Category Theory by Example

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Contents

Notations

$[\mathbf{C}, \mathbf{D}]$ **Fun** category (Example 4.2)

$\alpha \circ \beta$ Vertical composition of natural transformations (circle dot)

$\alpha \star \beta$ Horizontal composition of natural transformations (star dot)

αH Left whiskering

α, β Natural transformation (Greek small letters)

$\alpha : F \dot{\to} G$ Natural transformation (arrow with dot)

$\mathbf{C_M}$ Kleisli category

$\mathbf{C}$ Category (bold capital Latin letter)

$\mathbf{C}^{op}$ Opposite category

$\operatorname{cod} f$ Codomain

$\operatorname{dom} f$ Domain

$\hom(a, b)$ set of Morphisms between a and b

$\hom_{\mathbf{C}}(a, b)$ Set of morphisms

$\mathbf{1}_{\mathbf{C} \Rightarrow \mathbf{C}}$ Identity functor

$\mathbf{1}_{a \to a}$ Identity morphism

$\mathbf{1}_{F \dot{\to} F}$ Identity natural transformation

$\langle M, \mu, \eta \rangle$ Monad

$\mathcal{H}_n$ finite dimensional Hilbert space

$a \cong_f b$ there is an Isomorphism f between a and b

a, b Objects (Latin small letters)

a^b Exponential

$F \circ G$ Functor composition (circle dot)

$f \circ g$ Morphism composition (circle dot)

F, G Functor (capital Latin letter)

f, g, h Morphism (Latin small letter)

$F : \mathbf{C} \Rightarrow \mathbf{D}$ Functor (double arrow)

$f : a \to b$ Morphism (simple arrow)

$H\alpha$ Right whiskering

TBD To Be Defined (later)

Introduction

You just looked at yet another introduction to Category Theory. The subject mostly consists of a lot of definitions that are related each others and I wrote the book to collect all of them in one place to be easy checked and updated in future when I decide to refresh my knowledge about the field of math. Therefore the book was written mostly for my category theory studying purposes but I will appreciate if somebody else find it useful.

The topics(chapters) cover the base definitions (Object, Morphism and Category), Functor, Natural transformation, Monad and also include important results from the category theory such as Yoneda's lemma and Curry-Howard-Lambek correspondence.

There are a lot of examples in each chapter. The examples cover different category theory application areas. I assume that the reader is familiar with the corresponding area and the example(s) can be passed if not. I.e. anyone can choose the suitable example(s) for (s)he.

The most important examples are related to the set theory. The set theory and category theory are very close related. Each one can be considered as an alternative view to another one.

There are a lot of examples from programming languages which include Haskell, Scala, C++. The source files for programming languages examples (Haskell, C++, Scala) can be found on github repository [7].

The examples from physics are related to quantum mechanics that is the most known for me. For the examples I am inspired by the Bob Coecke article [1].

Chapter 1

Base definitions

1.1 Definitions

1.1.1 Object

Definition 1.1 (Class)**.** A class is a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share.

Definition 1.2 (Object)**.** In category theory object is considered as something that does not have internal structure (aka point) but has a property that makes different objects belong to the same Class

Remark 1.3 (Class of Objects)**.** The Class of Objects will be marked as $\mathrm{ob}(\mathbf{C})$ (see fig. 1.1).

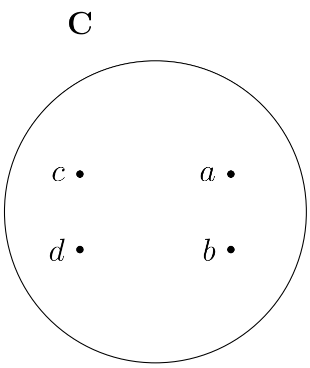

Figure 1.1: Class of objects $\mathrm{ob}(\mathbf{C}) = \{a, b, c, d\}$

1.1.2 Morphism

Morphism is a kind of relation between 2 Objects.

Definition 1.4 (Morphism)**.** A relation between two Objects a and b

$$f_{ab} : a \to b$$

is called *morphism*. Morphism assumes a direction i.e. one Object (a) is called *source* and another one (b) *target*.

The Set of all morphisms between objects a and b is denoted as $\hom(a, b)$.

The important remark about morphisms is below

Remark 1.5 (Morphism)**.** The morphism has to be considered as a relation between objects. We will avoid standard (from set theory) notation for morphisms: $f(a) = b$. The reason for this is the following. Let $f_1 : a \to b$ and $f_2 : a \to b$ are 2 different morphisms. The notation $f_1(a) = b, f_2(a) = b$ leads to incorrect conclusion that $f_1 = f_2$.

For instance if $a = b = \mathbb{R}$ then 2 functions $f_1(x) = x, f_2(x) = -x$ set 2 different ordering on $\mathbb{R}$ and as result have not to be considered as the same functions.

Definition 1.6 (Domain)**.** Given a Morphism $f : a \to b$, the Object a is called domain and denoted as $\operatorname{dom} f$.

Definition 1.7 (Codomain)**.** Given a Morphism $f : a \to b$, the Object b is called codomain and denoted as $\operatorname{cod} f$.

Morphisms have several properties. [1]

Axiom 1.8 (Composition)**.** *If we have 3 Objects a, b and c and 2 Morphisms*

$$f_{ab} : a \to b$$

and

$$f_{bc} : b \to c$$

then there exists Morphism

$$f_{ac} : a \to c$$

such that

$$f_{ac} = f_{bc} \circ f_{ab}$$

[1] The properties don't have any proof and postulated as axioms

Remark 1.9 (Composition)**.** The equation

$$f_{ac} = f_{bc} \circ f_{ab}$$

means that we apply f_{ab} first and then we apply f_{bc} to the result of the application i.e. if our objects are sets and $x \in a$ then

$$f_{ac}(x) = f_{bc}(f_{ab}(x)),$$

where $f_{ab}(x) \in b$.

Axiom 1.10 (Associativity)**.** *The Morphisms Composition (Axiom 1.8) s should follow associativity property:*

$$f_{ce} \circ (f_{bc} \circ f_{ab}) = (f_{ce} \circ f_{bc}) \circ f_{ab} = f_{ce} \circ f_{bc} \circ f_{ab}.$$

Definition 1.11 (Identity morphism)**.** For every Object a we define a special Morphism $\mathbf{1}_{a\to a} : a \to a$ with the following properties: $\forall f_{ab} : a \to b$

$$\mathbf{1}_{a\to a} \circ f_{ab} = f_{ab} \tag{1.1}$$

and $\forall f_{ba} : b \to a$

$$f_{ba} \circ \mathbf{1}_{a\to a} = f_{ba}. \tag{1.2}$$

This morphism is called as *identity morphism*.

Note that Identity morphism is unique, see Identity is unique (Theorem 2.3) below.

Definition 1.12 (Commutative diagram)**.** A commutative diagram is a diagram of Objects (also known as vertices) and Morphisms (also known as arrows or edges) such that all directed paths in the diagram with the same start and endpoint lead to the same result by composition

The following diagram commutes if $f_{ab} = f_{cb} \circ f_{ac}$.

$$\begin{tikzcd} a \arrow[rr, "f_{ab}"] \arrow[dr, "f_{ac}"'] & & b \\ & c \arrow[ur, "f_{cb}"'] & \end{tikzcd}$$

Remark 1.13 (Class of Morphisms)**.** The Class of Morphisms will be marked as $\hom(\mathbf{C})$ (see fig. 1.2)

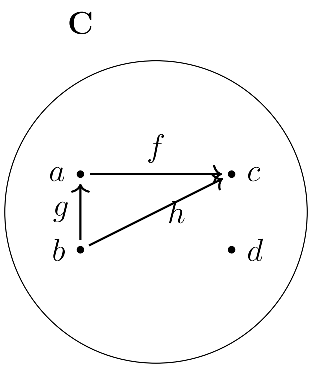

Figure 1.2: Class of morphisms $\hom(\mathbf{C}) = \{f, g, h\}$, where $h = f \circ g$

Definition 1.14 (Monomorphism)**.** If $\forall g_1, g_2$ the equation

$$f \circ g_1 = f \circ g_2$$

leads to

$$g_1 = g_2$$

then f is called *monomorphism*.

Definition 1.15 (Epimorphism)**.** If $\forall g_1, g_2$ the equation

$$g_1 \circ f = g_2 \circ f$$

leads to

$$g_1 = g_2$$

then f is called *epimorphism*.

Definition 1.16 (Isomorphism)**.** A Morphism $f : a \to b$ is called *isomorphism* if $\exists g : b \to a$ such that $f \circ g = \mathbf{1}_{a \to a}$ and $g \circ f = \mathbf{1}_{b \to b}$. If there is an isomorphism f between objects a and b then it is denoted as $a \cong_f b$.

Remark 1.17 (Isomorphism)**.** There are can be many different Isomorphisms between 2 Objects.

If there is an unique isomorphism between 2 objects then the objects can be treated as the same object.

1.1.3 Category

Definition 1.18 (Category)**.** A category $\mathbf{C}$ consists of

- Class of Objects $\mathrm{ob}(\mathbf{C})$

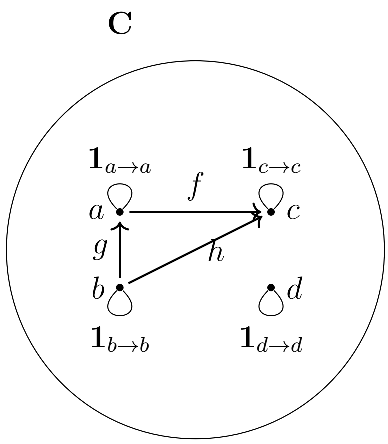

Figure 1.3: Category $\mathbf{C}$. It consists of 4 objects $\mathrm{ob}(\mathbf{C}) = \{a, b, c, d\}$ and 7 morphisms $\mathrm{ob}(\mathbf{C}) = \{f, g, h = f \circ g, \mathbf{1}_{a\to a}, \mathbf{1}_{b\to b}, \mathbf{1}_{c\to c}, \mathbf{1}_{d\to d}\}$

- Class of Morphisms $\hom(\mathbf{C})$ defined for $\mathrm{ob}(\mathbf{C})$, i.e. each morphism f_{ab} from $\hom(\mathbf{C})$ has both source a and target b from $\mathrm{ob}(\mathbf{C})$

For any Object a there should be unique Identity morphism $\mathbf{1}_{a\to a}$. Any morphism should satisfy Composition (Axiom 1.8) and Associativity (Axiom 1.10) . See fig. 1.3

Definition 1.19 (Set of morphisms)**.** The set of morphisms between objects a and b in the $\mathbf{C}$ will be denoted as $\hom_{\mathbf{C}}(a, b)$

The Category can be considered as a way to represent a structured data. Morphisms are the ones which form the structure.

Definition 1.20 (Opposite category)**.** If $\mathbf{C}$ is a Category then opposite (or dual) category $\mathbf{C}^{op}$ is constructed in the following way: Objects are the same, but the Morphisms are inverted i.e. if $f \in \hom(\mathbf{C})$ and $\mathrm{dom}\, f = a, \mathrm{cod}\, f = b$, then the corresponding morphism $f^{op} \in \hom(\mathbf{C}^{\mathbf{op}})$ has $\mathrm{dom}\, f^{op} = b, \mathrm{cod}\, f^{op} = a$ (see fig. 1.4)

Remark 1.21. Composition on C^{op} As you can see from fig. 1.4 the Composition (Axiom 1.8) is reverted for Opposite category. If $f, g, h = f \circ g \in \hom(\mathbf{C})$ then $f \circ g$ translated into $g^{op} \circ f^{op}$ in opposite category.

