Iván Mauricio Burbano Aldana

Perimeter Institute for Theoretical Physics

Homework 3: Energy levels of the anharmonic oscillator in λq^4.

1) Inserting the resolution of the identity corresponding to H_0°

$$K_0^\circ(q'',\beta;q',0) = \langle q''| e^{-\beta H_0^\circ} \sum_{n=0}^{\infty} |n\rangle\langle n| q'\rangle$$

$$= \sum_{n=0}^{\infty} \langle q''| e^{-\beta H_0^\circ} |n\rangle\langle n| q'\rangle$$

$$= \sum_{n=0}^{\infty} e^{-\beta E_n^\circ} \langle q''|n\rangle\langle n|q'\rangle$$

$$= \sum_{n=0}^{\infty} e^{-\beta E_n^\circ} \phi_n(q'')\phi_n^*(q').$$

2) Letting $q''=q'=q$ in the above and integrating over q,

$$Z_0^\circ(\beta) = \int dq\, K_0^\circ(q,\beta;q,0) = \sum_{n=0}^{\infty} e^{-\beta E_n^\circ} \cancelto{1}{\int dq |\phi_n(q)|^2}$$

$$= \sum_{n=0}^{\infty} e^{-\beta E_n^\circ}.$$

In a similar way, if $P_n^{(\lambda)}$ is the projector onto the Eigenspace corresponding to $E_n^{(\lambda)}$

$$Z_\lambda^\circ(\beta) = \int dq\ K_\lambda^\circ(q,\beta;q,0) = \int dq\ \langle q | e^{-\beta H_\lambda^\circ} | q \rangle$$

$$= \int dq\ \langle q | e^{-\beta H_\lambda^\circ} \sum_n P_n^{(\lambda)} | q \rangle$$

$$= \sum_n e^{-\beta E_n^{(\lambda)}} \int dq\ \langle q | P_n^{(\lambda)} | q \rangle$$

$$= \sum_n e^{-\beta E_n^{(\lambda)}} \operatorname{tr}(P_n^{(\lambda)}).$$

One can put the result into the form

$$Z_\lambda^\circ(\beta) = \sum_n e^{-\beta E_n^{(\lambda)}}$$

if one interprets the index n as listing the eigenvalues, each appearing as many times as its degeneracy, rather than as indexing different eigenvalues. Of course, this is not an issue in one dimension, where, under suitable conditions, there are no degeneracies. Indeed, since H_0° is rid of these, it is reasonable to ask the same of $\overset{\circ}{H}_\lambda$ for small enough λ.

3) To exercise, I think it is worthwhile to recompute this result. We have

$$K^{\circ}_{\circ}(q'', \beta, q', 0) = \int_M \mathcal{D}q \, e^{-\frac{1}{\hbar} S_E(q)}$$

where M is some suitable space of functions $q : [0, \beta] \longrightarrow \mathbb{R}$ s.t. $q(0) = q'$ and $q(\beta) = q''$. Consider instead the integral over perturbations

$$q(\tau) = q_c(\tau) + \sqrt{\hbar}\, u(\tau)$$

of the classical solution q_c. Such belong to some suitable space N of functions $u : [0, \beta) \longrightarrow \mathbb{R}$ satisfying $u(0) = 0 = u(\beta)$. By expanding the Euclidean action in such perturbations

$$-\frac{1}{\hbar} S_E(q_c + \sqrt{\hbar}\, u) = -\frac{1}{\hbar} S_E(q_c) - \frac{1}{\sqrt{\hbar}} \int d\tau \, \cancelto{0}{\frac{\delta S_E(q)}{\delta q(\tau)}(q_c)}\, u(\tau)$$

$$- \frac{1}{2} \int d\tau_1 d\tau_2 \, u(\tau_1) \frac{\delta^2 S_E(q)}{\delta q(\tau_1) \delta q(\tau_2)}(q_c)\, u(\tau_2)$$

$$- \frac{\sqrt{\hbar}}{3!} \int d\tau_1 d\tau_2 d\tau_3 \frac{\delta^3 S_E(q)}{\delta q(\tau_1) \delta q(\tau_2) \delta q(\tau_3)} u(\tau_1) u(\tau_2) u(\tau_3)$$

