Relativistic Quantum Information (March 25, 2020)

Tutorial 4: General Theory of Quantum Energy Teleportation

The System

a) Define

$$T_n' := T_n - \langle g|T_n|g\rangle,$$

and

$$H' = \sum_n T_n' = \sum_n T_n - \langle g|H|g\rangle.$$

Thus, since H' differs from H by a constant, they are equivalent.

b) Since $|g\rangle$ is an eigenstate of H, $H|g\rangle = E_0|g\rangle$.

Then

$$0 = \sum_n \langle g|T_n|g\rangle = \langle g|H|g\rangle = E_0\langle g|g\rangle = E_0.$$

Therefore $H|g\rangle = 0$.

c) Assume $T_n|g\rangle = e_n|g\rangle$. We then have

$$\langle g|T_n O_m|g\rangle = e_n \langle g|O_m|g\rangle = \langle g|T_n|g\rangle\langle g|O_m|g\rangle,$$

i.e. the state factorizes.

d) Consider the non-zero eigenvalues $\{e_{n,m}\}$ corresponding to eigenvectors $\{|n,m\rangle\}$ of T_n.

Then

$$0 = \langle g|T_n|g\rangle = \sum_m e_{n,m}|\langle g|n,m\rangle|^2.$$

Now, since $|g\rangle$ is not an eigenvector of T_n then $|g\rangle \notin \operatorname{Ker} T_n = \operatorname{span}\{|n,m\rangle\}^\perp$. Thus, at least one of the $\langle g|n,m\rangle \neq 0$. Therefore, at least one of te $e_{n,m} < 0$.

The QET Protocol

e) The probability of outcome α is given by

$$P(\alpha) = tr\,(\rho_0 \Pi_A(\alpha)) = \langle g | \Pi_A(\alpha) | g \rangle .$$

The corresponding resulting state is

$$\rho_\alpha = \frac{M_A(\alpha)\,\rho\, M_A(\alpha)^+}{\underbrace{tr(M_A(\alpha)\,\rho\, M_A(\alpha)^+)}_{tr(\Pi(\alpha)\rho)\, =\, P(\alpha).}}$$

f) $$\sum_\alpha P(\alpha)\,\rho_\alpha = \sum_\alpha M_A(\alpha)\rho\, M_A(\alpha)^+ = \rho_1$$

g) $$E_A = tr(H\rho_1) - \cancelto{0}{tr(H\rho_0)} = \sum_\alpha \langle g | M_A(\alpha)^+ H M_A(\alpha) | g \rangle$$

h) $$E_A = \sum_\alpha \langle g | M_A(\alpha)^+ H_A M_A(\alpha) | g \rangle$$

$$+ \sum_\alpha \underline{\langle g | M_A(\alpha)^+ H_{\bar{A}} M_A(\alpha) | g \rangle}$$

$$\langle g | M_A(\alpha)^+ M_A(\alpha) H_{\bar{A}} | g \rangle = \langle g | \Pi_A(\alpha) H_{\bar{A}} | g \rangle$$

$$= \cdots + \langle g | \mathbb{1}_n \cancelto{0}{H_{\bar{A}}} | g \rangle$$

i)

$$\rho_{QET} = P(\alpha)\, e^{-i\alpha\theta G_B} \rho_1 e^{i\alpha\theta G_B}$$

j)

$$E_B = \cancel{Tr(H_A \rho_1)}$$

$$- Tr(H_{\bar{A}\bar{B}} \rho_{QET}) - Tr(\cancel{H_A \rho_{QET}}) - Tr(H_B \rho_{QET})$$

$$Tr(H_A \rho_{QET}) = \sum_\alpha P(\alpha)\, Tr\left(H_A e^{i\alpha\theta G_B} \rho_1 e^{-i\alpha\theta G_B}\right)$$

$$= \sum_\alpha P(\alpha)\, Tr(H_A \rho_1) = Tr(H_A \rho_1) = E_A$$

$$Tr(H_{\bar{A}\bar{B}} \rho_{QET}) = tr(H_{\bar{A}\bar{B}} \rho_1)$$

$$= \sum_\alpha P(\alpha) \langle g | M_A(\alpha)^{+} H_{\bar{A}\bar{B}} M_A(\alpha) | g \rangle$$

$$= \sum_\alpha P \langle g | \Pi_A(\alpha) H_{\bar{A}\bar{B}} | g \rangle = \langle g | \cancelto{0}{H_{\bar{A}\bar{B}}} | g \rangle$$

$$Tr(H_B \rho_{QET}) = Tr(U_B(\alpha)^{+} H_B U_B(\alpha) \rho_1)$$

$$= \sum_\alpha \langle g | M_A(\alpha)^{+} U_B(\alpha)^{+} H_B U_B(\alpha) M_A(\alpha) | g \rangle$$

$$= \sum_\alpha \langle g | \Pi_A(\alpha) U_B(\alpha)^{+} H_B U_B(\alpha) | g \rangle$$

k) for small θ

$$E_B = -\sum_\alpha \langle g|\Pi_A(\alpha)(1+i\theta\alpha G_B)H_B(1-i\theta\alpha G_B)|g\rangle$$

$$= \underbrace{-\sum_\alpha \langle g|\Pi_A(\alpha)H_B|g\rangle}_{\langle g|H_B|g\rangle \to 0} - i\theta\sum_\alpha \alpha\langle g|\Pi_A(\alpha)G_B H_B|g\rangle$$

$$+ i\theta\sum_\alpha \alpha\langle g|\Pi_A(\alpha)G_B H_B|g\rangle$$

$$= i\theta\sum_\alpha \alpha\langle g|\Pi_A(\alpha)[G_B, H_B]|g\rangle$$

If $i\sum_\alpha \alpha\langle g|\Pi_A(\alpha)[G_B, H_B]|g\rangle < 0$, we

choose $\theta > 0$ and viceversa.