Definition 1.22 (Small category)**.** A category $\mathbf{C}$ is called *small* if both $\mathrm{ob}(\mathbf{C})$ and $\hom(\mathbf{C})$ are Sets

Definition 1.23 (Large category)**.** A category $\mathbf{C}$ is not Small category then it is called *large*. The example of large category is **Set** category

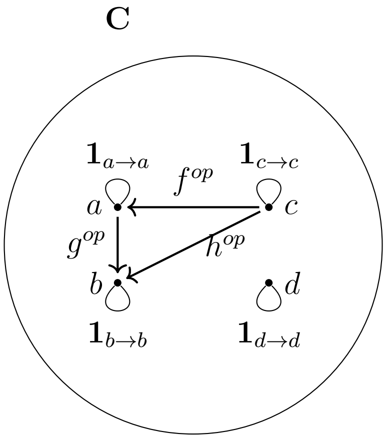

Figure 1.4: Opposite category C^{op} to the category from fig. 1.3 . It consists of 4 objects $\mathrm{ob}(\mathbf{C^{op}}) = \mathrm{ob}(\mathbf{C}) = \{a, b, c, d\}$ and 7 morphisms $\hom(\mathbf{C^{op}}) = \{f^{op}, g^{op}, h^{op} = g^{op} \circ f^{op}, \mathbf{1}_{a \to a}, \mathbf{1}_{b \to b}, \mathbf{1}_{c \to c}, \mathbf{1}_{d \to d}\}$

1.2 Set category example

There are several examples of categories that will also be used later

Definition 1.24 (Set)**.** Set is a collection of distinct object. The objects are called the elements of the set.

Definition 1.25 (Binary relation)**.** If A and B are 2 Sets then a subset of $A \times B$ is called binary relation R between the 2 sets, i.e. $R \subset A \times B$.

Definition 1.26 (Function)**.** Function f is a special type of Binary relation. I.e. if A and B are 2 Sets then a subset of $A \times B$ is called function f between the 2 sets if $\forall a \in A \, \exists! b \in B$ such that $(a, b) \in f$. In other words function definition does not allow "multi value".

Definition 1.27 (Cartesian product)**.** If A and B are two sets then we can define a new set $A \times B = \{(a, b) | a \in A, b \in B\}$ that is called as the *cartesian product.*

Definition 1.28 (**Set** category)**.** In the set category we consider a Set of Sets where Objects are the Sets and Morphisms are Functions between the sets.

The Identity morphism is trivial function such that $\forall x \in X : \mathbf{1}_{X \to X}(x) = x$.

In general case when we say **Set** category we assume the set of all sets. But the result is inconsistent because famous Russell's paradox [13] can be applied. To avoid such situations we consider a limitation that is applied on our construction, for instance ZFC [14]. If we apply the limitation we have that set of all sets is not a set itself and as result the **Set** category is a Large category

Remark 1.29 (Set vs Category)**.** There is an interesting relation between sets and categories. In both we consider objects(sets) and relations between them(morphisms/functions).

In the set theory we can get info about functions by looking inside the objects(sets) aka use "microscope" [5]

Contrary in the category theory we initially don't have any info about object internal structure but can get it using the relation between the objects i.e. using Morphisms. In other words we can use "telescope" [5] there.

Definition 1.30 (Categorical approach)**.** The description of a system via its communications we will call as *categorical approach*.

This description is contrary to an ordinary system description via its internal structure.

Definition 1.31 (Singleton)**.** The *singleton* is a Set with only one element.

Example 1.32 (Domain)**.** Given a function $f : X \to Y$, the set X is the domain. I.e. $\operatorname{dom} f = X$

Example 1.33 (Codomain)**.** Given a function $f : X \to Y$, the set Y is the codomain. I.e. $\operatorname{cod} f = Y$

Definition 1.34 (Surjection)**.** The function $f : X \to Y$ is surjective (or onto) if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$ (see figs. 1.5 and 1.9).

Remark 1.35 (Surjection vs Epimorphism)**.** Surjection and Epimorphism are related each other. Consider a non-surjective function $f : X \to Y' \subset Y$ (see fig. 1.6). One can conclude that there is not an Epimorphism because $\exists g_1 : Y' \to Y'$ and $g_2 : Y \to Y$ such that $g_1 \neq g_2$ because they operates on different Domains but from other hand $g_1(Y') = g_2(Y')$. For instance we can choose $g_1 = \mathbf{1}_{Y' \to Y'}, g_2 = \mathbf{1}_{Y \to Y}$. As soon as Y' is Codomain of f we always have $g_1(f(X)) = g_2(F(X))$.

As result we can say that an Surjection is a Epimorphism in the **Set** category. Moreover there is a proof [11] of that fact.

Definition 1.36 (Injection)**.** The function $f : X \to Y$ is injective (or one-to-one function) if $\forall x_1, x_2 \in X$, such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ (see figs. 1.7 and 1.9).

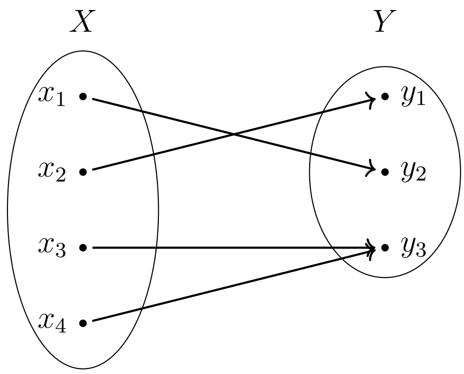

Figure 1.5: A surjective (non-injective) function from domain X to codomain Y

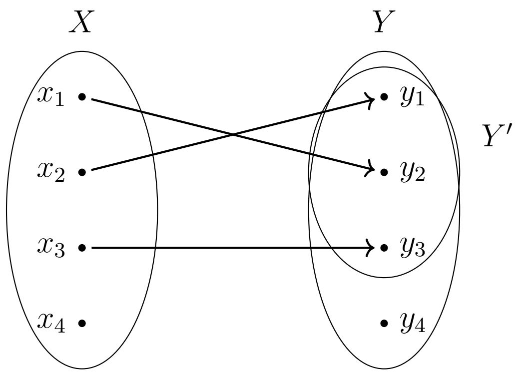

Figure 1.6: A non-surjective function f from domain X to codomain $Y' \subset Y$. $\exists g_1 : Y' \to Y', g_2 : Y \to Y$ such that $g_1(Y') = g_2(Y')$, but as soon as $Y' \neq Y$ we have $g_1 \neq g_2$. Using the fact that Y' is codomain of f we got $g_1 \circ f = g_2 \circ f$. I.e. the function f is not epimorphism.

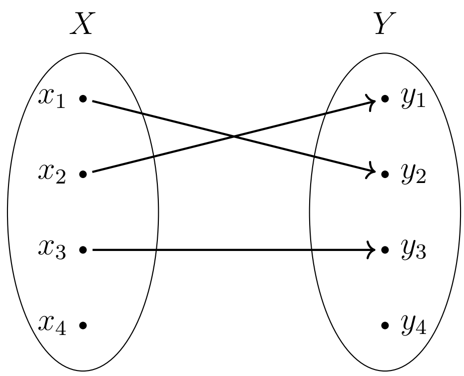

Figure 1.7: A injective (non-surjective) function from domain X to codomain Y

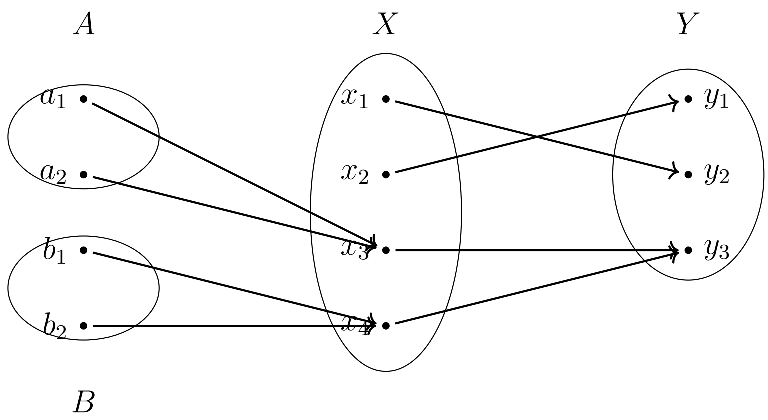

Figure 1.8: A non-injective function f from domain X to codomain Y. $\exists g_1 : A \to X, g_2 : B \to X$ such that $g_1 \neq g_2$ but $f \circ g_1 = f \circ g_2$. I.e. the function f is not monomorphism.

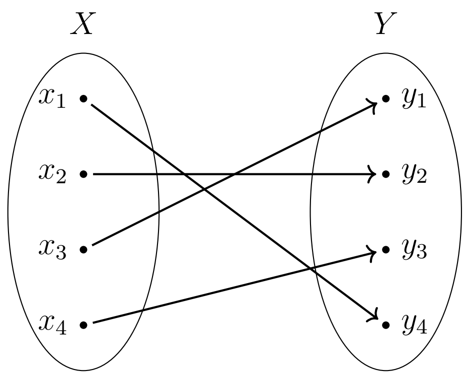

Figure 1.9: An injective and surjective function (bijection)

Remark 1.37 (Injection vs Monomorphism)**.** Injection and Monomorphism are related each other. Consider a non-injective function $f : X \to Y$ (see fig. 1.8). One can conclude that it is not monomorphism because $\exists g_1, g_2$ such that $g_1 \neq g_2$ and $f(g_1(a_1)) = y_3 = f(g_2(b_1))$.

As result we can say that an Injection is a Monomorphism in **Set** category. Moreover there is a proof [10] of that fact.

Definition 1.38 (Bijection)**.** The function $f : X \to Y$ is bijective (or one-to-one correspondence) if it is an Injection and a Surjection (see fig. 1.9).

There is a question what is the categorical analog of a single Set. Main characteristic of a category is a structure but the set by definition does not have a structure. Which category does not have any structure? The answer is Discrete category.

Definition 1.39 (Discrete category)**.** Discrete category is a Category where Morphisms are only Identity morphisms.

1.3 Programming languages examples

In the programming languages we consider types as Objects and functions as Morphisms. The critical requirements for such consideration is that the functions have to be pure functions (without side effects). This requirement mainly is satisfied by functional languages such as Haskell and Scala. From other side the functional languages use lazy evaluation to improve their performance. The laziness can also make category theory axiom invalid (see Haskell lazy evaluation (Remark 1.45)).

Strictly speaking neither Haskell (pure functional language) nor C++ can be considered as a category in general. For the first approximation a functional language (Haskell, Scala) can be considered as a category if we avoid to use functions with side effects (mainly for Scala) and use strict (for both Haskell and Scala) evaluations. Take the fact into consideration and define categories for 3 languages

Definition 1.40 (**Hask** category)**.** The objects in the **Hask** category are Haskell types and morphisms are functions

Definition 1.41 (Pure function)**.** The function is pure if it's execution give the same results independently from the environment.

Definition 1.42 (**Scala** category)**.** The objects in the **Scala** category are Scala types and morphisms are functions. We don't define functions that have a state in the category. I.e. the functions are Pure functions.

Definition 1.43 (**C++** category)**.** The objects in the **C++** category are Scala types and morphisms are functions. We don't define functions that have a state in the category. I.e. the functions are Pure functions.

In any case we can construct a simple toy category that can be easy implemented in any language. Particularly we will look into category with 3 objects that are types: Int, Bool, String. There are also several functions between them (see fig. 1.10).

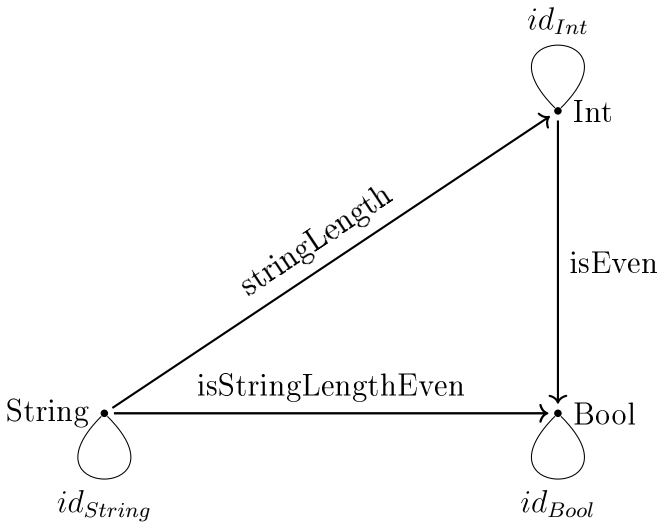

Figure 1.10: Programming language category example. Objects are types: Int, Bool, String. Morphisms are several functions

1.3.1 Hask toy category

Example 1.44 (**Hask** toy category)**.** Types in Haskell are considered as Objects. Functions are considered as Morphisms. We are going to implement Category from fig. 1.10.

