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The O(n->0) Model and SAWs

Quick Introduction to some Concepts of Graph Theory

Definition A *graph* is a pair $G=(V(G),E(G))$ where $V(G)$ and $E(G)$ are sets such that for all $e\in E(G)$ there exists different $p,q\in V(G)$ st $e=\{p,q\}$.

Definition A *multigraph* is a triple $G=(V(G),E(G),m_G)$ where $(V(G),E(G))$ is a graph and $m_G\ E(G)\longrightarrow \mathbb{N}^+$

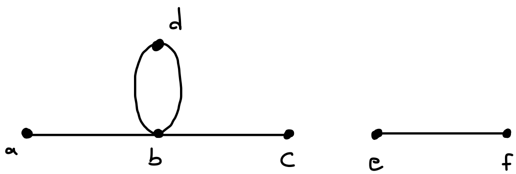
Example Consider the multigraph G with

$$V(G)=\{a,b,c,d,e,f\},\quad E(G)=\{\{a,b\},\{b,c\},\{b,d\},\{e,f\}\},$$

$$m_G\ E(G)\longrightarrow \mathbb{N}^+$$

$$e\longmapsto\begin{cases}1 & e\neq\{b,d\}\\ 2 & e=\{b,d\}\end{cases}$$

A useful pictorial representation is



Defn. Let L be a graph and G be a multigraph. Then G is said to be on L if $V(G)\subseteq V(L)$ and $E(G)\subseteq E(L)$

Defn A graph L is said to be finite if $V(L)$ is

Spherical Integration

Let $n \in \mathbb{N}^+$ and consider the parametrization (U, ψ) of $\mathbb{R}^n$ given by

$$\psi \; U = (0,\infty) \times (0,\pi)^{n-2} \times (0,2\pi) \longrightarrow \psi(U) \subseteq \mathbb{R}^n$$

where

$$\psi^1(r,\varphi) = r\cos(\varphi^1),$$

$$\psi^2(r,\varphi) = r\cos(\varphi^2)\sin(\varphi^1),$$

$$\psi^a(r,\varphi) = r\cos(\varphi^r) \prod_{\nu=1}^{a-1} \sin(\varphi^\nu),$$

$$\psi^n(r,\varphi) = r \prod_{\nu=1}^{n-1} \sin(\varphi^\nu)$$

One can check that this parametrization, commonly called hyperspherical coordinates, almost covers $\mathbb{R}^n$, i e $\mathbb{R}^n \setminus \psi(U)$ has measure 0 In these the metric tensor δ of $\mathbb{R}^n$ takes the form $(\mu,\nu \in \{0, \; ,n-1\})$

$$\delta_{(\psi^{-1})\mu\nu}(\psi\;(r,\varphi)) = \delta_{\psi(r,\varphi)}\left(\left(\frac{\partial}{\partial(\psi^{-1})^\mu}\right)_{\psi(r,\varphi)}, \left(\frac{\partial}{\partial(\psi^{-1})^\nu}\right)_{\psi(r,\varphi)}\right)$$

In terms of the chart $(\mathbb{R}^n, \pi = id_{\mathbb{R}^n})$ we have

$$\left(\frac{\partial}{\partial(\psi^{-1})^\mu}\right)_{\psi(r,\varphi)} = \left(\frac{\partial}{\partial(\psi^{-1})^\mu}\right)_{\psi(r,\varphi)}\pi^b \left(\frac{\partial}{\partial\pi^b}\right)_{\psi(r,\varphi)} = \partial_\mu(\pi^b\circ\psi)(r,\varphi)\left(\frac{\partial}{\partial\pi^b}\right)_{\psi(r,\varphi)}$$

$$= \partial_\mu\psi^b(r,\varphi)\left(\frac{\partial}{\partial\pi^b}\right)_{\psi(r,\varphi)}$$

Therefore

$$\delta_{(\psi^{-1})\mu\nu}(\psi(r,\varphi)) = \partial_\mu\psi^c(r,\varphi)\,\partial_\nu\psi^d(r,\varphi)\,\delta_{cd} = \sum_{c=1}^{n} \partial_\mu\psi^c(r,\varphi)\,\partial_\nu\psi^c(r,\varphi).$$

We calculate this by grouping the terms cleverly in each case

* $$\delta_{(\psi^{-1})00}(\psi(r,\varphi)) = \cos(\varphi^1)^2 + \sin(\varphi^1)^2\cos(\varphi^2)^2 + \sin(\quad ^1)^2\sin(\varphi^2)^2\cos(\varphi^3)^2$$

$$+ \sin(\varphi^1)^2\sin(\varphi^2)^2\sin(\varphi^3)^2\cos(\varphi^4)^2 + \quad + \sin(\varphi^1)^2 \quad \sin(\varphi^{n-2})^2\cos(\varphi^{n-1})^2$$

$$+ \sin(\varphi^1)^2 \quad \sin(\varphi^{n-1})^2$$

$$= \cos(\varphi^1)^2 + \sin(\varphi^1)^2\Big(\cos(\varphi^2)^2 + \sin(\varphi^2)^2\big(c \quad ^3)^2 + \sin(\varphi^3)^2\big($$

$$\sin(\varphi^{n-2})^2\big(\cos(\varphi^{n-1})^2 + \sin(\varphi^{n-1})^2\big)\,.\,\Big)$$

