# How to derive Feynman diagrams for finite-dimensional integrals directly from the BV formalism

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Consider $V_\bullet = \mathbb{K}[[x^1, \dots, x^N, \xi_1, \dots, \xi_N, \hbar]]$ and the operator

$$Q = a_{ij} x^i \frac{\partial}{\partial \xi_j} - \frac{\partial b(x)}{\partial x^i} \frac{\partial}{\partial \xi_i} - \hbar \frac{\partial^2}{\partial x^i \partial \xi_i}$$

For some fixed $N \in \mathbb{N}^+$, field of characteristic $0$ $\mathbb{K}$, invertible symmetric $a \in M_N(\mathbb{K})$ and power series $b(x) \in \mathbb{K}[[x^1, \dots, x^N]] \subseteq V_\bullet$ with only cubic or higher terms. In here $\xi_1, \dots, \xi_N$ are our usual symbols satisfying $\delta_{ij} = \{\xi_i, \xi_j\}$. By assigning them degree $1$, while assigning $x^1, \dots, x^N, \hbar$ degree $0$, we obtain a graded supercommutative algebra

$$V_\bullet = \bigoplus_{n=0}^{N} V_n .$$

Moreover, since $Q$ decreases the degree by $1$,

$(V_\bullet, Q)$ becomes a chain complex

$$\cdots \xrightarrow{Q_{N+2}} 0 \xrightarrow{Q_{N+1}} V_N \xrightarrow{Q_N} V_{N-1} \xrightarrow{Q_{N-1}} \cdots \xrightarrow{Q_1} V_0 \xrightarrow{Q_0} 0 \xrightarrow{Q_{-1}} \cdots$$

As we will see, understanding the homology of chain complexes will lead us to the Feynman's rules for finite dimensonal Gaussian integrals. Our main ansatz is that

$$H_0(V_\bullet, Q) = \frac{\ker Q_0}{\operatorname{im} Q_1} = V_0 / \operatorname{im} Q_1 \cong \mathbb{K}[\![\hbar]\!]$$

as a $\mathbb{K}[\![\hbar]\!]$-module. Noticing that $\operatorname{im} Q_1 \subseteq \langle x^1, \dots, x^N, \hbar \rangle$; it is clear $1 \notin \operatorname{im} Q_1$. Then, our ansatz translates to the requirement that for every $f \in V_0$ there exists a unique $\langle f \rangle \in \mathbb{K}[\![\hbar]\!]$ s.t. $[f] = \langle f \rangle [1]$. By finding such an $\langle f \rangle$ we verify the ansatz. Of course, I imagine the idea is that

$$\langle f \rangle = \int d^N x \, \exp\left(x \cdot Ax + b(x)\right) f(x).$$

At this point I don't know where the $\hbar$ comes in.

## 1. Example: Wick's Lemma

Let $N=1$ and $b(x)=0$. Then our complex is

$$\mathbb{K}[\![x,\hbar]\!]\xi \xrightarrow{Q_1} \mathbb{K}[\![x,\hbar]\!]$$

with

$$Q = ax\frac{\partial}{\partial \xi} - \hbar \frac{\partial^2}{\partial x \partial \xi}.$$

Let $f \in \mathbb{K}[\![x,\hbar]\!]$. Then

$$Q(f\xi) = axf(x,\hbar) - \hbar\frac{\partial f(x,\hbar)}{\partial x}.$$

We have

$$0 = Q(f\xi) = axf(x,\hbar) - \hbar\frac{\partial f(x,\hbar)}{\partial x}$$

implies with some formal calculus (for $f \neq 0$)

$$f(x,\hbar) = f(0,\hbar)\, e^{ax^2/2\hbar} \notin \mathbb{K}[\![x,\hbar]\!]$$

The $\hbar$ in the denominator?

Thus $\operatorname{Ker} Q_1 = \{0\}$. Then

$$H_1(V_\bullet, Q) = \frac{\operatorname{Ker} Q_1}{\operatorname{im} Q_2} = \{0\}/\{0\} = \{[0]\}.$$

Now,

$$Q(x^n \xi) = a x^{n+1} - \hbar n x^{n-1},$$

and $\operatorname{im} Q_1$ is the closure in the power series topology of the $\mathbb{K}[\hbar]$-span of such elements.