$$+ \mathcal{O}(\hbar).$$

For the harmonic oscillator

$$S_E(q) = \int d\tau \left(\frac{1}{2} m q'(\tau)^2 + \frac{1}{2} m\omega^2 q(\tau)^2 \right)$$

$$\frac{\delta S_E(q)}{\delta q(\tau)} = \int d\tau_1 \left(m q'(\tau_1) \frac{d}{d\tau_1} \delta(\tau_1 - \tau) + m\omega^2 q(\tau_1)\, \delta(\tau_1 - \tau) \right)$$

$$= - \int d\tau_1\, m q''(\tau_1)\, \delta(\tau_1 - \tau) + m \Big[q'(\tau_1)\, \delta(\tau_1 - \tau) \Big]_{\tau_1 = 0}^{\beta}$$

$$+ m\omega^2 q(\tau)$$

$$= - m q''(\tau) + m\omega^2 q(\tau) + m q'(\tau) \big(\delta(\beta - \tau) - \delta(0 - \tau) \big)$$

$$\frac{\delta^2 S_E(q)}{\delta q(\tau_1)\, \delta q(\tau_2)} = \frac{\delta}{\delta q(\tau_1)} \Big(- m q''(\tau_2) + m\omega^2 q(\tau_2) + m q'(\tau_2) \big(\delta(\beta - \tau_2) - \delta(-\tau_2) \big) \Big)$$

$$= - m \frac{d^2}{d\tau_2^2} \delta(\tau_2 - \tau_1) + m\omega^2 \delta(\tau_2 - \tau_1) + m \frac{d}{d\tau_2} \delta(\tau_2 - \tau_1) \big(\delta(\beta - \tau) - \delta(-\tau) \big)$$

$$\frac{\delta^3 S_E(q)}{\delta q(\tau_1)\, \delta q(\tau_2)\, \delta q(\tau_3)} = 0.$$

Thus, in fact, for the harmonic oscillator, and in general quadratic potentials, the expansion of $\hbar^{-1} S_E$ to $O(\sqrt{\hbar})$ is exact. We conclude

$$K_0^\circ(q'',\beta;q',0) = e^{-\frac{1}{\hbar}S_E(q_c)} \int_N \mathcal{D}u \exp\left(-\frac{1}{2}\int d\tau_1 d\tau_2\, u(\tau_1)\frac{\delta^2 S_E(q)}{\delta q(\tau_1)\delta q(\tau_2)}(q_c)u(\tau_2)\right)$$

Notice that

$$\int d\tau_1 d\tau_2\, u(\tau_1)\, m \frac{d}{d\tau_2}\, \delta(\tau_2-\tau_1)\left(\delta(\beta-\tau_2)-\delta(-\tau_2)\right) u(\tau_2) = 0$$

since the integrand is supported at $\tau_2 \in \{\beta, 0\}$, where u vanishes. Thus, letting

$$A(\tau_1,\tau_2) = m\left(-\frac{d^2}{d\tau_2^2} + m\omega^2\right)\delta(\tau_2-\tau_1),$$

$$K_0^\circ(q'',\beta;q',0) = e^{-\frac{1}{\hbar}S_E(q_c)} \int_N \mathcal{D}u \exp\left(-\frac{1}{2}\int d\tau_1 d\tau_2\, u(\tau_1)A(\tau_1,\tau_2)u(\tau_2)\right).$$

Now, q_c is determined by

$$\frac{\delta S_E(q)}{\delta q(\tau)}(q_c) = 0, \qquad q_c(0) = q', \quad q_c(\beta) = q''.$$

Thus, taking $\tau \in (0,\beta)$,

$$q_c''(\tau) = \omega^2 q_c(\tau),$$

i.e.

$$q_c(\tau) = Ae^{\omega\tau} + Be^{-\omega\tau}$$

for some appropriately chosen A and B.

