

# Category Theory

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# Introduction

There is an introduction to Category Theory.



# Chapter 1

## Base definitions

### 1.1 Definitions

#### 1.1.1 Object

**Definition 1.1** (Class). A class is a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share.

**Definition 1.2** (Object). In category theory object is considered as something that does not have internal structure (aka point) but has a property that makes different objects belong to the same [Class](#)

**Remark 1.3** (Class of Objects). The [Class](#) of [Objects](#) will be marked as  $\text{ob}(C)$  (see fig. 1.1).

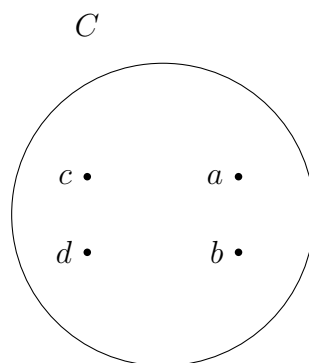


Figure 1.1: Class of objects  $\text{ob}(C) = \{a, b, c, d\}$

### 1.1.2 Morphism

Morphism is a kind of relation between 2 **Objects**.

**Definition 1.4** (Morphism). A relation between two **Objects**  $a$  and  $b$

$$f_{ab} : a \rightarrow b$$

is called *morphism*. Morphism assumes a direction i.e. one **Object** ( $a$ ) is called *source* and another one ( $b$ ) *target*.

**Morphisms** have several properties. <sup>1</sup>

**Property 1.5** (Composition). If we have 3 **Objects**  $a, b$  and  $c$  and 2 **Morphisms**

$$f_{ab} : a \rightarrow b$$

and

$$f_{bc} : b \rightarrow c$$

then there exists **Morphism**

$$f_{ac} : a \rightarrow c$$

such that

$$f_{ac} = f_{bc} \circ f_{ab}$$

**Remark 1.6** (Composition). The equation

$$f_{ac} = f_{bc} \circ f_{ab}$$

means that we apply  $f_{ab}$  first and then we apply  $f_{bc}$  to the result of the application i.e. if our objects are sets and  $x \in a$  then

$$f_{ac}(x) = f_{bc}(f_{ab}(x)),$$

where  $f_{ab}(x) \in b$ .

**Property 1.7** (Associativity). The **Morphisms Composition** (*Property 1.5*) should follow associativity property:

$$f_{ce} \circ (f_{bc} \circ f_{ab}) = (f_{ce} \circ f_{bc}) \circ f_{ab} = f_{ce} \circ f_{bc} \circ f_{ab}.$$

---

<sup>1</sup>The properties don't have any proof and postulated as axioms





Figure 1.2: Class of morphisms  $\text{hom } \text{ob}(C) = \{f, g, h\}$ , where  $h = f \circ g$

**Definition 1.8** (Identity morphism). For every **Object**  $a$  we define a special **Morphism**  $1_a : a \rightarrow a$  with the following properties:  $\forall f_{ab} : a \rightarrow b$

$$1_a \circ f_{ab} = f_{ab} \quad (1.1)$$

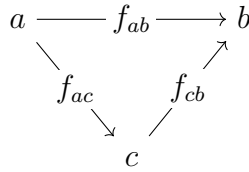
and  $\forall f_{ba} : b \rightarrow a$

$$f_{ba} \circ 1_a = f_{ba}. \quad (1.2)$$

This morphism is called *identity morphism*.

**Definition 1.9** (Commutative diagram). A commutative diagram is a diagram of **Objects** (also known as vertices) and **Morphisms** (also known as arrows or edges) such that all directed paths in the diagram with the same start and endpoints lead to the same result by composition

The following diagram commutes if  $f_{ab} = f_{cb} \circ f_{ac}$ .



**Remark 1.10** (Class of Morphisms). The **Class** of **Morphisms** will be marked as  $\text{hom}(C)$  (see fig. 1.2)

**Definition 1.11** (Monomorphism). If  $\forall g_1, g_2$  the equation

$$f \circ g_1 = f \circ g_2$$

leads to

$$g_1 = g_2$$

then  $f$  is called *monomorphism*.