Hence in $H_0(V_\bullet, Q) = \operatorname{Ker} Q_0 / \operatorname{im} Q_1 = V_0 / \operatorname{im} Q_1$

$$[x^{n+1}] = \frac{\hbar}{a} n [x^{n-1}].$$

By recursion then $\langle x^{2n+1} \rangle = 0$ and

$$\langle x^{2n} \rangle = \left(\frac{\hbar}{a}\right)^n (2n-1)(2n-3) \cdots 1$$

$$= \left(\frac{\hbar}{a}\right)^n (2n-1)!!$$

Now, $(2n-1)!!$ is the number of ways of joining $2n$ points. Each contributes $\hbar/a$.

Example: $n = 2$


This of course corresponds to the e.v. of $x^{2n}$

on a system with partition function

$$Z = \int dx \; e^{-\frac{a}{2\hbar} x^2}.$$

If $N>1$,

$$Q = a_{ij} x^i \frac{\partial}{\partial \xi_j} - \hbar \frac{\partial^2}{\partial x^i \partial \xi_i}.$$

We have for $f \in \mathbb{K}[\![x^1, \dots, x^N, \hbar]\!]$

$$Q(f \xi_1 \cdots \xi_N) = a_{ij} x^i f(x,\hbar)(-1)^{j-1} \xi_1 \cdots \hat{\xi}_j \cdots \xi_N$$

$$- \hbar \frac{\partial f(x,\hbar)}{\partial x^i} (-1)^{i-1} \xi_1 \cdots \hat{\xi}_i \cdots \xi_N.$$

$$= (-1)^{j-1} \left( a_{ij} x^i f(x,\hbar) - \hbar \frac{\partial f(x,\hbar)}{\partial x^j} \right) \xi_1 \cdots \hat{\xi}_j \cdots \xi_N.$$

With some formal calculus, this is 0 only when

$$\frac{\partial f(x,\hbar)}{\partial x^j} = \frac{1}{\hbar} a_{ij} x^i f(x,\hbar),$$

i.e.

$$f(x,\hbar) = f(0,\hbar) \exp\left( \frac{1}{2\hbar} a_{ij} x^i x^j \right).$$

If $f(0,\hbar) \neq 0$, this is not in $\mathbb{K}[x,\hbar]$. Thus, much like in the case $N=1$, $H_N(V_\bullet, Q) = \{[0]\}$.

As far as integration is concerned however (much like it was discussed in my previous note) the interesting object is the homology in degree 0. We have that the image of $Q_1$ is spanned by elements of the form

$$Q(x^{i_1}\cdots x^{i_n}\xi_j) = a_{ij}\, x^i x^{i_1}\cdots x^{i_n} - \hbar \frac{\partial}{\partial x^j}(x^{i_1}\cdots x^{i_n})$$

$$= a_{ji}\, x^i x^{i_1}\cdots x^{i_n} - \hbar \sum_{l=1}^{n} \delta_j^{i_l}\, x^{i_1}\cdots \hat{x}^{i_l}\cdots x^{i_n},$$

$a^{kj}$

We recognize this to be precisely the calculation done in the previous notes with $\frac{\partial}{\partial x^j} = \xi_j$ and $Q = \hbar^{-1}\mathrm{Div}_{\omega_{a/\hbar}}$. We conclude that in $H_0(Q_\bullet, V_\bullet)$

$$[x^{i_1}\cdots x^{i_n} x^{i_0}] = \hbar \sum_{l=1}^{n} a^{i_0 i_l}\,[x^{i_1}\cdots \hat{x}^{i_l}\cdots x^{i_n}].$$

Thus by induction, $\langle x^{i_1}\cdots x^{i_{2n+1}}\rangle = 0$ and

$$\langle x^{i_1}\cdots x^{i_{2n}}\rangle = \hbar^n \sum_{\substack{\text{pairings}\\ \{\{i_1,j_1\},\dots,\{i_n,j_n\}\}\\ \text{of } \{1,\dots,2n\}}} a^{i_1 j_1}\cdots a^{i_n j_n} \implies \text{Wick's theorem!}$$

Remark: This is beautiful!

