Chapter 1

Exercises: Chapter One

Exercise 1.1

As suggested in the hint, we begin by differentiating the constraints with respect to the velocities

$$0 = \frac{\partial \phi_m(q, p(q, \dot{q}))}{\partial \dot{q}^n} = \frac{\partial \phi_m}{\partial p_{n'}}(q, p(q, \dot{q}))\frac{\partial p_{n'}}{\partial \dot{q}^n} = \frac{\partial \phi_m}{\partial p_{n'}}(q, p(q, \dot{q}))\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}}. \tag{1.1}$$

These shows that indeed the vectors considered are null vectors for the Hessian matrix. Now, taking the derivative with respect to the positions,

$$\begin{aligned} 0 = \frac{\partial \phi_m(q, p(q, \dot{q}))}{\partial q^n} &= \frac{\partial \phi_m}{\partial q^n}(q, p(q, \dot{q})) + \frac{\partial \phi_m}{\partial p_{n'}}(q, p(q, \dot{q}))\frac{\partial p_{n'}}{\partial q^n} \\ &= \frac{\partial \phi_m}{\partial q^n}(q, p(q, \dot{q})) + \frac{\partial \phi_m}{\partial p_{n'}}(q, p(q, \dot{q}))\frac{\partial^2 L}{\partial q^n \partial \dot{q}^{n'}} \end{aligned} \tag{1.2}$$

Exercise 1.2

(a) Well, consider the function $\Pi_i(q^n, \dot{q}^n) := \frac{\partial L}{\partial \dot{q}^i}$. The constraint surface is then determined by $p_i = \Pi_i(q^n, \dot{q}^n)$. However, since the transformation $\dot{q}^n \mapsto \dot{q}^{m'}, p_\alpha$ is invertible, we can rewrite $\Pi_i(q^n, \dot{q}^n) = P_i(q^n, \dot{q}^{m'}, p_\alpha)$. Since the transformation $\dot{q}^n \mapsto \dot{q}^{m'}, p_\alpha$ is precisely constructed from the relation $p_i = \Pi_i(q^n, \dot{q}^n)$, the equations $\Pi_i(q^n, \dot{q}^n) = P_i(q^n, \dot{q}^{m'}, p_\alpha)$, for $i = \alpha$ reduce to trivial identities. On the other hand, the equations $p_{m'} = \Pi_{m'}(q^n, \dot{q}^n) = P_{m'}(q^n, \dot{q}^{m'}, p_\alpha)$ are expected to not be trivial. As a consequence, $P_{m'}$ cannot depend on the $\dot{q}^{m'}$. If this wasn't the case, one could use $p_{m'} = P_{m'}(q^n, \dot{q}^{m'}, p_\alpha)$ to express some of the $\dot{q}^{m'}$ in function of q^n, p_α and $p_{m'}$, meaning that the rank of the Hessian was bigger than $N - M'$.

(b) We have

$$\left(\frac{\partial H}{\partial \dot{q}^{m'}}\right)_{q^n, p_\alpha} = P_{m'} - \frac{\partial L}{\partial \dot{q}^{m'}} = P_{m'} - \Pi_{m'} = 0. \tag{1.3}$$

(c) We have

$$\left(\frac{\partial H}{\partial p^\alpha}\right)_{q^n} = \left(\frac{\partial H}{\partial p^\alpha}\right)_{q^n,\dot{q}^{m'}} = \cancel{p_\beta\frac{\partial \dot{q}^\beta}{\partial p^\alpha}} + \dot{q}^\alpha + \dot{q}^{m'}\frac{\partial P_{m'}}{\partial p^\alpha} - \cancel{\frac{\partial L}{\partial \dot{q}^\beta}}\frac{\partial \dot{q}^\beta}{\partial p_\alpha}. \tag{1.4}$$

(d) Under such variations

$$\begin{aligned}
0 &= \delta\int\left(p_\alpha\dot{q}^\alpha + P_{m'}\dot{q}^{m'} - H\right)\\
&= \int\Bigg(\delta p_\alpha\dot{q}^\alpha + p_\alpha\frac{\partial\dot{q}^\alpha}{\partial q^n}\delta q^n + p_\alpha\frac{\partial\dot{q}^\alpha}{\partial p_\beta}\delta p_\beta + \frac{\partial P_{m'}}{\partial q^n}\delta q^n\dot{q}^{m'}\\
&\qquad + \frac{\partial P_{m'}}{\partial p_\beta}\delta p_\beta\dot{q}^{m'} - \frac{\partial H}{\partial q^n}\delta q^n - \frac{\partial H}{\partial p^\alpha}\delta p^\alpha\Bigg)\\
&= \int\Bigg(\delta p_\alpha\left(\cancel{\dot{q}^\alpha} + p_\beta\frac{\partial\dot{q}^\beta}{\partial p_\alpha} + \cancel{\frac{\partial P_{m'}}{\partial p_\alpha}\dot{q}^{m'} - \frac{\partial H}{\partial p_\alpha}}\right)\\
&\qquad \delta q^n\left(p_\alpha\frac{\partial\dot{q}^\alpha}{\partial q^n} + \frac{\partial P_{m'}}{\partial q^n}\dot{q}^{m'} - \frac{\partial H}{\partial q^n}\right)\Bigg)\\
&= \int\left(\delta p_\alpha\left(p_\beta\frac{\partial\dot{q}^\beta}{\partial p_\alpha}\right) + \delta q^n\left(p_\alpha\frac{\partial\dot{q}^\alpha}{\partial q^n} + \frac{\partial P_{m'}}{\partial q^n}\dot{q}^{m'} - \frac{\partial H}{\partial q^n}\right)\right)
\end{aligned} \tag{1.5}$$

Thanks to Qmechanic for the post `https://physics.stackexchange.com/questions/59936/primary-constraints-for-constrained-hamiltonian-systems/59953#59953`, which is very much what inspired the above solution.