KMS states and Tomita-Takesaki Theory

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Chapter 1

Introduction

One of the most important problems in modern physics is that of the mathematical formulation of quantum field theory. Although used successfully throughout the community to study the most relevant problems of particle physics, solid state physics, cosmology, and the dark sector, among others, it still lacks a complete rigorous mathematical formulation. Indeed, divergences and ill-defined symbols rid the whole theory. As physicists, it is not only our duty to give predictions of the physical world but to understand its working principles. In particular, a complete understanding of these must be of a logical nature. It is our believe that mathematics serves this purpose, especially in the cases where those who claim to have an intuitive understanding are often wrong. The purpose of this monograph is to do a bibliographical revision of a result in the realm of mathematical physics and in the search for the correct mathematical framework of quantum theories with infinite degrees of freedom: to every thermodynamical equilibrium quantum state there is a canonical dynamical law governing the time evolution of the system.

In chapter 2 we do a quick revision of general frameworks in classical and quantum theories. This serves mainly to establish notation and point towards some common elements these theories have which will inspire the algebraic approach present throughout this monograph. Before we delve into this unifying scheme, in chapter 3 we attempt to understand the differences between these theories. We arrive at the conclusion that it has is roots in the mathematical structure of the propositions associated to a quantum theory. In particular, the structure will not be that of a boolean algebra[1]. In chapter

[1]This departure from the mathematical framework of classical propositions is the root

4 we present the general framework of algebraic quantum theory. This is inspired on the features noted in 2. In particular, we develop the theory of the GNS construction (which we exemplify by showing how it can aid in the calculation of entropies) and of dynamical systems, the two stepping stones in the path to the main result. In chapter 5 we study KMS states. These will be states characterized by certain analytic properties (the KMS condition) which we will interpret as those of quantum states in thermodynamic equilibrium. This will be inspired by studying the relationship between KMS states and Gibbs states (the canonical ensemble) in finite dimensional quantum systems. Chapter 6 develops Tomita-Takesaki theory. This theory will yield the mathematical objects that appear in the main result of this monograph. We follow the approach of [1] and [2] to avoid encountering unbounded operators and domain issues. Finally, in chapter 7 we gather the partial result obtained in 4, 5, and 6 and develop the final result which amounts to the connection between KMS states and Tomita-Takesaki theory.

towards my skepticism towards those who claim to have an intuitive understanding of quantum theory.

Chapter 2

Classical and Quantum Mechanics as Probability Theories

This chapter shows how both classical and quantum mechanics are probability theories. This is not intended as an axiomatization of these theories. Indeed the reader is assumed to be comfortable with these physical theories as well as the basic mathematical concepts of measure theory and functional analysis.

2.1 Classical Mechanics

The setting of classical mechanics is usually a locally compact Hausdorff space X. We consider the states of maximal knowledge (or pure states) to be the elements of X. Likewise, observables take the form of real valued functions on X. We call the points of X states of maximal knowledge because we interpret $f(p)$ as the value of the observable f in the state $p \in X$. Moreover, given that in principle we could make the value of an observable as precise as we want by improving our knowledge of the state, observables are self-adjoint elements of the set of continuous functions $C(X)$. Nonetheless the purpose of statistical mechanics is to treat systems in which total knowledge of a state is not practically possible. Instead we consider a probability measure[1] which

[1] Along with a σ-algebra which we won't mention explicitly to keep the notation simple but should always be kept in mind.

assigns to every measurable subset of X a probability of the system's state being in it. We may define the expected value of an observable $f \in C(X)$ through a probability measure P by

$$\langle f \rangle_P = \int f dP. \tag{2.1}$$

Notice that an element $p \in X$ can also be though as a probability measure by using the Dirac measure δ_p which assigns 1 to a set if it contains p and 0 otherwise. Indeed for every element $p \in X$ and observable $f \in C(X)$ we have $\langle f \rangle_{\delta_p} = f(p)$. This motivates us to broaden the definition of states to the probability measures on X. We will call Dirac measures (or equivalently the points in X) pure states.

This definition of state proves to be very helpful for the discussion of ensembles. Whenever the description of the state of a system as a pure state is not feasible, we may consider the set of outcomes Y of measurements we may perform on the system. Every element of Y gives us information of the system in the form of a finite measure. We may define an ensemble as the mapping from Y into the set of finite measures on X. Through normalization of finite measures every ensemble yields a mapping from Y into the set of states and we define the accessible (pure) states of an element $y \in Y$ to be the support of the corresponding state. Although the construction of an ensemble is in general a difficult task, for systems in statistical equilibrium[2] there are many standard procedures. In the case of these type of systems we define the partition function $Z : Y \to \mathbb{R}_0^+$ by assigning to every element y the measure of X given by the ensemble evaluated at y.

Example 2.1.1. In many physical systems the space of pure states has a natural notion of size which we may represent by giving it the structure of a measure space $(X, \mathcal{A}, \mu)$ where $\mathcal{A}$ contains the Borel σ-algebra[3] . We may consider $Y = \mathbb{R}$ to be the set of energy outcomes. If $H : X \to \mathbb{R}$ is a measurable function taking the interpretation of energy we define the micro-canonical ensemble to be the mapping $y \mapsto \mu_y$ where $\mu_y(\Sigma) = \mu(\Sigma \cap H^{-1}(y))$

[2] These are systems whose state does not change in time. We refer to the equilibrium as statistical because it may be that the pure state of the system is changing in time but noticing these changes is not feasible for us.

[3] Usually we take a countable set with the counting measure but another example would be a phase space with the Liouville measure. In the latter, it is common that $H^{-1}(y)$ is a set of measure zero so we actually have to take $X = H^{-1}(y)$ with the appropriate induced measure.

for all measurable Σ. The set $H^{-1}(y)$ is the set of accessible states and $\mu_y(\Sigma)$ measures the amount of pure states in Σ which are accessible. Notice that the normalization of μ_y yields a state P_y which assigns a uniform probability measure to X. This is called the equal a priori probabilities postulate. In this ensemble the partition function $Z(y) = \mu_y(X) = \mu(H^{-1}(y))$ is just the amount of accessible states. This ensemble is usually used to describe systems with constant energy and a fixed number of particles.

Example 2.1.2. Consider again a measure space $(X, \mathcal{A}, \mu)$ but let $Y = \mathbb{R}^+$ be the set of inverse temperatures of the system. If we have an energy function $H : X \to \mathbb{R}$ such that $x \mapsto \exp(-yH(x))$ is integrable for all $y \in Y$ the canonical ensemble assigns to every inverse temperature y a finite measure μ_y by

$$\mu_y(\Sigma) = \int_\Sigma e^{-yH(x)} d\mu(x) \tag{2.2}$$

for all measurable sets Σ. This ensemble is usually used to describe systems with a fixed number of particles in thermal equilibrium with a heat bath. Note that we could add to the description of the system the heat bath and we would be able to in principle use the microcanonical ensemble. The difficulty lies in that generally the counting of accessible states is more difficult than the application of the canonical ensemble.

Note that both of the ensembles discussed have images consisting of absolutely continuous measures μ_y with respect to the notion of size μ. The same is true for the induced states P_y. Moreover the Lebesgue-Radon-Nikodým derivative exits and we define the entropy of the ensemble in the state P_y by[**?**][4]

$$S(P_y) = -\int_{supp(P_y)} \log\left(\frac{dP_y}{d\mu}\right) dP_y = -\langle \log\left(\frac{dP_y}{d\mu}\right) \chi_{suppP_y} \rangle_{P_y}. \tag{2.3}$$

One can check that in the microcanonical ensemble

$$\frac{dP_y}{d\mu}(x) = \frac{\chi_{H^{-1}(y)}(x)}{Z(y)} \tag{2.4}$$

[4]In general if we start from a decomposable $(X, \mathcal{A}, \mu)$ and have an ensemble which yields absolutely continuous measures with respect to μ we can define entropy in this fashion. In particular, if μ comes from the Daniel extension of a positive linear functional the space is decomposable[3].

and in the canonical ensemble

$$\frac{dP_y}{d\mu}(x) = \frac{\exp(-yH(x))}{Z(y)}. \tag{2.5}$$

In the case we have a state of maximal knowledge δ_p we may collapse X to $(\{p\}, \{\{p\}, \emptyset\}, \delta_p)$ and define an ensemble $y \mapsto \delta_p$. Such an ensemble has zero entropy.

2.2 Quantum Mechanics

The setting of quantum mechanics is a separable Hilbert space $\mathcal{H}$. In this case the states are the non-negative self-adjoint operators of unit trace on $\mathcal{H}$ (called density operators) and observables take the form of self-adjoint operators on $\mathcal{H}$. The possible outcomes of an observable A are the elements of its spectrum and, if P_A is the unique projection-valued measure such that $A = \int id_{\mathbb{C}} dP_A$ given by the spectral theorem, we have that the probability of the measurement of the observable A yielding a value in the measurable subset $E \subseteq \mathbb{R}$ in the state ρ is $\operatorname{tr}(P_A(E)\rho)$. One can check that given this way of measuring probabilities we have that the expected value of an observable A in the state ρ is

$$\langle A \rangle_\rho = \operatorname{tr}(A\rho). \tag{2.6}$$

Moreover we define the entropy of a state ρ to be

$$S(\rho) = -\operatorname{tr}(\log(\rho)\rho) = -\langle \log(\rho) \rangle_\rho. \tag{2.7}$$

Inspired by the classical case we define a pure state ρ_ψ to be orthogonal projection on the span of ψ for $\psi \in \mathcal{H}$ of unit norm. Although such a state has null entropy as in the classical case

$$S(\rho_\psi) = -\operatorname{tr}(\log(\rho_\psi)\rho_\psi) = -\langle \psi, \log(1)\psi \rangle = 0, \tag{2.8}$$

we can't in general associate to an observable A a definite outcome unless ψ is an eigenvector of A corresponding to an eigenvalue λ where we have

$$\operatorname{tr}(P_A(\{\lambda\})\rho_\psi) = \langle \psi, P_A(\{\lambda\})\psi \rangle = \langle \psi, \psi \rangle = \|\psi\|^2 = 1. \tag{2.9}$$

Notice that we haven't inspired the definition of a state like we did in the classical case. A more detailed account of this quick summary of quantum mechanics may be found in [4] and [5]. The connection between states and probability is given through the study of quantum lattices of propositions in the next chapter.

Chapter 3

Quantum Probability

The previous chapter showed that there is a dictionary to understand in a very similar language quantum and classical theories. Before we develop the language of operator algebras to make this similarity more concrete we will first show how this two theories are different. To do this we will study the logical structure of quantum mechanics and show that it isn't boolean.

3.1 EPR paradox

Einstein, Podolsky and Rosen examined the completeness of quantum mechanics in their famous 1935 paper [6]. They considered that an element of physical reality was one whose outcome in a measurement could be predicted without actually performing the experiment. They defined that a physical theory was complete if to every element of physical reality there corresponded an object in the theory. One can prove that in quantum mechanics two observables A and B[1] satisfy for every state ρ the Heisenberg uncertainty relation

$$\Delta_\rho A \Delta_\rho B \geq \frac{1}{2}|\langle [A,B] \rangle_\rho| \tag{3.1}$$

where $[A,B] = AB - BA$ is the commutator and $\Delta_\rho A = \sqrt{\langle A^2 \rangle_\rho - \langle A \rangle_\rho^2}$ (for a proof see [4]). Therefore either quantum mechanics is incomplete or two non-commuting observables cannot have a simultaneous physical reality.

[1] We will assume them to be bounded to avoid technical difficulties since our example will be finite dimensional.

Assuming that quantum mechanics is indeed complete we are forced to accept that two non-commuting observables don't have a simultaneous physical reality.

Example 3.1.1. In order to examine the previous assertion let us follow an example developed in [7]. To describe the polarization of a photon we may consider the Hilbert space $\mathbb{C}^2$. We assign to the proposition "*the photon is linearly polarized at an angle θ (1 means that this is the case and 0 that it isn't)*" the operator $P(\theta)$ the orthogonal projection to the span of

$$|\theta\rangle = \cos(\theta)(1,0) + \sin(\theta)(0,1). \tag{3.2}$$

Therefore the vector $|\theta\rangle$ takes the interpretation of the state in which the photon is certain to have linear polarization at angle θ. Now consider a system with two photons in a state ρ_ψ where

$$\psi = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |\pi/2\rangle - |\pi/2\rangle \otimes |0\rangle\right). \tag{3.3}$$

We may prepare such a system by allowing a Calcium atom to decay into two photons and waiting till the photons are far apart. It's easy to see that

$$\psi = \frac{1}{\sqrt{2}} \left(|\pi/4\rangle \otimes |3\pi/4\rangle - |3\pi/4\rangle \otimes |\pi/4\rangle\right). \tag{3.4}$$

Therefore if we measure that the first photon has horizontal polarization we know the second one has a vertical polarization and if we measure that the first one has a polarization at an angle $\pi/4$ we know the second one has an angle of $3\pi/4$. But since the photons are far apart, measurements on the first one cannot affect the second one. Therefore both states $|\pi/2\rangle$ and $|3\pi/4\rangle$ describe the same physical reality and we are forced to conclude that $P(\pi/2)$ and $P(3\pi/4)$ have simultaneous realities. Nonetheless since $|\pi/2\rangle$ is not orthogonal to $|3\pi/4\rangle$ the two projections don't commute arriving to a contradiction.

Contradictions of the type shown above due to the use of coupled systems are often referred to nowadays as the EPR paradox. They led to the notion of entanglement. We are therefore, subject to the definitions given in the paper [6], forced to the conclusion that quantum theory must be an incomplete theory.

3.2 Lattices and Bell's Inequalities

We may continue EPR's agenda and try to find a complete theory of physical reality. In such a theory (much like in every physical theory) we must be able to ask true or false questions about a physical system. Studying these questions gives us an excuse to begin the discussion of lattices of propositions (using as a main source [8]). Although previous knowledge of logic is not essential, we will use our experience from classical logic to inspire the definitions we will use.

Definition 3.2.1. An order relation on a set P is a relation $\leq$ on X which satisfies for all $p, q, r \in P$:

- reflexivity: $p \leq p$;
- antisymmetry: $p \leq q$ and $q \leq p$ implies $p = q$;
- transitivity: $p \leq q$ and $q \leq r$ implies $p \leq r$.

The pair $(P, \leq)$ is called a partially ordered set or poset.

We may recognize these laws if we exchange the symbols $\leq$ for the implication symbol $\implies$ where the rules above evidently follow. Much like in this case, the rest of the definitions ahead will have a counterpart in propositional logic and are intended as an extension of it.

Definition 3.2.2. Let $(P, \leq)$ be a poset and $A \subseteq P$. A lower (upper) bound of A is an element $p \in P$ such that for all $a \in A$ we have $p \leq a$ ($a \leq p$). An infimum (supremum) of A is a lower (upper) bound p of A such that if $r \in P$ is a lower (upper) bound of A then $r \leq p$ ($p \leq r$).

Once again, through the symbol exchange made earlier we may note that the conjunction of two propositions $p \wedge q$ is the infimum of $\{p, q\}$ and the disjunction $p \vee q$ is the supremum of $\{p, q\}$. The infimum and sumpremum are closely related and we will in general carry the discussion only for the infimum leaving the details of the supremum in parenthesis just as we did in the previous definition.

Theorem 3.2.3. Let $(P, \leq)$ be a poset and $A \subseteq P$ such that its infimum (supremum) exists. Then the infimum (supremum) is unique.

Proof. Suppose p and q are infima (suprema) of A. Then since p is a lower (upper) bound we have $p \leq q$ ($q \leq p$). Similarly, since q is a lower (upper) bound $q \leq p$ ($p \leq q$). Therefore by antisymmetry $p = q$. $\square$

Notation 3.2.4. Let $(P, \leq)$ be a poset and $A \subseteq P$ such that its infimum (supremum) exists. We denote the infimum (supremum) of A by $\bigwedge A$ ($\bigvee A$). If $A = \{p, q\}$ then we denote $p \wedge q := \bigwedge A$ ($p \vee q := \bigvee A$). As is common in logic literature we will now use for infimum (supremum) the term meet (join).

Now we shall list some of the algebraic properties of posets.

