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Statistical Mechanics

Homework 2: 2D Ising Model

1. a) Let the edges of $\mathcal{L}$ be $E(\mathcal{L})$. Thus,

$$NN_{\vec{i}} = \{\vec{j} \in \mathcal{L} \mid \{\vec{i},\vec{j}\} \in E(\mathcal{L})\}.$$

It is thus clear that $\vec{i} \in NN_{\vec{j}}$ if and only if $\vec{j} \in NN_{\vec{i}}$. Therefore

$$H(s) = -J \sum_{\{\vec{i},\vec{j}\} \in E(\mathcal{L})} s_{\vec{i}} s_{\vec{j}} .$$

Thus, the partition function is

$$Z(\beta) = \sum_{s \in \{-1,1\}^{\mathcal{L}}} \exp\left(\beta J \sum_{\{\vec{i},\vec{j}\} \in E(\mathcal{L})} s_{\vec{i}} s_{\vec{j}}\right)$$

$$= \sum_{s \in \{-1,1\}^{\mathcal{L}}} \prod_{\{\vec{i},\vec{j}\} \in E(\mathcal{L})} e^{\beta J s_{\vec{i}} s_{\vec{j}}}$$

$$= \sum_{s \in \{-1,1\}^{\mathcal{L}}} \prod_{\{\vec{i},\vec{j}\} \in E(\mathcal{L})} \left(\cosh(\beta J s_{\vec{i}} s_{\vec{j}}) + \sinh(\beta J s_{\vec{i}} s_{\vec{j}})\right)$$

$$= \sum_{s\in\{-1,1\}^{L}} \prod_{\{\vec{i},\vec{j}\}\in E(L)} \left(\cosh(\beta J) + s_{\vec{i}} s_{\vec{j}} \sinh(\beta J)\right)$$

$$= \cosh(\beta J)^{|E(L)|} \sum_{s\in\{-1,1\}^{L}} \prod_{\{\vec{i},\vec{j}\}\in E(L)} \left(1 + v s_{\vec{i}} s_{\vec{j}}\right)$$

where $v = \tanh(\beta J)$. The term

$$\prod_{\{\vec{i},\vec{j}\}\in E(L)} \left(1 + v s_{\vec{i}} s_{\vec{j}}\right)$$

is obtained by summing over all possible way of choosing eithe 1 or $v s_{\vec{i}} s_{\vec{j}}$ at each edge $\{\vec{i},\vec{j}\}\in E(L)$. In other words

$$\prod_{\{\vec{i},\vec{j}\}\in E(L)} \left(1 + v s_{\vec{i}} s_{\vec{j}}\right) = \sum_{n\in\{0,1\}^{E(L)}} \prod_{\{\vec{i},\vec{j}\}\in E(L)} \left(v s_{\vec{i}} s_{\vec{j}}\right)^{n_{\vec{i}\vec{j}}},$$

$$= \sum_{n\in\{0,1\}^{E(L)}} v^{\sum_{\{\vec{i},\vec{j}\}\in E(L)} n_{\vec{i}\vec{j}}} \prod_{\vec{i}\in L} s_{\vec{i}}^{\sum_{\vec{j}\in NN_{\vec{i}}} n_{\vec{i}\vec{j}}}$$

$$= \sum_{n\in\{0,1\}^{E(L)}} v^{|n^{-1}(\{1\})|} \prod_{\vec{i}\in L} \left(s_{\vec{i}}\right)^{\sum_{\vec{j}\in NN_{\vec{i}}} n_{\vec{i}\vec{j}}}.$$

Thus

$$Z = \cosh(\beta J)^{|E(L)|} \sum_{n \in \{0,1\}^{E(L)}} v^{|n^{-1}(\{1\})|} \prod_{\vec{i} \in L} \sum_{s_{\vec{i}} \in \{\pm 1\}} (s_{\vec{i}})^{\sum_{\vec{j} \in NN_{\vec{i}}} n_{\vec{i}\vec{j}}} .$$

