Iván Mauricio Burbano Aldana

Perimeter Scholars International

Quantum Field Theory II

Homework 3: Renormalization of ϕ^4 theory

1.a) We know that at one loop

$$\Gamma(\varphi) = S(\varphi) + \frac{\hbar}{2} \operatorname{tr}\left(\log S''(\varphi)\right) + \mathcal{O}(\hbar^2).$$

For the action

$$S(\varphi) = \int d^4x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}\varphi^4\right)$$

$$= \int d^4x \left(\frac{1}{2}\varphi(-\Delta + m^2)\varphi + \frac{g}{4!}\varphi^4\right),$$

we have

$$\frac{\delta S(\varphi)}{\delta\varphi(x)} = \int d^4y \left(\frac{1}{2}\,\delta(y-x)(-\Delta+m^2)\,\varphi(y) + \frac{1}{2}\,\varphi(y)\,(-\Delta_y+m^2)\,\delta(y-x) + \frac{g}{3!}\,\varphi(y)^3\,\delta(y-x) \right)$$

$$= \frac{1}{2}\,(-\Delta+m^2)\,\varphi(x) + \int d^4y\ \frac{1}{2}\left(-\Delta_y+m^2\right)\varphi(y)\,\delta(y-x) + \frac{g}{3!}\,\varphi(x)^3$$

$$= (-\Delta+m^2)\,\varphi(x) + \frac{g}{3!}\,\varphi(x)^3 ,$$

and

$$\frac{\delta S(\varphi)}{\delta\varphi(x)\,\delta\varphi(y)} = (-\Delta_y+m^2)\,\delta(y-x) + \frac{g}{2}\,\varphi(y)^2\,\delta(x-y).$$

Therefor,

$$S''(\varphi)(\psi)(x) = \int d^4y\ \frac{\delta S(\varphi)}{\delta\varphi(x)\,\delta\varphi(y)}\ \psi(y)$$

$$= \int d^4y \left((-\Delta_y + m^2)\,\delta(y-x)\,\psi(y) + \frac{g}{2}\varphi(y)^2\delta(x-y)\,\psi(y) \right)$$

$$= (-\Delta + m^2)\,\psi(x) + \frac{g}{2}\varphi(x)^2\,\psi(x).$$

We conclude

$$S''(\varphi) = (-\Delta + m^2) + \frac{g}{2}\varphi^2.$$

Our effective action is then

$$\Gamma(\varphi) = S(\varphi) + \frac{\hbar}{2}\,tr\left(\log\left((-\Delta + m^2) + g/_2\,\varphi^2\right)\right)$$

$$= S(\varphi) + \frac{\hbar}{2}\,tr\left(\log\left((-\Delta+m^2)\left(\mathbb{1} + \frac{g}{2}(-\Delta+m^2)^{-1}\circ\varphi^2\right)\right)\right)$$

$$= S(\varphi) + \frac{\hbar}{2}\,tr\left(\log(-\Delta + m^2)\right)$$

$$+ \frac{\hbar}{2}\sum_{k=1}^{\infty}\left(\frac{g}{2}\right)^k \frac{(-1)^{k+1}}{k}\,tr\left[\left(\frac{g}{2}(-\Delta+m^2)^{-1}\circ\varphi^2\right)^k\right].$$

Since the Kernel of $(-\Delta + m^2)^{-1}\circ\varphi$ is

$$\left((-\Delta+m^2)^{-1}\circ\varphi^2\right)(x,y)=\int d^4z\, G_0(x-z)\,\varphi(z)^2\,\delta(z-y)$$

$$=G_0(x-y)\,\varphi(y)^2,$$

we get

$$\Gamma(\varphi)=S(\varphi)+\frac{\hbar}{2}\,\mathrm{tr}\left(\log(-\Delta+m^2)\right)$$

$$+\frac{\hbar}{2}\sum_{K=1}^{\infty}\left(\frac{g}{2}\right)^K\frac{(-1)^{K+1}}{K}\int d^4x_1\cdots d^4x_K\;\varphi(x_1)^2\,G(x_1-x_2)\cdots$$

