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Quantum Field Theory III

Homework 2: Maxwell, boundary and the axial anomaly

1. Conformal Invariance of Maxwell's Action for D=4

a) In general dimension the Maxwell action is proportional to

$$S(g, A) = \int d^D x \sqrt{|g(x)|}\, F(A)^{\mu\nu}(x) F(A)_{\mu\nu}(x)$$

for $A \in \Omega^1(M)$ and

$$F(A)_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).$$

b) Under a conformal transformation $x \longmapsto \tilde{x}$ we transform A as a one form $A \longmapsto A'$ where

$$A'_\mu(\tilde{x}) := \frac{\partial x^\nu}{\partial \tilde{x}^\mu}(x) A_\nu(x)$$

and afterwards perform a Weyl transformation

$A' \longmapsto \tilde{A}$, where

$$\tilde{A}_\mu(\tilde{x}) = \left|\frac{\partial \tilde{x}}{\partial x}(x)\right|^w A'_\mu(\tilde{x}) = \left|\frac{\partial \tilde{x}}{\partial x}(x)\right|^w \frac{\partial x^\nu}{\partial \tilde{x}^\mu}(x) A_\nu(x)$$

for some weight w. Considering the dilation $x \longmapsto \tilde{x} = \lambda x$, we have

$$\tilde{A}_\mu(\tilde{x}) = \lambda^{Dw} \frac{1}{\lambda} \delta^\nu{}_\mu A_\nu(x) = \lambda^{Dw-1} A_\mu(x).$$

We thus see that the Weyl weight w and the scaling dimension are related by

$$1 - Dw = \Delta,$$

i.e.

$$w = \frac{1-\Delta}{D}.$$

The transformed field has a curvature

$$F(\tilde{A})_{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{A}_\nu}{\partial \tilde{x}^\mu}(\tilde{x}) - \frac{\partial \tilde{A}_\mu}{\partial \tilde{x}^\nu}(\tilde{x}).$$

Then

$$\frac{\partial \tilde{A}_\nu}{\partial \tilde{x}^\mu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \frac{\partial}{\partial x^\alpha}\left(\left|\frac{\partial \tilde{x}}{\partial x}(x)\right|^w \frac{\partial x^\beta}{\partial \tilde{x}^\nu}(x) A_\beta(x)\right)$$

$$F(\tilde{A})_{\mu\nu} = \frac{1}{\lambda^2} \lambda^{Dw} F(A)_{\mu\nu}(x) = \lambda^{Dw-2} F(A)_{\mu\nu}(x).$$

Therefore it has scaling dimension

$$2 - Dw = 2 - \cancel{D}\,\frac{1-\Delta}{\cancel{D}} = 1 + \Delta.$$

Under an infinitesimal conformal transformation $\tilde{x} = x + \xi$ we have

$$\left|\frac{\partial \tilde{x}}{\partial x}(x)\right| = \left|\delta^{\cdot}{}_{\cdot} + \partial_{\cdot}\xi^{\cdot}(x)\right| = 1 + \partial_\mu \xi^\mu(x).$$

Thus

$$\tilde{A}_\mu(\tilde{x}) = \left(1 + w\,\partial_\alpha \xi^\alpha(x)\right)\left(\delta^\nu{}_\mu - \partial_\mu \xi^\nu(x)\right) A_\nu(x)$$

$$= \left(1 + w\,\partial_\alpha \xi^\alpha(x)\right)\left(A_\mu(x) - \partial_\mu \xi^\nu(x) A_\nu(x)\right)$$

$$= A_\mu(x) - \partial_\mu \xi^\nu(x) A_\nu(x) + w\,\partial_\alpha \xi^\alpha(x).$$

In here we have used that $\dfrac{\partial x^\mu}{\partial \tilde{x}^\nu}(x) = \delta^\mu{}_\nu - \partial_\nu \xi^\mu(x)$, as is easily seen by noting that

$$\left(\delta^\mu{}_\nu - \partial_\nu \xi^\mu(x)\right)\left(\delta^\nu{}_\sigma + \partial_\sigma \xi^\nu(x)\right) = \delta^\mu{}_\sigma \cancel{- \partial_\sigma \xi^\mu(x) + \partial_\sigma \xi^\mu(x)}$$

$$= \delta^\mu{}_\sigma = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}(x)\,\frac{\partial \tilde{x}^\nu}{\partial x^\sigma}(x).$$

$$= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \frac{\partial}{\partial x^\alpha} \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{W} \frac{\partial x^\beta}{\partial \tilde{x}^\nu}(x) A_\beta(x) +$$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{-W} \frac{\partial^2 x^\beta}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}(x) A_\beta(x) +$$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{W} \frac{\partial x^\beta}{\partial \tilde{x}^\nu}(x) \frac{\partial A_\beta}{\partial x^\alpha}(x).$$

