# KMS states and Tomita-Takesaki Theory

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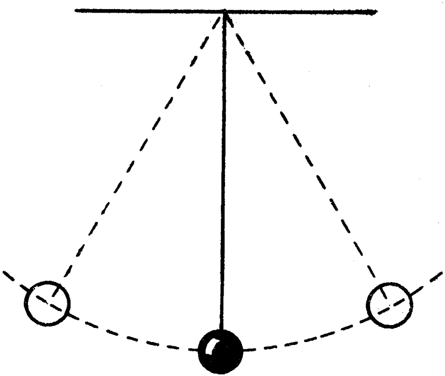
# Motivation



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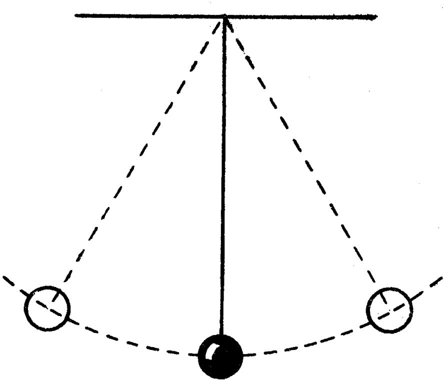

Can we obtain the equations of motion from the equilibrium state?

Maybe in quantum thermal systems.

$$e^{-iHt} \circlearrowright e^{-\beta H}$$

# Outline

# Elements of Classical and Quantum Theories

Classical theories

- Auxiliary space: locally compact Hausdorff space $X$;

Quantum theories

- Auxiliary space: separable Hilbert space $\mathcal{H}$

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- States: positive, self-adjoint, normalized and trace-class operators $\rho$ on $\mathcal{H}$;

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- Auxiliary space: separable Hilbert space $\mathcal{H}$
- Observables: self-adjoint operators on $\mathcal{H}$
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# Difference between Classical and Quantum theories

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\begin{tikzcd}
\text{EPR paradox [Einstein et al., 1935]} \arrow[r] \arrow[d] & \text{QM is incomplete} \arrow[d] \\
\text{QM is complete} \arrow[d] & {} \arrow[d] \\
{} \arrow[uur] & {} \arrow[d] \\
{} & {} \arrow[l]
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EPR paradox [Einstein et al., 1935] ↓ QM is complete ↓ Entangled states ↯ ⟶ QM is incomplete

QM is incomplete ↓ Search for complete theory: Boolean propositions ↓ Bell's inequalities [Bell, 1964] [Reyes-Lega, 2013] ↓

⟵

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\text{EPR paradox [Einstein et al., 1935]} \arrow[r] \arrow[d] & \text{QM is incomplete} \arrow[d] \\
\text{QM is complete} \arrow[d] & \begin{array}{c}\text{Search for complete theory:}\\ \text{Boolean propositions}\end{array} \arrow[d] \\
\text{Entangled states } \lightning \arrow[uur] & \begin{array}{c}\text{Bell's inequalities [Bell, 1964]}\\ \text{[Reyes-Lega, 2013]}\end{array} \arrow[d] \\
\text{Lattice of projections [Wilce, 2012]} & \text{Against experiment!} \arrow[l]
\end{tikzcd}
$$

# Algebraic Quantum Mechanics

- Observables: A $C^*(W^*)$-algebra $\mathcal{A}$:
  - Complete normed vector space with product and involution;
  - $C^*$ property: $\|A^*A\| = \|A\|^2$;
  - If $W^*$: $\mathcal{A}'' = \mathcal{A}$.
- States: Positive normalized linear functionals $\omega : \mathcal{A} \to \mathbb{C}$.

# GNS Construction

Start with a $C^*$-algebra $\mathcal{A}$ and a state $\omega$.

- $\mathcal{N}_\omega := \{A \in \mathcal{A} | \omega(AA^*) = 0\}$
- Hilbert space $\mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{N}_\omega}$ with $\langle [A], [B] \rangle := \omega(A^* B)$
- Define the representation extending

$$
\begin{aligned}
\pi_\omega : \mathcal{A} \to \quad & \mathcal{B}(\mathcal{H}_\omega) \\
A \mapsto & \pi_\omega(A) : \mathcal{H}_\omega \to \mathcal{H}_\omega \\
& \qquad [B] \mapsto [AB]
\end{aligned}
$$

- Cyclic vector $\Omega_\omega := [1]$
- Important facts:
  - $\omega(A) = \langle \Omega_\omega, A\Omega_\omega \rangle$
  - If $\mathcal{A}$ is a $W^*$-algebra, $\pi_\omega$ is a $^*$-isomorphism, $\pi_\omega(\mathcal{A})$ is a $W^*$-algebra, and $\Omega_\omega$ is cyclic and separating $\bigstar$.