$$\varphi(x_2)^2\,G(x_2-x_3)\cdots\varphi(x_K)^2\,G(x_K-x_1),$$

We can now compute the 4-point amputated irreducible diagrams

$$\Gamma^{(4)}(x_1,x_2,x_3,x_4)=\left.\frac{\delta^4\Gamma(\varphi)}{\delta\varphi(x_1)\delta\varphi(x_2)\delta\varphi(x_3)\delta\varphi(x_4)}\right|_{\varphi=0}$$

$$= \frac{\hbar}{2}\,\frac{g^2}{4}\,\frac{1}{2}\,(-1)\int d^4y_1\, d^4y_2\; G(y_1-y_2)\,G(y_2-y_1)$$

$$\underbrace{\frac{\delta^4\left(\varphi(y_1)^2\,\varphi(y_2)^2\right)}{\delta\varphi(x_1)\,\delta\varphi(x_2)\,\delta\varphi(x_3)\,\delta\varphi(x_4)}}$$

$$= \frac{\delta^3}{\delta\varphi(x_1)\,\delta\varphi(x_2)\,\delta\varphi(x_3)}\Big(2\varphi(y_1)\,\delta(y_1-x_4)\,\varphi(y_2)^2 + 2\varphi(y_1)^2\,\varphi(y_2)\,\delta(y_2-x_4)\Big)$$

$$= \frac{\delta^2}{\delta\varphi(x_1)\,\delta\varphi(x_2)}\Big(2\,\delta(y_1-x_3)\,\delta(y_1-x_4)\,\varphi(y_2)^2 + 4\,\varphi(y_1)\,\delta(y_1-x_4)\,\varphi(y_2)\,\delta(y_2-x_3)$$

$$+ 4\,\varphi(y_1)\,\delta(y_1-x_3)\,\varphi(y_2)\,\delta(y_2-x_4) + 2\,\varphi(y_1)^2\,\delta(y_2-x_3)\,\delta(y_2-x_4)\Big)$$

$$= 4\,\delta(y_1-x_3)\,\delta(y_1-x_4)\,\delta(y_2-x_2)\,\delta(y_2-x_1)$$

$$+ 4\,\delta(y_1-x_2)\,\delta(y_1-x_4)\,\delta(y_2-x_1)\,\delta(y_2-x_3)$$

$$+ 4\,\delta(y_1-x_1)\,\delta(y_1-x_4)\,\delta(y_2-x_2)\,\delta(y_2-x_3)$$

$$+ 4\,\delta(y_1-x_2)\,\delta(y_1-x_3)\,\delta(y_2-x_1)\,\delta(y_2-x_4)\,.$$

$$+ 4\,\delta(y_1-x_1)\,\delta(y_1-x_3)\,\delta(y_2-x_2)\,\delta(y_2-x_4)$$

$$+ 4\,\delta(y_1-x_2)\,\delta(y_1-x_1)\,\delta(y_2-x_3)\,\delta(y_2-x_4)$$

$$\Gamma^{(4)}(x_1,x_2,x_3,x_4) = -\hbar\frac{g^2}{4}\Big(G(x_3-x_2)^2\delta(x_3-x_4)\delta(x_2-x_1)$$

$$+G(x_2-x_1)^2\delta(x_2-x_4)\delta(x_1-x_3)+G(x_1-x_2)^2\delta(x_1-x_4)\delta(x_2-x_3)$$

$$+G(x_2-x_1)^2\delta(x_2-x_3)\delta(x_1-x_4)+G(x_1-x_2)^2\delta(x_1-x_3)\delta(x_2-x_4)$$

$$+G(x_2-x_3)^2\delta(x_2-x_1)\delta(x_3-x_4)\Big)$$

$+\cdots$

Well, let us just write the result

$$I(p,m_R;\Lambda)=\int_{\|k\|\le\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m_R^2}\frac{1}{(p+k)^2+m_R^2}$$

b) We have the trick

$$\frac{1}{AB}=\int_0^1 dx\,\frac{1}{(xA+(1-x)B)^2}$$

taken from physics.mcgill.ca/~jcline/qft1b.pdf.

