Iván Mauricio Burbano Aldana

Perimeter Scholars International

Gravitational Physics

Homework 1

1. Cartan's Formalism: FRW cosmology

a) By inspection we see a possible choice is

$$\omega^{\hat{t}} = dt, \qquad \omega^{\hat{r}} = \frac{a(t)}{\sqrt{1-kr^2}}\,dr, \qquad \omega^{\hat{\theta}} = a(t)\,r\,d\theta,$$

$$\omega^{\hat{\phi}} = a(t)\,r\sin(\theta)\,d\phi .$$

b) Computing the exterior derivatives yields

$$d\omega^{\hat{t}} = 0,$$

$$d\omega^{\hat{r}} = \frac{a'(t)}{\sqrt{1-kr^2}}\,dt\wedge dr = \frac{a'(t)}{a(t)}\,\omega^{\hat{t}}\wedge\omega^{\hat{r}}$$

$$d\omega^{\hat{\theta}} = a'(t)\,r\,dt\wedge d\theta + a(t)\,dr\wedge d\theta$$

$$= \frac{a'(t)}{a(t)}\,\omega^{\hat{t}}\wedge\omega^{\hat{\theta}} + \frac{\sqrt{1-kr^2}}{r\,a(t)}\,\omega^{\hat{r}}\wedge\omega^{\hat{\theta}}$$

$$d\omega^{\hat{\phi}} = a'(t)\,r\sin(\theta)\,dt\wedge d\phi + a(t)\sin(\theta)\,dr\wedge d\phi + a(t)\,r\cos(\theta)\,d\theta\wedge d\phi$$

$$= \frac{a'(t)}{a(t)}\,\omega^{\hat{t}}\wedge\omega^{\hat{\phi}} + \frac{\sqrt{1-kr^2}}{r\,a(t)}\,\omega^{\hat{r}}\wedge\omega^{\hat{\phi}} + \frac{1}{r\,a(t)}\cot(\theta)\,\omega^{\hat{\theta}}\wedge\omega^{\hat{\phi}}$$

From these we obtain the connection one-forms

$$\theta^{\hat{i}}{}_{\hat{t}} = \frac{a'(t)}{a(t)}\,\omega^{\hat{i}} = -\theta_{\hat{i}\hat{t}} = \theta_{\hat{t}\hat{i}} = \theta^{\hat{t}}{}_{\hat{i}}, \qquad \hat{i} \in \{\hat{r}, \hat{\theta}, \hat{\phi}\}$$

$$\theta^{\hat{i}}{}_{\hat{r}} = \frac{\sqrt{1-kr^2}}{r\,a(t)}\,\omega^{\hat{i}} = -\theta_{\hat{i}\hat{r}} = \theta_{\hat{r}\hat{i}} = -\theta^{\hat{r}}{}_{\hat{i}}, \qquad \hat{i} \in \{\hat{\theta}, \hat{\phi}\}$$

$$\theta^{\hat{\phi}}{}_{\hat{\theta}} = \frac{1}{r a(t)}\cot(\theta)\,\omega^{\hat{\phi}} = -\theta_{\hat{\phi}\hat{\theta}} = \theta_{\hat{\theta}\hat{\phi}} = -\theta^{\hat{\theta}}{}_{\hat{\phi}}.$$

All other connection one-forms vanish.

c) The curvature two-forms are given by

$$R^{\hat{t}}{}_{\hat{r}} = d\theta^{\hat{t}}{}_{\hat{r}} + \theta^{\hat{t}}{}_{\hat{\theta}} \wedge \theta^{\hat{\theta}}{}_{\hat{r}} + \theta^{\hat{t}}{}_{\hat{\phi}} \wedge \theta^{\hat{\phi}}{}_{\hat{r}}$$

$$= d\left(\frac{a'(t)}{\sqrt{1-kr^2}}\,dr\right) + \cancelto{0}{\frac{a'(t)}{a(t)}\,\omega^{\hat{\theta}} \wedge \frac{\sqrt{1-kr^2}}{r a(t)}\,\omega^{\hat{\theta}}}$$

