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Chern-Simons Theory

Homework 1: 3d Gravity as a Chern-Simons Theory

1. Euclidean Signature, Vanishing Cosmological Constant

1.1. Preliminaries

Q1: The orthonormality of the vielbein is given by

$$e_I{}^\mu e_J{}^\nu g_{\mu\nu} = \eta_{IJ}.$$

The inverse vielbein e^I is defined by

$$e^I{}_\mu e_J{}^\mu = \delta^I{}_J.$$

Since the left matrix inverse of a square matrix is also the right matrix inverse, we have

$$e_I{}^\mu e^I{}_\nu = \delta^\mu{}_\nu.$$

Therefore

$$e^I{}_\mu e^J{}_\nu \eta_{IJ} = e^I{}_\mu e^J{}_\nu e_I{}^\sigma e_J{}^\rho g_{\sigma\rho}$$

$$= \delta^\sigma{}_\mu \delta^\rho{}_\nu g_{\sigma\rho} = g_{\mu\nu},$$

proving (6). Now, let us define $e_{I\mu} := \eta_{IJ} e^J{}_\mu$.

Then

$$e_{I\mu} = g_{\sigma\rho} e_I{}^\sigma e_J{}^\rho e^J{}_\mu = g_{\sigma\rho} e_I{}^\sigma \delta^\rho{}_\mu = g_{\sigma\mu} e_I{}^\sigma,$$

proving the first half of (7). Notice then that we can also connect $e^I{}_\mu$ and $e_I{}^\mu$ by raising and lowering indices

$$\eta^{IJ} g_{\mu\nu} e_J{}^\nu = \eta^{IJ} e_{J\mu} = \eta^{IJ} \eta_{JK} e^K{}_\mu = \delta^I{}_K e^K{}_\mu = e^I{}_\mu,$$

or inversely

$$\eta_{IJ} g^{\mu\nu} e^J{}_\nu = \eta_{IJ} g^{\mu\nu} \eta^{JK} g_{\nu\sigma} e_K{}^\sigma = \delta^K{}_I \delta^\mu{}_\sigma e_K{}^\sigma = e_I{}^\mu.$$

Now, define $e^{I\mu} := \eta^{IJ} e_J{}^\mu$. We then have

$$e^{I\mu} = \eta^{IJ} \eta_{JK} g^{\mu\nu} e^K{}_\nu = \delta^I{}_K g^{\mu\nu} e^K{}_\nu = g^{\mu\nu} e^I{}_\nu,$$

showing the last half of 7.

Q2: Assume that $\lambda^I{}_J \in O(\mathbb{R}^D, \eta_{IJ})$. Then

$$g'_{\mu\nu} = \eta_{IJ} e'^I{}_\mu e'^J{}_\nu = \eta_{IJ} \lambda^I{}_K e^K{}_\mu \lambda^J{}_L e^L{}_\nu$$

$$= (\eta_{IJ} \lambda^I{}_K \lambda^J{}_L) e^K{}_\mu e^L{}_\nu = \eta_{KL} e^K{}_\mu e^L{}_\nu = g_{\mu\nu}.$$

1.2. Cartan's Structure Equations

Q3: Let us first study the transformation of $e_I{}^\mu$ under $\lambda^I{}_J \in O(\mathbb{R}^D, \eta_{IJ})$. We have

$$e'_I{}^\mu = \eta_{IJ} g^{\mu\nu} e'^J{}_\nu = \eta_{IJ} g^{\mu\nu} \lambda^J{}_K e^K{}_\nu = \lambda_{IK} e^{K\mu}$$

$$= \lambda_I{}^K e_K{}^\mu.$$

Of course, $\lambda_I{}^K$ is easily computed by noting that

$$\lambda_K{}^I \lambda^K{}_J = \eta_{KL} \eta^{IM} \lambda^L{}_M \lambda^K{}_J = \eta_{JM} \eta^{IM} = \delta^I{}_J.$$

We can now compute

$$\omega'_\mu{}^I{}_J = e'^I{}_\nu \nabla_\mu e'_J{}^\nu = \lambda^I{}_K e^K{}_\nu \nabla_\mu (\lambda_J{}^L e_L{}^\nu)$$

$$= \lambda^I{}_K e^K{}_\nu \partial_\mu \lambda_J{}^L e_L{}^\nu + \lambda^I{}_K e^K{}_\nu \lambda_J{}^L \partial_\mu e_L{}^\nu$$

$$+ \lambda^I{}_K e^K{}_\nu \Gamma^\nu{}_{\mu\rho} \lambda_J{}^L e_L{}^\rho$$

$$= \lambda^I{}_K \partial_\mu \lambda_J{}^L \delta^K{}_L + \lambda^I{}_K e^K{}_\nu \lambda_J{}^L \nabla_\mu e_L{}^\nu$$

$$= \lambda^I{}_K \omega_\mu{}^K{}_L \lambda_J{}^L + \lambda^I{}_K \partial_\mu \lambda_J{}^K .$$

In terms of the one-forms,

$$\omega'^I{}_J = \omega'_\mu{}^I{}_J dx^\mu = \lambda^I{}_K \omega^K{}_L \lambda_J{}^L + \lambda^I{}_K \partial_\mu \lambda_J{}^K dx^\mu$$

$$= \lambda^I{}_K \omega^K{}_L \lambda_J{}^L + \lambda^I{}_K d\lambda_J{}^K .$$

In matrix notation, the equation $\lambda_K{}^I \lambda^K{}_J = \delta^I{}_J$ means that the matrix $M^I{}_J := \lambda_J{}^I$ has $M^T \lambda = I_D$. Therefore $M^T = \lambda^{-1}$. We conclude

$$\omega' = \lambda \omega M^T + \lambda d M^T = \lambda \omega \lambda^{-1} + \lambda d \lambda^{-1} .$$