* $$\delta_{(\psi^{-1})11}(\psi(r,\varphi)) = r^2\Big(\sin(\varphi^1)^2 + \cos(\varphi^1)^2\big(\cos(\varphi^2)^2 + \sin(\varphi^2)^2\big(\quad \sin(\varphi^{n-2})^2(\cos(\varphi^{n-1})^2 + \sin(\varphi^{n-1})^2)\quad\Big)$$

$$= r^2$$

$\mu > 1$

* $$\delta_{(\psi^{-1})\mu\mu}(\psi(r,\varphi)) = r^2\sin(\varphi^1)^2 \quad \sin(\varphi^{\mu-1})^2\Big(\sin(\varphi^\mu)^2 + \cos(\varphi^\mu)^2\big(\cos(\varphi^{\mu+1})^2$$

$$+ \sin(\varphi^{\mu+1})^2\big(\cos(\varphi^{\mu+2})^2 + \quad \sin(\varphi^{n-2})^2\big(\cos(\varphi \quad + \sin(\varphi^{n-1})^2\big)\quad\Big)$$

$$= r^2\sin(\varphi^1)^2 \quad \sin(\varphi^{\mu-1})^2$$

$$* \ \delta_{(\psi^{-1})10}(\psi(r,\varphi)) = r\cos(\varphi^1)\sin(\varphi^1)\Big(-1+\cos(\varphi^2)^2+\sin(\varphi^2)^2\Big(\cos(\varphi^3)^2$$

$$+\sin(\varphi^3)^2\Big(\cos(\varphi^4)^2+\quad+\sin(\varphi^{n-2})^2\big(\cos(\varphi^{n-1})^2+\sin(\varphi^n)^2\big)\cdot\Big)$$

$$=0$$

$\mu>1$

$$* \ \delta_{(\psi^{-1})\mu 0}(\psi(r,\varphi)) = r\sin(\varphi^1)^2 \quad \sin(\varphi^{\mu-1})^2\cos(\varphi^\mu)\sin(\varphi^\mu)\Big($$

$$-1+\cos(\varphi^{\mu+1})^2+\sin(\varphi^{\mu+1})^2\Big(\cos(\varphi^{\mu+2})^2+\sin(\varphi^{\mu+2})(\cdot\ \cdot$$

$$\sin(\varphi^{n-2})^2\big(\cos(\varphi^{n-1})^2+\sin(\varphi^{n-1})^2\big)=0$$

$\mu>\nu>0$

$$* \ \delta_{(\psi^{-1})\mu\nu}(\psi(r,\varphi)) = r^2\cos(\varphi^\mu)\sin(\varphi^\mu)\cos(\varphi^\nu)\sin(\varphi^\nu)\prod_{\substack{\lambda=1\\ \lambda\notin\{\mu,\nu\}}}^{\mu}\sin(\varphi^\lambda)^2\Big(-1+\cos(\varphi^{\nu+1})^2$$

$$+\sin(\varphi^{\nu+1})^2\Big(\cos(\varphi^{\nu+2})^2+\sin(\varphi^{\nu+2})^2\Big(\quad\sin(\varphi^{n-2})^2\Big($$

$$\cos(\varphi^{n-1})^2+\sin(\varphi^{n-1})^2\Big)\quad\Big)=0$$

Therefore the metric tensor is

$$\delta_{(\psi^{-1})\mu\nu}(\psi(r,\varphi)) = \mathrm{diag}\big(1, r^2, r^2\sin(\varphi^1)^2, \ , r^2\sin(\varphi^1)^2\cdot\ \sin(\varphi^{\mu-1})^2, \ , r^2\sin(\varphi^1)^2\cdot\ \sin(\varphi^{n-2})^2\big)_{\mu\nu}$$

We now consider the sphere of radius R

$$S^{n-1} = \{\vec{x}\in\mathbb{R}^n \mid \|\vec{x}\|=R\}.$$

This is a smooth manifold of dimension $n-1$ We have a param

$(\tilde{U},\tilde{\psi})$ which almost covers S^{n-1} given by

$$\tilde{\psi}\ \tilde{U} = (0,\pi)^{n-2} \times (0,2\pi) \longrightarrow \tilde{\psi}(U) \subseteq S^{n-1} \subseteq \mathbb{R}^n$$

where $\tilde{\psi}^a(\varphi) = \psi^a(R,\varphi)$ S^{n-1} inherits a metric $g = \imath^* \delta$ through its inclusion $\imath\ S^{n-1} \hookrightarrow \mathbb{R}^n$ given in the chart $(\tilde{\psi}(\tilde{U}), \tilde{\psi}^{-1})$ by

$$g_{(\tilde{\psi}^{-1})\mu\nu}(\tilde{\psi}(\varphi)) = g_{\tilde{\psi}(\varphi)}\left(\left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)}, \left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\nu}}\right)_{\tilde{\psi}(\varphi)}\right)$$

$$= \delta_{\tilde{\psi}(\varphi)}\left(\imath^*\left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)}, \imath^*\left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\nu}}\right)_{\tilde{\psi}(\varphi)}\right)$$

where now $\mu,\nu \in \{1,..,n-1\}$

$$\imath^*\left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)} = \imath^*\left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)} (\psi^{-1})^{\nu} \left(\frac{\partial}{\partial(\psi^{-1})^{\nu}}\right)_{\tilde{\psi}(\varphi)}$$

$$= \left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)} (\psi^{-1})^{\nu} \circ \imath \left(\frac{\partial}{\partial(\psi^{-1})^{\nu}}\right)_{\tilde{\psi}(\varphi)}$$

$$= \partial_{\mu}((\psi^{-1})^{\nu} \circ \imath \circ \tilde{\psi})(\varphi)\left(\frac{\partial}{\partial(\psi^{-1})^{\nu}}\right)_{\tilde{\psi}}$$

