Quantum Field Theory I: Quiz 2

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a) We calculate directly

$$
\begin{aligned}
&i\int \mathrm{d}^3\mathbf{x}e^{-i\mathbf{k}\cdot\mathbf{x}}(\pi(\mathbf{x})-iE_{\mathbf{k}}\varphi(\mathbf{x}))=\\
&i\int\frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\Big(-iE_{\mathbf{p}}\Big(a(\mathbf{p})e^{i(-\mathbf{k}+\mathbf{p})\cdot\mathbf{x}}-a(\mathbf{p})^\dagger e^{i(-\mathbf{k}-\mathbf{p})\cdot\mathbf{x}}\Big)\\
&-iE_{\mathbf{k}}\Big(a(\mathbf{p})e^{i(-\mathbf{k}+\mathbf{p})\cdot\mathbf{x}}+a(\mathbf{p})^\dagger e^{i(-\mathbf{k}-\mathbf{p})\cdot\mathbf{x}}\Big)\Big)=\\
&\int\frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}}\big(E_{\mathbf{p}}\big(a(\mathbf{p})\delta(-\mathbf{k}+\mathbf{p})-a(\mathbf{p})^\dagger\delta(-\mathbf{k}-\mathbf{p})\big)\\
&+E_{\mathbf{k}}\big(a(\mathbf{p})\delta(-\mathbf{k}+\mathbf{p})+a(\mathbf{p})^\dagger\delta(-\mathbf{k}-\mathbf{p})\big)\big)=\\
&\frac{1}{2E_{\mathbf{k}}}(E_{\mathbf{k}}a(\mathbf{k})+E_{\mathbf{k}}a(\mathbf{k}))+\frac{1}{2E_{-\mathbf{k}}}\big(-E_{-\mathbf{k}}a(-\mathbf{k})^\dagger+E_{\mathbf{k}}a(-\mathbf{k})^\dagger\big)=\\
&\frac{1}{2E_{\mathbf{k}}}(E_{\mathbf{k}}a(\mathbf{k})+E_{\mathbf{k}}a(\mathbf{k}))+\frac{1}{2E_{\mathbf{k}}}\big(-E_{\mathbf{k}}a(-\mathbf{k})^\dagger+E_{\mathbf{k}}a(-\mathbf{k})^\dagger\big)=a(\mathbf{k}).
\end{aligned}
\tag{1}
$$

Similarly,

$$
\begin{aligned}
&i\int \mathrm{d}^3\mathbf{x}e^{i\mathbf{k}\cdot\mathbf{x}}(-\pi(\mathbf{x})-iE_{\mathbf{k}}\varphi(\mathbf{x}))=\\
&i\int\frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\Big(iE_{\mathbf{p}}\Big(a(\mathbf{p})e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}}-a(\mathbf{p})^\dagger e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}}\Big)\\
&-iE_{\mathbf{k}}\Big(a(\mathbf{p})e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}}+a(\mathbf{p})^\dagger e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}}\Big)\Big)=\\
&\int\frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}}\big(-E_{\mathbf{p}}\big(a(\mathbf{p})\delta(\mathbf{k}+\mathbf{p})-a(\mathbf{p})^\dagger\delta(\mathbf{k}-\mathbf{p})\big)\\
&+E_{\mathbf{k}}\big(a(\mathbf{p})\delta(\mathbf{k}+\mathbf{p})+a(\mathbf{p})^\dagger\delta(\mathbf{k}-\mathbf{p})\big)\big)=\\
&\frac{1}{2E_{-\mathbf{k}}}(-E_{-\mathbf{k}}a(-\mathbf{k})+E_{\mathbf{k}}a(-\mathbf{k}))+\frac{1}{2E_{\mathbf{k}}}\big(E_{\mathbf{k}}a(\mathbf{k})^\dagger+E_{\mathbf{k}}a(\mathbf{k})^\dagger\big)=\\
&\frac{1}{2E_{\mathbf{k}}}(-E_{\mathbf{k}}a(-\mathbf{k})+E_{\mathbf{k}}a(-\mathbf{k}))+\frac{1}{2E_{\mathbf{k}}}\big(E_{\mathbf{k}}a(\mathbf{k})^\dagger+E_{\mathbf{k}}a(\mathbf{k})^\dagger\big)=a(\mathbf{k})^\dagger.
\end{aligned}
\tag{2}
$$

b) We have

$$
\begin{aligned}
&= -\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} e^{i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{k}\cdot\mathbf{y})} \left([\pi(\mathbf{x}), \pi(\mathbf{y})] - iE_{\mathbf{k}}[\pi(\mathbf{x}), \varphi(\mathbf{y})]\right. \\
&\quad \left.+ iE_{\mathbf{p}}[\varphi(\mathbf{x}), \pi(\mathbf{y})] - E_{\mathbf{p}}E_{\mathbf{k}}[\varphi(\mathbf{x}), \varphi(\mathbf{y})]\right) \\
&= -\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} e^{i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{k}\cdot\mathbf{y})} (iE_{\mathbf{k}}[\varphi(\mathbf{y}), \pi(\mathbf{x})]. + iE_{\mathbf{p}}[\varphi(\mathbf{x}), \pi(\mathbf{y})]) \\
&= -\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} e^{i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{k}\cdot\mathbf{y})} i(E_{\mathbf{k}} + E_{\mathbf{p}}) i\delta(\mathbf{x}-\mathbf{y}) \\
&= \int \mathrm{d}^3\mathbf{x} e^{i(-\mathbf{p}+\mathbf{k})\cdot\mathbf{x}} (E_{\mathbf{k}} + E_{\mathbf{p}}) = (E_{\mathbf{k}} + E_{\mathbf{p}})(2\pi)^3\delta(\mathbf{k}-\mathbf{p}) \\
&= 2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{k})
\end{aligned} \tag{3}
$$