Now,

$$\sum_{s_{\vec{i}} \in \{\pm 1\}} (s_{\vec{i}})^{\sum_{\vec{j} \in NN_{\vec{i}}} n_{\vec{i}\vec{j}}} = 1^{\sum_{\vec{j} \in NN_i} n_{\vec{i}\vec{j}}} + (-1)^{\sum_{\vec{j} \in NN_i} n_{\vec{i}\vec{j}}}$$

$$= 1 + (-1)^{\sum_{\vec{j} \in NN_{\vec{i}}} n_{\vec{i}\vec{j}}}$$

$$= \begin{cases} 0 & \sum_{\vec{j} \in NN_{\vec{i}}} n_{\vec{i}\vec{j}} \in 2\mathbb{N}+1 \\ 2 & \sum_{\vec{j} \in NN_{\vec{i}}} n_{\vec{i}\vec{j}} \in 2\mathbb{N} . \end{cases}$$

Thus, if

$$\Gamma = \{ n \in \{0,1\}^{E(L)} \mid \sum_{\vec{j} \in NN_{\vec{i}}} n_{\vec{i}\vec{j}} \in 2\mathbb{N} \text{ for all } \vec{i} \in L \}$$

we have

$$Z = 2^{|L|} \cosh(\beta J)^{|E(L)|} \sum_{n \in \Gamma} v^{|n^{-1}(\{1\})|} .$$

Each $n \in \{0,1\}^{E(L)}$ offers a graphical interpretation. For each $\{i,j\} \in E(L)$, we put a stick on the edge if $n_{ij} = 1$ and leave it alone otherwise. Thus, every element of $\{0,1\}^{E(\mathcal{L})}$ corresponds to a configuration of sticks on $\mathcal{L}$. The subset Γ corresponds to those configurations of sticks where every vertex has an even number of sticks attached. This of course corresponds to closed chains of sticks on $\mathcal{L}$. In this intepretation, $|n^{-1}(\{1\})|$ is just the number of sticks in a given configuration $n \in \Gamma$, i.e. the total length of the closed chains.

b) We now restrict to $\Gamma = \mathbb{Z}^2$. Every configuration of spins can be mapped to a configuration of chains by

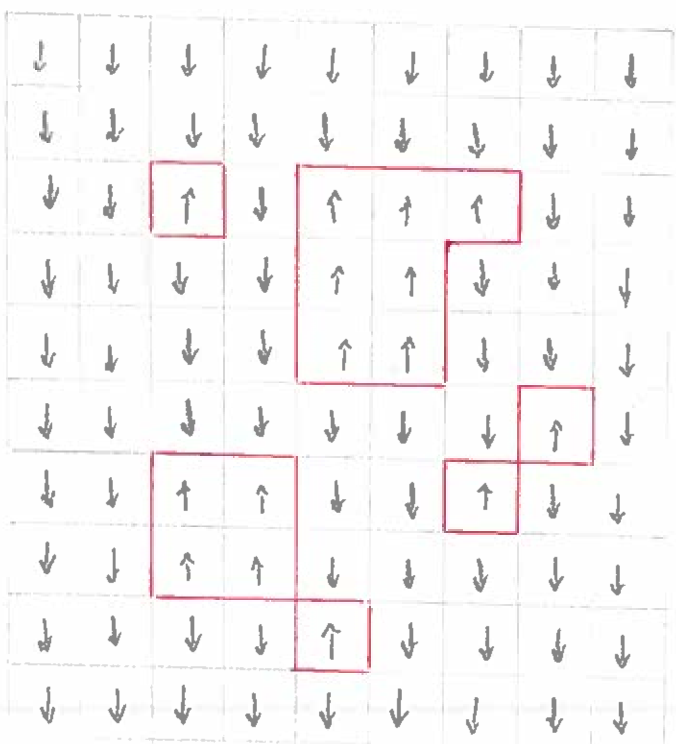
$$\text{chains}: \{-1,1\}^{\mathbb{Z}^d} \longrightarrow \Gamma \subseteq \{0,1\}^{E(\mathbb{Z}^2)}$$

$$s \longmapsto \text{chains}(s),$$

where

$$\text{chains}(s)_{\vec{\imath},\,\vec{\imath}+\hat{e}_1} = \begin{cases} 0 & s_{\vec{\imath}+\hat{e}_1} = s_{\vec{\imath}+\hat{e}_1+\hat{e}_2} \\ 1 & s_{\vec{\imath}+\hat{e}_1} \neq s_{\vec{\imath}+\hat{e}_1+\hat{e}_2} \end{cases},$$

