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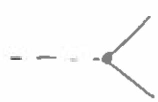
Quantum Field Theory I

Homework 3: Feynman Diagrams

1, a) 1. $\langle \Omega | T \varphi(x_1) \cdots \varphi(x_n) \bar{\Phi}(y_1), \ldots, \bar{\Phi}(y_m) | \Omega \rangle$ is the sum of all diagrams with n external legs of φ-type  and m external legs  of $\bar{\Phi}$-type, built from the vertex  and containing no vacuum subdiagrams. The contribution of each such diagram is obtained by multiplying the contribution of each connected subdigram as follows:

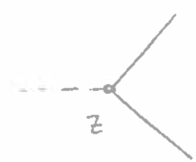
i) for every line  or 

we put a term $\Delta_F^m(x-y)$ or $\Delta_F^M(x-y)$

respectively. In here

$$\Delta_F^\mu(x-y) := \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{p^2-\mu^2+i\varepsilon}$$

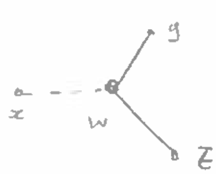
ii) for every internal vertex

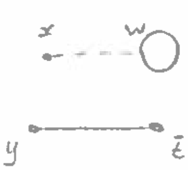


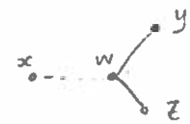
add an integral $-ig\int d^4z$

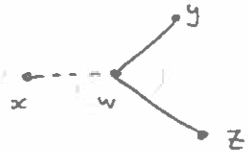
iii) Divide by the symmetry factor.

b) According to the above Feynman rules,

$G(x,y,z) =$  $= -ig\int d^4w\, \Delta_F^M(x-w)\, \Delta_F^m(y-w)\, \Delta_F^m(z-w)$

$+$  $= \frac{-ig}{2}\int d^4w\, \Delta_F^M(x-w)\, \Delta_F^m(0)\, \Delta_F^m(y-z)$

c) i) The Fourier transform of  is

$$\int d^4x\, d^4y\, d^4z \quad e^{ip_1\cdot x}\, e^{ip_2\cdot y}\, e^{ip_3\cdot z}$$ 

$$= -ig \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4}\, d^4x\, d^4y\, d^4z\, d^4w$$

$$e^{i(p_1-q_1)\cdot x}\, e^{iq_1\cdot w} \quad \frac{i}{q_1^2 - M^2 + i\varepsilon} \quad \times$$

$$e^{i(p_2-q_2)\cdot y}\, e^{iq_2\cdot w} \quad \frac{i}{q_2^2 - m^2 + i\varepsilon} \quad \times$$

$$e^{i(p_3-q_3)\cdot z}\, e^{iq_3\cdot w} \quad \frac{i}{q_3^2 - m^2 + i\varepsilon}$$

$$= -ig \int d^4q_1\, d^4q_2\, d^4q_3\, dw$$

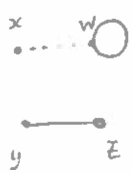
$$e^{ip_1\cdot w} \quad \frac{i}{p_1^2 - M^2 + i\varepsilon} \quad \delta(p_1 - q_1)$$

$$e^{ip_2\cdot w} \quad \frac{i}{p_2^2 - m^2 + i\varepsilon} \quad \delta(p_2 - q_2)$$

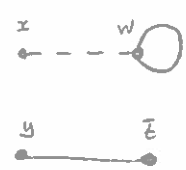
$$e^{ip_3\cdot w} \frac{i}{p_3^2 + m^2 + i\varepsilon} \delta(p_3 - q_3)$$

$$= -ig \int dw\, e^{i(p_1 + p_2 + p_3)\cdot w} \frac{i}{p_1^2 - M^2 + i\varepsilon} \frac{i}{p_2^2 - m^2 + i\varepsilon} \frac{i}{p_3^2 - m^2 + i\varepsilon}$$

$$= -ig (2\pi)^4 \delta(p_1 + p_2 + p_3) \frac{i}{p_1^2 - M^2 + i\varepsilon} \frac{i}{p_2^2 - m^2 + i\varepsilon} \frac{i}{p_3^2 - m^2 + i\varepsilon} .$$

