mu \overleftrightarrow{\partial}^\lambda - \gamma^\lambda \overleftrightarrow{\partial}^\mu) \psi + \frac{i}{8} \bar{\psi} [\gamma^\lambda, \gamma^\mu] (\partial^\nu + \overleftrightarrow{\partial}^\nu) \psi.$$

c

$$T^{\mu\nu} = \frac{i}{4} \bar{\psi} (2\gamma^\mu \overleftrightarrow{\partial}^\nu - 2\gamma^\mu \overleftrightarrow{\partial}^\nu - \cancel{\gamma^\mu \partial^\nu} + \gamma^\nu \partial^\mu + \cancel{\gamma^\mu \partial^\nu} - \gamma^\nu \partial^\mu) \psi$$

$$-g^{\mu\nu} \mathcal{L} + \frac{i}{8} \bar{\psi} [\gamma^\nu, \gamma^\mu] (\partial^\nu + \overleftrightarrow{\partial}^\nu) \psi.$$

Como $T^{\mu\nu} = \frac{1}{2} T^{(\mu\nu)}$ e $[\gamma^\nu, \gamma^\mu] = 0$, temos

$$T^{\mu\nu} = \frac{i}{4} \bar{\psi} (\gamma^\mu \overleftrightarrow{\partial}^\nu + \gamma^\nu \overleftrightarrow{\partial}^\mu) \psi - g^{\mu\nu} \mathcal{L}$$

$$= \frac{i}{8} \bar{\psi} \gamma^{(\mu} \overleftrightarrow{\partial}^{\nu)} \psi - g^{\mu\nu} \mathcal{L}.$$

Campo de Maxwell.

Temos $d(M_{\mu\nu}) = M_{\mu\nu}$ e pelo, tanto $\bar{I}_{\mu\nu} = -iM_{\mu\nu}$. Logo

$$H^{\nu\mu\lambda} = i(-F^{\mu\nu})(-iM^{\nu\lambda})_{\beta\sigma} A^\sigma$$

$$= -F^{\mu\nu} A^\lambda + F^{\mu\lambda} A^\nu$$

Então

$$\Omega^{\nu\mu\lambda} = \frac{1}{2} (F^{\mu\lambda} \cancel{A^\nu} - F^{\mu\nu} A^\lambda - \cancel{F^{\lambda\nu} A^\mu} + \cancel{F^{\lambda\mu} A^\nu} + F^{\nu\mu} A^\lambda - \cancel{F^{\nu\lambda} A^\mu})$$

$$= F^{\nu\mu} A^\lambda,$$

e $T^{\mu\nu} = -i\eta^{\mu\sigma} \partial^\nu A_\sigma - g^{\mu\nu} \mathcal{L} = -\partial^\mu \Pi^\nu - g^{\mu\nu} \mathcal{L}$

$$\partial_\rho \Omega^{\rho\mu\nu} = \partial_\rho (F^{\rho\mu} A^\nu) = \cancel{\partial_\rho F^{\rho\mu}} A^\nu + F^{\rho\mu} \partial_\rho A^\nu.$$

O tensor de Belifante então é

$$\begin{aligned} T^{\mu\nu} &= -F^{\mu\sigma} \partial^\nu A_\sigma - F^{\rho\mu} \partial_\rho A^\nu - g^{\mu\nu} \mathcal{L}(A, \partial A) \\ &= F^{\rho\mu} (\partial^\nu A_\rho - \partial_\rho A^\nu) - g^{\mu\nu} \mathcal{L}(A, \partial A) \\ &= F_\rho{}^\mu F^{\nu\rho} - g^{\mu\nu} \mathcal{L}(A, \partial A). \end{aligned}$$

2.2.3. Transformações Conformes

O grupo multiplicativo $(0, \infty)$ actua sobre o espacotempo por dilatações

$$R(\lambda) = \lambda \quad r(\lambda) = \lambda$$

$$A(\lambda) = 0 \quad a(\lambda) = 0.$$

Si $b \in \mathbb{R}^4$, e $x \in \mathbb{R}^4$ é t.q.

$$1 - 2b \cdot x + b^2 x^2 \neq 0,$$

então, a transformação conforme especial de x é

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}.$$

Note que esta não é uma transformação afim e não é global, no sentido que existe $x \in \mathbb{R}^4$ t.q.

$$1 - 2b \cdot x + b^2 x^2 = 0.$$

É claro que o grupo de Poincaré junto com as dilatações claramente preservam o cone de Luz.