$$\text{chains}(s)_{\vec{\imath},\,\vec{\imath}+\hat{e}_2} = \begin{cases} 0 & s_{\vec{\imath}+\hat{e}_2} = s_{\vec{\imath}+\hat{e}_2+\hat{e}_1} \\ 1 & s_{\vec{\imath}+\hat{e}_2} \neq s_{\vec{\imath}+\hat{e}_2+\hat{e}_1} \end{cases}.$$



This map can be pictorially viewed by identifying the lattices $\mathbb{Z}^2$ and $(\mathbb{Z}+\frac{1}{2})^2$ through $(p,q) \sim (p+\frac{1}{2}, q+\frac{1}{2})$. Then, if we draw s on $\mathbb{Z}^2$, the corresponding chains(s) on $(\mathbb{Z}+{}^1\!/_2)^2$ is precisely the one that separates regions of different spins.

Of course, this map is not injective. Indeed, much like H, chains is a $\mathbb{Z}_2 \cong O(1)$-invariant map.

where $O(1)$ acts on $\{-1,1\}^{\mathbb{Z}^2}$ by

$$(O(1)=\{-1,1\})\times\{-1,1\}^{\mathbb{Z}^2}\longrightarrow\{-1,1\}^{\mathbb{Z}^2}$$

$$(t\ ,\ s)\longmapsto ts:\mathbb{Z}^d\longrightarrow\{-1,1\}$$

$$\vec{i}\longmapsto ts_{\vec{i}}\,.$$

Moreover,

$$H(s)=-J\Big(1\,|\{\{\vec{i},\vec{j}\}\in E(\mathbb{Z}^2)\,|\,s_{\vec{i}}=s_{\vec{j}}\}|\ +$$

$$(-1)|\{\{\vec{i},\vec{j}\}\in E(\mathbb{Z}^2)\,|\,s_{\vec{i}}\neq s_{\vec{j}}\}|\Big)$$

$$=-J\left(|\mathrm{chains}(s)^{-1}(\{0\})|-|\mathrm{chains}(s)^{-1}(\{1\})|\right)$$

$$=-J\left(|E(\mathbb{Z}^2)|-2|\mathrm{chains}(s)^{-1}(\{1\})|\right).$$

Thus, noticing that chains is surjective and $\mathrm{chains}^{-1}(\{n\})$ is a $O(1)$ orbit (in particular, of cardinality 2), we have

$$Z=e^{\beta J|E(\mathbb{Z}^2)|}\sum_{s\in\{-1,1\}^{\mathbb{Z}^2}}e^{-2\beta J|\mathrm{chains}(s)^{-1}(\{1\})|}$$

$$= 2\, e^{\beta J |E(\mathbb{Z}^2)|} \sum_{n \in \Gamma} e^{-2\beta J |n^{-1}(\{1\})|},$$

showcasing the duality between the Ising model and a gas of closed chains with hamiltonianian

$$h : \Gamma \longrightarrow \mathbb{R}$$

$$n \longmapsto h(n) := -J|E(\mathbb{Z}^2)| + 2J|n^{-1}\{1\}|.$$

c) We have

$$\sinh(\beta J) = \frac{e^{\beta J} - e^{-\beta J}}{2}$$

$$= \frac{1}{2}\begin{pmatrix} \cancel{1} + \beta J + \frac{1}{2}\cancel{(\beta J)^2} + \frac{1}{6}(\beta J)^3 + \cancel{\frac{1}{24}(\beta J)^4} \\ -\left(\cancel{1} - \beta J + \frac{1}{2}\cancel{(\beta J)^2} - \frac{1}{6}(\beta J)^3 + \cancel{\frac{1}{24}(\beta J)^4}\right. \end{pmatrix} + \mathcal{O}(\beta^5)$$

$$= \beta J + \frac{1}{6}(\beta J)^3 + \mathcal{O}(\beta^5).$$

To order β^4, we are only interested in configurations of chains no larger than 4.