$$I(p,0;\Lambda) = \int_0^1 dx \int_{\|k\| \le \Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(x(p+k)^2 + (1-x)k^2\right)^2}$$

$$= \int_0^1 dx \int_{\|k\| \le \Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(xp^2 + 2x\, p\cdot k + k^2\right)^2}$$

$$= \int_0^1 dx \int_{\|k\| \le \Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{\left((k+xp)^2 - x^2p^2 + xp^2\right)^2}$$

$$= \int_0^1 dx \int_{B(0,\Lambda)+xp} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 + p^2x(1-x)\right)^2}$$

Taking $B(0,\Lambda) + xp \approx B(0,\Lambda)$ for large Λ,

$$I(p,0,\Lambda) = \int_0^1 dx \int_{S^3} d\Omega \int_0^\Lambda \frac{du}{(2\pi)^4} u^3 \frac{1}{\left(u^2 + p^2x(1-x)\right)^2}$$

$v = u^2 + p^2x(1-x)$

$dv = 2u\,du$

$$= \mathrm{area}(S^3) \int_0^1 dx \int_{p^2x(1-x)}^{\Lambda^2 + p^2x(1-x)} \frac{dv}{2(2\pi)^4} \frac{v - p^2x(1-x)}{v^2}.$$

Using $\mathrm{area}(S^3) = 2\pi^2$ from sjsu.edu/faculty/watkins/ndim.htm, we get

$$I(p,0;\Lambda) = \frac{\pi^2}{(2\pi)^4} \int_0^1 dx \left(\ln\left(\frac{\Lambda^2 + p^2 x(1-x)}{p^2 x(1-x)} \right) \right.$$

$$\left. - p^2 x(1-x) \left(-\frac{1}{\Lambda^2 + p^2 x(1-x)} + \frac{1}{p^2 x(1-x)} \right) \right)$$

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \left(\ln\left(\frac{\Lambda^2 + p^2 x(1-x)}{p^2 x(1-x)} \right) - \left(1 - \frac{p^2 x(1-x)}{\Lambda^2 + p^2 x(1-x)} \right) \right)$$

For big Λ,

$$I(p,0,\Lambda) \approx \frac{1}{(4\pi)^2} \int_0^1 dx \left(\ln\left(\frac{\Lambda^2}{p^2 x(1-x)} \right) - 1 \right)$$

$$\approx \frac{1}{(4\pi)^2} \int_0^1 dx \left(\ln\left(\frac{\Lambda^2}{p^2} \right) - \ln\left(x(1-x)\right) - 1 \right)$$

$$\approx \frac{1}{(4\pi)^2} \ln\left(\frac{\Lambda^2}{p^2} \right) + \text{Finite terms}$$

c) We have

$$\frac{\partial}{\partial (m_R)^2} I(p, m_R; \Lambda) = \frac{\partial}{\partial (m_R)^2} \int_{\|k\| \leq \Lambda} d^4k \, \frac{1}{k^2 + m_R^2} \, \frac{1}{(k+p)^2 + m_R^2}$$

$$= \int_{\|k\| \leq \Lambda} d^4k \left(- \frac{1}{(k^2+m_R^2)^2} \, \frac{1}{(k+p)^2 + m_R^2} - \frac{1}{k^2+m_R^2} \, \frac{1}{((k+p)^2 + m_R^2)^2} \right).$$

At large k this behaves like

$$-2 \int_{\|k\| \leq \Lambda} d^4k \, \frac{1}{k^6} \simeq -2 \, \mathrm{Area}(S^3) \underbrace{\int^{\Lambda} du \, \frac{1}{u^3}}_{\text{Finite!}}$$

d) For this choice of momenta

$$g_R = \Gamma^{(4)}(p_1, p_2, p_3, p_4) = g_R - \frac{\hbar g_R^2}{2} \frac{3}{(4\pi)^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$+ \hbar C_1 .$$