## 2. Example: Counting trivalent graphs

Let $N=1$, $a=1$ and $b(x) = x^3/6$. Then

$$Q = x\frac{\partial}{\partial \xi} - \frac{1}{2}x^2\frac{\partial}{\partial \xi} - \hbar\frac{\partial^2}{\partial x\,\partial \xi}.$$

We thus see that $\operatorname{im} Q_1$ is spanned by

$$Q(x^n\xi) = x^{n+1} - \frac{1}{2}x^{n+2} - \hbar n x^{n-1}.$$

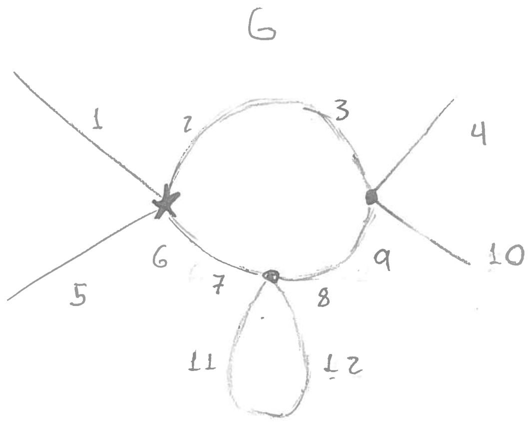
Thus

$$[x^{n+1}] = \frac{1}{2}[x^{n+2}] + \hbar n [x^{n-1}]$$

To proceed, recall

Definition: A Feynman graph $G$ is a finite set of half edges $H(G)$ together with a partition $V(G)$, a set $E(G)$ of disjoint pairs of half edges and a subset $M(G) \subseteq V(G)$. Moreover, each $p \in M(G)$ is totally ordered. The elements of $V(G)$ are called vertices, of $M(G)$, marked vertices, of $E(G)$, internal edges and of $H(G)\setminus \bigcup E(G)$ internal edges. For every $p \in V(G)$, $|p|$ is its valency.

Example:




$H(G) = \{1, \dots, 12\}$

$E(G) = \{\{2,3\}, \{6,7\}, \{8,9\}, \{11,12\}\}$

$V(G) = \{\{1,2,5,6\}, \{7,8,11,12\}, \{9,10,3,4\}\}$

Remark: This example is taken from K. Yeats, "A Combinatorial Perspective on Quantum Field Theory". I don't think his labelling is correct though.

$H(G) \setminus \cup E(G) = \{1, 4, 5, 10\}$

$M(G) = \{\{1, 2, 5, 6\}\}$.

Definition: Let $G$ be a Feynman graph. Then an automorphism of $G$ is a bijection $f : H(G) \to H(G)$ s.t.

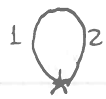
i) for all $\{e, e'\} \in E(G)$, we have $\{f(e), f(e')\} \in E(G)$,

ii) for all $e \in p \in V(G)$ we have $f(e) \in p$ (i.e. $\forall p \in V(G),\ f(p) = p$),

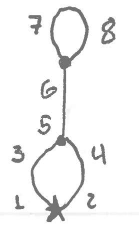
iii) For all $p \in M(G)$, $f|_p = id_p$.

The set of all automorphisms of $G$ is called $\mathrm{Aut}\, G$.

Example:

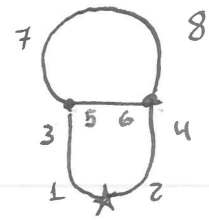



$|\mathrm{Aut}\, G| = 1$



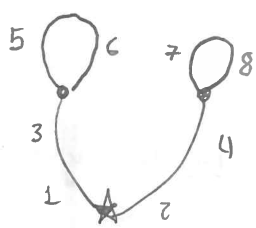

$|\mathrm{Aut}\, G| = 2$

$\begin{pmatrix} 7 \to 8 \\ 8 \to 7 \end{pmatrix}$




$|\mathrm{Aut}\, G| = 2$

$\begin{pmatrix} 5 \to 7 \\ 6 \to 8 \\ 7 \to 5 \\ 8 \to 6 \end{pmatrix}$




$|\mathrm{Aut}\, G| = 4$

$\begin{pmatrix} \begin{matrix} 5 \to 6 \\ 6 \to 5 \end{matrix}, & \begin{matrix} 7 \to 8 \\ 8 \to 7 \end{matrix}, & \begin{matrix} 5 \to 6 \\ 6 \to 5 \\ 7 \to 8 \\ 8 \to 7 \end{matrix} \end{pmatrix}$

Theorem: Let $G$ be a Feynman graph, $e \in p \in M(G)$ and $f \in \mathrm{Aut}\, G$. Then, for all $e' \in H(G)$ if $\{e, e'\} \in E(G)$ then $f(e') = e'$.