$$+ \cancelto{0}{\frac{a'(t)}{a(t)}\,\omega^{\hat{\phi}} \wedge \frac{\sqrt{1-kr^2}}{r a(t)}\,\omega^{\hat{\phi}}}$$

$$= \frac{a''(t)}{\sqrt{1-kr^2}}\,dt \wedge dr = \frac{a''(t)}{a(t)}\,\omega^{\hat{t}} \wedge \omega^{\hat{r}} = R_{\hat{t}\hat{r}} = -R_{\hat{r}\hat{t}} = R^{\hat{r}}{}_{\hat{t}}$$

$$R^{\hat{t}}{}_{\hat{\theta}} = d\theta^{\hat{t}}{}_{\hat{\theta}} + \theta^{\hat{t}}{}_{\hat{r}} \wedge \theta^{\hat{r}}{}_{\hat{\theta}} + \theta^{\hat{t}}{}_{\hat{\phi}} \wedge \theta^{\hat{\phi}}{}_{\hat{\theta}}$$

$$= d\left(a'(t)\,r\,d\theta\right) + \frac{a'(t)}{a(t)}\,\omega^{\hat{r}} \wedge \left(-\frac{\sqrt{1-kr^2}}{r a(t)}\right)\omega^{\hat{\theta}}$$

$$+ \cancelto{0}{\frac{a'(t)}{a(t)}\,\omega^{\hat\phi} \wedge \frac{1}{r a(t)}\cot(\theta)\,\omega^{\hat\phi}}$$

$$= a''(t)\, r\, dt \wedge d\theta + \cancel{a'(t)\, dr \wedge d\theta} - \cancel{\frac{a'(t)\sqrt{1-kr^2}}{r a(t)^2}}\,\omega^{\hat r} \wedge \omega^{\hat\theta}$$

$$= \frac{a''(t)}{a(t)}\,\omega^{\hat t} \wedge \omega^{\hat\theta} = -R^{\hat\theta}{}_{\hat t},$$

$$R^{\hat t}{}_{\hat\phi} = d\theta^{\hat t}{}_{\hat\phi} + \theta^{\hat t}{}_{\hat r} \wedge \theta^{\hat r}{}_{\hat\phi} + \theta^{\hat t}{}_{\hat\theta} \wedge \theta^{\hat\theta}{}_{\hat\phi}$$

$$= d\left(\frac{a'(t)}{a(t)}\,\omega^{\hat\phi}\right) + \frac{a'(t)}{a(t)}\,\omega^{\hat r} \wedge \left(-\frac{\sqrt{1-kr^2}}{r a(t)}\,\omega^{\hat\phi}\right)$$

$$+ \frac{a'(t)}{a(t)}\,\omega^{\hat\theta} \wedge \left(-\frac{1}{r a(t)}\cot(\theta)\right)\omega^{\hat\phi}$$

$$= d\Big(a'(t)\, r \sin(\theta)\, d\phi\Big) - \frac{a'(t)\sqrt{1-kr^2}}{r a(t)^2}\,\omega^{\hat r} \wedge \omega^{\hat\phi} - \frac{a'(t)\cot(\theta)}{r a(t)^2}\,\omega^{\hat\theta} \wedge \omega^{\hat\phi}$$

$$= a''(t)\, r \sin(\theta)\, dt \wedge d\phi + \cancel{a'(t)\sin(\theta)\, dr \wedge d\phi + a'(t)\, r\cos(\theta)\, d\theta \wedge d\phi}$$

$$\cancel{- \frac{a'(t)\sqrt{1-kr^2}}{r a(t)^2}\,\frac{a(t)}{\sqrt{1-kr^2}}\, dr \wedge a(t)\, r \sin(\theta)\, d\phi}$$

$$\cancel{- \frac{a'(t)\cot(\theta)}{r a(t)^2}\, a(t)\, r\, d\theta \wedge a(t)\, r \sin(\theta)\, d\phi} = \frac{a''(t)}{a(t)}\,\omega^{\hat t} \wedge \omega^{\hat\phi}$$

