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Quantum Field Theory I

Muon Decay

1. a) We have

$$x_1 + x_2 + x_3 = \frac{2(k_1 + k_2 + k_3)\cdot Q}{Q^2} = 2\frac{Q^2}{Q^2} = 2$$

and, if $i \neq j \neq k$,

$$dM_{jk}^2 = 2(k_j + k_k)\cdot(dk_j + dk_k) = 2(k_j + k_k)\cdot(dk_i + dk_j + dk_k - dk_i)$$

$$= 2(k_j + k_k)\cdot(\cancelto{0}{dQ} - dk_i) = -2(k_j + k_k)\cdot dk_i - \cancelto{0}{dm_i^2}$$

$$= -2(k_j + k_k)\cdot dk_i - 2k_i \cdot dk_i = -2(k_i + k_j + k_k)\cdot dk_i$$

$$= -2Q\cdot dk_i = -Q^2 dx_i .$$

Now, in the CM frame, the total spatial momentum is 0, so that $Q = (Q^0, 0, 0, 0)$. Now,

$$\cos(\theta_{ij}) = \frac{\vec{k}_i \cdot \vec{k}_j}{\|\vec{k}_i\| \|\vec{k}_j\|} = \frac{-k_i \cdot k_j + k_i^0 k_j^0}{\sqrt{(k_i^0 k_i^0 - m_i^2)(k_j^0 k_j^0 - m_j^2)}} .$$

We thus need to express $k_i \cdot k_j$ and k_i^0 as a function of x_1, x_2 and x_3. For this, we have, if $i \neq j \neq k$

$$k_i \cdot k_j = k_i \cdot (Q - k_i - k_k) = \frac{1}{2} Q^2 x_i - m_i^2 - k_i \cdot k_k$$

$$= \frac{1}{2} Q^2 x_i - m_i^2 - (Q - k_j - k_k) \cdot k_k$$

$$= \frac{1}{2} Q^2 x_i - m_i^2 - \frac{1}{2} Q^2 x_k + k_k \cdot k_j + m_k^2$$

$$= \frac{1}{2} Q^2 (x_i - x_k) + m_k^2 - m_i^2 + (Q - k_i - k_j) \cdot k_j$$

$$= \frac{1}{2} Q^2 (x_i - x_k) + m_k^2 - m_i^2 + \frac{1}{2} Q^2 x_j - k_i \cdot k_j - m_j^2$$

$$= \frac{1}{2} Q^2 (x_i + x_j - x_k) + (m_k^2 - m_i^2 - m_j)^2 - k_i \cdot k_j$$

Thus

$$k_i \cdot k_j = \frac{1}{4} Q^2 (x_i + x_j - x_k) + \frac{1}{2} (m_k^2 - m_i^2 - m_j^2).$$

On the other hand,

$$x_i = \frac{2 k_i^0 \cancel{Q^0}}{(Q^0)^{\cancel{2}}} = \frac{2}{Q^0} k_i^0.$$

Thus, we obtain

$$\cos(\theta_{ij}) = \frac{-\frac{1}{4}(Q^0)^2(x_i + x_j - x_K) - \frac{1}{2}\left(m_K^2 - m_i^2 - m_j^2\right) + \frac{(Q^0)^2}{4}x_i x_j}{\sqrt{\left(\frac{(Q^0)^2}{4}x_i^2 - m_i^2\right)\left(\frac{(Q^0)^2}{4}x_j^2 - m_j^2\right)}}$$

$$= \frac{\frac{(Q^0)^2}{4}\left(x_i x_j + x_K - x_i - x_j\right) + \frac{1}{2}\left(m_i^2 + m_j^2 - m_K^2\right)}{\sqrt{\left(\frac{(Q^0)^2}{4}x_i^2 - m_i^2\right)\left(\frac{(Q^0)^2}{4}x_j^2 - m_j^2\right)}}$$

Cite: Dalila helped me with this problem.

b) The three body phase space is

$$d\Phi_3 = (2\pi)^4 \delta(Q - K_1 - K_2 - K_3) \frac{d^3\vec{K}_1}{(2\pi)^3 2E_{1,\vec{K}_1}} \frac{d^3\vec{K}_2}{(2\pi)^3 2E_{2,\vec{K}_2}} \frac{d^3\vec{K}_3}{(2\pi)^3 2E_{3,\vec{K}_3}}.$$