For 

we have the Fourier transform

$$\int d^4x\, d^4y\, d^4z\; e^{ip_1\cdot x} e^{ip_2\cdot y} e^{ip_3\cdot z}$$ 

$$= -\frac{ig}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} d^4x\, d^4y\, d^4z\, d^4w$$

$$e^{ip_1\cdot x} e^{-iq_1\cdot(x-w)} \frac{i}{q_1^2 - M^2 + i\varepsilon} e^{ip_2\cdot y} \frac{i}{q_2^2 - m^2 + i\varepsilon}$$

$$e^{ip_3\cdot z} e^{-iq_3\cdot(y-z)} \frac{i}{q_3^2 - m^2 + i\varepsilon}$$

$$= -\frac{ig}{2} \int d^4q_1\, d^4q_2\, d^4q_3\, d^4w$$

$$e^{ip_1 \cdot w} \frac{i}{p_1^2 - M^2 + i\varepsilon} \delta(p_1 - q_1) \frac{i}{q_2^2 - m^2 + i\varepsilon}$$

$$\frac{i}{p_2^2 - m^2 + i\varepsilon} \delta(p_2 - q_3)\, \delta(p_3 + q_3)$$

$$= -\frac{ig}{2} (2\pi)^4 \delta(p_1) \frac{i}{p_1^2 - M^2 + i\varepsilon} \frac{i}{p_2^2 - m^2 + i\varepsilon} \delta(p_2 + p_3)$$

$$\times \int d^4q_2 \frac{i}{q_2^2 - m^2 + i\varepsilon} .$$

ii) The momentum space Feynman diagrams are

We obtain the same integrals as before by using the momentum space Feynman rules

i) For every internal line with momentum p of type f we add

$$\int d^4p \, \tilde{\Delta}_F^f(p)$$

ii) For every internal vertex with incoming momenta $p_1, \dots, p_n$ we add

$$-ig \, \delta\left(\sum_{i=1}^{n} p_i\right)(2\pi)^4$$

iii) For every external vertex of type f with momentum p we add $\tilde{\Delta}^+(p)$.

iv) We divide by the symmetry factor

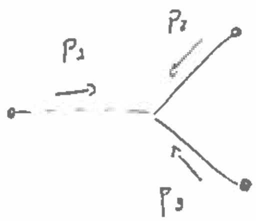
v) In the special case

we have $\dfrac{i}{p_2^2 - m^2 + i\varepsilon} \, \delta(p_2 + p_3)$.

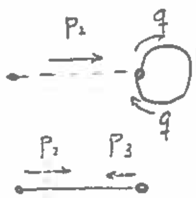
These are all found in T. Evans, "Feynman

Rules in Momentum Space". Indeed applying

them



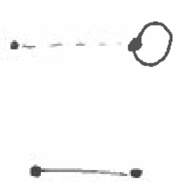
$$= -ig\,\delta(p_1+p_2+p_3)\,\tilde{\Delta}^M(p_1)\,\tilde{\Delta}^m(p_2)\,\tilde{\Delta}^m(p_3)\,(2\pi)^4,$$



$$= -\frac{ig}{2}\int d^4q\,\tilde{\Delta}_F^m(q)\,\delta(p_1+\cancel{q}-\cancel{q})\,\tilde{\Delta}_F^m(q)\,(2\pi)^4$$

$$\delta(p_2+p_3)\,\Delta_F^m(p_2).$$

iii) We clearly see the Feynman rules are not the same as in the calculation of $i\mathcal{M}$. Moreover, to $i\mathcal{M}$ disconnected diagrams do not contribute. Thus, we wouldn't consider



d) The LSZ reduction formula states

$$\langle f|S|i\rangle = i^3 \int d^4x\, d^4y\, d^4z\, e^{i(p\cdot x - k_1\cdot y - k_2\cdot z)}$$

$$(\partial_x^2 + m^2)(\partial_y^2 + m^2)(\partial_z^2 + m^2)\langle\Omega|T\,\Phi(x)\varphi(y)\varphi(z)|\Omega\rangle .$$

