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Statistical Mechanics

Homework 3: KT transition, the XY model and Renormalization

1.a) To understand this result we have to notice that given a spin $s \in S^1$ one cannot uniquely determine a $\theta \in \mathbb{R}$ s.t.

$$s = (\cos(\theta), \sin(\theta))$$

However the converse is true: given θ we can define

$$s := \pi(\theta) = (\cos(\theta), \sin(\theta)).$$

Thus, given a curve $\gamma\ [0,T] \longrightarrow \mathbb{R}^2$ on which the field s is smooth, we may ask whether there exists smooth $\theta: [0,T] \longrightarrow \mathbb{R}$ s.t $s \circ \gamma = \pi \circ \theta$.

as it turns out, once we fix a $\Theta_0 \in \mathbb{R}$ s.t.

$$s(\gamma(0)) = \pi(\Theta_0),$$

there exists a unique Θ s.t. $\Theta(0) = 0$. Thus

$$\int_\gamma d\Theta := \int_0^T dt\, \Theta'(t) = \Theta(T) - \Theta(0).$$

If γ is closed then

$$\pi(\Theta(T)) = \pi(\Theta(0)) = s(\gamma(T) = \gamma(0)).$$

Thus $\Theta(0) \equiv \Theta(T) \mod 2\pi$, i.e. there exists a $n \in \mathbb{Z}$ s.t. $\Theta(0) = \Theta(T) + 2\pi n$. We conclude

$$\int_\gamma d\Theta = 2\pi n.$$

b) Although, as argued above, there is no field Θ, the divergence $\nabla\Theta$ is well defined. Indeed, around every $x \in \mathbb{R}^2$ where s is smooth there exists a sufficiently small neighborhood of x where a smooth field Θ is defined s.t. $S = \pi \circ \Theta$. Moreover, this field is unique up to a constant multiple of 2π. Thus, all possibilities yield the same $\nabla\Theta(x)$. One can extend $\nabla\Theta$ then to the domain where S is smooth by patching up those results.

Now, the result being asked of us cannot be true. Indeed, consider the special case where we only have a singularity at the origin. Then

$$\text{"}\Delta\Theta(x) = 2\pi\delta(x)\text{"}$$

has the solution

$$\Theta(x) = \log(\|x\|).$$

If $\vec{x} \neq \vec{0}$ then

$$\nabla\theta(x) = \frac{1}{\|x\|}\frac{\partial x}{\partial\|x\|} = \frac{x}{\|x\|^2} .$$

This vector field is radial and thus

$$\int_{\partial B(0,r)} ds\,\hat{T}\cdot\nabla\theta = 0 \neq 2\pi .$$

The correct result however can be obtained considering Stoke's theorem. Let

$$\nabla_\perp = \left(-\frac{\partial}{\partial x^2} , \frac{\partial}{\partial x^1}\right) ,$$

so that, if $F = (F_1, F_2)$

$$\nabla\times F(x) = \left(0, 0, \frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2}\right)$$

$$= (0, 0, \nabla_\perp\cdot F(x)) .$$

Let x be a vortex of strength n. If Ω is a sufficiently small neighborhood of x,

$$\int_{\Omega} d^2x \ \nabla_{\perp} \cdot \nabla\theta = \int_{\Omega} d^2x \ \hat{k} \cdot (\vec{\nabla} \times (\nabla\theta))$$

$$= \int_{\partial\Omega} ds \ T \cdot \nabla\theta = 2\pi n$$

$$= \int_{\Omega} d^2x \ 2\pi\rho \, .$$

Thus

$$\nabla_{\perp} \cdot \nabla\theta = 2\pi\rho .$$

Notice of course that if χ is the harmonic conjugate of θ, i.e.

