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Homework 1: Equivalence principle at work: charge in a lab.

1.a) We have

$$U^{\mu}(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0)^{\mu}$$

and

$$\frac{dU^{\mu}(\tau)}{d\tau} = g(\sinh(g\tau), \cosh(g\tau), 0, 0)^{\mu} = g(U^{1}(\tau), U^{0}(\tau), 0, 0)'$$

This solves the EOM,

$$\frac{dU^{\mu}(\tau)}{d\tau} = \kappa F^{\mu\nu} U_{\nu}(\tau) = \kappa(F^{01} U_{1}(\tau), F^{10} U_{0}(\tau), 0, 0)^{\mu}$$

$$= \kappa(E U^{1}_{(\tau)}, E U^{0}_{(\tau)}, 0, 0)^{M}$$

if $g = \kappa E$. Moreover, the proper time is

$$\int_0^{\tau} d\tau \sqrt{U^0(\tau)^2 - U^1(\tau)^2} = \int_0^{\tau} d\tau \sqrt{\cosh(g\tau)^2 - \sinh(g\tau)^2}$$

$$= \int_0^{\tau} d\tau = \tau,$$

showing that τ is the proper time.

b) We have

$$a^{\mu} a_{\mu} = g^2 \left(-\sinh(g\tau)^2 + \cosh(g\tau)^2 \right) = g^2.$$

c) Exploring with the code we see that small four-acceleration correspond to very flat curves. As g increses the curvature becomes bigger and the particle comes closer to the origin.

d) The parametrization in (7) satisfies

$$x^2 - t^2 = \left(\frac{1}{g} + X \right)^2 \geq 0.$$

Thus $x^2 \geq t^2$, i.e. $|x| \geq |t|$. Thus, the Rindler coordinates only cover this region,

Known as the Rindler wedge. This is confirmed in the plots on the notebook. By writing X

$$(t, x) = \left(\frac{1}{g} + X\right)(\sinh(gT), \cosh(gT)),$$

it is clear that the constant T curves correspond to straight lines through the origin with a slope

$$\cotanh(gT).$$

This slope quickly stabilizes as T grows to 1. On the other hand, the lines at constant X, trace out to the trajectories of uniformly accelerated particles explored in c). This is seen by the replacement

$$\frac{1}{g'} := \frac{1}{g} + X, \qquad t' := \frac{gT}{g'} = gT\left(\frac{1}{g} + X\right),$$

s.t. $(t, x) = \frac{1}{g'}\left(\sinh(g' t'), \cosh(g' t')\right)$

The boundary of the Rindler wedge is the space where $|t| = |x|$, which is the light cone.

e) If $x^\mu = (t, x, \rho, \varphi)^\mu$ and $x'^\mu = (T, X, \rho, \varphi)^\mu$, we have

$$g'_{\mu\nu} = g_{\sigma\beta} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$

$$= -\frac{\partial t}{\partial x'^\mu} \frac{\partial t}{\partial x'^\nu} + \frac{\partial x}{\partial x'^\mu} \frac{\partial x}{\partial x'^\nu} + \frac{\partial \rho}{\partial x'^\mu} \frac{\partial \rho}{\partial x'^\nu}$$

$$+ \rho^2 \frac{\partial \varphi}{\partial x'^\mu} \frac{\partial \varphi}{\partial x'^\nu} ,$$

Thus

$$g'_{00} = -\left(\frac{1}{g} + X\right)^2 g^2 \cosh(gT)^2 + \left(\frac{1}{g} + X\right)^2 g^2 \sinh(gT)^2$$

$$= -\left(\frac{1}{g} + X\right)^2 g^2 = -\left(1 + gX\right)^2 ,$$

$$\boxed{g = a^\mu a_\mu > 0 .}$$

$$g'_{11} = -\sinh(gT)^2 + \cosh(gT)^2 = 1 ,$$

$$g'_{22} = 1 \quad , \qquad g'_{33} = 1 \quad ,$$

$$g'_{01} = - \left(\frac{1}{g} + X \right) g \cosh(gT) \sinh(gT) + \left(\frac{1}{g} + X \right) g \sinh(gT) \cosh(gT)$$