Proof: Suppose $e' \in H(G)$ and $\{e, e'\} \in E(G)$. Then $f(e) = e$ and $\{f(e), f(e')\} = \{e, f(e')\} \in E(G)$. However, since $E(G)$ is a partition, there doesn't exists an internal edge, other than $\{e, e'\}$ to which $e$ belongs to. Thus $\{e, f(e')\} = \{e, e'\}$ and we conclude $f(e') = e'$.

We need a final definition:

Definition: Let $G$ be a Feynman graph. Then its first Betti number is

$$\beta(G) = |E(G)| - |V(G) \setminus M(G)|$$

Now, let $\mathcal{G}_n$ be the set (up to automorphisms) of all Feynman diagrams for $\langle x^n \rangle$, i.e. $G \in \mathcal{G}$ if $|M(G)| = 1$, $G$ is connected (how to formalize this?), $|p| = 3$ for all $p \in V(G) \setminus M(G)$, $|\star| = n$ for the $\star \in M(G)$ and $G$ has no external edges.

Claim: $\langle x^n \rangle = c_n := \sum_{G \in \mathcal{G}_n} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}\, G|}$.

We have

$$c_0 = \frac{\hbar^{\beta(\star)}}{|\mathrm{Aut}(\star)|} = \frac{\hbar^0}{1} = 1 = \langle 1 \rangle.$$

Thus, we only need to proof

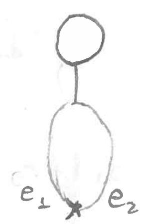
$$c_{n+1} = \frac{1}{2} c_{n+2} + \hbar n c_{n-1}$$

Remark: Don't we need to proof this for $\langle x \rangle$. Notice $c_1$ already has an infinite amount of terms.

Let $G \in \mathcal{G}_{n+1}$ and $M(G) = \{\star\}$ ~~where the ordering of $\star$ is given by the subindices.~~ Since our Feynman graphs have no internal edges, there is unique $Ed_G \in E(G)$ s.t. $\max \star \in Ed_G$. If the $e \in Ed_G \setminus \max \star$ is s.t. $e \in \star$, we say $G \in \mathcal{G}^{loop}_{n+1}$.

Example:

$\in G_2^{loop}$



$\notin G_2^{loop}$

Then $\{\mathcal{G}^{loop}_{n+1}, \mathcal{G}_{n+1} \setminus \mathcal{G}^{loop}_{n+1}\}$ is a partition of $\mathcal{G}_{n+1}$ and

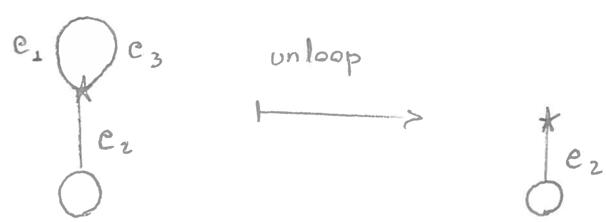
$$c_{n+1} = \sum_{G \in \mathcal{G}_{n+1} \setminus \mathcal{G}^{loop}_{n+1}} \frac{\hbar^{\beta(G)}}{|\operatorname{Aut} G|} + \sum_{G \in \mathcal{G}^{loop}_{n+1}} \frac{\hbar^{\beta(G)}}{|\operatorname{Aut} G|}.$$

Now consider the map

$U \equiv \text{unloop} : \mathcal{G}^{\text{loop}}_{n+1} \longrightarrow \mathcal{G}_{n-1}$

which removes the loop at the last half edge.

Example




In fact, $\tilde{G} = \text{unloop}(G)$, is defined by

$$H(\tilde{G}) = H(G) \setminus Ed_G,$$

$$E(\tilde{G}) = E(G) \setminus \{Ed_G\},$$

$$V(\tilde{G}) = (V(G) \setminus \{\star\}) \cup \{\tilde{\star} := \star \setminus Ed_G\}$$

$$M(\tilde{G}) = = \{\tilde{\star}\},$$

and $\tilde{\star} \subseteq \star$ has the induced order.

