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Quantum Field Theory II

Homework 4: The large N limit of U(N) Yang-Mills Theory

1. The action of U(N) Yang-Mills theory

a) First let us characterize the Lie algebra

$$u(N)=\{X\in M_N(\mathbb{C})\mid \forall t\in\mathbb{R}: e^{itX}\in U(N)\}.$$

Assume $X\in u(N)$. Then

$$0=\frac{d}{dt}\mathbb{1}_N\Big|_{t=0}=\frac{d}{dt}\left((e^{itX})^\dagger e^{itX}\right)\Big|_{t=0}=\frac{d}{dt}\left(e^{-itX^\dagger}e^{itX}\right)\Big|_{t=0}$$

$$=-iX^\dagger+iX=i(X-X^\dagger).$$

We conclude X is hermitian. On the other hand, all Hermitian $X\in M_N(\mathbb{C})$ exponentiate into unitaries

$$(e^{itX})^\dagger e^{itX}=e^{-itX}e^{itX}=\mathbb{1}_N.$$

We conclude

$$u(N)=\{X\in M_N(\mathbb{C})\mid X^\dagger=X\}.$$

The adjoint representation is

$$\begin{aligned} ad: u(N) &\longrightarrow \mathrm{End}\, u(N) \\ X &\longmapsto iad(X): u(N) \longrightarrow u(N) \\ &\qquad\qquad Y \longmapsto i[X,Y]. \end{aligned}$$

To even begin discussing whether the image of ρ is Hermitian, we need to equip $u(N)$ with an inner product. Since $U(N)$ is compact, one would always be able to do equip $u(N)$ with a $U(N)$-invariant inner product, making the image of ρ Hermitian. However, this inner product is non-unique. We will give an example later. One could, on the other hand, ask this question with respect to the standard inner product the adjoint representation is usually endowed with. This is given by the Killing form. However, once again due to the non-compactness of $U(N)$, this form is degenerate and doesn't yield an inner product.

Let us instead consider the Hilbert-Schmidt on inner product

$$\langle X, Y \rangle := tr(X^{\dagger} Y).$$

We then have

$$\langle X, ad(Y) Z \rangle = tr(X\,[Y, Z]) = tr(XYZ) - tr(XZY)$$

$$= tr(XYZ) - tr(YXZ) = tr([X,Y]Z)$$

$$= tr([Y, X]^{\dagger} Z) = \langle ad(Y) X, Z \rangle.$$

Thus, the generators in the adjoint representation $ad(X)$ are Hermitian.

Furthermore, they are traceless as well. Letting Tr be the trace on $B(u(N))$, the linear operators on $u(N)$.

Then, given an orthonormal basis $(X_1, \dots, X_{\dim u = N + 2\frac{N(N-1)}{2} = N^2})$ of $u(N)$ we have

$$Tr(ad(X)) = \sum_{a=1}^{N^2} \langle X_a, i[X, X_a] \rangle = \sum_{a=1}^{N^2} tr(X_a\, i[X, X_a])$$

$$= i \sum_{a=1}^{N^2} \Big(tr(X_a X X_a) - tr(X_a X_a X) \Big) = 0.$$

In the fundamental representation these generators are not usually traceless. It is enough to consider the case $N=1$, where $u(1) = \mathbb{R}$ and the fundamental representation acts on $\mathbb{C}$ via

$$\rho(\alpha)\phi = i\alpha\phi .$$

In particular, if we endow $\mathbb{C}$ with its usual inner product, $\mathrm{Tr}(\rho(\alpha)) = i\alpha \neq 0$ for $\alpha \neq 0$.

b) We have

$$
\begin{aligned}
i f^{(\bar{m}\, n)}{}_{(i\bar{j})(k\bar{l})} A_\mu{}^{\bar{i} j} A_\nu{}^{\bar{k} l} &= [A_\mu , A_\nu]^{\bar{m} n} \\
&= A_\mu{}^{\bar{m} r} A_\nu{}^{\bar{r} n} - A_\nu{}^{\bar{m} r} A_\mu{}^{\bar{r} n} \\
&= \left(\delta^{\bar{m}}{}_i \delta^r{}_{\bar{j}} \delta^{\bar{r}}{}_k \delta^n{}_{\bar{l}} - \delta^{\bar{m}}{}_k \delta^r{}_{\bar{l}} \delta^{\bar{r}}{}_i \delta^n{}_{\bar{j}} \right) \\
&\quad A_\mu{}^{\bar{i} j} A_\nu{}^{\bar{k} l}
\end{aligned}
$$