Performing the $\vec{K}_3$ integral in the CM frame we have

$$\int d\Phi_3 = \frac{2\pi}{2E_{3,\vec{K}_1+\vec{K}_2}} \delta(Q^0 - E_{1,\vec{K}_1} - E_{2,\vec{K}_2} - E_{3,\vec{K}_1+\vec{K}_2}) \frac{d^3\vec{K}_1}{(2\pi)^3 2E_{1,\vec{K}_1}} \frac{d^3\vec{K}_2}{(2\pi)^3 2E_{2,\vec{K}_2}}$$

$$= \frac{2\pi}{2E_{3,\vec{K}_1+\vec{K}_2}} \delta(Q^0 - E_{1,\vec{K}_1} - E_{2,\vec{K}_2} - E_{3,\vec{K}_1+\vec{K}_2}) \frac{d^3\vec{K}_1}{(2\pi)^3 2E_{1,\vec{K}_1}} \frac{\|\vec{K}_2\|^2 d\|\vec{K}_2\|\, d(\cos\theta_{21})\, d\varphi_2}{(2\pi)^3 2E_{2,\vec{K}_2}}$$

Now,

$$E_{3,\vec{K}_1+\vec{K}_2}^2 = \|\vec{K}_1\|^2 + \|\vec{K}_2\|^2 + 2\|\vec{K}_1\|\|\vec{K}_2\|\cos(\theta_{12}) + m_3^2.$$

So, fixing $\vec{K}_1$ and $\|\vec{K}_2\|$,

$$2E_{3,\vec{k}_1+\vec{k}_2}\, dE_{3,\vec{k}_1+\vec{k}_2} = 2\|\vec{k}_1\|\|\vec{k}_2\| d\cos(\theta_{12}).$$

Thus

$$\int d\bar{\Phi}_3 = \frac{2\pi}{\cancel{2E_{3,\vec{k}_1+\vec{k}_2}}} \delta(Q^0 - E_{1,\vec{k}_1} - E_{2,\vec{k}_2} - E_{3,\vec{k}_1+\vec{k}_2}) \frac{d^3\vec{k}_1}{(2\pi)^3 2E_{1,\vec{k}_1}} \frac{\|\vec{k}_2\|^{\cancel{2}} d\|\vec{k}_2\|}{(2\pi)^3 2E_{2,\vec{k}_2}} d\varphi_2 \frac{\cancel{E_{3,\vec{k}_1+\vec{k}_2}}}{\|\vec{k}_1\|\cancel{\|\vec{k}_2\|}} dE_{3,\vec{k}_1+\vec{k}_2}$$

By performing the $E_{3,\vec{k}_1+\vec{k}_2}$ and φ_2 integrals, we obtain

$$\int d\bar{\Phi}_3 = \frac{(2\pi)^2}{2} \frac{\|\vec{k}_2\|}{\|\vec{k}_1\|} \frac{d^3\vec{k}_1}{(2\pi)^3\, 2E_{1,\vec{k}_1}} \frac{d\|\vec{k}_2\|}{(2\pi)^3\, 2E_{2,\vec{k}_2}}.$$

The integrand now only depends on $\|\vec{k}_1\|$ and $\|\vec{k}_2\|$. We can thus, by setting $d^3\vec{k}_1 = \|\vec{k}_1\|^2 d\|\vec{k}_1\| d\cos\theta_1\, d\varphi_1$, perform the integrals over θ_1 and φ_1 to obtain a factor of 4π, i.e.

$$\int d\bar{\Phi}_3 = \cancel{(2\pi)^3} \|\vec{k}_1\|\|\vec{k}_2\| \frac{d\|\vec{k}_1\|}{\cancel{(2\pi)^3} 2E_{1,\vec{k}_1}} \frac{d\|\vec{k}_2\|}{(2\pi)^3 2E_{2,\vec{k}_2}}.$$

We now have to express our integrals over $\|\vec{k}_1\|$ and $\|\vec{k}_2\|$ in terms of x_1 and x_2.