$$= 0.$$

The other mixed elements vanish since none

of the components of (t, x, ρ, φ) that

depend on ρ or φ depend simultaneously

with another coordinate in (T, X, ρ, φ). Thus

$$ds^2 = -(1 + gX)^2 dT^2 + dX^2 + d\rho^2 + \rho^2 d\varphi^2.$$

2. a) In the Front End we showed that

$$\varphi(\vec{r}, t) = \frac{e}{\|\vec{r} - \vec{x}(t_R)\| - \vec{v}(t_R) \cdot (\vec{r} - \vec{x}(t_R))},$$

where t_R is the time for which $t = t_R + \|\vec{r} - \vec{x}(t_R)\|$

$\vec{x}$ is the trajectory of the particle, and e

its charge. One can show also that

$$\vec{A}(\vec{r},t) = \frac{e\,\vec{v}(t_R)}{\|\vec{r}-\vec{x}(t_R)\| - \vec{v}(t_R)\cdot(\vec{r}-\vec{x}(t_R))}$$

Let us consider a trayectory y^μ the particle as a function of an arbitrary parameter τ. This is any trayectory for which

$$\vec{y}(\tau) = \vec{x}(y^0(\tau)).$$

$\longrightarrow$ y^0 monotonically increasing

Define $\quad R^\mu(\tau) := z^\mu - y^\mu(\tau)\quad$ for some fixed z^μ.

Then, if $\quad U^\mu = (z^\mu)'$, then

$$R^\mu(\tau)\,U_\mu(\tau) = R^\mu(\tau)\frac{dy_\mu(\tau)}{dx^0}\frac{dy^0}{d\tau}(\tau)$$

$$= -\left(z^0 - y^0(\tau) - (\vec{z}-\vec{y}(\tau))\cdot\vec{v}(y^0(\tau))\right)\cdot\frac{dy^0}{d\tau}(\tau)$$

At the τ, for which $\quad z^0 = y^0(\tau) + \|\vec{z}-\vec{y}(\tau)\|,$

$$R^\mu(\tau)\,U_\mu(\tau) = -\left(\|\vec{z}-\vec{y}(\tau)\| - \vec{v}(y^0(\tau))\cdot(\vec{z}-\vec{y}(\tau))\right)\frac{dy^0}{d\tau}(\tau).$$

On the other hand

$$U^0(\tau) = \frac{dy^0}{d\tau}(\tau), \qquad \vec{U}(\tau) = \vec{v}(y^0(\tau))\frac{dy^0}{d\tau}(\tau).$$

Then, at this retarded time

$$A^\mu(z) = - \frac{Q \, U^\mu(\tau)}{R^\nu(\tau) U_\nu(\tau)}$$

$$= \frac{Q \, (1, \vec{v}(y^0(\tau)))}{\|\vec{z} - \vec{y}(\tau)\| - (\vec{z} - \vec{y}(\tau)) \cdot \vec{v}(y^0(\tau))} = \left(\varphi(\vec{z}, z^0), \vec{A}(z, z^0) \right)$$

setting $Q = e$ and realizing $t_R = y^0(\tau)$.

b) We already showed

$$U^\mu(\tau) = g(x_Q, t_Q, 0, 0).$$

Thus

$$U_\mu(\tau) = \eta_{\mu\nu} U^\nu(\tau) = g(-x_Q, t_Q, 0, 0).$$

We then conclude

$$A_\mu = \left(\frac{Q \, U_\mu}{R^\nu U_\nu} \right)_{t = t(\tau) + \|\vec{x} - \vec{x}(\tau)\|}$$

$$= - \frac{Q g(-x_Q, t_Q, 0, 0)_\mu}{(t - t_Q(\tau))g(-x_Q) + (\vec{x} - \vec{x}_Q(\tau)) g(t_Q, 0, 0)}$$

$$= \frac{Q(-x_Q, t_Q, 0, 0)_\mu}{t x_Q - t_Q x_Q - x t_Q + x_Q t_Q} = \frac{Q}{5}(-x_Q, t_Q, 0, 0).$$

