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Chern-Simons Theory

Homework 1: 3d Gravity as a Chern-Simons Theory

1. Euclidean Signature, Vanishing Cosmological Constant

1.1. Preliminaries

Q1: The orthonormality of the vielbein is given by

$$e_I{}^\mu e_J{}^\nu g_{\mu\nu} = \eta_{IJ}.$$

The inverse vielbein e^I is defined by

$$e^I{}_\mu e_J{}^\mu = \delta^I{}_J.$$

Since the left matrix inverse of a square matrix is also the right matrix inverse, we have

$$e_I{}^\mu e^I{}_\nu = \delta^\mu{}_\nu.$$

Therefore

$$e^I{}_\mu e^J{}_\nu \eta_{IJ} = e^I{}_\mu e^J{}_\nu e_I{}^\sigma e_J{}^\rho g_{\sigma\rho}$$

$$= \delta^\sigma{}_\mu \delta^\rho{}_\nu g_{\sigma\rho} = g_{\mu\nu},$$

proving (6). Now, let us define $e_{I\mu} := \eta_{IJ} e^J{}_\mu$. Then

$$e_{I\mu} = g_{\sigma\rho} e_I{}^\sigma e_J{}^\rho e^J{}_\mu = g_{\sigma\rho} e_I{}^\sigma \delta^\rho{}_\mu = g_{\sigma\mu} e_I{}^\sigma,$$

proving the first half of (7). Notice then that we can also connect $e^I{}_\mu$ and $e_I{}^\mu$ by raising and lowering indices

$$\eta^{IJ} g_{\mu\nu} e_J{}^\nu = \eta^{IJ} e_{J\mu} = \eta^{IJ} \eta_{JK} e^K{}_\mu = \delta^I{}_K e^K{}_\mu = e^I{}_\mu,$$

or inversely

$$\eta_{IJ} g^{\mu\nu} e^J{}_\nu = \eta_{IJ} g^{\mu\nu} \eta^{JK} g_{\nu\sigma} e_K{}^\sigma = \delta^K{}_I \delta^\mu{}_\sigma e_K{}^\sigma = e_I{}^\mu.$$

Now, define $e^{I\mu} := \eta^{IJ} e_J{}^\mu$. We then have

$$e^{I\mu} = \eta^{IJ} \eta_{JK} g^{\mu\nu} e^K{}_\nu = \delta^I{}_K g^{\mu\nu} e^K{}_\nu = g^{\mu\nu} e^I{}_\nu,$$

showing the last half of 7.

Q2: Assume that $\lambda^I{}_J \in O(\mathbb{R}^D, \eta_{IJ})$. Then

$$g'_{\mu\nu} = \eta_{IJ} e'^I{}_\mu e'^J{}_\nu = \eta_{IJ} \lambda^I{}_K e^K{}_\mu \lambda^J{}_L e^L{}_\nu$$

$$= (\eta_{IJ} \lambda^I{}_K \lambda^J{}_L) e^K{}_\mu e^L{}_\nu = \eta_{KL} e^K{}_\mu e^L{}_\nu = g_{\mu\nu}.$$

1.2. Cartan's Structure Equations

Q3: Let us first study the transformation of $e_I{}^\mu$ under $\lambda^I{}_J \in O(\mathbb{R}^D, \eta_{IJ})$. We have

$$e'_I{}^\mu = \eta_{IJ} g^{\mu\nu} e'^J{}_\nu = \eta_{IJ} g^{\mu\nu} \lambda^J{}_K e^K{}_\nu = \lambda_{IK} e^{K\mu}$$

$$= \lambda_I{}^K e_K{}^\mu.$$

Of course, $\lambda_I{}^K$ is easily computed by noting that

$$\lambda_K{}^I \lambda^K{}_J = \eta_{KL} \eta^{IM} \lambda^L{}_M \lambda^K{}_J = \eta_{JM} \eta^{IM} = \delta^I{}_J.$$

We can now compute

$$\omega'_\mu{}^I{}_J = e'^I{}_\nu \nabla_\mu e'_J{}^\nu = \lambda^I{}_K e^K{}_\nu \nabla_\mu (\lambda_J{}^L e_L{}^\nu)$$

$$= \lambda^I{}_K e^K{}_\nu \partial_\mu \lambda_J{}^L e_L{}^\nu + \lambda^I{}_K e^K{}_\nu \lambda_J{}^L \partial_\mu e_L{}^\nu$$

$$+ \lambda^I{}_K e^K{}_\nu \Gamma^\nu{}_{\mu\rho} \lambda_J{}^L e_L{}^\rho$$

$$= \lambda^I{}_K \partial_\mu \lambda_J{}^L \delta^K{}_L + \lambda^I{}_K e^K{}_\nu \lambda_J{}^L \nabla_\mu e_L{}^\nu$$

$$= \lambda^I{}_K \omega_\mu{}^K{}_L \lambda_J{}^L + \lambda^I{}_K \partial_\mu \lambda_J{}^K .$$

In terms of the one-forms,

$$\omega'^I{}_J = \omega'_\mu{}^I{}_J dx^\mu = \lambda^I{}_K \omega^K{}_L \lambda_J{}^L + \lambda^I{}_K \partial_\mu \lambda_J{}^K dx^\mu$$

$$= \lambda^I{}_K \omega^K{}_L \lambda_J{}^L + \lambda^I{}_K d\lambda_J{}^K .$$

In matrix notation, the equation $\lambda_K{}^I \lambda^K{}_J = \delta^I{}_J$ means that the matrix $M^I{}_J := \lambda_J{}^I$ has $M^T \lambda = I_D$.

Therefore $M^T = \lambda^{-1}$. We conclude

$$\omega' = \lambda \omega M^T + \lambda d M^T = \lambda \omega \lambda^{-1} + \lambda d \lambda^{-1} .$$

Q4: We have

$$D_\mu v^I := e^I{}_\nu \nabla_\mu v^\nu = e^I{}_\nu \nabla_\mu (e_J{}^\nu v^J)$$

$$= e^I{}_\nu e_J{}^\nu \partial_\mu v^J + e^I{}_\nu \nabla_\mu e_J{}^\nu v^J$$

$$= \partial_\mu v^I + \omega_\mu{}^I{}_J v^J .$$

Under a transformation $\lambda^I{}_J \in O(\mathbb{R}^D, \eta_{IJ})$, we have

$$v^I e_I = v = v'^I e'_I = v'^I \lambda_I{}^J e_J .$$

Thus

$$v'^I = \delta^I{}_J v'^J = \lambda^I{}_K \lambda_J{}^K v'^J = \lambda^I{}_K v^K .$$

We conclude

$$
\begin{aligned}
(D_\mu v)'^I &= \partial_\mu v'^I + \omega'_\mu{}^I{}_J v'^J \\
&= \partial_\mu (\lambda^I{}_J v^J) + \lambda^I{}_K \omega_\mu{}^K{}_L \lambda_J{}^L v'^J + \lambda^I{}_K \partial_\mu \lambda_J{}^K v'^J \\
&= \partial_\mu \lambda^I{}_J v^J + \lambda^I{}_J \partial_\mu v^J + \lambda^I{}_K \omega_\mu{}^K{}_L \lambda_J{}^L \lambda^J{}_M v^M \\
&\quad + \lambda^I{}_K \partial_\mu \lambda_J{}^K \lambda^J{}_M v^M \\
&= \lambda^I{}_J \partial_\mu v^J + \lambda^I{}_K \omega_\mu{}^K{}_M v^M + \cancel{\partial_\mu \lambda^I{}_J v^J} \\
&\quad + \partial_\mu \big(\cancel{\lambda^I{}_K \lambda_J{}^K}\big)^{0} \lambda^J{}_M v^M - \cancel{\partial_\mu \lambda^I{}_K \lambda_J{}^K \lambda^J{}_M v^M} \\
&= \lambda^I{}_J D_\mu v^J .
\end{aligned}
$$

Q5: Using the orthogonality condition

$$
\begin{aligned}
g^{\mu\nu} = g^{\mu\sigma} g^{\nu\rho} g_{\sigma\rho} &= g^{\mu\sigma} g^{\nu\rho} \eta_{IJ} e^I{}_\sigma e^J{}_\rho \\
&= \eta_{IJ} e^{I\mu} e^{J\nu} = \eta^{IJ} e_I{}^\mu e_J{}^\nu ,
\end{aligned}
$$

we can express the metricity condition as

$$0 = \nabla_\mu g^{\nu\sigma} = \nabla_\mu (\eta^{IJ} e_I{}^\nu e_J{}^\sigma) = \eta^{IJ} \nabla_\mu e_I{}^\nu e_J{}^\sigma + \eta^{IJ} e_I{}^\nu \nabla_\mu e_J{}^\sigma.$$