namely, the empty chain $(n=0)$ and translations of $\square$, i.e. the

$$n^{\vec{p}}_{\vec{i}\vec{j}} := \begin{cases} 1, & \vec{i}=\vec{p} \text{ and } \vec{j}=\vec{p}+\hat{e}_1, \text{ or} \\ & \vec{i}=\vec{p} \text{ and } \vec{j}=\vec{p}+\hat{e}_2, \text{ or} \\ & \vec{i}=\vec{p}+\hat{e}_1 \text{ and } \vec{j}=\vec{p}+\hat{e}_1+\hat{e}_2, \text{ or} \\ & \vec{i}=\vec{p}+\hat{e}_2 \text{ and } \vec{j}=\vec{p}+\hat{e}_1+\hat{e}_2, \text{ or} \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$Z = 2^{|L|}\cosh(\beta J)^{|E(\mathbb{Z}^2)|}\left(1+\sum_{\vec{p}\in\mathbb{Z}^d} v^4\right) + \mathcal{O}(\beta^5)$$

$$= 2^{|L|}\cosh(\beta J)^{|E(\mathbb{Z}^2)|}\left(1+|\mathbb{Z}^d|\,v^4\right). + \mathcal{O}(\beta^5)$$

We thus also need the expansions

$$\cosh(\beta J) = 1 + \frac{1}{2}(\beta J)^2 + \frac{1}{24}(\beta J)^4,$$

and

$$\tanh(\beta J) = \left(\beta J + \frac{1}{6}(\beta J)^3\right)\left(1+\frac{1}{2}(\beta J)^2 + \frac{1}{24}(\beta J)^4\right)^{-1} + \mathcal{O}(\beta^5)$$

$$= \left(\beta J + \frac{1}{6}(\beta J)^3\right)\left(1 - \frac{1}{2}(\beta J)^2 - \frac{1}{24}(\beta J)^4 + \left(\frac{1}{2}(\beta J)^2 + \frac{1}{24}(\beta J)^4\right)^2\right) + O(\beta^5)$$

$$= \left(\beta J + \frac{1}{6}(\beta J)^3\right)\left(1 - \frac{1}{2}(\beta J)^2 - \frac{1}{24}(\beta J)^4 + \frac{1}{4}(\beta J)^4\right) + O(\beta^5)$$

$$= \left(\beta J + \frac{1}{6}(\beta J)^3\right)\left(1 - \frac{1}{2}(\beta J)^2 + \frac{5}{24}(\beta J)^4\right) + O(\beta^5)$$

$$= \beta J + \frac{1}{6}(\beta J)^3 - \frac{1}{2}(\beta J)^3 + O(\beta^5)$$

$$= \beta J - \frac{1}{3}(\beta J)^3 + O(\beta^5).$$

Therefore

$$Z = 2^{|\mathbb{Z}^2|}\cosh(\beta J)^{|E(\mathbb{Z}^2)|}\left(1 + |\mathbb{Z}^d|\left(\beta J - \frac{1}{3}(\beta J)^3\right)^4\right) + O(\beta^5)$$

$$= 2^{|\mathbb{Z}^2|}\cosh(\beta J)^{|E(\mathbb{Z}^2)|}\left(1 + |\mathbb{Z}^d|\left((\beta J)^4 - \frac{4}{3}(\beta J)^4\right)\right) + O(\beta^5)$$

$$= 2^{|\mathbb{Z}^2|}\cosh(\beta J)^{|E(\mathbb{Z}^2)|}\left(1 - \frac{1}{3}|\mathbb{Z}^d|(\beta J)^4\right) + O(\beta^5)$$

d) We recognize that at low temperature, i.e. High β, $e^{-2\beta J}$ is small. Thus

$$Z = 2e^{\beta J |E(\mathbb{Z}^2)|} \sum_{n \in \Gamma} e^{-2\beta J |n^{-1}(\{1\})|}$$

yields the low temperature expansion of the Ising model. From the expression of the Hamiltonian that led us to this expansion, we see the vacuum states are those corresponding to the empty configuration, i.e. with aligned spins.