We have

$$dx_i = \frac{2}{Q^0} \frac{2\|\vec{K}_i\| d\|\vec{K}_i\|}{2\sqrt{\vec{K}_i^2 + m_i^2}} = \frac{2}{Q_0} \|\vec{K}_i\| \frac{d\|\vec{K}_i\|}{E_{i,\vec{K}_i}} .$$

Thus

$$\int d\bar{\Phi}_3 = \left(\frac{Q_0}{2}\right)^2 \frac{dx_1}{2} \frac{dx_2}{2} \frac{1}{(2\pi)^3}$$

$$= \frac{Q^2}{128\pi^3} dx_1 \, dx_2 .$$

Performing the last integrals

$$\int d\bar{\Phi}_3 = \frac{Q^2}{128\pi^3} \int dx_1 \, dx_2 ,$$

In here, the limits of integration have be dealt with care. Going back to

$$E_{3,\vec{K}_1+\vec{K}_2} = \sqrt{\|\vec{K}_1\|^2 + \|\vec{K}_2\|^2 + 2\|\vec{K}_1\|\|\vec{K}_2\| \cos\theta_{21} + m_3^2} ,$$

we see that the limits of integration are

$$\sqrt{(\|\vec{K}_1\| - \|\vec{K}_2\|)^2 + m_3^2} \leq E_{3,\vec{K}_1+\vec{K}_2} \leq \sqrt{(\|\vec{K}_1\| + \|\vec{K}_2\|)^2 + m_3^2} .$$

The delta function for the energies then adds the constraint on the $\|\vec{K}_1\|, \|\vec{K}_2\|$ integrals

$$\sqrt{(\|\vec{k}_1\|-\|\vec{k}_2\|)^2+m_3^2} \le Q^0 - \sqrt{\|\vec{k}_1\|^2+m_1^2} - \sqrt{\|\vec{k}_2\|^2+m_2^2} \le \sqrt{(\|\vec{k}_1\|+\|\vec{k}_2\|)^2+m_3^2}.$$

This later restricts

$$x_i = \frac{2E_{i,\vec{k}_i}}{Q^0} = \frac{2}{Q^0}\sqrt{\|\vec{k}_i\|^2+m_i^2}.$$

c) If $m_1 = m_2 = m_3 = 0$, the condition above becomes

$$Q^0 \le 2(\|\vec{k}_1\|+\|\vec{k}_2\|)$$

and, if $\|\vec{k}_1\| > \|\vec{k}_2\|$

$$2\|\vec{k}_1\| \le Q^0,$$

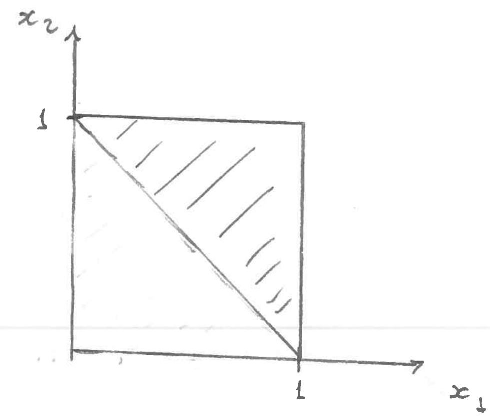
while if $\|\vec{k}_2\| \ge \|\vec{k}_1\|$,

$$2\|\vec{k}_2\| \le Q^0.$$

Multiplying this by $\frac{2}{Q^0}$ we obtain

$$\max\{x_1, x_2\} \le 1 \le x_1 + x_2$$

This precisely corresponds to the triangle bounded by the lines $x_1 = 1$, $x_2 = 1$, $x_1 + x_2 = 1$



