Relativistic Quantum Information (Tutorial 3) (March. 21, 2020)

Tutorial 3: The Unruh Effect

(Symmetries, entanglement, and thermality)

1. Squeezed states

a) Using (3) and noticing that $|0\rangle_1 \otimes |0\rangle_2$ is in the kernel of $\hat{a}_1 \otimes \hat{a}_2$, and

$$\hat{H}_{12} |0\rangle_1 \otimes |0\rangle_2 = \omega \left(a_1^{+} \cancel{a_1 |0\rangle_1} \otimes |0\rangle_2 + |0\rangle_1 \otimes a_2^{+} \cancel{a_2 |0\rangle_2} + |0\rangle_1 \otimes |0\rangle_2 \right)$$

$$= \omega |0\rangle_1 \otimes |0\rangle_2 ,$$

we have

$$\hat{S}(\zeta) |0\rangle_1 \otimes |0\rangle_2 = \exp\left(- \frac{\log(\cosh(r))}{\omega} \omega \right) \exp\left(e^{i\phi} \tanh(r)\, a_1^{+} \otimes a_2^{+} \right) |0\rangle_1 \otimes |0\rangle_2$$

$$= \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \frac{(e^{i\phi} \tanh(r))^n}{n!} (a_1^{+})^n |0\rangle_1 \otimes (a_2^{+})^n |0\rangle_2$$

$$= \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} (e^{i\phi} \tanh(r))^n\, |n\rangle_1 \otimes |n\rangle_2 .$$

b)

$$\rho(\zeta) = tr_2(|\zeta\rangle\langle\zeta|) = \frac{1}{\cosh(r)^2}\sum_{n,m=0}^{\infty}(e^{i\phi}\tanh(r))^n(e^{-i\phi}\tanh(r))^m$$

$$\underbrace{tr_2\left(|n\rangle_1\otimes|n\rangle_2\langle m|_1\otimes\langle m|_2\right)}$$

$$= tr_2\left(|n\rangle\langle m|_1\otimes|n\rangle\langle m|_2\right)$$

$$= \sum_K |n\rangle\langle m|_1 \otimes \underbrace{\langle K|n\rangle\langle m|K\rangle}_{\delta_{Kn}\delta_{Km}}$$

$$= |n\rangle\langle n|\,\delta_{nm}$$

$$= \frac{1}{\cosh(r)^2}\sum_{n=0}^{\infty}\tanh(r)^{2n}\,|n\rangle\langle n|.$$

Now note that

$$e^{-\beta(\zeta)\omega\hat{a}_1^\dagger\hat{a}_1} = \sum_{n=0}^{\infty} e^{-\beta(\zeta)\omega\hat{a}_1^\dagger\hat{a}_1}|n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty} e^{-\beta(\zeta)\omega n}|n\rangle\langle n|$$

We can thus take $e^{-\beta(\zeta)\omega} = \tanh(r)^2$, i.e.

$$\beta(\zeta) = -\frac{2}{\omega} \log\left(\tanh(r)\right)$$

c) Indeed, noticing that we already have $\rho(\varsigma)$ in its spectral decomposition, we have

$$S(\rho(\varsigma)) = -\sum_{n=0}^{\infty} \frac{\tanh(r)^{2n}}{\cosh(r)^2} \log\left(\frac{\tanh(r)^{2n}}{\cosh(r)^2}\right),$$

In order to relate it to temperature we have to find

$$E(\beta) = \cosh(r)^2$$

in terms of the temperature, For this we use that

$$1 = \tanh(r)^2 + \operatorname{sech}(r)^2 = e^{-\beta(\varsigma)\omega} + \frac{1}{\cosh(r)^2},$$

i.e.,

$$\bar{E}(\beta) = \cosh(r)^2 = \frac{1}{1 - e^{-\beta(\varsigma)\omega}}.$$

Therefore

$$S(\rho(\xi)) = -\sum_{n=0}^{\infty} e^{-\beta(\xi)\omega n}\left(1 - e^{-\beta(\xi)\omega}\right) \times$$

$$\left(-n\beta(\xi)\omega + \log\left(1 - e^{-\beta(\xi)\omega}\right)\right).$$

Plotted it in mathemathica and it seemed to be monotonically increasing with $1/\beta$.