As transformações conformes especiais também. Para

ver isso, observe

$$x'^2 = \frac{x^2 - 2b \cdot x x^2 + b^2 x^4}{\sigma(x)^2} = \frac{x^2 \cancel{\sigma(x)}}{\sigma(x)^2} = \frac{x^2}{\sigma(x)},$$

onde $\sigma(x) = 1 - 2b \cdot x + b^2 x^2$. Além mais,

$$(x' - y')^2 = x'^2 - 2x' \cdot y' + y'^2 = \frac{x^2}{\sigma(x)} + \frac{y^2}{\sigma(y)} - 2 \frac{(x - bx^2) \cdot (y - by^2)}{\sigma(x)\sigma(y)}$$

$$= \frac{1}{\sigma(x)\sigma(y)} (x^2 \sigma(y) + y^2 \sigma(x) - 2(x - bx^2)(y - by^2))$$

$$= \frac{1}{\sigma(x)\sigma(y)} (x^2 - 2b \cdot y x^2 + b^2 y^2 x^2 + y^2 - 2b \cdot x y^2 + b^2 x^2 y^2 - 2x \cdot y + 2x by^2 + 2b \cdot y x^2 - 2b^2 x^2 y^2)$$

$$= \frac{1}{\sigma(x)\sigma(y)} (x - y)^2.$$

Exercício 2.3.

Assuma que $D(\lambda) = \lambda^{-d}$. Então para $\lambda = 1 + \epsilon$ com ϵ

infinitesimal $d(\epsilon) = -d\epsilon$ e

$$\delta x = \epsilon x,$$

$$\delta \varphi(x) = -\epsilon d\varphi(x) - \partial_\mu \varphi(x) \epsilon x^\mu.$$

Logo, sob esta transformação, a variação da ação é

$$\delta S_R(\varphi) = \int d^4x \left(\left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) - \partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) \right) (-\varepsilon d\varphi(x) - \varepsilon x^\mu \partial_\mu \varphi(x)) \right. \\ \left. + \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi} (-\varepsilon d\varphi - \varepsilon x^\nu \partial_\nu \varphi) + \mathcal{L}(\varphi, \partial\varphi) \varepsilon x^\mu \right)(x) \right)$$

$$= \int d^4x \left(-\varepsilon d\varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) + \varepsilon d\varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x)} \right. \\ \left. - \varepsilon x^\mu \partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) + \varepsilon x^\nu \partial_\nu \varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x)} \right. \\ \left. - \varepsilon d\varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x)} - \varepsilon d \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) \partial_\mu \varphi(x) \right. \\ \left. - \varepsilon x^\nu \partial_\nu \varphi(x) \cancel{\partial_\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x)} - \varepsilon \partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) \right. \\ \left. - \varepsilon x^\nu \partial_\mu \partial_\nu \varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) + \varepsilon x^\mu \partial_\mu \mathcal{L}(\varphi, \partial\varphi)(x) \right. \\ \left. + 4 \varepsilon \mathcal{L}(\varphi(x), \partial\varphi(x)) \right)$$

$$= -\varepsilon \int d^4x \left(-d\varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) + (d+1) \partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) \right. \\ \left. - 4 \mathcal{L}(\varphi(x), \partial\varphi(x)) + \varepsilon x^\mu \partial_\mu \varphi(x) \cancel{\frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x)} \right. \\ \left. + \varepsilon x^\nu \partial_\mu \partial_\nu \varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) - \varepsilon x^\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) \partial_\mu \varphi(x) \right. \\ \left. - x^\mu \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\nu \varphi}(x) \partial_\mu \partial_\nu \varphi(x) \right)$$

$$= -\varepsilon \int d^4x \left(d\varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \varphi}(x) + (d+1) \partial_\mu \varphi(x) \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial \partial_\mu \varphi}(x) - 4 \mathcal{L}(\varphi(x), \partial\varphi(x)) \right)$$

Logo as dilatações são uma simetria da ação si o Lagrangiano satisfaz

$$d \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(x) \varphi(x) + (d+1) \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(x) \partial_\mu \varphi(x) = 4 \mathcal{L}(\varphi(x), \partial \varphi(x)).$$

Considere o Lagrangiano

$$\mathcal{L}(\varphi, \partial \varphi) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi.$$

Se tem

~~$$d \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \varphi}(x) \varphi(x) + (d+1) \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi}(x) \partial_\mu \varphi(x) = (d+1) \partial^\mu \varphi(x) \partial_\mu \varphi(x).$$~~

Logo, si $d = D-1 = 4-1 = 3$, as dilatações são uma simetria. Logo

$$j^\mu = \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} (-d\varphi(x) - x^\nu \partial_\nu \varphi(x)) + \mathcal{L}(\varphi, \partial \varphi) x^\mu$$

$$= -d\pi^\mu \varphi - x^\nu \partial^\mu \varphi \partial_\nu \varphi + \mathcal{L}(\varphi, \partial \varphi) x^\mu$$

$$= -d\pi^\mu \varphi - x_\nu (\partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}(\varphi, \partial \varphi))$$

$$= -d\pi^\mu \varphi - x_\nu \Theta^{\mu\nu}.$$

Agora procuremos a: dimensão de massa de φ .

