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Condensed Matter Core

Homework 3

1, a) Let us be careful so that we make sure we really understand. We define

$$BZ = \left(\frac{2\pi}{L_x} \mathbb{Z}/_{L_x\mathbb{Z}}\right) \times \left(\frac{2\pi}{L_y} \mathbb{Z}/_{L_y\mathbb{Z}}\right).$$

Then, for all $\vec{k} \in BZ$, we may define

$$c_{\vec{k}} := \frac{1}{\sqrt{N}} \sum_{\vec{n}\in\Lambda} e^{-i\vec{k}\cdot\vec{n}} c_{\vec{n}},$$

with $\Lambda := \left(\mathbb{Z}/_{L_x\mathbb{Z}}\right) \times \left(\mathbb{Z}/_{L_y\mathbb{Z}}\right)$ our original lattice.

We obtain our inversion formula by extension of

$$\delta_{nm} = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \frac{k}{N}(n-m)}.$$

Indeed

$$\frac{1}{\sqrt{N}} \sum_{\vec{k}\in BZ} e^{i\vec{k}\cdot\vec{n}} c_{\vec{k}} = \frac{1}{N} \sum_{\vec{m}\in\Lambda,\vec{k}\in BZ} e^{i\vec{k}\cdot(\vec{n}-\vec{m})} c_{\vec{m}}$$

$$= \frac{1}{N} \sum_{\vec{m}\in\Lambda} \sum_{l_x=1}^{L_x} \sum_{l_y=1}^{L_y} e^{2\pi i \frac{l_x}{L_x}(n_x-m_x)} e^{2\pi i \frac{l_y}{L_y}(n_y-m_y)} c_{\vec{m}} ,$$

$$= \sum_{\vec{m}\in\Lambda} \delta_{\vec{n},\vec{m}} c_{\vec{m}} = c_{\vec{n}} .$$

On the Hamiltonian we obtain

$$H = \frac{1}{N} \sum_{\vec{n}\in\Lambda,\vec{k},\vec{k}'\in BZ} \Big(e^{-i(\vec{k}-\vec{k}')\cdot\vec{n}} e^{i\vec{k}'\cdot\hat{x}} c^{\dagger}_{\vec{k}} \frac{\sigma_z - i\sigma_x}{2} c_{\vec{k}'}$$

$$+ e^{-i(\vec{k}-\vec{k}')\cdot\vec{n}} e^{i\vec{k}'\cdot\hat{y}} c^{\dagger}_{\vec{k}} \frac{\sigma_z - i\sigma_y}{2} c_{\vec{k}'}$$

$$+ h.c. + m\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{n}} c^{\dagger}_{\vec{k}} \sigma_z c_{\vec{k}'} \Big)$$

$$= \sum_{\vec{k}\in BZ} \Big(e^{ik_x} c^{\dagger}_{\vec{k}} \frac{\sigma_z - i\sigma_x}{2} c_{\vec{k}} + e^{ik_y} c^{\dagger}_{\vec{k}} \frac{\sigma_z - i\sigma_y}{2} c_{\vec{k}}$$

$$+ h.c. + m c^\dagger_{\vec{k}} \sigma_z c_{\vec{k}} \Big)$$

$$= \sum_{\vec{k} \in BZ} \Big(-\frac{i}{2} e^{ik_x} c^\dagger_{\vec{k}} \sigma_x c_{\vec{k}} - \frac{i}{2} e^{ik_y} c^\dagger_{\vec{k}} \sigma_y c_{\vec{k}}$$

$$+ \frac{1}{2} e^{ik_x} c^\dagger_{\vec{k}} \sigma_z c_{\vec{k}} + \frac{1}{2} e^{ik_y} c^\dagger_{\vec{k}} \sigma_z c_{\vec{k}} + h.c.$$