Note that $(\psi^{-1})^{\nu} \circ \imath \circ \tilde{\psi}(\varphi) = (\psi^{-1})^{\nu}(\tilde{\psi}(\varphi)) = (\psi^{-1})^{\nu}(\psi(R,\varphi)) = \varphi^{\nu}$ Then

$$\imath^*\left(\frac{\partial}{\partial(\tilde{\psi}^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)} = \left(\frac{\partial}{\partial(\psi^{-1})^{\mu}}\right)_{\tilde{\psi}(\varphi)}$$

Thus

$$g_{(\tilde{\psi}^{-1})\mu\nu}(\tilde{\psi}(\varphi)) = \delta_{(\psi^{-1})\mu\nu}(\tilde{\psi}(\varphi)),$$

ie

$$g_{(\tilde{\psi}^{-1})^{\mu\nu}}(\tilde{\psi}(\varphi)) = \mathrm{diag}\left(r^2, r^2\sin(\varphi^1)^2, \dots, r^2\sin(\varphi^1)^2 \cdots \sin(\varphi^{n-2})^2\right)$$

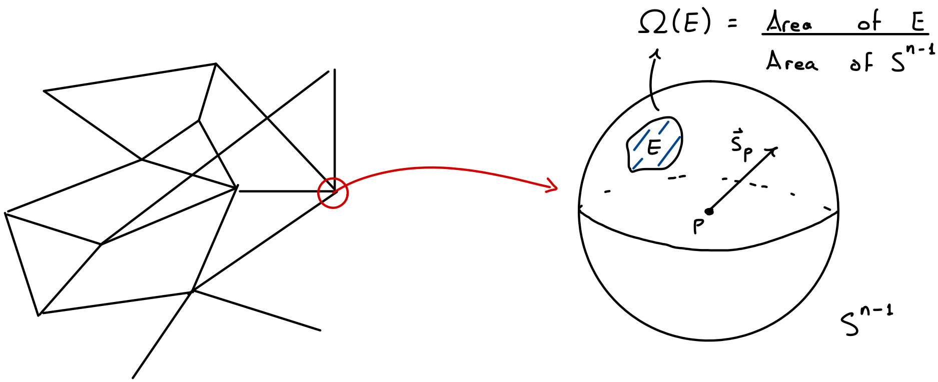
Thus

$$\det\left(g_{(\tilde{\psi}^{-1})^{\mu\nu}}(\tilde{\psi}(\varphi))\right) = r^{2(n-1)} \sin(\varphi^1)^{2(n-2)} \sin(\varphi^2)^{2(n-3)} \cdots \sin(\varphi^{n-2})^2$$
$$= \left(r^{n-1}\prod_{\nu=1}^{n-2} \sin(\varphi^\nu)^{(n-1-\nu)}\right)^2$$

and for any integrable function $f\ S^{n-1} \longrightarrow \mathbb{R}$ we have

$$\int_{S^{n-1}} f = \int_{[0,\pi]^{n-2}} \prod_{\mu=1}^{n-2} d\varphi^\mu \int_0^{2\pi} d\varphi^{n-1}\, r^{n-1}\prod_{\nu=1}^{n-2} \sin(\varphi^\nu)^{(n-1-\nu)}\, f(\tilde{\psi}^{-1}(\varphi))$$

The $O(n \to 0)$ Model



Let $n \in \mathbb{N}^+$. The $O(n)$ model describes a material with spins distributed along a lattice. These are treated classically and are n-dimensional. Although free to point in any direction, the magnitude of each spin will be fixed to $S \in (0, \infty)$ independently of its localization.

We model the lattice through a finite graph L. At each vertex $p \in V$ we have a spin whose possible states are the elements of the n-dimensional sphere of radius S

$$S^{n-1} = \{ \vec{n} \in \mathbb{R}^n \mid \|\vec{n}\| = S \} \subseteq \mathbb{R}^n$$

As it is usually done, we can equip S^{n-1} with a multiple

Ω of its volume measure for the purposes of state counting

We will choose Ω s.t. $\Omega(S^{n-1}) = 1$. Notice that we have an action of

$$O(n) = \{ T : \mathbb{R}^n \to \mathbb{R}^n \mid T \text{ is linear and } (Tu)\cdot(Tv) = u\cdot v \text{ for all } u, v \in \mathbb{R}^n \}$$

on S^{n-1} given by

$$O(n) \times S^{n-1} \longrightarrow S^{n-1}$$

$$(T, \vec{n}) \longmapsto T\vec{n}$$

It is a nice fact that Ω is the only invariant probability measure of S^{n-1} under this action (see Christensen's lemma in Christensen, JPR, On Some Measures Analogous to Haar Measure, Math Scand 26 (1970), 103-106)

The state of our system is then determined by assigning a spin to every site of the lattice. We thus take as our phase space

$$M = (S^{n-1})^V = \{ \vec{s} : V \to S^{n-1} : p \longmapsto \vec{s}_p := \vec{s}(p) \}$$