Theorem 3.2.5. Let $(P, \leq)$ be a poset. Then for all $p, q, r \in P$:

1. $p \leq q$ if and only if $p = p \wedge q$ if and only if $q = p \vee q$;
2. (idempotency) $p \wedge p = p$ and $p \vee p = p$;
3. (associativity) if the meet (join) of $\{p, q\}$, $\{q, r\}$, $\{p \wedge q, r\}$ ($\{p \vee q, r\}$), $\{p, q \wedge r\}$ ($\{p, q \vee r\}$) and $\{p, q, r\}$ exists then $(p \wedge q) \wedge r = p \wedge (q \wedge r) = \bigwedge\{p, q, r\}$ ($(p \vee q) \vee r = p \vee (q \vee r) = \bigvee\{p, q, r\}$);
4. (commutativity) if the meet (join) of $\{p, q\}$ exists then $p \wedge q = q \wedge p$ ($p \vee q = q \vee p$).

Proof. All of the statements are clear from the definitions. $\square$

All of these properties are familiar from propositional logic. Nevertheless, there are some properties of basic logic which we cannot prove with the definitions above and need to be added as additional properties of posets.

Definition 3.2.6.

- A poset $(P, \leq)$ is said to be a lattice if for every $p, q \in P$ there exists $p \wedge q$ and $p \vee q$.
- A lattice $(L, \leq)$ is said to be complete if for every $A \subseteq L$ there exists $\bigwedge A$ and $\bigvee A$.
- A lattice $(L, \leq)$ is said to be distributive if for every $p, q, r \in L$ we have $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee r) \wedge (p \vee r)$.
- A poset $(P, \leq)$ is said to be bounded if there exists $0 := \bigwedge P$ and $1 := \bigvee P$.

- A complement of an element $p \in P$ of a bounded poset $(P, \leq)$ is an element $q \in P$ such that $p \wedge q = 0$ and $p \vee q = 1$.
- A Boolean algebra is a distributive bounded lattice in which every element has a complement.

Comparison with propositional logic shows that we usually equip propositions with the structure of a Boolean algebra. In this case complements take the interpretation of negation and are unique due to the following theorem.

Theorem 3.2.7. In a distributive bounded lattice $(L, \leq)$ elements have at most one complement.

Proof. Suppose q and r are complements of $p \in L$. Then

$$q = q \wedge 1 = q \wedge (p \vee r) = (q \wedge p) \vee (q \wedge r) = 0 \vee (q \wedge r) = q \wedge r \tag{3.5}$$

and therefore $q \leq r$. Exchanging the roles of q and r one finds that $r \leq q$ and therefore by antisymmetry $q = r$. □

Now, we ask that the set of propositions in a complete theory of physical reality has the structure of classical propositions, that is of a Boolean algebra. Denoting the complement of a proposition p by p' we may consider the following logical function

$$f(p, q) = (p \wedge q) \vee (p' \wedge q'). \tag{3.6}$$

Note that for classical propositions p_1, q_1, p_2 and q_2 we have

$$\begin{gathered}(p_1 \wedge q_1) \wedge ((p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1)) = \\ (p_1 \wedge q_1 \wedge q_2) \vee (p_1 \wedge q_1 \wedge (p_2 \vee q_2)') \vee (p_1 \wedge q_1 \wedge p_2) = \\ (p_1 \wedge q_1) \wedge (q_2 \vee (p_2 \vee q_2)' \vee p_2) = p_1 \wedge q_1\end{gathered} \tag{3.7}$$

and therefore

$$p_1 \wedge q_1 \leq (p_1 \wedge q_2) \vee (p_2' \wedge q_2') \vee (p_2 \wedge q_1). \tag{3.8}$$

Similarly

$$p_1' \wedge q_1' \leq (p_1' \wedge q_2') \vee (p_2 \wedge q_2) \vee (p_2' \wedge q_1') \tag{3.9}$$

from which we conclude

$$f(p_1, q_1) \leq f(p_1, q_2) \vee f(p_2, q_2) \vee f(p_2, q_1). \tag{3.10}$$

Following Jaynes [9] we may assign to every proposition p a degree of plausibility $P(p) \in \mathbb{R}$. Every sensible way of assigning such degrees of plausibility must be such that if $p \leq q$ then $P(p) \leq P(q)$. Therefore we find what we will call Bell's inequalities following [7]

$$P(f(p_1, q_1)) \leq P(f(p_1, q_2) \vee f(p_2, q_2) \vee f(p_2, q_1)). \tag{3.11}$$

In order to study this inequalities in the setting of quantum mechanics, we must first explain how this machinery applies in the case of the theory.

3.3 Lattice of Projections on a Hilbert Space

To study the logical structure of quantum mechanics we first discuss the notion of proposition in the theory. Since propositions have to be observables with two possible outcomes "true" or "false", we identify them with the self-adjoints whose spectrum is $\{0, 1\}$. These are precisely the orthogonal projections on a Hilbert space $\mathcal{H}$.

Theorem 3.3.1. Every closed subspace of $\mathcal{H}$ is the image of an orthogonal projection. Conversely, the image of every orthogonal projection is closed.

Proof. Let $V \subseteq \mathcal{H}$ be a closed subspace. By the Orthogonal Decomposition Theorem we have $\mathcal{H} = V \bigoplus V^{\perp}$[3]. Therefore take the orthogonal projection $\psi \mapsto \xi$ where ξ is the unique element of V such that there exists a $\zeta \in V^{\perp}$ such that $\psi = \xi + \zeta$. Let P be an orthogonal projection. Then $P(\mathcal{H}) = (\ker P)^{\perp}$ and, since every orthogonal complement is closed, $P(\mathcal{H})$ is closed. $\square$

Therefore we see that the set of propositions can also be identified with the closed subspaces of $\mathcal{H}$. From now on we won't make a distinction between quantum propositions, orthogonal projections and closed subspaces and we will denote such an identification by $L(\mathcal{H})$. Both of these identifications will help us endow the quantum mechanical propositions with a logical structure.

Theorem 3.3.2. The set of closed subspaces of a Hilbert space $\mathcal{H}$ is naturally a poset when equipped with the relation of set inclusion. This is bounded by $\{0\}$ and $\mathcal{H}$. Moreover, it is a lattice where for every family of closed subspaces $\mathcal{C}$ we have $\bigwedge \mathcal{C} = \bigcap \mathcal{C}$ and $\bigvee \mathcal{C} = \overline{\operatorname{span}(\bigcup \mathcal{C})}$.

Proof. Note that if X is a set then $(P(X), \subseteq)$ is a poset and for every $A \subseteq P(X)$ it remains true that $(A, \subseteq)$ is a poset. The case of closed subspaces of a Hilbert space is a special case of this. Moreover, recall that in $(P(X), \subseteq)$ we have for $\mathcal{A} \in X$ that $\bigwedge A = \bigcap \mathcal{A}$ and since intersection of closed subspaces is a closed subspace, this remains true for our case of interest. Similarly $\bigvee \mathcal{A} = \bigcup \mathcal{A}$. But in general the union of subspaces is not a subspace. Nevertheless, the smallest subspace that contains a subset is its span. But we may still run into trouble because the span may not be closed. We can solve this by noticing that the smallest closed set that contains a subset is its closure, yielding the formula for the join in the theorem. Finally it is clear that application of the formulas for the meet and join yield $0 = \{0\}$ and $1 = \mathcal{H}$. □

In particular, in the case of two propositions P and Q that commute as projections, the projection onto the intersection of P and Q is given by the multiplication of the projections PQ. We can also see that we may interpret the expectation value of a proposition P as the degree of plausibility. Now, given that quantum mechanics seems to correctly predict the behavior of light polarization, we may go back to out previous example and test Bell's inequalities.

Example 3.3.3. Following up on example 3.1.1 suppose $P_A(\theta) = P(\theta) \otimes id_{\mathbb{C}^2}$ and $P_B(\theta) = id_{\mathbb{C}^2} \otimes P(\theta)$. This means that $P_A(\theta)$ measures on the first photon and $P_B(\theta)$ on the second. More precisely we may interpret $P_A(\theta)$ as the proposition"*the first photon has linear polarization at an angle* θ" and $P_B(\theta)$ playing the analogue role for "*the second photon has linear polarization at an angle* θ". Therefore we find that the degree of plausibility for the proposition $P_A(\alpha) \wedge P_B(\beta) = P_A(\alpha) P_B(\beta)$ is

$$\begin{aligned}
\operatorname{tr}(P_A(\alpha) P_B(\beta) \rho_\psi) = \\
\langle \psi, P_A(\alpha) P_B(\beta) \psi \rangle = \\
\frac{1}{\sqrt{2}} \langle \psi, P(\alpha)|0\rangle \otimes P(\beta)|\pi/2\rangle - P(\alpha)|\pi/2\rangle \otimes P(\beta)|0\rangle\rangle = \\
\frac{1}{\sqrt{2}} \langle \psi, \cos(\alpha)|\alpha\rangle \otimes \sin(\beta)|\beta\rangle - \sin(\alpha)|\alpha\rangle \otimes \cos(\beta)|\beta\rangle\rangle = \\
\frac{1}{2} \left(\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta)\right)^2 = \\
\frac{1}{2} \sin^2(\alpha - \beta).
\end{aligned} \tag{3.12}$$

It is clear that setting $P_A(\theta)' := P_A(\theta + \pi/2)$ and $P_B(\theta)' := P_A(\theta + \pi/2)$ indeed yields complements according to theorem 3.3.2. With this, we find that $P(f(P_A(\alpha), P_B(\beta))) = \sin^2(\alpha - \beta)$. Therefore we obtain through Bell's inequalities

$$\begin{aligned} 1 &= \sin^2(0 - \pi/2) = P(f(P_A(0), P_B(\pi/2))) \\ &\leq \sin^2(0 - \pi/6) + \sin^2(\pi/3 - \pi/6) + \sin^2(\pi/3 - \pi/2) \\ &= 3/4 \end{aligned} \tag{3.13}$$

which is clearly a contradiction.

We find thus through the contradiction between Bell's inequalities and experiment that we failed in our search of a complete theory of physics according to the definitions given by EPR. In his paper [10], Bell found his inequalities by assuming there was a hidden probability space (as in the classical case) from which we could assign degrees of plausibility to propositions. Of course such a view point falls within our discussion and makes it clear that there are no hidden variables. Nonetheless our exposition shows that the problem with the critique to quantum mechanics made by EPR lies in their definitions. Bell's inequalities show that no theory satisfying their requirements for completeness (which we interpreted as having a boolean logical structure) will ever be found. Moreover, our discussion yielded a clearer view on the root of the distinction between quantum mechanics and previous theories: *the logical structure*.

Notice now that in general a closed subspace of $\mathcal{H}$ has many different complements. For example in $\mathbb{C}^2$ we have that $\text{span}(\{(\cos(\theta), \sin(\theta))\})$ is a complement of $\text{span}(\{(1, 0)\})$ for all $\theta \in (0, \pi)$. Therefore by theorem 3.2.7 the lattice of quantum propositions cannot be boolean. This explains the root of the contradiction in Bell's inequalities as well as the EPR paradox.

Finally, we would like to make use of the logical structure of quantum mechanics to explain the objects appearing in section 2.2. First of all, notice that the operator $P_A(E)$ corresponding to an observable A and a Borel set E is the orthogonal projection corresponding to the proposition "*measurement of the observable A yields a value in the Borel set E.*" Secondly, a reasonable way to define a state in quantum mechanics would be as a mapping that assigned to every proposition a degree of plausibility. Precisely,

Definition 3.3.4. A probability measure on the lattice of propositions $L(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a map $\mu : L(\mathcal{H}) \to [0, 1]$ such that $\mu(H) = 1$

and for every sequence (P_n) of pairwise orthogonal projections we have $\mu(\bigoplus_{n=0}^{\infty} P_n) = \sum_{n=0}^{\infty} \mu(P_n)$.

To relate the definition above to the one we gave in section 2.2 we may note that for every density operator ρ on a Hilbert space $\mathcal{H}$ the function $\mu_\rho : L(\mathcal{H}) \to [0,1] : P \mapsto \mathrm{tr}(P\rho)$ is a probability measure on $L(\mathcal{H})$. Conversely,

Theorem 3.3.5 (Gleason's Theorem)**.** If $\mathcal{H}$ is a Hilbert space with dimension greater than 2 then every probability measure on $L(\mathcal{H})$ is of the form μ_ρ for some density operator ρ on $\mathcal{H}$.

Chapter 4

Algebraic Quantum Physics

Now that we've understood classical and quantum mechanics as probability theories and displayed their differences, we will now concern ourselves with the development of algebraic methods that will allow us to describe both classical and quantum mechanics in the same framework and to discuss equilibrium further. We will define the notions of C^*-algebras, von Neumann algebras, dynamical systems, and develop the GNS construction. Through examples we will see the physical importance of these concepts.

4.1 C*-algebras

We will start by getting acquainted with the notion of a C*-algebra. This is the mathematical structure we will endow our physical observables with. Even though the general need for this structure can be inspired by the abstract analysis of experimental apparatuses[4] we will instead give the abstract definition and then justify it through examples.

Definition 4.1.1. An (associative) algebra $\mathcal{A}$ is a set equipped with three operations:

$$\begin{aligned}
&\mathcal{A} \times \mathcal{A} \to \mathcal{A} \\
&\quad (x, y) \mapsto x + y \quad \text{addition;} \\
&\mathbb{C} \times \mathcal{A} \to \mathcal{A} \\
&\quad (\lambda, x) \mapsto \lambda x \quad \text{scalar multiplication;} \\
&\mathcal{A} \times \mathcal{A} \to \mathcal{A} \\
&\quad (x, y) \mapsto xy \quad \text{multiplication;}
\end{aligned} \tag{4.1}$$

such that with addition and scalar multiplication it forms a complex vector space, with addition and multiplication it forms a ring, and there is a compatibility condition between scalar multiplication and multiplication which is that for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ we have $(\lambda x)y = x(\lambda y) = \lambda(xy)$. If the ring is commutative the algebra is said to be commutative and if the ring is unital the algebra is said to be unital. A norm on an algebra $\mathcal{A}$ is a norm on the vector space structure $\|\cdot\| : \mathcal{A} \to \mathbb{R}_0^+$ such that for all $x, y \in \mathcal{A}$ we have $\|xy\| \leq \|x\|\|y\|$. An algebra endowed with a norm is called a normed algebra. If the normed vector space structure of an algebra is Banach, the algebra is called Banach. An involution on an algebra $\mathcal{A}$ is a map $^* : \mathcal{A} \to \mathcal{A} \quad x \mapsto x^*$ such that for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

$$\begin{aligned} (\lambda x + y)^* &= \bar{\lambda} x^* + y^*; \\ (xy)^* &= y^* x^*; \\ (x^*)^* &= x. \end{aligned} \tag{4.2}$$

An algebra equipped with an involution is said to be a *-algebra. A C^*-algebra is a Banach *-algebra where for all $x \in \mathcal{A}$

$$\|x^* x\| = \|x\|^2. \tag{4.3}$$

Example 4.1.2. The set of continuous functions vanishing at infinity on a locally compact Hausdorff space X, that is the set $C_0(X)$ of continuous $f : X \to \mathbb{C}$ such that for every $\epsilon \in \mathbb{R}^+$ there exists a compact set K such that $f(K^c) \subseteq B(0, \epsilon) \subseteq \mathbb{C}$ forms a C^*-algebra with the supremum norm

$$\|f\| = \sup\{|f(x)| | x \in X\}. \tag{4.4}$$

This algebra differs from the structure described in 2.1 in that the functions are admittedly complex and their behavior at infinity is restricted. Nevertheless $C_0(X)$ is unital if and only if X is compact. In that case $C_0(X) = C(X)$ and the observables coincide with the self-adjoint elements of the C^*-algebra. One can associate both the need for restricting behavior at infinity or making the space compact by noting that any real feasible experiment performed on a system should be localized. This has to do with the experimental motivation of C^*-algebras given in [4]. We now note that every commutative C^*-algebra can be realized as the space of continuous functions on a compact Hausdorff space.