Noting that the second therm is symmetric in $\mu \leftrightarrow \nu$ while $F(\tilde{A})_{\mu\nu}$ is antisymmetric, we have

$$F(\tilde{A})_{\mu\nu}(\tilde{x}) = \frac{\partial}{\partial x^\alpha} \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{W} A_\beta(x) \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \frac{\partial x^\beta}{\partial \tilde{x}^\nu}(x) - \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}(x) \frac{\partial x^\beta}{\partial \tilde{x}^\mu}(x) \right)$$

$$+ \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{W} \frac{\partial A_\beta}{\partial x^\alpha}(x) \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \frac{\partial x^\beta}{\partial \tilde{x}^\nu}(x) - \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}(x) \frac{\partial x^\beta}{\partial \tilde{x}^\mu}(x) \right)$$

$$= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}(x) \frac{\partial x^\beta}{\partial \tilde{x}^\nu}(x) \left(\frac{\partial}{\partial x^\alpha} \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{W} A_\beta(x) - \frac{\partial}{\partial x^\beta} \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{W} A_\alpha(x) \right.$$

$$\left. + \left| \frac{\partial \tilde{x}}{\partial x}(x) \right|^{-W} F(A)_{\alpha\beta}(x) \right).$$

We thus see that $F(A)$ is not primary under conformal transformations. We can still find its scaling dimension by noting that for $\tilde{x} = \lambda x$, we have

Similarly, we find that under such a transformation

$$F(\tilde{A})_{\mu\nu}(\tilde{x}) = \left(\delta^\alpha{}_\mu - \partial_\mu \xi^\alpha(x)\right)\left(\delta^\beta{}_\nu - \partial_\nu \xi^\beta(x)\right)\Big(w\,\partial_\alpha \partial_\sigma \xi^\sigma(x) A_\beta(x) - w\,\partial_\beta \partial_\sigma \xi^\sigma(x) A_\alpha(x)$$

$$+ \left(1 + w\,\partial_\sigma \xi^\sigma(x)\right) F(A)_{\alpha\beta}(x)\Big)$$

$$= w\,\partial_\mu \partial_\sigma \xi^\sigma(x) A_\nu(x) - w\,\partial_\nu \partial_\sigma \xi^\sigma(x) A_\mu(x) + w\,\partial_\sigma \xi^\sigma(x) F(A)_{\mu\nu}(x)$$

$$+ F(A)_{\mu\nu}(x) - F(A)_{\mu\beta}(x)\partial_\nu \xi^\beta(x) - F(A)_{\alpha\nu}(x)\partial_\mu \xi^\alpha(x)$$

c) I found this problem easier to solve using finite conformal transformations. In these the Killing equation reduces to the transformation behaviour of the metric under the change of coordinates $g \mapsto g'$, namely

$$g_{\alpha\beta}(x)\,\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\,\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x) =: g'_{\mu\nu}(\tilde{x}) = \Omega(x)^{-2} g_{\mu\nu}(x).$$

However, if done after a Weyl transformation,

$$g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu},$$

the metric is left invariant. Thus, for a Weyl transformation followed by a conformal transformation of coordinates we have that the action gets transformed into

$$\int d^D \tilde{x} \sqrt{|g(x)|} \; F(\tilde{A})^{\mu\nu}(\tilde{x}) \, F(\tilde{A})_{\mu\nu}(\tilde{x})$$

$$= \int d^D x \sqrt{|g(x)|} \left| \frac{\partial \tilde{x}}{\partial x}(x) \right| g^{\alpha\mu} g^{\beta\nu} F(\tilde{A})_{\alpha\beta}(\tilde{x}) \, F(\tilde{A})_{\mu\nu}(\tilde{x})$$

In particular, if we consider the dilation $\tilde{x} = \lambda x$, we have

$$\int d^D x \sqrt{|g(x)|} \; \lambda^D g^{\alpha\mu} g^{\beta\nu} \lambda^{-2(\Delta+1)} F(A)_{\alpha\beta}(x) F(A)_{\mu\nu}(x).$$