Se tem

$$1 = [h] = [S] = [d^D x] [\mathcal{L}] = L^D [\partial_\mu \psi \partial^\mu \psi] = L^{\cancel{D}^{D-2}} [\psi]^2$$

$$= L^{D-2} [\psi]^2 = M^{2-D} [\psi]^2$$

onde se usa $L = M^{-1}$ como é claro de

$$1 = [h] = [\vec{r} \times \vec{p}] = [\vec{r}] [m] [\cancel{r}]^1 = LM$$

Logo $[\psi] = M^{\frac{D-2}{2}}$. Para o campo de Dirac temos
analogamente

$$1 = [S] = [d^D x] [M \bar{\psi} \psi] = D^D M [\psi]^2 = M^{1-D} [\psi]^2,$$

é dizer

$$[\psi] = M^{\frac{D-1}{2}}.$$

Além das simetrias geométricas estudadas, uma teoria pode ter simetrias internas. Estas estão caracterizadas por $R(g) = I$, $A(g) = 0$ para todo $g \in G$. Pelo tanto $r(X) = a(X) = 0$ para todo $X \in \mathfrak{g}$. Por outra parte, D é não trivial e muitas vezes é a representação fundamental. Associada temos a corrente conservada

$$j^\mu = - \frac{\partial \mathcal{L}(\varphi, \partial \varphi)}{\partial \partial_\mu \varphi} d(X) \varphi$$

para todo $X \in \mathfrak{g}$.

Campo Escalar Complexo

Temos $G = U(1)$ e $d(e^{i\alpha}) = e^{i\alpha}$. Então $\mathfrak{g} = \mathbb{R}$ e $d(i\alpha) = i\alpha$. Logo

$$j^\mu = -i (\partial^\mu \varphi^* \varphi - \partial^\mu \varphi \varphi^*) = i (\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*)$$

Campo de Dirac

Temos o mesmo que arriba

$$j^\mu = -i \left(\frac{i}{2} \bar{\psi} \gamma^\mu \psi + \frac{i}{2} \bar{\psi} \gamma^\mu \psi \right) = \bar{\psi} \gamma^\mu \psi.$$

Considere $\psi = (\psi^1, \dots, \psi^n)$ com ψ^A espinores de Dirac.

Suponha que ψ pertence à representação fundamental de $SU(n)$. Logo a ação de $U \in SU(n)$ sobre ψ é

$$\psi' = U\psi \quad (\psi'^A = U^A_B \psi^B).$$

A álgebra de Lie $su(n)$ é definida por

$$I = (I + X)(I + X)^+ = I + X + X^+$$

$$1 = \det(I + X) = \det(I) + \det'(I)X = 1 + \text{tr}(X).$$

para toda $X \in su(n)$. Logo

$$su(n) = \{X \in M_n(\mathbb{C}) \mid X = X^+ \text{ e } \text{tr}(X) = 0\}.$$

Em particular, $\dim_{\mathbb{R}} su(n) = n^2 - 1$. Pegue geradores iT_A , $A \in \{1, \dots, n^2 - 1\}$ de $su(n)$. Então para ϵ^A infinitesimais

$$\begin{aligned} \bar{\psi} T_A \psi &\longrightarrow \bar{\psi} e^{-i\epsilon^B T_B} T_A e^{i\epsilon^C T_C} \psi = \bar{\psi} (I - i\epsilon^B T_B) T_A (I + i\epsilon^C T_C) \psi \\ &= \bar{\psi} T_A \psi - i\epsilon^B \bar{\psi} T_B T_A \psi + i\epsilon^B \bar{\psi} T_A T_B \psi \\ &= \bar{\psi} T_A \psi + i\epsilon^B \bar{\psi} [T_A, T_B] \psi \\ &= \bar{\psi} T_A \psi + i\epsilon^B i C^C_{AB} \bar{\psi} T_C \psi \\ &= (\delta^C_A + \epsilon^B C^C_{BA}) \bar{\psi} T_C \psi. \end{aligned}$$

Concluimos que $\bar{\psi} T_A \psi$ transforma com a representação adjunta. (26)