$$\begin{aligned} s_0^{\uparrow} : \mathbb{Z}^L &\longrightarrow \{-1,1\} \\ \vec{p} &\longmapsto 1, \\ s_0^{\downarrow} : \mathbb{Z}^L &\longrightarrow \{-1,1\} \\ \vec{p} &\longmapsto -1. \end{aligned}$$

These correspond to the vacuum energy

$$-J|E(\mathbb{Z}^2)|.$$

On the other hand, exitations correspond to kinks, i.e. interfaces between spins of different orientation. These are of course our sticks, each providing an energy contribution $2J$.

e) We already did.

f) We already did.

g) By comparing both examples, let $\varphi(\beta J)$ be s.t.

$$\tanh(\beta J) = e^{-2\varphi(\beta J)}.$$

Then

$$\frac{Z(\beta)}{Z(\varphi(\beta))} = \frac{2^{|\mathbb{Z}^2|} \cosh(\beta J)^{|E(\mathbb{Z}^2)|}}{2\, e^{\varphi(\beta J)|E(\mathbb{Z}^2)|}}$$

$$= 2^{|\mathbb{Z}^2|} \left(\frac{\cosh(\beta J)}{e^{\varphi(\beta J)}} \right)^{|E(\mathbb{Z}^2)|}$$

$$= \eta(\beta J).$$

h) We did above.

2. i) Consider the set of walks on $\mathbb{Z}^d$ that never back track. At each step, the walker has three possible directions. Moreover, the walker is free to start its walk at any point of $\mathbb{Z}^d$. There are thus

$$|\mathbb{Z}^2| \, 3^L$$

walks of this form. Now, every closed chain of length L can be walked in this fashion. Indeed, the condition of not having more than two sticks per edge leads to walks that never back track. There are thus, less than

$$|\mathbb{Z}^2| \, 3^L = e^{\ln(|\mathbb{Z}^2| 3^L)} = e^{\ln(3) L + \ln(|\mathbb{Z}^2|)}$$

closed chains of length L.

Note: This is a worsened version of an argument found in E. Brézin, "Introduction to Statistical Field Theory." This reference will guide us during the rest of this section. Another similar reference I found useful is M. Le Bellac, F. Mortessagne, G. Batrouni, "Equilibrium and Non-equilibrium Statistical Thermodynamics.

j) We've thus far worked on the Lattice $\mathcal{L} = \mathbb{Z}^2$. This has led to some impressions, namely of convergence of some of the quantities we have studied so far. This can of course be resolved by instead considering the finite lattice $\mathcal{L} = (\mathbb{Z}/_D)^2$, i.e. the square lattice of length D and periodic boundary conditions. All of our formulas translate exactly to this case. This is shown in an attached file. of a homework I did for a course a year and a half ago. I apologize for the spanish.

To answer this question, we will use the finite lattice. This allows us to impose a particular vacuum state, say $s^{\downarrow}$ by restricting our phase space to

$$X_{\downarrow} := \{ s \in \{-1,1\}^{L} \mid s_{(0,m)} = s_{(m,0)} = s_{(D-1,m)} = s_{(m,D-1)} = -1 \quad \text{for all} \quad m \in \mathbb{Z}/D \}.$$

Indeed, $s_0^{\uparrow} \notin X_{\downarrow}$. Now, the state $s_0^{\uparrow}$ can however be approximated by the state where all of the spins that are not fixed are $+1$. Via the chains map, this corresponds to a square chain of length $D-1$. This configuration has a probability

$$\frac{1}{Z} e^{-2\beta J \, 2(D-1)} \xrightarrow[\beta, D \to \infty]{} 0 .$$

Thus, if β is small enough and D large enough, this transition is neglible.

K) The first non trivial example is to order β^4, where we have only the chains $\square$

$$Z\langle s_i\rangle = e^{\beta J|E(\mathcal{L})|}\left(\sum_{s\in\{-1,1\}^{\mathcal{L}}} s_i\, e^{-2\beta J|\mathrm{chains}(s)^{-1}(\{1\})|}\right)$$

$$= e^{\beta J|E(\mathcal{L})|}\sum_{L=0}^{\infty}\ \sum_{s\in\{s\in\{-1,1\}^{\mathcal{L}}\,|\,|\mathrm{chains}(s)^{-1}(\{1\})|=L\}} s_i\, e^{-2\beta J L}$$

$$= e^{\beta J|E(\mathcal{L})|}\left(-1 + e^{-8\beta J}\sum_{s\in\{s\in\{-1,1\}^{\mathcal{L}}\,|\,|\mathrm{chains}(s)^{-1}(\{1\})|=4\}} s_i\right)$$