We conclude

$$f^{(\bar{m}\, n)}{}_{(i\bar{j})(k\bar{l})} = -i \left(\delta^{\bar{m}}{}_i \delta_{\bar{j} k} \delta^n{}_{\bar{l}} - \delta^{\bar{m}}{}_k \delta_{\bar{l} i} \delta^n{}_{\bar{j}} \right).$$

c) We get

$$(D_\mu \Phi_A)^{\bar{i}j} = \partial_\mu \Phi_A^{\bar{i}j} - i\,[A_\mu, \Phi_A]^{\bar{i}j}$$

$$= \partial_\mu \Phi_A{}^{\bar{i}j} - i A_\mu{}^{\bar{i}r} \Phi_A{}^{\bar{r}j} + i \Phi_A{}^{\bar{i}r} A_\mu{}^{\bar{r}j} .$$

d) We have

$$F_{\mu\nu}{}^{\bar{i}j} = g\partial_\mu \tilde{A}_\nu{}^{\bar{i}j} - g\partial_\nu \tilde{A}_\mu{}^{\bar{i}j}$$

$$-ig^2 (\delta^{\bar{i}}{}_m \delta_{\bar{n}r} \delta^j{}_{\bar{s}} - \delta^{\bar{i}}{}_r \delta_{\bar{s}m} \delta^j{}_{\bar{n}}) \tilde{A}_\mu{}^{m\bar{n}} \tilde{A}_\nu{}^{r\bar{s}} .$$

On the other hand,

$$\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = \mathrm{Tr}(2g^2 \partial_\mu \tilde{A}_\nu \partial^\mu \tilde{A}^\nu - 2g^2 \partial_\mu \tilde{A}_\nu \partial^\nu \tilde{A}^\mu$$

$$-2ig^3 \partial_\mu \tilde{A}_\nu [\tilde{A}^\mu, \tilde{A}^\nu] + 2ig^3 \partial_\nu \tilde{A}_\mu [\tilde{A}^\mu, \tilde{A}^\nu]$$

$$- g^4 [\tilde{A}_\mu, \hat{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu])$$

$$= 2g^2 \partial_\mu \tilde{A}_\nu{}^{\bar{i}j} \partial^\mu A^{\nu \bar{j} i} - 2g^2 \partial_\mu \tilde{A}_\nu{}^{\bar{i}j} \partial^\nu \tilde{A}^{\mu \bar{j} i}$$

$$-2ig^3 \partial_\mu \tilde{A}_\nu{}^{\bar{i}j} f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} \tilde{A}^{\mu \bar{r} s} \tilde{A}^{\nu \bar{p} q}$$

$$+2ig^3 \partial_\nu \tilde{A}_\mu{}^{\bar{i}j} f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} \tilde{A}^{\mu \bar{r} s} \tilde{A}^{\nu \bar{p} q}$$

$$- g^4 f^{(\bar{i}j)}{}_{(r\bar{s})(p\bar{q})} f^{(\bar{j}i)}{}_{(t\bar{u})(v\bar{w})} \tilde{A}_\mu{}^{\bar{r}s} \tilde{A}_\nu{}^{\bar{p}q} A^{\mu \bar{t} u} A^{\nu \bar{v} w} .$$

Therefore, the action is

$$S(A) = -\int d^4x \; \Big(\partial_\mu \tilde{A}_\nu{}^{\bar{i}j} \partial^\mu \tilde{A}^{\nu \bar{j} i} - \partial_\mu \tilde{A}_\nu{}^{\bar{i}j} \partial^\nu \tilde{A}^{\mu \bar{j} i}$$

$$- ig \left(\partial_\mu \tilde{A}_\nu{}^{\bar{i}j} - \partial_\nu \tilde{A}_\mu{}^{\bar{i}j} \right) f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} \tilde{A}^{\mu \bar{r} s} \tilde{A}^{\nu \bar{p} q}$$

$$- \frac{1}{2} g^2 f^{(\bar{i}j)}{}_{(r\bar{s})(p\bar{q})} f^{(\bar{j}i)}{}_{(t\bar{u})(v\bar{w})} \tilde{A}_\mu{}^{\bar{r}s} \tilde{A}_\nu{}^{\bar{p}q} A^{\mu \bar{t} u} A^{\nu \bar{v} w} \Big).$$

2. Feynman Rules

a) To obtain the propagator we need to understand the quadratic part of the action. For this purpose we note that up to boundary terms the quadratic part of the action is

$$\int d^4x \left(\tilde{A}_\nu{}^{\bar{i}j} \Box \tilde{A}^{\nu \bar{j} i} - \tilde{A}_\nu{}^{\bar{i}j} \partial_\mu \partial^\nu \tilde{A}^{\mu \bar{j} i} \right)$$

$$= \int d^4x \; \tilde{A}_\nu{}^{\bar{i}j} \left(g^{\nu\mu} \Box \delta^{\bar{j}}{}_r \delta^i{}_{\bar{s}} - \partial^\nu \partial^\mu \delta^{\bar{j}}{}_r \delta^i{}_{\bar{s}} \right) \tilde{A}_\mu{}^{\bar{r}s}$$

$$= \int d^4x \; \tilde{A}_\nu{}^{\bar{i}j} \left(g^{\nu\mu} \Box - \partial^\nu \partial^\mu \right) \delta^{\bar{j}}{}_r \delta^i{}_{\bar{s}} \tilde{A}_\mu{}^{\bar{r}s}.$$

