KMS states and Tomita-Takesaki Theory

30% advancement

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Outline

Elements of Classical Mechanics

A classical mechanical system has

- a locally compact Hausdorff space of pure states X;
- observables taking the form continuous real-valued functions $C(X)$ on X;
- states which take the form of probability measures on X.

With this structure we may define the expected value of $f \in C(X)$ by

$$\langle f \rangle_P = \int f dP. \tag{1}$$

And by the identification of points $p \in X$ with Dirac measures δ_p, we have a compatible notion of pure state in which

$$\langle f \rangle_{\delta_p} = f(p). \tag{2}$$

Ensembles

With this approach we may define ensembles to be mappings $y \mapsto \mu_y$ from a space of macroscopic measurements Y to the finite measure on X. Then we have

- the partition function $Z(y) = \mu_y(X)$ which we can use to normalize the measure μ_y and obtain a probability measure P_y;
- the entropy of the induced state

$$S(P_y) = -\int_{supp(P_y)} \log\left(\frac{dP_y}{d\mu}\right) dP_y = -\langle \log\left(\frac{dP_y}{d\mu}\right) \chi_{suppP_y} \rangle_{P_y} \quad (3)$$

given there is a natural measure μ on X and the Radon-Nikodým derivative exists.

A classical system described by $(\{p\}, \{\{p\}, \emptyset\}, \delta_p)$ equipped with an ensemble $y \mapsto \delta_p$, the state δ_p has null entropy.

Elements of Quantum Mechanics

A quantum mechanical system has:

- a separable Hilbert space $\mathcal{H}$;
- observables taking the form of self-adjoint operators on $\mathcal{H}$;
- states represented by density operators (non-negative self-adjoint of unit trace).

With this structure we may define the expected value of an observable A in a state ρ to be

$$\langle A \rangle_\rho = tr(A\rho) \tag{4}$$

and the entropy of a state ρ to be

$$S(\rho) = -tr(\log(\rho)\rho) = \langle \log \rho \rangle_\rho. \tag{5}$$

A pure state ρ_ψ is a projection onto the $span\{\psi\}$ for some $\psi \in \mathcal{H}$. These have null entropy.

Suppose Quantum Mechanics is Complete

- Einstein, Podolsky and Rosen considered that an element of physical reality was one whose outcome in a measurement could be predicted without actually performing the experiment. They defined that a physical theory was complete if to every element of physical reality there corresponded an object in the theory.
- Then two non-commuting observables A and B cannot have a simultaneous realities due to Heisenberg's uncertainty relations

$$\Delta_\rho A \Delta_\rho B \geq \frac{1}{2} |\langle [A, B] \rangle_\rho|. \tag{6}$$

Polarization of photons

- To describe the linear polarization of a photon we may consider the Hilbert space $\mathbb{C}^2$ on which the projection $P(\theta)$ onto the span of $|\theta\rangle = \cos(\theta)(1,0) + \sin(\theta)(0,1)$ representing the proposition "*the photon is linearly polarized at an angle θ (1 means that this is the case and 0 that it isn't)*" acts.
- We may consider the composite system represented by the tensor product and the state

$$\begin{aligned}\psi &= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\pi/2\rangle - |\pi/2\rangle \otimes |0\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|\pi/4\rangle \otimes |3\pi/4\rangle - |3\pi/4\rangle \otimes |\pi/4\rangle\right).\end{aligned} \tag{7}$$

Contradiction!

- If we measure that the first photon has horizontal polarization we know the second one has a vertical polarization.
- If we measure that the first one has a polarization at an angle $\pi/4$ we know the second one has an angle of $3\pi/4$.
- Since the photons are far apart, measurements on the first one cannot affect the second one.

Therefore both states $|\pi/2\rangle$ and $|3\pi/4\rangle$ describe the same physical reality and we are forced to conclude that $P(\pi/2)$ and $P(3\pi/4)$ have simultaneous realities. Nonetheless since $|\pi/2\rangle$ is not orthogonal to $|3\pi/4\rangle$ the two projections don't commute arriving to a contradiction.