Now, inverting the vielbein on the definition of the spin connection

$$e_I{}^\rho \omega_\mu{}^I{}_J = e_I{}^\rho e^I{}_\nu \nabla_\mu e_J{}^\nu = \delta^\rho{}_\nu \nabla_\mu e_J{}^\nu = \nabla_\mu e_J{}^\rho.$$

Then

$$\begin{aligned} 0 &= \eta^{IJ} e_K{}^\nu \omega_\mu{}^K{}_I e_J{}^\sigma + \eta^{IJ} e_I{}^\nu e_K{}^\sigma \omega_\mu{}^K{}_J \\ &= e_K{}^\nu e_J{}^\sigma \omega_\mu{}^{KI} + e_I{}^\nu e_K{}^\sigma \omega_\mu{}^{KI} \\ &= e_K{}^\nu e_J{}^\sigma (\omega_\mu{}^{KI} + \omega_\mu{}^{IK}). \end{aligned}$$

Since the veilbein is invertible, we conclude that the metricity condition is equivalent to

$$\omega^{IJ} = -\omega^{JI}.$$

For the Torsion free condition, note that we can invert $e^I := e^I{}_\mu dx^\mu$ to

$$dx^\mu = e_I{}^\mu e^I.$$

Moreover, recall that

$$[U,V]^\mu = \left(U^\nu \partial_\nu V^\sigma \partial_\sigma + U^\nu V^\sigma \cancel{\partial_\nu} \partial_\sigma - V^\nu \partial_\nu U^\sigma \partial_\sigma - V^\nu U^\sigma \cancel{\partial_\nu \partial_\sigma}\right)$$

$$= U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu V^\mu .$$

We then have

$$(de^I + \omega^I{}_J \wedge e^J)(e_K, e_L) = \partial_\mu e^I{}_\nu \, dx^\mu \wedge dx^\nu (e_K, e_L)$$

$$+ \omega_\mu{}^I{}_J \, dx^\mu \wedge e^J (e_K, e_L)$$

$$= \partial_\mu e^I{}_\nu e_K{}^\mu e_L{}^\nu - \partial_\mu e^I{}_\nu e_L{}^\mu e_K{}^\nu + \omega_\mu{}^I{}_J e_M{}^\mu e^M \wedge e^J (e_K, e_L)$$

$$= \partial_\mu (\cancel{e^I{}_\nu e_L{}^\nu}^{\,0}) e_K{}^\mu - e^I{}_\nu \partial_\mu e_L{}^\nu e_K{}^\mu - \partial_\mu (\cancel{e^I{}_\nu e_K{}^\nu}^{\,0}) e_L{}^\mu$$

$$+ e^I{}_\nu \partial_\mu e_K{}^\nu e_L{}^\mu + \omega_\mu{}^I{}_L e_K{}^\mu - \omega_\mu{}^I{}_K e_L{}^\mu$$

$$= e^I{}_\nu \nabla_\mu e_L{}^\nu e_K{}^\mu - e^I{}_\nu \nabla_\mu e_K{}^\nu e_L{}^\mu - e^I{}_\nu (e_K{}^\mu \partial_\mu e_L{}^\nu - e_L{}^\mu \partial_\mu e_K{}^\nu)$$

$$= e^I{}_\nu \left(\nabla_{e_K} e_L{}^\nu - \nabla_{e_L} e_K{}^\nu - [e_K, e_L]^\nu \right) = e^I{}_\nu T(e_K, e_L)^\nu$$

$$= e^I (T(e_K, e_L)).$$

Thus, the connection is free of torsion if and only if

$$de^I + \omega^I{}_J \wedge e^J = 0.$$

Q6: Using the result from Q4, we have

$$D_\mu D_\nu v^I = \partial_\mu (D_\nu v)^I + \omega_\mu{}^I{}_J (D_\nu v)^J$$

$$= \partial_\mu \partial_\nu v^I + \partial_\mu(\omega_\nu{}^I{}_J v^J) + \omega_\mu{}^I{}_J \partial_\nu v^J$$

$$+ \omega_\mu{}^I{}_J \omega_\nu{}^J{}_K v^K$$

$$= \partial_\mu \partial_\nu v^I + \partial_\mu \omega_\nu{}^I{}_J v^J + \omega_\nu{}^I{}_J \partial_\mu v^J$$

$$+ \omega_\mu{}^I{}_J \partial_\nu v^J + \omega_\mu{}^I{}_J \omega_\nu{}^J{}_K v^K$$

$$= \partial_\mu \partial_\nu v^I + \omega_{(\nu|}{}^I{}_J \partial_{|\mu)} v^J + \partial_\mu \omega_\nu{}^I{}_J v^J$$

$$+ \omega_\mu{}^I{}_J \omega_\nu{}^J{}_K v^K .$$

Therefore

$$[D_\mu, D_\nu] v^I = D_{[\mu} D_{\nu]} v^I = \partial_{[\mu} \omega_{\nu]}{}^I{}_J v^J + \omega_{[\mu|}{}^I{}_J \omega_{|\nu]}{}^J{}_K v^K$$

$$= \left(\partial_{[\mu} \omega_{\nu]}{}^I{}_J + \omega_{[\mu|}{}^I{}_K \omega_{|\nu]}{}^K{}_J \right) v^J .$$

Notice that

$$F^I{}_J := d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J = (\partial_\mu \omega_\nu{}^I{}_J + \omega_\mu{}^I{}_K \omega_\nu{}^K{}_J) dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2} \left(\partial_{[\mu} \omega_{\nu]}{}^I{}_J + \omega_{[\mu|}{}^I{}_K \omega_{|\nu]}{}^K{}_J \right) dx^\mu \wedge dx^\nu .$$

Thus we conclude

$$[D_\mu, D_\nu] v^I = F^I{}_{J\mu\nu} v^J .$$

To obtain the Bianchi identity, note that

$$dF^I{}_J = \underbrace{\cancel{d^2 \omega^I{}_J}}_{0} + d\omega^I{}_K \wedge \omega^K{}_J - \omega^I{}_K \wedge d\omega^K{}_J$$

$$= (F^I{}_K - \omega^I{}_L \cancel{\wedge} \omega^L{}_K) \wedge \omega^K{}_J$$

$$- \omega^I{}_K \wedge (F^K{}_J - \omega^K{}_L \cancel{\wedge} \omega^L{}_J)$$

$$= F^I{}_K \wedge \omega^K{}_J - \omega^I{}_K \wedge F^K{}_J .$$

Therefore

$$0 = dF^I{}_J + \omega^I{}_K \wedge F^K{}_J - \underbrace{\cancel{(-1)^{2\times 1}}}_{1} \omega^K{}_J \wedge F^I{}_K$$

$$= DF^I{}_J .$$

1.3. The Einstein-Hilbert action in the first order formalism

Q7: We have

$$D_\mu D_\nu v^I = e^I{}_\rho \nabla_\mu D_\nu v^\rho = e^I{}_\rho \nabla_\mu (e_J{}^\rho D_\nu v^J)$$

$$= e^I{}_\rho \nabla_\mu (e_J{}^\rho e^J{}_\sigma \nabla_\nu v^\sigma) = e^I{}_\rho \nabla_\mu \nabla_\nu v^\rho .$$

Therefore

$$F^I{}_{J\mu\nu} v^J = [D_\mu, D_\nu] v^I = D_{[\mu} D_{\nu]} v^I = e^I{}_\rho \nabla_{[\mu} \nabla_{\nu]} v^\rho$$

$$= e^I{}_\rho [\nabla_\mu, \nabla_\nu] v^\rho = e^I{}_\rho R^\rho{}_{\sigma\mu\nu} v^\sigma$$

$$= e^I{}_\rho e_J{}^\sigma R^\rho{}_{\sigma\mu\nu} v^J .$$

We conclude

$$F^I{}_{J\mu\nu} = e^I{}_\rho e_J{}^\sigma R^\rho{}_{\sigma\mu\nu} .$$

Q8: From

$$g_{\mu\nu} = \eta_{IJ} e^I{}_\mu e^J{}_\nu$$

we have

$$\det g = \cancelto{\pm 1}{\det \eta} \det(e)^2 ,$$

i.e.

$$\sqrt{|\det g|} = \sqrt{|\det(e)^2|} = |\det(e)| .$$

On the other hand

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\alpha{}_{\mu\alpha\nu}$$

$$= \eta^{IJ} e_I{}^\mu e_J{}^\nu e_M{}^\alpha e^N{}_\mu F^M{}_{N\alpha\nu}$$

$$= \eta^{NJ} e_J{}^{\nu} e_M{}^{\alpha} F^{M}{}_{N\alpha\nu} = e_J{}^{\nu} e_M{}^{\alpha} F^{MN}{}_{\alpha\nu}.$$

