Iván Mauricio Burbano Aldana

Perimeter Institute for Theoretical Physics

Homework 3: Light Bending in Newton's and Einstein's Gravity

1.a) In a centrally symmetric field, one can, in apropriate inertial reference frame, describe the force felt by a particle in the form

$$\vec{F}(t,\vec{r},\vec{v}) = f(t,\vec{r},\vec{v})\,\vec{r}$$

for some $f:\mathbb{R}\times U\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ with $U\subseteq\mathbb{R}^3$ open. Then, for every $\mathcal{C}^2$ trajectory $\vec{\gamma}:(t_0,t_f)\to\pi(U)$, whose angular momentum is $\vec{L}:(t_0,t_f)\longrightarrow\mathbb{R}^3: t\longmapsto m\vec{\gamma}(t)\times\vec{\gamma}'(t)$, satisfies

$$\vec{L}'(t) = m\underbrace{\cancel{\vec{\gamma}'(t)\times\vec{\gamma}'(t)}}_{\vec{0}} + m\vec{\gamma}(t)\times\vec{\gamma}''(t) = m\vec{\gamma}(t)\times\vec{F}(t,\vec{\gamma}(t),\vec{\gamma}'(t))$$

$$= m f(t,\vec{\gamma}(t),\vec{\gamma}'(t))\underbrace{\cancel{\vec{\gamma}(t)\times\vec{\gamma}(t)}}_{\vec{0}} = \vec{0}.$$

Thus, there exists $\vec{l}\in\mathbb{R}^3$ s.t. $\vec{L}(t)=\vec{l}$ for all $t\in(t_0,t_f)$. Since $\vec{\gamma}(t)\in\{\vec{L}(t)\}^\perp=\{\vec{l}\}^\perp$ for all $t\in(t_0,t_f)$, we conclude that, the trajectory is contained in $\{\vec{\ell}\}^{\perp}$, which is a plane as long as $\vec{\ell} \neq \vec{0}$. Even in the case $\vec{\ell} = 0$, we have that $\vec{\gamma}'(t) \propto \vec{\gamma}(t)$ for all $t \in (t_0, t_f)$. To stop being pedantic, this means the trajectory is contained in a line or a point.

b) The Lagrangian is

$$L : (0,\infty) \times (0, 2\pi) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(r, \phi, v_r, v_\phi) \longmapsto \frac{1}{2} m \left(v_r^2 + r^2 v_\phi^2 \right) + \frac{GMm}{r}$$

Since L is ϕ independent,

We will rather take L per unit mass

Sorry

$$L = \frac{\partial L}{\partial v_\phi}(r, \phi, v_r, v_\phi) = m r^2 v_\phi^2$$

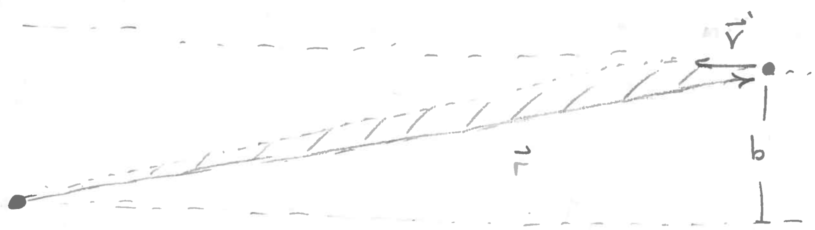
is conserved. Similarly, time independence guarantees the conservation of energy

$$E = \frac{\partial L}{\partial v_r} v_r + \frac{\partial L}{\partial v_\phi} v_\phi - L = m v_r^2 + m r^2 v_\phi - L$$

$$= \frac{1}{2} m (v_r^2 + r^2 v_\phi^2) - \frac{GMm}{r}.$$

In particular redefining E by E/m, we still have a conserved quantity.

c) E is the energy per unit mass. L is the area spanned by the parallelogram formed between the position vector and the the velocity



From the picture above we see

$$L = \|\vec{v}\| b = cb = 1b = b$$

since light moves at the speed of light far away from gravitational sources.

d) We have

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2} r^2 \left(\frac{d\phi}{dr}\right)^2 \left(\frac{dr}{dt}\right)^2 - \frac{M}{r}$$

$$= \frac{1}{2}\left(\frac{dr}{dt}\right)^2 \left(1 + r^2 \left(\frac{d\phi}{dr}\right)^2\right) - \frac{M}{r},$$

and thus

$$\frac{2\left(E + M/r\right)}{1 + r^2\left(\frac{d\phi}{dr}\right)^2} = \left(\frac{dr}{dt}\right)^2$$