2.3. Teoria de Yang-Mills.

Considere uma teoria com uma ação $S(\varphi) = \int d^4x \mathcal{L}(\varphi, \partial\varphi)(x)$ invariante baixo transformações globais de calibre. Seja o grupo G e a representação D . Assumamos agora que G é compacto ou, pelo menos, que T_A exponencial é sobrejectiva. Sejam iT_A geradores de \mathfrak{g} na representação D . Considere a ação induzida de:

$$G_\infty = \{f: \mathbb{R}^4 \rightarrow G \mid f \text{ é suave}\},$$

de dimensão infinita. Para todo $f \in G_\infty$ existe

$$\alpha_A: \mathbb{R}^4 \rightarrow \mathbb{R} \text{ suave t.q. } D(f(x)) = e^{i\alpha_A(x)T_A}. \text{ Considere}$$

agora a transformação $\partial_\mu \varphi$

$$\begin{aligned} \partial_\mu \varphi \mapsto \partial_\mu (e^{i\alpha_A T_A} \varphi) &= e^{i\alpha_A T_A} \partial_\mu \varphi + e^{i\alpha_A T_A} i \partial_\mu \alpha_A T_A \varphi \\ &\neq e^{i\alpha_A T_A} \partial_\mu \varphi. \end{aligned}$$

Definir

Logo em geral a teoria não é invariante baixo G_∞ .

Para corrigir isso introduzimos campos $A_\mu = \bar{A}_\mu^A T_A: \mathbb{R}^4 \rightarrow \mathfrak{d}(\mathfrak{g})$.

Considere

$$D_\mu = \partial_\mu - ig A_\mu,$$

onde g é conhecida como a carga de φ . Logo

$$D_\mu \psi \mapsto (\partial_\mu - ig A'_\mu)(U\psi) = U\partial_\mu \psi + (\partial_\mu U)\psi - ig A'_\mu U\psi = \partial_\mu U \psi + U\partial_\mu \psi - ig A'_\mu U\psi$$

Logo, si

$$A'_\mu = U A_\mu U^{-1} + \frac{1}{ig} (\partial_\mu U) U^{-1},$$

se tem

$$D_\mu \psi \mapsto U D_\mu \psi.$$

Então a ação $S(\psi, A) = \int d^4x \mathcal{L}(\psi, D\psi)$ é invariante baixo G_∞ .

Considere a curvatura

$$\begin{aligned} [D_\mu, D_\nu]\psi &= [\partial_\mu - ig A_\mu, \partial_\nu - ig A_\nu]\psi = (\cancel{[\partial_\mu, \partial_\nu]} - ig [\partial_\mu, A_\nu] - ig [A_\mu, \partial_\nu] - g^2 [A_\mu, A_\nu])\psi \\ &= (-ig \partial_\mu A_\nu - \cancel{ig A_\nu \partial_\mu} + \cancel{ig A_\nu \partial_\mu} - \cancel{ig A_\mu \partial_\nu} + ig \partial_\nu A_\mu + \cancel{ig A_\mu \partial_\nu} - g^2 [A_\mu, A_\nu])\psi \\ &= -ig F_{\mu\nu} \psi \end{aligned}$$

onde

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu].$$

A curvatura transforma na representação adjunta

$$\begin{aligned} F_{\mu\nu}^1 &= \partial_\mu (U A_\nu U^{-1} + \frac{1}{ig} \partial_\nu U U^{-1}) - \partial_\nu (U A_\mu U^{-1} + \frac{1}{ig} \partial_\mu U U^{-1}) \\ &= ig [U A_\mu U^{-1} + \frac{1}{ig} \partial_\mu U U^{-1}, U A_\nu U^{-1} + \frac{1}{ig} \partial_\nu U U^{-1}] \end{aligned}$$

$$\begin{aligned}
&= \cancel{\partial_\mu U A_\nu U^{-1}} + U \partial_\mu A_\nu U^{-1} + U A_\nu \cancel{\partial_\mu U^{-1}} + \frac{1}{ig} \cancel{\partial_\mu \partial_\nu U U^{-1}} U \partial_\nu A_\mu U^{-1} \\
&\quad + \frac{1}{ig} \cancel{\partial_\nu \partial_\mu U U^{-1}} - \cancel{\partial_\nu U A_\mu U^{-1}} - U \cancel{\partial_\nu A_\mu U^{-1}} - U A_\mu \cancel{\partial_\nu U^{-1}} \\
&\quad - \frac{1}{ig} \cancel{\partial_\mu \partial_\nu U U^{-1}} - \frac{1}{ig} \cancel{\partial_\mu U \partial_\nu U^{-1}} - ig (U A_\mu U^{-1} U A_\nu U^{-1} - U A_\nu U^{-1} U A_\mu U^{-1}) \\
&\quad - U A_\mu U^{-1} \cancel{\partial_\nu U U^{-1}} + \cancel{\partial_\nu U U^{-1}} U A_\mu U^{-1} - \cancel{\partial_\mu U U^{-1}} U A_\nu U^{-1} \\
&\quad + U A_\nu U^{-1} \cancel{\partial_\mu U U^{-1}} - \frac{1}{ig} (\cancel{\partial_\mu U U^{-1}} \cancel{\partial_\nu U U^{-1}} - \cancel{\partial_\nu U U^{-1}} \cancel{\partial_\mu U U^{-1}}) \\
&= U F_{\mu\nu} U^{-1} \quad \text{com } \partial_\mu (U U^{-1}) = \partial_\mu U U^{-1} + U \partial_\mu U^{-1}
\end{aligned}$$

