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Quantum Field Theory 1

Homework 4

1. a) In components,

$$\mathcal{L} = \bar{\psi}_a (i\gamma^{\mu a}{}_b \partial_\mu - m\delta^a{}_b)\psi^b$$

$$= i\gamma^{\mu a}{}_b \bar{\psi}_a \partial_\mu \psi^b - m\bar{\psi}_a \psi^a$$

$$= i\bar{\psi}_a \gamma^{0a}{}_b \dot{\psi}^b + i\bar{\psi}_a \vec{\gamma}^a{}_b \cdot \vec{\nabla}\psi^b - m\bar{\psi}_a\psi^a.$$

Thus

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^a} = i\bar{\psi}_b \gamma^{0b}{}_c \delta^c{}_a = i\psi^*_d \gamma^{0d}{}_b \gamma^{0b}{}_a$$

$$= i\psi^*_d \delta^d{}_a = i\psi^*_a.$$

In here $*$ corresponds to complex conjugation, i.e. $(\psi^\dagger)_d = (\psi_d)^* \equiv \psi^*_d$. On the other hand,

$$\bar{\pi}^a := \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}_a} = 0.$$

Thus

$$\mathcal{H} = \pi_a \dot{\psi}^a - \mathcal{L} = i\psi_a^* \dot{\psi}^a - \mathcal{L}$$

$$= i\psi^\dagger \dot{\psi} - \bar{\psi}(i\not{\partial} - m)\psi = i\psi^\dagger \gamma^0\gamma^0 \dot{\psi} - \bar{\psi}(i\not{\partial} - m)\psi$$

$$= \cancel{i\bar{\psi}\gamma^0\dot{\psi}} - \cancel{i\bar{\psi}\gamma^0\dot{\psi}} - i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi$$

$$= \bar{\psi}(-i\gamma^i\partial_i + m)\psi.$$

b) Now, noticing that all x dependence is on the exponentials

$$(-i\gamma^i\partial_i + m)\psi = \int dV_{\vec{p}} \sum_{s=1}^{2} \Big((-i\gamma^i(-ip_i) + m) u^{(s)}(\vec{p}) b^{(s)}(\vec{p}) e^{-ip\cdot x}$$

$$+ (-i\gamma^i(ip_i) + m) v^{(s)}(\vec{p}) c^{(s)}(\vec{p})^* e^{ip\cdot x} \Big)$$

$$\int dV_{\vec{p}} \sum_{s=1}^{2} \Big((-\gamma^i p_i + m) u^{(s)}(\vec{p}) b^{(s)}(\vec{p}) e^{-ip\cdot x}$$

$$+ (\gamma^i p_i + m) v^{(s)}(\vec{p}) c^{(s)}(\vec{p})^* e^{ip\cdot x} \Big).$$

We have

$$0 = (-\not{p} + m) u^{(s)}(\vec{p}) = (-\gamma^0 E_{\vec{p}} - \gamma^i p_i + m) u^{(s)}(\vec{p})$$

$$0 = (\not{p} + m) v^{(s)}(\vec{p}) = (\gamma^0 E_{\vec{p}} + \gamma^i p_i + m) v^{(s)}(\vec{p})$$

Therefore

$$(-i\gamma^i\partial_i + m)\,\psi = -\int dV_{\vec{p}}\ \gamma^0 E_{\vec{p}} \sum_{s=1}^{2}\Big(u^{(s)}(\vec{p})\, b^{(s)}(\vec{p})\, e^{-ip\cdot x}$$

$$- v^{(s)}(\vec{p})\, c^{(s)}(\vec{p})^{\dagger}\, e^{ip\cdot x}\Big).$$

Then

$$H = \int d^3\vec{x}\ \bar{\psi}(x)\,(-i\gamma^i\partial_i + m)\,\psi(x)$$

$$= \int \frac{d^3\vec{x}\, d^3\vec{q}\, d^3\vec{p}}{(2\pi)^3 2E_{\vec{q}}\,(2\pi)^3 2E_{\vec{p}}}\ E_{\vec{p}} \sum_{r,s=1}^{2}$$

$$\Big(\bar{u}^{(r)}(\vec{q})\, b^{(r)}(\vec{q})^{*}\, e^{iq\cdot x} + \bar{v}^{(r)}(\vec{q})\, c^{(r)}(\vec{q})\ e^{-iq\cdot x}\Big)\gamma^0$$