On the other hand, the gauge fixing term is as shown in class,

$$S_\xi(A) = -\frac{1}{2\xi}\int d^4x\ \partial_\mu \tilde{A}^\mu{}_a(x)\,\partial_\nu A^{\nu a}(x)$$

$$= -\frac{1}{2\xi}\int d^4x\ \partial_\mu \tilde{A}^\mu{}_a(x)\,\partial_\nu \tilde{A}^{\nu b}(x)\,\delta^a{}_b$$

$$= -\frac{1}{2\xi}\int d^4x\ \partial_\mu \tilde{A}^\mu{}_a(x)\,\partial_\nu \tilde{A}^{\nu b}(x)\ 2\,tr(T^a T_b)$$

$$= -\frac{1}{\xi}\int d^4x\ tr(\partial_\mu \tilde{A}^\mu\,\partial_\nu \tilde{A}^\nu)$$

$$= -\frac{1}{\xi}\int d^4x\ \partial_\mu \tilde{A}^{\mu\bar{i}j}\,\partial_\nu \tilde{A}^{\nu\bar{j}i}$$

$$= \frac{1}{\xi}\int d^4x\ \tilde{A}_\nu{}^{\bar{i}j}\,\partial^\nu\partial^\mu\,\delta^{\bar{j}}{}_{\bar{r}}\,\delta^i{}_{\bar{s}}\,\tilde{A}_\mu{}^{\bar{r}s}.$$

We conclude that the propagator is given by the Feynman's choice of Green's function of

$$D^{\mu\nu\bar{j}i}{}_{r\bar{s}} = \left(g^{\mu\nu}\,\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)\delta^{\bar{j}}{}_r\,\delta^i{}_{\bar{s}}.$$

We need to solve

$$D^{\mu\nu\bar{j}i}{}_{r\bar{s}}\,G_{\nu\sigma}{}^{\bar{r}s}{}_{p\bar{q}}(x) = i\delta(x)\,\delta^\mu{}_\sigma\,\delta^{\bar{j}}{}_p\,\delta^i{}_{\bar{q}}$$

Take

$$G_{\nu\sigma}{}^{\bar{r}s}{}_{p\bar{q}}(x) = \int \frac{d^4p}{(2\pi)^4}\,e^{ip\cdot x}\,\tilde{G}_{\nu\sigma}{}^{\bar{r}s}{}_{p\bar{q}}(p).$$

Therefore

$$i\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, \delta^{\mu}{}_{\sigma}\, \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}} = -i\delta(x)\, \delta^{\mu}{}_{\sigma}\, \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x} \left(g^{\mu\nu}(-p^2) + \left(1-\frac{1}{\xi}\right) p^{\mu} p^{\nu} \right) \tilde{G}_{\nu\sigma}{}^{\bar{j} i}{}_{p\bar{q}}(p).$$

We conclude that

$$\left(g^{\mu\nu}(-p^2) + \left(1-\frac{1}{\xi}\right) p^{\mu} p^{\nu} \right) \tilde{G}_{\nu\sigma}{}^{\bar{j} i}{}_{p\bar{q}}(p)$$

$$= i\, \delta^{\mu}{}_{\sigma}\, \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}.$$

Using our previous experience we propose

$$\tilde{G}_{\nu\sigma}{}^{\bar{j} i}{}_{p\bar{q}}(p) = \frac{-i}{p^2+i\varepsilon} \left(g_{\nu\sigma} - (1-\xi)\frac{p_\nu p_\sigma}{p^2} \right) \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}.$$

Indeed,

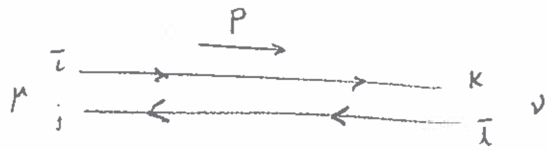
$$\left(g^{\mu\nu}(-p^2) + \left(1-\frac{1}{\xi}\right) p^{\mu} p^{\nu} \right) \frac{-i}{p^2+i\varepsilon} \left(g_{\nu\sigma} - (1-\xi)\frac{p_\nu p_\sigma}{p^2} \right) \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}$$

$$= \frac{-i}{p^2-i\varepsilon} \left(\delta^{\mu}{}_{\sigma}(-p^2) + \left(1-\frac{1}{\xi}\right) p^{\mu} p_{\sigma} + (1-\xi) p^{\mu} p_{\sigma} - \left(1-\frac{1}{\xi}\right)\overset{(1-\xi)}{\wedge} p^{\mu} p_{\sigma} \right) \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}$$

$$= \frac{-i}{p^2-i\varepsilon} \left(\delta^{\mu}{}_{\sigma}(-p^2) + p^{\mu} p_{\sigma} \left(\cancel{1} - \cancel{\frac{1}{\xi}} + \cancel{1} - \cancel{\xi} - \cancel{1} + \cancel{\xi} + \cancel{\frac{1}{\xi}} - \cancel{1} \right) \right) \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}$$