From the initial conditions,

$$q' = q_c(0) = A + B$$

$$q'' = q_c(\beta) = A e^{\omega\beta} + B e^{-\omega\beta}$$

$$= (q' - B) e^{\omega\beta} + B e^{-\omega\beta}$$

$$= q' e^{\omega\beta} - 2B \sinh(\omega\beta),$$

so that

$$B = \frac{1}{2\sinh(\omega\beta)} \left(q' e^{\omega\beta} - q'' \right)$$

and

$$A = q' - B = \frac{1}{2\sinh(\omega\beta)} \left(2\sinh(\omega\beta) q' - q' e^{\omega\beta} + q'' \right)$$

$$= - \frac{1}{2\sinh(\omega\beta)} \left(q' e^{-\omega\beta} - q'' \right).$$

Thus

$$q_c(\tau) = - \frac{1}{2\sinh(\omega\beta)} \left(q' e^{-\omega(\beta-\tau)} - q'' e^{\omega\tau} - q' e^{\omega(\beta-\tau)} + q'' e^{-\omega\tau} \right.$$

$$= \frac{1}{\sinh(\omega\beta)} \Big(q'' \sinh(\omega\tau) + q' \sinh(\omega(\beta-\tau)) \Big).$$

With this we can now calculate the action on the classical solution

$$S_E(q_c) = \int d\tau \left(\frac{1}{2} m q_c'(\tau)^2 + \frac{1}{2} m\omega^2 q_c(\tau)^2 \right)$$

$$= \int d\tau \left(- \frac{1}{2} m q_c''(\tau) + \cancel{\frac{1}{2} m\omega^2 q_c(\tau)} \right) q_c(\tau)$$

$$+ \frac{1}{2} m \int d\tau \frac{d}{d\tau} \left(q_c(\tau) q_c'(\tau) \right)$$

$$= \frac{1}{2} m \left(q'' q_c'(\beta) - q' q_c'(0) \right).$$

We have

$$q_c'(\tau) = \frac{\omega}{\sinh(\omega\beta)} \left(q'' \cosh(\omega\tau) - q' \cosh(\omega(\beta - \tau)) \right),$$

so that

$$q_c'(0) = \frac{\omega}{\sinh(\omega\beta)} \left(q'' - q' \cosh(\omega\beta) \right)$$

$$q_c'(\beta) = \frac{\omega}{\sinh(\omega\beta)} \left(q'' \cosh(\omega\beta) - q' \right).$$

Then

$$S_E(q_c) = \frac{m\omega}{2\sinh(\omega\beta)} \left((q'')^2 \cosh(\omega\beta) - q'' q' - q' q'' + (q')^2 \cosh(\omega\beta) \right)$$

$$= \frac{m\omega}{2\sinh(\omega\beta)} \left((q'^2 + q''^2) \cosh(\omega\beta) - 2 q' q'' \right),$$

To calculate the remaining path integral, we note that λ is an eigenvalue of A if

$$\lambda u(\tau) = \int d\tau_1 \, A(\tau,\tau_1) u(\tau_1) = m \int d\tau_1 \left(-\frac{d^2}{d\tau_1^2} + m\omega^2\right) \delta(\tau_1 - \tau) u(\tau_1)$$

$$= m \int d\tau_1 \, \delta(\tau_1 - \tau) \left(-\frac{d^2}{d\tau_1^2} + m\omega^2\right) u(\tau_1)$$

$$= -m u''(\tau) + m\omega^2 u(\tau),$$

i.e.

$$u''(\tau) = \left(\omega^2 - \frac{\lambda}{m}\right) u(\tau) = -\left(\frac{\lambda}{m} - \omega^2\right) u(\tau).$$

This is the EOM for a harmonic oscillator, and thus the positive eigenvalues are

$$\sqrt{\frac{\lambda_n}{m} - \omega^2} = \frac{n\pi}{\beta}$$

for the initial conditions $u(0) = u(\beta) = 0$. Then

$$\lambda_n = m\left(\frac{n^2\pi^2}{\beta^2} + \omega^2\right), \quad n \in \mathbb{N}^+.$$

We conclude

$$K_0^0(q'', \beta; q', 0) = e^{-\frac{1}{\hbar} S_E(q_c)} \prod_{n=0}^{\infty} \sqrt{\frac{2\pi}{\lambda_n}}.$$