Example 4.1.3. The set of bounded operators in a Hilbert space $\mathcal{H}$ forms a C^*-algebra with the operator norm

$$\|A\| = \sup\left\{ \frac{\|Ax\|}{\|x\|} \,\middle|\, x \in \mathcal{H} \setminus \{0\} \right\}. \tag{4.5}$$

Moreover, every closed self-adjoint subspace of the bounder operators $\mathcal{B}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is a C^*-algebra[1]. Once again, this algebra differs from the structure given in section 2.2 because we only consider bounded operators. Once again at a fundamental level this doesn't matter since we know through the spectral theorem or the logical structure presented in section 3.3 that we can describe all observables (bounded or unbounded) through their spectral decomposition into projections. In particular, we should be able to take the C^*-algebra generated by the projections associated to the observable we want to analyze. For example, instead of considering the position operator q on $L^2(\mathcal{H})$ given by $q\psi(x) = x\psi(x)$ for all $\psi \in \mathcal{H}$, we can consider the C^*-algebra generated by the characteristic functions of Borel sets $E \subseteq \mathbb{R}$ whose action on the Hilbert space is $\chi_E\psi(x) = \chi_E(x)\psi(x)$. Moreover, this problem, as in the classical case, is related to the fact that no experimental apparatus has an infinite display of outcomes. One indeed cannot measure infinitely large positions or momenta.

Another solution for the case of Schrödinger's mechanics is to consider the Weyl operators $U(a)$ and $U(b)$ for $a, b \in \mathbb{R}$ given by

$$\begin{aligned} U(a)\psi(x) &= \psi(x - \hbar a) \\ V(b)\psi(x) &= e^{-ibx}\psi(x). \end{aligned} \tag{4.6}$$

By Stone's theorem if q is the position operator and p is the momentum operator satisfying the canonical commutation relations $[x, p] = i\hbar$ we have $U(a) = e^{-iap}$ and $V(b) = e^{-ibq}$[4].

As mentioned before the above definition gives structure to the observables of a system. To get a complete kinematical description we need to also give structure to the notion of state. We can inspire the definition of a state by the fact that both in classical and quantum descriptions the statistically appropriate notion of state seemed to act on the observable either through equation 2.1 or 2.6.

[1] Although we won't need it, the Gelfand-Naimark theorem shows that every C^*-algebra is realizable as a closed self-adjoint subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$[11].

Definition 4.1.4. A state on a C^*-algebra $\mathcal{A}$ is a positive normalized linear functional $\omega : \mathcal{A} \to \mathbb{C}$, i.e. it is a linear map such that $\|\omega\| = 1$ (normalized) and for all $x \in \mathcal{A}$ we have $\omega(x^*x) \geq 0$ (positive). If $\omega(x^*x) > 0$ for all $x \in \mathcal{A} \setminus \{0\}$, the state is said to be faithful.

Note that for a unital C^*-algebra a positive linear functional ω is normalized if and only if $\omega(1) = 1$[12].

Some useful facts about states are included in the next theorem[12].

Theorem 4.1.5. Let ω be a state on a C^*-algebra $\mathcal{A}$. Then for all $A, B \in \mathcal{A}$ we have

$$\omega(AB^*) = \overline{\omega(BA^*)}. \tag{4.7}$$

Proof. Let $A, B \in \mathcal{A}$. Then for all $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} 0 &\geq \omega((\lambda A + B)(\lambda A + B)^*) = \omega(|\lambda|^2 AA^* + \lambda AB^* + \overline{\lambda} BA^* + BB^*) \\ &= |\lambda|^2 \omega(AA^*) + \lambda\omega(AB^*) + \overline{\lambda}\omega(BA^*) + \omega(BB^*) \geq \lambda\omega(AB^*) + \overline{\lambda}\omega(A^*B). \end{aligned} \tag{4.8}$$

Setting $\lambda = 1$ we see that $\omega(AB^*) + \omega(BA^*) \in \mathbb{R}$ and therefore $\operatorname{Im}\omega(AB^*) = -\operatorname{Im}\omega(BA^*)$. Setting $\lambda = i$ we have $i(\omega(AB^*) - \omega(BA^*)) \in \mathbb{R}$ and therefore $\operatorname{Re}\omega(AB^*) = \operatorname{Re}\omega(BA^*)$. The theorem follows. $\square$

Example 4.1.6. By Riesz's representation theorem [3] we have that for every state ω on $C(X)$ for X compact Hausdorff there exists a probability measure P on X such that

$$\omega(f) = \int f dP \tag{4.9}$$

for every f in $C(X)$. Indeed P is the measure induced by the Daniel extension of ω. Moreover, it turns out that every commutative C^*-algebra is isomorphic to an algebra $C_0(X)$ for X locally compact Hausdorff[12]. In particular, this final remark justifies that classical systems can be treated in the context of C^*-algebras.

Example 4.1.7. Given a density operator ρ on a Hilbert space $\mathcal{H}$, $\omega_\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ given by $\omega_\rho(A) = \operatorname{tr}(A\rho)$ is a state. More generally, a state constructed as the restriction of ω_ρ to a C^*-algebra viewed as a subalgebra of $\mathcal{B}(\mathcal{H})$ is called a normal state. Of particular importance is the generalization of the canonical ensemble discussed in 2.1.2. Consider a system with Hamiltonian H in equilibrium with a heat bath without exchange of particles at inverse temperature β. Then the state is

$$\rho_\beta = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})} \tag{4.10}$$

known as the β-Gibbs state[13][1].

4.2 GNS Construction

Although we won't prove the structure theorems mentioned above for the characterization of C^*-algebras we will indeed be interested in the representation of a C^*-algebra on a Hilbert space induced by a state. For this we will follow [12]

Definition 4.2.1. A representation of a a C^*-algebra $\mathcal{A}$ is a tuple $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism (i.e. an adjoint preserving homomorphism). If $\mathcal{H}$ has non trivial invariant subspaces under the action of $\pi(\mathcal{A})$ then the representation is said to be reducible. If π is a *-isomorphism onto its image, the representation is said to be faithful.

Definition 4.2.2. Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{B}(\mathcal{H})$ and $G \subseteq \mathcal{H}$. Then G is said to be cyclic for S if $\operatorname{span} SG$ is dense and separating for S if for every $A \in S$ if $AG = \{0\}$ then $A = 0$. A vector $x \in \mathcal{H}$ is said to be cyclic (separating) for S if $\{x\}$ is.

Theorem 4.2.3. If $\mathcal{A}$ is a unital[2] C^*-algebra and ω is a state on it, then there exists a unique representation (up to unitary equivalence) $(\mathcal{H}_\omega, \pi_\omega)$ with a cyclic unit vector Ω_ω and for all $x \in \mathcal{A}$ we have that $\omega(x) = \langle \Omega_\omega, \pi_\omega(x)\Omega_\omega \rangle = \mathrm{tr}(\pi_\omega(x)\rho_{\Omega_\omega})$ (omega is a vector state).

Proof. Notice that in particular $\mathcal{A}$ is a vector space. Consider the function

$$\begin{aligned} \mathcal{A} \times \mathcal{A} &\to \mathbb{C} \\ (x, y) &\mapsto \omega(x^* y). \end{aligned} \tag{4.11}$$

[2]From now onwards, we will always consider our algebras to be unital to avoid technical difficulties. The GNS construction can be done without this assumption but it requires to first extend to a unital algebra [12].

One can show that this function is an inner product except for the fact that there may be elements $x \in \mathcal{H} \setminus \{0\}$ such that $\omega(x^*x) = 0$. We may define $\mathcal{N}_\omega := \{x \in \mathcal{A} | \omega(x^*x) = 0\}$. Notice that if $x \in \mathcal{N}_\omega$ and $y \in \mathcal{A}$ then

$$\begin{aligned} |\omega((yx)^*(yx))|^2 = |\omega(x^*y^*yx)|^2 &= |\omega((y^*yx)^*x)|^2 \\ &\leq \omega((y^*yx)^*(y^*yx))\omega(x^*x) = 0, \end{aligned} \tag{4.12}$$

that is, $\mathcal{N}_\omega$ is a left ideal of $\mathcal{A}$. Notice that now the inner product

$$\begin{aligned} \mathcal{A}/\mathcal{N}_\omega \times \mathcal{A}/\mathcal{N}_\omega &\to \mathbb{C} \\ ([x],[y]) &\mapsto \langle [x],[y] \rangle := \omega(x^*y) \end{aligned} \tag{4.13}$$

is well defined and therefore take $\mathcal{H}_\omega = \overline{\mathcal{A}/\mathcal{N}_\omega}$. We define

$$\begin{aligned} \pi_\omega : \mathcal{A} &\to L(\mathcal{H}_\omega) \\ x &\mapsto \pi_\omega(x) \end{aligned} \tag{4.14}$$

by extension of $\pi_\omega(x)[y] := [xy]$ (which is bounded and therefore uniformly continuous) on $\mathcal{A}/\mathcal{N}_\omega$. We define at last $\Omega_\omega := [1]$. If $x \in \mathcal{A}$ we have

$$\langle \Omega_\omega, \pi_\omega(x)\Omega_\omega \rangle = \langle \Omega_\omega, [x] \rangle = \omega(x). \tag{4.15}$$

Moreover $\pi_\omega(\mathcal{A})\Omega_\omega = \mathcal{A}/\mathcal{N}_\omega$ and it is therefore verified that the vector Ω_ω is cyclic.

Now suppose we have another representation $(\mathcal{H}', \pi')$ that satisfies the conditions of the theorem. Let $\Omega' \in \mathcal{H}'$ such that $\overline{\pi'(\mathcal{A})\Omega'} = \mathcal{H}'$ and $\omega(A) = \langle \Omega', A\Omega' \rangle$. Define $U : \mathcal{H} \to \mathcal{H}'$ by extension of $U\pi_\omega(A)\Omega_\omega = \pi'(A)\Omega'$ which is unitary since

$$\begin{aligned} \langle U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega \rangle &= \langle \pi'(A)\Omega', \pi'(B)\Omega' \rangle \\ &= \langle \Omega', \pi'(A^*)\pi'(B)\Omega' \rangle = \omega(A^*B) \\ &= \langle \Omega_\omega, \pi_\omega(A^*)\pi_\omega(B)\Omega_\omega \rangle \\ &= \langle \pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega \rangle. \end{aligned} \tag{4.16}$$

Then $U^{-1}\pi'(A)U = \pi_\omega(A)$ and $U\Omega_\omega = \Omega'$. □

Example 4.2.4. Let's follow the GNS construction with the example of the C^*-algebra of 2×2 matrices with complex entries $M_2(\mathbb{C})$. This is of physical importance for 2 state systems. For example our recurring system in example

3.1.1 has this algebra of observables (the canonical matrix representations of the operators $P(0)$, $P(\pi/4)$, $P(\pi/2)$ and $P(3\pi/4)$ generate this algebra). Let the elementary matrices of $M_2(\mathbb{C})$ be $E_{ij} = ((\delta_{in}\delta_{jm})_{nm})$. Let's choose the state

$$\omega_\lambda(\alpha) = \lambda\alpha_{11} + (1-\lambda)\alpha_{22} \tag{4.17}$$

for some $\lambda \in [0,1]$. The parameter λ can be given interpretation by noting that $\omega_\lambda(P(0)) = \lambda$, that is, λ is the expectation value of the photon described to have polarization along the horizontal axis. We have that

$$\begin{aligned}\omega_\lambda(\alpha^*\alpha) = \omega_\lambda\left(\left(\sum_{i=1}^{2}(\alpha^*)_{ik}\alpha_{kj}\right)_{ij}\right) = \omega_\lambda\left(\left(\sum_{i=1}^{2}\overline{\alpha}_{ki}\alpha_{kj}\right)_{ij}\right) \\ = \lambda(|\alpha_{11}|^2 + |\alpha_{21}|^2) + (1-\lambda)(|\alpha_{12}|^2 + |\alpha_{22}|^2).\end{aligned} \tag{4.18}$$

Therefore the ideal $\mathcal{N}_\lambda := \mathcal{N}_{\omega_\lambda}$ will depend on the choice of λ.

- If $\lambda = 0$,

$$\mathcal{N}_0 = \{\alpha \in M_2(\mathbb{C}) | \alpha_{12} = \alpha_{22} = 0\}. \tag{4.19}$$

  Therefore it is clear that if $\mathcal{H}_\lambda := \mathcal{H}_{\omega_\lambda}$ we have

$$\mathcal{H}_0 = M_2(\mathbb{C})/\mathcal{N}_0 \simeq \left\{ \begin{bmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \middle| \alpha_{12}, \alpha_{22} \in \mathbb{C} \right\}. \tag{4.20}$$

- If $\lambda = 1$ we have the symmetric case and we conclude

$$\mathcal{H}_1 = M_2(\mathbb{C})/\mathcal{N}_1 \simeq \left\{ \begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{bmatrix} \middle| \alpha_{11}, \alpha_{21} \in \mathbb{C} \right\}. \tag{4.21}$$

- If $\lambda \in (0,1)$ we have that $\mathcal{N}_\lambda = \{0\}$ and therefore $M_2(\mathbb{C})/\mathcal{N}_\lambda \simeq M_2(\mathbb{C})$. We have in particular that this representation can be decomposed into the two previous representations

$$M_2(\mathbb{C}) = \left\{ \begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{bmatrix} \middle| \alpha_{11}, \alpha_{21} \in \mathbb{C} \right\} \oplus \left\{ \begin{bmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \middle| \alpha_{12}, \alpha_{22} \in \mathbb{C} \right\}. \tag{4.22}$$

  Moreover, if $\alpha \in M_2(\mathbb{C})$ we have

$$(\pi_{\Omega_{\omega_\lambda}}(\alpha)E_{ij})_{nm} = \sum_{k=1}^{2} \alpha_{nk}\delta_{ik}\delta_{jm} = \alpha_{ni}\delta_{jm} \tag{4.23}$$

and therefore the spaces in the decomposition are invariant under the action of the representation of the algebra. In particular, we check that the projection $\rho_{\Omega_{\omega_\lambda}}$ onto Ω_{ω_λ} cannot be of the form $\pi_{\Omega_{\omega_\lambda}}(\alpha)$ for some $\alpha \in M_2(\mathbb{C})$ since it doesn't respect that invariance

$$\rho_{\Omega_{\omega_\lambda}}(E_{ij}) = \langle \Omega_{\omega_\lambda}, E_{ij} \rangle \Omega_{\omega_\lambda} = \omega_\lambda(E_{ij}) I_2 = (\lambda \delta_{1i}\delta_{1j} + (1-\lambda)\delta_{2i}\delta_{2j}) I_2. \tag{4.24}$$

Given equation 4.15 one may feel tempted to associate to the system the orthogonal projection $\rho_{\Omega_{\omega_\lambda}}$ onto Ω_{ω_λ} as a state. This would yield according to equation 2.8 a state of zero entropy. We need to find a way around this. Now, examining our previous example where the state was conveniently written as a convex sum of states, we find that the extremal points of this sum (the cases $\lambda \in \{0, 1\}$) generate irreducible representations of the algebra while the other cases didn't. Moreover, the actual state $\rho_{\Omega_{\omega_\lambda}}$ was not in the image of the algebra of observables in the reducible representations considered. This inspires us to try to find a state ρ_{ω_λ} which also satisfies $\mathrm{tr}(\pi_{\omega_\lambda}(\alpha)\rho_{\omega_\lambda}) = \omega(\alpha)$ from the irreducible representations in the GNS construction. Such an agenda may also be found in [14] ,[15], [16] and[17].