c) We have by squaring the retarded time condition and (8)

$$t^2 - 2t\,t_Q + \cancel{t_Q^2} = \rho^2 + x^2 - 2x\sqrt{L^2 + t_Q^2} + L^2 + \cancel{t_Q^2} , \quad \text{i.e.}$$

$$4x^2(L^2 + t_Q^2) = (\delta + 2t\,t_Q)^2$$

$$= \delta^2 + 4t^2 t_Q^2 + 4t\delta t_Q .$$

We thus have the quadratic equation

$$4(x^2 - t^2)t_Q^2 - 4t\delta t_Q + 4x^2 L^2 - \delta^2 = 0,$$

whose two solutions are

$$t_Q = \frac{\cancel{4}t\delta \pm \sqrt{\cancel{16}t^2\delta^2 - \cancel{16}(x^2 - t^2)(4x^2 L^2 - \delta^2)}}{\cancel{8}(x^2 - t^2)}{}_2$$

$$= \frac{t\delta \pm \sqrt{\cancel{t^2\delta^2} - 4x^4 L^2 + x^2\delta^2 + 4x^2 t^2 L^2 - \cancel{t^2\delta^2}}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x|\sqrt{t^2 L^2 + \delta^2/4{}^2 - x^2 L^2}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x|\sqrt{t^2 L^2 - x^2 L^2 + (\rho^2 + x^2 + L^2 - t^2)^2/4}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x| \sqrt{t^2 L^2 - x^2 L^2 + (L^2 + t^2 - \rho^2 - x^2 - 2L^2)^2/4}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x| \sqrt{t^2 L^2 - x^2 L^2 + (L^2 + t^2 - \rho^2 - x^2)^2/4}}{2(x^2 - t^2)}$$

$$\frac{+ \frac{4L^4}{4} - 4L^2(L^2 + t^2 - \rho^2 - x^2)/4}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x| \sqrt{(L^2 + t^2 - \rho^2 - x^2)^2/4 + L^2(\cancel{t^2 - x^2 + 2L^2 - L^2 - t^2} + \rho^2 + x^2)}}{2(x^2 - t^2)}$$

$$= \frac{t\delta \pm 2|x|S}{2(x^2 - t^2)} .$$

If $x > 0$, we take the $(-)$ solution to

ensure we obtain the retarded instead of the

advanced time.   Thus

$$t_Q = \frac{t\delta - 2xS}{2(x^2 - t^2)} .$$

Repeating for x_Q, we have

$$x_Q^2 - 2xx_Q + x^2 = (t - t_Q)^2 - \rho^2$$

$$= t^2 - 2tt_Q + t_Q^2 - \rho^2$$

$$= t^2 - 2t\sqrt{x_Q^2 - L^2} + x_Q^2 - L^2 - \rho^2$$

$$4t^2(x_Q^2 - L^2) = (t^2 - x^2 - L^2 - \rho^2 + 2xx_Q)^2$$

$$= (-\delta + 2xx_Q)^2 = 4x^2 x_Q^2 + \delta^2 - 4x\delta x_Q$$

$$4(t^2 - x^2)x_Q^2 + 4x\delta x_Q - 4t^2 L^2 - \delta^2 = 0$$

$$x_Q = \frac{-4x\delta \pm \sqrt{16x^2\delta^2 - 16(t^2 - x^2)(4t^2L^2 - \delta^2)}}{8(t^2 - x^2)}$$

$$= \frac{x\delta \pm \xi t}{2(x^2 - t^2)}$$

noticing that the square root is obtained by

the replacement $x \leftrightarrow t$ on the previous one,

a replacement under which ξ is invariant.

Once agin, we take the $(-)$ soln

$$x_Q = \frac{x\delta - \xi t}{2(x^2 - t^2)}$$

d) Clearly $F_{\rho\phi} = 0$ since $A_\rho = A_\phi = 0$.