$$\frac{\partial\theta}{\partial x^1} = \frac{\partial\chi}{\partial x^2} \, , \quad \frac{\partial\theta}{\partial x^2} = -\frac{\partial\chi}{\partial x^1} \, ,$$

then

$$\Delta\chi = \nabla_{\perp} \cdot \nabla\theta = 2\pi\rho .$$

c) Indeed, if $\phi_{classical}$ is the minimum energy configuration satisfying (2), then for any Θ

$$\Delta(\Theta - \phi_{classical}) = 2\pi\rho - 2\pi\rho = 0,$$

i.e. $\Theta - \phi_{classical}$ is harmonic.

d) The Hamiltonian of our model is

$$H = -J \sum_{\langle ij \rangle}{}' \cos(\Theta_i - \Theta_j).$$

Thus, in a state of minimum energy $\Theta_i \approx \Theta_j$ for neighbors $\langle ij \rangle$. We may thus expand

$$H \approx -J \sum_{\langle ij \rangle}{}' \left(1 - \frac{1}{2}(\Theta_i - \Theta_j)^2\right)$$

$$= -J(\#\text{ Edges}) + \frac{1}{2} J \sum_{\langle ij \rangle}{}' (\Theta_i - \Theta_j)^2.$$

Let $E_0 = -J(\#\text{ edges})$ and N_i be the set of neighbors of i. Then, if $a_{ij} = \|i-j\|$ is the distance between sites i and j, we have

$$H - E_0 = \frac{J}{4} \sum_i \sum_{j \in N_i} \left(\frac{\theta_i - \theta_j}{a_{ij}} \right)^2 a_{ij}^2$$

Now, if $j \in N_i$ there exists a $\mu \in \{1,2\}$ s.t.

$j = i \pm a_{ij} e_\mu$. Then

$$\frac{\theta_i - \theta_j}{a_{ij}} = \frac{\theta(i) - \theta(i \pm a_{ij} e_\mu)}{a_{ij}} \xrightarrow[a_{ij} \to 0]{} \partial_\mu \theta(i).$$

On the other hand, in the continuum limit

$$\sum_i a_{ij}^2 \longrightarrow \int d^2x .$$

Thus, in this limit

$$H - E_0 = \frac{J}{\cancel{4}_{\,2}} \int d^2x \, \cancel{2} \sum_{\mu=1}^{2} \partial_\mu \theta(x)^2$$

$$= \frac{J}{2} \int d^2x \, (\nabla \theta(x))^2 .$$

e) We have

$$H = J \int d^2x \, (\nabla \phi_{\text{classical}} + \nabla \phi_{\text{fluctuations}})^2$$

$$= J \int d^2x \, \left((\nabla \phi_{\text{classical}})^2 + (\nabla \phi_{\text{fluctuations}})^2 + 2 \, \nabla \phi_{\text{classical}} \cdot \nabla \phi_{\text{fluctuations}} \right.$$

$$= J \int d^2 x \Big((\nabla \phi_{classical})^2 + (\nabla \phi_{fluctuations})^2$$

$$+ 2 \nabla \cdot (\phi_{classical} \nabla \phi_{fluctuations})$$

$$- 2 \cancel{\phi_{classical} \Delta \phi_{fluctuations}}^{\,0} \Big)$$

$$= J \int d^2 x \Big((\nabla \phi_{classical})^2 + (\nabla \phi_{fluctuations})^2 \Big)$$

$$+ 2 J \int ds\, \hat{n} \cdot \phi_{classical} \nabla \phi_{fluctuations}.$$

Thus, assuming that the fluctuations decay at infinity, we have that the boundary term vanishes and

$$H = J \int d^2 x \Big((\nabla \phi_{classical})^2 + (\nabla \phi_{fluctuations})^2 \Big).$$

Thus, $\phi_{fluctuations}$ decouples from $\phi_{classical}$ and indeed behaves like a massless gaussian field (not being subjected to the vorticity conditions of $\phi_{classical}$).