We thus obtain that the action is symmetric under these transformations if

$$0 = D - 2(\Delta + 1),$$

i.e. if

$$\Delta = \frac{D}{2} - 1 = \frac{D-2}{2}.$$

We the have for $D = 4$, $\Delta = 1$, i.e. $w = 0$. Then under Weyl transformations both A and $F(A)$ are invariant. In other words, under full conformal transformations A transforms as a 1-form and the general covariance of the action guarantees that it is left invariant

2. A space with a boundary

a) From translation invariance, for all $x \in \mathbb{R}^D$ we must have $\langle \Theta(x) \rangle = \langle \Theta(0) \rangle$. Moreover, if Θ has scaling dimension $\Delta \neq 0$, under a scaling $\tilde{x} = \lambda x$ we have the transformation $\Theta \longmapsto \tilde{\Theta}$ where $\tilde{\Theta}(\tilde{x}) = \lambda^{-\Delta} \Theta(x)$. Thus, invariance under such requires

$$\lambda^{-\Delta} \langle \Theta(0) \rangle = \lambda^{-\Delta} \langle \Theta(x) \rangle = \langle \tilde{\Theta}(\tilde{x}) \rangle = \langle \Theta(\tilde{x}) \rangle = \langle \Theta(0) \rangle,$$

i.e., we must have

$$\langle \Theta(x) \rangle = \langle \Theta(0) \rangle = 0.$$

b) We can't translate in the negative D axis anymore. Thus, the symmetry generated by P_D is broken.

Similarly, any rotation which doesn't leave the D-axis invariant is not a symmetry anymore. Therefore, the generators

$$M_{iD}\,, \quad i = 1, \dots, D-1$$

correspond to broken symmetries. Finally, (after debating with Jonas) we can study the special conformal transformations. Under such a transformation with parameter b^μ, we have

$$\tilde{x}^D = \frac{x^D - b^D x^2}{1 - 2\, b \cdot x + b^2 x^2}$$

In particular, if b^D is sufficiently big, $1 - 2b \cdot x + b^2 x^2 > 0$ and $x^D - b^D x^2 < 0$, i.e. $\tilde{x}^D < 0$. We conclude that K_D corresponds to a broken symmetry. Now, take $b^j = 0$ for all $j \neq i$ for some $i \neq D$. Then

$$\tilde{x}^b = \frac{x^D}{1 - 2b^i x^i + (b^i)^2 x^2}.$$

Consider the polynomial $f(b^i) = 1 - 2b^i x^i + (b^i)^2 x^2$. We will have $x^D < 0$ if $f(b^i) < 0$. Since $x^2 > 0$, there exists such a b^i if and only if the minimum value of f is negative. Such a minimum happens at the c s.t.

$$0 = f'(c) = -\cancel{2} x^i + \cancel{2} c x^2 ,$$

i.e. at $c = x^i / x^2$. Thus, the minimum value is

$$f(c) = 1 - \cancel{2}\frac{(x^i)^2}{x^2} + \cancel{\frac{(x^i)^2}{x^2}} = 1 - \frac{(x^i)^2}{x^2} \geq 1 - \cancel{\frac{(x^i)^2}{(x^i)^2}}^{\,1} = 0 .$$

We conclude that the rest of the special conformal symmetries remain conserved.

It is easy to see that dilations also remain conserved since $\tilde{x}^D = \lambda x^D \geq 0$ if $\lambda > 0$.

c) From the remaining translation symmetries we have

$$\langle \Theta(x) \rangle = f(x^D)$$

for some $f : \mathbb{R} \longrightarrow \mathbb{R}$. From scaling symmetry we further obtain that

$$\lambda^{-\Delta} f(x^D) = \lambda^{-\Delta} \langle \mathcal{O}(x) \rangle = \langle \tilde{\mathcal{O}}(\tilde{x}) \rangle = \langle \mathcal{O}(\tilde{x}) \rangle$$

$$= f(\tilde{x}^D) = f(\lambda x^D).$$

Differentiating wrt λ we obtain

$$x^D f'(\lambda x^D) = -\Delta \lambda^{-\Delta-1} f(x^D).$$

Choosing $\lambda = 1$, we have

$$(\log \circ f)'(x^D) = \frac{f'(x^D)}{f(x^D)} = -\Delta \frac{1}{x^D} = (-\Delta \log)'(x_D).$$

Therefore, there is a $c \in \mathbb{R}$ s.t.

$$\log(f(x^D)) = -\Delta \log(x^D) + c$$

$$= \log((x^D)^{-\Delta}) + c$$

$$= \log(e^c (x^D)^{-\Delta}).$$

We conclude $f(x^D) = f(1)(x^D)^{-\Delta}$. This is known as Euler's homogeneous function theorem.