Being concerned for the moment with finite dimensional representations, we will in general be able to write

$$\mathcal{H}_\omega = \bigoplus_{\beta \in I} \mathcal{H}_\omega^{(\beta)} \tag{4.25}$$

where $\{\mathcal{H}_\omega^{(\beta)} | \beta \in I\}$ is a set of irreducible representations of $\mathcal{A}$. The decomposition leaves the projection operators $P^{(\beta)}$ onto $\mathcal{H}_\omega^{(\beta)}$ such that

$$id_{\mathcal{H}_\omega} = \sum_{\beta \in I} P^{(\beta)}. \tag{4.26}$$

Therefore, we have if $\{e_1, \cdots, e_n\}$ is a basis for $\mathcal{H}_\omega$

$$\begin{aligned}
\omega(\alpha) &= \langle \Omega_\omega, \pi_\omega(\alpha)\Omega_\omega \rangle \\
&= \langle \Omega_\omega, \sum_{\beta \in I} P^{(\beta)} \pi_\omega(\alpha)\Omega_\omega \rangle \\
&= \langle \Omega_\omega, \sum_{\beta \in I} P^{(\beta)} \pi_\omega(\alpha) P^{(\beta)} \Omega_\omega \rangle \\
&= \langle \Omega_\omega, \sum_{m=1}^{n} \langle e_m, \sum_{\beta \in I} P^{(\beta)} \pi_\omega(\alpha) P^{(\beta)} \Omega_\omega \rangle e_m \rangle \\
&= \sum_{m=1}^{n} \langle e_m, \sum_{\beta \in I} P^{(\beta)} \pi_\omega(\alpha) P^{(\beta)} \langle \Omega_\omega, e_m \rangle \Omega_\omega \rangle \\
&= \sum_{m=1}^{n} \langle e_m, \sum_{\beta \in I} P^{(\beta)} \pi_\omega(\alpha) P^{(\beta)} \rho_{\Omega_\omega} e_m \rangle \\
&= \operatorname{tr}(\sum_{\beta \in I} P^{(\beta)} \pi_\omega(\alpha) P^{(\beta)} \rho_{\Omega_\omega}) \\
&= \operatorname{tr}(\pi_\omega(\alpha) \sum_{\beta \in I} P^{(\beta)} \rho_{\Omega_\omega} P^{(\beta)}).
\end{aligned} \tag{4.27}$$

Therefore we define

$$\rho_\omega := \sum_{\beta \in I} P^{(\beta)} \rho_{\Omega_\omega} P^{(\beta)} \tag{4.28}$$

as the induced state.

Example 4.2.5. Continuing with example 4.2.4 we find that since in the cases $\lambda \in \{0, 1\}$ since the representation is irreducible we have $\rho_{\omega_\lambda} = \rho_{\Omega_{\omega_\lambda}}$ and therefore the state is pure and has null entropy. In the case $\lambda \in (0, 1)$

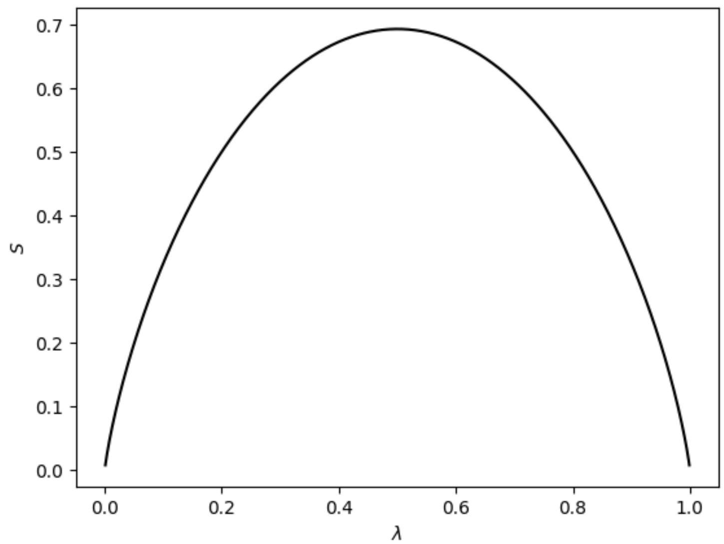

Figure 4.1: The entropy of equation 4.30 as a function of the probability that the photon has horizontal polarization.

we have for all $\alpha \in M_2(\mathbb{C})$

$$\begin{aligned}
\rho_{\omega_\lambda}\alpha &= \sum_{i=1}^{2} P^{(i)}\rho_{\Omega_{\omega_\lambda}} P^{(i)}\alpha \\
&= P^{(1)}\rho_{\Omega_{\omega_\lambda}} \begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{bmatrix} + P^{(2)}\rho_{\Omega_{\omega_\lambda}} \begin{bmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \\
&= P^{(1)}\omega\left(\begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{bmatrix}\right) I_2 + P^{(2)}\omega\left(\begin{bmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix}\right) I_2 \\
&= P^{(1)}\lambda\alpha_{11}I_2 + P^{(2)}(1-\lambda)\alpha_{22}I_2 = \lambda\alpha_{11}E_{11} + (1-\lambda)\alpha_{22}E_{22} \\
&= \lambda\rho_{E_{11}}\alpha + (1-\lambda)\rho_{E_{22}}\alpha = (\lambda\rho_{E_{11}} + (1-\lambda)\rho_{E_{22}})\alpha
\end{aligned} \tag{4.29}$$

and therefore $\rho_{\omega_\lambda} = \lambda\rho_{E_{11}} + (1-\lambda)\rho_{E_{22}}$. We conclude that the entropy is

$$S = -(\lambda\log(\lambda) + (1-\lambda)\log(1-\lambda)) \tag{4.30}$$

For completeness we state a theorem given by [12] that shows that every GNS representation can be decomposed into representations with a cyclic vector.

Theorem 4.2.6. Every non-degenerate representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$ (that is, one in which $\pi(\mathcal{A})x = 0$ implies $x = 0$) is a direct sum of representations with a cyclic vector.

4.3 Von Neumann Algebras

In this section we will explore the theory of von Neumann algebras. Although these are special cases of C^*-algebras, they will be the correct setting to develop Tomita-Takesaki theory and eventually connect it with KMS states. Moreover, their study is also important for the general theory of quantum systems with infinite degrees of freedom including quantum field theories[18]. In this case, to be concrete we will follow the presentation of [11].

Definition 4.3.1. Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The commutant of $\mathfrak{M}$ is

$$\mathfrak{M}' = \{A \in \mathcal{B}(\mathcal{H}) | AB = BA \text{ for all } B \in \mathfrak{M}\}. \tag{4.31}$$

We say $\mathfrak{M}$ is a von Neumann algebra (W^*-algebra) if $\mathfrak{M}'' = \mathfrak{M}$.

Example 4.3.2. It is clear that for all $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ we have $\mathfrak{M} \subseteq \mathfrak{M}''$. Therefore $\mathcal{B}(\mathcal{H})$ is a W^*-algebra.

We claimed that every W^*-algebras is a C^*-algebra. Indeed,

Theorem 4.3.3. Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ be self-adjoint. Then $\mathfrak{M}'$ is a C^*-algebra.

Proof. It is a matter of checking that $\mathfrak{M}'$ is a *-algebra. It is clear that for all $A \in \mathfrak{M}'$ we have $\|A^*A\| = \|A\|^2$ since this is true in $\mathcal{B}(\mathcal{H})$. Finally, if (A_n) is a Cauchy sequence in $\mathfrak{M}'$ then it converges to some $A \in \mathcal{B}(\mathcal{H})$. Since multiplication is continuous (a simple consequence of the compatibility between multiplication and the norm in C^*-algebras and therefore in $\mathcal{B}(\mathcal{H})$) we have that

$$\begin{aligned} AC &= (\lim_{n\to\infty} A_n)C = \lim_{n\to\infty} A_n C = \lim_{n\to\infty} CA_n \\ &= C \lim_{n\to\infty} A_n = CA. \end{aligned} \tag{4.32}$$

Therefore $\mathfrak{M}'$ is Banach and we conclude it is a C^*-algebra. $\square$

Corollary 4.3.4. A W^*-algebra is a C^*-algebra.

Theorem 4.3.5. Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $G \subseteq \mathcal{H}$. Then G is cyclic for $\mathfrak{M}$ if and only if G is separating for $\mathfrak{M}'$.

Proof. Suppose G is cyclic for $\mathfrak{M}$ and let $A \in \mathcal{M}'$ be such that $AG = \{0\}$. Then for all $\mathcal{B} \in \mathfrak{M}$ and $x \in G$ we have $ABx = BAx = B0 = 0$. By continuity $A = 0$. Conversely, suppose G is separating for $\mathfrak{M}'$ and let P be the orthogonal projection on $\overline{\mathfrak{M}G}$. We will prove that the projection onto its orthogonal complement is null. First note that $P \in \mathfrak{M}'$. Indeed, if $x \in \mathcal{H}$ there exists $y \in \overline{\mathfrak{M}G}$ and $z \in \overline{\mathfrak{m}G}^{\perp}$ such that $x = y + z$. If $A \in \mathfrak{M}$ then $Ay \in \overline{\mathfrak{M}G}$ and $APy = Ay = PAy$. On the other hand, $Az \in \overline{\mathfrak{m}G}^{\perp}$ since for all $v \in \overline{\mathfrak{m}G}$ we have $\langle v, Az\rangle = \langle A^*v, z\rangle = 0$. Therefore $PAz = 0 = A0 = APz$. We conclude that $PAx = APx$ and $P \in \mathfrak{M}'$.Then it is clear that $1 - P \in \mathfrak{M}'$ and we have $(1 - P)G = \{0\}$. Since G is separating for $\mathfrak{M}'$ we have $1 - P = 0$ and therefore $P = 1$ showing that G is cyclic for $\mathfrak{M}$. □

As it turns out, the GNS representation of a von Neumann algebra equipped with a faithful normal state will have properties which as we will see later are suitable for the application of Tomita-Takesaki theory.

Theorem 4.3.6. Let $\mathfrak{M}$ be a W^*-algebra and ω be a normal faithful state. Then the GNS representation $(\mathcal{H}_\omega, \pi_\omega)$ is faithful, $\pi_\omega(\mathfrak{M})$ is a W^*-algebra and the cyclic vector Ω_ω is separating for $\pi_\omega(\mathfrak{M})$.

Proof. The fact that $\pi_\omega(\mathfrak{M})$ is a W^*-algebra is given in [12] and relies on the topological properties of von Neumann algebras which we have not discussed. Let $A \in \mathfrak{M}$ be such that $\pi_\omega(A)\Omega_\omega = 0$ (which in particular is true if $A \in \ker \pi_\omega$). Then since ω is faithful and

$$\omega(A^*A) = \langle \Omega_\omega, \pi_\omega(A^*A)\Omega_\omega\rangle = \langle \pi_\omega(A)\Omega_\omega, \pi_\omega(A)\Omega_\omega\rangle = 0 \tag{4.33}$$

we conclude that $A^*A = 0$. Therefore by the C^*-property $0 = \|A^*A\| = \|A\|^2$ and $A = 0$. We conclude that the representation is faithful and Ω_ω is separating for $\pi_\omega(\mathfrak{M})$. □

4.4 Dynamical Systems

Up until now, we have only focused on the description of observables and states of a physical system. In now way have we yet discussed the dynamics of a system. In the context of algebraic physics we have that the concept of time evolution is implemented through automorphisms of the algebra of observables. To show this we will follow [1].

Definition 4.4.1. Let $\mathcal{A}$ be a C^*-algebra. A one-parameter automorphism group is a group homomorphism $\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ $t \mapsto \tau_t$. If τ is a one-parameter automorphism group where the map $\mathbb{R} \to \mathcal{A}$ given by $t \mapsto \tau_t(A)$ is continuous for all $A \in \mathcal{A}$, then $(\mathcal{A}, \tau)$ is called a C^*-dynamical system. If $\mathfrak{M}$ is a W^*-algebra on a Hilbert space $\mathcal{H}$ and τ is a one-parameter automorphism group on $\mathfrak{M}$ such that $\mathbb{R} \to \mathcal{H}$ given by $t \mapsto \tau_t(A)x$ is continuous for all $A \in \mathfrak{M}$ and $x \in \mathcal{H}$, then $(\mathfrak{M}, \tau)$ is called a W^*-dynamical system.

The distinction we make for von Neumann algebras reflects the fact that even though we have corollary 4.3.4, von Neumann algebras will be more general in physical applications. Indeed we have,

Theorem 4.4.2. A C^*-dynamical system whose underlying algebra is a W^*-algebra is a W^*-dynamical system.

Proof. Let $\mathfrak{M}$ be a W^*-algebra on a Hilbert space $\mathcal{H}$ and $(\mathfrak{M}, \tau)$ a C^*-dynamical system. Notice that for every $x \in \mathcal{H}$ the evaluation map $\mathrm{ev}_x : \mathfrak{M} \to \mathcal{H}$ given by $\mathrm{ev}_x(A) = Ax$ is continuous. Indeed if $\epsilon \in \mathbb{R}^+$, $\delta = \epsilon/\|x\|$, and $\|A - B\| < \delta$ for $A, B \in \mathfrak{M}$ then

$$\|Ax - Bx\| = \|(A - B)x\| \leq \|A - B\|\|x\| < \frac{\epsilon}{\|x\|}\|x\| = \epsilon. \tag{4.34}$$

Then the map $\mathbb{R} \to \mathcal{H}$ $t \mapsto \tau_t(A)x$, being ev_x after $\mathbb{R} \to \mathcal{A}$ $t \mapsto \tau_t(A)$ continuous by hypotheses, is continuous for all $A \in \mathfrak{M}$ and $x \in \mathcal{H}$. The conclusion follows. $\square$

Following [1] we will from now on restrict to finite dimensional Hilbert spaces to inspire notions which we will however generalize later by using definition 4.4.1.

Example 4.4.3. In quantum mechanics we already have a dynamical law given by Schrödinger's equation. Consider a finite dimensional Hilbert space $\mathcal{H}$ with a self-adjoint Hamiltonian H. The time evolution, as prescribed by Heisenberg's representation of Schrodinger's mechanics is the restriction of

$$\tau : \mathbb{C} \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$$

$$
\begin{aligned}
z \mapsto \quad \tau_z : \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathcal{H}) \\
A &\mapsto e^{iHz} A e^{-iHz}
\end{aligned} \tag{4.35}
$$

to the real numbers. The verification that this map is well defined and indeed restricts to a one-parameter automorphism group is routine. Let $\{e_1, \dots, e_N\}$ be an orthonormal basis of eigenvectors of H associated to the eigenvalues $E_1, \dots, E_N$ (whose existence is guaranteed by the spectral theorem). Then we have

$$
\begin{aligned}
\|e^{iHt} - 1\| &= \|\sum_{n=1}^{N} e^{iE_n t} \rho_{e_n} - \sum_{n=1}^{N} \rho_{e_n}\| \leq \sum_{n=1}^{N} \| \left(e^{iE_n t} - 1\right) \rho_{e_n}\| \\
&= \sum_{n=1}^{N} |e^{iE_n t} - 1| \|\rho_{e_n}\| = \sum_{n=1}^{N} |e^{iE_n t} - 1| \to 0
\end{aligned} \tag{4.36}
$$

as $t \to 0$. Therefore if $s \in \mathbb{R}$ we have

$$
\lim_{t \to s} \tau_t(A) = \lim_{t \to s} \tau_{t-s} \tau_s(A) = \lim_{t \to s} e^{iH(t-s)} \tau_s(A) e^{iH(s-t)} = \tau_s(A) \tag{4.37}
$$

for all $A \in \mathcal{B}(\mathcal{H})$. We conclude that $(\mathcal{B}(\mathcal{H}), \tau)$ is a C^* and W^*-dynamical system.

Note that even though the dynamics have been defined through automorphisms of the algebra resembling Heisenberg's quantum mechanics, we could've equally defined it analogously to Schrodinger's mechanics through the evolution of states

$$
\tau_t(\omega)(A) := \omega(\tau_t(A)). \tag{4.38}
$$

This is of course yields the same physics and we will occasionally use it to give a physical interpretation to mathematical results.

The following consequence given in [1] of having a state invariant under the dynamics of a system for its GNS representation will be useful later on to formulate the connection between KMS states and Tomita-Takesaki theory.