The function `isEven` converts`Int` type into Bool.

```
isEven :: Int -> Bool
isEven x = x `mod` 2 == 0
```

There is also Identity morphism that is defined as follows

```
id :: a -> a
id x = x
```

If we have an additional function

```
stringLength :: String -> Int
stringLength x = length x
```

then we can create a Composition (Axiom 1.8)

```
isStringLengthEven :: String -> Bool
isStringLengthEven = isEven . stringLength
```

Remark 1.45 (Haskell lazy evaluation)**.** Each Haskell type has a special value $\perp$. The fact that the value and lazy evaluations are part of the language, make several category law invalid, for instance Identity morphism behaviour become invalid in specific cases:

The following code

```
seq undefined True
```

produces *undefined* But the following

```
seq (id.undefined) True
seq (undefined.id) True
```

produces *True* in both cases. As result we have (we cannot compare compare functions in Haskell, but if we could we can get the following)

```
id . undefined /= undefined
undefined . id /= undefined,
```

i.e. (1.1) and (1.2) are not satisfied.

1.3.2 C++ toy category

Example 1.46 (C++ toy category)**.** We will use the same trick as in **Hask toy category** (Example 1.44) and will assume types in C++ as Objects, functions as Morphisms. We also are going to implement Category from fig. 1.10.

We also define 2 functions:

```
auto isEven = [](int x) {
  return x % 2 == 0;
};

auto stringLength = [](std::string s) {
  return static_cast<int>(s.size());
};
```

Composition can be defined as follows:

```
// h = g . f
template <typename A, typename B>
auto compose(A g, B f) {
  auto h = [f, g](auto a) {
    auto b = f(a);
```

```
      auto c = g(b);
      return c;
    };
    return h;
  };
```

The Identity morphism:

```
auto id = [](auto x) { return x; };
```

The usage examples are the following:

```
auto isStringLengthEven = compose<>(isEven, stringLength);

auto isStringLengthEvenL = compose<>(id, isStringLengthEven);

auto isStringLengthEvenR = compose<>(isStringLengthEven, id);
```

Such construction will always provides us the category as soon as we use pure function (functions without effects).

1.3.3 Scala toy category

Example 1.47 (**Scala** toy category)**.** We will use the same trick as in **Hask** toy category (Example 1.44) and will assume types in Scala as Objects, functions as Morphisms. We also are going to implement Category from fig. 1.10.

```
object Category {
  def id[A]: A => A = a => a
  def compose[A, B, C](g: B => C, f: A => B):
      A => C = g compose f

  val isEven = (i: Int) => i % 2  == 0
  val stringLength = (s: String) => s.length
  val isStringLengthEven = (s: String) =>
      compose(isEven, stringLength)(s)
}
```

The usage example is below

```
class CategorySpec extends Properties("Category") {
  import Category._
```

```
    import Prop.forAll

    property("composition") = forAll { (s: String) =>
      isStringLengthEven(s)  == isEven(stringLength(s))
    }

    property("right id") = forAll { (i: Int) =>
      isEven(i)  == compose(isEven, id[Int])(i)
    }

    property("left id") = forAll { (i: Int) =>
      isEven(i)  == compose(id[Boolean], isEven)(i)
    }
  }
```

1.4 Quantum mechanics examples

The most critical property of quantum system is the superposition principle. The **Set** category cannot be used for it because it does not satisfied the principle. but a simple modification of the **Set** category does.

Definition 1.48 (**Rel** category)**.** We will consider a set of sets (same as **Set** category) i.e. Sets as Objects. Instead of Functions we will use Binary relations as Morphisms.

The **Rel** category is similar to the finite dimensional Hilber space especially because it assumes some kind of superposition. Really consider **Rel** - the **Rel** category. $X, Y \in \mathrm{ob}(\mathbf{Rel})$ - 2 sets which consists of different elements. Let $f : X \to X$ - Morphism. Each element $x \in X$ is mapped to a subset $Y' \subset Y$. The Y' can be Singleton (in this case no differences with **Set** category) but there can be a situation when Y' consists of several elements. In the case we will get some kind of superposition that is analogiest to quantum systems.

In the quantum mechanics we say about Hilber spaces.

Definition 1.49 (Hilbert space)**.** The Hilbert space is a complex vector space with an inner product as a complex number ($\mathbb{C}$).

Later we will consider only finite dimensional Hilbert spaces. We will denote a Hilber space of dimensional n as $\mathcal{H}_n$. Obviously $\mathcal{H}_1 = \mathbb{C}$.

Definition 1.50 (Dual space)**.** Each Hilber space $\mathcal{H}$ has an associated with it dual space $\mathcal{H}^*$ that consists of linear functionals

Example 1.51 (Dirac notation)**.** Consider a ket-vector $|\psi\rangle \in \mathcal{H}$. Then the corresponding vector from Dual space is called bra-vector $\langle\psi| \in \mathcal{H}^*$. From the definition of dual space the bra-vector is a linear functional i.e.

$$\langle\psi| : \mathcal{H} \to \mathbb{C},$$

$\forall |\phi\rangle \in \mathcal{H}$ we have $\langle\psi| (|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$ - inner product that is often written as $\langle\psi|\phi\rangle$.

The transformation between 2 Hilbert spaces that preserves the structure is called linear map or linear transformations.

Definition 1.52 (Linear map)**.** The linear map between2 Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$ is a mapping $f : \mathcal{A} \to \mathcal{B}$ that preserves additions

$$f(a_1 + a_2) = f(a_1) + f(a_2),$$

and scalar multiplications:

$$f(c \cdot a) = c \cdot f(a)$$

where $a, a_{1,2} \in \mathcal{A}$ and $f(a), f(a_{1,2}) \in \mathcal{B}$.

Remark 1.53 (Linear map)**.** Note that Linear map does not preserve inner product. TBD (verify the statement ???)

If we want to combine 2 Hilbert spaces into one we use a notion of direct sum.

Definition 1.54 (Direct sum of Hilber spaces)**.** Let $\mathcal{A}, \mathcal{B}$ are 2 Hilber spaces. The direct sum $\mathcal{A} \oplus \mathcal{B}$ is defined as follows

$$\mathcal{A} \oplus \mathcal{B} = \{a \oplus b | a \in \mathcal{A}, b \in \mathcal{B}\}.$$

The inner product is defined as follows

$$\langle a_1 \oplus b_1 | a_2 \oplus b_2 \rangle = \langle a_1 | a_2 \rangle + \langle b_1 | b_2 \rangle .$$

Definition 1.55 (**FdHilb** category)**.** Most common case in quantum mechanics is the case of quantum states in the finite dimensional Hilbert space. We can consider the set of all finite dimensional Hilbert spaces as a category. The Objects in the category are finite dimensional Hilbert spaces and Morphisms are Linear maps. The category is denoted as **FdHilb**. It is very similar to **Rel** category. The brief relation is described in the table 1.1.

Table 1.1: Relations between **Set**, **Rel** and **FdHilb** categories

	Set	**Rel**	**FdHilb**
Object	Set	Set	finite dimensional Hilbert space
Morphism	Function	Binary relation	Linear map
Initial object	empty set	empty set	trivial Hilbert space of dimensional 0
Terminal object	Singleton	Singleton	$\mathbb{C}$
Product	Cartesian product	Cartesian product	Direct sum of Hilber spaces
Sum	Sum (Example 2.15)	Sum (Example 2.15)	Direct sum of Hilber spaces

Example 1.56 (Rabi oscillations)**.** For our example we consider a 2 level atom with states $|a\rangle$ - excited and $|b\rangle$ - ground. As soon as we consider a 2-level system we are in the 2 dimensional Hilbert space i.e. have only one Object. Lets call it as $|\psi\rangle$. The category in the example will be called as **Rabi**. I.e. ob(**Rabi**) $= \mathcal{H}_2\{|\psi\rangle\}$.

The atom interacts with light beam of frequency $\omega = \omega_{ab}$. The state of the system is described by the following equation [15]:

$$|\psi\rangle = \cos\frac{\omega_R t}{2}\,|a\rangle - i\sin\frac{\omega_R t}{2}\,|b\rangle\,,$$

where ω_R - Rabi frequency [15].

The interaction time t is fixed and corresponds to $\omega_R t = \pi$ i.e. the interaction can be described a linear operator $\hat{L}$.

There are 4 different states and as result 4 Morphisms:

$$\begin{aligned}
|\psi\rangle_0 &= |a\rangle\,,\\
|\psi\rangle_1 &= \hat{L}\,|\psi\rangle_0 = -i\,|b\rangle\,,\\
|\psi\rangle_2 &= \hat{L}^2\,|\psi\rangle_0 = -\,|a\rangle\,,\\
|\psi\rangle_3 &= \hat{L}^3\,|\psi\rangle_0 = i\,|b\rangle\,,
\end{aligned}$$

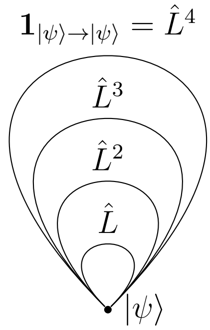

Figure 1.11: Rabi oscillations as a category **Rabi**

Chapter 2

Objects and morphisms

2.1 Equality

The important question is how can we decide whenever an object/morphism is equal to another object/morphism? The trivial answer is possible if an Object is a Set. In the case we can say that 2 objects are equal if they contain the equivalent collection of elements. Unfortunately we cannot do the same trick for categorical Objects as soon as they don't have any internal structure but can use a Categorical approach (see Set vs Category (Remark 1.29)): if we cannot use "microscope" lets use "telescope" and define the equality of objects and morphisms of a category $\mathbf{C}$ in the terms of whole $\hom(\mathbf{C})$.

Definition 2.1 (Objects equality)**.** Two Objects a and b in Category $\mathbf{C}$ are equal if there exists an unique Isomorphism $a \cong_f b$. This also means that also exist unique isomorphism $b \cong_g a$. These two Morphisms (f and g) are related each other via the following equations: $f \circ g = \mathbf{1}_{a \to a}$ and $g \circ f = \mathbf{1}_{b \to b}$.

Unlike Functions between Sets we don't have any additional info [1] about Morphisms except category theory axioms which the morphisms satisfy [2]. This leads us to the following definition of morphims equality:

Definition 2.2 (Morphisms equality)**.** Two Morphisms f and g in Category $\mathbf{C}$ are equal if the equality can be derived from the base axioms:

- Composition (Axiom 1.8)
- Associativity (Axiom 1.10)
- Identity morphism: (1.1), (1.2)

[1] for instance info about sets internals. i.e. which elements of the sets are connected by the considered functions

or Commutative diagrams which postulate the equality.

As an example lets proof the following theorem

Theorem 2.3 (Identity is unique)**.** *The Identity morphism is unique.*

Proof. Consider an Object a and it's Identity morphism $\mathbf{1}_{a\to a}$. Let $\exists f : a \to a$ such that f is also identity. In the case (1.1) for f as identity gives

$$f \circ \mathbf{1}_{a\to a} = \mathbf{1}_{a\to a}.$$

From other side (1.2) for $\mathbf{1}_{a\to a}$ satisfied

$$f \circ \mathbf{1}_{a\to a} = f$$

i.e.

$$f = f \circ \mathbf{1}_{a\to a} = \mathbf{1}_{a\to a}$$

or $f = \mathbf{1}_{a\to a}$. □

2.2 Initial and terminal objects

Definition 2.4 (Initial object)**.** Let $\mathbf{C}$ is a Category, the Object $i \in \mathrm{ob}(\mathbf{C})$ is called *initial object* if $\forall x \in \mathrm{ob}(\mathbf{C}) \exists! f_x : i \to x \in \mathrm{hom}(\mathbf{C})$.