This expression is regularized by taking the quotient with the corresponding free particle propagator. This is obtained by taking the $\omega = 0$ limit of our procedure thus far

$$\frac{K_0^\circ(q'',\beta;q',0)}{K_{free}(q'',\beta;q',0)} = e^{-\frac{1}{\hbar}S_E(q_c) + \frac{1}{\hbar}S_E^{free}(q_c^{free})} \prod_{n=0}^{\infty} \sqrt{\frac{\frac{n^2\pi^2}{\beta^2}}{\frac{n^2\pi^2}{\beta^2}+\omega^2}}$$

$$= e^{-\frac{1}{\hbar}S_E(q_c)} \prod_{n=0}^{\infty} \sqrt{\frac{1}{1+\frac{\beta^2\omega^2}{n^2\pi^2}}}$$

$$= e^{-\frac{1}{\hbar}S_E(q_c)} \sqrt{\frac{1}{\prod_{n=0}^{\infty}\left(1+\frac{\beta^2\omega^2}{n^2\pi^2}\right)}} .$$

$$= \sqrt{\frac{\beta\omega}{\sinh(\beta\omega)}}\, e^{-\frac{1}{\hbar}S_E(q_c)} .$$

Using the propagator calculated in the tutorial and restoring $\hbar = 1$ now that we don't need it anymore for power counting,

$$K_0^\circ(q'',\beta;q',0) = \sqrt{\frac{m\omega}{2\pi\sinh(\beta\omega)}}\, e^{\cancel{-\frac{(q''_{free}-q'_{free})^2}{2\beta}m}}\, e^{-S_E(q_c) + S_E^{free}\cancel{(q_c^{free})}}$$

In our particular case, $m=1$, $\omega=1$ and we obtain

$$K_0^0(q'', \beta, q', 0) = \sqrt{\frac{1}{2\pi \sinh(\beta)}}\, e^{-\frac{1}{2\sinh(\beta)}\left((q'^2 + q''^2)\cosh(\beta) - 2q'q''\right)}.$$

a) The term in the exponential just corresponds to the coefficient in the steepest descent method of $e^{-\frac{1}{2}S_E(q_c)}$ and the square root comes from the regularized calculation of the Gaussian integral

$$\int_N \mathcal{D}u \exp\left(-\frac{1}{2}\int d\tau_1\, d\tau_2\, u(\tau_1)\frac{\delta^2 S(q)}{\delta q(\tau_1)\delta q(\tau_2)} u(\tau_2)\right).$$

b) We have the trace

$$Z_0^0(\beta) = \int dq\ K_0^0(q, \beta; q, 0) = \sqrt{\frac{1}{2\pi \sinh(\beta)}} \int dq \exp\left(-\frac{1}{2} q^2 \frac{2\cosh(\beta) - 2}{\sinh(\beta)}\right)$$

$$= \sqrt{\frac{1}{\cancel{2\pi \sinh(\beta)}}} \sqrt{\frac{\cancel{2\pi \sinh(\beta)}}{2\cosh(\beta) - 2}} = \frac{1}{\sqrt{2(\cosh(\beta) - 1)}}.$$

We have

$$2(\cosh(\beta) - 1) = e^{\beta} + e^{-\beta} - 2 = e^{\beta} + e^{-\beta} - 2e^{\beta/2}e^{-\beta/2}$$

$$= (e^{\beta/2} - e^{-\beta/2})^2 = 4\sinh(\beta/2)^2,$$

Thus

$$Z_0^0(\beta) = \frac{1}{2\sinh(\beta/2)}.$$

4) We have

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = Z_0^0(\beta) = \frac{1}{2\sinh(\beta/2)} = \frac{1}{e^{\beta/2} - e^{-\beta/2}}$$

$$= e^{-\beta/2} \frac{1}{1 - e^{-\beta}} = e^{-\beta/2} \sum_{n=0}^{\infty} (e^{-\beta})^n$$

$$= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})} .$$

We thus obtain

$$E_n = n + \frac{1}{2} .$$

5) We have

$$Z_\lambda^J(\beta) = \mathrm{tr}\left(U_\lambda^J(\beta/2, -\beta/2)\right) = \int dq \, \langle q | U_\lambda^J(\beta/2, -\beta/2) | q \rangle$$

$$= \int dq \int_M \mathcal{D}\tilde{q} \, e^{-S_\lambda^J(\tilde{q})} = \int_N \mathcal{D}q \, e^{-S_\lambda^J(q)}$$

$$= \int_N \mathcal{D}q \, e^{-S_0^J(q)} \, e^{-\int d\tau \, \lambda q(\tau)^4}$$

$$= \int_N \mathcal{D}q \, e^{-S_0^0(q) + \int d\tau \, J(\tau) q(\tau)} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int d\tau_1 \cdots d\tau_n \, q(\tau_1)^4 \cdots q(\tau_n)^4 .$$

