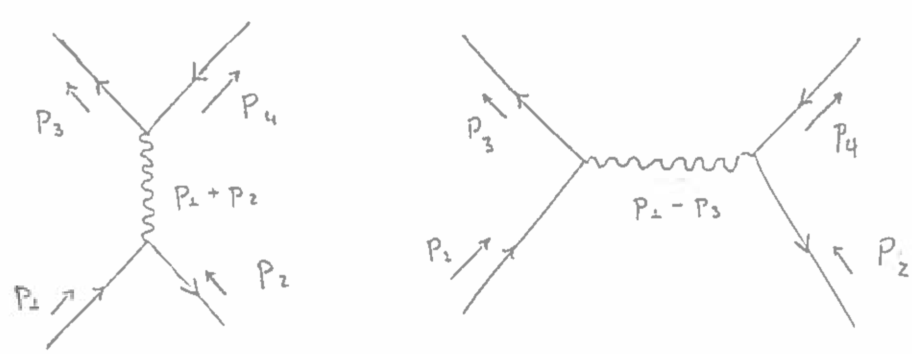
Iván Mauricio Burbano Aldana

Perimeter Scholars International

Quantum Field Theory I

Homework 5: Bhabha Scattering

a)



b)

$$i\mathcal{M} = i\mathcal{M}_1 + i\mathcal{M}_2,$$

$$i\mathcal{M}_1 = \bar{u}^{r'}(\vec{p}_3)(-ie\gamma^\mu)v^{s'}(\vec{p}_4)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i\varepsilon}\bar{v}^s(\vec{p}_2)(-ie\gamma^\nu)u^r(\vec{p}_1),$$

$$i\mathcal{M}_2 = -\bar{u}^{r'}(\vec{p}_3)(-ie\gamma^\mu)u^r(\vec{p}_1)\frac{-i\eta_{\mu\nu}}{(p_1-p_3)^2+i\varepsilon}\bar{v}^s(\vec{p}_2)(-ie\gamma^\nu)v^{s'}(\vec{p}_4).$$

c) We can simplify

$$i\mathcal{M}_1 = (-i)(-ie)^2\frac{\bar{u}^{r'}(\vec{p}_3)\gamma^\mu v^{s'}(\vec{p}_4)\bar{v}^s(\vec{p}_2)\gamma_\mu u^r(\vec{p}_1)}{(p_1+p_2)^2+i\varepsilon}$$

$$i\mathcal{M}_2 = -(-i)(-ie)^2 \frac{\bar{u}^{r'}(\vec{p}_3)\gamma^\mu u^r(\vec{p}_1)\bar{v}^s(\vec{p}_2)\gamma_\mu v^{s'}(\vec{p}_4)}{(p_1-p_3)^2 + i\varepsilon}.$$

Their conjugates are given by

$$(i\mathcal{M}_1)^* = i(ie)^2 \frac{u^r(\vec{p}_1)^\dagger \gamma^0 \gamma_\mu \cancel{\gamma^0\gamma^0} v^s(\vec{p}_2)\, v^{s'}(\vec{p}_4)^\dagger \gamma^0 \gamma^\mu \cancel{\gamma^0\gamma^0} u^{r'}(\vec{p}_3)}{(p_1+p_2)^2 - i\varepsilon}$$

$$= i(ie)^2 \frac{\bar{u}^r(\vec{p}_1)\gamma_\mu v^s(\vec{p}_2)\bar{v}^{s'}(\vec{p}_4)\gamma^\mu u^{r'}(\vec{p}_3)}{(p_1+p_2)^2 - i\varepsilon},$$

$$(i\mathcal{M}_2)^* = -i(ie)^2 \frac{\bar{v}^{s'}(\vec{p}_4)\gamma_\mu v^s(\vec{p}_2)\bar{u}^r(\vec{p}_1)\gamma^\mu u^{r'}(\vec{p}_3)}{(p_1-p_3)^2 - i\varepsilon}.$$

