Homework 2: Detection of Vacuum Fluctuations

1. Physical Parameters

Let us define the constants as indicated in the text

In[13]:=
```
a = 1000^-1;
λ = 10^-5;
```

a)

We begin by computing with our analytical result (★)

In[ ]:=
$$\text{TableForm}\left[\text{Table}\left[\left\{T, N\left[\frac{T^2 \lambda^2}{4 \pi} e^{-\frac{T^2}{2}} \frac{1}{\left(a^2+T^2\right)^{3/2}} \left(\sqrt{a^2+T^2} - \sqrt{\frac{\pi}{2}} T^2 e^{\frac{T^4}{2\left(a^2+T^2\right)}} \text{Erfc}\left[\frac{T^2}{\sqrt{2\left(a^2+T^2\right)}}\right]\right)\right]\right\},\right.\right.$$
$$\left.\left.\{T, \{0.1, 1, 10, 100, 1000\}\}\right], \text{TableHeadings} \to \left\{\text{None}, \left\{\text{"T}\left(\Omega^{-1}\right)\text{"}, \text{"}P_{g\to e}\text{"}\right\}\right\}\right]$$

General: Exp[-5000.] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[-500000.] is too small to represent as a normalized machine number; precision may be lost.

Out[ ]//TableForm=

$T\,(\Omega^{-1})$	$P_{g\to e}$
0.1	6.99949×10^{-12}
1	1.6619×10^{-12}
10	1.49096×10^{-35}
100	0.
1000	0.

We now proceed to confirm these result through the numerical computation of (●)

In[ ]:=
$$\text{TableForm}\left[\text{Table}\left[\left\{T, \frac{T^2 \lambda^2}{4\pi} \text{NIntegrate}\left[u\, e^{-\frac{1}{2} a^2 u^2} e^{-\frac{1}{2} T^2 (1+u)^2}, \{u, 0, \infty\}\right]\right\}, \{T, \{0.1, 1, 10, 100, 1000\}\}\right], \text{TableHeadings} \to \left\{\text{None}, \left\{\text{"T}(\Omega^{-1})\text{"}, \text{"}P_{g\to e}\text{"}\right\}\right\}\right]$$

NIntegrate: Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option.

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Out[ ]//TableForm=

$T(\Omega^{-1})$	$P_{g\to e}$
0.1	6.99949×10^{-12}
1	1.6619×10^{-12}
10	1.49096×10^{-35}
100	0.
1000	0.

and (■) of the Gaussian switching function

In[ ]:=
$$\text{TableForm}\left[\text{Table}\left[\left\{T, \text{NIntegrate}\left[\frac{\lambda^2}{(2\pi)^3\, 2\, \text{Norm}[\{kx, ky, kz\}]} e^{-\frac{1}{2} a^2 \text{Norm}[\{kx,ky,kz\}]^2} \text{Abs}\left[\text{FourierTransform}\left[e^{-t^2/T^2}, t, 1 + \text{Norm}[\{kx, ky, kz\}], \text{FourierParameters} \to \{1, -1\}\right]\right]^2, \{kx, -\infty, \infty\}, \{ky, -\infty, \infty\}, \{kz, -\infty, \infty\}\right]\right\}, \{T, \{0.1, 1, 10, 100, 1000\}\}\right], \text{TableHeadings} \to \left\{\text{None}, \left\{\text{"T}(\Omega^{-1})\text{"}, \text{"}P_{g\to e}\text{"}\right\}\right\}\right]$$

NIntegrate: Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option.

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Out[ ]//TableForm=

$T(\Omega^{-1})$	$P_{g\to e}$
0.1	6.99949×10^{-12}
1	1.6619×10^{-12}
10	1.49096×10^{-35}
100	0.
1000	0.

b)

The agreement of the results above gives us confidence that the probabilities for the sudden switching function are well described by the numerical computation of (◆)

In[ ]:=
```
TableForm[Table[{T, λ^2/π^2 NIntegrate[u/(1 + u)^2 e^(-1/2 a^2 u^2) Sin[((1 + u) T)/2]^2, {u, 0, ∞}]},
  {T, {0.1, 1, 10, 100, 1000}}], TableHeadings → {None, {"T (Ω^-1)", "P_g→e"}}]
```

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037×10^7}. NIntegrate obtained 2.544110747420445` and 0.000010207456582128827` for the integral and error estimates.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037×10^7}. NIntegrate obtained 3.110605021959676` and 0.00002701068655023394` for the integral and error estimates.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037×10^7}. NIntegrate obtained 2.979600838299077` and 0.000027185686614683625` for the integral and error estimates.