$$= -R^{\hat\phi}{}_{\hat t},$$

$$R^{\hat r}{}_{\hat\theta} = d\theta^{\hat r}{}_{\hat\theta} + \theta^{\hat r}{}_{\hat t}\wedge\theta^{\hat t}{}_{\hat\theta} + \cancel{\theta^{\hat r}{}_{\hat\phi}\wedge\theta^{\hat\phi}{}_{\hat\theta}} \xrightarrow{} \propto \omega^{\hat\phi}\wedge\omega^{\hat\phi} = 0$$

$$= -d\left(\frac{\sqrt{1-Kr^2}}{\cancel{r}\,a\cancel{(t)}}\,a\cancel{(t)}\,\cancel{r}\,d\theta\right) + \frac{a'(t)}{a(t)}\,\omega^{\hat r}\wedge\frac{a'(t)}{a(t)}\,\omega^{\hat\theta}$$

$$= \frac{Kr}{\sqrt{1-Kr^2}}\,dr\wedge d\theta + \frac{a'(t)^2}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\theta}$$

$$= \frac{K}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\theta} + \frac{a'(t)^2}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\theta} = \frac{a'(t)^2+K}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\theta}$$

$$= -R_{\hat r\hat\theta} = R_{\hat\theta\hat r} = -R^{\hat\theta}{}_{\hat r},$$

$$R^{\hat r}{}_{\hat\phi} = d\theta^{\hat r}{}_{\hat\phi} + \theta^{\hat r}{}_{\hat t}\wedge\theta^{\hat t}{}_{\hat\phi} + \theta^{\hat r}{}_{\hat\theta}\wedge\theta^{\hat\theta}{}_{\hat\phi}$$

$$= -d\left(\frac{\sqrt{1-Kr^2}}{\cancel{ra(t)}}\,\cancel{a(t)r}\sin(\theta)\,d\phi\right) + \frac{a'(t)}{a(t)}\,\omega^{\hat r}\wedge\frac{a'(t)}{a(t)}\,\omega^{\hat\phi}$$

$$- \frac{\sqrt{1-Kr^2}}{ra(t)}\,\omega^{\hat\theta}\wedge\left(-\frac{\cot(\theta)}{ra(t)}\,\omega^{\hat\phi}\right)$$

$$= \frac{Kr}{\sqrt{1-Kr^2}}\sin(\theta)\,dr\wedge d\phi \cancel{- \sqrt{1-Kr^2}\cos(\theta)\,d\theta\wedge d\phi}$$

$$+ \frac{a'(t)^2}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\phi} + \cancel{\frac{\sqrt{1-Kr^2}\,\cancel{\cot}(\theta)}{r^2a(t)^2}\,a(t)r\,d\theta\wedge a(t)r\sin(\theta)\,d\phi} \quad \cos(\theta)$$

$$= \frac{K}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\phi} + \frac{a'(t)^2}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\phi} = \frac{a'(t)^2+K}{a(t)^2}\,\omega^{\hat r}\wedge\omega^{\hat\phi} = -R^{\hat\phi}{}_{\hat r},$$

$$R^{\hat\theta}{}_{\hat\phi} = d\theta^{\hat\theta}{}_{\hat\phi} + \theta^{\hat\theta}{}_{\hat t}\wedge\theta^{\hat t}{}_{\hat\phi} + \theta^{\hat\theta}{}_{\hat r}\wedge\theta^{\hat r}{}_{\hat\phi}$$

$$= -d\left(\frac{1}{ra(t)}\cot(\theta)\omega^{\hat\phi}\right) + \frac{a'(t)^2}{a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$

$$- \frac{\sqrt{1-Kr^2}}{ra(t)}\,\omega^{\hat\theta}\wedge\frac{\sqrt{1-Kr^2}}{ra(t)}\,\omega^{\hat\phi}$$