Logo a Lagrangiana de Yang-Mills

$$\mathcal{L}_{Y.M.}(A, \partial A) = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu})$$

é invariante sob transformações de calibre. Logo

$$\begin{aligned}
\frac{\partial \mathcal{L}_{Y.M.}(A, \partial A)}{\partial \partial_\mu A_\nu^A} &= -\frac{1}{2} F^{\sigma\rho B} \frac{\partial F_{\sigma\rho}^B}{\partial \partial_\mu A_\nu^A} = -\frac{1}{2} F^{\sigma\rho B} \left(\delta_\sigma^\mu \delta_\rho^\nu \delta_A^B - \delta_\rho^\mu \delta_\sigma^\nu \delta_A^B \right) \\
&= -\frac{1}{2} F^{\mu\nu A} + \frac{1}{2} F^{\nu\mu A} = -F^{\mu\nu A}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_{Y.M.}(A, \partial A)}{\partial A_\nu^A} &= -\frac{1}{2} F^{\sigma\rho B} \frac{\partial F_{\sigma\rho}^B}{\partial A_\nu^A} = -\frac{1}{2} F^{\sigma\rho B} \left(g C_{DE}^B (\delta_\sigma^D \delta_\rho^E A_\nu^A + A_\sigma^D \delta_\rho^E A_\nu^A) \right) \\
&= -\frac{g}{2} F^{\sigma\rho B} C_{AE}^B A_\rho^E - \frac{g}{2} F^{\sigma\rho B} C_{DA}^B A_\sigma^D \\
&= -g F^{\nu\rho B} C_{AE}^B A_\rho^E
\end{aligned}$$

onde $\text{tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$, assumindo que G es semisimple. Logo

$$\begin{aligned}
\Theta &= -\partial_\mu F^{\mu\nu A} + g F^{\nu\rho B} C^B_{AE} A_\rho^E \\
&= -\partial_\mu F^{\mu\nu A} - g C^A_{EB} A_\mu^E F^{\mu\nu B} \\
&= -\partial_\mu F^{\mu\nu A} + ig [T_E, T_B]^A A_\mu^E F^{\mu\nu B} \\
&= -(\partial_\mu F^{\mu\nu A} + ig [A_\mu, F^{\mu\nu}]^A) = -(D_\mu F^{\mu\nu})^A
\end{aligned}$$

onde

$$D_\mu \Theta := \partial_\mu \Theta - ig [A_\mu, \Theta]$$

para Θ na representação adjunta. Aca usamos

$$\text{Tr}(T_A [T_B, T_C]) = i C^D_{BC} \text{Tr}(T_A T_D) = \frac{i}{2} C^A_{BC}$$

||

$$\begin{aligned}
\text{Tr}(T_A T_B T_C) - \text{Tr}(T_A T_C T_B) &= \text{Tr}([T_A, T_B] T_C) = i C^D_{AB} \text{Tr}(T_D T_C) \\
&= \frac{i}{2} C^C_{AB} \Rightarrow C^A_{BC} = C^C_{AB}
\end{aligned}$$

2.3.1. Exemplos

Modelo de Forças Nucleares

Como o próton e o nêutron tem spin $\frac{1}{2}$ e massas similares, nosso modelo tem um duplete $\psi = (\psi_p, \psi_n)$

com ψ_p e ψ_n espinores de Dirac. Asumimos que

ψ está na representação fundamental de $SU(2)$.

