Iván Mauricio Burbano Aldana

Perimeter Scholars International

Homework 2: Linearized Gravity

1. Let $\eta^{\mu\nu}$ be the inverse of η. Then, up to first order in h,

$$h^{\mu\nu} = g^{\mu\sigma} g^{\nu\rho} h_{\sigma\rho} = \eta^{\mu\sigma}\eta^{\nu\rho} h_{\sigma\rho}$$

Moreover

$$(\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\rho} + h_{\nu\rho}) = \delta^{\mu}{}_{\rho} + \eta^{\mu\nu} h_{\nu\rho} - h^{\mu\nu}\eta_{\nu\rho} + \mathcal{O}(h^2)$$

$$= \delta^{\mu}{}_{\rho} + \eta^{\mu\nu} h_{\nu\rho} - \eta^{\mu\sigma}\eta^{\nu\lambda} h_{\sigma\lambda}\eta_{\nu\rho} = \delta^{\mu}{}_{\rho} + \cancel{\eta^{\mu\nu} h_{\nu\rho}} - \cancel{\eta^{\mu\sigma} h_{\sigma\lambda}\delta^{\lambda}{}_{\rho}}$$

$$= \delta^{\mu}{}_{\rho}.$$

Thus $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Therefore, using (3.31), we have the Christoffel symbols

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}(\eta^{\alpha\delta} - h^{\alpha\delta})(h_{\beta\delta,\gamma} + h_{\gamma\delta,\beta} - h_{\beta\gamma,\delta})$$

$$= \frac{1}{2}\eta^{\alpha\delta}(h_{\beta\delta,\gamma} + h_{\gamma\delta,\beta} - h_{\beta\gamma,\delta}),$$

in our appropriate coordinates. Using (3.66) we can now calculate the components of the Riemann tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\delta}\left(h_{\nu\delta,\sigma\mu} + \cancel{h_{\sigma\delta,\nu\mu}} - h_{\nu\sigma,\delta\mu}\right)$$

$$- \frac{1}{2}\eta^{\rho\delta}\left(h_{\mu\delta,\sigma\nu} + \cancel{h_{\sigma\delta,\mu\nu}} - h_{\mu\sigma,\delta\nu}\right)$$

$$= \frac{1}{2}\eta^{\rho\delta}\left(h_{\nu\delta,\sigma\mu} - h_{\nu\sigma,\delta\mu} - h_{\mu\delta,\sigma\nu} + h_{\mu\sigma,\delta\nu}\right).$$

We can further calculate the components of the Ricci tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \frac{1}{2}\eta^{\rho\delta}\left(h_{\nu\delta,\mu\rho} - h_{\nu\mu,\delta\rho} - h_{\rho\delta,\mu\nu} + h_{\rho\mu,\delta\nu}\right),$$

$$= \frac{1}{2}\left(\partial^{\rho}\partial_{\mu}h_{\nu\rho} - \partial^{\rho}\partial_{\rho}h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h + \partial^{\rho}\partial_{\nu}h_{\mu\rho}\right),$$

and the Ricci scalar

$$R = R^{\mu}{}_{\mu} = \frac{1}{2}\left(\partial^{\rho}\partial^{\mu}h_{\mu\rho} - \partial^{\rho}\partial_{\rho}h - \partial^{\mu}\partial_{\mu}h + \partial^{\rho}\partial^{\mu}h_{\mu\rho}\right)$$

$$= \partial^{\rho}\partial^{\mu}h_{\mu\rho} - \partial^{\rho}\partial_{\rho}h.$$

It is important to notice that, up to order h^2 it doesn't matter whether we raise indices with η or g on the previous three tensors.