f) Let χ be the harmonic conjugate of $\phi_{classical}$. Then $(\nabla\phi_{classical})^2 = (\nabla\chi)^2$. The energy of the vortices in a region Ω is

$$H = J\int_\Omega d^2x\ (\nabla\chi)^2 = J\int_\Omega d^2x\left(\nabla\cdot(\chi\nabla\chi) - \chi\Delta\chi\right)$$

$$= J\int_{\partial\Omega} ds\ \hat{n}\cdot\chi\nabla\chi - J2\pi\int_\Omega d^2x\ \chi\rho.$$

Now, assume χ is a solution

$$\chi(x) = \sum_i n_i \log(\|x - x_i\|)$$

of $\Delta\chi = 2\pi\rho$. Then,

$$\nabla\chi(x) = \sum_i n_i \frac{1}{\|x - x_i\|}\ \frac{\cancel{2}(x - x_i)}{\cancel{2}\|x - x_i\|}$$

$$= \sum_i n_i \frac{x - x_i}{\|x - x_i\|^2}.$$

Let us take $\Omega = B(0,R)$ with R large enough so that all of the singularities are in Ω and we can approximate $x - x_i \approx x$ in the boundary integral. Then

$$H = J \cancel{R} \int_0^{2\pi} d\theta \left(\sum_i n_i \log\left(R \right) \right) \left(\sum_j n_j \frac{\cancel{R}}{\cancel{R^2}} \right)$$

$$- J 2\pi \int_{B(0,R)} d^2x \left(\sum_i n_i \log\left(\|x - x_i\| \right) \right) \left(\sum_j n_j \delta(x - x_j) \right)$$

$$= J 2\pi \left(\sum_i n_i \right)^2 \log\left(R \right)$$

$$- J 2\pi \sum_{i,j} n_i n_j \int_{B(0,R)} d^2x \log\left(\|x - x_i\| \right) \delta(x - x_j)$$

$$= J 2\pi \left(\sum_i n_i \right)^2 \log\left(\frac{R}{x_0} \right)$$

$$- J 2\pi \sum_{i,j}{}' n_i n_j \log\left(\|x_i - x_j\| \right).$$

We now see that configurations with

$$\sum_i n_i \neq 0$$

have infinite energy and are, thus, exponentially suppressed. For this reason they don't contribute to the thermodynamics of the system. We may thus assume $\sum_i n_i = 0$.

Now, let us assume that when $i=j$ we impose a minimum separation x_0 of the vortices. Thus, the terms with $i=j$ contribute $-J 2\pi\, n_i^2 \log(x_0) = 2\pi J n_i^2 \log\left(\frac{1}{x_0}\right)$.

Then, given that

$$\sum_{i,j} = 2 \sum_{\text{pairs } (i,j)} + \sum_i ,$$

we get

$$H = -4\pi J \sum_{\substack{\text{pairs } (i,j) \\ i \neq j}} n_i n_j \log\left(\|x_i - x_j\|\right)$$

$$+ 2\pi J \sum_i n_i^2 \log\left(\frac{1}{x_0}\right).$$

Citation: Bruno helped me a lot in the homework!

g) We have

$$Z = \sum_{\{s\}} e^{-\beta(E_{\text{vertex}} + E_{\text{massless gaussian}})}$$

$$= \sum_{\{s\}} e^{-\beta E_{\text{massless gaussian}}} e^{-\beta E_{\text{vertex}}}.$$

The arguments above shows that a configuration is determined by a configuration of the massless gaussian and one of vortices (up to permutation of the vortices of the same sign)

On the other hand, configurations of vortices are obtained by specifying their number and positions. Then

$$Z = Z_{\text{massless gaussian}} \sum_{m} \frac{1}{(m!)^2} e^{-2\pi\beta J \log(1/x_0) \sum_i n_i^2}$$