$$+ m c^\dagger_{\vec{k}} \sigma_z c_{\vec{k}} \Big)$$

$$= \sum_{\vec{k} \in BZ} \Big(-\sin(k_x) c^\dagger_{\vec{k}} \sigma_x c_{\vec{k}} + \sin(k_y) c^\dagger_{\vec{k}} \sigma_y c_{\vec{k}}$$

$$+ \big(\cos(k_x) + \cos(k_y) + m\big) c^\dagger_{\vec{k}} \sigma_z c_{\vec{k}} \Big)$$

$$= \sum_{\vec{k} \in BZ} c^\dagger_{\vec{k}} \, h(\vec{k}) \, c_{\vec{k}} .$$

In here we used that for all $z \in \mathbb{C}$,

$$z + h.c. = 2 \operatorname{Re} z .$$

Recalling that for all $\vec{a} \in \mathbb{R}^3 \setminus \{\vec{0}\}$, the operator $\vec{a} \cdot \vec{\sigma}$ has two eigenvalues $\pm\|\vec{a}\|$, we obtain the rest of the result.

b) Once we have obtained the diagonalization above, we can consider a first quantized language where we have a Hamiltonian

$$H(\vec{d}) = \vec{d} \cdot \vec{\sigma}.$$

The Chern number along the Brillouin zone is then

$$C_\pm = -\frac{1}{2\pi} \int d\vec{S} \cdot \vec{V}_\pm = \mp \frac{1}{2\pi} \int d\vec{S} \cdot \frac{\hat{b}}{2\|\vec{b}\|^2}$$

$$= \mp \frac{1}{4\pi} \int_{[-\pi,\pi]^2} d^2\vec{k} \, \frac{\hat{b}}{\|\vec{b}\|^2} \cdot \left(\frac{\partial \vec{b}}{\partial k_x} \times \frac{\partial \vec{b}}{\partial k_y} \right)$$

$$= \mp \frac{1}{4\pi} \int_{[-\pi,\pi]^2} d^2\vec{k} \quad \hat{b} \cdot \left[\left(\frac{\partial \hat{b}}{\partial k_x} - \frac{\partial (1/\|\vec{b}\|)}{\partial k_x} \hat{b} \right) \times \left(\frac{\partial \hat{b}}{\partial k_y} - \frac{\partial (1/\|\vec{b}\|)}{\partial k_y} \hat{b} \right) \right]$$

$$= \mp \frac{1}{4\pi} \int_{[-\pi,\pi]^2} d^2\vec{k} \quad \hat{b} \cdot \left(\frac{\partial \hat{b}}{\partial k_x} \times \frac{\partial \hat{b}}{\partial k_y} \right)$$

$$\pm \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \quad \hat{b} \cdot \cancelto{0}{\left(\frac{\partial \hat{b}}{\partial k_x} \times \hat{b} \right)} \frac{\partial (1/\|\vec{b}\|)}{\partial k_y}$$

$$\pm \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \quad \hat{b} \cdot \left(\hat{b} \times \frac{\partial \hat{b}}{\partial k_y} \right) \frac{\partial (1/\|\vec{b}\|)}{\partial k_x}$$