Thus

$$L^2 = r^4\left(\frac{d\phi}{dr}\right)^2\left(\frac{dr}{dt}\right)^2 = r^4\left(\frac{d\phi}{dr}\right)^2 \frac{2\left(E + M/r\right)}{1 + r^2\left(\frac{d\phi}{dr}\right)^2}$$

$$L^2 + r^2L^2\left(\frac{d\phi}{dr}\right)^2 = r^4\left(\frac{d\phi}{dr}\right)^2 2\left(E + M/r\right)$$

$$\frac{L^2}{2r^4\left(E + M/r\right) - r^2L^2} = \left(\frac{d\phi}{dr}\right)^2$$

$$\parallel$$

$$\frac{L^2}{r^4\, 2\left(E + \frac{M}{r} - \frac{L^2}{2r^2}\right)} = \left(\frac{d\phi}{dr}\right)^2.$$

Thus

$$\frac{d\phi}{dr} = \frac{L}{r^2\sqrt{2\left(E + \frac{M}{r} - \frac{L^2}{2r^2}\right)}}.$$

c). We can integrate this equation to

$$\Delta\phi = 2L \int_{r_0}^{\infty} \frac{dr}{r^2 \sqrt{2\left(E + \frac{M}{r} - \frac{L^2}{2r^2}\right)}}$$

$$= 2L \int_{0}^{1/r_0} \frac{du}{\sqrt{2\left(E + Mu - \frac{L^2}{2}u^2\right)}}$$

$$= 2L \int_{0}^{1/r_0} \frac{du}{\sqrt{2\left(E - \frac{L^2}{2}\left(u - \frac{M}{L^2}\right)^2 + \frac{M^2}{2L^2}\right)}}$$

$$= 2L \int_{-\frac{M}{L^2}}^{\frac{1}{r_0} - \frac{M}{L^2}} \frac{dv}{\sqrt{2\left(E + \frac{M^2}{2L^2} - \frac{L^2 v^2}{2}\right)}}$$

$$= \sqrt{2}\, L \int_{-\frac{M}{L^2}}^{\frac{1}{r_0} - \frac{M}{L^2}} \frac{dv}{\sqrt{\left(E + \frac{M^2}{2L^2}\right)\left(1 - \frac{L^2 v^2}{2(E + M^2/2L^2)}\right)}}$$

$$= \cancel{L} \sqrt{\frac{2}{\cancel{E + M^2/2L^2}}} \sqrt{\frac{2(\cancel{E + M^2/2L^2})}{\cancel{L}^2}} \int_{-\frac{M/L}{\sqrt{2(E + M^2/2L^2)}}}^{\frac{L\left(\frac{1}{r_0} - \frac{M}{L^2}\right)}{\sqrt{2(E + M^2/2L^2)}}} \frac{dw}{\sqrt{1 - w^2}}$$

$$= 2 \left[\arcsin\left(\frac{L\left(\frac{1}{r} - \frac{M}{L^2}\right)}{\sqrt{2(E + M^2/2L^2)}} \right) \right]_{r=\infty}^{r=r_0}$$

$$= 2 \left[\arcsin\left(\frac{b\left(\frac{1}{r} - \frac{M}{b^2}\right)}{\sqrt{2\left(E + \frac{M^2}{2b^2}\right)}} \right) \right]_{r=\infty}^{r_0}$$

$$= 2\left[\arcsin\left(\frac{\frac{M}{b}\left(\frac{b^2}{rM}-1\right)}{\sqrt{\frac{M^2}{b^2}}\sqrt{\frac{2b^2E}{M^2}+1}}\right)\right]_{r=\infty}^{r_0}$$

$$= 2\left[\arcsin\left(\frac{\frac{b^2}{rM}-1}{\sqrt{\frac{2b^2E}{M}+1}}\right)\right]_{r=\infty}^{r_0}$$

f) At the point of closest approach $\frac{dr}{dt}=0$. Thus

$$E = \frac{1}{2}r_0^2\left(\frac{d\phi}{dt}\right)^2 - \frac{M}{r_0} = \frac{1}{2}\frac{L^2}{r_0^2} - \frac{M}{r_0} = \frac{b^2}{2r_0^2} - \frac{M}{r_0},$$

so that

$$b^2 = 2r_0^2\left(E+\frac{M}{r_0}\right).$$

Thus,

$$\Delta\phi = 2\left[\arcsin\left(\frac{\frac{2r_0^2}{rM}\left(E+\frac{M}{r_0}\right)-1}{\sqrt{\frac{4Er_0^2}{M}\left(E+\frac{M}{r_0}\right)+1}}\right)\right]_{r=\infty}^{r_0}$$

$$= 2\arcsin\left[\frac{\frac{2r_0}{r}\left(\frac{r_0E}{M}+1\right)-1}{\sqrt{4Er_0\left(\frac{r_0E}{M}+1\right)+1}}\right]_{r=\infty}^{r_0}$$

$$= 2\arcsin\left(\frac{\frac{2}{x}\left(\frac{E}{\mu}+1\right)-1}{\sqrt{4Er_0\left(\frac{E}{\mu}+1\right)+1}}\right)$$