Example 2.5 (Initial object)**.** **[Set]** Note that there is only one function from empty set to any other sets [9] that makes the empty set as the Initial object in **Set** category.

Definition 2.6 (Terminal object)**.** Let $\mathbf{C}$ is a Category, the Object $t \in \mathrm{ob}(\mathbf{C})$ is called *terminal object* if $\forall x \in \mathrm{ob}(\mathbf{C}) \exists! g_x : x \to t \in \mathrm{hom}(\mathbf{C})$.

Example 2.7 (Terminal object)**.** **[Set]** Terminal object in **Set** category is a set with one element i.e Singleton.

As you can see the initial and terminal objects are opposite each other. I.e. if i is an Initial object in $\mathbf{C}$ then it will be Terminal object in the Opposite category $\mathbf{C^{op}}$.

Theorem 2.8 (Initial object is unique)**.** *Let $\mathbf{C}$ is a category and $i, i' \in \mathrm{ob}(\mathbf{C})$ two Initial objects then there exists an unique Isomorphism $u : i \to i'$ (see Objects equality)*

$$
\begin{tikzcd}
i \arrow[r, "u"] \arrow[dr, "\mathbf{1}_{i\to i}"'] & i' \arrow[d, "u^{-1}"] \arrow[dr, "\mathbf{1}_{i'\to i'}"] & \\
& i \arrow[r, "u"] & i'
\end{tikzcd}
$$

Figure 2.1: Commutative diagram for initial object uniqueness proof

$$
\begin{tikzcd}
t & t' \arrow[l, "v"'] & \\
& t \arrow[ul, "\mathbf{1}_{t\to t}"] \arrow[u, "v^{-1}"'] & t' \arrow[l, "v^{-1}"'] \arrow[ul, "\mathbf{1}_{t'\to t'}"']
\end{tikzcd}
$$

Figure 2.2: Commutative diagram for terminal object uniqueness proof

Proof. Consider the following Commutative diagram (see fig. 2.1). As soon as i initial object $\exists!\, u : i \to i'$. From other side i' is also initial object and therefore $\exists!\, u^{-1} : i' \to i$. Combining them together via composition we can get $u^{-1} \circ u : i \to i$ and $u \circ u^{-1} : i' \to i'$. From the fact that i is initial object one can get that there exists only one morphism $\mathbf{1}_{i\to i} : i \to i$. The same is the truth for i'. Therefore $u^{-1} \circ u = \mathbf{1}_{i\to i}$ and $u \circ u^{-1} = \mathbf{1}_{i'\to i'}$. These complete the commutative diagram build and finishes the proof. □

Theorem 2.9 (Terminal object is unique)**.** *Let* $\mathbf{C}$ *is a category and* $t, t' \in \mathrm{ob}(\mathbf{C})$ *two Terminal objects then there exists an unique Isomorphism* $v : t' \to t$ *(see Objects equality)*

Proof. Just got to the Opposite category and revert arrows in fig. 2.1. The result shown on fig. 2.2 and it proofs the theorem statement. □

Example 2.10 (Toy example)**.** In our toy example fig. 1.10 the type String is Initial object and type Bool is the Terminal object.

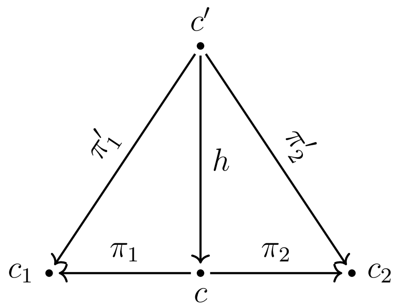

Figure 2.3: Product $c = c_1 \times c_2$. $\forall c, \exists! h \in \hom(\mathbf{C}) : \pi_1' = \pi_1 \circ h, \pi_2' = \pi_2 \circ h$.

2.3 Product and sum

The pair of 2 objects is defined via the universal property in the following way:

Definition 2.11 (Product)**.** Let we have a category $\mathbf{C}$ and $c_1, c_2 \in \mathrm{ob}(\mathbf{C})$ -two Objects then the product of the objects c_1, c_2 is another object in $\mathbf{C}$ $c = c_1 \times c_2$ with 2 Morphisms π_1, π_2 such that $c_1 = \pi_1(c), c_2 = \pi(c_2)$ and the following universal property is satisfied: $\forall c' \in \mathrm{ob}(\mathbf{C})$ and morphisms $\pi_1' : c' \to c_1, \pi_2' : c' \to c_2$, exists unique morphism h such that the following diagram (see fig. 2.3) commutes, i.e. $\pi_1' = \pi_1 \circ h, \pi_2' = \pi_2 \circ h$. In other words h factorizes $\pi_{1,2}'$.

Example 2.12 (Product)**.** **[Set]** The Product of two sets A and B in **Set** category is defined as a Cartesian product: $A \times B = \{(a, b) | a \in A, b \in B\}$.

If we invert arrows in Product we will got another object definition that is called sum

Definition 2.13 (Sum)**.** Let we have a category $\mathbf{C}$ and $c_1, c_2 \in \mathrm{ob}(\mathbf{C})$ -two Objects then the sum of the objects c_1, c_2 is another object in $\mathbf{C}$ $c = c_1 \oplus c_2$ with 2 Morphisms i_1, i_2 such that $c = i_1(c_1), c = i_2(c_2)$ and the following universal property is satisfied: $\forall c' \in \mathrm{ob}(\mathbf{C})$ and morphisms $i_1' : c_1 \to c', i_2' : c_2 \to c'$, exists unique morphism h such that the following diagram (see fig. 2.4) commutes, i.e. $i_1' = h \circ i_1, i_2' = h \circ i_2$. In other words h factorizes $i_{1,2}'$.

Definition 2.14 (Disjoint union)**.** Let $\{A_i : i \in I\}$ be a family of sets indexed by I. The *disjoint union* [12] of this family is the set

$$\sqcup_{i \in I} A_i = \cup_{i \in I} \{(x, i) : x \in A_i\}.$$

The elements of the disjoint union are ordered pairs (x, i). Here i serves as an auxiliary index that indicates which A_i the element x came from.

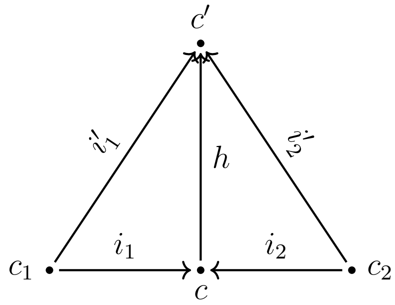

Figure 2.4: Sum $c = c_1 \oplus c_2$. $\forall c, \exists! h \in \hom(\mathbf{C}) : i'_1 = h \circ i_1, i'_2 = h \circ i_2$.

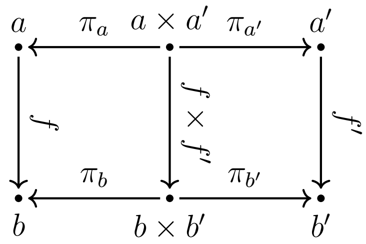

Figure 2.5: Product of morphisms.

Example 2.15 (Sum)**.** **[Set]** The Sum of two sets A and B in **Set** category is defined as Disjoint union.

The Product of objects will provide also a definition for product of morphisms

Definition 2.16 (Product of morphisms)**.** Let $\mathbf{C}$ is a category and $a, a' \in \mathrm{ob}(\mathbf{C})$ and $b, b' \in \mathrm{ob}(\mathbf{C})$ are 2 pairs of Objects that admit definition 2.11. Consider 2 morphisms that connects the objects: $f : a \to b, f' : a' \to b'$ then we can create a new unique morphism that connects the products: $f \times f' : a \times a' \to b \times b'$ and makes the diagram commute (see fig. 2.5).

2.4 Category as a monoid

Consider the following definition from abstract algebra

Definition 2.17 (Monoid)**.** The set of elements M with defined binary operation $\circ$ we will call as a monoid if the following conditions are satisfied.

1. Closure: $\forall a, b \in M$: $a \circ b \in M$
2. Associativity: $\forall a, b, c \in M$: $a \circ (b \circ c) = (a \circ b) \circ c$

$$
\begin{tikzcd}
z' \arrow[d, "h"'] & z' \times y \arrow[d, "h \times \mathbf{1}_{y \to y}"'] \arrow[dr, "e'"] & \\
z^y & z^y \times y \arrow[r, "e"] & z
\end{tikzcd}
$$

Figure 2.6: Exponential object

3. Identity element: $\exists e \in M$ such that $\forall a \in M$: $e \circ a = a \circ e = a$

We can consider 2 Monoids. The first one has Product as the binary operation and Terminal object as the identity element. As result we just got an analog of multiplication in the category theory. This is why the terminal object is often denoted as **1** and the operation is called as the product.

Another one is additional Monoid that has Initial object as the identity element and the Sum as the binary operation. The initial object in that case is often denoted as **0**. I.e. we can see a direct connection with addition in algebra.

2.5 Exponential

We are going to talk about functions (aka morphisms) as Objects.

Example 2.18 (Hom set)**.** Consider 2 sets A and B the set of functions between the 2 sets form a new set that is called as *Hom-set* and denoted as $A \to B$. Thus if $A, B \in \text{ob}(\mathbf{Set})$ then the Hom-set will also $A \to B \in \text{ob}(\mathbf{Set})$.

The construction of Hom set (Example 2.18) is applied to **Set** category but not to an arbitrary category because the Hom-set is a set and therefore the object in the **Set** category. I.e. if $\mathbf{C}$ is a category and $a, b \in \text{ob}(\mathbf{C})$ then the Hom-set $a \to b \in \text{ob}(\mathbf{Set})$ but we now want to construct something like to the Home-set but that is an object in $\mathbf{C}$. This will be called as the function object. We will use the universal construction for the object definition.

Definition 2.19 (Exponential)**.** Let $\mathbf{C}$ is a category and $z, y \in \text{ob}(\mathbf{C})$. We also assume that $\mathbf{C}$ allows all Products with y, i.e. $\forall z' \in \text{ob}(\mathbf{C}), \exists z' \times y$. An object z^y together with a Morphism $e : z^y \times y \to z$ is an *exponential object* if $\forall e' \in \mathrm{hom}(\mathbf{C})$ and $\forall z' \in \mathrm{ob}(\mathbf{C})$ exists an unique morphism $h : z' \to z$ such that the Commutative diagram shown in fig. 2.6 commutes:

$$e' = e \circ (h \times \mathbf{1}_{y \to y})$$

Example 2.20 (Exponential)**. [Set]** Lets look at the Exponential in **Set**. We want to show that the object corresponds to the function. Really if we want to define a function $f : X \to Y$ then we should look at the Hom set (Example 2.18) $F = X \to Y$. $f \in F$ - is an element of the Hom-set. For the function application we have to take the argument $x \in X$ and the function we want to apply $f \in F$. Then we construct the pair $(f, x) \in F \times X$. For the function application we have to call a Morphism $e : F \times X \to Z$. [2] I.e. the application $e(f, x)$ gives us $e(f, x) = y \in Y$ - the function value.

