

# Category Theory

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# Introduction

There is an introduction to Category Theory.



# Chapter 1

## Base definitions

### 1.1 Definitions

#### 1.1.1 Object

**Definition 1.1** (Class). A class is a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share.

**Definition 1.2** (Object). In category theory object is considered as something that does not have internal structure (aka point) but has a property that makes different objects belong to the same [Class](#)

**Remark 1.3** (Class of Objects). The [Class](#) of [Objects](#) will be marked as  $\text{ob}(C)$

#### 1.1.2 Morphism

Morphism is a kind of relation between 2 [Objects](#).

**Definition 1.4** (Morphism). A relation between two [Objects](#)  $a$  and  $b$

$$f_{ab} : a \rightarrow b$$

is called *morphism*. Morphism assumes a direction i.e. one [Object](#) ( $a$ ) is called *source* and another one ( $b$ ) *target*.

[Morphisms](#) have several properties. <sup>1</sup>

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<sup>1</sup>The properties don't have any proof and postulated as axioms

**Property 1.5** (Composition). *If we have 3 **Objects**  $a, b$  and  $c$  and 2 **Morphisms***

$$f_{ab} : a \rightarrow b$$

*and*

$$f_{bc} : b \rightarrow c$$

*then there exists **Morphism***

$$f_{ac} : a \rightarrow c$$

*such that*

$$f_{ac} = f_{bc} \circ f_{ab}$$

**Remark 1.6** (Composition). The equation

$$f_{ac} = f_{bc} \circ f_{ab}$$

means that we apply  $f_{ab}$  first and then we apply  $f_{bc}$  to the result of the application i.e. if our objects are sets and  $x \in a$  then

$$f_{ac}(x) = f_{bc}(f_{ab}(x)),$$

where  $f_{ab}(x) \in b$ .

**Property 1.7** (Associativity). *The **Morphisms Composition** (Property 1.5) should follow associativity property:*

$$f_{ce} \circ (f_{bc} \circ f_{ab}) = (f_{ce} \circ f_{bc}) \circ f_{ab} = f_{ce} \circ f_{bc} \circ f_{ab}.$$

**Definition 1.8** (Identity morphism). For every **Object**  $a$  we define a special **Morphism**  $\mathbf{1}_a : a \rightarrow a$  with the following properties:  $\forall f_{ab} : a \rightarrow b$

$$\mathbf{1}_a \circ f_{ab} = f_{ab}$$

and  $\forall f_{ba} : b \rightarrow a$

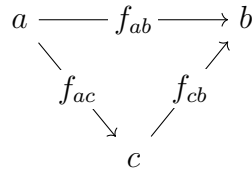
$$f_{ba} \circ \mathbf{1}_a = f_{ba}.$$

This morphism is called *identity morphism*.

**Definition 1.9** (Commutative diagram). A commutative diagram is a diagram of **Objects** (also known as vertices) and **Morphisms** (also known as arrows or edges) such that all directed paths in the diagram with the same start and endpoints lead to the same result by composition

The following diagram commutes if  $f_{ab} = f_{cb} \circ f_{ac}$ .





**Remark 1.10** (Class of Morphisms). The [Class](#) of [Morphisms](#) will be marked as  $\text{hom}(C)$

**Definition 1.11** (Monomorphism). If  $\forall g_1, g_2$  the equation

$$f \circ g_1 = f \circ g_2$$

leads to

$$g_1 = g_2$$

then  $f$  is called *monomorphism*.

**Definition 1.12** (Epimorphism). If  $\forall g_1, g_2$  the equation

$$g_1 \circ f = g_2 \circ f$$

leads to

$$g_1 = g_2$$

then  $f$  is called *epimorphism*.

### 1.1.3 Category

**Definition 1.13** (Category). A category  $\mathbf{C}$  consists of

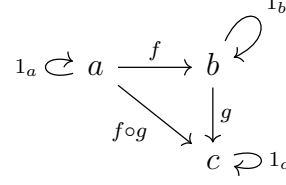
- [Class](#) of [Objects](#)  $\text{ob}(C)$
- [Class](#) of [Morphisms](#)  $\text{hom}(C)$  defined for  $\text{ob}(C)$ , i.e. each morphism  $f_{ab}$  from  $\text{hom}(C)$  has both source  $a$  and target  $b$  from  $\text{ob}(C)$

For any [Object](#)  $a$  there should be unique [Identity morphism](#)  $1_a$ . Any morphism should satisfy [Composition](#) ([Property 1.5](#)) and [Associativity](#) ([Property 1.7](#)) properties. See [fig. 1.1](#)

## 1.2 Examples

There are several examples of categories that will also be used later

Figure 1.1: Category example



### 1.2.1 Set category

**Definition 1.14** (Set). Set is a collection of distinct object. The objects are called the elements of the set.

**Definition 1.15** (Function). If  $A$  and  $B$  are 2 **Sets** then a subset of  $A \times B$  is called function  $f$  between the 2 sets, i.e.  $f \subset A \times B$ .

