Condensed Matter: Lecture 13

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1 Comments on Previous Results

1.1 Persistent Flow

Consider a fluid moving with velocity $\mathbf{v}$ inside a tube like the one depicted in figure 1. Moreover, assume that the velocity is small enough so that the problem can be treated in a Galilean fashion. We want to study the conditions under which the walls of the tube can slow down the fluid. To do this, we will go to the reference frame in which the liquid is stationary. In it the walls are moving with velocity $-\mathbf{v}$. What in our previous system corresponded to a slowing down of the fluid now corresponds to the same phenomena for the walls. However, if the walls are to slow down, conservation of energy would the require that the energy dissipated be deposited into the fluid. We will treat this increase in energy as the excitation of a quasiparticle.

Let us recall the transformation law for the energy of a Galilean particle. If the particle has a mass m and momentum $\mathbf{p}$, then its energy is $E(\mathbf{0},\mathbf{p})=\frac{\mathbf{p}^2}{2m}$. After a Galilean boost into a reference frame moving with velocity $-\mathbf{v}$, the particle has an energy

$$E(\mathbf{v},\mathbf{p})=\frac{(\mathbf{p}+m\mathbf{v})^2}{2m}=E(\mathbf{0},\mathbf{p})+\mathbf{p}\cdot\mathbf{v}+\frac{1}{2}m\mathbf{v}^2. \tag{1}$$

We will assume this equation to be true for our quasiparticle, even though it doesn't have to be subject to the dispersion relation above. Then, the energy increase in our fluid due to the excitation of our quasiparticle is

$$\Delta E=E(\mathbf{v},\mathbf{p})-E(\mathbf{v},\mathbf{0})=E(\mathbf{0},\mathbf{p})+\mathbf{p}\cdot\mathbf{v}, \tag{2}$$

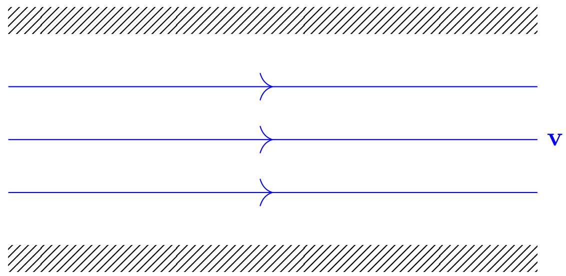

Figure 1: Cross-section of Galilean fluid moving with velocity $\mathbf{v}$ in blue. The black lines represent a wall that the fluid is in contact with.

in our original reference frame.

Remark. It was claimed that the transformation behaviour in (1) is a general result of the structure of the Galilean group. In particular, we should be able to obtain the transformation (2) from the commutation relations of the Lie algebra of the Galilean group. Although I haven't worked it out myself, I believe the commutation relation $[H, K_i] = iP_i$ gives the linear term of the transformation (1). On the other hand, the commutation relations $[K_i, K_j] = 0$ (which are not true in the case of the Poincaré algebra) should be responsible for the quadratic term. In particular, they should be responsible for the existence of the mass term m^2 independent of $\mathbf{p}$.

Thus, the creation of these quasiparticles is energetically favourable if $\Delta E < 0$. Given that $E(\mathbf{0}, \mathbf{p}) > 0$, this is only possible if $\mathbf{p} \cdot \mathbf{v} < 0$. Then, the condition for the creation of this particles reduces to

$$E(\mathbf{0}, \mathbf{p}) < |-\mathbf{p} \cdot \mathbf{v}| \leq \|\mathbf{p}\| \|\mathbf{v}\|, \tag{3}$$

by the triangle inequality, i.e.

$$\|\mathbf{v}\| > \frac{E(\mathbf{0}, \mathbf{p})}{\|\mathbf{p}\|}. \tag{4}$$

Therefore, if there is a lower bound different than 0 on $\frac{E(\mathbf{0},\mathbf{p})}{\|\mathbf{p}\|}$, and the speed of the fluid is below this upperbound, the creation of quasiparticles is always energetically disfavorable. In terms of our original problem, this corresponds to a fluid which doesn't not slow down due to its interaction with the walls. This phenomena is called persistent flow, one of the defining features of superfluids.

Examples.

