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Quantum Gravity

Homework 1: Spinors, SU(2) recoupling theory and intertwiners

a) Given a vector space V with a basis $\{v_1, \dots, v_n\}$, then $\otimes^m V = V^{\otimes m} = \underbrace{V \otimes \cdots \otimes V}_{m}$ has a basis $\{ v_{i_1} \otimes \cdots \otimes v_{i_m} \mid i_1, \dots, i_m \in \{1, \dots, n\}\}$. It thus have dimension n^m. The space $\mathrm{Sym}^m V$ is obtained by identifying

$$v_1 \otimes \cdots \otimes v_m \sim v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$$

for all $\sigma \in S_m$. A basis is then obtained by identifying the basis vectors of $\otimes^m V$ via this relation. So, while the $\dim \otimes^m V$ was given by the number of m-tuples of elements in $\{1, \ldots, n\}$, then $\dim \mathrm{Sym}^m V$ is given by the number of multisets of size m constructed from $\{1, \ldots, n\}$. To count this, consider a set of m dots separated by $n-1$ bars. This represents a multiset where 1 appears a number equal to the dots before the first bar, 2 appears a number equal to the dots between the first and second bar, etc.

Example: $n=6$, $m=7$

$$\bullet \,||\, \bullet \quad \bullet \,|\, \bullet \quad \bullet \quad \bullet \,||\, \bullet \,|$$

$$= \{1, 3, 3, 4, 4, 4, 5\}$$

Thus, a multiset is determined by choosing $n-1$ of $m+n-1$ characters to be stars. We conclude

$$\dim \mathrm{Sym}^m V = \left(\!\!\binom{n}{m}\!\!\right) = \binom{m+n-1}{n-1} = \frac{(m+n-1)!}{(n-1)!\, m!}$$

Reference: Thanks to Wikipedia and MathWorld entries on multisets.

In our case of interest, let us define $V^{(j)}$ to be the spin-j representation space. Then $\dim V^{(1/2)} = 2$ and $V^{(j)} = \mathrm{Sym}^{2j} V^{(1/2)}$. Thus

$$\dim V^{(j)} = \frac{(2j + 2 - 1)!}{1!\, 2j!} = \frac{(2j+1)!}{2j!} = 2j+1.$$

b) Indeed, recall that for a matrix $A = (A^i{}_j) \in M_n(\mathbb{C})$ we have

$$\det(A)\, \varepsilon^{i_1 \cdots i_n} = A^{i_1}{}_{j_1} \cdots A^{i_n}{}_{j_n}\, \varepsilon^{j_1 \cdots j_n}$$

where $\varepsilon^{i_1 \cdots i_n} = \mathrm{sgn}\,(i_1 \cdots i_n)$. We have $\varepsilon^{AB} = \mathrm{sgn}\,(AB)$.

Thus

$$U^A{}_C\, U^B{}_D\, \varepsilon^{CD} = \cancel{\det}^{\,1}(U)\, \varepsilon^{AB} = \varepsilon^{AB}$$

if $U \in SU(2)$.

c) It is clear that

$$\psi^{AB} = \frac{1}{2}\left(\psi^{AB} + \psi^{BA}\right) + \frac{1}{2}\left(\psi^{AB} - \psi^{BA}\right)$$

$$= \psi^{(AB)} + \psi^{[AB]}.$$

We thus only have to show $\psi^{[AB]} = \psi_0 \varepsilon^{AB}$.

Indeed $\psi_0 = \frac{1}{2}\psi^{CD}\varepsilon_{CD} = \frac{1}{2}\left(\psi^{12} - \psi^{21}\right) = \psi^{[12]}$, so that

$$\psi_0 \varepsilon^{12} = \psi_0 = \psi^{[12]}$$

$$\psi_0 \varepsilon^{21} = -\psi_0 = -\psi^{[12]} = \psi^{[21]}.$$

d) From the discussion above, we have that the symmetrization of $2(j_1+j_2-k)$ indices yields a representation of dimension

$$2j_3 + 1 = \left(\!\!\binom{2}{2(j_1+j_2-k)}\!\!\right) = \binom{2(j_1+j_2-k)+2-1}{2-1}$$