$$= -d\left(\cos(\theta)\,d\phi\right) + \frac{a'(t)^2}{a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi} - \frac{1-Kr^2}{r^2a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$

$$= \sin(\theta)\,d\theta\wedge d\phi + \frac{r^2a'(t)^2 - 1 + Kr^2}{r^2a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$

$$= \cancel{\frac{1}{r^2a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}} + \frac{\cancel{r^2}(a'(t)^2+K) - \cancel{1}}{\cancel{r^2}a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$

$$= \frac{a'(t)^2+K}{a(t)^2}\,\omega^{\hat\theta}\wedge\omega^{\hat\phi} = -R^{\hat\phi}{}_{\hat\theta}\,.$$

All other curvature two-forms vanish.

d) Let us first recover the Riemann tensor components in our orthonormal basis. Comparing with

$$R^{\hat a}{}_{\hat b} = \frac{1}{2}R^{\hat a}{}_{\hat b\hat c\hat d}\,\omega^{\hat c}\wedge\omega^{\hat d},$$

we obtain

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = \frac{a''(t)}{a(t)} = R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}},$$

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = \frac{a'(t)^2 + K}{a(t)^2} = R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} = R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}}.$$

All other components unrelated by symmetries of the Riemann tensor vanish.

To relate this to our coordinate basis we use (2.25)

$$R^{\alpha}{}_{\beta\gamma\delta} = e_{\hat{a}}{}^{\alpha}\, \omega^{\hat{b}}{}_{\beta}\, \omega^{\hat{c}}{}_{\gamma}\, \omega^{\hat{d}}{}_{\delta}\, R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}.$$

We notice that our one-forms are "diagonal", in the sense that $\omega^{\hat{i}}{}_{i} \propto \delta^{\hat{i}}{}_{i}$ with $i \in \{t, r, \theta, \phi\}$. Thus, so are their dual vectors. In particular

$$e^{t} = \frac{\partial}{\partial t}, \quad e^{r} = \frac{\sqrt{1-Kr^2}}{a(t)}\frac{\partial}{\partial r}, \quad e^{\theta} = \frac{1}{r a(t)}\frac{\partial}{\partial \theta}$$

We thus conclude

$$R^{t}{}_{rtr} = \frac{a(t)^2}{1-Kr^2} R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = \frac{a(t)a''(t)}{1-Kr^2},$$

$$R^{t}{}_{\theta t\theta} = a(t)^2 r^2 R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = r^2 a(t) a''(t),$$

$$R^{t}{}_{\phi t\phi} = a(t)^2 r^2 \sin(\theta)^2 R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} = r^2 \sin(\theta)^2 a(t) a''(t),$$

$$R^{r}{}_{\theta r\theta} = a(t)^2 r^2 R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = r^2 (a'(t)^2 + K),$$

$$R^{r}{}_{\phi r\phi} = a(t)^2 r^2 \sin(\theta)^2 R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} = r^2 \sin(\theta)^2 \left(a'(t)^2 + k\right),$$

$$R^{\theta}{}_{\phi\theta\phi} = a(t)^2 r^2 \sin(\theta)^2 R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = r^2 \sin(\theta)^2 \left(a'(t)^2 + k\right).$$

All other components unrelated by symmetries vanish.

We see this components agree with those presented in the Mathematica computation.

We proceed to compute the components of the Ricci tensor

$$\rightarrow g^{rr} R_{rtrt} = g^{rr} R_{trtr} = g^{rr} \cancel{g_{tt}}^{1} R^{t}{}_{rtr}$$

$$R_{tt} = R^{r}{}_{trt} + R^{\theta}{}_{t\theta t} + R^{\phi}{}_{t\phi t}$$

$$= -\frac{\cancel{1-kr^2}}{a(t)^2}\,\frac{a(t)a''(t)}{\cancel{1-kr^2}} - \frac{1}{\cancel{r^2}a(t)^2}\,\cancel{r^2}a(t)a''(t) - \frac{1}{\cancel{r^2}a(t)^2\cancel{\sin(\theta)^2}}\,\cancel{r^2\sin(\theta)^2}a(t)a''(t)$$

