Quantum Optics[1]

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Contents

Introduction

Quantum optics studies optical phenomena in which the quantum nature of light is manifested. It can be said that quantum optics considers optical phenomena for which light and the system interacting with it must be described by quantum equations.

Quantum optics deals with the frequency range roughly from $f_1 \simeq 10^{13}$Hz to $f_2 \simeq 10^{18}$Hz, i.e., from the infrared range to X-rays. The lower limit is determined by the condition that the energy of the quantum exceeds the thermal motion energy: $\omega_1\hbar > kT$. The upper limit is set based on the fact that in quantum optics non-relativistic electron energies are usually considered and, accordingly, the quantum energy should be much less than the electron rest energy: $\omega_2\hbar \ll mc^2$.

Currently, there are few educational materials devoted to quantum optics. Among domestic ones, note [50]. The presented material reflects a number of issues of quantum theory of light, presented in the course "Quantum Optics". The manual consists of three parts.

The first part of the manual gives an introduction to quantum electrodynamics. The simplest problems related to the interaction of the light field and matter are considered. The quantum theory of the laser is constructed on the simplest model. The quantum description of light coherence and its connection with the classical description are examined. Chapter 1 provides a brief introduction to quantum electrodynamics at the level necessary for the exposition of the quantum optics course. Various quantum states of the light field used in quantum optics are considered. Special attention is given to coherent states, which allow the quantum description of optical phenomena to be approximated as closely as possible to the classical one. Chapter 2 considers the simplest problems for the light field associated with both individual atoms and a large number of atoms in thermal equilibrium. Special attention is paid to the connection of the field with the thermostat, leading to relaxation processes that play an important role in quantum optics and quantum electronics.

The second part of the manual addresses some applied questions of quantum optics. Chapter 3 discusses the quantum theory of the laser on a simple model. An expression for the statistics of laser photons and a formula evaluating the

"natural" linewidth of the laser generation are obtained. Chapter 4 considers the quantum theory of the laser in the Heisenberg picture.

Chapter 5 is devoted to photon optics. The quantum description of light coherence and its connection with the classical description, coherence functions of various orders, the problem of photon statistics and its relation to the statistics of photo-detections (Mandel's formula) are presented. Numerous examples are given. Further, the connection between photon statistics and the spectral properties of light beams and its application to spectral measurements are considered.

Chapter 6 provides a quantum description of interferometric experiments. The Mach-Zehnder interferometer and the error of interference measurements are discussed. Later, in chapter 9, it will be shown how to increase measurement accuracy using squeezed states.

Chapter 7 is dedicated to the quantum description of polarization properties of light. Later, this material is used in the description of entangled states in chapter 10.

The third part of the manual discusses nonclassical states of light. A definition of a nonclassical state is given (see chapter 8). Squeezed (chapter 9) and entangled (see chapter 10) nonclassical states of light are described. For each type of nonclassical state, a definition, methods of obtaining, applications, and a proof of nonclassicality are provided.

In the fourth part of the manual, the quantum theory of information is considered. In particular, chapter 11 presents an introduction to quantum information theory. The concept of information quantity in both classical and quantum cases is defined. The classical Shannon coding theorem and its quantum analogue are discussed. Chapter 12 is devoted to information protection theory (cryptography): the classical cryptography theory and its main drawbacks are described. A description of quantum cryptography is given. Chapter 13 describes the main quantum algorithms: Shor's algorithm for integer factorization and Grover's algorithm for searching in an unstructured data array.

The appendices contain material that complements the main course and is mostly of a reference nature, allowing one to avoid resorting to special literature.

The manual is intended for senior students specializing in quantum electronics.

Part 1

Quantum Electrodynamics

Chapter 1

Quantization of the Electromagnetic Field

Quantum electrodynamics serves as the foundation of quantum optics. Below are the statements of the quantum theory of the electromagnetic field necessary for studying and understanding quantum optics.

1.1 Decomposition of the Electromagnetic Field into Modes (Types of Oscillations)

The electromagnetic field satisfies the system of Maxwell's equations:

$$rot\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}, \quad rot\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad div\vec{D} = \rho, \quad div\vec{B} = 0. \tag{1.1}$$

In free space ($\vec{j} = 0$, $\rho = 0$) we have a simpler system:

$$\begin{gathered} rot\vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \quad rot\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}, \\ div\vec{H} = 0, \quad div\vec{E} = 0, \\ \vec{B} = \mu_0 \vec{H}, \quad \vec{D} = \varepsilon_0 \vec{E}, \quad \mu_0 \varepsilon_0 = \frac{1}{c^2}. \end{gathered} \tag{1.2}$$

By eliminating the quantity $\vec{H}$ from (1.2), we obtain the equation:

$$rotrot\vec{E} = -\mu_0 \frac{\partial}{\partial t} rot\vec{H} = -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}.$$

Considering $rotrot\vec{E} = grad\, div\vec{E} - \Delta\vec{E} = -\Delta\vec{E}$ [1] we obtain the following system

[1] under the condition $div\vec{E} = 0$

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0;$$
$$div\vec{E} = 0, \quad \frac{\partial H}{\partial t} = -\frac{1}{\mu_0} rot\vec{E}$$
$$\vec{B} = \mu_0\vec{H}, \quad \vec{D} = \varepsilon_0\vec{E}, \quad \mu_0\varepsilon_0 = \frac{1}{c^2}. \tag{1.3}$$

which is fully equivalent to the original system of equations.

For quantizing the electromagnetic field, it is convenient to represent its equations in the so-called Hamiltonian form. The essence of the method, well known to radio physicists, is that the field is decomposed into modes, and solving the equations is reduced to solving a system of ordinary differential equations for the expansion coefficients that depend on time. For this purpose, a system of orthogonal vector functions is introduced, representing the field distributions corresponding to the electromagnetic field's inherent oscillations in a certain volume V bounded by a perfectly conducting surface S. The fields must satisfy the system of equations (1.3). On the surface S, certain boundary conditions must be met, for example:

$$\left[\vec{n}\vec{E}\right]\Big|_S = 0 \, or \, \left(\vec{n}\vec{H}\right)\Big|_S = 0, \tag{1.4}$$

corresponding to a perfectly conducting surface.

An arbitrary electromagnetic field within volume V is represented by the expansions:

$$\vec{E}\,(r,t) = \sum_{(n)} Q_n\,(t)\,\vec{E}_n\,(r)\,, \quad \vec{H}\,(r,t) = \sum_{(n)} P_n\,(t)\,\vec{H}_n\,(r)\,, \tag{1.5}$$

where $Q_n\,(t)\,\vec{E}_n\,(r)$ and $P_n\,(t)\,\vec{H}_n\,(r)$ are partial solutions of the equations satisfying the boundary conditions (1.4); $\vec{E}_n\,(r)$ and $\vec{H}_n\,(r)$ correspond to the n-th type of oscillation in volume V (mode). From the equations (1.3) we have:

$$Q_n\,(t)\;\Delta\vec{E}_n\,(r) = \frac{1}{c^2}\frac{d^2Q_n}{dt^2}\vec{E}_n\,(r)\,. \tag{1.6}$$

By separating variables, we obtain:

$$\frac{d^2Q_n}{dt^2} + \omega_n^2 Q_n = 0, \quad \Delta\vec{E}_n\,(r) = -\frac{\omega_n^2}{c^2}\vec{E}_n\,(r) \tag{1.7}$$

where ω_n is the separation constant, being the frequency of the inherent oscillations.

The second of the equations (1.7) takes the form:

$$\Delta\vec{E}_n(r) + k_n^2\vec{E}_n(r) = 0, \quad k_n^2 = \frac{\omega_n^2}{c^2}, \tag{1.8}$$

with boundary conditions $\left[\vec{n}\vec{E}_n\right]\Big|_S = 0$. This problem has a solution only at certain values $k_n(\omega_n)$.

To determine $\vec{H}_n(r)$, we use the equations (1.3):

$$Q_n(t)\, rot\vec{E}_n(r) = -\mu_0\frac{dP_n}{dt}\vec{H}_n(r). \tag{1.9}$$

Firstly, we make the following substitution:

$$Q_n(t) = \frac{\omega_n}{\sqrt{\varepsilon_0}}q_n(t), \quad P_n(t) = \frac{1}{\sqrt{\mu_0}}p_n(t). \tag{1.10}$$

For the new variable q_n we obviously have from (1.7)

$$\frac{d^2q_n}{dt^2} + \omega_n^2 q_n = 0.$$

The expansion (1.5) in new variables takes the form:

$$\vec{E}(r,t) = \sum_{(n)}\frac{q_n(t)\,\omega_n}{\sqrt{\varepsilon_0}}\vec{E}_n(r), \quad \vec{H}(r,t) = \sum_{(n)}\frac{p_n(t)}{\sqrt{\mu_0}}\vec{H}_n(r), \tag{1.11}$$

Considering the substitution (1.10) and the expression for the speed of light $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ (1.2) we can rewrite (1.9) in the following form

$$q_n(t)\, rot\vec{E}_n(r) = -\frac{1}{c\omega_n}\frac{dp_n(t)}{dt}\vec{H}_n(r) \tag{1.12}$$

The equations (1.12) will be satisfied if

$$p_n = \frac{dq_n}{dt}, \quad \frac{dp_n}{dt} = \frac{d^2q_n}{dt^2} = -\omega_n^2 q_n,$$

then

$$\begin{aligned} rot\vec{E}_n(r) &= k_n\vec{H}_n(r), \\ \vec{H}_n(r) &= \frac{1}{k_n}rot\vec{E}_n(r) \end{aligned} \tag{1.13}$$

From the course "Electromagnetic Oscillations," it is known that the system of functions $\vec{E}_n(r)$, $\vec{H}_n(r)$ is orthogonal and can be normalized:

$$\int_\nu \left(\vec{E}_n \vec{E}_m\right) d\nu = \delta_{nm}, \quad \int_\nu \left(\vec{H}_n \vec{H}_m\right) d\nu = \delta_{nm}, \quad \int_\nu \left(\vec{H}_n \vec{E}_m\right) d\nu = 0. \quad (1.14)$$

The solution to the original electrodynamic problem is reduced to finding the coefficients $q_n(t)$, $p_n(t)$ in the expansions (1.11).

1.2 Hamiltonian Form of the Equations for the Free Electromagnetic Field

Let us represent the field in the volume V in the form of expansions (1.11):

$$\vec{E}(r,t) = \sum_{(n)} \frac{q_n(t)\,\omega_n}{\sqrt{\varepsilon_0}} \vec{E}_n(r), \quad \vec{H}(r,t) = \sum_{(n)} \frac{p_n(t)}{\sqrt{\mu_0}} \vec{H}_n(r), \quad (1.15)$$

where $p_n = \frac{dq_n}{dt}$, and q_n satisfies the equation

$$\frac{d^2 q_n}{dt^2} + \omega_n^2 q_n = 0.$$

This is the equation of the harmonic oscillator. Its solution is:

$$q_n(t) = q_n(0)\cos\omega_n t + \frac{\dot{q_n}(0)}{\omega_n}\sin\omega_n t \quad (1.16)$$

$q_n(0)$ and $\dot{q_n}(0)$ are determined from the expansion of the initial field over $\vec{E_n}$, $\vec{H_n}$.

Now let us write the electrodynamics equations in Hamiltonian form. The Hamiltonian function for the electromagnetic field (the field energy)

$$\mathcal{H} = \frac{1}{2}\int_\nu \left(\varepsilon_0 \left(\vec{E}\right)^2 + \mu_0 \left(\vec{H}\right)^2\right) d\nu \quad (1.17)$$

Substituting expansions (1.15) into (1.17) and using the orthogonality of the eigen-

functions, we get

$$\mathcal{H} = \frac{1}{2}\int_{\nu}\left(\sum_{(n)}\sum_{(m)}\varepsilon_0\frac{q_n q_m\omega_n\omega_m}{\varepsilon_0}\left(\vec{E}_n\vec{E}_m\right)\right)d\nu + \\ +\frac{1}{2}\int_{\nu}\left(\sum_{(n)}\sum_{(m)}\mu_0\frac{p_n p_m}{\mu_0}\left(\vec{H}_n\vec{H}_m\right)\right)d\nu = \\ = \frac{1}{2}\sum_{(n)}\left(\omega_n^2 q_n^2 + p_n^2\right). \tag{1.18}$$

This expression corresponds to the Hamiltonian function for a collection of independent harmonic oscillators.

For a single mode, the Hamiltonian function is

$$\mathcal{H}_n = \frac{1}{2}\left(\omega_n^2 q_n^2 + p_n^2\right). \tag{1.19}$$

then

$$\mathcal{H} = \sum_{(n)}\mathcal{H}_n \tag{1.20}$$

The equations of motion are obtained from the Hamiltonian function by the standard procedure (14.49):

$$\dot{q}_n = \frac{\partial\mathcal{H}_n}{\partial p_n} = p_n, \quad \dot{p}_n = -\frac{\partial\mathcal{H}_n}{\partial q_n} = -\omega_n^2 q_n = \ddot{q}_n. \tag{1.21}$$

The resulting equations coincide with equations (1.7). The Hamiltonian function of a mechanical oscillator is

$$\mathcal{H} = \frac{1}{2}\left(M\Omega^2 x^2 + \frac{p^2}{M}\right)$$

Therefore, (1.19) formally corresponds to an oscillator with unit mass and eigenfrequency ω_n. The Hamiltonian formulation of the electromagnetic field equations is convenient for the field quantization procedure. This approach is often suitable for solving various classical problems as well. More details can be found in the book [41].

1.3 Quantization of the Free Electromagnetic Field

Using the analogy noted earlier, one can quantize the electromagnetic field similarly to how a simple mechanical harmonic oscillator is quantized.

Upon quantization, q_n and p_n become operators $\hat{q}_n$, $\hat{p}_n$, satisfying the same commutation relations as coordinate and momentum:

$$[\hat{q}_n, \hat{p}_n] = \hat{q}_n\hat{p}_n - \hat{p}_n\hat{q}_n = i\hbar \tag{1.22}$$

where $\hat{q}_n$ and $\hat{p}_n$ are Hermitian (self-adjoint) operators. In the Schrödinger representation they are time-independent, in the Heisenberg representation they are time-dependent. It is convenient to introduce new operators through these, which are not self-adjoint but are conjugates of each other:

$$\hat{a}_n = \frac{1}{\sqrt{2\hbar\omega_n}}\left(\omega_n\hat{q}_n + i\hat{p}_n\right), \quad \hat{a}_n^\dagger = \frac{1}{\sqrt{2\hbar\omega_n}}\left(\omega_n\hat{q}_n - i\hat{p}_n\right), \tag{1.23}$$

The inverse relations yield

$$\hat{q}_n = \sqrt{\frac{\hbar}{2\omega_n}}\left(\hat{a}_n + \hat{a}_n^\dagger\right), \quad \hat{p}_n = i\sqrt{\frac{\hbar\omega_n}{2}}\left(\hat{a}_n^\dagger - \hat{a}_n\right), \tag{1.24}$$

Using expressions (1.22), (1.23), (1.24) one can obtain commutation relations for the operators $\hat{a}_n$ and $\hat{a}_n^\dagger$:

$$\begin{aligned}\left[\hat{a}_n, \hat{a}_n^\dagger\right] &= \frac{1}{2\hbar\omega_n}\left(\omega_n\hat{q}_n + i\hat{p}_n\right)\left(\omega_n\hat{q}_n - i\hat{p}_n\right) - \\ &- \frac{1}{2\hbar\omega_n}\left(\omega_n\hat{q}_n - i\hat{p}_n\right)\left(\omega_n\hat{q}_n + i\hat{p}_n\right) = \\ &= \frac{1}{2\hbar\omega_n}\left(-2i\omega_n\left(\hat{q}_n\hat{p}_n - \hat{p}_n\hat{q}_n\right)\right) = 1.\end{aligned} \tag{1.25}$$

Thus,

$$\left[\hat{a}_n, \hat{a}_n^\dagger\right] = 1. \tag{1.26}$$

Moreover, it is obvious that

$$[\hat{a}_n, \hat{a}_n] = 0, \quad \left[\hat{a}_n^\dagger, \hat{a}_n^\dagger\right] = 0. \tag{1.27}$$

The Hamiltonian, expressed through the new operators, takes the form

$$\begin{aligned}\hat{\mathcal{H}}_n &= \frac{1}{2}\left(\omega_n^2\hat{q}_n^2 + \hat{p}_n^2\right) = \\ &= \frac{1}{2}\frac{\omega_n\hbar}{2}\left(\left(\hat{a}_n + \hat{a}_n^\dagger\right)\left(\hat{a}_n + \hat{a}_n^\dagger\right) - \left(\hat{a}_n^\dagger - \hat{a}_n\right)\left(\hat{a}_n^\dagger - \hat{a}_n\right)\right) = \\ &= \frac{\omega_n\hbar}{2}\left(\hat{a}_n\hat{a}_n^\dagger + \hat{a}_n^\dagger\hat{a}_n\right) = \frac{\omega_n\hbar}{2}\left(1 + \hat{a}_n^\dagger\hat{a}_n + \hat{a}_n^\dagger\hat{a}_n\right) = \\ &= \omega_n\hbar\left(\hat{a}_n^\dagger\hat{a}_n + \frac{1}{2}\right)\end{aligned} \tag{1.28}$$

Here the relations between $\hat{q}_n$, $\hat{p}_n$ and $\hat{a}_n^\dagger$, $\hat{a}_n$ (1.23) are used, as well as the commutation relation (1.26).

The energy $\frac{1}{2}\omega_n\hbar$ corresponds to zero-point oscillations. The total zero-point energy

$$\frac{1}{2}\sum_{(n)}\omega_n\hbar \to \infty,$$

since the number of modes is infinite. This does not lead to significant problems because we are only interested in energy differences, and the constant term can be discarded. Then the Hamiltonian takes the form

$$\hat{\mathcal{H}} = \sum_{(n)}\hat{\mathcal{H}}_n = \sum_{(n)}\hbar\omega_n\hat{a}_n^\dagger\hat{a}_n \tag{1.29}$$

The electric and magnetic field operators can be expressed as

$$\begin{aligned}\hat{\vec{E}} &= \sum_{(n)}\frac{\hat{q}_n\omega_n}{\sqrt{\varepsilon_0}}\vec{E}_n(r) = \sum_{(n)}\sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}}\left(\hat{a}_n^\dagger + \hat{a}_n\right)\vec{E}_n(r),\\ \hat{\vec{H}} &= \sum_{(n)}\frac{\hat{p}_n}{\sqrt{\mu_0}}\vec{H}_n(r) = i\sum_{(n)}\sqrt{\frac{\hbar\omega_n}{2\mu_0}}\left(\hat{a}_n^\dagger - \hat{a}_n\right)\vec{H}_n(r).\end{aligned} \tag{1.30}$$

Let us consider a simple example. In laser theory one considers a resonator formed by two parallel mirrors. Often, an approximate field representation is used, assuming the field depends only on the longitudinal coordinate (Figure 1.1). In this approximation, the normalized eigenfunction (mode) can be represented as:

$$E_{nx} = \sqrt{\frac{2}{V}}\sin k_n z, \quad H_{ny} = \sqrt{\frac{2}{V}}\cos k_n z, \quad k_n = \frac{\pi n}{L}, \quad \omega_n = \frac{c\pi n}{L}, \tag{1.31}$$

where $V = SL$, L is the resonator length, S is the cross-section of the light beam.

The operator of the mode's electric field in this case will be:

$$\begin{aligned}\hat{E}_{nx} = \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}}\left(\hat{a}_n^\dagger + \hat{a}_n\right)\sqrt{\frac{2}{V}}\sin k_n z = E_1\left(\hat{a}_n^\dagger + \hat{a}_n\right)\sin k_n z,\\ \hat{H}_{ny} = iE_1\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\hat{a}_n^\dagger - \hat{a}_n\right)\cos k_n z,\end{aligned} \tag{1.32}$$

where $E_1 = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$ is the electric field corresponding to one photon (quantum) in the mode.

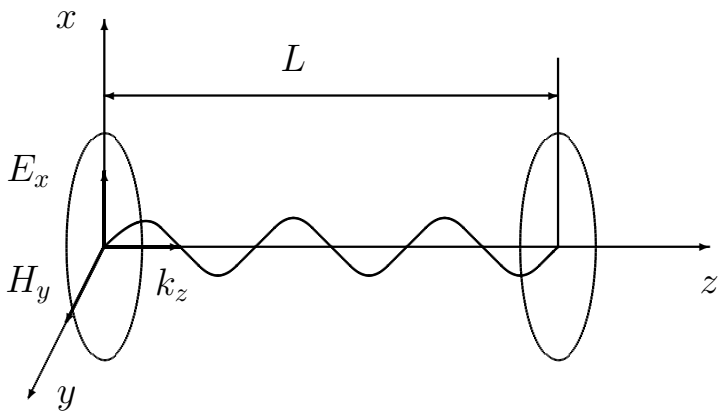


Figure 1.1: Electromagnetic field in a resonator formed by two mirrors. The length of the resonator is L. The wave vector of the electromagnetic field is directed along the z axis, the electric field along the x axis, and the magnetic field along the y axis.

1.4 Expansion of the Field into Plane Waves in Free Space

So far, we have considered the electromagnetic field in a physically distinguished volume—in a resonator. If we are dealing with free space, we can artificially distinguish a sufficiently large volume containing the region of space of interest, define modes (types of oscillations) for it with appropriate boundary conditions, and then proceed in the manner already considered. If necessary, the volume in the final result can be extended to infinity. Usually, a sufficiently large cubic volume with a side L (Figure 1.2) is used. In this case, it is customary to use periodic boundary conditions:

$$\begin{aligned}\vec{E}(0,y,z) &= \vec{E}(L,y,z),\\ \vec{E}(x,0,z) &= \vec{E}(x,L,z),\\ \vec{E}(x,y,0) &= \vec{E}(x,y,L).\end{aligned} \tag{1.33}$$

It is convenient to decompose into plane waves. As known from the course of electromagnetic oscillations, the solution corresponding to a plane wave has the form

$$\vec{E}_k(r,t) = A_k(t)\,\vec{e}_k e^{i\left(\vec{k}\vec{r}\right)} + (\text{c. c.}), \tag{1.34}$$

where $\vec{e}_k$ is the unit vector of wave polarization; k is the wave number; $\vec{k}$ is the wave vector; $A_k(t) = A_k(0)\,e^{-i\omega_k t}$.

The magnetic field is related to the electric field by the relation

$$\vec{H}_k(r,t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{1}{k}\left[\vec{k}\vec{e}_k\right] A_k(t)\,e^{i\left(\vec{k}\vec{r}\right)} + (\text{c. c.}). \tag{1.35}$$

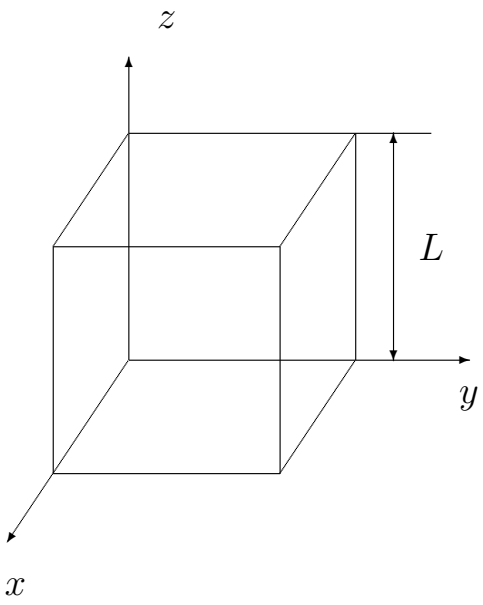


Figure 1.2: Decomposition of the electromagnetic field into plane waves in free space. The region of space in which the decomposition is performed is chosen in the form of a cube with side L. Periodic boundary conditions are used on the boundary of the cube.

The following equalities are valid:

$$\left(\vec{k}\vec{E}_k\right) = \left(\vec{k}\vec{H}_k\right) = \left(\vec{E}_k\vec{H}_k\right) = 0, \quad k^2 = \left(\vec{k}\vec{k}\right) = \frac{\omega_k^2}{c^2} \tag{1.36}$$

indicating that $\vec{E}_k$ and $\vec{H}_k$ are perpendicular to the direction of wave propagation and to each other (the wave is transverse).

The periodicity condition (1.33) will be satisfied if

$$\vec{k} = \frac{2\pi}{L}\left(n_x\vec{x}_0 + n_y\vec{y}_0 + n_z\vec{z}_0\right), \quad n_x, n_y, n_z = 0, \pm 1, \pm 2, \ldots. \tag{1.37}$$

Then we get

$$\left(\vec{k}\vec{r}\right) = \frac{2\pi}{L}\left(n_x x + n_y y + n_z z\right),$$
$$\left(\vec{k}\vec{r}\right)\Big|_{x=L} = 2\pi n_x + \frac{2\pi}{L}\left(n_y y + n_z z\right) = 2\pi n_x + \left(\vec{k}\vec{r}\right)\Big|_{x=0}. \tag{1.38}$$

Consequently, $e^{i\left(\vec{k}\vec{r}\right)}$ is periodic in x. In the same way, periodicity in y and z is shown.

In the future, instead of the real functions (1.34), (1.35) we will use complex eigenfunctions

$$\vec{E}_k(r) = \vec{e}_k e^{i\left(\vec{k}\vec{r}\right)}, \quad \vec{H}_k(r) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{1}{k}\left[\vec{k}\vec{E}_k(r)\right]. \tag{1.39}$$

The relations are valid:

$$\left(\vec{k}\vec{E}_k\right) = \left(\vec{k}\vec{H}_k\right) = \left(\vec{E}_k\vec{H}_k\right) = 0, \quad k^2 = \left(\vec{k}\vec{k}\right) = \frac{\omega_k^2}{c^2}.$$

The eigenfrequencies ω_k are determined by the equation (1.37). Indeed, from (1.37) taking into account $k^2 = \frac{\omega_k^2}{c^2}$ we have

$$\omega_k = c\sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}. \tag{1.40}$$

With linear polarization, the polarization vectors $\vec{e}_k$ are:

$$\vec{e}_{k_1} = \vec{\xi}_0, \quad \vec{e}_{k_2} = \vec{\eta}_0,$$

where $\vec{\xi}_0$, $\vec{\eta}_0$, are the unit vectors of directions, forming a rectangular system of directions with the vector $\vec{k}$, consequently,

$$\left(\vec{\xi}_0\vec{k}\right) = \left(\vec{\eta}_0\vec{k}\right) = \left(\vec{e}_{k_1}\vec{e}_{k_2}\right) = 0.$$

For circular polarization

$$\vec{e}_{k_1} = \frac{\vec{\xi}_0 + i\vec{\eta}_0}{\sqrt{2}}, \quad \vec{e}_{k_2} = \frac{\vec{\xi}_0 - i\vec{\eta}_0}{\sqrt{2}}.$$

In this case, the orthogonality condition holds: $\left(\vec{e}_{k_1}\vec{e}^{\,*}_{k_2}\right)$ and normalization conditions: $\left(\vec{e}_{k_1}\vec{e}^{\,*}_{k_1}\right) = \left(\vec{e}_{k_2}\vec{e}^{\,*}_{k_2}\right) = 1$.

An arbitrary electromagnetic field using complex functions (1.39) can be represented by decompositions

$$\begin{aligned} \vec{E}\,(r,t) &= \sum_{(k)} A_k\,(t)\,\vec{E}_k\,(r) + (\text{c. c.}), \\ \vec{H}\,(r,t) &= \sum_{(k)} A_k\,(t)\,\vec{H}_k\,(r) + (\text{c. c.}) \end{aligned} \tag{1.41}$$

1.5 Density of States

If the quantization volume L^3 is large, the density of states (the number of modes per unit frequency interval) will be quite large. The wave vector of the mode is determined by the relation (1.37)

$$\vec{k}_{n_x,n_y,n_z} = \frac{2\pi}{L}\left(n_x\vec{x}_0 + n_y\vec{y}_0 + n_z\vec{z}_0\right) \tag{1.42}$$

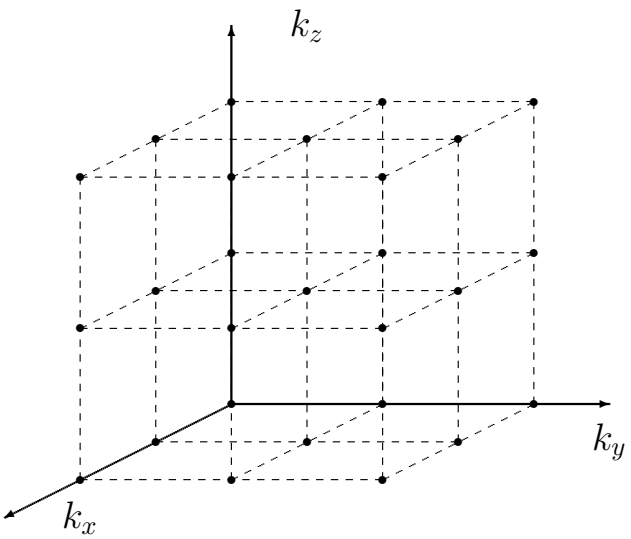


Figure 1.3: Allowed modes of the electromagnetic field (marked by dots) for a confined space in the form of a cube with side L. Rectangular coordinate system.

Each set of integers corresponds to two waves differing in polarization. The modes can be visually represented as points in a Cartesian coordinate system, with n_x, n_y, n_z as axes (Figure 1.3). The number of oscillations in the volume $\Delta n_x \Delta n_y \Delta n_z$ will obviously be $\Delta N = 2\Delta n_x \Delta n_y \Delta n_z$, or, considering the relationship between k and n (1.37),

$$\Delta N = 2\left(\frac{L}{2\pi}\right)^3 \Delta k_x \Delta k_y \Delta k_z. \tag{1.43}$$

The factor of two in formula (1.43) appears because two modes with different polarizations correspond to one value of k. For large L ($L \to \infty$), the distribution becomes quasi-continuous, and summation over modes, which is almost always required in solving quantum optics problems, can be replaced by integrating over k:

$$2\sum_{(n)} (\dots) = 2\left(\frac{L}{2\pi}\right)^3 \iiint_{-\infty}^{+\infty} (\dots)\, dk_x dk_y dk_z. \tag{1.44}$$

The transition from rectangular coordinates k_x, k_y, k_z to spherical coordinates k, θ, φ (Figure 1.4), gives

$$\begin{aligned} k_x = k\sin\theta\cos\varphi, \quad k_y = k\sin\theta\sin\varphi, \quad k_z = k\cos\theta, \\ dk_x dk_y dk_z = k^2 \sin\theta dk d\theta d\varphi = k^2 dk d\Omega, \end{aligned} \tag{1.45}$$

where $d\Omega = \sin\theta d\theta d\varphi$ is the solid angle element in the direction of $\vec{k}$. Thus, we obtain

$$dN = 2\left(\frac{L}{2\pi}\right)^3 k^2 dk d\Omega \tag{1.46}$$

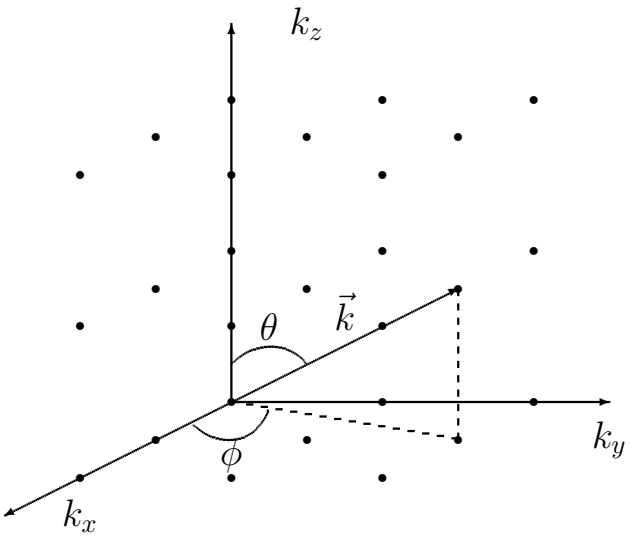


Figure 1.4: Allowed modes of the electromagnetic field (marked by dots) for a confined space region in the form of a cube with side L. Spherical coordinate system.

or, considering that $k^2c^2 = \omega^2$, i.e., $k^2dk = \frac{\omega^2 d\omega}{c^3}$:

$$dN = 2\left(\frac{L}{2\pi c}\right)^3 \omega^2 d\omega d\Omega \tag{1.47}$$

i.e., the number of states per unit volume, unit frequency, and unit solid angle

$$g(\omega) = \frac{2\omega^2}{(2\pi c)^3} \tag{1.48}$$

(spectral density of eigenstates).

1.6 Hamiltonian Form of Field Equations with Plane Wave Expansion

Eigenfunctions (1.39) are orthogonal, i.e.

$$\int_\nu \left(\vec{E}_k \vec{E}^*_{k'}\right) d\nu = C \int_\nu e^{i\left(\vec{k}\vec{r}\right)} e^{-i\left(\vec{k}'\vec{r}\right)} d\nu =$$

$$= C \int_0^L dx \int_0^L dy \int_0^L dz \left[e^{i\left((k_x - k'_x)x + \left(k_y - k'_y\right)y + (k_z - k'_z)z\right)}\right] = 0, \tag{1.49}$$

if at least one of the inequalities holds $k_x \neq k'_x$, $k_y \neq k'_y$, $k_z \neq k'_z$, (or $\vec{k} - \vec{k'} \neq 0$). This follows from the periodicity of the integrand functions.

Similarly, we obtain:

$$\int_{\nu}\left(\vec{E}_k\vec{E}_{k'}\right)d\nu=0,\quad \vec{k}\neq-\vec{k'}. \tag{1.50}$$

The eigenfunctions $\vec{E}_k$ can be normalized by setting

$$\int_{\nu}\left(\vec{E}_k\vec{E}_k^*\right)d\nu=1. \tag{1.51}$$

Then the eigenfunctions have the form

$$\vec{E}_k\left(r\right)=\frac{1}{\sqrt{\nu}}\vec{e}_k e^{i\left(\vec{k}\vec{r}\right)},\quad \vec{H}_k\left(r\right)=\frac{1}{k}\left[\vec{k}\vec{e}_k\left(r\right)\right]\sqrt{\frac{\varepsilon_0}{\nu\mu_0}}e^{i\left(\vec{k}\vec{r}\right)}.$$

Note that the index k refers to k_j, where $j=1,2$ correspond to two different polarizations. The functions $\vec{E}_k$ and $\vec{H}_k$ written in this form satisfy the following orthogonality and normalization conditions:

$$\begin{gathered}
\int_{\nu}\left(\vec{E}_k\vec{E}_{k'}^*\right)d\nu=0\quad\text{for }\vec{k}\neq\vec{k'},\\
\int_{\nu}\left(\vec{E}_k\vec{E}_{k'}\right)d\nu=0\quad\text{for }\vec{k}\neq-\vec{k'},\\
\int_{\nu}\left(\vec{E}_k\vec{E}_k^*\right)d\nu=1,\quad\int_{\nu}\left(\vec{E}_k\vec{E}_{-k}\right)d\nu=1,\\
\int_{\nu}\left(\vec{H}_k\vec{H}_{k'}^*\right)d\nu=0\quad\text{for }\vec{k}\neq\vec{k'},\\
\int_{\nu}\left(\vec{H}_k\vec{H}_{k'}\right)d\nu=0\quad\text{for }\vec{k}\neq-\vec{k'},\\
\int_{\nu}\left(\vec{H}_k\vec{H}_k^*\right)d\nu=\frac{\varepsilon_0}{\mu_0},\quad\int_{\nu}\left(\vec{H}_k\vec{H}_{-k}\right)d\nu=-\frac{\varepsilon_0}{\mu_0}.
\end{gathered} \tag{1.52}$$

Orthogonality of modes with the same k value, but different polarizations, follows from the equalities:

$$\left(\vec{e}_{k_1}\vec{e}_{k_2}^*\right)=0,\quad\left(\vec{e}_{k_1}\vec{e}_{-k_2}\right)=0.$$

The electric and magnetic fields are expressed through these functions using the expansion:

$$\begin{aligned}
\vec{E}\left(r,t\right)&=\sum_{(k)}A_k\left(t\right)\vec{E}_k\left(r\right)+\text{c.c.},\\
\vec{H}\left(r,t\right)&=\sum_{(k)}A_k\left(t\right)\vec{H}_k\left(r\right)+\text{c.c.}
\end{aligned} \tag{1.53}$$

Summation over $k = k_j$ implies summation over all values of k and two polarizations ($j = 1, 2$): $\sum_{(k)} \equiv \sum_{j=1,2} \sum_{(k_j)}$.

The Hamiltonian function of the electromagnetic field is given by:

$$\mathcal{H} = \frac{1}{2} \int_{\nu} \left(\varepsilon_0 \left(\vec{E}\,(r,t) \right)^2 + \mu_0 \left(\vec{H}\,(r,t) \right)^2 \right) d\nu \tag{1.54}$$

Substituting the series in (1.54), multiplying them and considering the conditions of orthogonality and normalization, we obtain

$$\begin{aligned}
\mathcal{H} = \frac{1}{2}\varepsilon_0 \int_{(\nu)} \left\{ \sum_{(k)} \sum_{(k')} \left(A_k \vec{E}_k + A_k^* \vec{E}_k^* \right) \left(A_{k'} \vec{E}_{k'} + A_{k'}^* \vec{E}_{k'}^* \right) \right\} d\nu + \\
+ \frac{1}{2}\mu_0 \int_{(\nu)} \left\{ \sum_{(k)} \sum_{(k')} \left(A_k \vec{H}_k + A_k^* \vec{H}_k^* \right) \left(A_{k'} \vec{H}_{k'} + A_{k'}^* \vec{H}_{k'}^* \right) \right\} d\nu = \\
\frac{\varepsilon_0}{2} \sum_{(k)} \left(A_k A_k^* + A_k A_{-k} + A_k^* A_{-k}^* + A_k^* A_k + \right. \\
\left. + A_k A_k^* - A_k A_{-k} - A_k^* A_{-k}^* + A_k^* A_k \right) = \varepsilon_0 \sum_{(k)} \left(A_k A_k^* + A_k^* A_k \right) .
\end{aligned} \tag{1.55}$$

1.7 Quantization of the Electromagnetic Field in the Expansion over Plane Waves

For the quantization of the electromagnetic field in this case, we pay attention to the analogy of (1.55) and (1.28):

$$\hat{\mathcal{H}}_n = \frac{\omega_n \hbar}{2} \left(\hat{a}_n \hat{a}_n^\dagger + \hat{a}_n^\dagger \hat{a}_n \right) = \omega_n \hbar \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right)$$

and

$$\mathcal{H} = \varepsilon_0 \sum_{(k)} \left(A_k A_k^* + A_k^* A_k \right) .$$

From this analogy it follows that the following quantization procedure can be used:

$$\sqrt{\varepsilon_0} A_k \to \sqrt{\frac{\omega_k \hbar}{2}} \hat{a}_k, \quad \sqrt{\varepsilon_0} A_k^* \to \sqrt{\frac{\omega_k \hbar}{2}} \hat{a}_k^\dagger .$$

Such substitution leads to the following expression for the Hamiltonian:

$$\hat{\mathcal{H}}_k = \frac{\omega_k \hbar}{2} \left(\hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k \right) .$$

Using the commutation relations

$$\left[\hat{a},\hat{a}^\dagger\right]=1,\quad \hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1,\quad \hat{a}\hat{a}^\dagger=\hat{a}^\dagger\hat{a}+1,$$

we have $\hat{\mathcal{H}}_k=\omega_k\hbar\left(\hat{a}_k^\dagger\hat{a}_k+\frac{1}{2}\right)$ - an expression that exactly coincides with (1.28). The full Hamiltonian is obtained by summing over all modes:

$$\hat{\mathcal{H}}=\sum_{(k)}\hat{\mathcal{H}}_k=\sum_{(k)}\omega_k\hbar\left(\hat{a}_k^\dagger\hat{a}_k+\frac{1}{2}\right). \tag{1.56}$$

In the Hamiltonian form, the momentum of the electromagnetic field can also be represented. The classical momentum of the electromagnetic field occupying the volume V is defined by the formula

$$\vec{G}=\frac{1}{c^2}\int_{(\nu)}\left[\vec{E}\times\vec{H}\right]d\nu. \tag{1.57}$$

Using the expansion of the field over plane waves and considering the orthogonality relations, we obtain

$$\vec{G}=\frac{1}{c^2}\int_{(\nu)}\left[\vec{E}\times\vec{H}\right]d\nu=$$

$$=\frac{1}{c^2}\int_{(\nu)}\left[\left(\sum_{(k)}A_k\vec{E}_k+\sum_{(k)}A_k^*\vec{E}_k^*\right)\times\left(\sum_{(k')}A_{k'}\vec{H}_{k'}+\sum_{(k')}A_{k'}^*\vec{H}_{k'}^*\right)\right]d\nu=$$

$$=\sum_{(k)}\frac{1}{c^2}\int_{(\nu)}\left(A_kA_k^*\left[\vec{E}_k\times\vec{H}_k^*\right]+A_k^*A_k\left[\vec{E}_k^*\times\vec{H}_k\right]\right)d\nu=$$

$$=\frac{1}{c^2}\sum_{(k)}\frac{\vec{k}}{k}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(A_kA_k^*+A_k^*A_k\right).$$

Passing to operators we have

$$\hat{\vec{G}}=\frac{1}{c^2}\sum_{(k)}\frac{\vec{k}}{k}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\hbar\omega_k}{2\varepsilon_0}\left(\hat{a}_k^\dagger\hat{a}_k+\hat{a}_k\hat{a}_k^\dagger\right)=$$

$$=\sum_{(k)}\hbar\vec{k}\frac{\omega_k}{ck}\frac{1}{c\sqrt{\varepsilon_0\mu_0}}\frac{\hat{a}_k^\dagger\hat{a}_k+\hat{a}_k\hat{a}_k^\dagger}{2}=\sum_{(k)}\vec{k}\hbar\left(\hat{a}_k^\dagger\hat{a}_k+\frac{1}{2}\right).$$

By symmetry it follows that $\sum_{(k)}\vec{k}\hbar=0$ and, consequently,

$$\hat{\vec{G}}=\sum_{(k)}\vec{k}\hbar\hat{a}_k^\dagger\hat{a}_k. \tag{1.58}$$

The operator of the electric field is now expressed as follows:

$$\hat{\vec{E}} = \sum_{(k)} \hat{a}_k \sqrt{\frac{\hbar\omega_k}{2\nu\varepsilon_0}} \vec{e}_k e^{i(\vec{k}\cdot\vec{r})} + \sum_{(k)} \hat{a}_k^\dagger \sqrt{\frac{\hbar\omega_k}{2\nu\varepsilon_0}} \vec{e}_k^* e^{-i(\vec{k}\cdot\vec{r})}. \tag{1.59}$$

In the following, we will use the Dirac formulation of quantum mechanical equations; therefore, section 14.1 provides a brief presentation of the Dirac formalism: the concept of the state vector is introduced, and methods of operating with it are outlined.

1.8 Properties of operators $\hat{a}$ and $\hat{a}^\dagger$ (annihilation and creation operators)

First of all, consider the properties of operators $\hat{a}$ and $\hat{a}^\dagger$. We show that

$$\left[\hat{\mathcal{H}}, \hat{a}\right] = -\hbar\omega\hat{a}, \quad \left[\hat{\mathcal{H}}, \hat{a}^\dagger\right] = \hbar\omega\hat{a}^\dagger. \tag{1.60}$$

Since

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

then, using the commutation relations

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1,$$

we obtain

$$\begin{aligned}\left[\hat{\mathcal{H}}, \hat{a}\right] &= \hbar\omega\left(\hat{a}^\dagger\hat{a}\hat{a} + \frac{1}{2}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}\right) = \\ &= \hbar\omega\left(\hat{a}^\dagger\hat{a}\hat{a} - \left(1 + \hat{a}^\dagger\hat{a}\right)\hat{a}\right) = -\hbar\omega\hat{a},\end{aligned}$$

The second equality in (1.60) is proved similarly.

In the Schrödinger picture, operators do not depend on time, while in the Heisenberg picture they do and satisfy the equations of motion:

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{a}\right] = -i\omega\hat{a}(t), \quad \dot{\hat{a}}^\dagger(t) = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{a}^\dagger\right] = i\omega\hat{a}^\dagger(t). \tag{1.61}$$

Here the equalities (1.60) are used. The solution to these equations is:

$$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)\, e^{i\omega t}, \tag{1.62}$$

where $\hat{a}(0)$, $\hat{a}^\dagger(0)$ are operators in the Schrödinger picture.

1.9 Quantum state of the electromagnetic field with a definite energy (with a definite number of photons)

First, let's consider a single mode of the electromagnetic field (a simple harmonic oscillator). The state vector with a definite energy $|E_n\rangle$ satisfies the following equation

$$\hat{\mathcal{H}}\,|E_n\rangle = E_n\,|E_n\rangle \tag{1.63}$$

Here and henceforth, we will use Dirac's formalism (see section 14.1). Using relation (1.60), we obtain

$$\hat{\mathcal{H}}\hat{a}\,|E_n\rangle = \left(\hat{a}\hat{\mathcal{H}} - \hbar\omega\hat{a}\right)|E_n\rangle = (E_n - \hbar\omega)\,\hat{a}\,|E_n\rangle \tag{1.64}$$

i.e., $\hat{a}\,|E_n\rangle$ is also a state vector with energy $E_n - \hbar\omega$. From this, it follows that

$$\hat{a}\,|E_n\rangle = |E_n - \hbar\omega\rangle$$

Thus, the operator $\hat{a}$ lowers the energy of the state by $\hbar\omega$, where ω is the frequency of the mode (oscillator). The operator $\hat{a}$ is often called the lowering operator, or the annihilation operator. The lowest energy must be positive, and reducing the energy cannot continue indefinitely. For an arbitrary state vector, the expected energy value

$$\langle\Phi|\,\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|\Phi\rangle = \hbar\omega\,\langle\Phi'\,|\Phi'\rangle + \frac{1}{2}\hbar\omega, \tag{1.65}$$

where $\hat{a}\,|\Phi\rangle = |\Phi'\rangle$, $\langle\Phi|\,\hat{a}^\dagger = \langle\Phi'|$. Since the norm of the state vector must be positive, the minimum energy value will be at $\langle\Phi'\,|\Phi'\rangle = 0$. This means $\hat{a}\,|0\rangle = 0$, where $|\Phi\rangle = |0\rangle$ is the state vector with the lowest energy. The lowest energy

$$E_0 = \frac{\hbar\omega}{2} \tag{1.66}$$

is called the zero-point energy. To verify this claim, we can write

$$\begin{aligned}\hat{\mathcal{H}}\,|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \\ = \hbar\omega\hat{a}^\dagger\hat{a}\,|0\rangle + \frac{\hbar\omega}{2}\,|0\rangle = \\ = \frac{\hbar\omega}{2}\,|0\rangle = E_0\,|0\rangle\,.\end{aligned} \tag{1.67}$$

Using (1.60), we can obtain

$$\hat{\mathcal{H}}\hat{a}^\dagger |0\rangle = \hat{\mathcal{H}} |1\rangle = \left(\hat{a}^\dagger \hat{\mathcal{H}} + \hbar\omega \hat{a}^\dagger\right) |0\rangle = \\ = \hbar\omega \left(1 + \frac{1}{2}\right) \hat{a}^\dagger |0\rangle = \hbar\omega \left(1 + \frac{1}{2}\right) |1\rangle \tag{1.68}$$

where $|1\rangle = \hat{a}^\dagger |0\rangle$ denotes the state with energy $\hbar\omega \left(1 + \frac{1}{2}\right)$.

By induction, we have

$$\hat{\mathcal{H}} \left(\hat{a}^\dagger\right)^n |0\rangle = \hat{\mathcal{H}} |n\rangle = \hbar\omega \left(n + \frac{1}{2}\right) \left(\hat{a}^\dagger\right)^n |0\rangle = \hbar\omega \left(n + \frac{1}{2}\right) |n\rangle \tag{1.69}$$

where $|n\rangle = \left(\hat{a}^\dagger\right)^n |0\rangle$ without normalization, which we will perform later - the state with energy $\hbar\omega \left(n + \frac{1}{2}\right)$, n is a positive integer.

We see that the operator $\hat{a}^\dagger$ raises the energy of the state by $\hbar\omega$. It can be considered as the creation operator of a photon with energy $\hbar\omega$. The photon is better understood as a particle with energy $\hbar\omega$ and momentum $\hbar\vec{k}$ when considering field decomposition into plane waves, as follows from (1.58).

The relations $\hat{a} |n\rangle = |n-1\rangle$ and $\hat{a}^\dagger |n\rangle = |n+1\rangle$ define unnormalized state vectors. Let's define the normalization factor. Assume that $\hat{a} |n\rangle = S_n |n-1\rangle$, where $|n\rangle$ and $|n-1\rangle$ are normalized to 1, and S_n is the normalization factor. From this, we find

$$S_n^2 \langle n-1|n-1\rangle = \langle n| \hat{a}^\dagger \hat{a} |n\rangle = n \langle n|n\rangle$$

since the operator $\hat{a}^\dagger \hat{a} = \hat{n}$ is the photon number operator, whose eigenvalue is the photon number. This is evident from the formula (1.69). Indeed, from the equality

$$\hat{\mathcal{H}} |n\rangle = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) |n\rangle = \hbar\omega \left(n + \frac{1}{2}\right) |n\rangle ,$$

we obtain:

$$\hat{n} |n\rangle = \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle .$$

From the normalization condition, it follows: $\langle n|n\rangle = 1$ and $S_n^2 = n$, thus $S_n = \sqrt{n}$ and hence:

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \tag{1.70}$$

Similarly, using the commutation relations (1.26), and 14.7 and 14.8 from section 14.1, we find

$$\hat{a}^\dagger |n\rangle = S_{n+1} |n+1\rangle , \quad \langle n| \hat{a} = S_{n+1} \langle n+1| , \\ \langle n| \hat{a}\hat{a}^\dagger |n\rangle = S_{n+1}^2 \langle n+1|n+1\rangle = \langle n| \hat{a}^\dagger \hat{a} + 1 |n\rangle = (n+1) \langle n|n\rangle , \\ S_{n+1}^2 = n+1. \tag{1.71}$$

Therefore, we have the equality

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{1.72}$$

The eigenstates of the photon number operator $\hat{n}$ are orthonormal. Indeed, from the fact that the operator $\hat{n}$ is Hermitian:

$$\hat{n}^\dagger = \left(\hat{a}^\dagger\hat{a}\right)^\dagger = \hat{a}^\dagger\left(\hat{a}^\dagger\right)^\dagger = \hat{a}^\dagger\hat{a} = \hat{n}$$

it follows (see section 14.1) that the eigenfunctions of this operator, corresponding to different eigenvalues, are orthogonal, i.e.

$$\langle n|n'\rangle = 0, \text{ if } n \neq n'. \tag{1.73}$$

Let's summarize the relations involving the operators $\hat{a}$ and $\hat{a}^\dagger$:

$$\begin{gathered}\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \quad \hat{a}\,|0\rangle = 0, \quad \hat{a}^\dagger\hat{a}\,|n\rangle = \hat{n}\,|n\rangle\,, \\ \left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \\ \hat{\mathcal{H}}\,|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle\,, \\ \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad |n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle\end{gathered} \tag{1.74}$$

and the conjugate equalities

$$\begin{gathered}\langle 0|\,\hat{a}^\dagger = 0, \quad \langle n|\,\hat{a} = \sqrt{n+1}\,\langle n+1| \\ \langle n|\,\hat{a}^\dagger = \sqrt{n}\,\langle n-1|\,, \quad \langle n| = \frac{1}{\sqrt{n!}}\langle 0|\left(\hat{a}^\dagger\right)^n.\end{gathered} \tag{1.75}$$

For the simplest resonator model, we have

$$\hat{E}\,(z,t) = E_1\left(\hat{a} + \hat{a}^\dagger\right)\sin k_n z$$

where $E_1 = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$ is the field, corresponding to one photon in the mode.

Consider some properties of energy states, i.e., states with a definite number of photons. We will show that the average value of the electric field in this state is zero:

$$\begin{gathered}\langle n|\,\hat{E}\,|n\rangle = E_1\sin k_n z\left(\langle n|\,\hat{a}\,|n\rangle + \langle n|\,\hat{a}^\dagger\,|n\rangle\right) = \\ = E_1\sin k_n z\left(\langle n|n-1\rangle\sqrt{n} + \langle n|n+1\rangle\sqrt{n+1}\right) = 0\end{gathered} \tag{1.76}$$

which follows from the orthogonality of states (1.73) $\langle n|n'\rangle = 0$.

The average value of the square of the electric field operator is non-zero:

$$\langle n|\,\hat{E}^2\,|n\rangle = E_1^2 \sin^2 k_n z\,\langle n|\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}\right)|n\rangle = \\ = 2E_1^2 \sin^2 k_n z\left(n+\frac{1}{2}\right). \tag{1.77}$$

The fields usually considered in quantum optics are not in a stationary state with a definite energy (with a definite number of photons). However, an arbitrary state can be represented as a superposition of states $|n\rangle$:

$$|\psi\rangle = \sum_{(n)} C_n\,|n\rangle \tag{1.78}$$

where $|C_n|^2$ is the probability of finding n photons in the mode upon measurement; $\sum_{(n)}|C_n|^2 = 1$. In section 14.1, it is shown that

$$C_n = \langle n|\,|\psi\rangle\,, \quad |\psi\rangle = \sum_{(n)}\langle n|\,|\psi\rangle\,|n\rangle = \sum_{(n)}|n\rangle\,\langle n|\,|\psi\rangle\,, \quad \sum_{(n)}|n\rangle\,\langle n| = \hat{I},$$

where $\hat{I}$ is the identity operator.

Remark 1.9.1 (On states with definite energy in quantum mechanics)**.** *It is worth noting that states with definite energy do not violate Heisenberg's uncertainty relation for the pair energy - time*

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

(see subsection 14.3.2 for more details).

Meanwhile, if we look at the energy operator of the harmonic oscillator, written as

$$\hat{\mathcal{H}} = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right)$$

and use expressions (1.24)*, we find that* $\langle n|\,\hat{q}\,|n\rangle = \langle n|\,\hat{p}\,|n\rangle = 0$*. Meanwhile,*

$$\langle n|\,\hat{q}^2\,|n\rangle = \frac{\hbar}{2\omega}\Big[\langle n|\,\hat{a}^2\,|n\rangle + \langle n|\left(\hat{a}^\dagger\right)^2|n\rangle + \\ + \langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle + \langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle\Big] = \\ = \frac{\hbar}{2\omega}\left[n + n + 1\right],$$

and also

$$\langle n|\,\hat{p}^2\,|n\rangle = -\frac{\hbar\omega}{2}\left[\langle n|\,\hat{a}^2\,|n\rangle + \langle n|\,\left(\hat{a}^\dagger\right)^2|n\rangle - \right.$$
$$\left. - \langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle - \langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle\right] =$$
$$= \frac{\hbar\omega}{2}\left[n + n + 1\right].$$

Thus,

$$\Delta p = \sqrt{\langle n|\,\hat{p}^2\,|n\rangle - \langle n|\,\hat{p}\,|n\rangle^2} = \sqrt{\frac{\hbar\omega}{2}\left(2n+1\right)}$$

and

$$\Delta q = \sqrt{\langle n|\,\hat{q}^2\,|n\rangle - \langle n|\,\hat{q}\,|n\rangle^2} = \sqrt{\frac{\hbar}{2\omega}\left(2n+1\right)}$$

or

$$\Delta p \Delta q = \frac{\hbar}{2}\left(2n+1\right) \geq \frac{\hbar}{2},$$

which is consistent with Heisenberg's uncertainty relations (14.47).

Thus, states with definite energy are states in which, despite the impossibility of precisely determining p and q, it is possible to determine the value of $\frac{1}{2}\left(p^2+\omega^2 q^2\right)$. This reflects the fact that for quantum systems, situations are possible where composite events exist in the absence of elementary ones (see section 15.3). In particular, such a situation occurs for the photon - states with definite energy $|1\rangle$.

This may lead us to consider that a photon is a virtual particle, which does not exist in the real physical world [17]. Meanwhile, the mathematical apparatus related to the concept of the photon such as creation operators $\hat{a}^\dagger$ and annihilation $\hat{a}$, electromagnetic field states with definite energy $\{|n\rangle\}$ (which can be used as basis states), is convenient for theoretical descriptions.

It is also worth noting that from the perspective of the formal definition of a non-classical state (8.5), *$|n\rangle$ is a non-classical state of light since from* (5.29) *it follows $G^{(2)} < 1$.*

On the other hand, if we consider the minimum possible energy of the electromagnetic field mode, from the relation (1.19)

$$\mathcal{H} = \frac{1}{2}\left(\omega^2 q^2 + p^2\right)$$

it follows that in the classical case, the minimum possible zero energy is reached at $p = 0, q = 0$, but due to (14.47), *zero values are impossible in the quantum*

case, and considering measurement uncertainties we find

$$\frac{1}{2}\left(\omega^2(\Delta q)^2+(\Delta p)^2\right)\geq$$
$$\geq\omega\Delta q\Delta p\geq\frac{\hbar\omega}{2}$$

that is, the minimum possible energy (vacuum energy) is determined by Heisenberg's inequalities for the coordinate - momentum pair.

1.10 Multimode States

So far, we have considered single-mode fields. The results are easily generalized to a multimode state. For an electromagnetic field in some volume, we have the expansion (1.11):

$$\vec{E}\left(r,t\right)=\sum_{(s)}\frac{q_s\left(t\right)\omega_s}{\sqrt{\varepsilon_0}}\vec{E}_s\left(r\right),\quad\vec{H}\left(r,t\right)=\sum_{(s)}\frac{p_s\left(t\right)}{\sqrt{\mu_0}}\vec{H}_s\left(r\right),\tag{1.79}$$

The Hamiltonian function is equal to the sum of the Hamiltonian functions of all modes, as the modes correspond to independent oscillators:

$$\mathcal{H}=\frac{1}{2}\sum_{(s)}\left(\omega_s^2q_s^2+p_s^2\right)\tag{1.80}$$

Quantization reduces to replacing q_s and p_s with operators $\hat{q}_s$ and $\hat{p}_s$ with commutation relations:

$$[\hat{q}_s,\hat{p}_{s'}]=i\hbar\delta_{ss'},\quad[\hat{q}_s,\hat{q}_{s'}]=[\hat{p}_s,\hat{p}_{s'}]=0.\tag{1.81}$$

The Hamiltonian function then becomes the Hamiltonian operator:

$$\hat{\mathcal{H}}=\frac{1}{2}\sum_{(s)}\left(\omega_s^2\hat{q}_s^2+\hat{p}_s^2\right)\tag{1.82}$$

As in section 1.3, we introduce creation and annihilation operators for each mode (1.23):

$$\hat{a}_s=\frac{1}{\sqrt{2\hbar\omega_s}}\left(\omega_s\hat{q}_s+i\hat{p}_s\right),\quad\hat{a}_s^\dagger=\frac{1}{\sqrt{2\hbar\omega_s}}\left(\omega_s\hat{q}_s-i\hat{p}_s\right),\tag{1.83}$$

Using them, the Hamiltonian can be represented as

$$\hat{\mathcal{H}}_s=\omega_n\hbar\left(\hat{a}_s^\dagger\hat{a}_s+\frac{1}{2}\right)$$

The general state, when there are n_1 photons in the first mode, n_2 photons in the second, and n_s photons in the s-th mode, etc., can be represented as a product of state vectors of each mode (see section 14.5):

$$|\{n_s\}\rangle = |n_1, n_2, \dots, n_s, \dots\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_s\rangle \otimes \dots,$$

where $\{n_s\}$ denotes the set of mode occupation numbers.

The action of the operators $\hat{a}_s^\dagger$ and $\hat{a}_s$ related to the s-th mode on the state vector is described by the equations

$$\begin{aligned} \hat{a}_s |n_1, n_2, \dots, n_s, \dots\rangle &= \sqrt{n_s}\, |n_1, n_2, \dots, n_s - 1, \dots\rangle, \\ \hat{a}_s^\dagger |n_1, n_2, \dots, n_s, \dots\rangle &= \sqrt{n_s + 1}\, |n_1, n_2, \dots, n_s + 1, \dots\rangle. \end{aligned} \tag{1.84}$$

The state vector in general can be represented as a linear superposition of states $|\{n_s\}\rangle$:

$$|\Psi\rangle = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_s} \dots C_{n_1,n_2,\dots,n_s,\dots} |n_1, n_2, \dots, n_s, \dots\rangle = \sum_{\{n_s\}} C_{\{n_s\}} |\{n_s\}\rangle \tag{1.85}$$

When decomposing the field into plane waves, the electric field operator has the form

$$\hat{\vec{E}} = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \left(\hat{a}_k(t)\, \vec{e}_k e^{i(\vec{k}\vec{r})} + \hat{a}_k^\dagger(t)\, \vec{e}_k^* e^{-i(\vec{k}\vec{r})} \right) \tag{1.86}$$

This expression is written in the Heisenberg representation (the operator depends on time). In this representation

$$\hat{a}_k(t) = \hat{a}_k(0)\, e^{-i\omega_k t}, \quad \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0)\, e^{i\omega_k t},$$

where $\hat{a}_k(0)$, $\hat{a}_k^\dagger(0)$ - operators in the Schrödinger representation. In the Schrödinger representation in expression (1.86), operators do not depend on time. For each mode, we have equations (1.76), (1.77):

$$\langle n_k | \hat{a}_k | n_k \rangle = \langle n_k | \hat{a}_k^\dagger | n_k \rangle = 0$$

From here for the full field, we obtain

$$\langle \{n_k\} | \hat{\vec{E}} | \{n_k\} \rangle = 0, \quad \langle \{n_k\} | \hat{\vec{E}}^2 | \{n_k\} \rangle = \sum_{(k)} \frac{\hbar\omega_k}{\varepsilon_0 V} \left(n + \frac{1}{2} \right) \tag{1.87}$$

Expression (1.86) can be broken into two terms: the positive frequency part, which includes annihilation operators, and the negative frequency part, which

includes creation operators:

$$\hat{\vec{E}} = \hat{\vec{E}}^{(+)} + \hat{\vec{E}}^{(-)},$$
$$\hat{\vec{E}}^{(+)} = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \hat{a}_k(t)\, \vec{e}_k e^{i(\vec{k}\vec{r})},$$
$$\hat{\vec{E}}^{(-)} = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \hat{a}_k^{\dagger}(t)\, \vec{e}_k^{*} e^{-i(\vec{k}\vec{r})} \tag{1.88}$$

$\hat{\vec{E}}^{(+)}$ corresponds to the analytic signal in the classical case.

1.11 Coherent States

States $|n\rangle$ do not tend with increasing n to the classical solution for the oscillator. This property is possessed by the so-called coherent states. They were introduced by E. Schrödinger and later developed in our time by R. Glauber [56] and E. Sudarshan [47] in relation to problems of quantum optics. A coherent state can be defined in various ways. We will use the definition in the form of a superposition of states $|n\rangle$:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{(n)} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \langle\alpha| = e^{-\frac{1}{2}|\alpha|^2} \sum_{(n)} \frac{\alpha^{*n}}{\sqrt{n!}} \langle n|, \tag{1.89}$$

where α is some complex parameter, the meaning of which will become clear later.

From the definition (1.89) follows a number of properties of coherent states. First of all,

$$\langle\alpha|\,\alpha\rangle = e^{-|\alpha|^2} \sum_{(n)} \frac{(\alpha\alpha^*)^n}{n!} = e^{|\alpha|^2} e^{-|\alpha|^2} = 1, \tag{1.90}$$

i.e., coherent states are normalized. However, these states are not orthogonal:

$$\langle\alpha|\,\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{(n)} \frac{(\alpha^*\beta)^n}{n!} = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta} \tag{1.91}$$

Hence

$$|\langle\alpha|\,\beta\rangle|^2 = e^{-|\alpha|^2 - |\beta|^2 + \alpha^*\beta + \alpha\beta^*} = e^{-|\alpha-\beta|^2}, \tag{1.92}$$

i.e., these states can be considered approximately orthogonal if $|\alpha - \beta|$ is sufficiently large.

The non-orthogonality of coherent states is a consequence of the over-completeness of the system (α is any complex number, whereas n are only integers).

Coherent states are eigenstates of the annihilation operator with an eigenvalue of α:

$$\hat{a}\,|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{(n)}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{(m=n-1)}\frac{\alpha\alpha^m}{\sqrt{m!}}\,|m\rangle = \alpha\,|\alpha\rangle \tag{1.93}$$

The conjugate relationship looks like

$$\langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha| \tag{1.94}$$

The probability of detecting (measuring) n photons in a coherent state

$$|\langle n|\,\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{\left(|\alpha|^2\right)^n}{n!} \tag{1.95}$$

corresponds to the Poisson distribution with an average number of photons $\langle n\rangle = |\alpha|^2$. Using the known relationship

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle \tag{1.96}$$

the coherent state can be represented as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{(n)}\frac{\left(\alpha\hat{a}^\dagger\right)^n}{n!}\,|0\rangle = e^{\alpha\hat{a}^\dagger-\frac{1}{2}|\alpha|^2}\,|0\rangle \tag{1.97}$$

Rewriting (1.97) in another form, we use the following relation

$$e^{-\alpha^*\hat{a}}\,|0\rangle = (1-\alpha^*\hat{a}+\dots)\,|0\rangle = |0\rangle\,,$$

resulting in

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger}e^{-\frac{1}{2}|\alpha|^2}e^{-\alpha^*\hat{a}}\,|0\rangle\,. \tag{1.98}$$

Next, we apply the operator identity (Baker-Hausdorff formula)

$$e^{\hat{c}+\hat{d}} = e^{-\frac{1}{2}\left[\hat{c},\hat{d}\right]}e^{\hat{c}}e^{\hat{d}} \tag{1.99}$$

valid under the condition $\left[\hat{c},\left[\hat{c},\hat{d}\right]\right] = \left[\hat{d},\left[\hat{c},\hat{d}\right]\right] = 0$. If we let $\hat{c} = \alpha\hat{a}^\dagger$, $\hat{d} = -\alpha^*\hat{a}$ then from (1.98) we get

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger}e^{-\frac{1}{2}|\alpha|^2}e^{-\alpha^*\hat{a}}\,|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}\,|0\rangle = \\ = e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}\,|0\rangle\,. \tag{1.100}$$

In order to vividly represent coherent states, in section 14.6 a coordinate representation of the coherent state is given.

Here are a few more useful relationships obtained from (1.97). Differentiating (1.97) by α, we have:

$$\frac{\partial}{\partial\alpha}|\alpha\rangle = \left(\hat{a}^\dagger - \frac{1}{2}\alpha^*\right)|\alpha\rangle,$$

or equivalently

$$\hat{a}^\dagger|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{1}{2}\alpha^*\right)|\alpha\rangle.$$

The conjugate equality looks like

$$\langle\alpha|\,\hat{a} = \left(\frac{\partial}{\partial\alpha^*} + \frac{1}{2}\alpha\right)\langle\alpha|$$

In general, the operator of the positive frequency part of the electric field looks like

$$\hat{\vec{E}}^{(+)} = \sum_{(k)}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{a}_k(t)\,\vec{e}_k e^{-i\omega_k t + i\left(\vec{k}\vec{r}\right)},$$

Suppose that the field (multimode) is in the state

$$|\{\alpha_s\}\rangle = |\alpha_1,\alpha_2,\ldots,\alpha_s,\ldots\rangle = |\alpha_1\rangle\otimes|\alpha_2\rangle\otimes\cdots\otimes|\alpha_s\rangle\otimes\ldots,$$

where $\{\alpha_s\} = \{\alpha_{k_s}\}$ denotes a certain set of parameters α. Applying the operator $\hat{\vec{E}}^{(+)}$ to the state $|\{\alpha_{k_s}\}\rangle$, we obtain

$$\hat{\vec{E}}^{(+)}|\{\alpha_{k_s}\}\rangle = \left\{\sum_{(k_s)}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{a}_{k_s}\vec{e}_k e^{-i\omega_k t + i\left(\vec{k}\vec{r}\right)}\right\}|\{\alpha_{k_s}\}\rangle =$$

$$= \sum_{(k_s)}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\alpha_{k_s}\vec{e}_k e^{-i\omega_k t + i\left(\vec{k}\vec{r}\right)}|\{\alpha_{k_s}\}\rangle, \qquad (1.101)$$

i.e., $|\{\alpha_{k_s}\}\rangle$ is an eigenvector of the positive-frequency part of the electric field operator, and the eigenvalue is a classical field (analytical signal), whose complex mode amplitudes are $\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\alpha_k$. From this, it follows that each classical field corresponds to some coherent state.

The average value of the electric field operator $\hat{\vec{E}} = \hat{\vec{E}}^{(+)} + \hat{\vec{E}}^{(-)}$ in a coherent state is obtained by first considering a single mode:

$$\langle\alpha_k|\,\hat{\vec{E}}_k\,|\alpha_k\rangle = 2\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,|\alpha_k|\cos\left(\omega_k t - \left(\vec{k}\vec{r} + \theta_k\right)\right),$$

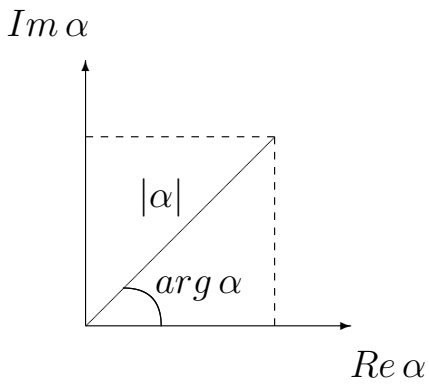


Figure 1.5: Complex plane on which coherent states are defined. Polar coordinates.

where $\alpha_k = |\alpha_k| e^{i\theta_k}$, which follows from the relations

$$\langle \alpha_k | \hat{a}_k | \alpha_k \rangle = \alpha_k, \quad \langle \alpha_k | \hat{a}_k^\dagger | \alpha_k \rangle = \alpha_k^*$$

Generalizing this to a multimode case, we obtain

$$\langle \{\alpha_k\} | \hat{\vec{E}} | \{\alpha_k\} \rangle = 2 \sum_{(k)} \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}} |\alpha_k| \cos\left(\omega_k t - \left(\vec{k}\vec{r}\right) + \theta_k\right), \tag{1.102}$$

which corresponds to a classical multimode field, i.e., the average value of $\hat{\vec{E}}$ in a coherent state is a classical field with corresponding mode amplitudes and phases.

Coherent states, as we have seen, are not orthogonal due to the over-completeness of the system of eigenvectors, but they satisfy the completeness condition:

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| \, d^2\alpha = \hat{I}, \tag{1.103}$$

Here $d^2\alpha = d\,Re\alpha\, d\,Im\alpha$, $\hat{I}$ is the identity operator, and the integration is carried out over the entire complex plane. We will prove this using polar coordinates, shown in Figure 1.5.

Using the decomposition of $|\alpha\rangle$ by $|n\rangle$ in polar coordinates, we get

$$\int d^2\alpha \, |\alpha\rangle \langle\alpha| =$$

$$= \int_0^\infty d\,|\alpha| \int_0^{2\pi} |\alpha| \sum_{(n)} \sum_{(m)} e^{-i(n-m)\varphi} e^{-|\alpha|^2} |m\rangle \langle n| \frac{|\alpha|^m |\alpha|^n}{\sqrt{n!m!}} d\varphi =$$

$$= \pi \sum_{(n)} \frac{1}{n!} \int_0^\infty 2\,|\alpha|\,|\alpha|^{2n} e^{-|\alpha|^2} |n\rangle \langle n|\, d\,|\alpha| =$$

$$= \pi \sum_{(n)} \frac{|n\rangle \langle n|}{n!} \int_0^\infty x^n e^{-x} dx = \pi \sum_{(n)} |n\rangle \langle n| = \pi \hat{I}, \quad (1.104)$$

where it is considered that

$$\int_0^{2\pi} e^{-i(n-m)\varphi} = 2\pi\delta_{nm},$$

a change of variables was made:

$$|\alpha|^2 = x, \quad 2\,|\alpha|\, d\,|\alpha| = dx,$$

and the integral representation of the factorial is used:

$$n! = \int_0^\infty x^n e^{-x} dx$$

Thus, the formula (1.103) is proved. With the help of this relationship, one coherent state can be expressed through all the others:

$$|\alpha\rangle = \frac{1}{\pi} \int |\beta\rangle \langle\beta|\, \alpha\rangle \, d^2\beta,$$

but in (1.91) we had $\langle\alpha|\, \beta\rangle = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta}$, thus,

$$|\alpha\rangle = \frac{1}{\pi} \int |\beta\rangle \, e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta} d^2\beta. \quad (1.105)$$

1.12 Mixed States of the Electromagnetic Field

So far, we have only considered pure states of the electromagnetic field, described by state vectors. One of the most distinctive properties of pure states is the principle of superposition.

Only isolated systems can be pure systems, whereas mixed systems are not isolated from their environment. The density matrix formalism is used to describe mixed states. The density operator can be introduced as follows.

Consider a mixed state, for which the probabilities P_M of the field being in each of the states $|M\rangle$ are known. We calculate the expectation value of the operator $\hat{O}$, corresponding to the observable O in this statistically mixed state. The expectation for the statistical mixture of states

$$\left\langle \hat{O} \right\rangle = \sum_{(M)} P_M \langle M| \hat{O} |M\rangle , \tag{1.106}$$

where, obviously,

$$\sum_{(M)} P_M = 1.$$

This expression can be written in another form. Using some complete set of states $|S\rangle$ and its completeness condition

$$\sum_{(S)} |S\rangle \langle S| = \hat{I},$$

we obtain

$$\left\langle \hat{O} \right\rangle = \sum_{(M)} P_M \sum_{(S)} \langle M| \hat{O} |S\rangle \langle S|M\rangle = \sum_{(M)} \sum_{(S)} P_M \langle S|M\rangle \langle M| \hat{O} |S\rangle . \tag{1.107}$$

From (1.107), a statistical operator $\hat{\rho}$ can be introduced using the expression

$$\hat{\rho} = \sum_{(M)} P_M |M\rangle \langle M| . \tag{1.108}$$

Then the equality (1.107) can be presented as

$$\left\langle \hat{O} \right\rangle = \sum_{(S)} \langle S| \hat{\rho}\hat{O} |S\rangle = Sp \left(\hat{\rho}\hat{O} \right) , \tag{1.109}$$

This relation does not depend on the choice of $|S\rangle$, as the trace of an operator is representation-independent. Note that

$$Sp\left(\hat{\rho}\right) = \sum_{(M)} \langle M| \hat{\rho} |M\rangle = \sum_{(M)} P_M = 1. \tag{1.110}$$

As an example, consider thermal photon excitation in one mode:

$$\hat{\rho} = \sum_{(n)} P_n |n\rangle \langle n| , \tag{1.111}$$

where P_n is determined by the Boltzmann distribution

$$P_n = e^{-\beta n\hbar\omega}\left(1 - e^{-\beta\hbar\omega}\right),$$

here $\beta = \frac{1}{k_b T}$, and $\left(1 - e^{-\beta\hbar\omega}\right)$ is a normalization factor.