The area of this region is $\frac{1}{2}$ and thus

$$\int d\bar{\Phi}_3 = \frac{Q^2}{256\pi^3}$$

d) Taking $p_1 = (m_\mu, \vec{0})$, $p_2 = k_3$, $p_3 = k_1$ and $p_4 = k_2$,

we have

$$p_1 \cdot p_2 = m_\mu E_{3,\vec{k}_1+\vec{k}_2} = m_\mu \|\vec{k}_1 + \vec{k}_2\|$$

$$= m_\mu \sqrt{\|\vec{k}_1\|^2 + \|\vec{k}_2\|^2 + 2\|\vec{k}_1\|\|\vec{k}_2\|\cos\theta_{12}}$$

$$= m_\mu \sqrt{\frac{(Q^0)^2}{4} x_1^2 + \frac{(Q^0)^2}{4} x_2^2 + 2\frac{(Q^0)^2}{4} \cancel{x_1 x_2} \frac{x_1 x_2 + x_3 - x_1 - x_2}{\cancel{x_1 x_2}}}$$

$$= \frac{Q^0 m_\mu}{2} \sqrt{x_1^2 + x_2^2 + 2x_1x_2 + 4 - 4x_1 - 4x_2}$$

$$= \frac{Q^0 m_\mu}{2} \sqrt{(2 - x_3)^2 + 4 - 4(2 - x_3)}$$

$$= \frac{Q^0 m_\mu}{2} \sqrt{(\cancel{2} - x_3 - \cancel{2})^2} = \frac{Q^0 m_\mu}{2} x_3 \quad \text{Duh!}$$

$$p_3 \cdot p_4 = E_{1,\vec{K}_1} E_{2,\vec{K}_2} - \vec{K}_1 \cdot \vec{K}_2$$

$$= \|\vec{K}_1\| \|\vec{K}_2\| \left(1 - \frac{x_1 x_2 + 2 - 2x_1 - 2x_2}{x_1 x_2} \right)$$

$$= \frac{(Q^0)^2}{4} \left(\cancel{x_1 x_2} - \cancel{x_1 x_2} - 2 + 2x_1 + 2x_2 \right)$$

$$= \frac{(Q^0)^2}{4} 2(x_1 + x_2 - 1)$$

In here we neglected m_e, m_{ν_μ}, m_{ν_e} and used the formula for $\cos(\theta_{ij})$ obtained in a),

Thus, with the formula obtained in class and noticing $Q^0 = m_\mu$,

$$d\Gamma = d\bar{\Phi} \frac{|M_{i \to f}|^2}{2m_\mu} = \frac{1}{\cancel{2 m_\mu}} \frac{m_\mu^{\cancel{2}}}{128 \pi^3} dx_1 dx_2 \cancel{2} \left(\frac{g_w}{M_w} \right)^4 \frac{m_\mu^2}{\cancel{2}} (2 - x_1 - x_2) \frac{m_\mu^2}{4} \times \cancel{2}(x_1 + x_2 - 1)$$

$$= \frac{m_\mu^5}{512 \pi^3} \left(\frac{g_w}{M_w} \right)^4 (2 - x_1 - x_2)(x_1 + x_2 - 1) \, dx_1 dx_2$$

Thus

$$\Gamma = \frac{m_\mu^5}{512 \pi^3} \left(\frac{g_w}{M_w} \right)^4 \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \, (2 - x_1 - x_2)(x_1 + x_2 - 1)$$

The marginal integral is

$$\int_{1-x_1}^{1} dx_2 \, (2 - x_1 - x_2)(x_1 + x_2 - 1)$$

$$= \int_{1-x_1}^{1} dx_2 \left(-2 + 2x_1 - x_1^2 + x_1 - (2x_1 + 3)x_2 - x_2^2 \right)$$

$$= \left(-2 + 3x_1 - x_1^2 \right) x_1 - (2x_1 - 3)\frac{1}{2}\left(\cancel{1} - \cancel{1} + 2x_1 - x_1^2 \right)$$

$$- \frac{1}{3}\left(\cancel{1} - \left(\cancel{1} - 3x_1 + 3x_1^2 - x_1^3 \right) \right)$$

$$= -2\cancel{x_1} + \cancel{3}x_1^2 - \cancel{x_1^3} - 2\cancel{x_1^2} + \cancel{x_1^3} + 3\cancel{x_1} - \frac{3}{2}x_1^2$$

$$- \cancel{x_1} + x_1^2 - \frac{1}{3}x_1^3$$

$$= \frac{x_1^2}{2} - \frac{x_1^3}{3} .$$

Thus,

$$\Gamma = \frac{(m_\mu)^5}{512\pi^3} \left(\frac{g_W}{M_W} \right)^4 \int_0^1 dx_1 \left(\frac{x_1^2}{2} - \frac{x_1^3}{3} \right) =$$

