

Category Theory by Example

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Contents

1	Base definitions	7
1.1	Definitions	7
1.1.1	Object	7
1.1.2	Morphism	8
1.1.3	Category	10
1.2	Examples	11
1.2.1	Set category	11
1.2.2	Programming languages	16
1.2.3	Quantum mechanics	19
2	Objects and morphisms	23
2.1	Equality	23
2.2	Initial and terminal objects	24
2.3	Product and sum	26
2.4	Category as monoid	26
2.5	Exponential	27
2.6	Programming languages and algebraic data types	27
2.7	Examples	27
2.7.1	Set category	27
2.7.2	Programming languages	28
2.7.3	Quantum mechanics	33
3	Functors	35
3.1	Definitions	35
3.2	Natural transformations	36
3.3	Monoidal category	36
3.4	Examples	36
3.4.1	Set category	36
3.4.2	Programming languages	37
3.4.3	Quantum mechanics	37

4 Monads**39****Index****41**

Introduction

There is an introduction to Category Theory. There are a lot of examples in each chapter. The examples covers different category theory application areas. I assume that the reader is familiar with the corresponding area if not the example(s) can be passed. Anyone can choose the most suitable example(s) for (s)he.

The most important examples are related to the set theory. The set theory and category theory are very close related. Each one can be considered as an alternative view to another one.

There are a lot of examples from programming languages. There are Haskell, Scala, C++. The source files for programming languages examples (Haskell, C++, Scala) can be found on github repo [\[4\]](#).

The examples from physics are related to quantum mechanics that is the most known for me. For the examples I am inspired by the Bob Coecke article [\[1\]](#).

Chapter 1

Base definitions

1.1 Definitions

1.1.1 Object

Definition 1.1 (Class). A class is a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share.

Definition 1.2 (Object). In category theory object is considered as something that does not have internal structure (aka point) but has a property that makes different objects belong to the same [Class](#)

Remark 1.3 (Class of Objects). The [Class](#) of [Objects](#) will be marked as $\text{ob}(\mathbf{C})$ (see fig. 1.1).



Figure 1.1: Class of objects $\text{ob}(\mathbf{C}) = \{a, b, c, d\}$

1.1.2 Morphism

Morphism is a kind of relation between 2 **Objects**.

Definition 1.4 (Morphism). A relation between two **Objects** a and b

$$f_{ab} : a \rightarrow b$$

is called *morphism*. Morphism assumes a direction i.e. one **Object** (a) is called *source* and another one (b) *target*.

The **Set** of all morphisms between objects a and b is called as $\text{hom}(a, b)$.

Definition 1.5 (Domain). Given a **Morphism** $f : a \rightarrow b$, the **Object** a is called domain and is denoted as $\text{dom } a$.

Definition 1.6 (Codomain). Given a **Morphism** $f : a \rightarrow b$, the **Object** b is called codomain and is denoted as $\text{cod } a$.

Morphisms have several properties.¹

Axiom 1.7 (Composition). If we have 3 **Objects** a, b and c and 2 **Morphisms**

$$f_{ab} : a \rightarrow b$$

and

$$f_{bc} : b \rightarrow c$$

then there exists **Morphism**

$$f_{ac} : a \rightarrow c$$

such that

$$f_{ac} = f_{bc} \circ f_{ab}$$

Remark 1.8 (Composition). The equation

$$f_{ac} = f_{bc} \circ f_{ab}$$

means that we apply f_{ab} first and then we apply f_{bc} to the result of the application i.e. if our objects are sets and $x \in a$ then

$$f_{ac}(x) = f_{bc}(f_{ab}(x)),$$

where $f_{ab}(x) \in b$.

¹The properties don't have any proof and postulated as axioms

Axiom 1.9 (Associativity). The *Morphisms Composition* (Axiom 1.7) should follow associativity property:

$$f_{ce} \circ (f_{bc} \circ f_{ab}) = (f_{ce} \circ f_{bc}) \circ f_{ab} = f_{ce} \circ f_{bc} \circ f_{ab}.$$

Definition 1.10 (Identity morphism). For every *Object* a we define a special *Morphism* $\mathbf{1}_a : a \rightarrow a$ with the following properties: $\forall f_{ab} : a \rightarrow b$

$$\mathbf{1}_a \circ f_{ab} = f_{ab} \tag{1.1}$$

and $\forall f_{ba} : b \rightarrow a$

$$f_{ba} \circ \mathbf{1}_a = f_{ba}. \tag{1.2}$$

This morphism is called *identity morphism*.

Note that *Identity morphism* is unique, see *Identity is unique* (Theorem 2.3) below.

Definition 1.11 (Commutative diagram). A commutative diagram is a diagram of *Objects* (also known as vertices) and *Morphisms* (also known as arrows or edges) such that all directed paths in the diagram with the same start and endpoints lead to the same result by composition

The following diagram commutes if $f_{ab} = f_{cb} \circ f_{ac}$.



Remark 1.12 (Class of Morphisms). The *Class* of *Morphisms* will be marked as $\text{hom}(\mathbf{C})$ (see fig. 1.2)

Definition 1.13 (Monomorphism). If $\forall g_1, g_2$ the equation

$$f \circ g_1 = f \circ g_2$$

leads to

$$g_1 = g_2$$

then f is called *monomorphism*.



Figure 1.2: Class of morphisms $\text{hom}(\mathbf{C}) = \{f, g, h\}$, where $h = f \circ g$

Definition 1.14 (Epimorphism). If $\forall g_1, g_2$ the equation

$$g_1 \circ f = g_2 \circ f$$

leads to

$$g_1 = g_2$$

then f is called *epimorphism*.

Definition 1.15 (Isomorphism). A **Morphism** $f : a \rightarrow b$ is called isomorphism if $\exists g : b \rightarrow a$ such that $f \circ g = \mathbf{1}_a$ and $g \circ f = \mathbf{1}_b$.

Remark 1.16 (Isomorphism). There are can be many different **Isomorphisms** between 2 **Objects**.

1.1.3 Category

Definition 1.17 (Category). A category **C** consists of

- **Class** of **Objects** $\text{ob}(\mathbf{C})$
- **Class** of **Morphisms** $\text{hom}(\mathbf{C})$ defined for $\text{ob}(\mathbf{C})$, i.e. each morphism f_{ab} from $\text{hom}(\mathbf{C})$ has both source a and target b from $\text{ob}(\mathbf{C})$

For any **Object** a there should be unique **Identity morphism** $\mathbf{1}_a$. Any morphism should satisfy **Composition** (**Axiom 1.7**) and **Associativity** (**Axiom 1.9**) properties. See fig. 1.3

The **Category** can be considered as a way to represent a structured data. **Morphisms** are the ones to form the structure.