1. Fermi liquids do not constitute superfluids. Indeed, consider a state in the Fermi sea which is infinitesimally close to the Fermi surface. This state can be excited to a state outside of the Fermi sea that is still infinitesimally close to the Fermi surface. This then costs an infinitesimal amount of energy. This can however be done with a finite change in momentum between this two states. Such an excitation is showed in figure 2. This shows that in these systems $\frac{E(\mathbf{0},\mathbf{p})}{\|\mathbf{p}\|}$ has 0 as a lower bound.

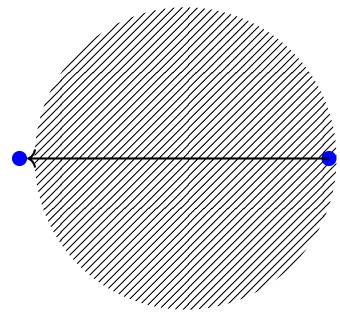

Figure 2: Excitation with finite momentum change but infinitesimal energy difference in Fermi liquid.

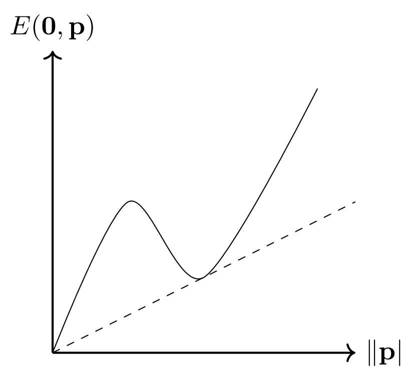

Figure 3: Dispersion relation of liquid helium. The dotted line corresponds to the least steep line connecting the origin and a point in the graph.

2. Free Bose-Einstein condensates do not constitute superfluids. Indeed, these systems obey the free particle dispersion relation, so that $\frac{E(\mathbf{0},\mathbf{p})}{\|\mathbf{p}\|} = \frac{\|\mathbf{p}\|}{2m}$, whose lower bound is 0.

3. Helium-4 can consitute a superfluid. This is clear from the form of its dispersion relation, shown in figure 3. To extract $\frac{E(\mathbf{0},\mathbf{p})}{\|\mathbf{p}\|}$ from a plot like this, it is necessary to realize that this quotient precisely corresponds to the slope of the line joining the origin to the point $(\|\mathbf{p}\|, E(\mathbf{0}, \mathbf{p}))$. In this case, these slopes are bounded from below by the slope of the dotted line.

4. Paired Fermi surfaces can constitute a superfluid. Although the process described in 2 can be realized, the energy difference is now bounded from below by $\frac{\Delta}{2k_F}$. This creates a lower bound for our quotient.

1.2 Anderson-Higgs Mechanism

The free energy that we studied before for a superconductor takes the form

$$f(\theta, \mathbf{A}) = \frac{1}{2}\rho_s \|\boldsymbol{\nabla}\theta - 2\mathbf{A}\|^2, \tag{5}$$

when expanded around a minimum energy solution. Under the identification $\tilde{\mathbf{A}} := \mathbf{A} - \frac{1}{2}\boldsymbol{\nabla}\theta$, this looks like a quadratic mass term

$$f(\tilde{\mathbf{A}}) = 2\rho_s \|\tilde{\mathbf{A}}\|^2. \tag{6}$$

For gauge fields, such as $\mathbf{A}$, these terms are forbidden since they break gauge invariance

$$m^2\|\mathbf{A}\|^2 \mapsto m^2\|\mathbf{A}+\boldsymbol{\nabla}\alpha\|^2 = m^2\|\mathbf{A}\|^2+2m^2\mathbf{A}\cdot\boldsymbol{\nabla}\alpha+m^2\|\boldsymbol{\nabla}\alpha\|^2 \neq m^2\|\mathbf{A}\|^2. \tag{7}$$

One then could mistakenly believe that gauge invariance has been broken. This is however not the case since $\tilde{\mathbf{A}}$ is not a gauge field. In fact, $\tilde{\mathbf{A}}$ is gauge invariant. To see this, recall that θ is the phase of our order parameter. Thus, it transforms under gauge transfomations like

$$\theta \mapsto \theta + 2\alpha. \tag{8}$$

Under a gauge transformation we then have

$$\tilde{\mathbf{A}} \mapsto \mathbf{A} + \boldsymbol{\nabla}\alpha - \frac{1}{2}\boldsymbol{\nabla}(\theta + 2\alpha) = \tilde{\mathbf{A}}. \tag{9}$$

The correct interpretation is that the gauge field $\mathbf{A}$ and the massless Goldstone boson θ combined to form a massive vector field $\tilde{\mathbf{A}}$.