The average number of photons in the mode under thermal excitation:

$$\begin{aligned}\bar{n} = \langle\hat{n}\rangle = Sp\left(\hat{\rho}\hat{n}\right) = Sp\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right) = \\ = \sum_{(m)}\sum_{(n)} P_n \langle m|n\rangle \langle n|\, \hat{a}^\dagger\hat{a}\, |m\rangle = \\ = \sum_{(n)} P_n \langle n|\, \hat{n}\, |n\rangle = \sum_{(n)} n e^{-\beta n\hbar\omega}\left(1 - e^{-\beta\hbar\omega}\right) = \frac{1}{e^{\beta\hbar\omega} - 1}\end{aligned} \tag{1.112}$$

This is the well-known expression derived by Planck. We used the following expression

$$\sum_{n=0}^{\infty} n r^{n-1} = \frac{d}{dr}\sum_{n=0}^{\infty} r^n = \frac{1}{(1-r)^2},$$

valid for $|r| < 1$.

Equations (1.111), (1.112) can be rewritten differently. From (1.112) we get

$$e^{-\beta\hbar\omega} = \frac{\bar{n}}{1+\bar{n}},$$

hence,

$$P_n = e^{-\beta\hbar\omega n}\left(1 - e^{-\beta\hbar\omega}\right) = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}. \tag{1.113}$$

Thus, the density matrix takes the form:

$$\hat{\rho} = \sum_{(n)} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle \langle n|.$$

This formula describes the density matrix of chaotic light via $\bar{n}$, the mean number of photons in the mode. Here, $\bar{n}$ does not necessarily depend on frequency according to Planck's formula (1.113). The dependence of $\bar{n}$ on frequency determines the line shape of chaotic light (different frequencies correspond to different modes). Generalizing the result, we can write the expression for the statistical operator of chaotic light:

$$\hat{\rho} = \sum_{\{n_k\}} P_{\{n_k\}} |\{n_k\}\rangle \langle\{n_k\}| = \sum_{\{n_k\}} |\{n_k\}\rangle \langle\{n_k\}| \prod_{\{k\}} \frac{\bar{n}_k^{n_k}}{(1+\bar{n}_k)^{n_k+1}}, \tag{1.114}$$

where $\bar{n}_k$ depends on frequency. For example, for chaotic light with a Lorentzian line,

$$\bar{n}_k = \frac{IS}{\hbar\omega_{k_0}} \frac{\gamma}{\left(\omega_{k_0} - \omega_k\right)^2 + \gamma^2},$$

where $\frac{IS}{\hbar\omega_{k_0}}$ is the average number of photons in the beam; I is the beam intensity (energy flux); S is the beam cross-section.

1.13 Representation of the Density Operator through Coherent States

An important aspect of quantum optics is the description of optical phenomena using coherent states. By using the double completeness condition for states with different numbers of photons, for a given statistical operator, we obtain:

$$\hat{\rho} = \hat{I}\hat{\rho}\hat{I} = \sum_{(n)}\sum_{(m)} |n\rangle \langle n| \hat{\rho} |m\rangle \langle m| = \sum_{(n)}\sum_{(m)} \rho_{mn} |n\rangle \langle m|$$

Similarly for coherent states

$$\hat{\rho} = \hat{I}\hat{\rho}\hat{I} = \frac{1}{\pi^2} \int d^2\alpha \int |\alpha\rangle \langle\alpha| \hat{\rho} |\beta\rangle \langle\beta| d^2\beta =$$
$$= \frac{1}{\pi^2} \int d^2\alpha \int R\left(\alpha^*, \beta\right) e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2} |\alpha\rangle \langle\beta| d^2\beta, \tag{1.115}$$

where

$$R\left(\alpha^*, \beta\right) = \langle\alpha| \hat{\rho} |\beta\rangle e^{\frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2}$$

Expressing $|\alpha\rangle$ and $|\beta\rangle$ through states $|n\rangle$, we obtain the formula

$$R\left(\alpha^*, \beta\right) = \sum_{(n)}\sum_{(m)} \frac{\left(\alpha^*\right)^n \left(\beta\right)^m}{\sqrt{n!m!}} \rho_{nm} \tag{1.116}$$

i.e., $R\left(\alpha^*, \beta\right)$ can be easily found if the density matrix is known in the Fock state (photon number) representation. For example, consider thermal excitation of a mode:

$$R\left(\alpha^*, \beta\right) = \sum_{(n)}\sum_{(m)} \frac{\left(\alpha^*\right)^n \left(\beta\right)^m}{\sqrt{n!m!}} e^{-\beta n\hbar\omega} \left(1 - e^{-\beta\hbar\omega}\right) \delta_{nm} =$$
$$= \sum_{(n)} \frac{\left(\alpha^*\beta\right)^n}{n!} e^{-\beta n\hbar\omega} \left(1 - e^{-\beta\hbar\omega}\right) = \left(1 - e^{-\beta\hbar\omega}\right) e^{\alpha^*\beta e^{-\beta\hbar\omega}}. \tag{1.117}$$

This representation depends on two parameters α^* and β. A more convenient representation of the density matrix, introduced by R. Glauber [56], called the diagonal representation, depends on one parameter. It has the form

$$\hat{\rho} = \int P(\alpha) |\alpha\rangle \langle\alpha| d^2\alpha, \tag{1.118}$$

where $P(\alpha)$ is a real function, satisfying the condition

$$\int P(\alpha) d^2\alpha = 1.$$

Furthermore, $P(\alpha)$ is a real function of a complex argument. This follows from the conditions $Sp(\hat{\rho}) = 1$ and $\hat{\rho} = \hat{\rho}^\dagger$.

Such decomposition is possible due to the overcompleteness of the coherent state system. If $P(\alpha)$ is a positive function, it can be interpreted as a probability distribution. This is applicable to some of the most practically interesting field states, such as a completely chaotic state, but it is not true in general. Sometimes $P(\alpha)$ may be negative in some limited range of values of α, in which case it cannot be interpreted as a probability distribution. $P(\alpha)$ can also be a generalized function (an example is a δ-function).

A rigorous justification of the possibility of representation (1.118) is contained in the literature [56], [47]. The possibility of introducing the diagonal representation (1.118) can be justified as follows. Assume the statistical operator $\hat{\rho}$, like any other operator acting on the electromagnetic field, can be represented as a function of creation and annihilation operators:

$$\hat{\rho} = \bar{f}\left(\hat{a}^\dagger, \hat{a}\right).$$

The operator can then be represented in ordered form: normal and antinormal. In the first case, creation operators are placed to the right of the annihilation operators, and in the second case, vice versa - to the left. For example,

$$\left(\hat{a}^\dagger\right)^m (\hat{a})^n$$

is a normally ordered operator. In an antinormally ordered operator, the order is reversed: creation operators are on the right of the annihilation operator. For example, the operator

$$(\hat{a})^n \left(\hat{a}^\dagger\right)^m$$

is antinormally ordered. Ordering can be achieved by repeatedly applying the commutation relation. For example, the operator

$$\left(\hat{a}^\dagger \hat{a}\right)^2 = \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}$$

is neither normal nor antinormal. Using the condition $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$, we bring it to normal form:

$$\left(\hat{a}^{\dagger}\hat{a}\right)^{2} = \hat{a}^{\dagger}\left(1 + \hat{a}^{\dagger}\hat{a}\right)\hat{a} = \hat{a}^{\dagger}\hat{a} + \left(\hat{a}^{\dagger}\right)^{2}\left(\hat{a}\right)^{2}$$

this operator can be presented in antinormal form:

$$\left(\hat{a}^{\dagger}\hat{a}\right)^{2} = \left(\hat{a}\hat{a}^{\dagger} - 1\right)\left(\hat{a}\hat{a}^{\dagger} - 1\right) = \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} - 2\hat{a}\hat{a}^{\dagger} + 1 =$$
$$= \hat{a}\left(\hat{a}\hat{a}^{\dagger} - 1\right)\hat{a}^{\dagger} - 2\hat{a}\hat{a}^{\dagger} + 1 = \left(\hat{a}\right)^{2}\left(\hat{a}^{\dagger}\right)^{2} - 3\hat{a}\hat{a}^{\dagger} + 1.$$

There are other, more efficient methods for ordering operators [61].

Let's represent the density operator in antinormal form (considering a single-mode case):

$$\hat{\rho}^{(a)}\left(\hat{a}, \hat{a}^{\dagger}\right) = \sum_{(n)}\sum_{(m)} C_{nm}^{(a)}\left(\hat{a}\right)^{n}\left(\hat{a}^{\dagger}\right)^{m} \tag{1.119}$$

Now use the decomposition (1.103) for the identity operator $\hat{I}$

$$\hat{I} = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^{2}\alpha$$

as a result, we obtain

$$\hat{\rho}^{(a)}\left(\hat{a}, \hat{a}^{\dagger}\right) = \sum_{(n)}\sum_{(m)} C_{nm}^{(a)}\left(\hat{a}\right)^{n}\hat{I}\left(\hat{a}^{\dagger}\right)^{m} =$$
$$= \frac{1}{\pi}\sum_{(n)}\sum_{(m)} C_{nm}^{(a)}\int d^{2}\alpha\left(\hat{a}\right)^{n}|\alpha\rangle\langle\alpha|\left(\hat{a}^{\dagger}\right)^{m} =$$
$$= \frac{1}{\pi}\sum_{(n)}\sum_{(m)}\int d^{2}\alpha C_{nm}^{(a)}\alpha^{n}\alpha^{*m}|\alpha\rangle\langle\alpha|. \tag{1.120}$$

Now denoting

$$P\left(\alpha, \alpha^{*}\right) = \frac{1}{\pi}\sum_{(n)}\sum_{(m)} C_{nm}^{(a)}\alpha^{n}\alpha^{*m}$$

we find that $\hat{\rho}$ can be represented as

$$\hat{\rho} = \int d^{2}\alpha P\left(\alpha, \alpha^{*}\right)|\alpha\rangle\langle\alpha|. \tag{1.121}$$

The average value of an operator using the diagonal representation

$$\left\langle \hat{O} \right\rangle = Sp\left(\hat{\rho}\hat{O}\right) =$$

$$= \sum_{(n)} \langle n| \int d^2\alpha P(\alpha,\alpha^*) |\alpha\rangle \langle\alpha| \hat{O} |n\rangle =$$

$$= \sum_{(n)} \int d^2\alpha P(\alpha,\alpha^*) \langle\alpha| \hat{O} |n\rangle \langle n| \alpha\rangle =$$

$$= \int d^2\alpha P(\alpha,\alpha^*) \langle\alpha| \hat{O} |\alpha\rangle , \qquad (1.122)$$

since $\sum_{(n)} |n\rangle \langle n| = \hat{I}$.

The expression $\langle\alpha| \hat{O} |\alpha\rangle$ is easily calculated if operator $\hat{O}$ is presented in normal form:

$$\hat{O} = \hat{O}^{(n)} = \sum_{(n)} \sum_{(m)} d_{nm} \left(\hat{a}^\dagger\right)^n (\hat{a})^m , \qquad (1.123)$$

then

$$\langle\alpha| \hat{O} |\alpha\rangle = \sum_{(n)} \sum_{(m)} d_{nm} \langle\alpha| \left(\hat{a}^\dagger\right)^n (\hat{a})^m |\alpha\rangle = \sum_{(n)} \sum_{(m)} d_{nm} \alpha^{*n} \alpha^m . \qquad (1.124)$$

From here follows a simple rule: to obtain the necessary matrix element, we replace operators in normal ordered form by $\hat{a} \to \alpha$, $\hat{a}^\dagger \to \alpha^*$.

As an example, we show that for thermal excitation of a mode

$$P(\alpha,\alpha^*) = \frac{1}{\pi\bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}} \qquad (1.125)$$

and, therefore

$$\hat{\rho} = \frac{1}{\pi\bar{n}} \int d^2\alpha e^{-\frac{|\alpha|^2}{\bar{n}}} |\alpha\rangle \langle\alpha| .$$

Now find matrix elements of operator $\hat{\rho}$ in the Fock state (photon number) representation

$$\langle n| \hat{\rho} |n\rangle = \frac{1}{\pi\bar{n}} \int d^2\alpha e^{-\frac{|\alpha|^2}{\bar{n}}} \langle n| \alpha\rangle \langle\alpha| |n\rangle =$$

$$= \frac{1}{\pi\bar{n}} \int d^2\alpha e^{-\frac{|\alpha|^2}{\bar{n}}} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

The resulting integral can be computed using polar coordinates:

$$\alpha = re^{i\varphi}, \quad d^2\alpha = rdrd\varphi, \quad \text{i.e. } |\alpha| = r, \quad arg\,\alpha = \varphi,$$

from where

$$\frac{1}{\pi\bar{n}}\int d^2\alpha e^{-\frac{|\alpha|^2}{\bar{n}}}\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} =$$
$$= \frac{1}{\pi\bar{n}}\int_0^{2\pi} d\varphi \int_0^{\infty} rdr\frac{exp\left(-r^2\frac{\bar{n}+1}{\bar{n}}\right)}{n!}r^{2n} =$$
$$= \frac{1}{\bar{n}}\int_0^{\infty} 2rdr\frac{exp\left(-r^2\frac{\bar{n}+1}{\bar{n}}\right)}{n!}r^{2n} =$$
$$= \frac{1}{\bar{n}}\int_0^{\infty}\frac{e^{-x}}{n!}\frac{\bar{n}^{n+1}}{(\bar{n}+1)^{n+1}}x^n dx =$$
$$= \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\frac{1}{n!}\int_0^{\infty} e^{-x}x^n dx = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}, \tag{1.126}$$

where a change of variable $x = r^2\frac{\bar{n}+1}{\bar{n}}$ is made.

Thus, we have obtained the well-known matrix element of the statistical operator of a chaotic field in the Fock state representation. Hence, the expression (1.125) is indeed the statistical operator of a chaotic field in the coherent state representation. Therefore, for thermal photon excitation in a mode we have

$$P(\alpha) = \frac{1}{\pi\bar{n}}e^{-\frac{|\alpha|^2}{\bar{n}}}, \text{ where } \bar{n} = \frac{1}{e^{\hbar\omega\beta}-1} \tag{1.127}$$

Generalizing the obtained expressions to the multimode case is straightforward:

$$\left\langle\hat{O}\right\rangle = \int_{\{\alpha_k\}}\cdots\int P^{(a)}(\{\alpha_k\})\,O^{(n)}(\{\alpha_k\})\prod_k d^2\alpha_k, \tag{1.128}$$
$$P^{(a)}(\{\alpha_k\}) = \prod_k \frac{1}{\pi\bar{n}_k}e^{-\frac{|\alpha_k|^2}{\bar{n}_k}}.$$

1.14 Exercises

1. Prove the orthogonality of the fields of eigen vibrations (1.14).
2. Prove the equalities for plane waves (1.52).
3. Based on equality (1.57), derive the expression (1.58) for the momentum operator of the quantized electromagnetic field.
4. Prove the Baker-Hausdorff formula (1.99)
5. How many modes of the electromagnetic field with wavelength $\lambda \geq 500$nm are in a quantization cube with a side of $L = 1$mm [14]

Chapter 2

Interaction of Light with an Atom

Issues related to the interaction of quantized electromagnetic fields with an atom are considered. A simplified model of an atom is used - a two-level atom. Such simplification is justified under resonant interactions and is widely used in problems of quantum electronics and quantum optics. Considerable attention is given to the interaction of an atom, a cavity mode (dynamic system) with a thermostat (dissipative system), which is responsible for the relaxation of the dynamic system.

2.1 Emission and Absorption of Light by an Atom

Consider the simplest problem of the interaction of single-mode radiation with a two-level atom. The simplifications are justified because laser radiation can often be considered single-mode, and in resonant interaction, all levels of the atom except two can be neglected, with the transition frequency between them close to the frequency of the mode interacting with the atom. This simplest situation is depicted in Figure 2.1

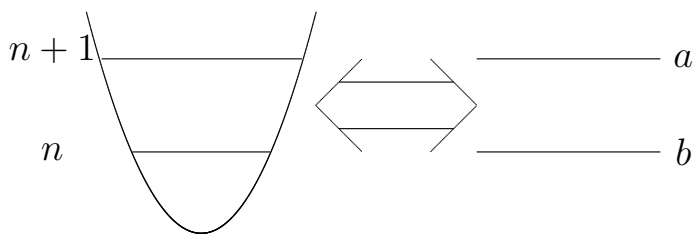


Figure 2.1: Interaction of an electromagnetic field mode with a two-level atom.

Assume that the atom and the field are initially in the states

$$|\psi_A\rangle = C_a |a\rangle + C_b |b\rangle , \; |\psi_F\rangle = \sum_{(n)} C_n |n\rangle \tag{2.1}$$

where: $|a\rangle$, $|b\rangle$ are vectors of the upper and lower states of the atom, respectively, $|n\rangle$ is the state vector of the mode containing n photons.

The complete state vector of the atom-field system is

$$|\psi_{AF}\rangle = \sum_{(n)} \left(C_{an} |a\rangle |n\rangle + C_{bn} |b\rangle |n\rangle \right) . \tag{2.2}$$

Here C_{an}, C_{bn} are the corresponding probability amplitudes. Due to the interaction between the atom and the field, the probability amplitudes change over time. For example, if the atom was initially in the upper level and the field mode contained n photons, i.e., $|\psi_{AF}\rangle = |a\rangle |n\rangle$, then at subsequent moments in time the field will be in the state

$$|\psi_{AF}\rangle = C_{an} |a\rangle |n\rangle + C_{b,n+1} |b\rangle |n+1\rangle . \tag{2.3}$$

2.2 Hamiltonian of the atom-field system

The Hamiltonian of the atom-field system can be represented as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}^{(A)} + \hat{\mathcal{H}}^{(F)} + \hat{V}$$

where $\hat{\mathcal{H}}^{(A)}$ is the Hamiltonian of the two-level atom, $\hat{\mathcal{H}}^{(F)}$ is the Hamiltonian of the electromagnetic field, and $\hat{V}$ is the interaction Hamiltonian.

For the Hamiltonian of the electromagnetic field, we have the expression (1.28):

$$\hat{\mathcal{H}}^{(F)} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) .$$

In the case of the Hamiltonian of the two-level atom, we use the following relations:

$$\begin{aligned} \hat{\mathcal{H}}^{(A)} |a\rangle &= \hbar\omega_a |a\rangle , \\ \hat{\mathcal{H}}^{(A)} |b\rangle &= \hbar\omega_b |b\rangle , \\ |a\rangle \langle a| + |b\rangle \langle b| &= \hat{I}, \end{aligned}$$

from which we obtain

$$\begin{aligned} \hat{\mathcal{H}}^{(A)} = \hat{\mathcal{H}}^{(A)} \hat{I} = \hat{\mathcal{H}}^{(A)} |a\rangle \langle a| + \hat{\mathcal{H}}^{(A)} |b\rangle \langle b| = \\ = \hbar\omega_a |a\rangle \langle a| + \hbar\omega_b |b\rangle \langle b| = \\ = \hbar\omega_a \hat{\sigma}_a + \hbar\omega_b \hat{\sigma}_b, \end{aligned} \tag{2.4}$$

where the following notations are introduced

$$\hat{\sigma}_a = |a\rangle \langle a| , \quad \hat{\sigma}_b = |b\rangle \langle b| .$$

The expression (2.4) can be rewritten in matrix form:

$$\hat{\mathcal{H}}^{(A)} = \hbar \begin{pmatrix} \omega_a & 0 \\ 0 & \omega_b \end{pmatrix} . \tag{2.5}$$

For the interaction Hamiltonian $\hat{V}$ within the semiclassical approach in the dipole approximation, we have:

$$\hat{V} = -e \left(\hat{\vec{r}} \vec{E}(t) \right) . \tag{2.6}$$

Applying the completeness condition twice to (2.6), we obtain

$$\begin{aligned} \hat{V} = \hat{I}\hat{V}\hat{I} = -e \left(\vec{E}(t)\, \hat{I}\hat{\vec{r}}\hat{I} \right) = \\ = -e \left(\vec{E}(t) \left(|a\rangle \langle a| + |b\rangle \langle b| \right) \hat{\vec{r}} \left(|a\rangle \langle a| + |b\rangle \langle b| \right) \right) = \\ = - \left(\vec{E}(t) \left(\vec{P}_{ab} |a\rangle \langle b| + \vec{P}_{ba} |b\rangle \langle a| \right) \right) , \end{aligned} \tag{2.7}$$

where

$$\vec{P}_{ab} = \vec{P}^*_{ba} = e \langle a| \hat{\vec{r}} |b\rangle$$

is the matrix element of the electric dipole moment. For simplicity, we assume $\vec{P}_{ab} = \vec{P}_{ba} = \vec{p}$ to be a real quantity. As a result, expression (2.7) can be written as

$$\begin{aligned} \hat{V} = - \left(\vec{E}(t)\, \vec{p} \right) \left(|a\rangle \langle b| + |b\rangle \langle a| \right) = \\ = - \left(\vec{E}(t)\, \vec{p} \right) \left(\hat{\sigma}^\dagger + \hat{\sigma} \right) , \end{aligned} \tag{2.8}$$

where $\hat{\sigma}^\dagger = |a\rangle \langle b|$ and $\hat{\sigma} = |b\rangle \langle a|$ are the transition operators. $\hat{\sigma}^\dagger$ is the raising operator corresponding to the transition from the lower to the upper state:

$$\hat{\sigma}^\dagger |b\rangle = |a\rangle \langle b|b\rangle = |a\rangle ,$$

$\hat{\sigma}$ is the lowering operator corresponding to the transition from the upper to the lower state:

$$\hat{\sigma} |a\rangle = |b\rangle \langle a|a\rangle = |b\rangle .$$

In addition, the following equalities hold:

$$\hat{\sigma}^\dagger |a\rangle = |a\rangle \langle b|a\rangle = 0$$

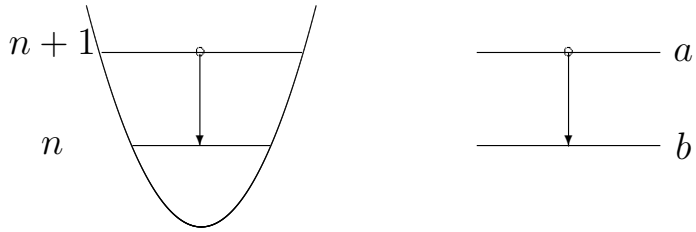


Figure 2.2: Hamiltonian of interaction. The process described by $\hat{a}\hat{\sigma}$ - absorption of a photon and the transition of an atom from the upper state to the lower state.

and

$$\hat{\sigma}\left|b\right\rangle = \left|b\right\rangle\left\langle a|b\right\rangle = 0.$$

For the operators $\hat{\sigma}$ and $\hat{\sigma}^{\dagger}$, the following relation can be written:

$$\begin{aligned}\hat{\sigma}\hat{\sigma}^{\dagger} + \hat{\sigma}^{\dagger}\hat{\sigma} = \\ = \left|b\right\rangle\left\langle a|a\right\rangle\left\langle b\right| + \left|a\right\rangle\left\langle b|b\right\rangle\left\langle a\right| = \\ = \left|b\right\rangle\left\langle b\right| + \left|a\right\rangle\left\langle a\right| = \hat{I},\end{aligned} \tag{2.9}$$

where $\hat{I}$ is the identity operator.

Expression (2.8) can also be written in matrix form:

$$\hat{V} = -\left(\vec{p}\vec{E}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.10}$$

Transition from the semiclassical interaction Hamiltonian to the fully quantum expression is made by replacing the classical electric field with the operator of the electric field. Thus, we obtain

$$\hat{V} = -pE_1 \sin kz \left(\hat{a} + \hat{a}^{\dagger}\right)\left(\hat{\sigma} + \hat{\sigma}^{\dagger}\right) \tag{2.11}$$

Here, the simplest expression for the mode electric field operator (1.32) is used.

The total Hamiltonian of the atom-field system has the form

$$\hat{\mathcal{H}}_{AF} = \hbar\omega_a\sigma_a + \hbar\omega_b\sigma_b + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \hbar g\left(\hat{a} + \hat{a}^{\dagger}\right)\left(\hat{\sigma} + \hat{\sigma}^{\dagger}\right)$$

where $g = -\frac{p}{\hbar}E_1 \sin kz$ is the interaction constant, depending on the spatial distribution of the mode field.

Not all terms entering the interaction Hamiltonian (2.11) are equivalent. When multiplying the brackets, we obtain terms:

1. $\hat{a}\hat{\sigma}$ - this term corresponds to the absorption of a photon and the transition of the atom from the upper state to the lower state. This process is conditionally depicted in Figure 2.2;

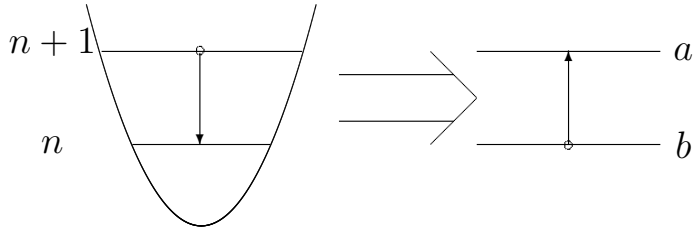


Figure 2.3: Interaction Hamiltonian. The process described by $\hat{a}\hat{\sigma}^{\dagger}$ - absorption of a photon and transition of the atom from the lower state to the upper state.

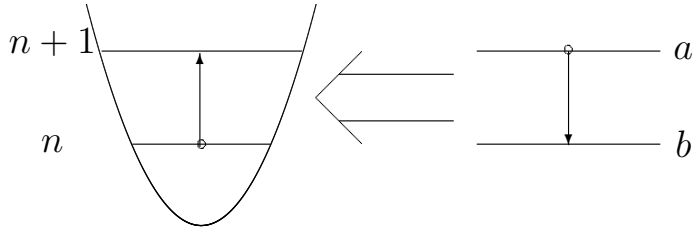


Figure 2.4: Hamiltonian of interaction. The process described by $\hat{a}^{\dagger}\hat{\sigma}$ - emission of a photon and transition of an atom from the upper state to the lower state.

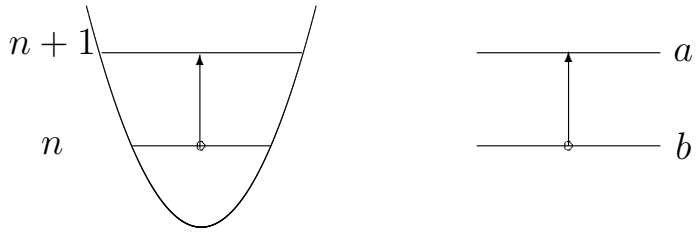


Figure 2.5: Interaction Hamiltonian. Process described by $\hat{a}^{\dagger}\hat{\sigma}^{\dagger}$ - emission of a photon and transition of the atom from the lower state to the upper state.

2. $\hat{a}\hat{\sigma}^\dagger$ - this term corresponds to the absorption of a photon and the transition of the atom from the lower state to the upper state (see Figure 2.3);

3. $\hat{a}^\dagger\hat{\sigma}$ - this term corresponds to the emission of a photon and the transition of the atom from the upper state to the lower state (see Figure 2.4);

4. $\hat{a}^\dagger\hat{\sigma}^\dagger$ - this term corresponds to the emission of a photon and the transition of the atom from the lower state to the upper state (see Figure 2.5).

Cases 1 and 4 correspond to processes (see Figure 2.2 and 2.5) where the energy conservation law is violated. These processes can be classified as so-called virtual processes. Their probability tends to zero as time increases, and their contribution to the interaction Hamiltonian can be neglected. Remaining are terms 2 and 3 (see Figure 2.3 and 2.4), corresponding to processes that occur without violation of the energy conservation law. Considering all the above, the interaction Hamiltonian takes the form:

$$\hat{V} = g\hbar\left(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}\right) \tag{2.12}$$

It is convenient to switch to the interaction picture (see section 14.2.1), using the well-known formula (14.31)

$$\hat{V}_I = e^{i\frac{\hat{\mathcal{H}}_0 t}{\hbar}}\hat{V}e^{-i\frac{\hat{\mathcal{H}}_0 t}{\hbar}} \tag{2.13}$$

where $\hat{\mathcal{H}}_0 = \hat{\mathcal{H}}^{(A)} + \hat{\mathcal{H}}^{(F)}$ is the unperturbed Hamiltonian of the atom-field system. It can be shown that in the interaction picture, the interaction Hamiltonian has the form

$$\hat{V}_I = g\hbar\left(\hat{a}^\dagger\hat{\sigma}e^{i(\omega-\omega_{ab})t} + \hat{\sigma}^\dagger\hat{a}e^{-i(\omega-\omega_{ab})t}\right). \tag{2.14}$$

This is most easily done by calculating the matrix elements $\langle n+1|\langle b|\hat{V}_I|a\rangle|n\rangle$ and $\langle n|\langle a|\hat{V}_I|b\rangle|n+1\rangle$ for the operator $\hat{V}_I$, which can be represented in the form (2.13) and (2.14), and verifying that they are the same in both cases.

2.3 Interaction of an atom with a mode of the electromagnetic field

The Schrödinger equation in the interaction representation has the form

$$i\hbar\frac{\partial}{\partial t}|\psi(r,t)\rangle_I = \hat{V}_I|\psi(r,t)\rangle_I, \tag{2.15}$$

where the wave function in the interaction representation is written as $|\psi(r,t)\rangle_I$:

$$|\psi(r,t)\rangle_I = e^{i\frac{\hat{\mathcal{H}}_0 t}{\hbar}} |\psi(r,t)\rangle . \tag{2.16}$$

Let us solve this equation for the initial state $|\psi(r,0)\rangle_I = |b,n+1\rangle_I$, i.e. we assume that at the initial moment the atom is in the lower state, and the field contains $n+1$ photons. The other case corresponds to the initial state of the system $|a,n\rangle_I$, i.e. the atom is in the upper state, and the field mode contains n photons. After some time, as a result of the interaction of the atom and the field, the system will be in a superposition state

$$|\psi(r,t)\rangle_I = C_{a,n}(t)_I |a,n\rangle_I + C_{b,n+1}(t)_I |b,n+1\rangle_I . \tag{2.17}$$

Using (2.16) on the left-hand side of expression (2.17), we get

$$\begin{aligned}
&C_{a,n}(t)_I |a,n\rangle_I + C_{b,n+1}(t)_I |b,n+1\rangle_I = \\
&= C_{a,n}(t)_I e^{i\frac{\hat{\mathcal{H}}_0 t}{\hbar}} |a,n\rangle + C_{b,n+1}(t)_I e^{i\frac{\hat{\mathcal{H}}_0 t}{\hbar}} |b,n+1\rangle = \\
&= C_{a,n}(t)_I \exp\left\{ i\left(\omega_a + \omega\left(n+\frac{1}{2}\right)\right) t \right\} |a,n\rangle + \\
&+ C_{b,n+1}(t)_I \exp\left\{ i\left(\omega_b + \omega\left(n+1+\frac{1}{2}\right)\right) t \right\} |b,n+1\rangle \\
&= C_{a,n}(t) |a,n\rangle + C_{b,n+1}(t) |b,n+1\rangle ,
\end{aligned}$$

where $C_{a,n}(t)$ and $C_{b,n+1}(t)$ are defined as

$$\begin{aligned}
C_{a,n}(t) &= C_{a,n}(t)_I \exp\left\{ i\left(\omega_a + \omega\left(n+\frac{1}{2}\right)\right) t \right\} , \\
C_{b,n+1}(t) &= C_{b,n+1}(t)_I \exp\left\{ i\left(\omega_b + \omega\left(n+1+\frac{1}{2}\right)\right) t \right\} .
\end{aligned}$$

Thus, equation (2.15) takes the following form

$$\begin{aligned}
&i\left(\dot{C}_{a,n}(t) |a,n\rangle + \dot{C}_{b,n+1}(t) |b,n+1\rangle\right) = \\
&= g\left(\hat{a}\hat{\sigma}^\dagger e^{-i(\omega-\omega_{ab})t} + \hat{a}^\dagger\hat{\sigma} e^{i(\omega-\omega_{ab})t}\right)\left(C_{a,n}|a,n\rangle + C_{b,n+1}|b,n+1\rangle\right) = \\
&= g\sqrt{n+1}\left(C_{b,n+1} e^{-i(\omega-\omega_{ab})t} |a,n\rangle + C_{a,n} e^{i(\omega-\omega_{ab})t} |b,n+1\rangle\right) .
\end{aligned} \tag{2.18}$$

Multiply this equation from the left by $\langle a,n|$ and $\langle b,n+1|$ respectively. Taking into account the properties of the operators $\hat{a}$, $\hat{a}^\dagger$, $\hat{\sigma}$, $\hat{\sigma}^\dagger$ and the orthogonality of the state vectors, we obtain a system of equations for probability amplitudes

$$\begin{aligned}
\dot{C}_{a,n}(t) &= -ig\sqrt{n+1} e^{-i(\omega-\omega_{ab})t} C_{b,n+1}(t) , \\
\dot{C}_{b,n+1}(t) &= -ig\sqrt{n+1} e^{i(\omega-\omega_{ab})t} C_{a,n}(t) .
\end{aligned} \tag{2.19}$$

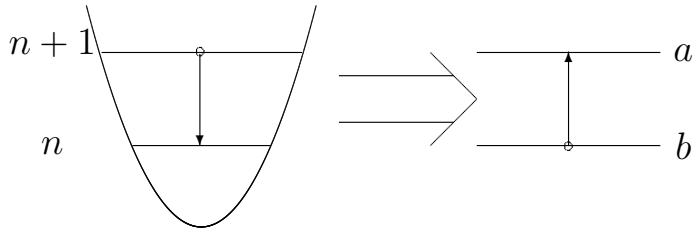


Figure 2.6: Absorption of a photon by a two-level atom. As a result of the interaction, the atom transitions from the lower level to the upper level, and the mode of the electromagnetic field contains one less photon.

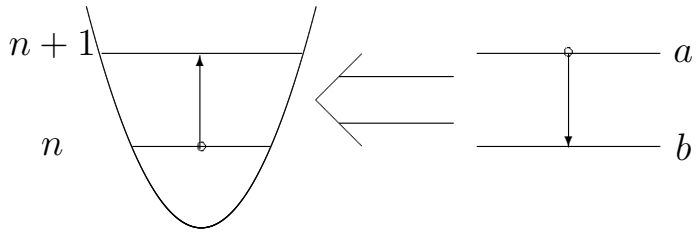


Figure 2.7: Emission of a photon by a two-level atom. As a result of the interaction, the atom transitions from the upper level to the lower one, and there is one more photon in the mode of the electromagnetic field.

For the case when the atom is in the lower state (case of photon absorption) we have at the initial moment $C_{a,n}(0) = 0$, $C_{b,n+1}(0) = 1$ (Figure 2.6). Formal integration of (2.19) gives

$$C_{a,n}(t) = -ig\sqrt{n+1}\int_0^t e^{-i(\omega-\omega_{ab})t'} C_{b,n+1}(t')\,dt' \tag{2.20}$$

In the first approximation for small times, one can put $C_{b,n+1}(t') = 1$ under the integral in (2.20). Integrating then yields

$$C_{a,n}(t) = -ig\sqrt{n+1}\frac{e^{-i(\omega-\omega_{ab})t} - 1}{-i(\omega-\omega_{ab})} \tag{2.21}$$

from which the probability of excitation of the atom and photon absorption is

$$|C_{a,n}(t)|^2 = g^2(n+1)\,t^2\frac{\sin^2((\omega-\omega_{ab})\,t/2)}{(\omega-\omega_{ab})^2\,(t/2)^2}. \tag{2.22}$$

In this approximation, one can also solve the second problem (Figure 2.7) $C_{a,n}(0) = 1$, $C_{b,n+1}(0) = 0$. We obtain an expression analogous to (2.22) for the probability of atom transition to the lower state and emission of a photon

$$|C_{b,n+1}(t)|^2 = g^2(n+1)\,t^2\frac{\sin^2((\omega-\omega_{ab})\,t/2)}{(\omega-\omega_{ab})^2\,(t/2)^2}. \tag{2.23}$$

In the case when the field contains no photons ($n = 0$), a nonzero probability is obtained $|C_{b,1}(t)|^2 \cong g^2t^2$ ($\frac{\sin^2 x}{x^2} \cong 1$, if $x \ll 1$), which shows that an excited atom can transition to the lower state even in the absence of photons. This process is known as spontaneous emission or spontaneous transition. It is related to the zero-point fluctuations of the field. Note that in the semiclassical treatment spontaneous emission does not follow from semiclassical equations and is introduced by additional considerations. From quantum equations, spontaneous emission naturally follows. The system of equations (2.19) can be solved exactly. For simplicity, we will consider the resonant case $\omega = \omega_{ab}$. Eliminate $C_{b,n+1}(t)$ from the equations:

$$\ddot{C}_{a,n}(t) = -ig\sqrt{n+1}\dot{C}_{b,n+1}(t) = -g^2(n+1)C_{a,n}(t).$$

The solution of the obtained equation:

$$C_{a,n}(t) = A\sin\left(g\sqrt{n+1}t\right) + B\cos\left(g\sqrt{n+1}t\right)$$

For absorption, the initial conditions ($C_{a,n}(0) = 0$, $C_{b,n+1}(0) = 1$) give $A = i$, $B = 0$, hence

$$C_{a,n}(t) = -i\sin\left(\frac{\omega_R t}{2}\right), \quad C_{b,n+1}(t) = \cos\left(\frac{\omega_R t}{2}\right), \tag{2.24}$$

where $\omega_R = 2g\sqrt{n+1}$ is the Rabi frequency in the quantum case. For emission, from initial conditions ($C_{a,n}(0) = 1$, $C_{b,n+1}(0) = 0$) we have $A = 0$, $B = 1$, hence

$$C_{a,n}(t) = \cos\left(\frac{\omega_R t}{2}\right), \quad C_{b,n+1}(t) = -i\sin\left(\frac{\omega_R t}{2}\right), \tag{2.25}$$

On Figure 2.8, the dependencies of probabilities $|C_{a,n}(t)|^2$ and $|C_{b,n+1}(t)|^2$ on time are shown for the case of photon emission.

In the semiclassical expression, the Rabi frequency ω_R equals $\frac{|pE|}{\hbar}$, while in the quantum case it is $2g\sqrt{n+1}$, i.e. E corresponds to $E_1\sqrt{n+1}$, where E_1 is the field corresponding to one photon in the mode. At large n, these two expressions practically coincide. However, for small n the difference can be significant. From the quantum expression it follows that even when there are no photons in the mode, the probabilities oscillate with frequency g. In fact, it is known that Rabi oscillations damp out. The discrepancy arises because the atom interacts with all modes of the space, but we considered only interaction with one mode.

Let's now try to find the general solution of the system of equations (2.19). The relations can be written as

$$\dot{C}_{a,n}(t) = -i\frac{\omega_R}{2}e^{-i\delta t}C_{b,n+1}(t),$$
$$\dot{C}_{b,n+1}(t) = -i\frac{\omega_R}{2}e^{i\delta t}C_{a,n}(t),$$

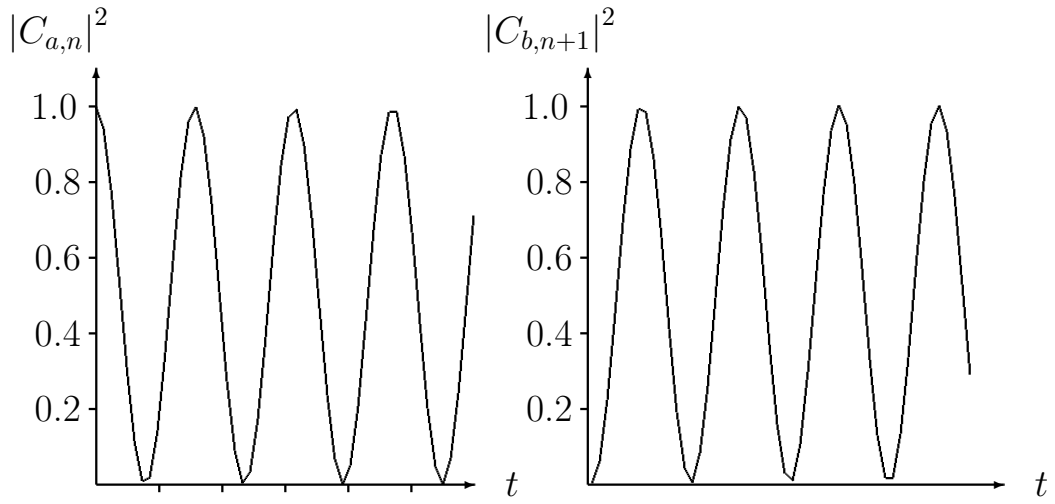


Figure 2.8: Dependencies of probabilities $|C_{a,n}|^2$ and $|C_{b,n+1}|^2$ on time for the case of photon emission.

where $\delta = \omega - \omega_{ab}$ is the frequency detuning. We will solve the problem only for the emission case, i.e., the initial conditions are

$$C_{a,n}(0) = 1,$$
$$C_{b,n+1}(0) = 0.$$

We are mainly interested in the behavior of the probabilities $|C_{b,n+1}(t)|^2$. Denoting $x(t) = C_{b,n+1}(t)$, we get

$$\ddot{x} = -i\frac{\omega_R}{2}\frac{d\left(e^{i\delta t}C_{a,n}(t)\right)}{dt} =$$
$$= i\delta\left(-i\frac{\omega_R}{2}e^{i\delta t}C_{a,n}(t)\right) - i\frac{\omega_R}{2}e^{i\delta t}\frac{dC_{a,n}(t)}{dt} =$$
$$= i\delta\dot{x} - \frac{\omega_R^2}{4}e^{-i\delta t}e^{i\delta t}x =$$
$$= i\delta\dot{x} - \frac{\omega_R^2}{4}x,$$

i.e.

$$\ddot{x} - i\delta\dot{x} + \frac{\omega_R^2}{4}x = 0. \tag{2.26}$$

Equation (2.26) must be solved with the following initial conditions:

$$x|_{t=0} = C_{b,n+1}(0) = 0,$$
$$\dot{x}|_{t=0} = -i\frac{\omega_R}{2}C_{a,n}(0) = -i\frac{\omega_R}{2}.$$

The result:

$$x(t) = -ie^{\frac{i\delta t}{2}}\frac{\omega_R}{\sqrt{\omega_R^2+\delta^2}}\sin\left(\frac{\sqrt{\omega_R^2+\delta^2}}{2}t\right),$$

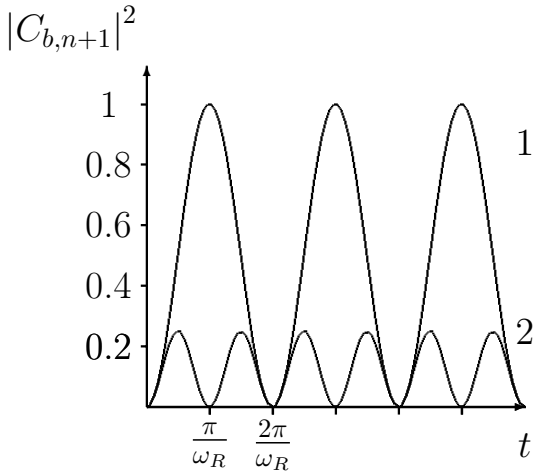


Figure 2.9: Graph 1: Rabi oscillations with zero detuning $\delta = 0$. Period of oscillations $\frac{2\pi}{\omega_R}$, $\Omega_R = \omega_R$, maximum probability amplitude $|C_{b,n+1}|^2 = 1$. Graph 2: Rabi oscillations with nonzero detuning $\delta = \sqrt{3}\omega_R$. Period of oscillations $\frac{\pi}{\omega_R}$, $\Omega_R = 2 \cdot \omega_R$, maximum probability amplitude $|C_{b,n+1}|^2 = \frac{\omega_R^2}{\Omega_R^2} = \frac{1}{4}$.

denoting $\Omega_R = \sqrt{\omega_R^2 + \delta^2}$ as the generalized Rabi frequency, for the desired probability $|C_{b,n+1}|^2$ we get (see Figure 2.9)

$$|C_{b,n+1}|^2 = \frac{\omega_R^2}{\Omega_R^2} \sin^2 \frac{\Omega_R}{2} t. \tag{2.27}$$

Remark 2.3.1 (AC Stark effect)**.** *It is worth noting that in the case $\delta \neq 0$ the states we used in expression 2.3 do not satisfy the law of energy conservation. Indeed, if the system initially was in the state $|a\rangle |n\rangle$ with energy $E_a + \hbar n\omega$, then after some time it, with nonzero probability, can be registered in the state $|b\rangle |n+1\rangle$ with energy*

$$E_b + \hbar(n+1)\omega \neq E_a + \hbar n\omega.$$

That is, a violation of the law of energy conservation appears. This contradiction is resolved as follows. Due to the AC Stark effect (also called the Autler-Townes effect in the English literature)[27], the atomic energy levels shift, which ensures that the law of conservation of energy is fulfilled.

For example, at small detunings $\delta \approx 0$, this shift equals [27]

$$\Delta E \approx \pm \frac{\hbar\omega_R}{2}.$$

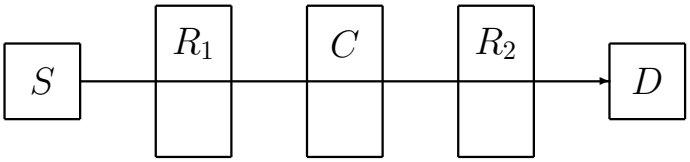


Figure 2.10: Ramsay Interferometer. An atom from the source S sequentially passes through 3 zones R_1, C, and R_2, and at the output D, the state of the atom is determined after passing through the 3 zones.

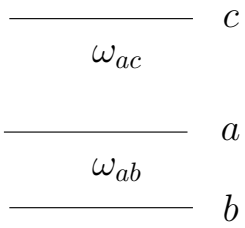


Figure 2.11: Three-level atom used in the experiment.

2.4 Ramsey Interferometer. Quantum Non-destructive Measurements

Observation of a photon usually implies its destruction, for example, when using a photodetector, the energy of the photon is converted into an electric signal, causing the photon itself to disappear.

An interesting experiment [20] using Rabi oscillations demonstrated the possibility of observing a photon without its destruction. A Ramsey interferometer was used (see Figure 2.10).

The operation scheme is as follows. Atoms with a level structure as shown in Figure 2.11 are used. The transition frequency ω_{ab} between levels a and b matches the radiation frequency in zones R_1 and R_2 of the interferometer (see Figure 2.10). Zone C contains the electromagnetic field under investigation, for which we want to answer whether there is a photon in it or not, i.e., whether it is in state $|1\rangle$ or in vacuum state $|0\rangle$. The field under investigation in zone C has a frequency matching the transition frequency ω_{ac} between levels a and c.

The size of zones $R_{1,2}$ is chosen so that the interaction time t_R of the atom with the electromagnetic field in them corresponds to the relation

$$\omega_R^{(R)} t_R = \frac{\pi}{2},$$

where $\omega_R^{(R)}$ is the Rabi frequency corresponding to the transition frequency ω_{ab}. The interaction time t_C in zone C is determined by the expression

$$\omega_R^{(C)} t_C = 2\pi,$$

where $\omega_R^{(C)}$ is the Rabi frequency corresponding to the transition frequency ω_{ac}.

For the interaction in zones $R_{1,2}$, the following relations hold (2.24), 2.25:

$$|a\rangle \to \frac{1}{\sqrt{2}}\left(|a\rangle - i\,|b\rangle\right),$$
$$|b\rangle \to \frac{1}{\sqrt{2}}\left(-i\,|a\rangle + |b\rangle\right). \tag{2.28}$$

Expression (2.28) can be rewritten in matrix form $|\psi\rangle \to R\,|\psi\rangle$, where

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Zone C does not affect the state $|b\rangle$, for the state $|a\rangle$ in the case of vacuum state $|0\rangle$ (absence of a photon)

$$|a\rangle \to |a\rangle,$$

i.e., in the absence of a photon, the state of the atom does not change. Indeed, if we consider a two-level system formed by states $|a\rangle$ and $|c\rangle$, then $|a\rangle\,|0\rangle$ will be the minimal possible energy state from which the system (the two-level atom and the electromagnetic field) cannot transition to another state. In this case for C, we have

$$C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the case of the presence of a photon from (2.24)

$$|a\rangle \to -\,|a\rangle,$$

and in this case C looks like

$$C_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, if initially the atom emitted by the source S is in state $|a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then in the absence of a photon in zone C on detector D, we get an atom in state

$$R_2 C_0 R_1\,|a\rangle =$$
$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$
$$= -i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\,|b\rangle,$$

i.e., the atom will be observed in the unexcited state $|b\rangle$.

For the case of the presence of one photon in zone C, we have

$$R_2 C_1 R_1 |a\rangle = \\ = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \\ = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -|a\rangle ,$$

i.e., the atom will be observed in the excited state $|a\rangle$. Thus, a conclusion can be made about the presence of a photon in zone C without destructive impact on it.

2.5 Interaction of an atom with a multimode field. Induced and spontaneous transitions

Above, we established that the probability of the atom transitioning from the lower to the upper state when interacting with a single mode of the field, which is in a state with n photons, is given by (2.22)

$$|C_{a,n}(t)|^2 = 4g^2(n+1)\frac{\sin^2\left((\omega-\omega_{ab})t/2\right)}{(\omega-\omega_{ab})^2}$$

Let us write this probability in the following form

$$P_a = |C_{a,n}(t)|^2 = P_a^{\text{ind}} + P_a^{\text{sp}}, \tag{2.29}$$

where

$$P_a^{\text{ind}} = 4g^2 n\frac{\sin^2\left((\omega-\omega_{ab})t/2\right)}{(\omega-\omega_{ab})^2}, \\ P_a^{\text{sp}} = 4g^2\frac{\sin^2\left((\omega-\omega_{ab})t/2\right)}{(\omega-\omega_{ab})^2} \tag{2.30}$$

are the probabilities of induced (P_a^{ind}) and spontaneous (P_a^{sp}) transitions.

2.5.1 Induced transitions

Further, we will focus on the probability of the induced transition P_a^{ind}. To obtain the transition probability considering interaction with all modes, one needs to sum the probabilities (2.30), corresponding to each mode.

In the case of an atom interacting with the free space field (expansion of the field into plane waves), the interaction constant is [61]:

$$g = -\frac{(\vec{p}\cdot\vec{e})}{\hbar}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \tag{2.31}$$

where $\vec{p}$ is the matrix element of the dipole moment operator, $\vec{e}$ is the polarization vector of the field.

If in expression (2.30) the time t is not very small (but significantly less than the characteristic time for the change of probabilities), the dependence of $|C_{a,n}(t)|^2$ on frequency will have a sharp peak at $\omega = \omega_{ab}$, which indicates energy conservation in the elementary act (the energy of the absorbed photon equals the change in the atom's energy). Let us assume that the field incident in the direction $\vec{k}$ within the solid angle $d\Omega$ has a spectrum that varies little near the frequency ω_{ab}.

The number of modes within the frequency interval $d\omega$ near ω_{ab} and the solid angle $d\Omega$ around the direction $\vec{k}$, as is known from (1.47), is

$$dN = 2\left(\frac{L}{2\pi c}\right)^3 \omega^2 d\omega d\Omega. \tag{2.32}$$

The probability of photon absorption during atom interaction with one mode is given by formula (2.30). The total probability can be obtained by summing over all modes. Treating the mode spectrum as quasi-continuous (see (1.44)), the summation can be replaced by integration

$$W_{b\to a} = \int_\Omega 8\,|g|^2\, n(\vec{k})\left(\frac{L}{2\pi c}\right)^3 \omega_{ab}^2 d\Omega \int_{-\infty}^{+\infty} \frac{\sin^2\left(\left(\omega-\omega_{ab}\right)t/2\right)}{\left(\omega-\omega_{ab}\right)^2} d\omega. \tag{2.33}$$

Here it is acknowledged that $\frac{\sin^2 x}{x^2}$ has a narrow peak near $x = 0$, and in expression (2.33) we assume that t is small compared to the characteristic time of change in C, but large enough for the filtering properties of the integral to manifest. All slowly varying terms are taken at $\omega = \omega_{ab}$ and pulled out of the integral. It is known that $\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2} dx = \pi$. Hence,

$$\int_{-\infty}^{+\infty} \frac{\sin^2\left(\left(\omega-\omega_{ab}\right)t/2\right)}{\left(\omega-\omega_{ab}\right)^2} d\omega = \frac{\pi t}{2}.$$

Thus, we have

$$W_{b\to a} = t\frac{\omega^2 2\pi\omega}{\hbar\varepsilon_0\left(2\pi c\right)^3}\int_\Omega n(\vec{k})\,|(\vec{p}\cdot\vec{e})|^2\, d\Omega. \tag{2.34}$$

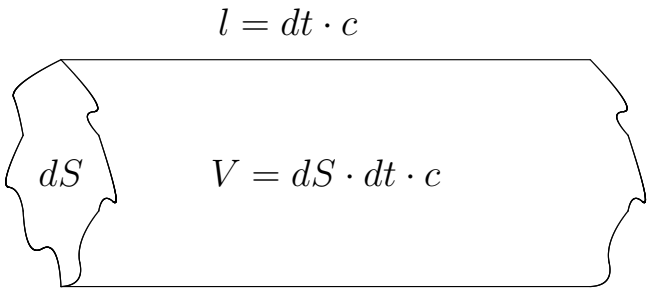


Figure 2.12: To the definition of the energy flow (photon flux). The energy flow is defined as the energy dH, transferred through the area dS in the unit of time dt. This energy can be defined through the number of photons in a cylinder of volume V.

Note that the quantization volume does not appear in the final expression, and the photon number n depends on direction, i.e. on solid angle Ω. Formula (2.34) shows that the transition probability is proportional to time. This allows introducing the concept of the transition rate, i.e., transition probability per unit time

$$w_{b\to a} = \frac{W_{b\to a}}{t} = \frac{2\pi\omega^3}{\hbar\varepsilon_0 \left(2\pi c\right)^3} \int_\Omega n(\vec{k}) \left|(\vec{p}\cdot\vec{e})\right|^2 d\Omega. \tag{2.35}$$

The transition rate or photon absorption rate can be expressed [61] through the energy flux (photon flux), propagating in direction $\vec{k}$ in the solid angle $d\Omega$. Energy flux is defined as energy transferred through a unit area dS per unit time dt, i.e.

$$dI = \frac{dH}{dSdt}.$$

From Figure 2.12 we have that through the area dS during unit time dt passes as many photons as are contained in the cylinder depicted in Figure 2.12, i.e., the transferred energy can be written as

$$dH = n\hbar\omega dN,$$

where dN is the number of modes within the interval $d\omega d\Omega$ in the volume $c\cdot dSdt$ (1.47):

$$dN = 2\left(\frac{1}{2\pi c}\right)^3 \omega^2 c \cdot dSdtd\omega d\Omega,$$

from which we have

$$dI = I\left(\omega, \vec{k}\right) d\omega d\Omega = \frac{2n\hbar\omega c}{\left(2\pi c\right)^3}\omega^2 d\omega d\Omega, \tag{2.36}$$

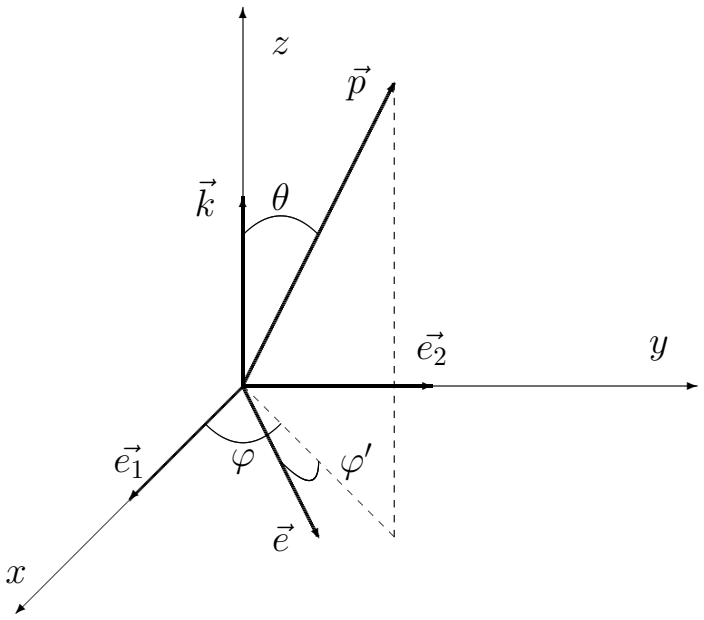


Figure 2.13: Coordinate system in which it is convenient to average the multimode field over all directions and polarizations. The wave vector of the electromagnetic wave $\vec{k}$ is directed along the z axis, and the position of the dipole moment matrix element $\vec{p}$ is determined by the angles θ and ϕ. The polarization vectors $\vec{e}_1$ and $\vec{e}_2$ are directed along the x and y axes. The angle ϕ' defines the direction of polarization of the electromagnetic wave.

where $I\left(\omega, \vec{k}\right)$ is the energy flux of photons in the direction $\vec{k}$ within unit frequency interval and unit solid angle. Thus, from (2.36) we obtain

$$I\left(\omega, \vec{k}\right) = \frac{2n\hbar\omega^3 c}{(2\pi c)^3}$$

Substituting this into expression (2.35), we get:

$$w_{ab} = \frac{\pi}{\hbar^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\int_\Omega I\left(\omega, \vec{k}\right)\left|(\vec{p}\cdot\vec{e})\right|^2 d\Omega. \tag{2.37}$$

Here the relation $\mu_0\varepsilon_0 = \frac{1}{c^2}$ is used.

To find the total absorption rate, one must integrate (2.37) over all propagation directions. Moreover, the incoming radiation is usually unpolarized. To account for this, averaging over all polarization directions is performed. We use the coordinate system shown in Figure 2.13. The polar axis z is chosen along $\vec{k}$. The polarization vectors $\vec{e}_1$ and $\vec{e}_2$ are directed along x and y respectively. The angles φ and θ define the direction of $\vec{p}$. The angle φ' defines the polarization direction of the incoming wave. From the figure, it follows that

$$\left|(\vec{p}\cdot\vec{e})\right|^2 = |p|^2 \sin^2\theta\cos^2\varphi'.$$

Averaging over all polarizations gives:

$$\frac{|p|^2}{2\pi}\int_0^{2\pi}\cos^2\varphi' d\varphi' = \frac{|p|^2}{2}. \tag{2.38}$$

Summing over all directions of wave incidence leads to the expression

$$w_{ab} = \frac{\pi}{2\hbar^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}|p|^2\int_\Omega I\left(\omega,\vec{k}\right)\sin^3\theta d\theta d\varphi. \tag{2.39}$$

Assuming the radiation arrives from all directions and is isotropic, we have:

$$w_{ab} = \frac{\pi}{2\hbar^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}|p|^2 I(\omega)\int_0^{2\pi}d\varphi\int_0^{\pi}\sin^3\theta d\theta = \frac{\pi}{\hbar^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{|p|^2}{3}I_0, \tag{2.40}$$

where $I_0 = 4\pi I(\omega)$ is the total energy flux incident on the atom. In deriving (2.40) we used the following relation:

$$\int_0^{\pi}\sin^3\theta d\theta = \frac{4}{3}.$$

If the radiation arrives from a small solid angular region $\Delta\Omega$ near angles θ_0, φ_0, then we can write:

$$\begin{aligned} w_{ab} &= \frac{\pi}{2\hbar^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}|p|^2\sin^2\theta_0\int_{\Delta\Omega} I(\theta,\varphi)\,d\Omega = \\ &= \frac{\pi}{2\hbar^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}|p|^2 I_0\sin^2\theta_0, \end{aligned} \tag{2.41}$$

where $I_0 = \int_{\Delta\Omega} I(\theta,\varphi)\,d\Omega$ is the total flux irradiating the atom.

2.5.2 Spontaneous transitions

Now consider a different problem. Define the probability of photon emission or transition of the excited atom to the lower state.

For the probability of photon emission by the atom into one mode we had expression (2.23):

$$|C_{b,n+1}(t)|^2 = 4g^2(n+1)\frac{\sin^2\left((\omega-\omega_{ab})t/2\right)}{(\omega-\omega_{ab})^2},$$

which similarly to (2.29) can be considered as a sum of two terms

$$P_b = |C_{b,n+1}(t)|^2 = P_b^{\text{ind}} + P_b^{\text{sp}}$$

where the first term

$$P_b^{\text{ind}} = 4g^2 n \frac{\sin^2\left(\left(\omega - \omega_{ab}\right) t/2\right)}{\left(\omega - \omega_{ab}\right)^2}$$

corresponds to induced emission, and the second

$$P_b^{\text{sp}} = 4g^2 \frac{\sin^2\left(\left(\omega - \omega_{ab}\right) t/2\right)}{\left(\omega - \omega_{ab}\right)^2}$$

to spontaneous emission. The term corresponding to induced emission is analyzed similarly to the case of photon absorption. The result will, of course, be the same. Corresponding formulas coincide with (2.40), (2.41). Hence, the probabilities of induced processes of absorption and emission are equal to each other. The probability of spontaneous emission per unit time will evidently be

$$w_{\text{sp}} = \frac{2\pi\omega^3}{\hbar\varepsilon_0 \left(2\pi c\right)^3} \int_\Omega \left|\left(\vec{p}\cdot\vec{e}\right)\right|^2 d\Omega \tag{2.42}$$

To obtain the total probability, one must integrate (2.42) over all directions, since spontaneous emission may occur into any mode. For polarizations, the average value (2.38) is used. Thus, the entire procedure reduces to computing the integral

$$w_{\text{sp}} = \frac{|p|^2 \omega^3}{\varepsilon_0 \hbar \left(2\pi\right)^2 c^3} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta d\theta = \frac{4\pi}{3} \frac{|p|^2 \omega^3}{\hbar\varepsilon_0 \left(2\pi\right)^2 c^3}. \tag{2.43}$$

Finally, we obtain

$$w_{\text{sp}} = \frac{|p|^2 \omega^3}{3\pi c^2 \hbar} \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{2.44}$$

Here the relation

$$\frac{1}{c} = \sqrt{\mu_0 \varepsilon_0}$$

was used. Formula (2.44) indicates a strong dependence of the spontaneous transition probability on frequency as ω^3. Thus, based on quantum electrodynamics equations, without invoking extraneous considerations, we directly obtained expressions for the probabilities of induced and spontaneous atomic transitions per unit time. A drawback of this consideration is the use of perturbation theory and, therefore, the necessity to restrict to small times. For example, when solving the problem of the lifetime of an excited atom, one cannot restrict to small times. In this case, another approximation called the Weisskopf-Wigner approximation [61] is used. Interestingly, the results obtained in these two cases are consistent with each other.

2.6 Spontaneous Emission. Weisskopf-Wigner Approximation.

Above we obtained a formula for the rate of spontaneous emission, valid for sufficiently short times. At the same time, when solving, for example, the problem of the lifetime of an excited atom, one cannot restrict to short times. In this case, a different method is applied, called the Weisskopf-Wigner approximation.

Let us consider this method. We have established that the atom transitions from the excited state to the ground state due to interaction with all modes of the space, even if they are not excited and are in the vacuum state. The interaction Hamiltonian in this case has the form

$$\hat{V}_I = \hbar \sum_k g_k \hat{\sigma}^\dagger \hat{a}_k e^{i(\omega_{ab}-\omega_k)t} + \text{h.c.}, \tag{2.45}$$

where g_k is the interaction constant with the k-th mode. Compared to the Hamiltonian (2.14) we used earlier, (2.45) differs by summation over all quantization modes of the space.

Assume that at the initial time $t = 0$ the atom is excited, and there are no photons in the field. Then the initial state reads

$$|\psi(0)\rangle = |a, \{0\}\rangle,$$

that is, the atom is excited and all modes are in the vacuum state. At later times, due to atom-field interaction, the state of the system will be

$$|\psi(t)\rangle = \sum_k C_{bk}(t)\,|b, 1_k\rangle + C_a(t)\,|a, \{0\}\rangle, \tag{2.46}$$

where $|b, 1_k\rangle$ is the state in which the atom is in the ground state and there is one photon in the k-th mode.

The Schrödinger equation in the interaction picture reads

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{V}_I\,|\psi(t)\rangle. \tag{2.47}$$

Substituting (2.46) here and projecting equation (2.47) sequentially onto $\langle a, \{0\}|$ and $\langle b, 1_k|$, we obtain a system of equations for the probability amplitudes $C_a(t)$ and $C_{bk}(t)$:

$$\begin{aligned} \dot{C}_a(t) &= -i\sum_k g_k e^{i(\omega_{ab}-\omega_k)t} C_{bk}(t), \\ \dot{C}_{bk}(t) &= -i g_k e^{-i(\omega_{ab}-\omega_k)t} C_a(t). \end{aligned} \tag{2.48}$$

Integrate the second equation (2.48) over time from 0 to t:

$$C_{bk}(t) = -ig_k \int_0^t e^{-i(\omega_{ab}-\omega_k)t'} C_a(t')\, dt'. \tag{2.49}$$

Substituting (2.49) into the first equation of the system (2.48), we obtain the following integro-differential equation:

$$\dot{C}_a(t) = -\sum_k g_k^2 \int_0^t e^{i(\omega_{ab}-\omega_k)(t-t')} C_a(t')\, dt'. \tag{2.50}$$

Now, let's make several simplifications (the Weisskopf-Wigner approximation). We will consider the mode distribution quasi-continuous and replace summation over k by integration using the relation (1.46):

$$dN = 2\left(\frac{L}{2\pi}\right)^3 k^2 dk d\Omega = 2\frac{V}{(2\pi)^3} k^2 dk d\Omega$$

Then we get (1.44)

$$\sum_k g_k^2 \int_0^t e^{i(\omega_{ab}-\omega_k)(t-t')} C_a(t')\, dt' =$$

$$= 2\frac{V}{(2\pi)^3} \int_{4\pi} d\Omega \int_0^\infty g_k^2 k^2 dk \int_0^t dt' e^{i(\omega_{ab}-\omega_k)(t-t')} C_a(t'). \tag{2.51}$$

In (2.51)

$$g_k^2 = \frac{\omega_k |p|^2 \sin^2\theta}{4\hbar\varepsilon_0 V}, \tag{2.52}$$

where θ is the angle between the directions $\vec{k}$ and $\vec{p}$. Expression (2.52) is obtained taking into account averaging over polarizations (**??**).

Transitioning in (2.51) from k to ω using the relations

$$k = \frac{\omega_k}{c}, \quad k^2 dk = \frac{\omega_k^2 d\omega_k}{c^3}$$

and denoting for convenience $\omega_k = \omega$, we get

$$\dot{C}_a(t) = -2\frac{V}{(2\pi c)^3} \int_{4\pi} d\Omega \int_0^\infty g^2(\omega)\, \omega^2 d\omega \int_0^t dt' e^{i(\omega_{ab}-\omega)(t-t')} C_a(t'). \tag{2.53}$$

For an approximate evaluation of the integral in (2.53) we additionally use a number of simplifying assumptions. We first integrate (2.53) over frequency (i.e.,

changing the order of integration, assuming this is possible). From the structure of (2.53) it is seen that the main contribution to the time integral comes from the frequency region $\omega \approx \omega_{ab}$. For this reason, we can take

$$\omega^2 g^2(\omega) \approx \omega_{ab}^2 g^2(\omega_{ab}).$$

In this approximation, the frequency integral looks like (see the integral representation of the delta-function (18.3)):

$$\int_0^\infty d\omega e^{i(\omega_{ab}-\omega)(t-t')} = |\nu = \omega - \omega_{ab}| = \int_{-\omega_{ab}}^\infty d\nu e^{-i\nu(t-t')} \approx$$
$$\approx \int_{-\infty}^\infty d\nu e^{-i\nu(t-t')} = \int_{-\infty}^\infty d\nu e^{i\nu(t'-t)} = 2\pi\delta(t'-t). \tag{2.54}$$

Substituting (2.54) into equation (2.53) and integrating over time, we get

$$\dot{C}_a(t) = -2\frac{V}{(2\pi c)^3}\int_{4\pi} d\Omega g^2(\omega_{ab})\omega_{ab}^2 2\pi C_a(t) = -\frac{\Gamma}{2}C_a(t). \tag{2.55}$$

Substituting here the expression for g_k^2 (2.52) and performing the angular integral (over $d\Omega$), we find an expression for the decay coefficient Γ:

$$\Gamma = \frac{\omega_{ab}^3 |p|^2}{3\pi c^2 \hbar}\sqrt{\frac{\mu_0}{\varepsilon_0}}. \tag{2.56}$$

Note that expression (2.56) coincides with that obtained earlier by another method (2.44).

From (2.55) it follows that Γ characterizes the rate of change of the probability $|C_a|^2$. Indeed, from (2.55) we have:

$$\dot{C}_a C_a^* = -\frac{\Gamma}{2}C_a C_a^*,$$
$$\dot{C}_a^* C_a = -\frac{\Gamma}{2}C_a^* C_a,$$

from which it follows that

$$\frac{dC_a C_a^*}{dt} = -\Gamma(C_a C_a^*),$$

or equivalently

$$\frac{d|C_a|^2}{dt} = -\Gamma|C_a|^2.$$

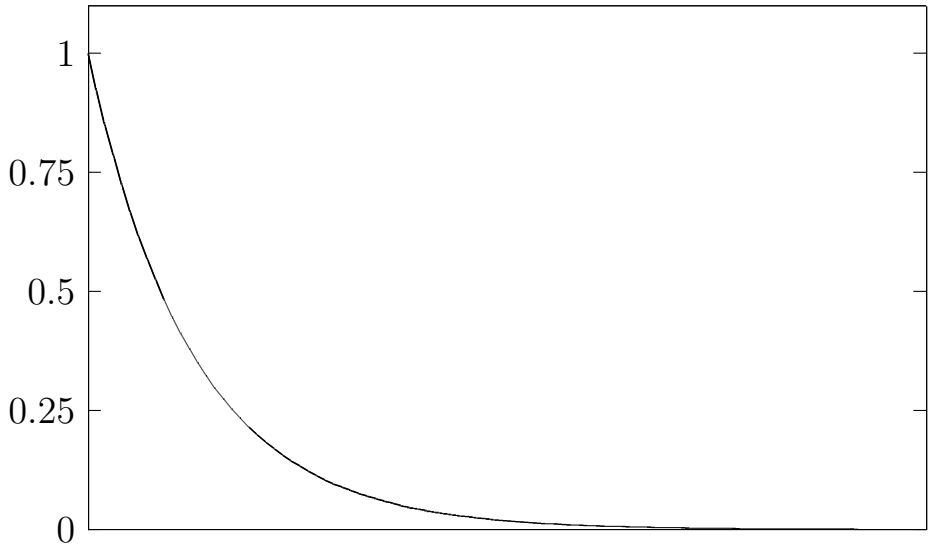


Figure 2.14: Dependence of the probability of finding an atom in an excited state $|C_a|^2$ in the Weisskopf-Wigner approximation

The solution has the form (see Figure 2.14)

$$|C_a|^2 = e^{-\Gamma t},$$

if the initial value $|C_a|^2\Big|_{t=0} = 1$, i.e. the atom was initially excited.

Remark 2.6.1 (Weisskopf-Wigner Approximation and Rabi Oscillations)**.** *If we compare Figure 2.8 and Figure 2.14, we can see that these two graphs differ significantly despite apparently describing the same model system. In this regard, the question arises when we can apply the single-mode approximation and correspondingly obtain Rabi oscillations (see Figure 2.8), and when it is necessary to use a multi-mode approximation like Weisskopf-Wigner (see Figure 2.14).*

Formally, we can consider that the interaction with each mode of the electromagnetic field gives its contribution to the resulting probabilities, while according to (2.23) the change of probability (for small t)

$$\Delta \left|C_{a,n}(t)\right|^2 = -\Delta \left|C_{b,n+1}(t)\right|^2 \sim (n+1).$$

Thus, those modes with a large number of photons give a greater contribution than those with fewer. Consequently, if we have a mode with a large number of photons $n \gg 1$, then the influence of vacuum modes with photon number $n = 0$ can be neglected. Otherwise, especially when all modes are equivalent, it is necessary to take into account all modes and apply approximations of the Weisskopf-Wigner type.

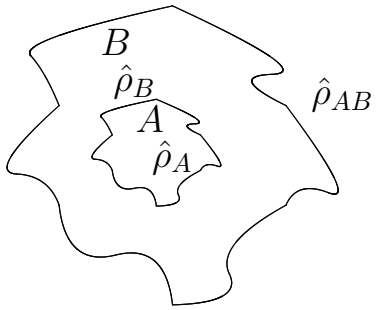


Figure 2.15: Interaction of dynamic system A with reservoir B

2.7 Relaxation of a dynamical system. Density matrix method

In quantum electronics and quantum optics, we deal with open systems. Such systems must be considered using the density matrix. We consider a small dynamical system (atom, molecules, resonator mode, etc.) weakly coupled to a large system called a reservoir (dissipative system, thermostat), consisting of a large number of atoms, molecules or oscillators, in equilibrium at a certain temperature. Thus, in the general case, we consider a dynamical system (A) interacting with a reservoir (B) (see Figure 2.15). The density operator of the combined system (AB) is denoted by $\hat{\rho}_{AB}$. The density operator traced over the variables of the reservoir (thermostat) describes the behavior of the dynamical system taking into account the averaged interaction with the thermostat. We have:

$$\hat{\rho}_A = Sp_B \hat{\rho}_{AB} = \sum_{(B)} \langle B| \hat{\rho}_{AB} |B\rangle . \tag{2.57}$$

At the initial time t_0, the dynamical system and the reservoir are considered independent (uncorrelated). Then

$$\hat{\rho}_{AB}(t_0) = \hat{\rho}_A(t_0) \otimes \hat{\rho}_B(t_0) \tag{2.58}$$

where $Sp_B \hat{\rho}_B = 1$, $\otimes$ denotes the external or tensor product.