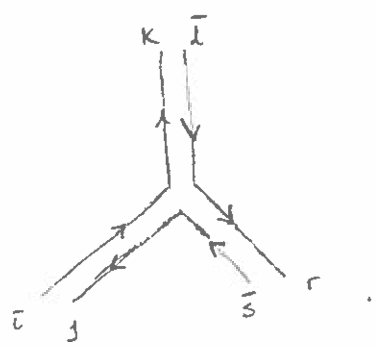
$$= i\, \delta^{\mu}{}_{\sigma}\, \delta^{\bar{j}}{}_{p}\, \delta^{i}{}_{\bar{q}}.$$

We conclude the Feynman rule



$$= \frac{-i}{p^2 - i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\delta^{\bar{i}}{}_{k}\,\delta^{j}{}_{\bar{l}}\,.$$

b) The three vertex is



To understand the associated Feynman rule we need to consider the Fourier transform of the cubic term in the action. For this, consider

$$\tilde{A}_\mu{}^{\bar{i}j}(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, \hat{A}_\mu{}^{\bar{i}j}(p).$$

Then our action has a cubic term

$$\int d^4x \left(ig\, f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} \tilde{A}^{\mu\bar{r}s} \tilde{A}^{\nu\bar{p}q} \partial_\mu \hat{A}_\nu{}^{\bar{\imath}j} \right.$$

$$\left. + ig\, f^{(\bar{j}i)}{}_{(p\bar{q})(r\bar{s})} \tilde{A}^{\mu\bar{r}s} \tilde{A}^{\nu\bar{p}q} \partial_\nu \hat{A}_\mu{}^{\bar{\imath}j} \right)$$

$$= 2ig \int d^4x\, f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} \tilde{A}^{\mu\bar{r}s} \hat{A}^{\nu\bar{p}q} \partial_\mu \tilde{A}_\nu{}^{\bar{\imath}j}$$

$$= 2ig \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \int d^4x\, e^{i(p_1+p_2+p_3)\cdot x} \cdots$$

$$f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} \hat{A}^{\mu\bar{r}s}(p_1) \hat{A}^{\nu\bar{p}q}(p_2)\; i(p_3)_\mu g_{\nu\sigma} \hat{A}^{\sigma\bar{\imath}j}(p_3)$$

$$= 2ig \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta(p_1+p_2+p_3)\; f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})}\, i(p_3)_\mu g_{\nu\sigma} \hat{A}^{\mu\bar{r}s}(p_1) \hat{A}^{\nu\bar{p}q}(p_2) \hat{A}^{\sigma\bar{\imath}j}(-p_1-p_2)$$

In its present form the interaction term cannot be interpreted in terms of the diagram above since it singles out the leg with momenta p_3.

We instead want to write it in the form

$$\int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta(p_1+p_2+p_3)\, V^{\mu\nu\sigma}_{(r\bar{s})(p\bar{q})(i\bar{j})}(p_1,p_2,p_3) \hat{A}_\mu{}^{\bar{r}s}(p_1) \hat{A}_\nu{}^{\bar{p}q}(p_2) \hat{A}_\sigma{}^{\bar{\imath}j}(p_3)$$

with $p_1 + p_2 + p_3 = 0$ and V symmetric under permutations of $\{((r\bar{s}), p_1, \mu), ((p\bar{q}), p_2, \nu), ((i\bar{j}), p_3, \sigma)\}$. We then have

$$V^{\mu\nu\sigma}_{(r\bar{s})(p\bar{q})(i\bar{j})}(p_1,p_2,p_3) = -\frac{2g}{3!}\Big(f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})}\, p_3{}^{\mu} g^{\nu\sigma} + f^{(\bar{j}i)}{}_{(p\bar{q})(r\bar{s})} p_3{}^{\nu} g^{\mu\sigma}$$

$$+ f^{(\bar{s}r)}{}_{(i\bar{j})(p\bar{q})}\, p_1{}^{\sigma} g^{\nu\mu} + f^{(\bar{s}r)}{}_{(p\bar{q})(i\bar{j})}\, p_1{}^{\nu} g^{\sigma\mu}$$

$$+ f^{(\bar{q}p)}{}_{(r\bar{s})(i\bar{j})}\, p_2{}^{\mu} g^{\sigma\nu} + f^{(\bar{q}p)}{}_{(i\bar{j})(r\bar{s})}\, p_2{}^{\sigma} g^{\mu\nu}\Big)$$

$$= -\frac{2g}{3!}\Big(F^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})}\, g^{\nu\sigma}(p_3-p_2)^{\mu} + F^{(\bar{q}p)}{}_{(i\bar{j})(r\bar{s})}\, g^{\mu\nu}(p_2-p_1)^{\sigma}$$

$$+ F^{(\bar{s}r)}{}_{(p\bar{q})(i\bar{j})}\, g^{\sigma\mu}(p_1-p_3)^{\nu}\Big),$$