$$= -3\,\frac{a''(t)}{a(t)},$$

$$\rightarrow g^{\theta\theta} R_{\theta r\theta r} = g^{\theta\theta} R_{r\theta r\theta} = g^{\theta\theta} g_{rr} R^{r}{}_{\theta r\theta}$$

$$R_{rr} = R^{t}{}_{rtr} + R^{\theta}{}_{r\theta r} + R^{\phi}{}_{r\phi r}$$

$$= \frac{a(t)a''(t)}{1-kr^2} + \frac{1}{\cancel{r^2}a(t)^2}\,\frac{a(t)^2}{1-kr^2}\,\cancel{r^2}\left(a'(t)^2+k\right)$$

$$+ \frac{1}{\cancel{r^2}a(t)^2\cancel{\sin(\theta)^2}}\,\frac{a(t)^2}{1-kr^2}\,\cancel{r^2\sin(\theta)^2}\left(a'(t)^2+k\right)$$

$$= \frac{a(t)a''(t) + 2\left(a'(t)^2+k\right)}{1-kr^2},$$

$$r^2 g^{\phi\phi} g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi}$$

$$R_{\theta\theta} = R^{t}{}_{\theta t\theta} + R^{r}{}_{\theta r\theta} + R^{\phi}{}_{\theta\phi\theta}$$

$$= r^2 a(t)a''(t) + r^2(a'(t)^2+k) + \frac{1}{\cancel{r^2}\, a(t)^2\, \cancel{\sin(\theta)^2}}\, \cancel{r^2}\, a(t)^2 r^2 \cancel{\sin(\theta)^2}\,(a'(t)^2+k)$$

$$= r^2\Big(a(t)a''(t) + 2(a'(t)^2+k)\Big),$$

$$R_{\phi\phi} = R^{t}{}_{\phi t\phi} + R^{r}{}_{\phi r\phi} + R^{\theta}{}_{\phi\theta\phi}$$

$$= r^2 \sin(\theta)^2 a(t)a''(t) + 2r^2\sin(\theta)^2(a'(t)^2+k)$$

$$= r^2\sin(\theta)^2\Big(a(t)a''(t) + 2(a'(t)^2+k)\Big).$$

We verify these again coincide with our Mathematica computation.

2. Explicit and Hidden Symmetries

a) The first equivalence is trivial. Namely,

$$0 = \nabla_{(\alpha}K_{\beta)} = \frac{1}{2}(\nabla_\alpha K_\beta + \nabla_\beta K_\alpha)$$

implies that $\nabla_\alpha K_\beta = -\nabla_\beta K_\alpha$, i.e.

$$\nabla_{[\alpha}K_{\beta]} = \frac{1}{2}(\nabla_\alpha K_\beta - \nabla_\beta K_\alpha) = \frac{1}{2}(\nabla_\alpha K_\beta + \nabla_\alpha K_\beta) = \nabla_\alpha K_\beta.$$

Conversely,

$$\nabla_\alpha K_\beta = \nabla_{[\alpha}K_{\beta]} = \frac{1}{2}(\nabla_\alpha K_\beta - \nabla_\beta K_\alpha)$$

Implies that $\nabla_\alpha K_\beta = -\nabla_\beta K_\alpha$, i.e. $0 = \nabla_{(\alpha}K_{\beta)}$. This equivalence is simply stating that if the symmetric part of a tensor vanishes, the tensor must be antisymmetric.