Theorem: For all $G \in \mathcal{G}^{\text{loop}}_{n+1}$, $\beta(\text{unloop}(G)) = \beta(G) - 1$

Proof:

$$\beta(U(G)) = |E(U(G))| - |V(U(G)) \setminus M(U(G))| = |E(G)| - 1 - |V(G) \setminus M(G)| = \beta(G) - 1.$$

Indeed if $M(U(G)) = \{\tilde{\star}\}$, and $\tilde{\star} \neq \emptyset$, $\tilde{\star} \notin V(G)$ since $\tilde{\star} \subsetneq \star$, $\star \in V(G)$.

and $V(G)$ is a partition. Thus

$$|V(\upsilon(G)) \setminus M(\upsilon(G))| = |V(\upsilon(G))| - 1$$
$$= |V(G)| - 1$$
$$= |V(G) \setminus M(G)|.$$

Theorem: $|\mathrm{Aut}(\upsilon(G))| = |\mathrm{Aut}\, G|$ for all $G \in \mathcal{G}^{\text{loop}}_{n+1}$.

Proof: Consider

$$\mathrm{Aut}\, G \longrightarrow \mathrm{Aut}(\upsilon(G))$$
$$f \longmapsto f|_{H(\upsilon(G)) = H(G) \setminus Ed_G}.$$

Notice that if $f, g \in \mathrm{Aut}\, G$ are s.t. $f|_{H(\upsilon(G))} = g|_{H(\upsilon(G))}$, then, in fact, $f = g$, since $E \subseteq \star_G$ (from now on $\{\star_G\} = M(G)$) and $f|_{\star_G} = id_{\star_G} = g|_{\star_G}$. We thus have an injective map. On the other hand, if $\varphi \in \mathrm{Aut}(\upsilon(G))$, we can define $f \in \mathrm{Aut}(G)$ by

$$f(e) = \begin{cases} \varphi(e), & e \in H(\upsilon(G)) \\ e, & e \in Ed_G \end{cases}$$

Then $f|_{H(\upsilon(G))} = \varphi$ and our map is surjective.

Theorem: unloop is surjective and, in fact, for all $\tilde{G} \in \mathcal{G}_{n-1}$, $|\mathrm{unloop}^{-1}(\{\tilde{G}\})| = n$.

Proof: Let $\tilde{G} \in \mathcal{G}_{n-1}$. Consider two new half edges $e, e_{n+1} \notin H(\tilde{G})$. Let $\star_{\tilde{G}} = \{e_1, \dots, e_{n-1}\}$ with $e_1 < e_2 < \dots < e_{n-1}$.

For all $j \in \{1, \dots, n\}$, let $\star_{G^j} := \star_{\tilde{G}} \cup \{e, e_{n+1}\}$ with the order

$$e_1 < \dots < e_{j-1} < e < e_{j+1} < \dots < e_{n+1}.$$

Define $Ed_{G^j} = \{e, e_{n+1}\}$. Then

$$H(G^j) = H(\tilde{G}) \dot{\cup} Ed_{G^j}$$

$$E(G^j) = E(\tilde{G}) \dot{\cup} \{Ed_{G^j}\}$$

$$V(G^j) = (V(\tilde{G}) \setminus M(\tilde{G})) \dot{\cup} \{\star_{G^j}\}$$

$$M(G^j) = \{\star_{G^j}\}$$

defines a $G^j \in \mathcal{G}^{\text{loop}}_{n+1}$ s.t. $\mathrm{unloop}(G^j) = \tilde{G}$. For differen $j$ the corresponding $\star^j$ are different since they have different orderings. Moreover, these are clearly, up to relabeling all of the graphs which unloop to $\tilde{G}$.