This is in turn equipped with the product measure

$$\Omega^{V(L)} = \prod_{p \in V(L)} \Omega$$

Likewise the action of $O(n)$ is extended to

$$O(n) \times M \longrightarrow M$$

$$(T, \vec{s}) \longmapsto T\vec{s} \quad V(L) \longrightarrow S^{n-1}$$

$$p \longmapsto T\vec{s}_p$$

$\Omega^{V(L)}$ is then the only $O(n)$-invariant measure of M. It is in fact invariant under the more general action

$$O(n)^V \times M \longrightarrow M$$

$$(T, \vec{s}) \longmapsto T\vec{s} \quad V(L) \longrightarrow S^{n-1}$$

$$p \longmapsto (T\vec{s})_p = T_p\vec{s}_p$$

We will consider the Hamiltonian

$$H_J \quad M \longrightarrow \mathbb{R}$$

$$\vec{s} \longmapsto -J \sum_{\{p_1,p_2\} \in E} \vec{s}_{p_1} s_{p_2}$$

for some $J \in \mathbb{R}$. Our system then has $O(n)$ symmetry, i.e.

$$H_J(T\vec{s}) = H_J(\vec{s})$$

for all $T \in O(n)$ and $\vec{s} \in M$.

Remark It is not always clear how to do state counting in some phase space. In this case there is a canonical choice due to the $O(n)$-symmetry

The partition function of our system at inverse temperature β is

$$Z(\beta, J) = \int d\Omega^{V}(\vec{s})\, e^{-\beta H_J(\vec{s})} = \int d\Omega^{V(L)}(\vec{s}) \exp\left(\beta J \sum_{\{p_1,p_2\}\in E(L)} \vec{s}_{p_1} \vec{s}_{p_2} \right)$$

$$= \int d\Omega^{V(L)}(\vec{s}) \sum_{K=0}^{\infty} \frac{(\beta J)^K}{K!} \left(\sum_{\{p_1,p_2\}\in E(L)} \vec{s}_{p_1} \vec{s}_{p_2} \right)^K$$

High temperature expansion

$$= \sum_{K=0}^{\infty} \frac{(\beta J)^K}{K!} \int d\Omega^{V(L)} \left(\sum_{\{p_1,p_2\}\in E(L)} \vec{s}_{p_1} \vec{s}_{p_2} \right)^K$$

$$= \sum_{K=0}^{\infty} \frac{(\beta J)^K}{K!} \int d\Omega^{V(L)} \sum_{\{p_1^{(1)}, p_2^{(1)}\}, \ldots, \{p_1^{(K)}, p_2^{(K)}\} \in E(L)} \prod_{i=1}^{K} \left(\vec{s}_{p_1^{(i)}} \vec{s}_{p_2^{(i)}} \right)$$

To symplify the notation, for every edge $e \in E(L)$ we choose $e_1 \in e$ and let $e_2 \in e \setminus \{e^1\}$. Then

$$Z(\beta, J) = \sum_{K=0}^{\infty} \frac{(\beta J)^K}{K!} \sum_{e^{(1)}, \ldots, e^{(K)} \in E(L)} \int d\Omega^{V(L)} \prod_{i=1}^{K} \vec{S}_{e_1^{(i)}} \vec{s}_{e_2^{(i)}}$$

For fixed $e^{(1)}, \ldots, e^{(K)} \in E(L)$ we can define a multigraph G on L given by

$$V(G) = \bigcup_{i=1}^{K} e^{(i)} = \bigcup_{i=1}^{K} \{e_1^{(i)}, e_2^{(i)}\},$$

$$E(G) = \{e^{(1)}, \ldots, e^{(K)}\},$$

$$m_G \quad E(G) \longrightarrow \mathbb{N}^+$$

$$e \longmapsto |I_e|, \qquad I_e = \{ i \in \underline{K} \mid e^{(i)} = e \}$$

The term in the high temperature expansion corresponding to this choice only depends on the corresponding graph

$$\int d\Omega^{V(L)} \prod_{i=1}^{K} (\vec{s}_{e_1^{(i)}} \vec{s}_{e_2^{(i)}}) = \int d\Omega^{V(G)} \prod_{i=1}^{K} (\vec{s}_{e_1^{(i)}} \vec{s}_{e_2^{(i)}})$$

$$= \int d\Omega^{V(G)} \prod_{e \in E(G)} (\vec{s}_{e_1} \vec{s}_{e_2})^{m_G(e)}$$

Another choice $\tilde{e}^{(1)}, , \tilde{e}^{(K)} \in E(L)$ will yield the same graph if and only if there is a permutation $\sigma \in S_K$ s t $(\tilde{e}^{(1)}, , \tilde{e}^{(K)}) = (e^{\sigma(1)}, , e^{\sigma(K)})$ There are $K!$ such permutations However, several permutations may relate the two choices above Indeed, if the two choices are related by $\sigma \in S_K$, another permutation $\tilde{\sigma} \in S_K$ will relate them as well if and only if there exist bijections $\mu_e \; I_e \longrightarrow I_e$ for all $e \in E(G)$ s t

$$\sigma(i) = \mu_{e_{\sigma(i)}}(\tilde{\sigma}(i))$$

for all $i \in \underline{K}$ We thus have $\prod_{e \in E(G)} |I_e|! = \prod_{e \in E(G)} m_G(e)!$ permutations which identify the choices $e^{(1)}, , e^{(K)}$ and $\tilde{e}^{(1)}, , \tilde{e}^{(K)}$ We thus conclude that the number of such choices which yield the same multigraph G is $\dfrac{K!}{\prod_{e \in E(G)} m_G(e)!}$