General: Further output of NIntegrate::ncvb will be suppressed during this calculation.

Out[ ]//TableForm=

$T\,(\Omega^{-1})$	$P_{g\to e}$
0.1	2.57772×10^{-11}
1	3.1517×10^{-11}
10	3.01897×10^{-11}
100	3.02358×10^{-11}
1000	3.02353×10^{-11}

and (■) of this switching

In[ ]:=
```
TableForm[
 Table[{T, NIntegrate[λ^2/((2 π)^3 2 Norm[{kx, ky, kz}]) ⅇ^(-1/2 a^2 Norm[{kx,ky,kz}]^2) Abs[FourierTransform[
       {1  Abs[t] ≤ T / 2
        0  Abs[t] > T / 2 , t, 1 + Norm[{kx, ky, kz}], FourierParameters → {1, -1}]]^2,
    {kx, -∞, ∞}, {ky, -∞, ∞}, {kz, -∞, ∞}]}, {T, {0.1, 1, 10, 100, 1000}}],
 TableHeadings → {None, {"T (Ω^-1)", "P_g→e"}}]
```

NIntegrate: Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

NIntegrate: The global error of the strategy GlobalAdaptive has increased more than 2000 times. The global error is expected to decrease monotonically after a number of integrand evaluations. Suspect one of the following: the working precision is insufficient for the specified precision goal; the integrand is highly oscillatory or it is not a (piecewise) smooth function; or the true value of the integral is 0. Increasing the value of the GlobalAdaptive option MaxErrorIncreases might lead to a convergent numerical integration. NIntegrate obtained 2.5777228696266398`*^-11 and 1.0468155125045452`*^-15 for the integral and error estimates.

NIntegrate: Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

NIntegrate: The global error of the strategy GlobalAdaptive has increased more than 2000 times. The global error is expected to decrease monotonically after a number of integrand evaluations. Suspect one of the following: the working precision is insufficient for the specified precision goal; the integrand is highly oscillatory or it is not a (piecewise) smooth function; or the true value of the integral is 0. Increasing the value of the GlobalAdaptive option MaxErrorIncreases might lead to a convergent numerical integration. NIntegrate obtained 3.151743783218376`*^-11 and 1.8302815905412002`*^-14 for the integral and error estimates.

NIntegrate: Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

General: Further output of NIntegrate::slwcon will be suppressed during this calculation.

NIntegrate: The global error of the strategy GlobalAdaptive has increased more than 2000 times. The global error is expected to decrease monotonically after a number of integrand evaluations. Suspect one of the following: the working precision is insufficient for the specified precision goal; the integrand is highly oscillatory or it is not a (piecewise) smooth function; or the true value of the integral is 0. Increasing the value of the GlobalAdaptive option MaxErrorIncreases might lead to a convergent numerical integration. NIntegrate obtained 3.019162323351664`*^-11 and 6.429668206306268`*^-14 for the integral and error estimates.

General: Further output of NIntegrate::eincr will be suppressed during this calculation.

Out[ ]//TableForm=

$T\ (\Omega^{-1})$	$P_{g\to e}$
0.1	2.57772×10^{-11}
1	3.15174×10^{-11}
10	3.01916×10^{-11}
100	3.54192×10^{-11}
1000	3.39258×10^{-11}

The matching between the first three values of the last two calculations gives us confidence that the simplifications done to arrive to (◆) from (■) were correct. Since the expression is much simpler and the run time is much smaller, we will use the expression (◆) for the rest of the computations.

d)

Now we redefine (★) and (◆) so that they depend on *a* instead of *T*.

In[3]:=
```
Clear[a];
T = 1;
```

In[12]:=

$$\text{TableForm}\Big[\text{Table}\Big[\Big\{a,\ \text{N}\Big[\frac{T^2\lambda^2}{4\pi}\,e^{-\frac{T^2}{2}}\,\frac{1}{(a^2+T^2)^{3/2}}\Big(\sqrt{a^2+T^2}-\sqrt{\frac{\pi}{2}}\,T^2\,e^{\frac{T^4}{2(a^2+T^2)}}\,\text{Erfc}\Big[\frac{T^2}{\sqrt{2(a^2+T^2)}}\Big]\Big)\Big],$$
$$\frac{\lambda^2}{\pi^2}\,\text{NIntegrate}\Big[\frac{u}{(1+u)^2}\,e^{-\frac{1}{2}a^2u^2}\,\text{Sin}\Big[\frac{(1+u)\,T}{2}\Big]^2,\ \{u,0,\infty\}\Big]\Big\},$$
$$\{a, \{0.001, 0.01, 0.1, 1, 10, 100, 1000\}\}\Big],$$
$$\text{TableHeadings}\to\{\text{None}, \{\text{"}a_0\,(\Omega^{-1})\text{"}, \text{"}P_{g\to e,\,\text{Gaussian}}\text{"}, \text{"}P_{g\to e,\,\text{Sudden}}\text{"}\}\}\Big]$$