We thus have

$$\sum_{G \in \mathcal{G}^{loop}_{n+1}} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}\, G|} = \sum_{\tilde{G} \in \mathcal{G}_{n-1}} \sum_{G \in U^{-1}(\{\tilde{G}\})} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}\, G|}$$

$$= \sum_{\tilde{G} \in \mathcal{G}_{n-1}} \sum_{G \in U^{-1}(\{\tilde{G}\})} \frac{\hbar^{\beta(\tilde{G})+1}}{|\mathrm{Aut}\, \tilde{G}|} = \hbar n c_{n-1} .$$

We thus are only left with showing

$$\sum_{G \in \mathcal{G}_{n+1} \setminus \mathcal{G}^{loop}_{n+1} =: \mathcal{G}^{\lambda}_{n+1}} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}\, G|} = \frac{1}{2} c_{n+2}$$

$$= \frac{1}{2} \sum_{\tilde{G} \in \mathcal{G}_{n+2}} \frac{\hbar^{\beta(\tilde{G})}}{|\mathrm{Aut}\, \tilde{G}|} ,$$

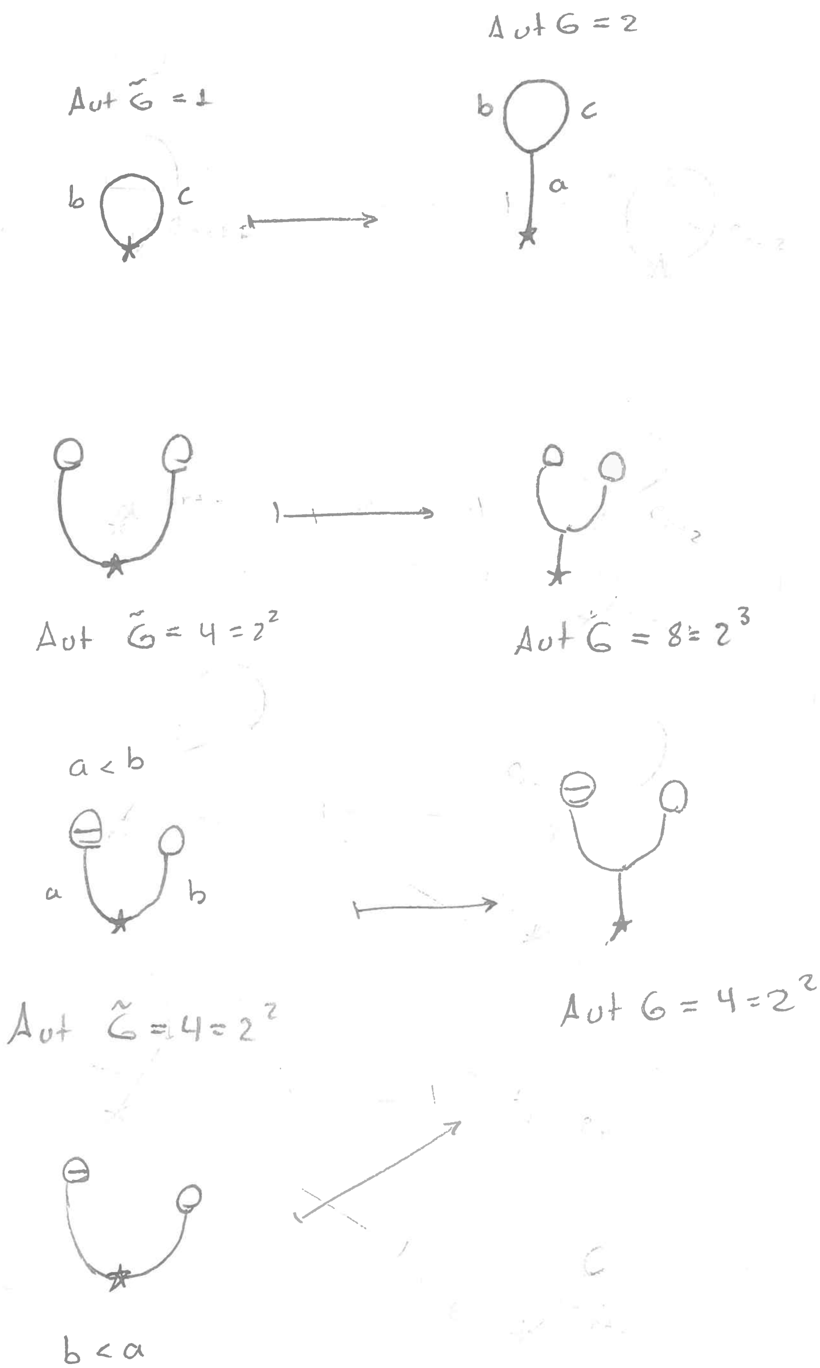
We follow the same strategy.