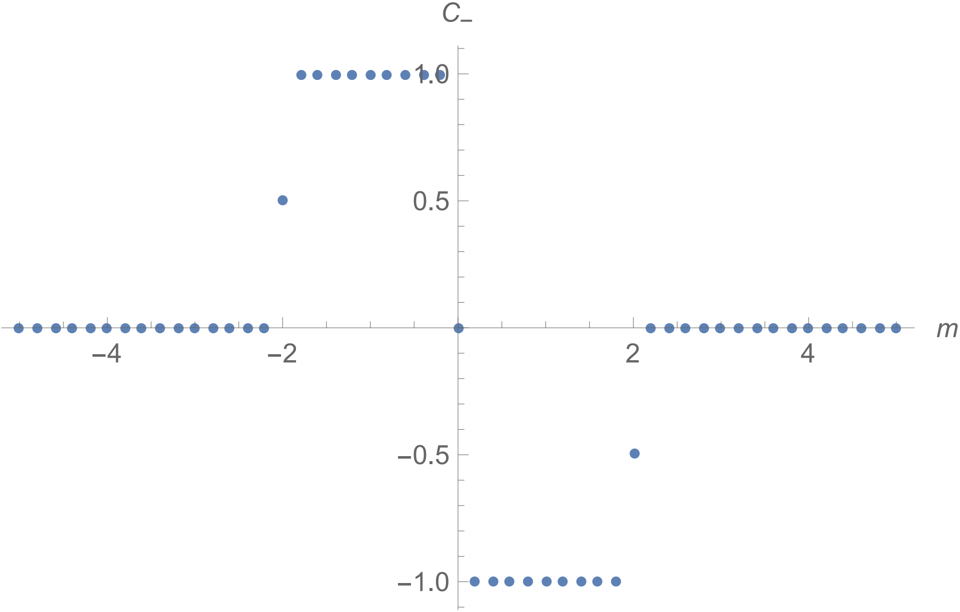
$$\mp \frac{1}{4\pi} \int_{[-\pi,\pi]} d^2\vec{k} \quad \hat{b} \cdot (\cancelto{0}{\hat{b} \times \hat{b}}) \frac{\partial (1/\|\vec{b}\|)}{\partial k_x} \frac{\partial (1/\|\vec{b}\|)}{\partial k_y}$$

$$= \mp \frac{1}{4\pi} \int d^2\vec{k} \quad \hat{b} \cdot \left(\frac{\partial \hat{b}}{\partial k_x} \times \frac{\partial \hat{b}}{\partial k_y} \right)$$

c) We find a Chern insulator for

$$m \in (-2,0) \cup (0,2),$$

as shown below



C_-
1.0
0.5
−0.5
−1.0
−4
−2
2
4
m

d) Now, for all $k_y \in \frac{2\pi}{L_y}(\mathbb{Z}/L_y\mathbb{Z})$ we can

define

$$c_{x,k_y} = \frac{1}{\sqrt{L_y}} \sum_{y \in \mathbb{Z}/L_y\mathbb{Z}} e^{-ik_y y} c_{(x,y)}.$$

Then, indeed

$$\frac{1}{\sqrt{L_y}} \sum_{k_y \in \frac{2\pi}{L_y} \mathbb{Z}/L_y\mathbb{Z}} e^{ik_y y} c_{x,k_y}$$

$$= \frac{1}{L_y} \sum_{y' \in \mathbb{Z}/L_y\mathbb{Z},\, k_y \in \frac{2\pi}{L_y} \mathbb{Z}/L_y\mathbb{Z}} e^{ik_y(y-y')} c_{(x,y')}$$

$$= \sum_{y' \in \mathbb{Z}/L_y\mathbb{Z}} \delta_{y,y'} c_{(x,y')} = c_{(x,y)}.$$

Using this, we have

$$H = \frac{1}{L_y} \sum_{x\in\{1,\dots,L_x\},\, y\in \mathbb{Z}/L_y\mathbb{Z},\, k_y,k_y'\in \frac{2\pi}{L_y}\mathbb{Z}/L_y\mathbb{Z}}$$

$$\Big(e^{-ik_y y} e^{ik_y' y} c^{+}_{x,k_y} \frac{\sigma_z - i\sigma_x}{2} c_{x+1,k_y'}$$

$$+ e^{-ik_y y} e^{ik_y'(y+1)} c^{+}_{x,k_y} \frac{\sigma_z - i\sigma_y}{2} c_{x,k_y'} + h.c.$$

$$+ m e^{-ik_y y} e^{ik_y' y} c^{+}_{x,k_y} \sigma_z c_{x,k_y'} \Big)$$

$$= \sum_{x\in\{1,\dots,L_x\},\, k_y\in\frac{2\pi}{L_y}\mathbb{Z}/L_y\mathbb{Z}}$$