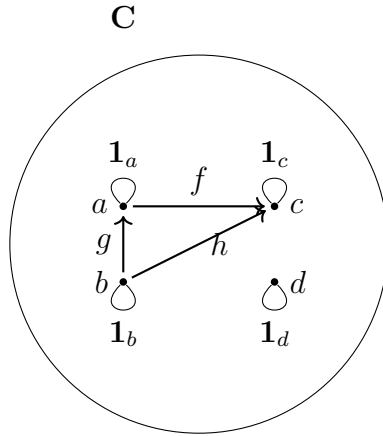


Figure 1.3: Category \mathbf{C} . It consists of 4 objects $\text{ob}(\mathbf{C}) = \{a, b, c, d\}$ and 7 morphisms $\text{ob}(\mathbf{C}) = \{f, g, h = f \circ g, 1_a, 1_b, 1_c, 1_d\}$

Definition 1.18 (Opposite category). If \mathbf{C} is a [Category](#) then opposite (or dual) category \mathbf{C}^{op} is constructed in the following way: [Objects](#) are the same, but the [Morphisms](#) are inverted i.e. if $f \in \text{hom}(\mathbf{C})$ and $\text{dom } f = a, \text{cod } f = b$, then the corresponding morphism $f^{op} \in \text{hom}(\mathbf{C}^{op})$ has $\text{dom } f^{op} = b, \text{cod } f^{op} = a$ (see fig. 1.4)

Remark 1.19. Composition on \mathbf{C}^{op} As you can see from fig. 1.4 the [Composition](#) ([Axiom 1.7](#)) is reverted for [Opposite category](#). If $f, g, h = f \circ g \in \text{hom}(\mathbf{C})$ then $f \circ g$ translated into $g^{op} \circ f^{op}$ in opposite category.

Definition 1.20 (Small category). A category \mathbf{C} is called *small* if both $\text{ob}(\mathbf{C})$ and $\text{hom}(\mathbf{C})$ are [Sets](#)

Definition 1.21 (Large category). A category \mathbf{C} is not [Small category](#) then it is called *large*. The example of large category is [Set category](#) ([Example 1.25](#))

1.2 Examples

There are several examples of categories that will also be used later

1.2.1 Set category

Definition 1.22 (Set). Set is a collection of distinct object. The objects are called the elements of the set.

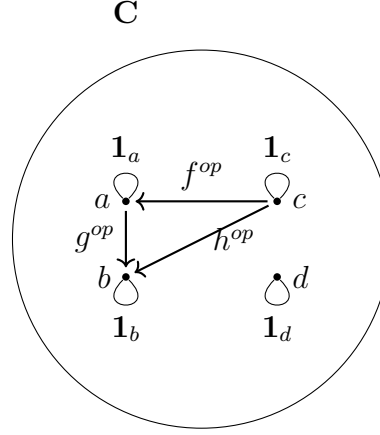


Figure 1.4: Opposite category C^{op} to the category from fig. 1.3 . It consists of 4 objects $\text{ob}(C^{op}) = \text{ob}(C) = \{a, b, c, d\}$ and 7 morphisms $\text{hom}(C^{op}) = \{f^{op}, g^{op}, h^{op} = g^{op} \circ f^{op}, 1_a, 1_b, 1_c, 1_d\}$

Definition 1.23 (Binary relation). If A and B are 2 **Sets** then a subset of $A \times B$ is called binary relation R between the 2 sets, i.e. $R \subset A \times B$.

Definition 1.24 (Function). Function f is a special type of **Binary relation**. I.e. if A and B are 2 **Sets** then a subset of $A \times B$ is called function f between the 2 sets if $\forall a \in A \exists! b \in B$ such that $(a, b) \in f$. In other words function does not allowed “multi value”.

Example 1.25 (**Set** category). In the set category we consider a **Set** of **Sets** where **Objects** are the **Sets** and **Morphisms** are **Functions** between the sets.

The **Identity morphism** is trivial function such that $\forall x \in X : 1_X(x) = x$.

In general case when we say **Set** category we assume the set of all sets. But the result is inconsistent because famous Russell’s paradox [9] can be applied. To avoid such situations we assume the some kind of limitations are applied on our construction, for instance ZFC [10]. If we apply the limitation we have that set of all sets is not a set itself and as result the **Set** category is a **Large category**

Remark 1.26 (Set vs Category). There is an interesting relation between sets and categories. In both we consider objects(sets) and relations between them(morphisms/functions).

In the set theory we can get info about functions by looking inside the objects(sets) aka use “microscope” [3]

Contrary in the category theory we initially don’t have info about object internal structure but can get it using the relation between the objects i.e. using **Morphisms**. In other words we can use “telescope” [3] there.



Figure 1.5: A surjective (non-injective) function from domain X to codomain Y

Definition 1.27 (Singleton). The *singleton* is a **Set** with only one element.

Definition 1.28 (Domain). Given a function $f : X \rightarrow Y$, the set X is the domain.

Definition 1.29 (Codomain). Given a function $f : X \rightarrow Y$, the set Y is the codomain.

Definition 1.30 (Surjection). The function $f : X \rightarrow Y$ is surjective (or onto) if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$ (see figs. 1.5 and 1.9).

Remark 1.31 (Surjection vs Epimorphism). **Surjection** and **Epimorphism** are related each other. Consider a non-surjective function $f : X \rightarrow Y' \subset Y$ (see fig. 1.6). One can conclude that there is not an **Epimorphism** because $\exists g_1 : Y' \rightarrow Y'$ and $g_2 : Y \rightarrow Y$ such that $g_1 \neq g_2$ because they operates on different **Domains** but from other hand $g_1(Y') = g_2(Y')$. For instance we can choose $g_1 = \mathbf{1}_{[Y']}$, $g_2 = \mathbf{1}_{[Y]}$. As soon as Y' is **Codomain** of f we always have $g_1(f(X)) = g_2(f(X))$.

As result we can say that an **Surjection** is a **Epimorphism** in **Set** category. Moreover there is a proof [7] of that fact.

Definition 1.32 (Injection). The function $f : X \rightarrow Y$ is injective (or one-to-one function) if $\forall x_1, x_2 \in X$, such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ (see figs. 1.7 and 1.9).