The Einstein tensor is finally

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R =$$

$$= \frac{1}{2}\left(\partial^\rho \partial_\mu h_{\nu\rho} - \partial^\rho \partial_\rho h_{\mu\nu} - \partial_\mu \partial_\nu h + \partial^\rho \partial_\nu h_{\mu\rho} - \eta_{\mu\nu}\partial^\rho \partial^\sigma h_{\rho\sigma} + \eta_{\mu\nu}\partial^\rho \partial_\rho h\right)$$

$$= -\frac{1}{2}\left(\partial^\rho \partial_\rho h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial_\rho h\right)$$

$$+ \frac{1}{2}\left(\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \partial_\mu \partial_\nu h\right)$$

$$- \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho \partial^\sigma h_{\rho\sigma} - \frac{1}{2}\partial^\rho \partial_\rho h\right).$$

$$= -\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \frac{1}{2}\left(\partial^\rho \partial_\mu h_{\nu\rho} - \frac{1}{2}\partial^\rho \partial_\mu g_{\nu\rho} h + \partial^\rho \partial_\nu h_{\mu\rho} - \frac{1}{2}\partial^\rho \partial_\nu g_{\mu\rho} h\right)$$

$$- \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho \partial^\sigma h_{\rho\sigma} - \frac{1}{2}\partial^\rho \partial^\sigma g_{\rho\sigma} h\right)$$

$$= -\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma}.$$

Therefore, the linearized Einstein equations are

$$-\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

2. a) Under such a coordinate transformation and up to first order on ξ and h, we have

$$\eta_{\mu\nu} + h_{\mu\nu} = g_{\mu\nu} = g'_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}$$

$$= \left(\eta_{\sigma\rho} + h'_{\rho\sigma}\right)\left(\delta^\rho{}_\mu - \partial_\mu \xi^\rho\right)\left(\delta^\sigma{}_\nu - \partial_\nu \xi^\sigma\right)$$

$$= \eta_{\mu\nu} - \eta_{\sigma\rho}\partial_\mu \xi^\rho \delta^\sigma{}_\nu - \eta_{\sigma\rho}\delta^\rho{}_\mu \partial_\nu \xi^\sigma + h'_{\mu\nu}$$

$$= \eta_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + h'_{\mu\nu}$$

Thus

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu .$$

b) Under such a transformation,

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}\left(h + 2\partial^\rho \xi_\rho\right)$$

$$= \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial^\rho \xi_\rho .$$

Thus

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \partial_\mu \xi_\nu + \cancel{\partial_\nu \partial^\mu \xi_\mu} - \cancel{\partial_\nu \partial^\rho \xi_\rho}$$

$$= \partial^\mu \bar{h}_{\mu\nu} + \Box \xi_\nu .$$

Therefore, by choosing ξ s.t.

$$\Box \xi_\nu = -\partial^\mu \bar{h}_{\mu\nu},$$

we obtain $\partial^\mu \bar{h}'_{\mu\nu} = 0$. This is, under suitable conditions, always possible, as is clear from the theory of electromagnetism. Indeed, this corresponds to the inhomogeneous Maxwell's equations in the Lorenz gauge for a four potential ξ_ν created by a four current $-\partial^\mu \bar{h}_{\mu\nu}$. Under this transformation

$$\partial^\rho \partial_{(\mu} \bar{h}_{\nu)\rho} = \frac{1}{2}\left(\partial_\mu \cancelto{0}{\partial^\rho \bar{h}_{\rho\nu}} + \partial_\nu \cancelto{0}{\partial^\rho \bar{h}_{\rho\mu}}\right) = 0,$$

$$\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = \eta_{\mu\nu}\partial^\rho \cancelto{0}{\partial^\sigma \bar{h}_{\sigma\rho}} = 0 .$$

Thus, eqn (2) becomes

$$-\frac{1}{2}\Box\bar{h}_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

i.e.