$$\Big(c^{+}_{x,k_y} \frac{\sigma_z - i\sigma_x}{2} c_{x+1,k_y} + e^{ik_y} c^{+}_{x,k_y} \frac{\sigma_z - i\sigma_y}{2} c_{x,k_y} + h.c.$$

$$+ m c^{+}_{x,k_y} \sigma_z c_{x,k_y} \Big)$$

$$= \sum_{k_y\in\frac{2\pi}{L_y}\mathbb{Z}/L_y\mathbb{Z},\, x\in\{1,\dots,L_x\}} \Big(c^{+}_{x,k_y} \frac{\sigma_z - i\sigma_x}{2} c_{x+1,k_y} + h.c.$$

$$+ \cos(k_y) c^{\dagger}_{x,k_y} \sigma_z c_{x,k_y} + \sin(k_y) c^{\dagger}_{x,k_y} \sigma_y c_{x,k_y}$$

$$+ m c^{\dagger}_{x,k_y} \sigma_z c_{x,k_y} \Big)$$

$$= \sum_{k_y \in \frac{2\pi}{L_y} \mathbb{Z}/L_y\mathbb{Z}} H_{1D}(k_y).$$

e) We see that for all $x \in \{1, \dots, L_x\}$ we have the term

$$c^{\dagger}_{x,k_y} F(k_y) c_{x,k_y},$$

where

$$F(k_y) = \sin(k_y)\sigma_y + (m + \cos(k_y))\sigma_z.$$

The other terms are

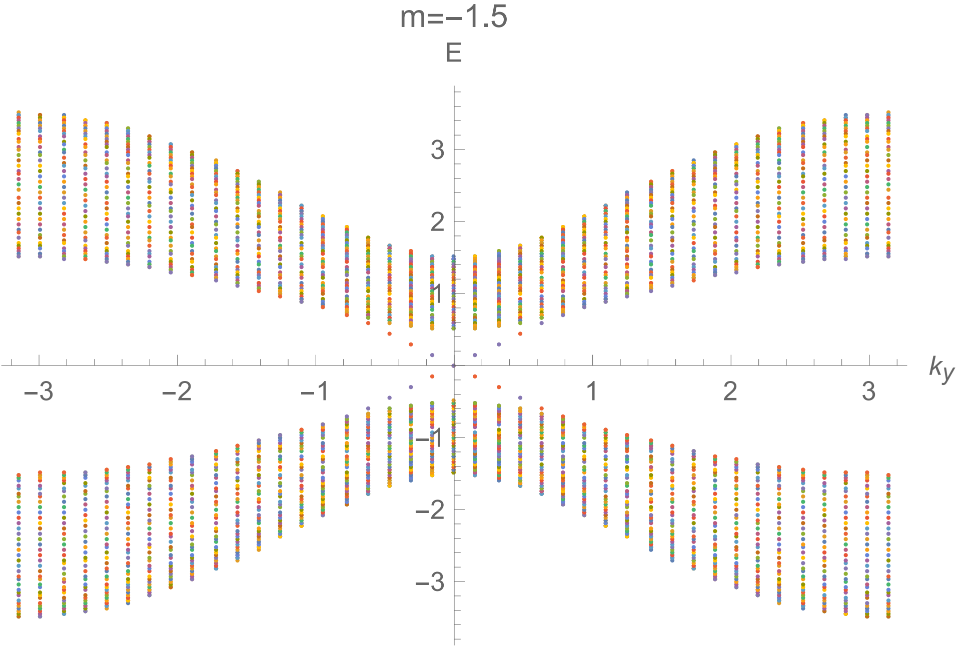
$$c^{\dagger}_{x,k_y} \frac{\sigma_z - i\sigma_x}{2} c_{x+1,k_y} + c^{\dagger}_{x+1,k_y} \frac{\sigma_z + i\sigma_x}{2} c_{x,k_y}$$