Consider the map

$$z \equiv zip : \mathcal{G}_{n+2} \longrightarrow \mathcal{G}_{n+1}$$

obtained by unzipping the $n+1$ edge

Examples




This map is defined as follows. Let $\tilde{G} \in \mathcal{G}_{2n+2}$ and take two new edges $e, e' \notin H(\tilde{G})$. We define $G = zip(\tilde{G})$ as follows:

$$H(G) := H(\tilde{G}) \dot{\cup} \{e, e'\},$$

$$E(G) := E(\tilde{G}) \dot{\cup} \{\{e, e'\}\},$$

$$V(G) := (V(\tilde{G}) \setminus \{\star_{\tilde{G}}\}) \dot{\cup} \{\star_G, \{e_{n+2}, e_{n+1}, e'\}\},$$

$$M(G) := \{\star_G\},$$

where $\star_G := (\star_{\tilde{G}} \setminus \{e_{n+2}, e_{n+1}\}) \dot{\cup} \{e\}, \quad a < e \quad \forall\, a \in \star_{\tilde{G}} \setminus \{e_{n+2}, e_{n+1}\}.$

$$e_{n+2} := \max \star_{\tilde{G}}$$

$$e_{n+1} := \max\left(\star_{\tilde{G}} \setminus \{e_{n+2}\}\right).$$

Theorem: $\mathrm{zip}(\mathcal{E}_{n+2}) = \mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}$

Proof: We will construct a right inverse

$$\mathrm{uzip}: \mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop} \longrightarrow \mathcal{E}_{n+2}$$

as follows. Take $G \in \mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}$. Let $Ed_G \in E(G)$ be the unique edge s.t. $\max \star_G \in Ed_G$. Let $Ed_G = \{\max \star_G, e\}$. Since $G \notin \mathcal{E}_{n+1}^{loop}$, $e \notin \star_G$. Since $V(G)$ is a partition, there is some $p \in V(G) \setminus \{\star_G\}$ s.t. $e \in p$. We are considering trivalent vertices and thus $p = \{e, a, b\}$ for some $a, b \in H(G)$.

Define $\tilde{G} = \mathrm{unzip}(G)$ by

$$H(\tilde{G}) = H(G) \setminus Ed_G$$

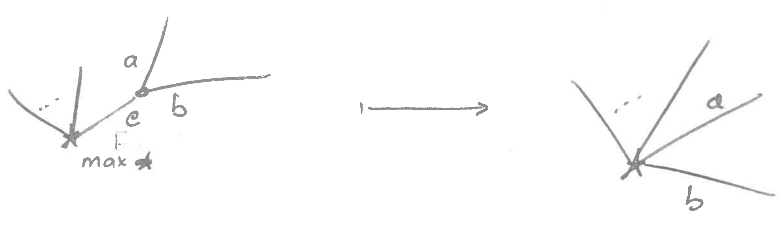
$$E(\tilde{G}) = E(G) \setminus \{Ed_G\}$$

$$V(\hat{G}) = (V(G) \setminus \{\star_G, p\}) \cup \{\star_{\tilde{G}}\}$$

$$M(\hat{G}) = \{\star_{\tilde{G}} := (\star_G \setminus \{\max \star_G\}) \cup \{a, b\}\}$$

where you choose to order $\star_{\tilde{G}}$ by $\tilde{e} < a < b$ for all $\tilde{e} \in \star_{\tilde{G}}$. Then $\mathrm{zip} \circ \mathrm{unzip} = \mathrm{id}_{\mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}}$.

We are thus only left with checking that $\mathrm{zip}(\mathcal{E}_{n+2}) \subseteq \mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}$. This is however clear.




Remark: In the above construction we could've chosen $b < a$ instead of $a < b$. This of course only makes a difference if there isn't a $\phi \in \mathrm{Aut}\, G$ s.t. $\phi(a) = b$ and $\phi(b) = a$.