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

3. a) We have $\partial_0\bar{h}_{\mu\nu}=0$ due to the static field condition. Thus, if we take the $\mu=\nu=0$ component of the linearized field eqns, we have

$$-16\pi G\rho = -16\pi G\, T_{00} = \Box\bar{h}_{00} = -\cancel{\partial_0^2\bar{h}_{00}}^{\,0} + \Delta\bar{h}_{00} = -4\,\Delta\phi.$$

Thus, we recover Poisson's eqn

$$\Delta\phi = 4\pi G\rho.$$

In this limit, the Christoffel symbols satisfy

$$\Gamma^i{}_{\mu\nu} = \frac{1}{2}\left(h_{\mu i,\nu} + h_{\nu i,\mu} - h_{\mu\nu,i}\right).$$

Moreover, for slow motion $\|\vec{v}\| \ll 1$. Thus

$$-1 = U^\mu U_\mu = -\left(\frac{dx^0}{d\tau}\right)^2 + \sum_{i=1}^{3}\left(\frac{dx^i}{d\tau}\right)^2$$

$$= -\left(\frac{dx^0}{d\tau}\right)^2\left(1-\|\vec{v}\|^2\right) \approx -\left(\frac{dx^0}{d\tau}\right)^2$$

and $\left|\frac{dx^0}{d\tau}\right| \gg \left\|\frac{d\vec{x}}{d\tau}\right\|$. Therefore, the autoparallel eqn is

$$\frac{d^2x^i}{dt^2} = \frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\cancel{\frac{d\tau}{dx^0}}\right)\cancel{\frac{d\tau}{dx^0}} = \frac{d^2x^i}{d\tau^2} = -\Gamma^i{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

$$= -\Gamma^i{}_{00}\cancel{\frac{dx^0}{d\tau}}\cancel{\frac{dx^0}{d\tau}} = \frac{1}{2}h_{00,i}$$

To continue we need to find $\bar{h}_{\mu\nu}$. The other Einstein equations become

$$\Delta\bar{h}_{i\nu} = \Box\bar{h}_{i\nu} = 0,$$

Thus $\bar{h}_{i\nu}$ is harmonic. Since it is assumed to be bounded (as it is a small perturbation), it has to be constant. Thus, if our spacetime is to approach Minkowski at infinity, $\bar{h}_{i\nu} = 0$. Thus

$$-\bar{h}_{00} = \bar{h} = h - 2h = -h$$

and

$$h_{00} = \bar{h}_{00} + \frac{1}{2}\eta_{00}h = \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = \frac{1}{2}\bar{h}_{00} = -2\phi.$$

Thus, indeed

$$\frac{d^2x^i}{dt^2} = -\phi_{,i} = -(\vec{\nabla}\phi)_i,$$

recovering Newton's second law. Finally, the components of the remaining components of the perturbation are

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} h = \bar{h}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \bar{h}_{00} ,$$

$$h_{i\nu} = \cancelto{0}{\bar{h}_{i\nu}} + \frac{1}{2} g_{i\nu} \bar{h}_{00}$$

$$= \frac{1}{2} \delta_{i\nu} \bar{h}_{00} = -2\delta_{i\nu}\phi .$$

We conclude

$$g_{\mu\nu} = \begin{bmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{bmatrix}_{\mu\nu} .$$

b) Taking the $\nu = 0$ components of our linearized Einstein equations we have

$$\Box A_\mu = -\frac{1}{4} \Box \bar{h}_{\mu 0} = -\frac{1}{4} (-16\pi T_{\mu 0}) = -4\pi J_\mu .$$

Taking this component for the De Donder gauge condition,

$$\partial^\mu A_\mu = -\frac{1}{4} \partial^\mu \bar{h}_{\mu 0} = 0 .$$

We still have $\Box \bar{h}_{ij} = 0$. Thus, as before, $\bar{h}_{ij} = 0$

and

$$h_{ij} = \cancelto{0}{\bar{h}_{ij}} - \frac{1}{2} g_{i\nu} \bar{h}_{00} = -2\delta_{i\nu}\phi .$$

However, we now have

$$h_{i0} = \bar{h}_{i0} + \frac{1}{2} \cancel{g_{i0}}^{\,0} \bar{h}_{00}$$

$$= -4A_i .$$

Yielding the metric

$$g_{\mu\nu} = \begin{bmatrix} -(1+2\phi) & -4A_1 & -4A_2 & -4A_3 \\ -4A_1 & 1-2\phi & 0 & 0 \\ -4A_2 & 0 & 1-2\phi & 0 \\ -4A_3 & 0 & 0 & 1-2\phi \end{bmatrix}_{\mu\nu} .$$