Noticing that

$$\frac{\delta}{\delta J(\tau_1)} e^{\int d\tau\, J(\tau)\, q(\tau)} = e^{\int d\tau\, J(\tau) q(\tau)} \int d\tau\, \delta(\tau-\tau_1)\, q(\tau)$$

$$= q(\tau_1)\, e^{\int d\tau\, J(\tau)\, q(\tau)},$$

we have

$$Z_\lambda^J(\beta) = \int_N \mathcal{D}q\, e^{-S_0^0(q)} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int d\tau_1 \cdots d\tau_n \prod_{i=1}^{n} \left(\frac{\delta^4}{\delta J(\tau_i)^4} e^{\int d\tau\, J(\tau) q(\tau)} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \prod_{i=1}^{n} \int d\tau_i \frac{\delta^4}{\delta J(\tau_i)^4} \int_N \mathcal{D}q\, e^{-S_0^0(q)}\, e^{\int d\tau\, J(\tau) q(\tau)}$$

$$= e^{-\lambda \int d\tau \frac{\delta^4}{\delta J(\tau)}} Z_0^J(\beta).$$

In here, clearly N is a suitable space of functions $q : [-\beta/2, \beta/2] \longrightarrow \mathbb{R}$ s.t. $q(-\beta/2) = q(\beta/2)$.

6) We have using eqn (3)

$$Z_0^J(\beta) = \int dq\, \langle q | U_0^J(\beta/2, -\beta/2) | q \rangle$$

$$= \int dq\, \langle q | U_0^0(\beta/2, -\beta/2) | q \rangle \exp\left(\int_{-\beta/2}^{\beta/2} du\, L(u) J(u) + \frac{1}{2} \int_{-\beta/2}^{\beta/2} du\, dv\, M(u,v) J(u) J(v) \right),$$

From (4)

$$\int_{-\beta/2}^{\beta/2} du\, L(u) J(u) = \int_{-\beta/2}^{\beta/2} du\, J(u) \frac{q}{\sinh(\beta)} \left(\sinh(u + \beta/2) + \sinh(\beta/2 - u) \right)$$

$$= \frac{2q}{\sinh(\beta)} \int_{-\beta/2}^{\beta/2} du\, J(u) \sinh(\beta/2) \cosh(u)$$

$$= \frac{q}{\cosh(\beta/2)} \int_{-\beta/2}^{\beta/2} du\, J(u) \cosh(u)$$

using $\sinh(\beta) = \sinh(2\beta/2) = 2\sinh(\beta/2)\cosh(\beta/2)$. On the other hand,

$$\sinh(\beta/2 - v)\sinh(u + \beta/2) = \left(\sinh(\beta/2)\cosh(v) - \cosh(\beta/2)\sinh(v)\right) \times$$

$$\left(\sinh(u)\cosh(\beta/2) + \cosh(u)\sinh(\beta/2)\right)$$

$$= \frac{1}{2}\sinh(\beta)\sinh(u)\cosh(v) - \cosh(\beta/2)^2 \sinh(u)\sinh(v)$$

$$+ \sinh(\beta/2)^2 \cosh(u)\cosh(v) - \frac{1}{2}\sinh(\beta)\cosh(u)\sinh(v)$$

$$= \frac{1}{2}\sinh(\beta)\sinh(u-v) + \cosh(\beta/2)^2\cosh(u-v)$$

$$- \cosh(u)\cosh(v),$$

so that

$$H(u,v) = \Theta(v-u)\frac{1}{2}\sinh(u-v) + \Theta(u-v)\frac{1}{2}\sinh(v-u)$$

$$+ \Theta(v-u)\frac{\cosh(\beta/2)^2}{\sinh(\beta)}\cosh(u-v) + \Theta(u-v)\frac{\cosh(\beta/2)^2}{\sinh(\beta)}\cosh(v-u)$$

$$- \Theta(v-u)\frac{1}{\sinh(\beta)}\cosh(u)\cosh(v) - \Theta(u-v)\frac{1}{\sinh(\beta)}\cosh(v)\cosh(u)$$

$$= -\frac{1}{2}\sinh(|u-v|) + \frac{\cosh(\beta/2)}{2\sinh(\beta/2)}\cosh(|u-v|)$$