Remark 1.33 (Injection vs Monomorphism). **Injection** and **Monomorphism** are related each other. Consider a non-injective function $f : X \rightarrow Y$ (see fig. 1.8). One can conclude that it is not monomorphism because $\exists g_1, g_2$ such that $g_1 \neq g_2$ and $f(g_1(a_1)) = y_3 = f(g_2(b_1))$.



Figure 1.6: A non-surjective function f from domain X to codomain $Y' \subset Y$. $\exists g_1 : Y' \rightarrow Y', g_2 : Y \rightarrow Y$ such that $g_1(Y') = g_2(Y')$, but as soon as $Y' \neq Y$ we have $g_1 \neq g_2$. Using the fact that Y' is codomain of f we got $g_1 \circ f = g_2 \circ f$. I.e. the function f is not epimorphism.

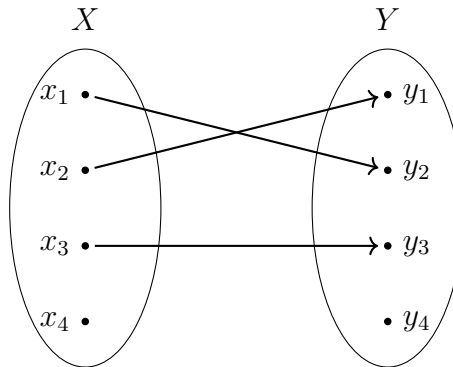


Figure 1.7: A injective (non-surjective) function from domain X to codomain Y



Figure 1.8: A non-injective function f from domain X to codomain Y . $\exists g_1 : A \rightarrow X, g_2 : B \rightarrow X$ such that $g_1 \neq g_2$ but $f \circ g_1 = f \circ g_2$. I.e. the function f is not monomorphism.



Figure 1.9: An injective and surjective function (bijection)

As result we can say that an **Injection** is a **Monomorphism** in **Set** category. Moreover there is a proof [6] of that fact.

Definition 1.34 (Bijection). The function $f : X \rightarrow Y$ is bijective (or one-to-one correspondence) if it is an **Injection** and a **Surjection** (see fig. 1.9).

There is a question what's analog of a single **Set**. Main characteristic of a category is a structure but the set by definition does not have a structure. Which category does not have any structure? The answer is **Discrete category**.

Definition 1.35 (Discrete category). Discrete category is a **Category** where **Morphisms** are only **Identity morphisms**.



Figure 1.10: Programming language category example. Objects are types: Int, Bool, String. Morphisms are several functions

1.2.2 Programming languages

In the programming languages we consider types as **Objects** and functions as **Morphisms**. The critical requirements for such consideration is that the functions have to be pure function (without side effects). This requirement mainly is satisfied by functional languages such as Haskell and Scala. From other side the functional languages use lazy evaluation to improve the performance. The laziness can also make category theory axiom invalid (see [Haskell lazy evaluation](#) ([Remark 1.37](#))).

Strictly speaking neither Haskell (pure functional language) nor C++ can be considered as a category in general. As a first approximation the functional language (Haskell, Scala) can be considered as a category if we avoid to use functions with side effects (mainly for Scala) and use strict (for both Haskell and Scala) evaluations.

In any case we can construct a simple toy category that can be easy implemented in any language. Particularly we will look into category with 3 objects that are types: Int, Bool, String. There are also several functions between them (see [fig. 1.10](#)).

Hask category

Example 1.36 (Hask category). Types in Haskell are considered as **Objects**. Functions are considered as **Morphisms**. We are going to implement **Category**

from fig. 1.10.

The function `isEven` that converts `Int` type into `Bool`.

```
isEven :: Int -> Bool
isEven x = x `mod` 2 == 0
```

There is also [Identity morphism](#) that is defined as follows

```
id :: a -> a
id x = x
```

If we have an additional function

```
stringLength :: String -> Int
stringLength x = length x
```

then we can create a [Composition](#) ([Axiom 1.7](#))

```
isStringLengthEven :: String -> Bool
isStringLengthEven = isEven . stringLength
```

Remark 1.37 (Haskell lazy evaluation). Each Haskell type has a special value \perp . The value presents and lazy evaluations make several category law invalid, for instance [Identity morphism](#) behaviour become invalid in specific cases:

The following code

```
seq undefined True
```

produces *undefined* But the following

```
seq (id.undefined) True
seq (undefined.id) True
```

produces *True* in both cases. As result we have (we cannot compare compare functions in Haskell, but if we could we can get the following)

```
id . undefined /= undefined
undefined . id /= undefined,
```

i.e. (1.1) and (1.2) are not satisfied.

C++ category

Example 1.38 (C++ category). We will use the same trick as in [Hask category](#) ([Example 1.36](#)) and will assume types in C++ as [Objects](#), functions as [Morphisms](#). We also are going to implement [Category](#) from [fig. 1.10](#).

We also define 2 functions:

```
auto isEven = [](int x) {
    return x % 2 == 0;
};

auto stringLength = [](std::string s) {
    return static_cast<int>(s.size());
};
```

Composition can be defined as follows:

```
// h = g . f
template <typename A, typename B>
auto compose(A g, B f) {
    auto h = [f, g](auto a) {
        auto b = f(a);
        auto c = g(b);
        return c;
    };
    return h;
};
```

The [Identity morphism](#):

```
auto id = [](auto x) { return x; };
```

The usage examples are the following:

```
auto isStringLengthEven = compose<>(isEven, stringLength);

auto isStringLengthEvenL = compose<>(id, isStringLengthEven);

auto isStringLengthEvenR = compose<>(isStringLengthEven, id);
```

Such construction will always provides us the category as soon as we use pure function (functions without effects).

Scala category

Example 1.39 (Scala category). We will use the same trick as in [Hask category](#) (Example 1.36) and will assume types in Scala as [Objects](#), functions as [Morphisms](#). We also are going to implement [Category](#) from fig. 1.10.

```
object Category {
  def id[A]: A => A = a => a
  def compose[A, B, C](g: B => C, f: A => B):
    A => C = g compose f

  val isEven = (i: Int) => i % 2 == 0
  val stringLength = (s: String) => s.length
  val isStringLengthEven = (s: String) =>
    compose(isEven, stringLength)(s)
}
```

The usage example is below

```
class CategorySpec extends Properties("Category") {
  import Category._
  import Prop.forAll

  property("composition") = forAll { (s: String) =>
    isStringLengthEven(s) == isEven(stringLength(s))
  }

  property("right id") = forAll { (i: Int) =>
    isEven(i) == compose(isEven, id[Int])(i)
  }

  property("left id") = forAll { (i: Int) =>
    isEven(i) == compose(id[Boolean], isEven)(i)
  }
}
```

1.2.3 Quantum mechanics

The most critical property of quantum system is superposition principle. The [Set category](#) (Example 1.25) cannot be used for it because it does not satisfied the principle.