But... but... you told me to integrate. In any case, I just realized that at infinity $E = \frac{1}{2}\|\vec{v}\|^2 = \frac{1}{2}c^2 = \frac{1}{2}$. Thus, the solution for e) could be simplified to

$$\Delta\phi = 2\ \arcsin\left(\frac{\frac{b^2}{rM}-1}{\sqrt{\frac{b^2}{M}+1}}\right)\Bigg]_{r=\infty}^{r_0}$$

while now for f)

$$\Delta\phi = 2\left[\arcsin\left(\frac{\frac{2r_0}{r}\left(\frac{r_0}{2M}+1\right)-1}{\sqrt{2r_0\left(\frac{r_0}{2M}+1\right)+1}}\right)\right]_{r=\infty}^{r_0}$$

$$= 2\left[\arcsin\left(\frac{\frac{2}{x}\left(\frac{1}{2\mu}+1\right)-1}{\sqrt{2r_0\left(\frac{1}{2\mu}+1\right)+1}}\right)\right]_{x=\infty}^{1}.$$

Indeed, in terms of our initial integral

$$\Delta\phi = 2b \int_{r_0}^{\infty} \frac{dr}{r^2 \sqrt{2\left(E + \frac{M}{r} - \frac{b^2}{2r^2}\right)}}$$

$$= 2\sqrt{2}\,\cancel{r_0} \sqrt{\frac{1}{2} + \mu} \int_{1}^{\infty} \frac{dx\, r_0}{x^2 \cancel{r_0^2} \sqrt{2\left(\frac{1}{2} + \mu/x - \frac{1}{x^2}\left(\frac{1}{2} + \mu\right)\right)}}$$

$$= 2\sqrt{2} \sqrt{1 + 2\mu}\; \frac{1}{2} \int_{1}^{\infty} \frac{dx}{x^2 \sqrt{\frac{1}{2} + \mu/x - \frac{1}{2x^2}(1+2\mu)}}$$

$$= 2\cancel{\sqrt{2}} \sqrt{1 + 2\mu}\; \frac{1}{\cancel{\sqrt{2}}} \int_{1}^{\infty} \frac{dx}{x^2 \sqrt{1 + 2\mu/x - \frac{1}{x^2}(1+2\mu)}}$$

$$= 2 \sqrt{1 + 2\mu} \int_{1}^{\infty} \frac{dx}{x^2 \sqrt{1 + 2\mu/x - \frac{1}{x^2}(1+2\mu)}}$$

g) We have

$$x^2 \sqrt{1 + \frac{2\mu}{x} - \frac{1}{x^2}(1+2\mu)} = x \sqrt{x^2 + 2\mu x - 1 - 2\mu}$$

$$= x \sqrt{x^2 - 1 + 2\mu(x-1)}$$

$$= x \sqrt{x^2 - 1} \sqrt{1 + \frac{2\mu}{x+1}},$$

so that for small μ

$$\Delta\phi = 2\sqrt{1+2\mu}\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}\left(1-\frac{\mu}{x+1}\right) + \mathcal{O}(\mu^2)$$

$$= 2\left(1+\mu+\mathcal{O}(\mu^2)\right)\left(\frac{\pi}{2} - \mu\int_1^\infty \frac{dx}{x(x+1)\sqrt{x^2-1}} + \mathcal{O}(\mu^2)\right).$$

Using

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Thus

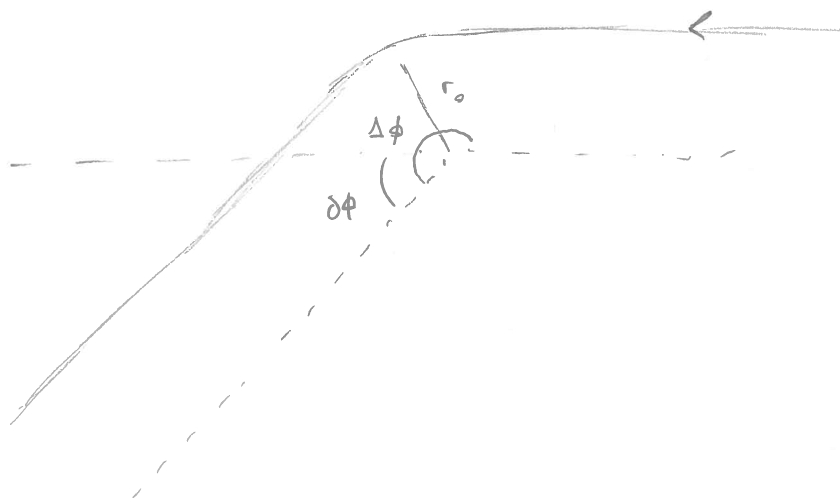
$$\Delta\phi = 2(1+\mu)\left(\frac{\pi}{2} - \mu\left(\frac{\pi}{2}-1\right)\right) + \mathcal{O}(\mu^2)$$