Now, to order g

$$\langle\Omega|T\,\Phi(x)\varphi(y)\varphi(z)|\Omega\rangle = \frac{\langle 0|T\,\Phi(x)\varphi(y)\varphi(z)\, e^{-\frac{ig}{2}\int d^4w\, \Phi(w)\varphi(w)^2}|0\rangle}{\langle 0|T\, e^{-\frac{ig}{2}\int d^4w\,\Phi(w)\varphi(w)^2}|0\rangle}$$

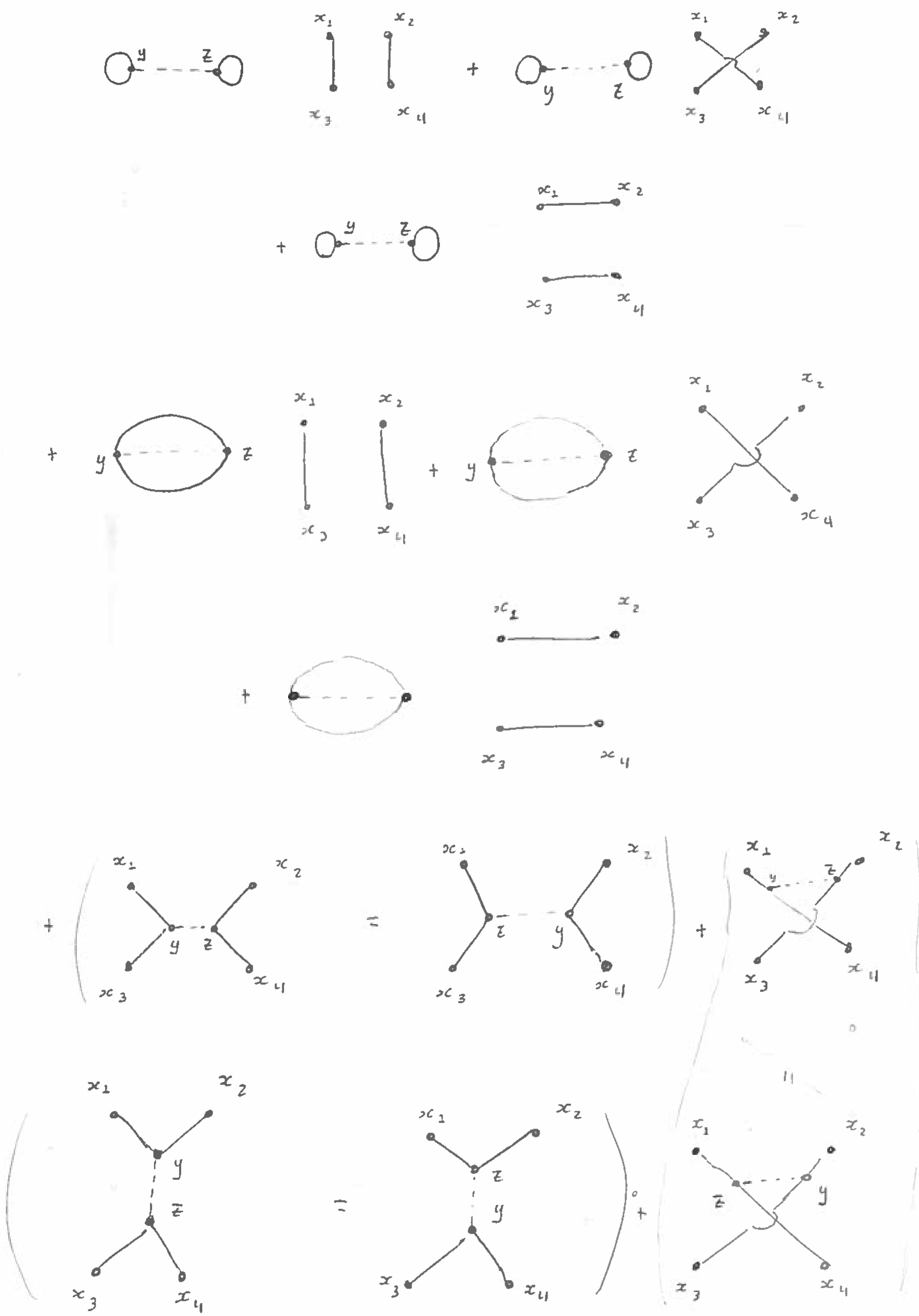
$$= \frac{\cancelto{0}{\langle 0|T\,\Phi(x)\varphi(y)\varphi(z)|0\rangle} - \frac{ig}{2}\int d^4w\,\langle 0|T\,\Phi(x)\Phi(w)\varphi(y)\varphi(z)\varphi(w)^2|0\rangle}{1 - \frac{ig}{2}\int d^4w\, \cancelto{0}{\langle 0|T\,\Phi(w)\varphi(w)^2|0\rangle}} + \mathcal{O}(g^2)$$

