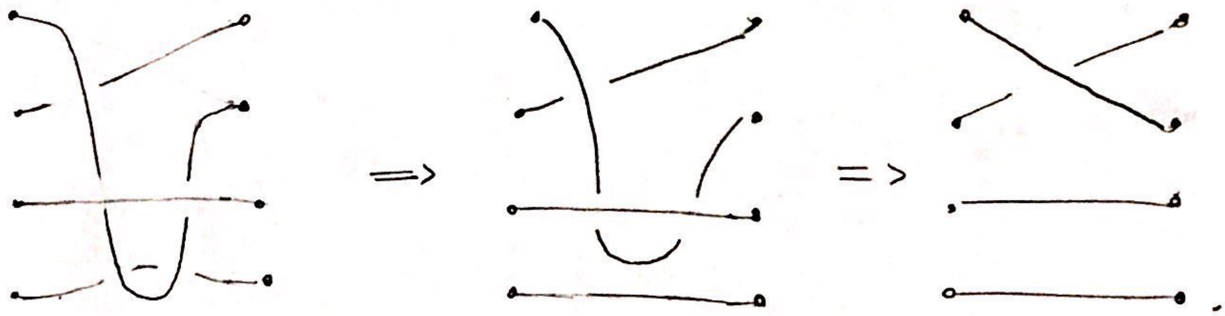
Iván Mauricio Burbano Aldana

Perimeter Scholars International

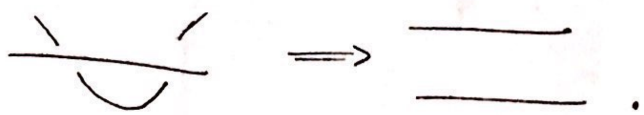
Chern - Simons Theory Part 1

Homework 2: Temperley - Lieb Algebra

Q0: This is because one can perform the following homotopy

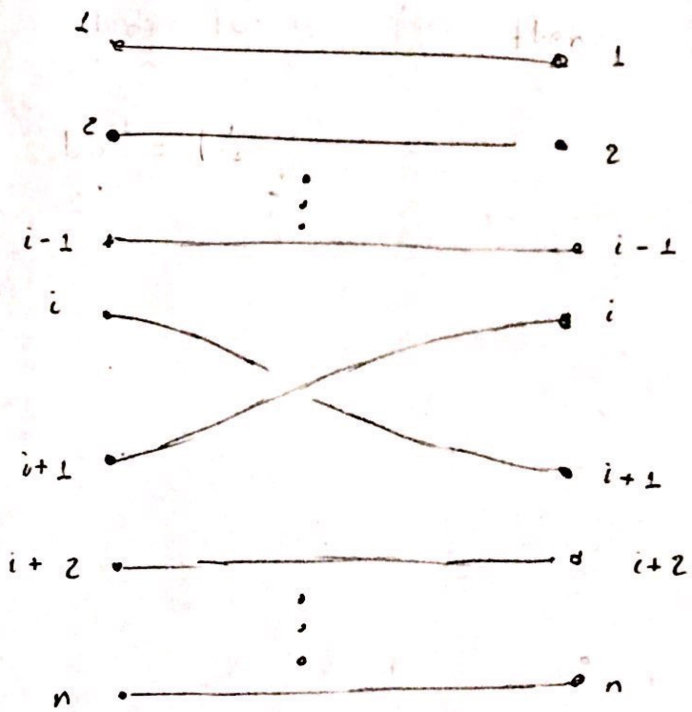


This is possible due to the movement

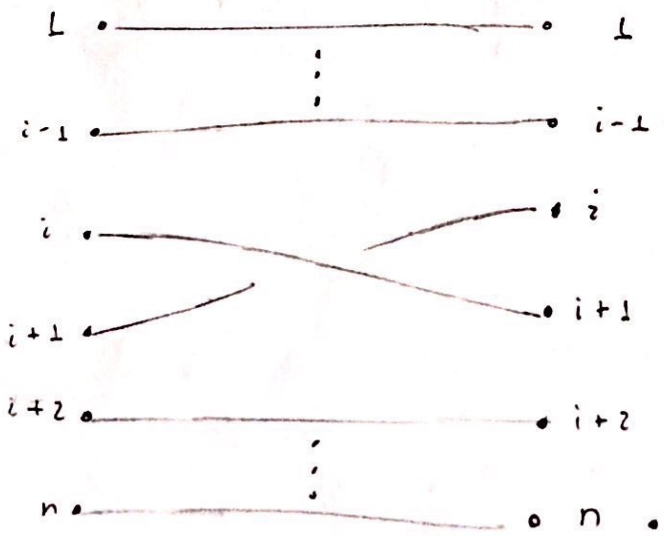


Q1: The inverse is obtained by performing all trajectories backwards.