Theorem 4.4.4. Let $\mathcal{A}$ be a C^*-algebra, τ a one-parameter group of automorphisms of $\mathcal{A}$, ω a state such that $\omega(\tau_t(A)) = \omega(A)$ for all $t \in \mathbb{R}$ and $A \in \mathcal{A}$, and $(\mathcal{H}_\omega, \pi_\omega)$ the induced GNS representation. Then there exists a unique one-parameter unitary group

$$
\begin{aligned}
U : \mathbb{R} &\to \{A \in \mathcal{B}(\mathcal{H}) | A \text{ is unitary}\} \\
t &\mapsto U_t
\end{aligned} \tag{4.39}
$$

such that $U_t\Omega_\omega = \Omega_\omega$ and $\pi_\omega(\tau_t(A)) = U_t\pi_\omega(A)U_{-t}$ for all $t \in \mathbb{R}$ and $A \in \mathcal{A}$. Furthermore, if $(\mathcal{A}, \tau)$ is a W^*-dynamical system and $\pi_\omega(\mathcal{A}$ is a von Neumann algebra, then U is strongly continuous.

Proof. Notice that $(\mathcal{H}_\omega, \pi \circ \tau_t)$ is a representation for all $t \in \mathbb{R}$ which satisfies the properties of the GNS representation, namely, Ω_ω is cyclic. Then, by uniqueness, we have the unique desired function U. We only need to show U is a group homomorphism. $\square$

Chapter 5

KMS States

Having developed the theory of algebraic quantum mechanics we are now in the correct setting to discuss the theory of KMS states as shown in [1]. Although starting with an abstract definition, we will use the case of finite dimensional quantum systems to inspire why this is a natural generalization of the Gibbs states presented in 4.1.7. In particular, they will be invariant under the dynamics of the system justifying therefore their usefulness given the generality of the dynamics we've defined.

5.1 Definition and Dynamical Invariance

Definition 5.1.1. Let $(\mathcal{A}, \tau)$ be a C^* or W^*-dynamical system, ω a state on $\mathcal{A}$ (in the W^* case we demand ω is normal), $\beta \in \mathbb{R}$,

$$\mathfrak{D}_\beta = \begin{cases} \{z \in \mathbb{C} | 0 < \operatorname{Im} z < \beta\} & \beta \geq 0 \\ \{z \in \mathbb{C} | \beta < \operatorname{Im} z < 0\} & \beta < 0 \end{cases}, \tag{5.1}$$

and $\overline{\mathfrak{D}_\beta}$ be the closure $\mathfrak{D}_\beta$ except for the case $\beta = 0$ where we set $\overline{\mathfrak{D}_\beta} = \mathbb{R}$ (we will keep using these sets during the rest of this work). ω is said to be a (τ, β)-KMS state if it satisfies the KMS conditions, that is, for every $A, B \in \mathcal{A}$ there exists a bounded continuous function $F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$ (which we will usually refer to as a witness to ω being a (τ, β)-KMS state) analytic on $\mathfrak{D}_\beta$ and such that for every $t \in \mathbb{R}$ it is true that

$$\begin{aligned} F_{A,B}(t) &= \omega(A\tau_t(B)) \\ F_{A,B}(t + i\beta) &= \omega(\tau_t(B)A). \end{aligned} \tag{5.2}$$

A $(\tau, -1)$-KMS state is called a τ-KMS state.

Although the definition of τ-KMS state may seem bizarre since it corresponds to negative temperatures, it is of great technical importance. Indeed the next theorem shows that for the most part everything we learn about τ-KMS states is true for (τ, β)-KMS states.

Theorem 5.1.2. Let $(\mathcal{A}, \tau)$ be a $C^*(W^*)$-dynamical system, ω a state on $\mathcal{A}$, and $\beta \in \mathbb{R}$. Define

$$\begin{aligned} \alpha : \mathbb{R} \to & \mathrm{Aut}(\mathcal{A}) \\ t \mapsto \ & \alpha_t : \mathcal{A} \to \mathcal{A} \\ & \qquad A \mapsto \alpha_t(A) := \tau_{-\beta t}(A). \end{aligned} \tag{5.3}$$

Then $(\mathcal{A}, \alpha)$ is a $C^*(W^*)$-dynamical system and:

- if ω is a (τ, β)-KMS state then it is an α-KMS state;
- if $\beta \neq 0$ then ω is a (τ, β)-KMS state if and only if it is an α-KMS state.

Proof. It is easy to see that $(\mathcal{A}, \alpha)$ is a $C^*(W^*)$-dynamical system. If ω is a (τ, β)-KMS state and $F_{A,B}$ is a witness to this then

$$\begin{aligned} G_{A,B} : \overline{\mathfrak{D}_{-1}} & \to \mathbb{C} \\ z & \mapsto F_{A,B}(-\beta z) \end{aligned} \tag{5.4}$$

clearly shows that it is an α-KMS state. Conversely assume $\beta \neq 0$ and ω is an α-KMS state. Suppose $G_{A,B}$ is a witness to ω being an α-KMS state. Then

$$\begin{aligned} F_{A,B} : \overline{\mathfrak{D}_{-1}} & \to \mathbb{C} \\ z & \mapsto F_{A,B}(-z/\beta) \end{aligned} \tag{5.5}$$

clearly shows it is a (τ, β)-KMS state. $\square$

The importance of KMS states becomes immediately obvious due to the next theorem since it shows that KMS states are constant in the dynamics.

Theorem 5.1.3. Let $(\mathcal{A}, \tau)$ be a C^* or W^* dynamical system where A is unital and ω a (τ, β)-KMS state for some $\beta \in \mathbb{R} \setminus \{0\}$. Then for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$ we have $\omega(\tau_t(A)) = \omega(A)$.

Proof. By the previous theorem, we might as well assume that ω is a τ-KMS state. Let $A \in \mathcal{A}$ be self-adjoint. If $F_{1,A}$ is a witness to ω being a τ-KMS state then for $t \in \mathbb{R}$

$$F_{1,A}(t) = \omega(\tau_t(A)) = F_{1,A}(t - i) \tag{5.6}$$

and since $\overline{\omega(\tau_t(A))} = \omega(\tau_t(A)^*) = \omega(\tau_t(A^*)) = \omega(\tau_t(A))$ we have that $F_{1,A}(\overline{\mathfrak{D}_{-1}} \setminus \mathfrak{D}_{-1}) \subseteq \mathbb{R}$. Given that $F_{1,A}$ is continuous, bounded and analytic on $\mathfrak{D}_{-1}$ we conclude that $F_{1,A}$ is constant (this is an application of Liouville's theorem, see [1]). Therefore the theorem follows for self-adjoint operators. If $A \in \mathcal{A}$ we have

$$\omega(\tau_t(A)) = \omega\left(\tau_t\left(\frac{A + A^*}{2}\right)\right) + i\omega\left(\tau_t\left(\frac{A - A^*}{2i}\right)\right) \tag{5.7}$$

and each of the terms in the sum are independent of t since the operators are self-adjoint. The theorem follows. □

5.2 Gibbs states

Although we've shown that KMS states are constant under the dynamics of a system, this isn't the only requirement for a description of statistical equilibrium. Other issues like stability should be studied. The step we will give into further justifying the study of this states is to show that they are equivalent to the Gibbs states in the case of a finite dimensional Hilbert space. This will be stated with the use of only one theorem inspired by the work in [1].

Theorem 5.2.1. Let $(\mathcal{B}(\mathcal{H}), \tau)$ be the $C^*(W^*)$-dynamical system discussed in example 4.4.3. Then a state ω on $\mathcal{B}(\mathcal{H})$ is a β-Gibbs state if and only if it is a (τ, β)-KMS state.

Proof. Assume ω is a β-Gibbs state. Define for $A, B \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} F_{A,B} &: \mathbb{C} \to \mathbb{C} \\ z &\mapsto \omega(A\tau_z(B)) \end{aligned} \tag{5.8}$$

Then, we have for $t \in \mathbb{R}$ that $F_{A,B}(t) = \omega(A\tau_t(B))$ and

$$\begin{aligned}
F_{A,B}(t+i\beta) = \omega(A\tau_{t+i\beta}(B)) &= \frac{\operatorname{tr}(e^{-\beta H} A e^{iH(t+i\beta)} B e^{-iH(t+i\beta)})}{\operatorname{tr} e^{-\beta H}} \\
&= \frac{\operatorname{tr}(e^{-\beta H} A e^{iHt} e^{-\beta H} B e^{-iHt} e^{\beta H})}{\operatorname{tr} e^{-\beta H}} \\
&= \frac{\operatorname{tr}(e^{iHt} e^{-\beta H} B e^{-iHt} e^{\beta H} e^{-\beta H} A)}{\operatorname{tr} e^{-\beta H}} \\
&= \frac{\operatorname{tr}(e^{iHt} e^{-\beta H} B e^{-iHt} A)}{\operatorname{tr} e^{-\beta H}} \\
&= \frac{\operatorname{tr}(e^{-\beta H} e^{iHt} B e^{-iHt} A)}{\operatorname{tr} e^{-\beta H}} = \omega(\tau_t(B)A).
\end{aligned} \tag{5.9}$$

Moreover, ω is continuous and doesn't depend on z, therefore $F_{A,B}$ is analytic (and continuous which easily follows) due to the product rule and the fact that the exponential is analytic. If $\{e_1, \ldots, e_N\}$ is an orthonormal basis of eigenvectors of H associated to the eigenvalues $E_1, \ldots E_N$ and $P_1, \ldots P_n$ are the corresponding projections on the span of each of the vectors, we have for $z \in \overline{\mathfrak{D}_\beta}$

$$\begin{aligned}
\|e^{\pm iHz}\| &= \|\sum_{n=1}^N e^{\pm iE_n z} P_n\| \leq \sum_{n=1}^N |e^{\pm iE_n z}| \|P_n\| \\
&= \sum_{n=1}^N |e^{\pm iE_n z}| = \sum_{n=1}^N |e^{\pm iE_n \operatorname{Re} z} e^{\mp E_n \operatorname{Im} z}| = \sum_{n=1}^N |e^{\mp E_n \operatorname{Im} z}| \\
&= \sum_{n=1}^N e^{\mp E_n \operatorname{Im} z} \leq \sum_{n=1}^N e^{|E_n \beta|}.
\end{aligned} \tag{5.10}$$

Since

$$\begin{aligned}
\|F_{A,B}(z)\| = \|\omega(A\tau_z(B))\| &\leq \|\omega\| \|A\| \|e^{iHz}\| \|B\| \|e^{-iHz}\| \\
&\leq \|A\| \sum_{n=1}^N e^{|E_n \beta|} \|B\| \sum_{n=1}^N e^{|E_n \beta|},
\end{aligned} \tag{5.11}$$

it follows that $F_{A,B}|_{\overline{\mathfrak{D}_\beta}}$ is bounded and that ω is a (τ, β)-KMS state.

Now assume that ω is a (τ, β)-KMS state and let $F_{A,B}$ be witness of this for $A, B \in \mathcal{B}(\mathcal{A})$. Define

$$\begin{aligned}
G_{A,B} : \mathbb{C} &\to \mathbb{C} \\
z &\mapsto \omega(A\tau_z(B))
\end{aligned} \tag{5.12}$$

We want to show that $F_{A,B} = G_{A,B}|_{\overline{\mathfrak{D}_\beta}}$ or, equivalently, that

$$\begin{aligned} f : \overline{\mathfrak{D}_\beta} &\to \mathbb{C} \\ z &\mapsto F_{A,B}(z) - G_{A,B}(z) \end{aligned} \tag{5.13}$$

(which is of course continuous and analytic in $\mathfrak{D}_\beta$) is null. In the case $\beta = 0$ this is obvious. Assume $\beta > 0$. Note that $D = \{z \in \mathbb{C} | -\beta < \operatorname{Im} z < \beta\}$ is a region (that is, open and connected) and $D^* = D$. It is clear that $f(\mathbb{R}) = \{0\}$. Then, by the Schwarz reflection principle[19] there exists an analytic function $g : D \to \mathbb{C}$ such that $g(z) = f(z)$ in $\mathfrak{D}_\beta \cup \mathbb{R}$. Then, since $g(\mathbb{R}) = 0$ we have that g is null[19]. We conclude that $f(z) = 0$ for $z \in \mathfrak{D}_\beta \cup \mathbb{R}$ and therefore is null by continuity. The case $\beta < 0$ is analogous.
In particular we now have that

$$\omega(A\tau_{i\beta}(B)) = G_{A,B}(i\beta) = F_{A,B}(i\beta) = \omega(\tau_0(B)A) = \omega(BA)^1. \tag{5.14}$$

Therefore, if $\{e_1, \dots, e_N\}$ is an orthonormal basis of eigenvectors of H associated to the eigenvalues $E_1, \dots E_N$ and $\{f_1, \dots, f_N\}$ is the dual basis, we have

$$\begin{aligned} \omega(A) &= \omega\left(\sum_{n,m=1}^{N} f_n(Ae_m) e_n \otimes e_m\right) = \sum_{n,m=1}^{N} f_n(Ae_m)\omega(e_n \otimes e_m) \\ &= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m=1}^{N} f_n(Ae_m) \operatorname{tr}(e^{-\beta H})\omega(e_n \otimes e_m) \\ &= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_k}\omega(e_n \otimes e_m) \\ &= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_k}\omega((e_n \otimes e_k)(e_k \otimes e_m)) \end{aligned} \tag{5.15}$$

where we have used that $(e_n \otimes e_k)(e_k \otimes e_m) = \langle e_k, e_k \rangle (e_n \otimes e_m) = e_n \otimes e_m$. Attempting to put the equation in a form where we can apply our previously

[1] A simple calculation shows that this equation is always true for β-Gibbs states. In our present situation we need to prove the converse.

derived condition we have

$$
\begin{aligned}
\omega(A) &= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega((e_n \otimes e_k) e^{-\beta E_k} (e_k \otimes e_m) e^{\beta E_m}) \\
&= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega((e_n \otimes e_k) e^{-\beta H} (e_k \otimes e_m) e^{\beta H}) \\
&= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega((e_n \otimes e_k) \tau_{i\beta}(e_k \otimes e_m)) \\
&= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega((e_k \otimes e_m)(e_n \otimes e_k)) \\
&= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m,k=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega(e_k \otimes e_k) \langle e_m, e_n \rangle \\
&= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega\left(\sum_{k=1}^{N} \rho_{e_k}\right) \delta_{mn}
\end{aligned}
\tag{5.16}
$$

by noticing that $e_k \otimes e_k = \rho_{e_k}$ using the notation presented in section 2.2. Finally, we have that

$$
\begin{aligned}
\omega(A) &= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n,m=1}^{N} f_n(Ae_m) e^{-\beta E_m} \omega(1) \delta_{mn} \\
&= \frac{1}{\operatorname{tr}(e^{-\beta H})} \sum_{n=1}^{N} f_n(Ae_n) e^{-\beta E_n} = \frac{\operatorname{tr}(Ae^{-\beta H})}{\operatorname{tr}(e^{-\beta H})}
\end{aligned}
\tag{5.17}
$$

from which follows that ω is a β-Gibbs state. $\square$

Chapter 6

The Modular Theory of Tomita-Takesaki

We are now going to develop the theory of Tomita-Takesaki following the approach of [1] and [2]. This approach is appropriate for our purposes since it allows us to introduce the main objects of the theory, the Tomita-Takesaki theory, and the connection to KMS states, while avoiding unbounded operators (and therefore domain issues) by making use of real Hilbert spaces. The study of this theory is mathematically important due to its use in the classification of type III factors (and therefore physically important). As we will see, it is also physically important due to the interpretation we will give to Δ^{it} and the modular group in connection to KMS states.

For the remaining of the discussion I will assume knowledge of real Hilbert spaces, Borel functional calculus and polar decomposition. A quick account of this matters may be found in [1]. A more detailed exposition of Borel calculus and polar decompositions may be found in [20].

6.1 Operators of the Theory

Let $\mathcal{K}$ and $\mathcal{L}$ be closed subspaces of a real Hilbert space $\mathcal{H}$ such that $\mathcal{K} \cap \mathcal{L} = \{0\}$ and $\mathcal{K} + \mathcal{L}$ is dense in $\mathcal{H}$. Define P and Q as the orthogonal projections on $\mathcal{K}$ and $\mathcal{L}$ respectively. Define $R = P + Q$ and let $JT = P - Q$ the polar decomposition of $P - Q$. J is called the modular conjugation due to its role in the Tomita-Takesaki theorem. We have that P, Q, and T are positive. We will start by proving some basic facts about this operators which we will use throughout the development of the theory.

