KMS states and Tomita-Takesaki Theory

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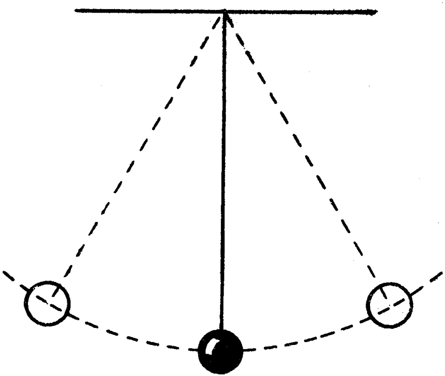
Motivation



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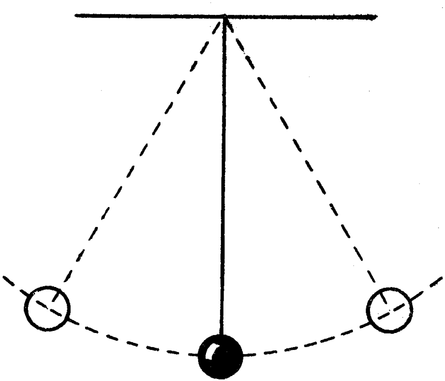

Can we obtain the equations of motion from the equilibrium state?

Maybe in quantum thermal systems.

$$e^{-\beta H} \circlearrowright e^{-iHt}$$

$$\text{temperature} \iff i \times \text{time}$$

Outline

1. Classical and Quantum Theories
2. Algebraic Quantum Mechanics
3. KMS States
4. Tomita-Takesaki Theory
5. The Canonical Time Evolution

Elements of Classical and Quantum Theories

Classical theories

- Auxiliary space: locally compact Hausdorff space X;

Quantum theories

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- States: positive, self-adjoint, normalized and trace-class operators ρ on $\mathcal{H}$;

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- Expected values: $\int f dP$.

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- Auxiliary space: separable Hilbert space $\mathcal{H}$
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- States: positive, self-adjoint, normalized and trace-class operators ρ on $\mathcal{H}$;
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Difference between Classical and Quantum theories

EPR paradox [Einstein et al., 1935]

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$$
\begin{tikzcd}
\text{EPR paradox [Einstein et al., 1935]} \arrow[rr] \arrow[d] & & \text{QM is incomplete} \arrow[d] \\
\text{QM is complete} \arrow[d] & & {} \arrow[d] \\
{} \arrow[uurr] & & {} \arrow[d] \\
& {} & {} \arrow[l]
\end{tikzcd}
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\text{Entangled states } \lightning \arrow[uur] & \substack{\text{Bell's inequalities [Bell, 1964]} \\ \text{[Reyes-Lega, 2013]}} \arrow[d] \\
\text{Lattice of projections [Wilce, 2012]} & \text{Against experiment!} \arrow[l]
\end{tikzcd}
$$

Algebraic Quantum Mechanics

- Observables: A $C^*(W^*)$-algebra $\mathcal{A}$:
  - Complete normed vector space with product and involution;
  - C^* property: $\|A^*A\| = \|A\|^2$;
  - If W^*: $\mathcal{A}'' = \mathcal{A}$.
- States: Positive normalized linear functionals $\omega : \mathcal{A} \to \mathbb{C}$.

GNS Construction

Start with a C^*-algebra $\mathcal{A}$ and a state ω.

- $\mathcal{N}_\omega := \{A \in \mathcal{A} | \omega(AA^*) = 0\}$
- Hilbert space $\mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{N}_\omega}$ with $\langle [A], [B] \rangle := \omega(A^* B)$
- Define the representation extending

$$\begin{aligned} \pi_\omega : \mathcal{A} \to \quad & \mathcal{B}(\mathcal{H}_\omega) \\ A \mapsto & \pi_\omega(A) : \mathcal{H}_\omega \to \mathcal{H}_\omega \\ & \qquad\qquad [B] \mapsto [AB] \end{aligned}$$

- Cyclic vector $\Omega_\omega := [1]$
- Important facts:
  - $\omega(A) = \langle \Omega_\omega, A\Omega_\omega \rangle$
  - If $\mathcal{A}$ is a W^*-algebra, π_ω is a *-isomorphism, $\pi_\omega(\mathcal{A})$ is a W^*-algebra, and Ω_ω is cyclic and separating $\bigstar$.

