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Homework 1: Equivalence principle at work: charge in a lab.

1.a) We have

$$u^\mu(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0)^\mu$$

and

$$\frac{du^\mu(\tau)}{d\tau} = g\,(\sinh(g\tau), \cosh(g\tau), 0, 0)^\mu = g\,(u^1(\tau), u^0(\tau), 0, 0)^\mu$$

This solves the EOM,

$$\frac{du^\mu(\tau)}{d\tau} = K F^{\mu\nu} u_\nu(\tau) = K\left(F^{01} u_1(\tau), F^{10} u_0(\tau), 0, 0\right)^\mu$$

$$= K\left(E u^1(\tau), E u^0(\tau), 0, 0\right)^\mu$$

if $g = KE$. Moreover, the proper time is

$$\int_0^\tau d\tau \sqrt{u^0(\tau)^2 - u^1(\tau)^2} = \int_0^\tau d\tau \sqrt{\cosh(g\tau)^2 - \sinh(g\tau)^2}$$

$$= \int_0^\tau d\tau = \tau,$$

showing that τ is the proper time.

b) We have

$$a^\mu a_\mu = g^2\left(-\sinh(g\tau)^2 + \cosh(g\tau)^2\right) = g^2.$$

c) Exploring with the code we see that small four-acceleration correspond to very flat curves. As g increses the curvature becomes bigger and the particle comes closer to the origin.

d) The parametrization in (7) satisfies

$$x^2 - t^2 = \left(\frac{1}{g} + X\right)^2 \geq 0.$$

Thus $x^2 \geq t^2$, i.e. $|x| \geq |t|$. Thus, the Rindler coordinates only cover this region, known as the Rindler wedge. This is confirmed in the plots on the notebook. By writing

$$(t, x) = \left(\frac{1}{g} + X\right)(\sinh(gT), \cosh(gT)),$$

it is clear that the constant T curves correspond to straight lines through the origin with a slope

$$\operatorname{cotanh}(gT).$$

This slope quickly stabilizes as T grows to 1. On the other hand, the lines of constant X, trace out the trajectories of uniformly accelerated particles explored in c). This is seen by the replacement

$$\frac{1}{g'} := \frac{1}{g} + X, \quad t' := \frac{gT}{g'} = gT\left(\frac{1}{g} + X\right),$$

s.t. $$(t, x) = \frac{1}{g'}\left(\sinh(g't'), \cosh(g't')\right)$$

The boundary of the Rindler wedge is the space where $|t| = |x|$, which is the light cone.

e) If $x^\mu = (t, x, \rho, \varphi)^\mu$ and $x'^\mu = (T, X, \rho, \varphi)^\mu$, we have

$$g'_{\mu\nu} = g_{\sigma\rho} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu}$$

$$= -\frac{\partial t}{\partial x'^\mu} \frac{\partial t}{\partial x'^\nu} + \frac{\partial x}{\partial x'^\mu} \frac{\partial x}{\partial x'^\nu} + \frac{\partial \rho}{\partial x'^\mu} \frac{\partial \rho}{\partial x'^\nu}$$

$$+ \rho^2 \frac{\partial \varphi}{\partial x'^\mu} \frac{\partial \varphi}{\partial x'^\nu} .$$

Thus

$$g'_{00} = -\left(\frac{1}{g} + X\right)^2 g^2 \cosh(gT)^2 + \left(\frac{1}{g} + X\right)^2 g^2 \sinh(gT)^2$$

$$= -\left(\frac{1}{g} + X\right)^2 g^2 = -(1 + gX)^2 ,$$

$(g = a^\mu a_\mu > 0.)$

$$g'_{11} = -\sinh(gT)^2 + \cosh(gT)^2 = 1 ,$$

$$g'_{22} = 1 , \quad g'_{33} = 1 ,$$

$$g'_{01} = - \left(\frac{1}{g} + X\right) \cancel{g \cosh(gT)} \sinh(gT) + \left(\frac{1}{g} + X\right) \cancel{g \sinh}(gT) \cosh(gT)$$

$$= 0 .$$

The other mixed elements vanish since none of the components of (t, x, ρ, φ) that depend on ρ or φ depend simultaneously with another coordinate in (T, X, ρ, φ). Thus

$$ds^2 = -(1 + gX)^2 \, dT^2 + dX^2 + d\rho^2 + \rho^2 d\varphi^2 .$$

2. a) In the Front End we showed that

$$\varphi(\vec{r}, t) = \frac{e}{\|\vec{r} - \vec{x}(t_R)\| - \vec{v}(t_R) \cdot (\vec{r} - \vec{x}(t_R))} ;$$

where t_R is the time for which $t = t_R + \|\vec{r} - \vec{x}(t_R)\|$, $\vec{x}$ is the trajectory of the particle, and e its charge. One can show also that

$$\vec{A}(\vec{r},t) = \frac{e\,\vec{v}(t_R)}{\|\vec{r}-\vec{x}(t_R)\| - \vec{v}(t_R)\cdot(\vec{r}-\vec{x}(t_R))}$$

Let us consider a trayectory y^μ the particle as a function of an arbitrary parameter τ. This is any trayectory for which

$$\vec{y}(\tau) = \vec{x}(\underbrace{y^0(\tau)}_{y^0 \text{ monotonically increasing}}).$$

Define $R^\mu(\tau) := z^\mu - y^\mu(\tau)$ for some fixed z^μ.