$|\mathcal{L}|-1$ of these chains correspond to $s_i=-1$, while the chain surrounding s_i corresponds to $s_i=+1$. Thus

$$Z\langle s_i\rangle = e^{\beta J|E(\mathcal{L})|}\left(-1 + e^{-8\beta J}\left(1-(|\mathcal{L}|-1)\right)\right)$$

$$= e^{\beta J|E(\mathcal{L})|}\left(-1 + e^{-8\beta J}\left(2-|\mathcal{L}|\right)\right).$$

Similarly,

$$Z = e^{\beta J|E(\mathcal{L})|}\left(1 + |\mathcal{L}|e^{-8\beta J}\right).$$

Therefore

$$\langle s_i \rangle = \left(-1 + e^{-8\beta J}(2 - |\Delta|)\right)\left(1 - |\Delta| e^{-8\beta J}\right)$$

$$= -1 + e^{-8\beta J}\left(2 - |\Delta| + |\Delta|\right)$$

$$= -1 + 2e^{-8\beta J}$$

l) Looking back at our derivation of the high temperature expansion, we see that the chains that contribute are those with only one stick incident on i but either 0 or 2 on every other vertex. There are no chains like that. Thus

$$\langle s_i \rangle = 0.$$

m) At leading order, $\langle s_i \rangle = -1$ at low temperature. It is however equal to 0 at big. We conclude there is a transition from a ferromagnetic to a paramagnetic phase.

3.

n) Note that

$$\beta f(\beta J) = -\frac{1}{|L|} \ln(Z(\beta J)) = -\frac{1}{|L|} \ln\Big(\eta(\beta J)\, Z(\varphi(\beta J))\Big)$$

$$= -\frac{1}{|L|} \ln(\eta(\beta J)) - \frac{1}{|L|} \ln(Z(\varphi(\beta J))).$$

Since $\frac{1}{|L|} \ln(\eta(\beta J))$ is analytic, $\beta f(\beta J)$ has discontinuities if and only if $\ln(Z(\beta J))$ or, equivalently $\ln(Z(\varphi(\beta J)))$ presents them. Now, if there is a unique phase transition at $(\beta J)_c$ we conclude

$$(\beta J)_c = \varphi((\beta J)_c),$$

i.e.

$$\tanh((\beta J)_c) = e^{-2(\beta J)_c}$$

o) We have

$$\frac{e^{(\beta J)_c} - e^{-(\beta J)_c}}{e^{(\beta J)_c} + e^{-(\beta J)_c}} = e^{-2(\beta J)_c},$$

$$e^{(\beta J)_c} - e^{-(\beta J)_c} = e^{-(\beta J)_c} + e^{-3(\beta J)_c},$$

$$e^{4(\beta J)_c} - 2e^{2(\beta J)_c} - 1 = 0,$$

This is a quadratic equation which can be solved to

$$e^{2(\beta J)_c} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

Since the exponential on $\mathbb{R}$ is positive, we conclude

$$e^{2(\beta J)_c} = 1 + \sqrt{2}.$$

Thus,

$$(\beta J)_c = \frac{1}{2} \ln(1 + \sqrt{2}),$$

which coincides with the critical temperature obtained from the exact solution.

4. p) Let $\psi(v) = \tanh(\varphi(\beta J))$. In terms of v,

$$\psi(v) = \tanh\left(-\frac{1}{2}\ln(v)\right)$$

$$= \frac{e^{-\frac{1}{2}\ln(v)} - e^{\frac{1}{2}\ln(v)}}{e^{-\frac{1}{2}\ln(v)} + e^{\frac{1}{2}\ln(v)}}$$

$$= -\frac{v-1}{v+1}.$$

Thus

$$\log\left((1+\psi(v)^2)^2 - 2\psi(v)(1-\psi(v)^2)\right)$$

$$= \log\left(\left(1+\left(\frac{v-1}{v+1}\right)^2\right)^2 + 2\,\frac{v-1}{v+1}\left(1-\left(\frac{v-1}{v+1}\right)^2\right)(\cos(p)+\cos(q))\right)$$