We conclude that

$$16\pi G\, S(g) = \int d^D x \sqrt{|\det g|}\; R = \int d^D x\, |\det(e)|\, e_I{}^{\mu} e_J{}^{\nu} F^{IJ}{}_{\mu\nu}$$

$$= 16\pi G\, S(e)$$

Q9: Specializing to 3D,

$$\varepsilon_{IJK}\, e^I \wedge F^{JK} = \tfrac{1}{2}\varepsilon_{IJK}\, e^I{}_{\mu} F^{JK}{}_{\nu\rho}\, dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho}$$

$$= \tfrac{1}{2}\varepsilon_{IJK}\, \varepsilon^{\mu\nu\rho} e^I{}_{\mu} F^{JK}{}_{\nu\rho}\, dx^1\wedge dx^2 \wedge dx^3$$

$$= \tfrac{1}{2}\varepsilon_{IJK}\, \varepsilon^{\mu\nu\rho} e^I{}_{\mu} e^M{}_{\nu} e^N{}_{\rho} e_M{}^{\alpha} e_N{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \tfrac{1}{2}\varepsilon_{IJK} \det(e)\, \varepsilon^{IMN} e_M{}^{\alpha} e_N{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \tfrac{1}{2}\delta^M{}_{[J}\delta^N{}_{K]} \det(e)\, e_M{}^{\alpha} e_N{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \tfrac{1}{2}\det(e)\, e_{[J}{}^{\alpha} e_{K]}{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \tfrac{1}{2}\det(e)\, e_J{}^{[\alpha} e_K{}^{\beta]} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= 2\det(e)\, e_J{}^{\alpha} e_K{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

Therefore

$$S(e) = \frac{1}{16\pi G} \int d^3x \, |\det(e)| \, e_I{}^\mu e_J{}^\nu F^{IJ}{}_{\mu\nu}$$

$$= \frac{1}{16\pi G} \int d^3x \, \mathrm{sgn}(\det(e)) \det(e) \, e_I{}^\mu e_J{}^\nu F^{IJ}{}_{\mu\nu}$$

$$= \frac{1}{16\pi G} \int \mathrm{sgn}(\det(e)) \, \varepsilon_{IJK} \, e^I \wedge F^{JK} .$$

Q10: Let us start by varying e

$$0 = \delta S(e,\omega) = \frac{1}{16\pi G} \int \varepsilon_{IJK} \, \delta e^I \wedge F^{JK}$$

$$= \frac{1}{16\pi G} \int \varepsilon_{IJK} \, \delta e^I{}_\mu \tfrac{1}{2} F^{JK}{}_{\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \frac{1}{16\pi G} \int d^3x \, \varepsilon_{IJK} \, \varepsilon^{\mu\nu\rho} \tfrac{1}{2} F^{JK}{}_{\nu\rho} \, \delta e^I{}_\mu .$$

We conclude

$$0 = \varepsilon_{IJK} \, \varepsilon^{\mu\nu\rho} F^{JK}{}_{\nu\rho} .$$

Of course, this then implies

$$0 = \varepsilon^{IMN} \varepsilon_{\mu\alpha\beta} \, \varepsilon_{IJK} \, \varepsilon^{\mu\nu\rho} F^{JK}{}_{\nu\rho}$$

$$= (\delta^M{}_J \delta^N{}_K - \delta^M{}_K \delta^N{}_J)(\delta^\nu{}_\alpha \delta^\rho{}_\beta - \delta^\rho{}_\alpha \delta^\nu{}_\beta) F^{JK}{}_{\nu\rho}$$

$$= (\delta^M{}_J \delta^N{}_K - \delta^M{}_K \delta^N{}_J)(F^{JK}{}_{\alpha\beta} - F^{JK}{}_{\beta\alpha})$$

$$= 2(\delta^M{}_J \delta^N{}_K - \delta^M{}_K \delta^N{}_J) F^{JK}{}_{\alpha\beta}$$

$$= 2(F^{MN}{}_{\alpha\beta} - F^{NM}{}_{\alpha\beta}) = 4F^{MN}{}_{\alpha\beta},$$

i.e. $F = 0$. Varying ω yields

$$0 = \delta S(e, \omega) = \frac{1}{32\pi G} \int \varepsilon_{IJK}\, e^I \wedge \delta F^{JK}.$$

Now,

$$\delta F^{IJ} = d\,\delta\omega^{IJ} + \delta\omega^I{}_K \wedge \omega^{KJ} + \omega^I{}_K \wedge \delta\omega^{KJ}.$$

Noting that

$$\delta\omega^I{}_K \wedge \omega^{KJ} = -\delta\omega^I{}_K \wedge \omega^{JK} = -\delta\omega^{IK} \wedge \omega^J{}_K$$

$$= \delta\omega^{KI} \wedge \omega^J{}_K = -\omega^J{}_K \wedge \delta\omega^{KI},$$

we have

$$\delta F^{IJ} = d\,\delta\omega^{IJ} - \omega^J{}_K \wedge \delta\omega^{KI} + \omega^I{}_K \wedge \delta\omega^{KJ}.$$

Now,

$$\varepsilon_{IJK}\, e^I \wedge d\,\delta\omega^{JK} = -d(\varepsilon_{IJK}\, e^I \wedge \delta\omega^{JK}) + \varepsilon_{IJK}\, de^I \wedge \delta\omega^{JK}$$

$$-\varepsilon_{IJK}\, e^I \wedge \omega^K{}_L \wedge \delta\omega^{LJ} = -\varepsilon_{IKL}\, e^I \wedge \omega^L{}_J \wedge \delta\omega^{JK}$$

$$\varepsilon_{IJK}\, e^I \wedge \omega^J{}_L \wedge \delta\omega^{LK} = \varepsilon_{ILK}\, e^I \wedge \omega^L{}_J \wedge \delta\omega^{JK},$$

Therefore

$$\delta S(e,\omega) = -\int_{\partial M} \varepsilon_{IJK}\, e^I \wedge \delta\omega^{JK}$$

$$+ \int_M \left(\varepsilon_{IJK}\, de^I - \varepsilon_{IKL}\, e^I \wedge \omega^L{}_J + \varepsilon_{ILK}\, e^I \wedge \omega^L{}_J\right) \wedge \delta\omega^{JK}$$

$$= -\int_{\partial M} \varepsilon_{IJK}\, e^I \wedge \delta\omega^{JK}$$

$$+ \int_M \left(\varepsilon_{IJK}\, de^I + 2\varepsilon_{ILK}\, e^I \wedge \omega^L{}_J\right) \wedge \delta\omega^{JK}.$$

Now, assume $\delta\omega = 0$ on ∂M. Then

$$0 = \delta S(e,\omega) = \int_M \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + 2\varepsilon_{ILK}\, e^I{}_\mu \omega_\nu{}^L{}_J\right) \delta\omega_\rho{}^{JK}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \int_M d^3x\, \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + 2\varepsilon_{ILK}\, e^I{}_\mu \omega_\nu{}^L{}_J\right) \delta\omega_\rho{}^{JK}$$

$$= \int_M d^3x\, \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + \varepsilon_{IL[K|}\, e^I{}_\mu \omega_\nu{}^L{}_{|J]}\right) \delta\omega_\rho{}^{JK}.$$

We conclude

$$0 = \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + \varepsilon_{IL[K|}\, e^I{}_\mu \omega_\nu{}^L{}_{|J]}\right)$$

Therefore,

$$0 = \varepsilon^{MJK} \varepsilon_{\alpha\beta\rho} \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + \varepsilon_{IL[K|}\, e^I{}_\mu \omega_\nu{}^L{}_{|J]}\right)$$

$$= \varepsilon_{\alpha\beta\rho} \varepsilon^{\mu\nu\rho} \left(2\partial_\mu e^M{}_\nu + 2\varepsilon^{MJK} \varepsilon_{ILK}\, e^I{}_\mu \omega_\nu{}^L{}_J\right)$$

$$= 2\varepsilon_{\alpha\beta\rho}\varepsilon^{\mu\nu\rho}\left(\partial_\mu e^M{}_\nu + e^M{}_\mu \omega_\nu{}^J{}_J - e^J{}_\mu \omega_\nu{}^M{}_J\right)$$

$$= 2\left(\delta^M{}_\alpha \delta^\nu{}_\beta - \delta^M{}_\beta \delta^\nu{}_\alpha\right)\left(\partial_\mu e^M{}_\nu + \underbrace{\cancel{e^M{}_\mu \eta_{JK}\omega_\nu{}^{JK}}}_{0} - e^J{}_\mu \omega_\nu{}^M{}_J\right)$$

$$= 2\left(\partial_\mu e^M{}_\nu - \partial_\nu e^M{}_\mu - \omega_\nu{}^M{}_J e^J{}_\mu + \omega_\mu{}^M{}_J e^J{}_\nu\right).$$

$$= 2T^M{}_{\mu\nu}.$$

Indeed

$$T^M = de^M + \omega^M{}_J \wedge e^J = \partial_\mu e^M{}_\nu \, dx^\mu \wedge dx^\nu + \omega_\mu{}^M{}_J e^J{}_\nu \, dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2}\left(\partial_\mu e^M{}_\nu - \partial_\nu e^M{}_\mu + \omega_\mu{}^M{}_J e^J{}_\nu - \omega_\nu{}^M{}_J e^J{}_\mu\right) dx^\mu \wedge dx^\nu.$$

We conclude the second EOM is $T = 0$.