Therefore, wrt multigraphs on L,

$$Z(\beta, J) = \sum_{K=0}^{\infty} \frac{(\beta J)^K}{K!} \sum_{G \in \mathcal{G}_K} \frac{K!}{\prod_{e \in E(G)} m_G(e)} \int d\Omega^{V(G)} \prod_{e \in E(G)} (\vec{s}_{e_1} \vec{s}_{e_2})^{m_G(e)}$$

$$= \sum_{K=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \frac{1}{\prod_{e \in E(G)} m_G(e)!} \int d\Omega^{V(G)} \prod_{e \in E(G)} (\vec{s}_{e_1} \vec{s}_{e_2})^{m_G(e)},$$

where $\mathcal{G}_K$ is the set of multigraphs G on L with

$$\sum_{e \in E(G)} m_G(e) = K$$

Moreover, since Ω is invariant under $O(n)$ and

$$P\ S^{n-1} \longrightarrow S^{n-1}$$

$$\vec{x} \longmapsto -\vec{x}$$

is in $O(n)$, we have that for each vertex $p \in V(G)$

$$Z(\beta, J) = \sum_{K=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \frac{(-1)^{m_G|_p}}{\prod_{e \in E(G)} m_G(e)!} \int d\Omega^{V(G)} \prod_{e \in E(G)} (\vec{s}_{e_1} \vec{s}_{e_2})^{m_G(e)}$$

where

$$m_G|_p := \sum_{e \in \{e \in E(G) \mid p \in E\} =: E(G)|_p} m_G(e)$$

We can thus restrict $\mathcal{G}_K$ to those graphs G s.t $m_G|_p \in 2\mathbb{N}$

Much like in the case of the Gaussian model, we now need a method for evaluating

$$\int d\Omega^{V(G)}(\vec{s}) \prod_{e\in E(G)} (\vec{s}_{e_1}\ \vec{s}_{e_2})^{m_G(e)} = \int d\Omega^{V(G)}(\vec{s}) \prod_{e\in E(G)} \left(\sum_{\mu=1}^{n} s^{\mu}_{e_1} s^{\mu}_{e_2}\right)^{m_G(e)}$$

$$= \int d\Omega^{V(G)}(\vec{s}) \prod_{e\in E(G)} \sum_{\mu_1, ,\mu_{m_G(e)}=1}^{n} \prod_{j=1}^{m_e(G)} s^{\mu_j}_{e_1} s^{\mu_j}_{e_2}$$

Consider the set

$$\mathcal{S} = \{\mu\ E(G) \longrightarrow \bigcup_{l=1}^{\infty} \underline{n}^{l}\ |\ \mu(e) \in \underline{n}^{m_G(e)}\}$$

Thus the above integral equals

$$\int d\Omega^{V(G)}(\vec{s}) \sum_{\mu\in\mathcal{S}} \prod_{e\in E(G)} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}_{e_1} s^{\mu(e)_j}_{e_2}$$

$$= \sum_{\mu\in\mathcal{S}} \int d\Omega^{V(G)}(\vec{s}) \prod_{e\in E(G)} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}_{e_1} s^{\mu(e)_j}_{e_2}$$

$$= \sum_{\mu\in\mathcal{S}} \int d\Omega^{V(G)}(\vec{s}) \left(\prod_{e\in E(G)} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}_{e_1}\right)\left(\prod_{e\in E(G)} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}_{e_2}\right)$$

We can now reinterprect the product over edges as a product over vertices Indeed

$$\left(\prod_{e\in E(G)} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}_{e_1}\right)\left(\prod_{e\in E(G)} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}_{e_2}\right) = \prod_{p\in V(G)} \prod_{e\in E(G)|_p} \prod_{j=1}^{m_G(e)} s_p^{\mu(e)_j}$$

The partition function then has the form

$$Z(\beta, J) = \sum_{k=0}^{\infty} (\beta J)^k \sum_{G \in \mathcal{G}_k} \frac{1}{\prod_{e \in E(G)} m_G(e)!} \sum_{\mu \in \mathcal{S}} \prod_{p \in V(G)} \int d\Omega(\vec{s}) \prod_{e \in E(G)|_p} \prod_{j=1}^{m_G(e)} s^{\mu(e)_j}$$

As a side observation, we note that this expression refines our previous graphical interpretation. Indeed, for each edge $e \in E(G)$ we are choosing a tuple of $m_G(e)$ numbers from 1 up to n. This choice can be encoded by drawing the $m_G(e)$ lines at edge e in different ways. Indeed, for every $\nu \in \underline{n}$ define $J^{\mu}_{\nu} = \{\lambda \in \underline{m_G(e)} \mid \mu^{(e)}_{\lambda} = \nu\}$ and $m^{\mu}_{\nu} = |J^{\mu}_{\nu}|$. Then we draw m^{μ}_{ν} of the lines at edge e with ν interruptions. The corresponding term only depends on this assignation at each edge. Furthermore, $O(n)$-invariance guarantees that we can exchange the type of lines.

Example $m_G(e) = 3$,

$\mu^{(e)} = (0,0,1)$ $\qquad$ $\mu^{(e)} = (0,1,0)$

$\wr$

$\mu^{(e)} = (0,1,1) \sim \mu^{(e)} = (2,2,3)$

Since we are going to study the limit $n \to 0$, we will however not develop more machinery towards this technique.