A simple modification of the [Set](#) category can satisfy the principle.

Example 1.40 (**Rel** category). We will consider a set of sets (same as **Set** category (Example 1.25)) i.e. **Sets** as **Objects**. Instead of **Functions** we will use **Binary relations** as **Morphisms**.

The **Rel** category is similar to the finite dimensional Hilber space especially because it assumes some kind of superposition. Really consider $\mathbf{C_R}$ - the **Rel** category. $X, Y \in \text{ob}(\mathbf{C_R})$ - 2 sets which consists of different elements. Let $f : X \rightarrow Y$ - **Morphism**. Each element $x \in X$ is mapped to a subset $Y' \subset Y$. The Y' can be **Singleton** (in this case no differences with **Set** category (Example 1.25)) but there can be a situation when Y' consists of several elements. In the case we will get some kind of superposition that is analogiest to quantum mechanics.

In the quantum mechanics we say about Hilber spaces.

Definition 1.41 (Hilbert space). The Hilbert space a complex vector space with an inner product as a complex number (\mathbb{C}).

Later we will consider only finite dimensional Hilber spaces. We will denote a Hilber space of dimensional n as \mathcal{H}_n . Obviously $\mathcal{H}_1 = \mathbb{C}$.

Definition 1.42 (Dual space). Each Hilber space \mathcal{H} has an associated with it so called dual space \mathcal{H}^* that consists of linear functionals

Example 1.43 (Dirac notation). Consider a so called ket-vector $|\psi\rangle \in \mathcal{H}$. Then the corresponding vector from **Dual space** is called bra-vector $\langle\psi| \in \mathcal{H}^*$. From the definition of dual space the bra-vector is a linear functional i.e.

$$\langle\psi| : \mathcal{H} \rightarrow \mathbb{C},$$

$\forall |\phi\rangle \in \mathcal{H}$ we have $\langle\psi|(|\phi\rangle) = (\langle\psi|, |\phi\rangle)$ - inner product that is often written as $\langle\psi|\phi\rangle$.

The transformation between 2 **Hilbert spaces** that preserves the structure is called linear map or linear transformations.

Definition 1.44 (Linear map). The linear map between 2 **Hilbert spaces** \mathcal{A} and \mathcal{B} is a mapping $f : \mathcal{A} \rightarrow \mathcal{B}$ that preserves additions

$$f(a_1 + a_2) = f(a_1) + f(a_2),$$

and scalar multiplications:

$$f(c \cdot a) = c \cdot f(a)$$

where $a, a_{1,2} \in \mathcal{A}$ and $f(a), f(a_{1,2}) \in \mathcal{B}$.

Table 1.1: Relations between **Set**, **Rel** and **FdHilb** categories

	Set	Rel	FdHilb
Object	Set	Set	finite dimensional Hilbert space
Morphism	Function	Binary relation	Linear map
Initial object	empty set	empty set	trivial Hilbert space of dimensional 0
Terminal object	Singleton	Singleton	\mathbb{C}
Product	Cartesian product	Cartesian product	Direct sum of Hilber spaces
Sum	Sum (Example 2.14)	Sum (Example 2.14)	Direct sum of Hilber spaces

If we want to combine 2 Hilbert spaces into one we use a notion of direct sum.

Definition 1.45 (Direct sum of Hilber spaces). Let \mathcal{A}, \mathcal{B} are 2 Hilber spaces. The direct sum $\mathcal{A} \oplus \mathcal{B}$ is defined as follows

$$\mathcal{A} \oplus \mathcal{B} = \{a \oplus b | a \in \mathcal{A}, b \in \mathcal{B}\}.$$

The inner product is defined as follows

$$\langle a_1 \oplus b_1 | a_2 \oplus b_2 \rangle = \langle a_1 | a_2 \rangle + \langle b_1 | b_2 \rangle.$$

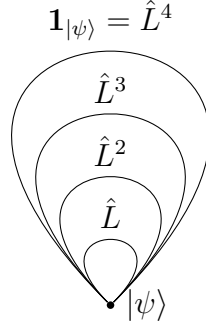
Example 1.46 (**FdHilb** category). Most common case in quantum mechanics is the case of quantum states in the finite dimensional Hilbert space. We can consider the set of all finite dimensional Hilbert spaces as a category. The **Objects** in the category are finite dimensional Hilbert spaces and **Morphisms** are Linear maps. The category is denoted as **FdHilb**. It is very similar to **Rel category** (Example 1.40). The brief relation is described in the table 1.1.

Definition 1.47 (Tensor product). TBD

The tensor product in quantum mechanics is used for representing a system that consists of multiple systems. For instance if we have an interaction between an 2 level atom (a is excited state b as a ground state) and one mode light then the atom has its own Hilber space \mathcal{H}_{at} with $|a\rangle$ and $|b\rangle$ as basis vectors. Light also has its own Hilber space \mathcal{H}_f with Fock state $\{|n\rangle\}$ as the basis.² The result system that describes both atom and light is represented as the tensor product $\mathcal{H}_{at} \otimes \mathcal{H}_f$.

The morphisms of **FdHilb** category have a connection with **Tensor product**. Consider the so called Hilbert-Schmidt correspondence for finite dimensional Hilbert spaces i.e. for given \mathcal{A} and \mathcal{B} there is a natural isomorphism

² Really the \mathcal{H}_f is infinite dimensional Hilber space and seems to be out of our assumption about **FdHilb** category as a collection of finite dimensional Hilber spaces only.

Figure 1.11: Rabi oscillations as a category \mathbf{R}

between the tensor product and linear maps (aka morphisms) between \mathcal{A} and \mathcal{B} :

$$\mathcal{A}^* \otimes \mathcal{B} \cong \text{hom}(\mathcal{A}, \mathcal{B})$$

where \mathcal{A}^* - [Dual space](#).