The second equivalence is less trivial. Notice that

$$\mathcal{L}_K\left(g\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right)\right) = (\mathcal{L}_K g)\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right)$$

$$+\, g\left(\left[K,\frac{\partial}{\partial x^\mu}\right],\frac{\partial}{\partial x^\nu}\right) + g\left(\frac{\partial}{\partial x^\mu},\left[K,\frac{\partial}{\partial x^\nu}\right]\right)$$

The commutator is computed by

$$\left[K,\frac{\partial}{\partial x^\mu}\right] = \cancel{K^\alpha\frac{\partial^2}{\partial x^\alpha\partial x^\mu}} - \frac{\partial K^\alpha}{\partial x^\mu}\frac{\partial}{\partial x^\alpha} - \cancel{K^\alpha\frac{\partial^2}{\partial x^\mu\partial x^\alpha}}.$$

Thus

$$(\mathcal{L}_K g)_{\mu\nu} = K^\alpha\frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial K^\alpha}{\partial x^\mu}g_{\alpha\nu} + \frac{\partial K^\alpha}{\partial x^\nu}g_{\mu\alpha}$$

$$= K^\alpha\nabla_\alpha g_{\mu\nu} + K^\alpha\Gamma^\beta{}_{\mu\alpha}g_{\beta\nu} + K^\alpha\Gamma^\beta{}_{\nu\alpha}g_{\mu\beta}$$

$$+\,\nabla_\mu K^\alpha g_{\alpha\nu} - \Gamma^\alpha{}_{\beta\mu}K^\beta g_{\alpha\nu}$$

$$+\,\nabla_\nu K^\alpha g_{\mu\alpha} - \Gamma^\alpha{}_{\beta\nu}K^\beta g_{\mu\alpha}.$$

If the connection is torsion-free we conclude

$$(\mathcal{L}_K g)_{\mu\nu} = K^\alpha\nabla_\alpha g_{\mu\nu} + \nabla_\mu K^\alpha g_{\alpha\nu} + \nabla_\nu K^\alpha g_{\mu\alpha}.$$

Finally, if the connection is metric compatible,

$$(\mathcal{L}_K g)_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu = \frac{1}{2} \nabla_{(\mu} K_{\nu)}.$$

This shows the second equivalence.

b) We have that along a geodesic

$$\frac{dc}{d\tau} = \frac{dx^\mu}{d\tau} \partial_\mu c = \frac{dx^\mu}{d\tau} \nabla_\mu c = U^\mu \nabla_\mu (K_\alpha U^\alpha)$$

$$= U^\mu \cancel{\nabla_\mu U^\alpha}^{\,0} K_\alpha + U^\mu U^\alpha \nabla_\mu K_\alpha$$

$$= U^\mu U^\alpha \cancel{\nabla_{(\mu} K_{\alpha)}}^{\,0} = 0.$$

c) The metric clearly has a Killing vector if if it is independent of x^μ for $\mu \in \{1, \dots, 4\}$. Indeed, in that case $\partial_\mu g_{\alpha\beta} = 0$. We can then take $K = \partial_\mu$, i.e. $K^\nu = \delta^\nu{}_\mu$. Thus

$$(\mathcal{L}_K g)_{\alpha\beta} = K^\nu \partial_\nu g_{\alpha\beta} + \cancel{\partial_\alpha K^\nu}^{\,0} g_{\nu\beta} + \cancel{\partial_\beta K^\nu}^{\,0} g_{\alpha\nu}$$

$$= \delta^\nu{}_\mu \partial_\nu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} = 0.$$

d) Indeed, we have

$$\frac{dK}{d\tau} = \frac{dx^\mu}{d\tau}\partial_\mu K = u^\mu \nabla_\mu K = u^\mu \nabla_\mu \left(K_{\alpha_1 \cdots \alpha_p} u^{\alpha_1} \cdots u^{\alpha_p} \right)$$

$$= u^\mu u^{\alpha_1} \cdots u^{\alpha_p} \nabla_\mu K_{\alpha_1 \cdots \alpha_p} + \sum_{r=1}^{p} \cancelto{0}{u^\mu \nabla_\mu u^{\alpha_r}} K_{\alpha_1 \cdots \alpha_p} u^{\alpha_1} \cdots \hat{u}^{\alpha_r} \cdots u^{\alpha_p}$$

$$= u^\mu u^{\alpha_1} \cdots u^{\alpha_p} \cancelto{0}{\nabla_{(\mu} K_{\alpha_1 \cdots \alpha_p)}} = 0 .$$