$\downarrow$

m positive and m negative

$$\int_{\|x_i - x_j\| > r_0} \prod_{i=1}^{2m} d^2 x_i \cdot e^{4\pi\beta J \sum_{\text{pairs } (i,j)} n_i n_j \log \|x_i - x_j\|}$$

The factor $\frac{1}{(m!)^2}$ takes care of the permutations of vortices with equal signs. The restriction $\|x_i - x_j\| > r_0$ is required to avoid having to vortices at the same point, which would off course correspond to a single vortex with the sum of their strengths.

h) We have

$$\langle s(x_i) \cdot s(x_j) \rangle = \langle \cos(\theta_i - \theta_j) \rangle = \operatorname{Re} \langle e^{i(\theta_i - \theta_j)} \rangle$$

Now

$$\langle e^{i(\theta_i-\theta_j)}\rangle = \frac{1}{Z}\int \mathcal{D}\theta\, e^{-\beta J\int d^2x\,(\nabla\theta)^2 + i\int d^2x\,(\delta(x-x_i)-\delta(x-x_j))\theta(x)}$$

$$= \frac{1}{Z}\int \mathcal{D}\theta\, e^{\int d^2x\,\left(-\beta J\theta(-\Delta)\theta + i(\delta(x-x_i)-\delta(x-x_j))\theta(x)\right)}.$$

Recalling the Gaussian integral

$$\frac{1}{Z}\int d^N\vec{x}\, e^{-\frac{1}{2}\vec{x}\cdot A\vec{x} + \vec{b}\cdot\vec{x}} = e^{\frac{1}{2}\vec{b}\cdot A^{-1}\vec{b}},$$

we obtan

$$\langle e^{i(\theta_i-\theta_j)}\rangle = e^{-\frac{1}{2}\int d^2x\, d^2x'\,(\delta(x-x_i)-\delta(x-x_j))G(x,x')\times(\delta(x'-x_i)-\delta(x'-x_j))}$$

$$= e^{-\frac{1}{2}\left(G(x_i,x_i) - G(x_j,x_i) - G(x_i,x_j) + G(x_j,x_j)\right)},$$

where $G(\cdot,\cdot)$ is the Green's function of the operator $-2\beta J\Delta$, i.e.

$$G(x,y) = -\frac{1}{4\pi\beta J}\log\left(\|x-y\|\right).$$

Thus, by translational invariance

$$\langle e^{i(\theta_i - \theta_j)} \rangle = e^{-G(x_i, x_i)} \, e^{G(x_i, x_j)}$$

$$\propto e^{-\frac{1}{4\pi\beta J} \log(\|x_i - x_j\|)}$$

$$= \frac{1}{\|x_i - x_j\|^{\Delta(\beta J)}}$$

with $\Delta(\beta J) = \frac{1}{4\pi\beta J}$.