It is clear that Jonas and I have discussed

d) The only symmetry that survives is the rotational symmetry. Thus, since for all x there is a rotation that maps into $(\|x\|, 0, \dots, 0)$. Then

$$\langle \mathcal{O}(x) \rangle = \langle \mathcal{O}(\|x\|, 0, \dots, 0) \rangle = f(\|x\|) .$$

3. Axial Anomaly

a) Thanks to Jonas, I figured the trick. The idea is that, thanks to Poincaré and Bose symmetry, the answer can only depend on $\eta_{\mu\nu}$ and $p_{\mu\nu}$ in a manner symmetric under the exchange $\mu \longleftrightarrow \nu$. Thus

$$\langle j^V_\mu(p) j^V_\nu(-p) \rangle = \eta_{\mu\nu} F(p^2) + p_\mu p_\nu G(p^2).$$

From conservation $\partial^\mu j^V{}_\mu = 0$, which in momentum space is $p^\mu j^V_\mu(p) = 0$, we have

$$0 = p^\mu \langle j^V_\mu(p) j^V_\nu(-p) \rangle = p_\nu F(p^2) + p^2 p_\nu G(p^2)$$

$$= p_\nu (F(p^2) + p^2 G(p^2)).$$

We conclude

$$C(p^2) = -\frac{1}{p^2}F(p^2),$$

i.e.

$$\langle j_\mu^V(p) j_\nu^V(-p)\rangle = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)F(p^2).$$

Finally, notice that, the kinetic term of the action for a fermionic field in general dimension is

$$S(\psi) = \int d^D x \sqrt{|g|}\, i\bar\psi \gamma^a e_a{}^\mu D_\mu \psi$$

For the theory to be a CFT, it has to be invariant under scale Weyl transformations

$$\psi \longmapsto \lambda^{-\Delta}\psi$$

$$g \longmapsto \lambda^2 g, \qquad e^a{}_\mu \longmapsto \lambda e^a{}_\mu$$

under which

$$S(\psi) \longmapsto \int d^D x\, \lambda^D \sqrt{|g|}\, i\lambda^{-\Delta}\bar\psi \gamma^a \lambda^{-1} e_a{}^\mu D_\mu(\lambda^{-\Delta}\psi)$$

$$= \lambda^{D-2\Delta-1} S(\psi),$$

that is

$$\Delta = \frac{D-1}{2}.$$

We conclude that the scaling dimension of $j^V_\mu = \bar{\psi}\gamma_\mu\psi$ is

$$2\Delta = D - 1 = 1$$

in $D=2$. Now,

$$j^V_\mu(p) = \int d^2x \; e^{ip\cdot x} j^V_\mu(x)$$

then has scaling dimension -1. We conclude that $\langle j^V_\mu(p) j^V_\nu(-p)\rangle$ has dimension -2. On a closer look, note that translation symmetry guarantees

$$\langle j^V_\mu(p) j^V_\nu(p')\rangle = \int d^3x \, d^3y \; e^{ip\cdot x} e^{ip'\cdot y} \langle j^V_\mu(x) j^V_\nu(y)\rangle$$

$$= \int d^3x \, d^3y \; e^{ip\cdot x} e^{ip'\cdot y} \langle j^V_\mu(x-y) j^V_\nu(0)\rangle$$

$$= \int d^3x \, d^3y \; e^{ip\cdot(x+y)} e^{ip'\cdot y} \langle j^V_\mu(x) j^V_\nu(0)\rangle$$

$$= \int d^3x\, d^3y\, e^{ip\cdot x} e^{i(p+p')\cdot y} \langle j_\mu^V(x) j_\nu^V(0)\rangle$$

$$= (2\pi)^D \delta(p+p') \int d^3x\, e^{ip\cdot x} \langle j_\mu^V(x) j_\nu^V(0)\rangle = (2\pi)^D \delta(p+p')\, F(p^2).$$