Q4: We have

$$D_\mu v^I := e^I{}_\nu \nabla_\mu v^\nu = e^I{}_\nu \nabla_\mu (e_J{}^\nu v^J)$$

$$= e^I{}_\nu e_J{}^\nu \partial_\mu v^J + e^I{}_\nu \nabla_\mu e_J{}^\nu v^J$$

$$= \partial_\mu v^I + \omega_\mu{}^I{}_J v^J .$$

Under a transformation $\lambda^I{}_J \in O(\mathbb{R}^D, \eta_{IJ})$, we have

$$v^I e_I = v = v'^I e'_I = v'^I \lambda_I{}^J e_J .$$

Thus

$$v'^I = \delta^I{}_J v'^J = \lambda^I{}_K \lambda_J{}^K v'^J = \lambda^I{}_K v^K .$$

We conclude

$$
\begin{aligned}
(D_\mu v)'^I &= \partial_\mu v'^I + \omega'_\mu{}^I{}_J v'^J \\
&= \partial_\mu(\lambda^I{}_J v^J) + \lambda^I{}_K \omega_\mu{}^K{}_L \lambda_J{}^L v'^J + \lambda^I{}_K \partial_\mu \lambda_J{}^K v'^J \\
&= \partial_\mu \lambda^I{}_J v^J + \lambda^I{}_J \partial_\mu v^J + \lambda^I{}_K \omega_\mu{}^K{}_L \lambda_J{}^L \lambda^J{}_M v^M \\
&\quad + \lambda^I{}_K \partial_\mu \lambda_J{}^K \lambda^J{}_M v^M \\
&= \lambda^I{}_J \partial_\mu v^J + \lambda^I{}_K \omega_\mu{}^K{}_M v^M + \cancel{\partial_\mu \lambda^I{}_J v^J} \\
&\quad + \partial_\mu \overset{\to 0}{\cancel{(\lambda^I{}_K \lambda_J{}^K)}} \lambda^J{}_M v^M - \cancel{\partial_\mu \lambda^I{}_K \lambda_J{}^K \lambda^J{}_M v^M} \\
&= \lambda^I{}_J D_\mu v^J .
\end{aligned}
$$

Q5: Using the orthogonality condition

$$
\begin{aligned}
g^{\mu\nu} &= g^{\mu\sigma} g^{\nu\rho} g_{\sigma\rho} = g^{\mu\sigma} g^{\nu\rho} \eta_{IJ} e^I{}_\sigma e^J{}_\rho \\
&= \eta_{IJ} e^{I\mu} e^{J\nu} = \eta^{IJ} e_I{}^\mu e_J{}^\nu ,
\end{aligned}
$$

we can express the metrizity condition as

$$0 = \nabla_\mu g^{\nu\sigma} = \nabla_\mu \left(\eta^{IJ} e_I{}^\nu e_J{}^\sigma \right) = \eta^{IJ} \nabla_\mu e_I{}^\nu e_J{}^\sigma + \eta^{IJ} e_I{}^\nu \nabla_\mu e_J{}^\sigma .$$

Now, inverting the vielbein on the definition of the spin connection

$$e_I{}^\rho \omega_\mu{}^I{}_J = e_I{}^\rho e^I{}_\nu \nabla_\mu e_J{}^\nu = \delta^\rho{}_\nu \nabla_\mu e_J{}^\nu = \nabla_\mu e_J{}^\rho .$$

Then

$$0 = \eta^{IJ} e_K{}^\nu \omega_\mu{}^K{}_I e_J{}^\sigma + \eta^{IJ} e_I{}^\nu e_K{}^\sigma \omega_\mu{}^K{}_J$$

$$= e_K{}^\nu e_J{}^\sigma \omega_\mu{}^{KI} + e_I{}^\nu e_K{}^\sigma \omega_\mu{}^{KI}$$

$$= e_K{}^\nu e_J{}^\sigma \left(\omega_\mu{}^{KI} + \omega_\mu{}^{IK} \right) .$$

Since the veilbein is invertible, we conclude that the metricity condition is equivalent to

$$\omega^{IJ} = -\omega^{JI} .$$

For the Torsion free condition, note that we can invert $e^I := e^I{}_\mu dx^\mu$ to

$$dx^\mu = e_I{}^\mu e^I .$$

Moreover, recall that

$$[U,V]^\mu = \left(U^\nu \partial_\nu V^\sigma \partial_\sigma + U^\nu V^\sigma \cancel{\partial_\nu} \partial_\sigma - V^\nu \partial_\nu U^\sigma \partial_\sigma - V^\nu U^\sigma \cancel{\partial_\nu} \partial_\sigma\right)$$

$$= U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu V^\mu .$$

We then have

$$(de^I + \omega^I{}_J \wedge e^J)(e_K, e_L) = \partial_\mu e^I{}_\nu \, dx^\mu \wedge dx^\nu (e_K, e_L)$$

$$+ \omega_\mu{}^I{}_J \, dx^\mu \wedge e^J (e_K, e_L)$$

$$= \partial_\mu e^I{}_\nu e_K{}^\mu e_L{}^\nu - \partial_\mu e^I{}_\nu e_L{}^\mu e_K{}^\nu + \omega_\mu{}^I{}_J e_M{}^\mu e^M \wedge e^J (e_K, e_L)$$

$$= \partial_\mu (\cancel{e^I{}_\nu e_L{}^\nu})^{0} e_K{}^\mu - e^I{}_\nu \partial_\mu e_L{}^\nu e_K{}^\mu - \partial_\mu (\cancel{e^I{}_\nu e_K{}^\nu})^{0} e_L{}^\mu$$

$$+ e^I{}_\nu \partial_\mu e_K{}^\nu e_L{}^\mu + \omega_\mu{}^I{}_L e_K{}^\mu - \omega_\mu{}^I{}_K e_L{}^\mu$$