The full field strength is calculated in the mathematica notebook. It describes an electric field in the $x-\rho$ plane and a magnetic field in the ϕ direction

3. a) Using mathematica, one finds

$$A_\mu' = \left(-\Phi, \; -\frac{Q}{L+X}, \; 0, \; Q \right)_\mu$$

with

$$\Phi = \frac{gQ(2L^2 + 2LX + X^2 + \rho^2)}{\sqrt{(X^2 + \rho^2)((2L+X)^2 + \rho^2)}}$$

$$= \frac{Q}{r} \frac{\left(\frac{2}{g} + 2X + g r^2 \right)}{\sqrt{4/g^2 + 4X/g + r^2}}$$

$$= \frac{Q}{r} \frac{1 + gX + g^2 r^2 /2}{\sqrt{1 + gX + g^2 r^2/4}}$$

b) Consider the gauge function

$$\Lambda = Q \ln (X + L).$$

Then

$$A_\mu' + \partial_\mu \Lambda = \left((-\Phi, \frac{-Q}{X+L}, 0, 0) + (0, \frac{Q}{X+L}, 0, 0) \right)_\mu$$

$$= (-\Phi, 0, 0, 0)_\mu$$

c) We have

$$\varphi = \frac{\Phi}{1 + gX} = \frac{Q}{r} \frac{1 + gX + g^2 \sigma^2/2}{\sqrt{1 + gX + g^2 r^2/4}} \cdot \frac{1}{1 + gX}$$

Using $(1+x)^n = 1 + nx + \frac{1}{2} n(n-1) x^2 + \mathcal{O}(x^3)$, we have

$$\varphi = \frac{Q}{r} \left(1 + gX + g^2 \sigma^2/2 \right) \left(1 - \frac{1}{2} \left(gX + g^2 r^2/4 \right) \right.$$

$$\left. + \frac{3}{8} \left(gX + g^2 r^2/4 \right)^2 \right) \times$$

$$\left(1 - gX + g^2 x^2 \right) \quad + \mathcal{O}(g^3)$$

$$= \frac{Q}{r}\left(1 + gX\left(\cancel{1} - \frac{1}{2} - \cancel{1}\right)\right.$$

$$+ g^2\left(\frac{r^2}{2} - \frac{1}{2}\,{}^{r^2}/_4 + \frac{3}{8}X^2 + \cancel{X^2}\right.$$

$$\left.\left. - \frac{1}{2}\cancel{X}^2 - X^2 + \frac{1}{2}\cancel{X^2}\right)\right) + O(g^3)$$

$$= \frac{Q}{r}\left(1 - \frac{1}{2}Xg + \frac{3}{8}(r^2 + X^2)g^2 + O(g^3)\right).$$

These where verified in the mathematica file

by explicit computation.

d) For small g we see the equipotentials expected

From the usual Coulomb potential. As g increases,

we see the potential lagging behind. In

e) The bottom of every graph we see the

Rindler horizon, the corresponding to the singularity at

$X = -\frac{1}{g}$. The equipotentials seem to flatten

horizontally near the horizon. This is a coordinate

artifact however, since the horizon is at ∞.

c) Checked in Mathematica.

Homework 1: Equivalence principle at work: charge in a lab

1. Uniformly accelerated charge

The trajectory of a uniformly accelerated charge in Minkowski spacetime is given by the following plots.

```
In[ ]:=  ParametricPlot[Evaluate@Table[{1/g Cosh[g τ], 1/g Sinh[g τ]}, {g, {0.35, 0.5, 1, 2, 14}}],
           {τ, -1, 1}, PlotRange → {{0, 3}, {-1, 1}},
           PlotLegends → LineLegend[{0.35, 0.5, 1, 2, 14}, LegendLabel → g], AxesLabel → {x, t}]
```