Thus,

$$|i\mathcal{M}_1|^2 = e^4 \frac{\bar{u}^r(\vec{p}_1)\gamma_\mu v^s(\vec{p}_2)\bar{v}^s(\vec{p}_2)\gamma_\nu u^r(\vec{p}_1)\bar{v}^{s'}(p_4)\gamma^\mu u^{r'}(\vec{p}_3)\bar{u}^{r'}(\vec{p}_3)\gamma^\nu v^{s'}(\vec{p}_4)}{(p_1+p_2)^4 + \varepsilon^2}$$

$$= e^4 \frac{\left(\begin{array}{l} u^r_b(\vec{p}_1)\bar{u}^r_a(\vec{p}_1)(\gamma_\mu)_{ac}\, v^s_c(\vec{p}_2)\,\bar{v}^s_d(\vec{p}_2)(\gamma_\nu)_{db} \times \\ v^{s'}_f(\vec{p}_4)\,\bar{v}^{s'}_e(\vec{p}_4)(\gamma^\mu)_{eg}\, u^{r'}(\vec{p}_3)_g\, \bar{u}^{r'}_h(\vec{p}_3)(\gamma^\nu)_{hf} \end{array} \right)}{(p_1+p_2)^4 + \varepsilon^2}.$$

After averaging over initial spins and summing over the final ones we obtain

$$|i\mathcal{M}_1|^2 = \frac{e^4}{4((p_1+p_2)^4+\varepsilon^2)}\begin{pmatrix}(\not{p}_1+m)_{ba}(\gamma_\mu)_{ac}(\not{p}_2-m)_{cd}(\gamma_\nu)_{db} \times / \\ (\not{p}_4-m)_{fe}(\gamma^\mu)_{eg}(\not{p}_3+m)_{gh}(\gamma^\nu)_{hf}\end{pmatrix}$$

$$= e^4 \frac{tr((\not{p}_1+m)\gamma_\mu(\not{p}_2-m)\gamma_\nu)\, tr((\not{p}_4-m)\gamma^\mu(\not{p}_3+m)\gamma^\nu)}{(p_1+p_2)^4+\varepsilon^2}.$$

In our regime we can ignore the electron mass m, obtaining

$$|i\mathcal{M}_1|^2 = \frac{1}{4}e^4 \frac{tr(\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu)\, tr(\not{p}_4\gamma^\mu\not{p}_3\gamma^\nu)}{(p_1+p_2)^4+\varepsilon^2}.$$

We now need to repeat this process to the rest of the terms in $|i\mathcal{M}|^2 = |i\mathcal{M}_1|^2 + |i\mathcal{M}_2|^2 + 2\mathrm{Re}((i\mathcal{M}_1)^* i\mathcal{M}_2)$

Charging through with our new experience, we have

$$|i\mathcal{M}_2|^2 = \frac{e^4}{(p_1-p_3)^4+\varepsilon^2}\begin{pmatrix}\bar{v}^{s'}(\vec{p}_4)\gamma_\mu v^{s}(\vec{p}_2)\,\bar{v}^{s}(\vec{p}_2)\gamma_\nu v^{s'}(\vec{p}_4) \times \\ \bar{u}^{r}(\vec{p}_1)\gamma^\mu u^{r'}(\vec{p}_3)\,\bar{u}^{r'}(\vec{p}_3)\gamma^\nu u^{r}(\vec{p}_1)\end{pmatrix}$$

$$\xrightarrow[\frac{1}{4}\sum_{r,s,r',s'}]{} \frac{e^4}{4} \frac{tr((\not{p}_4 - m)\gamma_\mu(\not{p}_2 - m)\gamma_\nu)\, tr((\not{p}_1 + m)\gamma^\mu(\not{p}_3 + m)\gamma^\nu)}{(p_1 - p_3)^4 + \varepsilon^2}.$$

We will stop keeping track of the ε.

$$(i\mathcal{M}_1)^*(i\mathcal{M}_2) = -e^4 \frac{\begin{pmatrix} \bar{u}^r(\vec{p}_1)\gamma_\mu v^s(\vec{p}_2)\bar{v}^s(\vec{p}_2)\gamma_\nu v^{s'}(\vec{p}_4)\, \times \\ \bar{v}^{s'}(\vec{p}_4)\gamma^\mu u^{r'}(\vec{p}_3)\bar{u}^{r'}(\vec{p}_3)\gamma^\nu u^r(\vec{p}_1) \end{pmatrix}}{(p_1 + p_2)^2 (p_1 - p_3)^2}.$$

$$\xrightarrow[\frac{1}{4}\sum_{r,r',s,s'}]{} = -\frac{e^4}{4} \frac{tr((\not{p}_1 + m)\gamma_\mu(\not{p}_2 - m)\gamma_\nu(\not{p}_4 - m)\gamma^\mu(\not{p}_3 + m)\gamma^\nu)}{(p_1 + p_2)^2 (p_1 - p_3)^2} \in \mathbb{R}.$$