$$= e^I{}_\nu \nabla_\mu e_L{}^\nu e_K{}^\mu - e^I{}_\nu \nabla_\mu e_K{}^\nu e_L{}^\mu - e^I{}_\nu (e_K{}^\mu \partial_\mu e_L{}^\nu - e_L{}^\mu \partial_\mu e_K{}^\nu)$$

$$= e^I{}_\nu \left(\nabla_{e_K} e_L{}^\nu - \nabla_{e_L} e_K{}^\nu - [e_K, e_L]^\nu \right) = e^I{}_\nu T(e_K, e_L)^\nu$$

$$= e^I (T(e_K, e_L)).$$

Thus, the connection is free of torsion if and only if

$$de^I + \omega^I{}_J \wedge e^J = 0.$$

Q6: Using the result from Q4, we have

$$D_\mu D_\nu v^I = \partial_\mu (D_\nu v)^I + \omega_\mu{}^I{}_J (D_\nu v)^J$$

$$= \partial_\mu \partial_\nu v^I + \partial_\mu(\omega_\nu{}^I{}_J v^J) + \omega_\mu{}^I{}_J \partial_\nu v^J$$

$$+ \omega_\mu{}^I{}_J \omega_\nu{}^J{}_K v^K$$

$$= \partial_\mu \partial_\nu v^I + \partial_\mu \omega_\nu{}^I{}_J v^J + \omega_\nu{}^I{}_J \partial_\mu v^J$$

$$+ \omega_\mu{}^I{}_J \partial_\nu v^J + \omega_\mu{}^I{}_J \omega_\nu{}^J{}_K v^K$$

$$= \partial_\mu \partial_\nu v^I + \omega_{(\nu|}{}^I{}_J \partial_{|\mu)} v^J + \partial_\mu \omega_\nu{}^I{}_J v^J$$

$$+ \omega_\mu{}^I{}_J \omega_\nu{}^J{}_K v^K .$$

Therefore

$$[D_\mu, D_\nu] v^I = D_{[\mu} D_{\nu]} v^I = \partial_{[\mu} \omega_{\nu]}{}^I{}_J v^J + \omega_{[\mu|}{}^I{}_J \omega_{|\nu]}{}^J{}_K v^K$$

$$= (\partial_{[\mu} \omega_{\nu]}{}^I{}_J + \omega_{[\mu|}{}^I{}_K \omega_{|\nu]}{}^K{}_J) v^J .$$

Notice that

$$F^I{}_J := d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J = (\partial_\mu \omega_\nu{}^I{}_J + \omega_\mu{}^I{}_K \omega_\nu{}^K{}_J) dx^\mu \wedge dx^\nu$$

$$= \tfrac{1}{2} (\partial_{[\mu} \omega_{\nu]}{}^I{}_J + \omega_{[\mu|}{}^I{}_K \omega_{|\nu]}{}^K{}_J) dx^\mu \wedge dx^\nu .$$

Thus we conclude

$$[D_\mu, D_\nu] v^I = F^I{}_{J\mu\nu} v^J .$$

To obtain the Bianchi identity, note that

$$dF^I{}_J = \cancelto{0}{d^2\omega^I{}_J} + d\omega^I{}_K \wedge \omega^K{}_J - \omega^I{}_K \wedge d\omega^K{}_J$$

$$= (F^I{}_K - \cancel{\omega^I{}_L \wedge \omega^L{}_K}) \wedge \omega^K{}_J$$

$$- \omega^I{}_K \wedge (F^K{}_J - \cancel{\omega^K{}_L \wedge \omega^L{}_J})$$

$$= F^I{}_K \wedge \omega^K{}_J - \omega^I{}_K \wedge F^K{}_J .$$

Therefore

$$0 = dF^I{}_J + \omega^I{}_K \wedge F^K{}_J - \cancelto{1}{(-1)^{2\times 1}} \omega^K{}_J \wedge F^I{}_K$$

$$= DF^I{}_J .$$

1.3. The Einstein-Hilbert action in the first order formalism

Q7: We have

$$D_\mu D_\nu v^I = e^I{}_\rho \nabla_\mu D_\nu v^\rho = e^I{}_\rho \nabla_\mu (e_J{}^\rho D_\nu v^J)$$

$$= e^I{}_\rho \nabla_\mu (e_J{}^\rho e^J{}_\sigma \nabla_\nu v^\sigma) = e^I{}_\rho \nabla_\mu \nabla_\nu v^\rho .$$

Therefore

$$F^I{}_{J\mu\nu} v^J = [D_\mu, D_\nu] v^I = D_{[\mu} D_{\nu]} v^I = e^I{}_\rho \nabla_{[\mu} \nabla_{\nu]} v^\rho$$

$$= e^I{}_\rho [\nabla_\mu, \nabla_\nu] v^\rho = e^I{}_\rho R^\rho{}_{\sigma\mu\nu} v^\sigma$$

$$= e^I{}_\rho e_J{}^\sigma R^\rho{}_{\sigma\mu\nu} v^J .$$

We conclude

$$F^I{}_{J\mu\nu} = e^I{}_\rho e_J{}^\sigma R^\rho{}_{\sigma\mu\nu} .$$

Q8: From

$$g_{\mu\nu} = \eta_{IJ} e^I{}_\mu e^J{}_\nu$$

we have

$$\det g = \overset{\pm 1}{\cancel{\det \eta}} \det(e)^2 ,$$

i.e.

$$\sqrt{|\det g|} = \sqrt{|\det(e)^2|} = |\det(e)| .$$

On the other hand

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\alpha{}_{\mu\alpha\nu}$$

$$= \eta^{IJ} e_I{}^\mu e_J{}^\nu e_M{}^\alpha e^N{}_\mu F^M{}_{N\alpha\nu}$$

$$= \eta^{NJ} e_J{}^{\nu} e_M{}^{\alpha} F^M{}_{N\alpha\nu} = e_J{}^{\nu} e_M{}^{\alpha} F^{MN}{}_{\alpha\nu}.$$