We have seen that the terms we will have to calculate are of the form

$$\int d\Omega(\vec{s})\, s^{\mu_1} \quad s^{\mu_\ell}$$

for $\mu_1, \ \mu_\ell \in \underline{n}$ To study the limit $n \to 0$ we introduce the generating fuction

$$\begin{aligned} f \ \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{x} &\longmapsto \int d\Omega(\vec{s})\, e^{\vec{x}\, \vec{s}} \end{aligned}$$

Then

$$\int d\Omega(\vec{s})\, s^{\mu_1} \quad s^{\mu_\ell} = \partial_{\mu_1} \cdot\cdot \partial_{\mu_\ell} f(\vec{0})$$

This function is rotationally invariant Indeed, for every $T \in O(n)$, the invariance of Ω and the fact that $T^t \in O(n)$, guarantees that

$$f(R\vec{x}) = \int d\Omega(\vec{s})\, e^{T\vec{x}\, \vec{s}} = \int d\Omega(\vec{s})\, e^{\vec{x}\ T^t \vec{s}} = \int d\Omega(\vec{s}) e^{\vec{x}\, \vec{s}} = f(\vec{x}).$$

for all $x \in \mathbb{R}^n$ Now, fix $\vec{x} \in \mathbb{R}^n$, and choose a $T \in O(n)$ 1. $T\vec{x} = \|\vec{x}\| \hat{e}_1$ Then, in hyperspherical coordinates

$$f(\vec{x}) = f(T\vec{x}) = \int d\Omega(\vec{s})\, e^{T\vec{x}\,\vec{s}} = \frac{\displaystyle\int_{\tilde{U}} \prod_{\nu=1}^{n-1} d\varphi^{\nu} \sqrt{\det\left(g_{(\tilde{\psi})^{-1}\mu\nu}(\tilde{\psi}(\varphi))\right)}\; e^{T\vec{x}\,\tilde{\psi}(\varphi)}}{\displaystyle\int_{\tilde{U}} \prod_{\nu=1}^{n-1} d\varphi^{\nu} \sqrt{\det\left(g_{(\tilde{\psi})^{-1}\mu\nu}(\tilde{\psi}(\varphi))\right)}}$$

$$= \frac{\displaystyle\int_{\tilde{U}} \prod_{\nu=1}^{n-1} d\varphi^{\nu}\, \cancel{S^{n-1}} \prod_{\nu=1}^{n-2} \sin(\varphi^{\nu})^{(n-1-\nu)}\; e^{\|\vec{x}\|S\cos(\varphi^{1})}}{\displaystyle\int_{\tilde{U}} \prod_{\nu=1}^{n-1} d\varphi^{\nu}\, \cancel{S^{n-1}} \prod_{\nu=1}^{n-2} \sin(\varphi^{\nu})^{(n-1-\nu)}}$$

$$= \frac{\displaystyle\int_0^{\pi} d\varphi^{1} \sin(\varphi^{1})^{n-2}\, e^{\|\vec{x}\|S\cos(\varphi^{1})}}{\displaystyle\int_0^{\pi} d\varphi^{1} \sin(\varphi^{1})^{n-2}} = \sum_{r=0}^{\infty} \frac{(S\|\vec{x}\|)^{r}}{r!} \frac{\displaystyle\int_0^{\pi} d\varphi^{1} \sin(\varphi^{1})^{n-2} \cos(\varphi^{1})^{r}}{\displaystyle\int_0^{\pi} \varphi^{1} \sin(\varphi^{1})^{n-2}}$$

When r is odd we have that $\sin(\varphi^{1})^{n-2}\cos(\varphi^{1})^{r}$ is odd around $\pi/2$ Thus these terms vanish When r is even however both $\sin(\varphi^{1})^{n-2}$ and $\sin(\varphi^{1})^{n-2}\cos(\varphi^{1})^{r}$ are odd Thus

$$f(\vec{x}) = \sum_{r=0}^{\infty} \frac{(S\|\vec{x}\|)^{2r}}{(2r)!} \frac{\displaystyle 2\int_0^{\pi/2} d\varphi^{1} \sin(\varphi^{1})^{n-2} \cos(\varphi^{1})^{2r}}{\displaystyle 2\int_0^{\pi/2} d\varphi^{1} \sin(\varphi^{1})^{n-2}} = \sum_{r=0}^{\infty} \frac{(S\|\vec{x}\|)^{2r}}{(2r)!} \frac{B\left(\frac{n-1}{2}, \frac{2r+1}{2}\right)}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)}$$

$$= \sum_{r=0}^{\infty} \frac{(S\|\vec{x}\|)^{2r}}{(2r)!} \frac{\cancel{\Gamma\left(\frac{n-1}{2}\right)}\,\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\cancel{\Gamma\left(\frac{n-1}{2}\right)}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+2r}{2}\right)}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(S\|\vec{x}\|)^{2r}}{(2r)!} \frac{\dfrac{(2r-1)!!\,\cancel{\sqrt{\pi}}}{\cancel{2^{r}}}\;\dfrac{(n-2)!!\,\cancel{\sqrt{\pi}}}{\cancel{2^{(n-1)/2}}}}{\cancel{\sqrt{\pi}}\;\dfrac{(n+2r-2)!!\,\cancel{\sqrt{\pi}}}{\cancel{2^{(n+2r-1)/2}}}} = 1 + \sum_{r=1}^{\infty} \frac{(S\|\vec{x}\|)^{2r}}{\cancel{(2r)!}} \frac{\dfrac{\cancel{(2r)!}}{2^{r}\,r!}\,(n-2)!!}{(n+2r-2)!!}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(S\|\vec{x}\|)^{2r}}{2^r r!} \frac{1}{(n+2r-2)(n+2r-4) \cdots n},$$

where we have used that for all $n \in \mathbb{N}^+$

$$\Gamma(n/2) = \frac{(n-2)!! \sqrt{\pi}}{2^{(n-1)/2}}, \qquad (2n-1)!! = \frac{(2n)!}{2^n n!}$$