$$= 2\left(\frac{\pi}{2} - \cancel{\mu\frac{\pi}{2}} + \mu + \cancel{\frac{\pi}{2}\mu}\right) + \mathcal{O}(\mu^2)$$

$$= \pi + 2\mu + \mathcal{O}(\mu^2).$$

Thus, since $\Delta\phi - \delta\phi = \pi$,

$$\delta\phi = 2\mu = \frac{2M}{r_0}.$$



Moreover,

$$b = \sqrt{2}\, r_0 \sqrt{\frac{1}{2} + \mu} = \sqrt{2}\, r_0 \sqrt{\frac{1}{2}}\,(1 + \mu), + \mathcal{O}(\mu^2)$$

so that

$$r_0 = \frac{b}{1 + \mu} \approx b$$

and

$$\delta\phi = \frac{2M}{b}.$$

For the sun and a ray scratching its surface

$$\mu = \frac{M_\odot}{R_\odot} =: \frac{M_\odot G}{R_\odot c^2} \approx \frac{2 \times 10^{30}\,\cancel{kg} \times \cancel{7} \times 10^{-11}\,\cancel{m^3}/\cancel{kg}\,s^2}{\cancel{7} \times 10^{8}\,\cancel{m}\,(3 \times 10^{8}\,\cancel{m}/\cancel{s})^2}$$

$$\approx \frac{2}{9} \times 10^{30-11-8-16}$$

$$\approx 2 \times 10^{-6},$$

justifying the assumption that μ is small. Moreover

$$\delta\phi = 2 \times 10^{-6}.$$

2.a) Because the trajectory of our light has to occur in the region $r > r_+ := \frac{2M}{r}$, where this line element is valid. In the case where the radius of the star R was less than r_+, if the light went inside that radius it would never escape back so that it can be observed.

b) Under an infinitesimal transformation $x \mapsto x' = x + \varepsilon\xi$, so that

$$g_{\mu\nu}(x) - \partial_\sigma g_{\mu\nu}(x)\varepsilon\xi^\sigma = g_{\mu\nu}(x - \varepsilon\xi) = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu} g'_{\sigma\rho}(x)$$

$$= (\delta^\sigma_\mu + \varepsilon\,\partial_\mu\xi^\sigma)(\delta^\rho_\nu + \varepsilon\,\partial_\nu\xi^\rho)\, g'_{\sigma\rho}(x)$$

$$= g'_{\mu\nu}(x) + \varepsilon g'_{\sigma\nu}(x)\,\partial_\mu \xi^\sigma + \varepsilon g'_{\mu\rho}(x)\,\partial_\nu \xi^\rho + \mathcal{O}((\xi)^2).$$

If this transformation is a symmetry of g,

we get

$$0 = g_{\sigma\nu}\,\partial_\mu \xi^\sigma + g_{\mu\sigma}\,\partial_\nu \xi^\sigma + \xi^\sigma \partial_\sigma g_{\mu\nu}.$$

If g is independent of x^α and $\xi^\mu = \delta^\mu_\alpha$,

$$g_{\sigma\nu}\cancel{\partial_\mu \xi^\sigma} + g_{\mu\sigma}\cancel{\partial_\nu \xi^\sigma} + \xi^\sigma \partial_\sigma g_{\mu\nu}$$

$$= \delta^\sigma_\alpha\,\partial_\sigma g_{\mu\nu} = \partial_\alpha g_{\mu\nu} = 0.$$

Thus ξ is a Killing vector.

We conclude that the independence of t and ϕ of $g_{\mu\nu}$ guarantees that

$$k^\mu = (1,0,0,0)^\mu, \quad m^\mu = (0,0,0,1)^\mu$$

are Killing vectors. We thus have the conserved quantities

$$e = -k_\mu l^\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}, \quad l = r^2 \sin(\theta)^2 \frac{d\phi}{d\lambda}$$

c) We will first argue that the motion is planar.