# Dynamical Systems

Time evolution is represented by a one-parameter group of automorphisms

$$\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$$
$$t \mapsto \tau_t.$$

Dynamical systems consist of an $C^*(W^*)$-algebra with a time evolutions which satisfy certain continuity properties.

# KMS States

**Definition**

Let $(\mathcal{A}, \tau)$ be a dynamical system. We say that a state $\omega$ is a $(\tau, \beta)$-KMS state if for all $A, B \in \mathcal{A}$ there exists a continuous bounded function $F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$ analytic on $\mathfrak{D}_\beta$ (the strip of the complex plain bounded by $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \beta$) such that

$$F_{A,B}(t) = \omega(A\tau_t(B))$$
$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A)$$

for all $t \in \mathbb{R}$.

# KMS states as Equilibrium states

KMS states are a candidate for a general definition of thermodynamic equilibrium in quantum systems[Haag, 1992][Duvenhage, 1999]:

- KMS states are invariant under the dynamics $\omega(\tau_t(A)) = \omega(A)$;
- In finite dimensional Hilbert spaces with Schrödinger's time evolution $\tau$, the only possible $(\tau, \beta)$-KMS states are the $\beta$-Gibbs states

$$\begin{aligned} \mathcal{B}(\mathcal{H}) &\to \mathbb{C} \\ A &\mapsto \frac{\mathrm{tr}(Ae^{-\beta H})}{\mathrm{tr}(e^{-\beta H})}. \end{aligned}$$

# Tomita-Takesaki Theory

For a $W^*$-algebra $\mathfrak{M}$ equipped with a cyclic and separating vector $\Omega$ Tomita-Takesaki theory yields:

- a one-parameter unitary group $t \mapsto \Delta^{it}$;
- a modular conjugation $J$.

**Theorem (Tomita-Takesaki)**

- $J\mathfrak{M}J = \mathfrak{M}'$;
- $\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}$ *for all* $t \in \mathbb{R}$.

**Proof.**

[Duvenhage, 1999] □

# Tomita-Takesaki, time evolution and KMS states

**Theorem (★)**

*$t \mapsto \Delta^{it}$ is the unique strongly continuous one-parameter unitary group on $\mathcal{H}$ that satisfies the KMS condition with respect to $\mathcal{K}$ such that $\Delta^{it}\mathcal{K} \subseteq \mathcal{K}$ for all $t \in \mathbb{R}$.*

**Theorem (★)**

*Let $\mathfrak{M}$ be a von Neuman algebra and $\omega$ a faithful normal state. Consider the unitary group $t \mapsto \Delta^{it}$ associated to the pair $(\pi_\omega(\mathfrak{M}), \Omega_\omega)$. Then the one-parameter group of automorphisms given by $\alpha_t = \pi_\omega^{-1}(\Delta^{it}\pi_\omega(A)\Delta^{-it})$ makes $(\mathfrak{M}, \alpha)$ a $W^*$-dynamical system.*

**Proof.**

[Duvenhage, 1999] □

# The Canonical Time Evolution

**Theorem (★★★)**

*Let $\mathfrak{M}$ be a von Neumann algebra and $\omega$ be a faithful normal state. Then $(\mathfrak{M}, \tau)$ with $\tau_t(A) = \alpha_{-t/\beta}(A)$ and $\alpha$ the modular group of $(\mathfrak{M}, \omega)$ is the unique $W^*$-dynamical system such that $\omega$ is an $(\alpha, \beta)$-KMS state.*

**Proof.**

[Duvenhage, 1999] □

# Further work

- Classical KMS states and Tomita-Takesaki theory.
- Understanding KMS states from "first principles":
  - stability;
  - passivity.
- Relativistic generalization of KMS states.
- Entropy ambiguities.

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