1.4. Global Symmetries of Euclidean 3-space

Q11: From the action

$$(R_2, a_2)(R_1, a_1)x = (R_2, a_2)(R_1 x + a_1) = R_2(R_1 x + a_1) + a_2$$

$$= R_2 R_1 x + R_2 a_1 + a_2 = (R_2 R_1, R_2 a_1 + a_2)x,$$

it is clear that the product structure on $ISO(3)$ is

$$(R_2, a_2)(R_1, a_1) = (R_2 R_1, R_2 a_1 + a_2).$$

From this it is clear that the identity element is $(I_3, 0)$ and the inverse of (R, a) is $(R^{-1}, -R^{-1}a)$.

Indeed

$$(I_3, 0)(R, a) = (R, I_3 a + 0) = (R, a)$$

and

$$(R^{-1}, -R^{-1}a)(R, a) = (\cancel{R^{-1}R}, \cancel{R^{-1}}a - \cancel{R^{-1}}a) = (I_3, 0).$$

Let us now study the Lie algebra $so(3)$. Assume we have a rotation $R = I_3 + \omega$ with ω infinitesimal.

Then

$$I_3 = R_3^T R_3 = (I_3 + \omega^T)(I_3 + \omega) = I_3 + \omega + \omega^T + \mathcal{O}(\omega^2).$$

Thus

$$\omega = -\omega^T.$$

Defining $(M_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}$, we have that $\{M_{12}, M_{23}, M_{31}\}$ is a basis for $so(3)$.

To study the full Lie algebra $iso(3)$, it is useful to recognize $ISO(3)$ as a matrix Lie group via the identification

$$\begin{aligned} ISO(3) &\longrightarrow M_4(\mathbb{R}) \\ (R,a) &\longmapsto \left[\begin{array}{c|c} R & a \\ \hline 0 & 1 \end{array}\right]. \end{aligned}$$

This is clearly injective. Moreover, it is a homomorphism since

$$\left[\begin{array}{c|c} R_2 & a_2 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_1 & a_1 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R_2 R_1 & R_2 a_1 + a_2 \\ \hline 0 & 1 \end{array}\right].$$

Thus, it's image is isomorphic to $ISO(3)$. We conclude that the Lie algebra is

$$iso(3) = \left\{ \left[\begin{array}{c|c} \omega & a \\ \hline 0 & 0 \end{array}\right] \middle| \ \omega = -\omega^T, \ a \in \mathbb{R}^3 \right\} \subseteq M_4(\mathbb{R}).$$

A basis for this is given by $\{M_{12}, M_{23}, M_{31}, e_1, e_2, e_3\}$ where $(M_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ and $(P_i)_{kl} = \delta_{4l}\delta_{ik}$.

Note that

$$\begin{aligned} (M_{ij} M_{kl})_{mn} &= (M_{ij})_{mr}(M_{kl})_{rn} = (\delta_{im}\delta_{jr} - \delta_{ir}\delta_{jm})(\delta_{kr}\delta_{ln} - \delta_{kn}\delta_{lr}) \\ &= \delta_{jk}\delta_{im}\delta_{ln} - \delta_{jl}\delta_{im}\delta_{kn} - \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn}, \end{aligned}$$

so that

$$[M_{ij}, M_{kl}]_{mn} = \delta_{jk}\delta_{im}\delta_{ln} - \delta_{jl}\delta_{im}\delta_{kn} - \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn}$$

$$- \delta_{li}\delta_{km}\delta_{jn} + \delta_{lj}\delta_{km}\delta_{in} + \delta_{ki}\delta_{lm}\delta_{jn} - \delta_{kj}\delta_{lm}\delta_{in}$$

$$= \delta_{jk}(\delta_{im}\delta_{ln} - \delta_{lm}\delta_{in}) - \delta_{jl}(\delta_{im}\delta_{kn} - \delta_{km}\delta_{in})$$

$$- \delta_{ik}(\delta_{jm}\delta_{ln} - \delta_{lm}\delta_{jn}) + \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn})$$

$$= \delta_{jk}(M_{il})_{mn} - \delta_{jl}(M_{ik})_{mn} - \delta_{ik}(M_{jl})_{mn} + \delta_{il}(M_{jk})_{mn}.$$

On the other hand, for all $i,j \in \{1,2,3\}$

$$(P_i P_j)_{mn} = (P_i)_{mr}(P_j)_{rn} = \delta_{im}\delta_{4r}\delta_{jr}\delta_{4n} = \delta_{im}\cancelto{0}{\delta_{4j}}\delta_{4n} = 0.$$

Finally, taking $i,j,k \in \{1,2,3\}$

$$(M_{ij}P_k)_{mn} = (M_{ij})_{mr}(P_k)_{rn} = (\delta_{im}\delta_{jr} - \delta_{ir}\delta_{jm})\delta_{kr}\delta_{4n}$$

$$= \delta_{im}\delta_{jk}\delta_{4n} - \delta_{ik}\delta_{jm}\delta_{4n} = \delta_{jk}(P_i)_{mn} - \delta_{ik}(P_j)_{mn}$$

and

$$(P_k M_{ij})_{mn} = (P_k)_{mr}(M_{ij})_{rn} = \delta_{km}\delta_{4r}(\delta_{ir}\delta_{jn} - \delta_{in}\delta_{jr})$$

$$= \delta_{km}\cancelto{0}{\delta_{4i}}\delta_{jn} - \delta_{km}\delta_{in}\cancelto{0}{\delta_{j4}} = 0,$$

We conclude the Lie algebra is

$$[M_{ij}, M_{kl}] = \delta_{jk} M_{il} + \delta_{il} M_{jk} - \delta_{jl} M_{ik} - \delta_{ik} M_{jl}$$

$$[M_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j$$

$$[P_i, P_j] = 0$$

For $\{M_{12}, M_{23}, M_{31}, P_1, P_2, P_3\}$. For future reference, define $J_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$. Then

$$[J_i, J_j] = \frac{1}{4}\varepsilon_{imn}\varepsilon_{jrs}[M_{mn}, M_{rs}]$$

$$= \frac{1}{4}\varepsilon_{imn}\varepsilon_{jrs}\left(\delta_{nr} M_{ms} + \delta_{ms} M_{nr} - \delta_{ns} M_{mr} - \delta_{mr} M_{ns}\right)$$

$$= \frac{1}{4}\left(\varepsilon_{imr}\varepsilon_{jrs} M_{ms} + \varepsilon_{isn}\varepsilon_{jrs} M_{nr} - \varepsilon_{ims}\varepsilon_{jrs} M_{mr} - \varepsilon_{irn}\varepsilon_{jrs} M_{ns}\right)$$

$$= \frac{1}{4}\left(-\delta_{ij}\cancelto{0}{M_{mm}} + M_{ji} - \delta_{ij}\cancelto{0}{M_{nn}} + M_{ji} - \delta_{ij}\cancelto{0}{M_{mm}} + M_{ji} - \delta_{ij}\cancelto{0}{M_{nn}} + M_{ji}\right)$$

$$= M_{ji} = \varepsilon_{jik} J_k .$$

Redefining instead $J_i = -\frac{1}{2}\varepsilon_{ijk} M_{jk}$, we have

$$[J_i, J_j] = \varepsilon_{ijk} J_k .$$

In terms of these we also have

$$[J_i, P_j] = -\frac{1}{2}\varepsilon_{imn}[M_{mn}, P_j] = -\frac{1}{2}\varepsilon_{imn}(\delta_{nj}P_m - \delta_{mj}P_n)$$

$$= -\frac{1}{2}(\varepsilon_{imj}P_m - \varepsilon_{ijn}P_n) = \varepsilon_{ijn}P_n .$$