Thus $$C_1 = g_R^2 \, \frac{3}{2} \, \frac{1}{(4\pi)^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right).$$

e) For general momenta

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = g_R - \frac{\hbar}{2} g_R^2 \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\Lambda^2}{(p_1+p_2)^2}\right) + \ln\left(\frac{\Lambda^2}{(p_1+p_3)^2}\right)\right.$$

$$\left. + \ln\left(\frac{\Lambda^2}{(p_1+p_4)^2}\right)\right)$$

$$+ \hbar g_R^2 \frac{3}{2} \frac{1}{(4\pi)^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$= g_R - \frac{\hbar}{2} g_R^2 \cancel{\frac{3}{(4\pi)^2}} \ln(\Lambda)^2 + \hbar g_R^2 \frac{3}{2} \cancel{\frac{1}{(4\pi)^2}} \ln(\Lambda^2)$$

$$- \frac{\hbar}{2} g_R^2 \frac{1}{(4\pi)^2}\left(3\ln(\mu^2) + \ln\left(\frac{1}{(p_1+p_2)^2}\right) + \ln\left(\frac{1}{(p_1+p_3)^2}\right)\right.$$

$$\left. + \ln\left(\frac{1}{(p_1+p_4)^2}\right)\right)$$

$$= g_R - \frac{\hbar}{2} g_R^2 \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\mu^2}{(p_1+p_2)^2}\right) + \ln\left(\frac{\mu^2}{(p_1+p_3)^2}\right) + \ln\left(\frac{\mu^2}{(p_1+p_4)^2}\right)\right).$$

f) We want

$$g_R' - \frac{\hbar}{2} g_R'^2 \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\mu'^2}{(p_1+p_2)^2}\right)+\ln\left(\frac{\mu'^2}{(p_1+p_3)^2}\right)+\ln\left(\frac{\mu'^2}{(p_1+p_4)^2}\right)\right)$$

$$= g_R - \frac{\hbar}{2} g_R^2 \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\mu^2}{(p_1+p_2)^2}\right)+\ln\left(\frac{\mu^2}{(p_1+p_3)^2}\right)+\ln\left(\frac{\mu^2}{(p_1+p_4)^2}\right)\right),$$

In particular, for

$$(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = \mu'^2,$$

$$g_R = g_R' + \frac{\hbar}{2} g_R^2 \frac{3}{(4\pi)^2} \ln\left(\frac{\mu^2}{\mu'^2}\right).$$

We see that,

$$g_R(\mu) = g_R^0 + \hbar (g_R^0)^2 \frac{3}{2}\frac{1}{(4\pi)^2} \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

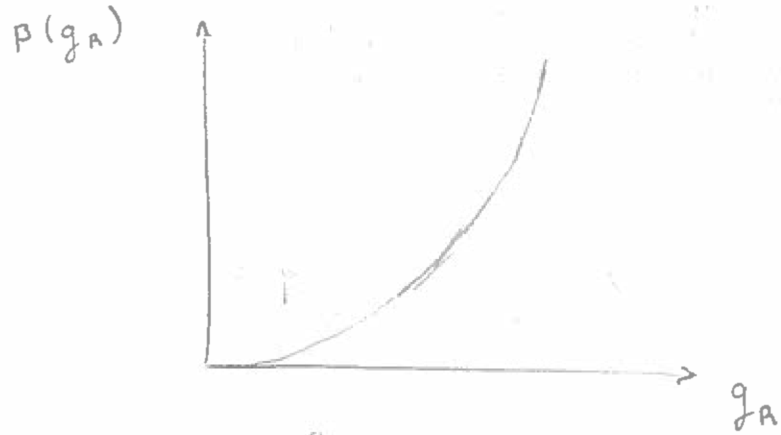
in fact works for all choices of momenta.

g) We have

$$\beta(g_R(\mu)) = \mu \frac{d g_R(\mu)}{d\mu}\Big|_{g_R(\mu_0) = g_R^0} .$$

$$= \hbar \, (g_R^0)^2 \, \frac{3}{\cancel{\epsilon}} \, \frac{1}{(4\pi)^2} \, \cancel{\epsilon}$$

$$= \frac{3}{(4\pi)^2} \hbar \, g_R(\mu)^2 + \mathcal{O}(g_R^{\,3})$$



h) We see that for small coupling, β is small. Thus, the effective coupling is insensitive to changes in energy. On the other hand, big coupling's are very sensitive to energy scales.

