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Homework 2: Generalized Measurement

1.a) We have a density matrix

$$\rho' = |\psi\rangle\langle\psi| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{\sqrt{2}} e^{-i\pi/3}|0\rangle\langle 1| + \frac{1}{\sqrt{2}} e^{i\pi/3}|1\rangle\langle 0|$$

$$+ \frac{1}{2}|1\rangle\langle 1|.$$

b) Recalling that

$$\sigma_x := |0\rangle\langle 1| + |1\rangle\langle 0|,$$

$$\sigma_y := -i|0\rangle\langle 1| + i|1\rangle\langle 0|,$$

we have

$$a_x = tr(\rho'\sigma_x) = tr\Big(\frac{1}{2}|0\rangle\langle 1| + \frac{1}{\sqrt{2}} e^{i\pi/3}|1\rangle\langle 1| + \frac{1}{\sqrt{2}} e^{-i\pi/3}|0\rangle\langle 0|$$

$$+ \frac{1}{2}|1\rangle\langle 0|\Big)$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\pi/3} + e^{-i\pi/3}\right) = \sqrt{2}\cos(\pi/3) = \frac{1}{\sqrt{2}},$$

$$a_y = tr(\rho'\sigma_y) = -\frac{i}{\sqrt{2}} e^{i\pi/3} + \frac{i}{\sqrt{2}} e^{-i\pi/3}$$

$$= -\frac{2i}{\sqrt{2}} \sin(\pi/3) = +\sqrt{\frac{3}{2}}.$$

$$a_z = tr(\rho' \sigma_z) = \frac{1}{2} - \frac{1}{2} = 0.$$

c) Consider the Stern-Gerlach set-up consists of subjecting the particles emitted from the source to a magnetic field along the $i \in \{x, y, z\}$ direction. Depending on its spin on this direction, a particle's trayectory is deviated to the $+i$ or $-i$ direction. The first case corresponds to the $+1$ eigenvalue of σ_i, while the second to $-i$. If N_+ is the number of particles deviated to the $+i$ direction, and N_- the corresponding number for the $-i$ direction, we have

$$a_i = \langle \sigma_i \rangle_{\rho'} = tr(\rho a_i) = \frac{N_+ - N_-}{N_+ + N_-}$$

d) Indeed, we have

$$E_1 = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|1'\rangle\langle 1'|.$$

Thus

$$E_0 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}\Big((1-\varepsilon)|0\rangle\langle 0| + \sqrt{\varepsilon(1-\varepsilon)}\,|0\rangle\langle 1|$$

$$+ \sqrt{\varepsilon(1-\varepsilon)}\,|1\rangle\langle 0| + \varepsilon|1\rangle\langle 1|\Big)$$

$$= \frac{1}{2}(2-\varepsilon)|0\rangle\langle 0| + \frac{1}{2}\varepsilon|1\rangle\langle 1| + \frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)}\,|0\rangle\langle 1|$$

$$+\frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)}\,|1\rangle\langle 0|.$$

Similarly

$$E_1 = \frac{1}{2}(2-\varepsilon)|1\rangle\langle 1| + \frac{1}{2}\varepsilon|0\rangle\langle 0| - \frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)}\,|0\rangle\langle 1|$$

$$-\frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)}\,|1\rangle\langle 0|.$$

f) We have

$$p'(0) = \mathrm{tr}(E_0\rho) = \langle\psi|E_0|\psi\rangle$$

$$= \langle\psi|\Big[\frac{1}{\sqrt{2}}\Big(\frac{1}{2}(2-\varepsilon)|0\rangle + \frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)}\,|1\rangle\Big)$$

$$+ \frac{e^{i\pi/3}}{\sqrt{2}}\Big(\frac{1}{2}\varepsilon|1\rangle + \frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)}\,|0\rangle\Big)\Big]$$

$$= \frac{1}{4}(2-\varepsilon) + \frac{e^{i\pi/3}}{4}\sqrt{\varepsilon(1-\varepsilon)} + \frac{e^{-i\pi/3}}{4}\sqrt{\varepsilon(1-\varepsilon)}$$

$$\cancel{\frac{1}{4}\varepsilon} = \frac{1}{2} + \frac{\sqrt{\varepsilon(1-\varepsilon)}}{2}\cos(\pi/3) = \frac{1}{2}\left(1 + \frac{\sqrt{\varepsilon(1-\varepsilon)}}{2}\right).$$

$$\rho = \frac{1}{2}\left(id_{\mathcal{H}} + \vec{a}\cdot\vec{\sigma} \right)$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\,(-i)\right)|0\rangle\langle 1| \right.$$

$$\left. + \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\,i\right)|1\rangle\langle 0| \right)$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{\sqrt{2}} e^{-i\pi/3}|0\rangle\langle 1| + \frac{1}{\sqrt{2}} e^{i\pi/3}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$= \rho'.$$

e) The probability of measuring a spin in the $+z$ direction is $tr(|0\rangle\langle 0|\rho)$ while that of the $+z'$ direction is $tr(|0'\rangle\langle 0'|\rho)$. If we measured on the first with probability $\frac{1}{2}$ and the second with the same, the probability of obtaining a vector in the $+$ direction is

$$\frac{1}{2} tr(|0\rangle\langle 0|\rho) + \frac{1}{2} tr(|0'\rangle\langle 0'|\rho)$$

$$= tr(E_0\rho),$$

with $E_0 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0'\rangle\langle 0'|$. Similarly