$$= \log\left(\left(\frac{(v+1)^2+(v-1)^2}{(v+1)^2}\right)^2 + 2\,\frac{v-1}{v+1}\,\frac{(v+1)^2-(v-1)^2}{(v+1)^2}(\cos(p)+\cos(q))\right)$$

$$= \log\left(\frac{(2v^2+2)^2}{(v+1)^4} + 2\,\frac{v-1}{v+1}\,\frac{4v}{(v+1)^2}(\cos(p)+\cos(q))\right)$$

$$= \log\left(4(1+v^2)^2 - 8(1-v^2)\,v\,(\cos(p)+\cos(q))\right) - 4\log(v+1)$$

$$= \log\left((1+v^2)^2 - 2v(1-v^2)(\cos(p)+\cos(q))\right) + \log\left(\frac{4}{(1+v)^4}\right).$$

Thus,

$$2^N \left(1-\phi(v)^2\right)^{-N} \exp\left(-\frac{N}{2}\int_{[-\pi,\pi]^2} \frac{dp\,dq}{(2\pi)^2} \log\left((1+v^2)^2 - 2v(1-v^2)(\cos p+\cos q)\right)\right)$$

$$= \left(\frac{1-\phi(v)^2}{1-v^2}\,\frac{2}{(1+v)^2}\right)^{-N} Z.$$

We thus conclude that the duality is exact.

9) We have the free energy per site

$$\beta f(\beta) = -\frac{1}{N}\log(Z) = -\log(2) + \log(1-v^2)$$

$$+ \frac{1}{2}\int_{[-\pi,\pi]^2} \frac{dp\,dq}{(2\pi)^2} \log\left((1+v^2)^2 - 2v(1-v^2)(\cos(p)+\cos(q))\right).$$

With this we obtain the average energy per site

$$U = -\frac{1}{N}\frac{\partial \log(Z)}{\partial\beta} = -\frac{\partial v}{\partial \beta}\frac{\partial \log(Z)}{\partial v} = -J(1-v^2)\times$$

$$\left[-\frac{2v}{1-v^2} + \frac{1}{2}\int_{[-\pi,\pi]^2}\frac{dp\,dq}{(2\pi)^2}\frac{4(1+v^2)v - 2(1-v^2-2v^2)(\cos(p)+\cos(q))}{(1+v^2)^2 - 2v(1-v^2)(\cos(p)+\cos(q))}\right].$$

$$= 2Jv - \frac{J}{2}\int_{[-\pi,\pi]^2}\frac{dp\,dq}{(2\pi)^2}\frac{4(1+v^2)v - 2(1-3v^2)(\cos(p)+\cos(q))}{\frac{(1+v^2)^2}{1-v^2} - 2v(\cos(p)+\cos(q))}.$$

We finally get a heat capacity per site

$$C = \frac{\partial U}{\partial T} = \frac{\partial v}{\partial T}\frac{\partial U}{\partial v} = \frac{\partial \beta}{\partial T}\frac{\partial v}{\partial \beta}\frac{\partial U}{\partial v} = -\frac{J^2}{k_B T^2}(1-v^2)\times\Big\{2 - $$

$$\frac{1}{2}\int_{[-\pi,\pi]^2}\frac{dp\,dq}{(2\pi)^2}\left[\left(4(1+3v^2)-2(1-6v)(\cos(p)+\cos(q))\right)\left(\frac{(1+v^2)^2}{1-v^2}-2v(\cos(p)+\cos(q))\right)\right.$$

$$\frac{\left.-\left(4(1+v^2)v-2(1-3v^2)(\cos(p)+\cos(q))\right)\left(\frac{4(1+v^2)v(1-v^2)+2v(1+v^2)^2}{(1-v^2)^2}-2(\cos(p)+\cos(q))\right)\right]}{\left(\frac{(1+v^2)^2}{1-v^2}-2v(\cos(p)+\cos(q))\right)^2}$$