The Lagrangian for a photon is

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$$= \left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2$$

$$- r^2\sin(\theta)^2\left(\frac{d\phi}{d\lambda}\right)^2.$$

The equation of motion for θ is then

$$0 = -2\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right) + 2r^2\sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\lambda}\right)^2,$$

i.e.

$$r^2\frac{d^2\theta}{d\lambda^2} + 2r\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - r^2\sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\lambda}\right)^2 = 0.$$

Orient our coordinates so that, for some $\tilde{\lambda}$, we have $\theta(\tilde{\lambda}) = \pi/2$ and $\left.\frac{d\theta}{d\lambda}\right|_{\tilde{\lambda}} = 0$. Thus $0 = \left.\frac{d^2\theta}{d\lambda^2}\right|_{\tilde{\lambda}}$ and we conclude the motion stays in the plane $\theta = \pi/2$.

We will assume this is the case from now on.

Thus $l = r^2 \frac{d\phi}{d\lambda}$. With this simplifying assumption, the null condition for the trajectory of the photon becomes

$$0 = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{1 - \frac{2M}{r}}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$= \frac{1}{\left(1 - \frac{2M}{r}\right)}\left(e^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right)\right),$$

i.e.

$$\frac{e^2}{l^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2} - \frac{2M}{r^3}.$$

d) In the long distance limit,

$$e \approx \frac{dt}{d\lambda}$$

and thus

$$b = \frac{l}{e} = r^2 \frac{d\phi/d\lambda}{dt/d\lambda} = r^2 \frac{d\phi}{dt},$$

corresponding to the classical angular momentum per unit mass. We thus identify it as the impact parameter.

e) We have

$$\left(\frac{d\phi}{dr}\right)^2 = \left(\frac{d\phi}{d\lambda}\right)^2 \left(\frac{d\lambda}{dr}\right)^2$$

$$= \frac{l^2}{r^4} \bigg/ \left(e^2 - \frac{l^2}{r^2} + \frac{2Ml^2}{r^3}\right)$$

$$= \frac{b^2}{r^4\left(1 - \frac{b^2}{r^2} + \frac{2Mb^2}{r^3}\right)},$$

so that

$$\frac{d\phi}{dr} = \frac{b}{r^2\sqrt{1 - \frac{b^2}{r^2} + \frac{2Mb^2}{r^3}}}$$

At the distance of closest approach, $\left.\frac{dr}{dt}\right|_{r=r_0} = 0$.

Thus

$$\frac{1}{b^2} = \frac{1}{r_0^2} - \frac{2M}{r_0^3} = \frac{1}{r_0^2}(1-2\mu)$$

with $\mu := M/r_0$ and $x := r/r_0$. Therefore

$$\frac{d\phi}{dx} = r_0\frac{d\phi}{dr} = \frac{1}{\sqrt{1-2\mu}} \frac{\cancel{r_0^2}}{x^2\cancel{r_0^2}\sqrt{1 - \cancel{r_0^2}(1-2\mu)^{-1}\left(\frac{1}{x^2\cancel{r_0^2}} - \frac{2\mu}{x^3\cancel{r_0^2}}\right)}}$$

$$= \cancel{\frac{1}{\sqrt{1-2\mu}}} \quad \frac{1}{x^2\sqrt{1-2\mu-\frac{1}{x^2}+\frac{2\mu}{x^3}}} \quad \cancel{\sqrt{1-2\mu}}$$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}+2\mu\left(1-\frac{1}{x^3}\right)}}$$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}\sqrt{1+2\mu\,\frac{1-\frac{1}{x^3}}{1-\frac{1}{x^2}}}}$$

$$= \frac{1}{x\sqrt{x^2-1}\sqrt{1+2\mu\,\frac{x^3-1}{x^3-x}}}$$

$$\approx \frac{1}{x\sqrt{x^2-1}}\left(1-\mu\,\frac{x^3-1}{x^3-x}+O(\mu^2)\right).$$

Therefore, to order μ^2

$$\Delta\phi = 2\int_1^\infty dx\,\frac{d\phi}{dx} = 2\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} - 2\mu\int_1^\infty \frac{dx\,(x^3-1)}{x^2\sqrt{x^2-1}\,(x^2-1)}$$

$$= \pi - 2\mu\int_1^\infty \frac{dx\,\cancel{(x-1)}(x^2+x+1)}{x^2\sqrt{x^2-1}\,(x+1)\cancel{(x-1)}} = \pi - 4\mu.$$

Then

$$\delta\phi = 4\mu .$$

Thus the GR prediction is twice that of the Newtonian!