We conclude that

$$16\pi G\, S(g) = \int d^D x \sqrt{|\det g|}\; R = \int d^D x\, |\det(e)|\, e_I{}^{\mu} e_J{}^{\nu} F^{IJ}{}_{\mu\nu}.$$

$$= 16\pi G\, S(e)$$

Q9: Specializing to 3D,

$$\epsilon_{IJK}\, e^I \wedge F^{JK} = \epsilon_{IJK}\, e^I{}_{\mu} F^{JK}{}_{\nu\rho}\, dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}$$

$$= \epsilon_{IJK}\, \epsilon^{\mu\nu\rho}\, e^I{}_{\mu} F^{JK}{}_{\nu\rho}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= \epsilon_{IJK}\, \epsilon^{\mu\nu\rho}\, e^I{}_{\mu} e^M{}_{\nu} e^N{}_{\rho} e_M{}^{\alpha} e_N{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \epsilon_{IJK} \det(e)\, \epsilon^{IMN} e_M{}^{\alpha} e_N{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \delta^M{}_{[J} \delta^N{}_{K]} \det(e)\, e_M{}^{\alpha} e_N{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \det(e)\, e_{[J}{}^{\alpha} e_{K]}{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= \det(e)\, e_J{}^{[\alpha} e_K{}^{\beta]} F^{JK}{}_{\alpha\beta}\, d^3x$$

$$= 2 \det(e)\, e_J{}^{\alpha} e_K{}^{\beta} F^{JK}{}_{\alpha\beta}\, d^3x$$

Therefore

$$S(e) = \frac{1}{16\pi G} \int d^3x \, |\det(e)| \, e_I{}^\mu e_J{}^\nu F^{IJ}{}_{\mu\nu}$$

$$= \frac{1}{16\pi G} \int d^3x \, \mathrm{sgn}(\det(e)) \det(e) \, e_I{}^\mu e_J{}^\nu F^{IJ}{}_{\mu\nu}$$

$$= \frac{1}{32\pi G} \int \mathrm{sgn}(\det(e)) \, \varepsilon_{IJK} \, e^I \wedge F^{JK}.$$

Q10: Let us start by varying e

$$0 = \delta S(e,\omega) = \frac{1}{32\pi G} \int \varepsilon_{IJK} \, \delta e^I \wedge F^{JK}$$

$$= \frac{1}{32\pi G} \int \varepsilon_{IJK} \, \delta e^I{}_\mu \, F^{JK}{}_{\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \frac{1}{32\pi G} \int d^3x \, \varepsilon_{IJK} \, \varepsilon^{\mu\nu\rho} \, F^{JK}{}_{\nu\rho} \, \delta e^I{}_\mu .$$

We conclude

$$0 = \varepsilon_{IJK} \, \varepsilon^{\mu\nu\rho} \, F^{JK}{}_{\nu\rho} .$$

Of course, this then implies

$$0 = \varepsilon^{IMN} \varepsilon_{\mu\alpha\beta} \, \varepsilon_{IJK} \, \varepsilon^{\mu\nu\rho} F^{JK}{}_{\nu\rho}$$

$$= (\delta^M{}_J \delta^N{}_K - \delta^M{}_K \delta^N{}_J)(\delta^\nu{}_\alpha \delta^\rho{}_\beta - \delta^\rho{}_\alpha \delta^\nu{}_\beta) F^{JK}{}_{\nu\rho}$$

$$= (\delta^M{}_J \delta^N{}_K - \delta^M{}_K \delta^N{}_J)(F^{JK}{}_{\alpha\beta} - F^{JK}{}_{\beta\alpha})$$

$$= 2(\delta^M{}_J \delta^N{}_K - \delta^M{}_K \delta^N{}_J) F^{JK}{}_{\alpha\beta}$$

$$= 2(F^{MN}{}_{\alpha\beta} - F^{NM}{}_{\alpha\beta}) = 4 F^{MN}{}_{\alpha\beta},$$

i.e. $F = 0$. Varying ω yields

$$0 = \delta S(e,\omega) = \frac{1}{32\pi G} \int \varepsilon_{IJK}\, e^I \wedge \delta F^{JK}.$$

Now,

$$\delta F^{IJ} = d\,\delta\omega^{IJ} + \delta\omega^I{}_K \wedge \omega^{KJ} + \omega^I{}_K \wedge \delta\omega^{KJ}.$$

Noting that

$$\delta\omega^I{}_K \wedge \omega^{KJ} = -\delta\omega^I{}_K \wedge \omega^{JK} = -\delta\omega^{IK} \wedge \omega^J{}_K$$

$$= \delta\omega^{KI} \wedge \omega^J{}_K = -\omega^J{}_K \wedge \delta\omega^{KI},$$

we have

$$\delta F^{IJ} = d\,\delta\omega^{IJ} - \omega^J{}_K \wedge \delta\omega^{KI} + \omega^I{}_K \wedge \delta\omega^{KJ}.$$

Now,

$$\varepsilon_{IJK}\, e^I \wedge d\,\delta\omega^{JK} = -d(\varepsilon_{IJK}\, e^I \wedge \delta\omega^{JK}) + \varepsilon_{IJK}\, de^I \wedge \delta\omega^{JK}$$

$$-\varepsilon_{IJK}\, e^I \wedge \omega^K{}_L \wedge \delta\omega^{LJ} = -\varepsilon_{IKL}\, e^I \wedge \omega^L{}_J \wedge \delta\omega^{JK}$$

$$\varepsilon_{IJK}\, e^I \wedge \omega^J{}_L \wedge \delta\omega^{LK} = \varepsilon_{ILK}\, e^I \wedge \omega^L{}_J \wedge \delta\omega^{JK},$$