The equation of motion for the density operator in the interaction picture is

$$\frac{d}{dt}\hat{\rho}_{AB} = -\frac{i}{\hbar}\left[\hat{V}_{AB}, \hat{\rho}_{AB}\right],$$

where $\hat{V}_{AB}$ is the interaction Hamiltonian between systems A and B. This equation can be solved using the following iterative process.

$$\hat{\rho}_{AB}(t) = \hat{\rho}^{(0)}_{AB}(t) + \hat{\rho}^{(1)}_{AB}(t) + \hat{\rho}^{(2)}_{AB}(t) + \ldots, \tag{2.59}$$

where the term $\hat{\rho}_{AB}^{(0)}(t)$ is found under the assumption of no interaction, i.e.

$$\hat{\rho}_{AB}^{(0)}(t) = \hat{\rho}_{AB}(t_0) = \hat{\rho}_A(t_0) \otimes \hat{\rho}_B(t_0).$$

The subsequent terms of the iterative relation (2.59) are given by

$$\hat{\rho}_{AB}^{(s)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} dt_s \left[\hat{V}_{AB}(t_s), \hat{\rho}_{AB}^{(s-1)}(t_s)\right] =$$

$$= \left(-\frac{i}{\hbar}\right)^s \int_{t_0}^{t} dt_s \int_{t_0}^{t_s} dt_{s-1} \ldots$$

$$\ldots \int_{t_0}^{t_2} dt_1 \left[\hat{V}_{AB}(t_s), \left[\hat{V}_{AB}(t_{s-1}), \ldots \left[\hat{V}_{AB}(t_1), \hat{\rho}_{AB}^{(0)}(t_0)\right] \ldots\right]\right] =$$

$$= \left(-\frac{i}{\hbar}\right)^s \int_{t_0}^{t} dt_s \int_{t_0}^{t_s} dt_{s-1} \ldots$$

$$\ldots \int_{t_0}^{t_2} dt_1 \left[\hat{V}_{AB}(t_s), \left[\hat{V}_{AB}(t_{s-1}), \ldots \left[\hat{V}_{AB}(t_1), \hat{\rho}_A(t_0) \otimes \hat{\rho}_B(t_0)\right] \ldots\right]\right]. \quad (2.60)$$

Substituting (2.60) into (2.59) and performing averaging over the thermostat variables, we obtain

$$\hat{\rho}_A(t) = \hat{\rho}_A(t_0) +$$

$$+ Sp_B \sum_{(s)} \left(-\frac{i}{\hbar}\right)^s \int_{t_0}^{t} dt_s \int_{t_0}^{t_s} dt_{s-1} \ldots$$

$$\ldots \int_{t_0}^{t_2} dt_1 \left[\hat{V}_{AB}(t_s), \left[\hat{V}_{AB}(t_{s-1}), \ldots \left[\hat{V}_{AB}(t_1), \hat{\rho}_A(t_0) \otimes \hat{\rho}_B(t_0)\right] \ldots\right]\right]. \quad (2.61)$$

Usually, only the first few terms of the expansion are considered. For example, the first two terms of the expansion are:

$$\hat{\rho}_A(t) = \hat{\rho}_A(t_0) - \frac{i}{\hbar} Sp_B \left\{\int_{t_0}^{t} \left[\hat{V}_{AB}(t_1), \hat{\rho}_A(t_0) \otimes \hat{\rho}_B(t_0)\right] dt_1\right\} -$$

$$-\frac{1}{\hbar^2} Sp_B \left\{\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \left[\hat{V}_{AB}(t_2), \left[\hat{V}_{AB}(t_1), \hat{\rho}_A(t_0) \otimes \hat{\rho}_B(t_0)\right]\right]\right\}.$$

2.8 Interaction of the resonator electromagnetic field with an atom reservoir at temperature T

Let us apply the general approach described above to a system consisting of a harmonic oscillator (electromagnetic field mode) interacting with a reservoir in

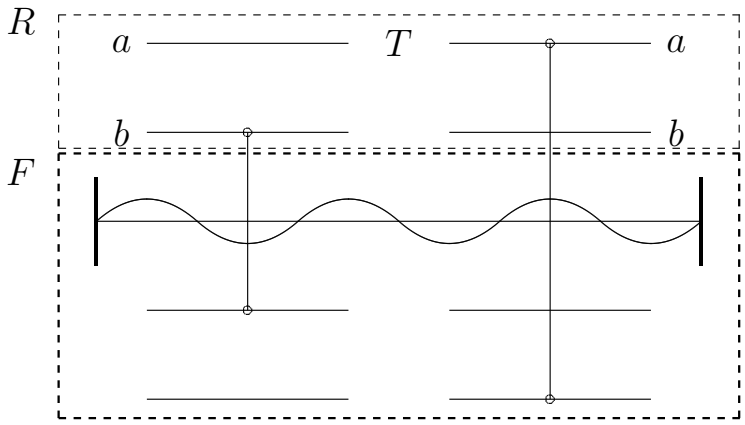


Figure 2.16: Interaction of the electromagnetic field of the resonator F with the reservoir of atoms R, at temperature T

the form of an atomic beam of two-level atoms in equilibrium at temperature T. As we will see later, the specific model of the reservoir does not affect the final result.[1] The considered model is shown in Figure 2.16.

The initial atomic density matrix corresponds to a Boltzmann distribution at temperature T:

$$\hat{\rho}_{at} = \begin{pmatrix} \rho_{aa} & 0 \\ 0 & \rho_{bb} \end{pmatrix} = z^{-1} \begin{pmatrix} e^{-\frac{\hbar\omega_a}{k_B T}} & 0 \\ 0 & e^{-\frac{\hbar\omega_b}{k_B T}} \end{pmatrix}. \tag{2.62}$$

Here $\hbar\omega_a = E_a$ is the energy of the upper level; $\hbar\omega_b = E_b$ is the energy of the lower level; $z = e^{-\frac{\hbar\omega_a}{k_B T}} + e^{-\frac{\hbar\omega_b}{k_B T}}$ is the partition function; T is the reservoir temperature. The interaction Hamiltonian is taken in the form (2.14). For simplicity, let $\omega_{ab} = \omega$, i.e., the transition frequency equals the mode frequency. Then

$$\hat{V} = \hbar g \left(\hat{\sigma}\hat{a}^\dagger + \hat{\sigma}^\dagger \hat{a} \right) = \hbar g \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix} \tag{2.63}$$

since $\hat{\sigma} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $\hat{\sigma}^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; g is the interaction constant.

We keep terms up to second order inclusive in the iterative series (2.61). In our case, after integration, a simple expression is obtained:

$$\hat{\rho}_f(t) = \hat{\rho}_f(t_0 + \tau) =$$
$$= \hat{\rho}_f(t_0) + Sp_{at} \left\{ -\frac{i}{\hbar}\tau \left[\hat{V}, \hat{\rho}_{at,f} \right] - \frac{1}{2} \left(\frac{\tau}{\hbar} \right)^2 \left[\hat{V}, \left[\hat{V}, \hat{\rho}_{at,f} \right] \right] \right\}, \tag{2.64}$$

where $\hat{\rho}_{at,f}$ is the statistical operator of the atom-field system; $\hat{\rho}_f$ is the statistical operator of the field traced over the atomic variables.

[1] If the interaction is weak, i.e., the binding energy is small compared to the energy of the dynamical system.

In obtaining (2.64), it was assumed that an atom from the beam interacts with the field during the time τ (the transit time through the resonator). From the initial conditions (at time t_0), we have:

$$\hat{\rho}_{at,f} = \hat{\rho}_f \begin{pmatrix} \rho_{aa} & 0 \\ 0 & \rho_{bb} \end{pmatrix} = \begin{pmatrix} \rho_{aa}\hat{\rho}_f & 0 \\ 0 & \rho_{bb}\hat{\rho}_f \end{pmatrix} \tag{2.65}$$

The first order term in (2.64) gives a matrix with zero diagonal elements with respect to reservoir variables. The trace of this matrix over these variables is obviously zero:

$$Sp_{at}\left[\hat{V}, \hat{\rho}_{at,f}\right] = \hbar g Sp_{at}\left\{ \begin{pmatrix} 0 & \hat{a}\rho_{bb}\hat{\rho}_f \\ \hat{a}^\dagger\rho_{aa}\hat{\rho}_f & 0 \end{pmatrix} - \right. \\ \left. - \begin{pmatrix} 0 & \rho_{aa}\hat{\rho}_f\hat{a} \\ \rho_{bb}\hat{\rho}_f\hat{a}^\dagger & 0 \end{pmatrix} \right\} = 0. \tag{2.66}$$

The matrix entering the second order term has nonzero diagonal elements, and its trace over reservoir variables is nonzero and can be computed:

$$Sp_{at}\left[\hat{V}, \left[\hat{V}, \hat{\rho}_{at,f}\right]\right] = \hbar^2 g^2 \cdot \\ \cdot Sp_{at}\begin{pmatrix} \hat{a}\hat{a}^\dagger\rho_{aa}\hat{\rho}_f - \hat{a}\rho_{bb}\hat{\rho}_f\hat{a}^\dagger & 0 \\ 0 & \hat{a}^\dagger\hat{a}\rho_{bb}\hat{\rho}_f - \hat{a}^\dagger\rho_{aa}\hat{\rho}_f\hat{a} \end{pmatrix} + \text{h.c.} \tag{2.67}$$

From here, using equations (2.64), (2.66), (2.67), we obtain the final equation for the field statistical operator $\hat{\rho}$:

$$\hat{\rho}_f(t_0+\tau) = \hat{\rho}_f(t_0) - \frac{1}{2}g^2\tau^2\left\{\left(\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}\right)\rho_{aa} + \right. \\ \left. + \left(\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f\hat{a}^\dagger\right)\rho_{bb}\right\} + \text{h.c.} \tag{2.68}$$

Thus, due to the interaction with one atom, the statistical operator of the mode changes by an amount $\delta\hat{\rho}_f = \hat{\rho}_f(t_0+\tau) - \hat{\rho}_f(t_0)$, given by (2.68). If the injection rate of atoms into the resonator is r atoms per second, then during the time Δt, $r\Delta t$ atoms will have interacted. The change in the density operator over this time will be $\Delta\hat{\rho}_f = \delta\hat{\rho}_f r\Delta t$. Define the average derivative as

$$\dot{\hat{\rho}}_f = \frac{\Delta\hat{\rho}_f}{\Delta t} = \delta\hat{\rho}_f r = r\left(\hat{\rho}_f(t_0+\tau) - \hat{\rho}_f(t_0)\right).$$

Denote $r_a = r\rho_{aa}$ as the injection rate of atoms in the upper state, and $r_b = r\rho_{bb}$ as the injection rate of atoms in the lower state. This leads to the final form of

the equation for the statistical operator of the resonator mode field:

$$\dot{\hat{\rho}}_f = -\frac{1}{2}R_a\left(\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}\right) - \\ -\frac{1}{2}R_b\left(\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f\hat{a}^\dagger\right) + \text{h.c.} \tag{2.69}$$

Here we denote

$$R_a = r_a g^2\tau^2, \\ R_b = r_b g^2\tau^2. \tag{2.70}$$

Equation (2.69) is written in operator form. It can be written in various representations.

2.9 Equation for the field density matrix in the number representation

In the number representation (photon number representation), the matrix elements are defined by the equality

$$\rho_{nm} = \langle n|\,\hat{\rho}\,|m\rangle\,.$$

To obtain the required equation, multiply (2.69) from the left by $\langle n|$ and from the right by $|m\rangle$. Using the properties of the operators $\hat{a}$ and $\hat{a}^\dagger$, we obtain a system of equations relating the matrix elements

$$\dot{\rho}_{nm} = -\frac{1}{2}\left(R_a\left(n+1+m+1\right) + R_b\left(n+m\right)\right)\rho_{nm} + \\ +R_a\sqrt{nm}\rho_{n-1,m-1} + R_b\sqrt{\left(n+1\right)\left(m+1\right)}\rho_{n+1,m+1}. \tag{2.71}$$

For diagonal elements $n = m$, hence

$$\dot{\rho}_{nn} = -\left(R_a\left(n+1\right) + R_b n\right)\rho_{nn} + \\ +R_a n\rho_{n-1,n-1} + R_b\left(n+1\right)\rho_{n+1,n+1}. \tag{2.72}$$

This equation can be considered as a balance of probability flows of photon numbers. This is schematically shown in Figure 2.17. At thermal equilibrium, the flows must be equal. Using the principle of detailed balance, we obtain equalities:

$$R_a n\rho_{n-1,n-1} = R_b n\rho_{nn}, \\ R_a\left(n+1\right)\rho_{nn} = R_b\left(n+1\right)\rho_{n+1,n+1} \tag{2.73}$$

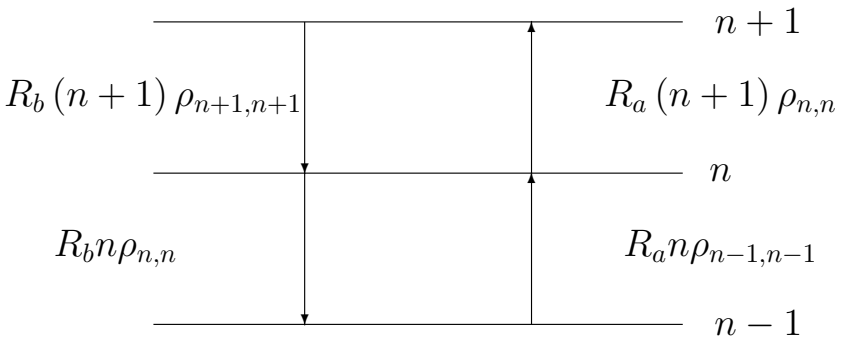


Figure 2.17: Balance of probability flows of photon numbers

From (2.73) we have

$$\rho_{n+1,n+1} = \frac{R_a}{R_b}\rho_{nn} = e^{-\frac{\hbar\omega}{k_B T}}\rho_{nn}, \tag{2.74}$$

since $\frac{R_a}{R_b} = \exp\left(-\frac{\hbar\omega}{k_B T}\right)$ (the reservoir is in equilibrium at temperature T). Applying (2.74) successively starting at $n = 0$, we get:

$$\rho_{nn} = \rho_{00} e^{-\frac{n\hbar\omega}{k_B T}} = \left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right) e^{-\frac{n\hbar\omega}{k_B T}}, \tag{2.75}$$

where ρ_{00} is found from the normalization condition $\sum_{(n)} \rho_{nn} = 1$.

The average photon number in the mode, as expected, is given by the Planck formula

$$\bar{n} = \langle n \rangle = \sum_{(n)} n\rho_{nn} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}. \tag{2.76}$$

It follows that over time the radiation temperature approaches the temperature of the atomic beam (reservoir). The time evolution of the average photon number can be obtained from equation (2.72)

$$\begin{aligned} \frac{d}{dt}\langle n\,(t)\rangle = \sum_{(n)} n\dot{\rho} = \\ = \sum_{(n)} \left(-R_a\left(n^2+n\right)\rho_{nn} - R_b n^2 \rho_{nn} + \right. \\ \left. + R_b\left(n^2+n\right)\rho_{n+1,n+1} + R_a n^2 \rho_{n-1,n-1}\right). \end{aligned} \tag{2.77}$$

Change the summation variables by $m = n + 1$ in the third sum and $m = n - 1$

in the fourth sum. We get:

$$\frac{d}{dt}\langle n\rangle = -R_a \sum_{(n)} \left(n^2+n\right)\rho_{nn} - R_b \sum_{(n)} n^2 \rho_{nn} +$$
$$+R_b \sum_{(m)} \left(m^2-m\right)\rho_{m,m} + R_a \sum_{(m)} \left(m^2+2m+1\right)\rho_{m,m} =$$
$$= R_a \sum_{(m)} (m+1)\,\rho_{m,m} - R_b \sum_{(m)} m\rho_{m,m} =$$
$$= (R_a - R_b)\,\langle n\rangle + R_a. \tag{2.78}$$

At equilibrium

$$\frac{d}{dt}\langle n\rangle = 0,$$

and we obtain the previous relation

$$\left\langle n_{(\infty)}\right\rangle = \bar{n} = \frac{R_a}{R_b - R_a} = \frac{1}{\frac{R_b}{R_a}-1} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}}-1}. \tag{2.79}$$

Definition 2.9.1 (Quality factor (Q factor))**.** The quality factor Q of an oscillatory system is defined as the number of oscillations performed by the system during the characteristic decay time τ [48]. That is, if the change in the energy of the system U (complex amplitude) satisfies the equation

$$\frac{dU}{dt} = -\frac{1}{\tau}\left(U - U_0\right), \tag{2.80}$$

then

$$U = \left(U|_{t=0} - U_0\right) e^{-\frac{t}{\tau}} + U_0.$$

Thus,

$$Q = \omega\tau,$$

and equation (2.80) can be rewritten as

$$\frac{dU}{dt} = -\frac{\omega}{Q}\left(U - U_0\right).$$

Equation (2.78) describes the change of the photon number (energy) over time due to relaxation (interaction with a dissipative system). The classical equation describing this process is (see def. 2.9.1)

$$\frac{d}{dt}\langle n\rangle = -\frac{\omega}{Q}\langle n\rangle + \frac{\omega}{Q}\langle n\rangle_{\text{eq}},$$

where $\langle n\rangle_{\text{eq}} = \bar{n}_T$ is the equilibrium value of $\langle n\rangle$ at temperature T, and Q is the quality factor of the resonator (see def. 2.9.1). Therefore, one can set

$$\begin{aligned} R_b - R_a &= \frac{\omega}{Q}, \\ R_a &= \bar{n}_T \frac{\omega}{Q}, \\ R_b &= \frac{\omega}{Q}\left(1 + \bar{n}_T\right), \end{aligned} \tag{2.81}$$

i.e., the introduced quantities R_a and R_b are expressed through the classical quantity $\frac{\omega}{Q}$ characterizing losses in the resonator.

Another quantity of interest is the average electromagnetic field

$$\begin{aligned} \bar{E}_{(f)} &= E_0 \sin kz \mathrm{Tr}\left(\hat{\rho}\left(\hat{a}^\dagger + \hat{a}\right)\right) = \\ &= E_0 \sin kz \mathrm{Tr}\left(\hat{\rho}\hat{a}\right) + \text{c.c.} = \\ &= E_1 \mathrm{Tr}\left(\hat{\rho}\hat{a}\right) + \text{c.c.} = \langle E\rangle + \text{c.c.}, \end{aligned}$$

where $\langle E\rangle$ is the analytic signal of the classical field. The equation satisfied by the field can be obtained using the equation of motion for the density matrix. Write the equation for the density operator of the mode field (2.69) and using the expressions for R_a, R_b through Q and $\bar{n}_T$ (2.81), we have

$$\begin{aligned} \dot{\hat{\rho}} = &-\frac{\omega}{2Q}\bar{n}_T\left(\hat{a}\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\right) - \\ &-\frac{\omega}{2Q}\left(\bar{n}_T + 1\right)\left(\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\right) \end{aligned} \tag{2.82}$$

Since

$$\langle E\rangle = E_1 \mathrm{Tr}\left(\hat{\rho}\hat{a}\right), \quad \frac{d\langle E\rangle}{dt} = E_1 \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{a}\right), \tag{2.83}$$

we multiply equation (2.82) by $E_1\hat{a}$ and take the trace. The result is:

$$\begin{aligned} \langle \dot{E}\rangle = &-\frac{\omega E_1}{2Q}\bar{n}_T\left\{\mathrm{Tr}\left(\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - 2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\right) + \right. \\ &\left. + \mathrm{Tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}\right)\right\} - \\ &-\frac{\omega E_1}{2Q}\mathrm{Tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}\right). \end{aligned} \tag{2.84}$$

It is known that under the trace one can perform cyclic permutations of operators.

Apply this to (2.84). For example,

$$\begin{aligned}
\mathrm{Tr}\left(\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - 2\hat{a}^{\dagger}\hat{\rho}\hat{a}\hat{a} + \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\right) = \\
= \mathrm{Tr}\left(\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - 2\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\right) = \\
= \mathrm{Tr}\left(\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - 2\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\left(\hat{a}\hat{a}^{\dagger} - 1\right)\right) = \\
\mathrm{Tr}\left(\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - 2\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} + \left(\hat{a}\hat{a}^{\dagger} - 1\right)\hat{\rho}\hat{a}\right) = -\mathrm{Tr}\left(\hat{\rho}\hat{a}\right).
\end{aligned} \tag{2.85}$$

Doing the same for the second bracket in (2.84), we get

$$\begin{aligned}
\mathrm{Tr}\left(\hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}\right) = \\
= \mathrm{Tr}\left(\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a} - 2\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a} + \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}\right) = \\
= \mathrm{Tr}\left(\hat{\rho}\left(\hat{a}^{\dagger}\hat{a} + 1\right)\hat{a} - 2\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a} + \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}\right) = \mathrm{Tr}\left(\hat{\rho}\hat{a}\right).
\end{aligned} \tag{2.86}$$

Substituting (2.85) and (2.86) into (2.84), we have

$$\begin{aligned}
\langle\dot{E}\rangle = -\frac{\omega E_1}{2Q}\bar{n}_T\left\{\mathrm{Tr}\left(\hat{\rho}\hat{a}\right) - \mathrm{Tr}\left(\hat{\rho}\hat{a}\right)\right\} - \\
-\frac{\omega E_1}{2Q}\mathrm{Tr}\left(\hat{\rho}\hat{a}\right) = -\frac{\omega}{2Q}\langle E\rangle.
\end{aligned} \tag{2.87}$$

Thus, we have related the reservoir parameters (beam parameters) to the classical quantity Q and to the average photon number in the mode at the reservoir temperature. This enables writing the density matrix equation of motion for the field in a general form, while the specific reservoir model does not matter

$$\begin{aligned}
\dot{\rho}_{nm} = -\frac{\omega}{2Q}\left(2\bar{n}_T\left(n + m + 1\right) + n + m\right)\rho_{nm} + \\
+\frac{\omega\bar{n}_T}{Q}\sqrt{nm}\rho_{n-1,m-1} + \frac{\omega}{Q}\left(\bar{n}_T + 1\right)\sqrt{\left(n + 1\right)\left(m + 1\right)}\rho_{n+1,m+1}
\end{aligned} \tag{2.88}$$

In operator form, this equation reads

$$\begin{aligned}
\dot{\rho} = -\frac{\omega}{2Q}\left\{\bar{n}_T\left(\hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{a}^{\dagger}\hat{\rho}\hat{a}\right) + \right. \\
\left. + \left(\bar{n}_T + 1\right)\left(\hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^{\dagger}\right)\right\} + \mathrm{h.c.}
\end{aligned} \tag{2.89}$$

The equations (2.69), (2.89) in number representation are one of many possible. Often it is convenient to use other representations.

2.10 Equation of motion for the statistical operator of the mode field in the coherent state representation

In quantum optics and quantum electronics, it is often convenient to use coherent states $|\alpha\rangle$. In this case, the statistical operator can be written in diagonal representation (1.118)

$$\hat{\rho}(t) = \int P(\alpha, t) |\alpha\rangle \langle\alpha| \, d^2\alpha, \tag{2.90}$$

Substituting this expression into the equation of motion for the statistical operator (2.69), 2.89, we obtain:

$$\begin{aligned} \int \dot{P}(\alpha, t) |\alpha\rangle \langle\alpha| \, d^2\alpha = \\ = -\frac{1}{2} \int P(\alpha, t) \left[R_a \left(\hat{a}\hat{a}^\dagger |\alpha\rangle \langle\alpha| - \hat{a}^\dagger |\alpha\rangle \langle\alpha| \hat{a} \right) + \right. \\ \left. + R_b \left(\hat{a}^\dagger \hat{a} |\alpha\rangle \langle\alpha| - \hat{a} |\alpha\rangle \langle\alpha| \hat{a}^\dagger \right) \right] d^2\alpha + \text{h.c.} \end{aligned} \tag{2.91}$$

The following equalities hold

$$\begin{aligned} \hat{a}^\dagger |\alpha\rangle \langle\alpha| = \left(\frac{\partial}{\partial \alpha} + \alpha^* \right) |\alpha\rangle \langle\alpha| , \\ |\alpha\rangle \langle\alpha| \hat{a} = \left(\frac{\partial}{\partial \alpha^*} + \alpha \right) |\alpha\rangle \langle\alpha| , \end{aligned} \tag{2.92}$$

These equalities can be easily proven by using one of the definitions of the coherent state

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \frac{1}{2}\alpha\alpha^*} |0\rangle .$$

From this we have

$$|\alpha\rangle \langle\alpha| = e^{\alpha\hat{a}^\dagger - \frac{1}{2}\alpha\alpha^*} |0\rangle \langle 0| e^{\alpha^*\hat{a} - \frac{1}{2}\alpha^*\alpha}.$$

Differentiating this expression with respect to α, we obtain:

$$\begin{aligned} \frac{\partial}{\partial \alpha} |\alpha\rangle \langle\alpha| = \left(\hat{a}^\dagger - \alpha^* \right) e^{\alpha\hat{a}^\dagger - \frac{1}{2}\alpha\alpha^*} |0\rangle \langle 0| e^{\alpha^*\hat{a} - \frac{1}{2}\alpha^*\alpha} = \\ = \left(\hat{a}^\dagger - \alpha^* \right) |\alpha\rangle \langle\alpha| \end{aligned}$$

therefore,

$$\hat{a}^\dagger |\alpha\rangle \langle\alpha| = \left(\frac{\partial}{\partial \alpha} + \alpha^* \right) |\alpha\rangle \langle\alpha| . \tag{2.93}$$

In the same way, the validity of the second equality is shown

$$\frac{\partial}{\partial\alpha^*}\left|\alpha\right\rangle\left\langle\alpha\right| = e^{\alpha\hat{a}^\dagger-\frac{1}{2}\alpha\alpha^*}\left|0\right\rangle\left\langle 0\right| e^{\alpha^*\hat{a}-\frac{1}{2}\alpha^*\alpha}\left(\hat{a}-\alpha\right).$$

Hence,

$$\left|\alpha\right\rangle\left\langle\alpha\right|\hat{a} = \left(\frac{\partial}{\partial\alpha^*}+\alpha\right)\left|\alpha\right\rangle\left\langle\alpha\right|, \tag{2.94}$$

Using (2.92) and known relations for creation and annihilation operators $\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$, $\left\langle\alpha\right|\hat{a}^\dagger = \alpha^*\left\langle\alpha\right|$, we reduce equation (2.91) to the form:

$$\begin{aligned}\int \dot{P}\left(\alpha,t\right)\left|\alpha\right\rangle\left\langle\alpha\right| d^2\alpha = \\ = -\frac{1}{2}R_a\int P\left(\alpha,t\right)\left\{-\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}+\alpha^*\frac{\partial}{\partial\alpha^*}\right)\left|\alpha\right\rangle\left\langle\alpha\right|\right\} d^2\alpha - \\ -\frac{1}{2}R_b\int P\left(\alpha,t\right)\alpha\frac{\partial}{\partial\alpha}\left|\alpha\right\rangle\left\langle\alpha\right| d^2\alpha + \text{h.c.}\end{aligned} \tag{2.95}$$

We need to consider the integrals appearing in (2.95), transforming them by integration by parts. In the integrals in (2.95), $d^2\alpha = dxdy$, if $\alpha = x+iy$, $x = \operatorname{Re}\alpha$, $y = \operatorname{Im}\alpha$. Therefore, to perform integration by parts, one needs to express the integrand in terms of x, y, integrate by parts, and then return to the original variables. However, we can proceed differently. Formally, consider α and α^* as independent variables and express $dxdy$ through $d\alpha d\alpha^*$.

$$d^2\alpha = dxdy = d\alpha d\alpha^*\left|J\right|,$$

where $\left|J\right|$ is the Jacobian of the transformation, which in our case is constant $\left|J\right| = \frac{1}{2}$. Now we can integrate by parts, and in the final expression revert to the previous notation

$$d\alpha d\alpha^*\left|J\right| \to d^2\alpha.$$

We have

$$\begin{aligned}\int P\alpha\frac{\partial}{\partial\alpha}\left|\alpha\right\rangle\left\langle\alpha\right| d^2\alpha = \\ = \alpha P\left|\alpha\right\rangle\left\langle\alpha\right|\Big|_{-\infty}^{+\infty} - \int\frac{\partial}{\partial\alpha}\left(\alpha P\right)\left|\alpha\right\rangle\left\langle\alpha\right| d^2\alpha = \\ = -\int\frac{\partial}{\partial\alpha}\left(\alpha P\right)\left|\alpha\right\rangle\left\langle\alpha\right| d^2\alpha.\end{aligned} \tag{2.96}$$

Here we assume that $\alpha P(\alpha,t) \to 0$ as $|\alpha| \to \infty$. The integral with the second derivative after double integration by parts transforms to

$$\int P \frac{\partial^2}{\partial\alpha\partial\alpha^*} |\alpha\rangle\langle\alpha| \, d^2\alpha = \\ = \int \left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} P\right) |\alpha\rangle\langle\alpha| \, d^2\alpha \tag{2.97}$$

Substituting all this into equality (2.95), we obtain:

$$\int \dot{P}(\alpha,t) |\alpha\rangle\langle\alpha| \, d^2\alpha = \\ = -\int \left\{ \frac{1}{2}(R_a - R_b) \left[\frac{\partial}{\partial\alpha}(P\alpha) + c.c.\right] - \right. \\ \left. - R_a \frac{\partial^2 P}{\partial\alpha\partial\alpha^*} \right\} |\alpha\rangle\langle\alpha| \, d^2\alpha \tag{2.98}$$

This equality will hold true if the factor multiplying $|\alpha\rangle\langle\alpha|$ is set to zero. Thus, we get

$$\dot{P}(\alpha,t) = -\frac{1}{2}(R_a - R_b)\left(\frac{\partial}{\partial\alpha}(\alpha P) + c.c.\right) + R_a \frac{\partial^2 P}{\partial\alpha\partial\alpha^*} \tag{2.99}$$

This equation is of the Fokker-Planck type, which is widely used to solve classical statistical physics problems, for example, problems of Brownian motion and other similar problems. By solving (2.99) with appropriate initial and boundary conditions, one can easily write an expression for the statistical operator using (2.90)

$$\hat{\rho}(t) = \int P(\alpha,t) |\alpha\rangle\langle\alpha| \, d^2\alpha$$

and compute average values of observables related to the considered system using (1.122) and (1.128)

$$\langle O\rangle = \int P(\alpha,t) \langle\alpha| \hat{O} |\alpha\rangle \, d^2\alpha = \int P(\alpha,t) O^{(n)}(\alpha,\alpha^*) \, d^2\alpha,$$

where $O^{(n)}$ is the normal representation of the operator $\hat{O}$ (1.123). The coefficients R_a and R_b appearing in (2.99) can be, as before (2.81), expressed through Q and $\bar{n}_T$, then the equation takes the form

$$\frac{\partial}{\partial t} P(\alpha,t) = \frac{1}{2}\frac{\omega}{Q}\left[\frac{\partial}{\partial\alpha}(\alpha P(\alpha,t)) + \text{c.c.}\right] + \frac{\omega\bar{n}_T}{Q}\frac{\partial^2 P(\alpha,t)}{\partial\alpha\partial\alpha^*}. \tag{2.100}$$

As an example, let us calculate the average value of the operator of the positive frequency part of the electric field $E_1\hat{a}$.

$$\frac{d\left\langle E_1\hat{a}\right\rangle}{dt} = E_1\int \alpha\dot{P}d^2\alpha =$$
$$= E_1\frac{1}{2}\frac{\omega}{Q}\int\left[\frac{\partial}{\partial\alpha}\left(\alpha P\right) + \text{c.c.}\right]\alpha d^2\alpha +$$
$$+E_1\frac{\omega\bar{n}_T}{Q}\int\alpha\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}d^2\alpha \qquad (2.101)$$

Integrating the second term over α^*, we obtain

$$\int\alpha\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}d^2\alpha = \frac{\partial P}{\partial\alpha}\alpha\Big|_{-\infty}^{\infty} -$$
$$-\int\frac{\partial P}{\partial\alpha}\alpha\frac{\partial\alpha}{\partial\alpha^*} = 0,$$

since $\frac{\partial P}{\partial\alpha}$ is assumed to go to zero fast enough at infinity, and $\frac{\partial\alpha}{\partial\alpha^*} = 0$, because α and α^* are independent variables. The first term can be written as follows

$$\int\alpha\left(\frac{\partial}{\partial\alpha}\left(\alpha P\right) + \frac{\partial}{\partial\alpha^*}\left(\alpha^* P\right)\right)d^2\alpha =$$
$$= -\int\alpha P\frac{\partial\alpha}{\partial\alpha}d^2\alpha - \int\alpha^* P\frac{\partial\alpha}{\partial\alpha^*}d^2\alpha = -\int\alpha P d^2\alpha = -\left\langle\hat{a}\right\rangle.$$

Thus, finally we have:

$$\frac{d\left\langle E_1\hat{a}\right\rangle}{dt} = -\frac{\omega}{2Q}\left\langle E_1\hat{a}\right\rangle \qquad (2.102)$$

- the result we obtained earlier by another method. Equation (2.100) can be written in various coordinate systems, for example, in polar coordinates $\alpha = re^{i\theta}$, $\alpha^* = re^{-i\theta}$. Performing the corresponding transformations, we get

$$\frac{\partial}{\partial t}P\left(r,\theta,t\right) = \frac{1}{2}\frac{\omega}{Q}\frac{\partial}{\partial r}\left(r^2 P\left(r,\theta,t\right)\right) +$$
$$+\frac{1}{2}\frac{\omega}{Q}\frac{\bar{n}_T}{2r^2}\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2}\right)P\left(r,\theta,t\right). \qquad (2.103)$$

If one uses coordinates x and y, $\alpha = x + iy$, $\alpha^* = x - iy$, i.e. $x = \frac{1}{2}\left(\alpha + \alpha^*\right)$, $y = -\frac{i}{2}\left(\alpha - \alpha^*\right)$, the equation can be rewritten as

$$\frac{\partial}{\partial t}P\left(x,y,t\right) = \frac{1}{2}\frac{\omega}{Q}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)P\left(x,y,t\right) +$$
$$+\frac{1}{4}\frac{\omega}{Q}\bar{n}_T\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)P\left(x,y,t\right). \qquad (2.104)$$

For solving a specific problem, it is convenient to use the equation written either in rectangular or polar coordinate systems.

2.11 General Theory of Interaction of a Dynamic System with a Thermostat

So far, we have dealt with a particular model of the interaction of a resonator mode (dynamic system) with an atomic beam in equilibrium at temperature T (with a reservoir). Here we will consider a more general approach to the problem of interaction of a dynamic system with a reservoir [51]. We are dealing with two interacting systems: A - a dynamic system, and B - a reservoir (dissipative system). Systems A and B are weakly coupled, meaning the coupling energy is small compared to the energy of the dynamic system. The interaction Hamiltonian is denoted by $\hat{V}$. The dissipative system B is an extensive system in thermodynamic equilibrium at temperature T. Initially, the systems A and B will be considered on equal footing. Denote the statistical operator of the whole system by $\hat{\rho}_{AB}$.

Introduce statistical operators $\hat{\rho}_A$ and $\hat{\rho}_B$ referring respectively only to systems A and B. These operators are related to operator $\hat{\rho}_{AB}$ by the relations

$$\hat{\rho}_A = Sp_B\left\{\hat{\rho}_{AB}\right\}, \quad \hat{\rho}_B = Sp_A\left\{\hat{\rho}_{AB}\right\}, \tag{2.105}$$

The equation of motion for the statistical operator of the whole combined system in the interaction representation is

$$i\hbar\dot{\hat{\rho}}_{AB} = \left[\hat{V}, \hat{\rho}_{AB}\right]. \tag{2.106}$$

The equations of motion for the reduced operators are clearly

$$\begin{aligned} i\hbar\dot{\hat{\rho}}_A &= Sp_B\left[\hat{V}, \hat{\rho}_{AB}\right], \\ i\hbar\dot{\hat{\rho}}_B &= Sp_A\left[\hat{V}, \hat{\rho}_{AB}\right]. \end{aligned} \tag{2.107}$$

The statistical operator $\hat{\rho}_{AB}$ can be represented in the form

$$\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B + \hat{\rho}_C, \tag{2.108}$$

where $\hat{\rho}_C$ arises from the interaction between the systems, $\otimes$ denotes the external (tensor) product. At the initial time t_0 the systems are assumed to be non-interacting, so the total statistical operator equals the tensor product

$$\hat{\rho}_{AB}|_{t=t_0} = \hat{\rho}_A \otimes \hat{\rho}_B,$$

and

$$\hat{\rho}_C|_{t=t_0} = 0.$$

Now transform the term $\left[\hat{V}, \hat{\rho}_A \otimes \hat{\rho}_B\right]$ appearing in equation (2.107), and obtain

$$\left[\hat{V}, \hat{\rho}_A \otimes \hat{\rho}_B\right] = \hat{V}\hat{\rho}_A \otimes \hat{\rho}_B - \hat{\rho}_A \otimes \hat{\rho}_B\hat{V} = \\ = \hat{V}\hat{\rho}_B \otimes \hat{\rho}_A - \hat{\rho}_A \otimes \hat{\rho}_B\hat{V}. \tag{2.109}$$

because $\hat{\rho}_A \otimes \hat{\rho}_B = \hat{\rho}_B \otimes \hat{\rho}_A$ due to the fact that $\hat{\rho}_A$ and $\hat{\rho}_B$ act on different variables. From here, applying the partial trace Sp_B, we get

$$Sp_B\left[\hat{V}, \hat{\rho}_A \otimes \hat{\rho}_B\right] = Sp_B\left(\hat{V}\hat{\rho}_B\right)\hat{\rho}_A - \\ -\hat{\rho}_A Sp_B\left(\hat{\rho}_B\hat{V}\right) = \left[\hat{V}_A, \hat{\rho}_A\right], \tag{2.110}$$

where $\hat{V}_A = Sp_B\left(\hat{V}\hat{\rho}_B\right)$. Similarly, we obtain

$$Sp_A\left[\hat{V}, \hat{\rho}_A \otimes \hat{\rho}_B\right] = \left[\hat{V}_B, \hat{\rho}_B\right], \tag{2.111}$$

where $\hat{V}_B = Sp_A\left(\hat{V}\hat{\rho}_A\right)$. Taking into account the equalities (2.108), (2.109), (2.110), (2.111), the equations of motion for $\hat{\rho}_A$ and $\hat{\rho}_B$ can be written as

$$i\hbar\frac{d}{dt}\hat{\rho}_A(t) = \left[\hat{V}_A(t), \hat{\rho}_A\right] + Sp_B\left[\hat{V}, \hat{\rho}_C\right], \\ i\hbar\frac{d}{dt}\hat{\rho}_B(t) = \left[\hat{V}_B(t), \hat{\rho}_B\right] + Sp_A\left[\hat{V}, \hat{\rho}_C\right]. \tag{2.112}$$

From the form of (2.112) it is clear that $\hat{V}_A$ and $\hat{V}_B$ represent the energy shifts of one system associated with its interaction with the other, for example, with the interaction of the dynamic system with the thermostat. These changes are small and in our case may be neglected. Thus (2.112) reduces to

$$i\hbar\frac{d}{dt}\hat{\rho}_A(t) = Sp_B\left[\hat{V}, \hat{\rho}_C\right], \\ i\hbar\frac{d}{dt}\hat{\rho}_B(t) = Sp_A\left[\hat{V}, \hat{\rho}_C\right]. \tag{2.113}$$

For the operator $\hat{\rho}_C$, using that

$$\hat{\rho}_C = \hat{\rho}_{AB} - \hat{\rho}_A \otimes \hat{\rho}_B$$

we can write the equation of motion as

$$i\hbar\frac{d\hat{\rho}_C}{dt} = i\hbar\frac{d\hat{\rho}_{AB}}{dt} - i\hbar\frac{d}{dt}\left(\hat{\rho}_A \otimes \hat{\rho}_B\right) = \\ = i\hbar\left(\frac{d\hat{\rho}_{AB}}{dt} - \frac{d\hat{\rho}_A}{dt}\otimes\hat{\rho}_B - \hat{\rho}_A \otimes \frac{d\hat{\rho}_B}{dt}\right). \tag{2.114}$$

Using the equations of motion for $\hat{\rho}_{AB}$, $\hat{\rho}_A$, and $\hat{\rho}_B$ (2.113), for (2.114) we get

$$i\hbar\frac{d\hat{\rho}_C}{dt} = \left[\hat{V}, \left(\hat{\rho}_A \otimes \hat{\rho}_B + \hat{\rho}_C\right)\right] - \\ - \left(Sp_B\left[\hat{V}, \hat{\rho}_C\right]\right)\hat{\rho}_B - \hat{\rho}_A\left(Sp_A\left[\hat{V}, \hat{\rho}_C\right]\right) \tag{2.115}$$

Neglecting terms of second order smallness, we obtain a simple equation:

$$i\hbar\frac{d\hat{\rho}_C}{dt} = \left[\hat{V}, \left(\hat{\rho}_A \otimes \hat{\rho}_B\right)\right]. \tag{2.116}$$

First order small terms are $\hat{V}$ and $\hat{\rho}_C$. This follows from the assumption that interaction between the dynamic system and the thermostat is small. Terms containing products like $\hat{V}\hat{\rho}_C$ are considered second order small. The formal solution of equation (2.116) is

$$\hat{\rho}_C(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\left[\hat{V}(t'), \left(\hat{\rho}_A \otimes \hat{\rho}_B\right)\right]dt' \tag{2.117}$$

with initial condition $\hat{\rho}_C(t_0) = 0$.

So far, we have considered systems A and B as equal. Now take into account that the dynamic system A is small, while the dissipative system B is so large that its interaction with the dynamic system cannot significantly disturb its equilibrium. Incorporate this by setting $\frac{d\hat{\rho}_B}{dt} = 0$:

$$\hat{\rho}_B(t') = \hat{\rho}_B(t_0). \tag{2.118}$$

Substituting into the equation for $\hat{\rho}_A$ (2.112) the expressions (2.115) and (2.117), we get

$$\frac{d\hat{\rho}_A}{dt} = -\frac{1}{\hbar^2}\int_{t_0}^{t} Sp_B\left[\hat{V}(t), \left[\hat{V}(t'), \hat{\rho}_A(t') \otimes \hat{\rho}_B(t_0)\right]\right]dt' = \\ = -\frac{1}{\hbar^2}\int_{t_0}^{t} dt' Sp_B\left\{\left[\hat{V}(t)\hat{V}(t')\hat{\rho}_A(t') \otimes \hat{\rho}_B(t_0) - \right.\right. \\ \left. - \hat{V}(t)\hat{\rho}_A(t') \otimes \hat{\rho}_B(t_0)\hat{V}(t')\right] - \\ \left. - \left[\hat{V}(t')\hat{\rho}_A(t') \otimes \hat{\rho}_B(t_0)\hat{V}(t) - \hat{\rho}_A(t') \otimes \hat{\rho}_B(t_0)\hat{V}(t')\hat{V}(t)\right]\right\}. \tag{2.119}$$

This equation is for the density operator of the dynamic system A interacting with the dissipative system B. For the extensive system B, the correlation time of $\hat{V}(t)$ and $\hat{V}(t')$ is very short, much less than the characteristic time of change of $\hat{\rho}_A$. Therefore, the significant integration region is near $t' = t$. Then in many cases during integration we can put

$$\hat{\rho}_A(t') = \hat{\rho}_A(t).$$

A more detailed exposition of the general theory of interaction of a dynamic system with a thermostat can be found in [51]. Let us apply the general theory to specific cases.

2.12 Relaxation of an atom in the case of the simplest reservoir consisting of harmonic oscillators

Consider a multimode electromagnetic field described by creation and annihilation operators $\hat{a}_k^\dagger$, $\hat{a}_k$. The frequency of the k-th mode is denoted by ω_k.

This electromagnetic field interacts with a certain two-level atom. The atom can be in two states: excited $|a\rangle$ and non-excited $|b\rangle$. The transition operators remain $\hat{\sigma}$ and $\hat{\sigma}^\dagger$. The transition frequency is denoted by Ω. The interaction Hamiltonian in this case has the form

$$\hat{V}(t) = \hbar \sum_{(k)} g_k \hat{a}_k^\dagger \hat{\sigma} e^{-i(\Omega-\omega_k)t} + \text{h.c.} \tag{2.120}$$

where g_k is the interaction constant.

Thus, the system B (the reservoir) is the multimode electromagnetic field, and the dynamic system A under study is a two-level atom. The density operator of the reservoir (electromagnetic field) is denoted by $\hat{\rho}_R$. The density operator of the dynamic system (atom) under study is denoted by $\hat{\rho}_{at}$.

The following relations hold:

$$Sp_R\left(\hat{a}_k^\dagger \hat{a}_k \hat{\rho}_R\right) = \bar{n}_k \tag{2.121}$$

where $\bar{n}_k$ is the average number of photons in the reservoir mode;

$$\begin{gathered} Sp_R\left(\hat{a}_k \hat{a}_k^\dagger \hat{\rho}_R\right) = \bar{n}_k + 1, \\ Sp_R\left(\hat{a}_k \hat{a}_k \hat{\rho}_R\right) = Sp_R\left(\hat{a}_k^\dagger \hat{a}_k^\dagger \hat{\rho}_R\right) = 0. \end{gathered} \tag{2.122}$$

Using these equalities, the general equation (2.119) can be rewritten as:

$$\dot{\hat{\rho}}_{at}(t) = -\int_{t_0}^{t} \sum_{(k)} g_k^2 \left\{ \hat{\sigma}^\dagger \hat{\sigma} \hat{\rho}_{at}(t') (\bar{n}_k + 1) e^{i(\Omega-\omega_k)(t-t')} + \right.$$
$$+ \hat{\sigma}\hat{\sigma}^\dagger \hat{\rho}_{at}(t') \bar{n}_k e^{-i(\Omega-\omega_k)(t-t')} - \hat{\sigma}^\dagger \hat{\rho}_{at}(t') \hat{\sigma} \bar{n}_k e^{i(\Omega-\omega_k)(t-t')} -$$
$$\left. - \hat{\sigma} \hat{\rho}_{at}(t') \hat{\sigma}^\dagger (\bar{n}_k + 1) e^{-i(\Omega-\omega_k)(t-t')} \right\} dt' + \text{h.c.} \qquad (2.123)$$

Summation over k, taking into account the quasi-continuous spectrum of states of the dissipative system, can be replaced by integrals (1.44)

$$\sum_{(k)} \to \int D(\omega)\, d\omega,$$

where $D(\omega)$ is the spectral density of states. We replace $D(\omega)$, $\bar{n}(\omega)$, $g^2(\omega)$ with D, $\bar{n}$, g^2 — their values at $\omega = \Omega$, assuming that these functions slowly depend on frequency. We use the equality (see the integral representation of the delta function (18.3))

$$\int_0^\infty e^{\pm i(\Omega-\omega)(t-t')} d\omega = |\nu = \omega - \Omega| =$$
$$= \int_{-\Omega}^\infty e^{-i\nu(t\mp t')} d\nu \approx \int_{-\infty}^\infty e^{-i\nu(t\mp t')} d\nu =$$
$$= 2\pi\delta(t \mp t') \qquad (2.124)$$

and integrate over time. This leads us to the equation

$$\dot{\hat{\rho}}_{at} = -\frac{1}{2}\gamma \left\{ \bar{n} \left[\hat{\sigma}\hat{\sigma}^\dagger \hat{\rho}_{at} - \hat{\sigma}^\dagger \hat{\rho}_{at} \hat{\sigma} \right] + \right.$$
$$\left. + (\bar{n} + 1) \left[\hat{\sigma}^\dagger \hat{\sigma} \hat{\rho}_{at} - \hat{\sigma} \hat{\rho}_{at} \hat{\sigma}^\dagger \right] \right\} + \text{h.c.} \qquad (2.125)$$

where $\gamma = 4\pi D g^2$, with $\bar{n}$ taken at the mode frequency Ω. Equation (2.125) has the same form as equation (2.89) obtained earlier. The equations completely coincide if one takes $\gamma = \frac{\omega}{Q}$ and makes the substitution $\hat{\sigma} \to \hat{a}$, $\hat{\sigma}^\dagger \to \hat{a}^\dagger$. This once again confirms that the final result does not depend on the nature of the reservoir.

At reservoir temperature $T = 0$ photons are absent and $\bar{n} = 0$. Assume that at the initial moment the atom is excited

$$\hat{\rho}_{at}(t_0) = 1.$$

Using the properties (2.10) - (2.9) of the operators $\hat{\sigma}$ and $\hat{\sigma}^\dagger$, we obtain

$$\dot{\rho}_{aa} = \langle a| \dot{\hat{\rho}}_{at} |a\rangle = -\frac{1}{2}\gamma \langle a| \hat{\sigma}^\dagger \hat{\sigma} \hat{\rho}_{at} |a\rangle + \\ + \frac{1}{2}\gamma \langle a| \hat{\sigma} \hat{\rho}_{at} \hat{\sigma}^\dagger |a\rangle + \text{c.c.} = \\ = -\frac{1}{2}\gamma \langle a| \hat{\rho}_{at} |a\rangle + \text{c.c.} = -\gamma \rho_{aa}. \tag{2.126}$$

Here we used the relations $\hat{\sigma}^\dagger |a\rangle = 0$, $\hat{\sigma}^\dagger |b\rangle = |a\rangle$, $\hat{\sigma} |a\rangle = |b\rangle$, $\hat{\sigma} |b\rangle = 0$, and their conjugates. The final result is

$$\dot{\hat{\rho}}_{aa} = -\gamma \rho_{aa}. \tag{2.127}$$

This means that as a result of spontaneous emission the probability of finding the atom in the upper level decreases exponentially.

2.13 Damping of the Resonator Mode. Heisenberg-Langevin Equation.

2.13.1 Problem Statement.

We consider the problem of relaxation of the resonator field (mode) using the density matrix method, i.e., the Schrödinger representation. There is another approach belonging to Langevin, in which the equations of the dynamical system are supplemented with random forces, the statistical properties of which are known, to take into account the influence of the dissipative system. In the quantum case, the dynamical equations are the Heisenberg equations for the operators of observable quantities, and the random effects associated with the thermostat are taken into account by adding some noise operator. The resulting equations are called the Heisenberg-Langevin equations.

Let us consider the problem of relaxation of the resonator mode (harmonic oscillator) interacting with a thermostat by this method. In the Heisenberg representation, operators depend on time and satisfy the Heisenberg equations:

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{O}\right],$$

where $\hat{\mathcal{H}}$ is the Hamiltonian of the system under consideration. For example, for the free field we previously obtained equations (1.61):

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{a}\right] = -i\omega\hat{a}, \quad \frac{d\hat{a}^\dagger}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{a}^\dagger\right] = i\omega\hat{a}^\dagger,$$

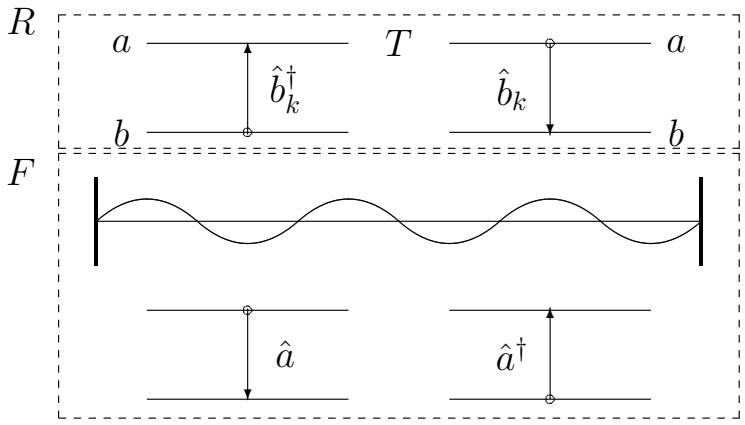


Figure 2.18: Interaction of the electromagnetic field mode of the resonator F with a reservoir of atoms R at temperature T. $\hat{a}^\dagger$ and $\hat{a}$ - creation and annihilation operators of the studied electromagnetic field mode. $\hat{b}_k^\dagger$ and $\hat{b}_k$ - creation and annihilation operators of the k-th mode of the reservoir (phonons).

where ω is the mode frequency, and $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators.

The problem under consideration is schematically shown in Figure 2.18. The electromagnetic field mode, considered as a dynamical system, interacts with a large reservoir consisting of a large number of harmonic oscillators (phonons) in equilibrium at temperature T. The Hamiltonian of such a system can be represented as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V},$$

where

$$\hat{\mathcal{H}}_0 = \hbar\omega\hat{a}^\dagger\hat{a} + \sum_{(k)} \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k$$

is the Hamiltonian of the system without interaction between its parts,

$$\hat{V} = \hbar \sum_{(k)} g_k \left(\hat{b}_k^\dagger \hat{a} + \hat{a}^\dagger \hat{b}_k \right) \tag{2.128}$$

is the Hamiltonian of the interaction between the dynamical system and the thermostat. The constant of interaction for the k-th reservoir mode is denoted by g_k in (2.128). Here, $\hat{a}$, $\hat{a}^\dagger$ are annihilation and creation operators for the field mode, and $\hat{b}_k$, $\hat{b}_k^\dagger$ are annihilation and creation operators for the k-th reservoir mode (phonons). Vacuum terms in the Hamiltonian are neglected since they cancel out during the derivation of the equations.

2.13.2 Heisenberg Equations of Motion for Operators.

The Heisenberg equations of motion for $\hat{a}$ and $\hat{b}_k$:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{a}\right],$$
$$\frac{d\hat{b}_k}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{b}_k\right]$$

using the commutation relations $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ and $\left[\hat{b}_k, \hat{b}_k^\dagger\right] = 1$ transform to the form

$$\frac{d\hat{a}(t)}{dt} = -i\omega\hat{a}(t) - i\sum_{(k)} g_k\hat{b}_k(t),$$
$$\frac{d\hat{b}_k(t)}{dt} = -i\omega_k\hat{b}_k(t) - ig_k\hat{a}(t). \tag{2.129}$$

From system (2.129) one can exclude $\hat{b}_k$ by integrating the second equation and substituting the result into the first equation. Integration is simplest using operational calculus. We obtain

$$\hat{b}_k(t) = \hat{b}_k(0)\, e^{-i\omega_k t} - ig_k\int_0^t dt'\hat{a}(t')\, e^{-i\omega_k(t-t')}$$

and then

$$\frac{d\hat{a}(t)}{dt} = -i\omega\hat{a}(t) - \sum_{(k)} g_k^2\int_0^t dt'\hat{a}(t')\, e^{-i\omega_k(t-t')} + \hat{f}(t),$$
$$\hat{f}(t) = -i\sum_{(k)} g_k\hat{b}_k(0)\, e^{-i\omega_k t}. \tag{2.130}$$

The operator $\hat{f}(t)$ depends on thermostat variables and is a noise operator describing the influence of the reservoir on the dynamical system.

Equation (2.130) is integro-differential. Under some approximations, it can be transformed into a purely differential equation. First, we switch to slow variables by writing

$$\hat{a}(t) = \hat{A}(t)\, e^{-i\omega t},$$
$$\hat{f}(t) = \hat{F}(t)\, e^{-i\omega t},$$

where ω is the mode frequency. From this we have

$$\frac{d\hat{A}(t)}{dt} = -\sum_{(k)} g_k^2\int_0^t dt'\hat{A}(t')\, e^{i(\omega-\omega_k)(t-t')} + \hat{F}(t). \tag{2.131}$$

The first term on the right-hand side of (2.131) resembles the term encountered when considering spontaneous emission using the Weisskopf-Wigner method. The approach applied there can be applied here as well. From (2.131) it is clear that the main contributions come from terms with frequencies close to the mode frequency: $\omega_k \approx \omega$. For this reason, all slowly varying terms can be taken at frequency ω and taken out of the integral. Summation over k, assuming a quasi-continuous phonon spectrum, can be replaced by integration over frequency:

$$\sum_{(k)} g_k^2 \int_0^t dt' \hat{A}(t') e^{i(\omega-\omega_k)(t-t')} =$$

$$= \int_0^t dt' \hat{A}(t') \int_0^\infty g^2(\omega) D(\omega) e^{i(\omega-\omega')(t-t')} d\omega' =$$

$$= g^2(\omega) D(\omega) \int_0^t dt' \hat{A}(t') \int_0^\infty e^{i(\omega-\omega')(t-t')} d\omega',$$

where $D(\omega)$ is the density of states. $D(\omega)$ and $g^2(\omega)$ are taken at the frequency of the field mode and factored out of the integral.

Consider the integral over frequency. Acting as when considering spontaneous emission by the Weisskopf-Wigner method, we get

$$\int_0^\infty e^{-i(\omega'-\omega)(t-t')} d\omega' = \int_{-\omega}^\infty e^{-i\nu(t-t')} d\nu \approx \int_{-\infty}^\infty e^{-i\nu(t-t')} d\nu =$$

$$= \int_{-\infty}^\infty e^{i\nu(t'-t)} d\nu = 2\pi\delta(t'-t).$$

Therefore, one can write

$$\frac{d\hat{A}(t)}{dt} = -2\pi g^2(\omega) D(\omega) \int_0^t dt' \hat{A}(t') \delta(t-t') + \hat{F}(t) =$$

$$= -2\pi g^2(\omega) D(\omega) \hat{A}(t) + \hat{F}(t).$$

Define $2\pi D g^2\big|_\omega = \frac{\gamma}{2}$ as the field damping coefficient. We obtain

$$\frac{d\hat{A}(t)}{dt} = -\frac{\gamma}{2}\hat{A}(t) + \hat{F}(t), \tag{2.132}$$

where $\hat{F}(t)$ is the random noise operator whose properties must be determined. As will be seen later, the noise operator ensures the preservation of commutation relations for the field mode operator.

Let us now study the statistical properties characterized by the correlation functions of $\hat{F}$. Assume the reservoir is in thermal equilibrium at temperature T.

Then we have:

$$\left\langle \hat{b}_k(0) \right\rangle_R = \left\langle \hat{b}_k^\dagger(0) \right\rangle_R = 0,$$
$$\left\langle \hat{b}_k^\dagger(0)\, \hat{b}_{k'}(0) \right\rangle_R = \bar{n}_k \delta_{kk'},$$
$$\left\langle \hat{b}_k(0)\, \hat{b}_{k'}^\dagger(0) \right\rangle_R = (\bar{n}_k + 1)\, \delta_{kk'},$$
$$\left\langle \hat{b}_k(0)\, \hat{b}_{k'}(0) \right\rangle_R = \left\langle \hat{b}_k^\dagger(0)\, \hat{b}_{k'}^\dagger(0) \right\rangle_R = 0, \tag{2.133}$$

where $\langle \dots \rangle_R$ denotes averaging over thermostat variables, and $\bar{n}_k$ is the average number of phonons in mode k. The noise operator was defined by the formula

$$\hat{F}(t) = -i \sum_{(k)} g_k \hat{b}_k(0)\, e^{-i(\omega_k - \omega)t}. \tag{2.134}$$

Using (2.133), it is easy to see

$$\left\langle \hat{F}(t) \right\rangle_R = \left\langle \hat{F}^\dagger(t) \right\rangle_R = 0,$$
$$\left\langle \hat{F}^\dagger(t)\, \hat{F}(t') \right\rangle_R =$$
$$= \sum_k \sum_{k'} g_k g_{k'} \left\langle \hat{b}_k^\dagger(0)\, \hat{b}_{k'}(0) \right\rangle_R e^{i(\omega_k - \omega)t} e^{-i(\omega_{k'} - \omega)t'} =$$
$$= \sum_k g_k^2 \bar{n}_k e^{i(\omega_k - \omega)(t - t')} =$$
$$= \int_0^\infty g^2(\omega)\, D(\omega)\, \bar{n}_T e^{i(\omega' - \omega)(t - t')} d\omega' =$$
$$= 2\pi g^2(\omega)\, D(\omega)\, \bar{n}_T \delta(t - t') = \frac{\gamma \bar{n}_T}{2} \delta(t - t'), \tag{2.135}$$

where $\bar{n}_T$ is the average number of phonons in the reservoir mode at temperature T, determined by Planck's formula, $\gamma = 4\pi g^2(\omega)\, D(\omega)$. If, according to Langevin, we introduce the diffusion coefficient $\mathcal{D} = \frac{\gamma \bar{n}_T}{2}$, then the last equation of system (2.135) can be written as

$$\left\langle \hat{F}^\dagger(t)\, \hat{F}(t') \right\rangle_R = \mathcal{D} \delta(t - t').$$

Note that $\langle \dots \rangle_R$ denotes averaging over thermostat variables, so the result of averaging is a correlation function depending only on the dynamical system variables (field).

Similarly, one can prove

$$\left\langle \hat{F}(t)\, \hat{F}^\dagger(t') \right\rangle_R = \frac{\gamma(\bar{n}_T + 1)}{2} \delta(t - t') \tag{2.136}$$

and

$$\left\langle \hat{F}(t)\,\hat{F}(t')\right\rangle_R = \left\langle \hat{F}^\dagger(t)\,\hat{F}^\dagger(t')\right\rangle_R = 0. \tag{2.137}$$

Thus, in our approximation, the noise is δ-correlated.

We still need the correlation function of the type $\left\langle \hat{F}^\dagger(t)\,\hat{A}(t')\right\rangle_R$ which is necessary for deriving the equation satisfied by $\left\langle \hat{A}^\dagger(t)\,\hat{A}(t')\right\rangle_R$. For this, integrate the equation for $\hat{A}(t)$ (2.132), and using operational calculus, obtain

$$\hat{A}(t) = \hat{A}(0)\,e^{-\frac{\gamma}{2}t} + \int_0^t dt' e^{-\frac{\gamma}{2}(t-t')}\hat{F}(t').$$

Multiply this expression by $\hat{F}^\dagger(t)$ and average over the thermostat. We get

$$\left\langle \hat{F}^\dagger(t)\,\hat{A}(t)\right\rangle_R = \int_0^t \left\langle \hat{F}^\dagger(t)\,\hat{F}(t')\right\rangle_R e^{-\frac{\gamma}{2}(t-t')}dt' + \\ + \left\langle \hat{F}^\dagger(t)\,\hat{A}(0)\right\rangle_R e^{-\frac{\gamma}{2}t}. \tag{2.138}$$

The last term in (2.138) is zero since $\hat{F}^\dagger(t)$ and $\hat{A}(0)$ are independent, and their averages are zero. Taking into account the second equation of system (2.135), i.e., $\left\langle \hat{F}^\dagger(t)\,\hat{F}(t')\right\rangle_R = \frac{\gamma\bar{n}_T}{2}\delta(t-t')$, we obtain

$$\left\langle \hat{F}^\dagger(t)\,\hat{A}(t)\right\rangle_R = \frac{\gamma\bar{n}_T}{2}.$$

Now consider the correlation function $\left\langle \hat{A}^\dagger(t)\,\hat{A}(t)\right\rangle_R$. Differentiating this function with respect to time, we get

$$\frac{d\left\langle \hat{A}^\dagger(t)\,\hat{A}(t)\right\rangle_R}{dt} = \left\langle \frac{d\hat{A}^\dagger(t)}{dt}\hat{A}(t)\right\rangle_R = \left\langle \frac{d\hat{A}^\dagger(t)}{dt}\hat{A}(t)\right\rangle_R + \left\langle \hat{A}^\dagger(t)\,\frac{d\hat{A}(t)}{dt}\right\rangle_R. \tag{2.139}$$

Then use equation (2.132) and its conjugate, resulting in

$$\frac{d\left\langle \hat{A}^\dagger(t)\,\hat{A}(t)\right\rangle_R}{dt} = -\gamma\left\langle \hat{A}^\dagger(t)\,\hat{A}(t)\right\rangle_R + \left\langle \hat{F}^\dagger(t)\,\hat{A}(t)\right\rangle_R + \left\langle \hat{A}^\dagger(t)\,\hat{F}(t)\right\rangle_R.$$

Considering the expressions

$$\left\langle \hat{F}^\dagger(t)\,\hat{A}(t)\right\rangle_R = \left\langle \hat{A}^\dagger(t)\,\hat{F}(t)\right\rangle_R = \frac{\gamma\bar{n}_T}{2},$$

we finally get

$$\frac{d\left\langle \hat{A}^{\dagger}(t)\,\hat{A}(t)\right\rangle_{R}}{dt}=-\gamma\left\langle \hat{A}^{\dagger}(t)\,\hat{A}(t)\right\rangle_{R}+\gamma\bar{n}_{T}.$$

Similarly, one can derive

$$\frac{d\left\langle \hat{A}(t)\,\hat{A}^{\dagger}(t)\right\rangle_{R}}{dt}=-\gamma\left\langle \hat{A}(t)\,\hat{A}^{\dagger}(t)\right\rangle_{R}+\gamma\left(\bar{n}_{T}+1\right). \tag{2.140}$$

Subtracting these expressions, we obtain for the commutator

$$\frac{d\left\langle \left[\hat{A}(t),\hat{A}^{\dagger}(t)\right]\right\rangle_{R}}{dt}=\gamma\left(1-\left\langle \left[\hat{A}(t),\hat{A}^{\dagger}(t)\right]\right\rangle_{R}\right). \tag{2.141}$$

From equation (2.141) it follows that if at the initial moment

$$\left\langle \left[\hat{A}(0),\hat{A}^{\dagger}(0)\right]\right\rangle_{R}=1,$$

i.e., commutation relations hold, then these relations will always hold. In other words, commutation relations averaged over thermostat variables are preserved over time.

The picture would be completely different if quantum noise were not taken into account. In that case, equation (2.141) would take the form

$$\frac{d\left\langle \left[\hat{A}(t),\hat{A}^{\dagger}(t)\right]\right\rangle_{R}}{dt}=-\gamma\left\langle \left[\hat{A}(t),\hat{A}^{\dagger}(t)\right]\right\rangle_{R},$$

the solution of which is

$$\left\langle \left[\hat{A}(t),\hat{A}^{\dagger}(t)\right]\right\rangle_{R}=\left\langle \left[\hat{A}(0),\hat{A}^{\dagger}(0)\right]\right\rangle_{R}e^{-\gamma t}=e^{-\gamma t}.$$

This means that after sufficiently long time, operators $\hat{A}$ and $\hat{A}^{\dagger}$ would commute, which is actually not observed. For this reason, accounting for the quantum noise term is mandatory.

The equation of motion for the field averaged over thermostat variables is easily obtained from (2.132). First, write the averaged field

$$\left\langle \hat{E}(t)\right\rangle_{R}=\left\langle E_{0}\sin kz\left(\hat{a}(t)+\hat{a}^{\dagger}(t)\right)\right\rangle_{R}=$$
$$=E_{0}\sin kz\left(\left\langle \hat{A}(t)\right\rangle_{R}e^{-i\omega t}+\left\langle \hat{A}^{\dagger}(t)\right\rangle_{R}e^{i\omega t}\right),$$

taking into account $\hat{a} = \hat{A}e^{-i\omega t}$. From (2.132), we have

$$\frac{d\left\langle \hat{A}(t)\right\rangle_R}{dt} = -\frac{\gamma}{2}\left\langle \hat{A}(t)\right\rangle_R,$$

since $\left\langle \hat{F}(t)\right\rangle_R = 0$. Thus, over time $\left\langle \hat{A}(t)\right\rangle_R$ and $\left\langle \hat{A}^\dagger(t)\right\rangle_R$, and with them $\left\langle \hat{E}(t)\right\rangle_R$, tend to zero:

$$\left\langle \hat{E}(t)\right\rangle_R \to 0.$$

2.13.3 Fluctuation-Dissipation Formula

We obtained expression (2.135):

$$\left\langle \hat{F}^\dagger(t)\,\hat{F}(t')\right\rangle_R = \frac{\gamma\bar{n}_T}{2}\delta(t-t').$$

Integrating both sides over dt', we get

$$\gamma = \frac{2}{\bar{n}_T}\int_0^\infty \left\langle \hat{F}^\dagger(t)\,\hat{F}(t')\right\rangle_R dt', \tag{2.142}$$

i.e., the decay rate and noise fluctuations are related by equation (2.142). Greater decay corresponds to greater noise.

2.14 Exercises

1. Based on (2.13), derive the expression (2.14) for the operator $\hat{V}$ in the interaction representation.
2. Derive the system of equations (2.19).
3. Obtain formula (2.64).
4. Derive from (2.69) the equation for the density matrix in the occupation number representation (2.71).
5. Reduce formula (2.91) to the form (2.95).
6. By integrating by parts, obtain the expressions (2.96), (2.97).
7. Write the general equation (2.99) in polar coordinates (2.103).

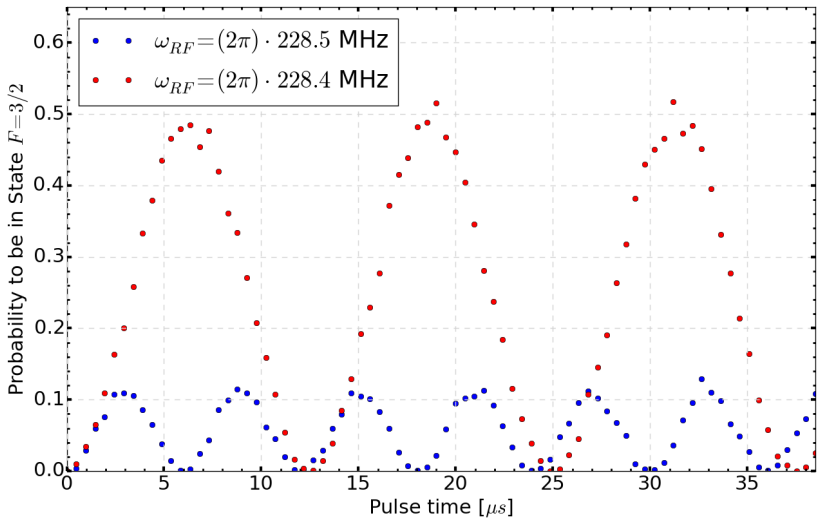


Figure 2.19: You want to measure the transition frequency $f_0 = \frac{\omega_o}{2\pi}$ between two states of a lithium atom by assessing Rabi oscillations. You prepare a cold cloud of lithium atoms in the state $|1\rangle$ (the so-called $|F = 1/2\rangle$ state) and you can apply an electromagnetic pulse of fixed intensity and variable duration τ. Finally, you measure the number of atoms in the state $|2\rangle$ (the so-called $|F = 3/2\rangle$ state). You know that the frequency difference between these two levels is close to $2\pi \cdot 228.5$MHz, therefore you apply an electromagnetic pulse with a frequency close to this value: initially with a frequency $\omega_1 = 2\pi \cdot 228.5$MHz, then you repeat the measurements with frequency $\omega_2 = 2\pi \cdot 228.4$MHz. The result is shown in the presented figure. What is the transition frequency $f_0 = \frac{\omega_o}{2\pi}$ between the two states of the lithium atom?

8. Write the general equation (2.99) in rectangular coordinates (2.104).

9. Derive the relations (2.121) and (2.122).

10. Derive the equation (2.123).

11. Prove (2.136) and (2.137).

12. Obtain the expression (2.140).

13. What is the transition frequency $f_0 = \frac{\omega_o}{2\pi}$ between two states of a lithium atom if the data obtained from experiments is shown in Figure 2.19 [2]

[2] Taken from the assignments [14]

Part 2

Quantum Optics

Chapter 3

Quantum Theory of the Laser

The semiclassical laser theory cannot answer all questions arising in connection with its operation. According to this theory, the laser does not generate at all before reaching the threshold, and upon exceeding the threshold begins to generate a classical electromagnetic field (light).

In reality, well below the threshold the laser generates chaotic light, and significantly above the threshold its radiation is close to classical. At the threshold and near it lies the transitional region from chaotic light to ordered radiation. Only a fully quantum theory can adequately describe this.

Another problem, also requiring quantization of the electromagnetic field, is determining the ultimate (natural) linewidth of the laser radiation, when the linewidth is determined by quantum fluctuations of the field, while various external influences, in principle removable, are not taken into account.

The laser is an open system in which the active atoms and the resonator field are connected with large external systems, which we will call reservoirs, providing pumping and losses. It follows that the laser, as an open system, should be considered using the density matrix.

3.1 Laser Model

Earlier in (2.69) we considered the interaction of a resonator mode (harmonic oscillator) with a reservoir in thermal equilibrium at temperature T. The same model can be used to study the operation of a laser, but it needs to be supplemented by an additional reservoir that provides pumping. See Figure 3.1: the first reservoir, R_1, contains atoms at temperature T. The second, R_2, contains atoms (of a different type) in an inverted state.

In the linear approximation (to second order perturbation theory) we already solved the problem of the interaction of the resonator mode with a reservoir. For the density matrix describing the state of the resonator mode field, the equations

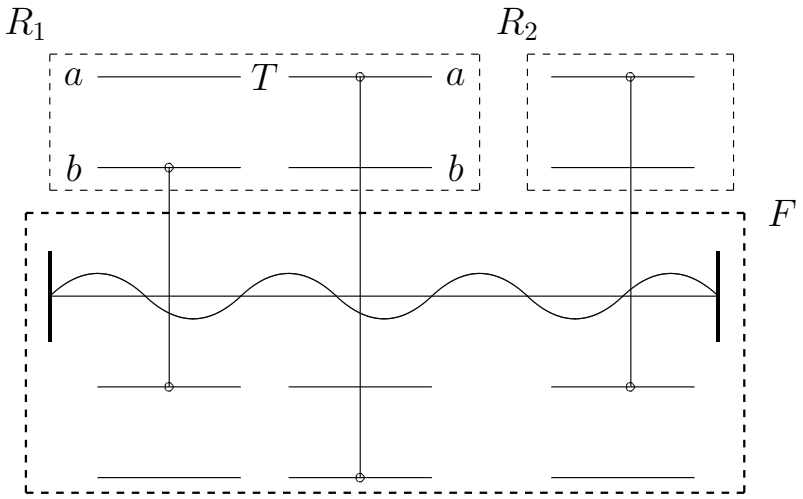


Figure 3.1: Considered laser model. The resonator mode (electromagnetic field) F interacts with reservoirs R_1 and R_2. R_1 contains atoms in thermal equilibrium at temperature T (losses). R_2 contains atoms in an inverted state (pumping).

obtained were:

$$\dot{\hat{\rho}} = -\frac{1}{2}\frac{\omega}{Q}\left\{\bar{n}_T\left(\hat{a}\hat{a}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{a}\right) + \right.$$
$$\left. + \left(\bar{n}_T + 1\right)\left(\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\right)\right\} + \text{h.c.} \tag{3.1}$$

or in another form (2.69)

$$\dot{\hat{\rho}} = -\frac{1}{2}R_a\left(\hat{a}\hat{a}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{a}\right) -$$
$$-\frac{1}{2}R_b\left(\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\right) + \text{h.c.} \tag{3.2}$$

If we limit ourselves to the linear approximation (in the field), the equation for the laser density matrix can be written using these equations. To describe the action of the reservoir introducing losses, it is convenient to use (3.1), taking into account that at not too high temperatures $\bar{n}_T \ll 1$ and $\bar{n}_T$ can be neglected compared to unity.

To describe the reservoir providing pumping, it is convenient to use (3.2), discarding the second term, since for simplicity it is assumed that all atoms of the pumping reservoir are in the upper state.

All this leads to the following equation:

$$\dot{\hat{\rho}} = -\frac{1}{2}\frac{\omega}{Q}\left(\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\right) -$$
$$-\frac{1}{2}A\left(\hat{a}\hat{a}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{a}\right) + \text{h.c.} \tag{3.3}$$

where $A = R_a$ is determined by the pumping intensity.

Equation (3.3) is obtained in the second-order perturbation approximation. It can describe the behavior of the laser below the generation threshold, allows determining threshold conditions, but cannot describe the laser above the generation threshold.

Let us assume losses are linear, and the first term in (3.3) accurately describes the losses. The second term in (3.3) should be found in the next non-zero approximation (fourth order perturbation theory).

The procedure to find this approximation is analogous to the one used earlier in the derivation of equations (3.1) - (3.2). One just needs to continue it to higher order terms. Third-order terms will give zero, since the resulting matrix has zero diagonal elements and its trace is zero. The next non-zero term is fourth order and has the form:

$$Sp_{at}\left\{\left(-\frac{i}{\hbar}\right)^4\int_t^{t+\tau}dt_1\int_t^{t_1}dt_2\int_t^{t_2}dt_3 \right.$$
$$\left.\int_t^{t_3}\left[\hat{V},\left[\hat{V},\left[\hat{V},\left[\hat{V},\hat{\rho}_{at}\left(t\right)\otimes\hat{\rho}_f\left(t\right)\right]\right]\right]\right]dt_4\right\} \tag{3.4}$$

Calculations similar to those done earlier lead to the expression (see [49]):

$$\frac{1}{8}B\left[\left(\hat{a}\hat{a}^\dagger\right)^2\hat{\rho}+3\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger-\right.$$
$$\left.-4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\right]+\text{h.c.} \tag{3.5}$$

Taking (3.5) into account, the equation for the density matrix (statistical operator) of the laser mode takes the form:

$$\dot{\hat{\rho}}\left(t\right)=-\frac{1}{2}\frac{\omega}{Q}\left(\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{a}\hat{\rho}\hat{a}^\dagger\right)-\frac{1}{2}A\left(\hat{a}\hat{a}^\dagger\hat{\rho}-\hat{a}^\dagger\hat{\rho}\hat{a}\right)+$$
$$+\frac{1}{8}B\left[\left(\hat{a}\hat{a}^\dagger\right)^2\hat{\rho}+3\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger-4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\right]+\text{h.c.} \tag{3.6}$$

where A is the linear (unsaturated) gain, $B=\frac{g^2\tau^2A}{3}=\frac{1}{3}r_ag^4\tau^4$ is the saturation parameter.