$$= \frac{(2(j_1+j_2-k)+2-1)!}{1!\,(2(j_1+j_2-k))!} = 2(j_1+j_2-k)+1.$$

We conclude that

$$j_3 = j_1 + j_2 - k.$$

Now, notice that $j_1, j_2 \in \mathbb{N}/2$ and the procedure only makes sense if $k \in \{0, 1, \dots, 2\min\{j_1, j_2\}\}$. Thus $k \leq 2j_1$ and $k \leq 2j_2$. We conclude

$$j_1 + j_2 + j_3 = 2j_1 + 2j_2 - k \in \mathbb{N}.$$

On the other hand,

$$|j_1 - j_2| = j_1 + j_2 - 2\min\{j_1, j_2\} \leq j_3 \leq j_1 + j_2.$$

Now let

$$a := k = j_1 + j_2 - j_3 \in \mathbb{N}$$

$$b := 2j_2 - a \in \mathbb{N}$$

$$c := 2j_1 - a \in \mathbb{N},$$

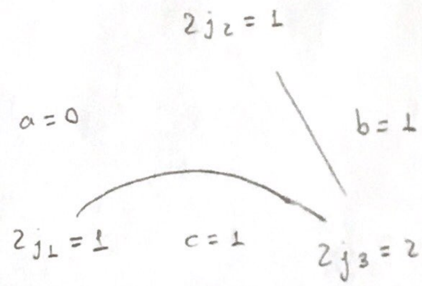
so that

$$a + b = 2j_1, \qquad a + c = 2j_2, \quad \text{and}$$

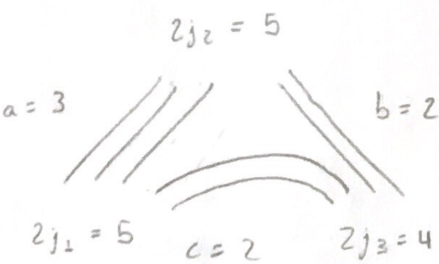
$$b + c = 2j_1 - a + 2j_2 - a = 2j_1 + 2j_2 - 2a = 2j_3.$$

We then have the drawings

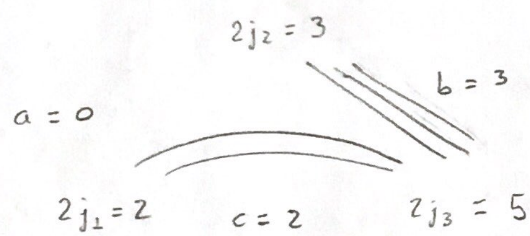
i)



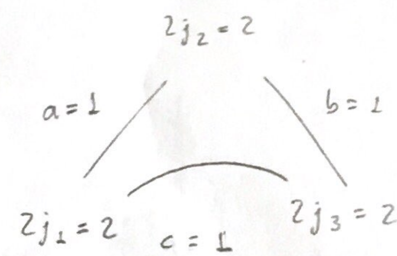
ii)



iii)



iv)



v) $2j_2 = 4$

$2j_1 = 5$ $\quad$ $2j_3 = 6$

Impossible because

$$j_1 + j_2 + j_3 = \frac{15}{2} \notin \mathbb{N}$$

vi) $2j_2 = 5$

$2j_1 = 5$ $\quad$ $2j_3 = 7$

Impossible because

$$j_1 + j_2 + j_3 = \frac{17}{2} \notin \mathbb{N}.$$

e) We have

$$\iota^{AB0} \propto e^{A}_{\ A_1}\, e^{B}_{\ B_1}\, 1_{\iota}^{\ A_1 B_1 0} = e^{A}_{\ A_1}\, e^{B}_{\ B_1}\, \varepsilon^{A_1 B_1} = \delta^{A}_{\ A_1}\, \delta^{B}_{\ B_1}\, \varepsilon^{A_1 B_1}$$

$$= \varepsilon^{AB} .$$

Then

$$\left(\iota^{AB0}\right)^{*} \iota_{AB0} \propto \varepsilon^{AB}\, \varepsilon_{AB} = tr\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{T} \right)$$

$$= tr \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 ,$$

so that

$$\iota^{AB0} = \frac{1}{\sqrt{2}}\, \varepsilon^{AB} .$$