One quickly checks

$$E_0 + E_1 = \frac{1}{2}(2-\varepsilon+\varepsilon)|0\rangle\langle 0| + \frac{1}{2}(\varepsilon+2-\varepsilon)|1\rangle\langle 1|$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = id_{\mathcal{H}} .$$

Thus

$$p'(1) = tr(E_1\rho) = tr(\rho) - tr(E_0\rho) = 1 - p'(0)$$

$$= \frac{1}{2}\left(1 - \frac{\sqrt{\varepsilon(1-\varepsilon)}}{2}\right).$$

g) We have

$$a_z = 1\,p'(0) - 1\,p'(1) = \frac{1}{2}\sqrt{\varepsilon(1-\varepsilon)} .$$

Thus, she would measure

$$\rho_n = \rho' + \frac{1}{2}a_z\sigma_z = \frac{1}{2}\left(1+\sqrt{\varepsilon(1-\varepsilon)}\right)|0\rangle\langle 0| + \frac{1}{\sqrt{2}}e^{-i\pi/3}|0\rangle\langle 1|$$

$$+ \frac{1}{\sqrt{2}}e^{i\pi/3}|1\rangle\langle 0| + \frac{1}{2}\left(1-\sqrt{\varepsilon(1-\varepsilon)}\right)|1\rangle\langle 1|$$

h) Quite literally,

$$\rho_n - \rho' = \frac{1}{2}a_z\sigma_z .$$

However, physically,

$$\mathrm{tr}\left(|0\rangle\langle 0|\rho_n\right) = \frac{1}{2}\left(1 + \sqrt{\varepsilon(1-\varepsilon)}\right) > \frac{1}{2}\left(1 - \sqrt{\varepsilon(1-\varepsilon)}\right)$$
$$= \mathrm{tr}\left(|1\rangle\langle 1|\rho_n\right),$$

i.e. Alice would think that the orientation $+z$ has a higher probability than $-z$.

i) We have

$$\mathrm{tr}\left(\rho_n^2\right) = \mathrm{tr}\left(\cancelto{\rho'}{\rho'^2}\right) + \frac{1}{4} a_z^2 \,\mathrm{tr}\left(\cancelto{id_{\mathcal{H}}}{\sigma_z^2}\right) + \frac{1}{2} a_z \left(\mathrm{tr}\left(\rho'\sigma_z\right) + \mathrm{tr}\left(\sigma_z\rho'\right)\right)$$

$$= 1 + \frac{1}{2} a_z^2 + a_z \cancelto{0}{\langle \sigma_z \rangle_{\rho'}}$$

$$= 1 + \frac{1}{2}\frac{1}{4}\varepsilon(1-\varepsilon) = 1 + \frac{1}{8}\varepsilon(1-\varepsilon) > 1,$$

something impossible. Thus Alice would realize ρ_n is not a density matrix.

2. a) The projectors are $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$.

b) If the outcome is $m=0$, the resulting state is

$$|\psi_0\rangle = \frac{P_0|\psi\rangle}{\sqrt{\mathrm{Prob}(0)}} = \frac{1/\sqrt{2}\,|0\rangle}{\sqrt{1/2}} = |0\rangle.$$

Similarly, after the outcome $m=1$, the state is

$$|\psi_1\rangle = |1\rangle.$$

c) Indeed,

$$P_0 P_0 = |0\rangle\cancel{\langle 0|0\rangle}^{1}\langle 0| = |0\rangle\langle 0|,$$

$$P_1 P_1 = |1\rangle\cancel{\langle 1|1\rangle}^{1}\langle 1| = |1\rangle\langle 1|.$$

d) Indeed

$$\mathrm{Prob}_{|\psi_m\rangle}(m) = \langle \psi_m | P_m | \psi_m \rangle = \langle m | P_m | m \rangle = \cancel{\langle m|m\rangle}^{1}\cancel{\langle m|m\rangle}^{1} = 1.$$

e) Because such a projective measurement has the wrong update rule: Whatever the outcome of the measurement is, the post-measurent state is a state with no particles

f) We need to expand the Hilbert space to a three-dimensional vector space spanned by the orthonormal basis $\{|0\rangle, |1\rangle, |vac\rangle\}$. Then, we let