Definition 2.21 (Cartesian closed category)**.** If a category **C** satisfies the following conditions then it is called *Cartesian closed category*

1. It has Terminal object
2. $\forall a, b \in \mathrm{ob}(\mathbf{C})$ exists Product $a \times b \in \mathrm{ob}(\mathbf{C})$.
3. $\forall a, b \in \mathrm{ob}(\mathbf{C})$ exists Exponential $a^b \in \mathrm{ob}(\mathbf{C})$

2.6 Type algebra and Curry-Howard-Lambek correspondence

There is an interesting correspondence between computer programs and mathematical proofs. First of all consider a category of proofs

2.6.1 Proof category

Definition 2.22 (Proposition)**.** TBD

Definition 2.23 (Proof)**.** TBD

Definition 2.24 (**Proof** category)**.** The **Proof** category is a category where Propositions are Objects and Proofs are Morphisms. I.e. proofs are used as connectors between different propositions.

Consider different objects and constructions of the proof (logic) theory from the categorical point of view

[2] e from the word "eval"

Table 2.1: Relation between logic proofs and programming languages

Proof category	Programming language	Cartesian closed category
Proposition	Type	Object
Proof	Function type	Exponential
Conjunction	Product type	Product
Disjunction	Sum type	Sum
true	unit type	Terminal object
false	botom type	Initial object

Example 2.25 (Initial object)**. [Proof]** The *false* statement can be considered as the initial object because for any other statement exists only one proof from the false statement to that one.

Example 2.26 (Terminal object)**. [Proof]** The *true* statement can be considered as the terminal object

Example 2.27 (Product)**. [Proof]** TBD

Example 2.28 (Sum)**. [Proof]** TBD

The correspondence can be written in the form of the table 2.1

2.6.2 Hask category

Example 2.29 (Initial object)**. [Hask]** If we avoid lazy evaluations in Haskell (see Haskell lazy evaluation (Remark 1.45)) then we can found several types as candidates for initial and terminal object in Haskell. Initial object in **Hask** category is a type without values

```
data Void
```

i.e. you cannot construct a object of the type.

There is only one function from the initial object:

```
absurd :: Void -> a
```

The function is called absurd because it does absurd action. Nobody can proof that it does not exist. For the existence proof the following absurd argument can be used: "Just provide me an object type `Void` and I will provide you the result of evaluation".

There is no function in opposite direction because it would had been used for the `Void` object creation.

Example 2.30 (Terminal object)**.** [**Hask**] Terminal object (unit) in **Hask** category keeps only one element

```
data () = ()
```

i.e. you can create only one element of the type. You can use the following function for the creation:

```
unit :: a -> ()
unit _ = ()
```

Example 2.31 (Product)**.** [**Hask**] The Product in **Hask** category keeps a pair and the constructor defined as follows

```
(,) :: a -> b -> (a, b)
(,) x y = (x, y)
```

There are 2 projectors:

```
fst :: (a, b) -> a
fst (x, _) = x
snd :: (a, b) -> b
snd (_, y) = y
```

Example 2.32 (Sum)**.** [**Hask**] The Sum in **Hask** category defined as follows

```
data Either a b = Left a | Right b
```

The typical usage is via pattern matching for instance

```
factor :: (a -> c) -> (b -> c) -> Either a b -> c
factor f _ (Left x) = f x
factor _ g (Right y) = g y
```

2.6.3 C++ category

Example 2.33 (Initial object)**.** [**C++**] In C++ exists a special type that does not hold any values and as result cannot be created: `void`. You cannot create an object of that type i.e. you will get a compiler error if you try.

Example 2.34 (Terminal object)**.** [**C++**] C++ 17 introduced a special type that keeps only one value - `std::monostate`:

```
namespace std {
  struct monostate {};
}
```

Example 2.35 (Product)**.** [**C++**] The Product in **C++** category keeps a pair and the constructor defined as follows

```
namespace std {
  template< class A, class B > struct pair {
    A first;
    B second;
  };
}
```

There is a simple usage example

```
  std::pair<int, bool> p(0, false);

  std::cout << "First projector: " << p.first << std::endl;
  std::cout << "Second projector: " << p.second << std::endl;
```

Really any `struct` or `class` can be considered as a product.

Example 2.36 (Sum)**.** [**C++**] If we consider Objects as types then Sum is an object that can be either one or another type. The corresponding C/C++ construction that provides an ability to keep one of two types is `union`.

C++17 suggests `std:variant` as a safe replacement for `union`. The example of the `factor` function is below

```
template <typename A, typename B, typename C, typename D>
auto factor(A f, B g, const std::variant<C, D>& either) {
  try {
    return f(std::get<C>(either));
  }
  catch(...) {
    return g(std::get<D>(either));
  }
};
```

The simple usage as follows:

```
std::variant<std::string, int> var = std::string("abc");
std::cout << "String length:" <<
factor<>(stringLength, id, var) << std::endl;
var = 4;
std::cout << "id(int):" <<
factor<>(stringLength, id, var) << std::endl;
```

TBD

2.6.4 Scala category

Example 2.37 (Initial object). **[Scala]** We used a same trick as for Initial object (Example 2.29) in **Hask** category and define Initial object in **Scala** category as a type without values

```
sealed trait Void
```

i.e. you cannot construct a object of the type.

Example 2.38 (Terminal object). **[Scala]** We used a same trick as for Terminal object (Example 2.30) in **Hask** category and define Terminal object in **Scala** category as a type with only one value

```
abstract final class Unit extends AnyVal
```

TBD i.e. you can create only one element of the type.

TBD

2.7 Quantum mechanics

Example 2.39 (Initial object). **[FdHilb]** We will use a Hilber space of dimensional 0 as the Initial object. I.e. the set that does not have any states in it.

Example 2.40 (Terminal object). **[FdHilb]** We will use a Hilber space of dimensional 1 as the Terminal object. I.e. the set of complex numbers $\mathbb{C}$.

Example 2.41 (Product). **[FdHilb]** The Product in **FdHilb** category is a Direct sum of Hilber spaces.

Example 2.42 (Sum). **[FdHilb]** The Sum in **FdHilb** category is a Direct sum of Hilber spaces.

TBD

Chapter 3

Functors

3.1 Definitions

Definition 3.1 (Functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ are 2 categories. A mapping $F : \mathbf{C} \Rightarrow \mathbf{D}$ between the categories is called *functor* if it preserves the internal structure (see fig. 3.1):

- $\forall a_C \in \mathrm{ob}(\mathbf{C}), \exists a_D \in \mathrm{ob}(\mathbf{D})$ such that $a_D = F(a_C)$
- $\forall f_C \in \hom(\mathbf{C}), \exists f_D \in \hom(\mathbf{D})$ such that $\mathrm{dom}\, f_D = F(\mathrm{dom}\, f_C), \mathrm{cod}\, f_D = F(\mathrm{cod}\, f_C)$. We will use the following notation later: $f_D = F(f_C)$.
- $\forall f_C, g_C$ the following equation holds:

$$F\left(f_C \circ g_C\right) = F\left(f_C\right) \circ F\left(g_C\right) = f_D \circ g_D.$$

- $\forall x \in \mathrm{ob}(\mathbf{C}) : F(\mathbf{1}_{x \to x}) = \mathbf{1}_{F(x) \to F(x)}$.

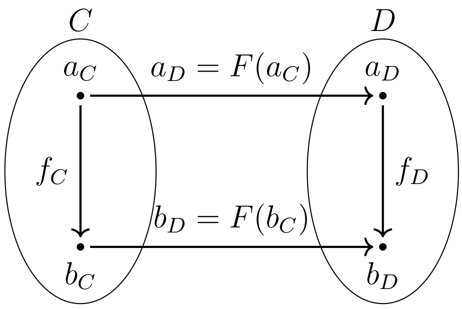

Figure 3.1: Functor $F : \mathbf{C} \Rightarrow \mathbf{D}$ definition

Remark 3.2 (Functor)**.** When we say that functor preserve internal structure we assume that the functor is not just mapping between Objects but also between Morphisms.

Thus functor is something that allows map one category into another. The initial category can be considered as a pattern thus the mapping is some kind of searching of the pattern inside another category.

Programming languages can be considered as a good platform for the functor examples. The functor can be defined in Haskell as follows [1]

Example 3.3 (Functor)**. [Hask]**

```
class Functor f where
fmap :: (a -> b) -> f a -> f b
```

The concrete example is the following

Example 3.4 (Maybe as a functor)**. [Hask]** Lets show how the `Maybe a` type can be constructed from different Functors and as result show that the `Maybe a` is also a Functor.

```
data Maybe a = Nothing | Just a
-- This is equivalent to
data Maybe a = Either () (Identity a)
-- Either is a bifunctor and () == Const () a
-- Thus Maybe is a composition of 2 functors
```

In Scala it can be defined in the same way

Example 3.5 (Functor)**. [Scala]**

```
trait Functor[F[_]] {
  def fmap[A, B](f: A => B): F[A] => F[B]
```

Definition 3.6 (Endofunctor)**.** Let $\mathbf{C}$ is a Category. The Functor $E : \mathbf{C} \Rightarrow \mathbf{C}$ i.e. the functor from a category to the same category is called *endofunctor*.

Definition 3.7 (Identity functor)**.** Let $\mathbf{C}$ is a Category. The Functor $\mathbf{1}_{\mathbf{C}\Rightarrow\mathbf{C}} : \mathbf{C} \Rightarrow \mathbf{C}$ is called *identity functor* if for every object $a \in \mathrm{ob}(\mathbf{C})$

$$\mathbf{1}_{\mathbf{C}\Rightarrow\mathbf{C}}(a) = a$$

and for every Morphism $f \in \mathrm{hom}(\mathbf{C})$

$$\mathbf{1}_{\mathbf{C}\Rightarrow\mathbf{C}}(f) = f$$

[1] the real definition is quite different from the current one

Remark 3.8 (Identity functor)**.** First of all notice that Identity functor is an Endofunctor.

There is difference between identity functor and Identity morphism because the first one has deal with both Objects and Morphisms while the second one with the objects only.

Definition 3.9 (Functor composition)**.** If we have 3 categories $\mathbf{C}, \mathbf{D}, \mathbf{E}$ and 2 functors between them: $F : \mathbf{C} \Rightarrow \mathbf{D}$ and $G : \mathbf{D} \Rightarrow \mathbf{E}$ then we can construct a new functor $H : \mathbf{C} \Rightarrow \mathbf{E}$ that is called *functor composition* and denoted as $H = G \circ F$. TBD

3.2 Cat category

The Functor composition is associative by definition. Therefore Identity functor with the associative composition allow us to define a category where other categories are considered as objects and functors as morphisms:

Definition 3.10 (**Cat** category)**.** The category of small categories (see Small category) denoted as **Cat** is the Category where objects are small categories and morphisms are Functors between them.

We can construct an extension of Cartesian product as follows

Definition 3.11 (Category Product)**.** If we have 2 categories $\mathbf{C}$ and $\mathbf{D}$ then we can construct a new category $\mathbf{C} \times \mathbf{D}$ with the following components:

- Objects are the pairs (c, d) where $c \in \text{ob}(\mathbf{C})$ and $d \in \text{ob}(\mathbf{D})$
- Morphisms are the pair (f, g) where $f \in \text{hom}(\mathbf{C})$ and $g \in \text{hom}(\mathbf{D})$
- Composition (Axiom 1.8) is defined as follows $(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$
- Identity is defined as follows: $\mathbf{1}_{C \times D \to C \times D} = (\mathbf{1}_{C \to C}, \mathbf{1}_{D \to D})$

Definition 3.12 (Terminal object in **Cat** category)**.** Let consider Δ_c is a trivial functor from Category $\mathbf{A}$ to category $\mathbf{C}$ such that $\forall a \in \text{ob}(\mathbf{A}) : \Delta_c a = c$ -fixed object in $\mathbf{C}$ and $\forall f \in \text{hom}(\mathbf{A}) : \Delta_c f = \mathbf{1}_{c \to c}$.

The good example can be found in **Hask** category.