We conclude that in our limit

$$|i\mathcal{M}|^2 = \frac{e^4}{4(p_1 + p_2)^4} tr(\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu)\, tr(\not{p}_4\gamma^\mu\not{p}_3\gamma^\nu)$$

$$+ \frac{e^4}{4(p_1 - p_3)^4} tr(\not{p}_4\gamma_\mu \not{p}_2\gamma_\nu)\, tr(\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu)$$

$$- \frac{e^4}{2(p_1 + p_2)^2(p_1 - p_3)^2} tr(\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu\not{p}_4\gamma^\mu\not{p}_3\gamma^\nu)$$

Following the lecture notes from the QED class in the course Quantum Field Theory I held in 2009 by Prof Douglas Ross in Southampton University, found in

southampton.ac.uk/~doug/ft1/ft115.pdf, and wikipedia

we begin by noting that

- $$\gamma^\mu \gamma_\nu \gamma_\mu = (2\delta^\mu{}_\nu - \gamma_\nu \gamma^\mu)\gamma_\mu$$
$$= 2\gamma_\nu - \gamma_\nu \delta^\mu{}_\mu = -2\gamma_\nu$$

- $$\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\mu = (2\delta^\mu{}_\nu - \gamma_\nu \gamma^\mu)\gamma_\rho \gamma_\mu$$
$$= 2\gamma_\rho \gamma_\nu - \gamma_\nu(-2\gamma_\rho)$$
$$= 2(2\eta_{\rho\nu} - \cancel{\gamma_\nu}\gamma_\rho) + 2\cancel{\gamma_\nu}\gamma_\rho$$
$$= 4\eta_{\rho\nu}$$

- $$\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu = (2\delta^\mu{}_\nu - \gamma_\nu \gamma^\mu)\gamma_\rho \gamma_\sigma \gamma_\mu$$
$$= 2\gamma_\rho \gamma_\sigma \gamma_\nu - \gamma_\nu 4\eta_{\rho\sigma}$$
$$= 2(2\cancel{\eta_{\rho\sigma}} - \gamma_\sigma \gamma_\rho)\gamma_\nu - 4\cancel{\eta_{\rho\sigma}}\gamma_\nu$$
$$= -2\gamma_\sigma \gamma_\rho \gamma_\nu .$$

Therefore

$$\gamma^\nu \not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu = p_1^\rho p_2^\sigma \gamma^\nu \gamma_\rho \gamma_\mu \gamma_\sigma \gamma_\nu$$

$$= -2 p_1^\rho p_2^\sigma \gamma_\sigma \gamma_\mu \gamma_\rho$$

$$= -2 \not{p}_2 \gamma_\mu \not{p}_1$$

and

$$\mathrm{tr}(\not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4 \gamma^\mu \not{p}_3 \gamma^\nu) = \mathrm{tr}(\gamma^\nu \not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4 \gamma^\mu \not{p}_3)$$

$$= -2\,\mathrm{tr}(\not{p}_2 \gamma_\mu \not{p}_1 \not{p}_4 \gamma^\mu \not{p}_3).$$

Moreover

$$\gamma_\mu \not{p}_1 \not{p}_4 \gamma^\mu = p_1^\rho p_4^\sigma \gamma_\mu \gamma_\rho \gamma_\sigma \gamma^\mu$$

$$= 4 p_1^\rho p_4^\sigma \eta_{\rho\sigma} = 4 p_1 \cdot p_4 .$$

Thus

$$\mathrm{tr}(\not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4 \gamma^\mu \not{p}_3 \gamma^\nu) = -8 p_1 \cdot p_4 \,\mathrm{tr}(\not{p}_2 \not{p}_3).$$

Finally,

$$\mathrm{tr}(\gamma_\mu \gamma_\nu) = \frac{1}{2}\mathrm{tr}(\{\gamma_\mu, \gamma_\nu\}) = \frac{1}{2}\mathrm{tr}(8\eta_{\mu\nu}) = 4\eta_{\mu\nu},$$

i.e. $\mathrm{tr}(\not{p}_2 \not{p}_3) = p_2^\mu p_3^\nu \,\mathrm{tr}(\gamma_\mu \gamma_\nu) = 4 p_2 \cdot p_3 .$ We

conclude

$$\mathrm{tr}(\not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4 \gamma^\mu \not{p}_3 \gamma^\nu) = -32(p_1 \cdot p_4)(p_2 \cdot p_3).$$

We conclude the cross term is

$$+ \frac{e^4}{(p_1+p_2)^2 (p_1-p_3)^2} \, 16(p_1 \cdot p_4)(p_2 \cdot p_3).$$