Equation (3.6) is the motion equation for the density matrix of the laser field interacting with a nonlinear medium consisting of active atoms and linear losses.

Equation (3.6) can be written in various representations. Here we restrict ourselves to the photon number (photon number states) representation and the coherent state representation.

In the first case, from (3.6) it is easy to obtain a system of equations for the matrix elements $\langle m|\,\hat{\rho}\,|n\rangle=\rho_{mn}$.

For the diagonal elements we have:

$$\dot{\rho}_{nn}(t) = -[A-(n+1)B](n+1)\rho_{nn} + \\ +(A-nB)n\rho_{n-1,n-1} - \frac{\omega}{Q}n\rho_{nn} + \frac{\omega}{Q}(n+1)\rho_{n+1,n+1}. \tag{3.7}$$

Note that only diagonal terms enter this equation.

Equations for off-diagonal terms are obtained in the same way. They are somewhat more complicated and are not presented here.

To write equation (3.6) in the coherent state representation, one can proceed as before when considering the decay of the resonator mode. Write

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$$

and substitute it into equation (3.6). Then perform the familiar procedure used previously for resonator mode relaxation. For the linear terms characterizing losses and unsaturated gain, the previously obtained results can be used. Additionally, the terms characterizing saturation must be considered. The calculations are more cumbersome here, although not fundamentally different from those done before. The resulting equation is quite complex, but recalling that perturbation theory (small field) was used, one can neglect smaller terms, leaving only the main ones [49]. In conclusion, one obtains an equation of the Fokker-Planck type for the quasi-probability $P(\alpha, t)$:

$$\frac{\partial}{\partial t}P(\alpha,t) = -\frac{1}{2}\left\{\frac{\partial}{\partial\alpha}\left[A - \frac{\omega}{Q} - B|\alpha|^2\right]\alpha P + \text{c.c.}\right\} + A\frac{\partial^2 P}{\partial\alpha\partial\alpha^*} \tag{3.8}$$

That is, compared to formula (2.100), $-\frac{\omega}{Q}$ is replaced by $A-\frac{\omega}{Q}-B|\alpha|^2$, where $A-\frac{\omega}{Q} = G$ is the unsaturated gain minus losses, and B characterizes gain reduction due to saturation.

3.2 Theory of Laser Generation

Using equation (3.7), one can obtain the equation for the average number of photons in the mode

$$\langle n(t)\rangle = \sum_{(n)} n\rho_{nn}.$$

To do this, multiply equation (3.7) by n and sum over all n. We have

$$\frac{d}{dt}\langle n\rangle = \frac{d}{dt}\sum_n \rho_{nn}n = \sum_n \dot{\rho}_{nn}n.$$

Thus from (3.7)

$$\frac{d}{dt}\langle n\rangle = -\sum_n n\left[A-(n+1)B\right](n+1)\rho_{nn} + \\ +\sum_n n\left(A-nB\right)n\rho_{n-1,n-1} - \sum_n n\frac{\omega}{Q}n\rho_{nn} + \\ +\sum_n n\frac{\omega}{Q}(n+1)\rho_{n+1,n+1}. \tag{3.9}$$

Replace the summation indices in (3.9). In the second sum, let $n-1=m$, i.e. $n=m+1$. In the last sum let $n+1=m$, i.e. $n=m-1$. Substituting this into (3.9) gives

$$\frac{d}{dt}\langle n\rangle = -\sum_n n\left[A-(n+1)B\right](n+1)\rho_{nn} + \\ +\sum_m (m+1)\left(A-(m+1)B\right)(m+1)\rho_{m,m} - \sum_n n\frac{\omega}{Q}n\rho_{nn} + \\ +\sum_m (m-1)\frac{\omega}{Q}m\rho_{m,m} = \sum_m \left(A-(m+1)B\right)(m+1)\rho_{m,m} - \sum_m \frac{\omega}{Q}m\rho_{m,m} = \\ = \sum_n \left(A-(n+1)B\right)(n+1)\rho_{n,n} - \sum_n \frac{\omega}{Q}n\rho_{n,n}.$$

Here, we changed the summation index back from m to n and combined the sums. From this, we have

$$\frac{d}{dt}\langle n(t)\rangle = \left(A-\frac{\omega}{Q}\right)\langle n(t)\rangle + A - B\left\langle (n+1)^2\right\rangle \tag{3.10}$$

Here, the first term $\left(A-\frac{\omega}{Q}\right)$ corresponds to unsaturated gain minus losses. The term $B\left\langle (n+1)^2\right\rangle = B\left[\langle n^2\rangle + 2\langle n\rangle + 1\right]$ characterizes gain reduction due to saturation. The term A accounts for spontaneous emission, which is absent in the semiclassical treatment. Apart from that, (3.10) resembles the classical laser equation [56]:

$$\dot{I}_n = 2I_n\left(\alpha_n - \beta_n I_n\right).$$

3.3 Statistics of Laser Photons

For equation (3.7), one can visually represent the probability flows shown in Figure 3.2.

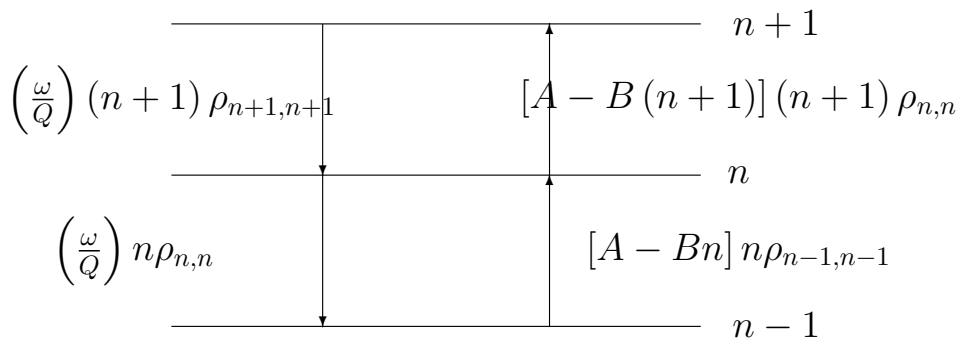


Figure 3.2: Balance of probability flows of photon numbers.

The term $\frac{\omega}{Q}n\rho_{nn}$ represents the flow from the state $|n\rangle$ to the state $|n-1\rangle$ due to photon absorption by atoms of the first reservoir. The same can be said for the term $\frac{\omega}{Q}\,(n+1)\,\rho_{n+1,n+1}$ — the probability flow from the state $|n+1\rangle$ to the state $|n\rangle$ for the same reasons.

The term $[A-(n+1)\,B]\,(n+1)\,\rho_{nn}$ represents the flow from the state $|n\rangle$ to the state $|n+1\rangle$ and characterizes the birth of photons due to interaction with active atoms, taking saturation effects into account. The same applies to the term $-\,(A-nB)\,n\rho_{n-1,n-1}$ — the probability flow from the state $|n-1\rangle$ to the state $|n\rangle$.

ρ_{nn} has the meaning of the probability of detecting n photons in the laser mode. In the transient regime, $\rho_{nn}\,(t)$ depends on time. In the steady-state regime, $\dot{\rho}_{nn}=0$, and from equation Equation 3.7 one can establish the photon statistics in the steady generation regime [62].

Equating, for example, the probability flows between the states $|n+1\rangle$ and $|n\rangle$ (using the principle of detailed balance), we obtain:

$$\frac{\omega}{Q}\,(n+1)\,\rho_{n+1,n+1}=[A-(n+1)\,B]\,(n+1)\,\rho_{nn} \tag{3.11}$$

Equality (3.11) gives an iterative relation

$$\rho_{n+1,n+1}=\frac{A-(n+1)\,B}{\omega/Q}\rho_{nn} \tag{3.12}$$

which allows expressing ρ_{nn} through ρ_{00}. We have

$$\rho_{nn}=\rho_{00}\prod_{k=1}^{n}\frac{A-kB}{\omega/Q} \tag{3.13}$$

ρ_{00} can be found from the normalization condition $\sum_{(n)}\rho_{nn}=1$.

Let us qualitatively consider using (3.13) ρ_{nn} as a function of n, when the laser operates above threshold, i.e., when $A > \frac{\omega}{Q}$. In the steady-state regime, the gain equals the losses (on average), that is

$$A - \bar{n}_{st}B = \frac{\omega}{Q},$$

whence the average number of photons in the steady-state regime equals

$$\bar{n}_{st} = \frac{A - \omega/Q}{B}.$$

For $n \ll \bar{n}_{st}$, $Bn \ll A$ the formula (3.13) can be represented as

$$\rho_{nn} \approx \left(\frac{AQ}{\omega}\right)^n.$$

This quantity grows exponentially with increasing n, since $\frac{AQ}{\omega} > 1$, meaning that for small n the probability increases with n, but the growth slows down as n increases. For large n the factor $\frac{A-nB}{\frac{\omega}{Q}}$ approaches unity when n approaches $\bar{n}_{st}$. At $n \approx \bar{n}_{st}$ the factor becomes equal to 1, and the distribution reaches a maximum:

$$\left.\frac{A - Bk}{\omega/Q}\right|_{k=\bar{n}_{st}} \to \frac{A - B\bar{n}_{st}}{\omega/Q} = \frac{A - A + \omega/Q}{\omega/Q} = 1.$$

For $n > \bar{n}_{st}$ and $k > \bar{n}_{st}$, the factors decrease, reaching zero at $k = \frac{A}{B}$. Note that for $k = \frac{A}{B}$ the factor becomes approximately zero, and terms with larger k should be discarded, since ρ_{nn} cannot be negative. The arising difficulty is related to the fact that we used perturbation theory and therefore assumed that the number of photons is not too large. Our theory is valid while $n < \frac{A}{B}$. These difficulties disappear if one uses the large signal theory developed in [62]. From the above, it follows that ρ_{nn} as a function of n (the photon distribution in the laser field) first increases, reaching a maximum near $n = \bar{n}_{st}$, and then decreases to zero.

In Figure 3.3 a qualitative curve of the photon distribution is shown for the case of threshold excess. At threshold, $A = \frac{\omega}{Q}$. Then the factor $\frac{A-Bk}{\omega/Q} = 1 - \left(\frac{B}{A}\right)k$ is less than unity for any k. The distribution curve will be monotonically decreasing. Below threshold, $A < \frac{\omega}{Q}$, then $\frac{A-Bk}{\omega/Q} \approx \frac{AQ}{\omega} \ll 1$ and ρ_{nn} decays exponentially, as $\left(\frac{AQ}{\omega}\right)^n$. The dependence of ρ_{nn} on n for all three cases can be numerically calculated using formula (3.13). The results of such calculation are shown in Figure 3.4.

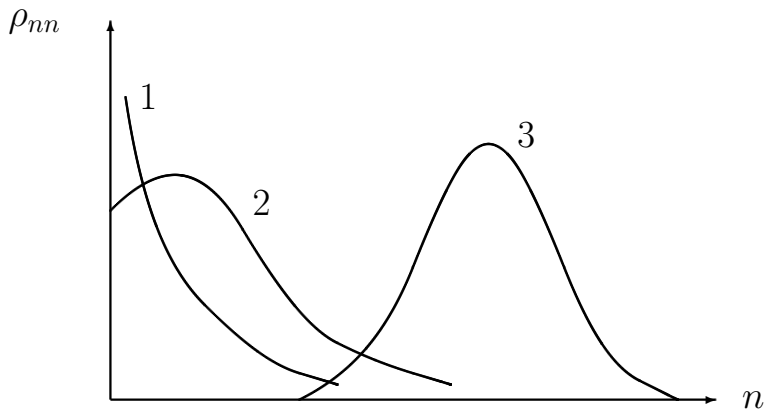


Figure 3.3: Qualitative view of the photon distribution for three cases. Plot 1: $A < \frac{\omega}{Q}$ - below the generation threshold. Plot 2: $A \approx \frac{\omega}{Q}$ - at the generation threshold. Plot 3: $A > \frac{\omega}{Q}$ - above the generation threshold.

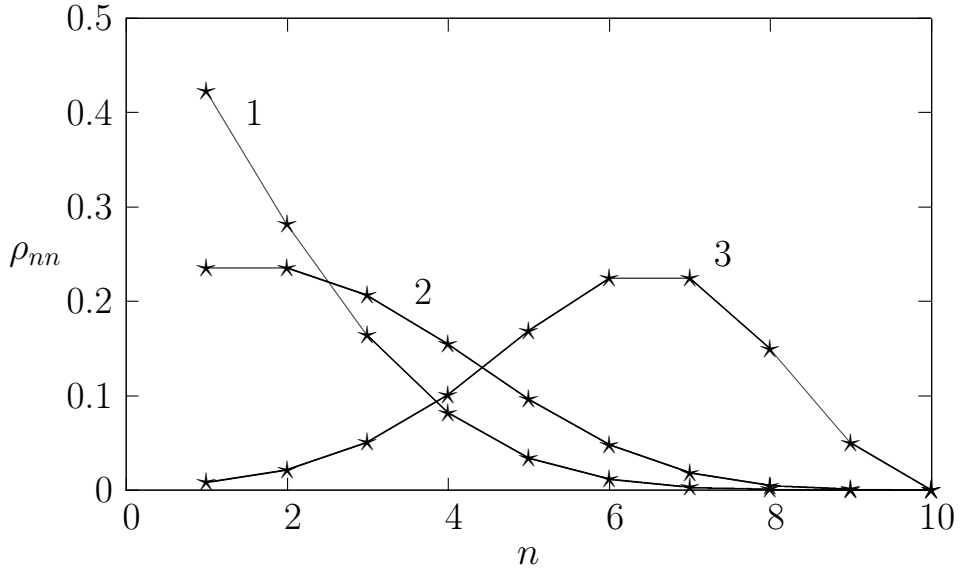


Figure 3.4: Results of the numerical calculation of the photon distribution. Curve 1, which describes the photon distribution below the generation threshold, was obtained with the following parameters: $A = 10$, $\frac{\omega}{Q} = 12$, $B = 1$. Curve 2, corresponding to the generation threshold, was obtained at $A = 10$, $\frac{\omega}{Q} = 8$, $B = 1$. For curve 3, the parameters were $A = 10$, $\frac{\omega}{Q} = 3$, $B = 1$, which corresponds to exceeding the threshold.

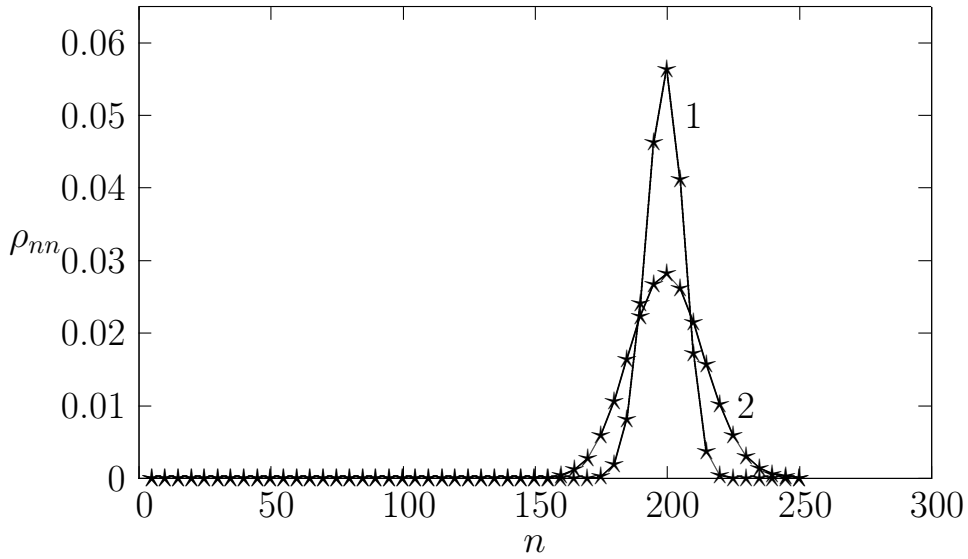


Figure 3.5: Difference in the photon distribution in the laser field under moderate pumping (curve 1) from the Poisson distribution (curve 2).

From the large signal theory [62], which we do not consider here, it follows that for very large threshold excess, $\rho_{nn} \to e^{-\bar{n}} \frac{\bar{n}^n}{n!}$, i.e., tends to the distribution characteristic of a coherent state. For moderate threshold excess, the photon distribution in laser radiation noticeably differs from the photon distribution in a coherent state. The difference between the photon distribution in the laser field and the Poisson distribution (corresponding to a coherent state) under moderate pumping is clearly shown in Figure 3.5.

3.4 Laser theory. Representation of coherent states

It is necessary to use the equation for the quasiprobability function $P(\alpha, t)$ (3.8):

$$\frac{\partial}{\partial t} P(\alpha, t) = -\frac{1}{2} \left\{ \frac{\partial}{\partial \alpha} \left[A - \frac{\omega}{Q} - B |\alpha|^2 \right] \alpha P + \text{c.c.} \right\} + A \frac{\partial^2 P}{\partial \alpha \partial \alpha^*}.$$

For our purposes, it is convenient to represent this equation in polar coordinates $\alpha = re^{i\theta}$. We have

$$\frac{\partial}{\partial t} P(r, \theta, t) = -\frac{1}{2r} \cdot \frac{\partial}{\partial r} \left[r^2 \left(A - \frac{\omega}{Q} - Br^2 \right) P(r, \theta, t) \right] + \\ + \frac{A}{4r^2} \left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) P(r, \theta, t) \qquad (3.14)$$

Suppose we are interested in the steady state. Then $\frac{\partial P}{\partial t} = 0$, and the phase θ is uniformly distributed from 0 to 2π. It follows that $\frac{\partial^2 P}{\partial \theta^2} = 0$ and the quasiprobability $P(r, \theta, t)$ does not depend on t or θ. Equation (3.14) then takes the form

$$0 = -\frac{\partial}{\partial r}\left[r^2\left(A - \frac{\omega}{Q} - Br^2\right)P(r)\right] + \frac{1}{2}A\frac{\partial}{\partial r}r\frac{\partial}{\partial r}P(r). \tag{3.15}$$

This equation is easily integrated. The first integration gives

$$r\left(A - \frac{\omega}{Q} - Br^2\right)P(r) = \frac{1}{2}A\frac{\partial}{\partial r}P(r) + C. \tag{3.16}$$

The constant $C = 0$, since P and $\frac{\partial P}{\partial r}$ must tend sufficiently quickly to zero as $r \to \infty$.

From here follows the equation

$$\frac{dP}{P} = \frac{2}{A}r\left(A - \frac{\omega}{Q} - Br^2\right)dr. \tag{3.17}$$

Its solution is

$$\begin{aligned} P = N\exp\left(\frac{1}{A}\left[r^2\left(A - \frac{\omega}{Q}\right) - \frac{Br^4}{2}\right]\right) = \\ = N\exp\left(\frac{1}{A}\left[r^2G - \frac{Br^4}{2}\right]\right), \end{aligned} \tag{3.18}$$

where N is the normalization factor, equal to

$$\frac{1}{N} = 2\pi\int_0^\infty \exp\left(\frac{1}{A}\left[r^2G - \frac{B}{2}r^4\right]\right)rdr.$$

Distribution (3.18) is a function of $r^2 = |\alpha|^2 = \bar{n}$. When threshold is exceeded $A > \frac{\omega}{Q}$, i.e. when $G > 0$, $P(r)$ first increases with increasing r^2, reaching a maximum at $r^2 = \frac{A - \omega/Q}{B} = \frac{G}{B}$, and then decreases.

This same result was obtained earlier. The obtained value $r = \sqrt{\frac{G}{B}}$ corresponds, at $G > 0$, to the most probable value of $|\alpha|$.

For $A < \frac{\omega}{Q}$, $G < 0$, the laser is below threshold. With increasing r^2 the curve monotonically decreases. At threshold $A = \frac{\omega}{Q}$, $G = 0$, the curve also decreases, but at $r^2 = 0$ we have an extremum.

On Figure 3.6 are shown dependencies of the quasiprobability on $r^2 = |\alpha|^2$, for different values of parameter G, corresponding to below-threshold, threshold, and above-threshold regimes.

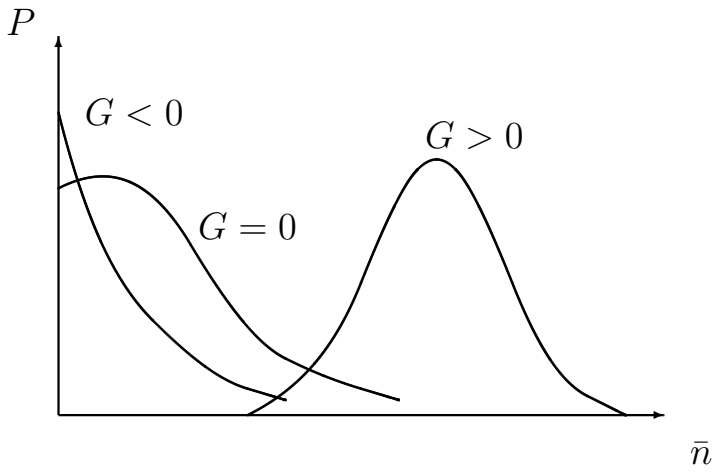


Figure 3.6: Qualitative view of the dependence of the quasiprobability P on the average number of photons $\bar{n} = r^2$ at three different values of the parameter G. $G < 0$ below the generation threshold; $G \approx 0$ - at the generation threshold; $G > 0$ - above the generation threshold.

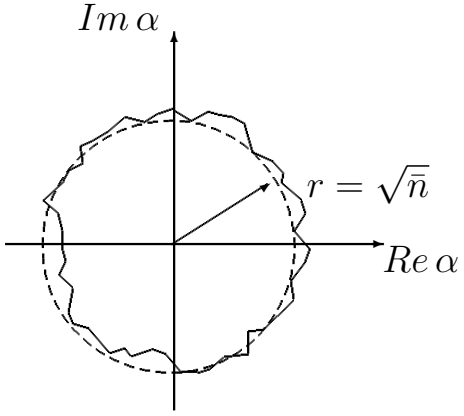


Figure 3.7: Fluctuation of the field amplitude generated by the laser in a narrow region near the circle of radius $r = \sqrt{\bar{n}_{st}}$.

On the complex plane $\alpha = re^{i\theta}$, when the threshold is substantially exceeded, the region occupied by the generated field can be visually represented as shown on Figure 3.7. The broken curve characterizes the distribution of the oscillation amplitude, i.e., corresponds to amplitude noise. The phase θ is uniformly distributed in the interval $0 - 2\pi$, therefore the broken curve lies close to the circle $r = \sqrt{\bar{n}_{st}}$ with the radius corresponding to the most probable value of $|\alpha|$.

From Figure 3.7 it is seen that the amplitude fluctuates in a narrow region around the circle of radius $r = \sqrt{\bar{n}_{st}}$, while the phase freely diffuses (wanders) along this circle.

Phase diffusion in the laser occurs slowly, therefore, if we consider not too long time intervals, it can be assumed that the laser generates radiation with a well-defined phase. When averaged over a long time interval, during which the

phase can assume any value in the range $0-2\pi$ with equal probability, the mean value of the laser field over this interval will be zero.

Let us consider the phase diffusion process more carefully. Suppose the amplitude distribution is established and $P(r)$ corresponds to the stationary amplitude distribution. Then

$$P(r,\theta,t) = P(r)\,P(\theta,t)\,.$$

In this case, equation (3.14) takes the form

$$\frac{\partial P(\theta,t)}{\partial t} = \frac{A}{4r^2}\frac{\partial^2}{\partial\theta^2}P(\theta,t) \tag{3.19}$$

The function $P(r)$ cancels out because it satisfies the stationary equation (3.17).

Equation (3.19) can be used to obtain the equation satisfied by the average field. The average electric field of the laser mode, expressed through P, has the form

$$\left\langle \hat{E} \right\rangle = E_1 \int P(\alpha,t)\,\alpha d^2\alpha.$$

In polar coordinates this looks like:

$$\left\langle \hat{E} \right\rangle = E_1 \int_0^\infty r dr \int_0^{2\pi} r e^{i\theta} P(r,\theta,t)\,d\theta.$$

Using now equation (3.19):

$$\begin{aligned}\frac{\partial\left\langle \hat{E} \right\rangle}{\partial t} &= E_1\frac{\partial}{\partial t}\int_0^\infty r^2 dr\int_0^{2\pi} e^{i\theta}P(r,\theta,t)\,d\theta = \\ &= E_1\int_0^\infty r^2 dr\int_0^{2\pi} e^{i\theta}\frac{\partial P}{\partial t}d\theta = \\ &= \frac{E_1}{\bar{n}_{st}}\frac{A}{4}\int_0^\infty r^2 dr\int_0^{2\pi} e^{i\theta}\frac{\partial^2}{\partial\theta^2}P(r,\theta,t)\,d\theta.\end{aligned} \tag{3.20}$$

Here we approximately replaced $\frac{1}{r^2}\approx\frac{1}{\bar{n}_{st}}$, assuming the amplitude distribution is sufficiently narrow.

We integrate the inner integral twice by parts. We have

$$\begin{aligned}\int_0^{2\pi} e^{i\theta}\frac{\partial^2 P}{\partial\theta^2}d\theta &= e^{i\theta}\left.\frac{\partial P}{\partial\theta}\right|_0^{2\pi} - i\int_0^{2\pi} e^{i\theta}\frac{\partial P}{\partial\theta}d\theta = \\ &= -ie^{i\theta}\left.P\right|_0^{2\pi} - \int_0^{2\pi} e^{i\theta}P d\theta.\end{aligned} \tag{3.21}$$

The first terms in (3.21) equal zero due to periodicity of P and $\frac{\partial P}{\partial \theta}$. Using (3.21), equation (3.20) can be written as

$$\frac{\partial \langle E \rangle}{\partial t} = E_1 \int_0^\infty r^2 dr \int_0^{2\pi} e^{i\theta} \frac{\partial P}{\partial t} d\theta = \\ = -\frac{AE_1}{4\bar{n}_{st}} \int_0^\infty r dr \int_0^{2\pi} r e^{i\theta} P d\theta = -\frac{A}{4\bar{n}_{st}} \langle E \rangle . \tag{3.22}$$

The solution of this equation is

$$\langle E(t) \rangle = \langle E(0) \rangle e^{-\frac{D}{2}t}, \tag{3.23}$$

where

$$D = \frac{A}{2\bar{n}_{st}}. \tag{3.24}$$

Thus, we have obtained that the mean field indeed tends to zero with the characteristic time $\tau = \frac{1}{D}$. In lasers this interval can be quite long compared to the oscillation period.

To determine the natural linewidth of the laser emission it is necessary to find the correlation function of laser radiation; its spectrum by the Wiener-Khinchin theorem (Theorem 15.2.1) will be the energy spectrum of laser radiation. We have

$$\left\langle \hat{E}^{(-)}(0) \hat{E}^{(+)}(t) \right\rangle = E_1^2 \int d^2\alpha P(\alpha) \alpha^*(0) \alpha(t) e^{-i\omega_0 t},$$

where $P(\alpha)$ is the quasiprobability of the laser state, defined by formula (3.14). When switching to polar coordinates (amplitude-phase) $\alpha = re^{i\theta}$ we assume that the amplitude is already established and does not change, while the phase changes slowly, so that

$$\alpha^*(0) = re^{i\theta(0)} = re^{i0} = r, \\ \alpha(t) = re^{i\theta(t)} = re^{i\theta}, \tag{3.25}$$

In (3.25) it is assumed that at the initial time $\theta(0) = 0$ and at time t - $\theta(t) = \theta$. Thus we have:

$$\left\langle \hat{E}^{(-)}(0) \hat{E}^{(+)}(t) \right\rangle = E_1^2 e^{-i\omega_0 t} \int_0^\infty r^3 dr \int_0^{2\pi} d\theta P\left(re^{i\theta}, t\right) e^{i\theta}.$$

Let us derive an equation satisfied by $\left\langle \hat{E}^{(-)}(0) \hat{E}^{(+)}(t) \right\rangle$. From equation (3.19)

$\frac{\partial P(\theta,t)}{\partial t} = \frac{A}{4\bar{n}_{st}} \frac{\partial^2}{\partial \theta^2} P(\theta, t)$, therefore

$$\frac{d}{dt}\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle = -i\omega_0 \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle + \\ + E_1^2 e^{-i\omega_0 t}\int_0^\infty r^3 dr \int_0^{2\pi} d\theta \frac{\partial}{\partial t} P\left(re^{i\theta}, t\right) e^{i\theta} = \\ = -i\omega_0 \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle + \\ + \frac{E_1^2 A}{4\bar{n}_{st}}\int_0^\infty r^3 dr \int_0^{2\pi} d\theta \frac{\partial^2}{\partial \theta^2} P\left(re^{i\theta}, t\right) e^{i\theta}. \tag{3.26}$$

Performing two integrations by parts by θ in (3.26), we obtain:

$$\int_0^{2\pi} d\theta \frac{\partial^2}{\partial \theta^2} P e^{i\theta} = \frac{\partial}{\partial \theta} P e^{i\theta}\Big|_0^{2\pi} - i\int_0^{2\pi} d\theta \frac{\partial}{\partial \theta} P e^{i\theta} = \\ = -i\int_0^{2\pi} d\theta \frac{\partial}{\partial \theta} P e^{i\theta} = -i\, P e^{i\theta}\Big|_0^{2\pi} - \int_0^{2\pi} d\theta P e^{i\theta} = -\int_0^{2\pi} d\theta P e^{i\theta}.$$

Substituting the obtained result into the original expression (3.26), we get:

$$\frac{d}{dt}\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle = -i\omega_0 \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle - \\ - \frac{E_1^2 A}{4\bar{n}_{st}}\int_0^\infty r^3 dr \int_0^{2\pi} d\theta P\left(re^{i\theta}, t\right) e^{i\theta} = \\ = -i\omega_0 \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle - \frac{A}{4\bar{n}_{st}} \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle.$$

The solution of this equation is

$$r_+(t) = \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle = \left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(0)\right\rangle e^{-i\omega_0 t - \frac{D}{2}t}, \tag{3.27}$$

where

$$\frac{D}{2} = \frac{A}{4\bar{n}_{st}}.$$

Expression (3.27) defines the correlation function $r(t)$ in the domain $t \geq 0$. For $t \leq 0$, taking into account (15.5), the correlation function can be written as

$$r_-(t) = r_-(-|t|) = r_+^*(|t|). \tag{3.28}$$

Using expressions (3.27) and (3.28), one can write the following expression for the Fourier transform of the correlation function, which according to the Wiener-

Khinchin theorem (Theorem 15.2.1) defines the energy spectrum $S(\omega)$:

$$\begin{aligned}
S(\omega) = \tilde{r}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} r(t)\, dt = \\
= \frac{1}{2\pi}\int_{-\infty}^{0} e^{i\omega t} r_-(t)\, dt + \frac{1}{2\pi}\int_{0}^{+\infty} e^{i\omega t} r_+(t)\, dt = \\
= \frac{1}{2\pi}\int_{0}^{+\infty} e^{-i\omega t} r_+^*(t)\, dt + \frac{1}{2\pi}\int_{0}^{+\infty} e^{i\omega t} r_+(t)\, dt = \\
= \frac{1}{\pi}\mathrm{Re}\int_{0}^{+\infty} e^{i\omega t} r_+(t)\, dt = \\
= \frac{1}{\pi}\mathrm{Re}\int_{0}^{+\infty} e^{i\omega t}\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(t)\right\rangle dt = \\
= \frac{\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(0)\right\rangle}{\pi}\mathrm{Re}\int_{0}^{+\infty} e^{\left(i(\omega-\omega_0)-\frac{D}{2}\right)t} dt = \\
= \frac{\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(0)\right\rangle}{\pi}\mathrm{Re}\frac{1}{\frac{D}{2}-i(\omega-\omega_0)} = \\
= \frac{\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(0)\right\rangle}{\pi}\mathrm{Re}\frac{\frac{D}{2}+i(\omega-\omega_0)}{(\omega-\omega_0)^2+(D/2)^2} = \\
= \frac{\left\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(0)\right\rangle}{\pi}\frac{D/2}{(\omega-\omega_0)^2+(D/2)^2}.
\end{aligned} \tag{3.29}$$

In the derivation of (3.29), the definition of the Fourier transform (18.1) from section 18.1 was used.

It follows that the generation line is Lorentzian, and the linewidth equals D (Figure 3.8). Since we considered only the "natural", fundamentally unavoidable cause of line broadening, associated with quantum fluctuations, and neglected fundamentally removable technical causes, the obtained linewidth is extremely narrow.

Here we considered the simplest laser model. A more realistic model of a three-level system is considered in [44]. Results obtained for various laser models are in good agreement with each other and correspond to the results obtained with the simple model used here.

3.5 Exercises

1. Derive (3.5) from (3.4).

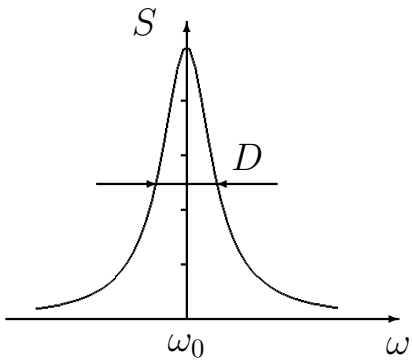


Figure 3.8: Calculation of the laser generation line width. Dependence of the spectral power density $S(\omega)$ on the frequency ω. The line width is determined at half power, i.e., those values of ω for which $\frac{S(\omega)}{S(\omega_0)} \geq \frac{1}{2}$ are considered. From (3.29), it follows that the interval boundaries correspond to frequencies $\omega = \omega_0 \pm \frac{D}{2}$. That is, the line width is D.

2. From (3.6) obtain the expression for the diagonal elements of the density matrix (3.7).

3. From (3.6) obtain the expression for the density operator in the coherent states representation (3.8).

4. Represent the equation (3.8) in polar coordinates.

5. Derive equation (3.19) from the general equation (3.14).

6. Using the formulas (3.24) and (3.29), estimate the "natural" linewidth of the laser generation for $A \approx \frac{\omega}{Q}$ (small excess over threshold), $\frac{\omega}{Q} = \Delta\omega = 10^6$Hz, $n_{st} = 10^6 - 10^7$ photons in the mode.

Chapter 4

Quantum Theory of the Laser in the Heisenberg Representation

So far we have considered the quantum theory of the laser using the Schrödinger representation, in which the density matrix depends on time, while operators do not depend on time. As we have seen from the example of the problem of the decay of the resonator mode, another approach is possible, using the Heisenberg representation [52], in which operators depend on time, while the density matrix does not depend on time. In some cases, this approach can be more convenient for investigating subtle questions of the quantum theory of lasers, such as laser generation of the field in a squeezed state.

4.1 Laser Model

The laser level scheme is shown in Figure 4.1. Level c is the ground level, level b is the lower working level, and a is the upper working level. In fact, the scheme is four-level. The pumping is done by incoherent light through a sufficiently broad absorption line, depicted in the figure by a dashed line. But since a high rate of non-radiative transition to the upper working level is assumed, it can be considered that the pumping occurs directly to level a. The presence of the fourth level greatly reduces the reverse transition from level a to the ground level c. In Figure 4.1, the transitions considered in the theory are indicated. γ_a and γ_b characterize the population relaxation from levels a and b due to coupling with the dissipative system. r_a is the pumping rate of the upper working level a due to incoherent optical pumping. The transition $a \to c$ is a stimulated transition caused by the laser field being generated.

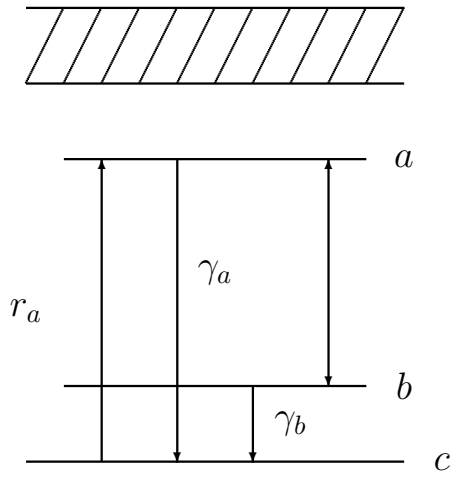


Figure 4.1: Diagram of laser levels. Level c is the ground level, level b is the lower working level, and a is the upper working level. γ_a and γ_b characterize the relaxation of populations from levels a and b due to coupling with the dissipative system. r_a is the pumping rate of the upper working level a due to incoherent optical pumping.

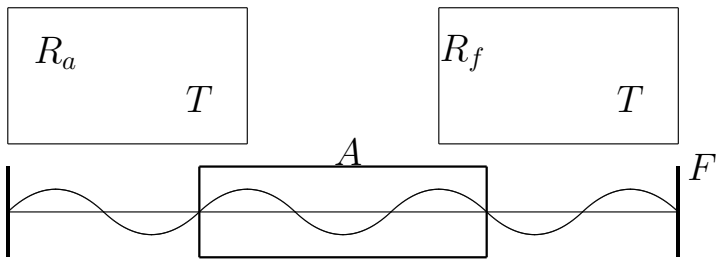


Figure 4.2: Laser model. The resonator in which the generated mode is excited, interacting with the active atoms of the working medium A, is denoted by F. Additionally, there are two reservoirs at temperature T: R_a, which is connected to the active atoms, and R_F, connected to the mode.

The laser scheme is presented in Figure 4.2. The scheme contains a resonator F in which the generated mode is excited, interacting with the active atoms of the working medium, whose scheme is shown in Figure 4.1. In addition, there are two reservoirs at temperature T: R_a (associated with active atoms) and R_F (associated with the mode field), which cause relaxation of the atoms and the mode field.

4.2 Heisenberg equation for the laser system

The considered system consists of 4 subsystems, which are connected to each other through 3 interaction Hamiltonians. The first subsystem represents the

mode of the electromagnetic field with frequency ω (denoted by F in Figure 4.2), which is described by creation and annihilation operators of the mode: $\hat{a}^\dagger$ and $\hat{a}$.

The electromagnetic field mode is connected with the reservoir R_F (see Figure 4.2), which is a collection of harmonic oscillators with frequencies $\omega_k^{(F)}$, described by creation and annihilation operators $\hat{b}_k^\dagger$, $\hat{b}_k$. The following commutation relations hold:

$$\begin{aligned}\left[\hat{b}_k, \hat{b}_{k'}^\dagger\right] &= \delta_{kk'},\\ \left[\hat{b}_k, \hat{b}_{k'}\right] &= 0,\\ \left[\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger\right] &= 0.\end{aligned}$$

The atomic subsystem consists of a large number of atoms. The j-th atom can be in the following ground states: $|a\rangle^j$, $|b\rangle^j$, and $|c\rangle^j$. The behavior of this atom can be described using projection operators onto the states $|a\rangle^j$ and $|b\rangle^j$:

$$\hat{\sigma}_a^j = |a\rangle^j \langle a|^j \,,\ \hat{\sigma}_b^j = |b\rangle^j \langle b|^j \,,$$

as well as transition operators

$$\begin{aligned}\hat{\sigma}_{a\to b}^j = \hat{\sigma}_{ab}^j = |b\rangle^j \langle a|^j \,,\ \hat{\sigma}_{b\to a}^j = \hat{\sigma}_{ba}^j = \left(\hat{\sigma}_{ab}^j\right)^\dagger,\\ \hat{\sigma}_{a\to c}^j = \hat{\sigma}_{ac}^j = |c\rangle^j \langle a|^j \,,\ \hat{\sigma}_{c\to a}^j = \hat{\sigma}_{ca}^j = \left(\hat{\sigma}_{ac}^j\right)^\dagger,\\ \hat{\sigma}_{b\to c}^j = \hat{\sigma}_{bc}^j = |c\rangle^j \langle b|^j \,,\ \hat{\sigma}_{c\to b}^j = \hat{\sigma}_{cb}^j = \left(\hat{\sigma}_{bc}^j\right)^\dagger.\end{aligned}$$

The atomic subsystem is connected to reservoir R_A, which consists of a large number of harmonic oscillators, each with frequency $\omega_k^{(A)}$, described by creation and annihilation operators $\hat{c}_k^\dagger$, $\hat{c}_k$ satisfying the following commutation relations:

$$\begin{aligned}\left[\hat{c}_k, \hat{c}_{k'}^\dagger\right] &= \delta_{kk'},\\ \left[\hat{c}_k, \hat{c}_{k'}\right] &= 0,\\ \left[\hat{c}_k^\dagger, \hat{c}_{k'}^\dagger\right] &= 0.\end{aligned}$$

In the general case, the Heisenberg equation has the form:

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{O}\right], \tag{4.1}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian of the system, and $\hat{O}$ is an operator of some observable related to this system. For simplicity, we will use the interaction picture, where

equation (4.1) is rewritten as:

$$\frac{d\hat{O}_I}{dt} = \frac{i}{\hbar}\left[\hat{V}_I, \hat{O}_I\right], \tag{4.2}$$

where the interaction operators $\hat{V}_I$ and observable $\hat{O}_I$ in the interaction picture are defined as:

$$\begin{aligned} \hat{O}_I &= e^{i\frac{\hat{\mathcal{H}}_0 t}{\hbar}} \hat{O} e^{-i\frac{\hat{\mathcal{H}}_0 t}{\hbar}}, \\ \hat{V}_I &= e^{i\frac{\hat{\mathcal{H}}_0 t}{\hbar}} \hat{V} e^{-i\frac{\hat{\mathcal{H}}_0 t}{\hbar}}. \end{aligned} \tag{4.3}$$

Here, $\hat{\mathcal{H}}_0$ denotes the full Hamiltonian of the considered system without including the interaction between its parts.

In what follows, all operators will be assumed to be written in the interaction picture (4.3), and thus the index I will be omitted.

The interaction Hamiltonian between the j-th atom and the mode of the electromagnetic field in the interaction picture is (see 2.14):

$$\hat{V}^j_{AF} = \Delta\left(t - t_j\right) g\hbar \left(\hat{a}^\dagger \hat{\sigma}^j_{ab} e^{i(\omega-\omega_{ab})t} + \left(\hat{\sigma}^j_{ab}\right)^\dagger \hat{a} e^{-i(\omega-\omega_{ab})t}\right), \tag{4.4}$$

where $\Delta\left(t - t_j\right)$ is a step function describing the process of atom injection, i.e. taking into account the fact that the interaction between the j-th atom and the field begins at time t_j:

$$\Delta\left(t - t_j\right) = \begin{cases} 1;\ t \geq t_j \\ 0;\ t < t_j \end{cases}$$

The full interaction Hamiltonian is the sum of Hamiltonians of individual atoms (4.4):

$$\hat{V}_{AF} = g\hbar \sum_j \Delta\left(t - t_j\right) \left(\hat{a}^\dagger \hat{\sigma}^j_{ab} + \left(\hat{\sigma}^j_{ab}\right)^\dagger \hat{a}\right). \tag{4.5}$$

In (4.5) we assume that the electromagnetic field mode frequency coincides with the transition frequency: $\omega = \omega_{ab}$.

The interaction Hamiltonian of the field with reservoir R_F is written as:

$$\hat{V}_{FR_F} = \hbar \sum_k g^{(F)}_k \left(\hat{b}^\dagger_k \hat{a} e^{i\left(\omega^{(F)}_k - \omega\right)t} + \hat{a}^\dagger \hat{b}_k e^{-i\left(\omega^{(F)}_k - \omega\right)t}\right). \tag{4.6}$$

The case is more complicated for the interaction Hamiltonian between the atomic subsystem and reservoir R_A. Based on the considered model (see Figure 4.2) we deal with two transitions: $a \to c$ and $b \to c$. Accordingly, the full Hamiltonian can be written as

$$\hat{V}_{AR_A} = \hat{V}^{a\to c}_{AR_A} + \hat{V}^{b\to c}_{AR_A}.$$

Assuming that the interaction constants for the considered transitions are the same and real, we get

$$\hat{V}_{AR_A}^{a\to c} = \hbar \sum_{k,j} g_k^{(A)} \left(\hat{c}_k^\dagger \hat{\sigma}_{ac}^j e^{i\left(\omega_k^{(A)} - \omega_{ac}\right)t} + \left(\hat{\sigma}_{ac}^j\right)^\dagger \hat{c}_k e^{-i\left(\omega_k^{(A)} - \omega_{ac}\right)t} \right),$$
$$\hat{V}_{AR_A}^{b\to c} = \hbar \sum_{k,j} g_k^{(A)} \left(\hat{c}_k^\dagger \hat{\sigma}_{bc}^j e^{i\left(\omega_k^{(A)} - \omega_{bc}\right)t} + \left(\hat{\sigma}_{bc}^j\right)^\dagger \hat{c}_k e^{-i\left(\omega_k^{(A)} - \omega_{bc}\right)t} \right). \tag{4.7}$$

In (4.7) summation is assumed over all atoms (index j) and all reservoir modes (k), where ω_{ac} and ω_{bc} denote transition frequencies.

4.2.1 Equations of motion of the atomic subsystem

First, consider the equations satisfied by operators $\hat{c}_k$ and $\hat{c}_k^\dagger$. Since these operators enter only in the Hamiltonian $\hat{V}_{AR_A}$, the equation of motion can be written as

$$\frac{d\hat{c}_k}{dt} = \frac{i}{\hbar}\left[\hat{V}_{AR_A}, \hat{c}_k\right] =$$
$$= \frac{i}{\hbar}\hbar \sum_{k',j} g_{k'}^{(A)} \left[\hat{c}_{k'}^\dagger, \hat{c}_k\right] \left(\hat{\sigma}_{ac}^j e^{i\left(\omega_{k'}^{(A)} - \omega_{ac}\right)t} + \hat{\sigma}_{bc}^j e^{i\left(\omega_{k'}^{(A)} - \omega_{bc}\right)t} \right) +$$
$$+ \frac{i}{\hbar}\hbar \sum_{k',j} g_{k'}^{(A)} \left(\left(\hat{\sigma}_{ac}^j\right)^\dagger e^{-i\left(\omega_{k'}^{(A)} - \omega_{ac}\right)t} + \left(\hat{\sigma}_{bc}^j\right)^\dagger e^{-i\left(\omega_{k'}^{(A)} - \omega_{bc}\right)t} \right) \left[\hat{c}_{k'}, \hat{c}_k\right] =$$
$$= -i \sum_{k',j} g_{k'}^{(A)} \delta_{kk'} \left(\hat{\sigma}_{ac}^j e^{i\left(\omega_{k'}^{(A)} - \omega_{ac}\right)t} + \hat{\sigma}_{bc}^j e^{i\left(\omega_{k'}^{(A)} - \omega_{bc}\right)t} \right) =$$
$$= -i g_k^{(A)} \sum_j \left(\hat{\sigma}_{ac}^j e^{i\left(\omega_k^{(A)} - \omega_{ac}\right)t} + \hat{\sigma}_{bc}^j e^{i\left(\omega_k^{(A)} - \omega_{bc}\right)t} \right). \tag{4.8}$$

Formal integration of (4.8) gives

$$\hat{c}_k(t) = \hat{c}_k(0) -$$
$$- i g_k^{(A)} \sum_j \int_0^t dt' \left(\hat{\sigma}_{ac}^j(t') e^{i\left(\omega_k^{(A)} - \omega_{ac}\right)t'} + \hat{\sigma}_{bc}^j(t') e^{i\left(\omega_k^{(A)} - \omega_{bc}\right)t'} \right), \tag{4.9}$$

from which for the Hermitian conjugate operator $\hat{c}_k^\dagger$ we have

$$\hat{c}_k^\dagger(t) = \hat{c}_k^\dagger(0) +$$
$$+ i g_k^{(A)} \sum_j \int_0^t dt' \left(\hat{\sigma}_{ca}^j(t') e^{-i\left(\omega_k^{(A)} - \omega_{ac}\right)t'} + \hat{\sigma}_{cb}^j(t') e^{-i\left(\omega_k^{(A)} - \omega_{bc}\right)t'} \right), \tag{4.10}$$

where we introduced the notation $\hat{\sigma}_{ca}^{j} = \left(\hat{\sigma}_{ac}^{j}\right)^{\dagger}$ and $\hat{\sigma}_{cb}^{j} = \left(\hat{\sigma}_{bc}^{j}\right)^{\dagger}$.

The next step is to derive the equations satisfied by operators $\hat{\sigma}_{a}^{j}$ and $\hat{\sigma}_{b}^{j}$. For $\hat{\sigma}_{a}^{j}$ we have

$$\frac{d\hat{\sigma}_{a}^{j}}{dt} = \frac{i}{\hbar}\left[\hat{V}_{AF}, \hat{\sigma}_{a}^{j}\right] + \frac{i}{\hbar}\left[\hat{V}_{AR_A}, \hat{\sigma}_{a}^{j}\right]. \tag{4.11}$$

For the first term of the sum (4.11) it holds

$$\frac{i}{\hbar}\left[\hat{V}_{AF}, \hat{\sigma}_{a}^{j}\right] = \frac{i}{\hbar} g\hbar \sum_{i} \Delta\left(t - t_i\right)\left(\hat{a}^{\dagger}\left[\hat{\sigma}_{ab}^{i}, \hat{\sigma}_{a}^{j}\right] + \left[\left(\hat{\sigma}_{ab}^{i}\right)^{\dagger}, \hat{\sigma}_{a}^{j}\right]\hat{a}\right),$$

from which, taking into account the commutation relations

$$\begin{aligned}\left[\hat{\sigma}_{ab}^{i}, \hat{\sigma}_{a}^{j}\right] &= \left|b^i\right\rangle\left\langle a^i\middle|a^j\right\rangle\left\langle a^j\right| - \left|a^j\right\rangle\left\langle a^j\middle|b^i\right\rangle\left\langle a^i\right| = \delta_{ij}\hat{\sigma}_{ab}^{j},\\ \left[\left(\hat{\sigma}_{ab}^{i}\right)^{\dagger}, \hat{\sigma}_{a}^{j}\right] &= \left|a^i\right\rangle\left\langle b^i\middle|a^j\right\rangle\left\langle a^j\right| - \left|a^j\right\rangle\left\langle a^j\middle|a^i\right\rangle\left\langle b^i\right| = -\delta_{ij}\left(\hat{\sigma}_{ab}^{j}\right)^{\dagger}\end{aligned}$$

we get

$$\frac{i}{\hbar}\left[\hat{V}_{AF}, \hat{\sigma}_{a}^{j}\right] = ig\Delta\left(t - t_j\right)\left(\hat{a}^{\dagger}\hat{\sigma}_{ab}^{j} - \left(\hat{\sigma}_{ab}^{j}\right)^{\dagger}\hat{a}\right). \tag{4.12}$$

For the second term of the sum (4.11) we have

$$\begin{aligned}\frac{i}{\hbar}\left[\hat{V}_{AR_A}, \hat{\sigma}_{a}^{j}\right] =\\ = i\sum_{k,i} g_k^{(A)}\hat{c}_k^{\dagger}\left(\left[\hat{\sigma}_{ac}^{i}, \hat{\sigma}_{a}^{j}\right]e^{i\left(\omega_k^{(A)} - \omega_{ac}\right)t} + \left[\hat{\sigma}_{bc}^{i}, \hat{\sigma}_{a}^{j}\right]e^{i\left(\omega_k^{(A)} - \omega_{bc}\right)t}\right) +\\ +i\sum_{k,i} g_k^{(A)}\left(\left[\left(\hat{\sigma}_{ac}^{i}\right)^{\dagger}, \hat{\sigma}_{a}^{j}\right]e^{-i\left(\omega_k^{(A)} - \omega_{ac}\right)t} + \left[\left(\hat{\sigma}_{bc}^{i}\right)^{\dagger}, \hat{\sigma}_{a}^{j}\right]e^{-i\left(\omega_k^{(A)} - \omega_{bc}\right)t}\right)\hat{c}_k,\end{aligned}$$

which, using the commutation relations

$$\begin{aligned}\left[\hat{\sigma}_{ac}^{i}, \hat{\sigma}_{a}^{j}\right] &= \delta_{ij}\hat{\sigma}_{ac}^{j},\\ \left[\left(\hat{\sigma}_{ac}^{i}\right)^{\dagger}, \hat{\sigma}_{a}^{j}\right] &= -\delta_{ij}\left(\hat{\sigma}_{ac}^{j}\right)^{\dagger},\\ \left[\hat{\sigma}_{bc}^{i}, \hat{\sigma}_{a}^{j}\right] &= 0,\\ \left[\left(\hat{\sigma}_{bc}^{i}\right)^{\dagger}, \hat{\sigma}_{a}^{j}\right] &= 0\end{aligned} \tag{4.13}$$

is rewritten as

$$\begin{aligned}\frac{i}{\hbar}\left[\hat{V}_{AR_A}, \hat{\sigma}_{a}^{j}\right] = i\sum_{k} g_k^{(A)}\hat{c}_k^{\dagger}\hat{\sigma}_{ac}^{j}e^{i\left(\omega_k^{(A)} - \omega_{ac}\right)t} -\\ -i\sum_{k} g_k^{(A)}\left(\hat{\sigma}_{ac}^{i}\right)^{\dagger}\hat{c}_k e^{-i\left(\omega_k^{(A)} - \omega_{ac}\right)t}.\end{aligned} \tag{4.14}$$

Combining (4.12) and (4.14) together we obtain

$$\frac{d\hat{\sigma}_a^j}{dt} = ig\Delta\left(t - t_j\right)\hat{a}^\dagger\hat{\sigma}_{ab}^j + i\sum_k g_k^{(A)}\hat{c}_k^\dagger\hat{\sigma}_{ac}^i e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)t} + \text{h.c.} \tag{4.15}$$

Using (4.10) we can compute the second term of (4.15):

$$\begin{aligned} i\sum_k g_k^{(A)}\hat{c}_k^\dagger\hat{\sigma}_{ac}^i e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)t} &= i\sum_k g_k^{(A)}\hat{c}_k^\dagger\left(0\right)\hat{\sigma}_{ac}^i e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)t} - \\ &- \sum_{k,i}\left(g_k^{(A)}\right)^2\int_0^t dt'\hat{\sigma}_{ca}^i\left(t'\right)\hat{\sigma}_{ac}^j\left(t\right)e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)\left(t-t'\right)} - \\ &- \sum_{k,i}\left(g_k^{(A)}\right)^2\int_0^t dt'\hat{\sigma}_{cb}^i\left(t'\right)\hat{\sigma}_{ac}^j\left(t\right)e^{i\left(\omega_k^{(A)}-\omega_{bc}\right)\left(t-t'\right)}e^{i\left(\omega_{bc}-\omega_{ac}\right)t}. \end{aligned}$$

Denoting

$$\hat{f}_a^j\left(t\right) = i\sum_k g_k^{(A)}\hat{c}_k^\dagger\left(0\right)\hat{\sigma}_{ac}^i e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)t} \tag{4.16}$$

we get

$$\begin{aligned} i\sum_k g_k^{(A)}\hat{c}_k^\dagger\hat{\sigma}_{ac}^i e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)t} &= \hat{f}_a^j - \\ -\sum_i\int_0^t dt'\sum_k\left(g_k^{(A)}\right)^2\hat{\sigma}_{ca}^i\left(t'\right)\hat{\sigma}_{ac}^j\left(t\right)e^{i\left(\omega_k^{(A)}-\omega_{ac}\right)\left(t-t'\right)} - \\ -D\left(\omega_{bc}\right)\left(g^{(A)}\left(\omega_{bc}\right)\right)^2 e^{i\left(\omega_{bc}-\omega_{ac}\right)t}\cdot \\ \cdot\sum_i\int_0^t dt'\int_0^\infty d\omega'\hat{\sigma}_{cb}^i\left(t'\right)\hat{\sigma}_{ac}^j\left(t\right)e^{i\left(\omega'-\omega_{bc}\right)\left(t-t'\right)} \approx \\ \approx \hat{f}_a^j - \frac{\gamma_a}{2}\sum_i\int_0^t dt'\hat{\sigma}_{ca}^i\left(t'\right)\hat{\sigma}_{ac}^j\left(t\right)\delta\left(t-t'\right) - \\ -\frac{\gamma_b}{2}e^{i\left(\omega_{bc}-\omega_{ac}\right)t}\sum_i\int_0^t dt'\hat{\sigma}_{cb}^i\left(t'\right)\hat{\sigma}_{ac}^j\left(t\right)\delta\left(t-t'\right) = \\ = \hat{f}_a^j - \frac{\gamma_a}{2}\hat{\sigma}_a^j\left(t\right) - \frac{\gamma_b}{2}e^{i\left(\omega_{bc}-\omega_{ac}\right)t}\hat{\sigma}_{ba}^j\left(t\right) \approx \\ \approx \hat{f}_a^j - \frac{\gamma_a}{2}\hat{\sigma}_a^j\left(t\right), \end{aligned} \tag{4.17}$$

where

$$\gamma_a = 4\pi D\left(\omega_{ac}\right)\left(g^{(A)}\left(\omega_{ac}\right)\right)^2$$

and

$$\gamma_b = 4\pi D\left(\omega_{bc}\right)\left(g^{(A)}\left(\omega_{bc}\right)\right)^2.$$

In deriving (4.17) we also neglected rapidly oscillating term $\frac{\gamma_b}{2}e^{i(\omega_{bc}-\omega_{ac})t}\hat{\sigma}^j_{ba}(t)$ compared to $\frac{\gamma_a}{2}\hat{\sigma}^j_a(t)$. Thus, as a result we get

$$\frac{d\hat{\sigma}^j_a}{dt} = -\gamma_a\hat{\sigma}^j_a + ig\Delta\left(t-t_j\right)\left(\hat{a}^\dagger\hat{\sigma}^j_{ab} - \left(\hat{\sigma}^j_{ab}\right)^\dagger\hat{a}\right) + \hat{f}^j_a. \tag{4.18}$$

By analogy, one can obtain the expression for $\hat{\sigma}^j_b$:

$$\frac{d\hat{\sigma}^j_b}{dt} = -\gamma_b\hat{\sigma}^j_b - ig\Delta\left(t-t_j\right)\left(\hat{a}^\dagger\hat{\sigma}^j_{ab} - \left(\hat{\sigma}^j_{ab}\right)^\dagger\hat{a}\right) + \hat{f}^j_b, \tag{4.19}$$

where

$$\hat{f}^j_b(t) = i\sum_k g^{(A)}_k\hat{c}^\dagger_k(0)\,\hat{\sigma}^i_{bc}e^{i\left(\omega^{(A)}_k-\omega_{bc}\right)t} + \text{h.c.} \tag{4.20}$$

For $\hat{\sigma}^j_{ab}$ we have

$$\frac{d\hat{\sigma}^j_{ab}}{dt} = \frac{i}{\hbar}\left[\hat{V}_{AF},\hat{\sigma}^j_{ab}\right] + \frac{i}{\hbar}\left[\hat{V}_{AR_A},\hat{\sigma}^j_{ab}\right]. \tag{4.21}$$

For the first term of the sum (4.21) it holds

$$\frac{i}{\hbar}\left[\hat{V}_{AF},\hat{\sigma}^j_{ab}\right] = \frac{i}{\hbar}g\hbar\sum_i\Delta\left(t-t_i\right)\left(\hat{a}^\dagger\left[\hat{\sigma}^i_{ab},\hat{\sigma}^j_{ab}\right] + \left[\left(\hat{\sigma}^i_{ab}\right)^\dagger,\hat{\sigma}^j_{ab}\right]\hat{a}\right),$$

from which, taking into account

$$\left[\left(\hat{\sigma}^i_{ab}\right)^\dagger,\hat{\sigma}^j_{ab}\right] = \left|a^i\right\rangle\left\langle b^i\middle|b^j\right\rangle\left\langle a^j\right| - \left|b^j\right\rangle\left\langle a^j\middle|a^i\right\rangle\left\langle b^i\right| = -\delta_{ij}\left(\hat{\sigma}^j_a - \hat{\sigma}^j_b\right)$$

we obtain

$$\frac{i}{\hbar}\left[\hat{V}_{AF},\hat{\sigma}^j_{ab}\right] = ig\Delta\left(t-t_j\right)\left(\hat{\sigma}^j_a - \hat{\sigma}^j_b\right)\hat{a}. \tag{4.22}$$

For the second term of the sum (4.21) we have

$$\begin{aligned}\frac{i}{\hbar}\left[\hat{V}_{AR_A},\hat{\sigma}^j_{ab}\right] = \\ = i\sum_{k,i}g^{(A)}_k\hat{c}^\dagger_k\left(\left[\hat{\sigma}^i_{ac},\hat{\sigma}^j_{ab}\right]e^{i\left(\omega^{(A)}_k-\omega_{ac}\right)t} + \left[\hat{\sigma}^i_{bc},\hat{\sigma}^j_{ab}\right]e^{i\left(\omega^{(A)}_k-\omega_{bc}\right)t}\right) + \\ +i\sum_{k,i}g^{(A)}_k\left(\left[\left(\hat{\sigma}^i_{ac}\right)^\dagger,\hat{\sigma}^j_{ab}\right]e^{-i\left(\omega^{(A)}_k-\omega_{ac}\right)t} + \left[\left(\hat{\sigma}^i_{bc}\right)^\dagger,\hat{\sigma}^j_{ab}\right]e^{-i\left(\omega^{(A)}_k-\omega_{bc}\right)t}\right)\hat{c}_k,\end{aligned}$$

which, using commutation relations

$$\begin{aligned}\left[\hat{\sigma}_{ac}^{i},\hat{\sigma}_{ab}^{j}\right]&=0,\\ \left[\left(\hat{\sigma}_{ac}^{i}\right)^{\dagger},\hat{\sigma}_{ab}^{j}\right]&=-\delta_{ij}\left(\hat{\sigma}_{bc}^{j}\right)^{\dagger},\\ \left[\hat{\sigma}_{bc}^{i},\hat{\sigma}_{ab}^{j}\right]&=\delta_{ij}\hat{\sigma}_{ac}^{j},\\ \left[\left(\hat{\sigma}_{bc}^{i}\right)^{\dagger},\hat{\sigma}_{ab}^{j}\right]&=0\end{aligned}\tag{4.23}$$

is rewritten as

$$\begin{aligned}\frac{i}{\hbar}\left[\hat{V}_{AR_A},\hat{\sigma}_{ab}^{j}\right]&=i\sum_{k}g_{k}^{(A)}\hat{c}_{k}^{\dagger}\hat{\sigma}_{ac}^{j}e^{i\left(\omega_{k}^{(A)}-\omega_{bc}\right)t}-\\ &-i\sum_{k}g_{k}^{(A)}\left(\hat{\sigma}_{bc}^{j}\right)^{\dagger}\hat{c}_{k}e^{-i\left(\omega_{k}^{(A)}-\omega_{ac}\right)t},\end{aligned}\tag{4.24}$$

Combining (4.22) and (4.24) together we get

$$\begin{aligned}\frac{d\hat{\sigma}_{ab}^{j}}{dt}&=ig\Delta\left(t-t_{j}\right)\left(\hat{\sigma}_{a}^{j}-\hat{\sigma}_{b}^{j}\right)\hat{a}+\\ +i\sum_{k}g_{k}^{(A)}\hat{c}_{k}^{\dagger}\hat{\sigma}_{ac}^{j}e^{i\left(\omega_{k}^{(A)}-\omega_{bc}\right)t}&-i\sum_{k}g_{k}^{(A)}\left(\hat{\sigma}_{bc}^{j}\right)^{\dagger}\hat{c}_{k}e^{-i\left(\omega_{k}^{(A)}-\omega_{ac}\right)t}\end{aligned}\tag{4.25}$$

Substituting into (4.25) the expressions for $\hat{c}_k$ (4.9) and ${\hat{c}_k}^{\dagger}$ (4.10) and neglecting rapidly oscillating terms, we have:

$$\begin{aligned}\frac{d\hat{\sigma}_{ab}^{j}}{dt}&=ig\Delta\left(t-t_{j}\right)\left(\hat{\sigma}_{a}^{j}-\hat{\sigma}_{b}^{j}\right)\hat{a}-\\ -\frac{\gamma_b}{2}\int_{0}^{\infty}dt'\hat{\sigma}_{cb}^{j}\left(t'\right)\hat{\sigma}_{ac}^{j}\left(t\right)\delta\left(t-t'\right)&-\\ -\frac{\gamma_a}{2}\int_{0}^{\infty}dt'\hat{\sigma}_{cb}^{j}\left(t\right)\hat{\sigma}_{ac}^{j}\left(t'\right)\delta\left(t-t'\right)+\hat{f}_{ab}^{j},&\end{aligned}\tag{4.26}$$

where

$$\begin{aligned}\hat{f}_{ab}^{j}\left(t\right)&=i\sum_{k}g_{k}^{(A)}\hat{c}_{k}^{\dagger}\left(0\right)\hat{\sigma}_{ac}^{j}e^{i\left(\omega_{k}^{(A)}-\omega_{bc}\right)t}-\\ &-i\sum_{k}g_{k}^{(A)}\left(\hat{\sigma}_{bc}^{j}\right)^{\dagger}\hat{c}_{k}\left(0\right)e^{-i\left(\omega_{k}^{(A)}-\omega_{ac}\right)t}.\end{aligned}\tag{4.27}$$

In what follows, we will use the assumption that

$$\gamma_a=\gamma_b=\gamma.\tag{4.28}$$

Taking into account

$$\hat{\sigma}_{cb}^{j}\hat{\sigma}_{ac}^{j}=\hat{\sigma}_{ab}^{j},$$

as a result from (4.26) we get

$$\begin{aligned}\frac{d\hat{\sigma}_{ab}^{j}}{dt}&=ig\Delta\left(t-t_{j}\right)\left(\hat{\sigma}_{a}^{j}-\hat{\sigma}_{b}^{j}\right)\hat{a}-\gamma\hat{\sigma}_{cb}^{j}\hat{\sigma}_{ac}^{j}+\hat{f}_{ab}^{j}=\\&=ig\Delta\left(t-t_{j}\right)\left(\hat{\sigma}_{a}^{j}-\hat{\sigma}_{b}^{j}\right)\hat{a}-\gamma\hat{\sigma}_{ab}^{j}+\hat{f}_{ab}^{j}\end{aligned}\tag{4.29}$$

For the noise operator $\hat{f}_{ab}^{j}$ one can write the following correlation functions:

$$\begin{aligned}\left\langle\left(\hat{f}_{ab}^{i}\left(t_{1}\right)\right)^{\dagger}\hat{f}_{ab}^{j}\left(t_{2}\right)\right\rangle&=\delta_{ij}\frac{\gamma}{2}\left\langle\hat{\sigma}_{a}^{j}\right\rangle\delta\left(t_{1}-t_{2}\right),\\\left\langle\hat{f}_{ab}^{j}\left(t_{1}\right)\left(\hat{f}_{ab}^{i}\left(t_{2}\right)\right)^{\dagger}\right\rangle&=\delta_{ij}\frac{\gamma}{2}\left\langle\hat{\sigma}_{b}^{j}\right\rangle\delta\left(t_{1}-t_{2}\right).\end{aligned}\tag{4.30}$$

Indeed, for the first relation (4.30), assuming the average phonon number in the reservoir mode at the initial time is zero:

$$\left\langle\hat{c}_{k_{1}}^{\dagger}\left(0\right)\hat{c}_{k_{2}}\left(0\right)\right\rangle=0,$$

so that

$$\left\langle\hat{c}_{k_{1}}\left(0\right)\hat{c}_{k_{2}}^{\dagger}\left(0\right)\right\rangle=\delta_{k_{1},k_{2}},$$

we obtain

$$\begin{aligned}&\left\langle\left(\hat{f}_{ab}^{i}\left(t_{1}\right)\right)^{\dagger}\hat{f}_{ab}^{j}\left(t_{2}\right)\right\rangle=\\&=\sum_{k_{1},k_{2}}g_{k_{1}}^{(A)}g_{k_{2}}^{(A)}\left\langle\hat{c}_{k_{1}}\left(0\right)\hat{c}_{k_{2}}^{\dagger}\left(0\right)\right\rangle\left\langle\left(\hat{\sigma}_{ac}^{i}\right)^{\dagger}\hat{\sigma}_{ac}^{j}\right\rangle e^{-i\left(\omega_{k_{1}}^{(A)}-\omega_{bc}\right)t_{1}}e^{i\left(\omega_{k_{2}}^{(A)}-\omega_{bc}\right)t_{2}}+\\&+\sum_{k_{1},k_{2}}g_{k_{1}}^{(A)}g_{k_{2}}^{(A)}\left\langle\hat{c}_{k_{1}}^{\dagger}\left(0\right)\hat{c}_{k_{2}}\left(0\right)\right\rangle\left\langle\hat{\sigma}_{bc}^{i}\left(\hat{\sigma}_{bc}^{j}\right)^{\dagger}\right\rangle e^{i\left(\omega_{k_{1}}^{(A)}-\omega_{ac}\right)t_{1}}e^{-i\left(\omega_{k_{2}}^{(A)}-\omega_{ac}\right)t_{2}}=\\&=\sum_{k}\left(g_{k}^{(A)}\right)^{2}\left\langle\left(\hat{\sigma}_{ac}^{i}\right)^{\dagger}\hat{\sigma}_{ac}^{j}\right\rangle e^{-i\left(\omega_{k}^{(A)}-\omega_{bc}\right)\left(t_{1}-t_{2}\right)}=\\&=\delta_{ij}\sum_{k}\left(g_{k}^{(A)}\right)^{2}\left\langle\hat{\sigma}_{a}^{j}\right\rangle e^{-i\left(\omega_{k}^{(A)}-\omega_{bc}\right)\left(t_{1}-t_{2}\right)}=\delta_{ij}\frac{\gamma}{2}\left\langle\hat{\sigma}_{a}^{j}\right\rangle\delta\left(t_{1}-t_{2}\right).\end{aligned}$$

4.2.2 Equations of motion for macroscopic variables

From the equations of motion describing the behavior of individual atoms (4.18, 4.19, 4.29), we proceed to equations describing the behavior of the entire

atomic subsystem collectively. For this, from operators $\hat{\sigma}_a^j$, $\hat{\sigma}_b^j$, and $\hat{\sigma}_{ab}^j$ we go to new operators which represent the collective effect of all atoms:

$$\begin{aligned}\hat{N}_a &= \sum_j \Delta\left(t-t_j\right)\hat{\sigma}_a^j,\\ \hat{N}_b &= \sum_j \Delta\left(t-t_j\right)\hat{\sigma}_b^j,\\ \hat{N}_{ab} &= \sum_j \Delta\left(t-t_j\right)\hat{\sigma}_{ab}^j.\end{aligned} \tag{4.31}$$

Next, consider $\frac{d\hat{N}_a}{dt}$:

$$\begin{aligned}\frac{d\hat{N}_a}{dt} &= \sum_j \frac{d\Delta\left(t-t_j\right)}{dt}\hat{\sigma}_a^j + \sum_j \Delta\left(t-t_j\right)\frac{d\hat{\sigma}_a^j}{dt} =\\ &= \sum_j \delta\left(t-t_j\right)\hat{\sigma}_a^j + \sum_j \Delta\left(t-t_j\right)\frac{d\hat{\sigma}_a^j}{dt}.\end{aligned} \tag{4.32}$$

Substituting (4.18) into (4.32) and using (4.28) we obtain:

$$\begin{aligned}\frac{d\hat{N}_a}{dt} &= \sum_j \delta\left(t-t_j\right)\hat{\sigma}_a^j - \gamma\sum_j \Delta\left(t-t_j\right)\hat{\sigma}_a^j +\\ +ig\sum_j \Delta\left(t-t_j\right)&\left(\hat{a}^\dagger\hat{\sigma}_{ab}^j - \left(\hat{\sigma}_{ab}^j\right)^\dagger\hat{a}\right) + \sum_j \Delta\left(t-t_j\right)\hat{f}_a^j =\\ &= \sum_j \delta\left(t-t_j\right)\hat{\sigma}_a^j - \gamma\hat{N}_a +\\ &+ig\left(\hat{a}^\dagger\hat{N}_{ab} - \hat{N}_{ab}^\dagger\hat{a}\right) + \sum_j \Delta\left(t-t_j\right)\hat{f}_a^j\end{aligned} \tag{4.33}$$

Next, consider the average value of the first term in (4.33):

$$\begin{aligned}\left\langle\sum_j \delta\left(t-t_j\right)\hat{\sigma}_a^j\left(t\right)\right\rangle &= \left\langle\sum_j \delta\left(t-t_j\right)\hat{\sigma}_a^j\left(t_j\right)\right\rangle =\\ &\left\langle\sum_j \delta\left(t-t_j\right)\left\langle\hat{\sigma}_a^j\left(t_j\right)\right\rangle\right\rangle_T,\end{aligned} \tag{4.34}$$

where $\langle\dots\rangle_T$ means averaging over injection times of atoms:

$$\langle\dots\rangle_T = r_a\int_{-\infty}^{\infty}\left(\dots\right)dt_j, \tag{4.35}$$

where r_a is the atom injection rate. For the average value of operator $\hat{\sigma}_a^j$ assuming that at time t_j, the j-th atom is injected in the excited state, we have

$$\left\langle \hat{\sigma}_a^j \left(t_j\right)\right\rangle = \left\langle a\right| \hat{\sigma}_a^j \left|a\right\rangle = 1. \tag{4.36}$$

Substituting (4.35) and (4.36) into (4.34) we get

$$\left\langle \sum_j \delta\left(t - t_j\right) \hat{\sigma}_a^j \left(t\right)\right\rangle = r_a. \tag{4.37}$$

Next, add and subtract the average (4.37) from (4.33), resulting in:

$$\frac{d\hat{N}_a}{dt} = r_a - \gamma \hat{N}_a + ig\left(\hat{a}^\dagger \hat{N}_{ab} - \hat{N}_{ab}^\dagger \hat{a}\right) + \hat{F}_a, \tag{4.38}$$

where a new noise operator $\hat{F}_a$ is introduced, which accounts for not only quantum fluctuations but also fluctuations of injection times of active atoms:

$$\hat{F}_a = \sum_j \delta\left(t - t_j\right) \hat{\sigma}_a^j - r_a + \sum_j \Delta\left(t - t_j\right) \hat{f}_a^j, \tag{4.39}$$

and it is obvious that the average of this operator is zero:

$$\left\langle \hat{F}_a \right\rangle = \left\langle \sum_j \delta\left(t - t_j\right) \hat{\sigma}_a^j \right\rangle - r_a + \left\langle \sum_j \Delta\left(t - t_j\right) \hat{f}_a^j \right\rangle = 0.$$

From (4.19, 4.29) one can get the following relations for $\hat{N}_b$ and $\hat{N}_{ab}$:

$$\begin{aligned} \frac{d\hat{N}_b}{dt} &= -\gamma \hat{N}_b - ig\left(\hat{a}^\dagger \hat{N}_{ab} - \hat{N}_{ab}^\dagger \hat{a}\right) + \hat{F}_b, \\ \frac{d\hat{N}_{ab}}{dt} &= -\gamma \hat{N}_{ab} + ig\left(\hat{N}_a - \hat{N}_b\right) \hat{a} + \hat{F}_{ab}, \end{aligned} \tag{4.40}$$

where the following noise operators are introduced:

$$\begin{aligned} \hat{F}_b &= \sum_j \Delta\left(t - t_j\right) \hat{f}_b^j + \sum_j \delta\left(t - t_j\right) \hat{\sigma}_b^j, \\ \hat{F}_{ab} &= \sum_j \Delta\left(t - t_j\right) \hat{f}_{ab}^j + \sum_j \delta\left(t - t_j\right) \hat{\sigma}_{ab}^j. \end{aligned} \tag{4.41}$$

Considering that

$$\left\langle \hat{\sigma}_b^j \left(t_j\right)\right\rangle = \left\langle \hat{\sigma}_{ab}^j \left(t_j\right)\right\rangle = 0,$$

these operators have zero average.