Example 1.48 (Rabi oscillations). For our example we consider a 2 level atom with states $|a\rangle$ - excited and $|b\rangle$. As soon as we consider a 2-level system we are in the 2 dimensional [Hilbert space](#) i.e. have only one [Object](#). Lets call it as $|\psi\rangle$. The category will be called as \mathbf{R} . I.e. $\text{ob}(\mathbf{R}) = \mathcal{H}_2\{|\psi\rangle\}$.

The atom interacts with light beam of frequency $\omega = \omega_{ab}$. The state of the system is described by the following equation [11]:

$$|\psi\rangle = \cos \frac{\omega_R t}{2} |a\rangle - i \sin \frac{\omega_R t}{2} |b\rangle ,$$

where ω_R - Rabi frequency [11].

The interaction time t is fixed and corresponds to $\omega_R t = \pi$ i.e. the interaction can be described a linear operator \hat{L} .

There are 4 different states and as result 4 [Morphisms](#):

$$\begin{aligned} |\psi\rangle_0 &= |a\rangle , \\ |\psi\rangle_1 &= \hat{L} |\psi\rangle_0 = -i |b\rangle , \\ |\psi\rangle_2 &= \hat{L}^2 |\psi\rangle_0 = -|a\rangle , \\ |\psi\rangle_3 &= \hat{L}^3 |\psi\rangle_0 = i |b\rangle , \end{aligned}$$

Chapter 2

Objects and morphisms

2.1 Equality

The important question is how can we decide whenever an object/morphism is equal to another object/morphism? The trivial answer is possible for if an **Object** is a **Set**. In this case we can say that 2 objects are equal if they contains the same elements. Unfortunately we cannot do the same for default objects as soon as they don't have any internal structure. We can use the same trick as in **Set vs Category** (**Remark 1.26**) : if we cannot use “microscope” lets use “telescope” and define the equality of objects and morphisms of a category **C** in the terms of whole $\text{hom}(\mathbf{C})$.

Definition 2.1 (Objects equality). Two **Objects** a and b in **Category** C are equal if there exists an unique **Isomorphism** $f : a \rightarrow b$. This also means that also exist unique isomorphism $g : b \rightarrow a$. These two **Morphisms** are related each other via the following equations: $f \circ g = \mathbf{1}_a$ and $g \circ f = \mathbf{1}_b$.

Unlike **Functions** between **Sets** we don't have any additional info ¹ about **Morphisms** except category theory axioms which the morphisms satisfied [2]. This leads us to the following definition for morphisms equality:

Definition 2.2 (Morphisms equality). Two **Morphisms** f and g in **Category** C are equal if the equality can be derived from the base axioms:

- **Composition** (**Axiom 1.7**)
- **Associativity** (**Axiom 1.9**)
- **Identity morphism**: (1.1), (1.2)

¹ for instance info about sets internals. i.e. which elements of the sets are connected by the considered functions

or [Commutative diagrams](#) which postulate the equality.

As an example lets proof the following theorem

Theorem 2.3 (Identity is unique). *The [Identity morphism](#) is unique.*

Proof. Consider an [Object](#) a and it's [Identity morphism](#) 1_a . Let $\exists f : a \rightarrow a$ such that f is also identity. In the case (1.1) for f as identity gives

$$f \circ 1_a = 1_a.$$

From other side (1.2) for 1_a satisfied

$$f \circ 1_a = f$$

i.e. $f = 1_a$. □

2.2 Initial and terminal objects

Definition 2.4 (Initial object). Let \mathbf{C} is a [Category](#), the [Object](#) $i \in \text{ob}(\mathbf{C})$ is called *initial object* if $\forall x \in \text{ob}(\mathbf{C}) \exists ! f_x : i \rightarrow x \in \text{hom}(\mathbf{C})$.

Definition 2.5 (Terminal object). Let \mathbf{C} is a [Category](#), the [Object](#) $t \in \text{ob}(\mathbf{C})$ is called *terminal object* if $\forall x \in \text{ob}(\mathbf{C}) \exists ! g_x : x \rightarrow t \in \text{hom}(\mathbf{C})$.

As you can see the initial and terminal objects are opposite each other. I.e. if i is an [Initial object](#) in \mathbf{C} then it will be [Terminal object](#) in the [Opposite category](#) \mathbf{C}^{op} .

Theorem 2.6 (Initial object is unique). *Let \mathbf{C} is a category and $i, i' \in \text{ob}(\mathbf{C})$ two [Initial objects](#) then there exists an unique [Isomorphism](#) $u : i \rightarrow i'$ (see [Objects equality](#))*

Proof. Consider the following [Commutative diagram](#) (see fig. 2.1) □

Theorem 2.7 (Terminal object is unique). *Let \mathbf{C} is a category and $t, t' \in \text{ob}(\mathbf{C})$ two [Terminal objects](#) then there exists an unique [Isomorphism](#) $v : t' \rightarrow t$ (see [Objects equality](#))*

Proof. Just got to the [Opposite category](#) and revert arrows in fig. 2.1. The result shown on fig. 2.2 and it proofs the theorem statement. □

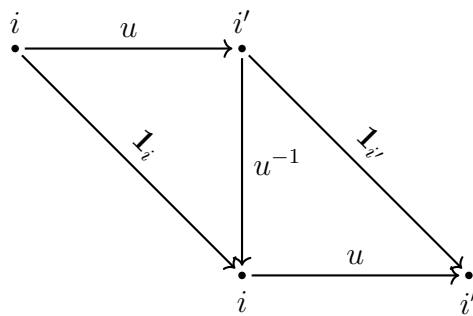


Figure 2.1: Commutative diagram for initial object unique proof

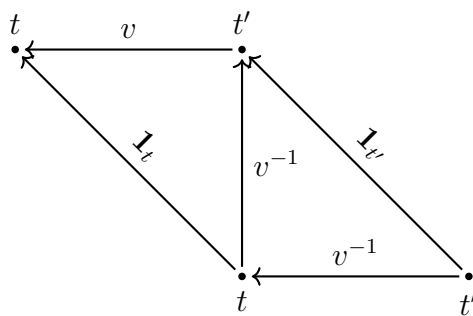


Figure 2.2: Commutative diagram for terminal object unique proof



Figure 2.3: Product $c = c_1 \times c_2$. $\forall c, \exists! h \in \text{hom}(\mathbf{C}) : \pi'_1 = \pi_1 \circ h, \pi'_2 = \pi_2 \circ h$.