Let us formulate the previous remark more formally. Given the theorem we proved first in this notes, for all $\phi \in \operatorname{Aut} G$, $\phi(e)=e$. But

$$\{a,b,e\} = \phi(\{a,b,e\}) = \{\phi(a), \phi(b), \phi(e)\} =$$

$$= \{\phi(a), \phi(b), e\}.$$

Thus, either

$$\phi(a)=a, \quad \phi(b)=b,$$

or

$$\phi(a)=b, \quad \phi(b)=a.$$

There is always an automorphism that satisfies the first (the identitity). If there is one which satisfies the second, we will say $G \in \mathcal{G}_{n+1}^{sym}$.

For all $G \in \mathcal{G}_{n+1}^{sym}$ we have $zip^{-1}(\{G\}) = \{unzip(G)\}$ and $|\operatorname{Aut} G| = 2\,|\operatorname{Aut} unzip(G)|$. For all $G \in (\mathcal{G}_{n+1} \setminus \mathcal{G}_{n+1}^{loop}) \setminus \mathcal{G}_{n+1}^{sym}$, $zip^{-1}(\{G\}) = \{unzip(G), \tilde{G}_2\}$ s.t. $|\operatorname{Aut} G| = |\operatorname{Aut} unzip(G)| = |\operatorname{Aut} \tilde{G}_2|$.

In all cases

$$\beta(G) = \beta(unzip(G)) = \beta(\tilde{G}_2).$$

Thus

$$\sum_{G \in \mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}(G)|} = \sum_{G \in \mathcal{E}_{n+1}^{sym}} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}(G)|} + \sum_{G \in (\mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}) \setminus \mathcal{E}_{n+1}^{sym}} \frac{\hbar^{\beta(G)}}{|\mathrm{Aut}(G)|}$$

$$= \sum_{G \in \mathcal{E}_{n+1}^{sym}} \frac{\hbar^{\beta(\mathrm{unzip} G)}}{2|\mathrm{Aut}\,\mathrm{unzip}\, G|} + \sum_{G \in (\mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}) \setminus \mathcal{E}_{n+1}^{sym}} \frac{2\,\hbar^{\beta(\mathrm{unzip}(G))}}{2|\mathrm{Aut}(\mathrm{unzip}\, G)|}$$

$$= \sum_{\tilde{G} \in \mathrm{zip}^{-1}(\mathcal{E}_{n+1}^{sym})} \frac{\hbar^{\beta(\tilde{G})}}{2|\mathrm{Aut}\,\tilde{G}|} + \sum_{\tilde{G} \in \mathrm{zip}^{-1}((\mathcal{E}_{n+1} \setminus \mathcal{E}_{n+1}^{loop}) \setminus \mathcal{E}_{n+1}^{sym})} \frac{\hbar^{\beta(\tilde{G})}}{2|\mathrm{Aut}\,\tilde{G}|}$$

$$= \frac{1}{2} \sum_{\tilde{G} \in \mathcal{E}_{n+2}} \frac{\hbar^{\beta(\tilde{G})}}{|\mathrm{Aut}\,\tilde{G}|},$$

completing the proof.

Why is all of this working?

Under the identification of $\xi_i$ with $\frac{\partial}{\partial x^i}$ (and thus of $\frac{\partial}{\partial \xi_i}$ with $dx^i$), we identify $V_\bullet$ and $\mathrm{Vect}(\mathbb{R}^n)$ (roughly). We further identify $Q$ and

$$-\hbar\,\mathrm{Div}_\omega = \frac{\partial S}{\partial x^i} dx^i - \hbar \frac{\partial}{\partial x^i} \circ dx^i = \left(a_{ij} x^j - \frac{\partial b}{\partial x^i}\right) dx^i - \hbar \frac{\partial}{\partial x^i} \circ dx^i$$

Thus

$$S = \frac{1}{2} a_{ij} x^i x^j - b$$

and we are solving

$$\langle f \rangle = \int d^N x \, e^{-S/\hbar} f .$$

## 3. The General Case

First of all, let us go back to the problem of defining connectedness of a graph. For this we need an appropriate notion of path on such graphs.

Definition: A path on a graph $G$ is a map

$$\gamma : \{1, \dots, m\} \longrightarrow H(G)$$

s.t.: i) for all $n \in (2\mathbb{N}+1)^{<m}$ $\{\gamma_n, \gamma_{n+1}\} \in E(G)$,

ii) for all $n \in (2\mathbb{N}^+)^{<m}$ $\{\gamma_n, \gamma_{n+1}\} \in p$ for some $p \in V(G)$.