Homework 3: Free Fermion and Null States

Null States

d)

First, let us be careful so that we can use subindices

In[103]:=
```
<< Notation`
```

In[104]:=
```
Notation[ L_{i_} ⟺ L[i_] ]
```

We will be working with a CFT of central charge $c = 1/2$

In[105]:=
```
c = 1 / 2
```

Out[105]=
$$\frac{1}{2}$$

Let us define rules for evaluation of expressions in the Virasoro algebra. We will reserve greek letters for numbers.

In[106]:=
```
rules = {
   before___ · (number_ operator_) · after___ /; NumericQ[number] →
    number (before · operator · after),
   before___ · number_ · operator_ · after___ /; NumericQ[number] →
    number (before · operator · after),
   before___ · (α operator_) · after___ → α (before · operator · after),
   before___ · (β operator_) · after___ → β (before · operator · after),
   before___ · (op1_ · op2_) · after___ → before · op1 · op2 · after,
   before___ · (op0_ · op1_ · op2_) · after___ → before · op0 · op1 · op2 · after,
   before___ · (op1_ + op2_) · after___ → before · op1 · after + before · op2 · after,
   before___ · L_i_ · L_j_ · after___ /; (i ≥ 0 ∧ j < 0) →
    before · (L_j · L_i + (i - j) L_(i+j) + c/12 i (i^2 - 1) KroneckerDelta[i + j, 0]) · after,
   before___ · L_0 · |h_⟩ → h (before · |h⟩),
   before___ · L_i_ · |h_⟩ /; i > 0 → 0,
   CenterDot[a_] → a,
   before___ · a_ · |h_⟩ /; NumericQ[a] → a (before · |h⟩)
  };
```

We now compute the conditions for a null state of level 3

In[107]:=
```
L_1 · (L_-3 + α (L_-2 · L_-1) + β (L_-1 · L_-1 · L_-1)) · |h⟩ //. rules // Expand // Simplify
```

Out[107]= $2\,(2 + h\,\alpha)\,L_{-2} \cdot |h\rangle + 3\,(\alpha + 2\,(1 + h)\,\beta)\,L_{-1} \cdot L_{-1} \cdot |h\rangle$

In[108]:=
```
L_2 · (L_-3 + α (L_-2 · L_-1) + β (L_-1 · L_-1 · L_-1)) · |h⟩ //. rules // Expand // Simplify
```

Out[108]= $\frac{1}{4}\,(20 + 17\,\alpha + 16\,h\,\alpha + 24\,\beta + 72\,h\,\beta)\,L_{-1} \cdot |h\rangle$

In[109]:=
```
L_3 · (L_-3 + α (L_-2 · L_-1) + β (L_-1 · L_-1 · L_-1)) · |h⟩ //. rules // Expand // Simplify
```

Out[109]= $(1 + 2\,h\,(3 + 5\,\alpha + 12\,\beta))\,|h\rangle$

In[110]:=
```
Solve[2 (2 + h α) == 0 ∧ 3 (α + 2 (1 + h) β) == 0 ∧
  1/4 (20 + 17 α + 16 h α + 24 β + 72 h β) == 0 ∧ (1 + 2 h (3 + 5 α + 12 β)) == 0, {α, β, h}]
```

Out[110]= $\left\{\left\{\alpha \to -4,\ \beta \to \frac{4}{3},\ h \to \frac{1}{2}\right\},\ \left\{\alpha \to -\frac{6}{5},\ \beta \to \frac{9}{40},\ h \to \frac{5}{3}\right\}\right\}$

For a solution to lead to a unitary CFT we need the value of *h* to be in the following table

In[111]:=
```
Table[{p, q, ((4 p - 3 q)^2 - 1)/48}, {p, 1, 2}, {q, 1, 2}]
```

Out[111]=

$$\left\{\left\{\{1, 1, 0\}, \left\{1, 2, \frac{1}{16}\right\}\right\}, \left\{\left\{2, 1, \frac{1}{2}\right\}, \left\{2, 2, \frac{1}{16}\right\}\right\}\right\}$$