Out[ ]=
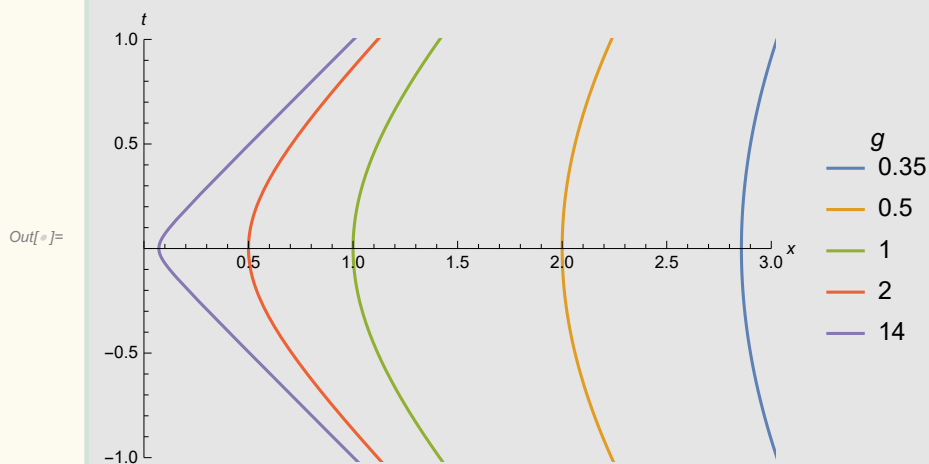
Let us now fix the value of the four acceleration to be some suitable parameter

```
In[ ]:=  g = 2
```

Out[ ]= 2

Then the solutions of constant X are

In[ ]:=
```
ParametricPlot[
  Evaluate@Table[{(1/g + X) Cosh[g T], (1/g + X) Sinh[g T]}, {X, {0.4, 1, 2, 5, 8}}],
  {T, -1, 1}, PlotRange → {{0, 10}, {-3, 3}},
  PlotLegends → LineLegend[{0.4, 1, 2, 5, 8}, LegendLabel → X], AxesLabel → {x, t}]
```
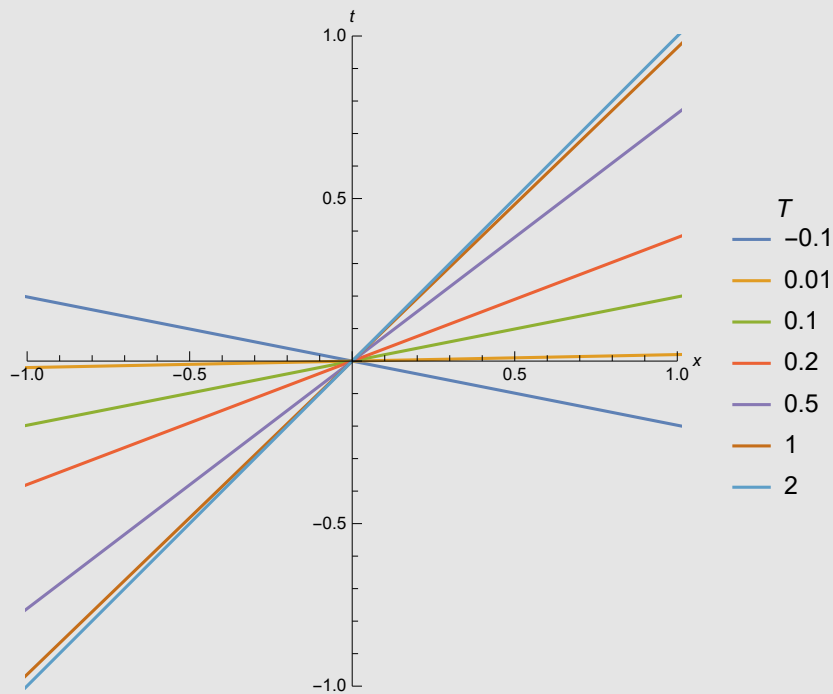
Out[ ]=
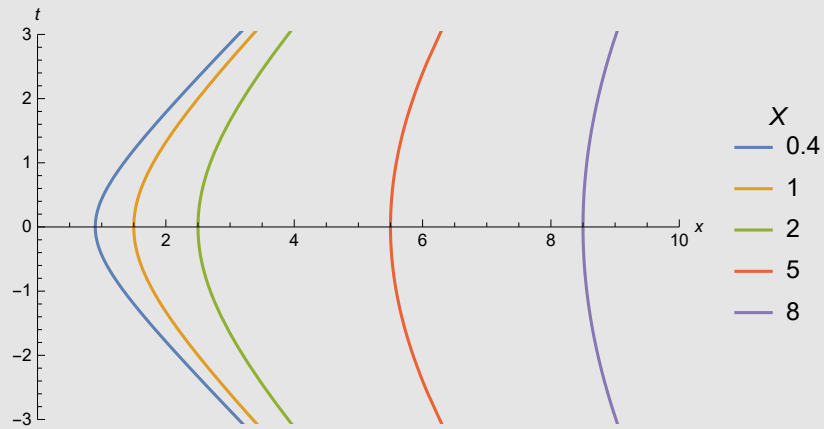
while those of constant T are

