Statistical Mechanics: Dualities & RG

1. Dualities

The notion of duality in statistical mechanics is rather broad and vague although extremely useful. By its very nature I don't think it is subject to a precise mathematical definition. We however may try to discuss some of its properties.

We say a system A is dual to a system B if some statistical properties of A can be calculated by reinterpreting them in terms of statistical properties of B.

Example: The $O(n \to 0)$ Model and SAWs

Consider the $O(n)$ model on some

connected lattice $\mathcal{L}$. As we showed in class, the two point function, in the limit $n \longrightarrow 0$ can be obtained by counting the number of SAWs between the points. Indeed, if $p, q \in V(\mathcal{L})$ and $\mathcal{G}_{p \to q, K}$ is the set of SAWs on $\mathcal{L}$ with endpoints p and q (without distinguishing orientation) and length ℓ, then

$$\langle s_p s_q \rangle_{\beta, J} = \sum_{\ell=0}^{\infty} (\beta J)^{\ell} \, |\mathcal{G}_{p,q,\ell}|.$$

Example: The Electric-Magnetic Duality

Consider the vacuum Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{B} = 0,$$
$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \qquad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0.$$

If $(\vec{E}, \vec{B})$ is a solution, so is $(\vec{B}, -\vec{E})$. This exemplifies a trivial duality at the classical level. It's trivialness is however due to its exactness. Usually non-trivial dualities have to be messier.

We will now show a hint of this duality at the quantum level. Consider the action

$$S(A) = -\frac{1}{4e^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right).$$

and the path integral

$$Z = \int \mathcal{D}A\, e^{iS(A)}.$$

Heuristically it is clear that one should be able to compute it by replacing the sum over fields A by a sum over the Maxwell tensors satisfying the homogeneous Maxwell eqns. This is specially true if the above integral is taken only over gauged fields.

In any case,

$$Z = \int \mathcal{D}A\, e^{iS(A)} \propto \int \mathcal{D}F \prod_x \delta(\partial_\mu \tilde{F}^{\mu\nu}(x))\, e^{-\frac{i}{4e^2}\int d^4x\, F^2},$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Analogously to

$$\delta(y) \propto \int_{-\infty}^{\infty} dx\, e^{ixy},$$

A usefull but arbitrary scaling of V

we have

$$\prod_x \delta(\partial_\mu \tilde{F}^{\mu\nu}(x)) \propto \int \mathcal{D}V\, e^{-\frac{i}{2\pi}\underbrace{\int d^4x\, V_\nu \partial_\mu \tilde{F}^{\mu\nu}}}$$

$$= -\int d^4x\, \partial_\mu V_\nu \tilde{F}^{\mu\nu}$$

$$= -\frac{1}{2}\int d^4x\, (\partial_\mu V_\nu - \partial_\nu V_\mu)\tilde{F}^{\mu\nu}.$$

Thus

$$\bigstar \quad Z \propto \int \mathcal{D}F\,\mathcal{D}V\, e^{-i\int d^4x \left(\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4\pi}(\partial_\mu V_\nu - \partial_\nu V_\mu)\tilde{F}^{\mu\nu}\right)}$$

We integrate over F via the usual

$$G_{\mu\nu} = \frac{e^2}{2\pi} (\partial_\mu V_\nu - \partial_\nu V_\mu) .$$

Thus, this is precisely the duality discussed at the beginning.

Taken from:
Polchinski, J. "Dualities of Fields and Strings" arXiv: 1412.5704

2. Renormalization

Renormalization consists on two steps

1. Coarse graining
2. Rescaling.

Coarse graining consists on averaging the local properties of the system to obtain a "birds eye view". Afterwards we rescale to compare the obtained system to the original one.

Example Random walks.

Consider some walk on $\mathbb{R}^d$ following a probability distribution P with finite moments. For our coarse graining we condense N steps. So, the probability of moving $\vec{r}$ in total is

$$\tilde{P}(\vec{r}) = \int d^d\vec{r}_1 \cdots d^d\vec{r}_N \, P(\vec{r}_1) \cdots P(\vec{r}_N) \, \delta\left(\vec{r} - \sum_{i=1}^{N} \vec{r}_i\right)$$

$$= \int \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \left(\int d^d\vec{x} \, P(\vec{x}) \, e^{-i\vec{k}\cdot\vec{x}} \right)^N$$

$$= \int \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \underbrace{\left\langle e^{-i\vec{k}\cdot\vec{x}} \right\rangle}_{1 - i\vec{k}\cdot\langle\vec{x}\rangle - \frac{1}{2}k_\mu k_\nu \langle x^\mu x^\nu\rangle + O(\vec{k}^3)}^N$$

$$= e^{-i\vec{k}\cdot\langle\vec{x}\rangle - \frac{1}{2}k_\mu k_\nu \langle x^\mu x^\nu\rangle + \frac{1}{2}k_\mu k_\nu \langle x^\mu \rangle\langle x^\nu\rangle + O(\vec{k}^3)}$$

$$= e^{-i\vec{k}\cdot\langle\vec{x}\rangle - \frac{1}{2}k_\mu k_\nu \left(\langle x^\mu x^\nu\rangle - \langle x^\mu\rangle\langle x^\nu\rangle\right) + O(\vec{k}}$$