We just finally need to check that indeed

$M_{ij} = -\varepsilon_{ijk}P_k$. Indeed

$$-\varepsilon_{ijk}P_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}M_{lm} = \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})M_{lm}$$

$$= \frac{1}{2}(M_{ij} - M_{ji}) = M_{ij} .$$

Optional Exercise: We have that

$$f_{ab}{}^{c} f_{cd}{}^{e} = f_{ab}{}^{c}[t_c^{ad}]^e{}_d = [[t_a^{ad}],[t_b^{ad}]]^e{}_d$$

$$= [t_a^{ad}]^e{}_c[t_b^{ad}]^c{}_d - [t_b^{ad}]^e{}_c[t_a^{ad}]^c{}_d$$

$$= f_{ac}{}^{e} f_{bd}{}^{c} - f_{bc}{}^{e} f_{ad}{}^{c} ,$$

i.e.

$$f_{ab}{}^{c} f_{cd}{}^{e} + f_{bd}{}^{c} f_{ca}{}^{e} + f_{da}{}^{c} f_{cb}{}^{e} = 0 .$$

This coincides with the Jacobi identity. Indeed

$$[[t_a, t_b], t_d] = f_{ab}{}^c [t_c, t_d] = f_{ab}{}^c f_{cd}{}^e\, t_e,$$

and

$$0 = [[t_a, t_b], t_d] + [[t_b, t_d], t_a] + [[t_d, t_a], t_b]$$

$$= f_{ab}{}^c f_{cd}{}^e + f_{bd}{}^c f_{ca}{}^e + f_{da}{}^c f_{cb}{}^e .$$

Q 12: We have

$$[J_1^{ad}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad [J_2^{ad}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$[J_3^{ad}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and

$$[[J_1^{ad}], [J_2^{ad}]] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [J_3^{ad}],$$

$$[[J_2^{ad}], [J_3^{ad}]] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = [J_1^{ad}],$$

$$[[J_3^{ad}], [J_1^{ad}]] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = [J_2^{ad}].$$

Q 13: We indeed have

$$F^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J = d\omega^M [J^{ad}_M]^I{}_J + \omega^M \wedge \omega^N [J^{ad}_M]^I{}_K [J^{ad}_N]^K{}_J$$

$$= d\omega^M [J^{ad}_M]^I{}_J + \omega^M \wedge \omega^N [J^{ad}_M J^{ad}_N]^I{}_J$$

$$= d\omega^M [J^{ad}_M]^I{}_J + \frac{1}{2}\omega^M \wedge \omega^N [[J^{ad}_M, J^{ad}_N]]^I{}_J$$

$$= d\omega^M [J^{ad}{}_M]^I{}_J + \frac{1}{2}\varepsilon_{KMN}\,\omega^M \wedge \omega^N [J^{ad}_K]^I{}_J$$

$$= \left(d\omega^K + \frac{1}{2}\varepsilon_{KMN}\,\omega^M \wedge \omega^N\right)[J^{ad}_K]^I{}_J .$$

Q14: Let us check that the pairing is indeed invariant.

We have

- $\langle [J_I, J_J], J_K\rangle + \langle J_J, [J_I, J_K]\rangle = \varepsilon_{IJL}\overset{0}{\cancel{\langle J_L, J_K\rangle}} + \varepsilon_{IKL}\overset{0}{\cancel{\langle J_J, J_L\rangle}}$

$$= 0,$$

- $\langle [P_I, J_J], J_K\rangle + \langle J_J, [P_I, J_K]\rangle = -\varepsilon_{IJL}\,\delta_{LK} - \varepsilon_{IKL}\,\delta_{JL}$

$$= -\varepsilon_{IJK} - \varepsilon_{IKJ} = 0$$

- $\langle [J_I, J_J], P_K\rangle + \langle J_J, [J_I, P_K]\rangle = \varepsilon_{IJL}\,\delta_{LK} + \varepsilon_{IKL}\,\delta_{JL}$

$$= \varepsilon_{IJK} + \varepsilon_{IKJ} = 0$$

- $\langle [P_I, J_J], P_K\rangle + \langle J_J, \overset{0}{\cancel{[P_I, P_K]}}\rangle = -\varepsilon_{IJL}\overset{0}{\cancel{\langle P_L, P_K\rangle}} = 0$

- $\langle [J_I, P_J], P_K\rangle + \langle P_J, [J_I, P_K]\rangle = \varepsilon_{IJL}\overset{0}{\cancel{\langle P_L, P_K\rangle}} + \varepsilon_{IKL}\overset{0}{\cancel{\langle P_J, P_L\rangle}} = 0$

- $\langle [P_I, P_J], P_K \rangle + \langle P_J, [P_I, P_K] \rangle = 0$.

Q15: We clearly have

$$f_{abc} = d_{dc} f_{ab}{}^d = -d_{dc} f_{ba}{}^d = -f_{bac}.$$

Thus, we only need to check $f_{abc} = -f_{acb}$. Indeed

$$f_{abc} = d_{dc} f_{ab}{}^d = \langle t_d, t_c \rangle f_{ab}{}^d = \langle [t_a, t_b], t_c \rangle$$

$$= -\langle t_b, [t_a, t_c] \rangle = -f_{ac}{}^d \langle t_b, t_d \rangle = -d_{db} f_{ac}{}^d = -f_{acb}.$$

Q16: For the moment, let us denote $(J_1, J_2, J_3, P_1, P_2, P_3)_{a=1}^6$ with $(J_1, J_2, J_3)_{I=1}^3$ and $(P_1, P_2, P_3)_{\tilde{I}=1}^3$. With these indices we have the non-vanishing structure constants

$$f_{IJ}{}^K = \varepsilon_{IJK}, \quad f_{I\tilde{J}}{}^{\tilde{K}} = \varepsilon_{I\tilde{J}\tilde{K}}.$$

Noticing that

$$f_{abI} = d_{dI} f_{ab}{}^d = f_{ab}{}^{\tilde{I}}, \quad f_{ab\tilde{I}} = d_{d\tilde{I}} f_{ab}{}^d = f_{ab}{}^I,$$

we have the non-vanishing constants

$$f_{I\tilde{J}K} = \varepsilon_{I\tilde{J}\tilde{K}}, \quad f_{IJ\tilde{K}} = f_{IJ}{}^K = \varepsilon_{IJK},$$

which are of course related by symmetry

We thus have

$$\frac{1}{3} f_{abc} A^a \wedge A^b \wedge A^c = \frac{3}{3} \varepsilon_{IJK} \, \omega^I \wedge e^J \wedge \omega^K = \varepsilon_{IJK} \, e^I \wedge \omega^J \wedge \omega^K .$$

Moreover

$$d_{ab} A^a \wedge dA^b = \omega^I \wedge de^I + e^I \wedge d\omega^I = -d(\omega^I \wedge e^I) + d\omega^I \wedge e^I + e^I \wedge d\omega^I$$

$$= -d(\omega^I \wedge e^I) + 2 e^I \wedge d\omega^I .$$

Therefore,

$$S_{cs}(A) = \frac{K}{4\pi} \left(-\int_{\partial M} \omega^I \wedge e^I + 2 \int_M e^I \wedge \left(d\omega^I + \frac{1}{2} \varepsilon_{IJK} \omega^J \wedge \omega^K \right) \right) .$$

Thus, up to boundary terms,

$$S_{cs}(A) = \frac{K}{2\pi} \int_M e^I \wedge F^I = \frac{1}{8\pi G} \int_M e^I \wedge F^I .$$

Notice that boundary terms do not affect the EOMs because to obtain the latter we consider variations that vanish at the boundary. This is possible since we are in a first order formalism.