In[ ]:=
```
ParametricPlot[Evaluate@
   Table[{(1/g + X) Cosh[g T], (1/g + X) Sinh[g T]}, {T, {-0.1, 0.01, 0.1, 0.2, 0.5, 1, 2}}],
  {X, -2, 2}, PlotRange → {{-1, 1}, {-1, 1}}, PlotLegends →
   LineLegend[{-0.1, 0.01, 0.1, 0.2, 0.5, 1, 2}, LegendLabel → T], AxesLabel → {x, t}]
```

Out[ ]=

2. Field of a uniformly accelerated charge

We begin by defining our coordinates and the value of the retarded time with the corresponding position of the particle.

In[ ]:=
```
coord = {t, x, ρ, ϕ};
δ = ρ² + x² + L² - t²;

ξ = √((L² + t² - ρ² - x²)²/4 + L² ρ²) ;

tQ = (t δ - 2 x ξ)/(2 (x² - t²));

xQ = (x δ - 2 t ξ)/(2 (x² - t²));
```

We can now create our four potential

In[ ]:=
```
A = ———————— {-xQ, tQ, 0, 0};
    t xQ - x tQ
```

Our field strength tensor can now be defined

In[ ]:=
```
F = Table[
    D[A[[n]], coord[[m]]] - D[A[[m]], coord[[n]]] // FullSimplify, {m, 1, 4}, {n, 1, 4}];
F // MatrixForm
```

Out[ ]//MatrixForm=

$$
\begin{pmatrix}
0 & \dfrac{4 L^2 \left(L^2+t^2-x^2+\rho^2\right)}{\left(L^4+2 L^2 \left(t^2-x^2+\rho^2\right)+\left(-t^2+x^2+\rho^2\right)^2\right)^{3/2}} & -\dfrac{8 L^2 x \rho}{\left(L^4+2 L^2 \left(t^2-x^2+\rho^2\right)+\left(-t^2+x^2+\rho^2\right)^2\right)^{3/2}} & 0 \\
-\dfrac{4 L^2 \left(L^2+t^2-x^2+\rho^2\right)}{\left(L^4+2 L^2 \left(t^2-x^2+\rho^2\right)+\left(-t^2+x^2+\rho^2\right)^2\right)^{3/2}} & 0 & \dfrac{8 L^2 t \rho}{\left(L^4+2 L^2 \left(t^2-x^2+\rho^2\right)+\left(-t^2+x^2+\rho^2\right)^2\right)^{3/2}} & 0 \\
\dfrac{8 L^2 x \rho}{\left(L^4+2 L^2 \left(t^2-x^2+\rho^2\right)+\left(-t^2+x^2+\rho^2\right)^2\right)^{3/2}} & -\dfrac{8 L^2 t \rho}{\left(L^4+2 L^2 \left(t^2-x^2+\rho^2\right)+\left(-t^2+x^2+\rho^2\right)^2\right)^{3/2}} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

3. Sitting on the charge

In[ ]:=
```
Clear[g]
```

The modified Coulomb potential is

In[ ]:=
```
r = Sqrt[X^2 + ρ^2] ;
```

$$
\Phi = \frac{Q}{r} \frac{1 + g X + \frac{g^2 r^2}{2}}{\sqrt{1 + g X + \frac{g^2 r^2}{4}}};
$$