For the correlator $\left\langle \hat{F}_{ab}^{\dagger}(t_1)\,\hat{F}_{ab}(t_2)\right\rangle$ from (4.30) we obtain the following relation

$$\left\langle \hat{F}_{ab}^{\dagger}(t_1)\,\hat{F}_{ab}(t_2)\right\rangle = \sum_{ij}\Delta(t_1-t_i)\,\Delta(t_2-t_j)\left\langle \left(\hat{f}_{ab}^{i}\right)^{\dagger}\hat{f}_{ab}^{j}\right\rangle + \\ + \sum_{ij}\delta(t_1-t_i)\,\delta(t_2-t_j)\left\langle \left(\hat{\sigma}_{ab}^{i}\right)^{\dagger}\hat{\sigma}_{ab}^{j}\right\rangle = \\ = \frac{\gamma}{2}\delta(t_1-t_2)\sum_{j}\Delta(t_1-t_j)\left\langle\hat{\sigma}_{a}^{j}\right\rangle + \delta(t_1-t_2)\sum_{j}\delta(t_1-t_j)\left\langle\hat{\sigma}_{a}^{j}\right\rangle = \\ = \left(\frac{\gamma}{2}\bar{N}_a + r_a\right)\delta(t_1-t_2), \quad (4.42)$$

where $\bar{N}_a$ is the average number of atoms in the excited state at the considered time t_1

$$\bar{N}_a(t_1) = \sum_{j}\delta(t_1-t_j)\left\langle\hat{\sigma}_{a}^{j}\right\rangle$$

Similarly, for the correlator $\left\langle \hat{F}_{ab}(t_1)\,\hat{F}_{ab}^{\dagger}(t_2)\right\rangle$ from (4.30) we obtain the following relation

$$\left\langle \hat{F}_{ab}(t_1)\,\hat{F}_{ab}^{\dagger}(t_2)\right\rangle = \sum_{ij}\Delta(t_1-t_i)\,\Delta(t_2-t_j)\left\langle \hat{f}_{ab}^{i}\left(\hat{f}_{ab}^{j}\right)^{\dagger}\right\rangle + \\ + \sum_{ij}\delta(t_1-t_i)\,\delta(t_2-t_j)\left\langle \hat{\sigma}_{ab}^{j}\left(\hat{\sigma}_{ab}^{i}\right)^{\dagger}\right\rangle = \\ = \frac{\gamma}{2}\delta(t_1-t_2)\sum_{j}\Delta(t_1-t_j)\left\langle\hat{\sigma}_{b}^{j}\right\rangle + \delta(t_1-t_2)\sum_{j}\delta(t_1-t_j)\left\langle\hat{\sigma}_{b}^{j}\right\rangle = \\ = \frac{\gamma}{2}\bar{N}_b\delta(t_1-t_2), \quad (4.43)$$

where again the average number of atoms in the unexcited state is introduced:

$$\bar{N}_b(t_1) = \sum_{j}\delta(t_1-t_j)\left\langle\hat{\sigma}_{b}^{j}\right\rangle$$

It will be convenient to use the operator of population difference $\hat{N}_z = \hat{N}_a - \hat{N}_b$, for which from (4.38) and (4.40) we get:

$$\frac{d\hat{N}_z}{dt} = r_a - \gamma\hat{N}_z + 2ig\left(\hat{a}^{\dagger}\hat{N}_{ab} - \hat{N}_{ab}^{\dagger}\hat{a}\right) + \hat{F}_z, \quad (4.44)$$

where

$$\hat{F}_z = \hat{F}_a - \hat{F}_b. \quad (4.45)$$

4.2.3 Equations of motion for the electromagnetic field

Similarly to the atomic subsystem, we start by considering the equations satisfied by reservoir R_A operators: $\hat{b}_k$. Since these operators enter only in the Hamiltonian $\hat{V}_{FR_F}$, the equation of motion takes the form

$$\frac{d\hat{b}_k}{dt} = \frac{i}{\hbar}\left[\hat{V}_{FR_F}, \hat{b}_k\right]. \tag{4.46}$$

Substituting into (4.46) the expression for the Hamiltonian (4.6) we get

$$\begin{aligned} \frac{d\hat{b}_k}{dt} &= \frac{i}{\hbar}\left[\hat{V}_{FR_F}, \hat{b}_k\right] = \\ = \frac{i}{\hbar}\hbar\sum_{k',j} g_{k'}^{(F)}\left[\hat{b}_{k'}^{\dagger}, \hat{b}_{k'}\right]\hat{a}e^{i\left(\omega_{k'}^{(F)}-\omega\right)t} &= -ig_k^{(F)}\hat{a}e^{i\left(\omega_k^{(F)}-\omega\right)t}. \end{aligned} \tag{4.47}$$

Formal integration of (4.47) gives

$$\hat{b}_k(t) = \hat{b}_k(0) - ig_k^{(F)}\int_0^t dt'\hat{a}(t')\,e^{i\left(\omega_k^{(F)}-\omega\right)t'}. \tag{4.48}$$

The operator of the electromagnetic field mode $\hat{a}$ appears in the Hamiltonians $\hat{V}_{AF}$ and $\hat{V}_{FR_F}$; thus, taking into account the expressions for these Hamiltonians (4.5) and (4.6) and the definition (4.31), we have

$$\begin{aligned} \frac{d\hat{a}}{dt} &= \frac{i}{\hbar}\left[\hat{V}_{AF}, \hat{a}\right] + \frac{i}{\hbar}\left[\hat{V}_{FR_F}, \hat{a}\right] = \\ &= \frac{i}{\hbar}\hbar g\sum_j \Delta(t-t_j)\left[\hat{a}^{\dagger}, \hat{a}\right]\hat{\sigma}_{ab}^j + \\ &+ \frac{i}{\hbar}\hbar\sum_k g_k^{(F)}\left[\hat{a}^{\dagger}, \hat{a}\right]\hat{b}_k e^{-i\left(\omega_k^{(F)}-\omega\right)t} = \\ = -ig\sum_j \Delta(t-t_j)\,\hat{\sigma}_{ab}^j - i\sum_k g_k^{(F)}\hat{b}_k e^{-i\left(\omega_k^{(F)}-\omega\right)t} &= \\ = -ig\hat{N}_{ab} - i\sum_k g_k^{(F)}\hat{b}_k e^{-i\left(\omega_k^{(F)}-\omega\right)t}. \end{aligned} \tag{4.49}$$

Substituting (4.48) into (4.49) we get

$$\begin{aligned} \frac{d\hat{a}}{dt} = -ig\hat{N}_{ab} - 2\pi D(\omega)\left(g^{(F)}(\omega)\right)^2\int_0^t dt'\hat{a}(t')\,\delta(t-t') - \\ -i\sum_k g_k^{(F)}\hat{b}_k(0)\,e^{-i\left(\omega_k^{(F)}-\omega\right)t}. \end{aligned}$$

Denoting

$$\gamma^{(F)} = 4\pi D\left(\omega\right)\left(g^{(F)}\left(\omega\right)\right)^2 = \frac{\omega}{Q}$$

and introducing the noise operator

$$\hat{F}_F\left(t\right) = -i\sum_k g_k^{(F)}\hat{b}_k\left(0\right)e^{-i\left(\omega_k^{(F)}-\omega\right)t}, \tag{4.50}$$

we finally obtain

$$\frac{d\hat{a}}{dt} = -\frac{1}{2}\frac{\omega}{Q}\hat{a} - ig\hat{N}_{ab} + \hat{F}_F. \tag{4.51}$$

Since the definition of the noise operator (4.50) coincides with the definition (2.134), we can write the correlation relations as in (2.135) and (2.136):

$$\begin{aligned}\left\langle \hat{F}_F^{\dagger}\left(t_1\right)\hat{F}_F\left(t_2\right)\right\rangle &= \frac{\gamma\bar{n}_T}{2}\delta\left(t_1-t_2\right) = \frac{1}{2}\frac{\omega}{Q}\bar{n}_T\delta\left(t_1-t_2\right),\\ \left\langle \hat{F}_F\left(t_1\right)\hat{F}_F^{\dagger}\left(t_2\right)\right\rangle &= \frac{1}{2}\frac{\omega}{Q}\left(\bar{n}_T+1\right)\delta\left(t_1-t_2\right),\end{aligned} \tag{4.52}$$

where $\bar{n}_T$ is the average number of phonons in the reservoir mode at temperature T.

4.3 Heisenberg-Langevin Equation System Describing the Laser

Combining the equations we obtained (4.51), (4.40), and (4.44) we can write the following system:

$$\begin{aligned}\frac{d\hat{a}}{dt} &= -\frac{1}{2}\frac{\omega}{Q}\hat{a} - ig\hat{N}_{ab} + \hat{F}_F,\\ \frac{d\hat{N}_{ab}}{dt} &= -\gamma\hat{N}_{ab} + ig\hat{N}_z\hat{a} + \hat{F}_{ab},\\ \frac{d\hat{N}_z}{dt} &= r_a - \gamma\hat{N}_z + 2ig\left(\hat{a}^{\dagger}\hat{N}_{ab} - \hat{N}_{ab}^{\dagger}\hat{a}\right) + \hat{F}_z,\end{aligned} \tag{4.53}$$

which we will call the Heisenberg-Langevin equation system describing the laser operation.

By their form, these equations (4.53) are very similar to the laser equations obtained in the semiclassical approximation. The difference is that instead of classical quantities, the equations use operators, which may not commute with

each other, so the order of factors must be preserved during transformations. Moreover, the equations include Langevin terms describing quantum noise present in the laser system.

The same approximations applied in the semiclassical case can be applied to equations (4.53).

If, as is usually the case, the adiabatic approximation can be applied, neglecting in the second equation (4.53) the derivative $\frac{d\hat{N}_{ab}}{dt}$ compared to the term $\gamma\hat{N}_{ab}$, we obtain

$$\hat{N}_{ab} = \frac{1}{\gamma}\left(ig\hat{N}_z\hat{a} + \hat{F}_{ab}\right).$$

Substituting this expression for $\hat{N}_{ab}$ into the remaining two equations of system (4.53) yields a new system of equations:

$$\begin{aligned}\frac{d}{dt}\hat{a} &= -\frac{1}{2}\left(\frac{\omega}{Q}\right)\hat{a} + \frac{g^2}{\gamma}\hat{N}_z\hat{a} + \hat{F}_F - i\frac{g}{\gamma}\hat{F}_{ab},\\ \frac{d}{dt}\hat{N}_z &= r_a - \gamma\hat{N}_z - \frac{4g^2}{\gamma}\hat{N}_z\hat{a}^\dagger\hat{a} +\\ &+\hat{F}_z + \frac{2ig}{\gamma}\left(\hat{a}^\dagger\hat{F}_{ab} - \hat{F}^\dagger_{ab}\hat{a}\right).\end{aligned} \tag{4.54}$$

Introducing a new notation

$$\hat{F}_\sum = -\left(\frac{g}{\gamma}\hat{F}_{ab} + i\hat{F}_F\right), \tag{4.55}$$

the system (4.54) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\hat{a} &= -\frac{1}{2}\left(\frac{\omega}{Q}\right)\hat{a} + \frac{g^2}{\gamma}\hat{N}_z\hat{a} + i\hat{F}_\sum,\\ \frac{d}{dt}\hat{N}_z &= r_a - \gamma\hat{N}_z - \frac{4g^2}{\gamma}\hat{N}_z\hat{a}^\dagger\hat{a} +\\ &+\hat{F}_z + \frac{2ig}{\gamma}\left(\hat{a}^\dagger\hat{F}_{ab} - \hat{F}^\dagger_{ab}\hat{a}\right).\end{aligned} \tag{4.56}$$

For the noise operator (4.55), using (4.42), (4.43), and (4.52), the following correlation relations can be written:

$$\begin{aligned}\left\langle \hat{F}^\dagger_\sum(t_1)\,\hat{F}_\sum(t_2)\right\rangle &= \frac{g^2}{\gamma^2}\left\langle \hat{F}^\dagger_{ab}(t_1)\,\hat{F}_{ab}(t_2)\right\rangle + \left\langle \hat{F}^\dagger_F(t_1)\,\hat{F}_F(t_2)\right\rangle =\\ &= \frac{1}{2}\left(\frac{g^2}{\gamma}\bar{N}_a + 2\frac{g^2}{\gamma^2}r_a + \frac{\omega}{Q}\bar{n}_T\right)\delta(t_1 - t_2),\\ \left\langle \hat{F}_\sum(t_1)\,\hat{F}^\dagger_\sum(t_2)\right\rangle &= \frac{1}{2}\left(\frac{g^2}{\gamma}\bar{N}_b + \frac{\omega}{Q}(\bar{n}_T + 1)\right)\delta(t_1 - t_2)\end{aligned} \tag{4.57}$$

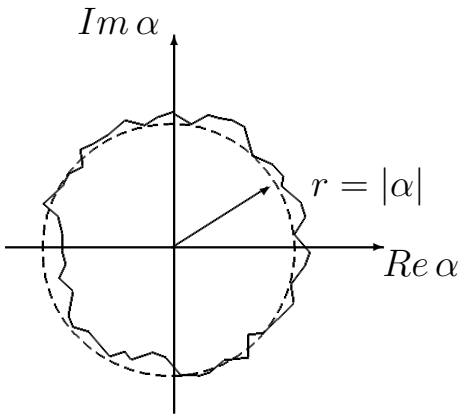


Figure 4.3: In the presence of noise sources (Langevin forces), the amplitude uncertainty occupies a narrow region near the circle $|\alpha| = const$, while the phase freely diffuses (wanders) along this circle.

Equations (4.56) resemble the semiclassical laser equation and can, to a large extent, be solved using the same approximations and techniques as those used in the semiclassical problem.

4.4 Natural linewidth of emission

Equations (4.56) are suitable for solving many problems related to the quantum nature of the laser: determining the statistics of laser photons in various operating modes, determining the linewidth of emission, and with some additions for solving more subtle issues, for example, the study of direct laser generation of squeezed states (for more on squeezed states see Ch. 9).

Here we will consider only the problem of the natural linewidth of laser emission. We will consider the steady-state laser operation mode that is sufficiently far from the generation threshold. In this case, the problem can be greatly simplified since the linewidth depends little on fluctuations of the population difference and amplitude fluctuations of the generated radiation, as we established earlier by considering this problem using the density matrix method. The main influence on the linewidth is due to phase fluctuations (drift). This can be visually represented on the complex amplitude plane, as shown in Figure 4.3.

If there were no noise sources (Langevin forces), the generated field would have a definite amplitude and phase. In reality, with their presence, the amplitude uncertainty occupies a narrow region near the circle $|\alpha| = const$, while the phase freely diffuses (wanders) along this circle.

In light of this, the simplification of the equations reduces to the following.

The equation for the population will be replaced by the averaged equation

$$r_a - \gamma \bar{N}_z - \frac{4g^2}{\gamma} \bar{N}_z \bar{n} = 0, \tag{4.58}$$

where $\bar{N}_z$ is the averaged population value, $\bar{n} = \left\langle \hat{a}^\dagger \hat{a} \right\rangle$ is the average photon number in the generated mode. Zero on the right means that a steady-state regime is considered.

From (4.58) we find $\bar{N}_z$

$$\bar{N}_z = \frac{r_a}{\gamma \left(1 + \left(\frac{2g}{\gamma} \right)^2 \bar{n} \right)} \approx \frac{r_a}{\gamma} \left(1 - \left(\frac{2g}{\gamma} \right)^2 \bar{n} \right). \tag{4.59}$$

The final value in (4.59) is obtained under the approximation that the second term in parentheses is small compared to 1 (laser is not at threshold). Substituting this result into the field equation, we get

$$\frac{d}{dt} \hat{a} = -\frac{1}{2} \left(\frac{\omega}{Q} \right) \hat{a} + r_a \frac{g^2}{\gamma^2} \hat{a} - \\ -4 r_a \frac{g^4}{\gamma^4} \bar{n} \hat{a} + i \hat{F}_\sum, \tag{4.60}$$

where the first term on the right corresponds to losses in the resonator, the second - gain by the active medium, the third characterizes the level saturation degree.

The field correlation function we need to determine is expressed by the equality

$$I(t, 0) = \left\langle \hat{a}^\dagger(t) \, \hat{a}(0) \right\rangle,$$

where $\hat{a}(t)$ satisfies equation (4.60).

In our approximation, when we neglect amplitude fluctuations, but do not neglect phase diffusion, the operator $\hat{a}$ can be represented in the form

$$\hat{a}(t) = A e^{i\hat{\varphi}(t)}, \\ \hat{a}^\dagger(t) = A e^{-i\hat{\varphi}(t)}, \tag{4.61}$$

where A is the average amplitude, and $\hat{\varphi}(t)$ is the phase operator. Then the correlation function can be presented as

$$\left\langle \hat{a}^\dagger(t) \, \hat{a}(0) \right\rangle = |A|^2 \left\langle e^{-i(\hat{\varphi}(t) - \hat{\varphi}(0))} \right\rangle = \\ = |A|^2 \left\langle e^{-\frac{1}{2}(\hat{\varphi}(t) - \hat{\varphi}(0))^2} \right\rangle. \tag{4.62}$$

The last follows from the fact that

$$\left\langle e^{-i(\Delta\varphi)}\right\rangle = \left\langle e^{-\frac{1}{2}(\Delta\varphi)^2}\right\rangle,$$

where $\Delta\varphi = \hat{\varphi}(t) - \hat{\varphi}(0)$, and it can be proved by expanding the exponential in a series, averaging each series term, and then summing the result.

We have

$$\Delta\varphi = \int_0^t \frac{d\varphi}{dt} dt,$$

then

$$\left\langle (\Delta\varphi)^2\right\rangle = \int_0^t dt_1 \left\langle \frac{d\varphi}{dt}\right\rangle \int_0^t dt_2 \left\langle \frac{d\varphi}{dt}\right\rangle,$$

where $\frac{d\varphi}{dt}$ can be obtained from equation (4.60) through the relations

$$\begin{aligned} i\frac{d\varphi}{dt} = \frac{1}{\hat{a}}\frac{d\hat{a}}{dt} = -\frac{1}{2}\left(\frac{\omega}{Q}\right) + r_a\frac{g^2}{\gamma^2} - \\ -4r_a\frac{g^4}{\gamma^4}\bar{n} + i\frac{\hat{F}_\sum}{\hat{a}}, \\ -i\frac{d\varphi}{dt} = \frac{1}{\hat{a}^\dagger}\frac{d\hat{a}^\dagger}{dt} = -\frac{1}{2}\left(\frac{\omega}{Q}\right) + r_a\frac{g^2}{\gamma^2} - \\ -4r_a\frac{g^4}{\gamma^4}\bar{n} - i\frac{\hat{F}^\dagger_\sum}{\hat{a}^\dagger}, \end{aligned} \tag{4.63}$$

thus subtracting the second equation (4.63) from the first we obtain

$$2\frac{d\varphi}{dt} = \frac{\hat{F}_\sum}{\hat{a}} + \frac{\hat{F}^\dagger_\sum}{\hat{a}^\dagger}. \tag{4.64}$$

Substituting now (4.61) into (4.64) we have:

$$\frac{d\varphi}{dt} = \frac{1}{2A}\left\{e^{i\hat{\varphi}(t)}\hat{F}^\dagger_\sum(t) + e^{-i\hat{\varphi}(t)}\hat{F}_\sum(t)\right\}.$$

It remains to calculate $\left\langle (\Delta\varphi)^2\right\rangle$. This leads to a double integral

$$\begin{aligned} \left\langle (\Delta\varphi)^2\right\rangle = \frac{1}{4A^2}\int_0^t dt_1 \int_0^t dt_2 \left\langle \left(e^{i\hat{\varphi}(t_1)}\hat{F}^\dagger_\sum(t_1) + e^{-i\hat{\varphi}(t_1)}\hat{F}_\sum(t_1)\right)\right. \\ \left.\left(e^{i\hat{\varphi}(t_2)}\hat{F}^\dagger_\sum(t_2) + e^{-i\hat{\varphi}(t_2)}\hat{F}_\sum(t_2)\right)\right\rangle = \\ = \frac{1}{4}\frac{1}{\bar{n}}\int_0^t dt_1 \int_0^t dt_2 \left\langle \hat{F}^\dagger_\sum(t_1)\,\hat{F}_\sum(t_2)\,e^{-i(\hat{\varphi}(t_1)-\hat{\varphi}(t_2))} + \text{h.c.}\right\rangle. \end{aligned}$$

From here, using (4.57) we obtain

$$\left\langle (\Delta\varphi)^2 \right\rangle = \frac{1}{4}\frac{1}{\bar{n}} \left(\frac{g^2}{2\gamma} \left(\bar{N}_a + \bar{N}_b \right) + \frac{g^2}{\gamma^2} r_a + \frac{\omega}{Q} \left(\bar{n}_T + \frac{1}{2} \right) \right) t. \tag{4.65}$$

From equations (4.38) and (4.40) one can write the equation for $\bar{N}_a + \bar{N}_b$:

$$\frac{d\left(\bar{N}_a + \bar{N}_b\right)}{dt} = r_a - \gamma \left(\bar{N}_a + \bar{N}_b \right),$$

hence in the steady-state regime:

$$r_a = \gamma \left(\bar{N}_a + \bar{N}_b \right).$$

Substituting this relation into (4.65) we get:

$$\left\langle (\Delta\varphi)^2 \right\rangle = \frac{1}{4}\frac{1}{\bar{n}} \left(\frac{3g^2}{2\gamma} \left(\bar{N}_a + \bar{N}_b \right) + \frac{\omega}{Q} \left(\bar{n}_T + \frac{1}{2} \right) \right) t. \tag{4.66}$$

Consider the laser at threshold:

$$A = R_a \approx \frac{\omega}{Q},$$

substituting here the expression for R_a (2.70) in which we take $\tau \approx \frac{1}{\gamma}$:

$$r_a \frac{g^2}{\gamma^2} \approx \frac{\omega}{Q}.$$

Using this relation from (4.59) we have:

$$\bar{N}_z \approx \frac{r_a}{\gamma} \approx \frac{\omega}{Q}\frac{\gamma}{g^2},$$

hence

$$\frac{g^2}{\gamma} \approx \frac{\omega}{Q}\frac{1}{\bar{N}_z}.$$

Thus (4.66) can be rewritten as

$$\left\langle (\Delta\varphi)^2 \right\rangle = \frac{1}{4}\frac{1}{\bar{n}}\frac{\omega}{Q} \left(\frac{3}{2}\frac{\bar{N}_a + \bar{N}_b}{\bar{N}_z} + \bar{n}_T + \frac{1}{2} \right) t. \tag{4.67}$$

If now in (4.67) we assume complete inversion: $\bar{N}_b = 0$, and also neglect $\bar{n}_T$ compared to 1, then we obtain

$$\left\langle (\Delta\varphi)^2 \right\rangle \approx \frac{1}{2}\frac{1}{\bar{n}}\frac{\omega}{Q} t. \tag{4.68}$$

Substituting this into (4.62) we get

$$\left\langle \hat{a}^{\dagger}(t)\,\hat{a}(0)\right\rangle = |A|^2\, e^{-\frac{1}{4}\frac{1}{\bar{n}}\frac{\omega}{Q}t}, \tag{4.69}$$

which coincides with expression (3.27) previously obtained, if we accept the threshold generation mode:

$$A \approx \frac{\omega}{Q}.$$

Thus the expression for the natural linewidth will be the same:

$$D \approx \frac{1}{2}\frac{1}{\bar{n}}\frac{\omega}{Q}.$$

4.5 Exercises

1. Prove the commutation relations (4.13) and (4.23).
2. Derive the equations of motion for the operator $\hat{\sigma}_b^j$: (4.19) and (4.20).
3. Derive the relations (4.40) for the operators $\hat{N}_b$ and $\hat{N}_{ab}$.
4. Prove the second relation (4.30).
5. Prove (4.62).
6. Prove (4.65).

Chapter 5

Photon Optics (Quantum Phenomena in Optics)

We will consider those optical phenomena in which the quantum properties of light manifest to some degree.

Although a large number of optical phenomena can be analyzed from a classical perspective, many phenomena can only be fully understood and described within the framework of a fully quantum description.

The quantum consideration allows a more complete understanding of the essence of interference experiments and on this basis to understand the connection between classical and quantum descriptions. In addition, the quantum approach allows for the consideration of new types of experiments that study the statistics of photons in light beams and its connection with the spectral properties of light.

5.1 Photoeffect

The photoeffect is used for photon detection, in which, as is known, upon absorption of a photon a bound electron transitions to a free state and is registered accordingly. One of the devices used for these purposes is the photomultiplier, schematically shown in Figure 5.1. Its principle of operation is well known. Semiconductor avalanche photodiodes with an amplifier may also be used. Let us consider the relationship between the rate of photoelectron transitions and the light field. The electric field operator can be decomposed into frequency-positive and frequency-negative parts (1.88):

$$\hat{\vec{E}}\left(\vec{r},t\right)=\hat{\vec{E}}^{(+)}\left(\vec{r},t\right)+\hat{\vec{E}}^{(-)}\left(\vec{r},t\right), \tag{5.1}$$

where the frequency-positive part includes annihilation operators, and the frequency-negative part includes creation operators.

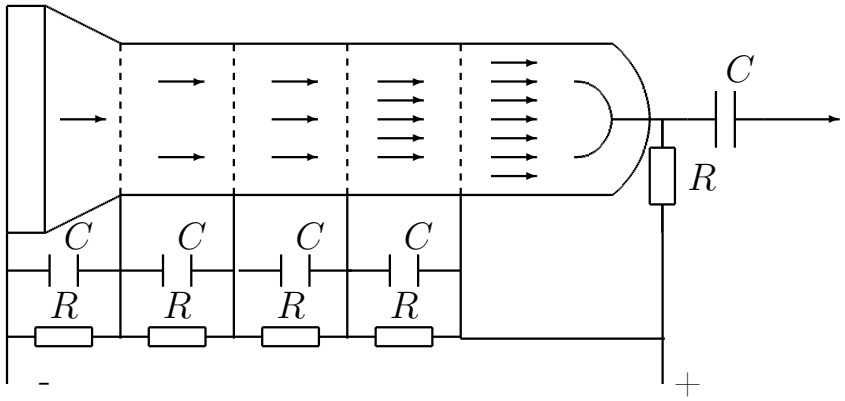


Figure 5.1: Photomultiplier

When decomposing the field into plane waves, we have:

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{a}_k \vec{e}_k e^{-i\omega_k t + i(\vec{k}\vec{r})},$$
$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{a}_k^\dagger \vec{e}_k^* e^{i\omega_k t - i(\vec{k}\vec{r})}. \tag{5.2}$$

We consider the electric dipole interaction approximation, that is, we assume that the size of the atom (or other electronic system) interacting with the light is much smaller than the wavelength. The interaction Hamiltonian in this case has the form:

$$\hat{\mathcal{H}}_{ED} = -\left(\hat{\vec{p}}\hat{\vec{E}}\right), \tag{5.3}$$

where $\hat{\vec{p}}$ is the dipole moment operator of the system. Furthermore, we assume that only processes accompanied by photon annihilation are considered. Thus, in (5.3), one must retain only the annihilation operators $\hat{a}$, i.e., replace $\hat{\vec{E}}$ by $\hat{\vec{E}}^{(+)}$:

$$\hat{\mathcal{H}}_{ED} = -\left(\hat{\vec{p}}\hat{\vec{E}}^{(+)}\right). \tag{5.4}$$

Assume that initially the atom is in the ground state, and the multimode field contains $\{n_k\}$ photons, so that the initial state vector is

$$|i\rangle = |\{n_k\}\rangle\,|b\rangle \tag{5.5}$$

After the interaction, the atom-field system will be in state $|f\rangle$ (we will not detail it). The probability of transition per unit time in first-order perturbation theory according to Fermi's golden rule (Theorem 14.9.1) is given by the square

modulus of the matrix element of the transition:

$$\left|\langle f|\,\hat{\mathcal{H}}_{ED}\,|i\rangle\right|^2 = \langle f|\left(\hat{\vec{p}}\hat{\vec{E}}^{(+)}\right)|i\rangle\,\langle f|\left(\hat{\vec{p}}\hat{\vec{E}}^{(+)}\right)|i\rangle^* =$$
$$= \langle f|\left(\hat{\vec{p}}\hat{\vec{E}}^{(+)}\right)|i\rangle\,\langle i|\left(\hat{\vec{p}}\hat{\vec{E}}^{(-)}\right)|f\rangle = \langle i|\left(\hat{\vec{p}}\hat{\vec{E}}^{(-)}\right)|f\rangle\,\langle f|\left(\hat{\vec{p}}\hat{\vec{E}}^{(+)}\right)|i\rangle \qquad (5.6)$$

The final state is unknown. It can be any state, and it is necessary to sum (5.6) over all final states (sum the probabilities). We consider the system of final states complete, i.e., $\sum_{(f)}|f\rangle\langle f| = \hat{I}$. From this we have:

$$\sum_{(f)}\langle i|\left(\hat{\vec{p}}\hat{\vec{E}}^{(-)}\right)|f\rangle\,\langle f|\left(\hat{\vec{p}}\hat{\vec{E}}^{(+)}\right)|i\rangle = \langle i|\,\hat{p}_E^2\hat{E}^{(-)}\hat{E}^{(+)}\,|i\rangle =$$
$$= \langle\{n_k\}|\,\hat{E}^{(-)}\hat{E}^{(+)}\,|\{n_k\}\rangle\,\langle b|\,\hat{p}_E^2\,|b\rangle = \alpha\,\langle\{n_k\}|\,\hat{E}^{(-)}\hat{E}^{(+)}\,|\{n_k\}\rangle\,, \qquad (5.7)$$

where $\hat{p}_E$ is the projection of the dipole moment onto the direction of the field, and α is some interaction constant that does not depend on the field magnitude. Now let us generalize (5.7) to the case of a mixed initial state. Suppose we know the probability $P_{\{n_k\}}$ of the state $|\{n_k\}\rangle$. The density matrix of this state can be represented as

$$\hat{\rho} = \sum_{\{n_k\}} P_{\{n_k\}}\,|\{n_k\}\rangle\,\langle\{n_k\}|\,. \qquad (5.8)$$

To obtain the result in the mixed state case, it is necessary to average (5.7) using the probabilities $P_{\{n_k\}}$. For the photo-count rate, we have:

$$W = \alpha\sum_{\{n_k\}} P_{\{n_k\}}\,\langle\{n_k\}|\,\hat{E}^{(-)}\hat{E}^{(+)}\,|\{n_k\}\rangle =$$
$$= \alpha Sp\left(\hat{\rho}\hat{E}^{(-)}\hat{E}^{(+)}\right) \qquad (5.9)$$

- in an arbitrary representation, since Sp does not depend on the representation.

Thus, in the general case of a statistically mixed state, the average photon count rate is proportional to the expectation value of the operators $\hat{E}^{(-)}\hat{E}^{(+)}$ in the initial field state, where $\hat{\rho}$ is the statistical operator of the initial field state.

We considered an atom or a similar microsystem as a photodetector. A real photodetector contains many atoms. If we assume that the detector contains N non-interacting atoms, and its size is small enough so that all atoms can be regarded as being in identical conditions with good accuracy, then the probability (5.9) can simply be increased by a factor of N.

Let us determine the meaning of the operator $\hat{E}^{(-)}\hat{E}^{(+)}$. Introduce the operator

$$\hat{\vec{J}} = 2\varepsilon_0\left(\hat{E}^{(-)}\hat{E}^{(+)}\right)c\vec{k}_0, \qquad (5.10)$$

where $\vec{k}_0$ is the unit vector in the propagation direction of the wave; the vector operator $\hat{\vec{J}}$ can be considered as the operator of energy flux in the direction $\vec{k}_0$. For example, for a single-mode field, this operator equals

$$\hat{\vec{J}} = \frac{c\hbar\omega_k}{V}\vec{k}_0\hat{n}_k, \tag{5.11}$$

where $\vec{k}_0$ is the unit vector in the direction $\vec{k}$. The mean value of the operator $\hat{\vec{J}}$ in a state with a definite photon number is

$$\left\langle \hat{\vec{J}} \right\rangle = \frac{c\hbar\omega_k}{V} n_k \vec{k}_0,$$

In a single-mode mixed state, we have

$$\left\langle \hat{\vec{J}} \right\rangle = \frac{c\hbar\omega_k}{V}\bar{n}\vec{k}_0, \text{ where } \bar{n} = \sum_{(n_k)} P_{n_k} n_k.$$

From the examples given, it is clear that $\hat{\vec{J}}$ indeed has the meaning of the energy flux operator. In the single-mode case, the intensity does not depend on time. In the multimode case, the intensity can vary in time and space. From all the above, it can be concluded that the photoemission rate is proportional to the average photon flux, i.e., the light intensity. In a semiclassical treatment, this corresponds to the average energy flux over a period.

5.2 Coherent Properties of Light

The coherent properties of light determine the interference phenomena in optics, which are well described by classical wave theory. In classical wave optics, the coherent properties of light are described by the corresponding coherence functions [59, 49, 60]. Let us consider the phenomenon of interference from a quantum point of view. To this end, consider Young's interference experiment (see Figure 5.2). On the screen (photographic plate), the arrival of a photon is recorded as a dark spot. To obtain an interference pattern similar to the classical one, a sufficient amount of time is needed. Then individual points merge into fringes and the pattern will not differ from the classical one (Figure 5.3).

Field operators on the screen can be represented as a superposition of fields passing through holes 1 and 2:

$$\hat{E}^{(+)}(\vec{r}, t) = \gamma\left[\hat{E}^{(+)}(\vec{r}_1, t_1) + \hat{E}^{(+)}(\vec{r}_2, t_2)\right], \tag{5.12}$$

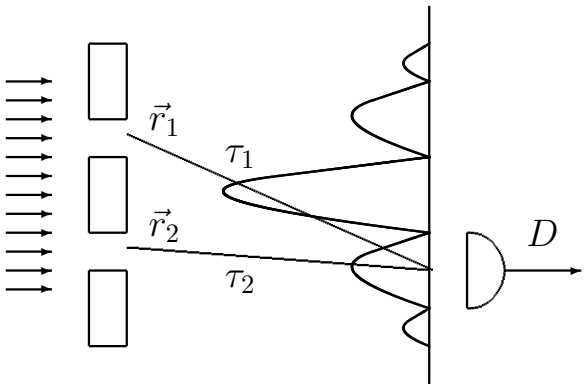


Figure 5.2: Young's interference experiment. $\vec{r}_{1,2}$ - coordinates of the two apertures, $\tau_{1,2}$ - delays in the propagation of light from the apertures to the screen, D - photodetector.

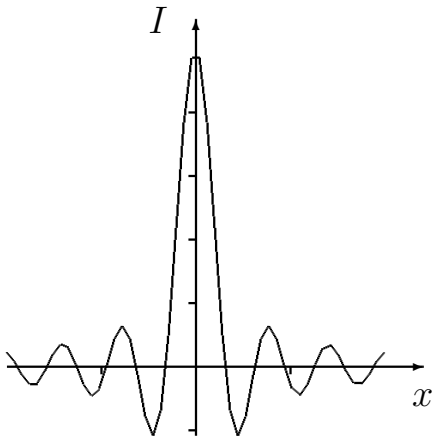


Figure 5.3: Classical interference pattern. Dependence of light intensity I on the coordinate on the screen x.

where $\vec{r}_1$, $\vec{r}_2$ are the coordinates of the holes; $t_1 = t - \tau_1$, $t_2 = t - \tau_2$; τ_1, τ_2 are the delays due to propagation of light from the holes to the screen; the coefficient γ characterizes the attenuation of the field during propagation from the holes to the screen. Using photodetector D, we examine the intensity of light at different points on the screen.

The photon counting rate will be, as we know, proportional to

$$W = \alpha Sp\left(\hat{\rho}\hat{E}^{(-)}\left(\vec{r},t\right)\hat{E}^{(+)}\left(\vec{r},t\right)\right) =$$
$$\alpha\left|\gamma\right|^2 Sp\left[\hat{\rho}\left(\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)+\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\right)\left(\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)+\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right)\right].$$

Multiplying out term by term, we obtain:

$$W = gSp\left[\hat{\rho}\left(\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)+\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)+\right.\right.$$
$$\left.\left.+\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)+\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\right)\right], \quad (5.13)$$

where the following notation is introduced: $g = \alpha\left|\gamma\right|^2$.

The first two terms give the intensity of the fields passing through the first and second holes when the other hole is closed. The last two terms describe interference. Since

$$\left(\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\right)^{\dagger} = \left(\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right)$$

then $Sp\left(\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\right)$ and $Sp\left(\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right)$ will be complex conjugates. The term

$$Sp\left[\hat{\rho}\left(\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)+\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\right)\right]$$

gives the oscillating interference term (Figure 5.3). The envelope of the interference term is proportional to the expression

$$\left|Sp\left[\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right]\right| \quad (5.14)$$

From this follows the definition of the first-order coherence function

$$G\left(\vec{r}_1,t_1,\vec{r}_2,t_2\right) = G_{12}^1 =$$
$$= \frac{\left|Sp\left[\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right]\right|}{\sqrt{Sp\left[\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\right]Sp\left[\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right]}} \quad (5.15)$$

This expression resembles the classical definition of the coherence function, but here instead of the analytic signal we have field operators and quantum averaging is performed using the density matrix.

Let us consider some particular cases as examples. We start with a single-mode field in the states $|n\rangle$ or $|\alpha\rangle$. In this case, we have:

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{a}_k\vec{e}_k e^{-i\omega_k t+i\left(\vec{k}\vec{r}\right)},$$
$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{a}_k^\dagger\vec{e}_k^* e^{i\omega_k t-i\left(\vec{k}\vec{r}\right)} = \left(\hat{\vec{E}}^{(+)}(\vec{r},t)\right)^\dagger. \tag{5.16}$$

Using these expressions to calculate the coherence function (5.15), we get $G_{12}^1 = 1$ since $\langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle = n$ and $\left|e^{-ix}\right| = 1$. The same result is obtained for the coherent state case: $G_{12}^1 = 1$, as $\langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle = |\alpha|^2$.

In general, a single-mode field excited in an arbitrary pure state possesses full first-order coherence. Moreover, a single-mode field excited in an arbitrary statistical mixed state also exhibits full coherence. In this case, one needs to compute $Sp\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right)$, where $\hat{\rho} = \sum_{(m)}\sum_{(l)}\rho_{ml}\,|m\rangle\langle n|$. We have

$$Sp\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right) = \sum_{(n)}\sum_{(m)}\sum_{(l)}\rho_{ml}\langle n|m\rangle\langle l|\,\hat{a}^\dagger\hat{a}\,|n\rangle =$$
$$= \sum_{(n)}\sum_{(l)}\rho_{nl}\langle l|n\rangle\, n = \sum_{(n)}\rho_{nn}n = \bar{n}. \tag{5.17}$$

A similar expression appears in the denominator, therefore, $G_{12}^{(1)} = 1$.

More commonly, one deals with multimode states of the field, so let us consider the coherence of light in this case. Consider two limiting cases: light in a coherent state and fully chaotic light.

In the first case, the state vector is

$$|\{\alpha_k\}\rangle = |\{\alpha_{k_1}\}\rangle\,|\{\alpha_{k_2}\}\rangle\ldots|\{\alpha_{k_s}\}\rangle\ldots$$

Using the properties of coherent states,

$$\hat{E}^{(+)}(\vec{r},t)\,|\{\alpha_k\}\rangle = E(\vec{r},t)\,|\{\alpha_k\}\rangle$$

and

$$\langle\{\alpha_k\}|\,\hat{E}^{(-)}(\vec{r},t) = E^*(\vec{r},t)\,\langle\{\alpha_k\}|\,,$$

we get:

$$\langle\{\alpha_k\}|\,\hat{E}^{(-)}(\vec{r}_1,t_1)\,\hat{E}^{(+)}(\vec{r}_2,t_2)\,|\{\alpha_k\}\rangle = E^*(\vec{r}_1,t_1)\,E(\vec{r}_2,t_2)$$

where E is the eigenvalue of the operator $\hat{E}^{(+)}$, being the analytic signal of the classical field. From this, we have

$$G_{12}^{(1)} = \frac{\left|E^*\left(\vec{r}_1, t_1\right) E\left(\vec{r}_2, t_2\right)\right|}{\sqrt{E^*\left(\vec{r}_1, t_1\right) E\left(\vec{r}_1, t_1\right) E^*\left(\vec{r}_2, t_2\right) E\left(\vec{r}_2, t_2\right)}} = 1 \tag{5.18}$$

that is, a multimode field in a coherent state is fully coherent.

In the case of an arbitrary multimode field state, we have only partial coherence. Consider the most common case of chaotic light (radiation from a heated body, gas discharge, etc.), where light is emitted by many independent sources (atoms, ions, molecules). The expression for the density matrix in this case has the form (1.114):

$$\hat{\rho} = \sum_{\{n_k\}} |\{n_k\}\rangle \langle\{n_k\}| \prod_{\{n_k\}} \frac{\bar{n}_k^{n_k}}{\left(1+\bar{n}_k\right)^{n_k+1}} = \\ = \sum_{\{n_k\}} \rho_{\{n_k\},\{n_k\}} |\{n_k\}\rangle \langle\{n_k\}| \tag{5.19}$$

where

$$\rho_{\{n_k\},\{n_k\}} = P_{\{n_k\}} = \prod_k \frac{\bar{n}_k^{n_k}}{\left(1+\bar{n}_k\right)^{n_k+1}} = \prod_k P_{n_k}, \\ \sum_{\{n_k\}} \cdots = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_k} \cdots .$$

The electric field operators are

$$\hat{\vec{E}}^{(+)} = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \vec{e}_k \hat{a}_k e^{-i\omega_k t + i\left(\vec{k}\vec{r}\right)}, \\ \hat{\vec{E}}^{(-)} = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \vec{e}_k^* \hat{a}_k^\dagger e^{i\omega_k t - i\left(\vec{k}\vec{r}\right)}, \tag{5.20}$$

from which it follows:

$$Sp\left(\hat{\rho}\hat{E}^{(-)}\left(x_1\right)\hat{E}^{(+)}\left(x_2\right)\right) = \\ = \sum_{\{n_{k'}\}} \sum_{\{n_k\}} \langle\{n_{k'}\}| \{n_k\}\rangle P_{\{n_k\}} \langle\{n_k\}| \hat{E}^{(-)}\left(x_1\right)\hat{E}^{(+)}\left(x_2\right)|\{n_{k'}\}\rangle = \\ = \sum_{\{n_k\}} \langle\{n_k\}| \hat{E}^{(-)}\left(x_1\right)\hat{E}^{(+)}\left(x_2\right)|\{n_k\}\rangle P_{\{n_k\}}. \tag{5.21}$$

The product of operators equals

$$\hat{E}^{(-)}(x_1)\,\hat{E}^{(+)}(x_2) = \sum_{(k)}\sum_{(k')} \frac{\hbar\sqrt{\omega_k\omega_{k'}}}{2\varepsilon_0 V}\,(\vec{e}_k\vec{e}_{k'})\,\hat{a}_k^\dagger \hat{a}_{k'} e^{-ix_2+ix_1}, \tag{5.22}$$

where $x_1 = \omega_k t_1 - \left(\vec{k}\vec{r}_1\right)$, $x_2 = \omega_k t_2 - \left(\vec{k}\vec{r}_2\right)$. The operator $\hat{a}_k^\dagger \hat{a}_{k'}$ is averaged. Since

$$|\{n_k\}\rangle = |n_{k_1}\rangle\,|n_{k_2}\rangle\ldots,$$

we have

$$\langle\{n_k\}|\,\hat{a}_k^\dagger \hat{a}_{k'}\,|\{n_k\}\rangle = n_k\delta_{kk'},$$

and, therefore,

$$Sp\left(\hat{\rho}\hat{E}^{(-)}(x_1)\,\hat{E}^{(+)}(x_2)\right) =$$

$$= \sum_k\sum_{\{n_k\}} \frac{\hbar\omega_k}{2\varepsilon_0 V} n_k e^{-i(x_2-x_1)} P_{\{n_k\}} =$$

$$= \sum_k \frac{\hbar\omega_k}{2\varepsilon_0 V}\sum_{\{n_k\}} n_k \prod_k P_{n_k} = \sum_k \frac{\hbar\omega_k}{2\varepsilon_0 V}\bar{n}_k e^{-i(x_2-x_1)},$$

since

$$\sum_{\{n_k\}} n_k \prod_k P_{n_k} = \sum_{n_1} P_{n_1}\sum_{n_2} P_{n_2}\cdots\sum_{n_k} n_k P_{n_k}\cdots =$$

$$= \sum_{n_k} n_k P_{n_k} = \bar{n}_k,$$

because

$$\sum_{n_k} P_{n_k} = 1.$$

From this we get

$$G_{12}^{(1)} = \frac{\left|\sum_k \frac{\hbar\omega_k}{2\varepsilon_0 V}\bar{n}_k e^{-i\omega_k(t_2-t_1)+i\left(\vec{k},\vec{r}_2-\vec{r}_1\right)}\right|}{\sum_k \frac{\hbar\omega_k}{2\varepsilon_0 V}\bar{n}_k} \tag{5.23}$$

The summation over k can be replaced by integration over frequency. Using the expression for the density of states, we get:

$$\sum_k (\ldots) = \frac{L^3}{(2\pi)^3}\int\int\int (\ldots)\,dk_x dk_y dk_z =$$

$$= \frac{V}{(2\pi)^3}\int_\Omega d\Omega \int \omega^2 (\ldots)\,d\omega.$$

This leads us to the expression

$$G_{12}^{(1)} = \frac{\left| \int d\Omega \int \omega^3 \bar{n}\left(\omega, \Omega\right) e^{-i\omega(t_2-t_1)+i\left(\vec{k},\vec{r}_2-\vec{r}_1\right)} d\omega \right|}{\int d\Omega \int \omega^3 \bar{n}\left(\omega, \Omega\right) d\omega}. \tag{5.24}$$

If the line is narrow compared to the carrier frequency, that is $\bar{n}\left(\omega, \Omega\right)$ has a narrow peak near the frequency $\sim \omega_0$, the function ω^3 can be taken out of the integral and canceled; moreover, if the light beam is narrow, that is, the integration domain over Ω is small, the expression (5.24) can be simplified to

$$G_{12}^{(1)} = \frac{\left| \int \bar{N}\left(\omega\right) e^{-i\omega(t_2-t_1)+i\left(\vec{k},\vec{r}_2-\vec{r}_1\right)} d\omega \right|}{\int \bar{N}\left(\omega\right) d\omega}, \tag{5.25}$$

where $\bar{N}\left(\omega\right) = \int_{\Delta\Omega} \bar{n}\left(\omega, \Omega\right) d\Omega$. For example, when the spectral line is Lorentzian, we have:

$$\bar{N}\left(\omega\right) = \bar{N}_0 \frac{\gamma}{\left(\omega_0 - \omega\right)^2 + \gamma^2}.$$

The problem considered can be solved using the representation of coherent states. Then

$$\hat{\rho} = \int \cdots \int P\left(\{\alpha_k\}\right) \left|\{\alpha_k\}\right\rangle \left\langle\{\alpha_k\}\right| d^2\left\{\alpha_k\right\},$$

where

$$P\left(\{\alpha_k\}\right) = \prod_k \frac{1}{\pi \bar{n}_k} e^{-\frac{\left|\alpha_k\right|^2}{\bar{n}_k}} = \prod_k P_k\left(\alpha_k\right).$$

We have

$$Sp\left(\hat{\rho}\hat{E}^{(-)}\left(x_1\right)\hat{E}^{(+)}\left(x_2\right)\right) =$$

$$= \int \cdots \int P\left(\{\alpha_k\}\right) \left\langle\{\alpha_k\}\right| \hat{E}^{(-)}\left(x_1\right)\hat{E}^{(+)}\left(x_2\right) \left|\{\alpha_k\}\right\rangle d^2\left\{\alpha_k\right\}. \tag{5.26}$$

The product of operators equals

$$\hat{E}^{(-)}\left(x_1\right)\hat{E}^{(+)}\left(x_2\right) = \sum_k \sum_{k'} \frac{\hbar\sqrt{\omega_k \omega_{k'}}}{2\varepsilon_0 V} \left(\vec{e}_k \vec{e}_{k'}\right) \hat{a}_k^{\dagger} \hat{a}_{k'} e^{-ix_2+ix_1}.$$

The product of operators is written in normal order (annihilation operators to the right of creation operators), so the matrix element appearing in (5.26) can be

easily written, replacing $\hat{a}_{k'} \to \alpha_{k'}$, $\hat{a}_k^\dagger \to \alpha_k^*$. From this, we have

$$Sp\left(\hat{\rho}\hat{E}^{(-)}(x_1)\,\hat{E}^{(+)}(x_2)\right) =$$
$$= \sum_{k\neq k'} \frac{\hbar\sqrt{\omega_k\omega_{k'}}}{2\varepsilon_0 V}\left(\vec{e}_k\vec{e}_{k'}\right) e^{-ix_2+ix_1}\cdot$$
$$\cdot \int P(\alpha_k)\,\alpha_k^* d^2\alpha_k \int P(\alpha_{k'})\,\alpha_{k'} d^2\alpha_{k'} +$$
$$+ \sum_k \frac{\hbar\omega_k}{2\varepsilon_0 V} e^{-ix_2+ix_1} \int |\alpha_k|^2\, d^2\alpha_k. \tag{5.27}$$

In the derivation of (5.27), the following relation was used:

$$\int P(\alpha_k)\, d^2\alpha_k = 1.$$

Further, the following relations are used:

$$\int P(\alpha_k)\,\alpha_k d^2\alpha_k = 0,$$
$$\int |\alpha_k|^2\, d^2\alpha_k = \bar{n}_k,$$

which can be easily verified by switching to polar coordinates:

$$\alpha = |\alpha|\, e^{i\theta} = re^{i\theta}.$$

As a result, for (5.27) we obtain

$$Sp\left(\hat{\rho}\hat{E}^{(-)}(x_1)\,\hat{E}^{(+)}(x_2)\right) = \sum_k \frac{\hbar\omega_k}{2\varepsilon_0 V} e^{-ix_2+ix_1}\bar{n}_k.$$

Using these results, which coincide with those obtained previously using the number state representation n, we get the same final expressions (5.24) and (5.25).

5.3 Second-Order Coherence

Second-order coherence can be introduced based on the analysis of various experiments where the rate of simultaneous photon registration by two detectors is measured. The analysis leads to the following definition of the second-order

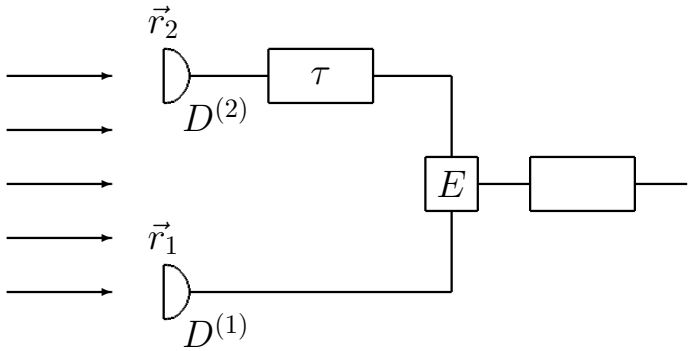


Figure 5.4: Scheme for measuring second-order coherence. Two photodetectors $D^{(1)}$ and $D^{(2)}$ are located at points in space defined by vectors $\vec{r}_1$ and $\vec{r}_2$. The signal from $D^{(2)}$ passes through a delay τ, so the coincidence circuit E registers the simultaneous absorption of two photons, one at point $\vec{r}_1$ at time $t_1 = t$, and the other at point $\vec{r}_2$ at time $t_2 = t - \tau$.

degree of coherence:

$$G^{(2)}\left(\vec{r}_1, t_1, \vec{r}_2, t_2\right) = G_{12}^{(2)} = \\ = \frac{\left\langle \hat{E}^{(-)}\left(\vec{r}_2, t_2\right) \hat{E}^{(-)}\left(\vec{r}_1, t_1\right) \hat{E}^{(+)}\left(\vec{r}_1, t_1\right) \hat{E}^{(+)}\left(\vec{r}_2, t_2\right) \right\rangle}{\left\langle \hat{E}^{(-)}\left(\vec{r}_1, t_1\right) \hat{E}^{(+)}\left(\vec{r}_1, t_1\right) \right\rangle \left\langle \hat{E}^{(-)}\left(\vec{r}_2, t_2\right) \hat{E}^{(+)}\left(\vec{r}_2, t_2\right) \right\rangle}. \tag{5.28}$$

Angle brackets denote quantum mechanical averaging over the ensemble using the statistical operator (density matrix)

$$\langle (\dots) \rangle = Sp\left\{\hat{\rho}\,(\dots)\right\}$$

Expression (5.28) resembles the classical expression that defines second-order coherence, but instead of classical fields, operators are used, and averaging is carried out using the density matrix.

Formula (5.28) can be justified as follows. Consider the operator $\hat{E}^{(+)}\left(\vec{r}_1, t_1\right) \hat{E}^{(+)}\left(\vec{r}_2, t_2\right)$. It corresponds to the absorption of one photon at point $\vec{r}_1$ and time $t_1 = t$, and a second photon at point $\vec{r}_2$ at time $t_2 = t - \tau$. This procedure can be realized using the setup shown in Figure 5.4. Points $\vec{r}_1$ and $\vec{r}_2$ are defined by the positions of the photodetectors. Times $t_1 = t$ and $t_2 = t - \tau$ are determined by coincident photo counts. τ is an adjustable delay. Signals from both detectors are fed into a coincidence circuit.

Applying the procedure used for the photoelectric effect to the operator $\hat{E}^{(+)}\left(\vec{r}_1, t_1\right) \hat{E}^{(+)}$ the probability of detecting the first photon at time $t_1 = t$ and the second at time $t_2 = t - \tau$ within the interval Δt is:

$$w\left(\vec{r}_1, \vec{r}_2, t_1, t_2\right) \Delta t = \alpha \left| \langle f | \, \hat{E}^{(+)}\left(\vec{r}_1, t_1\right) \hat{E}^{(+)}\left(\vec{r}_2, t_2\right) |i\rangle \right|^2 \Delta t,$$

where $|i\rangle$ is the initial state of the system, $|f\rangle$ is the final state of the system, α is a quantity depending on the properties of the photodetectors. Then, the counting rate (number of counts per unit time) after summation over final states is given by:

$$w\left(\vec{r}_1,\vec{r}_2,t_1,t_2\right)=\alpha\left\langle i\right|\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\left|i\right\rangle .$$

Here, only the field part is of interest; the atomic part is included in the coefficient α.

If the initial field is in a statistically mixed state, averaging must be done using the statistical operator of the initial field. Then we have:

$$w\left(\vec{r}_1,\vec{r}_2,t_1,t_2\right)=\alpha Sp\left\{\hat{\rho}\hat{E}^{(-)}\left(\vec{r}_2,t_2\right)\hat{E}^{(-)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_1,t_1\right)\hat{E}^{(+)}\left(\vec{r}_2,t_2\right)\right\}.$$

Using this expression for the normalized degree of coherence, we obtain formula (5.28).

Let us provide some examples of calculating the second-order degree of coherence. We start with a simple one. Find the second-order coherence for a single-mode state with a definite number of photons. For this, we need to consider the matrix element

$$\left\langle n\right|\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\left|n\right\rangle=n\left(n-1\right).$$

This expression will be in the numerator. In the denominator, we have

$$\left(\left\langle n\right|\hat{a}^{\dagger}\hat{a}\left|n\right\rangle\right)^2=n^2.$$

The numerical coefficients in numerator and denominator cancel out. We get:

$$\begin{aligned} G_{12}^{(2)}=\frac{n\left(n-1\right)}{n^2}=\frac{n-1}{n}\ \text{for}\ n>2,\\ G_{12}^{(2)}=0\ \text{for}\ n=1,\\ G_{12}^{(2)}\ \text{is undefined for}\ n=0. \end{aligned} \tag{5.29}$$

The obvious fact in (5.29) is worth noting: if there is one photon, the probability of detecting two photons equals zero, which leads to $G_{12}^{(2)}=0$ at $n=1$.

Consider another simple case: a single-mode field in a coherent state. We have

$$\begin{aligned} \left\langle\alpha\right|\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\left|\alpha\right\rangle=\left(\alpha^{*}\alpha\right)^2\ \text{- in numerator},\\ \left(\left\langle\alpha\right|\hat{a}^{\dagger}\hat{a}\left|\alpha\right\rangle\right)^2=\left(\alpha^{*}\alpha\right)^2\ \text{- in denominator}. \end{aligned}$$

Altogether this gives

$$G_{12}^{(2)}=1.$$

So far, we have considered the coherence of pure states. Let us proceed to the consideration of mixed states. Consider single-mode chaotic light, whose density matrix is

$$\hat{\rho} = \sum_n \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} |n\rangle \langle n| = \sum_n \rho_{nn} |n\rangle \langle n| .$$

From this, we obtain

$$Sp\left\{\hat{\rho}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\right\} = \sum_n \sum_m \langle n|m\rangle \langle m| \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} |m\rangle \rho_{mm} = \\ = \sum_n \rho_{nn} (n-1) n = \bar{n^2} - \bar{n}. \tag{5.30}$$

For chaotic light, the relation holds

$$\bar{n^2} = 2(\bar{n})^2 + \bar{n}. \tag{5.31}$$

Indeed,

$$\bar{n^2} = \sum_n n^2 \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} = \frac{\bar{n}}{(\bar{n}+1)} \sum_n n^2 \frac{\bar{n}^{n-1}}{(\bar{n}+1)^n} = \\ = \frac{\bar{n}}{(\bar{n}+1)} \sum_{m=n-1} (m+1)^2 \frac{\bar{n}^m}{(\bar{n}+1)^{m+1}} = \\ = \frac{\bar{n}}{(\bar{n}+1)} \sum_m \left(m^2+2m+1\right) \frac{\bar{n}^m}{(\bar{n}+1)^{m+1}} = \\ = \frac{\bar{n}}{(\bar{n}+1)} \left(\bar{n^2} + 2\bar{n} + 1\right),$$

thus we get

$$\bar{n^2}(\bar{n}+1) = \bar{n}\left(\bar{n^2} + 2\bar{n} + 1\right)$$

from which follows the desired expression (5.31):

$$\bar{n^2} = 2(\bar{n})^2 + \bar{n}.$$

Thus, from (5.30) and (5.31), we have

$$Sp\{\dots\} = 2(\bar{n})^2$$

In the denominator, we have $(\bar{n})^2$. Hence, for chaotic light,

$$G_{12}^{(2)} = 2, \tag{5.32}$$

which means photon pairs are registered more frequently than observed for more ordered light. Physically, this is related to fluctuations of chaotic light. Because of this, it is sometimes said that photons tend to bunch. Generally, for an arbitrary single-mode field, the second-order coherence is

$$G^{(2)} = \frac{\bar{n^2} - \bar{n}}{(\bar{n})^2}. \tag{5.33}$$

We will consider this issue in more detail in the third part of this book (see ch. 8 Non-classical light).

In practice, multimode fields are more common. Define the second-order degree of coherence in two limiting cases: multimode coherent state and multimode chaotic light. The first case is simple to consider. The state vector can be represented as

$$|\{\alpha_k\}\rangle = |\{\alpha_{k_1}\}\rangle\,|\{\alpha_{k_2}\}\rangle \ldots |\{\alpha_{k_s}\}\rangle \ldots . \tag{5.34}$$

Using the equalities

$$\hat{E}^{(+)}(x)\,|\{\alpha_k\}\rangle = E(x)\,|\{\alpha_k\}\rangle$$

and

$$\langle\{\alpha_k\}|\,\hat{E}^{(-)}(x) = \langle\{\alpha_k\}|\,E^*(x),$$

where $x = (t, r)$, $E(x)$ is the analytic signal (positive-frequency part) of the classical field obtained from the operator $\hat{E}^{(+)}$ by replacing $\hat{a}_k \to \alpha_k$. From this, we get:

$$\langle\{\alpha_k\}|\,\hat{E}^{(-)}(x_2)\,\hat{E}^{(-)}(x_1)\,\hat{E}^{(+)}(x_1)\,\hat{E}^{(+)}(x_2)\,|\{\alpha_k\}\rangle = \\ = E^*(x_2)\,E^*(x_1)\,E(x_1)\,E(x_2).$$

Similarly, we see that the denominator has the quantity $E^*(x_1)\,E(x_1)\,E^*(x_2)\,E(x_2)$. Therefore, in this case,

$$G^{(2)}_{12} = 1, \tag{5.35}$$

The case of a chaotic multimode light field is more difficult. The statistical operator in this case will have the form (1.114)

$$\hat{\rho} = \sum_{\{n_k\}} P_{\{n_k\}}\,|\{n_k\}\rangle\,\langle\{n_k\}| = \sum_{\{n_k\}} |\{n_k\}\rangle\,\langle\{n_k\}|\prod_{\{n_k\}}\frac{\bar{n}_k^{n_k}}{(1+\bar{n}_k)^{n_k+1}} = \\ = \sum_{\{n_k\}} |\{n_k\}\rangle\,\langle\{n_k\}|\prod_{\{n_k\}} P_{\{n_k\}}. \tag{5.36}$$

When multiplying the electric field operators, we obtain a fourfold sum, since $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ are expanded in plane waves:

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \vec{e}_k \hat{a}_k e^{-i\omega_k t + i\left(\vec{k}\vec{r}\right)},$$
$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sum_{(k)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \vec{e}_k^{\,*} \hat{a}_k^{\dagger} e^{i\omega_k t - i\left(\vec{k}\vec{r}\right)}. \tag{5.37}$$

General term in the product of sums (5.37) will contain operator products of the form

$$\hat{a}_{k^I}^{\dagger} \hat{a}_{k^{II}}^{\dagger} \hat{a}_{k^{III}} \hat{a}_{k^{IV}}. \tag{5.38}$$

Averaging this term using the statistical operator (5.36) leads to

$$Sp\left(\hat{\rho}\hat{a}_{k^I}^{\dagger}\hat{a}_{k^{II}}^{\dagger}\hat{a}_{k^{III}}\hat{a}_{k^{IV}}\right) = \sum_{\{n_k\}} P_{\{n_k\}} \left\langle \{n_k\} \right| \hat{a}_{k^I}^{\dagger}\hat{a}_{k^{II}}^{\dagger}\hat{a}_{k^{III}}\hat{a}_{k^{IV}} \left| \{n_k\} \right\rangle.$$

It is easy to see that terms in which all modes are different:

$$k^I \neq k^{II} \neq k^{III} \neq k^{IV},$$

will be zero. Non-zero terms satisfy the conditions:

$$k^I = k^{III} = k_1, \quad k^{II} = k^{IV} = k_2$$

or

$$k^I = k^{IV} = k_1, \quad k^{II} = k^{III} = k_2$$

or

$$k^I = k^{II} = k^{III} = k^{IV} = k.$$

We have: in the first case -

$$\sum_{n_{k_1}} \sum_{n_{k_2}} P_{n_{k_1}} P_{n_{k_2}} n_{k_1} n_{k_2} = \bar{n}_{k_1} \bar{n}_{k_2}, \tag{5.39}$$

in the second -

$$\sum_{n_{k_1}} \sum_{n_{k_2}} P_{n_{k_1}} P_{n_{k_2}} n_{k_1} n_{k_2} = \bar{n}_{k_1} \bar{n}_{k_2}. \tag{5.40}$$

The last case was considered earlier (5.33), which gives

$$\sum_{n_k} P_{n_k} \left\langle n_k \right| \hat{a}_k^{\dagger}\hat{a}_k^{\dagger}\hat{a}_k\hat{a}_k \left| n_k \right\rangle = 2\left(\bar{n}_k\right)^2. \tag{5.41}$$

All the above allows writing the coherence function as

$$G_{12}^{(2)} = \frac{\sum_{k_1}\sum_{k_2\neq k_1}\bar{n}_{k_1}\bar{n}_{k_2}\omega_{k_1}\omega_{k_2}e^{i\omega_{k_1}\tau}e^{-i\omega_{k_2}\tau}}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} + \\ + \frac{2\sum_k \bar{n}_k^2\omega_k^2}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} + \\ + \frac{\sum_{k_1}\sum_{k_2\neq k_1}\bar{n}_{k_1}\bar{n}_{k_2}\omega_{k_1}\omega_{k_2}}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} \tag{5.42}$$

where

$$\omega\tau = \omega\left(t_2 - t_1\right) - \left(\vec{k}, \vec{r}_2 - \vec{r}_1\right) = \omega\left[\left(t_2 - t_1\right) - \frac{1}{c}\left(\vec{k}_0, \vec{r}_2 - \vec{r}_1\right)\right],$$

if we have a narrow light beam in which all modes propagate approximately in the same direction.

Half of the average sum $2\sum_k \bar{n}_k^2\omega_k^2$ can be combined with the left sum, and the other half with the right one. We get:

$$G_{12}^{(2)} = \frac{\sum_{k_1}\sum_{k_2}\bar{n}_{k_1}\bar{n}_{k_2}\omega_{k_1}\omega_{k_2}e^{i\omega_{k_1}\tau}e^{-i\omega_{k_2}\tau}}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} + \\ + \frac{\sum_{k_1}\sum_{k_2}\bar{n}_{k_1}\bar{n}_{k_2}\omega_{k_1}\omega_{k_2}}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} = \\ = \frac{\sum_{k_1}\bar{n}_{k_1}\omega_{k_1}e^{i\omega_{k_1}\tau}\sum_{k_2}\bar{n}_{k_2}\omega_{k_2}e^{-i\omega_{k_2}\tau} + \left(\sum_{(k)}\bar{n}_k\omega_k\right)^2}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} = \\ = \frac{\left|\sum_{(k)}\bar{n}_k\omega_k e^{i\omega_k\tau}\right|^2 + \left(\sum_{(k)}\bar{n}_k\omega_k\right)^2}{\left(\sum_{(k)}\bar{n}_k\omega_k\right)^2} = \left(G_{12}^{(1)}\right)^2 + 1, \tag{5.43}$$

where $G_{12}^{(1)}$ is the first-order coherence function.

An important result has been obtained: the second-order coherence for chaotic light is expressed via the first-order coherence. This fact forms the basis of so-called intensity interferometry. It was first observed in the experiments by Hanbury Brown and Twiss, whose setup is shown in Figure 5.5.

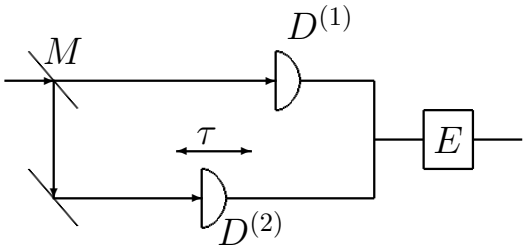


Figure 5.5: Experiment on observing second-order coherence. Using a 50% mirror M, the light beam is directed to two photodetectors $D^{(1)}$ and $D^{(2)}$. The adjustable delay τ is implemented by moving the photodetector $D^{(2)}$.

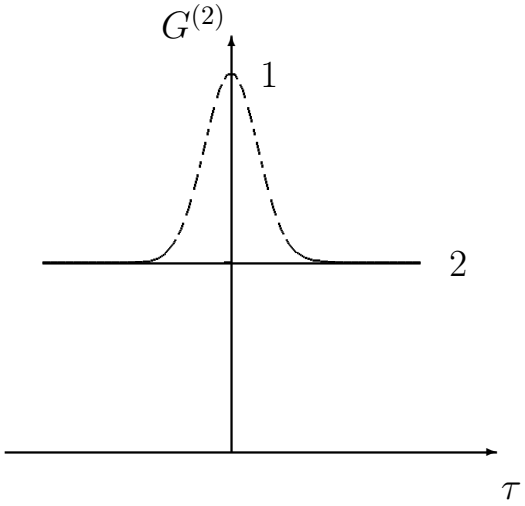


Figure 5.6: Dependence of the second-order coherence $G^{(2)}$ on the delay τ. Graph 1 describes the case of chaotic light. Graph 2 describes a coherent state (ideal laser).

As seen from Figure 5.5, using a 50% beam splitter, the light beam is directed to two photodetectors. Delay is introduced by moving one of the photodetectors. The coincidence circuit records detection of two photons with a given time delay.

Figure 5.6 shows the dependence of the number of coincidences on the delay. In the classical case, the experimental setup remains the same, only the coincidence circuit is replaced by a correlator. The experimental result in this case has the form shown in Figure 5.6.

In these experiments, intensity correlation was measured, i.e., the second-order coherence function. Using the relation between correlation functions of different orders, one can calculate the first-order function, and then, using the relation between the correlation function and the power spectrum (Wiener-Khinchin theorem), determine the emission spectrum. Thus, by "counting photons," spectral measurements can be indirectly performed. This is widely used nowadays. As will be seen later, this procedure allows measurement of spectra of chaotic light with high resolution, complementing traditional spectral measurement methods

(using spectral devices such as diffraction gratings, prisms, etc.).

Expression (5.43) can also be obtained using the coherent state representation. In this case,

$$\hat{\rho} = \int \cdots \int P\left(\{\alpha_k\}\right) |\{\alpha_k\}\rangle \langle\{\alpha_k\}| \, d^2\{\alpha_k\},$$

where

$$P\left(\{\alpha_k\}\right) = \prod_k \frac{1}{\pi \bar{n}_k} e^{-\frac{|\alpha_k|^2}{\bar{n}_k}} = \prod_k P_k\left(\alpha_k\right).$$

The operator product to be averaged is

$$\hat{a}^\dagger_{k^I} \hat{a}^\dagger_{k^{II}} \hat{a}_{k^{III}} \hat{a}_{k^{IV}}.$$

This leads to the integral

$$Sp\left\{\hat{\rho}\hat{a}^\dagger_{k^I} \hat{a}^\dagger_{k^{II}} \hat{a}_{k^{III}} \hat{a}_{k^{IV}}\right\} =$$

$$= \int \cdots \int d^2\{\alpha_k\}, P\left(\{\alpha_k\}\right) \left(\alpha^*_{k^I} \alpha^*_{k^{II}} \alpha_{k^{III}} \alpha_{k^{IV}}\right).$$

Again, non-zero contributions come from paired products satisfying

$$k^I = k^{III} = k_1, \quad k^{II} = k^{IV} = k_2$$

or

$$k^I = k^{IV} = k_1, \quad k^{II} = k^{III} = k_2$$

and also

$$k^I = k^{II} = k^{III} = k^{IV} = k.$$

It is easy to show by integrating in polar coordinates $\alpha = re^{i\theta}$ that the non-zero integrals are:

$$\int d^2\alpha_{k_1} P\left(\alpha_{k_1}\right) \alpha^*_{k_1} \alpha_{k_1} \int d^2\alpha_{k_2} P\left(\alpha_{k_2}\right) \alpha^*_{k_2} \alpha_{k_2} = \bar{n}_{k_1} \bar{n}_{k_2}.$$

In the third case, given

$$P\left(\alpha_k\right) = \frac{1}{\pi \bar{n}_k} e^{-\frac{r^2}{\bar{n}_k}}$$

we have

$$\int d^2\alpha_k P\left(\alpha_k\right) \left(\alpha^*_k \alpha_k\right)^2 = \int_0^\infty r dr \int_0^{2\pi} P\left(\alpha_k\right) r^4 d\theta =$$

$$= \frac{2\pi}{\pi \bar{n}_k} \int_0^\infty r^4 r dr e^{-\frac{r^2}{\bar{n}_k}} = \bar{n}_k^2 \int_0^\infty x^2 dx e^{-x} = 2\bar{n}_k^2.$$

Here, the substitution $x = \frac{r^2}{\bar{n}_k}$, $dx = \frac{2rdr}{\bar{n}_k}$ is made. Thus, we arrive at the same results as previously obtained. Therefore, the final result (5.43) remains unchanged.

5.4 Higher-Order Coherence

First- and second-order coherence functions are special, although fundamental cases of coherence functions. The nth-order coherence function in the quantum case is defined as follows

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \\ = \frac{\left\langle \hat{E}^{(-)}(x_1)\,\hat{E}^{(-)}(x_2)\ldots\hat{E}^{(-)}(x_n)\,\hat{E}^{(+)}(x_n)\,\hat{E}^{(+)}(x_{n-1})\ldots\hat{E}^{(+)}(x_1)\right\rangle}{\left\langle \hat{E}^{(-)}(x_1)\,\hat{E}^{(+)}(x_1)\right\rangle\ldots\left\langle \hat{E}^{(-)}(x_n)\,\hat{E}^{(+)}(x_n)\right\rangle}. \quad (5.44)$$

Angle brackets denote quantum-mechanical averaging using the statistical operator (density matrix), x denotes the set of variables t, $\vec{r}$. The degree of nth-order coherence determines the counting rate in an experiment where n photons are registered in some way.

5.5 Photon Counting and Statistics

We have confirmed that the quantization of the optical field and the existence of quanta of light (photons) preserve the overall pattern of interference phenomena, provided that the observation is conducted long enough for an averaged pattern to emerge. In our formulas, this corresponds to averaging using the density matrix. However, the existence of photons enables experiments of a new kind, based on photon counting and the investigation of their statistical regularities. These methods are called photon counting methods. Their essence is as follows: the studied light is directed onto a photodetector connected to a counter that counts the number of photoelectrons recorded over a certain time interval. A shutter before the photodetector (or a circuit lock) controls the counting duration. With the counter reset to zero, the shutter opens for a time T, and the number of photoelectrons is recorded. After a time longer than the correlation time τ_c, the process repeats many times. Based on the measurement results, one can determine $P_m(T)$ — the probability of registering m photoelectron counts during time T:

$$P_m(T) = \frac{N_m}{N}, \quad (5.45)$$

where N_m is the number of measurements in which m photoelectrons were recorded, and N is the total number of measurements, which must be large. It is assumed, of course, that the light flux is stationary. The obtained distribution contains information about the spectral properties of light beams. The first task here is how, based on the statistics of photo counts that we can measure, to find the photon statistics needed to obtain information about the properties of light beams.

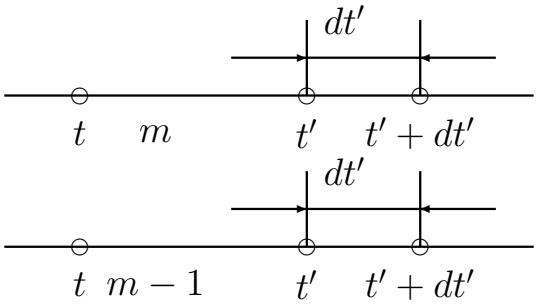


Figure 5.7: To the derivation of $P_m(t, t'+dt')$. There are two possibilities of obtaining m counts in the time interval $t, t'+dt'$: m counts are made in the interval t, t', and none in the interval dt; $m-1$ counts are made in the interval t, t', and one in the interval dt'.

5.6 Connection of photon statistics with photoelectron count statistics

Let us start with a semiclassical consideration and then generalize the results to a fully quantum treatment. We have seen that the photoemission rate is proportional to the intensity of the light flux, which is determined by the mean value of the operator $\left\langle \hat{E}^{(-)}\hat{E}^{(+)} \right\rangle$. Since the quantum-mechanical intensity is analogous, in the classical case, to the intensity $I(t)$ averaged over a period, the electron emission rate can be considered proportional to $I(t)$ in the semiclassical approximation. Let $P(t)\,dt$ denote the probability of the appearance of a photoelectron in the time interval t to $t+dt$. Then, $P(t)\,dt = \xi I(t)\,dt$, where ξ characterizes the efficiency of the photocathode. Denote the probability of registering m photo counts in the time interval t to $t'+dt'$ by $P_m(t, t'+dt')$. There are two possibilities to obtain m counts in the given time interval, shown in Figure 5.7:

1. In the interval t to t', m counts were made, and in the interval dt - none.
2. In the interval t to t', $m-1$ counts were made, and in the interval dt' - one count.