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037×10^7}. NIntegrate obtained 3.110605021959676` and 0.000027010686550233 94` for the integral and error estimates.

Out[12]//TableForm=

$a_0\ (\Omega^{-1})$	$P_{g\to e,\ \text{Gaussian}}$	$P_{g\to e,\ \text{Sudden}}$
0.001	1.6619×10^{-12}	3.1517×10^{-11}
0.01	1.66181×10^{-12}	1.99627×10^{-11}
0.1	1.65286×10^{-12}	9.25005×10^{-12}
1	1.09651×10^{-12}	1.61583×10^{-12}
10	4.22738×10^{-14}	2.27607×10^{-14}
100	4.76613×10^{-16}	2.32387×10^{-16}
1000	4.82057×10^{-18}	2.32836×10^{-18}

2. Experimental Implementation

a)

We will fix $a_0 = 0.0001\,\Omega^{-1} = 1\,\text{x}10^{-18}\,s \simeq 1\,\text{Å}$, the order of atomic radii. Let us compute the probabilities by looking at what happens from $T = 10^{-15}\,s = 10^{-15}\,s\,10^{14}\,\text{Hz}\,\Omega^{-1} = 0.1\,\Omega^{-1}$ to $T = 1\,s = 10^{14}\,\Omega^{-1}$.