since we have an odd number of fields. Thus

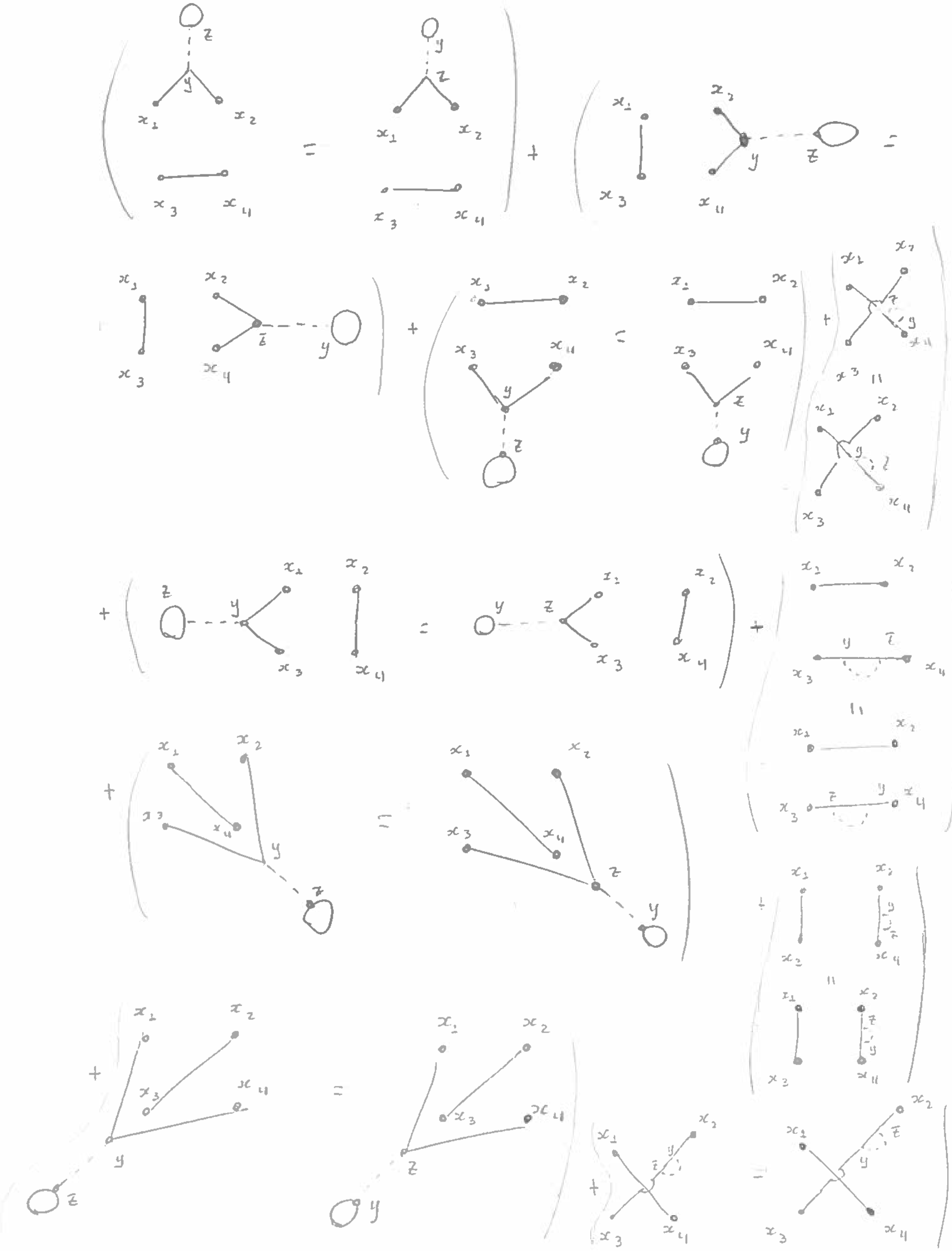
$$\langle f|S|i\rangle = -\frac{g}{2}\int d^4x\, d^4y\, d^4z\, d^4w\, e^{i(p\cdot x - k_1\cdot y - k_2\cdot z)}$$