Note: the interval is so small that the probabilities of two or more counts are negligibly small. For these two cases, we can write

$$\begin{aligned} P_m^{(1)}(t, t'+dt') &= P_m(t, t')\,(1 - P(t')\,dt'), \\ P_m^{(2)}(t, t'+dt') &= P_{m-1}(t, t')\,P(t')\,dt', \end{aligned} \tag{5.46}$$

where $P(t)\,dt = \xi I(t)\,dt$. The total probability will be the sum of the probabilities of these two events:

$$\begin{aligned} P_m(t, t'+dt') = P_m(t, t')\,(1 - P(t')\,dt') + \\ + P_{m-1}(t, t')\,P(t')\,dt'. \end{aligned} \tag{5.47}$$

From this, we get a chain of differential recurrence equations

$$\frac{dP_m}{dt'} = \xi I\left(t'\right)\left\{P_{m-1}\left(t'\right) - P_m\left(t'\right)\right\}, \tag{5.48}$$

which can be solved by sequentially integrating, starting with $m = 0$. The initial equation has the form:

$$\frac{dP_0}{dt'} = -\xi I\left(t'\right) P_0\left(t'\right), \tag{5.49}$$

with obvious initial conditions $P_0\left(t'\right) = 1$ at $t' = t$. The solution of the equation under these initial conditions is

$$P_0\left(t, T\right) = e^{-\xi \int_t^{t+T} I(t')dt'}, \tag{5.50}$$

where T is the counting time. If we introduce the intensity averaged over the counting time

$$\bar{I}\left(t, T\right) = \frac{1}{T}\int_t^{t+T} I\left(t'\right) dt',$$

then equation (5.50) can be rewritten as

$$P_0\left(t, T\right) = e^{-\xi \bar{I} T} \tag{5.51}$$

The other probabilities can be successively expressed through $P_0\left(t, T\right)$. By induction it is easy to show that

$$P_m\left(t, T\right) = \frac{\left(\xi \bar{I} T\right)^m}{m!} e^{-\xi \bar{I} T} \tag{5.52}$$

One can verify the validity of this solution by substituting it into the original equation (5.48). Generally, the intensity $\bar{I}\left(t, T\right)$ fluctuates from one counting interval to another. To account for this, an averaging over the ensemble of measurements needs to be done. The result is Mandel's formula:

$$P_m\left(T\right) = \left\langle P_m\left(t, T\right)\right\rangle = \left\langle \frac{\left(\xi \bar{I}\left(t, T\right) T\right)^m}{m!} e^{-\xi \bar{I}(t,T) T} \right\rangle. \tag{5.53}$$

Alternatively, this formula can be written as

$$P_m\left(T\right) = \left\langle P_m\left(t, T\right)\right\rangle = \int_0^\infty P\left(\bar{I}\right) \frac{\left(\xi \bar{I} T\right)^m}{m!} e^{-\xi \bar{I} T} d\bar{I}, \tag{5.54}$$

where $P\left(\bar{I}\right)$ is the probability density for $\bar{I}$.

5.7 Photocount distribution for coherent and chaotic light

To get an idea of how the statistics of photo counts are related to the statistics of light incident on the photodetector, we apply formula (5.54) to two limiting cases: light constant in amplitude (intensity) and chaotic light with fluctuating intensity. Let us start with the simplest case when the intensity of the incident light is constant. In this case, $\bar{I}(t,T)$ does not depend on t and T, $\bar{I}(t,T) = I_0 = const$, so the second averaging is not necessary. The same result can be obtained if in (5.54) we consider $P\left(\bar{I}\right) = \delta\left(\bar{I} - I_0\right)$. The final expression for the case of intensity-constant light is:

$$P_m = \frac{\bar{m}^m}{m!} e^{-\bar{m}}. \tag{5.55}$$

where $\bar{m} = \xi I_0 T$ is the mean number of photo counts registered by the photodetector during time T. Thus, for constant intensity, the photo count distribution is a Poisson distribution. Note that for sufficiently large T (much greater than the correlation time τ_c), $\bar{I}(t,T)$ in any case[1] will tend to a constant value, and the photo count distribution approaches a Poisson distribution.

In the other limiting case, when $T \ll \tau_c$, one can consider $\bar{I}(t,T) = I(t)$ - the instantaneous intensity. In the case of chaotic light, $P(I) = \frac{1}{\bar{I}} e^{-\frac{I}{\bar{I}}}$ [57]. According to Mandel's formula we have:

$$\begin{aligned} P_m(T) &= \frac{1}{\bar{I}} \int_0^\infty e^{-\frac{I}{\bar{I}}} \frac{(\xi I T)^m}{m!} e^{-\xi I T} dI = \\ &= \int_0^\infty \frac{y^m \left(\bar{I}\xi T\right)^m}{m! \left(1+\bar{m}\right)^{m+1}} e^{-y} dy. \end{aligned} \tag{5.56}$$

Here a change of variables was made:

$$y = I\left(\frac{1}{\bar{I}} + \xi T\right),$$

$$I = \frac{y\bar{I}}{1 + \xi \bar{I} T},$$

$$\bar{m} = \xi \bar{I} T.$$

Finally, we obtain

$$P_m(T) = \frac{\bar{m}^m}{\left(1+\bar{m}\right)^{m+1} m!} \int_0^\infty y^m e^{-y} dy = \frac{\bar{m}^m}{\left(1+\bar{m}\right)^{m+1}} \tag{5.57}$$

[1] including for chaotic light

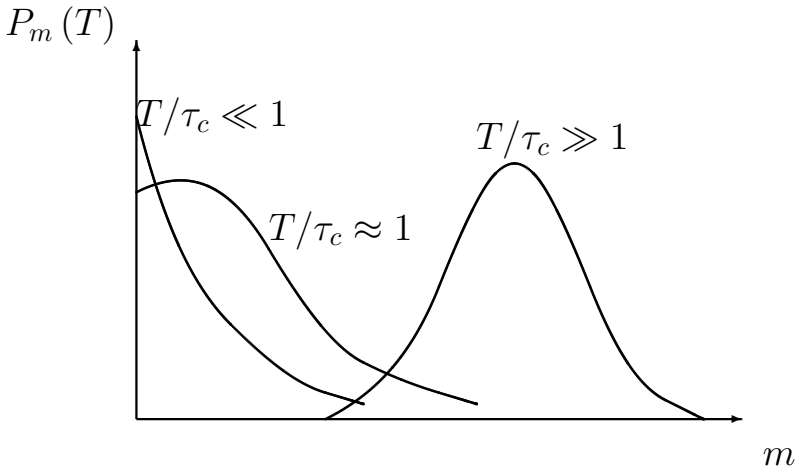


Figure 5.8: Qualitative view of the distribution of photo counts $P_m(T)$ for three values of T/τ_c.

since

$$\int_0^\infty y^m e^{-y} dy = m!.$$

We have obtained that the photo count distribution replicates the photon statistics for chaotic fields, but with a modified scale. It is much more complicated to determine the photo count distribution for a counting time T corresponding to the intermediate case $T \approx \tau_c$. Only numerical calculations are possible here. The results of such calculations are shown in Figure 5.8 [57]. They allow judging the nature of changes in the photo count distribution as the ratio T/τ_c increases. From the graphs, it is clear that the character of the distribution changes near $T = \tau_c$. This allows, by conducting measurements for various T, to estimate τ_c and the width of the spectrum of the incident light $\sim 1/\tau_c$.

5.8 Determination of photon statistics through the distribution of photocounts

Usually, the main interest is the photon statistics characterizing the light beam. To determine it from the measured $P_m(T)$, one should invert (5.54), expressing $P\left(\bar{I}\right)$ through $P_m(T)$.

There are several methods to find $P\left(\bar{I}\right)$. (See [43] for inversion methods.) We will focus on one of them. Consider the expression (5.54)

$$P_m(T) = \int_0^\infty P\left(\bar{I}\right) \frac{\left(\xi \bar{I} T\right)^m}{m!} e^{-\xi \bar{I} T} d\bar{I}.$$

From this equality, given the measured $P_m(T)$, one needs to determine the probability density $P\left(\bar{I}\right)$ that characterizes the statistics of the incident light.

Let us change the variable to $u = \xi \bar{I} T$ and define a new probability density

$$P_T(u) = \frac{1}{\xi T} P\left(\bar{I}\right).$$

We have:

$$P_T(u)\, du = \frac{1}{\xi T} P\left(\bar{I}\right) \xi T d\bar{I} = P\left(\bar{I}\right) d\bar{I}.$$

Hence:

$$\int_0^\infty P_T(u)\, du = \int_0^\infty P\left(\bar{I}\right) d\bar{I} = 1,$$

as it should be. Rewrite equation (5.54) using these variables:

$$P_m(T) = \int_0^\infty P_T(u) \frac{u^m}{m!} e^{-u} du. \tag{5.58}$$

Next, we use the orthogonal Laguerre polynomials

$$L_n(y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-y)^k}{k!},$$
$$\binom{n}{k} = C_k^n = \frac{n!}{k!\,(n-k)!} \tag{5.59}$$

which satisfy the orthogonality conditions

$$\int_0^\infty L_p(y)\, L_q(y)\, e^{-y} dy = \delta_{pq}, \tag{5.60}$$

which can be put into the form by substituting $y \to 2y$:

$$2\int_0^\infty L_p(2y)\, L_q(2y)\, e^{-2y} dy = \delta_{pq}. \tag{5.61}$$

Now, expand $P_T(u)$ in a series of Laguerre polynomials:

$$P_T(u) = \sum_{n=0}^{\infty} A_n L_n(2u)\, e^{-u} \tag{5.62}$$

Using the orthogonality conditions (5.61), we get for the expansion coefficients A_n the expression

$$A_n = 2\int_0^\infty L_n(2u)\, P_T(u)\, e^{-u} du. \tag{5.63}$$

Substituting $L_n(2u)$ in the form of the series (5.59) into (5.63), we obtain:

$$A_n = 2\sum_{k=0}^{k=n}\binom{n}{k}(-2)^k\int_0^\infty P_T(y)\frac{y^k}{k!}e^{-y}dy = \\ = 2\sum_{k=0}^{k=n}\binom{n}{k}(-2)^k P_k(T), \tag{5.64}$$

where the expression

$$\int_0^\infty P(y)\frac{y^k}{k!}e^{-y}dy = P_k(T)$$

was used.

Assume that from measurements we know a sufficient number of values $P_k(T)$. This will allow us to compute a sufficient number of expansion coefficients A_n using formula (5.64). Substituting them into the series (5.62) for $P_T(u)$ and reverting to the original variables, we get

$$P(\bar{I}) = \xi T\sum_{k=0}^{\infty}A_n L_n(2\xi T\bar{I})e^{-\xi T\bar{I}}. \tag{5.65}$$

Using this expression, it is in principle possible to invert Mandel's formula. The difficulty here lies in the fact that only a finite number of $P_n(T)$ are known with limited accuracy.

5.9 Quantum expression for the photocount distribution

The expression for the photocount distribution obtained above is based on the semiclassical approach. A fully quantum expression is given by [57]:

$$P_m(T) = Sp\left\{\hat{\rho}\hat{N}\left[\frac{\left(\beta\hat{\bar{I}}(t)T\right)^m}{m!}e^{-\beta\hat{\bar{I}}(t)T}\right]\right\} \tag{5.66}$$

where

$$\hat{\bar{I}}(t) = \frac{1}{T}\int_t^{t+T}2\varepsilon_0 c\hat{\bar{E}}^{(-)}(t')\hat{\bar{E}}^{(+)}(t')dt' = \\ = \frac{2}{T}\int_t^{t+T}\sqrt{\frac{\varepsilon_0}{\mu_0}}\hat{\bar{E}}^{(-)}(t')\hat{\bar{E}}^{(+)}(t')dt'$$

corresponds to the operator of the energy flux averaged over the counting period, β is the quantum efficiency of the photodetector. Formula (5.66) looks very similar to Mandel's formula. The difference is that here the classical fields are replaced by their operators, averaging is performed quantum mechanically using the statistical operator, and there is the normal ordering operator $\hat{N}$. The action of this operator reduces to rearranging the operators so that annihilation operators stand to the right of creation operators. When expanding the operator inside (5.66) in a power series, terms of the form $\left(\hat{\hat{E}}^{(-)}\hat{\hat{E}}^{(+)}\right)^n$ appear. The action of $\hat{N}$ on them is given by the expression

$$\hat{N}\left(\hat{\hat{E}}^{(-)}\hat{\hat{E}}^{(+)}\right)^n = \left(\hat{\hat{E}}^{(-)}\right)^n\left(\hat{\hat{E}}^{(+)}\right)^n.$$

Such terms correspond to higher-order coherences, meaning the photocount distribution depends on coherences of all orders. Formula (5.66) can be justified as follows: Assume that the field is in a coherent state $|\{\alpha_k\}\rangle$. Then the probability of electron emission in the time interval dt is

$$Pdt = \sigma\left\langle\{\alpha_k\}\right|\hat{E}^{(-)}\hat{E}^{(+)}\left|\{\alpha_k\}\right\rangle dt = \sigma\left(E^*E\right)dt,$$

where σ characterizes the efficiency of the photocathode, and E is the classical field corresponding to the state $|\{\alpha_k\}\rangle$

$$\hat{E}^{(+)}\left|\{\alpha_k\}\right\rangle = E\left|\{\alpha_k\}\right\rangle,$$
$$\left\langle\{\alpha_k\}\right|\hat{E}^{(-)} = E^*\left\langle\{\alpha_k\}\right|.$$

Further consideration proceeds in the same way as in the semiclassical approach. One obtains a similar formula:

$$P_m\left(T\right|\left\{\alpha_k\right\}) = \frac{\left(\sigma\overline{E^*E}T\right)^m}{m!}e^{-\sigma T\overline{E^*E}} \tag{5.67}$$

where

$$\overline{E^*E} = \frac{1}{T}\int_t^{t+T} E^*\left(t'\right)E\left(t'\right)dt'.$$

This expression can be written in operator form:

$$P_m\left(T\right|\left\{\alpha_k\right\}) = \left\langle\{\alpha_k\}\right|\hat{N}\left\{\frac{\left(\sigma\overline{\hat{E}^{(-)}\hat{E}^{(+)}}T\right)^m}{m!}e^{-\sigma T\overline{\hat{E}^{(-)}\hat{E}^{(+)}}}\right\}\left|\{\alpha_k\}\right\rangle, \tag{5.68}$$

where $P_m\left(T\right|\left\{\alpha_k\right\})$ is the probability of counting m photoelectrons in time T if the field is in the state $|\{\alpha_k\}\rangle$. The operator inside the curly braces can be

considered as the operator counting m photoelectrons in time T. If the field is in a statistically mixed state with statistical operator, defined in the diagonal representation by the function $P_m(T|\{\alpha_k\})$,

$$\hat{\rho} = \int_{\{\alpha_k\}} P(\{\alpha_k\}) |\{\alpha_k\}\rangle \langle\{\alpha_k\}| \prod_k d^2\alpha_k,$$

the average value of the operator in expression (5.68), according to (1.122), will be

$$P_m(T) = \int_{\{\alpha_k\}} P(\{\alpha_k\}) \langle\{\alpha_k\}| \cdot$$
$$\cdot \hat{N} \left\{ \frac{\left(\sigma \overline{\hat{E}^{(-)}\hat{E}^{(+)}} T\right)^m}{m!} e^{-\sigma T \overline{\hat{E}^{(-)}\hat{E}^{(+)}}} |\{\alpha_k\}\rangle \right\} \prod_k d^2\alpha_k \tag{5.69}$$

The obtained formula can be written in any representation, since the operation Sp does not depend on the representation. We have

$$P_m = Sp\left(\hat{\rho}\hat{N}\left\{\frac{\left(\sigma \overline{\hat{E}^{(-)}\hat{E}^{(+)}} T\right)^m}{m!} e^{-\sigma T \overline{\hat{E}^{(-)}\hat{E}^{(+)}}}\right\}\right), \tag{5.70}$$

where

$$\overline{\hat{E}^{(-)}\hat{E}^{(+)}} = \frac{1}{T}\int_t^{t+T} \hat{E}^{(-)}(t')\,\hat{E}^{(+)}(t')\,dt'.$$

Formula (5.70) fully corresponds to formula (5.66) taken from [57]. Calculations using formula (5.70) are somewhat more complicated than using Mandel's formula. As an example, let's consider a few simple cases. First, consider single-mode states. Then (5.70) can be written as

$$P_m = Sp\left(\hat{\rho}\hat{N}\left\{\frac{\left(\gamma \overline{\hat{a}^\dagger \hat{a}} T\right)^m}{m!} e^{-\gamma T \overline{\hat{a}^\dagger \hat{a}}}\right\}\right)$$

where γ is a coefficient characterizing the photocathode efficiency in this notation. The operator

$$\hat{N}\left\{\frac{\left(\gamma \overline{\hat{a}^\dagger \hat{a}} T\right)^m}{m!} e^{-\gamma T \overline{\hat{a}^\dagger \hat{a}}}\right\}$$

has only diagonal elements in the number representation (photon number). Then $Sp(\dots)$ will be determined only by the diagonal elements of the density matrix, which we denote as $P_n = \langle n|\,\hat{\rho}\,|n\rangle$.

Then (5.70) can be represented as:

$$P_m = \sum_n P_n \langle n| \hat{N} \frac{\left(\gamma \hat{a}^\dagger \hat{a} T\right)^m}{m!} e^{-\gamma T \hat{a}^\dagger \hat{a}} |n\rangle =$$

$$= \sum_n P_n \frac{(\gamma T)^m}{m!} \langle n| \sum_l (-1)^l \frac{(\gamma T)^l}{l!} \left(\hat{a}^\dagger\right)^{l+m} (\hat{a})^{l+m} |n\rangle =$$

$$= \sum_{n=m} P_n \frac{(\gamma T)^m}{m!} \sum_{l=0}^{n-m} (-1)^l \frac{(\gamma T)^l}{l!} \frac{n!}{(n-m-l)!}, \tag{5.71}$$

which follows from the relation:

$$\langle n| \left(\hat{a}^\dagger\right)^{l+m} (\hat{a})^{l+m} |n\rangle =$$

$$= \{n\,(n-1)\,(n-2)\ldots(n-m-l+1)\} = \frac{n!}{(n-m-l)!}.$$

Expression (5.71) can be further transformed. The known expansion (Newton's binomial) is:

$$(1-\gamma T)^{n-m} = \sum_{l=0}^{n-m} (-1)^l (\gamma T)^l \frac{(n-m)!}{l!\,(n-m-l)!}.$$

From here we have:

$$P_m(T) = \sum_{n=m}^{\infty} P_n \frac{n!}{m!\,(n-m)!} (\gamma T)^m (1-\gamma T)^{n-m}. \tag{5.72}$$

Note that, as known from probability theory (see Bernoulli distribution), the general term of the sum gives the probability that out of n photons m are registered, and $n-m$ remain unregistered. Let's apply the general formula (5.71) to the case of chaotic light. Then

$$P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}},$$

and from (5.71) we get

$$P_m(T) = \sum_{n=m}^{\infty} \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} (\gamma T)^m (1-\gamma T)^{n-m} \frac{n!}{m!\,(n-m)!} =$$

$$= (\gamma T)^m \sum_{l=0}^{\infty} \frac{\bar{n}^{l+m}}{(\bar{n}+1)^{l+m+1}} (1-\gamma T)^l \frac{(l+m)!}{m!l!}. \tag{5.73}$$

In the transformation, the summation index substitution is made: $l = n - m$, $n = l + m$.

The known expansion [43] is:

$$(1-x)^{-(m+1)} = \sum_{l=0}^{\infty} \frac{x^l (m+l)!}{m! l!}.$$

Using it, we get:

$$P_m(T) = \frac{(\gamma T \bar{n})^m}{(1+\bar{n})^{m+1}} \left(1 - \frac{\bar{n}}{1+\bar{n}}(1-\gamma T)\right)^{-(m+1)} =$$
$$= \frac{(\gamma T \bar{n})^m}{(1+\bar{n})^{m+1}} \left\{\frac{1-\gamma T \bar{n}}{1+\bar{n}}\right\}^{-(m+1)} = \frac{(\gamma T \bar{n})^m}{(1+\gamma T \bar{n})^{m+1}} = \frac{\bar{m}^m}{(\bar{m}+1)^{m+1}}, \quad (5.74)$$

where $\bar{m} = \gamma T \bar{n}$. We obtained the same result as in the semiclassical consideration. The distribution $P_m(T)$ mirrors P_n, but at a different scale. Instead of $\bar{n}$, there is $\bar{m} = \gamma T \bar{n}$. Now consider another limiting case: the field is in a coherent state. It is convenient to consider it in the representation of coherent states. We have

$$P_m(T) = \langle \alpha | \hat{N} \frac{\left(\gamma \hat{a}^\dagger \hat{a} T\right)^m}{m!} e^{-\gamma T \hat{a}^\dagger \hat{a}} |\alpha\rangle =$$
$$= \frac{\left(\gamma |\alpha|^2 T\right)^m e^{-\gamma |\alpha|^2 T}}{m!} = \frac{\bar{m}^m}{m!} e^{-\bar{m}}, \quad (5.75)$$

where $\bar{m} = \gamma |\alpha|^2 T = \gamma \bar{n} T$.

The photon number distribution for a coherent state is known (1.95):

$$P_n = \frac{\bar{n}^n e^{-\bar{n}}}{n!}.$$

Thus, in this case as well, the photocount distribution mirrors the photon distribution but at a different scale. Instead of $\bar{n}$, there is $\bar{m}$. In the general case, such a simple relation does not hold. From the derivation of the quantum photocount formula it follows that as long as $P(\{\alpha_k\})$ in the expression for the statistical operator can be interpreted as a probability distribution, there is no discrepancy between semiclassical results (Mandel's formula) and the fully quantum treatment. However, if $P(\{\alpha_k\})$ is negative in some regions, the results will differ.

5.10 Experiments on photon counting. Application of photon counting technique for spectral measurements

We have seen that the distribution of photo counts depends on the counting time T. When the counting time is compared with the correlation time, the nature of the distribution changes, a maximum appears on the curve. With further increase in counting time, the nature of the curve is preserved, and at large counting times the distribution tends to the Poisson distribution. Therefore, by measuring $P_m(T)$ at different T, one can estimate τ_c and the spectral width of the light beam $\sim 1/\tau_c$. Such methods, in which spectral parameters of light are determined from photon counting experiments, are called intensity fluctuation spectroscopy. The minimum counting time is determined by the resolution of the photodetector, which is on the order of $10^{-8} - 10^{-9}$ sec, corresponding to frequencies of $10^8 \boldsymbol{\nabla} \cdot 10^9$ Hz. This is the upper frequency limit of the method. The lower limit is determined by the maximum counting time, which is usually 1 sec, corresponding to a resolution of 1 Hz.

Thus, the photon counting method can be used to study the frequency range from 1 Hz to 10^8 Hz. Therefore, this method complements conventional spectroscopy, which operates in the frequency range from 10^7 Hz to 10^{15} Hz. It is more convenient to use photon counting experiments of another type. They measure the correlation between photon numbers m_1 and m_2, i.e., $\langle m_1 m_2 \rangle$, recorded in two short time intervals $\Delta t_1 = \Delta t_2 = \Delta t$, delayed relative to each other by time τ. Both intervals have the same duration Δt, which is less than τ and the correlation time τ_c. In this case, the second-order coherence is measured, defined by the formula

$$G_{12}^{(2)} = \frac{\langle m_1 m_2 \rangle}{(\bar{m})^2} = 1 + \left(G_{12}^{(1)} \right)^2$$

where $\bar{m}$ is the average number of counts in time Δt.

Knowing $G_{12}^{(2)}$ for chaotic light, the first-order coherence function can be calculated as

$$G_{12}^{(1)} = \sqrt{G_{12}^{(2)} - 1}$$

and via its Fourier transform, the shape and width of the spectral line of the light beam can be obtained. For chaotic light with a Lorentzian spectral line we have:

$$\langle m_1 m_2 \rangle = \bar{m}^2 \left(e^{-2\gamma|\tau|} + 1 \right), \tag{5.76}$$

and for a Doppler line -

$$\langle m_1 m_2 \rangle = \bar{m}^2 \left(e^{-\delta^2 \tau^2} + 1 \right). \tag{5.77}$$

Here δ and γ determine the line widths in these cases. By measuring these parameters experimentally, one can determine the spectral width of the lines.

5.11 Exercises

1. Show that formula (5.52) indeed provides the solution of the problem.
2. Derive expression (5.57) from formula (5.56).
3. Obtain formula (5.74) using the coherent states representation.
4. Prove the orthogonality conditions for Laguerre polynomials (5.60) and (5.61).
5. Expand $P_T(u)$ in a series of Laguerre polynomials: (5.62)-(5.63).

Chapter 6

Quantum Description of Optical Interference Experiments

6.1 General Considerations

In quantum optics, as in classical optics, one of the main devices used for studying optical fields (light) is the interferometer.

In quantum physics, the device is described based on classical physics concepts, while the quantum object interacting with the device is described by quantum equations.

The boundary between the device and the quantum object is somewhat conditional and can be chosen depending on the problem to be solved.

In quantum optics, the interferometer itself is described by classical equations, just as in classical optics. The light field is described by quantum equations for field operators, usually using the Heisenberg picture.

The most important element of any interferometer is a mirror. According to the above, we consider the mirror as part of the device and describe it classically, introducing reflection coefficients r, r' and transmission coefficients t, t' (see Figure 6.1). The figure shows coefficients r, t and r', t' corresponding to different sides of the mirror. Also shown are the annihilation operators of two input field modes $\hat{a}_0$ and $\hat{a}_1$ and two output mode operators $\hat{a}_2$ and $\hat{a}_3$.

If the mirror is lossless, then the "scattering matrix" is unitary, as in the classical case:

$$\hat{S} = \begin{bmatrix} t' & r \\ r' & t \end{bmatrix},$$

where r, t and r', t' are the transmission and reflection coefficients from different

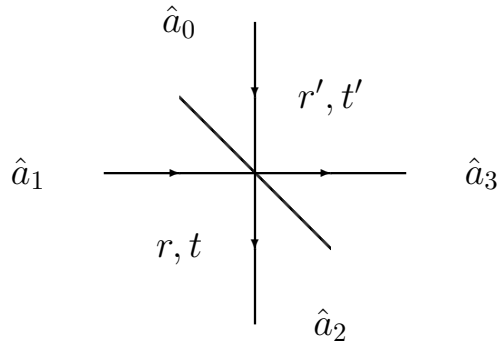


Figure 6.1: Mirror model, $\hat{a}_0$, $\hat{a}_1$ - input field mode annihilation operators, $\hat{a}_2$, $\hat{a}_3$ - output field mode annihilation operators, r, r', t and t' - reflection and transmission coefficients from different sides of the mirror.

sides of the mirror.

The unitarity condition is written as follows:

$$\begin{aligned}
\hat{S}^{-1} = \hat{S}^{\dagger} = \begin{bmatrix} t'^{*} & r'^{*} \\ r^{*} & t^{*} \end{bmatrix}, \\
\hat{S}\hat{S}^{\dagger} = \begin{bmatrix} t' & r \\ r' & t \end{bmatrix} \begin{bmatrix} t'^{*} & r'^{*} \\ r^{*} & t^{*} \end{bmatrix} = \\
= \begin{bmatrix} |t'|^2 + |r|^2 & t'r'^{*} + rt^{*} \\ r't'^{*} + tr^{*} & |r'|^2 + |t|^2 \end{bmatrix} = \hat{I}, \\
\hat{S}^{\dagger}\hat{S} = \begin{bmatrix} t'^{*} & r'^{*} \\ r^{*} & t^{*} \end{bmatrix} \begin{bmatrix} t' & r \\ r' & t \end{bmatrix} = \\
= \begin{bmatrix} |t'|^2 + |r'|^2 & t'^{*}r + r'^{*}t \\ r^{*}t' + t^{*}r' & |r|^2 + |t|^2 \end{bmatrix} = \hat{I}
\end{aligned} \tag{6.1}$$

For the unitarity condition (6.1) to hold, the energy conservation law and the reciprocity theorem must be satisfied. This leads to certain conditions that the coefficients r, r', t, and t' must satisfy:

$$\begin{aligned}
|r| = |r'|, \; |t| = |t'|, \\
|r|^2 + |t|^2 = 1, \\
r't'^{*} + tr^{*} = 0, \\
t'r^{*} + r't^{*} = 0.
\end{aligned} \tag{6.2}$$

It should be noted that the last expression is not independent. Indeed, assuming $t \neq 0$ (otherwise one can consider the relation for the reflection coefficients) and from the relation $|t| = |t'|$ we have $tt^{*} = t't'^{*}$ from which

$$t^{*} = \frac{t't'^{*}}{t}.$$

Therefore, using $r't'^* + tr^* = 0$, we have

$$t'r^* + r't^* = t'r^* + r'\frac{t't'^*}{t} = \\ = \frac{t'}{t}\left(tr^* + r't'^*\right) = 0.$$

The reference phase plane can be shifted slightly with respect to the mirror plane, especially since a real mirror is not infinitely thin. In some cases, the reference plane can be moved by a distance of the order of the wavelength. By choosing the reference plane, the scattering matrix can be brought to a simpler form, with conditions (6.2) automatically satisfied. Assuming $t = t' = Te^{i\theta}$ and $r = r' = Re^{i\phi}$, from (6.2) we get

$$tr^* + r't^* = TRe^{i\theta}e^{-i\phi} + TRe^{i\phi}e^{-i\theta} = e^{i(\phi-\theta)}TR\left(e^{-2i(\phi-\theta)} + 1\right) = 0,$$

i.e., $\phi - \theta = \frac{\pi}{2}$. Taking $\theta = 0$ and redefining $t = T, r = R$ we have

$$\hat{S} = \begin{pmatrix} t & ir \\ ir & t \end{pmatrix}. \tag{6.3}$$

If we take $t, t', r, r' \in \mathbb{R}$, then obviously the necessary conditions will also be satisfied for $t' = t$ and $r' = -r$:

$$\hat{S} = \begin{pmatrix} t & r \\ -r & t \end{pmatrix}. \tag{6.4}$$

In equations (6.3) and (6.4), the coefficients r and t are real and positive.

In expression (6.3) the phase of the reflection coefficient is shifted by $\frac{\pi}{2}$ relative to the phase of the transmission coefficient. In the second variant (6.4), all coefficients are real, but the reflection coefficients from different sides have different signs. Since by choosing the reference plane the general case can be reduced either to (6.3) or to (6.4), we will subsequently use one of these expressions.

Example 6.1.1 (Symmetric Splitter. Classical Case)**.** *Suppose our splitter is symmetric, i.e., $r = r'$ and $t = t'$ (see Figure 6.2). Obviously, the law of energy conservation must hold:*

$$|E_0|^2 + |E_1|^2 = |E_2|^2 + |E_3|^2,$$

or

$$|E_0|^2 + |E_1|^2 = |tE_0 + rE_1|^2 + |rE_0 + tE_1|^2 = \\ = \left(|t|^2 + |r|^2\right)\left(|E_0|^2 + |E_1|^2\right) + \\ + E_0E_1^*\left(tr^* + rt^*\right) + E_1E_0^*\left(tr^* + rt^*\right)$$

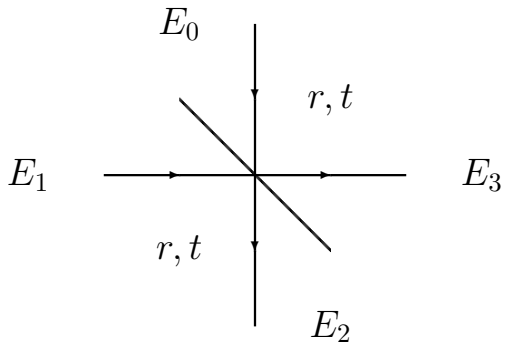


Figure 6.2: Symmetric splitter. Classical case. E_0, E_1 - two input light beams, E_2, E_3 - output fields, annihilation operators of the output mode fields, r and t - reflection and transmission coefficients.

That is,

$$\begin{aligned} |t|^2 + |r|^2 &= 1, \\ tr^* + rt^* &= 0 \end{aligned} \tag{6.5}$$

The last equality will hold if the phases of t *and* r *differ by* $\frac{\pi}{2}$*, in particular if we take* $t = |t|$*,* $r = i\,|r|$*, which corresponds to expression* (6.3)*.*

Moreover, expressions (6.5) *correspond to* (6.1)*, indeed*

$$\hat{S}\hat{S}^\dagger = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} t^* & r^* \\ r^* & t^* \end{pmatrix} =$$

$$= \begin{pmatrix} |t|^2 + |r|^2 & tr^* + rt^* \\ t^*r + r^*t & |t|^2 + |r|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}.$$

Thus the unitarity condition (6.1) *represents the law of energy conservation.*

6.2 Operator relations

We will consider the incident and transmitted fields as two-dimensional, see Figure 6.3. The operators $\hat{a}_0$ and $\hat{a}_1$ are operators of the incident field, corresponding to two different modes (zero and first). The operators of the transmitted field are $\hat{a}_2$ and $\hat{a}_3$ (second and third modes). Naturally, the operators must satisfy the commutation relations:

$$\begin{aligned} \left[\hat{a}_0, \hat{a}_0^\dagger\right] = \left[\hat{a}_1, \hat{a}_1^\dagger\right] = \left[\hat{a}_2, \hat{a}_2^\dagger\right] = \left[\hat{a}_3, \hat{a}_3^\dagger\right] = 1, \\ \left[\hat{a}_0, \hat{a}_1^\dagger\right] = \left[\hat{a}_2, \hat{a}_3^\dagger\right] = 0. \end{aligned} \tag{6.6}$$

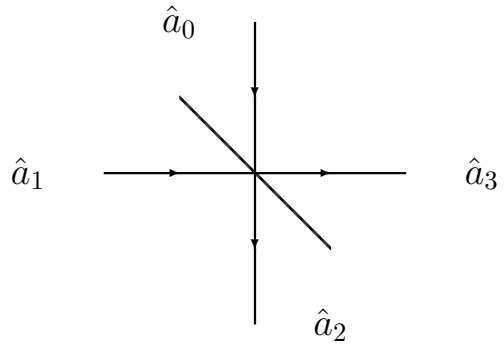


Figure 6.3: Mirror model. Operators $\hat{a}_0$ and $\hat{a}_1$ are operators of the incident field, corresponding to two different modes (zero and first). Operators of the transmitted field $\hat{a}_2$ and $\hat{a}_3$ (second and third modes).

The relation between the modes arises upon reflection from the mirror and passing through it. It is described by the following equations:

$$\begin{aligned} \hat{a}_2 &= t'\hat{a}_0 + r\hat{a}_1, \\ \hat{a}_3 &= r'\hat{a}_0 + t\hat{a}_1. \end{aligned} \tag{6.7}$$

The incident and transmitted fields must satisfy the commutation relations (6.6), from which, taking into account (6.2), we have

$$\begin{aligned} \left[\hat{a}_2, \hat{a}_2^\dagger\right] = \left[t'\hat{a}_0 + r\hat{a}_1, t'^*\hat{a}_0^\dagger + r^*\hat{a}_1^\dagger\right] = \\ = |r|^2 + |t|^2 = 1. \end{aligned} \tag{6.8}$$

Similarly, we get

$$\left[\hat{a}_3, \hat{a}_3^\dagger\right] = 1 \tag{6.9}$$

and further

$$\left[\hat{a}_2, \hat{a}_3^\dagger\right] = \left[t'\hat{a}_0 + r\hat{a}_1, r'^\dagger\hat{a}_0^\dagger + t^*\hat{a}_1^\dagger\right] = t'r'^* + rt^* = 0. \tag{6.10}$$

Thus, we have obtained the fulfillment of all the required commutation relations. Note that if only mode 1 (input 1) is excited, then the field of mode zero (input 0) cannot be canceled as is done in the classical case. In the quantum case, the mode is in the vacuum state, and its field is not zero. If this is not taken into account, then the commutation relations (6.6) will be violated.

Let us assume that the mirror (beam splitter) shown in Figure 6.4 receives at input 1 (first mode) - a coherent state $|\alpha\rangle$, and at the zero input (zero mode) a vacuum state $|0\rangle$, i.e.

$$\begin{aligned} \hat{a}_0 |0\rangle &= 0, \\ \hat{a}_1 |\alpha\rangle &= \alpha |\alpha\rangle . \end{aligned}$$

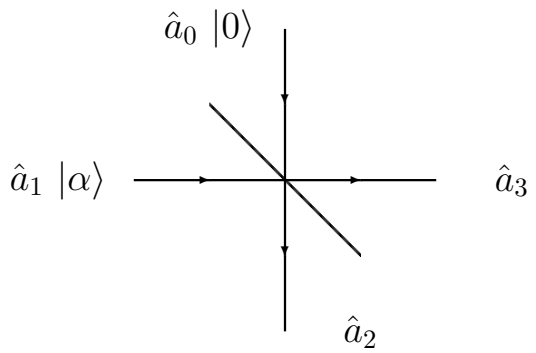


Figure 6.4: Mirror model. A coherent state $|\alpha\rangle$ is input at port 1 (first mode), and a vacuum state $|0\rangle$ is input at port zero (zero mode)

Thus, at the input, we have a two-mode state

$$|\psi\rangle = |0\rangle_0 |\alpha\rangle_1 .$$

From (6.7) it follows

$$\hat{a}_3 |\psi\rangle = (r'\hat{a}_0 + t\hat{a}_1) |0\rangle_0 |\alpha\rangle_1 = t\alpha |\psi\rangle ,$$

from which it is seen that the state at the output 3 is a coherent state with a reduced amplitude $\alpha_3 = t\alpha$. Similarly, we get

$$\hat{a}_2 |\psi\rangle = r\alpha |\psi\rangle , \ \alpha_2 = r\alpha. \tag{6.11}$$

Thus, the change in amplitude α is the same as in the classical case, and the state remains coherent.

6.3 Mach-Zehnder Interferometer. Error of Phase Measurements.

6.3.1 Interferometer Equation

Let us now consider in more detail the operation of the Mach-Zehnder interferometer, the schematic of which is shown in Figure 6.5.

Assume that the input M_1 and output M_2 mirrors are partially transparent (50:50), i.e.

$$t = r = \frac{1}{\sqrt{2}},$$

and the other two mirrors are fully reflective

$$t = 0,\ r = 1.$$

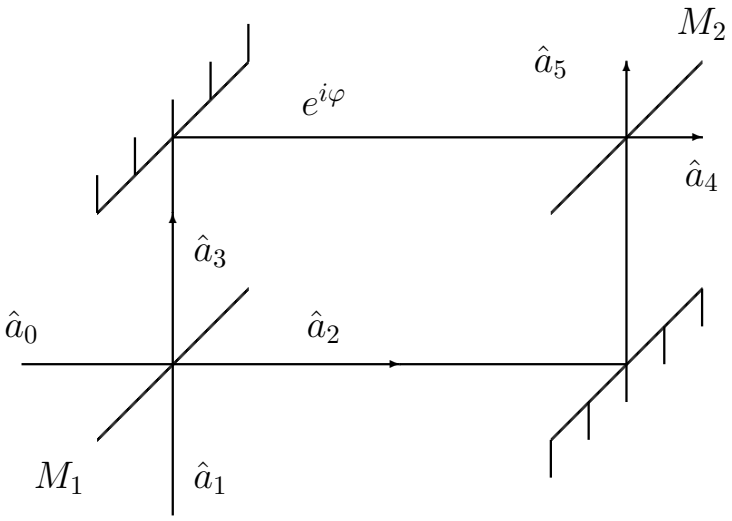


Figure 6.5: Mach-Zehnder interferometer. We assume that the input M_1 and output M_2 mirrors are semi-transparent (50:50), and the other two mirrors are fully reflective.

The "scattering matrix" of the beam splitters has the form

$$\hat{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Here we used the notation (6.3). The interferometer equation is:

$$\hat{a}_2 = \frac{1}{\sqrt{2}} \left(\hat{a}_0 + i\hat{a}_1 \right), \; \hat{a}_3 = \frac{1}{\sqrt{2}} \left(i\hat{a}_0 + \hat{a}_1 \right),$$
$$\hat{a}_4 = \frac{1}{\sqrt{2}} \left(i\hat{a}_2 + e^{i\varphi}\hat{a}_3 \right) = \frac{1}{2} \left[i \left(1 + e^{i\varphi} \right) \hat{a}_0 - \left(1 - e^{i\varphi} \right) \hat{a}_1 \right],$$
$$\hat{a}_5 = \frac{1}{\sqrt{2}} \left(\hat{a}_2 + ie^{i\varphi}\hat{a}_3 \right) = \frac{1}{2} \left[\left(1 - e^{i\varphi} \right) \hat{a}_0 + i \left(1 + e^{i\varphi} \right) \hat{a}_1 \right], \qquad (6.12)$$

where φ is the difference in optical path lengths of the interferometer arms (the phase difference accrued in the arms).

Formulas (6.12) connect the input fields with the output fields. Let us assume that the input field at port zero is in the vacuum state $|0\rangle$, and the input field at port one is in a coherent state $|\alpha\rangle$. Thus the input field is in a two-mode state $|\psi\rangle = |0\rangle_0 |\alpha\rangle_1$. Next, one can find the mean number of photons at outputs 4 and 5 depending on φ. The result will be indistinguishable from the classical one. Indeed,

$$\hat{a}_5^\dagger \hat{a}_5 = \frac{1}{4} \left[\left(1 - e^{-i\varphi} \right) \hat{a}_0^\dagger - i \left(1 + e^{-i\varphi} \right) \hat{a}_1^\dagger \right]$$
$$\left[\left(1 - e^{i\varphi} \right) \hat{a}_0 + i \left(1 + e^{i\varphi} \right) \hat{a}_1 \right] =$$
$$= \frac{1}{2} \left[(1 - \cos\varphi) \, \hat{a}_0^\dagger \hat{a}_0 + (1 + \cos\varphi) \, \hat{a}_1^\dagger \hat{a}_1 \right] - \frac{\sin\varphi}{2} \left[\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0 \right] \qquad (6.13)$$

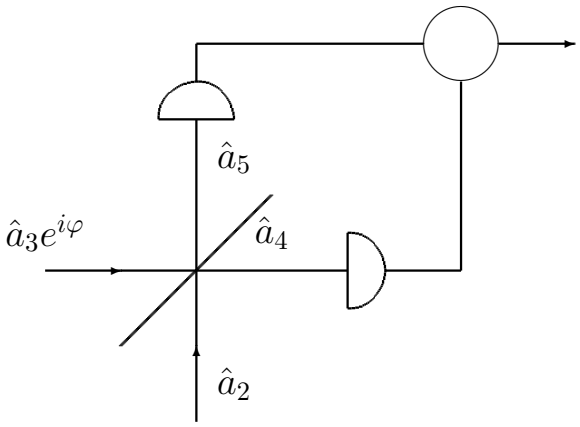


Figure 6.6: Balanced detector

Thus, for the state $|\psi\rangle = |0\rangle_0 |\alpha\rangle_1$ the detector at port 5 produces the following signal

$$\langle\psi| \hat{a}_5^\dagger \hat{a}_5 |\psi\rangle = \frac{1}{2}(1+\cos\varphi)|\alpha|^2 .$$

For the photodetector at port 4 similarly we obtain

$$\hat{a}_4^\dagger \hat{a}_4 = \frac{1}{4}\left[-i\left(1+e^{-i\varphi}\right)\hat{a}_0^\dagger - \left(1-e^{-i\varphi}\right)\hat{a}_1^\dagger\right]$$
$$\left[i\left(1+e^{i\varphi}\right)\hat{a}_0 - \left(1-e^{i\varphi}\right)\hat{a}_1\right] =$$
$$= \frac{1}{2}\left[(1+\cos\varphi)\,\hat{a}_0^\dagger\hat{a}_0 + (1-\cos\varphi)\,\hat{a}_1^\dagger\hat{a}_1\right] + \frac{\sin\varphi}{2}\left[\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\right] \qquad (6.14)$$

At photodetector 4 we have the following signal

$$\langle\psi| \hat{a}_4^\dagger \hat{a}_4 |\psi\rangle = \frac{1}{2}(1-\cos\varphi)|\alpha|^2 .$$

We consider a more complex balanced detection scheme, shown in Figure 6.6. Each channel is detected by its own photodetector. The signals coming from the photodetectors are subtracted and recorded. This detection scheme essentially is a synchronous detector (homodyne detector), where the local oscillator is the field in the coherent state, and the signal consists of vacuum fluctuations.

Using (6.14, 6.13) one can calculate the signal at the input of the balanced detector, which is defined as the expectation value of the operator

$$\hat{R} = \hat{a}_5^\dagger\hat{a}_5 - \hat{a}_4^\dagger\hat{a}_4 = \left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_0^\dagger\hat{a}_0\right)\cos\varphi - \left(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\right)\sin\varphi. \qquad (6.15)$$

In deriving (6.15) the relations (6.12) were used.

If the mode of the local oscillator (input 1, mode $\hat{a}_1$) is in a coherent state with a large amplitude α, while the signal mode (input 0, mode $\hat{a}_0$) is in the vacuum

state, then the output signal is

$$\left\langle \hat{R} \right\rangle = \langle\psi| \hat{R} |\psi\rangle = \langle 0|_0 \langle\alpha|_1 \left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_0^\dagger \hat{a}_0\right) |0\rangle_0 |\alpha\rangle_1 \cos\varphi - \\ - \langle 0|_0 \langle\alpha|_1 \left(\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0\right) |0\rangle_0 |\alpha\rangle_1 = \\ = |\alpha|^2 \cos\varphi. \qquad (6.16)$$

In deriving (6.16) it was taken into account that the operators of the first and zero modes act only on the state of their own mode. Therefore,

$$\hat{a}_0 |0\rangle = 0, \ \langle 0| \hat{a}_0^\dagger = 0,$$

and hence

$$\langle 0|_0 \langle\alpha|_1 \left(\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0\right) |0\rangle_0 |\alpha\rangle_1 = 0,$$

while

$$\langle\psi| \hat{R} |\psi\rangle = \langle 0|_0 \langle\alpha|_1 \left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_0^\dagger \hat{a}_0\right) |0\rangle_0 |\alpha\rangle_1 = \\ = \langle 0|_0 \langle\alpha|_1 \left(\hat{a}_1^\dagger \hat{a}_1\right) |0\rangle_0 |\alpha\rangle_1 = |\alpha|^2 = \langle\hat{n}\rangle$$

is equal to the mean photon number in the local oscillator mode.

If $\varphi = \frac{\pi}{2}$, then $\left\langle \hat{R} \right\rangle = 0$ and the output signal is absent. At the same time, the operator $\hat{R} \neq 0$:

$$\hat{R} = -\left(\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0\right).$$

Although the mean of this operator is zero, the mean of its square will be nonzero and will describe the noise which limits the accuracy of measuring φ, which itself describes the difference in optical path lengths of the interferometer arms.

6.3.2 Measurement Accuracy of the Interferometer

First, let us find the mean square of the noise term:

$$\langle\psi| \hat{R}\hat{R}^\dagger |\psi\rangle = \langle\psi| \left(\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0\right) \left(\hat{a}_0 \hat{a}_1^\dagger + \hat{a}_1 \hat{a}_0^\dagger\right) |\psi\rangle = \\ \langle\psi| \left(\hat{a}_0^\dagger \hat{a}_1 \hat{a}_0 \hat{a}_1^\dagger + \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_0^\dagger + \hat{a}_1^\dagger \hat{a}_0 \hat{a}_0 \hat{a}_1^\dagger + \hat{a}_1^\dagger \hat{a}_0 \hat{a}_1 \hat{a}_0^\dagger\right) |\psi\rangle . \qquad (6.17)$$

Of the four terms in (6.17), the first three vanish upon averaging, while the last one is nonzero. For example, for the first term we have

$$\langle\psi| \hat{a}_0^\dagger \hat{a}_1 \hat{a}_0 \hat{a}_1^\dagger |\psi\rangle = \langle 0|_0 \langle\alpha|_1 \hat{a}_0^\dagger \hat{a}_1 \hat{a}_0 \hat{a}_1^\dagger |0\rangle_0 |\alpha\rangle_1 = \\ = \langle 0|_0 \langle\alpha|_1 \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \hat{a}_1^\dagger |0\rangle_0 |\alpha\rangle_1 = \langle 0|_0 \hat{a}_0^\dagger \hat{a}_0 |0\rangle_0 \langle\alpha|_1 \hat{a}_1 \hat{a}_1^\dagger |\alpha\rangle_1 = 0.$$

The last term equals

$$\langle\psi|\,\hat{a}_1^\dagger\hat{a}_0\hat{a}_1\hat{a}_0^\dagger\,|\psi\rangle = \langle 0|_0\,\langle\alpha|_1\,\hat{a}_1^\dagger\hat{a}_0\hat{a}_1\hat{a}_0^\dagger\,|0\rangle_0\,|\alpha\rangle_1 =$$
$$= \langle 0|_0\,\langle\alpha|_1\,\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\hat{a}_0^\dagger\,|0\rangle_0\,|\alpha\rangle_1 = \langle\alpha|_1\,\hat{a}_1^\dagger\hat{a}_1\,|\alpha\rangle_1\,\langle 0|_0\,\hat{a}_0\hat{a}_0^\dagger\,|0\rangle_0 =$$
$$= \langle\alpha|_1\,\hat{a}_1^\dagger\hat{a}_1\,|\alpha\rangle_1\,\langle 1|1\rangle_0 = \langle\alpha|_1\,\hat{a}_1^\dagger\hat{a}_1\,|\alpha\rangle_1 = |\alpha|^2\,. \qquad (6.18)$$

Thus, we have found that the root mean square value of the noise is equal to the mean photon number in the mode that is in the coherent state:

$$|\bar{R}|^2 = \left\langle \hat{R}\hat{R}^\dagger \right\rangle = |\alpha|^2 = \bar{n}.$$

The standard deviation is

$$\sqrt{|\bar{R}|^2} = \sqrt{\bar{n}}.$$

The signal arises due to the variation

$$\varphi = \frac{\pi}{2} + \Delta\varphi.$$

Its magnitude is

$$\bar{R} = \bar{n}\cos\left(\frac{\pi}{2} + \Delta\varphi\right) \approx -\bar{n}\Delta\varphi.$$

The minimal distinguishable signal, i.e., the threshold signal, will be considered the signal equal to the noise standard deviation:

$$|\bar{n}\Delta\varphi| \geq \sqrt{|\bar{R}|^2} = \sqrt{\bar{n}},$$

from which it follows that

$$\Delta\varphi \geq \frac{1}{\sqrt{\bar{n}}}.$$

Equality corresponds to the threshold value of $\Delta\varphi$.

As we will see later (see ch. 9), the measurement accuracy can be significantly improved if the "squeezed vacuum" is supplied to the signal arm of the interferometer.

6.4 Exercises

1. Verify the validity of the commutation relations (6.8), (6.9), and (6.10).
2. Prove the validity of (6.11).
3. Derive the interferometer equation (6.12).

Chapter 7

Quantum Description of the Polarization Properties of Light

7.1 Classical description of polarization properties of light

The field of an arbitrary monochromatic plane wave can be represented as follows:

$$\vec{E} = \vec{E}_0 e^{-i\left(\omega t - \vec{k}\vec{r}\right)},$$

where the vector

$$\vec{E}_0 = E_x \vec{e}_x + E_y \vec{e}_y$$

defines the intensity and polarization properties of the electromagnetic radiation. For describing polarization properties, the complex polarization vector— the Jones vector—is used:

$$\vec{e} = \alpha \vec{e}_x + \beta \vec{e}_y. \tag{7.1}$$

The components of this vector, α and β, can be represented as points on a certain sphere called the Poincaré sphere. The coordinates of these points are determined by two angles θ and φ

$$\alpha = \frac{E_x}{|E_x|^2 + |E_y|^2} = \cos\frac{\theta}{2},$$
$$\beta = \frac{E_y}{|E_x|^2 + |E_y|^2} = e^{i\varphi}\sin\frac{\theta}{2}.$$

To measure the components of the Jones vector α and β, the scheme shown in Figure 7.1 can be used. Here, a beam of polarized light is directed onto a Nicol

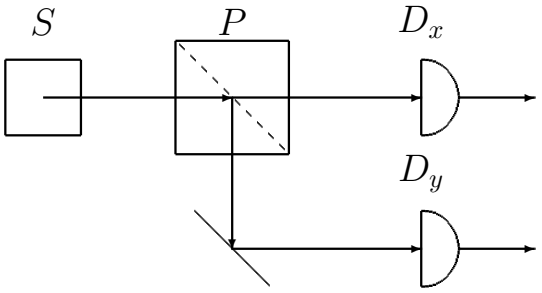


Figure 7.1: Scheme for measuring the Jones vector parameters. S - light source, P - Nicol prism, D_x and D_y - photodetectors.

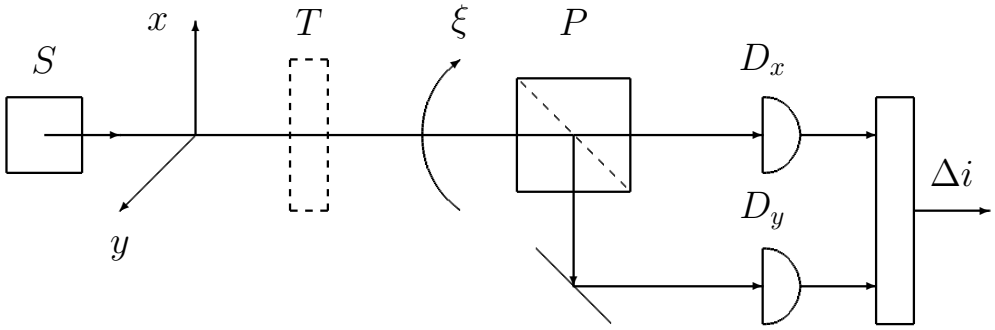


Figure 7.2: Scheme for measuring the Stokes parameters. S - light source, P - Nicol prism, ξ - angle of its orientation relative to the x axis, equal to 0 when measuring S_0 and S_1, $\frac{\pi}{4}$ when measuring S_2, T - additional quarter-wave plate $\frac{\lambda}{4}$ for measuring S_3, D_x and D_y - photodetectors, Δi - difference of the detector currents.

prism P, which separates the x and y components of this beam, which are then sent to two photodetectors D_x and D_y.

In addition to the Jones vector, the Stokes vector is often used to describe polarization properties. Its four components have the dimension of intensity and can be easily measured experimentally. The Stokes vector can be defined as follows:

$$\begin{aligned} S_0 &= |E_x|^2 + |E_y|^2 \,, \\ S_1 &= |E_x|^2 - |E_y|^2 \,, \\ S_2 &= E_x^* E_y + E_x E_y^* = 2\mathrm{Re}\left(E_x^* E_y\right) \,, \\ S_3 &= \frac{E_x^* E_y - E_x E_y^*}{i} = 2\mathrm{Im}\left(E_x^* E_y\right) \,, \end{aligned} \tag{7.2}$$

where the amplitude values $E_{x,y}$ are taken at some moment in time t. The parameter S_0 in (7.2) defines the intensity of the wave at time t, and the other three parameters S_1, S_2, and S_3 describe the polarization properties.

To measure the Stokes parameters, the scheme shown in Figure 7.2 can be used [45]. To measure S_1, S_2, and S_3, it is necessary to split the original beam

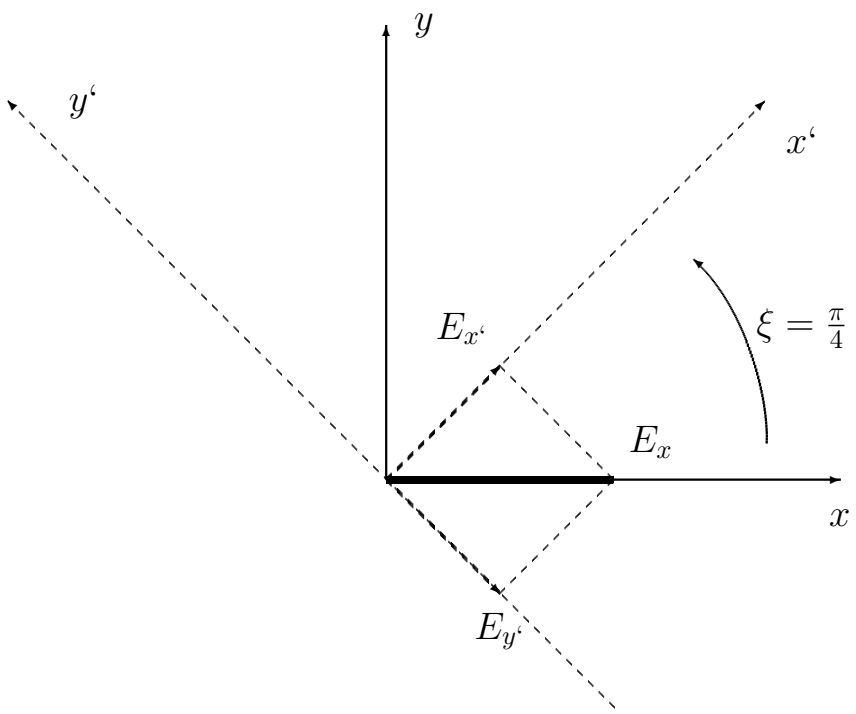


Figure 7.3: Polarization in the rotated coordinate system. $E_{x‘} = \frac{E_y+E_x}{\sqrt{2}}$, $E_{y‘} = \frac{E_y-E_x}{\sqrt{2}}$, thus $|E_{x‘}|^2 - |E_{y‘}|^2 = E_{x‘}E^*_{x‘} - E_{y‘}E^*_{y‘}$ or $|E_{x‘}|^2 - |E_{y‘}|^2 = \frac{|E_x|^2+|E_y|^2+E_xE^*_y+E^*_xE_y}{2} - \frac{|E_x|^2+|E_y|^2-E_xE^*_y-E^*_xE_y}{2}$, i.e., $|E_{x‘}|^2 - |E_{y‘}|^2 = E^*_xE_y + E_xE^*_y = 2Re\,(E^*_xE_y) = S_2$

into three parts, each of which is sent to a detector depicted in Figure 7.2.

To measure S_1, a Nicol prism P is used, which separates the x and y polarized components of the beam, so that the difference of the currents of the two photodetectors is proportional to S_1, while the sum of the currents is proportional to S_0. To measure S_2, the prism is rotated by an angle $\xi = \frac{\pi}{4}$, so that the difference of the currents is proportional to S_2 (see Figure 7.3). To measure S_3, a quarter-wave plate $\frac{\lambda}{4}$ is placed before the prism with an orientation at $\frac{\pi}{4}$, resulting in the difference of the photodetector currents being proportional to S_3.

The detector readings are averaged over time. Thus, one can speak of measuring time-averaged Stokes parameters - $\langle S_k \rangle$, which can also be used to describe partially polarized light. For quantitative characterization of such radiation, a quantity called the degree of polarization is used:

$$P = \frac{\sqrt{\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2}}{\langle S_0 \rangle}.$$

The degree of polarization of fully polarized light is $P = 1$. For completely unpolarized light, $P = 0$.

7.2 Quantum description of the polarization properties of light

When describing the polarization properties of a single-photon state, it is convenient to use the Jones vector, in which case the wave function will have the form:

$$|\psi\rangle = \alpha\,|x\rangle + \beta\,|y\rangle\,, \tag{7.3}$$

where $|\alpha|^2+|\beta|^2 = 1$, and $|x\rangle = |1\rangle_x \otimes |0\rangle_y$ denotes a single-photon state polarized along x. Similarly, $|y\rangle = |0\rangle_x \otimes |1\rangle_y$ is a single-photon state polarized along y.

The measurable quantities for us will be the Stokes parameters, so the variables (7.2) must be replaced by operators. For this, we replace the electric field operators $E_{x,y}$ with annihilation operators $\hat{a}_{x,y}$, resulting in

$$\begin{aligned}
\hat{S}_0 &= \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y,\\
\hat{S}_1 &= \hat{a}_x^\dagger \hat{a}_x - \hat{a}_y^\dagger \hat{a}_y,\\
\hat{S}_2 &= \hat{a}_x^\dagger \hat{a}_y + \hat{a}_x \hat{a}_y^\dagger,\\
\hat{S}_3 &= \frac{\hat{a}_x^\dagger \hat{a}_y - \hat{a}_x \hat{a}_y^\dagger}{i}.
\end{aligned} \tag{7.4}$$

Further, we will be primarily interested in the two operators $\hat{S}_1$ and $\hat{S}_2$, for which we will find the commutator, eigenvalues, and eigenvectors.

For the commutator $\left[\hat{S}_1, \hat{S}_2\right]$, we have:

$$\begin{aligned}
\left[\hat{S}_1, \hat{S}_2\right] &= \hat{S}_1\hat{S}_2 - \hat{S}_2\hat{S}_1 =\\
&= \hat{a}_x^\dagger \hat{a}_x \hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_y \hat{a}_y \hat{a}_x^\dagger +\\
&+ \hat{a}_x^\dagger \hat{a}_x \hat{a}_x \hat{a}_y^\dagger - \hat{a}_y^\dagger \hat{a}_y \hat{a}_y^\dagger \hat{a}_x -\\
&- \hat{a}_x^\dagger \hat{a}_x^\dagger \hat{a}_x \hat{a}_y + \hat{a}_y \hat{a}_y^\dagger \hat{a}_y \hat{a}_x^\dagger -\\
&- \hat{a}_x \hat{a}_x^\dagger \hat{a}_x \hat{a}_y^\dagger + \hat{a}_y^\dagger \hat{a}_y^\dagger \hat{a}_y \hat{a}_x =\\
&= \hat{a}_x^\dagger \left[\hat{a}_x \hat{a}_x^\dagger\right] \hat{a}_y - \left[\hat{a}_x \hat{a}_x^\dagger\right] \hat{a}_x \hat{a}_y^\dagger +\\
&+ \left[\hat{a}_y \hat{a}_y^\dagger\right] \hat{a}_y \hat{a}_x^\dagger - \hat{a}_y^\dagger \left[\hat{a}_y \hat{a}_y^\dagger\right] \hat{a}_x =\\
&= \hat{a}_x^\dagger \hat{a}_y + \hat{a}_y \hat{a}_x^\dagger - \hat{a}_x \hat{a}_y^\dagger - \hat{a}_y^\dagger \hat{a}_x =\\
&= 2\left(\hat{a}_x^\dagger \hat{a}_y - \hat{a}_x \hat{a}_y^\dagger\right) = 2i\hat{S}_3 \neq 0.
\end{aligned} \tag{7.5}$$

In the derivation of (7.5), we used the commutation relations for creation and annihilation operators:

$$\left[\hat{a}_x, \hat{a}_x^\dagger\right] = \left[\hat{a}_y, \hat{a}_y^\dagger\right] = 1,$$

as well as the fact that operators acting on different polarization components x and y commute with each other.

To find the eigenvalues and eigenvectors of the operators $\hat{S}_1$ and $\hat{S}_2$, it is convenient to represent them in matrix form. To do this, we will use the basis formed by the vectors $|x\rangle$ and $|y\rangle$:

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

For the operator $\hat{S}_1$, we have:

$$\begin{aligned} \hat{S}_1 |x\rangle &= \hat{a}_x^\dagger \hat{a}_x |1\rangle_x \otimes |0\rangle_y - \hat{a}_y^\dagger \hat{a}_y |1\rangle_x \otimes |0\rangle_y = \\ &= \hat{a}_x^\dagger \hat{a}_x |1\rangle_x \otimes |0\rangle_y = |1\rangle_x \otimes |0\rangle_y = |x\rangle , \\ \hat{S}_1 |y\rangle &= \hat{a}_x^\dagger \hat{a}_x |0\rangle_x \otimes |1\rangle_y - \hat{a}_y^\dagger \hat{a}_y |0\rangle_x \otimes |1\rangle_y = \\ &= -\hat{a}_y^\dagger \hat{a}_y |0\rangle_x \otimes |1\rangle_y = - |0\rangle_x \otimes |1\rangle_y = - |y\rangle , \end{aligned} \tag{7.6}$$

from which we obtain the following matrix representation:

$$\hat{S}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.7}$$

From (7.7) one can write the eigenvalue equation:

$$(1 - s)(1 + s) = 0,$$

from which two eigenvalues can be found, $s_1 = 1$ and $s_2 = -1$. As can be easily verified, the eigenvector for $s_1 = 1$ will be $|s_1\rangle = |x\rangle$. Indeed, from (7.6):

$$\hat{S}_1 |s_1\rangle = \hat{S}_1 |x\rangle = 1 \cdot |s_1\rangle . \tag{7.8}$$

For the second eigenvalue, the eigenvector will be $|s_2\rangle = |y\rangle$:

$$\hat{S}_1 |s_2\rangle = - |y\rangle = -1 \cdot |s_2\rangle . \tag{7.9}$$

For the operator $\hat{S}_2$, we have:

$$\begin{aligned} \hat{S}_2 |x\rangle &= \hat{a}_x^\dagger \hat{a}_y |1\rangle_x \otimes |0\rangle_y + \hat{a}_y^\dagger \hat{a}_x |1\rangle_x \otimes |0\rangle_y = \\ &= \hat{a}_y^\dagger \hat{a}_x |1\rangle_x \otimes |0\rangle_y = |0\rangle_x \otimes |1\rangle_y = |y\rangle , \\ \hat{S}_2 |y\rangle &= \hat{a}_x^\dagger \hat{a}_y |0\rangle_x \otimes |1\rangle_y + \hat{a}_y^\dagger \hat{a}_x |0\rangle_x \otimes |1\rangle_y = \\ &= \hat{a}_x^\dagger \hat{a}_y |0\rangle_x \otimes |1\rangle_y = |1\rangle_x \otimes |0\rangle_y = |x\rangle . \end{aligned} \tag{7.10}$$

From (7.10) we obtain the following matrix representation of the operator $\hat{S}_2$:

$$\hat{S}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.11}$$

Analogous to the operator $\hat{S}_1$, from (7.10) and (7.11) one can find two eigenvalues $s_1 = 1$ and $s_2 = -1$. For the first eigenvalue, the eigenvector is

$$|s_1\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle + |y\rangle \right), \tag{7.12}$$

and for the second

$$|s_2\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle - |y\rangle \right). \tag{7.13}$$

For the operator $\hat{S}_3$, we have:

$$\begin{aligned} \hat{S}_3 |x\rangle &= \frac{1}{i} \left(\hat{a}_x^\dagger \hat{a}_y |1\rangle_x \otimes |0\rangle_y - \hat{a}_y^\dagger \hat{a}_x |1\rangle_x \otimes |0\rangle_y \right) = \\ &= -\frac{1}{i} \hat{a}_y^\dagger \hat{a}_x |1\rangle_x \otimes |0\rangle_y = -\frac{1}{i} |0\rangle_x \otimes |1\rangle_y = -\frac{|y\rangle}{i} = i |y\rangle, \\ \hat{S}_2 |y\rangle &= \frac{1}{i} \left(\hat{a}_x^\dagger \hat{a}_y |0\rangle_x \otimes |1\rangle_y + \hat{a}_y^\dagger \hat{a}_x |0\rangle_x \otimes |1\rangle_y \right) = \\ &= \frac{1}{i} \hat{a}_x^\dagger \hat{a}_y |0\rangle_x \otimes |1\rangle_y = \frac{1}{i} |1\rangle_x \otimes |0\rangle_y = -i |x\rangle. \end{aligned} \tag{7.14}$$

Thus, from (7.14) we obtain the following matrix representation of the operator $\hat{S}_3$:

$$\hat{S}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{7.15}$$

From (7.15), one can find that the eigenvalues are again $s_1 = 1$ and $s_2 = -1$. At the same time, the eigenstates of the operator $\hat{S}_3$ are the states with left and right circular polarizations:

$$\begin{aligned} |s_1\rangle = |-\rangle &= \frac{1}{\sqrt{2}} \left(|x\rangle - i |y\rangle \right), \\ |s_2\rangle = |+\rangle &= \frac{1}{\sqrt{2}} \left(|x\rangle + i |y\rangle \right). \end{aligned} \tag{7.16}$$

TBD: Add the connection with Pauli matrices

7.3 Exercises

1. TBD

Part 3

Nonclassical Light

Chapter 8

Nonclassical Light

Quantum states of light can be divided into two groups: states that asymptotically approach a classical state (classical light) as the intensity of the light increases, and states that do not possess this property. For example, among the states we considered earlier, the coherent state has a classical limit, while the number (Fock) state does not.

Earlier we defined the concept of the second-order coherence function $G^{(2)}$. For single-mode light (5.33)

$$G^{(2)} = \frac{\left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \right\rangle}{\left\langle \hat{a}^\dagger \hat{a} \right\rangle^2} = \\ = \frac{\left\langle \hat{a}^\dagger \hat{a} \left(\hat{a}^\dagger \hat{a} - 1 \right) \right\rangle}{\left\langle \hat{a}^\dagger \hat{a} \right\rangle^2} = \frac{\left\langle n^2 \right\rangle - \left\langle n \right\rangle}{\left\langle n \right\rangle^2}, \tag{8.1}$$

where the commutation relations $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, $\hat{n} = \hat{a}^\dagger \hat{a}$ is the particle number operator, have been used.

In the classical case, $\left\langle \hat{a}^\dagger \hat{a} \right\rangle = |\alpha|^2$, where $|\alpha|^2 = I$ is the intensity of the field (the squared modulus of the field amplitude). Since classical quantities commute, formula (8.1) in the classical case can be replaced by

$$G^{(2)}_{\text{cl}} = \frac{\left\langle |\alpha|^4 \right\rangle}{\left\langle |\alpha|^2 \right\rangle^2} = \frac{\left\langle I^2 \right\rangle}{\left\langle I \right\rangle^2}, \tag{8.2}$$

where averaging is performed with the help of a positive definite classical distribution function $P(\alpha)$

$$\left\langle |\alpha|^{2n} \right\rangle = \int_0^\infty P\left(|\alpha|^2 \right) |\alpha|^{2n} \, d^2\alpha, \quad n = 1, 2.$$

Consider the difference $G^{(2)}_{\rm cl} - G^{(1)}_{\rm cl} = G^{(2)}_{\rm cl} - 1$, since for a single-mode field, as we know, $G^{(1)}_{\rm cl} = 1$ always. From (8.2) it follows

$$\begin{aligned} G^{(2)}_{\rm cl} - G^{(1)}_{\rm cl} = G^{(2)}_{\rm cl} - 1 = \\ = \frac{\left\langle |\alpha|^4 \right\rangle}{\left\langle |\alpha|^2 \right\rangle^2} - 1 = \frac{\left\langle |\alpha|^4 \right\rangle - \left\langle |\alpha|^2 \right\rangle^2}{\left\langle |\alpha|^2 \right\rangle^2} = \\ = \frac{\int_0^\infty P\,|\alpha|^4\, d^2\alpha - 2\left\langle |\alpha|^2 \right\rangle^2 + \left\langle |\alpha|^2 \right\rangle^2}{\left\langle |\alpha|^2 \right\rangle^2} = \\ = \frac{\int_0^\infty P\left(|\alpha|^4 - 2\,|\alpha|^2 \left\langle |\alpha|^2 \right\rangle + \left\langle |\alpha|^2 \right\rangle^2\right) d^2\alpha}{\left\langle |\alpha|^2 \right\rangle^2} = \\ = \frac{\int_0^\infty P\left(|\alpha|^2 - \left\langle |\alpha|^2 \right\rangle\right)^2 d^2\alpha}{\left\langle |\alpha|^2 \right\rangle^2} \geq 0. \end{aligned} \tag{8.3}$$

From inequality (8.3) it follows that

$$G^{(2)}_{\rm cl} \geq 1.$$

In the quantum case, due to operator noncommutativity, the situation will be different. Formula (8.1) can be rewritten in the following form:

$$\begin{aligned} G^{(2)} = \frac{\left\langle \hat{n}^2 \right\rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} = \\ = 1 + \frac{\left(\left\langle \hat{n}^2 \right\rangle - \langle \hat{n} \rangle^2\right) - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} = 1 + \frac{\sigma^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}, \end{aligned} \tag{8.4}$$

where σ^2 is the variance, i.e., σ is the standard deviation.

From (8.4) it follows that $G^{(2)}$ can be either greater or less than 1, depending on whether σ^2 or $\langle \hat{n} \rangle$ is larger. In the classical case $G^{(2)} \geq 1$, so a criterion of nonclassicality can be taken as

$$G^{(2)} < 1. \tag{8.5}$$

From (8.4, 8.5) it follows that the condition for nonclassicality is $\sigma^2 - \langle \hat{n} \rangle < 0$, i.e., the variance must be less than the mean photon number. It is known that a

Figure 8.1: Examples of photo count realizations corresponding to different statistical properties of light: a) bunching and super-Poissonian statistics, b) Poissonian statistics, c) antibunching and sub-Poissonian statistics (based on the data from [46]).

photon flux for which $\sigma^2 - \langle \hat{n} \rangle = 0$, has Poissonian statistics (an example is the coherent state). The case $\sigma^2 - \langle \hat{n} \rangle < 0$ corresponds to a more regular flux, called sub-Poissonian. This is the case of photon antibunching. An example of a state for which this holds is, for instance, a state resulting from parametric scattering, which we will consider later. The case $\sigma^2 - \langle \hat{n} \rangle > 0$ corresponds to a less regular process where photons bunch (taken as the thermal excitation of light). Such a state is called super-Poissonian. All three cases are shown in Figure 8.1, where the timing of photodetection events is illustrated.

We conclude that nonclassical states correspond to the sub-Poissonian threshold. The number $G^{(2)}$ is simply related to other parameters characterizing photon number fluctuations: the variance and the Fano factor, which frequently appear in the literature.

$$\Phi = \frac{\sigma^2}{\langle \hat{n} \rangle},$$
$$G^{(2)} - 1 = \frac{\Phi - 1}{\langle \hat{n} \rangle}$$

The statistical nature of the experiment can be established from an experiment in which the second-order correlation function is measured (the Brown-Twiss experiment). The experimental setup is shown in Figure 8.2.

The adjustable delay τ is implemented by changing the length of one of the arms.

Figure 8.3 shows three curves obtained from the experiment. Curve (b) corresponds to the coherent state. Curve (a) corresponds to photon bunching and super-Poissonian statistics. Curve (c) corresponds to antibunching and sub-Poissonian statistics. Such a curve is characteristic of nonclassical light. The

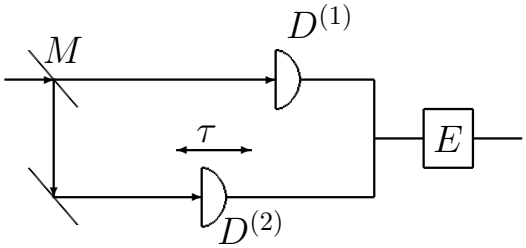


Figure 8.2: Scheme of the Brown-Twiss experiment. Using a 50% mirror M, the light beam is directed to two photodetectors $D^{(1)}$ and $D^{(2)}$. The adjustable delay τ is accomplished by moving the photodetector $D^{(2)}$. The output from the photodetectors is fed into the coincidence circuit E.