Homework 1

1. Cartan's Formalism: FRW Cosmology

e)

We start up by setting up our coordinates and the FRW metric

In[43]:=

```
coord = {t, r, θ, ϕ};
g = {{1, 0, 0, 0}, {0, -a[t]^2/(1 - k r^2), 0, 0}, {0, 0, -a[t]^2 r^2, 0}, {0, 0, 0, -a[t]^2 r^2 Sin[θ]^2}};
g // MatrixForm
```

Out[45]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a[t]^2}{1-k r^2} & 0 & 0 \\ 0 & 0 & -r^2 a[t]^2 & 0 \\ 0 & 0 & 0 & -r^2 a[t]^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

We compute the Christoffel symbols

```
Γ = Table[Sum[1/2 Inverse[g][[μ, σ]]
      (D[g[[σ, ν]], coord[[ρ]]] + D[g[[σ, ρ]], coord[[ν]]] - D[g[[ρ, ν]], coord[[σ]]]),
    {σ, 1, 4}], {μ, 1, 4}, {ν, 1, 4}, {ρ, 1, 4}]
```

Out[48]=

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{a[t]\,a'[t]}{1-k r^2},0,0\right\},\left\{0,0,r^2 a[t]\,a'[t],0\right\},\right.\right.$$
$$\left.\left\{0,0,0,r^2 a[t]\,\mathrm{Sin}[\theta]^2 a'[t]\right\}\right\},\left\{\left\{0,\frac{a'[t]}{a[t]},0,0\right\},\left\{\frac{a'[t]}{a[t]},\frac{k r}{1-k r^2},0,0\right\},\right.$$
$$\left.\left\{0,0,-r\left(1-k r^2\right),0\right\},\left\{0,0,0,-r\left(1-k r^2\right)\mathrm{Sin}[\theta]^2\right\}\right\},$$
$$\left\{\left\{0,0,\frac{a'[t]}{a[t]},0\right\},\left\{0,0,\frac{1}{r},0\right\},\left\{\frac{a'[t]}{a[t]},\frac{1}{r},0,0\right\},\{0,0,0,-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\}\right\},$$
$$\left.\left\{\left\{0,0,0,\frac{a'[t]}{a[t]}\right\},\left\{0,0,0,\frac{1}{r}\right\},\{0,0,0,\mathrm{Cot}[\theta]\},\left\{\frac{a'[t]}{a[t]},\frac{1}{r},\mathrm{Cot}[\theta],0\right\}\right\}\right\}$$

We may now compute the Riemann tensor

In[50]:=

```
riem = Table[D[Γ[[α, δ, β]], coord[[γ]]] - D[Γ[[α, γ, β]], coord[[δ]]] + Sum[
    Γ[[α, γ, λ]] × Γ[[λ, δ, β]], {λ, 1, 4}] - Sum[Γ[[α, δ, λ]] × Γ[[λ, γ, β]], {λ, 1, 4}],
  {α, 1, 4}, {β, 1, 4}, {γ, 1, 4}, {δ, 1, 4}] // Simplify
```