2. The Unruh effect

a) We have

$$\hat{\phi}(\eta, -\xi, \vec{x}) = \sum_{k}\left(u_{Ik}(\eta, -\xi, \vec{x})\hat{b}_{Ik} + u^{*}_{Ik}(\eta, -\xi, \vec{x})\hat{b}^{+}_{Ik}\right)$$

$$+ \sum_{k}\left(u_{IIk}(\eta, -\xi, \vec{x})\hat{b}_{IIk} + u^{*}_{IIk}(\eta, -\xi, \vec{x})\hat{b}^{+}_{IIk}\right.$$

$$= \sum_{k}\left(u^{*}_{II\tilde{k}}(\eta, \xi, \vec{x})\hat{b}_{Ik} + u_{II\tilde{k}}(\eta, \xi, \vec{x})\hat{b}^{+}_{Ik}\right)$$

$$+ \sum_{k}\left(u^{*}_{I\tilde{k}}(\eta, \xi, \vec{x})\hat{b}_{IIk} + u_{I\tilde{k}}(\eta, \xi, \vec{x})\hat{b}^{+}_{IIk}\right.$$

$$= \sum_{K} \left(U^{*}_{\mathrm{II}K}(\eta,\xi,\vec{x})\hat{b}_{\mathrm{I}\tilde{K}} + U_{\mathrm{II}K}(\eta,\xi,\vec{x})\hat{b}^{+}_{\mathrm{I}\tilde{K}} \right)$$

$$+ \sum_{K} \left(U^{*}_{\mathrm{I}K}(\eta,\xi,\vec{x})\hat{b}_{\mathrm{II}\tilde{K}} + U_{\mathrm{I}K}(\eta,\xi,\vec{x}) b^{+}_{\mathrm{II}\tilde{K}} \right),$$

since $\tilde{\tilde{K}} = K$. On the other hand

$$J(\hat{\phi}(\eta,\xi,\vec{x})) = \sum_{K} \left(U^{*}_{\mathrm{I}K}(\eta,\xi,\vec{x}) J(\hat{b}_{\mathrm{I}K}) + U_{\mathrm{I}K}(\eta,\xi,\vec{x}) J(\hat{b}^{+}_{\mathrm{I}K}) \right)$$

$$+ \sum_{K} \left(U^{*}_{\mathrm{II}K}(\eta,\xi,\vec{x}) J(\hat{b}_{\mathrm{II}K}) + U_{\mathrm{II}K}(\eta,\xi,\vec{x}) J(b^{+}_{\mathrm{II}K}) \right).$$

Comparing the coefficients in our basis we confirm that

$$J(\hat{b}_{\mathrm{I}K}) = b_{\mathrm{II}\tilde{K}}.$$

b) Well, we might as well expand (11)

$$0 = \sum_{K} \left(U_{\mathrm{I}K}(\eta,\xi,\vec{x}) \left(e^{-i\Omega\pi} \hat{b}_{\mathrm{I}K} - \hat{b}^{+}_{\mathrm{II}\tilde{K}} \right) \right.$$

$$+ U^{*}_{\mathrm{I}K}(\eta,\xi,\vec{x}) \left(e^{-i\Omega\pi} \hat{b}^{+}_{\mathrm{I}K} - \hat{b}_{\mathrm{II}\tilde{K}} \right)$$

$$+ U_{\mathrm{II}K}(\eta,\xi,\vec{x}) \left(e^{-i\Omega\pi} \hat{b}_{\mathrm{II}K} - \hat{b}^{+}_{\mathrm{I}\tilde{K}} \right)$$

$$+ \, u^*_{\mathrm{II}\,k}(\eta, \xi, \vec{x}) \left(e^{i\Omega\pi} \, \hat{b}^{\dagger}_{\mathrm{II}\,k} - \hat{b}_{\mathrm{I}\,\tilde{k}} \right) \Big) \, |0\rangle_M .$$

Since this equation must be valid on I + II and the coefficients have disjoint support, we see that, letting (being very schematic)

$$\hat{d}_{\mathrm{I}\,k} = \sum_{\eta, \xi, \vec{x}} \Big(u_{\mathrm{I}\,k}(\eta, \xi, \vec{x}) \left(e^{i\Omega\pi} \, \hat{b}_{\mathrm{I}\,k} - \hat{b}^{\dagger}_{\mathrm{II}\,\tilde{k}} \right)$$

$$+ \, u^*_{\mathrm{I}\,k}(\eta, \xi, \vec{x}) \left(e^{-i\Omega\pi} \, \hat{b}^{\dagger}_{\mathrm{I}\,k} - \hat{b}_{\mathrm{II}\,\tilde{k}} \right) \Big)$$

and

$$\hat{d}_{\mathrm{II}\,k} = \sum_{\eta, \xi, \vec{x}} \Big(u_{\mathrm{II}\,k}(\eta, \xi, \vec{x}) \left(e^{-i\Omega\pi} \, \hat{b}_{\mathrm{II}\,k} - \hat{b}^{\dagger}_{\mathrm{I}\,\tilde{k}} \right)$$

$$+ \, u^*_{\mathrm{II}\,k}(\eta, \xi, \vec{x}) \left(e^{i\Omega\pi} \, \hat{b}^{\dagger}_{\mathrm{II}\,k} - \hat{b}_{\mathrm{I}\,\tilde{k}} \right) \Big) .$$

Then

$$\hat{d}_{\mathrm{I}\,k} \, |0\rangle_M = 0$$

and

$$\hat{d}_{\mathrm{II}\,k} \, |0\rangle_M = 0 .$$