**Example 1.16** (Set category). In the set category we consider a **Set** of **Sets** where **Objects** are the **Sets** and **Morphisms** are **Functions** between the sets.

The **Identity morphism** is trivial function such that  $\forall x \in X : \mathbf{1}_X(x) = x$ .

**Remark 1.17** (Set vs Category). There is an interesting relation between sets and categories. In both we consider objects(sets) and relations between them(morphisms/functions).

In the set theory we can get info about functions by looking inside the objects(sets) aka use “microscope” [1]

Contrary in the category theory we initially don’t have info about object internal structure but can get it using the relation between the objects i.e. using **Morphisms**. In other words we can use “telescope” [1] there.

**Definition 1.18** (Domain). Given a function  $f : X \rightarrow Y$ , the set  $X$  is the domain.

**Definition 1.19** (Codomain). Given a function  $f : X \rightarrow Y$ , the set  $Y$  is the codomain.

**Definition 1.20** (Surjection). The function  $f : X \rightarrow Y$  is surjective (or onto) if  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$  (see figs. 1.2 and 1.3).

**Remark 1.21** (Surjection vs Epimorphism). **Surjection** and **Epimorphism** are related each other. Consider a non-surjective function  $f : X \rightarrow Y' \subset Y$  (see fig. 1.4). One can conclude that there is not an **Epimorphism** because  $\exists g_1 : Y' \rightarrow Y'$  and  $g_2 : Y \rightarrow Y$  such that  $g_1 \neq g_2$  because they operates on different **Domains** but from other hand  $g_1(Y') = g_2(Y')$ . For instance we can

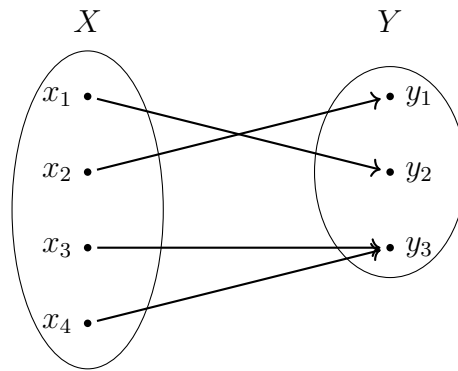


Figure 1.2: A surjective (non-injective) function from domain  $X$  to codomain  $Y$

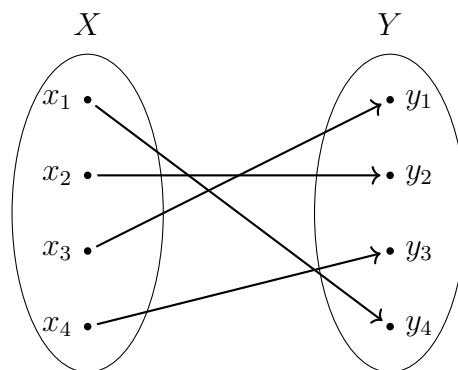


Figure 1.3: An injective and surjective function (bijection)

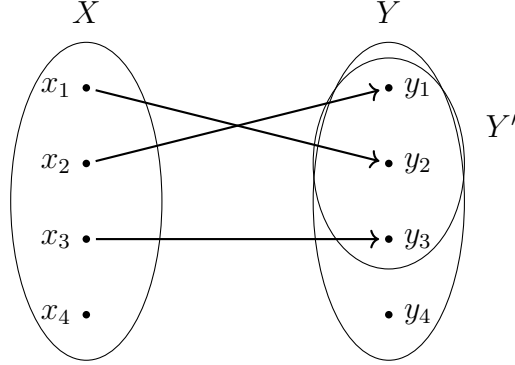


Figure 1.4: A non-surjective function  $f$  from domain  $X$  to codomain  $Y' \subset Y$ .  $\exists g_1 : Y' \rightarrow Y', g_2 : Y \rightarrow Y$  such that  $g_1(Y') = g_2(Y')$ , but as soon as  $Y' \neq Y$  we have  $g_1 \neq g_2$ . Using the fact that  $Y'$  is codomain of  $f$  we got  $g_1 \circ f = g_2 \circ f$ . I.e. the function  $f$  is not epimorphism.