Q2: Let σ_i be the braid where all strings are straight except for the i-th and the $(i+1)$-th. Instead, the i-th string will go underneath the $(i+1)$-th, ending up at the site $i+1$. Pictorially, σ_i is

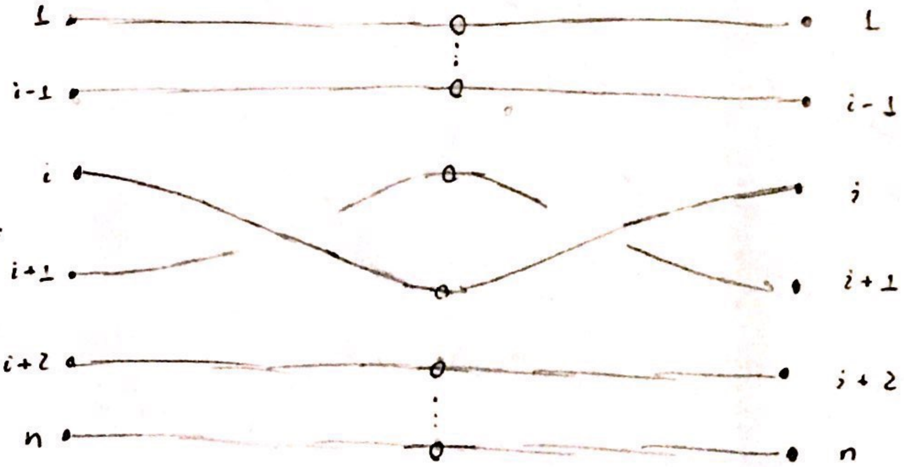


Note that σ_i^{-1} is given by



We indeed have

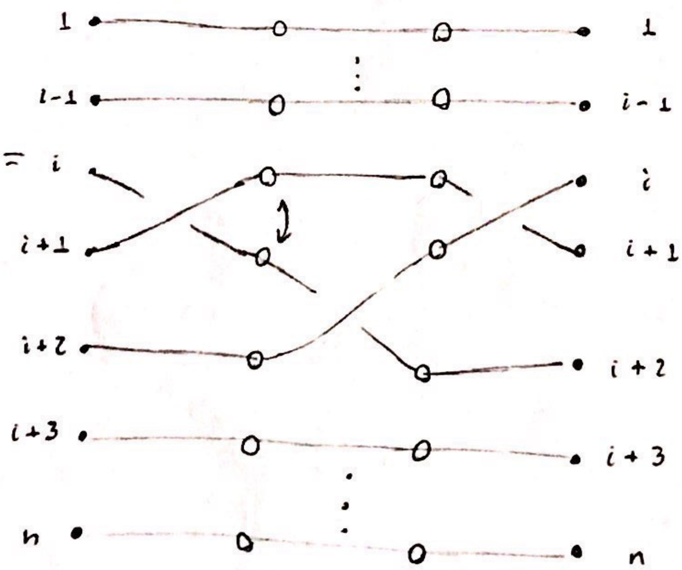
$\sigma_i \sigma_i^{-1} =$



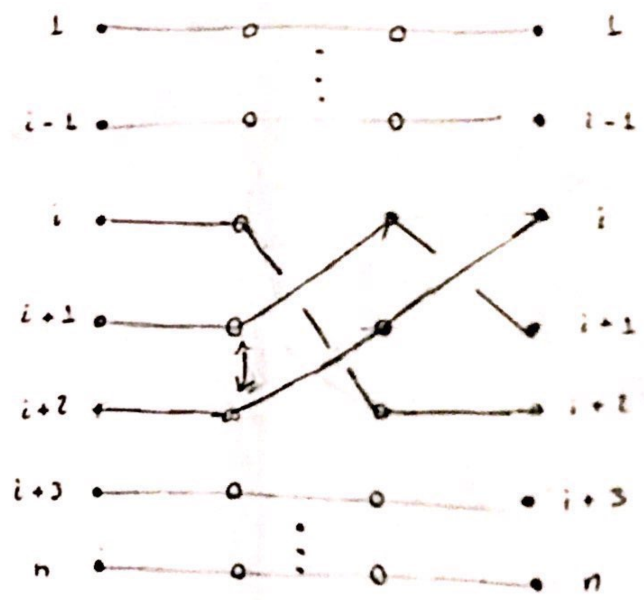
R II

$= \mathbb{1}_n$

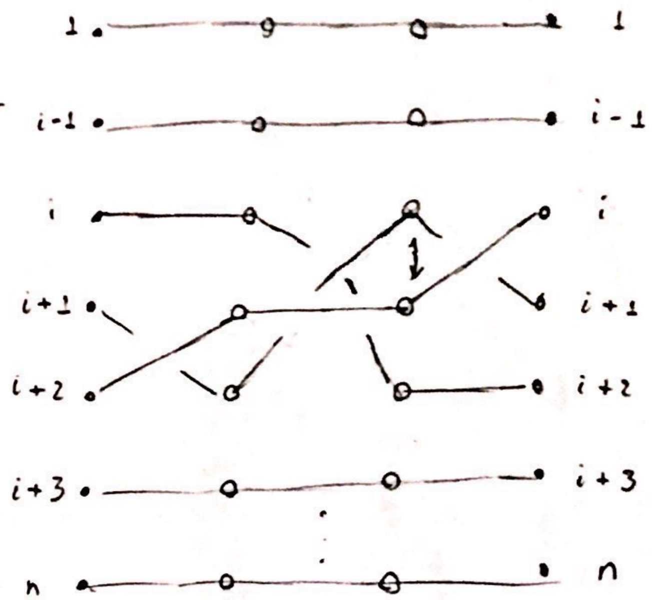
$\sigma_i \sigma_{i+1} \sigma_i =$



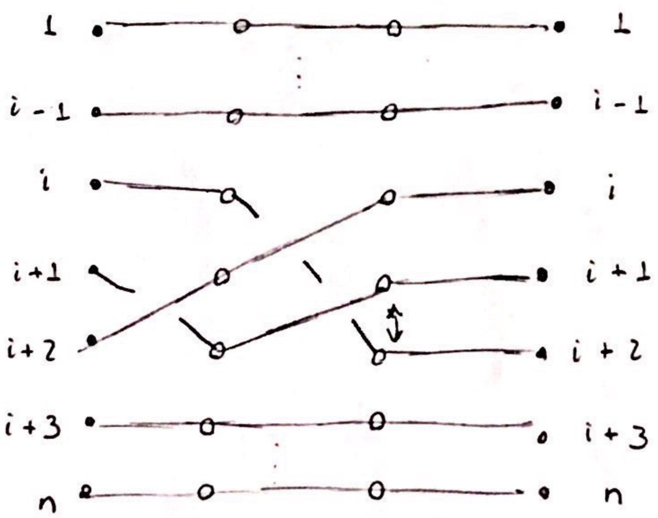
$=$


1
i-1
i
i+1