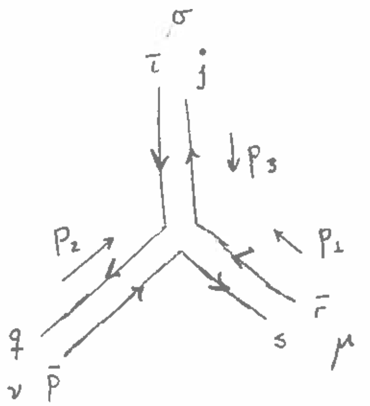
with

$$F^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} = f^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})} - f^{(\bar{q}p)}{}_{(r\bar{s})(i\bar{j})},$$

which constitutes the antisymmetrization of the structure constants. Then we get

$$V^{\mu\nu\sigma}_{(r\bar{s})(p\bar{q})(i\bar{j})}(p_1,p_2,p_3) = -\frac{2g}{3!} F^{(\bar{j}i)}{}_{(r\bar{s})(p\bar{q})}\Big(g^{\nu\sigma}(p_3-p_2)^{\mu}$$

$$+ g^{\mu\nu}(p_2-p_1)^{\sigma} + g^{\sigma\mu}(p_1-p_3)^{\nu}\Big)$$



$$= \quad i V^{\mu\nu\sigma}{}_{(r\bar{s})(p\bar{q})(i\bar{j})}(p_1, p_2, p_3)$$

with an integral $\int \frac{d^4 p}{(2\pi)^4}$ over any undetermined

momenta and $p_1 + p_2 + p_3$.

For the four vertex we need to understand the

quartic term of the action. We want to

write in the form

$$\int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} \frac{d^4 p_4}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4)$$

$$W^{\mu\nu\rho\sigma}{}_{(r\bar{s})(p\bar{q})(t\bar{u})(v\bar{w})}(p_1, p_2, p_3, p_4)\, \hat{A}_\mu{}^{\bar{r}s}(p_1) A_\nu{}^{\bar{p}q}(p_2) A_\rho{}^{\bar{t}u}(p_3) A_\sigma{}^{\bar{v}w}(p_4)$$

with W invariant under permutations of

$$\{((r\bar{s}), \mu, p_1), ((p\bar{q}), \nu, p_2), ((t\bar{u}), \rho, p_3), ((v\bar{w}), \sigma, p_4)\}.$$

We see then that

it at least contains

$$\frac{g^2}{2} f^{(\bar{i}j)}{}_{(r\bar{s})(p\bar{q})} f^{(j\bar{i})}{}_{(t\bar{u})(v\bar{w})}\, g^{\mu\rho} g^{\nu\sigma}.$$

We have antisymmetry under the changes $(r\bar{s}) \leftrightarrow (p\bar{q})$ and $(t\bar{u}) \leftrightarrow (v\bar{w})$. We must thus build antisymmetry under $\mu \longleftrightarrow \nu$ and $\rho \leftrightarrow \sigma$.

Then we get

$$\frac{g^2}{4} f^{(\bar{i}j)}{}_{(r\bar{s})(p\bar{q})} f^{(j\bar{i})}{}_{(t\bar{u})(v\bar{w})} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma} \right).$$

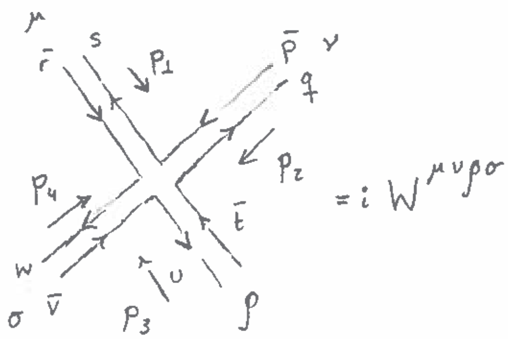
The remaining permutations are built by composition of $((r\bar{s}),\mu) \longrightarrow ((t\bar{u}),\rho)$, $((r\bar{s}),\mu) \rightarrow ((v\bar{w}),\sigma)$. Thus

$$W^{\mu\nu\rho\sigma}{}_{(r\bar{s})(p\bar{q})(t\bar{u})(v\bar{w})} = \frac{g^2}{12} \Big(f^{(\bar{i}j)}{}_{(r\bar{s})(p\bar{q})} f^{(j\bar{i})}{}_{(t\bar{u})(v\bar{w})} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma} \right)$$

$$+ f^{(\bar{i}j)}{}_{(t\bar{u})(p\bar{q})} f^{(j\bar{i})}{}_{(r\bar{s})(v\bar{w})} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma} \right)$$

$$+ f^{(\bar{i}j)}{}_{(v\bar{w})(p\bar{q})} f^{(j\bar{i})}{}_{(t\bar{u})(r\bar{s})} \left(g^{\sigma\rho} g^{\nu\mu} - g^{\nu\rho} g^{\mu\sigma} \right) \Big).$$

Then, our Feynman rule is



$= i W^{\mu\nu\rho\sigma}{}_{(r\bar{s})(p\bar{q})(t\bar{u})(v\bar{w})}$

with an integral over any undetermined momenta and $p_1 + p_2 + p_3 + p_4 = 0$.