por outro lado, os piones $\vec{\varphi} = (\pi^+, \pi^0, \pi^-)$ (30)

estão na representação adjunta, que tem dimensão

$\dim \mathfrak{su}(2) = 2^2 - 1 = 3$. A interação é descrita por o escalar

$$i \bar{\psi} \gamma^5 \vec{e} \psi \cdot \vec{\varphi}.$$

Modelo $CP(n-1)$

Temos

$$\mathcal{L}(Z, Z^*, \partial Z, \partial Z^*) = \partial_\mu Z^* \partial^\mu Z + \frac{f}{2n} (Z^* \overleftrightarrow{\partial}_\mu Z)^2$$

com Z_a , $a \in \{1, \dots, n\}$ satisfazendo $Z^\dagger Z = \frac{n}{2f}$. É claro

que esta teoria é invariante de calibre $U(1)$ com

Z_a na representação fundamental. Bajo transformações

locais

$$\mathcal{L}(e^{i\alpha} Z, e^{-i\alpha} Z^*, \partial(e^{i\alpha} Z), \partial(e^{-i\alpha} Z^*)) =$$

$$(e^{-i\alpha} \partial_\mu Z^* - i \partial_\mu \alpha e^{-i\alpha} Z^*) (e^{i\alpha} \partial^\mu Z + i \partial^\mu \alpha e^{i\alpha} Z)$$

$$+ \frac{f}{2n} (e^{-i\alpha} Z^* (e^{i\alpha} \partial_\mu Z + i \partial_\mu \alpha e^{i\alpha} Z) - (e^{-i\alpha} \partial_\mu Z^* - i \partial_\mu \alpha e^{-i\alpha} Z^*) e^{i\alpha} Z)^2 =$$

$$\partial_\mu Z^* \partial^\mu Z + i Z \partial^\mu \alpha \partial_\mu Z^* - i Z^* \partial_\mu \alpha \partial^\mu Z + \partial_\mu \alpha \partial^\mu \alpha Z^* Z$$

$$+ \frac{f}{2n} (Z^* \partial_\mu Z + i \partial_\mu \alpha Z^* Z - Z \partial_\mu Z^* + i \partial_\mu \alpha Z^* Z)^2 =$$

$$\partial_\mu Z^* \partial^\mu Z + i Z \partial^\mu \alpha \partial_\mu Z^* - i Z^* \cancel{\partial_\mu \alpha \partial^\mu Z} + \frac{n}{2f} \cancel{\partial_\mu \alpha \partial^\mu \alpha}$$

$$+ \frac{f}{2n} (Z^* \overleftrightarrow{\partial}_\mu Z)^2 + i \frac{f}{n} (\cancel{Z^* \partial_\mu Z}) \cancel{\partial^\mu \alpha \frac{n}{f}} - \frac{2f}{n} (\cancel{\partial_\mu \alpha})^2 \left(\frac{n}{2f}\right)^2$$

$$= \mathcal{L}(Z, Z^*, \partial Z, \partial Z^*).$$

Logo a Lagrangiana é invariante sim introduzir o campo

calibrante. Assim, assim,

$$\mathcal{L}(\bar{Z}, \partial Z) = (D_\mu \bar{Z})^\dagger D^\mu Z,$$

como $D_\mu Z = (\partial_\mu + i A_\mu) Z$ e $A_\mu = \frac{if}{n} Z^\dagger \overleftrightarrow{\partial}_\mu Z$. Em efeito

$$D_\mu Z = \partial_\mu Z - \frac{f}{n} (Z^\dagger \overleftrightarrow{\partial}_\mu Z) \bar{Z}.$$

Logo

$$\begin{aligned} (D_\mu \bar{Z})^\dagger D^\mu Z &= \partial_\mu \bar{Z}^\dagger \partial^\mu Z - \partial_\mu \bar{Z}^\dagger \frac{f}{n} (Z^\dagger \overleftrightarrow{\partial}^\mu Z) \bar{Z} \\ &\quad - \frac{f}{n} (Z^\dagger \overleftrightarrow{\partial}_\mu Z) \bar{Z}^\dagger \partial^\mu Z + \frac{f^2}{n^2} (Z^\dagger \overleftrightarrow{\partial}_\mu Z) (Z^\dagger \overleftrightarrow{\partial}^\mu Z) \bar{Z} \bar{Z}^\dagger \\ &= \partial_\mu \bar{Z}^\dagger \partial^\mu Z - \frac{f}{2n} (Z^\dagger \overleftrightarrow{\partial}_\mu Z)^2 + \frac{f}{n} (Z^\dagger \overleftrightarrow{\partial}_\mu Z) (\bar{Z}^\dagger \partial^\mu Z - Z^\dagger \partial^\mu \bar{Z}) \\ &= \partial_\mu \bar{Z}^\dagger \partial^\mu Z + \frac{f}{2n} (Z^\dagger \overleftrightarrow{\partial}_\mu Z)^2 = \mathcal{L}(\bar{Z}, Z^\dagger, \partial Z, \partial \bar{Z}^\dagger). \end{aligned}$$