Theorem 6.1.1. 1. R and $2-R$ are injective and $2 \geq R \geq 0$;

2. $T = R^{1/2}(2-R)^{1/2}$ and is injective;

3. J is self-adjoint, isometric, $J^2 = 1$ and is bijective;

4. T commutes with P, Q, R, and J;

5. $JP = (1-Q)J$, $JQ = (1-P)J$, and $JR = (2-R)J$.

6. $J\mathcal{K} = \mathcal{L}^{\perp}$

Proof. 1. It is clear that the sum of positive operators is positive. Therefor $R \geq 0$. Moreover $1-P$ and $1-Q$ are also projections so that $2-R = (1-P)+(1-Q) \geq 0$. We conclude $2 \geq R$.
Let $x \in \ker R$. Then

$$\begin{aligned}\|Px\|^2 + \|Qx\|^2 &= \langle Px, Px\rangle + \langle Qx, Qx\rangle = \langle x, P^*Px\rangle + \langle x, Q^*Qx\rangle \\ &= \langle x, P^2x\rangle + \langle x, Q^2x\rangle = \langle x, Px\rangle + \langle x, Qx\rangle \\ &= \langle x, Rx\rangle = 0\end{aligned} \tag{6.1}$$

Therefore $x \in \ker P \cap \ker Q = \mathcal{K}^{\perp} \cap \mathcal{K}^{\perp} = (\mathcal{K}+\mathcal{L})^{\perp} = \mathcal{H}^{\perp} = \{0\}$ and we conclude $\ker R = 0$, that is, R is injective. Notice now that $\mathcal{K}^{\perp}$ and $\mathcal{L}^{\perp}$ satisfy the same properties that we imposed on $\mathcal{K}$ and $\mathcal{L}$ at the beginning. That is $\mathcal{K}^{\perp} \cap \mathcal{L}^{\perp} = \{0\}$ and

$$\overline{\mathcal{K}^{\perp} + \mathcal{L}^{\perp}} = (\mathcal{K}^{\perp} + \mathcal{L}^{\perp})^{\perp\perp} = (\mathcal{K}^{\perp\perp} \cap \mathcal{L}^{\perp\perp})^{\perp} = (\mathcal{K} \cap \mathcal{L})^{\perp} = \{0\}^{\perp} = \mathcal{H}. \tag{6.2}$$

Since $1-P$ and $1-Q$ are the projections on $\mathcal{K}^{\perp}$ and $\mathcal{L}^{\perp}$ we have by repeating the previous arguments that $2-R$ is injective.

2. We first note that

$$\begin{aligned}T^2 &= (P-Q)^*(P-Q) = (P-Q)(P-Q) = P - PQ - QP + Q \\ &= (P+Q)(2-P-Q) = R(2-R) = (R^{1/2})^2((2-R)^{1/2})^2 \\ &= (R^{1/2}(2-R)^{1/2})^2\end{aligned} \tag{6.3}$$

Therefore, if $x \in \ker T$ the $x \in \ker T^2 = \ker R(2-R) = \{0\}$. We conclude T is injective. Since the product of positive elements is positive we have $T = R^{1/2}(2-R)^{1/2}$ by the uniqueness of the positive square root.

3. J is an isometry on the orthogonal complement of

$$\begin{aligned} \ker J = \ker(P-Q) \subseteq \ker(P-Q)^2 = \ker(P-Q)^*(P-Q) \\ = \ker T^2 = \ker R(2-R) = \{0\}. \end{aligned} \tag{6.4}$$

   We conclude that J is an isometry and is injective. $J^* = J$ and $J^2 = 1$ follows from $P - Q$ being self adjoint and the properties of the polar decomposition. Since $J^2 = 1$ we have that it is surjective and we conclude it is a bijection.

4. T commutes with J because $P-Q$ is self-adjoint. We also have $T^2P = (P-Q)^2P = P - PQP = PT^2$ and therefore the positive square root must also commute $TP = PT$. Analogously $TQ = QT$ and one concludes that T and R commute.

5. Since $TJP = JTP = (P-Q)P = P - QP = (1-Q)(P-Q) = (1-Q)TJ = T(1-Q)J$ and T is injective we have $JP = (1-Q)J$. We also have $PJ = (J^*P^*)^* = (JP)^* = ((1-Q)J)^* = J(1-Q)$ and we can solve for $JQ = (1-P)J$. We conclude $JR = J(P+Q) = (2-R)J$.

6. Since J is surjective ($J^2 = 1$) and $1 - Q$ is the projection on $\mathcal{L}^\perp$

$$J\mathcal{K} = JP\mathcal{H} = (1-Q)J\mathcal{H} = (1-Q)\mathcal{H} = \mathcal{L}^\perp. \tag{6.5}$$

□

To obtain now the operators of the theory let $\mathcal{K}$ be a closed real subspace of a Hilbert space $\mathcal{H}$ such that $\mathcal{K} \cap i\mathcal{K} = \{0\}$ and $\mathcal{K} + i\mathcal{K}$ is dense in $\mathcal{H}$. Then the replacement of $\mathcal{L}$ by $i\mathcal{K}$ allows us to define the operators P, Q, R, and T as functions on $\mathcal{H}$. We will now give some properties of these as functions on the complex structure.

Theorem 6.1.2. 1. $R, 2-R$, and T are positive in $\mathcal{H}$

2. J is a conjugate linear isometry in $\mathcal{H}$.

3. $\langle x, Jy\rangle = \langle y, Jx\rangle$ for all $x, y \in \mathcal{H}$

4. $(JAJ)^* = JA^*J$ for all $A \in \mathcal{B}(\mathcal{H})$

Proof. 1. We only need to prove these functions are $\mathcal{B}(\mathcal{H})$. Note that for this proving linearity is enough since the topologies in the real and complex case coincide. Let $x \in \mathcal{H}$ and since $\mathcal{H}_{\mathbb{R}} = \mathcal{K} \oplus \mathcal{K}^{\perp}$ there exist unique $y \in \mathcal{K}$ and $z \in \mathcal{K}^{\perp}$ such that $x = y + z$. It is clear that $iz \in (i\mathcal{K})^{\perp}$ and therefore $iPx = iy = Q(ix)$. We conclude that $iP = Qi$ and therefore $Rix = P(ix) + Q(ix) = iQx + iPx = iRx$. We conclude that R is linear. It is the clear too that $2 - R$ is also linear. By uniqueness of the positive square root that $T = R^{1/2}(2 - R)^{1/2}$ is also linear.

2. Given that T is injective and

$$TJ(ix) = (P - Q)(ix) = iQx - iPx = -i(P - Q)x = T(-iJx) \quad (6.6)$$

for all $x \in \mathcal{H}$, we conclude that J is conjugate linear. Moreover, since the norm in $\mathcal{H}_{\mathbb{R}}$ is the same as $\mathcal{H}$ we have that J is an isometry.

3. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned}
\langle x, Jy\rangle &= \langle x, Jy\rangle_{\mathbb{R}} - i\langle x, iJy\rangle_{\mathbb{R}} \\
&= \langle x, Jy\rangle_{\mathbb{R}} - i\langle x, J(-iy)\rangle_{\mathbb{R}} \\
&= \langle Jx, y\rangle_{\mathbb{R}} - i\langle Jx, -iy\rangle_{\mathbb{R}} \\
&= \langle y, Jx\rangle_{\mathbb{R}} - i\langle -iy, Jx\rangle_{\mathbb{R}} \\
&= \langle y, Jx\rangle_{\mathbb{R}} - i\langle y, iJx\rangle_{\mathbb{R}} \\
&= \langle y, Jx\rangle.
\end{aligned} \quad (6.7)$$

4. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned}
\langle x, JAJy\rangle &= \langle AJy, Jx\rangle = \langle Jy, A^*Jx\rangle \\
&= \overline{\langle A^*Jx, Jy\rangle} = \overline{\langle y, JA^*Jx\rangle} \\
&= \langle JA^*Jx, y\rangle
\end{aligned} \quad (6.8)$$

We conclude that $(JAJ)^* = JA^*J$. □

Theorem 6.1.3. The spectra of $2 - R$ and R are equal.

Proof. We have that J is a bijection and for all $\lambda \in \mathbb{C}$

$$J(\lambda - R)J = J\lambda J - JRJ = J^2\lambda - (2-R)J^2 = \lambda - (2-R). \tag{6.9}$$

Therefore, $\lambda - R$ is bijective if an only if $\lambda - (2-R)$ is. Moreover, by the open mapping theorem this implies that $\lambda - R$ is invertible by a bounded operator if and only if $\lambda - (2-R)$ is. We conclude $\sigma(R) = \sigma(2-R)$. $\square$

Since the spectrum of a positive operator $A \in \mathcal{B}(\mathcal{H})$ is in $[0,\infty)$ and compact and the function

$$\begin{aligned} [0,\infty) &\to \mathbb{C} \\ \lambda &\mapsto \lambda^z \end{aligned} \tag{6.10}$$

(and $0^0 = 0$)is bounded and measurable we have that $A^z \in \mathcal{B}(\mathcal{H})$ is bounded.

Theorem 6.1.4. We have that $JR^{it}J = (2-R)^{-it}$.

Proof. We have that if P_R is the resolution of the identity for R,

$$\begin{aligned} 2 - R = 2\cdot 1 - R = 2\int dP_R - \int \mathrm{id}_{\mathbb{C}}\, dP_R = \int 2dP_R - \int \mathrm{id}_{\mathbb{C}}\, dP_R \\ = \int (2 - \mathrm{id}_{\mathbb{C}}) dP_R, \end{aligned} \tag{6.11}$$

from which we conclude $2 - R = (2 - \mathrm{id}_{\mathbb{C}})(R)$. Therefore

$$(2 - \mathrm{id}_{\mathbb{C}})\sigma(R) = \sigma((2-\mathrm{id}_{\mathbb{C}})(R)) = \sigma(2-R) = \sigma(R) \tag{6.12}$$

and we have that $2 - \mathrm{id}_{\mathbb{C}}$ is an homeomorphism from $\sigma(R)$ onto itself. We may therefore define the function

$$\begin{aligned} F : \Sigma &\to L(\mathcal{H}) \\ E &\mapsto JP_R((2-\mathrm{id}_{\mathbb{C}})(E))J \end{aligned} \tag{6.13}$$

where Σ is the Borel σ-algebra on $\sigma(R)$. We are now gonna prove this is a spectral valued measure.

Calculating we have $F(\varnothing) = JP_R(\varnothing)J = J0J = 0$ and $F(\sigma(R)) = JJ = 1$. Let $E, D \in \Sigma$. Then

$$\begin{aligned} F(E\cap D) &= JP_R((2 - id_{\mathbb{C}})(E\cap D))J \\ &= JP_R((2 - id_{\mathbb{C}})(E) \cap (2 - \mathrm{id}_{\mathbb{C}})(D))J \\ &= JP_R((2 - id_{\mathbb{C}})(E))P_R((2 - \mathrm{id}_{\mathbb{C}})(D))J \quad . \\ &= JP_R((2 - id_{\mathbb{C}})(E))JJP_R((2 - \mathrm{id}_{\mathbb{C}})(D))J \\ &= F(E)F(D) \end{aligned} \tag{6.14}$$

Finally, let $x, y \in \mathcal{H}$. Then for all $E \in \Sigma$

$$\begin{aligned} F_{x,y}(E) &= \langle x, F(E)y \rangle = \langle x, JP_R((2 - \mathrm{id}_{\mathbb{C}})(E))Jy \rangle \\ &= \langle P_R((2 - \mathrm{id}_{\mathbb{C}})(E))Jy, Jx \rangle \\ &= \langle Jy, P_R((2 - \mathrm{id}_{\mathbb{C}})(E))Jx \rangle = P_{R_{Jy,Jx}}((2 - \mathrm{id}_{\mathbb{C}})(E)) \end{aligned} \tag{6.15}$$

and we conclude $F_{x,y} = P_{R_{Jy,Jx}} \circ (2 - \mathrm{id}_{\mathbb{C}})$ and since $(2 - \mathrm{id}_{\mathbb{C}})$ is a bijection on $\sigma(R)$, $F_{x,y}$ is a complex measure. Now, notice that for all $x, y \in \mathcal{H}$ we have that

$$\begin{aligned} \langle x, Ry \rangle &= \langle x, J(2 - R)Jy \rangle = \langle (2 - R)Jy, Jx \rangle \\ &= \langle Jy, (2 - R)Jx \rangle = \left\langle Jy, \int (2 - \mathrm{id}_{\mathbb{C}}) dP_R Jx \right\rangle \\ &= \int (2 - \mathrm{id}_{\mathbb{C}}) dP_{R_{x,y}} \end{aligned} \tag{6.16}$$

and since $2 - \mathrm{id}_{\mathbb{C}}$ is a bijection,

$$\begin{aligned} \langle x, Ry \rangle &= \int (2 - \mathrm{id}_{\mathbb{C}}) \circ (2 - \mathrm{id}_{\mathbb{C}}) d(P_{R_{x,y}} \circ (2 - \mathrm{id}_{\mathbb{C}})) \\ &= \int \mathrm{id}_{\mathbb{C}}\, d(P_{R_{x,y}} \circ (2 - \mathrm{id}_{\mathbb{C}})) = \int \mathrm{id}_{\mathbb{C}}\, dF_{x,y}. \end{aligned} \tag{6.17}$$

$$= \left\langle x, \int \mathrm{id}_{\mathbb{C}}\, dFy \right\rangle \tag{6.18}$$

Therefore, by the uniqueness of the spectral resolution guaranteed by the spectral theorem we have that $P_R = F$. We may repeat a similar analysis to show that

$$\begin{aligned} G : \Sigma &\to L(\mathcal{H}) \\ E &\mapsto P_R((2 - \mathrm{id}_{\mathbb{C}})(E)) \end{aligned} \tag{6.19}$$

is a resolution of the identity on $\sigma(R)$ and it is clear that $G_{x,y} = P_{R_{x,y}} \circ (2 - \mathrm{id}_{\mathbb{C}})$. Notice that for all $x, y \in \mathcal{H}$

$$\begin{aligned} \langle x, (2 - R)y \rangle &= \left\langle x, \int (2 - \mathrm{id}_{\mathbb{C}}) dP_R y \right\rangle = \int (2 - \mathrm{id}_{\mathbb{C}}) dP_{R_{x,y}} \\ &= \int (2 - \mathrm{id}_{\mathbb{C}}) \circ (2 - \mathrm{id}_{\mathbb{C}}) d(P_{R_{x,y}} \circ (2 - \mathrm{id}_{\mathbb{C}})) \\ &= \int \mathrm{id}_{\mathbb{C}}\, dG_{x,y}, \end{aligned} \tag{6.20}$$

that is, G is the resolution of the identity of $2-R$. Finally, we calculate for all $x, y \in \mathcal{H}$ and $t \in \mathbb{R}$

$$\begin{aligned}
\langle x, JR^{it}Jy\rangle &= \langle R^{it}Jy, Jx\rangle = \langle Jy, R^{-it}Jx\rangle \\
&= \left\langle Jy, \int \lambda^{-it} dP_R(\lambda)Jx \right\rangle = \int \lambda^{-it} dP_{R_{Jy,Jx}}(\lambda) \\
&= \int (2-\lambda)^{-it} d(P_{R_{Jy,Jx}} \circ (2 - \mathrm{id}_{\mathbb{C}}))(\lambda) \\
&= \int (2-\lambda)^{-it} dF_{x,y}(\lambda) = \int (2-\lambda)^{-it} dE_{x,y}(\lambda) \\
&= \int (2-\lambda)^{-it} d(E_{x,y} \circ \mathrm{id}_{\mathbb{C}})(\lambda) \\
&= \int (2-\lambda)^{-it} d(E_{x,y} \circ (2 - \mathrm{id}_{\mathbb{C}}) \circ (2 - \mathrm{id}_{\mathbb{C}}))(\lambda) \\
&= \int (2-\lambda)^{-it} d(G_{x,y} \circ (2 - \mathrm{id}_{\mathbb{C}}))(\lambda) \\
&= \int \lambda^{-it} d(G_{x,y})(\lambda) = \langle x, (2-R)^{-it}y\rangle.
\end{aligned} \tag{6.21}$$

We conclude $JR^{it}J = (2-R)^{-it}$. ◻

We are now ready to define the main operator in the theory and give some of the properties which will proof to be useful in connecting the theory with KMS states.