Q17: We have, ignoring boundary terms,

$$\delta S_{cs}(A) = \frac{k}{4\pi}\int_M \Big(d_{ab}\,\delta A^a \wedge dA^b + d_{ab}A^a \wedge d\delta A^b + \tfrac{1}{3} f_{abc}\,\delta A^a \wedge A^b \wedge A^c$$

$$+ \tfrac{1}{3} f_{abc}\,A^a \wedge \delta A^b \wedge A^c + \tfrac{1}{3} f_{abc}\,A^a \wedge A^b \wedge \delta A^c\Big)$$

$$= \frac{k}{4\pi}\int_M \Big(d_{ab}\,\delta A^a \wedge dA^b - d_{ab}\,d(A^a \wedge \delta A^b) + d_{ab}\,dA^a \wedge \delta A^b$$

$$+ f_{abc}\,A^a \wedge A^b \wedge \delta A^c\Big)$$

$$= \frac{k}{4\pi}\int_M \Big(2 d_{ab}\,\delta A^a \wedge dA^b + d_{ab} f_{cd}{}^{b} A^c \wedge A^d \wedge \delta A^a\Big)$$

$$= \frac{k}{4\pi} 2 d_{ab} \int_M \Big(dA^b + \tfrac{1}{2} f_{cd}{}^{b} A^c \wedge A^d\Big) \wedge \delta A^a$$

$$= \frac{k}{2\pi} \Big\langle \int_M \Big(dA + \tfrac{1}{2}[t_c, t_d]\, A^c \wedge A^d\Big) \wedge \delta A^a ,\ t_a \Big\rangle$$

$$= \frac{k}{2\pi} \Big\langle \int_M (dA + A \wedge A) \wedge \delta A^a ,\ t_a \Big\rangle .$$

Now, eventhough our pairing is not an inner product, it wasn't degenerate. Therefore

$$0 = \int_M (dA + A \wedge A) \wedge \delta A^a = \int_M \mathfrak{F}(A) \wedge \delta A^a$$

This implies $\mathfrak{F}(A) = 0$. To see this,

$$0 = \int_M \mathfrak{F}(A)_{\mu\nu} \delta A^a{}_\rho \, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \int_M d^3x \; \varepsilon^{\mu\nu\rho} \mathfrak{F}(A)_{\mu\nu} \, \delta A_\rho$$

implies $\quad \varepsilon^{\mu\nu\rho} \mathfrak{F}(A)_{\mu\nu} = 0. \quad$ Then

$$0 = \varepsilon_{\alpha\beta\rho} \varepsilon^{\mu\nu\rho} \mathfrak{F}(A)_{\mu\nu} = (\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha) \mathfrak{F}(A)_{\mu\nu}$$

$$= \mathfrak{F}(A)_{\alpha\beta} - \mathfrak{F}(A)_{\beta\alpha} = 2\, \mathfrak{F}(A)_{\alpha\beta}.$$

Furthermore [illegible]

$$\mathfrak{F}(A) = \mathfrak{F}^a(A) t_a = \left(dA^I + \tfrac{1}{2} f_{ab}{}^I A^a \wedge A^b\right) t_I$$

$$+ \left(dA^{\tilde{I}} + \tfrac{1}{2} f_{ab}{}^{\tilde{I}} A^a \wedge A^b\right) t_{\tilde{I}}$$

$$= \left(d\omega^I + \tfrac{1}{2} \varepsilon_{JKI} \, \omega^J \wedge \omega^K\right) J_I + \left(de^I + \tfrac{1}{2} \varepsilon_{JKI} \, \omega^J \wedge e^K\right.$$

$$= F \; [illegible] \; + \tfrac{1}{2} \varepsilon_{JKI} \, e^J \wedge \omega^K \big) P_I$$

$$= F + \left(de^I + \varepsilon_{JKI} \, \omega^J \wedge e^K\right) P_I = F + \left(de^I - \omega^{KI} \wedge e^K\right) P_I$$

$$= F + \left(de^I + \omega^I{}_K \wedge e^K\right) P_I = F + T.$$

Thus, $\quad \mathfrak{F}(A) = 0 \quad$ implies the EOMs $\; F = 0 \;$ and $\; T = 0.$

2. Euclidean Signature, Negative Cosmological Constant

2.1. Preliminaries

Q18. Recalling that

$$\sqrt{|\det g|} = |\det(e)|,$$

and the solution to Q8, we have

$$S(g) = \frac{1}{16\pi G} \int_M d^3x \sqrt{|\det g|} \, (R - 2\Lambda)$$

$$= \frac{1}{16\pi G} \int_M d^3x \sqrt{|\det g|} \, R - \frac{1}{16\pi G} \int_M d^3x \sqrt{|\det g|} \, 2\Lambda$$

$$= \frac{1}{16\pi G} \int_M d^3x \, |\det(e)| \, e_I{}^{\mu} e_J{}^{\nu} F^{IJ}{}_{\mu\nu} - \frac{1}{16\pi G} \int_M d^3x \, |\det(e)| 2\Lambda$$

$$= \frac{1}{16\pi G} \int_M d^3x \, |\det(e)| \left(e_I{}^{\mu} e_J{}^{\nu} F^{IJ}{}_{\mu\nu} - 2\Lambda \right)$$

$$= S(e).$$

Also note that

$$\mathrm{sgn}(\det(e)) \, \varepsilon_{IJK} \frac{\Lambda}{3} e^I \wedge e^J \wedge e^K$$

$$= \mathrm{sgn}(\det(e))\, \varepsilon_{IJK} \frac{\Lambda}{3} e^I{}_\mu e^J{}_\nu e^K{}_\sigma \, dx^\mu \wedge dx^\nu \wedge dx^\sigma$$

$$= \mathrm{sgn}(\det(e))\, \varepsilon_{IJK}\, e^I{}_\mu e^J{}_\nu e^K{}_\sigma \frac{\Lambda}{3} \varepsilon^{\mu\nu\sigma} d^3x$$

$$= \mathrm{sgn}(\det(e))\, \varepsilon_{\mu\nu\sigma} \det(e) \frac{\Lambda}{3} \varepsilon^{\mu\nu\sigma} d^3x$$

$$= |\det(e)| \frac{\Lambda}{3} 3!\, d^3x = d^3x\, |\det(e)|\, 2\Lambda .$$

Thus, from the result of Q9, we have

$$S(e) = \frac{1}{16\pi G} \int_M \mathrm{sgn}(\det(e))\, \varepsilon_{IJK} \left(e^I \wedge F^{JK} - \frac{\Lambda}{3} e^I \wedge e^J \wedge e^K \right)$$

Q19. Recalling the results of Q10, we have that the new term $\frac{\Lambda}{3} e^I \wedge e^J \wedge e^K$ is independent of ω. Thus, the variation with respect to ω is left invariant and we obtain the EOM

$$T = 0.$$

On the other hand, varying e we have

$$\delta \int \varepsilon_{IJK} \frac{\Lambda}{3} e^I \wedge e^J \wedge e^K = \int \varepsilon_{IJK} \frac{\Lambda}{3} \left(\delta e^I \wedge e^J \wedge e^K + e^I \wedge \delta e^J \wedge e^K + e^I \wedge e^J \wedge \delta e^K \right)$$

$$= \int \Lambda\, \varepsilon_{IJK}\, \delta e^I \wedge e^J \wedge e^K = \int d^3x\, \varepsilon_{IJK}\, \varepsilon^{\mu\nu\rho}\, e^J{}_\nu e^K{}_\rho\, \delta e^I{}_\mu .$$

Thus

$$\delta S(e,\omega) = \frac{1}{16\pi G} \int d^3x\, \varepsilon_{IJK}\, \varepsilon^{\mu\nu\rho} \left(\tfrac{1}{2} F^{JK}{}_{\nu\rho} - e^J{}_{[\nu} e^K{}_{\rho]} \right) \delta e^I{}_\mu .$$

We can eliminate the ε factors precisely as before to obtain the EOMs

$$F^{IJ} - \Lambda e^I \wedge e^J = \tfrac{1}{2} F^{JK}{}_{\nu\rho}\, dx^\nu \wedge dx^\rho - \Lambda e^I{}_{[\nu} e^J{}_{\rho]}\, dx^\nu \wedge dx^\rho$$

$$= \left(\tfrac{1}{2} F^{JK}{}_{\nu\rho} - \Lambda e^I{}_{[\nu} e^J{}_{\rho]} \right) dx^\nu \wedge dx^\rho = 0,$$

i.e.

$$F = \Lambda\, e \wedge e .$$

Using our result from Q7, we have

$$e^I{}_\rho e^{J\sigma} R^\rho{}_{\sigma\mu\nu} = F^{IJ}{}_{\mu\nu} = 2\Lambda e^I{}_{[\mu} e^J{}_{\nu]}$$

$$= 2\Lambda\, \delta^\rho{}_{[\mu} g_{\nu]\sigma}\, e^I{}_\rho\, e^{J\sigma}$$

We conclude

$$R^{\rho}{}_{\sigma\mu\nu} = 2\Lambda g_{\sigma[\nu}\delta^{\rho}{}_{\mu]}.$$

We conclude that for the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = 2\Lambda g_{\mu[\nu}\delta^{\alpha}{}_{\alpha]} = 2\Lambda \tfrac{1}{2}\left(3g_{\mu\nu} - g_{\mu\alpha}\delta^{\alpha}{}_{\nu}\right)$$

$$= 2\Lambda \tfrac{1}{\cancel{2}}\cancel{2} g_{\mu\nu} = 2\Lambda g_{\mu\nu},$$

i.e., we obtain Einstein's equations in 3D with a cosmological constant.