Dynamical Systems

Time evolution is represented by a one-parameter group of automorphisms

$$\begin{aligned}\tau : \mathbb{R} &\to \mathrm{Aut}(\mathcal{A}) \\ t &\mapsto \tau_t.\end{aligned}$$

Dynamical systems consist of an $C^*(W^*)$-algebra with a time evolutions which satisfy certain continuity properties.

KMS States

Definition

Let $(\mathcal{A}, \tau)$ be a dynamical system. We say that a state ω is a (τ, β)-KMS state if for all $A, B \in \mathcal{A}$ there exists a continuous bounded function $F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$ analytic on $\mathfrak{D}_\beta$ (the strip of the complex plain bounded by $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \beta$) such that

$$\begin{aligned} F_{A,B}(t) &= \omega(A\tau_t(B)) \\ F_{A,B}(t + i\beta) &= \omega(\tau_t(B)A) \end{aligned}$$

for all $t \in \mathbb{R}$.

KMS states as Equilibrium states

KMS states are a candidate for a general definition of thermodynamic equilibrium in quantum systems[Haag, 1992][Duvenhage, 1999]:

- KMS states are invariant under the dynamics $\omega(\tau_t(A)) = \omega(A)$;
- In finite dimensional Hilbert spaces with Schrödinger's time evolution τ, the only possible (τ, β)-KMS states are the β-Gibbs states

$$\begin{aligned} \mathcal{B}(\mathcal{H}) &\to \mathbb{C} \\ A &\mapsto \frac{\mathrm{tr}(Ae^{-\beta H})}{\mathrm{tr}(e^{-\beta H})}. \end{aligned}$$

Tomita-Takesaki Theory

For a W^*-algebra $\mathfrak{M}$ equipped with a cyclic and separating vector Ω Tomita-Takesaki theory yields:

- a one-parameter unitary group $t \mapsto \Delta^{it}$;
- a modular conjugation J.

Theorem (Tomita-Takesaki)

- $J\mathfrak{M}J = \mathfrak{M}'$;
- $\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}$ *for all* $t \in \mathbb{R}$.

Proof.

[Duvenhage, 1999] □

Tomita-Takesaki, Time Evolution and KMS States

Theorem (★)

$t \mapsto \Delta^{it}$ is the unique strongly continuous one-parameter unitary group on $\mathcal{H}$ that satisfies the KMS condition with respect to $\mathcal{K}$ such that $\Delta^{it}\mathcal{K} \subseteq \mathcal{K}$ for all $t \in \mathbb{R}$.

Theorem (★)

Let $\mathfrak{M}$ be a von Neuman algebra and ω a faithful normal state. Consider the unitary group $t \mapsto \Delta^{it}$ associated to the pair $(\pi_\omega(\mathfrak{M}), \Omega_\omega)$. Then the one-parameter group of automorphisms given by $\alpha_t = \pi_\omega^{-1}(\Delta^{it}\pi_\omega(A)\Delta^{-it})$ makes $(\mathfrak{M}, \alpha)$ a W^*-dynamical system.

Proof.

[Duvenhage, 1999] $\square$

The Canonical Time Evolution

Theorem ($\bigstar\bigstar\bigstar$)

Let $\mathfrak{M}$ be a von Neumann algebra and ω be a faithful normal state. Then $(\mathfrak{M}, \tau)$ with $\tau_t(A) = \alpha_{-t/\beta}(A)$ and α the modular group of $(\mathfrak{M}, \omega)$ is the unique W^*-dynamical system such that ω is a (τ, β)-KMS state.

Proof.

[Duvenhage, 1999] □

Further work

- Classical KMS states and Tomita-Takesaki theory.
- Understanding KMS states from "first principles":
  - stability;
  - passivity.
- Relativistic generalization of KMS states.
- Entropy ambiguities.

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