We now fix $S = \sqrt{n}$. Then

$$f(x) = 1 + \sum_{r=1}^{\infty} \frac{\|\vec{x}\|^{2r}}{2^r r!} \frac{n^r}{(n+2r-2)(n+2r-4) \cdots n}$$

$$= 1 + \frac{\|\vec{x}\|^2}{2} + \sum_{r=2}^{\infty} \frac{\|\vec{x}\|^{2r}}{2^r r!} \frac{n^{r-1}}{(n+2r-2)(n+2r) \cdots (n+2)}$$

$$\longrightarrow 1 + \frac{\|\vec{x}\|^2}{2}$$

as $n \longrightarrow 0$. Since f is quadratic the integral of the product of more than two components is null. Thus, in this limit, only graphs for which

$$m_G|_p = \sum_{e \in E(G)|_p} m_G(e) \in \{0, 2\},$$

for every $p \in V(G)$, contribute to the partition function. Thus, in this limit, $\mathcal{G}_X$ can be restricted to those closed graphs that avoid themselves.

To calculate the partition function we note that for $n \in \mathbb{N}^+$

$$\int d\Omega(\vec{s}) = 1,$$

$$\int d\Omega(\vec{s})\, s_\mu s_\nu = \delta_{\mu\nu}$$

To see the last, start by considering $\mu \neq \nu$. Note that $r_\mu \in O(n)$ where

$$r_\mu \ \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$(x_1, .., x_\mu, \ , x_n) \longmapsto (x_1, \ , -x_\mu, .., x_n)$$

Then

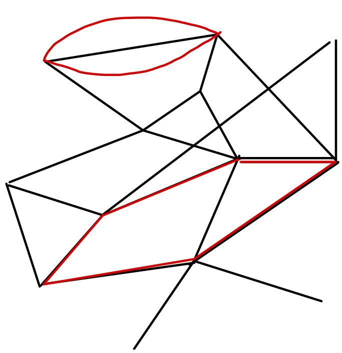
$$\int d\Omega(\vec{s}) s^\mu s^\nu = -\int d\Omega(\vec{s}) s^\mu s^\nu = 0$$

On the other hand

$$\int d\Omega(\vec{s}) (s^\mu)^2 = \frac{1}{n} \sum_{\mu=1}^{n} \int d\Omega(\vec{s}) (s^\mu)^2 = \frac{1}{n} \int d\Omega(\vec{s})\, \vec{s}^{\,2} = \frac{1}{n} \int d\Omega(\vec{s})\, n = 1$$

Now consider a multgraph G which satisfies the $n \longrightarrow 0$ restriction

Example



Let $G^1, \ldots, G^{N_G}$ be its connected components. The corresponding contribution will be the product of the contribution from each component

$$\int d\Omega^{V(G)}(\vec{s}) \prod_{e\in E(G)} (\vec{s}_{e_1} \cdot \vec{s}_{e_2})^{m_G(e)} = \int d\Omega^{V(G)}(\vec{s}) \prod_{a=1}^{N_G} \prod_{e\in E(G^a)} (\vec{s}_{e_1} \cdot \vec{s}_{e_2})^{m_G(e)}$$

$$= \prod_{a=1}^{N_G} \int d\Omega^{V(G^a)} \prod_{e\in E(G^a)} (\vec{s}_{e_1} \cdot \vec{s}_{e_2})^{m_G(e)}$$

Thus, for the time being, let us assume that G is connected. Let $e \in E(G)$. The $n \to 0$ restriction then requires that $m_G(e) \in \{1,2\}$. Assume that $m_G(e) = 2$. Then $V(G) = e$ and $E(G) = \{e\}$. Thus the contribution is

$$\int d\Omega^e(\vec{s}) (\vec{s}_{e_1} \cdot \vec{s}_{e_2})^2 = \sum_{\mu,\nu=1}^{n} \int d\Omega^e(\vec{s})\, s^\mu_{e_1} s^\mu_{e_2} s^\nu_{e_1} s^\nu_{e_2} = \sum_{\mu,\nu=1}^{n} \int d\Omega(\vec{s})\, s^\mu s^\nu \int d\Omega(\vec{s})\, s^\mu s^\nu$$

$$= \sum_{\mu,\nu=1}^{n} \delta^{\mu\nu}\delta^{\mu\nu} = \sum_{\mu=1}^{n} 1 = n \longrightarrow 0$$

in the limit $n \to 0$. On the other hand, If $m_G(e) = 1$, then for all $\tilde{e} \in E(G)$ $m_G(\tilde{e}) = 1$. Moreover, for each $p \in V(G)$, $|E(G)|_p| = 2$. Therefore the corresponding contribution is