$$(\partial_x^2 + m^2)(\partial_y^2 + m^2)(\partial_z^2 + m^2)\langle 0|T\,\Phi(x)\Phi(w)\varphi(y)\varphi(z)\varphi(w)^2|0\rangle .$$

e) i)



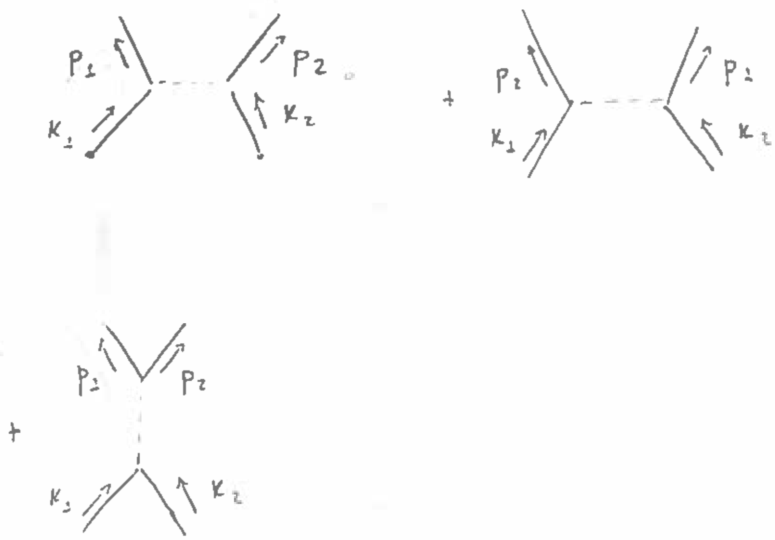
y
z
x1
x2
x3
x4



x1
x2
x3
x4
y
z

ii) All of the diagrams we drew before except for the first G, which contained vacuum subdiagrams.

iii)



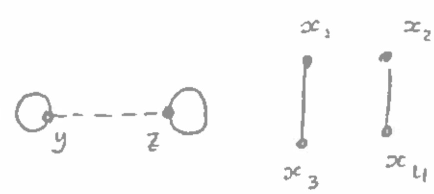
f) The green's function is obtained by

$$\langle \Omega | T \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) | \Omega \rangle$$

$$= \frac{\langle 0 | T \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\, e^{\int dw\, \Phi(w)\varphi(w)^2} | 0 \rangle}{\langle 0 | e^{\int dw\, \Phi(w)\varphi(w)^2} | 0 \rangle}.$$

The denominator cancels all of the vacuum diagrams from the numerator. Since at $\mathcal{O}(g^2)$ the numerator coincides with i), the diagrams of ii) coincides with those of i) that do not contain vacuum bubbles. On the other hand, the diagrams on iii) are obtained through the LSZ formula. Diagrams from ii) which do not scatter do not contribute to $\mathcal{M}$. Thus the diagrams for iii) are obtained from the scattering diagrams in ii) in momentum space.

g) Consider the diagram



corresponds to the Wick contraction

$$\frac{g^2}{4}\langle 0|T\varphi(y)\varphi(y)|0\rangle\langle 0|T\varphi(z)\varphi(z)|0\rangle\langle 0|T\Phi(y)\Phi(z)|0\rangle\langle 0|T\varphi(x_1)\varphi(x_3)|0\rangle\langle 0|T\varphi(x_2)\varphi(x_4)\rangle$$