Then, if $u^\mu = (x^\mu)'$, then

$$R^\mu(\tau)\,u_\mu(\tau) = R^\mu(\tau)\frac{dy_\mu(\tau)}{dx^0}\,\frac{dy^0}{d\tau}(\tau)$$

$$= -\left(z^0 - y^0(\tau) - (\vec{z}-\vec{y}(\tau))\cdot\vec{v}(y^0(\tau))\right)\cdot\frac{dy^0}{d\tau}(\tau)$$

At the τ, for which $z^0 = y^0(\tau) + \|\vec{z}-\vec{y}(\tau)\|$,

$$R^\mu(\tau)\,u_\mu(\tau) = -\left(\|\vec{z}-\vec{y}(\tau)\| - \vec{v}(y^0(\tau))\cdot(\vec{z}-\vec{y}(\tau))\right)\frac{dy^0}{d\tau}(\tau)$$

On the other hand

$$u^0(\tau) = \frac{dy^0}{d\tau}(\tau), \qquad \vec{u}(\tau) = \vec{v}(y^0(\tau))\,\frac{dy^0}{d\tau}(\tau).$$

Then, at this retarded time

$$A^\mu(z) = -\frac{Q\,u^\mu(\tau)}{R^\nu(\tau)u_\nu(\tau)}$$

$$= \frac{Q\,(1, \vec{v}(y^0(\tau)))}{\|\vec{z}-\vec{y}(\tau)\| - (\vec{z}-\vec{y}(\tau))\cdot\vec{v}(y^0(\tau))} = (\varphi(\vec{z}, z^0), \vec{A}(z, z^0))$$

setting $Q=e$ and realizing $t_R = y^0(\tau)$.

b) We already showed

$$u^\mu(\tau) = g(-x_Q, t_Q, 0, 0).$$

Thus

$$u_\mu(\tau) = \eta_{\mu\nu}u^\nu(\tau) = g(-x_Q, t_Q, Q, Q).$$

We then conclude

$$A_\mu \equiv \left(\frac{Q\,u_\mu}{R^\nu u_\nu}\right)_{t = t(\tau) + \|\vec{x} - \vec{x}(\tau)\|}$$

$$= -\frac{Q\,g\,(-x_Q, t_Q, 0, 0)_\mu}{(t - t_Q(\tau))g(-x_Q) + (\vec{x}-\vec{x}_Q(\tau))\,g(t_Q, 0, 0)}$$

$$= \frac{Q\,(-x_Q, t_Q, 0, 0)_\mu}{t x_Q - \cancel{t x_Q} - x t_Q + x_Q \cancel{t_Q}} = \frac{Q}{\xi}(-x_Q, t_Q, 0, 0).$$

c) We have by squaring the retarded time condition and (8)

$$t^2 - 2t\,t_Q + \cancel{t_Q^2} = \rho^2 + x^2 - 2x\sqrt{L^2 + t_Q^2} + L^2 + \cancel{t_Q^2}, \quad \text{i.e.}$$

$$4x^2(L^2 + t_Q^2) = (\delta + 2t\,t_Q)^2$$

$$= \delta^2 + 4t^2t_Q^2 + 4t\delta t_Q .$$

We thus have the quadratic equation

$$4(x^2 - t^2)t_Q^2 - 4t\delta t_Q + 4x^2L^2 - \delta^2 = 0,$$

whose two solutions are

$$t_Q = \frac{\cancel{4}t\delta \pm \sqrt{\cancel{16}t^2\delta^2 - \cancel{16}(x^2-t^2)(4x^2L^2 - \delta^2)}}{\cancel{8}\,2(x^2 - t^2)}$$

$$= \frac{t\delta \pm \sqrt{\cancel{t^2\delta^2} - 4x^4L^2 + x^2\delta^2 + 4x^2t^2L^2 - \cancel{t^2\delta^2}}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x|\sqrt{t^2L^2 + \delta^2/4 - x^2L^2}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x|\sqrt{t^2L^2 - x^2L^2 + (\rho^2 + x^2 + L^2 - t^2)^2/4}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x| \sqrt{t^2L^2 - x^2L^2 + (L^2 + t^2 - \rho^2 - x^2 - 2L^2)^2/4}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x| \sqrt{t^2L^2 - x^2L^2 + (L^2 + t^2 - \rho^2 - x^2)^2/4 + \cancel{4}L^4/\cancel{4} - \cancel{4}L^2(L^2 + t^2 - \rho^2 - x^2)/\cancel{4}}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x| \sqrt{(L^2 + t^2 - \rho^2 - x^2)^2/4 + L^2(\cancel{t^2} \cancel{- x^2} + \cancel{4L^2} - \cancel{4L^2} \cancel{- t^2} + \rho^2 \cancel{+ x^2})}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x|\xi}{2(x^2 - t^2)}.$$