Up to linear order in the velocity the geodesic eqn becomes

$$\frac{d^2x^i}{dt^2} = \frac{d}{d\tau}\left(\frac{dx^i}{d\tau} \cancel{\frac{d\tau}{dx^0}}^{\,1}\right) \cancel{\frac{d\tau}{dx^0}}^{\,1} = \frac{d^2x^i}{d\tau^2} = -\Gamma^i{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

$$= -\Gamma^i{}_{00} - 2\Gamma^i{}_{0j} v^j .$$

Much like before, $\Gamma^i{}_{00} = -\frac{1}{2} h_{00,i} = -(\vec{\nabla}\phi)_i$. However, now

$$\Gamma^i{}_{0j} = \frac{1}{2}\left(h_{0i,j} + \cancel{h_{ji,0}}^{\,0} - h_{0j,i}\right) = 2(\partial_i A_j - \partial_j A_i)$$

$$= 2\,\varepsilon_{ijk} B_k .$$

Thus,

$$\frac{d^2 x^i}{dt^2} = -(\vec{\nabla}\phi)_i - 4\,\varepsilon_{ijk} B_k v^j ,$$

i.e.

$$\frac{d^2 \vec{x}}{dt^2} = -\vec{E} - 4\,\vec{v} \times \vec{B} .$$

4. Notice that

$$M_{\alpha\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\mu{}_{\beta\alpha} g_{\mu\gamma} - \Gamma^\mu{}_{\gamma\alpha} g_{\beta\mu} .$$

Thus

$$\delta_\Gamma M_{\alpha\beta\gamma} = -\delta\Gamma^\mu{}_{\beta\alpha} g_{\mu\gamma} - \delta\Gamma^\mu{}_{\gamma\alpha} g_{\beta\mu} = -2\,\delta\Gamma^\mu{}_{(\beta|\alpha} g_{\mu|\gamma)}$$

Then

$$\delta_\Gamma S_M = -\frac{1}{2}\int d^4x \sqrt{-g}\; 2 M^{\alpha\beta\gamma} \delta M_{\alpha\beta\gamma}$$

$$= 2\int d^4x \sqrt{-g}\; M^{\alpha\beta\gamma} \delta\Gamma^\mu{}_{(\beta|\alpha} g_{\mu|\gamma)}$$

$$= \int d^4x \sqrt{-g}\; M^{\alpha\beta\gamma} g_{\mu\gamma} \delta\Gamma^\mu{}_{\beta\alpha} .$$

Thus, the equations obtained through the variation of Γ are precisely the metricity conditions $M_{\alpha\beta\gamma} = 0$. The variation wrt g is more complicated. Recalling

(4.28)

$$\delta_g S_M = -\frac{1}{2}\int d^4x \Big(\delta\sqrt{-g}\, M^{\alpha\beta\gamma} M_{\alpha\beta\gamma} +$$

$$\delta g_{\rho\alpha}\, g_{\sigma\beta}\, g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda} +$$

$$g_{\rho\alpha}\, \delta g_{\sigma\beta}\, g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda} +$$

$$g_{\rho\alpha}\, g_{\sigma\beta}\, \delta g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda} +$$

$$2\sqrt{-g}\, M^{\alpha\beta\gamma}\, \delta_g M_{\alpha\beta\gamma}\Big),$$

$$\delta\sqrt{-g}\, M^{\alpha\beta\gamma} M_{\alpha\beta\gamma} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} M^{\alpha\beta\gamma} M_{\alpha\beta\gamma}$$

$$\delta g_{\rho\alpha}\, g_{\sigma\beta}\, g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda} = -g_{\rho\mu}\, g_{\alpha\nu}\, \delta g^{\mu\nu}\, g_{\sigma\beta}\, g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda}$$