Therefore

$$\delta S(e,\omega) = -\int_{\partial M} \varepsilon_{IJK}\, e^I \wedge \delta\omega^{JK}$$

$$+ \int_M \left(\varepsilon_{IJK}\, de^I - \varepsilon_{IKL}\, e^I \wedge \omega^L{}_J + \varepsilon_{ILK}\, e^I \wedge \omega^L{}_J\right) \wedge \delta\omega^{JK}$$

$$= -\int_{\partial M} \varepsilon_{IJK}\, e^I \wedge \delta\omega^{JK}$$

$$+ \int_M \left(\varepsilon_{IJK}\, de^I + 2\varepsilon_{ILK}\, e^I \wedge \omega^L{}_J\right) \wedge \delta\omega^{JK}.$$

Now, assume $\delta\omega = 0$ on ∂M. Then

$$0 = \delta S(e,\omega) = \int_M \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + 2\varepsilon_{ILK}\, e^I{}_\mu \omega_\nu{}^L{}_J\right) \delta\omega_\rho{}^{JK}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \int_M d^3x\, \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + 2\varepsilon_{ILK}\, e^I{}_\mu \omega_\nu{}^L{}_J\right) \delta\omega_\rho{}^{JK}$$

$$= \int_M d^3x\, \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + \varepsilon_{IL[K|}\, e^I{}_\mu \omega_\nu{}^L{}_{|J]}\right) \delta\omega_\rho{}^{JK}.$$

We conclude

$$0 = \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + \varepsilon_{IL[K|}\, e^I{}_\mu \omega_\nu{}^L{}_{|J]}\right)$$

Therefore,

$$0 = \varepsilon^{MJK} \varepsilon_{\alpha\beta\rho}\, \varepsilon^{\mu\nu\rho} \left(\varepsilon_{IJK}\, \partial_\mu e^I{}_\nu + \varepsilon_{IL[K|}\, e^I{}_\mu\, \omega_\nu{}^L{}_{|J]}\right)$$

$$= \varepsilon_{\alpha\beta\rho}\, \varepsilon^{\mu\nu\rho} \left(2\partial_\mu e^M{}_\nu + 2\varepsilon^{MJK} \varepsilon_{ILK}\, e^I{}_\mu\, \omega_\nu{}^L{}_J\right)$$

$$= 2\varepsilon_{\alpha\beta\rho}\varepsilon^{\mu\nu\rho}\left(\partial_\mu e^M{}_\nu + e^M{}_\mu \omega_\nu{}^J{}_J - e^J{}_\mu \omega_\nu{}^M{}_J\right)$$

$$= 2(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha)\left(\partial_\mu e^M{}_\nu + \underbrace{\cancel{e^M{}_\mu \eta_{JK}\omega_\nu{}^{JK}}}_{0} - e^J{}_\mu \omega_\nu{}^M{}_J\right)$$

$$= 2\left(\partial_\mu e^M{}_\nu - \partial_\nu e^M{}_\mu - \omega_\nu{}^M{}_J e^J{}_\mu + \omega_\mu{}^M{}_J e^J{}_\nu\right).$$

$$= 2T^M{}_{\mu\nu}.$$

Indeed

$$T^M = de^M + \omega^M{}_J \wedge e^J = \partial_\mu e^M{}_\nu \, dx^\mu \wedge dx^\nu + \omega_\mu{}^M{}_J e^J{}_\nu \, dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2}\left(\partial_\mu e^M{}_\nu - \partial_\nu e^M{}_\mu + \omega_\mu{}^M{}_J e^J{}_\nu - \omega_\nu{}^M{}_J e^J{}_\mu\right) dx^\mu \wedge dx^\nu.$$

We conclude the second EOM is $T = 0$.

1.4. Global Symmetries of Euclidean 3-space

Q11: From the action

$$(R_2, a_2)(R_1, a_1)x = (R_2, a_2)(R_1 x + a_1) = R_2(R_1 x + a_1) + a_2$$

$$= R_2 R_1 x + R_2 a_1 + a_2 = (R_2 R_1, R_2 a_1 + a_2)x,$$

it is clear that the product structure on $ISO(3)$ is

$$(R_2, a_2)(R_1, a_1) = (R_2 R_1, R_2 a_1 + a_2).$$

From this it is clear that the identity element is $(I_3, 0)$ and the inverse of (R, a) is $(R^{-1}, -R^{-1}a)$.

Indeed

$$(I_3, 0)(R, a) = (R, I_3 a + 0) = (R, a)$$

and

$$(R^{-1}, -R^{-1}a)(R, a) = (\cancel{R^{-1}R}, \cancel{R^{-1}}a - \cancel{R^{-1}}a) = (I_3, 0).$$

Let us now study the Lie algebra $so(3)$. Assume we have a rotation $R = I_3 + \omega$ with ω infinitesimal.

Then

$$I_3 = R_3^T R_3 = (I_3 + \omega^T)(I_3 + \omega) = I_3 + \omega + \omega^T + \mathcal{O}(\omega^2).$$

Thus

$$\omega = -\omega^T.$$

Defining $(M_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}$, we have that $\{M_{12}, M_{23}, M_{31}\}$ is a basis for $so(3)$.

To study the full Lie algebra $iso(3)$, it is useful to recognize $ISO(3)$ as a matrix Lie group via the identification

$$ISO(3) \longrightarrow M_4(\mathbb{R})$$

$$(R, a) \longmapsto \left[\begin{array}{c|c} R & a \\ \hline 0 & 1 \end{array}\right].$$

This is clearly injective. Moreover, it is a homomorphism since

$$\left[\begin{array}{c|c} R_2 & a_2 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_1 & a_1 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R_2 R_1 & R_2 a_1 + a_2 \\ \hline 0 & 1 \end{array}\right].$$

Thus, it's image is isomorphic to $ISO(3)$. We conclude that the Lie algebra is

$$iso(3) = \left\{ \left[\begin{array}{c|c} \omega & a \\ \hline 0 & 0 \end{array}\right] \,\middle|\, \omega = -\omega^T,\ a \in \mathbb{R}^3 \right\} \subseteq M_4(\mathbb{R}).$$

A basis for this is given by $\{M_{12}, M_{23}, M_{31}, e_1, e_2, e_3\}$ where $(M_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ and $(P_i)_{kl} = \delta_{4l}\delta_{ik}$.