i+2
i+3
n
1
i-1
i
i+1
i+2
i+3
n

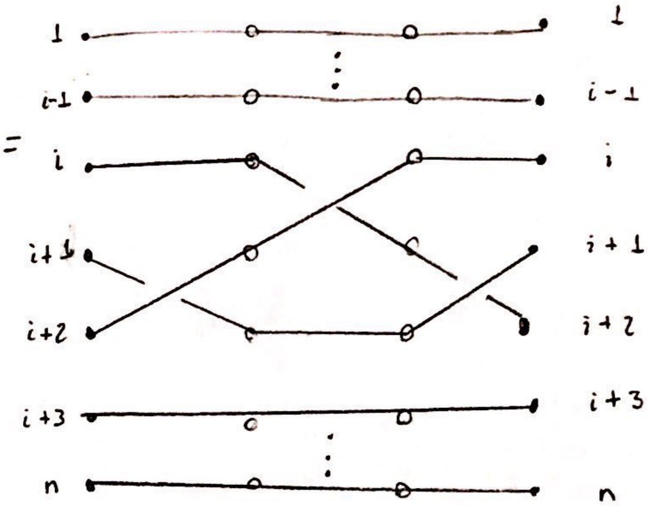
=


1
i-1
i
i+1
i+2
i+3
n
1
i-1
i
i+1
i+2
i+3
n

=

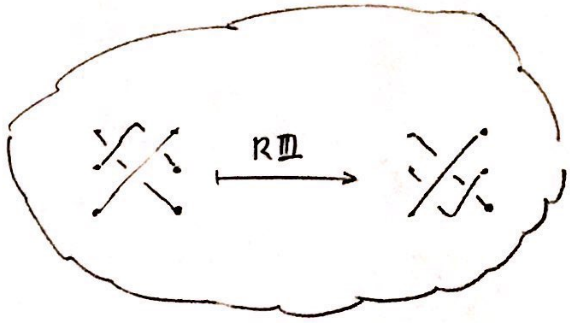

1
i-1
i
i+1
i+2
i+3
n
1
i-1
i
i+1
i+2
i+3
n

$=$

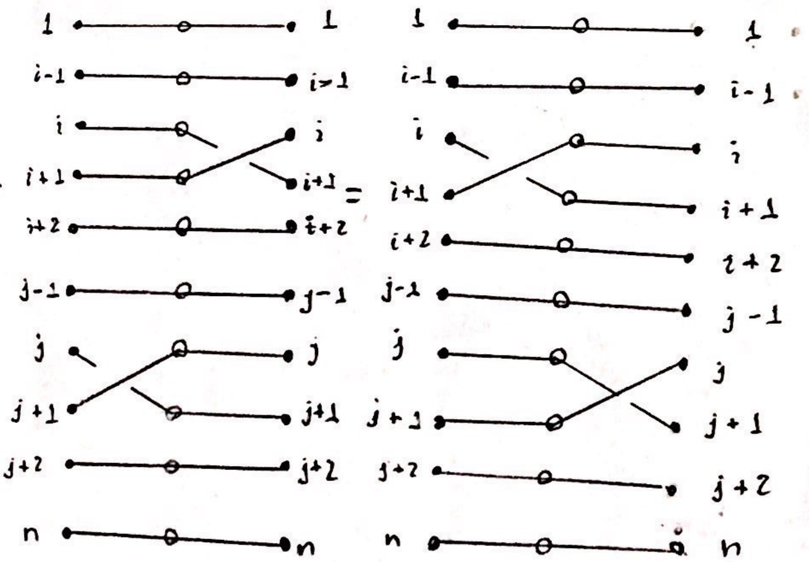


$= \sigma_{i+1}\sigma_i\sigma_{i+1}$

$\vdots$



$\sigma_i\sigma_j =$



$= \sigma_j\sigma_i$

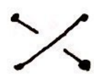
Q3. An even stronger assertion can be made. Let $a, b \in B_n$. Consider the infinite sequence of braids

$$\cdots ababab \cdots$$