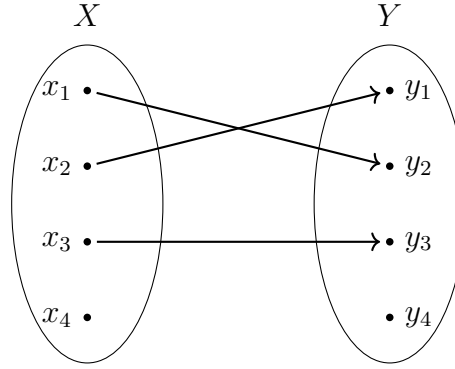


Figure 1.5: A injective (non-surjective) function from domain  $X$  to codomain  $Y$

choose  $g_1 = \mathbf{1}_{Y'}, g_2 = \mathbf{1}_Y$ . As soon as  $Y'$  is **Codomain** of  $f$  we always have  $g_1(f(X)) = g_2(f(X))$ .

As result we can say that an **Surjection** is a **Epimorphism** in **Set** category. Moreover there is a proof [3] of that fact.

**Definition 1.22** (Injection). The function  $f : X \rightarrow Y$  is injective (or one-to-one function) if  $\forall x_1, x_2 \in X$ , such that  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$  (see figs. 1.3 and 1.5).

**Remark 1.23** (Injection vs Monomorphism). **Injection** and **Monomorphism** are related each other. Consider a non-injective function  $f : X \rightarrow Y$  (see fig. 1.6). One can conclude that it is not monomorphism because  $\exists g_1, g_2$  such that  $g_1 \neq g_2$  and  $f(g_1(a_1)) = y_3 = f(g_2(b_1))$ .

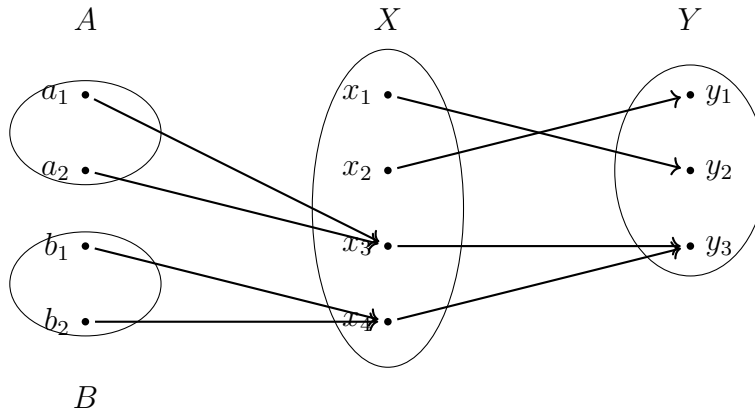


Figure 1.6: A non-injective function  $f$  from domain  $X$  to codomain  $Y$ .  $\exists g_1 : A \rightarrow X, g_2 : B \rightarrow X$  such that  $g_1 \neq g_2$  but  $f \circ g_1 = f \circ g_2$ . I.e. the function  $f$  is not monomorphism.

As result we can say that an **Injection** is a **Monomorphism** in **Set** category. Moreover there is a proof [2] of that fact.

### 1.2.2 Hask category

**Example 1.24** (Hask category). *Types in Haskell are considered as **Objects** Functions are considered as **Morphisms***

*For instance consider the function `even` that converts `Int` type into `Bool`.*

```
isEven :: Int -> Bool
isEven x = x `mod` 2 == 0
```

*There is also **Identity morphism** that is defined as follows*

```
id :: a -> a
id x = x
```

*If we have an additional function*

```
stringLength :: String -> Int
stringLength x = length x
```

*then we can create a **Composition** (Property 1.5)*

```
isStringLengthEven :: String -> Bool
isStringLengthEven = isEven . stringLength
```

*If we consider pure (without effects) functions then **Associativity** (Property 1.7) is also satisfied.*



# Chapter 2

## Objects and morphisms

### 2.1 Equality

#### 2.1.1 Equality of objects

via unique isomorphism

#### 2.1.2 Equality of morphisms

TBD

### 2.2 Initial and terminal objects

TBD

### 2.3 Product and sum

TBD

### 2.4 Examples

#### 2.4.1 Set category

TBD

#### 2.4.2 Hask category

TBD





# Chapter 3

## Functors

TBD



# Chapter 4

## Monads

TBD



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