In[26]:=
```
a = 0.0001;
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In[27]:=
$$\text{TableForm}\left[\text{Table}\left[\left\{T, N\left[\frac{T^2\lambda^2}{4\pi} e^{-\frac{T^2}{2}} \frac{1}{(a^2+T^2)^{3/2}} \left(\sqrt{a^2+T^2} - \sqrt{\frac{\pi}{2}}\, T^2\, e^{\frac{T^4}{2(a^2+T^2)}}\, \text{Erfc}\left[\frac{T^2}{\sqrt{2(a^2+T^2)}}\right]\right)\right],\right.\right.$$
$$\left.\left.\frac{\lambda^2}{\pi^2}\text{NIntegrate}\left[\frac{u}{(1+u)^2} e^{-\frac{1}{2}a^2u^2} \text{Sin}\left[\frac{(1+u)\,T}{2}\right]^2, \{u, 0, \infty\}\right]\right\}, \{T, \text{Table}[10^n, \{n, -1, 14\}]\}\right],$$
$$\text{TableHeadings} \to \{\text{None}, \{\text{"T}(\Omega^{-1})\text{"}, \text{"P}_{g\to e,\,\text{Gaussian}}\text{"}, \text{"P}_{g\to e,\,\text{Sudden}}\text{"}\}\}\Big]$$

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037 × 10^7}. NIntegrate obtained 3.6942017177362554` and 0.021974725456850606` for the integral and error estimates.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037 × 10^7}. NIntegrate obtained 4.260674211087046` and 0.02038555463982336` for the integral and error estimates.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037 × 10^7}. NIntegrate obtained 4.24302805190276` and 0.0284865212567462` for the integral and error estimates.

General: Further output of NIntegrate::ncvb will be suppressed during this calculation.

General: Exp[−5000.] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−500000.] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−5. × 10^7] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

General: Overflow occurred in computation.

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General: Overflow occurred in computation.

General: Further output of General::ovfl will be suppressed during this calculation.

Out[27]//TableForm=

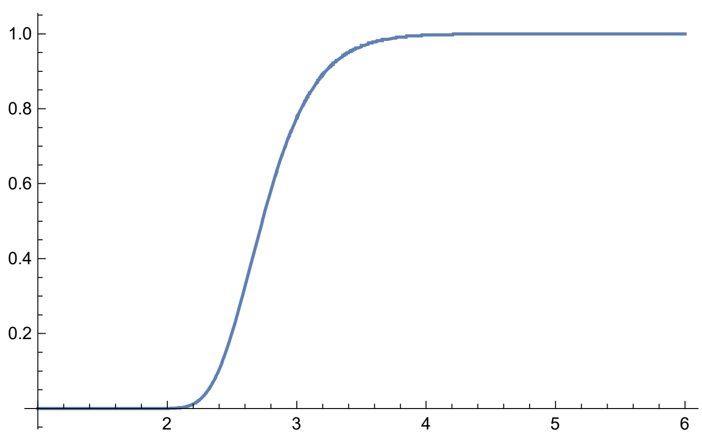
$T\ (\Omega^{-1})$	$P_{g\to e,\ \text{Gaussian}}$	$P_{g\to e,\ \text{Sudden}}$
$\frac{1}{10}$	7.00014×10^{-12}	3.74301×10^{-11}
1	1.6619×10^{-12}	4.31697×10^{-11}
10	1.49096×10^{-35}	4.29909×10^{-11}
100	0.	4.18615×10^{-11}
1000	0.	4.1888×10^{-11}
10 000	0.	4.1888×10^{-11}
100 000	0.	4.18883×10^{-11}
1 000 000	0.	4.18866×10^{-11}
10 000 000	0.	2.90664×10^{-11}
100 000 000	0.	2.65697×10^{-11}
1 000 000 000	0.	2.65448×10^{-11}
10 000 000 000	0.	2.65447×10^{-11}
100 000 000 000	0.	1.26499×10^{-11}
1 000 000 000 000	0.	3.22351×10^{-12}
10 000 000 000 000	0.	9.78495×10^{-13}
100 000 000 000 000	0.	9.78494×10^{-13}

b)

We will be satisfied by having an excitation per second with 90% chance. If the time scale of the experiment is T, we will assume that in one second we can perform the experiment $\frac{1\,s}{T} = \frac{10^{14}\,\Omega^{-1}}{T}$ times in a second. The probability of not having an excitation in the time T is given by $1 - P_{g\to e}$. Thus, the probability of not being excited after $1\,s$ is given by $(1 - P_{g\to e})^{10^{14}\,\Omega^{-1}/T}$. We will look for a time scale for which this probability is 10%. As is shown in the plot below, this is possible. (Thanks to Dalila for helping me understand what this question was about).

In[70]:= $$\text{Plot}\left[\left(1 - \frac{T^2 \lambda^2}{4\pi} e^{-\frac{T^2}{2}} \frac{1}{(a^2+T^2)^{3/2}} \left(\sqrt{a^2+T^2} - \sqrt{\frac{\pi}{2}}\, T^2\, e^{\frac{T^4}{2(a^2+T^2)}}\, \text{Erfc}\left[\frac{T^2}{\sqrt{2(a^2+T^2)}}\right]\right)\right)^{10^{14}/T}, \{T, 1, 6\}\right]$$

Out[70]=



However, for some reason NSolve doesn't seem to be able to find a solution.

In[71]:= $\text{NSolve}\left[\left(1-\frac{T^2\lambda^2}{4\pi}e^{-\frac{T^2}{2}}\frac{1}{(a^2+T^2)^{3/2}}\left(\sqrt{a^2+T^2}-\sqrt{\frac{\pi}{2}}T^2 e^{\frac{T^4}{2(a^2+T^2)}}\text{Erfc}\left[\frac{T^2}{\sqrt{2(a^2+T^2)}}\right]\right)\right)^{10^{14}/T}==0.1, T\right]$

NSolve: Inverse functions are being used by NSolve, so some solutions may not be found; use Reduce for complete solution information.

Out[71]= { }

We can however, graphically guess an approximate solution $T = 2.4\,\Omega^{-1} = 2.4\,\text{x}10^{-14}\,s$

In[103]:= $\left(1-\frac{T^2\lambda^2}{4\pi}e^{-\frac{T^2}{2}}\frac{1}{(a^2+T^2)^{3/2}}\left(\sqrt{a^2+T^2}-\sqrt{\frac{\pi}{2}}T^2 e^{\frac{T^4}{2(a^2+T^2)}}\text{Erfc}\left[\frac{T^2}{\sqrt{2(a^2+T^2)}}\right]\right)\right)^{10^{14}/T}$ /. T → 2.4

Out[103]= 0.104134

c)

We now repeat the computations of a) with the time scales now between $T = 10^{-6}\,s = 10^3\,\Omega^{-1}$ to $T = 1\,s = 10^9\,\Omega^{-1}$ and the new coupling constant

In[106]:= λ = 0.1;

In[107]:=