This has period 2 in the time direction. The corresponding link is obtained by identifying the points at this distance. However, this chain can be shifted to

$$\cdots bababababa \cdots .$$

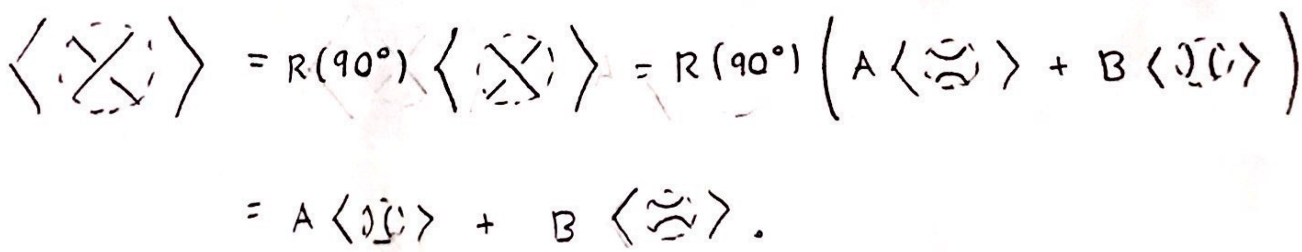
Therefore the knot defined by ab is the same as ba. In particular, the one defined by aba^{-1} is the same as $a^{-1}(ab) = b$.

Q4. The braid  in B_1 leads to the knot . while the braid  leads to

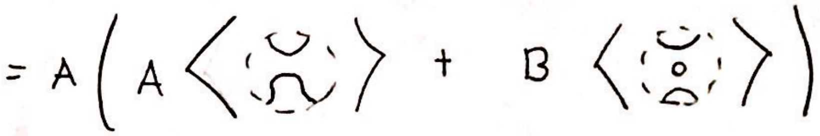
. These knots are isotopic.

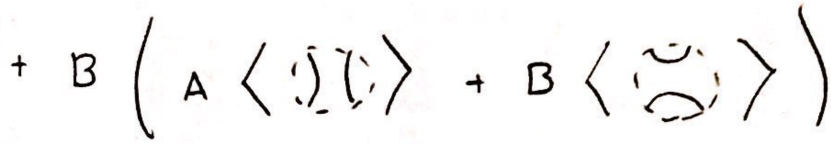
Q5: Markovs move relies on RI, as is easily seen from the example above.

