Quantum Gravity Tutorial 3 (March 19, 2020)

Tutorial 3: Free Parametrized Particle (again)

Q1: Indeed, we have

$$p_t = \frac{\partial L}{\partial \dot{t}} = -\frac{m}{2}\left(\frac{\dot{q}}{\dot{t}}\right)^2 , \quad p_q = m\frac{\dot{q}}{\dot{t}} ,$$

which leads to the constraint

$$\phi = \frac{p_q^2}{2m} + p_t = 0 .$$

We obtain a Hamiltonian

$$H = \dot{t}p_t + \dot{q}p_q - \frac{m}{2}\frac{\dot{q}^2}{\dot{t}}$$

$$= \dot{t}p_t + \frac{1}{m}\dot{t}p_q^2 - \frac{m}{2}\dot{t}\left(\frac{p_q}{m}\right)^2$$

$$= \dot{t}\left(p_t + \frac{p_q^2}{2m}\right) = \dot{t}\phi = 0 = N\phi$$

Q2: Since $\{\phi, \phi\} = 0$ trivially and

$$\{\phi, H\} = \{\phi, N\}\phi \approx 0$$

trivially, ϕ is the only constraint and it is first class. It thus generates gauge transformations.

Since

$$\{q, \phi\} = \frac{1}{2m}\{q, p_q^2\} = \frac{1}{2m}(2p_q) = \frac{p_q}{m}$$

$$\{t, \phi\} = \{t, p_t\} = 1$$

$$\{p_q, \phi\} = \{p_t, \phi\} = 0,$$

we have that the gauge transformations generate orbits of the form

$$p_q(s) = p_q(0), \qquad p_t(s) = p_t(0),$$

$$q(s) = q(0) + \frac{p_q}{2m} s, \qquad t(s) = t(0) + s.$$

Q3: Our gauge fixing condition is $q = \tau$. Take a point $(t_0, q_0, p_{t_0}, p_{q_0})$ satisfying this. Its gauge orbit is given by all points satisfying

$$q = q_0 + \frac{p_{q_0}}{2m} s, \qquad t = t_0 + s$$

$$= \tau + \frac{p_{q_0}}{2m} s$$

$$p_q = p_{q_0} \qquad p_t = p_{t_0}.$$

We thus obtain

$$S = \frac{m}{p_{q_0}} (q - \tau),$$

i.e.

$$q_0 = \tau, \qquad t_0 = t - \frac{m}{p_q} (q - \tau)$$

$$p_{q_0} = p_q \qquad p_{t_0} = p_t .$$

Thus

$$F_t(\tau)(t, q, p_t, p_q) = t + \frac{m}{p_q} (\tau - q)$$

$$F_q(\tau)(t, q, p_t, p_q) = \tau$$

$$F_{p_t}(\tau)(t, q, p_t, p_q) = p_t$$

$$F_{p_q}(\tau)(t, q, p_t, p_q) = p_q \approx \pm \sqrt{-2m p_t}$$