$$M_0 = |vac\rangle\langle 0|$$

$$M_1 = |vac\rangle\langle 1|$$

$$M_{vac} = |vac\rangle\langle vac|$$

and $E_i = M_i^{\dagger} M_i$. Then the measurement $\{E_0, E_1, E_{vac}\}$ has $\mathrm{Prob}(0) = \frac{1}{2}$, $\mathrm{Prob}(1) = \frac{1}{2}$, $\mathrm{Prob}(vac) = 0$ but independent of the outcome, the resulting state is

$$\frac{M_i |\psi\rangle\langle\psi| M_i^\dagger}{\text{Prob}(i)} = \frac{|vac\rangle\langle i \cancel{|\psi\rangle\langle\psi|} i\rangle\langle vac|}{\cancel{\langle\psi|i\rangle\langle i|\psi\rangle}} = |vac\rangle\langle vac|.$$

(there is the problem that Prob (vac) = 0, but... boh!).

3. a) We have

$$\text{Prob}_{|\psi_1\rangle}(1) = \langle\psi_1|E_1|\psi_1\rangle = 0,$$

$$\text{Prob}_{|\psi_2\rangle}(2) = \langle\psi_2|E_2|\psi_2\rangle = 0.$$

Thus, an outcome 1 yields that the state was $|\psi_2\rangle$, while 2 yields $|\psi_1\rangle$.

b) $$\text{Prob}_{|\psi_1\rangle}(2) = 1 - \langle\psi_1|E_2|\psi_1\rangle$$

$$= 1 - \frac{1}{4}|\langle -|\psi_1\rangle|^2 = 1 - \frac{1}{8} = \frac{7}{8}$$

$$\text{Prob}_{|\psi_2\rangle}(1) = 1 - \frac{1}{4}|\langle 1|\psi_2\rangle|^2 = 1 - \frac{1}{8} = \frac{7}{8}.$$

Thus, outcome 3 gives no information of whether the state was on $|\psi_1\rangle$ or $|\psi_2\rangle$.

c) We wish to obtain $|\phi\rangle \in \mathbb{C}^2$, $U : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ unitary and a PVM $\{P_1, P_2, P_3\}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$, such that for any density matrix $\rho : \mathbb{C}^2 \to \mathbb{C}^2$ and $i \in \{1, 2, 3\}$

$$tr(\rho E_i) = tr(U \rho \otimes |\phi\rangle\langle\phi| U^+ P_i).$$

By the linearity of the equation above on ρ, it is clear that it is enough to demand it only for pure states $|\psi\rangle \in \mathbb{C}^2$, where it reduces to

$$\langle\psi|E_i|\psi\rangle = tr(U|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| U^+ P_i).$$

Moreover, let $(|\Phi_I\rangle)_{I \in \{1,\dots,4\}}$ be an ON basis adapted to $\{P_1, P_2, P_3\}$. This means that for every $i \in \{1, 2, 3\}$, there is a $\mathcal{I}_i \subseteq \{1, \dots, 4\}$ s.t.

$$P_i = \sum_{I \in \mathcal{I}_i} |\Phi_I\rangle\langle\Phi_I|.$$

Of course, by noting that none of the elements of the POVM are trivial and counting dimensions, one sees that out of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, two are singletons while the other contains two elements. In terms of this basis we have

$$\langle\psi|E_i|\psi\rangle = \sum_{I \in \mathcal{I}_i} \langle\Phi_I|U|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| U^+ |\Phi_I\rangle.$$

Moreover, by the linearity of the requirement on on the PVM, it is clear that satisfying it for $i \in \{1,2\}$ is sufficient. Indeed, in this case, if $P_3 = id_{\mathbb{C}^2 \otimes \mathbb{C}^2} - P_1 - P_2$, then the requirement is fulfilled for $i = 3$. As a final general comment, by redefining P_i as $U^+ P_i U$, it is clear we may assume $U = id_{\mathbb{C}^2}$. In summary, we want to find a ON basis $(|\Phi_I\rangle)_{I \in \{1,2,3,4\}}$, disjoint subsets $\mathcal{G}_1, \mathcal{G}_2 \subseteq \{1,2,3,4\}$ s.t. $\mathcal{G}_3 := \{1,2,3,4\} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2) \neq \emptyset$, and a vector $|\phi\rangle \in \mathbb{C}^2$ s.t.

$$\langle \psi | E_i | \psi \rangle = \sum_{I \in \mathcal{G}_i} \langle \Phi_I | (|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|) | \Phi_I \rangle$$

for all $i \in \{1,2\}$ and $|\psi\rangle \in \mathbb{C}^2$.