The arguments used were adopted from Field Theory: A Modern Primer by Pierre Ramond. We see a simplification in that the coefficients are products of dirac deltas.

c) We have

$$S_{ghost}(A,\bar c,c) = \int d^4x\, \bar c\,(-\partial^\mu)\left(\partial_\mu c - i[A_\mu, c]\right)$$

$$= \int d^4x\left(\bar c\,(-\Box)\,c + i\,\bar c\,\partial_\mu [A_\mu, c]\right)$$

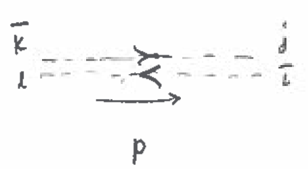
$$= \int d^4x\left(\bar c\,(-\Box)\,c - i\,\partial_\mu \bar c\,[A_\mu, c]\right)$$

$$= \int d^4x\left(-\bar c^{\bar i j}\,\Box\, c^{\bar j i} - i\,\partial_\mu \bar c^{\bar i j}\left(A_\mu{}^{\bar j k} c^{\bar k i} - c^{\bar j k} A_\mu{}^{\bar k i}\right)\right)$$

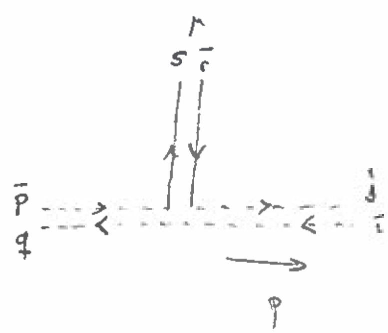
$$= \int d^4x\left(-\bar c^{\bar i j}\,\Box\, c^{\bar j i} + \partial_\mu \bar c^{\bar i j}\, f^{(j\bar i)}{}_{(r\bar s)(p\bar q)}\, A_\mu{}^{\bar r s}\, c^{\bar p q}\right)$$

$$= \int d^4x\left(-\bar c^{\bar i j}\,\Box\, c^{\bar j i} + f^{(j\bar i)}{}_{(r\bar s)(p\bar q)}\,\partial_\mu \bar c^{\bar i j} A^{\mu\, \bar r s}\, c^{\bar p q}\right)$$

d) We are immediately able to write the propagator



$$= -\frac{i}{p^2+i\varepsilon}\,\delta^{\bar i}{}_{l}\,\delta^{j}{}_{\bar k}$$



$$= i g\, f^{(j\bar i)}{}_{(r\bar s)(p\bar q)}\, p_\mu$$

Remark: At the cost of adding more diagrams we can simplify our Feynman rules. To see this, no that

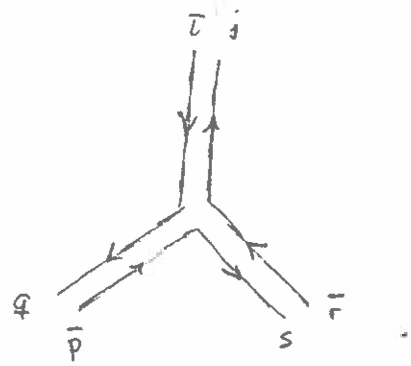
$$F^{(\bar{j} i)}{}_{(r\bar{s})(p\bar{q})} = -i \Big(\delta^{\bar{j}}{}_{r}\, \delta_{\bar{s} p}\, \delta^{i}{}_{\bar{q}} - \delta^{\bar{j}}{}_{p}\, \delta_{\bar{q} r}\, \delta^{i}{}_{\bar{s}}$$

$$- \delta^{\bar{q}}{}_{r}\, \delta_{\bar{s} i}\, \delta^{p}{}_{\bar{j}} + \delta^{\bar{q}}{}_{i}\, \delta_{r \bar{j}}\, \delta^{p}{}_{\bar{s}} \Big),$$

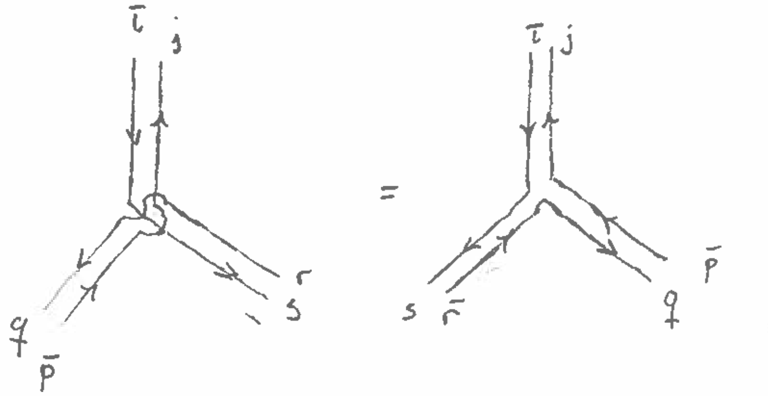
$$= i\, f^{(\bar{j} i)}{}_{(r\bar{s})(p\bar{q})} \quad \ldots$$