$$= \frac{(m_\mu)^5}{\underset{64}{{}_2\cancel{512}}\pi^3} \left(\frac{g_W}{M_W} \right)^4 \frac{\cancel{2}}{m_\mu} \int_0^{\frac{m_\mu}{2}} dE_4 \, \frac{\cancel{4}}{m_\mu^2} E_4^2 \left(\frac{1}{2} - \frac{2E_4}{3m_\mu} \right)$$

$$= \frac{(m_\mu)^3}{64\pi^3} \left(\frac{g_W}{M_W} \right)^4 \int_0^{m_\mu/2} dE_4 \, E_4^2 \left(\frac{1}{2} - \frac{2E_4}{3m_\mu} \right) .$$

We thus conclude

$$\frac{d\Gamma}{dE_4} = \frac{(m_\mu)^2}{64\pi^3}\left(\frac{g_w}{M_w}\right)^4 E_4^2\left(\frac{1}{2} - \frac{2E_4}{3m_\mu}\right)$$

c) The total decay rate is

$$\Gamma = \frac{m_\mu^5}{512\pi^3}\left(\frac{g_w}{M_w}\right)^4\left(\frac{1}{6} - \frac{1}{12}\right)$$

$$= \frac{m_\mu^5}{6144\pi^3}\left(\frac{g_w}{M_w}\right)^4.$$

Thus, the energy spectrum is

$$P(E_4) = \frac{96}{m_\mu^3} E_4^2\left(\frac{1}{2} - \frac{2E_4}{3m_\mu}\right).$$

Let $x = E_4/m_\mu$. Then $0 \le x \le \frac{1}{2}$. Now,

$$\frac{dP}{dx} = \frac{96}{m_\mu}\left(2x\left(\frac{1}{2} - \frac{2}{3}x\right) - \frac{2}{3}x^2\right)$$

$$= \frac{96}{m_\mu} x(1 - 2x).$$

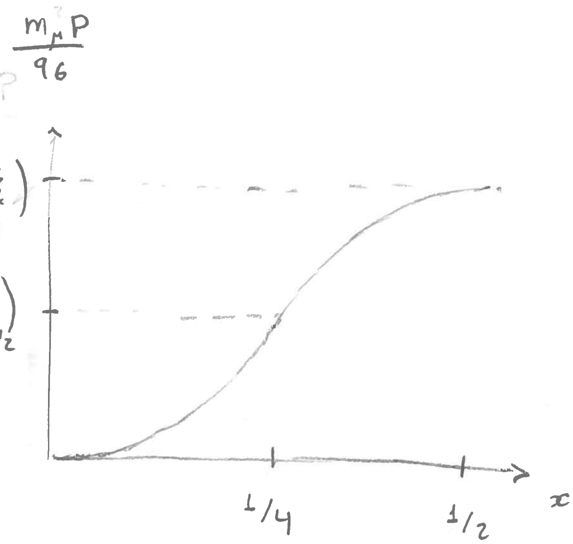
Thus, we se that P is alway increasing, having slope 0 at $x = 0$ and $x = \frac{1}{2}$. Moreover

$$\frac{d^2P}{dx^2} = \frac{96}{m_\mu}\left(1-2x-2x\right) = \frac{96}{m_\mu}\left(1-4x\right),$$

indicating an inflection at $x = \frac{1}{4}$. We thus have

$$\frac{1}{24} = \frac{1}{4}\,\frac{1}{6} = \frac{1}{4}\left(\frac{1}{2} - \frac{2}{3}\,\frac{1}{8}\right)$$

$$\frac{1}{48} = \frac{1}{16}\,\frac{2}{6} = \frac{1}{16}\left(\frac{1}{2} - \frac{2}{3}\,\frac{1}{4}\right)$$



This clearly shows that the highest energies of emission of the electron are the most probable. This of course corresponds to the fact that these decays occupy a bigger phase space than those with low energies.

Cite: This last observation was made by dmckee as a response to "Electron energy from Muon decay" by

Sesquipedal in Physics StackExchange.