$$-\frac{1}{\sinh(\beta)}\cosh(u)\cosh(v)$$

$$= \frac{1}{2\sinh(\beta/2)}\Big(\cosh(\beta/2)\cosh(|u-v|) - \sinh(\beta/2)\sinh(|u-v|)\Big)$$

$$-\frac{1}{\sinh(\beta)}\cosh(u)\cosh(v)$$

$$= \frac{\cosh(\beta/2-|u-v|)}{2\sinh(\beta/2)} - \frac{1}{\sinh(\beta)}\cosh(u)\cosh(v)$$

$$= \Delta(u-v) - \frac{1}{\sinh(\beta)}\cosh(u)\cosh(v).$$

Thus

$$Z_0^J(\beta) = \int dq \langle q|\, U_0^0(-\beta/2,\beta/2)\,|q\rangle \exp\Bigg(\frac{q}{\cosh(\beta/2)}\int_{-\beta/2}^{\beta/2} du\, J(u)\cosh(u)$$

$$+\frac{1}{2}\int_{-\beta/2}^{\beta/2} du\,dv\, \Delta(u-v)J(u)J(v) - \frac{1}{2\sinh(\beta)}\left(\int_{-\beta/2}^{\beta/2} du\, J(u)\cosh(u)\right)^2\Bigg)$$

From now on, let $a^J = \int_{-\beta/2}^{\beta/2} du\, J(u)\cosh(u)$. Then, with (2)

$$\int dq \langle q|U_0^0(-\beta/2,\beta/2)|q\rangle\, e^{q\frac{a^J}{\cosh(\beta/2)}} =$$

$$= \sqrt{\frac{1}{2\pi \sinh(\beta)}} \int dq \, \exp\left(- \frac{q^2}{\sinh(\beta)} \left(\cosh(\beta) - 1\right) + q \frac{a^J}{\cosh(\beta/2)} \right)$$

$$= \sqrt{\frac{1}{2\pi \sinh(\beta)}} \int dq \exp\left(- \frac{q^2}{\sinh(\beta)} \, 2 \sinh\left(\beta/2\right)^2 + q \frac{a^J}{\cosh(\beta/2)} \right)$$

$$= \sqrt{\frac{1}{2\pi \sinh(\beta)}} \int dq \exp\left(- \frac{\sinh(\beta/2)}{\cosh(\beta/2)} \left(q - \frac{a^J}{2\sinh(\beta/2)} \right)^2 \right)$$

$$\exp\left(\frac{{a^J}^2}{4 \sinh(\beta/2) \cosh(\beta/2)} \right)$$

$$= \sqrt{\frac{1}{2\pi \sinh(\beta)}} \sqrt{\frac{\pi \cosh(\beta/2)}{\sinh(\beta/2)}} \exp\left(\frac{(a^J)^2}{4 \sinh(\beta/2) \cosh(\beta/2)} \right)$$

$$= \frac{1}{2 \sinh(\beta/2)} \exp\left(\frac{(a^J)^2}{2 \sinh(\beta)} \right).$$

Recalling 3 b) we have

$$Z_0^J(\beta) = Z_0^0(\beta) \exp\left(\frac{1}{2} \int_{-\beta/2}^{\beta/2} du \, dv \, \Delta(u-v) \, J(u) \, J(v) \right).$$

7) Through 5) we have

$$Z_\lambda^0(\beta) = e^{-\lambda \int d\tau \frac{\delta^4}{\delta J(\tau)}} Z_0^J(\beta) \Big|_{J=0}$$

$$= Z_0^0(\beta) - \lambda \int d\tau \frac{\delta^4}{\delta J(\tau)} Z_0^J(\beta) \Big|_{J=0} + \mathcal{O}(\lambda^2).$$

Using (5)

$$Z^{o}_{\lambda}(\beta) = Z^{o}_{0}(\beta)\left(1 - \lambda\int d\tau \frac{\delta^4}{\delta J(\tau)^4} \exp\left(\frac{1}{2}\int_{-\beta/2}^{\beta/2} du\, dv\, \Delta(u-v) J(u) J(v)\right)\right)\Bigg|_{J=0}$$

$$= Z^{o}_{0}(\beta)\left(1 - \lambda\int d\tau \frac{\delta^4}{\delta J(\tau)^4} \frac{1}{8}\int_{-\beta/2}^{\beta/2} du_1 dv_1 du_2 dv_2\, \Delta(u_1-v_1)\Delta(u_2-v_2) J(u_1)J(v_1) J(u_2)J(v_2)\right)$$