2. i) One has by dimensional analysis

$$\exp(-F_{pair}) \propto L^4 \, e^{-4\pi\beta J \log L}$$

$$= e^{4\log L - 4\pi\beta J \log L} = e^{4(1 - \pi\beta J)\log L},$$

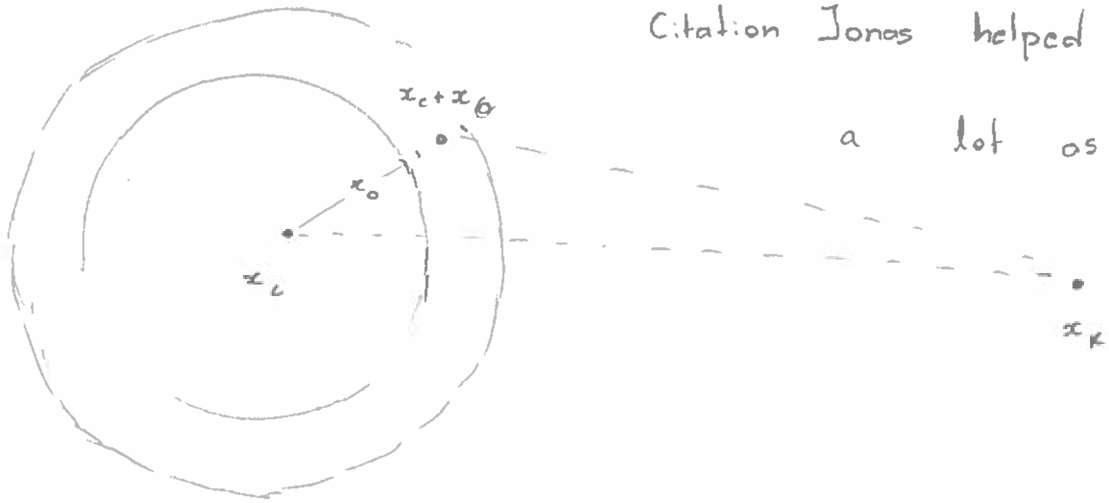
i.e.

$$-F_{pair} \propto 4(1 - \pi\beta J)\log L$$

j) At large distances, a state of bounded vortices looks like a state without vortices.

Thus, we can think of the phase transition as a transition from a state without vortices to one with. In particular, vortex creation to relieve the spin field between to vortex, is yet another mechanism to create vortices. Thus, compared to (9), where we only consider the entropical advantage of vortex creation, the real critical temperature should be lower.

3. K)



Citation Jonas helped me a lot as well!

We have from eqn (4)

$$E_{dipole} = -4\pi J \sum_{K}^{'} \left(n_c n_K \log \|x_c - x_K\| - n_c n_K \log \|x_c + x_0 - x_K\| \right)$$

$$= -4\pi J \sum_{K}^{'} n_c n_K \log \frac{\|x_c - x_K\|}{\|x_c - x_K + x_0\|}$$

l) Consider

$$f : (0,\infty) \longrightarrow \mathbb{R}$$
$$x \longmapsto \log\left(\frac{x}{x+a}\right)$$

for $a>0$. Then

$$\lim_{x\to\infty} f(x) = \log\left(\lim_{x\to\infty} \frac{1}{1+a/x}\right) = 0,$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \log(x) = -\infty,$$

$$f'(x) = \frac{\cancel{x+a}}{x} \frac{\cancel{x}+a-\cancel{x}}{(x+a)^{\cancel{2}}} = \frac{a}{x(x+a)} > 0.$$

Thus f is on, alway negative monotonically increasing function of x. We conclude that the energy decays to 0 as $\|x_c - x_i\|$ increases.

m) Notice that in $E[x_o + dx_o, y, J]$ we are summing over the same configurations as $Z[x_0, y, J]$ except for those in which the vertices are of a distance

$$x_0 < \|x_i - x_j\| < x_0 + dx_0 .$$

We will, considering regimes with low density of vortices, only take those configurations with one vortex pair satisfying this. Then

$$-\delta Z_{xy} = Z[x_0, y, J] - Z[x_0 + \delta x_0, y, J]$$

$$= Z_{\text{massless gaussian}} \times$$

$$\sum_m \frac{1}{m!^2} y^{2m} m^2 \int\limits_{x_0 < \|x_c - x_\theta\| < x_0 + dx_0} dx_c \int dx_\theta \int\limits_{\|x_i - x_j\| < x_0} \prod_{j=3}^{2m} dx_j$$

$$e^{4\pi\beta J \sum_{\substack{\text{pairs } (i,j) \\ i,j \neq c,\theta}} n_i n_j \log\|x_i - x_j\|} \, e^{-\beta E_{\text{dipole}}}.$$

The factor of m^2 comes from the fact that the vertices are indistinguishable and we have to allow for m choices of x_c and m of x_σ.