Example 3.13 (Terminal object in **Cat** category)**.** [**Hask**]

```haskell
data Const c a = Const c
fmap :: (a -> b) -> Const c a -> Const c b
fmap f (Const c a) = Const c
```

3.3 Bifunctors

Definition 3.14 (Bifunctor)**.** Bifunctor is a Functor whose Domain is a Category Product. I.e. if $\mathbf{C_1}, \mathbf{C_2}, \mathbf{D}$ are 3 categories then the Functor $F : \mathbf{C_1} \times \mathbf{C_2} \Rightarrow \mathbf{D}$ is called *bifunctor*.

Example 3.15 (Bifunctor)**.** **[Set]** Lets A, B, C and D are sets and $f : A \to C, g : B \to D$ are two Functions. Then the Cartesian product with Product of morphisms form a Bifunctor $\times$.

Definition 3.16 (Contravariant functor)**.** If we have categories $\mathbf{C}$ and $\mathbf{D}$ then the Functor $\mathbf{C^{op}} \Rightarrow \mathbf{D}$ is called *contravariant functor*.

Example 3.17 (Contravariant functor)**.** **[Hask]** TBD

```
class Contravariant f where
contramap :: (a -> b) -> f b -> f a
```

Definition 3.18 (Profunctor)**.** If we have a category $\mathbf{C}$ then the Bifunctor $\mathbf{C^{op}} \times \mathbf{C} \Rightarrow \mathbf{C}$ is called *profunctor*.

Example 3.19 (Profunctor)**.** **[Hask]** TBD

```
class Profunctor p where
dimap :: (a' -> a) -> ( b -> b' ) -> p a b -> p a' b'
-- p a b == a -> b
dimap f g h = g . h . f
```

Chapter 4

Natural transformation

Natural transformation is the most important part of the category theory. It provides a possibility to compare Functors via a standard tool.

4.1 Definitions

The natural transformation is not an easy concept compare other ones and requires some additional preparations before we can give the formal definition.

Consider 2 categories $\mathbf{C}, \mathbf{D}$ and 2 Functors $F : \mathbf{C} \Rightarrow \mathbf{D}$ and $G : \mathbf{C} \Rightarrow \mathbf{D}$. If we have an Object $a \in \mathrm{ob}(\mathbf{C})$ then it will be translated by different functors into different objects of category $\mathbf{D}$: $a_F = F(a), a_G = G(a) \in \mathrm{ob}(\mathbf{D})$ (see fig. 4.1). There are 2 options possible

1. There is not any Morphism that connects a_F and a_G.
2. $\exists \alpha_a \in \hom(a_F, a_G) \subset \hom(\mathbf{D})$.

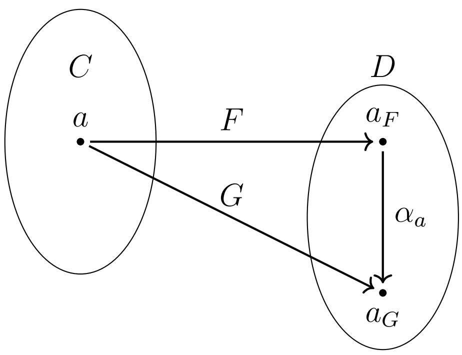

Figure 4.1: Natural transformation: object mapping

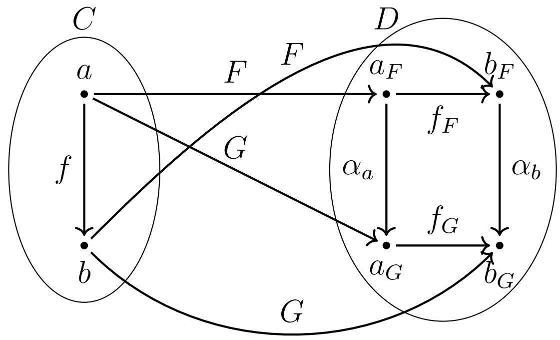

Figure 4.2: Natural transformation: morphisms mapping

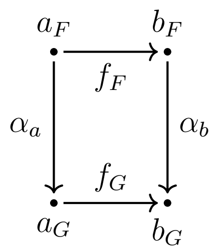

Figure 4.3: Natural transformation: commutative diagram

We can of course to create an artificial morphism that connects the objects but if we use *natural* morphisms [1] then we can get a special characteristic of the considered functors and categories. For instance if we have such morphisms then we can say that the considered functors are related each other. Opposite example if there are no such morphisms then the functors can be considered as unrelated each other.

The functor is not just the object mapping but also the morphisms mapping. If we have 2 objects a and b in the category $\mathbf{C}$ then we potentially can have a morphism $f \in \hom_{\mathbf{C}}(a, b)$. In this case the morphism is mapped by the functors F and G into 2 morphisms f_f and f_G in the category $\mathbf{D}$. As result we have 4 morphisms: $\alpha_a, \alpha_b, f_F, f_G \in \hom(\mathbf{D})$. It is natural to impose additional conditions on the morphisms especially that they form a Commutative diagram (see fig. 4.3):

$$f_f \circ \alpha_b = \alpha_a \circ f_G.$$

[1] the word natural means that already existent morphisms from category $\mathbf{D}$ are used

Definition 4.1 (Natural transformation)**.** Let F and G are 2 Functors from category $\mathbf{C}$ to the category $\mathbf{D}$. The *natural transformation* is a set of Morphisms $\alpha \subset \hom(\mathbf{D})$ which satisfy the following conditions:

- For every Object $a \in \mathrm{ob}(\mathbf{C})$ $\exists \alpha_a \in \hom(a_F, a_G)$ [2] - Morphism in category $\mathbf{D}$. The morphism α_a is called the component of the natural transformation.
- For every morphism $f \in \hom(\mathbf{C})$ that connects 2 objects a and b, i.e. $f \in \hom_{\mathbf{C}}(a, b)$ the corresponding components of the natural transformation $\alpha_a, \alpha_b \in \alpha$ should satisfy the following conditions

$$f_G \circ \alpha_a = \alpha_b \circ f_F, \tag{4.1}$$

  where $f_F = F(f), f_G = G(f)$. In other words the morphisms form a Commutative diagram shown on the fig. 4.3.

We use the following notation (arrow with a dot) for the natural transformation between functors F and G: $\alpha : F \dot{\to} G$.

4.2 Operations with natural transformations

Example 4.2 (**Fun** category)**.** The functors can be considered as objects in a special category **Fun**. The morphisms in the category are Natural transformations.

To define a category we need to define composition operation that satisfied Composition (Axiom 1.8) , identity morphism and verify Associativity (Axiom 1.10) .

For the composition consider 2 Natural transformations α, β and consider how they act on an object $a \in \mathrm{ob}(\mathbf{C})$ (see fig. 4.4). We always can construct the composition $\beta_a \circ \alpha_a$ i.e. we can define the composition of natural transformations α, β as $\beta \circ \alpha = \{\beta_a \circ \alpha_a | a \in \mathrm{ob}(\mathbf{C})\}$.

The natural transformation is not just object mapping but also morphism mapping. We will require that all morphisms shown on fig. 4.5 commute. The composition defined in such way is called Vertical composition.

The functor category between categories $\mathbf{C}$ and $\mathbf{D}$ is denoted as $[\mathbf{C}, \mathbf{D}]$.

Definition 4.3 (Vertical composition)**.** Let F, G, H are functors between categories $\mathbf{C}$ and $\mathbf{D}$. Also we have $\alpha : F \dot{\to} G, \beta : G \dot{\to} H$ - natural transformations. We can compose the α and β as follows

$$\alpha \circ \beta : F \dot{\to} H.$$

[2] $a_F = F(a), a_G = G(a)$

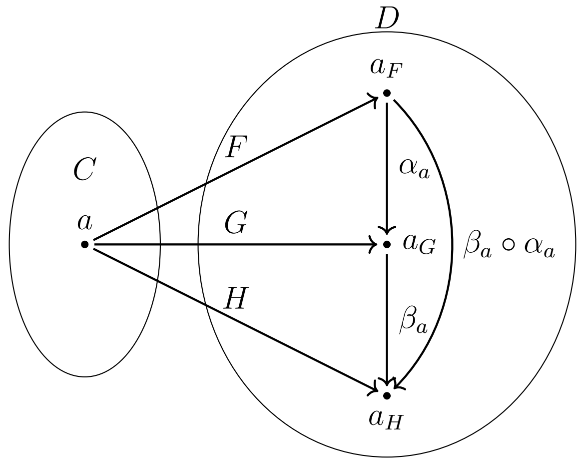

Figure 4.4: Natural transformation vertical composition: object mapping

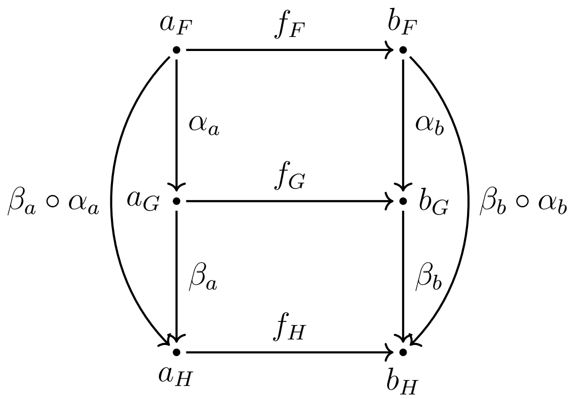

Figure 4.5: Natural transformation vertical composition: morphism mapping - commutative diagram

This composition is called *vertical composition*.

Definition 4.4 (Horizontal composition)**.** If we have 2 pairs of functors. The first one $F, G : \mathbf{C} \to \mathbf{D}$ and another one $J, K : \mathbf{D} \Rightarrow \mathbf{E}$. We also have a natural transformation between each pair: $\alpha : F \dot{\to} G$ for the first one and $\beta : J \dot{\to} K$ for the second one. We can create a new transformation

$$\alpha \star \beta : F \circ J \dot{\to} G \circ K$$

that is called *horizontal composition*. Note that we use a special symbol $\star$ for the composition.

Remark 4.5 (Bifunctor in category of functors)**.** If we have the same pair of functors as in definition 4.4 then we can consider the functors as objects of 3 categories: $\mathcal{A} = [\mathbf{C}, \mathbf{D}], \mathcal{B} = [\mathbf{D}, \mathbf{E}]$ and $\mathcal{C} = [\mathbf{C}, \mathbf{E}]$

Next we want to construct a Bifunctor $\otimes : \mathcal{A} \times \mathcal{B} \Rightarrow \mathcal{C}$ where for each pair of objects $F \in \mathrm{ob}(\mathcal{A}), J \in \mathrm{ob}(\mathcal{B})$ we got another object from $\mathcal{C}$. The used operation is an ordinary functor's composition. I.e.

$$\otimes : F \times G \to F \circ G \in \mathrm{ob}(\mathcal{C}).$$

The bifunctor is not just a map for objects. There is also a map between morphisms. Thus if we have 2 Morphisms: $\alpha : F \to G$ and $\beta : J \to K$ then we can construct the following mapping

$$\otimes : \alpha \times \beta \to \alpha \star \beta \in \hom(\mathcal{C}).$$

As result we have the introduced mapping $\otimes$ as a bifunctor.

Definition 4.6 (Left whiskering)**.** If we have 3 categories $\mathbf{B}, \mathbf{C}, \mathbf{D}$, Functors $F, G : \mathbf{C} \Rightarrow \mathbf{D}$, $H : \mathbf{B} \to \mathbf{C}$ and Natural transformation $\alpha : F \dot{\to} G$ then we can construct a new natural transformations:

$$\alpha H : F \circ H \dot{\to} G \circ H$$

that is called *left whiskering* of functor and natural transformation [8].

Definition 4.7 (Right whiskering)**.** If we have 3 categories $\mathbf{C}, \mathbf{D}, \mathbf{E}$, Functors $F, G : \mathbf{C} \Rightarrow \mathbf{D}$, $H : \mathbf{D} \to \mathbf{E}$ and Natural transformation $\alpha : F \dot{\to} G$ then we can construct a new natural transformations:

$$H\alpha : H \circ F \dot{\to} H \circ G$$

that is called *right whiskering* of functor and natural transformation [8].