$$= -M_\nu{}^{\beta\gamma} M_{\mu\beta\gamma}\, \delta g^{\mu\nu},$$

$$g_{\rho\alpha}\, \delta g_{\sigma\beta}\, g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda} = -g_{\rho\alpha}\, g_{\sigma\mu}\, g_{\beta\nu}\, \delta g^{\mu\nu}\, g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda}$$

$$= -M^{\alpha}{}_{\nu}{}^{\gamma} M_{\alpha\mu\gamma}\, \delta g^{\mu\nu},$$

$$g_{\rho\alpha}\, g_{\sigma\beta}\, \delta g_{\lambda\gamma}\, M^{\alpha\beta\gamma} M^{\rho\sigma\lambda} = -g_{\rho\alpha}\, g_{\sigma\beta}\, g_{\lambda\mu}\, g_{\gamma\nu}\, \delta g^{\mu\nu} M^{\alpha\beta\gamma} M^{\rho\sigma\lambda}$$

$$= -M^{\alpha\beta}{}_{\nu} M_{\alpha\beta\mu}\, \delta g^{\mu\nu}.$$

Moreover

$$\delta_g M_{\alpha\beta\gamma} = -\partial_\alpha (g_{\beta\mu}\, g_{\gamma\nu}\, \delta g^{\mu\nu}) + \Gamma^{\rho}{}_{\beta\alpha}\, g_{\rho\mu}\, g_{\gamma\nu}\, \delta g^{\mu\nu}$$

$$+ \Gamma^{\rho}{}_{\gamma\alpha}\, g_{\beta\mu}\, g_{\rho\nu}\, \delta g^{\mu\nu}$$

$$\delta_g S_M = -\frac{1}{2}\int d^4x \sqrt{-g}\Big(-\frac{1}{2} M^{\alpha\beta\gamma} M_{\alpha\beta\gamma}\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$- M_\nu{}^{\beta\gamma} M_{\mu\beta\gamma}\, \delta g^{\mu\nu} - M^\alpha{}_\nu{}^\gamma M_{\alpha\mu\gamma}\, \delta g^{\mu\nu}$$

$$- M^{\alpha\beta}{}_\nu M_{\alpha\beta\mu}\, \delta g^{\mu\nu}\Big)$$

$$+ \underbrace{\int d^4x \sqrt{-g}\, M^{\alpha\beta\gamma}\, \partial_\alpha\left(g_{\beta\mu} g_{\gamma\nu}\, \delta g^{\mu\nu}\right)}$$

$$- \int d^4x\, g_{\beta\mu} g_{\gamma\nu}\, \delta g^{\mu\nu}\, \partial_\alpha\left(\sqrt{-g}\, M^{\alpha\beta\gamma}\right) + \text{boundary}$$

$$- \int d^4x \sqrt{-g}\, M^{\alpha\beta\gamma}\, \Gamma^\rho{}_{\beta\alpha}\, g_{\rho\mu} g_{\gamma\nu}\, \delta g^{\mu\nu}$$

$$- \int d^4x \sqrt{-g}\, M^{\alpha\beta\gamma}\, \Gamma^\rho{}_{\gamma\alpha}\, g_{\beta\mu} g_{\rho\nu}\, \delta g^{\mu\nu}.$$

We thus obtain the EOMs

$$- g_{\beta\mu} g_{\gamma\nu}\, \partial_\alpha\left(\sqrt{-g}\, M^{\alpha\beta\gamma}\right) = \frac{1}{4} M^{\alpha\beta\gamma} M_{\alpha\beta\gamma}\, g_{\mu\nu}$$

$$+ \frac{1}{2}\left(M_\nu{}^{\beta\gamma} M_{\mu\beta\gamma} + M^\alpha{}_\gamma{}^\gamma M_{\alpha\mu\gamma} + M^{\alpha\beta}{}_\nu M_{\alpha\beta\mu}\right)$$

$$- M^{\alpha\beta}{}_\nu\, \Gamma_{\mu\beta\alpha} - M^\alpha{}_\mu{}^\beta\, \Gamma_{\nu\beta\alpha}.$$