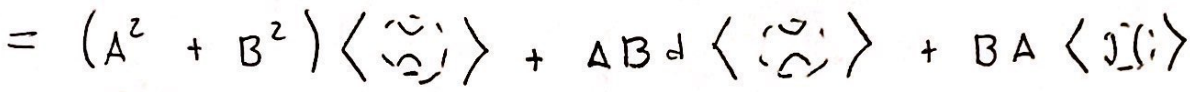
Q6: Indeed

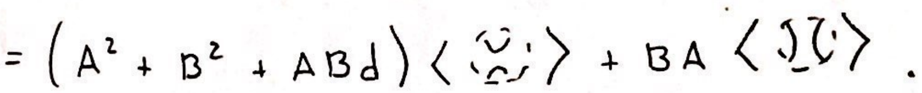


Q7:

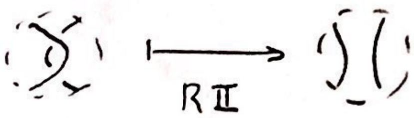




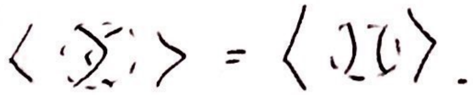




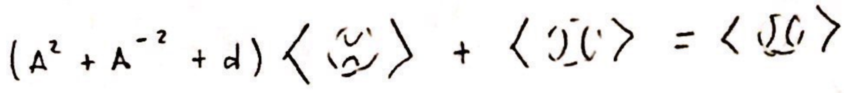
However, if this is to be a Knot invariant,