2.3 Product and sum

The pair of 2 objects is defined via so called universal property in the following way:

Definition 2.8 (Product). Let we have a category \mathbf{C} and $c_1, c_2 \in \text{ob}(\mathbf{C})$ -two [Objects](#) the product of the objects c_1, c_2 is another object in \mathbf{C} $c = c_1 \times c_2$ with 2 [Morphisms](#) π_1, π_2 such that $a = g_a c, b = g_b c$ and the following universal property is satisfied: $\forall c' \in \text{ob}(\mathbf{C})$ and morphisms $\pi'_1 : \pi'_2 c' = c_1, \pi'_2 : \pi'_2 c' = c_2$, exists unique morphism h such that the following diagram (see fig. 2.3) commutes, i.e. $\pi'_1 = \pi_1 \circ h, \pi'_2 = \pi_2 \circ h$. In other words h factorizes $\pi'_{1,2}$.

If we invert arrows in [Product](#) we will got another object definition that is called sum

Definition 2.9 (Sum). Let we have a category \mathbf{C} and $c_1, c_2 \in \text{ob}(\mathbf{C})$ -two [Objects](#) the sum of the objects c_1, c_2 is another object in \mathbf{C} $c = c_1 \oplus c_2$ with 2 [Morphisms](#) i_1, i_2 such that $c = i_1 c_1, c = i_2 c_2$ and the following universal property is satisfied: $\forall c' \in \text{ob}(\mathbf{C})$ and morphisms $i'_1 : i'_1 x_1 = c', i'_2 : i'_2 x_2 = c'$, exists unique morphism h such that the following diagram (see fig. 2.4) commutes, i.e. $i'_1 = h \circ i_1, i'_2 = h \circ i_2$. In other words h factorizes $i'_{1,2}$.

2.4 Category as monoid

Consider the following definition from abstract algebra

Definition 2.10 (Monoid). The set of elements M with defined binary operation \circ we will call as a monoid if the following conditions are satisfied.

1. Closure: $\forall a, b \in M: a \circ b \in M$



Figure 2.4: Sum $c = c_1 \oplus c_2$. $\forall c, \exists! h \in \text{hom}(\mathbf{C}) : i'_1 = h \circ i_1, i'_2 = h \circ i_2$.

2. Associativity: $\forall a, b, c \in M: a \circ (b \circ c) = (a \circ b) \circ c$

3. Identity element: $\exists e \in M$ such that $\forall a \in M: e \circ a = a \circ e = a$

We can consider 2 **Monoids**. The first one has **Product** as the binary operation and **Terminal object** as the identity element. As result we just got an analog of multiplication in the category theory. This is why the terminal object is often called as **1** and the operation as the product.

Another one is additional **Monoid** that has **Initial object** as the identity element and the **Sum** as the binary operation. The initial object in that case is often called as **0**. I.e. we can see a direct connection with addition in algebra.

2.5 Exponential

TBD

2.6 Programming languages and algebraic data types

TBD

2.7 Examples

2.7.1 Set category

Example 2.11 (Initial object). **[Set]** Note that there is only one function from empty set to any other sets [5] that makes the empty set as the **Initial**

object in **Set** category (Example 1.25) .

Example 2.12 (Terminal object). **[Set]** Terminal object in **Set** category (Example 1.25) is a set with one element i.e Singleton.

Example 2.13 (Product). **[Set]** The Product of two sets A and B in **Set** category (Example 1.25) is defined as a Cartesian product: $A \times B = \{(a, b) | a \in A, b \in B\}$.

Example 2.14 (Sum). **[Set]** The Sum of two sets A and B in **Set** category (Example 1.25) is defined as disjoint union [8]. Let $\{A_i : i \in I\}$ be a family of sets indexed by I . The disjoint union of this family is the set

$$\sqcup_{i \in I} A_i = \cup_{i \in I} \{(x, i) : x \in A_i\}.$$

The elements of the disjoint union are ordered pairs (x, i) . Here i serves as an auxiliary index that indicates which A_i the element x came from.

2.7.2 Programming languages

In our toy example fig. 1.10 the type `String` is Initial object and type `Bool` is the Terminal object. From other side there are types in different programming languages that satisfies the definitions of initial and terminal objects.

Hask category

Example 2.15 (Initial object). **[Hask]** If we avoid lazy evaluations in Haskell (see Haskell lazy evaluation (Remark 1.37)) then we can found the following types as candidates for initial and terminal object in haskell. Initial object in **Hask** category (Example 1.36) is a type without values

```
data Empty
```

i.e. you cannot construct a object of the type.

There is only one function from the initial object:

```
absurd :: Empty -> a
```

The function is called absurd because it does absurd action. Nobody can proof that it does not exist. For the existence proof can be used the following absurd argument: “Just provide me an object type `Empty` and I will provide you the result of evaluation”.

There is no function in opposite direction because it would had been used for the `Empty` object creation.

Example 2.16 (Terminal object). `[Hask]` Terminal object (unit) in `Hask category` (Example 1.36) keeps only one element

```
data () = ()
```

i.e. you can create only one element of the type. You can use the following function for the creation:

```
unit :: a -> ()
unit _ = ()
```

Example 2.17 (Product). `[Hask]` The `Product` in `Hask category` (Example 1.36) keeps a pair and the constructor defined as follows

```
(,) :: a -> b -> (a, b)
(,) x y = (x, y)
```

There are 2 projectors:

```
fst :: (a, b) -> a
fst (x, _) = x
snd :: (a, b) -> b
snd (_, y) = y
```

Example 2.18 (Sum). `[Hask]` The `Sum` in `Hask category` (Example 1.36) defined as follows

```
data Either a b = Left a | Right b
```

The typical usage is via pattern matching for instance

```
factor :: (a -> c) -> (b -> c) -> Either a b -> c
factor f _ (Left x) = f x
factor _ g (Right y) = g y
```

C++ category

Example 2.19 (Initial object). `[C++]` In C++ exists a special type that does not hold any values and as result that cannot be created: `void`. You cannot create an object of that type: you will get a compiler error if you try.

Example 2.20 (Terminal object). `[C++]` C++ 17 introduced a special type that keeps only one value - `std::monostate`:

```
namespace std {
    struct monostate {};
}
```

Example 2.21 (Product). [C++] The **Product** in **C++ category** (Example 1.38) keeps a pair and the constructor defined as follows

```
namespace std {
    template< class A, class B > struct pair {
        T1 first;
        T2 second;
    };
}
```

There is a simple usage example

```
std::pair<int, bool> p(0, false);

std::cout << "First projector: " << p.first << std::endl;
std::cout << "Second projector: " << p.second << std::endl;
```

Really any **struct** or **class** can be considered as the product.