Taking $\vec{r}_0 = \langle \vec{x} \rangle$ and $C^{\mu\nu} = \langle (x^\mu - \langle x^\mu \rangle)(x^\nu - \langle x^\nu \rangle) \rangle$

$$= \langle x^\mu x^\nu \rangle - \langle x^\mu \rangle \langle x^\nu \rangle - \cancel{\langle x^\mu \rangle \langle x^\nu \rangle} + \cancel{\langle x^\mu \rangle \langle x^\nu \rangle}$$

we have

$$\tilde{P}(\vec{r}) = \int \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}\cdot\vec{r}_0 - \frac{1}{2}k_\mu k_\nu C^{\mu\nu}} + \mathcal{O}(\vec{k}^3)$$

Let C^{-1} be the inverse of $C^{\cdot\cdot}$. Thus

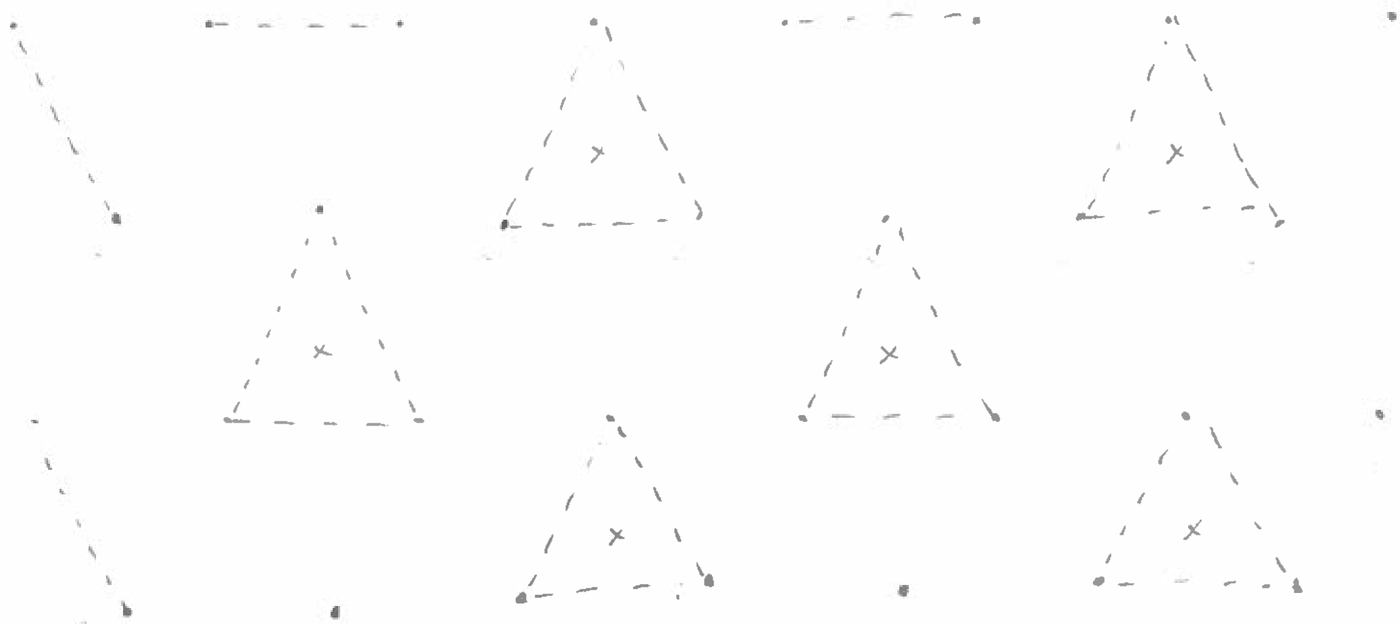
$$\tilde{P}(\vec{r}) = \int \frac{d^d\vec{k}}{(2\pi)^d} e^{-\frac{1}{2}k_\mu C^{\mu\nu} k_\nu + i\vec{k}\cdot(\vec{r}-\vec{r}_0)}$$

$$= \frac{1}{(2\pi)^d} \frac{(2\pi)^{d/2}}{\sqrt{\det(C)}} e^{-\frac{1}{2}(\vec{r}-\vec{r}_0)\cdot C^{-1}(\vec{r}-\vec{r}_0)}$$

$$= \frac{1}{\sqrt{(2\pi)^d \det(C)}} e^{-\frac{1}{2}(\vec{r}-\vec{r}_0)\cdot C^{-1}(\vec{r}-\vec{r}_0)}.$$

Thus, every distribution flows to the Gaussian

Example: Ising Model on Triangular Lattice



Consider an Ising model on a triangular lattice. Our coarse graing will be by averaging groups of three spins (as denoted by the dotted lines) and collapsing them into the x. We thus have a rescaling factor λ s.t.

$$\lambda^2 = 3.$$

The averaging is done by a rule of majority

$$f : (s_1, s_2, s_3) \longmapsto \operatorname{sgn}\left(\sum_{i=1}^{3} s_i\right)$$

We obtain a new Hamiltonian by requiring

$$P_{new}(\{s\}) = P_{previous}\left(f^{-1}(\{s\})\right).$$

After much calculation we obtain the renormalization

$$(K,h) \longmapsto (K',h') = \left(2K\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)^2,\ 3h\,\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right).$$

By evaluating the critical points we find the critical temperature

$$K_c = \frac{1}{4} \ln\left(\frac{3 - \sqrt{2}}{\sqrt{2} - 1}\right) \approx 0.336,$$

which can be compared to that of the

usual Ising

$$K_c = \frac{1}{2} \ln(1+\sqrt{2}) \approx 0.441 .$$