Lattices of Propositions

Definition

An order relation on a set P is a relation $\leq$ on X which satisfies for all $p, q, r \in P$:

- reflexivity: $p \leq p$;
- antisymmetry: $p \leq q$ and $q \leq p$ implies $p = q$;
- transitivity: $p \leq q$ and $q \leq r$ implies $p \leq r$.

The pair $(P, \leq)$ is called a partially ordered set or poset. Given $p, q \in P$ we define the meet $p \wedge q$ to be the supremum of $\{p, q\}$ and the join $p \vee q$ to be the infimum of $\{p, q\}$. If the infimum of P is 0 and the supremum is 1, we define a complement of $p \in P$ to be an element $q \in P$ such that $p \wedge q = 0$ and $p \vee q = 1$

Lattices of Propositions

Definition

A poset $(P, \leq)$ is said to be a lattice if for every $p, q \in P$ there exists $p \wedge q$ and $p \vee q$. It is a distributive lattice if for every $p, q, r \in L$ we have $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee r) \wedge (p \vee r)$.

Theorem

In a distributive bounded lattice $(L, \leq)$ elements have at most one complement.

Proof.

Suppose q and r are complements of $p \in L$. Then

$$q = q \wedge 1 = q \wedge (p \vee r) = (q \wedge p) \vee (q \wedge r) = 0 \vee (q \wedge r) = q \wedge r \quad (8)$$

and therefore $q \leq r$. Exchanging the roles of q and r one finds that $r \leq q$ and therefore by antisymmetry $q = r$. □

Bell's inequalities

We ask that the set of propositions in a complete theory of physical reality has the structure of classical propositions, that is of a Boolean algebra (a distributive bounded lattice). Denoting the complement of a proposition p by p' we may consider the following logical function

$$f(p,q) = (p \wedge q) \vee (p' \wedge q'). \tag{9}$$

If we assign to every proposition p a degree of plausibility $P(p) \in \mathbb{R}$ such that if $p \leq q$ then $P(p) \leq P(q)$ we find the Bell inequalities

$$P(f(p_1,q_1)) \leq P(f(p_1,q_2) \vee f(p_2,q_2) \vee f(p_2,q_1)). \tag{10}$$

Lattice of Projections on a Hilbert space

- Since propositions on a Hilbert space should have a spectrum $\{0, 1\}$ then they are represented by the orthogonal projections.
- We have that $\{$Orthogonal Projections$\}$ are in correspondence with $\{$closed subspaces$\}$. These yields the lattice structure with the inclusion relation.

A Correct Physical Theory Cannot be Complete!

Continuing with the photon example consider the operators of the measurements of Alice and Bob

- $P_A(\theta) = P(\theta) \otimes id_{\mathbb{C}^2}$ and $P_B(\theta) = id_{\mathbb{C}^2} \otimes P(\theta)$
- $tr(P_A(\alpha)P_B(\beta)\rho_\psi) = \frac{1}{2}\sin^2(\alpha - \beta)$
- $P(f(P_A(\alpha), P_B(\beta))) = \sin^2(\alpha - \beta)$

We can test Bell's inequalities

$$\begin{aligned} 1 &= \sin^2(0 - \pi/2) = P(f(P_A(0), P_B(\pi/2))) \\ &\leq P(f(P_A(0), P_B(\pi/6))) + P(f(P_A(\pi/3), P_B(\pi/6))) + P(f(P_A(\pi/3), P_B(\pi \\ &= \sin^2(0 - \pi/6) + \sin^2(\pi/3 - \pi/6) + \sin^2(\pi/3 - \pi/2) = 3/4! \end{aligned} \tag{11}$$

No correct physical theory can satisfy EPR requirements for being a complete physical theory.