```
TableForm[Table[{T, N[(T^2 λ^2)/(4 π) e^(-T^2/2) 1/(a^2 + T^2)^(3/2) (Sqrt[a^2 + T^2] - Sqrt[π/2] T^2 e^(T^4/(2 (a^2+T^2))) Erfc[T^2/Sqrt[2 (a^2 + T^2)]])],
   λ^2/π^2 NIntegrate[u/(1 + u)^2 e^(-1/2 a^2 u^2) Sin[((1 + u) T)/2]^2, {u, 0, ∞}]}, {T, Table[10^n, {n, 3, 9}]}],
 TableHeadings → {None, {"T (Ω^-1)", "P_(g→ e, Gaussian)", "P_(g→ e, Sudden)"}}]
```

General: Exp[−500000.] is too small to represent as a normalized machine number; precision may be lost.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037 × 10^7}. NIntegrate obtained 4.134178366057366` and 0.011010671603865707` for the integral and error estimates.

General: Exp[−5. × 10^7] is too small to represent as a normalized machine number; precision may be lost.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037 × 10^7}. NIntegrate obtained 4.134175065764332` and 0.011013681195073122` for the integral and error estimates.

General: Exp[−5. × 10^9] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {4.81037 × 10^7}. NIntegrate obtained 4.134213418021526` and 0.0109754878928362` for the integral and error estimates.

General: Further output of NIntegrate::ncvb will be suppressed during this calculation.

General: Overflow occurred in computation.

General: Overflow occurred in computation.

Out[107]//TableForm=

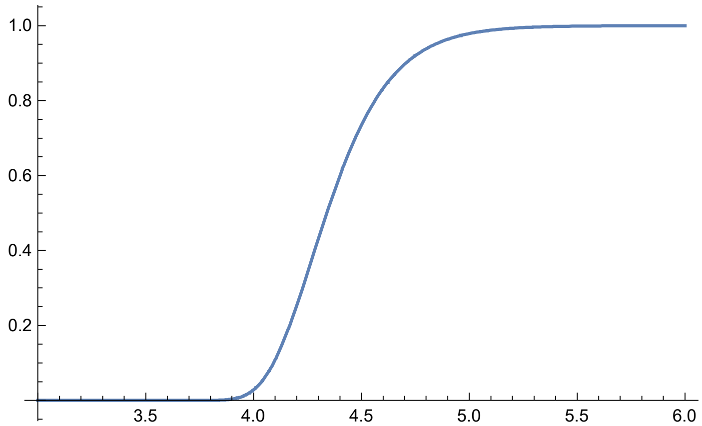
$T\ (\Omega^{-1})$	$P_{g\to e,\ \text{Gaussian}}$	$P_{g\to e,\ \text{Sudden}}$
1000	0.	0.0041888
10 000	0.	0.0041888
100 000	0.	0.00418883
1 000 000	0.	0.00418866
10 000 000	0.	0.00290664
100 000 000	0.	0.00265697
1 000 000 000	0.	0.00265448

d)

Repeating the same technique as before, we have

In[125]:=
$$\text{Plot}\left[\left(1-\frac{T^2\,\lambda^2}{4\,\pi}\,e^{-\frac{T^2}{2}}\,\frac{1}{\left(a^2+T^2\right)^{3/2}}\left(\sqrt{a^2+T^2}-\sqrt{\frac{\pi}{2}}\;T^2\,e^{\frac{T^4}{2\left(a^2+T^2\right)}}\,\text{Erfc}\left[\frac{T^2}{\sqrt{2\left(a^2+T^2\right)}}\right]\right)\right)^{10^9/T},\ \{T,3,6\}\right]$$

Out[125]=



We thus guess $T = 4.1\,\Omega^{-1} = 4.1\,\text{x}10^{-9}\,s$.

In[129]:=
$$\left(1-\frac{T^2\,\lambda^2}{4\,\pi}\,e^{-\frac{T^2}{2}}\,\frac{1}{\left(a^2+T^2\right)^{3/2}}\left(\sqrt{a^2+T^2}-\sqrt{\frac{\pi}{2}}\;T^2\,e^{\frac{T^4}{2\left(a^2+T^2\right)}}\,\text{Erfc}\left[\frac{T^2}{\sqrt{2\left(a^2+T^2\right)}}\right]\right)\right)^{10^9/T}\ /.\ T\to 4.1$$

Out[129]= 0.108461