Note that

$$(M_{ij} M_{kl})_{mn} = (M_{ij})_{mr}(M_{kl})_{rn} = (\delta_{im}\delta_{jr} - \delta_{ir}\delta_{jm})(\delta_{kr}\delta_{ln} - \delta_{kn}\delta_{lr})$$

$$= \delta_{jk}\delta_{im}\delta_{ln} - \delta_{jl}\delta_{im}\delta_{kn} - \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn},$$

so that

$$[M_{ij}, M_{kl}]_{mn} = \delta_{jk}\delta_{im}\delta_{ln} - \delta_{jl}\delta_{im}\delta_{kn} - \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn}$$

$$- \delta_{li}\delta_{km}\delta_{jn} + \delta_{lj}\delta_{km}\delta_{in} + \delta_{ki}\delta_{lm}\delta_{jn} - \delta_{kj}\delta_{lm}\delta_{in}$$

$$= \delta_{jk}(\delta_{im}\delta_{ln} - \delta_{lm}\delta_{in}) - \delta_{jl}(\delta_{im}\delta_{kn} - \delta_{km}\delta_{in})$$

$$- \delta_{ik}(\delta_{jm}\delta_{ln} - \delta_{lm}\delta_{jn}) + \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn})$$

$$= \delta_{jk}(M_{il})_{mn} - \delta_{jl}(M_{ik})_{mn} - \delta_{ik}(M_{jl})_{mn} + \delta_{il}(M_{jk})_{mn}.$$

On the other hand, for all $i,j \in \{1,2,3\}$

$$(P_i P_j)_{mn} = (P_i)_{mr}(P_j)_{rn} = \delta_{im}\delta_{4r}\delta_{jr}\delta_{4n} = \delta_{im}\cancelto{0}{\delta_{4j}}\delta_{4n} = 0.$$

Finally, taking $i,j,k \in \{1,2,3\}$

$$(M_{ij}P_k)_{mn} = (M_{ij})_{mr}(P_k)_{rn} = (\delta_{im}\delta_{jr} - \delta_{ir}\delta_{jm})\delta_{kr}\delta_{4n}$$

$$= \delta_{im}\delta_{jk}\delta_{4n} - \delta_{ik}\delta_{jm}\delta_{4n} = \delta_{jk}(P_i)_{mn} - \delta_{ik}(P_j)_{mn}$$

and

$$(P_k M_{ij})_{mn} = (P_k)_{mr}(M_{ij})_{rn} = \delta_{km}\delta_{4r}(\delta_{ir}\delta_{jn} - \delta_{in}\delta_{jr})$$

$$= \delta_{km}\cancelto{0}{\delta_{4i}}\delta_{jn} - \delta_{km}\delta_{in}\cancelto{0}{\delta_{j4}} = 0,$$

We conclude the Lie algebra is

$$[M_{ij}, M_{kl}] = \delta_{jk} M_{il} + \delta_{il} M_{jk} - \delta_{jl} M_{ik} - \delta_{ik} M_{jl}$$

$$[M_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j$$

$$[P_i, P_j] = 0$$

For $\{M_{12}, M_{23}, M_{31}, P_1, P_2, P_3\}$. For future reference, define $J_i = \frac{1}{2} \varepsilon_{ijk} M_{jk}$. Then

$$[J_i, J_j] = \frac{1}{4} \varepsilon_{imn} \varepsilon_{jrs} [M_{mn}, M_{rs}]$$

$$= \frac{1}{4} \varepsilon_{imn} \varepsilon_{jrs} \left(\delta_{nr} M_{ms} + \delta_{ms} M_{nr} - \delta_{ns} M_{mr} - \delta_{mr} M_{ns} \right)$$

$$= \frac{1}{4} \left(\varepsilon_{imr} \varepsilon_{jrs} M_{ms} + \varepsilon_{isn} \varepsilon_{jrs} M_{nr} - \varepsilon_{ims} \varepsilon_{jrs} \overset{M_{mr}}{-} \varepsilon_{irn} \varepsilon_{jrs} M_{ns} \right)$$

$$= \frac{1}{4} \left(-\delta_{ij} \cancel{M_{mm}}^{0} + M_{ji} - \delta_{ij} \cancel{M_{nn}}^{0} + M_{ji} - \delta_{ij} \cancel{M_{mm}}^{0} + M_{ji} - \delta_{ij} \cancel{M_{nn}}^{0} + M_{ji} \right)$$

$$= M_{ji} = \varepsilon_{jik} J_k .$$

Redefining instead $J_i = -\frac{1}{2} \varepsilon_{ijk} M_{jk}$, we have

$$[J_i, J_j] = \varepsilon_{ijk} J_k .$$

In terms of these we also have

$$[J_i, P_j] = -\frac{1}{2}\varepsilon_{imn}[M_{mn}, P_j] = -\frac{1}{2}\varepsilon_{imn}(\delta_{nj}P_m - \delta_{mj}P_n)$$

$$= -\frac{1}{2}(\varepsilon_{imj}P_m - \varepsilon_{ijn}P_n) = \varepsilon_{ijn}P_n .$$