Example 2.22 (Sum). [C++] If we consider **Objects** as types then **Sum** is an object that can be either one or another type. The corresponding C/C++ construction that provides an ability to keep one of two types is **union**.

There is an **Either** implementation from **Sum** (Example 2.18)

```
template <typename A, typename B> class Either
{
public:
    Either(const Either& e) : is_left_(e.is_left_){
        if (is_left_){
            data.l = e.data.l;
        } else {
            data.r = e.data.r;
        }
    }
    ~Either(){
        if (is_left_){
            data.l.~A();
        } else {
            data.r.~B();
        }
    }
    Either(const A& l) : data(l), is_left_(true){
    }
}
```

```

Either(const B& r) : data(r), is_left_(false){
}
const A& left() const {
    if (!is_left_){
        throw std::logic_error("no left");
    }
    return data.l;
}
const B& right() const {
    if (is_left_){
        throw std::logic_error("no right");
    }
    return data.r;
}
private:
    union Data {
        Data() {}
        Data( const A& a) : l(a) {}
        Data (const B& b) : r(b) {}
        ~Data() {}
        A l;
        B r;
    } data;
    bool is_left_;
};

```

The usage example:

```

template <typename A, typename B, typename C, typename D>
auto factor(A f, B g, const Either<C, D>& either) {
    try {
        return f(either.left());
    }
    catch(...) {
        return g(either.right());
    }
};

auto stringLength = [](std::string s) {
    return static_cast<int>(s.size()); };
auto id = [](auto x) { return x; };

```

```

Either<std::string, int> str = std::string("abc");
std::cout << "String length:" <<
factor<>(stringLength, id, str) << std::endl;
Either<std::string, int> i = 4;
std::cout << "id(int):" <<
factor<>(stringLength, id, i) << std::endl;

```

C++17 suggests `std::variant` as a safe replacement for `union`. The example of the `factor` function is below

```

template <typename A, typename B, typename C, typename D>
auto factor(A f, B g, const std::variant<C, D>& either) {
    try {
        return f(std::get<C>(either));
    }
    catch(...) {
        return g(std::get<D>(either));
    }
};

```

The usage example is the same:

```

std::variant<std::string, int> var = std::string("abc");
std::cout << "String length:" <<
factor<>(stringLength, id, var) << std::endl;
var = 4;
std::cout << "id(int):" <<
factor<>(stringLength, id, var) << std::endl;

```

TBD

Scala category

Example 2.23 (Initial object). [Scala] We used a same trick as for [Initial object](#) ([Example 2.15](#)) and define [Initial object](#) in [Scala category](#) ([Example 1.39](#)) as a type without values

```
sealed trait Empty
```

i.e. you cannot construct a object of the type.

Example 2.24 (Terminal object). [Scala] We used a same trick as for [Terminal object](#) ([Example 2.16](#)) and define [Terminal object](#) in [Scala category](#) ([Example 1.39](#)) as a type with only one value


```
abstract final class Unit extends AnyVal
```

TBD i.e. you can create only one element of the type.

TBD

2.7.3 Quantum mechanics

Example 2.25 (Initial object). **[FdHilb]** We will use a Hilber space of dimensional 0 as the **Initial object**. I.e. the set that does not have any states in it.

Example 2.26 (Terminal object). **[FdHilb]** We will use a Hilber space of dimensional 1 as the **Terminal object**. I.e. the set of complex numbers \mathbb{C} .

Example 2.27 (Product). **[FdHilb]** The **Product** in **FdHilb** category (**Example 1.46**) is a **Direct sum of Hilber spaces**.

Example 2.28 (Sum). **[FdHilb]** The **Sum** in **FdHilb** category (**Example 1.46**) is a **Direct sum of Hilber spaces**.

TBD

Chapter 3

Functors

3.1 Definitions

Definition 3.1 (Functor). Let \mathbf{C} and \mathbf{D} are 2 categories. A mapping $F : \mathbf{C} \rightarrow \mathbf{D}$ between the categories is called *functor* is it preserves the internal structure (see fig. 3.1):

- $\forall a_C \in \text{ob}(\mathbf{C}), \exists a_D \in \text{ob}(\mathbf{D})$ such that $a_D = F(a_C)$
- $\forall f_C \in \text{hom}(\mathbf{C}), \exists f_D \in \text{hom}(\mathbf{D})$ such that $\text{dom } f_D = F(\text{dom } f_C), \text{cod } f_D = F(\text{cod } f_C)$. We will use the following notation later: $f_D = F(f_C)$.
- $\forall f_C, g_C$ the following equation holds:

$$F(f_C \circ g_C) = F(f_C) \circ F(g_C) = f_D \circ g_D.$$

- $\forall x \in \text{ob}(\mathbf{C}) : F(1_x) = 1_{F(x)}$.



Figure 3.1: Functor $F : \mathbf{C} \rightarrow \mathbf{D}$ definition

Remark 3.2 (Functor). When we say that functor preserve internal structure means that functor is not just mapping between [Objects](#) but also between [Morphisms](#).

Definition 3.3 (Category Composition). TBD

Definition 3.4 (Category Identity). TBD

Definition 3.5 (Cat category). TBD

Definition 3.6 (Category Product). TBD

Definition 3.7 (Bifunctor). TBD

Definition 3.8 (Terminal object in **Cat** category). Let consider Δ_c is a trivial functor from [Category](#) **A** to category **C** such that $\forall a \in \text{ob}(\mathbf{A}) : \Delta_c a = c$ -fixed object in **C** and $\forall f \in \text{hom}(\mathbf{A}) : \Delta_c f = \mathbf{1}_c$.

Definition 3.9 (Contravariant functor). If we have a categories **C** and **D** then the [Functor](#) $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ is called *contravariant functor*.

Definition 3.10 (Profunctor). If we have a category **C** then the [Bifunctor](#) $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}$ is called *profunctor*.

3.2 Natural transformations

TBD

3.3 Monoidal category

Definition 3.11 (Monoid). The set of elements M with defined binary operation \circ we will call as a monoid if the following conditions are satisfied.