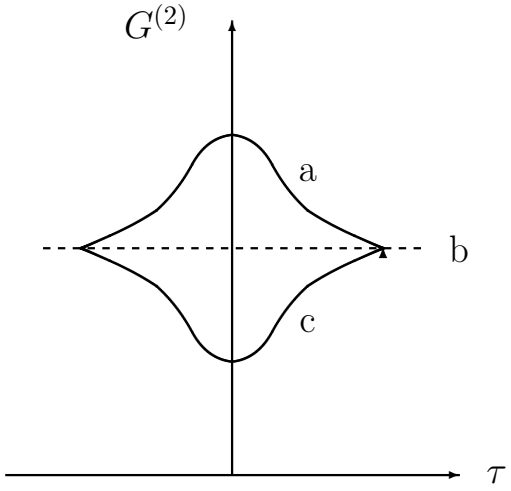


Figure 8.3: Qualitative form of the dependence of the correlation function $G^{(2)}$ on the delay τ for 3 different cases: a) bunching and super-Poissonian statistics, b) Poissonian statistics, c) antibunching and sub-Poissonian statistics.

explanation of the behavior of the curves is simple. If τ is large, photodetection events are random and independent of each other. Therefore, the behavior of all three curves is determined by random coincidences and will be the same for all three. For small τ the behavior of the curves differs. In the case of photon bunching, there is a maximum at $\tau = 0$, and for antibunching – a minimum.

Another criterion for nonclassicality is the absence of positive definiteness of the quasiprobability distribution $P(\alpha)$ when using the coherent state representation.

In the classical case, the function $P(\alpha)$ must be everywhere positive definite, i.e., $P(\alpha) \geq 0$ for all values of α. Let us consider this issue in more detail. The condition of nonclassicality $G^{(2)} < 1$, using (8.1), can be represented in the form

$$\left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \right\rangle - \left\langle \hat{a}^\dagger \hat{a} \right\rangle^2 < 0. \tag{8.6}$$

Consider the expression

$$\int P(\alpha) \left(|\alpha|^2 - \langle \hat{n} \rangle \right)^2 d^2\alpha < 0 \tag{8.7}$$

$$\int P(\alpha) \left(|\alpha|^2 - \langle \hat{n} \rangle \right)^2 d^2\alpha = \int P(\alpha) \left(|\alpha|^4 - 2 |\alpha|^2 \langle \hat{n} \rangle + \langle \hat{n} \rangle^2 \right) d^2\alpha =$$
$$= \left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \right\rangle - \left\langle \hat{a}^\dagger \hat{a} \right\rangle^2 < 0,$$

i.e., expressions (8.6) and (8.7) are equivalent. This follows from the following relations

$$\left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \right\rangle = \mathrm{Tr}\left(\hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\rho \right) =$$
$$= \sum_n \langle n | \left(\hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \int P(\alpha) |\alpha\rangle \langle \alpha| \, d^2\alpha \right) |n\rangle =$$
$$= \int P(\alpha) \langle \alpha | \, \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \, |\alpha\rangle \, d^2\alpha = \int P(\alpha) |\alpha|^4 \, d^2\alpha,$$
$$\int P(\alpha) |\alpha|^2 \, d^2\alpha = \left\langle \hat{a}^\dagger \hat{a} \right\rangle = \langle \hat{n} \rangle .$$

We have found that expression (8.7) must be negative. Since $\left(|\alpha|^2 - \langle \hat{n} \rangle \right)^2$ is always positive, a negative result will be obtained if $P(\alpha)$ is negative in at least part of the domain of α. Thus, for nonclassical light, the quasiprobability $P(\alpha)$ is not a positive definite function.

Chapter 9

Squeezed States

In squeezed states, the possibility allowed by the Heisenberg uncertainty relation is realized: to reduce the uncertainty of one of the conjugate observables at the expense of increasing the uncertainty of the other. Such states may prove useful in quantum optics for practical needs. For example, to improve the accuracy of interference experiments.

Next, the theory of squeezed states in parametric processes and the applications of squeezed states in interference experiments are considered.

9.1 Heisenberg Uncertainty Relation

In quantum mechanics, two conjugate observables A and B correspond to non-commuting operators $\hat{A}$ and $\hat{B}$, satisfying the commutation relation

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C},$$

where $\hat{C}$ is some Hermitian operator. In this case, the observables A and B can only be measured with some uncertainty, expressed by the Heisenberg uncertainty relation (see in more detail subsection 14.3.1)

$$(\Delta A \Delta B) \geq \frac{1}{2}\left|\left\langle \hat{C} \right\rangle\right|, \tag{9.1}$$

where

$$(\Delta A)^2 = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2$$

and

$$(\Delta B)^2 = \left\langle \hat{B}^2 \right\rangle - \left\langle \hat{B} \right\rangle^2$$

are the mean square deviations from the mean value. The average $\langle \dots \rangle$ corresponds to the quantum state in which the quantum object is located.

Relation (9.1) limits only the product of uncertainties, allowing states for which the uncertainty of one of the observables is significantly less than the uncertainty of the other. If

$$(\Delta A)^2 < \frac{1}{2}\left|\left\langle \hat{C} \right\rangle\right|,$$

such a state is called a squeezed state for the observable A. If, in addition, the minimal uncertainty condition

$$(\Delta A \Delta B) = \frac{1}{2}\left|\left\langle \hat{C} \right\rangle\right|,$$

is observed, such a state is called an ideally squeezed state.

9.2 Operators of Quadrature Components of the Electromagnetic Field

Consider the single-mode electromagnetic field operator

$$\hat{\vec{E}} = E_1 \vec{e}\left[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right]$$

and introduce two new operators related to it

$$\hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right),$$
$$\hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right),$$

which are proportional to operators $\hat{q}$ and $\hat{p}$ (coordinate, momentum) that we encountered earlier.

The operators $\hat{X}_1$ and $\hat{X}_2$ are Hermitian operators. Their commutation relations are as follows.

$$\left[\hat{X}_1, \hat{X}_2\right] = \left(\hat{X}_1\hat{X}_2 - \hat{X}_2\hat{X}_1\right) =$$
$$= \frac{1}{4i}\left\{\left(\hat{a} + \hat{a}^\dagger\right)\left(\hat{a} - \hat{a}^\dagger\right) - \left(\hat{a} - \hat{a}^\dagger\right)\left(\hat{a} + \hat{a}^\dagger\right)\right\} =$$
$$\frac{1}{4i}\left\{\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right\} = \frac{1}{2i}\left\{\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right\} =$$
$$= \frac{1}{2i}\left[\hat{a}^\dagger, \hat{a}\right] = \frac{i}{2}$$

Using the operators $\hat{X}_1$ and $\hat{X}_2$, the electric field operator can be represented as

$$E = 2E_1\vec{e}\left(\hat{X}_1 \cos\omega t + \hat{X}_2 \sin\omega t,\right)$$

i.e., these operators can be considered as operators of the quadrature components of the electromagnetic field. From the uncertainty relation (9.1) we have

$$(\Delta X_1 \Delta X_2) \geq \frac{1}{4}.$$

The condition for a squeezed state is

$$(\Delta X_i) < \frac{1}{4},\ i = 1, 2.$$

If at the same time the minimal uncertainty relation holds

$$(\Delta X_1 \Delta X_2) = \frac{1}{4},$$

then the squeezing is ideal.

As an example, consider the uncertainties of $\hat{X}_1$, $\hat{X}_2$ in the case when the field is in a coherent state:

$$\begin{aligned}(\Delta X_1)^2 &= \langle\alpha|\, \hat{X}_1^2\, |\alpha\rangle - \langle\alpha|\, \hat{X}_1\, |\alpha\rangle^2 = \\ &= \frac{1}{4}\left\{\langle\alpha|\left(\hat{a}+\hat{a}^\dagger\right)^2|\alpha\rangle - \langle\alpha|\left(\hat{a}+\hat{a}^\dagger\right)|\alpha\rangle^2\right\} = \\ &= \frac{1}{4}\left\{\alpha^{*2}+\alpha^2+2|\alpha|^2+1-\left(\alpha^{*2}+\alpha^2+2|\alpha|^2\right)\right\} = \frac{1}{4}.\end{aligned}$$

Similarly, one can obtain

$$(\Delta X_2)^2 = \frac{1}{4} \tag{9.2}$$

and

$$(\Delta X_1 \Delta X_2) = \frac{1}{4}. \tag{9.3}$$

Thus, we have found that the considered case is a state with minimal uncertainty, but there is no squeezing.

Now consider what the energy eigenstate $|n\rangle$ gives. We have

$$\begin{aligned}(\Delta X_1)^2 &= \frac{1}{4}\left\{\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^2|n\rangle - \langle n|\left(\hat{a}+\hat{a}^\dagger\right)|n\rangle^2\right\} = \\ &= \frac{1}{4}\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^2|n\rangle = \frac{1}{4}\langle n|\left(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)|n\rangle = \\ &= \frac{1}{4}\langle n|\left(1+2\hat{a}^\dagger\hat{a}\right)|n\rangle = \frac{1}{4}(2n+1).\end{aligned}$$

Similarly,

$$(\Delta X_2)^2 = \frac{1}{4}(2n+1) \tag{9.4}$$

and

$$(\Delta X_1 \Delta X_2) = \frac{1}{4}(2n+1). \tag{9.5}$$

Therefore, the energy eigenstate is neither a state with minimal uncertainty nor a squeezed state for the observables X_1 or X_2.

9.3 Squeezed Quadrature State

First, let's consider the theoretical possibility of creating squeezed quadrature states. Consider constructing a squeezed state from a coherent state, in which one of the quadrature components will be squeezed. The state transformation is performed using a certain unitary transformation.

Consider the unitary operator known as the squeezing operator (the reason for this name will become clear later)

$$\hat{S}(z) = e^{\frac{1}{2}z^*\hat{a}^2 - \frac{1}{2}z\hat{a}^{+2}},$$

where $z = re^{i\theta}$ is an arbitrary complex number. The conjugate operator has the form

$$\hat{S}^\dagger(z) = e^{\frac{1}{2}z\hat{a}^{+2} - \frac{1}{2}z^*\hat{a}^2},$$

from which it follows that

$$\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = \hat{I},$$

where $\hat{I}$ is the identity operator. Thus, $\hat{S}^\dagger = \hat{S}^{-1}$ is indeed a unitary operator.

Consider the action of the operator $\hat{S}(z)$ on the operators $\hat{a}$ and $\hat{a}^\dagger$ and on the coherent state. For this purpose, we apply the operator decomposition theorem (see detailed derivation in the theorem on operator identity (Theorem 17.1.1)):

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \ldots \tag{9.6}$$

Using (9.6), it is easy to show, by taking

$$e^{\hat{A}} = \hat{S}(z), \quad \hat{B} = \hat{a},$$

that

$$\hat{A} = \hat{S}(z)\,\hat{a}\hat{S}^\dagger(z) = \hat{a} + z\hat{a}^\dagger + \frac{|z|^2\hat{a}}{2!} + \frac{z|z|^2\hat{a}^\dagger}{3!} + \cdots = \\ = \hat{a}\cosh r + \hat{a}^\dagger e^{i\theta}\sinh r = \mu\hat{a} + \nu\hat{a}^\dagger, \tag{9.7}$$

where $\mu = \cosh r$, $\nu = e^{i\theta} \sinh r$, $|\mu|^2 - |\nu|^2 = 1$. Similarly, it can be shown that for the reverse order of operators $\hat{S}^\dagger \ldots \hat{S}$ the formula differs from (9.7) only by the sign between the terms:

$$\begin{aligned} \hat{S}^\dagger(z)\, \hat{a} \hat{S}(z) &= \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r, \\ \hat{S}^\dagger(z)\, \hat{a}^\dagger \hat{S}(z) &= \hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r. \end{aligned} \tag{9.8}$$

Let's act with the operator $\hat{S}$ on the coherent state vector. Then we obtain a new state:

$$|\alpha, z\rangle = \hat{S}(z) |\alpha\rangle . \tag{9.9}$$

Now show that the state $|\alpha, z\rangle$ is an eigenstate of the operator $\hat{A}$:

$$\begin{aligned} \hat{A} |\alpha, z\rangle = \hat{S}(z)\, \hat{a} \hat{S}^\dagger(z)\, \hat{S}(z) |\alpha\rangle = \hat{S}(z)\, \hat{a} |\alpha\rangle = \\ = \alpha \hat{S}(z) |\alpha\rangle = \alpha |\alpha, z\rangle . \end{aligned} \tag{9.10}$$

Here the definition of $\hat{A}$ and $\hat{S}\hat{S}^\dagger = \hat{I}$ were used. From (9.10) it follows that the state $|\alpha, z\rangle$ is an eigenstate of the operator $\hat{A}$, and the eigenvalue coincides with the eigenvalue of the coherent state from which the state $|\alpha, z\rangle$ is obtained. The conjugate equality is

$$\langle \alpha, z| \hat{A}^\dagger = \alpha^* \langle \alpha, z| \tag{9.11}$$

Based on relations (9.10) and (9.11), by analogy with the operators $\hat{a}$ and $\hat{a}^\dagger$, $\hat{A}$ and $\hat{A}^\dagger$ are called quasiparticle creation and quasiparticle annihilation operators.

It remains to determine whether the state $|\alpha, z\rangle$ is a squeezed state. To this end, the uncertainty relation for ΔX_1 and ΔX_2 should be written. We have

$$(\Delta X_{1,2})^2 = \langle \alpha, z| \hat{X}_{1,2}^2 |\alpha, z\rangle - \langle \alpha, z| \hat{X}_{1,2} |\alpha, z\rangle^2 .$$

This expression, using (9.9), can be rewritten as

$$(\Delta X_{1,2})^2 = \langle \alpha| \hat{S}^\dagger(z)\, \hat{X}_{1,2}^2 \hat{S}(z) |\alpha\rangle - \langle \alpha| \hat{S}^\dagger(z)\, \hat{X}_{1,2} \hat{S}(z) |\alpha\rangle^2 . \tag{9.12}$$

The operators $\hat{X}_{1,2}$ can be expressed through $\hat{a}$ and $\hat{a}^\dagger$:

$$\begin{aligned} \hat{X}_1 &= \frac{1}{2} \left(\hat{a} + \hat{a}^\dagger \right), \\ \hat{X}_2 &= \frac{1}{2i} \left(\hat{a} - \hat{a}^\dagger \right). \end{aligned}$$

Substituting this into (9.12) and transforming the operators $\hat{a}$ and $\hat{a}^\dagger$ using (9.8), we obtain the final expressions for ΔX_1 and ΔX_2: (under the condition $z = r$,

i.e. $\theta = 0$)

$$(\Delta X_1)^2 = \frac{1}{4}e^{-2r},$$
$$(\Delta X_2)^2 = \frac{1}{4}e^{2r},$$
$$(\Delta X_1 \Delta X_2) = \frac{1}{4}, \tag{9.13}$$

where $r = |z|$ is the squeezing parameter. From (9.13) it follows that the state $|\alpha, z\rangle$ is indeed a squeezed state for one of the quadrature components. Since the uncertainty product has the minimum value, the state is an ideally squeezed state. We will call this state the ideally squeezed quadrature state.

If the operator $\hat{S}(z)$ acts on the vacuum state ($\alpha = 0$), we obtain the squeezed vacuum state (squeezed vacuum)

$$\hat{S}(z)|0\rangle = |z, 0\rangle .$$

The average photon number in the quadrature squeezed coherent state is given by the expression:

$$\begin{aligned}\langle \alpha, z| \hat{a}^\dagger \hat{a} |\alpha, z\rangle = \langle \alpha| \hat{S}^\dagger \hat{a}^\dagger \hat{a} \hat{S} |\alpha\rangle = \\ = \langle \alpha| \hat{S}^\dagger \hat{a}^\dagger \hat{S} \hat{S}^\dagger \hat{a} \hat{S} |\alpha\rangle = \\ = \langle \alpha| \left(\hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r\right) \left(\hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r\right) |\alpha\rangle = \\ = \left(|\alpha|^2 \left(\cosh^2 r + \sinh^2 r\right) - (\alpha^*)^2 e^{i\theta} \sinh r \cosh r - \alpha^2 \sinh r \cosh r e^{-i\theta} + \sinh^2 r\right)\end{aligned} \tag{9.14}$$

In deriving (9.14) we used relation (9.8).

For the squeezed vacuum state, $\alpha = 0$. Then the average photon number in this state is

$$\langle 0, z| \hat{a}^\dagger \hat{a} |0, z\rangle = \sinh^2 r \neq 0, \tag{9.15}$$

where r is the squeezing parameter.

Obviously, the squeezed vacuum state is not a true vacuum state, since it can be detected by a photodetector using the photoelectric effect, which is impossible for a true vacuum state.

Visually, quadrature squeezed states are shown in Figures 9.1-9.3, where initial states and time evolution of oscillations in various squeezed states are presented. The graphs on the right depict the uncertainty regions, which have the shape of ellipses. Such an image could be seen if we had something like an optical stroboscopic oscilloscope, which in reality does not exist.

Let's examine the obtained states in more detail. The formula (9.9) can be written somewhat differently by using the representation of the coherent state via

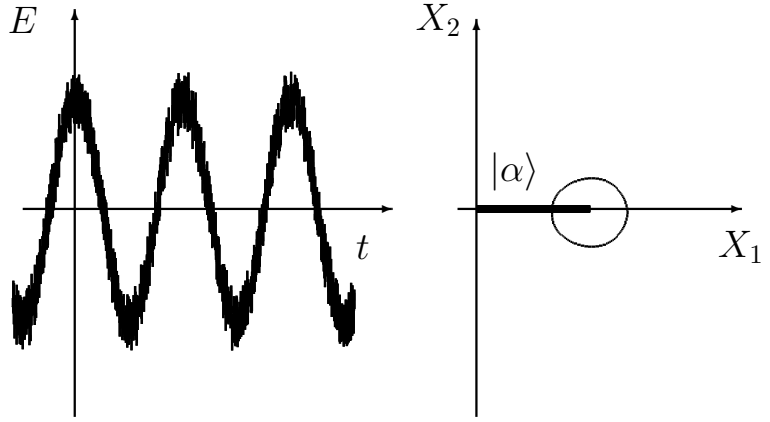


Figure 9.1: Uncertainty region and the corresponding dependence of the electric field on time for a coherent state. The complex number α, corresponding to the considered state $|\alpha\rangle$ on the complex plane defined by X_1 and X_2, lies along the X_1 axis. In this case, the number α is real, the average value $\langle\alpha|\,\hat{X}_1\,|\alpha\rangle = \frac{\alpha+\alpha^*}{2} = Re\alpha = \alpha$. Thus, the uncertainty ΔX_1 can be interpreted as the amplitude uncertainty, and ΔX_2 as the phase uncertainty. In the case of a coherent (unsqueezed) state shown in the figure, the uncertainty in phase and amplitude is the same.

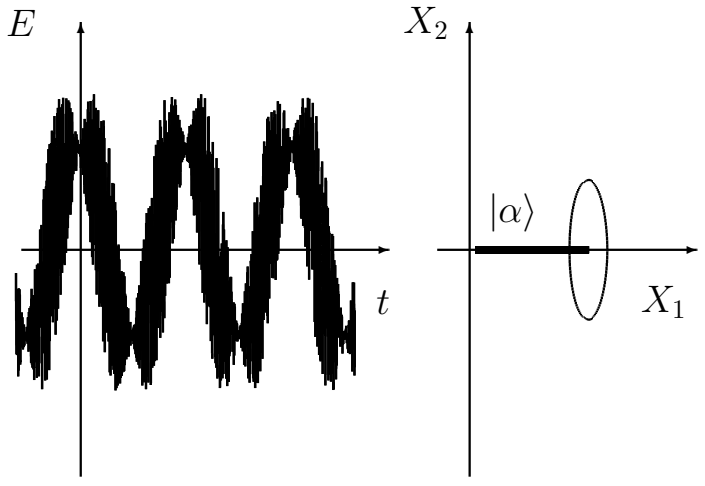


Figure 9.2: Uncertainty region and the corresponding dependence of the electric field on time for the coherent state squeezed along X_1. The uncertainty in amplitude is minimal, which allows for highly precise measurements of the peaks of the electric field intensity.

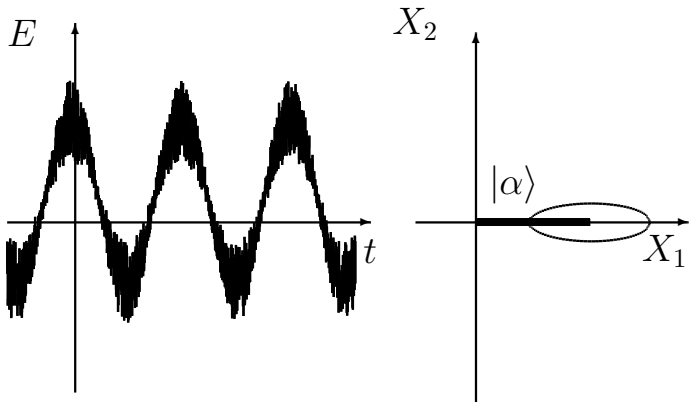


Figure 9.3: Uncertainty region and the corresponding time dependence of the electric field for a coherent state squeezed along X_2. The phase uncertainty is minimal, which allows for high-precision measurement of the oscillation period: the times at which the electric field intensity equals zero.

the vacuum state in the form (1.100):

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\,|0\rangle = \hat{D}\,|0\rangle\,,$$

where $\hat{D} = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ is called the displacement operator. $\hat{D}$ is a unitary operator, since

$$\hat{D}^\dagger = e^{-\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)},$$

from which

$$\hat{D}^\dagger\hat{D} = \hat{D}\hat{D}^\dagger = \hat{I},$$

i.e. $\hat{D}^\dagger = \hat{D}^{-1}$.

The name of the operator is related to the fact that the action of $\hat{D}\,(\alpha)$ on the operators $\hat{a}$ and $\hat{a}^\dagger$ leads to their displacement by the amount α (α^*):

$$\begin{aligned}\hat{D}^\dagger\,(\alpha)\,\hat{a}\hat{D}\,(\alpha) &= \hat{a} + \alpha,\\ \hat{D}^\dagger\,(\alpha)\,\hat{a}^\dagger\hat{D}\,(\alpha) &= \hat{a}^\dagger + \alpha^*,\end{aligned} \tag{9.16}$$

This result follows from the operator decomposition theorem (9.6) when taking $\hat{A} = -\alpha\hat{a}^\dagger + \alpha^*\hat{a}$, $B = \hat{a}, \hat{a}^\dagger$.

Thus, the coherent state $|\alpha\rangle$ is a vacuum state displaced by α (see Figure 9.4). Based on the above, a squeezed state can be considered as a result of two actions: displacement of the vacuum state and its subsequent squeezing, as shown in Figure 9.5.

The squeezing procedure considered here is not unique. One can apply another sequence of operations – first squeeze the vacuum state, then displace it to obtain a new squeezed state (see Figure 9.6). In operator form, this operation can be written as:

$$|\alpha, z\rangle = \hat{D}\,(\alpha)\,\hat{S}\,(z)\,|0\rangle\,. \tag{9.17}$$

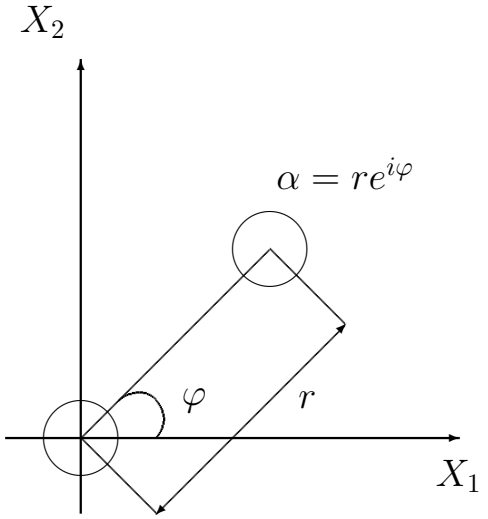


Figure 9.4: Coherent state as a displaced vacuum state. The uncertainties of observables $\hat{X}_1$ and $\hat{X}_2$ are: $\Delta X_1 = \Delta X_2 = \frac{1}{2}$.

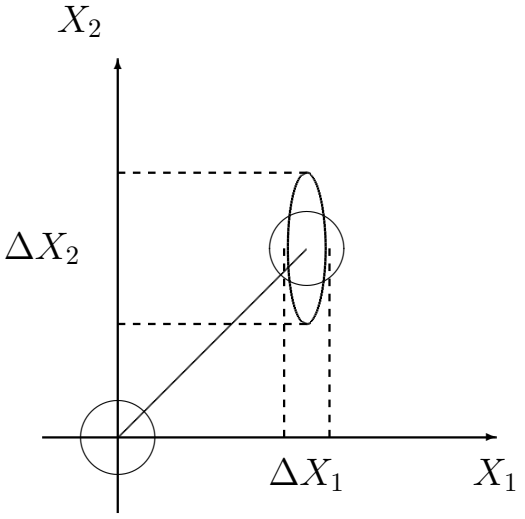


Figure 9.5: State squeezed along X_1 as a result of two actions: displacement of the vacuum state followed by its squeezing.

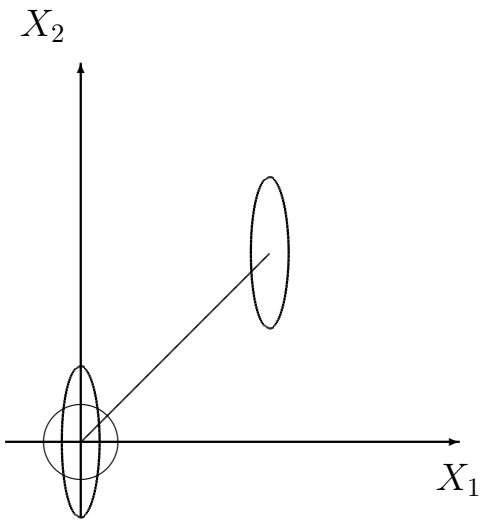


Figure 9.6: Squeezed state as a result of squeezing the vacuum state followed by its displacement.

To distinguish the states, the parameter order here is reversed compared to

$$|z, \alpha\rangle = \hat{S}(z)\,\hat{D}(\alpha)\,|0\rangle\,.$$

The state (9.17) also corresponds to a quadrature squeezed state, but they are not entirely identical. We will not consider this case in detail here.

The above results and illustrations shown in Figure 9.1-9.3 were obtained for the case when parameters $z = re^{i\theta}$ and $\alpha = |\alpha|\,e^{i\varphi}$ are real ($\varphi = \theta = 0$). If the parameters are complex, the picture changes somewhat (see Figure 9.7): the squeezing axis is rotated by an angle $\frac{\theta}{2}$ relative to the X_1, X_2 axes, and the position of the uncertainty region center is rotated by an angle φ relative to X_1. By changing the phase of the light beams used in the generation and detection process, we can move to new quadratures Y_1, Y_2 as shown in Figure 9.7. If we further compensate the phase shift φ, we return to the situation depicted in Figures 9.1-9.3. From Figure 9.7, it is clear that by measuring the angles θ and φ it is possible, for example, to transform a state squeezed in X_1 into a state squeezed in X_2, and vice versa.

9.4 Generation of Quadrature-Squeezed States

So far, we have considered the theoretical possibility of the existence and generation of squeezed states. The question arises — in what real processes can one obtain light in a squeezed state? Here we consider the possibility of obtaining a quadrature squeezed state through parametric interaction occurring when a strong pump field passes through a medium with quadratic nonlinearity. In such

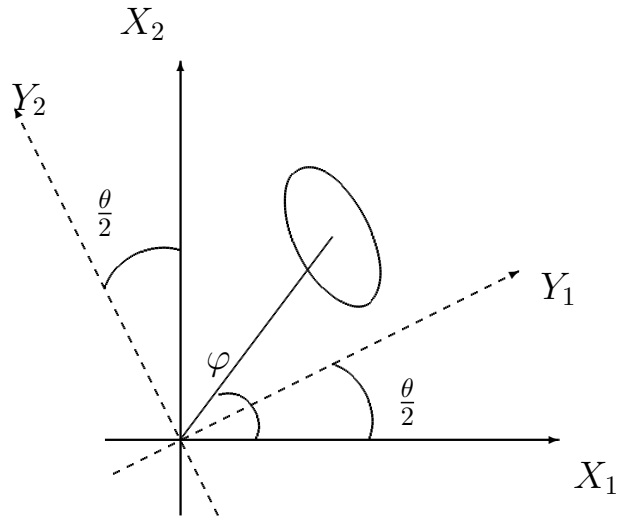


Figure 9.7: Squeezed coherent state. The squeezing axis is rotated by an angle $\frac{\theta}{2}$ relative to the axes X_1, X_2, and the position of the center of the uncertainty region is rotated by an angle φ relative to X_1.

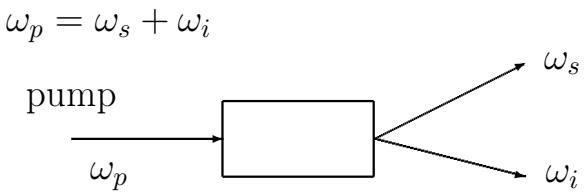


Figure 9.8: Generation of a squeezed state during parametric interaction

an interaction, two waves arise, related to the pump frequency by the relations:

$$\omega_p = \omega_s + \omega_i,$$

where ω_p is the pump frequency, ω_s is the signal frequency, ω_i is the idler frequency. This is schematically shown in Figure 9.8.

The interaction Hamiltonian between the pump, signal, and idler waves in the interaction picture is given by:

$$\hat{V} = \hbar\kappa \left(\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{b} + \hat{a}_s \hat{a}_i \hat{b}^\dagger \right), \tag{9.18}$$

where $\hat{a}_s^\dagger$, $\hat{a}_i^\dagger$, $\hat{a}_s$, and $\hat{a}_i$ are creation and annihilation operators for the signal and idler waves, and κ is the interaction constant.

In the degenerate regime, the signal and idler waves have the same frequency equal to half the pump frequency (see Figure 9.9):

$$\omega_s = \omega_i = \frac{\omega_p}{2} = \omega.$$

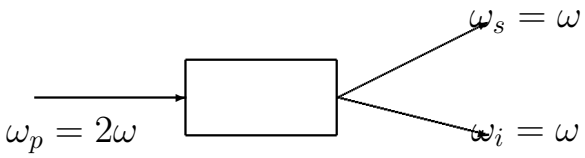


Figure 9.9: Generation of a squeezed state in degenerate parametric interaction

For the degenerate parametric process, the interaction Hamiltonian (9.18) simplifies to:

$$\hat{V} = \hbar\kappa \left(\left(\hat{a}^\dagger\right)^2 \hat{b} + \hat{a}^2 \hat{b}^\dagger \right).$$

If the pump field is in a coherent state with large amplitude $\alpha_p = A_p e^{i\varphi}$, $|\alpha_p| \gg 1$, the pump can be replaced by a classical field. Then the interaction Hamiltonian further simplifies to:

$$\hat{V} = \hbar\kappa A_p \left(\left(\hat{a}^\dagger\right)^2 e^{-i\varphi} + \hat{a}^2 e^{i\varphi} \right),$$

where A_p and φ are the real amplitude and phase of the pump. Obviously, in this approximation, we neglect pump depletion. This is valid as long as the amplitude of the signal (idler) wave remains small compared to the pump field amplitude. The Heisenberg equation for the operator $\hat{a}$ takes the form:

$$\begin{aligned}
\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{V}, \hat{a}\right] = i\kappa A_p \left[\left(\hat{a}^\dagger\right)^2 e^{-i\varphi} + \hat{a}^2 e^{i\varphi}, \hat{a} \right] = \\
= i\kappa A_p e^{-i\varphi} \left(\left(\hat{a}^\dagger\right)^2 \hat{a} - \hat{a} \left(\hat{a}^\dagger\right)^2 \right) = \\
= i\kappa A_p e^{-i\varphi} \left(\left(\hat{a}^\dagger\right)^2 \hat{a} - \left(\hat{a}^\dagger \hat{a} + 1\right) \hat{a}^\dagger \right) = \\
= i\kappa A_p e^{-i\varphi} \left(\hat{a}^\dagger \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \right) = \\
= i\kappa A_p e^{-i\varphi} \hat{a}^\dagger \left(\hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger - 1 \right) = i\kappa A_p e^{-i\varphi} \hat{a}^\dagger \left((-1) - 1 \right) = \\
= -2i\kappa A_p e^{-i\varphi} \hat{a}^\dagger.
\end{aligned}$$

Thus we have

$$\frac{d\hat{a}}{dt} = -i\Omega_p e^{-i\varphi} \hat{a}^\dagger, \tag{9.19}$$

where

$$\Omega_p = 2\kappa A_p$$

is the effective Rabi frequency (energy expressed through frequency). Similarly,

$$\frac{d\hat{a}^\dagger}{dt} = i\Omega_p e^{i\varphi} \hat{a}. \tag{9.20}$$

Eliminating $\hat{a}^{\dagger}$ from the system (9.19) and (9.20), we get

$$\frac{d^2\hat{a}\,(t)}{dt^2} = -i\Omega_p e^{-i\varphi}\frac{d\hat{a}^{\dagger}\,(t)}{dt} = \Omega_p^2\hat{a}\,(t)\,. \tag{9.21}$$

The initial conditions for this equation are

$$\hat{a}\,(0) = \hat{a}_0,$$
$$\hat{a}^{\dagger}\,(0) = \hat{a}_0^{\dagger},$$

from which

$$\hat{a}|_{t=0} = \hat{a}_0,$$
$$\left.\frac{d\hat{a}}{dt}\right|_{t=0} = -i\Omega_p e^{-i\varphi}\hat{a}_0^{\dagger}, \tag{9.22}$$

where $\hat{a}_0$ and $\hat{a}_0^{\dagger}$ are the initial values of the operators at $t = 0$. The solution of (9.21) with the initial conditions (9.22) is:

$$\hat{a}\,(t) = \hat{a}_0 ch\,(\Omega_p t) - i\hat{a}_0^{\dagger} sh\,(\Omega_p t)\,e^{-i\varphi}. \tag{9.23}$$

Similarly,

$$\hat{a}^{\dagger}\,(t) = \hat{a}_0^{\dagger} ch\,(\Omega_p t) + i\hat{a}_0 sh\,(\Omega_p t)\,e^{i\varphi}. \tag{9.24}$$

Note that at $\varphi = \frac{\pi}{2}$ expressions (9.23) and (9.24) reduce to the relations (9.8):

$$\hat{a}\,(t) = \hat{a}_0 ch\,(\Omega_p t) - \hat{a}_0^{\dagger} sh\,(\Omega_p t)\,,$$
$$\hat{a}^{\dagger}\,(t) = \hat{a}_0^{\dagger} ch\,(\Omega_p t) - \hat{a}_0 sh\,(\Omega_p t)\,.$$

As we see, the operator $\hat{a}\,(t)$ results from a unitary transformation (9.8) which in our case has the form:

$$\hat{a}\,(t) = \hat{S}^{\dagger}\,(t)\,\hat{a}\,(0)\,\hat{S}\,(t) = \hat{a}_0 ch\,(\Omega_p t) - \hat{a}_0^{\dagger} sh\,(\Omega_p t)\,. \tag{9.25}$$

Thus, moving from the Heisenberg picture (14.36) to the Schrödinger picture (14.34), one can conclude that the initial coherent state undergoes the same unitary transformation as the annihilation operator (9.25):

$$\hat{S}\,(t)\,|\alpha\rangle = |t, \alpha\rangle\,,$$

i.e., the parametric interaction is equivalent to the action of the squeezing operator. In this case, for the quadrature components

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^{\dagger}}{2}$$

and

$$\hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}$$

a squeezed state is realized with uncertainties

$$(\Delta X_1)^2 = \frac{1}{4} e^{-2\Omega_p t},$$
$$(\Delta X_2)^2 = \frac{1}{4} e^{2\Omega_p t},$$
$$(\Delta X_1 \Delta X_2) = \frac{1}{4}.$$

The squeezing parameter $r = \Omega_p t$ grows unbounded in time and, therefore, the degree of squeezing increases indefinitely. In reality, this does not happen. Unlimited squeezing is a consequence of overly idealized assumptions: pump depletion, deviation of the real pump from a monochromatic classical light, and several other factors were neglected. All these lead to limitations on the degree of squeezing. The closer the real conditions to ideal, the higher the degree of squeezing that can be obtained.

The case considered above of obtaining squeezed states via parametric processes pertains to the generation of squeezed vacuum, since the generation of the squeezed state occurs from the always-present vacuum seed field. This is sometimes called parametric scattering of the pump field. As we have found, the "squeezed vacuum" is not a vacuum field in the strict sense. The mean photon number in this field is not zero but depends on the degree of squeezing according to formula (9.15) and can be quite large at high degrees of squeezing.

9.5 Observation of the squeezed state. Measurement of the squeezing degree

In quadrature squeezing of light, the fluctuations of one of the quadrature components can be significantly smaller than the other. If this component is selected and used for measurement purposes, one can expect a strong reduction of noise that limits measurement accuracy.

Selection can be carried out using a homodyne (synchronous) detector. The simplest scheme of a homodyne detector is shown in Figure 9.10. The beam splitter has the following parameters: the transmission coefficient is close to 1 ($t \approx 1$) and, consequently, $r \ll 1$, since $|t|^2 + |r|^2 = 1$. This is necessary so that the signal $\hat{a}_0$ is not significantly attenuated. At the same time, the amplitude of the local oscillator, whose field is in a coherent state $|\alpha_H\rangle$, is so large that the local oscillator field intensity strongly exceeds the signal field. For the homodyne detector to

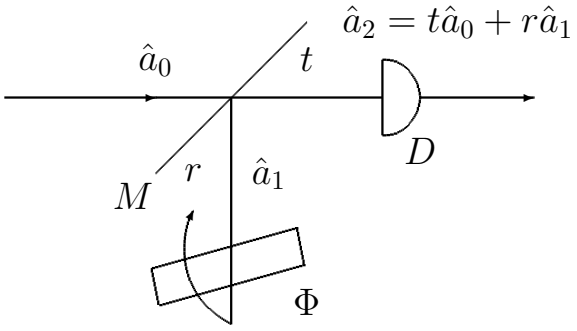


Figure 9.10: Scheme of a homodyne detector. M - mirror with reflection coefficient r and transmission coefficient t. D- detector. Φ - phase shifter.

operate, the local oscillator field and signal must have the same frequency. This is achieved by forming the signal and local oscillator fields from the same specified laser. For example, if $\hat{a}_0$ corresponds to the squeezed vacuum, the pumping of the parametric generator is done by the second harmonic of the master laser $2\omega_p$. During degenerate parametric interaction, a signal with frequency ω_p is generated, which coincides with the frequency of the local oscillator ω_p. Thus, the condition of frequency equality is satisfied.

The field operator incident on the photodetector $\hat{a}_2$ can be expressed through the operators $\hat{a}_0$ and $\hat{a}_1$ — the signal field and local oscillator field operators:

$$\hat{a}_2 = t\hat{a}_0 + r\hat{a}_1.$$

Then the photon number operator is

$$\hat{a}_2^\dagger \hat{a}_2 = \left(t\hat{a}_0^\dagger + r\hat{a}_1^\dagger\right)(t\hat{a}_0 + r\hat{a}_1) =$$
$$= t^2\hat{a}_0^\dagger\hat{a}_0 + rt\left(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\right) + r^2\hat{a}_1^\dagger\hat{a}_1.$$

Assume that the local oscillator is in a coherent state, and the signal is in a squeezed vacuum state. Then the input state is two-mode

$$|\psi\rangle = |\alpha_p\rangle\,|r,0\rangle\,.$$

The expectation value of the photon number operator $\hat{n}_2 = \hat{a}_2^\dagger\hat{a}_2$ in this case is:

$$\langle\hat{n}_2\rangle = \langle\psi|\,\hat{a}_2^\dagger\hat{a}_2\,|\psi\rangle =$$
$$= t^2\,\langle r,0|\,\hat{a}_0^\dagger\hat{a}_0\,|r,0\rangle + r^2\,|\alpha_p|^2 + 2tr\,|\alpha_p|\,\langle r,0|\,\hat{X}\,(\varphi)\,|r,0\rangle\,, \qquad (9.26)$$

where

$$\hat{X}\,(\varphi) = \frac{1}{2}\left(\hat{a}_0 e^{-i\varphi} + \hat{a}_0^\dagger e^{i\varphi}\right),$$
$$\alpha_p = |\alpha_p|\,e^{i\varphi}.$$

Here φ is the local oscillator phase. Then at $\varphi = 0$ and at $\varphi = \frac{\pi}{2}$ we have

$$\hat{X}(0) = \frac{1}{2}\left(\hat{a}_0 + \hat{a}_0^\dagger\right) = \hat{X}_1,$$
$$\hat{X}\left(\frac{\pi}{2}\right) = \frac{1}{2i}\left(\hat{a}_0 - \hat{a}_0^\dagger\right) = \hat{X}_2,$$

i.e., by changing the local oscillator phase with a phase shifter, one can select the corresponding quadrature component.

Equation (9.26) contains three terms: those containing $\left\langle \hat{a}_0^\dagger \hat{a}_0 \right\rangle$ — the average photon number in the signal mode, $r^2 |\alpha_p|^2$ — the average photon number in the local oscillator mode, and the interference term $\langle r, 0| \hat{X}(\varphi) |r, 0\rangle$, which selects the respective quadrature component. Assume that the local oscillator intensity is so high that

$$r^2 |\alpha_p|^2 \gg \left\langle \hat{a}_0^\dagger \hat{a}_0 \right\rangle,$$

then the first term in (9.26) can be neglected. In this case, the average photon number incident on the photodetector is

$$\langle \hat{n}_2 \rangle = r^2 |\alpha_p|^2 + 2tr |\alpha_p| \langle r, 0| \hat{X}(\varphi) |r, 0\rangle,$$

where the first term is a known number and can be subtracted. Only the term containing the signal quadrature remains. Next, the photon number fluctuations (uncertainty) can be determined using the formula

$$(\Delta n_2)^2 = \left\langle \hat{n}_2^2 \right\rangle - \langle \hat{n}_2 \rangle^2.$$

Calculations lead to the formula

$$(\Delta n_2)^2 = r^2 |\alpha_p|^2 \left\{ r^2 + 4t^2 (\Delta X(\varphi))^2 \right\}. \tag{9.27}$$

From (9.27) it follows that noise has two components, one of which $r^2 |\alpha_p|^2 r^2$ — expresses the local oscillator noise, the other $r^2 |\alpha_p|^2 4t^2 (\Delta X(\varphi))^2$ — is related to the signal noise. If the signal is vacuum fluctuations,

$$(\Delta X(\varphi))^2 = \frac{1}{4},$$

then the squeezing condition is

$$(\Delta X_1)^2 < \frac{1}{4}$$

for the squeezed component.

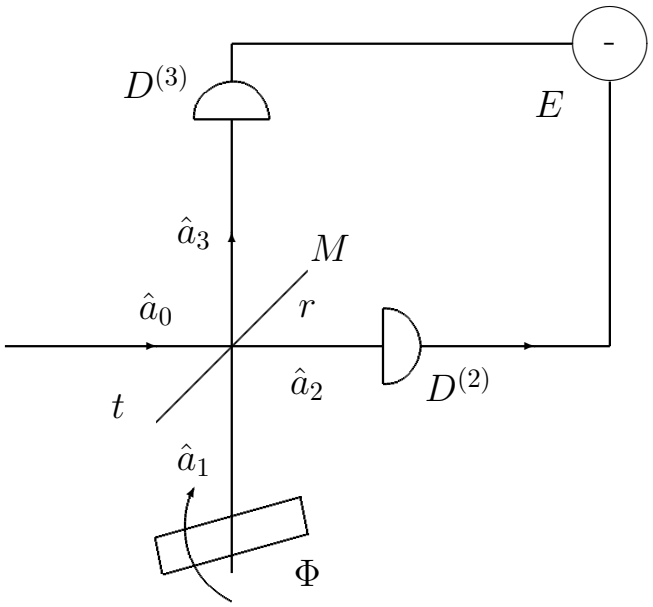


Figure 9.11: Balanced scheme of a homodyne detector. The tested signal ($\hat{a}_0$) and the homodyne mode signal ($\hat{a}_1$) are fed into the mirror M with reflection coefficient $r = \frac{1}{\sqrt{2}}$ and transmission coefficient $t = \frac{1}{\sqrt{2}}$. The signals after passing through the mirror are registered by two photodetectors $D^{(2)}$ and $D^{(3)}$, the outputs of which are fed into a subtractor E.

The squeezing effect can be detected using a homodyne detector in the following experiment. First, only the vacuum state is present at the input, and the noise level is measured. Then, the studied oscillation is fed to the input, and the noise level is measured depending on φ. If a squeezed state is applied at the input, at some value $\varphi = \varphi^{(1)}$, the noise will be minimal (less than in the case of vacuum fluctuations at the input). At another value $\varphi = \varphi^{(2)}$, differing by π, the noise will be maximal (higher than in the case of vacuum fluctuations). Such experiments have been carried out repeatedly. At the same time, the squeezing effect was observed and its degree measured (see for example [30]).

9.5.1 Balanced homodyne detector scheme

A more complex homodyne receiver scheme is the balanced scheme, shown in Figure 9.11. It allows substantial reduction of the local oscillator noise. It differs from the previously discussed scheme by having two photodetectors, one for each output beam. The signals from each photodetector are subtracted, and the difference is recorded. The beam splitter is semi-transparent with $t = r = \frac{1}{\sqrt{2}}$ (this time we assume t and r are real, but the reflection coefficient on different sides of the beam splitter has opposite sign).

In the scheme, $\hat{a}_0$ is the signal operator, and $\hat{a}_1$ is the local oscillator mode operator. It is assumed that the local oscillator is in a coherent state with a large

amplitude. The fields of modes 2 and 3 (operators $\hat{a}_2$ and $\hat{a}_3$) are detected by separate photodetectors $D^{(2)}$ and $D^{(3)}$, and the signals from each photodetector are subtracted. Thus, the output signal is defined by the expectation value of the operator

$$\hat{n}_{23} = \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3.$$

The operators $\hat{a}_2$ and $\hat{a}_3$ are related to the input field operators $\hat{a}_0$ and $\hat{a}_1$ by the beam splitter through the relations:

$$\hat{a}_2 = \frac{1}{\sqrt{2}} \left(\hat{a}_0 + \hat{a}_1 \right),$$
$$\hat{a}_3 = \frac{1}{\sqrt{2}} \left(-\hat{a}_0 + \hat{a}_1 \right),$$

so

$$\hat{n}_{23} = \frac{1}{2} \left(\left(\hat{a}_0^\dagger + \hat{a}_1^\dagger \right) \left(\hat{a}_0 + \hat{a}_1 \right) - \left(-\hat{a}_0^\dagger + \hat{a}_1^\dagger \right) \left(-\hat{a}_0 + \hat{a}_1 \right) \right) =$$
$$= \frac{1}{2} \left(\hat{a}_0^\dagger \hat{a}_0 + \hat{a}_1^\dagger \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_1 - \left(\hat{a}_0^\dagger \hat{a}_0 - \hat{a}_1^\dagger \hat{a}_0 - \hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_1 \right) \right) =$$
$$= \hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0 = \left(\hat{a}_0^\dagger e^{i\varphi} + \hat{a}_0 e^{-i\varphi} \right) |\alpha_H| ,$$

where it was used that the local oscillator is in a coherent state with large amplitude

$$\alpha_H = |\alpha_H| \, e^{i\varphi},$$

and $\hat{a}_1$ is replaced by the classical field $|\alpha_H| \, e^{i\varphi}$ ($|\alpha_H| \gg 1$).

Let us find the expectation value of the operator $\hat{n}_{23}$ in the case of the signal in a squeezed vacuum state $|z, 0\rangle$, where for simplicity let $z = r$ be real ($r = |z|$). Also for simplicity, set $\varphi = 0$. In this case

$$\hat{n}_{23} = \left(\hat{a}_0 + \hat{a}_0^\dagger \right) |\alpha_H| = 2\hat{X}_1 \, |\alpha_H| .$$

This means we consider squeezing of the quadrature component $\hat{X}_1$. The expectation value of $\hat{n}_{23}$ writes as

$$\langle \hat{n}_{23} \rangle = 2 \, |\alpha_H| \left\langle \hat{X}_1 \right\rangle = |\alpha_H| \, \langle r, 0| \, \hat{a}_0 + \hat{a}_0^\dagger \, |r, 0\rangle =$$
$$= |\alpha_H| \, \langle 0| \, \hat{S}^\dagger (r) \left(\hat{a}_0 + \hat{a}_0^\dagger \right) \hat{S} (r) \, |0\rangle =$$
$$= |\alpha_H| \left(\langle 0| \, \hat{a}_0 ch \, r - \hat{a}_0^\dagger sh \, r \, |0\rangle + \langle 0| \, \hat{a}_0^\dagger ch \, r - \hat{a}_0 sh \, r \, |0\rangle \right) = 0,$$

since $\hat{a}_0 \, |0\rangle = 0$, $\langle 0| \, \hat{a}_0^\dagger = 0$. Thus, the mean value $\langle \hat{n}_{23} \rangle = 0$. However, the mean square is not zero; it reflects noise limiting measurement accuracy.

Calculate the uncertainty $(\Delta X_{1,2})^2$, which determines the noise level for the considered balanced scheme:

$$(\Delta X_{1,2})^2 = \langle r,0| \frac{1}{4}\left(\hat{a}_0 \pm \hat{a}_0^\dagger\right)^2 |r,0\rangle - \\ - \langle r,0| \frac{1}{2}\left(\hat{a}_0 \pm \hat{a}_0^\dagger\right) |r,0\rangle^2 . \tag{9.28}$$

Since we showed that the last term in (9.28) equals zero, we obtain:

$$(\Delta X_{1,2})^2 = \langle 0| \hat{S}^\dagger(r,0) \frac{1}{4}\left(\hat{a}_0 \pm \hat{a}_0^\dagger\right)^2 \hat{S}(r,0) |0\rangle = \\ = \langle 0| \hat{S}^\dagger(r,0) \frac{1}{4}\left(\hat{a}_0 \pm \hat{a}_0^\dagger\right) \hat{S}^\dagger(r,0) \hat{S}(r,0) \left(\hat{a}_0 \pm \hat{a}_0^\dagger\right) \hat{S}(r,0) |0\rangle . \tag{9.29}$$

To find $(\Delta X_{1,2})^2$, calculate the averages:

$$\langle \hat{a}_0 \rangle = \langle 0| \hat{S}^\dagger(r,0) \hat{a}_0 \hat{S}(r,0) |0\rangle = \langle 0| \hat{a}_0 ch\, r - \hat{a}_0^\dagger sh\, r |0\rangle = 0, \\ \left\langle \hat{a}_0^\dagger \right\rangle = \langle 0| \hat{a}_0^\dagger ch\, r - \hat{a}_0 sh\, r |0\rangle = 0, \tag{9.30}$$

since $\hat{a}_0 |0\rangle = 0$, $\langle 0| \hat{a}_0^\dagger = 0$.

Similarly, one can show

$$\left\langle \hat{a}_0^\dagger \hat{a}_0 \right\rangle = sh^2 r, \\ \left\langle \hat{a}_0 \hat{a}_0^\dagger \right\rangle = 1 + \left\langle \hat{a}_0^\dagger \hat{a}_0 \right\rangle = 1 + sh^2 r = ch^2 r, \\ \langle \hat{a}_0 \hat{a}_0 \rangle = \left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \right\rangle^* = -ch\, r sh\, r. \tag{9.31}$$

Transforming (9.29) using (9.30) and (9.31), we get the following expression:

$$(\Delta X_{1,2})^2 = ch\, 2r \mp sh\, 2r = e^{\mp 2r},$$

from which

$$(\Delta n_{2,3})^2 = n_H e^{-2r}$$

for the X_1 component. The variance is

$$\Delta n_{2,3} = \sqrt{n_H} e^{-r},$$

where $n_H = |\alpha_H|^2$ is the photon number in the local oscillator field. Using the balanced scheme, it is possible to measure the degree of squeezing in the same way as with the simplest homodyne scheme. The advantage here is that the local oscillator noise is compensated, and measurement accuracy increases.

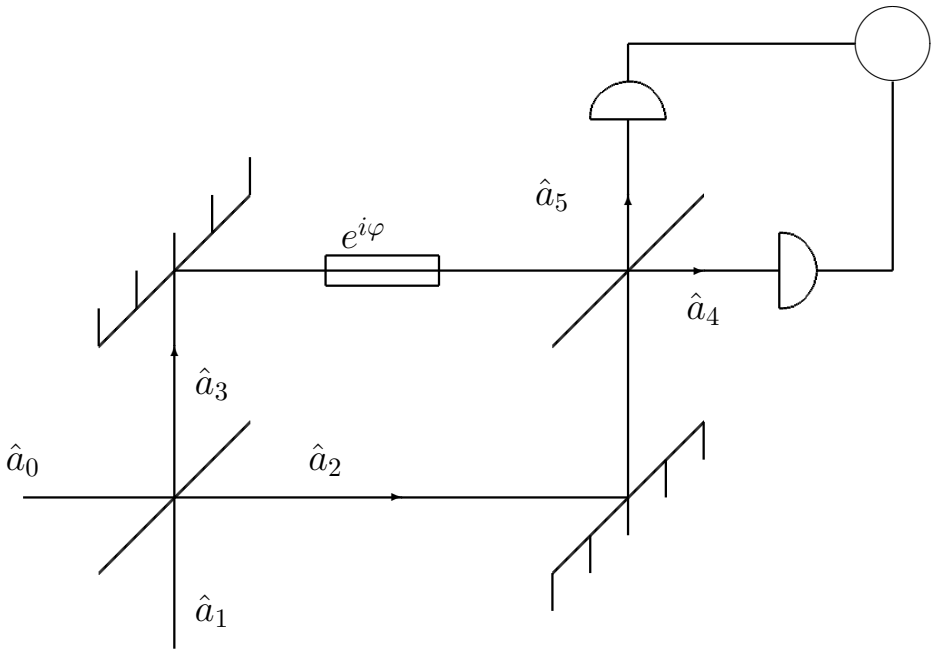


Figure 9.12: Mach-Zehnder Interferometer

9.6 Interference Measurements Using Squeezed Light

Consider the Mach-Zehnder interferometer scheme shown in Figure 9.12. This scheme has been previously examined during the study of interference measurement errors. The difference is that at the zero input, squeezed vacuum radiation is fed instead of vacuum radiation. We still assume that the input and output mirrors are partially transparent ($t = r = \frac{1}{\sqrt{2}}$), and the angular mirrors are opaque ($r = 1$, $t = 0$). The scattering matrix of the mirrors is

$$\hat{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The interferometer is considered as a sensor of some physical quantity affecting the optical length of one arm. On the scheme, the phase shifter included in the upper arm can respond to the action of this quantity.

The interferometer equations are given by (6.12):

$$\hat{a}_2 = \frac{1}{\sqrt{2}} \left(\hat{a}_0 + i\hat{a}_1\right), \; \hat{a}_3 = \frac{1}{\sqrt{2}} \left(i\hat{a}_0 + \hat{a}_1\right),$$

$$\hat{a}_4 = \frac{1}{\sqrt{2}} \left(i\hat{a}_2 + e^{i\varphi}\hat{a}_3\right) = \frac{1}{2} \left[i\left(1 + e^{i\varphi}\right)\hat{a}_0 - \left(1 - e^{i\varphi}\right)\hat{a}_1\right],$$

$$\hat{a}_5 = \frac{1}{\sqrt{2}} \left(\hat{a}_2 + ie^{i\varphi}\hat{a}_3\right) = \frac{1}{2} \left[\left(1 - e^{i\varphi}\right)\hat{a}_0 + i\left(1 + e^{i\varphi}\right)\hat{a}_1\right]. \qquad (9.32)$$

Let the input signal field (input 0, operator $\hat{a}_0$) be in a squeezed vacuum state.

The second input field (input 1, operator $\hat{a}_1$) is in a coherent state with a large amplitude. So large that the input field is two-mode

$$|\psi\rangle_{\text{in}} = |\alpha\rangle \left|re^{i\theta}, 0\right\rangle,$$

where $|\alpha\rangle$ is the coherent state, $\left|re^{i\theta}, 0\right\rangle$ is the squeezed vacuum state. Channels 4 and 5 are each detected by their own photodetector. The signals from the photodetectors are subtracted and recorded.

Thus, the output signal is determined by the average value of the operator

$$\begin{aligned} \hat{n}_{54} = \hat{a}_5^\dagger \hat{a}_5 - \hat{a}_4^\dagger \hat{a}_4 = \\ = \left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_0^\dagger \hat{a}_0\right) \cos\varphi - \left(\hat{a}_0^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_0\right) \sin\varphi \end{aligned} \tag{9.33}$$

Here the equality (9.32) was used.

Let us find the average value of the operator $\langle\psi| \hat{n}_{54} |\psi\rangle$. We take into account that each operator acts only on its own mode. We have:

$$\begin{aligned} \langle\psi| \hat{n}_{54} |\psi\rangle = \left(\langle\alpha| \hat{a}_1^\dagger \hat{a}_1 |\alpha\rangle - \left\langle re^{i\theta}, 0\right| \hat{a}_0^\dagger \hat{a}_0 \left|re^{i\theta}, 0\right\rangle\right) \cos\varphi - \\ - \left(\left\langle re^{i\theta}, 0\right| \hat{a}_0^\dagger \left|re^{i\theta}, 0\right\rangle \alpha + \alpha \left\langle re^{i\theta}, 0\right| \hat{a}_0 \left|re^{i\theta}, 0\right\rangle\right) \sin\varphi. \end{aligned} \tag{9.34}$$

The first term in the first bracket equals the average number of pump photons $|\alpha|^2$, and the second term, as we know, equals the number of photons in the squeezed state, which is sh^2r. The second bracket on the right, as we found when examining the balanced detector, is zero. Thus we get

$$\langle\psi| \hat{n}_{54} |\psi\rangle = \left(|\alpha|^2 - sh^2r\right) \cos\varphi.$$

If we set $\varphi = \frac{\pi}{2}$, then the average value of the output signal is zero:

$$\langle\psi| \hat{n}_{54} |\psi\rangle = 0.$$

The average value of the second term in (9.34) is zero, but its mean square is not zero. This means it is a source of noise which limits measurement accuracy.

Let us calculate the mean square of $\hat{n}_{54}$:

$$(\Delta n_{54})^2 = \langle\psi| \hat{n}_{54}^2 |\psi\rangle - \langle\psi| \hat{n}_{54} |\psi\rangle^2.$$

The mean in our case is zero, so we have

$$\begin{aligned} (\Delta n_{54})^2 = \langle\psi| \hat{n}_{54}^2 |\psi\rangle = \\ = |\alpha|^2 \left\langle re^{i\theta}, 0\right| \left(\hat{a}_0^\dagger + \hat{a}_0\right)^2 \left|re^{i\theta}, 0\right\rangle = \\ = |\alpha|^2 \left\langle re^{i\theta}, 0\right| \left(\hat{a}_0^\dagger\right)^2 + (\hat{a}_0)^2 + \hat{a}_0^\dagger \hat{a}_0 + \hat{a}_0 \hat{a}_0^\dagger \left|re^{i\theta}, 0\right\rangle. \end{aligned}$$

Previously, we had (9.30):

$$\langle \hat{a}_0 \rangle = \langle 0 | \hat{S}^{\dagger}(r,0)\, \hat{a}_0 \hat{S}(r,0) | 0 \rangle = \langle 0 | \hat{a}_0 ch\, r - \hat{a}_0^{\dagger} sh\, r | 0 \rangle = 0,$$
$$\left\langle \hat{a}_0^{\dagger} \right\rangle = \langle 0 | \hat{a}_0^{\dagger} ch\, r - \hat{a}_0 sh\, r | 0 \rangle = 0,$$

from which we get

$$(\Delta n_{54})^2 = \langle \psi | \hat{n}_{54}^2 | \psi \rangle = |\alpha|^2 (ch\, 2r - sh\, 2r) = |\alpha|^2 e^{-2r}.$$

The root mean square noise value will be

$$|\Delta n_{54}| = |\alpha| e^{-r}.$$

We take this value as the threshold below which we will not be able to detect or measure the signal. Hence, the signal must be above the threshold.

If the initial value $\varphi = \frac{\pi}{2}$ changes by an amount $\Delta\varphi$, a signal equal to

$$\Delta n_{54} = |\alpha|^2 \Delta\varphi$$

will appear. It is necessary that

$$|\alpha|^2 \Delta\varphi > |\alpha| e^{-r},$$

from which we obtain

$$\Delta\varphi > \frac{1}{|\alpha|} e^{-r}, \tag{9.35}$$

where $|\alpha| = \sqrt{\bar{n}}$, and $\bar{n}$ is the average number of photons in the homodyne field.

Expression (9.35) gives a smaller value than the one obtained earlier when considering the accuracy of interference measurements. Using the squeezed state can increase measurement accuracy.

9.7 Squeezed light in nonlinear phenomena

The efficiency of higher harmonic generation can be increased if squeezed light is used instead of a coherent state as the electromagnetic field. In particular, the $2N$th harmonic is generated $(2N-1)!!$ times more efficiently [42, pp. 72-76], where

$$n!! = \prod_{k=1}^{\frac{n}{2}} (2k)$$

denotes the double factorial.

TBD

9.8 Nonclassicality of the Squeezed State

In conclusion, we show that the squeezed state is nonclassical. The condition for a squeezed state is

$$(\Delta X_1)^2 < \frac{1}{4},$$

where

$$(\Delta X_1)^2 = \left\langle \hat{X}_1^2 \right\rangle - \left\langle \hat{X}_1 \right\rangle^2.$$

The operator

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2},$$

then

$$\hat{X}_1^2 = \frac{\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}{4} = \\ = \frac{\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 + 2\hat{a}^\dagger\hat{a} + 1}{4}.$$

At the same time,

$$\left(\hat{N}\hat{X}_1^2\right) = \frac{\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 + 2\hat{a}^\dagger\hat{a}}{4},$$

where $\hat{N}\hat{X}_1^2$ is the operator subjected to the action of the operator $\hat{N}$, whose action consists in setting the normal ordering of the operators $\hat{a}^\dagger$ and $\hat{a}$, regardless of the commutation relation.

From this we get

$$(\Delta N X_1)^2 = (\Delta X_1)^2 - \frac{1}{4},$$

or

$$(\Delta N X_1)^2 < 0,$$

since

$$(\Delta X_1)^2 < \frac{1}{4}.$$

Suppose the squeezed state is described by a density operator, which is written in the coherent state representation as

$$\hat{\rho} = \int P(\alpha) |\alpha\rangle \langle\alpha| \, d^2\alpha,$$

then for the expectation value

$$(\Delta N X_1)^2 = \left\langle \hat{N}\hat{X_1}^2 \right\rangle - \left\langle \hat{N}\hat{X_1} \right\rangle^2 \tag{9.36}$$

we can write

$$\begin{aligned}(\Delta N X_1)^2 = \int P(\alpha) \langle\alpha| \hat{N}\hat{X_1}^2 |\alpha\rangle \, d^2\alpha - \int P(\alpha) \langle X_1\rangle^2 d^2\alpha = \\ = \int P(\alpha) \left(\frac{\alpha^2 + \alpha^{*2} + 2\alpha\alpha^*}{4} - \langle X_1\rangle^2 \right) d^2\alpha = \\ = \int P(\alpha) (\Delta X_1)^2 d^2\alpha \end{aligned} \tag{9.37}$$

where $(\Delta X_1)^2$ is obtained from (9.36) if all $\hat{a}^\dagger$ are replaced by α^*, and $\hat{a}$ by α. It can be shown that

$$(\Delta X_1)^2 = (\Delta\alpha^* + \Delta\alpha)^2$$

which is clearly non-negative. By the squeezing condition, the integral must be less than zero:

$$(\Delta N X_1)^2 = \int P(\alpha) (\Delta X_1(\alpha^*\alpha))^2 d^2\alpha < 0.$$

This is obviously possible if at least on part of the plane (α), $P(\alpha) < 0$. This, as we know, is the condition for nonclassicality.

9.9 Second-Order Coherence for the Squeezed Vacuum

Let us calculate the second-order coherence for the squeezed vacuum $|z, 0\rangle$. From (8.1) we have

$$G^{(2)} = \frac{\langle z, 0| \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} |z, 0\rangle - \langle n\rangle}{\langle n\rangle^2}.$$

The expression $\langle z, 0| \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} |z, 0\rangle$ can be rewritten as

$$\begin{aligned}\langle z, 0| \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} |z, 0\rangle = \\ = \langle 0| \hat{a}^\dagger\hat{S}\hat{S}^\dagger\hat{a}\hat{S}\hat{S}^\dagger\hat{a}^\dagger\hat{S}\hat{S}^\dagger\hat{a}\hat{S} |0\rangle = \langle\phi |\phi\rangle ,\end{aligned}$$

where

$$|\phi\rangle = \hat{S}^\dagger\hat{a}^\dagger\hat{S}\hat{S}^\dagger\hat{a}\hat{S} |0\rangle .$$

Using (9.8) one obtains

$$|\phi\rangle = \left(\hat{a}^{\dagger} ch\, r - \hat{a} e^{-i\theta} sh\, r\right)\left(\hat{a} ch\, r - \hat{a}^{\dagger} e^{i\theta} sh\, r\right)|0\rangle =$$
$$= e^{i\theta} sh\, r\left(\hat{a}^{\dagger} ch\, r - \hat{a} e^{-i\theta} sh\, r\right)\hat{a}^{\dagger}|0\rangle = sh^2 r\,|0\rangle + \sqrt{2} e^{i\theta} sh\, r ch\, r\,|2\rangle .$$

Thus,

$$\langle\phi\,|\phi\rangle = sh^4 r + 2sh^2 r ch^2 r.$$

Therefore, taking into account (9.15),

$$G^{(2)} = \frac{sh^4 r + 2sh^2 r ch^2 r - sh^2 r}{sh^4 r} = 1 + 2\frac{ch^2 r}{sh^2 r} - \frac{1}{sh^2 r} =$$
$$= 1 + \frac{ch^2 r}{sh^2 r} + \frac{ch^2 r - 1}{sh^2 r} = 1 + \frac{ch^2 r}{sh^2 r} + \frac{sh^2 r}{sh^2 r} = 2 + \frac{ch^2 r}{sh^2 r} =$$
$$= 2 + \frac{1 + sh^2 r}{sh^2 r} = 3 + \frac{1}{\langle n\rangle}$$

9.10 Exercises

1. Derive the uncertainty relations (9.2) and (9.3).
2. Derive the uncertainty relations (9.4) and (9.5).
3. Prove (9.7) and (9.8).
4. Prove the uncertainty relations (9.13).
5. Prove expressions (9.16) for displacement operators $\hat{D}(\alpha)$.
6. Prove that (9.23) and (9.24) are solutions of equation (9.21) with initial conditions (9.22).

Chapter 10

Entangled States

Since the advent of quantum mechanics, the question of the completeness of this theory has arisen. The accuracy of quantum mechanical predictions is very high, but only probabilities of certain events are predicted. In particular, it is impossible to measure the coordinate and momentum of a particle with arbitrary precision. It seems that the probabilities inherent in the quantum mechanical description reflect its incompleteness, and perhaps there exists another theory that would possess the accuracy of quantum mechanics and at the same time would not use a probabilistic approach.

It turned out that the probabilities underlying quantum mechanics have a deep physical meaning and there are no theories in which they can be discarded, and in which, for example, it would be possible to measure coordinate and momentum with arbitrary precision. Entangled states play a special role in this context. They describe systems consisting of several particles, while the behavior of such a composite system is described by a common wave function.

Because entangled states are purely quantum, i.e., have no classical analogues, it is possible using them to observe phenomena that seem completely impossible from a classical point of view, such as quantum teleportation. In addition, recently practical applications of entangled states have appeared, such as quantum dense coding (see 11.2.3) and quantum cryptography (see 12.3).

There are several ways to obtain entangled photons [34], i.e., related to quantum optics, among which polarization-entangled states should be highlighted. This is because there are many methods for controlling polarization characteristics, as well as methods for measuring the polarization properties of light.

10.1 EPR Paradox for Stokes Parameters and Entangled States

Consider the single-photon state (7.3). To explain the physical meaning of the parameters α and β, one needs to refer to the experimental setup scheme shown in Figure 7.1. Suppose that the source S emits photons in the state (7.3). Then the average current of photodetector D_x will be proportional to $|\alpha|^2$, and that of photodetector D_y - $|\beta|^2$, i.e., $P_x = |\alpha|^2$ and $P_y = |\beta|^2$ describe the probabilities of detecting a photon in a state polarized along x - $|x\rangle$ or y - $|y\rangle$. The following question arises - "what is behind these probabilities?". Two possible answers exist. The first one assumes that we actually have incomplete knowledge about the source S, i.e., if we knew everything about this source, we could predict which photon, polarized along x or along y, is created at an arbitrary moment in time and, correspondingly, the readings of photodetectors D_x and D_y. From this point of view, quantum mechanics represents some intermediate theory, which can later be replaced by a more precise one, describing quantum properties of objects with absolute accuracy.

The second answer variant assumes that it is impossible to determine all parameters describing some quantum system with arbitrary precision because the measurement result is not predetermined in advance (as in our case by properties of source S), but rather defined at the moment of measurement as a result of the interaction between the quantum system and the macroscopic measuring apparatus.

The first attempt to answer this question was made by A. Einstein, B. Podolsky, and N. Rosen in 1935 in a paper often called EPR [8]. In EPR, a physical theory is considered complete if every element of physical reality has a counterpart in the physical theory. The following was adopted as a definition of an element of physical reality: "if, without disturbing a system, we can predict with certainty (probability equal to unity) the value of some physical quantity, then there exists an element of reality corresponding to this quantity" [40]. Non-commuting operators are of particular interest here. In our case, these could be operators $\hat{S}_1$ and $\hat{S}_2$. Since these operators do not commute (7.5), the values of the corresponding observables cannot be measured with arbitrary precision. From this, two assumptions can be made:

1. The quantum mechanical description of reality is incomplete

2. The quantities defined by operators $\hat{S}_1$ and $\hat{S}_2$ cannot be simultaneously real

As a test system, following EPR, we consider a complex system consisting of

several particles. At the same time, some overall wave function is used to describe the entire system.

To describe the polarization properties of a complex system consisting of two photons, one must use a wave function of the following form:

$$|\psi\rangle = \sum_{i,j=x,y} c_{ij} |i\rangle_1 |j\rangle_2 =$$
$$= c_{xx} |x\rangle_1 |x\rangle_2 + c_{xy} |x\rangle_1 |y\rangle_2 + c_{yx} |y\rangle_1 |x\rangle_2 + c_{yy} |y\rangle_1 |y\rangle_2 . \quad (10.1)$$

If two or more coefficients c_{ij} in (10.1) are nonzero, then the wave function is non-factorizable, i.e.,

$$|\psi\rangle \neq |\psi\rangle_1 |\psi\rangle_2 .$$

Thus, no single wave function can be assigned to each separate photon. Such states are called entangled.

Remark 10.1.1. *Consider the following pure entangled state*

$$|\psi_{12}\rangle = \frac{|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2}{\sqrt{2}},$$

whose density matrix has the form

$$\hat{\rho}_{12} = |\psi_{12}\rangle \langle\psi_{12}| =$$
$$\frac{1}{2} (|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2) (\langle x|_1 \langle y|_2 - \langle y|_1 \langle x|_2) =$$
$$= \frac{|x\rangle_1 |y\rangle_2 \langle y|_2 \langle x|_1 + |y\rangle_1 |x\rangle_2 \langle x|_2 \langle y|_1}{2} -$$
$$- \frac{|x\rangle_1 |y\rangle_2 \langle x|_2 \langle y|_1 + |y\rangle_1 |x\rangle_2 \langle y|_2 \langle x|_1}{2}.$$

At the same time, the density matrix of the second photon has the form (see section 14.5)

$$\hat{\rho}_2 = \mathrm{Tr}_1 \hat{\rho}_{12} = \langle x|_1 \hat{\rho}_{12} |x\rangle_1 + \langle y|_1 \hat{\rho}_{12} |y\rangle_1 =$$
$$= \frac{|y\rangle_2 \langle y|_2 + |x\rangle_2 \langle x|_2}{2}.$$

Thus, the state of the second photon is mixed.

This astonishing fact greatly distinguishes quantum systems from classical ones: if we have a pure state of two particles, then the corresponding states of individual particles are not necessarily pure. In particular, if the state of the Universe is a pure quantum state [12], then the states of individual parts of this Universe (stars, planets, etc.) are not necessarily pure and can be mixed, i.e., classical.

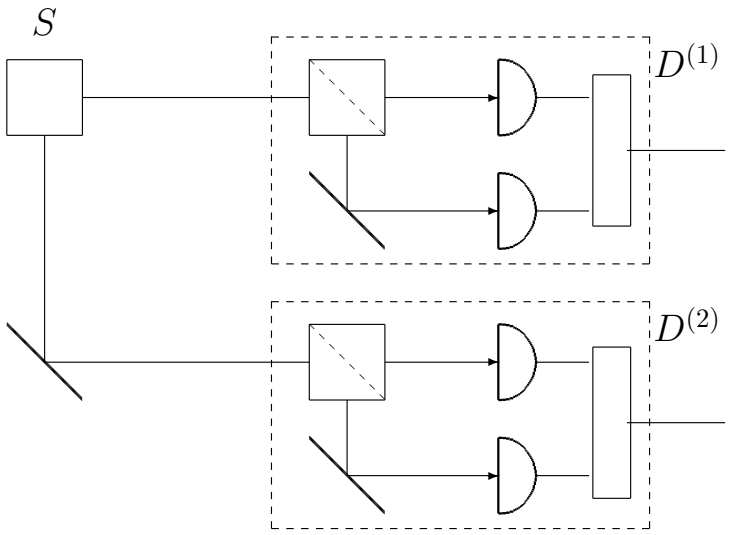


Figure 10.1: Scheme for measuring the Stokes parameters for an entangled two-photon state. Entangled photon source S. Entangled photon detectors $D^{(1,2)}$

As an example, we study the following wave function:

$$|\psi\rangle = \frac{|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2}{\sqrt{2}}, \tag{10.2}$$

i.e., all we know about the photons is that they have different circular polarizations. With 50% probability the first photon has left circular polarization, and in this case the second photon has right circular polarization. Conversely, with 50% probability the first photon has right circular polarization, and the second photon has left circular polarization.

To measure the polarization properties of the state (10.2), the scheme shown in Figure 10.1 can be used. In this scheme, entangled photons from the source S are fed to two detectors of Stokes parameters $D^{(1,2)}$. Detector $D^{(1)}$ measures the average values of the Stokes operators for the first photon - $\hat{S}_k^{(1)}$, and $D^{(2)}$ measures the average values of the Stokes operators $\hat{S}_k^{(2)}$ for the second photon. The setup of each detector is identical to the scheme shown in Figure 7.2.