Sorry for not seeing this earlier

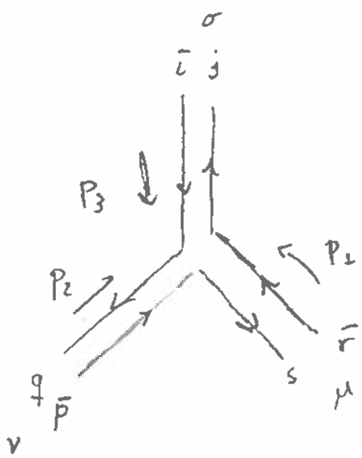
The first term can be associated quite naturally to our diagram



The other can be associated to



Then, we refine our Feynman rule to



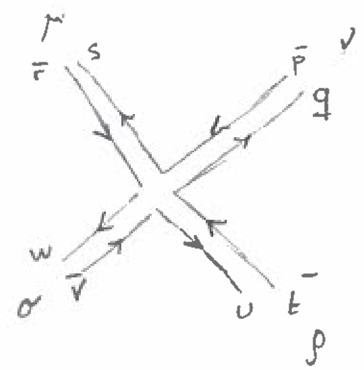
$$= -\frac{2g}{3}\,\delta_{iq}\,\delta_{ps}\,\delta_{jr}\Big(g^{\nu\sigma}(p_3-p_2)^\mu + g^{\mu\nu}(p_2-p_1)^\sigma + g^{\sigma\mu}(p_1-p_3)^\nu\Big)$$

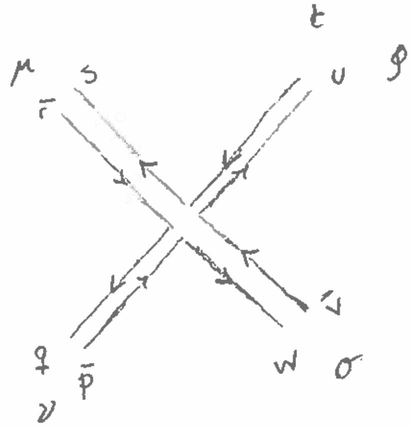
at the cost of adding one more diagram. For the four point vertex

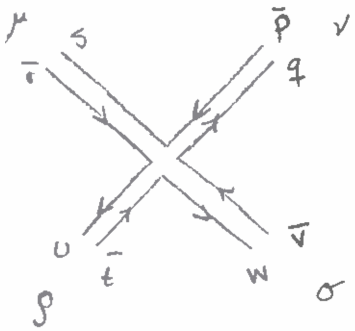
$$f^{(\bar{i}j)}{}_{(r\bar{s})(p\bar{q})}\, f^{(\bar{j}i)}{}_{(t\bar{u})(v\bar{w})} = -\left(\delta_{\bar{i}r}\delta_{j\bar{q}}\delta_{\bar{s}p} - \delta_{\bar{i}p}\delta_{j\bar{s}}\delta_{r\bar{q}}\right)\left(\delta_{\bar{j}t}\delta_{i\bar{w}}\delta_{\bar{u}v} - \delta_{\bar{j}v}\delta_{i\bar{u}}\delta_{t\bar{w}}\right)$$

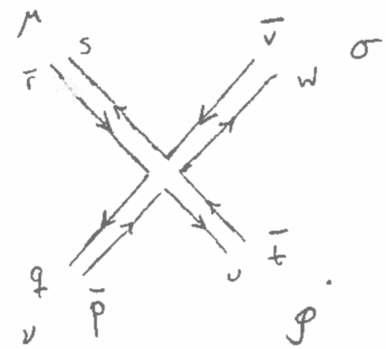
$$= -\Big(\delta_{r\bar{w}}\delta_{t\bar{q}}\delta_{\bar{s}p}\delta_{\bar{u}v} - \delta_{p\bar{w}}\delta_{t\bar{s}}\delta_{r\bar{q}}\delta_{v\bar{u}} - \delta_{r\bar{u}}\delta_{v\bar{q}}\delta_{\bar{s}p}\delta_{t\bar{w}} + \delta_{p\bar{u}}\delta_{v\bar{s}}\delta_{r\bar{q}}\delta_{t\bar{w}}\Big)$$

This terms are naturally associated with the diagrams









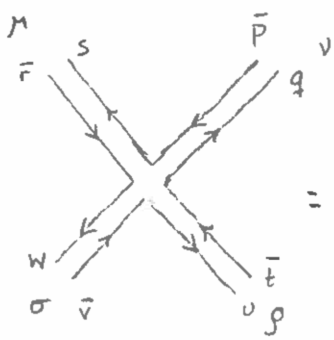
One can check there are another two configuration (these are built by choosing an index to pair r with, for which there are three possibilities, and then one for s, for which there remain two). Since there are 12 diagrams in the expansion of W, we conclude each diagram must appear twice. For example, our first diagram reappears in

$$f^{(\bar{i}j)}{}_{(t\bar{u})(p\bar{q})}\, f^{(\bar{j}i)}{}_{(r\bar{s})(v\bar{w})} = \cdots + \left(-\delta^{\bar{i}}{}_{p}\,\delta^{j}{}_{\bar{u}}\,\delta_{t\bar{q}}\right)\left(-\delta^{\bar{j}}{}_{v}\,\delta^{i}{}_{\bar{s}}\,\delta_{r\bar{w}}\right)$$

$$+ \cdots$$

$$= \cdots + \delta_{\bar{s}p}\,\delta_{\bar{u}v}\,\delta_{t\bar{q}}\,\delta_{r\bar{w}} + \cdots$$