Cromodinâmica Quântica $\frac{1}{2n} (A_\mu^a)^2$

Esta é uma teoria com $G = SU(3)$. Se tem

$\psi = (\psi_{up}, \psi_{down}, \psi_{charm}, \psi_{strange}, \psi_{top}, \psi_{bottom})$ e cada entrada é um elemento de representação fundamental de $SU(3)$. A campo de calibre tem 8 componentes pois $\dim_{\mathbb{C}} su(3) = 3^2 - 1 = 8$. Logo

$$\mathcal{L}(\psi, \partial\psi, \bar{\psi}, \partial\bar{\psi}, A, \partial A) = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) + i \bar{\psi} \gamma_\mu D^\mu \psi - \bar{\psi} M \psi$$

onde $M = \text{diag.}(M_{up}, M_{down}, M_{charm}, M_{strange}, M_{top}, M_{bottom})$. Logo

a equação de movimento é

$$(D^\mu F_{\mu\nu})^A = -g \bar{\psi} \gamma_\nu T^A \psi.$$

—————> Ver o final *

Modelo Sigma Linear

Considere a Lagrangiana

$$\mathcal{L} = \frac{1}{2} \sum_{A=1}^4 \partial_\mu \phi_A \partial^\mu \phi_A$$

onde $\phi = (\varphi_1, \varphi_2, \varphi_3, \sigma)$. Considere o grupo de calibre

$O(4)$. Então $\mathfrak{o}(4) = \{X \in M_4(\mathbb{C}) \mid X^t = -X\}$. Uma base são

$$(M_{AB})_{CD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}, \quad A \leq B$$

Logo

$$\int \mathcal{L}^{AB} = -\frac{1}{2} \sum_{E=1}^4 \partial^\mu \phi_E (M_{AB})_{ED} \phi_D$$

$$= -(\partial^\mu \phi_A \phi^B - \partial^\mu \phi^B \phi_A) = \partial^\mu \phi^{[B} \phi^{A]}$$

Agora considere as transformações geradas pelo

$\mathfrak{E}_{24} \propto M_{04}$ infinitesimal. Se pegamos $\vec{\epsilon} = (\epsilon_{14}, \epsilon_{24}, \epsilon_{34})$, temos

$$\delta \sigma = -\vec{\epsilon} \cdot \vec{\varphi} \quad \text{e} \quad \delta \vec{\varphi} = \vec{\epsilon} \cdot \sigma.$$

Se um espinor de Dirac transforma-se tal que

$$\delta \psi = -\frac{i}{2} \gamma_5 (\vec{\epsilon} \cdot \vec{\tau}) \psi,$$

podemos ver que $(\bar{\psi} \psi, i \bar{\psi} \gamma_5 \vec{\tau} \psi)$ é um vector.

De fato,

(33)

$$\begin{aligned}\delta \bar{\psi} &= \left(-\frac{i\gamma_5}{2} (\vec{\varepsilon} \cdot \vec{\tau}) \psi \right)^\dagger \gamma_0 = \frac{i}{2} \psi^\dagger (\vec{\varepsilon} \cdot \vec{\tau})^\dagger \gamma_5^\dagger \gamma_0 \\ &= -\frac{i}{2} \psi^\dagger \gamma_0 (\vec{\varepsilon} \cdot \vec{\tau}) \gamma_5^\dagger = -\frac{i}{2} \bar{\psi} \gamma_5 (\vec{\varepsilon} \cdot \vec{\tau}),\end{aligned}$$

lembrando que $\tau_i^\dagger = \tau_i$, $\gamma_5^\dagger = \gamma_5$, $\{\gamma_5, \gamma^\mu\} = 0$, e as matrizes gamma e de Pauli actuam em espacos distintos. Entao

$$\begin{aligned}\delta(\bar{\psi}\psi) &= -\frac{i}{2} \bar{\psi} \gamma_5 (\vec{\varepsilon} \cdot \vec{\tau}) \psi + \frac{i}{2} \bar{\psi} \frac{\gamma_5}{2} (\vec{\varepsilon} \cdot \vec{\tau}) \psi \\ &= -\vec{\varepsilon} \cdot \left(i \bar{\psi} \gamma_5 \vec{\tau} \psi \right),\end{aligned}$$

$$\begin{aligned}\delta \left(i \bar{\psi} \gamma_5 \vec{\tau} \psi \right)^i &= \frac{1}{2} \bar{\psi} \cancel{\gamma_5} (\vec{\varepsilon} \cdot \vec{\tau}) \cancel{\gamma_5} \tau^i \psi + \frac{1}{2} \bar{\psi} \cancel{\gamma_5} \tau^i \cancel{\gamma_5} (\vec{\varepsilon} \cdot \vec{\tau}) \psi \\ &= \frac{1}{2} \bar{\psi} \{ \vec{\varepsilon} \cdot \vec{\tau}, \tau^i \} \psi = \varepsilon^j \bar{\psi} \delta^{ji} \psi = \varepsilon^i \bar{\psi} \psi.\end{aligned}$$