n) We have, following Jonas's neat trick,

$$E_{dipole} = 2\pi J \sideset{}{'}\sum_i n_c n_i \log\left(\frac{\|x_c - x_K + x_0\|^2}{\|x_c - x_K\|^2} \right)$$

$$= 2\pi J \sideset{}{'}\sum_i n_c n_i \log\left(1 + \frac{\|x_c - x_K + x_0\|^2 - \|x_c - x_K\|^2}{\|x_c - x_K\|^2} \right)$$

$$= 2\pi J \sideset{}{'}\sum_i n_c n_i \log\left(1 + \frac{\left(\cancel{\|x_c - x_K\|^2} + \|x_0\|^2 + 2(x_c - x_K)\cdot x_0 - \cancel{\|x_c - x_K\|^2} \right)}{\|x_c - x_K\|^2} \right)$$

$$\approx 2\pi J \sideset{}{'}\sum_i n_c n_i \left(\frac{\|x_0\|^2 + 2(x_c - x_i)\cdot x_0}{\|x_c - x_i\|^2} - \frac{(\|x_0\|^2 + 2(x_c - x_i)\cdot x_0)^2}{2\|x_c - x_i\|^4} + \mathcal{O}\left(\frac{\|x_0\|^3}{\|x_c - x_i\|^3} \right) \right)$$

$$\approx 2\pi J \sum_i n_c n_i \left(\frac{\|x_0\|^2 + 2(x_c - x_i)\cdot x_0}{\|x_c - x_i\|^2} - \frac{2((x_c - x_i)\cdot x_0)^2}{\|x_c - x_i\|^4} + \Theta\left(\frac{\|x_0\|^3}{\|x_c - x_i\|^3}\right)\right).$$

Thus

$$e^{-\beta E_{dipole}} \approx 1 - E_{dipole} + \frac{1}{2} E_{dipole}^2 + \Theta\left(\frac{\|x_0\|^3}{\|x_c - x_i\|^3}\right)$$

$$= 1 - \beta 2\pi J \sum_i n_c n_i \left(\frac{\|x_0\|^2 + 2(x_c - x_i)\cdot x_0}{\|x_c - x_i\|^2} - \frac{2((x_c - x_i)\cdot x_0)^2}{\|x_c - x_i\|^4}\right)$$

$$+ \beta 2\pi^2 J^2 \sum_{i,j} n_c^2 n_i n_j \times$$

$$\left(\frac{\|x_0\|^2 + 2(x_c - x_i)\cdot x_0}{\|x_c - x_i\|^2} - \frac{2((x_c - x_i)\cdot x_0)^2}{\|x_c - x_i\|^4}\right) \times$$

$$\left(\frac{\|x_0\|^2 + 2(x_c - x_j)\cdot x_0}{\|x_c - x_j\|^2} - \frac{2((x_c - x_j)\cdot x_0)^2}{\|x_c - x_j\|^4}\right)$$

$$+ \Theta\left(\frac{\|x_0\|^3}{\|x_c - x_i\|^3}\right)$$

$$\simeq 1 - 2\pi J_\beta \sum_i n_c n_i \left(\frac{\|x_o\|^2 + 2(x_c - x_i)\cdot x_o}{\|x_c - x_i\|^2} - 2\frac{((x_c - x_i)\cdot x_o)^2}{\|x_c - x_i\|^4} \right)$$

$$+ 8\pi^2 J_\beta^2 \sum_{i,j} \cancel{n_c^2} n_i n_j \frac{((x_c - x_i)\cdot x_o)((x_c - x_j)\cdot x_o)}{\|x_c - x_i\|^2 \|x_c - x_j\|^2}$$

$$+ \mathcal{O}\left(\frac{\|x_o\|^3}{\|x_c - x_i\|^3} \right).$$

Now, consider the integral over the linear term, which is proportional to

$$\int_{x_o < \|x_c - x_\sigma\| < x_o + dx_o} dx_\sigma \; x_\sigma = 0,$$

due to rotational symmetry (the integral is in particular invariant under a rotation by π of x_σ, i.e. $x_\sigma \longmapsto -x_\sigma$).

