Homework 1

1. Cartan’s Formalism: FRW Cosmology

e)

We start up by setting up our coordinates and the FRW metric

In[43]:=

```
coord = {t, r, θ, ϕ};
g = {{1, 0, 0, 0}, {0, -a[t]^2/(1 - k r^2), 0, 0}, {0, 0, -a[t]^2 r^2, 0}, {0, 0, 0, -a[t]^2 r^2 Sin[θ]^2}};
g // MatrixForm
```

Out[45]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a[t]^2}{1-k r^2} & 0 & 0 \\ 0 & 0 & -r^2 a[t]^2 & 0 \\ 0 & 0 & 0 & -r^2 a[t]^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

We compute the Christoffel symbols

```
Γ = Table[Sum[1/2 Inverse[g][[μ, σ]]
     (D[g[[σ, ν]], coord[[ρ]]] + D[g[[σ, ρ]], coord[[ν]]] - D[g[[ρ, ν]], coord[[σ]]]),
    {σ, 1, 4}], {μ, 1, 4}, {ν, 1, 4}, {ρ, 1, 4}]
```

Out[48]=

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{a[t]\,a'[t]}{1-k r^2},0,0\right\},\left\{0,0,r^2 a[t]\,a'[t],0\right\},\left\{0,0,0,r^2 a[t]\,\mathrm{Sin}[\theta]^2 a'[t]\right\}\right\},\left\{\left\{0,\frac{a'[t]}{a[t]},0,0\right\},\left\{\frac{a'[t]}{a[t]},\frac{k r}{1-k r^2},0,0\right\},\left\{0,0,-r\left(1-k r^2\right),0\right\},\left\{0,0,0,-r\left(1-k r^2\right)\mathrm{Sin}[\theta]^2\right\}\right\},\left\{\left\{0,0,\frac{a'[t]}{a[t]},0\right\},\left\{0,0,\frac{1}{r},0\right\},\left\{\frac{a'[t]}{a[t]},\frac{1}{r},0,0\right\},\{0,0,0,-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\}\right\},\left\{\left\{0,0,0,\frac{a'[t]}{a[t]}\right\},\left\{0,0,0,\frac{1}{r}\right\},\{0,0,0,\mathrm{Cot}[\theta]\},\left\{\frac{a'[t]}{a[t]},\frac{1}{r},\mathrm{Cot}[\theta],0\right\}\right\}\right\}$$

We may now compute the Riemann tensor

In[50]:=
```
riem = Table[D[Γ[[α, δ, β]], coord[[γ]]] - D[Γ[[α, γ, β]], coord[[δ]]] + Sum[
    Γ[[α, γ, λ]] × Γ[[λ, δ, β]], {λ, 1, 4}] - Sum[Γ[[α, δ, λ]] × Γ[[λ, γ, β]], {λ, 1, 4}],
   {α, 1, 4}, {β, 1, 4}, {γ, 1, 4}, {δ, 1, 4}] // Simplify
```

Out[50]=

$$\left\{\left\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\right.\right.$$
$$\left\{\left\{0,\frac{a[t]\,a''[t]}{1-k\,r^2},0,0\right\},\left\{\frac{a[t]\,a''[t]}{-1+k\,r^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},$$
$$\left\{\left\{0,0,r^2\,a[t]\,a''[t],0\right\},\{0,0,0,0\},\left\{-r^2\,a[t]\,a''[t],0,0,0\right\},\{0,0,0,0\}\right\},$$
$$\left\{\left\{0,0,0,r^2\,a[t]\,\mathrm{Sin}[\theta]^2\,a''[t]\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{-r^2\,a[t]\,\mathrm{Sin}[\theta]^2\,a''[t],0,0,0\right\}\right\}\Big\},$$
$$\left\{\left\{\left\{0,\frac{a''[t]}{a[t]},0,0\right\},\left\{-\frac{a''[t]}{a[t]},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\right.$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,r^2\left(k+a'[t]^2\right),0\right\},\left\{0,-r^2\left(k+a'[t]^2\right),0,0\right\},\{0,0,0,0\}\right\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\left(k+a'[t]^2\right)\right\},\{0,0,0,0\},\left\{0,-r^2\,\mathrm{Sin}[\theta]^2\left(k+a'[t]^2\right),0,0\right\}\right\}\Big\},$$
$$\left\{\left\{\left\{0,0,\frac{a''[t]}{a[t]},0\right\},\{0,0,0,0\},\left\{-\frac{a''[t]}{a[t]},0,0,0\right\},\{0,0,0,0\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,0,\frac{k+a'[t]^2}{-1+k\,r^2},0\right\},\left\{0,\frac{k+a'[t]^2}{1-k\,r^2},0,0\right\},\{0,0,0,0\}\right\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\Big\{\{0,0,0,0\},\{0,0,0,0\},$$
$$\left\{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\left(k+a'[t]^2\right)\right\},\left\{0,0,-r^2\,\mathrm{Sin}[\theta]^2\left(k+a'[t]^2\right),0\right\}\Big\}\Big\},$$
$$\left\{\left\{\left\{0,0,0,\frac{a''[t]}{a[t]}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{-\frac{a''[t]}{a[t]},0,0,0\right\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,\frac{k+a'[t]^2}{-1+k\,r^2}\right\},\{0,0,0,0\},\left\{0,\frac{k+a'[t]^2}{1-k\,r^2},0,0\right\}\right\},$$
$$\left\{\{0,0,0,0\},\{0,0,0,0\},\left\{0,0,0,-r^2\left(k+a'[t]^2\right)\right\},\left\{0,0,r^2\left(k+a'[t]^2\right),0\right\}\right\},$$
$$\left.\left.\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\right\}\right\}$$

In particular, we see that they agree with the ones computed. Presenting the output in the same order as they were written in the homework, we have

In[66]:=
```
riem[[1, 2, 1, 2]]
riem[[1, 3, 1, 3]]
riem[[1, 4, 1, 4]]
riem[[2, 3, 2, 3]]
riem[[2, 4, 2, 4]]
riem[[3, 4, 3, 4]]
```

Out[66]= $\frac{a[t]\, a''[t]}{1 - k\, r^2}$

Out[67]= $r^2\, a[t]\, a''[t]$

Out[68]= $r^2\, a[t]\, \mathrm{Sin}[\theta]^2\, a''[t]$

Out[69]= $r^2 \left(k + a'[t]^2\right)$

Out[70]= $r^2\, \mathrm{Sin}[\theta]^2 \left(k + a'[t]^2\right)$

Out[71]= $r^2\, \mathrm{Sin}[\theta]^2 \left(k + a'[t]^2\right)$

Finally, we compute the Ricci tensor, which agrees with the one presented in the homework

In[77]:=
```
ricci = Table[Sum[riem[[α, μ, α, ν]], {α, 1, 4}], {μ, 1, 4}, {ν, 1, 4}] // Simplify;
ricci // MatrixForm
```

Out[78]//MatrixForm=

$$\begin{pmatrix} -\frac{3\, a''[t]}{a[t]} & 0 & 0 & 0 \\ 0 & \frac{2\,k+2\,a'[t]^2+a[t]\,a''[t]}{1-k\,r^2} & 0 & 0 \\ 0 & 0 & r^2 \left(2 \left(k + a'[t]^2\right) + a[t]\, a''[t]\right) & 0 \\ 0 & 0 & 0 & r^2\, \mathrm{Sin}[\theta]^2 \left(2 \left(k + a'[t]^2\right) + a[t]\, a''[\right. \end{pmatrix}$$