Remark. As a cute note, people sometimes say that the gauge field *ate* the Goldstone boson.

2 Abrikosov Vortex Lattice

As we have already studied before, the behaviour of magnetic field in superconductors is like the one described in figure 4. Notice that, given the absense of free currents, the $\mathbf{H}$-field remains constant inside the superconductor. Recalling the definition of this field

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \tag{10}$$

with $\mathbf{M}$ the magnetization, we arrive at the constitutive relation for the bulk of a superconductor

$$\mathbf{M} = -\frac{\mathbf{H}}{4\pi}. \tag{11}$$

This is the reason why superconductors are sometimes called perfect diamagnets.

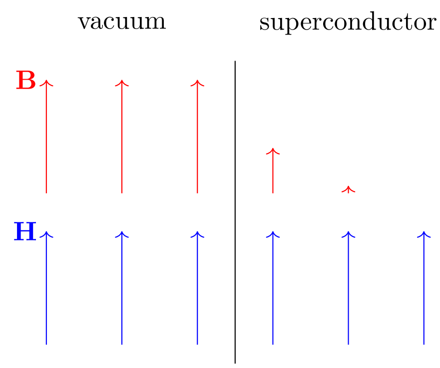

Figure 4: Behaviour of the **B**-fields and **H**-fields when a superconductor is placed in a uniform magnetic field.

Our description for superconductors started from the free energy density

$$f(\mathbf{B}) = \frac{1}{2}\|(\boldsymbol{\nabla} - i2\mathbf{A})\Phi\|^2 - \frac{1}{\xi^2}|\Phi|^2 + b|\Phi|^4 + \frac{1}{8\pi}\|\mathbf{B}\|^2 = f(\mathbf{0}) + \frac{1}{8\pi}\|\mathbf{B}\|^2. \quad (12)$$

However, given the previous discussion, it seems better to describe our system in terms of the constant **H**-field. For this we do a Legendre transform

$$g(\mathbf{H}) = \inf_{\mathbf{B}}\left(f(\mathbf{B}) - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}\right). \quad (13)$$

Since

$$\frac{\partial}{\partial \mathbf{B}}\left(f(\mathbf{B}) - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}\right) = \frac{\partial f(\mathbf{B})}{\partial \mathbf{B}} - \frac{\mathbf{H}}{4\pi}, \quad (14)$$

we see that we can equivalently write

$$g(\mathbf{H}) = f(\mathbf{B}) - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}, \quad (15)$$

with

$$\mathbf{H} = 4\pi\frac{\partial f(\mathbf{B})}{\partial \mathbf{B}} = \mathbf{B}, \quad (16)$$

which is the more familiar form from the theory of mechanics. Inverting this relation we obtain

$$g(\mathbf{H}) = f(\mathbf{B} = \mathbf{H}) - \frac{\|\mathbf{H}\|^2}{4\pi} = f(\mathbf{B} = 0) - \frac{\mathbf{H}^2}{8\pi}. \quad (17)$$

The fact that inside of a superconductor $\mathbf{B} = \mathbf{0}$, shows us that inside these $g(\mathbf{H}) = f_{\text{sc}}(\mathbf{B} = \mathbf{0})$. Meanwhile, in a normal metal, we would have the result (17) with $f(\mathbf{B} = \mathbf{0}) \mapsto f_{\text{nm}}(\mathbf{B} = \mathbf{0})$. Thus, the difference between these energy densities is given by $f_{\text{sc}}(\mathbf{B} = \mathbf{0}) - f_{\text{nm}}(\mathbf{B} = \mathbf{0}) + \frac{\mathbf{H}^2}{8\pi}$. Now, we have a transition between the normal and the superconducting phases precisely when this energy densities agree, i.e. when

$$\frac{\|\mathbf{H}_c\|^2}{8\pi} = f_{\mathrm{sc}}(\mathbf{B}=\mathbf{0}) - f_{\mathrm{nm}}(\mathbf{B}=\mathbf{0}) = \frac{n_e \Delta^2}{E_F} = \left(\frac{\Phi_0}{\lambda_L \xi}\right)^2. \tag{18}$$