2.2. Global Symmetries of 3d Einstein Manifolds

Q20. Consider

$$N = \{(x^0, x^1, x^2, x^3) \in M^4 \mid (x^0)^2 - (x^1)^2 - (x^3)^2 - (x^4)^2 = \alpha\}$$

Being the level set of a smooth function, N is a regular submanifold of M^4 of dimension $4-1=3$. Moreover, this function is invariant under the action of $SO(1,3)$ on M^4 since it is simply

$$x \longmapsto g(x,x) - \alpha.$$

We must have

that for all $x \in N$

$$0 < (x^1)^2 + (x^2)^2 + (x^3)^2 = (x^0)^2 - \alpha .$$

This form inspires the use of the coordinates

$y = (x^0, \theta, \phi)$ so that

$$x^1 = \sqrt{(x^0)^2 - \alpha}\ \sin(\theta)\cos(\phi),$$

$$x^2 = \sqrt{(x^0)^2 - \alpha}\ \sin(\theta)\sin(\phi),$$

$$x^3 = \sqrt{(x^0)^2 - \alpha}\ \cos(\theta).$$

The metric on N is given by $\gamma = i^* g$, with $i : N \longrightarrow M^4$, i.e.

$$\gamma_{ab} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b} g_{\mu\nu} = \begin{bmatrix} \frac{\alpha}{(x^0)^2 - \alpha} & 0 & 0 \\ 0 & (x^0)^2 - \alpha & 0 \\ 0 & 0 & (x^0)^2 - \alpha \end{bmatrix},$$

as shown in the mathematica file attached

we thus have $(+,+,+)$, corresponding to H^3, for all $\alpha > 0$. On the other hand, for

$\alpha < 0$, we have $(-,+,+)$, corresponding to dS_3.

2.3. The Chern-Simons Formolation of Euclideon Gravity with Negative Cosmological Constant

Q21: We have

- $\langle [J_I, J_J], J_K \rangle + \langle J_J, [J_I, J_K] \rangle = \varepsilon_{IJL} \cancelto{0}{\langle J_L, J_K \rangle} + \varepsilon_{IKL} \cancelto{0}{\langle J_J, J_L \rangle} = 0,$

- $\langle [K_I, J_J], J_K \rangle + \langle J_J, [K_I, J_K] \rangle = \varepsilon_{IJL} \langle K_L, J_K \rangle + \varepsilon_{IKL} \langle J_J, K_L \rangle$

  $= \varepsilon_{IJK} + \varepsilon_{IKJ} = 0,$

- $\langle [J_I, J_J], K_K \rangle + \langle J_J, [J_I, K_K] \rangle = \varepsilon_{IJL} \langle J_L, K_K \rangle + \varepsilon_{IKL} \langle J_J, K_K \rangle$

  $= \varepsilon_{IJK} + \varepsilon_{IKJ} = 0,$

- $\langle [K_I, J_J], K_K \rangle + \langle J_J, [K_I, K_K] \rangle = \varepsilon_{IJL} \cancelto{0}{\langle K_L, K_K \rangle} + \Lambda \varepsilon_{IKL} \cancelto{0}{\langle J_J, J_L \rangle} = 0,$

- $\langle [J_I, K_J], K_K \rangle + \langle K_J, [J_I, K_K] \rangle = \varepsilon_{IJL} \cancelto{0}{\langle K_L, K_K \rangle} + \varepsilon_{IKL} \cancelto{0}{\langle K_J, K_L \rangle} = 0,$

- $\langle [K_I, K_J], K_K \rangle + \langle K_J, [K_I, K_K] \rangle = \Lambda \varepsilon_{IJL} \langle J_L, K_K \rangle + \Lambda \varepsilon_{IKL} \langle K_J, J_L \rangle$

  $= \Lambda \varepsilon_{IJK} + \Lambda \varepsilon_{IKJ} = 0.$

Q22: We already did this in Q15.

Q 23:

Since the term $d_{ab} A^a \wedge A^b$ only depends on d_{ab}, which gets mapped to that of Q16 by $K_I \mapsto P_I$, we already have

$$d_{ab} A^a \wedge A^b = -d(\omega^I \wedge e^I) + 2 e^I \wedge d\omega^I .$$

The commutation relatios <u>do</u> change though. However, separating our indices the way we did before, we have the non-vanishing structure constants

$$f_{IJ}{}^{K} = \varepsilon_{IJK} , \quad f_{I\tilde{J}}{}^{\tilde{K}} = \varepsilon_{I\tilde{J}\tilde{K}} , \quad f_{\tilde{I}\tilde{J}}{}^{K} = \Lambda \varepsilon_{\tilde{I}\tilde{J}K} ,$$

along with the ones obtained by their symmetries. As before, we have

$$f_{abI} = d_{dI} f_{ab}{}^{d} = f_{ab}{}^{\tilde{I}} ,$$

$$f_{ab\tilde{I}} = d_{d\tilde{I}} f_{ab}{}^{\tilde{I}} = f_{ab}{}^{I} ,$$

so that the non-vanishing structure constants are

$$f_{IJ\tilde{K}} = \varepsilon_{IJK} , \quad f_{I\tilde{J}K} = \varepsilon_{I\tilde{J}\tilde{K}} , \quad f_{\tilde{I}\tilde{J}\tilde{K}} = \Lambda \varepsilon_{\tilde{I}\tilde{J}K} ,$$

along with their antisymmetrizations. We conclude

$$\frac{1}{3} f_{abc} A^a \wedge A^b \wedge A^c = \frac{1}{3}\Big(f_{\tilde{I}JK} A^{\tilde{I}} \wedge A^J \wedge A^K + f_{I\tilde{J}K} A^I \wedge A^{\tilde{J}} \wedge A^K$$

$$+ f_{IJ\tilde{K}} A^I \wedge A^J \wedge A^{\tilde{K}} + f_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{I}} \wedge A^{\tilde{J}} \wedge A^{\tilde{K}} \Big)$$

$$= f_{\tilde{I}JK} A^{\tilde{I}} \wedge A^J \wedge A^K + \frac{1}{3} f_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{I}} \wedge A^{\tilde{J}} \wedge A^{\tilde{K}}$$

$$= \varepsilon_{IJK}\, e^I \wedge \omega^J \wedge \omega^K + \frac{1}{3} \Lambda\, \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K .$$

The action, up to total derivatives, ends up taking the form

$$S_{cs} = \frac{2k}{4\pi} \int_M e^I \wedge \Big(d\omega^I + \frac{1}{2} \varepsilon_{IJK}\, \omega^J \wedge \omega^K + \frac{1}{6} \Lambda\, \varepsilon_{IJK}\, e^J \wedge e^K \Big)$$

$$= \frac{k}{2\pi} \int_M e^I \wedge \Big(F^I + \frac{1}{6} \Lambda\, \varepsilon_{IJK}\, e^J \wedge e^K \Big)$$

Comparing (24) and (37) we see

$$S_{cs}(A) = \frac{k}{4\pi} \int_M \varepsilon_{IJK}\, e^I \wedge \Big(F^{JK} + \frac{1}{3} \Lambda\, e^J \wedge e^K \Big)$$

$$= S(\omega, e)$$

if we set $k = \dfrac{1}{4G}$.

Q24. We already proved that the EOMs of Chern-Simons are $\mathcal{F}(A)=0$. We thus are only left with checking

$$\mathcal{F}(A) = \mathcal{F}(A)^a t_a = \left(dA^I + \tfrac{1}{2} f_{ab}{}^I A^a \wedge A^b\right) t_I + \left(dA^{\tilde{I}} + \tfrac{1}{2} f_{ab}{}^{\tilde{I}} A^a \wedge A^b\right) t_{\tilde{I}}$$

$$= \left(d\omega^I + \tfrac{1}{2}\varepsilon_{IJK}\,\omega^J \wedge \omega^K + \tfrac{1}{2}\Lambda\varepsilon_{IJK}\,e^J \wedge e^K\right) J_I$$

$$+ \left(de^I + \tfrac{1}{2}\varepsilon_{IJK}\,\omega^J \wedge e^K + \tfrac{1}{2}\varepsilon_{IJK}\,e^J \wedge \omega^K\right) K_I$$

$$= \left(F^I + \tfrac{1}{2}\Lambda\varepsilon_{IJK}\,e^J \wedge e^K\right) J_I + T.$$

The EOMs are then

$$T = 0$$

$$0 = \left(F^I + \tfrac{1}{2}\Lambda\varepsilon_{IJK}\,e^J \wedge e^K\right)_I,$$

as in Q19. Indeed

$$F^I J_I = F = \Lambda e \wedge e = \Lambda e^J \wedge e^K J_J J_K = \Lambda e^J \wedge e^K \tfrac{1}{2}[J_J, J_K]$$

$$= \Lambda e^J \wedge e^K \tfrac{1}{2}\varepsilon_{IJK} J_I .$$

Q25. We clearly get back the previous theory without cosmological constant. In particular

$$[K_I, K_J] \to 0,$$

i.e. $K_I = P_I$.