c) To perform the integrals we first see that $$\int\limits_{x_0<\|x_c-x_\theta\|<x_0+dx_0} d^2x_0 = \int_{x_0}^{x_0+dx_0} dr \int_0^{2\pi} d\theta\, r = 2\pi \frac{(x_0+dx_0)^2 - x_0^2}{2}$$

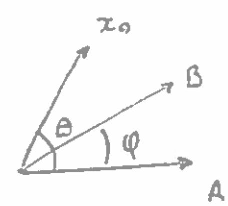
$$= 2\pi x_0\, dx_0,$$

$$\int\limits_{x_0<\|x_c-x_\theta\|<x_0+dx_0} d^2x_\theta\, \|x_\theta\|^2 = \int_{x_0}^{x_0+dx_0} dr \int_0^{2\pi} d\theta\, r\, r^2$$

$$= 2\pi \frac{(x_0+dx_0)^4 - x_0^4}{4}$$

$$= 2\pi \frac{\cancel{x_0^4} + 4x_0^3\,dx_0 - \cancel{x_0^4} + \mathcal{O}(dx_0^2)}{\cancel{4}}$$

$$= 2\pi x_0^3\, dx_0,$$



$$\int\limits_{x_0<\|x_c-x_\theta\|<x_0+dx_0} d^2x_\theta\, (x_\theta\cdot A)(x_\theta\cdot B)$$

$$= \int_{x_0}^{x_0+dx_0} dr \int_0^{2\pi} d\theta\, r\, r^2 \|A\|\|B\| \cos(\theta)\cos(\varphi-\theta)$$

$$= \|A\|\|B\|\, x_0^3\, dx_0 \int_0^{2\pi} d\theta\, \Big(\cos(\theta)\cos(\varphi)\cos(\theta)$$

$$+ \cos(\theta)\cancelto{0}{\sin(\varphi)}\sin(\theta)\Big)$$

$$= A\cdot B\, x_0^3\, dx_0 \int_0^{2\pi} d\theta\, \frac{1+\cancelto{0}{\cos(2x)}}{2}$$

Since $\cos(\theta)\sin(\theta)$ is odd around π

$$= A\cdot B\, \pi\, x_0^3\, dx_0.$$

Then

$$\int_{x_0 < \|x_c - x_\sigma\| < x_0 + dx_0} d^2 x_\sigma \; e^{-\beta E_{\text{dipole}}} = 2\pi \, x_0 \, dx_0$$

$$- 2\pi J_\beta \sum_i n_c n_i \left(\frac{2\pi \cancel{x_0^3} \, dx_0}{\|x_c - \cancel{x_i}\|^2} - 2 \frac{\cancel{\|x_c - x_i\|^2} \, \pi x_0^3 \, dx_0}{\|x_c - \cancel{x_i}\|^{\cancel{4}\,2}} \right)$$

$$+ 8\pi^2 J_\beta^2 \sum_{i,j} n_i n_j \frac{(x_c - x_i) \cdot (x_c - x_j) \, \pi x_0^3 \, dx_0}{\|x_c - x_i\|^2 \, \|x_c - x_j\|^2} .$$

Thus

$$\int d^2 x_c \int_{x_0 < \|x_c - x_\sigma\| < x_0 + dx_0} d^2 x_\sigma \; e^{-\beta E_{\text{dipole}}} = 2\pi x_0 \, dx_0 \, A$$

$$+ 8\pi^3 J^2 \beta \, x_0^3 \, dx_0 \sum_{i,j} n_i n_j \int d^2 y \, \frac{y \cdot (y + x_i - x_j)}{\|y\|^2 \, \|y + x_i - x_j\|^2} .$$