1. Closure: $\forall a, b \in M: a \circ b \in M$
2. Associativity: $\forall a, b, c \in M: a \circ (b \circ c) = (a \circ b) \circ c$
3. Identity element: $\exists e \in M$ such that $\forall a \in M: e \circ a = a \circ e = a$

TBD

3.4 Examples

3.4.1 Set category

TBD

3.4.2 Programming languages

Hask category

TBD

Example 3.12 (Terminal object in **Cat** category). **[Hask]**

```
data Const c a = Const c
fmap :: (a -> b) -> Const c a -> Const c b
fmap f (Const c a) = Const c
```

Example 3.13 (Maybe as a functor). **[Hask]** Lets show how the **Maybe** a type can be constructed from different **Functors** and as result show that the **Maybe** a is also **Functor**.

```
data Maybe a = Nothing | Just a
-- This is equivalent to
data Maybe a = Either () (Identity a)
-- Either is a bifunctor and () == Const () a
-- Thus Maybe is a composition of 2 functors
```

Example 3.14 (Contravariant functor). **[Hask]** TBD

```
class Contravariant f where
    contramap :: (a -> b) -> f b -> f a
```

Example 3.15 (Profunctor). **[Hask]** TBD

```
class Profunctor p where
    dimap :: (a' -> a) -> ( b -> b' ) -> p a b -> p a' b'
    -- p a b == a -> b
    dimap f g h = g . h . f
```

C++ category

TBD

Scala category

TBD

3.4.3 Quantum mechanics

TBD

Chapter 4

Monads

TBD

Index

- C++** category
 - example, [18](#)
- C++** category example, [30](#)
- Cat** category
 - definition, [36](#)
- FdHilb** category
 - example, [21](#)
- FdHilb** category example, [33](#)
- Hask** category
 - example, [16](#)
- Hask** category example, [18](#), [19](#), [28](#), [29](#)
- Rel** category
 - example, [20](#)
- Rel** category example, [21](#)
- Scala** category
 - example, [19](#)
- Scala** category example, [32](#)
- Set** category
 - example, [12](#)
- Set** category example, [11](#), [19](#), [20](#), [28](#)
- Associativity axiom, [10](#), [23](#)
 - declaration, [9](#)
- Bifunctor, [36](#)
 - definition, [36](#)
- Bijection
 - definition, [15](#)
- Binary relation, [12](#), [20](#), [21](#)
 - definition, [12](#)
- Category, [10](#), [11](#), [15](#), [16](#), [18](#), [19](#), [23](#), [24](#), [36](#)
 - definition, [10](#)
 - dual, [11](#)
 - large, [11](#), [12](#)
 - opposite, [11](#)
 - small, [11](#)
- Category Composition
 - definition, [36](#)
- Category Identity
 - definition, [36](#)
- Category Product
 - definition, [36](#)
- Class, [7](#), [9](#), [10](#)
 - definition, [7](#)
- Class of Morphisms
 - remark, [9](#)
- Class of Objects
 - remark, [7](#)
- Codomain, [13](#)
 - definition, [8](#), [13](#)
- Commutative diagram, [24](#)
 - definition, [9](#)
- Composition
 - opposite category, [11](#)
 - remark, [8](#)
- Composition axiom, [9–11](#), [17](#), [23](#)
 - declaration, [8](#)
- Contravariant functor

- Hask** example, 37
- definition, 36
- Dirac notation
 - example, 20
- Direct sum of Hilber spaces, 21, 33
 - definition, 21
- Discrete category, 15
 - definition, 15
- Disjoint union, 28
- Domain, 13
 - definition, 8, 13
- Dual space, 20, 22
 - definition, 20
- Epimorphism, 13
 - definition, 10
- Function, 12, 20, 21, 23
 - definition, 12
- Functor, 36, 37
 - definition, 35
 - remark, 36
- Haskell lazy evaluation
 - remark, 17
- Haskell lazy evaluation remark, 16, 28
- Hilbert space, 20–22
 - definition, 20
- Identity is unique theorem, 9
 - declaration, 24
- Identity morphism, 9, 10, 12, 15, 17, 18, 23, 24
 - definition, 9
- Initial object, 21, 24, 27, 28, 32, 33
 - C++** example, 29
 - FdHilb** example, 33
 - Hask** example, 28
 - Scala** example, 32
 - Set** example, 27
- definition, 24
- Initial object example, 32
- Initial object is unique theorem
 - declaration, 24
- Injection, 13, 15
 - definition, 13
- Injection vs Monomorphism
 - remark, 13
- Isomorphism, 10, 23, 24
 - definition, 10
 - remark, 10
- Large category, 12
 - definition, 11
- Linear map, 21
 - definition, 20
- Maybe as a functor
 - Hask** example, 37
- Monoid, 27
 - definition, 26, 36
- Monomorphism, 13, 15
 - definition, 9
- Morphism, 8–12, 15, 16, 18–23, 26, 36
 - C++** example, 18
 - FdHilb** example, 21
 - Hask** example, 16
 - Rel** example, 20
 - Scala** example, 19
 - Set** example, 12
 - definition, 8
- Morphisms equality
 - definition, 23
- Object, 7–12, 16, 18–24, 26, 30, 36
 - C++** example, 18
 - FdHilb** example, 21
 - Hask** example, 16
 - Rel** example, 20
 - Scala** example, 19
 - Set** example, 12

- definition, 7
- Objects equality, 24
 - definition, 23
- Opposite category, 11, 24
 - definition, 11
- Product, 21, 26–30, 33
 - C++** example, 30
 - FdHilb** example, 33
 - Hask** example, 29
 - Set** example, 28
 - definition, 26
- Profunctor
 - Hask** example, 37
 - definition, 36
- Rabi oscillations
 - example, 22
- Set, 8, 12, 13, 15, 20, 21, 23
 - definition, 11
- Set vs Category
 - remark, 12
- Set vs Category remark, 23
- Singleton, 20, 21, 28
 - definition, 13
- Small category, 11
 - definition, 11
- Sum, 21, 27–30, 33
 - C++** example, 30
 - FdHilb** example, 33
 - Hask** example, 29
 - Set** example, 28
 - definition, 26
- Sum example, 21, 30
- Surjection, 13, 15
 - definition, 13
- Surjection vs Epimorphism
 - remark, 13
- Tensor product, 21
 - definition, 21
- Terminal object, 21, 24, 27, 28, 32, 33
 - C++** example, 29
 - FdHilb** example, 33
 - Hask** example, 29
 - Scala** example, 32
 - Set** example, 28
 - Cat** category, 36
 - definition, 24
- Terminal object example, 32
- Terminal object in **Cat** category
 - Hask** example, 37
 - definition, 36
- Terminal object is unique theorem
 - declaration, 24

Bibliography

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