Although this expression could be further simplified, lets try to study its behaviour at

$$v_c=\tanh((\beta J)_c)=\tanh\left(\frac{1}{2}\ln(1+\sqrt{2})\right)=\frac{e^{\frac{1}{2}\ln(1+\sqrt{2})}-e^{-\frac{1}{2}\ln(1+\sqrt{2})}}{e^{\frac{1}{2}\ln(1+\sqrt{2})}+e^{-\frac{1}{2}\ln(1+\sqrt{2})}}$$

$$=\frac{\sqrt{1+\sqrt{2}}-\frac{1}{\sqrt{1+\sqrt{2}}}}{\sqrt{1+\sqrt{2}}+\frac{1}{\sqrt{1+\sqrt{2}}}}=\frac{\cancel{1}+\sqrt{2}-\cancel{1}}{1+\sqrt{2}+1}=\frac{\sqrt{2}}{2+\sqrt{2}}$$

$$=\frac{\sqrt{2}(2-\sqrt{2})}{4-2}=\frac{2\sqrt{2}-2}{2}=\sqrt{2}-1.$$

The coefficients are then

$$\frac{J_c^2}{k_B T_c^2}(1-v_c^2) = k_B(\beta J)_c^2 (1-v_c^2) = k_B \frac{1}{4}\ln(1+\sqrt{2})^2 \left(\cancel{1}-(2-2\sqrt{2}+\cancel{1})\right)$$

$$= \frac{1}{2} k_B \ln(1+\sqrt{2})^2 (\sqrt{2}-1),$$

$$4(1+3v^2) = 4(1+3(2-2\sqrt{2}+1)) = 4(1+9-6\sqrt{2})$$

$$= 4(10-6\sqrt{2}) = 8(5-3\sqrt{2}),$$

$$2(1-6v) = 2(1-6(\sqrt{2}-1)) = 2(1-6\sqrt{2}+6) = 2(7-6\sqrt{2}),$$

$$\frac{(1+v^2)^2}{1-v^2} = \frac{(1+2-2\sqrt{2}+1)^2}{\cancel{1}-\cancel{2}+2\sqrt{2}+\cancel{1}} = \frac{(4-2\sqrt{2})^2}{2\sqrt{2}} = \frac{2(2-\sqrt{2})^2}{\sqrt{2}}$$

$$= \frac{2(4-4\sqrt{2}+2)}{\sqrt{2}} = \frac{4(2-2\sqrt{2}+1)}{\sqrt{2}} = \frac{\overset{2}{\cancel{4}}\sqrt{2}(3-2\sqrt{2})}{\cancel{2}}$$

$$= 6\sqrt{2} - 4\cdot 2 = 6\sqrt{2} - 8 = 2(3\sqrt{2}-4)$$

This is going to be complicated. However, we see that in the denominator we have

$$\frac{(1+v^2)^2}{1-v^2} = \frac{(1+2-2\sqrt{2}+1)^2}{\cancel{1}-2+2\sqrt{2}-\cancel{1}} = \frac{\overset{2}{\cancel{4}}(2-\sqrt{2})^2}{\cancel{2}(\sqrt{2}-1)}$$

$$= 2 \, \frac{4 - 4\sqrt{2} + 2}{\sqrt{2} - 1} = 2 \, \frac{6 - 4\sqrt{2}}{\sqrt{2} - 1} = 4 \, \frac{3 - 2\sqrt{2}}{\sqrt{2} - 1}$$

$$= 4 \, \frac{(3 - 2\sqrt{2})(\sqrt{2} + 1)}{2 - 1} = 4 \left(3\sqrt{2} + 3 - 4 - 2\sqrt{2} \right)$$

$$= 4 \left(\sqrt{2} - 1 \right) = 4v.$$

Thus, at $p = q = 0$ the denominator becomes null.

The singularity is of the form

$$\int d^2\vec{x} \; \frac{a + \vec{b}\cdot\vec{x} + c\vec{x}^{\,2}}{(1 + \vec{d}\cdot\vec{x})^2} \sim \int d^2x \, \frac{1}{\vec{x}^{\,2}}$$

$$\sim \int dr \, r \, \frac{1}{r^2} = \int dr \, \frac{1}{r},$$

i.e. its logarithmic.