$$
\varphi := \frac{\Phi}{1 + g X};
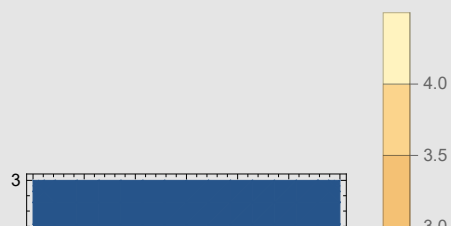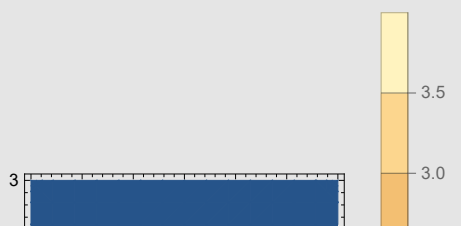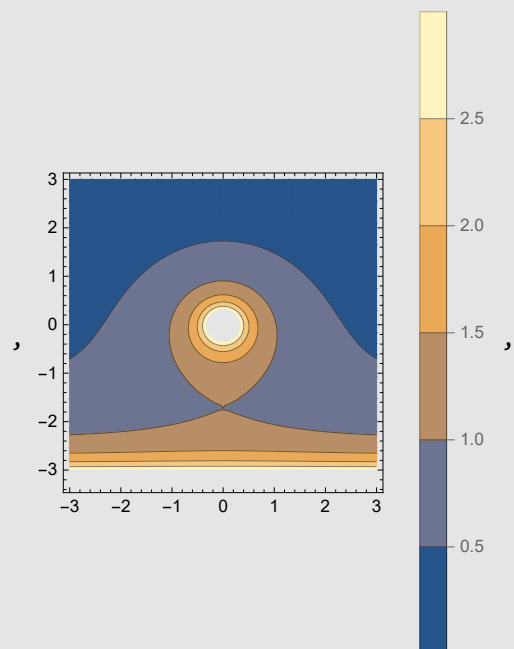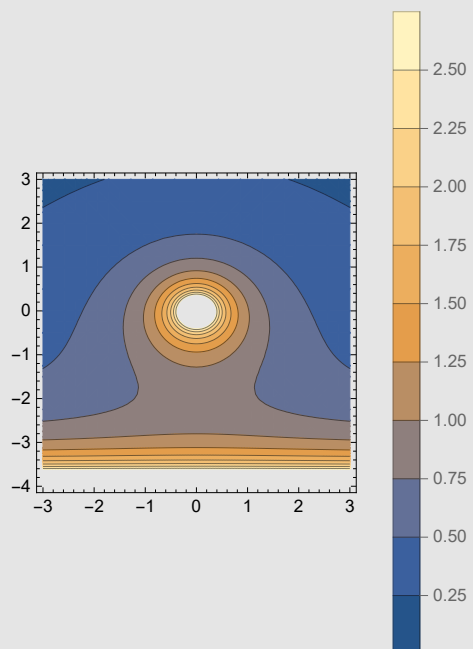$$

We normalize by the charge.