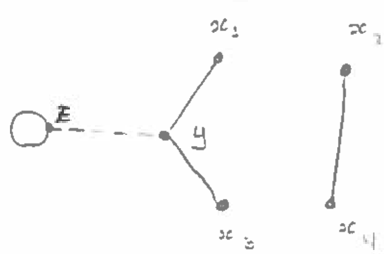
of

$$\langle 0 | T\, \Phi(y)\, \varphi(y)^2\, \Phi(z)\, \varphi(z)^2\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4) | 0 \rangle$$

This diagram has a symmetry factor of 4, corresponding to the fact that this Wick contraction is unambiguous

$$\langle 0 | T\, \Phi(y)\, \Phi(z)\, \varphi(y)\, \varphi(y)\, \varphi(z)\, \varphi(z)\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4) | 0 \rangle$$

On the other hand



corresponds to the contraction

$$\langle 0 | T\, \Phi(y)\, \Phi(z) | 0 \rangle \langle 0 | \varphi(z)^2 | 0 \rangle \langle 0 | \varphi(x_1)\, \varphi(y) | 0 \rangle \langle 0 | \varphi(y)\, \varphi(x_3) | 0 \rangle$$
$$\times \langle 0 | \varphi(x_2)\, \varphi(x_4) | 0 \rangle .$$

This has a symmetry factor of 2, such

that is weight is $\frac{g^2}{2} = 2\frac{g^2}{4}$. This corresponds to the two contractions

$$\langle 0|T\,\Phi(y)\Phi(z)\,\varphi(y)\varphi(y)\varphi(z)\varphi(z)\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$$

$$\langle 0|T\,\Phi(y)\Phi(z)\,\varphi(y)\varphi(y)\varphi(z)\varphi(z)\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle .$$

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d) I just realized we computed the leading order term, not the next-to-leading. For this we have

$$\langle\Omega|T\,\Phi(x)\varphi(y)\varphi(z)|\Omega\rangle = \frac{-\frac{ig}{2}\int d^4w\,\langle 0|T\,\Phi(x)\Phi(w)\varphi(y)\varphi(z)\varphi(w)^2|0\rangle + \frac{ig^3}{2^3}\int d^4w_1 d^4w_2 d^4w_3 \langle 0|T\,\Phi(x)\Phi(w_1)\Phi(w_2)\Phi(w_3)\times \varphi(y)\varphi(z)\varphi(w_1)^2\varphi(w_2)^2\varphi(w_3)^2|0\rangle}{1 + \frac{ig^2}{2^2}\int d^4w_1 d^4w_2\,\langle 0|T\,\Phi(w_1)\Phi(w_2)\varphi(w_1)^2\varphi(w_2)^2|0\rangle}$$

$$+ \mathcal{O}(g^4)$$

In here we used that the $\mathcal{O}(g^2)$ term in the numerator vanishes since we would have an odd number of Φ fields. The same happens with the $\mathcal{O}(g^3)$ term in the denominator. We thus have

$$\langle \Omega | T \Phi(x) \varphi(y) \varphi(z) | \Omega \rangle =$$

$$- i \frac{g}{2} \int d^4w \, \langle 0 | T \Phi(x) \Phi(w) \varphi(y) \varphi(z) \varphi(w)^2 | 0 \rangle$$

$$+ \frac{i g^3}{2^3} \Big[\int d^4w_1 \, d^4w_2 \, d^4w_3 \, \langle 0 | T \Phi(x) \Phi(w_1) \Phi(w_2) \Phi(w_3) \varphi(y) \varphi(z) \varphi(w_1)^2 \varphi(w_2)^2 \varphi(w_3)^2 | 0 \rangle$$

$$+ i \int d^4w_1 \, d^4w_2 \, d^4w_3 \, \langle 0 | T \Phi(w_1) \Phi(w_2) \varphi(w_1)^2 \varphi(w_2)^2 | 0 \rangle \langle 0 | T \Phi(x) \Phi(w_3) \varphi(y) \varphi(z) \varphi(w_3)^2 | 0 \rangle$$

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