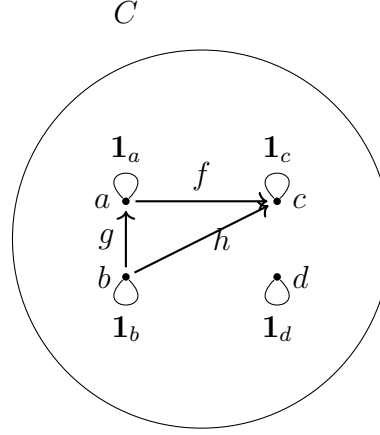


Figure 1.3: Category  $C$ . It consists of 4 objects  $\text{ob}(C) = \{a, b, c, d\}$  and 7 morphisms  $\text{ob}(C) = \{f, g, h = f \circ g, 1_a, 1_b, 1_c, 1_d\}$

**Definition 1.12** (Epimorphism). If  $\forall g_1, g_2$  the equation

$$g_1 \circ f = g_2 \circ f$$

leads to

$$g_1 = g_2$$

then  $f$  is called *epimorphism*.

### 1.1.3 Category

**Definition 1.13** (Category). A category  $\mathbf{C}$  consists of

- **Class** of **Objects**  $\text{ob}(C)$
- **Class** of **Morphisms**  $\text{hom}(C)$  defined for  $\text{ob}(C)$ , i.e. each morphism  $f_{ab}$  from  $\text{hom}(C)$  has both source  $a$  and target  $b$  from  $\text{ob}(C)$

For any **Object**  $a$  there should be unique **Identity morphism**  $1_a$ . Any morphism should satisfy **Composition** (**Property 1.5**) and **Associativity** (**Property 1.7**) properties. See fig. 1.3

## 1.2 Examples

There are several examples of categories that will also be used later

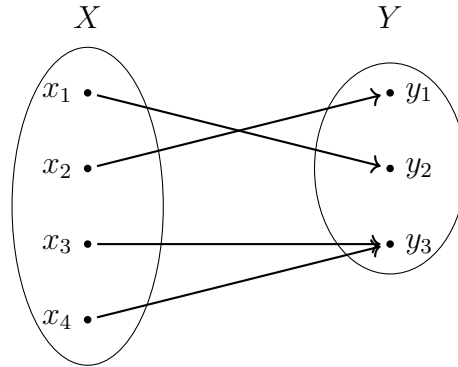


Figure 1.4: A surjective (non-injective) function from domain  $X$  to codomain  $Y$

### 1.2.1 Set category

**Definition 1.14** (Set). Set is a collection of distinct object. The objects are called the elements of the set.

**Definition 1.15** (Function). If  $A$  and  $B$  are 2 **Sets** then a subset of  $A \times B$  is called function  $f$  between the 2 sets, i.e.  $f \subset A \times B$ .

**Example 1.16** (Set category). In the set category we consider a **Set** of **Sets** where **Objects** are the **Sets** and **Morphisms** are **Functions** between the sets.

The **Identity morphism** is trivial function such that  $\forall x \in X : \mathbf{1}_X(x) = x$ .

**Remark 1.17** (Set vs Category). There is an interesting relation between sets and categories. In both we consider objects(sets) and relations between them(morphisms/functions).

In the set theory we can get info about functions by looking inside the objects(sets) aka use “microscope” [1]

Contrary in the category theory we initially don’t have info about object internal structure but can get it using the relation between the objects i.e. using **Morphisms**. In other words we can use “telescope” [1] there.

**Definition 1.18** (Domain). Given a function  $f : X \rightarrow Y$ , the set  $X$  is the domain.

**Definition 1.19** (Codomain). Given a function  $f : X \rightarrow Y$ , the set  $Y$  is the codomain.

**Definition 1.20** (Surjection). The function  $f : X \rightarrow Y$  is surjective (or onto) if  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$  (see figs. 1.4 and 1.8).

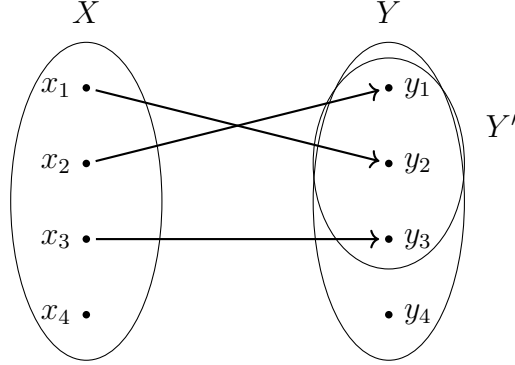


Figure 1.5: A non-surjective function  $f$  from domain  $X$  to codomain  $Y' \subset Y$ .  $\exists g_1 : Y' \rightarrow Y', g_2 : Y \rightarrow Y$  such that  $g_1(Y') = g_2(Y')$ , but as soon as  $Y' \neq Y$  we have  $g_1 \neq g_2$ . Using the fact that  $Y'$  is codomain of  $f$  we got  $g_1 \circ f = g_2 \circ f$ . I.e. the function  $f$  is not epimorphism.