O termo cinetico da Lagrangiana de Dirac e invariante

$$\begin{aligned}\delta(\bar{\psi} \gamma^\mu \partial_\mu \psi) &= -\bar{\psi} \frac{i\gamma_5}{2} \vec{\varepsilon} \cdot \vec{\tau} \gamma^\mu \partial_\mu \psi - \bar{\psi} \gamma^\mu \partial_\mu \frac{i\gamma_5}{2} (\vec{\varepsilon} \cdot \vec{\tau}) \psi \\ &= -\bar{\psi} \frac{i}{2} \vec{\varepsilon} \cdot \vec{\tau} \{ \cancel{\gamma_5}, \gamma^\mu \} \partial_\mu \psi = 0.\end{aligned}$$

Mais o termo de massa não e. Entao a Lagrangiana do modelo sigma limed originalmente proposto por Gell-Mann e Levy e

$$\mathcal{L}(\phi = (\vec{\varphi}, \sigma), \psi, \partial\phi, \partial\psi) = i\bar{\psi} \not{\partial} \psi + \frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2} \phi^2 - \lambda \phi^4 + g(\sigma \bar{\psi} \psi + i\bar{\psi} \gamma_5 \vec{\tau} \psi \cdot \vec{\varphi}).$$

Modelo sigma não linear

O modelo linear tem 4 campos escalares, mais só temos 3 píons. Logo campo σ não está associado a nenhuma partícula física. Para eliminar o campo σ pegamos um vínculo invariante por $O(4)$

$$\vec{\phi}^2 + \sigma^2 = f^2.$$

Então $\vec{\phi} \cdot \partial_\mu \vec{\phi} + \sigma \partial_\mu \sigma = 0$! Logo

$$\begin{aligned} \mathcal{L}(\phi, \partial\phi) &= \frac{1}{2} \sum_{i=1}^3 \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma \\ &= \frac{\sigma \partial_\mu \sigma \partial^\mu \sigma}{\sigma^2} = \frac{(\vec{\phi} \cdot \partial_\mu \vec{\phi})(\vec{\phi} \cdot \partial^\mu \vec{\phi})}{f^2 - \phi^2} \end{aligned}$$

Esta é invariante pela transformação não linear

$$\delta \vec{\phi} = \vec{\epsilon} \sigma = \vec{\epsilon} \sqrt{f^2 - \phi^2}.$$

★ A corrente $-g \bar{\psi} \gamma_\nu T^A \psi$ é conservada em a Cromodinâmica Quântica pois $D^\nu D^\mu F_{\mu\nu} = 0$. De fato, pela identidade de Jacobi

$$\begin{aligned} 0 &= [A^\mu, [A^\nu, F_{\mu\nu}]] + [A^\nu, [F_{\mu\nu}, A^\mu]] + [F_{\mu\nu}, [A^\mu, A^\nu]] \\ &= 2[A^\mu, [A^\nu, F_{\mu\nu}]] + [F_{\mu\nu}, [A^\mu, A^\nu]]. \end{aligned}$$

$$\text{Logo } [A^\mu, [A^\nu, F_{\mu\nu}]] = -\frac{1}{2} [F_{\mu\nu}, [A^\mu, A^\nu]]$$

$$D^\nu D^\mu F_{\mu\nu} = \cancel{\partial^\nu \partial^\mu F_{\mu\nu}} - ig \partial^\nu [A^\mu, F_{\mu\nu}] - ig [A^\nu, \cancel{\partial^\mu F_{\mu\nu}}] \\ + g^2 [A^\nu, [A^\mu, F_{\mu\nu}]]$$

$$= -ig [\partial^\nu A^\mu, F_{\mu\nu}] - ig [\cancel{A^\mu \partial^\nu F_{\mu\nu}}] - ig [\cancel{A^\nu \partial^\mu F_{\mu\nu}}] \\ + \frac{g^2}{2} [F_{\nu\mu}, [A^\nu, A^\mu]]$$

$$= -\frac{ig}{2} [\partial^{[\nu} A^{\mu]} = ig [A^\nu, A^\mu], F_{\mu\nu}]$$

$$= \frac{ig}{2} [F^{\mu\nu}, F_{\mu\nu}] = 0$$