We just finally need to check that indeed

$M_{ij} = -\varepsilon_{ijk}P_k$. Indeed

$$-\varepsilon_{ijk}P_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}M_{lm} = \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})M_{lm}$$

$$= \frac{1}{2}(M_{ij} - M_{ji}) = M_{ij} .$$

Optional Exercise: We have that

$$f_{ab}{}^{c} f_{cd}{}^{e} = f_{ab}{}^{c}[t_c^{ad}]^e{}_d = [[t_a^{ad}], [t_b^{ad}]]^e{}_d$$

$$= [t_a^{ad}]^e{}_c [t_b^{ad}]^c{}_d - [t_b^{ad}]^e{}_c [t_a^{ad}]^c{}_d$$

$$= f_{ac}{}^{e} f_{bd}{}^{c} - f_{bc}{}^{e} f_{ad}{}^{c} ,$$

i.e.

$$f_{ab}{}^{c} f_{cd}{}^{e} + f_{bd}{}^{c} f_{ca}{}^{e} + f_{da}{}^{c} f_{cb}{}^{e} = 0 .$$

This coincides with the Jacobi identity. Indeed

$$[[t_a, t_b], t_d] = f_{ab}{}^{c}[t_c, t_d] = f_{ab}{}^{c} f_{cd}{}^{e}\, t_e,$$

and

$$0 = [[t_a, t_b], t_d] + [[t_b, t_d], t_a] + [[t_d, t_a], t_b]$$

$$= f_{ab}{}^{c} f_{cd}{}^{e} + f_{bd}{}^{c} f_{ca}{}^{e} + f_{da}{}^{c} f_{cb}{}^{e}.$$

Q 12: We have

$$[J_1^{ad}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad [J_2^{ad}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$[J_3^{ad}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and

$$[[J_1^{ad}], [J_2^{ad}]] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [J_3^{ad}],$$

$$[[J_2^{ad}], [J_3^{ad}]] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = [J_1^{ad}],$$

$$[[J_3^{ad}], [J_1^{ad}]] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = [J_2^{ad}].$$

Q 13: We indeed have

$$F^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J = d\omega^M [J^{ad}_M]^I{}_J + \omega^M \wedge \omega^N [J^{ad}_M]^I{}_K [J^{ad}_N]^K{}_J$$

$$= d\omega^M [J^{ad}_M]^I{}_J + \omega^M \wedge \omega^N [J^{ad}_M J^{ad}_N]^I{}_J$$

$$= d\omega^M [J^{ad}_M]^I{}_J + \tfrac{1}{2} \omega^M \wedge \omega^N [[J^{ad}_M, J^{ad}_N]]^I{}_J$$

$$= d\omega^M [J^{ad}{}_M]^I{}_J + \tfrac{1}{2} \varepsilon_{KMN} \omega^M \wedge \omega^N [J^{ad}_K]^I{}_J$$

$$= \left(d\omega^K + \tfrac{1}{2} \varepsilon_{KMN} \omega^M \wedge \omega^N \right) [J^{ad}_K]^I{}_J .$$

Q14: Let us check that the pairing is indeed invariant.

We have

- $\langle [J_I, J_J], J_K \rangle + \langle J_J, [J_I, J_K] \rangle = \varepsilon_{IJL} \cancelto{0}{\langle J_L, J_K \rangle} + \varepsilon_{IKL} \cancelto{0}{\langle J_J, J_L \rangle}$

$$= 0,$$

- $\langle [P_I, J_J], J_K \rangle + \langle J_J, [P_I, J_K] \rangle = -\varepsilon_{IJL} \delta_{LK} - \varepsilon_{IKL} \delta_{JL}$

$$= -\varepsilon_{IJK} - \varepsilon_{IKJ} = 0$$

- $\langle [J_I, J_J], P_K \rangle + \langle J_J, [J_I, P_K] \rangle = \varepsilon_{IJL} \delta_{LK} + \varepsilon_{IKL} \delta_{JL}$

$$= \varepsilon_{IJK} + \varepsilon_{IKJ} = 0$$

- $\langle [P_I, J_J], P_K \rangle + \langle J_J, \cancelto{0}{[P_I, P_K]} \rangle = -\varepsilon_{IJL} \cancelto{0}{\langle P_L, P_K \rangle} = 0$

- $\langle [J_I, P_J], P_K \rangle + \langle P_J, [J_I, P_K] \rangle = \varepsilon_{IJL} \cancelto{0}{\langle P_L, P_K \rangle} + \varepsilon_{IKL} \cancelto{0}{\langle P_J, P_L \rangle} = 0$

- $\langle [P_I, P_J], P_K \rangle + \langle P_J, [P_I, P_K] \rangle = 0$.

Q15: We clearly have

$$f_{abc} = d_{dc} f_{ab}{}^{d} = -d_{dc} f_{ba}{}^{d} = -f_{bac}.$$

Thus, we only need to check $f_{abc} = -f_{acb}$. Indeed

$$f_{abc} = d_{dc} f_{ab}{}^{d} = \langle t_d, t_c \rangle f_{ab}{}^{d} = \langle [t_a, t_b], t_c \rangle$$

$$= -\langle t_b, [t_a, t_c] \rangle = -f_{ac}{}^{d} \langle t_b, t_d \rangle = -d_{db} f_{ac}{}^{d} = -f_{acb}.$$

Q16: For the moment, let us denote $(J_1, J_2, J_3, P_1, P_2, P_3)_{a=1}^{6}$ with $(J_1, J_2, J_3)_{I=1}^{3}$ and $(P_1, P_2, P_3)_{\tilde{I}=1}^{3}$. With these indices we have the non-vanishing structure constants

$$f_{IJ}{}^{K} = \varepsilon_{IJK}, \quad f_{I\tilde{J}}{}^{\tilde{K}} = \varepsilon_{I\tilde{J}\tilde{K}}.$$