shows that

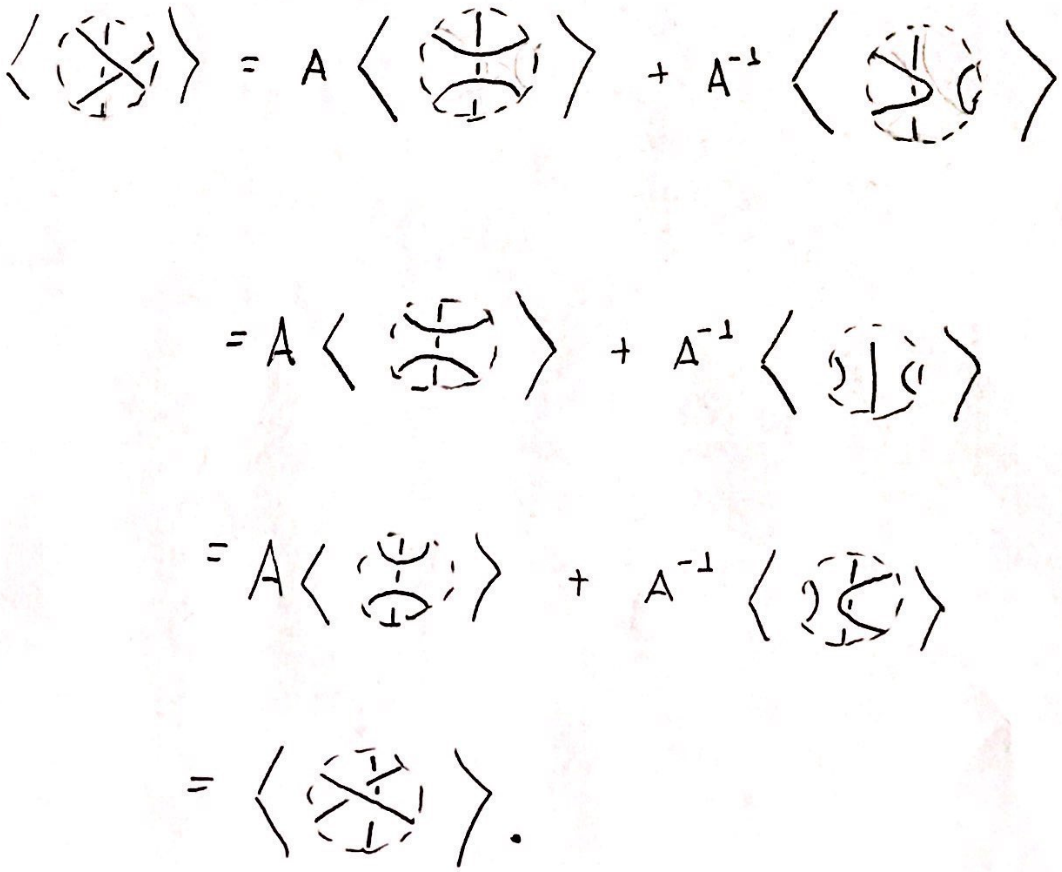


We in particular see that the choice $B = A^{-1}$ leads to



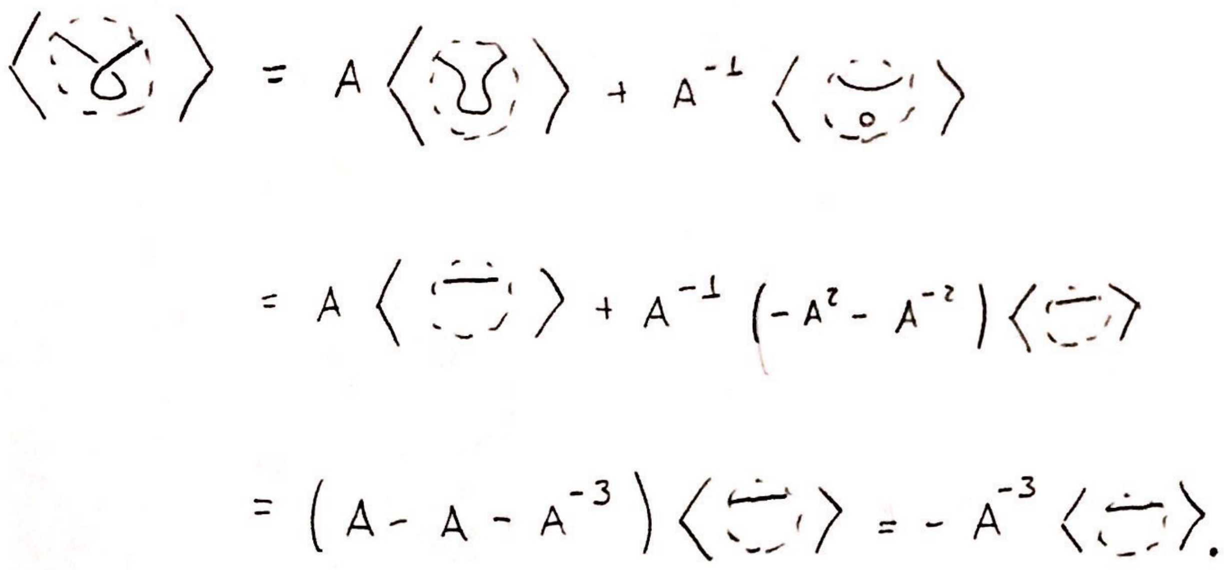
while $d = -A^2 - A^{-2}$ makes the equality true. I don't know why this <u>needs</u> to be the case.

Q8: With the choices above and using RII we have

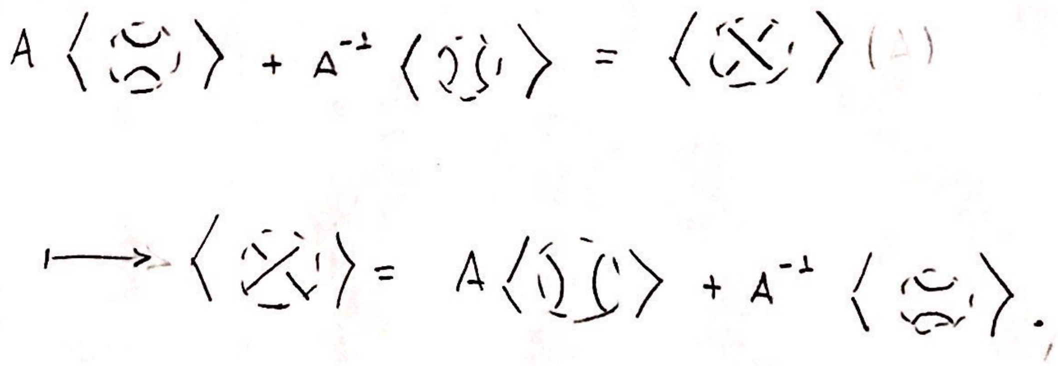


Thus the bracket is is invariant under R III

Q9:

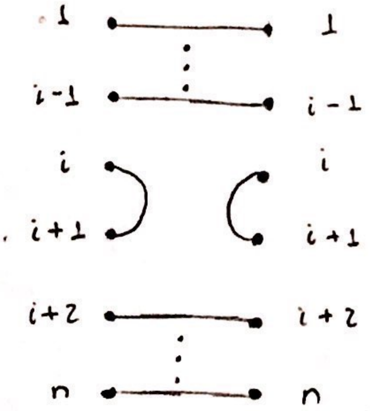


Q10: Note that under the reflection we have

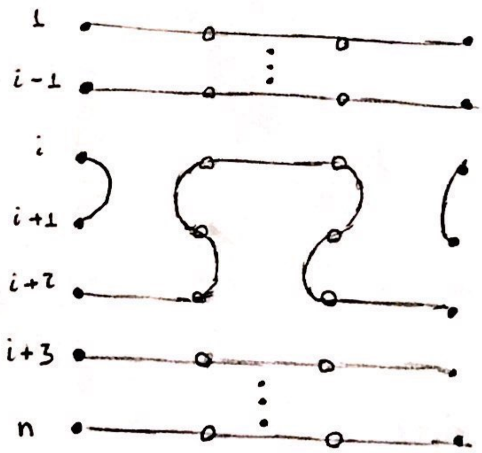


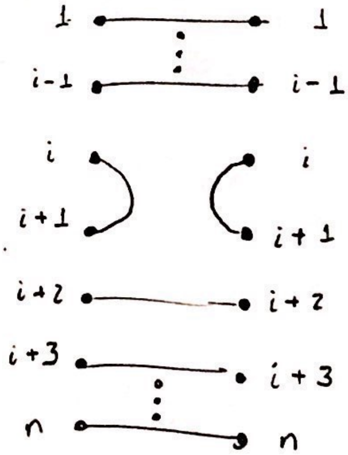
Now, the Kauffman bracket of every knot can be evaluated into combinations of unknots via application of the above to every crossing. We have thus confirmed $\langle !L \rangle (A) = \langle L \rangle (A^{-1})$.

Q11. It consists of

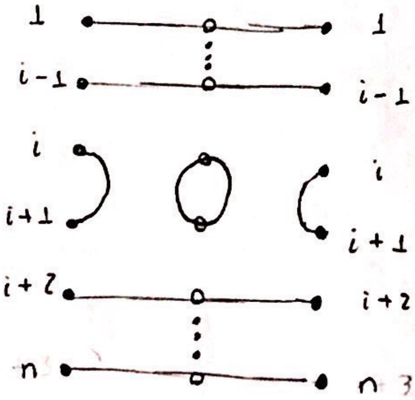
$U_i =$ 

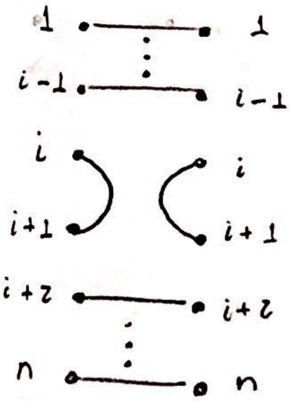
We then have

$U_i U_{i+1} U_i =$ 

$=$ 

$= U_i$,

$U_i^2 =$ 

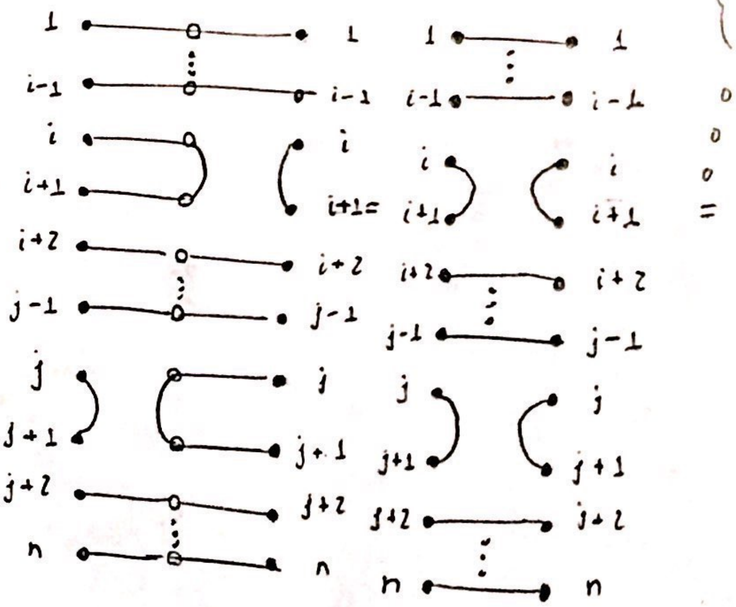
$= d$ 

$= \delta U_i$,

and

Does not distinguish between i and j

$$U_i U_j =$$



$$= U_j U_i$$

Q12. Indeed we have

$$\rho_n(\sigma_i)\rho_n(\sigma_i^{-1}) = (A + A^{-1}U_i)(A^{-1} + AU_i)$$

$$= \mathbb{1}_n + A^2 U_i + A^{-2} U_i + U_i^2$$

$$= \mathbb{1}_n - dU_i + U_i^2 = \mathbb{1}_n - dU_i + dU_i = \mathbb{1}_n,$$

$$\rho_n(\sigma_i)\rho_n(\sigma_{i+1})\rho_n(\sigma_i) = (A + A^{-1}U_i)(A + A^{-1}U_{i+1})(A + A^{-1}U_i)$$

$$= A^3 + AU_i + AU_{i+1} + A^{-1}U_{i+1}U_i + AU_i + A^{-1}U_i^2 + A^{-1}U_iU_{i+1} + A^{-3}U_iU_{i+1}U_i$$

$$= A^3 + 2AU_i + AU_{i+1} + A^{-1}dU_i + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1} + A^{-3}U_i$$