Out[50]=

$$
\begin{aligned}
&\{\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\\
&\quad\left\{\left\{0,\frac{a[t]\,a''[t]}{1-k\,r^2},0,0\right\},\left\{\frac{a[t]\,a''[t]}{-1+k\,r^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\\
&\quad\{\{0,0,r^2\,a[t]\,a''[t],0\},\{0,0,0,0\},\{-r^2\,a[t]\,a''[t],0,0,0\},\{0,0,0,0\}\},\\
&\quad\{\{0,0,0,r^2\,a[t]\,\mathrm{Sin}[\theta]^2\,a''[t]\},\{0,0,0,0\},\\
&\quad\ \{0,0,0,0\},\{-r^2\,a[t]\,\mathrm{Sin}[\theta]^2\,a''[t],0,0,0\}\}\},\\
&\ \left\{\left\{\left\{0,\frac{a''[t]}{a[t]},0,0\right\},\left\{-\frac{a''[t]}{a[t]},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\right.\\
&\quad\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\\
&\quad\{\{0,0,0,0\},\{0,0,r^2\,(k+a'[t]^2),0\},\{0,-r^2\,(k+a'[t]^2),0,0\},\{0,0,0,0\}\},\\
&\quad\{\{0,0,0,0\},\{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\,(k+a'[t]^2)\},\\
&\quad\ \{0,0,0,0\},\{0,-r^2\,\mathrm{Sin}[\theta]^2\,(k+a'[t]^2),0,0\}\}\},\\
&\ \left\{\left\{\left\{0,0,\frac{a''[t]}{a[t]},0\right\},\{0,0,0,0\},\left\{-\frac{a''[t]}{a[t]},0,0,0\right\},\{0,0,0,0\}\right\},\right.\\
&\quad\left\{\{0,0,0,0\},\left\{0,0,\frac{k+a'[t]^2}{-1+k\,r^2},0\right\},\left\{0,\frac{k+a'[t]^2}{1-k\,r^2},0,0\right\},\{0,0,0,0\}\right\},\\
&\quad\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\\
&\quad\ \{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\,(k+a'[t]^2)\},\{0,0,-r^2\,\mathrm{Sin}[\theta]^2\,(k+a'[t]^2),0\}\}\},\\
&\ \left\{\left\{\left\{0,0,0,\frac{a''[t]}{a[t]}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{-\frac{a''[t]}{a[t]},0,0,0\right\}\right\},\right.\\
&\quad\left\{\{0,0,0,0\},\left\{0,0,0,\frac{k+a'[t]^2}{-1+k\,r^2}\right\},\{0,0,0,0\},\left\{0,\frac{k+a'[t]^2}{1-k\,r^2},0,0\right\}\right\},\\
&\quad\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,-r^2\,(k+a'[t]^2)\},\{0,0,r^2\,(k+a'[t]^2),0\}\},\\
&\quad\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\}\}
\end{aligned}
$$

In particular, we see that they agree with the ones computed. Presenting the output in the same order as they were written in the homework, we have

In[66]:=
```
riem[[1, 2, 1, 2]]
riem[[1, 3, 1, 3]]
riem[[1, 4, 1, 4]]
riem[[2, 3, 2, 3]]
riem[[2, 4, 2, 4]]
riem[[3, 4, 3, 4]]
```

Out[66]= $\frac{a[t]\, a''[t]}{1 - k\, r^2}$

Out[67]= $r^2\, a[t]\, a''[t]$

Out[68]= $r^2\, a[t]\, \mathrm{Sin}[\theta]^2\, a''[t]$

Out[69]= $r^2\, \left(k + a'[t]^2\right)$

Out[70]= $r^2\, \mathrm{Sin}[\theta]^2\, \left(k + a'[t]^2\right)$

Out[71]= $r^2\, \mathrm{Sin}[\theta]^2\, \left(k + a'[t]^2\right)$

Finally, we compute the Ricci tensor, which agrees with the one presented in the homework

In[77]:=
```
ricci = Table[Sum[riem[[α, μ, α, ν]], {α, 1, 4}], {μ, 1, 4}, {ν, 1, 4}] // Simplify;
ricci // MatrixForm
```

Out[78]//MatrixForm=

$$\begin{pmatrix} -\frac{3\, a''[t]}{a[t]} & 0 & 0 & 0 \\ 0 & \frac{2\,k+2\,a'[t]^2+a[t]\, a''[t]}{1-k\, r^2} & 0 & 0 \\ 0 & 0 & r^2\, \left(2\, \left(k + a'[t]^2\right) + a[t]\, a''[t]\right) & 0 \\ 0 & 0 & 0 & r^2\, \mathrm{Sin}[\theta]^2\, \left(2\, \left(k + a'[t]^2\right) + a[t]\, a''[\end{pmatrix}$$