**Remark 1.21** (Surjection vs Epimorphism). [Surjection](#) and [Epimorphism](#) are related each other. Consider a non-surjective function  $f : X \rightarrow Y' \subset Y$  (see fig. 1.5). One can conclude that there is not an [Epimorphism](#) because  $\exists g_1 : Y' \rightarrow Y'$  and  $g_2 : Y \rightarrow Y$  such that  $g_1 \neq g_2$  because they operates on different [Domains](#) but from other hand  $g_1(Y') = g_2(Y')$ . For instance we can choose  $g_1 = 1_{Y'}, g_2 = 1_Y$ . As soon as  $Y'$  is [Codomain](#) of  $f$  we always have  $g_1(f(X)) = g_2(f(X))$ .

As result we can say that an [Surjection](#) is a [Epimorphism](#) in **Set** category. Moreover there is a proof [3] of that fact.

**Definition 1.22** (Injection). The function  $f : X \rightarrow Y$  is injective (or one-to-one function) if  $\forall x_1, x_2 \in X$ , such that  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$  (see figs. 1.6 and 1.8).

**Remark 1.23** (Injection vs Monomorphism). [Injection](#) and [Monomorphism](#) are related each other. Consider a non-injective function  $f : X \rightarrow Y$  (see fig. 1.7). One can conclude that it is not monomorphism because  $\exists g_1, g_2$  such that  $g_1 \neq g_2$  and  $f(g_1(a_1)) = y_3 = f(g_2(b_1))$ .

As result we can say that an [Injection](#) is a [Monomorphism](#) in **Set** category. Moreover there is a proof [2] of that fact.

**Definition 1.24** (Bijection). The function  $f : X \rightarrow Y$  is bijective (or one-to-one correspondence) if it is an [Injection](#) and a [Surjection](#) (see fig. 1.8).

**Definition 1.25** (Discrete category). TBD

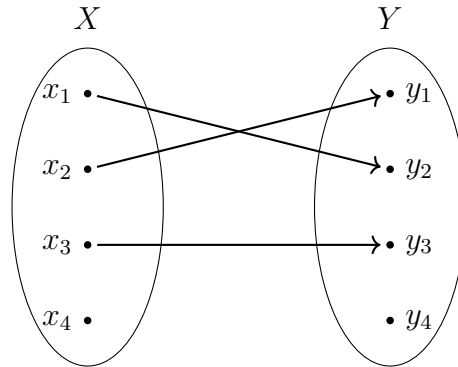


Figure 1.6: A injective (non-surjective) function from domain  $X$  to codomain  $Y$

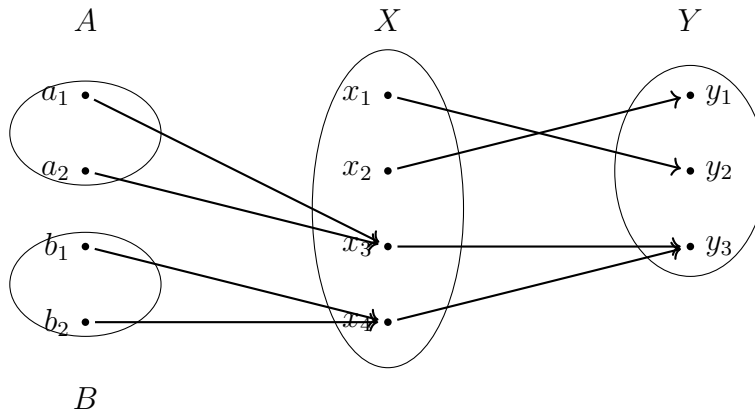


Figure 1.7: A non-injective function  $f$  from domain  $X$  to codomain  $Y$ .  $\exists g_1 : A \rightarrow X, g_2 : B \rightarrow X$  such that  $g_1 \neq g_2$  but  $f \circ g_1 = f \circ g_2$ . I.e. the function  $f$  is not monomorphism.

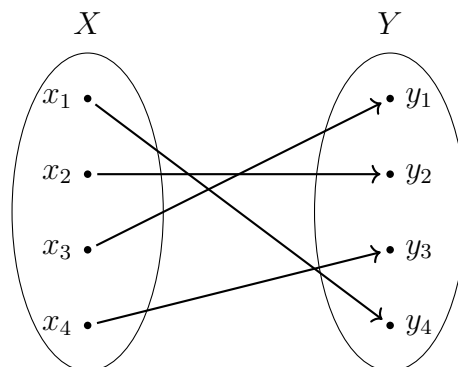


Figure 1.8: An injective and surjective function (bijection)