The second equality was stated without proof but is a consequence of BCS theory. The third inequality is a rewritting of the second in terms of the more familiar scales. In here, n_e is the electron density, Δ is the gap from BCS theory, E_F is the Fermi energy, Φ_0 is the magnetic flux quantum, λ_L is the London penetration depth, and ξ is the correlation length. Therefore, when $\|\mathbf{H}\| < \|\mathbf{H}_c\|$, we have that the energy density of the superconductor is less than that of the normal metal. Thus, in this case it is energetically favourable to be in the superconducting phase. On the other hand, when $\|\mathbf{H}\| > \|\mathbf{H}_c\|$ the normal metal phase is more favourable.

Let us now study the situation close to our critical $\mathbf{H}$-field. In it both faces will coexist and interfaces between them will form. By definition of the critical field, the energy densities of both phases are the same. However, the creation of the interfaces has its own energy cost. This cost is described by the surface tension, which is given by the energy per unit area of such an interface. Let us start by studying this surface tension for superconductors where $\xi \gg \lambda_L$. These are known as Type I superconductors and are depicted in figure 5. In these, the surface tension at the interfaces is given by

$$\sigma = \xi(f_{\mathrm{nm}}(\mathbf{0}) - f_{\mathrm{sc}}(\mathbf{0})) \sim \xi \frac{\|\mathbf{H_c}\|^2}{8\pi} > 0. \tag{19}$$

This means that the surface area of an interface will try to be minimized. Thus, if two metallic domains encounter each other, they will merge in an attempt to minimize their surface area. Therefore, above the critical field the metallic domains quickly populate the superconductor producing a phase transition into a normal metal, as predicted with our previous analysis.

The situation for Type II superconductors is rather different. In these we have $\lambda_L \gg \xi$. To see how the surface tension behaves in these case, consider the behaviour of the magnetic field $\mathbf{B}(x) = \mathbf{B}_0 e^{-x/\lambda_L}$ inside the superconductor. In here x is the distance from the interface. Then, since the order parameter attains its ground state Φ_0 before the magnetic field vanishes, we may assume that inside the superconductor the order parameter has this value. In particular, due to the definition of ground state, the energy density is given by

$$g(\mathbf{H}) = \frac{1}{2}\rho_s \|\boldsymbol{\nabla}\theta - 2\mathbf{A}\|^2 + \frac{\|\mathbf{B}\|^2}{8\pi} - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}. \tag{20}$$

Now, as we saw before $\mathbf{j} = \frac{1}{2\rho_s}(\boldsymbol{\nabla}\theta - 2\mathbf{A})$. However, by the equations of motion $\mathbf{j} = \frac{1}{4\pi}\boldsymbol{\nabla}\times\mathbf{B}$. If $\mathbf{B}_0$ is orthogonal to the x-direction, our anzats for the $\mathbf{B}$-field yields $\|\mathbf{j}\| = \frac{1}{4\pi\lambda_L}\|\mathbf{B}\|$. Finally, recalling that in the ground state $\lambda_L^{-2} = 16\pi\rho_s$, we obtain

$$\frac{1}{2}\rho_s \|\boldsymbol{\nabla}\theta - 2\mathbf{A}\|^2 = \frac{1}{8\rho_s}\|\mathbf{j}\|^2 = \frac{1}{8\rho_s}\frac{1}{16\pi^2\lambda_L^2}\|\mathbf{B}\|^2 = \frac{\|\mathbf{B}\|^2}{8\pi}, \tag{21}$$