Independently however, β is always positive.

Thus, the coupling grows as the energy scale grows

i)

$$T(m_R : \Lambda) = \int_{\|K\| \le \Lambda} \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 + m_R^2}$$

j) We have already used $d^D x = d\Omega_D \, dr \, r^{D-1}$.

Thus

$$\pi^{D/2} = \int d^D x \, e^{-x^2} = \int d\Omega_D \int_0^\infty dr \, r^{D-1} e^{-r^2} \qquad \begin{aligned} u &= r^2 \\ du &= 2r\,dr \end{aligned}$$

$$= \int d\Omega_D \int_0^\infty du \, \frac{1}{2} \, u^{\frac{D-2}{2}} e^{-u} = \int d\Omega_D \, \frac{1}{2} \Gamma\left(\frac{D-2}{2} + 1\right).$$

Thus

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}.$$

k) We have

$$T(m_R;\Lambda) = \frac{1}{(2\pi)^4}\int d\Omega_4 \int_0^{\Lambda} du\, u^3 \,\frac{1}{u^2+m_R^2}$$

$v = u^2 + m_R^2$

$dv = 2u\,du$

$$= \frac{8\pi^2}{(2\pi)^4 \underbrace{\cancel{\Gamma(2)}}_{1}} \frac{1}{8}\int_{m_R^2}^{\Lambda^2+m_R^2} dv \quad \frac{v - m_R^2}{v}$$

$$= \frac{\pi^2}{(2\pi)^4}\left(\Lambda^2 + \cancel{m_R^2} - \cancel{m_R^2} - m_R^2 \ln\left(\frac{\Lambda^2+m_R^2}{m_R^2}\right)\right)$$

$$= \frac{1}{(4\pi)^2}\left(\Lambda^2 - m_R^2 \ln\left(\frac{\Lambda^2}{m_R^2}\right) - m_R^2 \ln\left(1 + \frac{m_R^2}{\Lambda^2}\right)\right).$$

l) We have

$$m_R^2 = \tilde{\Gamma}_R^{(2)}(0, m_R, g_R, \mu) = m_R^2 + \hbar \frac{g_R}{2} T(m_R;\Lambda)$$

$$+ \hbar B_{1,0}(g_R, \mu, \Lambda)$$

$$+ \hbar m_R^2 B_{1,1}(g_R, \mu, \Lambda)\Big|_{\mu = m_R}$$

Thus

$$0 = \frac{1}{2(4\pi)^2} \hbar g_R \Lambda^2 + \hbar B_{1,0}(g_R, m_R; \Lambda)$$

$$- \hbar m_R^2 \left(\frac{1}{2} g_R \frac{1}{(4\pi)^2} \ln\left(\frac{\Lambda^2}{m_R^2}\right) - B_{1,1}(g_R, m_R; \Lambda) \right)$$

$$- \hbar g_R \frac{1}{2(4\pi)^2} m_R^2 \ln\left(1 + \frac{m_R^2}{\Lambda}\right).$$

Since we are interested in large Λ, we can take the last term as vanishing.

Thus

$$B_{1,0}(g_R, \mu; \Lambda) = -\frac{1}{2(4\pi)^2} g_R \Lambda^2$$

$$B_{1,1}(g_R, \mu; \Lambda) = g_R \frac{1}{2(4\pi)^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right).$$

m) We indeed have

$$\tilde{\Gamma}^{(2)}_R(p, m_R, g_R, \mu) = p^2 + m_R^2$$

$$+ \frac{1}{2}\hbar g_R \frac{1}{(4\pi)^2}\left(\cancel{\Lambda^2} - m_R^2 \ln\left(\frac{\Lambda^2}{m_R^2}\right) - m_R^2 \ln\left(1 + \frac{m_R^2}{\Lambda^2}\right)\right)$$