$$\Big(u^{(s)}(\vec{p})\, b^{(s)}(\vec{p})\, e^{-ip\cdot x} - v^{(s)}(\vec{p})\, c^{(s)}(\vec{p})^{*}\, e^{ip\cdot x}\Big)$$

$$= \int \frac{d^3\vec{x}\, d^3\vec{q}\, d^3\vec{p}}{(2\pi)^3 2E_{\vec{q}}\,(2\pi)^3 2E_{\vec{p}}}\ E_{\vec{p}} \sum_{r,s=1}^{2}$$

$$\Big(u^{(r)}(\vec{q})^{+} u^{(s)}(\vec{p})\, b^{(r)}(\vec{q})^{*}\, b^{(s)}(\vec{p})\, e^{i(q-p)\cdot x}$$

$$- v^{(r)}(\vec{q})^{+} v^{(s)}(\vec{p})\ c^{(r)}(\vec{q})\, c^{(s)}(\vec{p})^{*}\ e^{-i(q-p)\cdot x}$$

$$+ v^{(r)}(\vec{q})^{+} u^{(s)}(\vec{p})\, c^{(r)}(\vec{q})\, b^{(s)}(\vec{p})\, e^{-i(q+p)\cdot x}$$

$$- u^{(r)}(\vec{q})^{+} v^{(s)}(\vec{p})\, b^{(r)}(\vec{q})^{*} c^{(s)}(\vec{p})^{*}\ e^{i(q+p)\cdot x}\Big)$$

$$= \int \frac{d^3\vec{q}\, d^3\vec{p}}{2E_{\vec{q}}(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \sum_{r,s=1}^{2}$$

$$\Big(u^{(r)}(\vec{p})^+ u^{(s)}(\vec{p})\, b^{(r)}(\vec{p})^* b^{(s)}(\vec{p})\, \delta^3(\vec{q}-\vec{p})$$

$$- v^{(r)}(\vec{p})^+ v^{(s)}(\vec{p})\, c^{(r)}(\vec{p})\, c^{(s)}(\vec{p})^*\, \delta^3(\vec{q}-\vec{p})$$

$$+ v^{(r)}(\vec{q})^+ u^{(s)}(-\vec{q})\, c^{(r)}(\vec{q})\, b^{(s)}(-\vec{q})\, \delta^3(\vec{q}+\vec{p})$$

$$- u^{(r)}(\vec{q})^+ v^{(s)}(-\vec{q})\, b^{(r)}(\vec{q})^* c^{(s)}(-\vec{q})^*\, \delta^3(\vec{q}+\vec{p}) \Big)$$

$$= \int \frac{d^3\vec{q}\, d^3\vec{p}}{2E_{\vec{q}}(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \sum_{r,s=1}^{2} \Big(2E_{\vec{p}}\, \delta^{rs}\, b^{(r)}(\vec{p})^* b^{(s)}(\vec{p}) - 2E_{\vec{p}}\, \delta^{rs}\, c^{(r)}(\vec{p})\, c^{(s)}(\vec{p})^* \Big)\, \delta^3(\vec{q}-\vec{p})$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \sum_{s=1}^{2} \Big(b^{(s)}(\vec{p})^* b^{(s)}(\vec{p}) - c^{(s)}(\vec{p})\, c^{(s)}(\vec{p})^* \Big).$$

In this calculation Tong's notes were essential to avoid mistakes.

c) After quantization this operator is not positive definite! Indeed

$$\langle\psi|H|\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \sum_{s=1}^{2} \Big(\| b^{(s)}(\vec{p})|\psi\rangle \|^2 - \| c^{(s)}(\vec{p})|\psi\rangle \|^2 \Big).$$

The solution will be to quantize making use of anticommutation relations. Then, if

$$\{ c^{(s)}(\vec{p}), c^{(s)}(\vec{q})^{+} \} = (2\pi)^3 2E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q}),$$

we have a Hamiltonian

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \sum_{s=1}^{2} \left(b^{(s)}(\vec{p})^{+} b^{(s)}(\vec{p}) + c^{(s)}(\vec{p})^{+} c^{(s)}(\vec{p}) \right)$$

$$- \int d^3\vec{p}\, E_{\vec{p}}\, \delta(\vec{0}).$$

The first term is now positive. The second is a negative infinity which may be adjudicated, in the spirit of Dirac's proposal, to the Dirac sea. Once again, Tong's lecture notes were very useful.