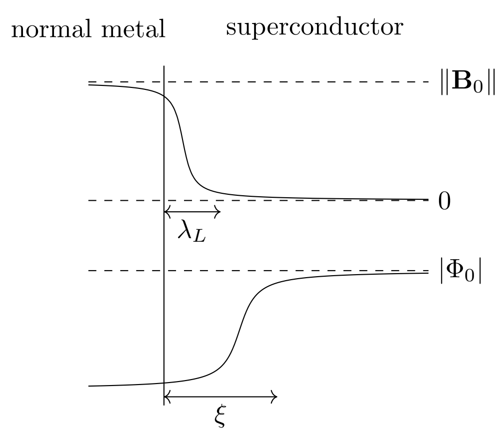

Figure 5: A cross-section of the interface between a normal metal and a superconductor is shown. On the top, the behaviour of the **B**-field is shown. On the bottom, the behaviour of the order parameter is shown. The meaning of ξ and λ_L is also illustrated. The particular behaviour shown most closely resembles that of a Type I superconductor

i.e.

$$g(\mathbf{H}) = \frac{\|\mathbf{B}\|^2}{4\pi} - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}. \tag{22}$$

Above, we have an energy density. Thus, if we want to obtain the surface tension, which corresponds to the energy density per unit area of the interface, we need to integrate along x. This yields

$$\begin{aligned}\sigma = \int_0^\infty \mathrm{d}x\, g(\mathbf{H}) = \int_0^\infty \mathrm{d}x \left(\frac{\|\mathbf{B}_0\|^2}{4\pi} e^{-2x/\lambda_L} - \frac{\mathbf{B}_0\cdot\mathbf{H}}{4\pi} e^{-x/\lambda_L}\right)\\ = \lambda_L \frac{\|\mathbf{B}_0\|^2}{8\pi} - \lambda_L \frac{\mathbf{B}_0\cdot\mathbf{H}}{4\pi}.\end{aligned} \tag{23}$$

Recalling that $\mathbf{B}_0$ is the magnetic field outside the superconductor, we have $\mathbf{B}_0 = \mathbf{H}$. Thus, since we are near the critical field,

$$\sigma = -\lambda_L \frac{\|\mathbf{H}_c\|^2}{8\pi} < 0. \tag{24}$$

Thus, interfaces in Type II superconductors try to maximize their surface area! This means, that metallic domains tend to separate into subdomains. The behaviour found for both types is summarized in figure 6.

This process however cannot continue indefinitely. Indeed, through metallic domains there would be no Meissner effect, allowing for the flow of magnetic flux. However, as we've seen before, magnetic flux through superconductors is quantized. Thus, the metallic domains separate until they allow for only the fundamental flux. These elementary domains are called vortexes. The

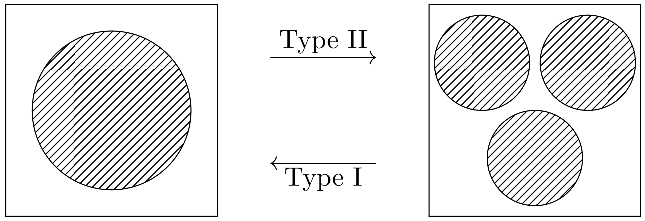

Figure 6: Comparison of the behaviour of domains between Type I and Type II. Domains in the normal metal phase are shaded while the superconducting domains are not.

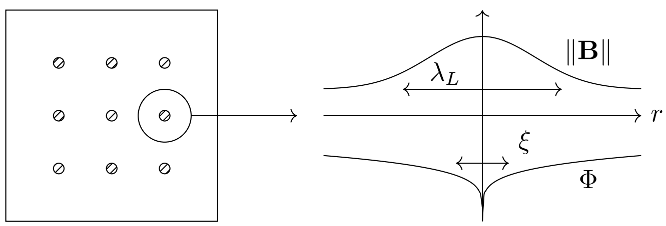

Figure 7: An example of an Abrikosov vortex lattice. The behaviour of the magnetic field $\mathbf{B}$ and the order parameten Φ as a function of the distance from the center of the vortex r is also shown.

separation of these vortices amount to a repulsion between them. Thus, much like particles subject to a repulsive interaction in a 2D box, these vortices will align themselves in a lattice structure. This is known as the Abrikosov vortex lattice. An illustration is found in figure 7.

Remark. I want to thank Bruno for many fruitful discussions during the creation of these notes. His insight helped me clarified many doubts I had throughout the lecture.