Definition 6.1.5. Define the one-parameter unitary group $t \mapsto \Delta^{it} = (2-R)^{it}R^{-it}$ for all $t \in \mathbb{R}$.

Note that Δ^{it} is to be understood as a symbol by its own rather than a function of an operator Δ. Although such an operator $\Delta = (2-R)R^{-1}$ is usually defined, it may be unbounded and therefore the definition of Δ^{it} would require a measurable calculus for unbounded operators. Our approach using real Hilbert spaces avoided just this.

Theorem 6.1.6. $t \mapsto \Delta^{it}$ is a strongly continuous one-parameter unitary group which satisfies $J\Delta^{it} = \Delta^{it}J$, $T\Delta^{it} = \Delta^{it}T$, and $\Delta^{it}\mathcal{K} = \mathcal{K}$ for every $t \in \mathbb{R}$.

Proof. Using the fact that R and $2-R$ commute and are self-adjoint (since they are positive) along with the properties of exponentiation prove $t \mapsto \Delta^{it}$

is a one-parameter unitary group. The fact that T and R commute with R and $2-R$ also shows they commute with Δ^{it}. Note

$$\begin{aligned} J\Delta^{it} &= J(2-R)^{it}R^{-it} = J(2-R)^{it}JJR^{-it} = R^{-it}JR^{-it} \\ &= R^{-it}JR^{-it}JJ = R^{-it}(2-R)^{it}J = (2-R)^{it}R^{-it}J = \Delta^{it}J. \end{aligned} \tag{6.22}$$

□

The connection between KMS states and Tomita-Takesaki theory follows from a certain KMS type condition that $t \mapsto \Delta^{it}$ satisfies.

Definition 6.1.7. Let $\mathcal{L}$ be a real subspace of $\mathcal{H}$. A one-parameter unitary group $t \mapsto U_t$ satisfies the KMS condition with respect to $\mathcal{L}$ if for all $x, y \in \mathcal{H}$ there exists a bounded continuous function $f : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$ analytic on $\mathfrak{D}_{-1}$ such that

$$f(t) = \langle x, U_t y \rangle \tag{6.23}$$

and

$$f(t-i) = \langle U_t y, x \rangle \tag{6.24}$$

for all $t \in \mathbb{R}$.

The relationship between this version of the KMS condition and the one given before will become apparent due to the connection between the inner product and the state in a GNS representation. The function f in the previous definition has to be unique [1]. There is a useful alternative formulation of this.

Theorem 6.1.8. A one-parameter unitary group $t \mapsto U_t$ satisfies the KMS condition with respect to a real subspace $\mathcal{L}$ of $\mathcal{H}$ if and only if for all $x, y \in \mathcal{L}$ there exists a bounded continuous function $f : \overline{\mathfrak{D}_{-1/2}} \to \mathbb{C}$ analytic on $\mathfrak{D}_{-1/2}$ such that

$$f(t) = \langle x, U_t y \rangle \tag{6.25}$$

and

$$f(t-i/2) \in \mathbb{R} \tag{6.26}$$

for all $t \in \mathbb{R}$

Proof. Let $x, y \in \mathcal{L}$. Suppose there exists a function $f : \overline{\mathfrak{D}_{-1/2}} \to \mathbb{C}$ analytic on $\mathfrak{D}_{-1/2}$ such that

$$f(t) = \langle x, U_t y \rangle \tag{6.27}$$

and

$$f(t - i/2) \in \mathbb{R} \tag{6.28}$$

for all $t \in \mathbb{R}$. By the Schwarz Reflection Principle the function $g : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$ defined by

$$g(z) = f(z) \tag{6.29}$$

for all $z \in \overline{\mathfrak{D}_{-1/2}}$ and

$$g(z) = \overline{f(\overline{z + i/2})} \tag{6.30}$$

for all $z \in \overline{\mathfrak{D}_{-1}} \setminus \overline{\mathfrak{D}_{-1/2}}$ is analytic on $\mathfrak{D}_{-1}$. Moreover

$$g(t - i/2) = \overline{f(t)} = \langle U_t y, x \rangle \tag{6.31}$$

for all $t \in \mathbb{R}$. The function g shows that $t \mapsto U_t$ is a KMS state with respect to $\mathcal{L}$.

Assume now that $t \mapsto U_t$ satisfies the KMS condition with respect to $\mathcal{L}$ and let f be a witness of this. Consider the function

$$\begin{aligned} g : \overline{\mathfrak{D}_{-1}} &\to \mathbb{C} \\ z &\mapsto \overline{f(\overline{z + i})}. \end{aligned} \tag{6.32}$$

Then it is clear the g is also a witness of $t \mapsto U_t$ satisfying the KMS condition with respect to $\mathcal{L}$ and by uniqueness $f = g$. But then it is clear that $f(t - i/2) \in \mathbb{R}$. $\square$

To be able to prove the uniqueness of our first result stating the relation between KMS states and Tomita-Takesaki theory we need the notion of a weak entire vector.

Definition 6.1.9. Let $t \mapsto U_t$ be a one-parameter unitary group on $\mathcal{H}$. $x \in \mathcal{H}$ is a weak entire vector for $t \mapsto U_t$ if there exists a function $h : \mathbb{C} \to \mathcal{H}$ such that $h(t) = U_t x$ for all $t \in \mathbb{R}$, the function $z \mapsto \langle y, h(z) \rangle$ is an entire function for every $y \in \mathcal{H}$, and h is bounded on every bounded subset of $i\mathbb{R}$. We denote the set of weak entire vector of $t \mapsto U_t$ by $W(U_t)$.

Theorem 6.1.10. Let $\mathcal{L}$ be a closed real subspace of $\mathcal{H}$ and $t \mapsto U_t$ a strongly continuous one-parameter unitary group in $\mathcal{H}$ such that $U_t \mathcal{L} \subseteq \mathcal{L}$. Then $\mathcal{L} \cap W(U_t)$ is dense in $\mathcal{L}$.

Proof. Let $x \in \mathcal{L}$. Given that $\mathcal{L}$ is closed and $t \mapsto U_t$ is strongly continuous and bounded we can define the sequence (x_n) in $\mathcal{L}$ where

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} U_t x dt. \tag{6.33}$$

Having in mind the standard Gaussian integral we have that $x_n \to x$ (the analysis required for this result can be found in [1]). Now define for every $n \in \mathbb{N}^*$

$$\begin{aligned} h_n &: \mathbb{C} \to \mathcal{H} \\ z &\mapsto \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} U_t x dt. \end{aligned} \tag{6.34}$$

By the definition of a Riemann integral and continuity of the inner product

$$\langle y, h_n(z) \rangle = z\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \langle y, U_t x \rangle dt. \tag{6.35}$$

It can be shown that $z \mapsto \langle y, h_n(z) \rangle$ is an entire function (the type of argument required for this result can be found in [1]). By explicit calculation one can show that h_n is witness of $x_n \in W(U_t)$ from which the theorem follows. $\square$

We are now ready to show the key to understand the connection between KMS states and Tomita-Takesaki theory.

Theorem 6.1.11. $t \mapsto \Delta^{it}$ is the unique strongly continuous one-parameter unitary group on $\mathcal{H}$ that satisfies the KMS condition with respect to $\mathcal{K}$ such that $\Delta^{it}\mathcal{K} \subseteq \mathcal{K}$ for all $t \in \mathbb{R}$.

Proof. We know that $t \mapsto \Delta^{it}$ is a strongly continuous one parameter unitary group. Let $x, y \in \mathcal{K}$ and

$$\begin{aligned} f &: \overline{\mathfrak{D}_{-1/2}} \to \mathbb{C} \\ z &\mapsto \langle x, (2-R)^{iz} R^{-iz+1/2} (R^{1/2} + (2-R)^{1/2} J) y/2 \rangle. \end{aligned} \tag{6.36}$$

This function is continuous, bounded and analytic on $\mathfrak{D}_{-1/2}$. Let $t \in \mathbb{R}$. Then

$$\begin{aligned} f(t) &= \langle x, (2-R)^{it} R^{-it+1/2} (R^{1/2} + (2-R)^{1/2} J) y/2 \rangle \\ &= \langle x, \Delta^{it} (R + TJ) y/2 \rangle = \langle x, \Delta^{it} (R + JT) y/2 \rangle \\ &= \langle x, \Delta^{it} (P + Q + P - Q) y/2 \rangle = \langle x, \Delta^{it} P y \rangle = \langle x, \Delta^{it} y \rangle \end{aligned} \tag{6.37}$$

and

$$\begin{aligned}
\operatorname{Im} f(t-i/2) &= \operatorname{Im}\langle x, (2-R)^{it+1/2}R^{-it}(R^{1/2}+(2-R)^{1/2}J)y/2\rangle \\
&= \operatorname{Im}\langle \Delta^{-it}x, (2-R)^{1/2}(R^{1/2}+(2-R)^{1/2}J)y/2\rangle \\
&= \operatorname{Im}\langle \Delta^{-it}x, (T+(2-R)J)y/2\rangle \\
&= \operatorname{Im}\langle \Delta^{-it}x, (T+JR)y/2\rangle \\
&= \operatorname{Im}\langle \Delta^{-it}x, J(JT+R)y/2\rangle = \operatorname{Im}\langle \Delta^{-it}x, JPy\rangle \\
&= \operatorname{Im}(i\langle i\Delta^{-it}x, (1-Q)Jy\rangle) \\
&= \operatorname{Im}(i\langle i\Delta^{-it}x, (1-Q)Jy\rangle) = 0
\end{aligned} \tag{6.38}$$

since $\Delta^{-it}x \in \mathcal{K}$ and $(1-Q)Jy \in (i\mathcal{K})^{\perp}$. Therefore $t \mapsto \Delta^{it}$ satisfies the KMS condition with respect to $\mathcal{K}$.

Now assume that $t \mapsto U_t$ is a strongly continuous one-parameter group that satisfies the KMS condition with respect to $\mathcal{K}$ and such that $U_t\mathcal{K} \subseteq \mathcal{K}$ for all $t \in \mathbb{R}$. Let $x \in \mathcal{K} \cap W(U_t)$ and h a witness of x being an entire vector. We have that h is bounded on all strips of the form $\{z \in \mathbb{C} | a < \operatorname{Im} z < b\}$ with $a, b \in \mathbb{R}$ such that $a < b$. Indeed this is because the functions $z \mapsto \langle y, h(t+iz)\rangle$ and $z \mapsto \langle y, U_t h(iz)\rangle$ with $y \in \mathcal{H}$ are entire and coincide on $i\mathbb{R}$. Therefore they coincide in $\mathbb{C}$ for all $y \in \mathcal{H}$ and we have

$$\|h(t+is)\| \leq \|U_t h(is)\| \leq \|h(is)\|. \tag{6.39}$$

Boundedness on the strips come from h being bounded on bounded subsets of $i\mathbb{R}$. Let $y \in \mathcal{K}$ and define the bounded, continuous function analytic on $\mathfrak{D}_{-1/2}$ (see [1])

$$\begin{aligned}
g : \overline{\mathfrak{D}_{-1/2}} &\to \mathbb{C} \\
z &\mapsto \langle J(2-R)^{iz}R^{-iz+1/2}(R^{1/2}+(2-R)^{1/2}J)y/2, h(z)\rangle.
\end{aligned} \tag{6.40}$$

With a similar argument as the one presented in equation (6.38) we can show that $g(\mathbb{R}) \subseteq \mathbb{R}$ and

$$g(t-i/2) = \langle \Delta^{it}y, h(t-i/2)\rangle. \tag{6.41}$$

Let $f : \overline{\mathfrak{D}_{-1/2}} \to \mathbb{C}$ be a witness of $t \mapsto U_t$ satisfying the KMS condition with respect to $\mathcal{K}$ for x and $\Delta^{is}y$ and define for $s \in \mathbb{R}$

$$\begin{aligned}
F : \overline{\mathfrak{D}_{-1/2}} &\to \mathbb{C} \\
z &\mapsto \langle \Delta^{is}y, h(z)\rangle - f(z)
\end{aligned} \tag{6.42}$$

Using the Schwarz Reflection Principle to extend F and noticing that $F(\mathbb{R}) = \{0\}$ we have that $F = 0$. We therefore conclude that since $f(t - i/2) \in \mathbb{R}$

$$g(t - i/2) = \langle \Delta^{it} y, h(t - i/2) \rangle \in \mathbb{R} \tag{6.43}$$

for all $t \in \mathbb{R}$ and g is constant. In particular

$$\langle \Delta^{it} J y, U_t x \rangle = g(t) = g(0) = \langle J y, x \rangle. \tag{6.44}$$

Given that $\operatorname{span} \mathcal{K}$ is dense in $\mathcal{H}$ since $\mathcal{K} + i\mathcal{K}$ is dense we have that $\operatorname{span}(\mathcal{K} \cap W(U_t))$ is dense and therefore we conclude that $\Delta^{it} J y = U_t J y$ for all $t \in \mathbb{R}$ and $y \in \mathcal{K}$. Now, span $J\mathcal{K}$ is dense since J is surjective, continuous, and

$$\mathcal{H} = J\overline{\mathcal{K} + i\mathcal{K}} \subseteq \overline{J(\mathcal{K} + i\mathcal{K})} = \overline{J\mathcal{K} + iJ\mathcal{K}} \tag{6.45}$$

. Therefore $U_t = \Delta^{it}$. □

With these we have now constructed all of the operators of the theory and stated their most important properties. Our last task for the section will be to show how the closed real subspace $\mathcal{K}$ arises in a natural fashion from a von Neumann algebra and a cyclic and separating vector.

Theorem 6.1.12. Let $\mathfrak{M}$ be a W^*-algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ be a cyclic and separating vector for $\mathfrak{M}$. Then $\mathcal{K} = \overline{\{A\Omega | A \in \mathfrak{M} \text{ is self adjoint}\}}$ is a closed real subspace of $\mathcal{H}$ such that $\mathcal{K} \cap i\mathcal{K} = \varnothing$ and $\mathcal{K} + i\mathcal{K}$ is dense in $\mathcal{H}$.

Proof. It is clear that $\mathcal{K}$ is closed real subspace. By decomposition of bounded operators (as in the proof of theorem 5.1.3) we have $\mathfrak{M}\Omega \subseteq \mathcal{K} + i\mathcal{K}$ and therefore we have that $\mathcal{K} + i\mathcal{K}$ is dense. Now let $A \in \mathfrak{M}'$ be self-adjoint. Then if $B \in \mathfrak{M}$ is self-adjoint we have that $\langle B\Omega, A\Omega \rangle \in \mathbb{R}$ and therefore $0 = \langle B\Omega, iA\Omega \rangle_{\mathbb{R}}$. We conclude due to decomposition of bounded operators that $\mathfrak{M}'\Omega \subseteq (i\mathcal{K})^\perp + \mathcal{K}^\perp \subseteq (\mathcal{K} \cap i\mathcal{K})^\perp$. Since Ω is cyclic for $\mathfrak{M}'$ we have that $\mathcal{K} \cap i\mathcal{K} = \varnothing$. □

With this theorem we can define for every pair $(\mathfrak{M}, \Omega)$ with $\mathfrak{M}$ a W^*-algebra and Ω a cyclic and separating vector for $\mathfrak{M}$ the functions P, Q, R, T, an associated unitary group $t \mapsto \Delta^{it}$, and a modular conjugation J. Theorem 4.3.6 shows that we will always be able to do Tomita-Takesaki theory on the GNS representation of a von Neumann algebra equipped with a faithful normal state. The operators of the theory behave nicely with respect to the cyclic vector Ω.

Theorem 6.1.13. $J\Omega = \Omega$ and $\Delta^{it}\Omega = \Omega$ for all $t \in \mathbb{R}$.