$$= A^3 + (2A - \cancel{A^{-3}} - A)U_i + AU_{i+1} + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1} + \cancel{A^{-3}U_i}$$

$$= A^3 + AU_i + AU_{i+1} + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1}$$

$$= A^3 + AU_i + AU_{i+1} + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1} + A^{-3}U_{i+1} - A^{-3}U_{i+1}$$

$$= A^3 + AU_i + (2A - A^{-3} - A)U_{i+1} + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1} + A^{-3}U_{i+1}$$

$$= A^3 + AU_i + 2AU_{i+1} + A^{-1}dU_{i+1} + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1} + A^{-3}U_{i+1}$$

$$= A^3 + AU_i + 2AU_{i+1} + A^{-1}U_{i+1}^2 + A^{-1}U_{i+1}U_i + A^{-1}U_iU_{i+1} + A^{-3}U_{i+1}$$

$$= (A + A^{-1}U_{i+1})(A + A^{-1}U_i)(A + A^{-1}U_{i+1}) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1}), \quad \text{and}$$

$$\rho(\sigma_i)\rho(\sigma_j) = (A + A^{-1}U_i)(A + A^{-1}U_j) = A^2 + U_j + U_i + A^{-2}U_iU_j$$

$$= A^2 + U_i + U_j + A^{-2}U_jU_i = (A + A^{-1}U_j)(A + A^{-1}U_i) = \rho(\sigma_j)\rho(\sigma_i).$$

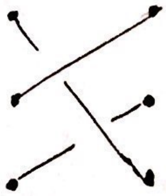
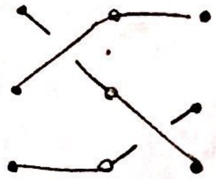
For $|i-j|>1$.

Q13: For the same reason as for braids we must have $\overline{t_1t_2} = \overline{t_2t_1}$ for all tangles t_1 and t_2. This relation can be extended to all TL_n. Then

$$\langle \overline{\rho(aba^{-1})} \rangle = \langle \overline{\rho(a)\rho(b)\rho(a^{-1})} \rangle = \langle \overline{\rho(a^{-1})\rho(a)\rho(b)} \rangle = \langle \overline{\rho(\cancel{a^{-1}a}b)} \rangle$$

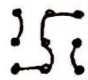
$$= \langle \overline{\rho(b)} \rangle.$$

Q14: From

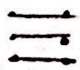
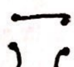
$b =$  $=$  $= \sigma_2^{-1}\sigma_1$,

we get

$$\rho_3(b) = (A^{-1} + AU_2)(A + A^{-1}U_1) = \mathbb{1}_3 + A^{-2}U_1 + A^2 U_2 + U_2 U_1$$

$=$  $+ A^{-2}$  $+ A^2$  $+$  .

Thus, if

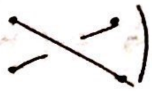
$t_0 =$  , $t_1 =$  , $t_2 =$  , $t_3 =$  ,

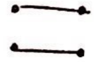
we have:

$$\langle b|0\rangle = 1, \quad \langle b|1\rangle = A^{-2}, \quad \langle b|2\rangle = A^2, \quad \langle b|3\rangle = 1.$$

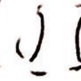
Q.15 Note that, drawing only the non-trivial strands,

(I will change $\sigma_i \longmapsto \sigma_i^{-1}$ to make things nicer)

$\rho_n($  $) = \rho_n(\sigma_i) = A\,\mathbb{1}_n + A^{-1}U_i$

$=A$  $+\ A^{-1}$  ,

which compared to

$\langle$  $\rangle = A\,\langle$  $\rangle + A^{-1}\langle$  $\rangle$,

suggests

$$\langle \overline{\rho(b)} \rangle = \langle \bar{b} \rangle$$

for all braids $b \in \mathcal{B}_n$. We thus have have

$$\langle \bar{b} \rangle = (-A)^{-3\omega(\bar{b})}\langle \bar{b} \rangle = (-A)^{-3\omega(b)}\langle \overline{\rho(b)} \rangle$$

$$= (-A)^{-3\omega(\bar{b})} \sum_{\ell} \langle b|\ell \rangle \langle \bar{t}_\ell \rangle$$

Now, it is clear from their definitions that $\omega(\bar{b}) = W(b)$. On the other hand, if t_ℓ is elementary, $\bar{t}_\ell$ is only composed of unknots. It is then clear that $\langle \bar{t}_\ell \rangle = d^{\|t_\ell\|}\langle O \rangle$. Thus,

$$\langle \bar{b} \rangle = (-A)^{-3W(b)} \sum_{l} \langle b \mid l \rangle d^{\|L_l\|}.$$