2.4. The Self-Dual / Anti-Self-Dual Formulation

Q26: We have

$$[J_I^\pm, J_J^\pm] = \frac{1}{4}\Big([J_I, J_J] \pm \frac{i}{\sqrt{-\Lambda}}[J_I, K_J] \pm \frac{i}{\sqrt{-\Lambda}}[K_I, J_J]$$

$$- \frac{1}{-\Lambda}[K_I, K_J] \Big)$$

$$= \frac{1}{4}\Big(\varepsilon_{IJK} J_K \pm \frac{2i}{\sqrt{-\Lambda}} \varepsilon_{IJK} K_K + \frac{1}{\cancel{\Lambda}} \cancel{\Lambda} \varepsilon_{IJK} J_K \Big)$$

$$= \varepsilon_{IJK} \frac{1}{2}\Big(J_K \pm \frac{i}{\sqrt{-\Lambda}} K_K \Big) = \varepsilon_{IJK} J_K^\pm ,$$

$$[J_I^+, J_J^-] = \frac{1}{4}\Big([J_I, J_J] - \frac{i}{\sqrt{-\Lambda}}[J_I, K_J] + \frac{i}{\sqrt{-\Lambda}}[K_I, J_J]$$

$$+ \frac{1}{-\Lambda}[K_I, K_J] \Big)$$

$$= \frac{1}{4}\Big(\cancel{\varepsilon_{IJK} J_K} - \cancel{\frac{i}{\sqrt{-\Lambda}} \varepsilon_{IJK} K_K} + \cancel{\frac{i}{\sqrt{-\Lambda}} \varepsilon_{IJK} K_K} - \frac{1}{\cancel{\Lambda}} \cancel{\Lambda} \cancel{\varepsilon_{IJK} K_K} \Big)$$

$$= 0.$$

Moreover, recalling

$$[\sigma_I, \sigma_J] = 2i\varepsilon_{IJK}\,\sigma_K ,$$

we have

$$[[J_I^{1/2}],[J_J^{1/2}]] = -\frac{1}{4}[\sigma_I, \sigma_J] = -\frac{1}{2}\,i\,\varepsilon_{IJK}\,\sigma_K$$

$$= \varepsilon_{IJK}\,[J_K^{1/2}] ,$$

$$[[K_I^{\pm 1/2}],[K_J^{\pm 1/2}]] = \frac{-\Lambda}{4}[\sigma_I, \sigma_J] = \frac{-\Lambda}{\cancel{4}\,2}\cancel{2}\,i\,\varepsilon_{IJK}\,\sigma_K$$

$$= \Lambda\,\varepsilon_{IJK}\,[J_K^{1/2}] ,$$

$$[[J_I^{1/2}],[K_J^{\pm 1/2}]] = \mp\frac{i}{2}\frac{\sqrt{-\Lambda}}{2}[\sigma_I, \sigma_J] = \cancel{\frac{i}{2}}\left(\pm\frac{\sqrt{-\Lambda}}{2}\right)\cancel{2i}\,\varepsilon_{IJK}\,\sigma_K$$

$$= \varepsilon_{IJK}\,[K_K^{\pm 1/2}] .$$

Finally, in this representation

$$\left[(J_I^{\pm_1})^{\pm_2 1/2}\right] = \frac{1}{2}\left(-\frac{i}{2}\sigma_I \pm_1 \frac{i}{\cancel{\sqrt{-\Lambda}}}\left(\pm_2\frac{\cancel{\sqrt{-\Lambda}}}{2}\sigma_I\right)\right)$$

$$= -\frac{i}{4}\sigma_I\left(1 \mp_1 (\pm_2)\right) .$$

In other words,

$$\left[(J_I^+)^{1/2} \right] = 0 \qquad \left[(J_I^-)^{1/2} \right] = -\frac{i}{2}\sigma_I$$

$$\left[(J_I^+)^{-1/2} \right] = -\frac{i}{2}\sigma_I \qquad \left[(J_I^-)^{-1/2} \right] = 0$$

Q27. In this theory we have $f_{IJ}{}^K = \varepsilon_{IJK}$, and

$f_{IJK} = \delta_{KL} f_{IJ}{}^L = f_{IJ}{}^K = \varepsilon_{IJK}$. Therefore

$$d_{IJ} A^I \wedge dA^J = \delta_{IJ} A^I \wedge dA^J = A^I \wedge dA^I$$

$$= \omega^I \wedge d\omega^I - i\sqrt{-\Lambda}\, \omega^I \wedge de^I - i\sqrt{-\Lambda}\, e^I \wedge d\omega^I + \Lambda\, e^I \wedge de^I,$$

whose imaginary part is

$$-\sqrt{-\Lambda}\left(\omega^I \wedge de^I + e^I \wedge d\omega^I \right) = -2\sqrt{-\Lambda}\, e^I \wedge d\omega^I,$$

up to boundary terms. On the other hand,

$$\frac{1}{3} f_{IJK} A^I \wedge A^J \wedge A^K = \frac{1}{3} \varepsilon_{IJK} A^I \wedge A^J \wedge A^K,$$

whose imaginary part is

$$\frac{1}{3}\varepsilon_{IJK}\left(-\sqrt{-\Lambda}\, e^I \wedge \omega^J \wedge \omega^K + \text{permutations} \right) =$$

$$-\sqrt{-\Lambda}\, \varepsilon_{IJK}\, e^I \wedge \omega^J \wedge \omega^K.$$

The imaginary part of this Chern-Simons action is then

$$-\frac{k}{\underset{2}{4\pi}}\, 2\sqrt{-\Lambda} \int_M e^I \wedge \left(d\omega^I + \frac{1}{2}\,\varepsilon_{IJK}\, \omega^J \wedge \omega^K \right)$$

$$= S(\omega, e)$$

if $\quad -k\sqrt{-\Lambda} = \dfrac{1}{4G}$.

Homework 1: 3d Gravity as a Chern-Simons Theory

2.2 Global Symmetries of 3d Einstein Manifolds

We define the coordinates on Minkowski space.

In[7]:=
```
coordM = {x0, x1, x2, x3}
η = {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Out[7]= $\left\{x0, \sqrt{x0^2 - \alpha}\ \mathrm{Cos}[\phi]\ \mathrm{Sin}[\theta], \sqrt{x0^2 - \alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi], \sqrt{x0^2 - \alpha}\ \mathrm{Cos}[\theta]\right\}$

Out[8]= {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

We then take the coordinates on *N* with

In[2]:=
```
coordN = {x0, θ, ϕ}
```

Out[2]= {x0, θ, ϕ}

We now stablish the relations between them

In[3]:=

$$x1 = \sqrt{x0^2 - \alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Cos}[\phi]$$
$$x2 = \sqrt{x0^2 - \alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi]$$
$$x3 = \sqrt{x0^2 - \alpha}\ \mathrm{Cos}[\theta]$$

Out[3]= $\sqrt{x0^2 - \alpha}\ \mathrm{Cos}[\phi]\ \mathrm{Sin}[\theta]$

Out[4]= $\sqrt{x0^2 - \alpha}\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi]$

Out[5]= $\sqrt{x0^2 - \alpha}\ \mathrm{Cos}[\theta]$

We compute the Jacobian matrix

In[14]:=
```
J = Table[D[coordM[[μ]], coordN[[ν]]], {μ, 1, 4}, {ν, 1, 3}];
J // MatrixForm
```

Out[15]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{x0\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]}{\sqrt{x0^2-\alpha}} & \sqrt{x0^2-\alpha}\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi] & -\sqrt{x0^2-\alpha}\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] \\ \frac{x0\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]}{\sqrt{x0^2-\alpha}} & \sqrt{x0^2-\alpha}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi] & \sqrt{x0^2-\alpha}\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] \\ \frac{x0\,\mathrm{Cos}[\theta]}{\sqrt{x0^2-\alpha}} & -\sqrt{x0^2-\alpha}\,\mathrm{Sin}[\theta] & 0 \end{pmatrix}$$

Finally, we compute the induced metric

In[16]:=
```
gamma = Jᵀ.η.J // Simplify;
gamma // MatrixForm
```

Out[17]//MatrixForm=

$$\begin{pmatrix} \frac{\alpha}{x0^2-\alpha} & 0 & 0 \\ 0 & x0^2-\alpha & 0 \\ 0 & 0 & \left(x0^2-\alpha\right)\mathrm{Sin}[\theta]^2 \end{pmatrix}$$