For this last integral we have

$$\int d^2 y \, \frac{y \cdot (y + x_i - x_j)}{\|y\|^2 \, \|y + x_i - x_j\|^2} = \int_0^\infty dr \int_0^{2\pi} d\theta \, \cancel{r} \, \frac{r^{\cancel{2}} + \cancel{r} \, \|x_i - x_j\| \cos(\theta)}{\cancel{r^2} \left(r^2 + 2r \|x_i - x_j\| \cos(\theta) + \|x_i - x_j\|^2 \right)}$$

$$= \int_0^{2\pi} d\theta \int_0^{r} dr \frac{r + \|x_i - x_j\|\cos(\theta)}{(r + \|x_i - x_j\|\cos(\theta))^2 + \|x_i - x_j\|^2 \sin(\theta)^2}$$

$$= \int_0^{2\pi} d\theta \int_{\|x_i - x_j\|\cos(\theta)}^{\infty} du \frac{u}{u^2 + \|x_i - x_j\|^2 \sin(\theta)^2}$$

$$\begin{cases} = \int_0^{2\pi} d\theta \int_{\|x_i - x_j\|\cos(\theta)}^{\infty} \dfrac{du}{\|x_i - x_j\| \sin(\theta)} \dfrac{\frac{u}{\|x_i - x_j\| \sin(\theta)}}{\left(\frac{u}{\|x_i - x_j\|\sin(\theta)}\right)^2 + 1} \\[2ex] = \int_0^{2\pi} d\theta \int_{\cot(\theta)}^{\infty} dv \dfrac{v}{v^2 + 1}, & i \neq j \\[2ex] = \int_0^{2\pi} d\theta \int_0^{\infty} du \dfrac{1}{u}, & i = j \end{cases}$$

The integrals do not depend on $x_i - x_j$. Something went wrong.

r) We have

$$\frac{dn}{dl} = x_0 \frac{dn}{dx_0} = x_0 \left(\frac{d(2\pi\beta J)}{dx_0} \right)$$

$$= x_0 \frac{(2\pi\beta J)' - (2\pi\beta J)}{dx_0}$$

$$= \frac{\cancel{x_0}}{\cancel{dx_0}} (2\pi\beta J) \left(\cancel{1} - (2\pi\beta J) \left(4\pi^2 (y x_0^2)^2\right) \cancel{dx_0}/\cancel{x_0} - \cancel{1} \right)$$

$$= -(2\pi\beta J)^2 \left(4\pi^2 (y x_0^2)^2\right)$$

$$= -(n+2)^2 \frac{4\pi^2}{(4\pi)^2} (4\pi y x_0^2)^2 = -\frac{1}{4}(n+2)^2 m^2,$$

$$\frac{dm}{dl} = \frac{\cancel{x_0}}{\cancel{dx_0}} 4\pi y x_0^2 \left(\cancel{1} - (2\pi\beta J - 2) \frac{\cancel{dx_0}}{\cancel{x_0}} - \cancel{1} \right)$$

$$= -mn.$$

s) Check Mathematica file attached

t) We indeed have to $\Theta(m)$

$$\frac{d}{dl}(n^2 - m^2) = 2\left(n\frac{dn}{dl} - m\frac{dm}{dl}\right) = 2m^2 n\left(1 - \frac{1}{4}(n+2)^2\right)$$

$$= 0 + \mathcal{O}(m^2).$$

u) The eqn $n^2 - m^2 = c$ allows for solutions $m = 0$ iff $c \geq 0$. Thus, the critical point is at $c = 0$, which corresponds to

$$2\pi\beta_c J - 2 = n = \pm m = 4\pi y x_0^2 \longrightarrow 0$$

i.e.

$$\beta_c = \frac{1}{\pi J}.$$

v)

w) For $C < 0$, we see that our theory flows to large m in our Mathematica graph

x) We've learned that the XY model is dual to both, the solid-on-solid model and, through the identification of vertices, to a 2D Coulomb gas.