Now, $\delta(p+p')$ has dimensions -2. We conclude that, after removing the divergent $\delta(p-p)=0$, then

$$\langle j_\mu^V(p) j_\nu^V(-p)\rangle = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) F(p^2)$$

has dimension 0. This implies that $F(p^2)$ is a constant by the Euler homogeneous function theorem. We conclude

$$\langle j_\mu^V(p) j_\nu^V(-p)\rangle = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) c.$$

b) Note that

$$\gamma^0\gamma^5 = \gamma^0\gamma^0\gamma^1 = \gamma^1 = \varepsilon^{0\mu}\gamma_\mu$$

$$\gamma^1\gamma^5 = \gamma^1\gamma^0\gamma^1 = -\gamma^0\gamma^1\gamma^1 = -\gamma^0 = \varepsilon^{1\mu}\gamma_\mu.$$

We conclude that $\gamma^\mu\gamma^5 = \varepsilon^{\mu\nu}\gamma_\nu$. Therefore the axial and vector current are connected by

$$j^A_\mu = \varepsilon_{\mu\nu}\, j^{V,\nu} = \varepsilon_{\mu\nu}\bar{\psi}\gamma^\nu\psi = \bar{\psi}\gamma_\mu\gamma^5\psi$$

To be completed...

c) We have

$$p^\mu \langle j^A_\mu(p) j^V_\nu(-p)\rangle = p^\mu \varepsilon_\mu{}^\sigma \langle j^V_\sigma(p) j^V_\nu(-p)\rangle$$

$$= p^\mu \varepsilon_\mu{}^\sigma \left(\eta_{\sigma\nu} + \frac{p_\sigma p_\nu}{p^2}\right)c = \left(p^\mu \varepsilon_{\mu\nu} + \cancelto{0}{p^\sigma \varepsilon^\mu{}_\sigma \frac{p_\mu p_\nu}{p^2}}\right)c$$

$$= c\, p^\mu \varepsilon_{\mu\nu}.$$

Thus, its Fourier transform indicates that

$$\langle \partial^\mu j^A_\mu(x) j^V_\nu(0)\rangle = c\, \varepsilon_{\mu\nu}\partial^\mu \delta(x).$$

d) We conclude that

$$\langle \partial^\mu j^A_\mu(0)\rangle_{A_\mu} = \langle \partial_\mu j^{A,\mu}(0)\, e^{\int d^2x\, A^\nu(x) j^V_\nu(x)}\rangle$$

$$= \cancelto{0}{\langle \partial_\mu j^{A,\mu}(0)\rangle} + \int d^2x\, A^\nu(x)\langle \partial_\mu j^{A,\mu}(0) j^V_\nu(x)\rangle$$

$$= \int d^2x\, A^\nu(x) \langle \partial_\mu j^{A,\mu}(x)\, j^V_\nu(0)\rangle = c\int d^2x\, A^\nu(x)\, \varepsilon'_{\mu\nu}\, \partial^\mu \delta(x)$$

$$= -c\int d^2x\, \partial^\mu A^\nu(x)\, \varepsilon_{\mu\nu}\, \delta(x) = -c\, \partial^\mu A^\nu(x)\, \varepsilon_{\mu\nu}$$

$$= -c\left(\partial^0 A^1(x) - \partial^1 A^0(x)\right) = -c F^{01}(x).$$

Continuation of 3b)

We have Dirac's equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

and its conjugate

$$i\,\partial_\mu \bar\psi \gamma^\mu + m\bar\psi = 0.$$

Moreover, note that

$$\gamma^5\gamma^0 = \gamma^0\gamma^1\gamma^0 = -\gamma^1,$$
$$\gamma^5\gamma^1 = \gamma^0\gamma^1\gamma^1 = \gamma^0,$$

i.e. $\gamma^\mu\gamma^5 = \gamma^5\gamma^\mu$. Then

$$\partial_\mu j^{A,\mu} = \partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = \partial_\mu\bar\psi\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= im\bar\psi\gamma^5\psi - \bar\psi\gamma^5\gamma^\mu\partial_\mu\psi = im\bar\psi\gamma^5\psi + im\bar\psi\gamma^5\psi = 2im\bar\psi\gamma^5\psi = 0$$

if $m=0$. Note that this is enforced by scaling invariance. Indeed, as we already saw, the kinetic part of the action has scaling dimension 0 if $\Delta = \frac{D-1}{2}$. However, under a Weyl scale transformation, the massive part of the action scales like

$$\int d^D x \sqrt{|g|}\, m \bar{\psi} \psi \longmapsto \int d^D x\, \lambda^D \sqrt{|g|}\, m \lambda^{-2\Delta} \bar{\psi} \psi ,$$

i.e. has scaling dimension $D - 2\Delta = D - D + 1 = 1 \neq 0$.

We must thus have $m = 0$ to retain scale invariance