Here is where we have to be creative (guess). Let $|\Phi_1\rangle = |1\rangle \otimes |+\rangle$ so that

$$\langle \Phi_1 | (|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|) | \Phi_1 \rangle = |\langle 1 | \psi \rangle|^2 \, |\langle + | \phi \rangle|^2 .$$

Similarly, let $|\Phi_2\rangle = |-\rangle \otimes |0\rangle$ so that

$$\langle \Phi_2 | (|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|) | \Phi_2 \rangle = |\langle - | \psi \rangle|^2 \, |\langle 0 | \phi \rangle|^2 .$$

We can make these coincide with

$$\langle \psi | E_1 | \psi \rangle = \frac{1}{4} |\langle 1 | \psi \rangle|^2 ,$$

and

$$\langle \psi | E_2 | \psi \rangle = \frac{1}{4} |\langle - | \psi \rangle|^2$$

respectively, by ensuring that $|\langle + | \phi \rangle|^2 = |\langle 0 | \phi \rangle|^2 = \frac{1}{4}$.

Recall also

$$\langle + | \phi \rangle = \frac{1}{\sqrt{2}} \left(\langle 0 | \phi \rangle + \langle 1 | \phi \rangle \right)$$

Demanding $\langle 0 | \phi \rangle = \frac{1}{2}$, $|\phi\rangle$ takes the form

$$|\phi\rangle = \frac{1}{2} |0\rangle + e^{i\theta} b |1\rangle$$

with $\theta \in [0, 2\pi)$ and $b \in \mathbb{R}^+$. Normalization (which we forgot to list above) requires

$$1 = \langle \phi | \phi \rangle = \frac{1}{4} + b^2 ,$$

i.e. $b = \frac{\sqrt{3}}{2}$. Thus

$$\langle + | \phi \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{2} + e^{i\theta} b \right).$$

Therefore

$$\frac{1}{2} = |\langle + | \phi \rangle|^2 = \frac{1}{2} \left(\frac{1}{2} + e^{i\theta} b \right) \left(\frac{1}{2} + e^{-i\theta} b \right)$$

$$= \frac{1}{2} \left(\frac{1}{4} + e^{i\theta} \frac{\sqrt{3}}{4} + e^{-i\theta} \frac{\sqrt{3}}{4} + \frac{3}{4} \right)$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\cos(\theta)\right)$$

We thus want $\cos(\theta) = 0$. For this we choose $\theta = \pi/2$.

Finally, we have

$$|\phi\rangle = \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle,$$

$$P_1 = |\Phi_1\rangle\langle\Phi_1| = (|1\rangle \otimes |+\rangle)(\langle 1| \otimes \langle +|) = |1\rangle\langle 1| \otimes |+\rangle\langle +|,$$

$$P_2 = |\Phi_2\rangle\langle\Phi_2| = (|-\rangle \otimes |0\rangle)(\langle -| \otimes \langle 0|) = |-\rangle\langle -| \otimes |0\rangle\langle 0|,$$

$$P_3 = id_{\mathbb{C}^2 \otimes \mathbb{C}^2} - P_1 - P_2,$$

$$U = id_{\mathbb{C}^2 \otimes \mathbb{C}^2}.$$

We finally check that indeed

$$\mathrm{tr}(U|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|U^\dagger P_1) = \langle\Phi_1|(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|)|\Phi_1\rangle$$

$$= |\langle 1|\psi\rangle|^2 |\langle +|\phi\rangle|^2 = \mathrm{tr}(|\psi\rangle\langle\psi|1\rangle\langle 1|)\left|\frac{1}{2\sqrt{2}} + i\frac{\sqrt{3}}{2\sqrt{2}}\right|^2$$

$$= \mathrm{tr}(|\psi\rangle\langle\psi|1\rangle\langle 1|)\left(\frac{1}{8} + \frac{3}{8}\right) = \mathrm{tr}(|\psi\rangle\langle\psi|E_1),$$

$$\mathrm{tr}(U|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|U^\dagger P_2) = \mathrm{tr}(|\psi\rangle\langle\psi|-\rangle\langle -|)\left(\frac{1}{2}\right)^2 = \mathrm{tr}(|\psi\rangle\langle\psi|E_2),$$

$$\mathrm{tr}(U|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|U^\dagger P_3) = \underbrace{\mathrm{tr}(U|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|U^\dagger)}_{1} - \mathrm{tr}(|\psi\rangle\langle\psi|E_1) - \mathrm{tr}(|\psi\rangle\langle\psi|E_2)$$

$$= \mathrm{tr}(|\psi\rangle\langle\psi| E_3)$$

for all $|\psi\rangle \in \mathbb{C}^2$. Moreover it is easy to see that P_1 and P_2 are mutually orthogonal projections, making $(P_1, P_2, P_3 = id_{\mathbb{C}^2 \otimes \mathbb{C}^2} - P_1 - P_2)$ a PVM.