$$\cancel{- \frac{1}{2}\hbar \frac{1}{(4\pi)^2} g_R \Lambda^2} + \hbar m_R^2 g_R \frac{1}{2(4\pi)^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$= p^2 + m_R^2 - \frac{1}{2(4\pi)^2}\hbar g_R m_R^2 \ln\left(\frac{\mu^2}{m_R^2}\right)$$

$$- \frac{1}{2(4\pi)^2}\hbar g_R m_R^2 \ln\left(1 + \frac{m_R^2}{\Lambda^2}\right),$$

which is finite for all Λ and in $\Lambda \to \infty$.

n) We have, in $\Lambda \to \infty$

$$\cancel{p^2} + m_R(\mu)^2 - \frac{1}{2(4\pi)^2}\hbar g_R m_R(\mu)^2 \ln\left(\frac{\mu^2}{m_R(\mu)^2}\right)$$

$$= \cancel{p^2} + m_R(\mu')^2 - \frac{1}{2(4\pi)^2}\hbar g_R m_R(\mu')^2 \ln\left(\frac{\mu'^2}{m_R(\mu')^2}\right).$$

Following Bruno,

$$m_R(\mu)^2 = m_R(\mu')^2 \; \frac{1 - \frac{1}{2(4\pi)^2}\hbar g_R \ln\left(\frac{\mu'^2}{m_R(\mu')^2}\right)}{1 - \frac{1}{2(4\pi)^2}\hbar g_R \ln\left(\frac{\mu^2}{m_R(\mu)^2}\right)}$$

$$= m_R(\mu')^2 \left(1 - \frac{1}{2(4\pi)^2}\hbar g_R \ln\left(\frac{\mu'^2}{m_R(\mu')^2}\right)\right) \times$$

$$\left(1 + \frac{1}{2(4\pi)^2}\hbar g_R \ln\left(\frac{\mu^2}{m_R(\mu)^2}\right) + O(\hbar^2)\right)$$

$$= m_R(\mu')^2 \left(1 - \frac{1}{2(4\pi)^2}\hbar g_R \left(\ln\left(\frac{\mu'^2}{m_R(\mu')^2}\right) - \ln\left(\frac{\mu^2}{m_R(\mu)^2}\right)\right)\right.$$

$$\left. + O(\hbar^2)\right)$$

$$= m_R(\mu')^2 \left(1 - \frac{1}{2(4\pi)^2}\hbar g_R \ln\left(\frac{m_R(\mu)^2}{m_R(\mu')^2}\,\frac{\mu'^2}{\mu^2}\right) + O(\hbar^2)\right).$$

From the above result however,

$$\frac{m_R(\mu)^2}{m_R(\mu')^2} = 1 + O(\hbar),$$ so that

$$m_R(\mu)^2 = m_R(\mu')^2 \left(1 + \frac{1}{2(4\pi)^2} \hbar g_R \ln\left(\frac{\mu^2}{\mu'^2}\right)\right)$$

o) We have

$$m_{phys}^2 := m_R^2 - \frac{1}{2(4\pi)^2} \hbar g_R m_R^2 \ln\left(\frac{\mu^2}{m_R^2}\right)$$

$$= m_R^2$$

when $\mu^2 = m_R^2$.

p) We have

$$\gamma_{m^2}(g_R) := \mu \frac{\partial}{\partial \mu} \ln\left(m_R^2(\mu)\right)$$

$$= \mu \frac{\partial}{\partial \mu} \ln\left(1 + \frac{1}{2(4\pi)^2} \hbar g_R \ln\left(\frac{\mu^2}{\mu_0^2}\right)\right)$$

$$= \cancel{\mu} \frac{1}{1 + \frac{1}{2(4\pi)^2} \hbar g_R \ln\left(\frac{\mu^2}{\mu_0^2}\right)} \frac{1}{\cancel{2}(4\pi)^2} \hbar g_R \cancel{2} \frac{1}{\cancel{\mu}}$$

$$= \frac{\hbar g_R}{(4\pi)^2}.$$