Definition 4.8 (Identity natural transformation)**.** If $F : \mathbf{C} \Rightarrow \mathbf{D}$ is a Functor then we can define *identity natural transformation* $\mathbf{1}_{F \xrightarrow{\cdot} F}$ that maps any Object $a \in \mathrm{ob}(\mathbf{C})$ into Identity morphism $\mathbf{1}_{F(a) \to F(a)} \in \mathrm{hom}(\mathbf{D})$.

Remark 4.9 (Whiskering)**.** With Identity natural transformation we can redefine Left whiskering and Right whiskering via Horizontal composition as follows.

For left whiskering:

$$\alpha H = \alpha \star \mathbf{1}_{H \xrightarrow{\cdot} H} \tag{4.2}$$

For right whiskering:

$$H\alpha = \mathbf{1}_{H \xrightarrow{\cdot} H} \star \alpha \tag{4.3}$$

4.3 Polymorphism and natural transformation

Polymorphism plays a certain role in programming languages. Category theory provides several facts about polymorphic functions which are very important.

Definition 4.10 (Parametrically polymorphic function)**.** Polymorphism is parametric if all function instances behave uniformly i.e. have the same realization. The functions which satisfy the parametric polymorphism requirements are parametrically polymorphic.

Definition 4.11 (Ad-hoc polymorphism)**.** Polymorphism is parametric if the function instances can behave differently dependently on the type they are being instantiated with.

Theorem 4.12 (Reynolds)**.** *Parametrically polymorphic functions are Natural transformations*

Proof. TBD □

4.3.1 Hask category

In Haskell most of functions are Parametrically polymorphic functions [3].

Example 4.13 (Parametrically polymorphic function)**.** [**Hask**] Consider the following function

```
safeHead :: [a] -> Maybe a
safeHead [] = Nothing
safeHead (x:xs) = Just x
```

[3] really in the run-time the functions are not Parametrically polymorphic functions

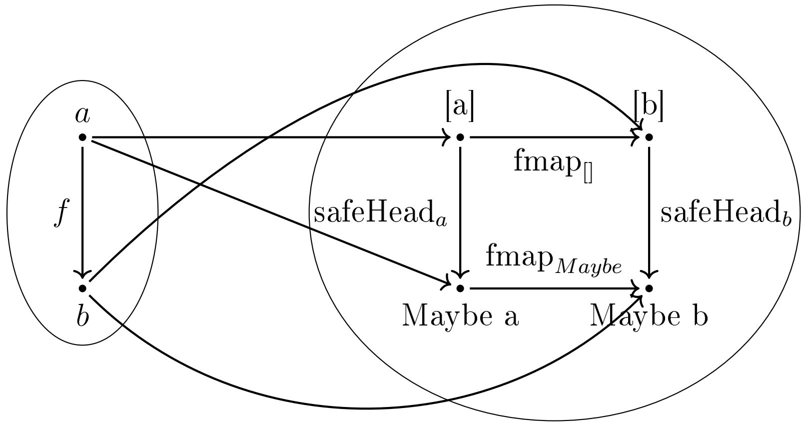

Figure 4.6: Haskell parametric polymorphism as a natural transformation

The function is parametricaly polymorphic and by Reynolds (Theorem 4.12) is Natural transformation (see fig. 4.6).

Therefore from the definition of the natural transformation (4.1) we have `fmap f . safeHead = safeHead . fmap f`. I.e. it does not matter if we initially apply `fmap f` and then `safeHead` to the result or initially `safeHead` and then `fmap f`.

The statement can be verified directly. For empty list we have

```
fmap f . safeHead []
-- equivalent to
fmap f Nothing
-- equivalent to
Nothing
```

from other side

```
safeHead . fmap f []
-- equivalent to
safeHead []
-- equivalent to
Nothing
```

For a non empty list we have

```
fmap f . safeHead (x:xs)
-- equivalent to
fmap f (Just x)
```

```
-- equivalent to
Just (f x)
```

from other side

```
safeHead . fmap f (x:xs)
-- equivalent to
safeHead (f x: fmap f xs )
-- equivalent to
Just ( f x )
```

Using the fact that `fmap f` is an expensive operation if it is applied to the list we can conclude that the second approach is more productive. Such transformation allows compiler to optimize the code. [4]

[4]It is not directly applied to Haskell because it has lazy evaluation that can perform optimization before that one

Chapter 5

Monads

Monads are very important for pure functional programming languages such as Haskell. We will start with Monoid consideration, continue with the formal mathematical definition for monad and will finish with programming languages examples later.

5.1 Monoid in Set category

We are going to consider Monoid in the terms of Set theory and will try to give the definition that is based rather on morphisms then on internal set structure i.e. we will use Categorical approach. Let M is a set and by the monoid definition (definition 2.17) $\forall m_1, m_2 \in M$ we can define a new element of the set $\mu(m_1, m_2) \in M$. Later we will use the following notation for the μ:

$$\mu(m_1, m_2) \equiv m_1 \cdot m_2.$$

If the $(M, \cdot)$ is monoid then the following 2 conditions have to be satisfied. The first one (associativity) declares that $\forall m_1, m_2, m_3 \in M$

$$m_1 \cdot (m_2 \cdot m_3) = (m_1 \cdot m_2) \cdot m_3.$$

The second one (identity presence) says that $\exists e \in M$ such that $\forall m \in M$:

$$m \cdot e = e \cdot m = m. \tag{5.1}$$

Lets start with the first one we can define μ as Morphism in the following way $\mu : M \times M \to M$ where $M \times M$ is Product (Example 2.12) in **Set** category. I.e. $M, M \times M \in \mathrm{ob}(\mathbf{Set})$ and $\mu \in \mathrm{hom}(\mathbf{Set})$. Consider another objects of **Set**: $A = M \times (M \times M)$ and $A' = (M \times M) \times M$. They are not the same but there is a trivial Isomorphism between them $A \cong_\alpha A'$, where

$$\alpha(x, (y, z)) = ((x, y), z).$$

$$\begin{tikzcd}
M \times (M \times M) \arrow[rr, "\alpha"] \arrow[d, "\mathbf{1}_{M\to M} \times \mu"'] & & (M \times M) \times M \arrow[d, "\mu \times \mathbf{1}_{M\to M}"] \\
M \times M \arrow[r, "\mu"] & M & M \times M \arrow[l, "\mu"']
\end{tikzcd}$$

Figure 5.1: Commutative diagram for $\mu \circ (\mu \times \mathbf{1}_{M\to M}) = \mu \circ (\mathbf{1}_{M\to M} \times \mu) \circ \alpha$.

Consider the action of Product of morphisms $\mathbf{1}_{M\to M} \times \mu$ on A:

$$\mathbf{1}_{M\to M} \times \mu\,(x, (y, z)) = (\mathbf{1}_{M\to M}(x), \mu\,(y, z)) = (x, y \cdot z) \in M \times M$$

i.e. $\mathbf{1}_{M\to M} \times \mu : M \times (M \times M) \to M \times M$. If we act μ on the result we will get:

$$\begin{aligned}\mu\,(\mathbf{1}_{M\to M} \times \mu\,(x, (y, z))) = (\mathbf{1}_{M\to M}(x), \mu\,(y, z)) = \\ = \mu\,(x, y \cdot z) = x \cdot (y \cdot z) \in M,\end{aligned}$$

i.e. $\mu \circ (\mathbf{1}_{M\to M} \times \mu) : M \times (M \times M) \to M$.

For A' we have the following one:

$$\mu \circ (\mu \times \mathbf{1}_{M\to M})\,((x, y)\,, z) = \mu\,(x \cdot y, z) = (x \cdot y) \cdot z.$$

Monoid associativity requires

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

i.e. the corresponding morphisms commute:

$$\mu \circ (\mu \times \mathbf{1}_{M\to M}) = \mu \circ (\mathbf{1}_{M\to M} \times \mu) \circ \alpha.$$

This corresponding diagram is shown in fig. 5.1.

Very often the isomorphism α is omitted i.e.

$$M \times (M \times M) = (M \times M) \times M = M^3$$

and the morphism equality is written as follow

$$\mu \circ (\mu \times \mathbf{1}_{M\to M}) = \mu \circ (\mathbf{1}_{M\to M} \times \mu)$$

$$
\begin{tikzcd}
M^3 \arrow[r, "\mathbf{1}_{M\to M} \times \mu"] \arrow[d, "\mu \times \mathbf{1}_{M\to M}"'] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"] & M
\end{tikzcd}
$$

Figure 5.2: Commutative diagram for $\mu \circ (\mu \times \mathbf{1}_{M\to M}) = \mu \circ (\mathbf{1}_{M\to M} \times \mu)$.

The corresponding commutative diagram is shown in fig. 5.2.

For (5.1) consider a morphism η from Singleton [1] $I = \{0\}$ to a special element $e \in M$ such that $\forall m \in M : e \cdot m = m \cdot e = m$. I.e. $\eta : I \to M$ and $e = \eta(0)$. Consider 2 sets $B = I \times M$ and $B' = M \times I$. We have 2 Isomorphisms: $B \cong_\lambda M$ and $B' \cong_\rho M$ where

$$\lambda(m) = 0 \times m$$

and

$$\rho(m) = m \times 0.$$

If we apply Product of morphisms $\eta\times\mu$ and $\mu\times\eta$ on B and B' respectively then we get

$$
\begin{aligned}
\eta \times \mathbf{1}_{M\to M} (0 \times m) = e \times m, \\
\mathbf{1}_{M\to M} \times \eta\, (m \times 0) = m \times e.
\end{aligned}
$$

If we apply μ on the result then we get

$$
\begin{aligned}
\mu\,(\eta \times \mathbf{1}_{M\to M} (0 \times m)) = \mu\,(e \times m) = e \cdot m, \\
\mu\,(\mathbf{1}_{M\to M} \times \eta\,(m \times 0)) = \mu\,(m \times e) = m \cdot e.
\end{aligned}
$$

The (5.1) leads to the following equation for morphisms

$$\mu \circ (\eta \times \mathbf{1}_{M\to M}) \circ \rho = \mu \circ (\mathbf{1}_{M\to M} \times \mu) \circ \lambda = \mathbf{1}_{M\to M}$$

or the commutative diagram show on fig. 5.3.

Before given a formal definition lets look at the operations were used for the construction. The first one is the product of 2 objects:

$$M \times M.$$

[1] It also is called [4] as a one point set

$$\begin{tikzcd}
M \arrow[r, "\lambda"'] \arrow[d, "\rho"] \arrow[dr, equal] & M \times M \arrow[d, "\mu"'] \\
M \times M \arrow[r, "\mu"] & M
\end{tikzcd}$$

Figure 5.3: Commutative diagram for $\mu \circ (\eta \times \mathbf{1}_{M \to M}) \circ \lambda = \mu \circ (\mathbf{1}_{M \to M} \times \mu) \circ \rho = \mathbf{1}_{M \to M}$.

We also have 2 pairs of morphisms:

$$\begin{aligned} \mu &: M \times M \to M, \\ \mathbf{1}_{M \to M} &: M \to M. \end{aligned}$$

and

$$\begin{aligned} \eta &: I \to M, \\ \mathbf{1}_{M \to M} &: M \to M. \end{aligned}$$

The pairs can be combined into one using Product of morphisms as follows:

$$\begin{aligned} \mu \times \mathbf{1}_{M \to M} &: (M \times M) \times M \to M \times M, \\ \mathbf{1}_{M \to M} \times \mu &: M \times (M \times M) \to M \times M \end{aligned}$$

and

$$\begin{aligned} \eta \times \mathbf{1}_{M \to M} &: I \times M \to M \times M, \\ \mathbf{1}_{M \to M} \times \eta &: M \times I \to M \times M. \end{aligned}$$

The same structure [2] is used by Functor and especially by Bifunctor (Example 3.15) .

Now we are ready to provide the monoid definition in the terms of morphisms.