$$= Z^{o}_{0}(\beta)\left(1 - \frac{\lambda}{8} 4! \int d\tau du_1 dv_1 du_2 dv_2\, \Delta(u_1-v_1)\Delta(u_2-v_2)\delta(u_1-\tau)\delta(v_1-\tau)\delta(u_2-\tau)\delta(v_2-\tau)\right)$$

$$= Z^{o}_{0}(\beta)\left(1 - 3\lambda\int d\tau\, \Delta(0)\Delta(0)\right)$$

$$= Z^{o}_{0}(\beta)\left(1 - 3\lambda\beta\left(\frac{1}{2}\coth(\beta/2)\right)^2\right)$$

$$= Z^{o}_{0}(\beta)\left(1 - \frac{3\lambda\beta}{4}\coth(\beta/2)^2\right)$$

8) We have

$$\sum_{n=0}^{\infty} \exp\left(-\beta\left(n + \frac{1}{2} + \frac{3}{2}\lambda\left(n^2+n+\frac{1}{2}\right) + \mathcal{O}(\lambda^2)\right)\right)$$

$$= \sum_{n=0}^{\infty} e^{-\beta(n+1/2)} \exp\left(\frac{3}{2}\lambda\left(n^2+n+\frac{1}{2}\right)\right) + \mathcal{O}(\lambda^2)$$

$$= \sum_{n=0}^{\infty} e^{-\beta(n+1/2)}\left(1 + \frac{3}{2}\lambda\left(n^2+n+\frac{1}{2}\right) + \mathcal{O}(\lambda^2)\right) + \mathcal{O}(\lambda^2)$$

$$= Z^0_0(\beta) + \frac{3}{2}\lambda\left(\sum_{n=0}^{\infty}\left(n^2+n+1/2\right)e^{-\beta(n+1/2)}\right)$$

We have

$$\frac{\partial}{\partial\beta}\sum_{n=0}^{\infty}e^{-\beta(n+1/2)} = \sum_{n=0}^{\infty}e^{-\beta(n+1/2)}\left[-(n+1/2)\right]$$

$$\frac{\partial}{\partial\beta}\frac{1}{2\sinh(\beta/2)} = -\frac{\cosh(\beta/2)}{4\sinh(\beta/2)^2} = -Z^0_0(\beta)\coth(\beta/2)$$

and

$$\frac{\partial^2}{\partial\beta^2}\sum_{n=0}^{\infty}e^{-\beta(n+1/2)} = \sum_{n=0}^{\infty}e^{-\beta(n+1/2)}\left(n+1/2\right)^2$$

$$= \sum_{n=0}^{\infty}e^{-\beta(n+1/2)}\left(n^2+n+1/4\right)$$

$$= \sum_{n=0}^{\infty}e^{-\beta(n+1/2)}\left(n^2+n+1/2\right) - \frac{1}{4}Z^0_0(\beta)$$

$$\frac{\partial^2}{\partial\beta^2}\frac{1}{2\sinh(\beta/2)} = -\frac{\partial}{\partial\beta}\frac{\cosh(\beta/2)}{4\sinh(\beta/2)^2}$$

$$= -\frac{1}{4}\left(\frac{\cancel{\sinh(\beta/2)}}{2\sinh(\beta/2)^{\cancel{2}}} - 2\,\frac{\cosh(\beta/2)^2}{2\sinh(\beta/2)^3}\right)$$

$$= -\frac{1}{4}Z^0_0(\beta)\left(1 - 2\coth(\beta/2)^2\right).$$

Thus

$$\sum_{n=0}^{\infty}e^{-\beta E_n^{(\lambda)}} = Z^0_0(\beta) + \frac{3}{2}\lambda\left(\frac{1}{4}Z^0_0(\beta) - \frac{1}{4}Z^0_0(\beta)\left(1-2\coth(\beta/2)^2\right)\right)$$

$$= Z_0^0(\beta)\left(1 + \frac{3}{2}x\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} + \frac{1}{2}\coth(\beta/2)^2\right)\right)$$

$$= Z_0^0(\beta)\left(1 + \frac{3}{4}x\coth(\beta/2)^2\right) = Z_x^0(\beta).$$