Proof. It is clear that $\Omega \in \mathcal{K} \cap (i\mathcal{K})^{\perp}$. Therefore $P\Omega = \Omega$ and $Q\Omega = 0$. We immediately have then that $R\Omega = \Omega$ and $\Delta^{it}\Omega = \Omega$. On the other hand $T^2\Omega = \Omega$ and therefore $J\Omega = J(P - Q)\Omega = JJT\Omega = T\Omega = \Omega$. $\square$

6.2 Tomita-Takesaki Theorem

This section is completely devoted to the Tomita-Takesaki theorem. Although an effort was made in the previous section to be thorough with the construction of the operators involved in the theorem, we will now only assume the result $\Delta^{it} J\mathfrak{M}' J\Delta^{-it} \subseteq \mathfrak{M}$ given in [1][2]. This is in order to keep the mathematics as concise as possible since this theorem is not easy to prove.

Theorem 6.2.1 (Tomita-Takesaki Theorem)**.** Let $\mathfrak{M}$ be a W^*-algebra and Ω be a cyclic and separating vector for $\mathfrak{M}$. Let $t \mapsto \Delta^{it}$ and J be the unitary group and modular conjugation associated to $(\mathfrak{M}, \Omega)$. Then:

- $J\mathfrak{M}J = \mathfrak{M}'$;
- $\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}$ for all $t \in \mathbb{R}$.

Proof. For all $A \in \mathfrak{M}'$ and $t \in \mathbb{R}$ we have that $\Delta^{it} JAJ\Delta{-}it \in \mathfrak{M}$ [1][2]. Then by setting $t = 0$ we have that $J\mathfrak{M}'J \subseteq \mathfrak{M}$ and therefore $\mathfrak{M}' = J^2\mathfrak{M}'J^2 \subseteq J\mathfrak{M}J$. On the other hand, it is clear that $\Omega \in \mathcal{K} \cap (i\mathcal{K})^{\perp}$. Therefore $P\Omega = \Omega$ and $Q\Omega = 0$. Then $JT\Omega = \Omega$ and $T^2\Omega = \Omega$. By the Borel calculus $T^2\Omega = \Omega$ implies that $T\Omega = \Omega$[1] and we conclude $J\Omega = JJT\Omega = T\omega = \Omega$. Since $J\mathcal{K} \subseteq (i\mathcal{K})^{\perp}$ we have that for all $A, B \in \mathfrak{M}$ self-adjoint $\langle JA\Omega, B\Omega\rangle \in \mathbb{R}$. Therefore by the properties of J we have that

$$\begin{aligned}\langle BJAJ\Omega, \Omega\rangle &= \langle BJA\Omega, \Omega\rangle = \langle JA\Omega, B\Omega\rangle = \langle B\Omega, JA\Omega\rangle \\ &= \langle A\Omega, JB\Omega\rangle = \langle \Omega, AJB\Omega\rangle = \langle \Omega, AJBJ\Omega\rangle\end{aligned} \tag{6.46}$$

[1]This step is not trivial. Indeed we want to prove something along the lines of: if $Ax = x$ then $f(A)x = f(1)x$. This is given in [1] and requires decomposing f into positive functions, approximating positive functions by simple ones, simple functions by continuous ones, and continuous functions by polynomials using the Stone-Weierstrass theorem.

and by linearity this extends to arbitrary $B \in \mathfrak{M}$. If $C \in \mathfrak{M}'$ we have that $JCJ \in \mathfrak{M}$ since $J\mathfrak{M}'J \subseteq \mathfrak{M}$ and therefore we have for $A, B \in \mathfrak{M}$ self-adjoint recalling the properties of J that

$$\begin{aligned}\langle AJBJ\Omega, C\Omega\rangle &= \langle JBJ\Omega, AC\Omega\rangle = \langle \Omega, JBJAC\Omega\rangle = \langle BJAC\Omega, J\Omega\rangle \\ &= \langle BJAC\Omega, \Omega\rangle = \langle BJCA\Omega, \Omega\rangle = \langle BJCJJA\Omega, \Omega\rangle \\ &= \langle BJCJJAJ\Omega, \Omega\rangle = \langle \Omega, AJBJCJJ\Omega\rangle = \langle \Omega, AJBJC\Omega\rangle \\ &= \langle A\Omega, JBJC\Omega\rangle = \langle JBJA\Omega, C\Omega\rangle.\end{aligned} \tag{6.47}$$

Since Ω is cyclic for $\mathfrak{M}'$ by linearity we have that $JBJA\Omega = AJBJ\Omega$ for all $A, B \in \mathfrak{M}$. Therefore if $C \in \mathfrak{M}$ it is true that $JBJAC\Omega = ACJBJ\Omega = AJBJC\Omega$ and since Ω is cyclic for $\mathfrak{M}$ we have that $JBJ \in \mathfrak{M}'$ by continuity. We conclude that $J\mathfrak{M}J \subseteq \mathfrak{M}'$ proving the first part of the theorem.

For the second part note that for all $t \in \mathbb{R}$

$$\Delta^{it}\mathfrak{M}\Delta^{-it} = \Delta^{it}J\mathfrak{M}'J\Delta^{-it} \subseteq \mathfrak{M} \tag{6.48}$$

and therefore

$$\mathfrak{M} = \Delta^0\mathfrak{M}\Delta^0 = \Delta^{it}\Delta^{-it}\mathfrak{M}\Delta^{it}\Delta^{-it} \subseteq \Delta^{it}\mathfrak{M}\Delta^{-it} \tag{6.49}$$

proving the second part.

$\square$

The Tomita-Takesaki theorem will be of great importance to us since it yields a one-parameter automorphism group that defines a dynamical system.

Theorem 6.2.2. Let $\mathfrak{M}$ be a von Neuman algebra and ω a faithful normal state. Consider the unitary group $t \mapsto \Delta^{it}$ associated to the pair $(\pi_\omega(\mathfrak{M}), \Omega_\omega)$. Then the map

$$\begin{aligned}\alpha : \mathbb{R} \to &\mathrm{Aut}(\mathcal{M}) \\ t \mapsto \ &\alpha_t : \mathcal{M} \to \mathcal{M} \\ &\qquad A \mapsto \alpha_t(A) := \pi_\omega^{-1}(\Delta^{it}\pi_\omega(A)\Delta^{-it}).\end{aligned} \tag{6.50}$$

makes $(\mathfrak{M}, \alpha)$ a W^*-dynamical system.

Proof. Because of theorem 6.2.1 we have that $\Delta^{it}\pi_\omega(A)\Delta^{-it} \in \pi_\omega(\mathfrak{M})$ and therefore by theorem 4.3.6 α is well defined. Since $t \mapsto \Delta^{it}$ is a one-parameter unitary group and the representation is faithful it is clear that α is a one-parameter group of automorphisms. We have that since $t \mapsto \Delta^{it}$ is strongly continuous for all $x \in \mathcal{H}_\omega$ and Δ^{it} is unitary that α satisfies the correct continuity conditions and the theorem follows. □

Definition 6.2.3. The map α of the previous theorem is called the modular group of $(\mathfrak{M}, \omega)$.

Chapter 7

KMS States and Tomita-Takesaki Theory

Now that we've discussed KMS states and the mathematical tool of Tomita-Takesaki theory we are ready to discuss their relationship. In this chapter we will prove the most important theorem of this work. This will show that the modular group induced by a state yields the unique time evolution that makes the state an equilibrium state. Actually most of the mathematical work required has already been done in theorems 6.1.11 and 6.2.2. Our only task now is to translate these results into a deep physical result.

7.1 The Modular Group and KMS States

Theorem 7.1.1. Let $\mathfrak{M}$ be a von Neumann algebra and ω be a normal faithful state. Then $(\mathfrak{M}, \alpha)$ is the only W^*-dynamical system with respect to which ω is a α-KMS state where α is the modular group of $(\mathfrak{M}, \omega)$.

Proof. First of all let us see that ω is a α-KMS state with α the modular group of $(\mathfrak{M}, \omega)$. Let $A, B \in \pi_\omega(\mathfrak{M})$ be self-adjoint. We know by 6.1.11 that there exists a bounded continuous function $f : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$ analytic on $\mathfrak{D}_{-1}$ such that

$$f(t) = \langle A\Omega_\omega, \Delta^{it} B\Omega_\omega\rangle = \langle \Omega_\omega, A\Delta^{it} B\Delta^{-it}\Omega_\omega\rangle \tag{7.1}$$

and

$$f(t-i) = \langle \Delta^{it} B\Omega_\omega, A\Omega_\omega\rangle = \langle \Omega_\omega, \Delta^{it} B\Delta^{-it} A\Omega_\omega\rangle. \tag{7.2}$$

We may extend this result through linearity to arbitrary $A, B \in \pi_\omega(\mathfrak{M})$. In particular, for all $A, B \in \mathfrak{M}$ there is a continuous bounded function $f : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$ such that for all $t \in \mathbb{R}$

$$\begin{aligned} f(t) &= \langle \Omega_\omega, \pi_\omega(A)\Delta^{it}\pi_\omega(B)\Delta^{-it}\Omega_\omega\rangle = \langle \Omega_\omega, \pi_\omega(A)\pi_\omega(\alpha_t(B))\Omega_\omega\rangle \\ &= \omega(A\alpha_t(B)) \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} f(t-i) &= \langle \Omega_\omega, \Delta^{it}\pi_\omega(B)\Delta^{-it}\pi_\omega(A)\Omega_\omega\rangle = \langle \Omega_\omega, \pi_\omega(\alpha_t(B))\pi_\omega(A)\Omega_\omega\rangle \\ &= \omega(\alpha_t(B)A) \end{aligned} \tag{7.4}$$

We conclude that ω is an α-KMS state.

Now let $(\mathfrak{M}, \tau)$ be any W^*-dynamical system such that ω is a τ-KMS state. Given that ω is invariant under τ, there exists a one-parameter unitary group $t \mapsto U_t$ such that $U_t\Omega_\omega = \Omega_\omega$ and $\pi_\omega(\tau_t(A)) = U_t\pi_\omega(A)U_{-t}$. Now let $\mathcal{K} = \overline{\{A\Omega_\omega | A \in \pi_\omega(\mathfrak{M}) \text{ is self-adjoint}\}}$. Since for all $A \in \pi_\omega(\mathfrak{M})$ self-adjoint we have $U_tAU_{-t} = (U_tAU_{-t})^*$ then $U_tA\Omega = U_tAU_{-t}\Omega \in \mathcal{K}$. We conclude that $U_t\mathcal{K} \subseteq \mathcal{K}$. On the other hand, for all A, B in $\pi_\omega(\mathfrak{M})$ self-adjoint we have since ω is an τ-KMS state that there exists a bounded continuous function $f : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$ analytic on $\mathfrak{D}_{-1}$ such that

$$\begin{aligned} f(t) &= \omega(\pi_\omega^{-1}(A)\tau_t(\pi_\omega^{-1}(B))) = \omega(\pi_\omega^{-1}(AU_tBU_{-t})) \\ &= \langle \Omega_\omega, AU_tBU_{-t}\Omega_\omega\rangle = \langle A\Omega_\omega, U_tB\Omega\rangle \end{aligned} \tag{7.5}$$

and

$$f(t-i) = \omega(\tau_t(\pi_\omega^{-1}(B)\pi_\omega^{-1}(A)) = \omega(\pi_\omega^{-1}(U_tBU_{-t}A)) \tag{7.6}$$
$$= \langle \Omega_\omega, U_tBU_{-t}A\Omega_\omega\rangle = \langle U_tB\Omega_\omega, A\Omega_\omega\rangle. \tag{7.7}$$

By considering sequences of such functions using the fact that the set of self-adjoint elements in $\pi_\omega(\mathfrak{M})$ is dense in $\mathcal{K}$, for all $x, y \in \mathcal{K}$ we can construct a bounded continuous function $f : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$ analytic on $\mathfrak{D}_{-1}$ such that

$$f(t) = \langle x, U_ty\rangle \tag{7.8}$$

and

$$f(t-i) = \langle U_ty, x\rangle. \tag{7.9}$$

Therefore U_t satisfies the KMS condition with respect to $\mathcal{K}$ and by theorem 6.1.11 we have $U_t = \Delta^{it}$. $\square$

Corollary 7.1.2. Let $\mathfrak{M}$ be a von Neumann algebra and ω be a faithful normal state. Then $(\mathfrak{M}, \tau)$ with $\tau_t(A) = \alpha_{-t/\beta}(A)$ and α the modular group of $(\mathfrak{M}, \omega)$ is the unique W^*-dynamical system such that ω is a (τ, β)-KMS state.

7.2 The Canonical Dynamical Law of Equilibrium

After the amount of machinery we developed in chapters 4, 5, and especially in 6, which allowed us to obtain result 7.1.2, we must now sit back and give a physical interpretation to this work. Through the use of C^* and W^*-algebras we managed to give a general definition to the notions of observables, states, dynamical laws, and dynamical systems. In this framework we agreed to identify quantum states in thermal equilibrium with KMS states. Then, we studied the Tomita-Takesaki theory of an algebra of observables and a state. Surprisingly, this study yielded a canonical dynamical law which we identified as the modular group. Even more surprisingly the result 7.1.2 showed that this dynamical law yielded the only possible dynamical law such that the state was an equilibrium state at some inverse temperature β!

In the context of modern theoretical physics this result cannot be overstated. If the models presented above are correct, this result means that in the quantum setting instead of searching for a dynamical law in the construction of new theories, we can instead only look for the equilibrium states. Once we are certain that we've identified the correct equilibrium states, result 7.1.2 immediately yields the dynamical law. We conclude that this result represents a new technique for model building and a change in the paradigm of the construction of physical theories.

An example of this assertion is given by the case of Schrödinger's mechanics. In chapter 5 we showed that the only KMS states in a finite dimensional dynamical system given by Schrödinger's equation are the Gibbs states. Now assume that we don't know Schrodinger's equation and we want to study some finite dimensional system, say, a finite spin lattice. Through our knowledge of statistical physics we may arrive at the conclusion that the equilibrium state at inverse temperature β is the β-Gibbs state. Then result 7.1.2 guarantees that by postulating the dynamical law to be given by the modular group associated to this state, we would be bound to find

Schrodinger's equation.

Chapter 8

Final Remarks and Further Work

In this last chapter we will point out some of the topics related to this work that weren't addressed. This is in the hopes that an interested reader will start thinking about them and begin working in this beautiful subject.

- During the first chapters a great emphasis was put on classical mechanics. This was done because the algebraic theory presented is also suitable for the study of classical systems. Nevertheless, KMS states do not seem as an appropriate description of classical thermodynamical equilibrium. Indeed, in the classical case where algebras are commutative the modular group is static: observables do not evolve. Therefore result 7.1.2 would yield an undesirable result. Still, works like [21] and [22] have explored non-trivial classical analogues of the KMS condition and Tomita-Takesaki theory obtaining partial results.

- During this work we proved two main results which gave us confidence on the use of KMS states for the description of equilibrium. We showed that KMS states where invariant under the dynamical law 5.1.3 and that in the finite dimensional case they where equivalent to Gibbs states 5.2.1. Nevertheless, the arguments in favor of KMS states are multiple. In particular [18] discusses the appearance of KMS states through the grand canonical ensemble of systems with finite volume (inspired through arguments of ergodicity or the principle of maximal entropy) and through more direct arguments by making use of dynamical stability or passivity. The last two in particular make the case for KMS states without ever considering the thermodynamic limit.

- The work in [18] shows the importance of Tomita-Takesaki theory for relativistic quantum theories. Although this theory was used here in the context of statistical physics, it is a very useful tool in the general study and classification of von Neumann algebras. A simple argument puts forth the importance of these structures for relativistic theories. We should be able to assign to regions of spacetime local algebras of observables. The algebras corresponding to causally disconnected regions should commute to ensure the principle of causality. Then the commutant of a local algebra in a region should be intimately connected to the algebra corresponding to the causal complement of the region.

- Notice that the definition of a KMS state 5.1.1 seems to distinguish between time and space. This isn't the case precisely since the dynamical law doesn't have to represent time evolution. Nevertheless works like [23] have extended the KMS condition to the relativistic realm by giving a Lorentz covariant formulation suitable for quantum field theories.

- Works like [7], [14], and [16] show the vast amount of applications algebraic quantum theory has. We think that of particular importance are the ambiguities in the entropy calculated through the state 4.28 exhibited in [15] and [17].

Chapter 9

Acknowledgements

9.1 English

9.2 Español

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