If $x > 0$, we take the $(-)$ solution to ensure we obtain the retarded instead of the advanced time. Thus

$$t_Q = \frac{t\delta - 2x\xi}{2(x^2 - t^2)}.$$

Repeating for x_Q, we have

$$x_Q^2 - 2 x x_Q + x^2 = (t - t_Q)^2 - \rho^2$$

$$= t^2 - 2 t t_Q + t_Q^2 - \rho^2$$

$$= t^2 - 2t\sqrt{x_Q^2 - L^2} + x_Q^2 - L^2 - \rho^2$$

$$4t^2 (x_Q^2 - L^2) = (t^2 - x^2 - L^2 - \rho^2 + 2 x x_Q)^2$$

$$= (-\delta + 2 x x_Q)^2 = 4 x^2 x_Q^2 + \delta^2 - 4 x \delta x_Q$$

$$4(t^2 - x^2) x_Q^2 + 4 x \delta x_Q - 4 t^2 L^2 - \delta^2 = 0$$

$$x_Q = \frac{-\cancel{4} x \delta \pm \sqrt{\cancel{16} x^2 \delta^2 - \cancel{16} (t^2 - x^2)(4t^2 L^2 - \delta^2)}}{\underset{2}{\cancel{8}}(t^2 - x^2)}$$

$$= \frac{x\delta \pm \xi t}{2(x^2 - t^2)}$$

noticing that the square root is obtained by the replacement $x \leftrightarrow t$ on the previous one, a replacement under which ξ is invariant. Once agin, we take the (-) soln

$$x_Q = \frac{x\delta - \xi t}{2(x^2 - t^2)} .$$

d) Clearly $F_{\rho\phi} = 0$ since $A_\rho = A_\phi = 0$.

The full field strength is calculated in the mathematica notebook. It describes an electric field in the X-ρ plane and a magnetic field in the φ direction

3. a) Using mathematica, one finds

$$A'_\mu = \left(-\Phi, -\frac{Q}{L+X}, 0, 0\right)_\mu$$

with

$$\Phi = \frac{gQ(2L^2 + 2LX + X^2 + \rho^2)}{\sqrt{(X^2+\rho^2)((2L+X)^2+\rho^2)}}$$

$$= \frac{Q}{r} \frac{(2/g + 2X + gr^2)}{\sqrt{4/g^2 + 4X/g + r^2}}$$

$$= \frac{Q}{r} \frac{1 + gX + g^2r^2/2}{\sqrt{1 + gX + g^2r^2/4}}.$$

b) Consider the gauge function

$$\Lambda = Q \ln (X + L).$$

Then

$$A'_\mu + \partial_\mu \Lambda = \left(\left(-\Phi, \frac{-Q}{X+L}, 0, 0\right) + \left(0, \frac{Q}{X+L}, 0, 0\right) \right)_\mu$$

$$= (-\Phi, 0, 0, 0)_\mu$$

c) We have

$$\varphi = \frac{\Phi}{1 + gX} = \frac{Q}{r} \frac{1 + gX + g^2 r^2/2}{\sqrt{1 + gX + g^2 r^2/4}} \frac{1}{1 + gX}$$

Using $(1+x)^n = 1 + nx + \frac{1}{2} n(n-1) x^2 + O(x^3)$, we have

$$\varphi = \frac{Q}{r} \left(1 + gX + g^2 r^2/2\right)\left(1 - \frac{1}{2}\left(gX + g^2 r^2/4\right) + \frac{3}{8}\left(gX + g^2 r^2/4\right)^2\right) \times$$

$$\left(1 - gX + g^2 X^2\right) + O(g^3)$$

$$= \frac{Q}{r}\left(1 + gX\left(\cancel{1} - \frac{1}{2} - \cancel{1}\right)\right.$$

$$+ g^2\left(\frac{r^2}{2} - \frac{1}{2}\,{}^{r^2}\!/_4 + \frac{3}{8}X^2 + \cancel{X^2}\right.$$

$$\left.\left. - \cancel{\frac{1}{2}X^2} - X^2 + \cancel{\frac{1}{2}X^2}\right)\right) + \mathcal{O}(g^3)$$

$$= \frac{Q}{r}\left(1 - \frac{1}{2}Xg + \frac{3}{8}(r^2 + X^2)g^2 + \mathcal{O}(g^3)\right).$$

These where verified in the mathematica file by explicit computation.

d) For small g we see the equipotentials expected from the usual Coulomb potential. As g increases, we see the potential lagging behind. In the bottom of every graph we see the Rindler horizon, corresponding to the singularity at $X = -\frac{1}{g}$. The equipotentials seem to flatten horizontally near the horizon. This is a coordinate artifact however, since the horizon is at ∞.

c) Checked in Mathematica.