Noticing that

$$f_{abI} = d_{dI} f_{ab}{}^{d} = f_{ab}{}^{\tilde{I}}, \quad f_{ab\tilde{I}} = d_{d\tilde{I}} f_{ab}{}^{d} = f_{ab}{}^{I},$$

we have the non-vanishing constants

$$f_{I\tilde{J}K} = \varepsilon_{I\tilde{J}\tilde{K}}, \quad f_{IJ\tilde{K}} = f_{IJ}{}^{K} = \varepsilon_{IJK},$$

which are of course related by symmetry

We thus have

$$\frac{1}{3} f_{abc} A^a \wedge A^b \wedge A^c = \frac{3}{3} \epsilon_{IJK}\, \omega^I \wedge e^J \wedge \omega^K = \epsilon_{IJK}\, e^I \wedge \omega^J \wedge \omega^K .$$

Moreover

$$d_{ab} A^a \wedge dA^b = \omega^I \wedge de^I + e^I \wedge d\omega^I = -d(\omega^I \wedge e^I) + d\omega^I \wedge e^I + e^I \wedge d\omega^I$$

$$= -d(\omega^I \wedge e^I) + 2 e^I \wedge d\omega^I .$$

Therefore,

$$S_{cs}(A) = \frac{K}{4\pi} \left(-\int_{\partial M} \omega^I \wedge e^I + 2 \int_M e^I \wedge \left(d\omega^I + \frac{1}{2} \epsilon_{IJK}\, \omega^J \wedge \omega^K \right) \right) .$$

Thus, up to boundary terms,

$$S_{cs}(A) = \frac{K}{2\pi} \int_M e^I \wedge F^I = \frac{1}{8\pi G} \int_M e^I \wedge F^I .$$

Notice that boundary terms do not affect the EOMs because to obtain the latter we consider variations that vanish at the boundary. This is possible since we are in a first order formalism.

Q17: We have, ignoring boundary terms,

$$\delta S_{cs}(A) = \frac{k}{4\pi}\int_M \Big(d_{ab}\,\delta A^a \wedge dA^b + d_{ab}A^a \wedge d\delta A^b + \tfrac{1}{3} f_{abc}\,\delta A^a \wedge A^b \wedge A^c$$

$$+ \tfrac{1}{3} f_{abc} A^a \wedge \delta A^b \wedge A^c + \tfrac{1}{3} f_{abc} A^a \wedge A^b \wedge \delta A^c \Big)$$

$$= \frac{k}{4\pi}\int_M \Big(d_{ab}\,\delta A^a \wedge dA^b - d_{ab}\, d(A^a \wedge \delta A^b) + d_{ab}\, dA^a \wedge \delta A^b$$

$$+ f_{abc} A^a \wedge A^b \wedge \delta A^c \Big)$$

$$= \frac{k}{4\pi}\int_M \Big(2 d_{ab}\,\delta A^a \wedge dA^b + d_{ab} f_{cd}{}^{b} A^c \wedge A^d \wedge \delta A^a \Big)$$

$$= \frac{k}{4\pi} 2 d_{ab} \int_M \Big(dA^b + \tfrac{1}{2} f_{cd}{}^{b} A^c \wedge A^d \Big) \wedge \delta A^a$$

$$= \frac{k}{2\pi} \Big\langle \int_M \Big(dA + \tfrac{1}{2}[t_c, t_d] A^c \wedge A^d \Big) \wedge \delta A^a \,,\, t_a \Big\rangle$$

$$= \frac{k}{2\pi} \Big\langle \int_M (dA + A \wedge A) \wedge \delta A^a \,,\, t_a \Big\rangle .$$

Now, eventhough our pairing is not an inner product, it wasn't degenerate. Therefore

$$0 = \int_M (dA + A \wedge A) \wedge \delta A^a = \int_M F(A) \wedge \delta A^a$$

This implies $F(A) = 0$. To see this,

$$0 = \int_M \mathcal{F}(A)_{\mu\nu} \delta A^a{}_\rho \, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \int_M d^3x \; \varepsilon^{\mu\nu\rho} \mathcal{F}(A)_{\mu\nu} \, \delta A_\rho$$

implies $\qquad \varepsilon^{\mu\nu\rho} \mathcal{F}(A)_{\mu\nu} = 0. \qquad$ Then

$$0 = \varepsilon_{\alpha\beta\rho} \varepsilon^{\mu\nu\rho} \mathcal{F}(A)_{\mu\nu} = (\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha) \mathcal{F}(A)_{\mu\nu}$$

$$= \mathcal{F}(A)_{\alpha\beta} - \mathcal{F}(A)_{\beta\alpha} = 2 \mathcal{F}(A)_{\alpha\beta}.$$

Furthermore [illegible]

$$\mathcal{F}(A) = \mathcal{F}^a(A) t_a = \left(dA^I + \tfrac{1}{2} f_{ab}{}^I A^a \wedge A^b\right) t_I$$

$$+ \left(dA^{\tilde{I}} + \tfrac{1}{2} f_{ab}{}^{\tilde{I}} A^a \wedge A^b\right) t_{\tilde{I}}$$

$$= \left(d\omega^I + \tfrac{1}{2} \varepsilon_{JKI} \omega^J \wedge \omega^K\right) J_I + \left(de^I + \tfrac{1}{2} \varepsilon_{JKI} \omega^J \wedge e^K\right.$$

$$= F \; [illegible] \; + \tfrac{1}{2} \varepsilon_{JKI} e^J \wedge \omega^K \Big) P_I$$

$$= F + \left(de^I + \varepsilon_{JKI} \omega^J \wedge e^K\right) P_I = F + \left(de^I - \omega^{KI} \wedge e^K\right) P_I$$

$$= F + \left(de^I + \omega^I{}_K \wedge e^K\right) P_I = F + T.$$

Thus, $\quad \mathcal{F}(A) = 0 \quad$ implies the EOMs $\; F = 0 \;$ and $\; T = 0.$