Homework 1: 3d Gravity as a Chern-Simons Theory

2.2 Global Symmetries of 3d Einstein Manifolds

We define the coordinates on Minkowski space.

In[7]:=
```
coordM = {x0, x1, x2, x3}
η = {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Out[7]= $\left\{x0, \sqrt{x0^2-\alpha}\ \mathrm{Cos}[\phi]\ \mathrm{Sin}[\theta], \sqrt{x0^2-\alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi], \sqrt{x0^2-\alpha}\ \mathrm{Cos}[\theta]\right\}$

Out[8]= {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

We then take the coordinates on *N* with

In[2]:=
```
coordN = {x0, θ, ϕ}
```

Out[2]= {x0, θ, ϕ}

We now stablish the relations between them

In[3]:=

$$x1 = \sqrt{x0^2-\alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Cos}[\phi]$$
$$x2 = \sqrt{x0^2-\alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi]$$
$$x3 = \sqrt{x0^2-\alpha}\ \mathrm{Cos}[\theta]$$

Out[3]= $\sqrt{x0^2-\alpha}\ \mathrm{Cos}[\phi]\ \mathrm{Sin}[\theta]$

Out[4]= $\sqrt{x0^2-\alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi]$

Out[5]= $\sqrt{x0^2-\alpha}\ \mathrm{Cos}[\theta]$

We compute the Jacobian matrix

In[14]:=
```
J = Table[D[coordM[[μ]], coordN[[ν]]], {μ, 1, 4}, {ν, 1, 3}];
J // MatrixForm
```
Out[15]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{x0\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]}{\sqrt{x0^2-\alpha}} & \sqrt{x0^2-\alpha}\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi] & -\sqrt{x0^2-\alpha}\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] \\ \frac{x0\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]}{\sqrt{x0^2-\alpha}} & \sqrt{x0^2-\alpha}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi] & \sqrt{x0^2-\alpha}\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] \\ \frac{x0\,\mathrm{Cos}[\theta]}{\sqrt{x0^2-\alpha}} & -\sqrt{x0^2-\alpha}\,\mathrm{Sin}[\theta] & 0 \end{pmatrix}$$

Finally, we compute the induced metric

In[16]:=
```
gamma = Jᵀ.η.J // Simplify;
gamma // MatrixForm
```
Out[17]//MatrixForm=

$$\begin{pmatrix} \frac{\alpha}{x0^2-\alpha} & 0 & 0 \\ 0 & x0^2-\alpha & 0 \\ 0 & 0 & \left(x0^2-\alpha\right)\mathrm{Sin}[\theta]^2 \end{pmatrix}$$