By the same reason as above

$$\int d\Omega^{V(G)} \prod_{e\in E(G)} \vec{s}_{e_1} \cdot \vec{s}_{e_2} = \int d\Omega^{V(G)} \prod_{e\in E(G)} \sum_{\mu=1}^{n} s^\mu_{e_1} s^\mu_{e_2} = \sum_{\mu=1}^{n} \int d\Omega^{V(G)} \prod_{e\in E(G)} s^\mu_{e_1} s^\mu_{e_2}$$

$$= \sum_{\mu=1}^{n} \left(\int d\Omega \, (s^\mu)^2 \right)^{|V(G)|} = \sum_{\mu=1}^{n} 1 = n \longrightarrow 0$$

as $n \to 0$ We conclude that in the high temperature expansion only the $K=0$ (no graph) term survives Thus $Z(\beta, J) = 1$ in the $O(n \to 0)$ model

Perhaps more interestingly choose $p, q \in G$ with $p \neq q$ and let us evaluate

$$\langle s_p^1 \, s_q^1 \rangle_{\beta, J} = \sum_{K=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \frac{1}{\prod_{e \in E(G)} m_G(e)!} \int d\Omega^{V(G) \cup \{p,q\}}(\vec{s}) \, s_p^1 \, s_q^1 \prod_{e \in E(G)} \left(\vec{s}_{e_1} \, \vec{s}_{e_2} \right)^{m_G(e)}$$

Much like before, due to the $O(n)$ symmetry, only graphs with $m_G|_r \in 2\mathbb{N}$ for all $r \in V(G) \setminus \{p,q\}$ and $m_G|_p, m_G|_q \in 2\mathbb{N}+1$, contribute to the partition function In the limit $n \to 0$ this requiremes that $m_G|_r = 2$ for all $r \in V(G) \setminus \{p,q\}$ and $m_G|_p = m_G|_q = 1$ On the other hand, graphs which do not contain p don't contribute since

$$\int d\Omega^{V(G) \cup \{p,q\}}(\vec{s}) \, s_p^1 \, s_q^1 \prod_{e \in E(G)} \left(\vec{s}_{e_1} \, \vec{s}_{e_2} \right)^{m_G(e)} = \cancelto{0}{\int d\Omega(\vec{s}) \, s} \times \int d\Omega^{V(G) \cup \{p,q\}}(\vec{s}) \, s_p^1 \, s_q^1 \prod_{e \in E(G)} \left(\vec{s}_{e_1} \, \vec{s}_{e_2} \right)^{m_G(e)}$$

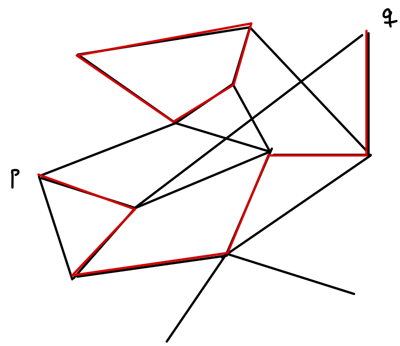
$$= 0$$

The same happens for graphs that don't contain q We restrict

$\mathcal{G}_K$ accordingly so that

$$\langle s_p^1 s_q^1 \rangle_{\beta, J} = \sum_{K=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \frac{1}{\prod_{e \in E(G)} m_G(e)!} \int d\Omega^{V(G)}(\vec{s})\, s_p^1 s_q^1 \prod_{e \in E(G)} (\vec{s}_{e_1} \vec{s}_{e_2})^{m_G(e)}$$

Example



Let $W(G)$ be the connected component of G which contains both p and q This is the image of a self avoiding walk from p to q Let $G \setminus W(G) = G^\circ$ be the rest of the graph. This is a graph which contributed to the partition function Then

$$\langle s_p^1 s_q^1 \rangle_{\beta, J} = \sum_{\beta=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \frac{1}{\prod_{e \in E(G)} m_G(e)!} \int d\Omega^{V(W(G))}(\vec{s})\, s_p^1 s_q^1 \prod_{e \in E(W(G))} \vec{s}_{e_1} \cdot \vec{s}_{e_2} \times$$

$$\int d\Omega^{V(G^\circ)}(\vec{s}) \prod_{e \in E(G^\circ} (\vec{s}_{e_1} \vec{s}_{e_2})^{m_G(e)}$$

As we saw before, the second term is null in general Thus only graphs $G = W(G)$ contribute We restrict $\mathcal{G}_K$ accordingly Then

$$\langle s_p^1 s_q^1 \rangle_{\beta,J} = \sum_{\beta=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \int d\Omega^{V(G)}(\vec{s})\, s_p^1 s_q^1 \prod_{e \in E(G)} \vec{s}_{e_1} \cdot \vec{s}_{e_2}$$

$$= \sum_{\beta=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \sum_{\mu=1}^{n} \left(\int d\Omega(\vec{s}) (s^\mu)^2 \right)^{|V(G)|-2} \left(\int d\Omega(s)\, s^1 s^\mu \right)^2$$

$$= \sum_{\beta=0}^{\infty} (\beta J)^K \sum_{G \in \mathcal{G}_K} \left(\int d\Omega(\vec{s}) (s^1)^2 \right)^{|V(G)|} = \sum_{\beta=0}^{\infty} (\beta J)^K \, |\mathcal{G}_K|,$$

where $|\mathcal{G}_K|$ is the number of self-avoiding walks from p to q

Thus the propagator of the $O(n \to 0)$ model is a generating function for these numbers