for all $x \in \{1, \dots, L_x - 1\}$. We conclude that

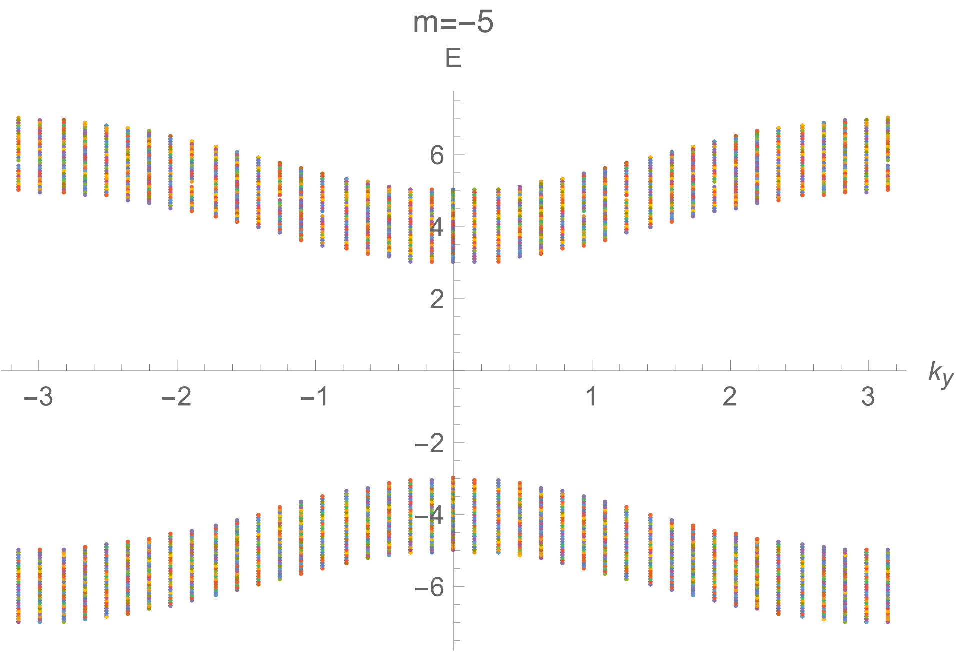
$$M(k_y)_{x,x'} = \begin{cases} \sin(k_y)\sigma_y + (m + \cos(k_y))\sigma_z , & x = x' \\ \dfrac{\sigma_z - i\sigma_x}{2} , & x' = x + 1 \\ \dfrac{\sigma_z + i\sigma_x}{2} , & x' = x - 1. \end{cases}$$

f) Found in the next page.

g) From our results in c) we expect to have a chern insulator at $m = -1.5$. Indeed, we see that the bulk of the band structure is gapped. This trend is only broken by two bands which cross. We then conclude they correspond to edge states.


m=−1.5
E
3
2
1
−1
−2
−3
k_y
−3
−2
−1
1
2
3

h) The graph is shown below. Unlike the previous system, we see that the band structure is fully gapped. There are no gapless edge modes. We thus don't have a Chern insulator. This however was expected from our results in c). At $m = -5$ we didn't have a Chern insulator!


m=−5
E
6
4
2
−2
−4
−6
k_y
−3
−2
−1
1
2
3

2.a) Well, in Landau's Fermi liquid theory we do not consider free fermions. Most crucially, there can now be momentum exchange between different fermions. This exchange allows for the possibility of an excited electron outside of the fermi surface to excite others.

b) Landau's Fermi liquid theory consists of an "almost" free Fermi gas. This almost is reflected on the following assumption: by turning on the interactions slowly enough, the initial states can be adiabatically connected to the final states. In particular, the final states can be described with the same quantum numbers as the initial states. Thus, much like the initial states, the final states also look like particles!. Thus, in the Fermi liquid, the low energy excitations can, over large periods of time, be described in terms of quasiparticles.