We may thus assign to that diagram



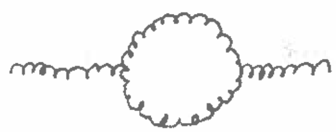
$$= \frac{g^2}{12}\,\delta_{\bar{r}w}\,\delta_{\bar{v}u}\,\delta_{\bar{t}q}\,\delta_{\bar{p}s}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma}\right)$$

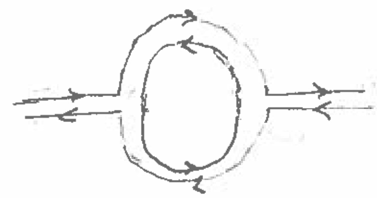
$$= \frac{g^2}{12}\,\delta_{\bar{r}w}\,\delta_{\bar{v}u}\,\delta_{\bar{t}q}\,\delta_{\bar{p}s}\left(2g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} - g^{\sigma\rho}g^{\mu\nu}\right)$$

3. One loop Correction to the Gluon Propagator

a) Let us draw the previous diagrams and the new ones side by side

planar g^0, N^0

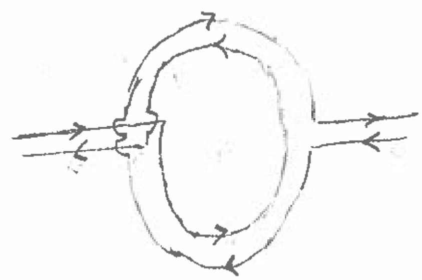




planar

$Ng^2 = \frac{\lambda^2}{N} N = \lambda^2$

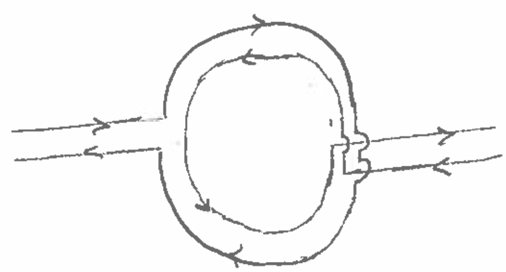
=



Non planar

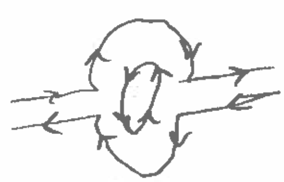
$g^2 = \frac{\lambda^2}{N}$

=



Non planar

$g^2 = \frac{\lambda^2}{N}$



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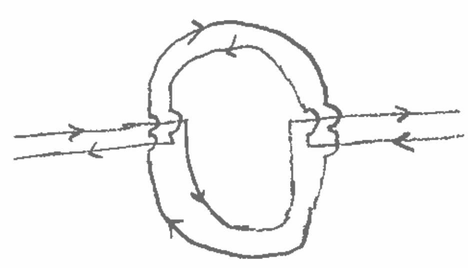
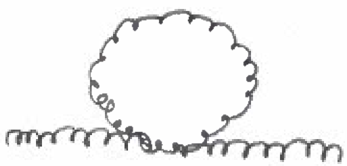
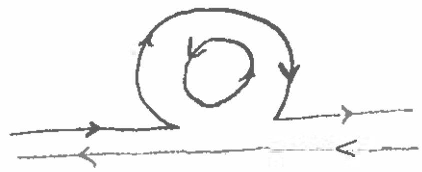


planar

$Ng^2 = \frac{\lambda^2}{N} N = \lambda^2$



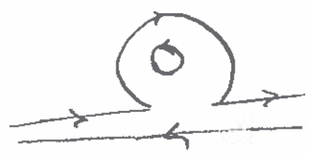


planar

$Ng^2 = \lambda^2$

Non planar

$g^2 = \frac{\lambda^2}{N}$



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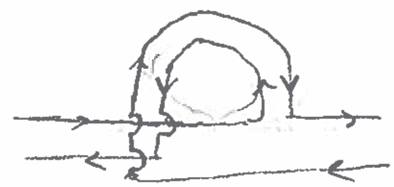
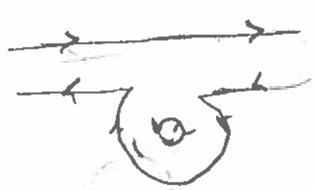
planar

$Ng^2 = \lambda^2$



Non planar

$g^2 = \frac{\lambda^2}{N}$



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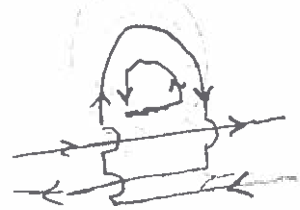
planar

$Ng^2 = \lambda^2$



planar

$Ng^2 = \lambda^2$

d) All index loops correspond to a factor of N

4. The Large N limit

a) We write its λ and N dependance next to each diagram

b) We see that only the planar diagrams survive

Remark: Dalila helped me understand these N dependences.