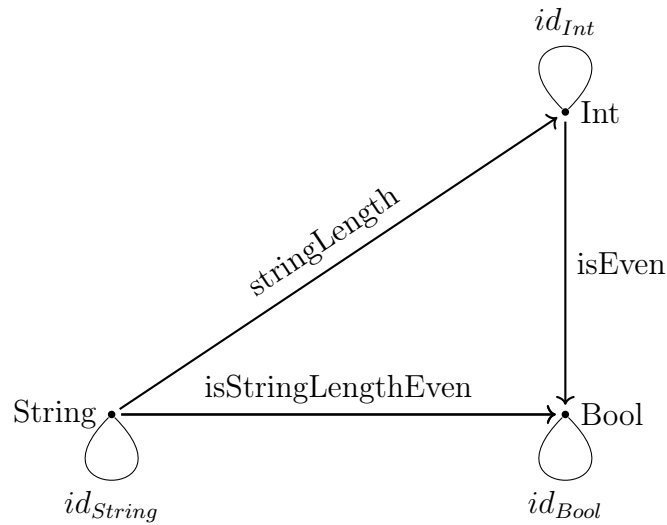


Figure 1.9: Programming language category example. Objects are types: Int, Bool, String. Morphisms are several functions

### 1.2.2 Programming languages

In the programming languages we consider types as **Objects** and functions as **Morphisms**. Particularly we will look into category with 3 objects that are types: Int, Bool, String. There are also several functions between them (see fig. 1.9).

#### Hask category

**Example 1.26** (Hask category). Types in Haskell are considered as **Objects**. Functions are considered as **Morphisms**. We are going to implement **Category** from fig. 1.9.

The function isEven that converts Int type into Bool.

```
isEven :: Int -> Bool
isEven x = x `mod` 2 == 0
```

There is also **Identity morphism** that is defined as follows

```
id :: a -> a
id x = x
```

If we have an additional function

```
stringLength :: String -> Int
stringLength x = length x
```

then we can create a [Composition](#) ([Property 1.5](#))

```
isStringLengthEven :: String -> Bool
isStringLengthEven = isEven . stringLength
```

**Remark 1.27** (Haskell lazy evaluation). Each Haskell type has a special value  $\perp$ . The value presents and lazy evaluations make several category law invalid, for instance [Identity morphism](#) behaviour become invalid in specific cases:

The following code

```
seq undefined True
```

produces *undefined* But the following

```
seq (id.undefined) True
seq (undefined.id) True
```

produces *True* in both cases. As result we have (we cannot compare compare functions in Haskell, but if we could we can get the following)

```
id . undefined /= undefined
undefined . id /= undefined,
```

i.e. [\(1.1\)](#) and [\(1.2\)](#) are not satisfied.

## C++ category

**Example 1.28** (C++ category). We will use the same trick as in [Hask category](#) ([Example 1.26](#)) and will assume types in C++ as [Objects](#), functions as [Morphisms](#). We also are going to implement [Category](#) from fig. 1.9.

We also define 2 functions:

```
std::function<bool(int)> isEven =
[] (int x)
{
    return x % 2 == 0;
};

std::function<int(std::string)> stringLength =
[] (std::string s)
{
    return static_cast<int>(s.size());
};
```

Composition can be defined as follows:

```
// h = g . f
template < typename A, typename B, typename C> std::function<C(A)>
compose(std::function<C(B)> g, std::function<B(A)> f)
{
    auto h = [f,g](A a)
    {
        B b = f(a);
        C c = g(b);
        return c;
    };
    return h;
};
```

The [Identity morphism](#):

```
std::function<bool(bool)> id_bool =
[] (bool x)
{
    return x;
};

std::function<std::string(std::string)> id_string =
[] (std::string x)
{
    return x;
};
```

The usage examples are the following:

```
std::function<bool(std::string)> isStringLengthEven =
compose<>(isEven, stringLength);

std::function<bool(std::string)> isStringLengthEvenLeft =
compose<>(id_bool, isStringLengthEven);

std::function<bool(std::string)> isStringLengthEvenRight =
compose<>(isStringLengthEven, id_string);
```

Such construction will always provides us the category as soon as we use pure function (functions without effects).



# Chapter 2

## Objects and morphisms

### 2.1 Equality

#### 2.1.1 Equality of objects

via unique isomorphism

#### 2.1.2 Equality of morphisms

TBD

### 2.2 Initial and terminal objects

TBD

### 2.3 Product and sum

TBD

### 2.4 Examples

#### 2.4.1 Set category

TBD

#### 2.4.2 Hask category

TBD



# Chapter 3

## Functors

TBD



# Chapter 4

## Monads

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