In[ ]:=
```
Q = 1
```
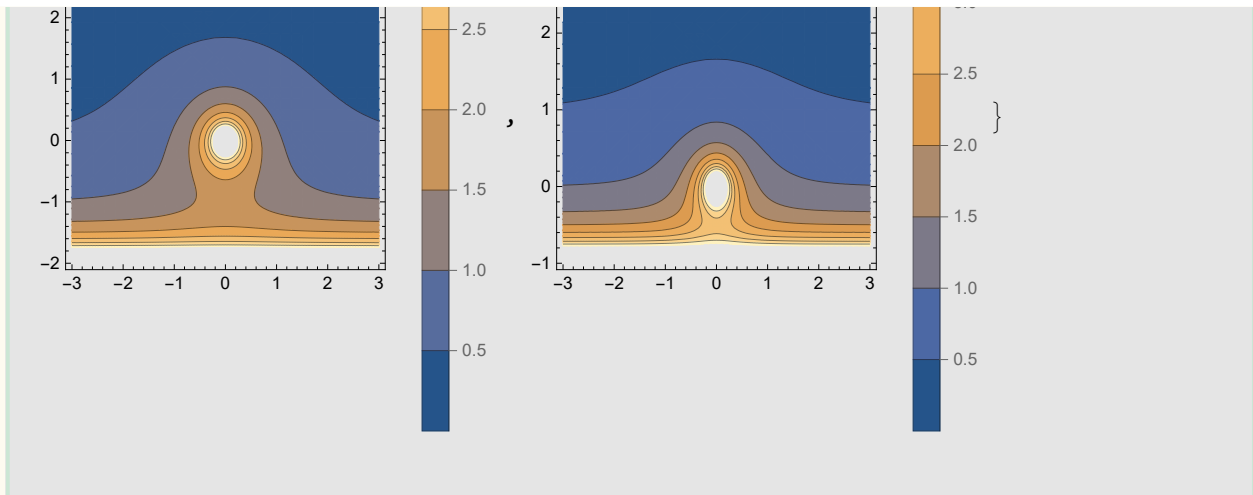
Out[ ]=
```
1
```

Then the contour plots of the potential are the following.

In[ ]:=
```
Table[ContourPlot[φ, {ρ, -3, 3}, {X, -1/g, 3}, PlotLegends → Automatic],
    {g, {0.1, 0.25, 0.3, 0.5, 1}}]
```

Out[•]= {

,

,

,
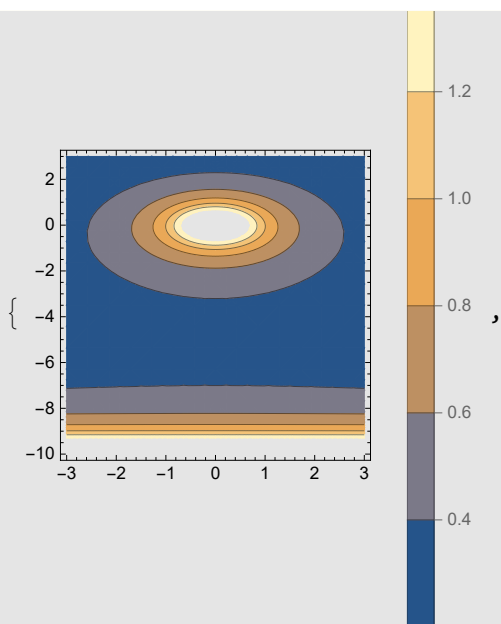
We can now define our new coordinates and the metric, potential and field strength in them.

```
In[ ]:=   coordp = {T, X, ρ, ϕ};
          gp = {{- (1 + g X)², 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, ρ²}};
          Ap = {-Φ, 0, 0, 0};
          Fp = Table[D[Ap[[n]], coordp[[m]]] - D[Ap[[m]], coordp[[n]]] // FullSimplify,
              {m, 1, 4}, {n, 1, 4}] ;
```

Then our field strength is.

```
In[ ]:=   Fp // MatrixForm
```

Out[ ]//MatrixForm=

$$
\begin{pmatrix}
0 & -\dfrac{4\,(1+g\,X)\,\left(X\,(2+g\,X)-g\,\rho^2\right)}{\left(X^2+\rho^2\right)^{3/2}\left((2+g\,X)^2+g^2\,\rho^2\right)^{3/2}} & -\dfrac{8\,(1+g\,X)^2\,\rho}{\left(X^2+\rho^2\right)^{3/2}\left((2+g\,X)^2+g^2\,\rho^2\right)^{3/2}} & 0 \\
\dfrac{4\,(1+g\,X)\,\left(X\,(2+g\,X)-g\,\rho^2\right)}{\left(X^2+\rho^2\right)^{3/2}\left((2+g\,X)^2+g^2\,\rho^2\right)^{3/2}} & 0 & 0 & 0 \\
\dfrac{8\,(1+g\,X)^2\,\rho}{\left(X^2+\rho^2\right)^{3/2}\left((2+g\,X)^2+g^2\,\rho^2\right)^{3/2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

We can quickly check that this indeed satisfies Maxwell' s equations.

```
In[ ]:=   Table[∑ₘ₌₁⁴ D[√-Det[gp] (Inverse[gp].Fp.Inverse[gp])[[m, n]], coordp[[m]]] // Simplify,
              {n, 1, 4}]
```

Out[ ]=   {0, 0, 0, 0}