Definition 5.1 (Monoid)**.** Consider **Set** category $\mathbf{C}$ with a Singleton $t \in \mathrm{ob}(\mathbf{C})$. The Cartesian product with Product of morphisms form a Bifunctor $\times$ (see example 3.15). The object $m \in \mathrm{ob}(\mathbf{C})$ is called *monoid* if the following conditions satisfied:

[2] not only objects mapping but also morphisms mapping

1. there is a Morphism $\mu : m \times m \to m$ in the category
2. there is another morphism $\eta : t \to m$
3. the morphisms satisfy the following conditions:

$$\mu \circ (\mu \times \mathbf{1}_{M \to M}) = \mu \circ (\mathbf{1}_{M \to M} \times \mu) \circ \alpha,$$
$$\mu \circ (\eta \times \mathbf{1}_{M \to M}) \circ \lambda = \mu \circ (\mathbf{1}_{M \to M} \times \mu) \circ \rho = \mathbf{1}_{M \to M} \tag{5.2}$$

   where α (associator) is an Isomorphism between $m \times (m \times m)$ and $(m \times m) \times m$. λ, ρ are another isomorphisms:

$$m \cong_{\lambda} t \times m$$

   and

$$m \cong_{\rho} m \times t$$

5.2 Monoidal category

As we saw in the categorical definition for monoid (see definition 5.1) the category **C** should satisfy several conditions to have an object as monoid. Lets formalise the conditions.

Definition 5.2 (Monoidal category)**.** A category **C** is called *monoidal category* if it is equipped with a Monoid structure i.e. there are

- Bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \Rightarrow \mathbf{C}$ called *monoidal product*
- an Object e called unit object or identity object

The elements should satisfy (up to Isomorphism) several conditions: associativity:

$$a \otimes (b \otimes c) \cong_{\alpha} (a \otimes b) \otimes c,$$

where α is called associator. e can be treated as left and right identity:

$$a \cong_{\lambda} e \otimes a,$$
$$a \cong_{\rho} a \otimes e,$$

where λ, ρ are called as left and right unitors respectively.

In the **Set** category we have $\times$ as the monoidal product (see example 3.15). There is a morphism η from terminal object t to e [3] (see definition 5.1).

Definition 5.3 (Strict monoidal category)**.** A Monoidal category $\mathbf{C}$ is said to be strict if the associator, left unitor and right unitors are all identity morphisms i.e.

$$\alpha = \lambda = \rho = \mathbf{1}_{C \to C}.$$

Remark 5.4 (Monoidal product)**.** The monoidal product is a binary operation that specifies the exact monoidal structure. Often it is called as *tensor product* but we will avoid the naming because it is not always the same as the Tensor product introduced for Hilbert spaces. We also note that the monoidal product is a Bifunctor.

5.3 Category of endofunctors

The **Fun** category (Example 4.2) is an example of a category. We can apply additional limitation and consider only Endofunctors i.e. we will look at the category $[\mathbf{C}, \mathbf{C}]$ - category of functors from category $\mathbf{C}$ to the same category. One of the most popular math definition of a monad is the following: "All told, a monad in X is just a monoid in the category of endofunctors of X"[4]. Later we will give an explanation for that one.

We start with the formal definition of category of endofunctors and a tensor product in the category

Definition 5.5 (Category of endofunctors)**.** Let $\mathbf{C}$ is a category, then $[\mathbf{C}, \mathbf{C}]$ - category of functors from category $\mathbf{C}$ to the same category is called the category of endofunctors.

The monoidal product in the category is the functor composition.

Definition 5.6 (Monad)**.** The monad M is an Endofunctor with 2 Natural transformations:

1. $\eta : \mathbf{1}_{\mathbf{C} \Rightarrow \mathbf{C}} \dot{\to} M$
2. $\mu : M \circ M \dot{\to} M$

where $\mathbf{1}_{\mathbf{C} \Rightarrow \mathbf{C}}$ is Identity functor.

The η, μ should satisfy the following conditions:

$$\begin{aligned} \mu \circ M\mu &= \mu \circ \mu M, \\ \mu \circ M\eta &= \mu \circ \eta M = \mathbf{1}_{M \dot{\to} M}, \end{aligned} \tag{5.3}$$

where $M\mu, M\eta$ - Right whiskerings, $\mu M, \eta M$ - Left whiskerings, $\mathbf{1}_{M \dot{\to} M}$ - Identity natural transformation for M. Vertical composition is used in the equations.

The monad will be denoted later as $\langle M, \mu, \eta \rangle$.

Lets look at the requirements (5.3) more closely. Notice that the functor composition is associative:

$$M \circ (M \circ M) = (M \circ M) \circ M = M^3.$$

Secondly all rewrite it with (4.2) and (4.3) as follows

$$\begin{aligned} \mu \circ (\mathbf{1}_{M \dot{\to} M} \star \mu) = \mu \circ (\mu \star \mathbf{1}_{M \dot{\to} M}), \\ \mu \circ (\mathbf{1}_{M \dot{\to} M} \star \eta) = \mu \circ (\eta \star \mathbf{1}_{M \dot{\to} M}) = \mathbf{1}_{M \dot{\to} M}. \end{aligned} \tag{5.4}$$

Thus we can notice that the pair of operations (composition $\circ$ and Horizontal composition $\star$) forms the bifunctor (see Bifunctor in category of functors (Remark 4.5)).

The morphism $\mathbf{1}_{M \dot{\to} M} \star \mu$ acts on $M \circ (M \circ M)$ as

$$\mathbf{1}_{M \dot{\to} M} \star \mu : M \circ (M \circ M) \dot{\to} M \circ (M \otimes M)$$

thus

$$\mu \circ (\mathbf{1}_{M \dot{\to} M} \star \mu) : M \circ (M \circ M) \dot{\to} M \otimes (M \otimes M).$$

Similarly

$$\mu \circ (\mu \star \mathbf{1}_{M \dot{\to} M}) : (M \circ M) \circ M \dot{\to} (M \otimes M) \otimes M.$$

I.e. the both morphisms start at the same object M^3 and finish also at the same point. The equality

$$\mu \circ (\mathbf{1}_{M \dot{\to} M} \star \mu) = \mu \circ (\mu \star \mathbf{1}_{M \dot{\to} M}) \tag{5.5}$$

is similar to the conditions on the fig. 5.2 and can be rewritten on fig. 5.4. Thus if we compare (5.5) and (5.2) then we can say that they are same if we replace $\star$ sign with $\times$ one. I.e. in the case we can say that the monad looks like a Monoid.

For the identity element consider the same trick: replace in (5.2) tensor product $\times$ with Horizontal composition $\star$ and identity morphism $\mathbf{1}_{M \to M}, \rho, \lambda$ with identity natural transformation $\mathbf{1}_{M \dot{\to} M}$. The result will e the following:

$$\mu \circ (\eta \times \mathbf{1}_{M \to M}) \circ \lambda = \mu \circ (\mathbf{1}_{M \to M} \times \mu) \circ \rho = \mathbf{1}_{M \to M}$$

will be replaced with

$$\mu \circ (\eta \star \mathbf{1}_{M \dot{\to} M}) = \mu \circ (\mathbf{1}_{M \dot{\to} M} \star \mu) = \mathbf{1}_{M \dot{\to} M}$$

that is the exact we want to get (see second equation of (5.4)).

$$\begin{tikzcd}
M^3 \arrow[r, "\mu \star \mathbf{1}_{M \dot{\to} M}"] \arrow[d, "\mathbf{1}_{M \dot{\to} M} \star \mu"'] & M^2 \arrow[d, "\mu"] \\
M^2 \arrow[r, "\mu"] & M
\end{tikzcd}$$

Figure 5.4: Monad as monoid in the category of endofunctors.

5.4 Kleisli category. Monads in programming languages

Definition 5.7 (Kleisli category)**.** Let $\mathbf{C}$ is a category, M is an Endofunctor and $\langle M, \mu, \eta \rangle$ is a Monad. Then we can construct a new category $\mathbf{C_M}$ that is called as *Kleisli category* as follows:

$$\begin{gathered}
\mathrm{ob}(\mathbf{C_M}) = \mathrm{ob}(\mathbf{C}), \\
\hom_{\mathbf{C_M}}(a, b) = \hom_{\mathbf{C}}(a, M(b))
\end{gathered}$$

i.e. objects of categories $\mathbf{C}$ and $\mathbf{C_M}$ are the same but morphisms from $\mathbf{C_M}$ form a subset of morphisms $\mathbf{C_M}$: $\hom(\mathbf{C_M}) \subset \hom(\mathbf{C})$.

5.4.1 Programming languages

Kleisli category widely spread in programming especially it provides good description for different types of computations, for instance [6, 5]

- **Partiality** i.e. then a function not defined for each input, for instance the following expression is undefined (or partially defined) for $x = 0$: $f(x) = \frac{1}{x}$
- **Non-Determinism** i.e. then multiply output are possible
- **Side-effects** i.e. TBD
- **Exception** i.e. TBD
- **Continuation** i.e. TBD

- **Interactive input** i.e. TBD
- **Interactive output** i.e. TBD

TBD

5.5 Examples

5.5.1 Programming languages

Haskell

Example 5.8 (Monad). **[Hask]** In Haskell monad can be defined from Functor (Example 3.3) as follows [3]

```
class Functor m => Monad m where
    return :: a -> m a
    (>>=)  :: m a -> (a -> m b) -> m b
```

To show how this one can be get we can start from a definition that is similar to the math definition:

```
class Functor m => Monad m where
    return :: a -> m a
    join  :: m (m a) -> m a
```

where `return` can be treated as η and `join` as μ. In the case the bind operator >>= can be implemented as follows

```
(>>=)  :: m a -> (a -> m b) -> m b
ma >>= f = join ( f ma )
```

Example 5.9 (Maybe monad). **[Hask]** Consider the monad definition that is most close to math definition first:

```
class Functor m => Monad m where
    return :: a -> m a
    join  :: m (m a) -> m a
```

the Maybe monad can be implemented as follows

```
instance Monad Maybe where
  return = Just
  join Just( Just x) = Just x
  join _ = Nothing
```

TBD

[3] real definition is quite different from the presented one

C++

TBD

Scala

Example 5.10 (Monad)**.** **[Scala]** The monad concept is Scala is more close to formal math definition for Monad. It can be defined as follows [4]

```
trait M[A] {
  def flatMap[B](f: A => M[B]): M[B]
}

def unit[A](x: A): M[A]
```

I.e. `flatMap` can be considered as μ and `unit` as η.

TBD

5.5.2 Quantum mechanics

Definition 5.11 (Tensor product)**.** TBD

The tensor product in quantum mechanics is used for representing a system that consists of multiple systems. For instance if we have an interaction between an 2 level atom (a is excited state b as a ground state) and one mode light then the atom has its own Hilber space $\mathcal{H}_{at}$ with $|a\rangle$ and $|b\rangle$ as basis vectors. Light also has its own Hilber space $\mathcal{H}_f$ with Fock state $\{|n\rangle\}$ as the basis. [5] The result system that describes both atom and light is represented as the tensor product $\mathcal{H}_{at} \otimes \mathcal{H}_f$.

The morphisms of **FdHilb** category have a connection with Tensor product. Consider the so called Hilbert-Schmidt correspondence for finite dimensional Hilbert spaces i.e. for given $\mathcal{A}$ and $\mathcal{B}$ there is a natural isomorphism between the tensor product and linear maps (aka morphisms) between $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}^* \otimes \mathcal{B} \cong \hom(\mathcal{A}, \mathcal{B})$$

where $\mathcal{A}^*$ - Dual space.

TBD

[4] real definition is quite different from the presented one

[5] Really the $\mathcal{H}_f$ is infinite dimensional Hilber space and seems to be out of our assumption about **FdHilb** category as a collection of finite dimensional Hilber spaces only.

Chapter 6

Yoneda's lemma

TBD

6.1 Examples

6.1.1 Quantum mechanics

Flori interpretation of quantum mechanics

TBD

Index

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