Since the eigenvalues of the Stokes operators (7.4) are $s = \pm 1$, at the detector outputs we will obtain either $+1$ or -1. Suppose we measure the Stokes parameter $\hat{S}_1^{(1)}$ for the first photon. Suppose the measurement outcome is $+1$. At this point, the wave function (10.2) undergoes reduction. The reduction can be described by the projection operator (see subsection 14.1.6) onto the state $|x\rangle_1$:

$$\hat{P}_{|x\rangle_1} = |x\rangle_1 \langle x|_1 ,$$

and the wave function (10.2) transforms as follows:

$$\hat{P}_{|x\rangle_1}|\psi\rangle = |x\rangle_1 \langle x|_1 |\psi\rangle =$$
$$= |x\rangle_1 \langle x|_1 \frac{(|x\rangle_1 + i|y\rangle_1)|-\rangle_2 - (|x\rangle_1 - i|y\rangle_1)|+\rangle_2}{2} =$$
$$= |x\rangle_1 \frac{|-\rangle_2 - |+\rangle_2}{2} = -\frac{1}{2\sqrt{2}}|x\rangle_1 (2i)|y\rangle_2 = -\frac{i}{\sqrt{2}}|x\rangle_1 |y\rangle_2 .$$

Therefore, after measuring $\hat{S}_1^{(1)}$, the wave function of the entire system will be written as

$$|\psi\rangle_{red} = |x\rangle_1 |y\rangle_2 .$$

This means that the second photon will be in the state $|y\rangle_2$, i.e., the reading of detector $D^{(2)}$ will be -1.

Similarly, if the reading of the first detector was -1, the wave function after measurement will be equal to

$$P_{|y\rangle_1}|\psi\rangle = |y\rangle_1 \langle y|_1 |\psi\rangle =$$
$$= |y\rangle_1 \langle y|_1 \frac{(|x\rangle_1 + i|y\rangle_1)|-\rangle_2 - (|x\rangle_1 - i|y\rangle_1)|+\rangle_2}{2} =$$
$$= |y\rangle_1 i\frac{|-\rangle_2 + |+\rangle_2}{2} = \frac{i}{2\sqrt{2}}|y\rangle_1 (2)|x\rangle_2 = \frac{i}{\sqrt{2}}|y\rangle_1 |x\rangle_2 ,$$

i.e.,

$$|\psi\rangle_{red} = |x\rangle_2 |y\rangle_1 ,$$

the state of the second photon will be described by the state $|x\rangle_2$ and the reading of detector $D^{(2)}$ will be $+1$. Thus, for any measurement of $\hat{S}_1^{(1)}$, we can reliably (with probability equal to 1) predict the measurement result of $\hat{S}_1^{(2)}$, without directly affecting the second photon, i.e., the quantity $\hat{S}_1^{(2)}$ should be considered an element of physical reality.

At the same time, the Stokes parameters for the second particle, $\hat{S}_1^{(2)}$ and $\hat{S}_2^{(2)}$ have different eigenvectors: $|x\rangle_2 , |y\rangle_2$ (7.8, 7.9) and $\frac{1}{\sqrt{2}}(|x\rangle_2 \pm |y\rangle_2)$ (7.12, 7.13), respectively. Thus, the first and second Stokes parameters for the second particle cannot be measured simultaneously; indeed, otherwise, the measured value would correspond to a certain state vector that would be an eigenvector of both operators $\hat{S}_1^{(2)}$ and $\hat{S}_2^{(2)}$ (see also subsection 14.3.3).

Now, if we assume that the results of experiments are predetermined in advance, i.e., do not depend on which quantity (S_1 or S_2) was measured, and since measurements on the first photon are made in arbitrary order, it follows that for the second photon $\hat{S}_1^{(2)}$ and $\hat{S}_2^{(2)}$ must simultaneously be elements of physical reality. This entails the thesis about the incompleteness of quantum mechanics.

We have a different situation when we accept the assumption that the result of some measurement is determined at the moment the experiment is performed. In this case, we can no longer claim that $\hat{S}_1^{(2)}$ and $\hat{S}_2^{(2)}$ should be **simultaneously** considered as elements of physical reality, so there are no contradictions with the basic principles of quantum mechanics.

10.2 Bell Inequality for Stokes Parameters

The answer to the question about the completeness of quantum mechanics can be given by the following experiment [4, 34, 9].

Suppose that the system under consideration consists of two photons with the joint wave function of the form

$$|\psi\rangle = \frac{|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2}{\sqrt{2}}. \tag{10.3}$$

Let detector $D^{(1)}$ (Figure 10.1) measure the quantity

$$\hat{A} = \hat{S}_1^{(1)}, \ (\xi = 0)$$

or

$$\hat{A}' = \hat{S}_2^{(1)}, \ (\xi = \frac{\pi}{4}).$$

As noted earlier, the eigenvalues of operators $\hat{A}$ and $\hat{A}'$ will be ± 1, i.e., the readings of detector $D^{(1)}$ can only be two numbers: ± 1.

Receiver $D^{(2)}$ will measure the following quantities

$$\hat{B} = \frac{1}{\sqrt{2}} \left(\hat{S}_1^{(2)} + \hat{S}_2^{(2)} \right), \ (\xi = \frac{\pi}{8})$$

or

$$\hat{B}' = \frac{1}{\sqrt{2}} \left(\hat{S}_1^{(2)} - \hat{S}_2^{(2)} \right), \ (\xi = -\frac{\pi}{8}).$$

To find the possible readings of $D^{(2)}$, it is necessary to find the eigenvalues of operators $\hat{B}$ and $\hat{B}'$. Using the matrix representations of operators $\hat{S}_1$ (7.7) and $\hat{S}_2$ (7.11), we obtain for operator $\hat{B}$ the following matrix representation:

$$\hat{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{10.4}$$

From (10.4) we can obtain the characteristic equation:

$$\left(1 - \sqrt{2}b\right) \left(-1 - \sqrt{2}b\right) - 1 = 0, \tag{10.5}$$

from which the eigenvalues are: $b = \pm 1$. In the case of operator $\hat{B}'$, we can obtain from (7.7) and (7.11) the following matrix representation:

$$\hat{B}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{10.6}$$

and the corresponding characteristic equation

$$\left(1 - \sqrt{2}b\right)\left(-1 - \sqrt{2}b\right) - 1 = 0. \tag{10.7}$$

As is easy to see, equation (10.7) has the same solutions as (10.5), thus the readings of $D^{(2)}$, just like for $D^{(1)}$, will be only two values ± 1.

Four series of experiments are conducted, each with N trials, in which pairs of operators $\left(\hat{A}, \hat{B}\right)$, $\left(\hat{A}', \hat{B}\right)$, $\left(\hat{A}, \hat{B}'\right)$, and $\left(\hat{A}', \hat{B}'\right)$ are measured. As a result, the following sets of numbers (a_i, b_i), (a'_i, b_i), (a_i, b'_i), and (a'_i, b'_i) are obtained. As was just shown, each of the obtained numbers can be either $+1$ or -1.

Next, from these pairs, the following value is calculated

$$f_i = \frac{1}{2}\left(a_i b_i + a'_i b_i + a_i b'_i - a'_i b'_i\right)$$

for which the average is computed as

$$\langle F \rangle_N = \frac{1}{N} \sum_i f_i. \tag{10.8}$$

As $N \to \infty$ one can assume

$$\langle F \rangle_N \to \langle F \rangle.$$

Remark 10.2.1 (About the Bell Experiment)**.** *As noted above, in the experiment we perform measurements of the following pairs of physical quantities* $\left(\hat{A}, \hat{B}\right)$, $\left(\hat{A}', \hat{B}\right)$, $\left(\hat{A}, \hat{B}'\right)$, *and* $\left(\hat{A}', \hat{B}'\right)$, *and for computing* (10.8) *it is necessary to assume that all these measurements are made in one go, i.e., simultaneously. In the classical case, unlike the quantum case, this is allowed. In addition, from the classical point of view,* $\hat{A}$ *and* $\hat{A}'$ *are two independent observables, as are* $\hat{B}$ *and* $\hat{B}'$*. At the same time, from the quantum perspective, they are no longer independent, for example, for* $\hat{A}, \hat{A}'$ *we have* (7.5)*:*

$$\left[\hat{A}, \hat{A}'\right] = 2i\hat{S}_3.$$

Furthermore, from the classical point of view, the measurements of quantities $\hat{A}, \hat{A}'$ *on one hand and* $\hat{B}, \hat{B}'$ *on the other are also independent because they are*

performed at different points in space. Thus, in the classical case the four pairs $\left(\hat{A},\hat{B}\right)$, $\left(\hat{A}',\hat{B}\right)$, $\left(\hat{A},\hat{B}'\right)$, *and* $\left(\hat{A}',\hat{B}'\right)$ *are independent, and it does not matter whether they are measured in the same experiment or in several.*

It should be noted that although $\hat{A}$ *and* $\hat{A}'$ *are dependent from the quantum point of view, we cannot measure them in a single experiment simultaneously (see subsection 14.3.3); at the same time, measurements at different points in space are connected through the joint wave function, which leads to correlation between the quantities* $\hat{A},\hat{A}'$ *on the one hand and* $\hat{B},\hat{B}'$ *on the other, and consequently to differing results of the averaging for the quantity* (10.8).

The quantum mechanical approach gives the following expression for the average in the considered state (10.3)

$$\begin{aligned}\langle F\rangle \sim \langle F\rangle_{quant} &= \frac{1}{2}\langle\psi|\,\hat{A}\hat{B}+\hat{A}'\hat{B}+\hat{A}\hat{B}'-\hat{A}'\hat{B}'\,|\psi\rangle = \\ &= \frac{1}{2}\langle\psi|\,\hat{A}\left(\hat{B}+\hat{B}'\right)+\hat{A}'\left(\hat{B}-\hat{B}'\right)|\psi\rangle = \\ &= \frac{1}{\sqrt{2}}\langle\psi|\,\hat{S}_1^{(1)}\hat{S}_1^{(2)}+\hat{S}_2^{(1)}\hat{S}_2^{(2)}\,|\psi\rangle = \\ &= \frac{1}{\sqrt{2}}(-1-1) = -\sqrt{2}. \end{aligned} \tag{10.9}$$

If we accept the thesis of the incompleteness of quantum mechanics, then we assume that the polarization of photons, and hence the Stokes parameters, are defined immediately after the two photons leave the source S. Thus, there exist a priori values a, a', b, and b', and the properties of the source can be described using classical probability theory. Thus, there exist 16 elementary probabilities $p\,(a,b,a',b')$ such that the average value can be written as

$$\langle F\rangle \sim \langle F\rangle_{class} = \sum_{a,b,a',b'=\pm 1} p\,(a,b,a',b')\,f\,(a,b,a',b')\,, \tag{10.10}$$

where

$$\begin{aligned} f\,(a,b,a',b') &= \frac{1}{2}\,(ab+a'b+ab'-a'b') = \\ &= \frac{1}{2}\,(a\,(b+b')+a'\,(b-b'))\,. \end{aligned} \tag{10.11}$$

Function (10.11) can only take two values: $f=\pm 1$. Indeed, there are two possibilities: $b=b'$ or $b=-b'$. In the first case,

$$f = \frac{1}{2}\,(2ab) = \pm 1.$$

In the second case,

$$f = \frac{1}{2}(2a'b) = \pm 1.$$

Thus, function f can take values within the following interval $f_{min} = -1 \leq f \leq f_{max} = +1$ (unlike the measurable quantity $f_i = 0, \pm 1, \pm 2$). It is clear that in the classical case

$$|\langle F \rangle_{class}| \leq 1. \tag{10.12}$$

Comparing expressions (10.9) and (10.12), which are called Bell inequalities, one can see that quantum correlations have a greater quantitative value. This quantitative difference can be tested in an experiment. The first experiment was conducted in 1972 [9] and its results confirm the completeness of the quantum description. The experiment verifying the completeness of quantum mechanics can have not only theoretical importance but also practical applications, for instance, in quantum cryptography, which we will consider below (see 12.3).

It should be noted that expression (10.9) can be obtained using formula (10.10), provided it is accepted that some of the probabilities $p(a, b, a', b') < 0$. Thus we can speak about the non-classicality of the entangled state.

10.3 Bell Basis States

Any two-photon entangled state of the form (10.1):

$$\begin{aligned} |\psi\rangle = c_{xx} |x\rangle_1 |x\rangle_2 + c_{xy} |x\rangle_1 |y\rangle_2 + \\ + c_{yx} |y\rangle_1 |x\rangle_2 + c_{yy} |y\rangle_1 |y\rangle_2 \end{aligned}$$

can be decomposed into the following Bell basis states:

$$\begin{aligned} \left|\psi^{\dagger}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |y\rangle_2 + |y\rangle_1 |x\rangle_2\right), \\ \left|\psi^{-}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2\right), \\ \left|\phi^{\dagger}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |x\rangle_2 + |y\rangle_1 |y\rangle_2\right), \\ \left|\phi^{-}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |x\rangle_2 - |y\rangle_1 |y\rangle_2\right). \end{aligned} \tag{10.13}$$

To prove the decomposability, it is sufficient to show that the wavefunctions (10.13) are orthogonal. For this, we write each of these functions in matrix form

in the basis formed by $|x\rangle_1 |x\rangle_2$, $|x\rangle_1 |y\rangle_2$, $|y\rangle_1 |x\rangle_2$, and $|y\rangle_1 |y\rangle_2$, resulting in

$$
\begin{aligned}
\left|\psi^{\dagger}\right\rangle_{12} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \\
\left|\psi^{-}\right\rangle_{12} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \\
\left|\phi^{\dagger}\right\rangle_{12} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
\left|\phi^{-}\right\rangle_{12} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.
\end{aligned}
\tag{10.14}
$$

When checking the orthogonality of $\left|\psi^{\dagger}\right\rangle_{12}$ and $|\psi^{-}\rangle_{12}$, we have

$$
\left\langle\psi^{\dagger}\right|_{12} \psi^{-}\rangle_{12} = \left|\psi^{\dagger}\right\rangle_{12} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \\
= \frac{1}{2}(1-1) = 0.
$$

Similarly, the orthogonality of the other states (10.13) can be shown.

Moreover, using (10.14), it can be shown that the states (10.13) are normalized; for example, for $\left|\psi^{\dagger}\right\rangle_{12}$ we get

$$
\left\langle\psi^{\dagger}\right|_{12} \psi^{\dagger}\rangle_{12} = \left|\psi^{\dagger}\right\rangle_{12} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \\
= \frac{1}{2}(1+1) = 1.
$$

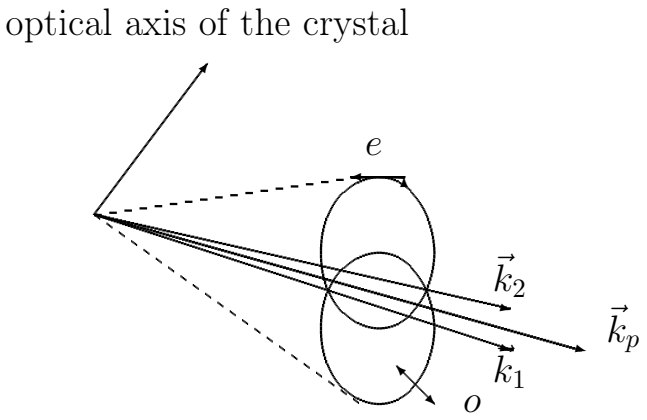


Figure 10.2: Spontaneous parametric scattering with type II synchronism. $\vec{k}_p$ - pump wave vector, e - cone along which extraordinarily polarized photons propagate, o - cone along which ordinarily polarized photons propagate, $\vec{k}_{1,2}$ - directions in which entangled photons propagate

10.4 Generation of Bell States

The process of parametric light scattering, which we considered earlier for describing the generation of squeezed states of the electromagnetic field (see 9.4), can be used to obtain entangled states. Then we considered the first kind of phase matching, where the polarizations of the two photons are the same. In the second kind of phase matching, the polarizations of the photons are different. In this case, the photons propagate along two cones, as shown in Figure 10.2.

Along one cone, the radiation is polarized as ordinary waves, and along the other as extraordinary waves. Thus, at the intersection points of the cones (see Figure 10.3), the state of light is written as

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle_1 |y\rangle_2 + e^{i\alpha} |y\rangle_1 |x\rangle_2 \right),$$

where the phase difference α arises due to the difference in refractive indices for ordinary and extraordinary photons. Using an additional birefringent phase plate, any value of the phase difference α can be obtained, for example 0 or π. As a result, we get the following Bell states:

$$\begin{aligned} \left|\psi^+\right\rangle_{12} &= \frac{1}{\sqrt{2}} \left(|x\rangle_1 |y\rangle_2 + |y\rangle_1 |x\rangle_2 \right), \\ \left|\psi^-\right\rangle_{12} &= \frac{1}{\sqrt{2}} \left(|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2 \right). \end{aligned} \tag{10.15}$$

Now, if a half-wave phase plate is placed before one of the beams, for example the first one, which changes the polarization to the orthogonal:

$$|x\rangle_2 \to |y\rangle_2, \; |y\rangle_2 \to |x\rangle_2,$$

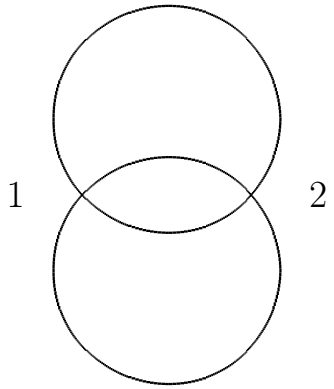


Figure 10.3: Schematic image obtained when photographing the radiation emitted by a nonlinear crystal. Numbers 1 and 2 indicate the directions in which the entangled photons propagate.

then from (10.15) we obtain

$$\left|\phi^{\dagger}\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(\left|x\right\rangle_1\left|x\right\rangle_2 + \left|y\right\rangle_1\left|y\right\rangle_2\right),$$
$$\left|\phi^{-}\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(\left|x\right\rangle_1\left|x\right\rangle_2 - \left|y\right\rangle_1\left|y\right\rangle_2\right). \tag{10.16}$$

Thus, combining (10.15) and (10.16), we obtain the complete basis (10.13).

10.5 Registration of Bell States

Of the four basis states (10.13), only $\left|\psi^{-}\right\rangle_{12}$ is antisymmetric, i.e., it changes its sign upon particle (photon) permutation. This allows us to distinguish this state from all others. For this purpose, the scheme shown in Figure 10.4 can be used, where two photons will be detected in different photodetectors only for the antisymmetric state; for all other states, both photons are detected either in $D^{(1)}$ or $D^{(2)}$.

To prove this, let us consider how a beamsplitter acts on a single-photon state. A photon can impinge on the beamsplitter from half-plane (1) (Figure 10.5); in this case, with probability 0.5 the photon ends up in half-plane (1) and with probability 0.5 in (2). If the photon comes from half-plane (2), it again can end up in either half-plane with equal probability. Thus, to describe the beamsplitter we will use the scattering matrix expression in the form (6.4), where $r = t = \frac{1}{\sqrt{2}}$, i.e., the action of the beamsplitter can be expressed by the following operator

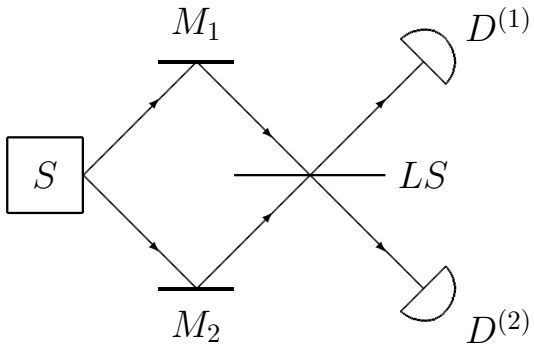


Figure 10.4: Scheme for Bell state registration. S - entangled photon source, M_1 and M_2 - two fixed mirrors, LS - beamsplitter, $D^{(1)}$ and $D^{(2)}$ - two photodetectors

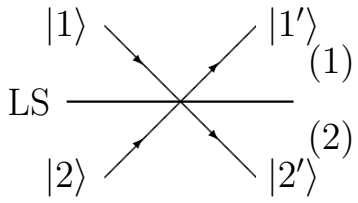


Figure 10.5: Photon passing through a beamsplitter

(Hadamard[34]):

$$\hat{H}\,|1\rangle = \frac{1}{\sqrt{2}}\left(|1'\rangle + |2'\rangle\right),$$
$$\hat{H}\,|2\rangle = \frac{1}{\sqrt{2}}\left(|1'\rangle - |2'\rangle\right). \tag{10.17}$$

In our case, two photons fall on the beamsplitter from different sides, so only two wavefunctions are possible, describing the spatial part of our initial state:

$$|S\rangle_{12} = \frac{1}{\sqrt{2}}\left(|1\rangle_1\,|2\rangle_2 + |2\rangle_1\,|1\rangle_2\right),$$
$$|A\rangle_{12} = \frac{1}{\sqrt{2}}\left(|1\rangle_1\,|2\rangle_2 - |2\rangle_1\,|1\rangle_2\right). \tag{10.18}$$

As seen from (10.18), the state $|S\rangle_{12}$ is symmetric, i.e., it remains unchanged under photon permutation, while the state $|A\rangle_{12}$ is antisymmetric (changes sign upon photon permutation).

Let us consider how the states (10.18) transform upon passing through the beamsplitter. Since the operator $\hat{H}$ acts independently on each photon, for $|S\rangle_{12}$ we have

$$\hat{H}\,|S\rangle_{12} = |S'\rangle_{12} = \frac{1}{\sqrt{2}}\left(\hat{H}\,|1\rangle_1\,\hat{H}\,|2\rangle_2 + \hat{H}\,|2\rangle_1\,\hat{H}\,|1\rangle_2\right),$$

from which, using (10.17), we obtain

$$\begin{aligned}
\hat{H}\,|S\rangle_{12} &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\left(|1'\rangle_1 + |2'\rangle_1\right)\left(|1'\rangle_2 - |2'\rangle_2\right) + \right. \\
&\left. + \frac{1}{2}\left(|1'\rangle_1 - |2'\rangle_1\right)\left(|1'\rangle_2 + |2'\rangle_2\right)\right) = \\
&= \frac{1}{2\sqrt{2}}\left(|1'\rangle_1\,|1'\rangle_2 + |2'\rangle_1\,|1'\rangle_2 - |1'\rangle_1\,|2'\rangle_2 - |2'\rangle_1\,|2'\rangle_2 + \right. \\
&\left. + |1'\rangle_1\,|1'\rangle_2 - |2'\rangle_1\,|1'\rangle_2 + |1'\rangle_1\,|2'\rangle_2 - |2'\rangle_1\,|2'\rangle_2\right) = \\
&= \frac{1}{2\sqrt{2}}\left(|1'\rangle_1\,|1'\rangle_2 - |2'\rangle_1\,|2'\rangle_2 + |1'\rangle_1\,|1'\rangle_2 - |2'\rangle_1\,|2'\rangle_2\right) = \\
&= \frac{1}{\sqrt{2}}\left(|1'\rangle_1\,|1'\rangle_2 - |2'\rangle_1\,|2'\rangle_2\right).
\end{aligned}$$

Thus, for the state $|S'\rangle_{12}$ both photons appear either in half-plane (1) or in half-plane (2).

For the antisymmetric state, we have

$$\begin{aligned}
\hat{H}\,|A\rangle_{12} = |A'\rangle_{12} &= \frac{1}{\sqrt{2}}\left(\hat{H}\,|1\rangle_1\,\hat{H}\,|2\rangle_2 - \hat{H}\,|2\rangle_1\,\hat{H}\,|1\rangle_2\right) = \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\left(|1'\rangle_1 + |2'\rangle_1\right)\left(|1'\rangle_2 - |2'\rangle_2\right) - \right. \\
&\left. - \frac{1}{2}\left(|1'\rangle_1 - |2'\rangle_1\right)\left(|1'\rangle_2 + |2'\rangle_2\right)\right) = \\
&= \frac{1}{2\sqrt{2}}\left(|1'\rangle_1\,|1'\rangle_2 + |2'\rangle_1\,|1'\rangle_2 - |1'\rangle_1\,|2'\rangle_2 - |2'\rangle_1\,|2'\rangle_2 - \right. \\
&\left. - |1'\rangle_1\,|1'\rangle_2 + |2'\rangle_1\,|1'\rangle_2 - |1'\rangle_1\,|2'\rangle_2 + |2'\rangle_1\,|2'\rangle_2\right) = \\
&= \frac{1}{2\sqrt{2}}\left(|2'\rangle_1\,|1'\rangle_2 - |1'\rangle_1\,|2'\rangle_2 + |2'\rangle_1\,|1'\rangle_2 - |1'\rangle_1\,|2'\rangle_2\right) = \\
&= -\frac{1}{\sqrt{2}}\left(|1'\rangle_1\,|2'\rangle_2 - |2'\rangle_1\,|1'\rangle_2\right).
\end{aligned}$$

Thus, in the case of the antisymmetric state, after passing through the beamsplitter, the photons will be in different half-planes.

Since photons are bosons, the total wavefunction describing both polarization and spatial properties must be symmetric [58] (see section 14.8). Thus, from the

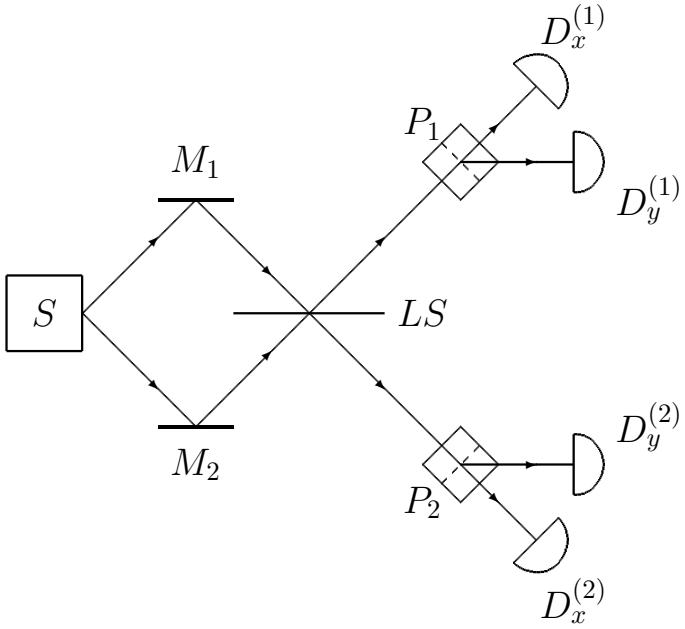


Figure 10.6: Scheme for registering Bell states. S - source of entangled photons, M_1 and M_2 - two fixed mirrors, LS - beam splitter, P_1 and P_2 - Nicol prisms, $D_x^{(1)}$ and $D_x^{(2)}$ - photodetectors for x polarization, $D_y^{(1)}$ and $D_y^{(2)}$ - photodetectors for y polarization

combination of (10.13) and (10.18), only the following combinations are possible:

$$\begin{gathered}\left|\psi^{\dagger}\right\rangle_{12}|S\rangle_{12}\,,\\ \left|\psi^{-}\right\rangle_{12}|A\rangle_{12}\,,\\ \left|\phi^{\dagger}\right\rangle_{12}|S\rangle_{12}\,,\\ \left|\phi^{-}\right\rangle_{12}|S\rangle_{12}\,.\end{gathered}$$

Consequently, for Bell states $\left|\psi^{\dagger}\right\rangle_{12}$, $\left|\phi^{\dagger}\right\rangle_{12}$, and $|\phi^{-}\rangle_{12}$, both photons after passing through the beamsplitter will hit the same photodetector, while for $|\psi^{-}\rangle_{12}$ they will go to different ones, which allows us to uniquely distinguish the state $|\psi^{-}\rangle_{12}$ from all others.

The scheme in Figure 10.4 can be modified so that it can also register the state $\left|\psi^{\dagger}\right\rangle_{12}$. To do this, note that among all symmetric Bell states only in this one do the photons have different polarizations. Thus, this state can be singled out by measuring the photon polarizations.

Figure 10.6 shows a scheme that allows registration of $|\psi^{-}\rangle_{12}$ and $\left|\psi^{\dagger}\right\rangle_{12}$. For the state $\left|\psi^{\dagger}\right\rangle_{12}$, photodetectors $D_x^{(1)}$ and $D_y^{(1)}$ or $D_x^{(2)}$ and $D_y^{(2)}$ will fire simultaneously. For $|\psi^{-}\rangle_{12}$, it is $D_x^{(1)}$ and $D_y^{(2)}$ or $D_x^{(2)}$ and $D_y^{(1)}$. For $\left|\phi^{\dagger}\right\rangle_{12}$ and $|\phi^{-}\rangle_{12}$, both photons will be detected simultaneously in one of the 4 photodetectors. Recently, works have appeared [16] where registration of all four Bell states has been performed.

10.6 No-Cloning Theorem and Quantum Teleportation.

In quantum mechanics, there exists the so-called no-cloning theorem [29]. The meaning of this theorem can be explained using our example of a photon with two mutually orthogonal polarizations, whose wave function is given by relation (7.3). The no-cloning theorem states that it is impossible to create a device where a particle in the state (7.3) is input and two particles in the same state are output.

Indeed, suppose we have a device that performs state cloning. The action of this device on a photon polarized along x is described by the cloning operator $\hat{D}$ as follows:

$$\hat{D}\,|D_I\rangle\,|x\rangle = |D_{F_x}\rangle\,|x\rangle\,|x\rangle\,,$$

where $|D_I\rangle$ is the wave function describing the initial state of the cloning device, and $|D_{F_x}\rangle$ is the wave function describing the state of the device after cloning the photon with vertical x-polarization. For a photon polarized along y, we have

$$\hat{D}\,|D_I\rangle\,|y\rangle = \left|D_{F_y}\right\rangle|y\rangle\,|y\rangle\,,$$

where $\left|D_{F_y}\right\rangle$ is the wave function describing the state of the device after cloning this photon.

If we apply the cloning operator $\hat{D}$ to a photon in an arbitrary state (7.3), we get

$$\hat{D}\,|D_I\rangle\,|\psi\rangle = \hat{D}\,|D_I\rangle\,(\alpha\,|x\rangle + \beta\,|y\rangle) = \alpha\,|D_{F_x}\rangle\,|x\rangle\,|x\rangle + \beta\left|D_{F_y}\right\rangle|y\rangle\,|y\rangle\,,$$

from which it is clearly impossible to obtain the expected result

$$\hat{D}\,|D_I\rangle\,|\psi\rangle = \left|D_{F_{xy}}\right\rangle(\alpha\,|x\rangle + \beta\,|y\rangle)\,(\alpha\,|x\rangle + \beta\,|y\rangle)\,.$$

But if the state of a photon cannot be cloned, it turns out that it can be transmitted from one point in space to another (naturally, with destruction of the original state), as demonstrated by quantum teleportation experiments.

The scheme of the quantum teleportation protocol is shown in Figure 10.7. In this figure, Alice wants to send to Bob the state of photon 1, which is described by the wave function (7.3).

$$|\psi\rangle_1 = \alpha\,|x\rangle_1 + \beta\,|y\rangle_1\,,$$

There is also an entangled photon source S that emits pairs of photons 2 and 3 in the state described by the following wave function:

$$|\psi\rangle_{23} = \left|\psi^-\right\rangle_{23} = \frac{1}{\sqrt{2}}\,(|x\rangle_2\,|y\rangle_3 - |y\rangle_2\,|x\rangle_3)\,.$$

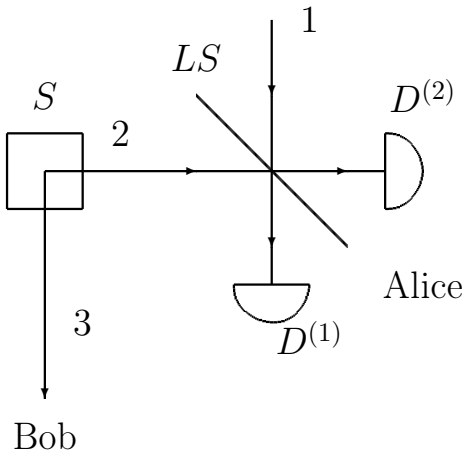


Figure 10.7: Scheme of the quantum teleportation protocol. S is the source of entangled photon pairs, LS - beam splitter, $D^{(1)}$ and $D^{(2)}$ - two photodetectors, 1 - the photon to be teleported, 2 and 3 - the entangled photon pair.

Alice mixes photons 1 and 2 on a beamsplitter LS and then registers a certain Bell state using detectors $D^{(1,2)}$. The easiest registration is of the state $|\psi^-\rangle_{12}$, in which both photodetectors must trigger simultaneously.

The total state of the three particles is written as

$$|\psi\rangle_{123} = |\psi\rangle_1 |\psi\rangle_{23} ,$$

which can be expanded into Bell states of photons 1 and 2:

$$\begin{aligned} |\psi\rangle_{123} = c_{|\psi^\dagger\rangle_{12}} \left|\psi^\dagger\right\rangle_{12} + c_{|\psi^-\rangle_{12}} \left|\psi^-\right\rangle_{12} + \\ + c_{|\phi^\dagger\rangle_{12}} \left|\phi^\dagger\right\rangle_{12} + c_{|\phi^-\rangle_{12}} \left|\phi^-\right\rangle_{12} . \end{aligned} \tag{10.19}$$

In the expansion (10.19), we are interested in the coefficient $c_{|\psi^-\rangle_{12}}$, since it describes the state of the third photon when Alice registers the Bell state $|\psi^-\rangle_{12}$. For the desired coefficient, we obtain:

$$\begin{aligned} c_{|\psi^-\rangle_{12}} = \left\langle\psi^-\right|_{12} \psi\rangle_{123} = \\ = \frac{1}{\sqrt{2}} \left(\langle x|_1 \langle y|_2 - \langle y|_1 \langle x|_2\right) \left(\alpha |x\rangle_1 + \beta |y\rangle_1\right) \frac{1}{\sqrt{2}} \left(|x\rangle_2 |y\rangle_3 - |y\rangle_2 |x\rangle_3\right) = \\ = \frac{1}{2} \left(\alpha \langle y|_2 - \beta \langle x|_2\right) \left(|x\rangle_2 |y\rangle_3 - |y\rangle_2 |x\rangle_3\right) = -\frac{1}{2} \left(\alpha |x\rangle_3 + \beta |y\rangle_3\right) . \end{aligned}$$

This means that whenever Alice registers the pair of photons 1 and 2 in the state $|\psi^-\rangle_{12}$, i.e., when both detectors $D^{(1)}$ and $D^{(2)}$ simultaneously trigger, photon 3 on Bob's side ends up in the state identical to the original photon 1 state, that is, teleportation of photon 1 to Bob occurs.

10.7 Quantum Pseudo-Telepathic Games

The nonclassicality of entangled states can be vividly demonstrated in the so-called quantum pseudo-telepathic games [7, 3]

10.7.1 Game Description

Consider the Mermin-Peres game in which two players, Alice and Bob, play against a casino. Alice and Bob fill a 3×3 square with numbers ± 1. The casino tells Alice the number of the row that Alice must fill, and Bob the number of the column. Alice's product of all digits must be equal to $+1$, while Bob's must be -1. The players win if the number at the intersection of the chosen row and column matches. They lose otherwise.

Alice and Bob are isolated from each other until they receive from the casino the row and column numbers, respectively. Thus, they can only agree on a strategy before this moment.

Example 10.7.1 (Mermin-Peres Game)**.** *Suppose Alice (A) received row number 1, and Bob (B) column number 3. In this case, a winning combination may look like*

$$A = \begin{pmatrix} +1 & -1 & -1 \\ * & * & * \\ * & * & * \end{pmatrix},$$
$$B = \begin{pmatrix} * & * & -1 \\ * & * & +1 \\ * & * & +1 \end{pmatrix}.$$

This combination is winning because both Alice and Bob have the same number -1 at the intersection of the selected row and column.

In the opposite case (row 1, column 1)

$$A = \begin{pmatrix} +1 & -1 & -1 \\ * & * & * \\ * & * & * \end{pmatrix},$$
$$B = \begin{pmatrix} -1 & * & * \\ +1 & * & * \\ +1 & * & * \end{pmatrix},$$

the players lose, since Alice obtains $+1$ at the intersection, while Bob has -1.

10.7.2 Classical Strategy

In the classical strategy, Alice and Bob cannot agree on all elements of the matrix because Alice's and Bob's conditions conflict: according to Alice, the product of all elements of the matrix equals $(+1)^3 = 1$, while according to Bob it equals $(-1)^3 = -1$. Thus, sometimes Alice and Bob win, but there will always be games in which they lose, since the probability of winning $p < 1$.

10.7.3 Quantum Strategy

As we have found, the classical strategy does not exist, but on the other hand, if Alice and Bob can create in advance some entangled state and agree on the measurements they will perform on this state [1], then there is a way to ensure a winning probability $p = 1$.

The entangled state used by Alice and Bob:

$$|\psi\rangle = |\psi\rangle_{A_1,B_1} \otimes |\psi\rangle_{A_2,B_2}$$

The matrix used for measurements

$$X = \begin{pmatrix} \hat{S}_0 \otimes \hat{S}_3 & \hat{S}_3 \otimes \hat{S}_0 & \hat{S}_3 \otimes \hat{S}_3 \\ \hat{S}_1 \otimes \hat{S}_0 & \hat{S}_0 \otimes \hat{S}_1 & \hat{S}_1 \otimes \hat{S}_1 \\ -\hat{S}_1 \otimes \hat{S}_3 & -\hat{S}_3 \otimes \hat{S}_1 & \hat{S}_2 \otimes \hat{S}_2 \end{pmatrix} \tag{10.20}$$

Since

$$\hat{S}_1^2 = \hat{S}_2^2 = \hat{S}_3^2 = \hat{S}_0^2 = \hat{I}$$

for each element $\hat{x}_{ij}$ of the matrix (10.20) we have $\hat{x}_{ij}^2 = \hat{I}$, and therefore upon measurement we get eigenvalues ± 1.

Moreover, Alice and Bob can perform measurements since the operators in the same row or the same column commute, i.e., can be measured simultaneously. The values Alice obtains satisfy the requirements imposed on the row, and Bob's on the column.

Finally, the measurements performed by Alice and Bob at the matching position also coincide. This follows from the relation

$$\left(\hat{x}_{ij}^A \otimes \hat{x}_{ij}^B\right) |\psi\rangle = |\psi\rangle \,,$$

where $\hat{x}_{ij}^A$ is Alice's measurement, and $\hat{x}_{ij}^B$ is Bob's measurement. Thus the product of the results obtained by Alice and Bob is equal to 1, or, since the results can only be ± 1, Alice's and Bob's results coincide.

TBD

[1] naturally, the measurements are performed by Alice and Bob separately, each in their own laboratory

10.8 Exercises

1. Prove the orthonormality of the Bell basis states (10.13).

2. Which Bell state should Alice register, in the scheme shown in Figure 10.7, to confirm the fact of teleportation in the case when the source of entangled photon pairs S produces photons in the state

$$|\psi\rangle_{23} = \left|\psi^{\dagger}\right\rangle_{23} = \frac{1}{\sqrt{2}}\left(|x\rangle_2 |y\rangle_3 + |y\rangle_2 |x\rangle_3\right).$$

3. Obtain expressions for the coefficients $c_{|\psi^{\dagger}\rangle_{12}}$, $c_{|\phi^{\dagger}\rangle_{12}}$, and $c_{|\phi^{-}\rangle_{12}}$ in the decomposition (10.19).

Part 4

Quantum Information Theory

Chapter 11

Introduction to Quantum Information Theory

Classical information theory was developed by Claude Shannon [23] applied to communication theory problems and is currently finding increasing application. In this section, we will consider the basic concepts and results of information theory, including its application to quantum phenomena.

11.1 Information and Entropy

The very concept of information is closely related to the measure of uncertainty of a system, i.e., what in statistical physics is called entropy. The more disordered a system is, the less information we know about it. Acquiring information reduces the measure of uncertainty. Accordingly, the more disordered the original system was, the more information we acquire about it as a result of the outcome of some experiment. Thus, as a measure of information, one can take the complexity (uncertainty) of the system. Following Shannon, we will call this measure entropy, while we will not distinguish between "amount of information measure" and "entropy," i.e., we will consider these concepts synonyms.

11.1.1 Entropy in Classical Information Theory

We begin the definition of entropy with an experiment with equally probable outcomes. Suppose we have an event $\mathcal{A}$, which has a equally likely outcomes. As entropy (a measure of uncertainty), it is reasonable to use some positive function that depends on the number of outcomes a and increases with increasing a:

$$H_{\mathcal{A}} = f(\mathcal{A}) = f(a).$$

In this case, it is reasonable to assume that $f(1) = 0$, since the result of an experiment with a single possible value is fully determined.

Now consider another experiment $\mathcal{B}$ with a number of outcomes b. Obviously, for it the relation

$$H_{\mathcal{B}} = f(b).$$

holds. If we consider the event $\mathcal{AB}$, which consists of simultaneous occurrence of $\mathcal{A}$ and $\mathcal{B}$, then the number of equally likely outcomes of such an event is $a \cdot b$, so we can write

$$H_{\mathcal{AB}} = f(ab).$$

On the other hand, we would like to express $H_{\mathcal{AB}}$ through $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$. To do this, recall the property of entropy increase with increasing system complexity, i.e., the following inequality holds:

$$H_{\mathcal{AB}} \geq H_{\mathcal{A}}, H_{\mathcal{B}}. \tag{11.1}$$

If we use only the four simplest arithmetic operations to find the desired relation, then only addition and multiplication satisfy property (11.1). Moreover, if we consider the situation where the number of outcomes of event $\mathcal{B}$ equals 1, then for the desired relation only the operation of addition can be used:

$$H_{\mathcal{AB}} = H_{\mathcal{A}} + H_{\mathcal{B}},$$

from which

$$f(ab) = f(a) + f(b).$$

Now, let's combine what we know about the desired function f

$$\begin{aligned} f(1) &= 0, \\ f(a) > f(b), &\text{ if } a > b, \\ f(ab) &= f(a) + f(b). \end{aligned} \tag{11.2}$$

There is only one continuous function that satisfies the requirements (11.2) (proof see [32]):

$$f(a) = K \log a. \tag{11.3}$$

The coefficient K in (11.3) determines the logarithm base and the unit of information measurement. For example, if we use as the unit of entropy the experiment with two equally probable outcomes (such as heads or tails in coin tossing), then (11.3) uses the logarithm base 2:

$$f(a) = \log_2 a, \tag{11.4}$$

and the corresponding unit of information is called a **bit**[1]. This unit choice for measuring information is explained by the fact that often information is encoded using objects which have only two stable states (e.g., signal power exceeding a certain threshold or not). Along with the logarithm base 2, the natural logarithm is often used in (11.3):

$$f(a) = \ln a, \tag{11.5}$$

where the corresponding unit of information is called a **nat**. In what follows, we will use entropy definition in the form (11.3) with the logarithm base taken as 2.

Expression (11.3) was obtained under the assumption of equally probable outcomes. As is known, the probability of event $\mathcal{A}$ is defined as

$$p_{\mathcal{A}} = \frac{1}{a},$$

from which entropy $H_{\mathcal{A}}$ can be written as

$$H_{\mathcal{A}} = -\log p_{\mathcal{A}}. \tag{11.6}$$

Now consider the case when event $\mathcal{A}$ has unequal probabilities of outcomes. In this case, each outcome ξ of event $\mathcal{A}$ should be associated with the probability of that outcome, p_ξ. To define the measure of information acquired upon realization of outcome ξ, we will still use expression (11.6):

$$H_\xi = -\log p_\xi. \tag{11.7}$$

The quantity H_ξ in (11.7) characterizes only one particular outcome ξ of the event and, accordingly, cannot be used as a measure of uncertainty of the whole event $\mathcal{A}$. As a measure, it is reasonable to use the averaged value of entropy:

$$H_{class} = H_{\mathcal{A}} = \sum_{\xi} -p_\xi \log p_\xi. \tag{11.8}$$

In expression (11.8) it is taken that

$$0 \cdot \log 0 = 0. \tag{11.9}$$

Despite the seeming convention in introducing (11.8), this expression indeed describes the amount of information; in particular, living organisms process information at a speed proportional to (11.8). In [32] experiments on human reaction speed to external stimuli are described. As the stimulus, one of several light bulbs illuminated, each with different frequency. Before the experiment, examinees were

[1] from English **bi**nary digi**t** - binary digit

given time to train, after which it was found that reaction speed is determined precisely by expression (11.8).

Entropy (11.8) has the following property:

$$H_{min} \leq H_{\mathcal{A}} \leq H_{max}, \tag{11.10}$$

where $H_{min} = 0$ and is achieved when the probability of one of the outcomes of event $\mathcal{A}$ equals 1 and the rest are 0 (minimal system uncertainty). $H_{max} = \log a$ is achieved when all outcomes are equally probable (maximum system uncertainty).

11.1.2 Entropy in Quantum Systems

The analog of the probability distribution $\{p_\xi\}$ in the quantum case is the density matrix $\hat{\rho}$. The averaging here uses the trace operation in some basis. That is, (11.8) in quantum systems corresponds to:

$$H_{quant} = -\mathrm{Tr}\left(\hat{\rho}\,\log\,\hat{\rho}\right). \tag{11.11}$$

When analyzing relation (11.11), the following questions arise:

- How to compute the entropy (11.11)
- In which cases the entropy will be maximal and minimal, i.e., what is the quantum analog of the classical expression 11.10.

To answer the first question, recall that the trace operation does not depend on the representation (see subsection 14.1.7), so it is enough to consider the density matrix in the basis where it is diagonal. If we denote the diagonal elements of the density matrix by ρ_{nn}, then using (14.27) we can rewrite (11.11) as

$$H_{quant} = -\mathrm{Tr}\left(\hat{\rho}\,\log\,\hat{\rho}\right) = -\sum_n \rho_{nn}\,\log\,\rho_{nn}. \tag{11.12}$$

Let's start analyzing expression (11.12) by considering pure states. If we take an arbitrary pure state $|\psi\rangle$, written in some basis $\{|n\rangle\}$:

$$|\psi\rangle = \sum_n c_n\,|n\rangle\,,$$

then one can always find a basis $\{|m\rangle\}$ in which $|\psi\rangle$ is one of the basis states, for example the first:

$$|\psi\rangle = \sum_m c_m\,|m\rangle = |m\rangle|_{m=1}\,,$$

thus, the diagonal density matrix will have only one non-zero element $\rho_{11} = 1$, and the entropy will be

$$H_{pure} = 0, \tag{11.13}$$

where we used expression (11.9). This expression tells us that the pure state is completely determined.

If we consider a mixed state, the density matrix will be diagonal, with diagonal elements equal to the probabilities of the system being in the corresponding states:

$$\rho_{nn} = p_n.$$

If now we use (11.12), we get that the expression for entropy of the mixed state coincides with the classical entropy expression (11.8):

$$H_{misc} = -\sum_n p_n \log p_n = H_{class}, \tag{11.14}$$

which, as mentioned earlier, is maximal for uniform probability distribution and minimal for degenerate one (one outcome probability 1, the others 0).

Along with expressions (11.12) and (11.14), in which the unit of information is nat, it is convenient to use a logarithm base two - then the unit is called a qubit (**q**uantum **bit**).

11.2 Information Transmission. Communication Channel.

Such general characteristics of information as entropy do not take into account the concrete content of information, which may also be important, and therefore may not always be applicable. Nevertheless, there are situations when the specific value is unimportant, and what matters is the amount of information. As an example of such systems we can consider information transmission systems.

11.2.1 Classical Communication Channel

A typical information transmission system is shown in Figure 11.1. As seen in this figure, before information is transmitted, it must be encoded, i.e., represented in states that can be transmitted through the channel. At the channel output, information must be decoded, i.e., transformed back into the form used at the input.

Encoding information with classical objects that have only two stable states (e.g., signal power exceeding or not a threshold) is the most convenient. Obviously, classical states are chosen so that they can be easily detected.

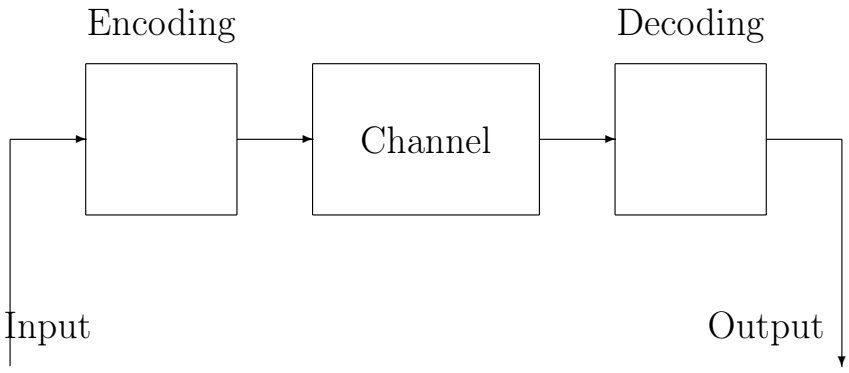


Figure 11.1: Information transmission. Communication channel

11.2.2 Quantum Communication Channel

Using quantum objects for information transmission significantly expands the capabilities of the communication channel because a quantum system has a much larger choice of states that can be used for transmission.

At the same time, the quantum channel itself has some special properties. For example, consider transmitting information encoded using pure quantum states. As mentioned earlier (see 10.6), an arbitrary quantum state cannot be copied, whence it follows that in such a system, unlike the classical communication channel, not only substitution of information is impossible, but even unauthorized reading. This is particularly important for information security.

11.2.3 Quantum Dense Coding

As an example of a quantum communication channel, consider the quantum dense coding protocol proposed by Bennett and Wiesner [5], [18]. This protocol uses the polarization-entangled two-photon state, considered earlier by us (see chapter 10).

Each of the photons individually has two orthogonal states $|x\rangle_{1,2}$ and $|y\rangle_{1,2}$. Thus, from the classical point of view using these two photons, one can encode four bits of information using the following states: $|x\rangle_1 |x\rangle_2$, $|x\rangle_1 |y\rangle_2$, $|y\rangle_1 |x\rangle_2$, and $|y\rangle_1 |y\rangle_2$.

In the quantum dense coding protocol, one can also encode 4 bits of information, but while operating on *only one* of the two photons. The scheme of this protocol is shown in Figure 11.2. Source S produces a pair of entangled photons in some Bell state, for example $\left|\Psi^{\dagger}\right\rangle_{12}$. The first photon of this pair is sent to Alice, and the second to Bob. By changing the properties of his (second) photon,

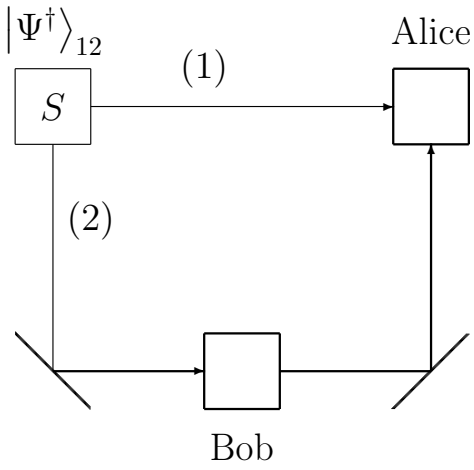


Figure 11.2: Quantum dense coding protocol

Bob transforms the initial state into one of the 4 Bell states (10.13):

$$\begin{aligned}
|\psi^\dagger\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |y\rangle_2 + |y\rangle_1 |x\rangle_2\right), \\
|\psi^-\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2\right), \\
|\phi^\dagger\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |x\rangle_2 + |y\rangle_1 |y\rangle_2\right), \\
|\phi^-\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|x\rangle_1 |x\rangle_2 - |y\rangle_1 |y\rangle_2\right).
\end{aligned} \tag{11.15}$$

In the case of the state $|\Psi^\dagger\rangle_{12}$ no additional action is required since it is the initial:

$$|y\rangle_2 \to |y\rangle_2, \; |x\rangle_2 \to |x\rangle_2. \tag{11.16}$$

To obtain the state $|\psi^-\rangle_{12}$ Bob must produce a polarization-dependent phase shift:

$$|y\rangle_2 \to |y\rangle_2, \; |x\rangle_2 \to e^{i\pi} |x\rangle_2. \tag{11.17}$$

In the case of the state $|\phi^\dagger\rangle_{12}$ Bob performs polarization flip to the orthogonal

$$|y\rangle_2 \to |x\rangle_2, \; |x\rangle_2 \to |y\rangle_2. \tag{11.18}$$

Finally, for the state $|\phi^-\rangle_{12}$ Bob performs both polarization flip and polarization-dependent phase shift:

$$|y\rangle_2 \to |x\rangle_2, \; |x\rangle_2 \to e^{i\pi} |y\rangle_2. \tag{11.19}$$

After Bob performs one of the transformations (11.16) - 11.19, he sends his particle to Alice. Alice, in turn, measures the received Bell state (11.15), for example using the detector described in 10.5, and determines which of the four actions Bob took. Thus, Bob can transmit 4 bits of information to Alice.

11.3 Information Encoding

11.3.1 Shannon's Coding Theorem

Suppose we have a message that can consist of a certain limited set of symbols (an alphabet). Assume that our alphabet has k symbols:

$$\{a_1, \ldots a_k.\} \tag{11.20}$$

Each symbol appears with some known probability $p_k = p(a_k)$. Next, consider a message consisting of $n \gg 1$ symbols. We are interested in the question: is it possible to compress this message, i.e. is it possible to encode this message in such a way that the result contains fewer than n symbols and at the same time it would be possible to recover the original message without loss of information?

The answer was given by Shannon, and it states that the maximally compressed message will consist of nH bits, where $H = -\sum_k p_k \log p_k$ is the quantity of information characterizing our alphabet. This result is known as Shannon's theorem for a noiseless channel.

As an example, it is convenient to use the binary alphabet, in which 0 occurs with probability p, and 1 with probability $1 - p$. Thus, if we have a message of length n bits, then it is possible to compress it to nH bits. It is obvious that

$$n \geq nH > 0. \tag{11.21}$$

Equality in (11.21) holds only when the probability distribution is uniform, $p = \frac{1}{2}$. Indeed, in this case

$$nH = n\left(-\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right) = n.$$

11.3.2 Quantum Coding Theorem

What would the quantum analogue of Shannon's theorem for a noiseless channel look like? First of all, consider the quantum analogue of the alphabet (11.20). A quantum message is encoded by some set of states

$$\{|a_1\rangle, \ldots |a_k\rangle .\} \tag{11.22}$$

Each of the states appears with some probability $p_k = p\left(|a_k\rangle\right)$. Thus, we can write the density matrix for each symbol of our message:

$$\hat{\rho} = \sum_k p_k \left|a_k\right\rangle \left\langle a_k\right| . \tag{11.23}$$

A message consisting of n symbols has the following density matrix:

$$\hat{\rho}^n = \hat{\rho} \otimes \cdots \otimes \hat{\rho}.$$

Can this message be compressed, and if yes, to what limit? The answer is given by the quantum coding theorem for a noiseless channel (Schumacher's theorem). The limit is reached when using

$$nH = -n \operatorname{Tr}\left(\hat{\rho} \log \hat{\rho}\right),$$

As an example, consider encoding using the polarization states of a photon. Suppose our alphabet consists of two states:

1. $|x\rangle$ - photon polarized along the x-axis
2. $|y\rangle$ - photon polarized along the y-axis

Assume that the state $|x\rangle$ occurs with probability $p_{|x\rangle} = p$, and the state $|y\rangle$ with probability $p_{|y\rangle} = 1 - p_{|x\rangle} = 1 - p$. Thus, the density matrix (11.23) will have the form

$$\hat{\rho} = p \left|x\right\rangle \left\langle x\right| + (1-p) \left|y\right\rangle \left\langle y\right| . \tag{11.24}$$

In this case, it is convenient to use qubits as the unit of information and, accordingly, if the original message is encoded by n photons, then it can be compressed to $nH = -n \operatorname{Tr}\left(\hat{\rho} \log \hat{\rho}\right)$ qubits, i.e. fewer than n photons can be used to transmit this message. The main difference between the quantum case and the classical one is that compression of information is possible even if $p = \frac{1}{2}$.

11.4 Exercises

1. Prove the quantum expression for entropy (11.12).

Chapter 12

Cryptography

From the moment the importance of information was realized, means of its protection began to appear.

New encryption methods were invented, such as the Caesar cipher, where each letter of the alphabet was replaced by another (for example, the one three positions after it in the alphabet). Along with new encryption methods, techniques for breaking these ciphers appeared, for example, for the Caesar cipher, one can use the statistical properties of the language in which the original message was written.

Very often, the security of a cipher was ensured by keeping the encryption algorithm secret, as was the case with the Caesar cipher mentioned above. In modern classical cryptography, algorithms are usually published and available for study by anyone. Security is ensured by mixing the message itself with a secret key according to some public algorithm.

Suppose we need to transmit a message from Alice to Bob via some secure communication channel. The message must be represented in some digital form. The protocol describing such transmission consists of several stages. In the first, Alice and Bob must obtain a common random sequence of numbers, which will be called the key. This procedure is called key distribution.

At the next stage, Alice should use some algorithm E to obtain from the original message P and key K the encrypted message C. This procedure can be described by the following relation:

$$E_K(P) = C. \tag{12.1}$$

At the third stage, the resulting encrypted message must be transmitted to Bob.

In the last stage, Bob, using the known algorithm D and the key K obtained in the first stage, should recover the original message P from the received encrypted

message C. This procedure can be described by the following relation

$$D_K(C) = P. \tag{12.2}$$

Analyzing this protocol raises the following questions. How to securely implement key distribution. Second - does an absolutely secure algorithm exist. And finally - is it possible to securely transmit an encrypted message without it being eavesdropped on or tampered with.

Classical cryptography gives a definitive answer only to the second question. An absolutely secure algorithm exists - it is called the one-time pad. A detailed description of this algorithm is given below.

12.1 One-Time Pad

The one-time pad scheme was proposed in 1917 by Major J. Moborn and G. Vernam. The classic one-time pad represents some set of random keys, each of which is the same size as the message being sent and is used only once.

Suppose we want to encrypt a message in some language (for example, English). Let the number of symbols (letters) used in the alphabet be denoted by X. For the English language (without punctuation marks and case differences) $X = 26$. Next, we assign each character of the language a number c, such that $0 \leq c \leq X$. For example, for the English language one can write

$$\begin{gathered} A \to 0 \\ B \to 1 \\ \dots \\ Z \to 25 \end{gathered}$$

The encryption procedure (12.1) is described by the following expression

$$E_{K_i}(P_i) = P_i + K_i \mod X = C_i, \tag{12.3}$$

where i is the index of the encrypted character.

The decryption procedure (12.2) is described by the following expression

$$D_{K_i}(C_i) = C_i - K_i \mod X = P_i, \tag{12.4}$$

where i is the index of the encrypted character.

This procedure is easily generalized to the case of binary data, where instead of addition modulo X, the XOR operation $(a \oplus b)$ is used for both encryption and decryption:

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

Table 12.1: XOR $a \oplus b$

Claude Shannon showed [24] that if the key is truly random, has the same length as the original message, and is never reused, then the proposed one-time pad scheme is perfectly secure.

According to Shannon, perfect security can be defined as follows.

Definition 12.1.1. A cipher (E, D) is perfectly secure if for any two messages of the same length m_0 and m_1, some ciphertext c, and key $k \leftarrow_R K$, the probabilities that the original texts are m_0 or m_1 are equal:

$$P\left(E\left(m_0, k\right) = c\right) = P\left(E\left(m_1, k\right) = c\right)$$

Rephrasing this definition, one can say that no information about the original message can be derived from the statistics of the ciphertext.

Theorem 12.1.1 (Security of the One-Time Pad)**.** *The one-time pad scheme has perfect security.*

Proof. Let $|K|$ denote the number of all possible keys of length l. Where l is also the length of the original messages: $|m_{0,1}| = l$. Because the key by which the message is encrypted is uniquely defined:

$$k_{0,1} = c \oplus m_{0,1},$$

we get for the probabilities

$$P\left(E\left(m_0, k\right) = c\right) = P\left(E\left(m_1, k\right) = c\right) = \frac{1}{|K|}.$$

□

12.2 Problems of Classical Cryptography

If there exists a perfectly secure cryptographic system (one-time pad), then what is wrong with classical cryptography? The problem lies in obtaining keys that satisfy the requirements of the one-time pad (key length equal to the message

length, key consists of random data and is never reused) and transmitting these keys to Bob and Alice.

Problems arise both at the key generation stage, [1] as well as during the transmission of these keys.

To transmit keys in classical cryptography, so-called public key algorithms are used. There are several key exchange protocols based on public key cryptographic systems. All of them are based on the fact that there are two keys, the first called the public key, used only for encryption, and the second—the private key—for decryption. To obtain the private key from the public key, one must perform some complex mathematical operation. For example, the security of one of the most popular public key systems—RSA (see 19.6), is based on the difficulty of factoring[2] large numbers.

The key distribution protocol scheme based on public key cryptography can be described as follows. In the first step, Alice creates a public and private key and sends the first to Bob. Bob, on his part, creates the key that both Alice and Bob would like to have (the key to be distributed). This key is encrypted (for example, using RSA) with Alice's public key and sent to her. Alice, upon receiving this encrypted key, can decrypt it using her private key.

If an intruder (Eve) wants to learn the transmitted key, she must solve the difficult mathematical problem of factoring large numbers. It is believed, but not proven, that the complexity of factoring grows exponentially with the number of digits in the number [34].[3] Thus, as the number of digits increases, the problem quickly becomes intractable.

There are several problems in this scheme. The first is that the complexity of factorization has not been proven. Moreover, there are algorithms for quantum computers—the Shor's algorithm (see 13.3), which solve the factoring problem for a number N in time $O(\log N)$, i.e., in time proportional to the number of digits in N. Therefore, at the moment when a quantum computer is built, all systems based on RSA will lose their relevance.

12.3 Quantum Cryptography

The randomness inherent in quantum objects suggests the idea of using them for key distribution. There exist many different secure key distribution schemes based on the use of quantum objects. We will explore a scheme based on the

[1] obtaining large sequences of random numbers is a non-trivial mathematical problem

[2] decomposing into prime factors

[3] The fastest known algorithms solve the factoring problem for a number N in time on the order of $O\left(\exp\left(\log^{\frac{1}{3}} N\,(\log\log N)^{\frac{2}{3}}\right)\right)$.

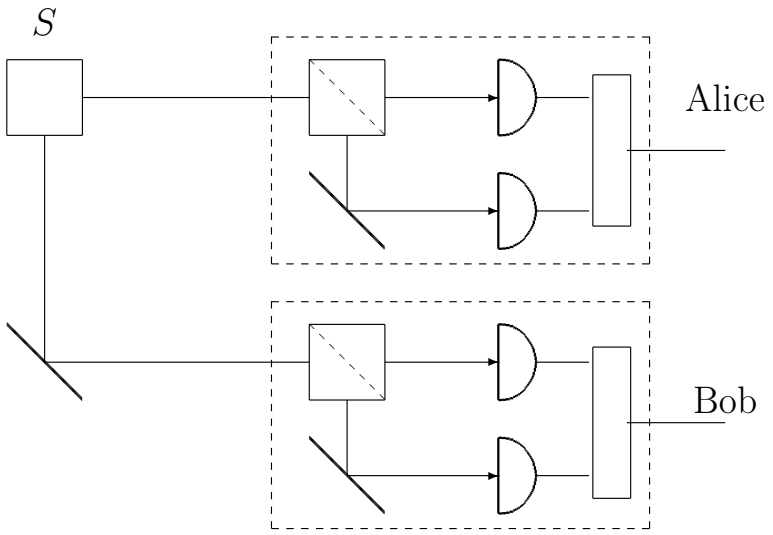


Figure 12.1: Key distribution scheme based on the verification of Bell inequalities

verification of Bell inequalities (see 10.2). The considered scheme is shown in Figure 12.1.

A source of entangled photons S generates pairs of photons, one of which is sent to Alice and the other to Bob, who perform measurements of the Stokes parameters of their photons.

Alice randomly measures either $\hat{A} = \hat{S}_1^{(1)}$ or $\hat{A}' = \hat{S}_2^{(1)}$. Bob randomly measures the following quantities

$$\hat{B} = \frac{1}{\sqrt{2}}\left(\hat{S}_1^{(2)} + \hat{S}_2^{(2)}\right),$$
$$\hat{B}' = \frac{1}{\sqrt{2}}\left(\hat{S}_1^{(2)} - \hat{S}_2^{(2)}\right),$$
$$\hat{C} = \hat{S}_1^{(2)},$$
$$\hat{C}' = \hat{S}_2^{(2)}.$$

Thus, as a result of the experiment, we obtain 8 pairs of values, which can be grouped into three sets.

The first set contains correlated combinations of operators $\hat{A}$, $\hat{A}'$, $\hat{C}$, and $\hat{C}'$: (a, c) and (a', c'). For these combinations, we can say that if one of the numbers a or a' equals ± 1, then the other (c or c') equals ∓ 1. Thus, these numbers can be used to obtain a random sequence of numbers, which will subsequently be used as the key.

The second group contains 4 pairs of values, which will be used to verify Bell inequalities: (a, b), (a', b), (a, b'), and (a', b').

The last group contains pairs of values (a, c') and (a', c). This group of values is discarded in the future.

At the initial stage, Bob and Alice independently perform measurements in a random order. Upon completion of the measurements, they communicate to each other (over a conventional open communication channel) which quantities they measured in each particular trial, without revealing the measurement results themselves. Subsequently, those trials in which either failed to register a photon are discarded, as well as results related to the third group of measurements. The results of the second group of measurements are openly published and the average value $\langle F \rangle$ (10.8) is calculated.

If the obtained value in magnitude is close to the predictions of quantum mechanics (10.9)

$$\langle F \rangle_{quant} = -\sqrt{2},$$

then the results of the first group of measurements can be interpreted as the key.

If an eavesdropper Eve attempts to learn the results of the first group of measurements, which constitute the distributed key, one way to do this is to replace the entangled photon pair source herself and send Alice and Bob pairs of photons with predetermined polarization properties, so that the outcomes of Alice's and Bob's experiments are predetermined in advance. However, in this case, $\langle F \rangle$, according to (10.12), will lie in the interval:

$$-1 \leq \langle F \rangle_{class} \leq 1.$$

Thus, Alice and Bob can determine the presence of Eve's intervention by checking Bell inequalities, and consider the obtained key compromised if the test detects the presence of Eve.

Chapter 13

Quantum Computing

Algorithms play a significant role in computer science. An algorithm is a sequence of steps necessary to obtain the solution to a certain problem. Each problem is characterized by a certain number that defines its size. The complexity of an algorithm is evaluated as the number of elementary operations needed to solve the given problem. Obviously, in most cases (but not always), this number grows with the size of the problem.

Example 13.0.1. Finding an element in an array *The task is to find an element in an array that satisfies certain conditions. The size of the problem is the number of elements in the array N.*

In the general case (an unstructured data array), the search is performed by simple enumeration. This search requires a number of operations (comparisons) that grows linearly with the size of the array $O(N)$.

In the case of structured data, the number of operations required for the search can be reduced. For example, in the case of a sorted array, the complexity of the problem grows as $O(\log N)$.

At the same time, the existence of an algorithm does not yet guarantee its practical feasibility. In particular, algorithms requiring an exponential number of steps relative to the size of the original problem are considered practically unrealizable despite the fact that, from a theoretical point of view, a solution exists.

One example is the problem of factorizing a natural number, i.e., the problem of decomposing it into prime factors (see example 13.0.2).

Example 13.0.2. Factorization of natural numbers *The task is to find the decomposition of a number into prime factors. The size of the problem is the number of digits of the original number. For example, in the case of digit length $r = 4$: $1 \leq N = 15 \leq 2^r = 2^4 = 16$. The result can be found easily and quickly:* $15 = 3 \cdot 5$.

As the number of digits r grows, the number of operations needed for factorization in classical algorithms grows as $O\left(2^r\right)$, which for $r = 1000 - 2000$ means the practical impossibility of factorizing such numbers.

Quantum objects possess properties differing from classical objects, accordingly algorithms built on the basis of quantum objects can, in some cases, have characteristics unavailable to classical algorithms. For example, Grover's quantum algorithm [11] solves the problem of search in an unstructured data array (see example 13.0.1) using $O\left(\sqrt{N}\right)$ operations. Shor's algorithm [25] allows solving the problem of number factorization (see example 13.0.2) using a linear number of operations $O\left(r\right)$.

13.1 Basic Principles of Quantum Computing

13.1.1 Information Representation. Classical and Quantum States

The main difference between quantum and classical computers lies in how they store information.

In the classical case, information is stored in some memory cells. The state of each memory cell is described by a single number which can take the value 0 or 1. When m memory cells are combined, the overall state of the classical system (which it can assume at a given moment in time) is described by m numbers.

In the quantum case, a memory cell is represented by a qubit, described by two complex numbers α_0 and α_1 [1]:

$$|\psi\rangle_1 = \alpha_0 |0\rangle + \alpha_1 |1\rangle .$$

To describe a composite system consisting of m qubits, 2^m complex numbers are required. In other words, the quantum state contains all possible classical states as a superposition. As an example, consider a system consisting of 3 qubits:

$$\begin{aligned} |\psi\rangle_3 = \alpha_0 |000\rangle + \alpha_1 |001\rangle + \alpha_2 |010\rangle + \alpha_3 |011\rangle + \\ + \alpha_4 |100\rangle + \alpha_5 |101\rangle + \alpha_6 |110\rangle + \alpha_7 |111\rangle . \end{aligned} \tag{13.1}$$

As can be seen, any classical state of a system of 3 bits is represented as one of the terms in the superposition (13.1). For example, the number $5_{10} = 101_2$ appears in (13.1) with coefficient α_5.

[1] More precisely, three real numbers are needed, because the following constraint acts on $\alpha_{0,1}$: $|\alpha_0|^2 + |\alpha_1|^2 = 1$, from which, considering $\alpha_{0,1} = r_{0,1} e^{i\theta_{0,1}}$, we get $r_0^2 = 1 - r_1^2$

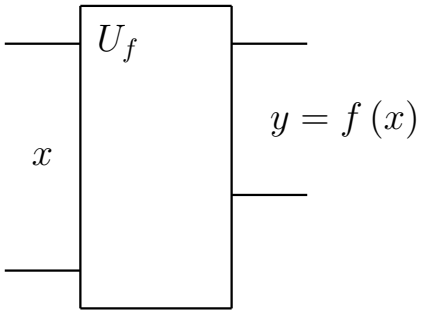


Figure 13.1: Classical computation. The input is a number x consisting of n bits, and the output is the result $y = f(x)$ described by m bits

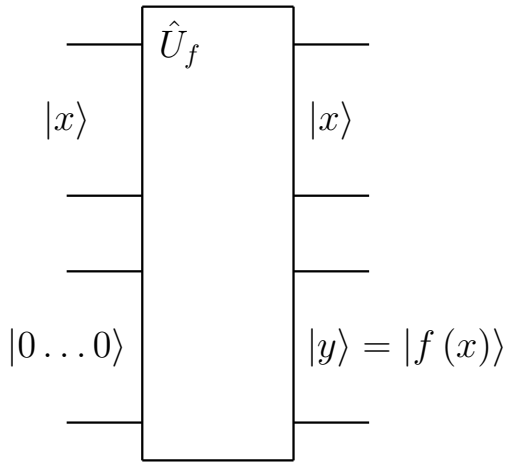


Figure 13.2: Quantum reversible computations. The input is a number $|x\rangle$ consisting of n qubits and a seed of zero states (m qubits), and the output is the result $|y\rangle = |f(x)\rangle$ described by m qubits and the original state $|x\rangle$

13.1.2 Reversible Computation

In the classical case, computation consists of transforming the initial n bits into a result described by m bits (see Figure 13.1). The transformation is defined by some function $f(x)$. A typical example is addition modulo 2 (see Table 12.1), where the input consists of 2 bits ($n = 2$), and the output is 1 bit ($m = 1$).

Such a scheme will not work in the quantum case, primarily because the evolution of pure quantum states over time must be carried out by a unitary evolution operator (14.35), i.e. it must be reversible, which is impossible for our classical example[2]. Therefore, quantum computations use a different scheme (see Figure 13.2) in which reversible computations are possible.

The input consists of the original data x, described by n qubits, along with m qubits in the state $|0\rangle$, such that the total number of inputs and outputs match.

[2]It is impossible to obtain two bits of original information from one bit (the result)

Hence, the relation between input and output can be described as [3]

$$\underbrace{|x\rangle}_{n}\underbrace{|f(x)\rangle}_{m} = \hat{U}_f \underbrace{|x\rangle}_{n}\underbrace{|0\ldots0\rangle}_{m}. \tag{13.2}$$

13.2 Quantum Logic Elements

How can an element performing the transformation $\hat{U}_f$ (13.2) be constructed? There exists a set of elements from which one can build, with a given accuracy, an element performing the required transformation $\hat{U}_f$. Such sets are called universal.

13.2.1 Universal Set of Quantum Gates

Identity Transformation

$$\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Negation

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Phase Shift

$$\hat{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hadamard Transformation

One of the basic quantum logic elements is the Hadamard transformation (see Figure 13.3), which is defined by the following relations

$$\hat{H}|0\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$
$$\hat{H}|1\rangle = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

[3] More accurately, it can be written in general form as $\underbrace{|x\rangle}_{n}\underbrace{|f(x)\rangle}_{m}\underbrace{|r\rangle}_{k} = \hat{U}_f \underbrace{|x\rangle}_{n}\underbrace{|0\ldots0\rangle}_{m+k}$, where $|r\rangle$ is a leftover register of size k qubits which is not used in the computation and serves to ensure that the operator $\hat{U}_f$ is unitary

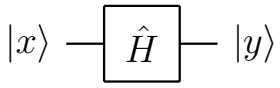


Figure 13.3: Hadamard transformation on a single qubit

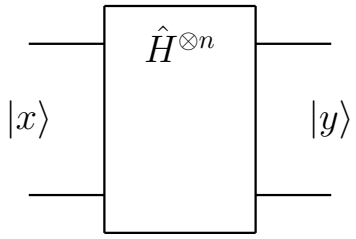


Figure 13.4: Hadamard transform $\hat{H}^{\otimes n}$ on multiple qubits

In matrix form, this transformation can be written as

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{13.3}$$

where the basis vectors are chosen as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From (13.3), the following property of the operator $\hat{H}$ can be obtained:

$$\hat{H}\hat{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13.4}$$

This transformation is used to obtain superpositions of states containing all possible values of the argument of the computed function (see Figure 13.4).

CNOT

The transformation matrix has the form

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This gate (see Figure 13.5) is applied to two qubits and flips the state of the second qubit only if the first qubit equals one.

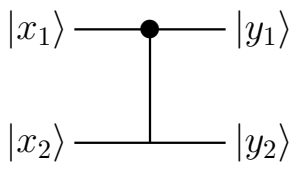


Figure 13.5: Control element CNOT

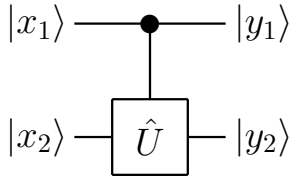


Figure 13.6: Control Element

Thus, if our initial state of two qubits was

$$|\psi_i\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

then it transforms into

$$CNOT\,|\psi_i\rangle = |\psi_f\rangle = a\,|00\rangle + b\,|01\rangle + c\,|11\rangle + d\,|10\rangle$$

Universal Set

Definition 13.2.1 (Universal Set of Quantum Gates)**.** A set of quantum gates is called universal if any unitary transformation can be approximated with given accuracy by a finite sequence of gates from this set.

Theorem 13.2.1 (Kitaev)**.** *The set $\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2, \hat{H}, CNOT$ is universal.*

Proof. TBD □

13.2.2 Controlled Elements

13.3 Shor's Algorithm

One of the most popular encryption algorithms, RSA (see section 19.6) is based on the assumption of the difficulty of factorization (decomposition into

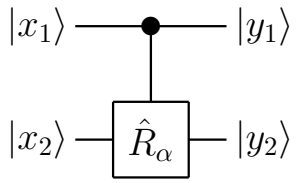


Figure 13.7: Controlled phase shift $\hat{R}_\alpha$

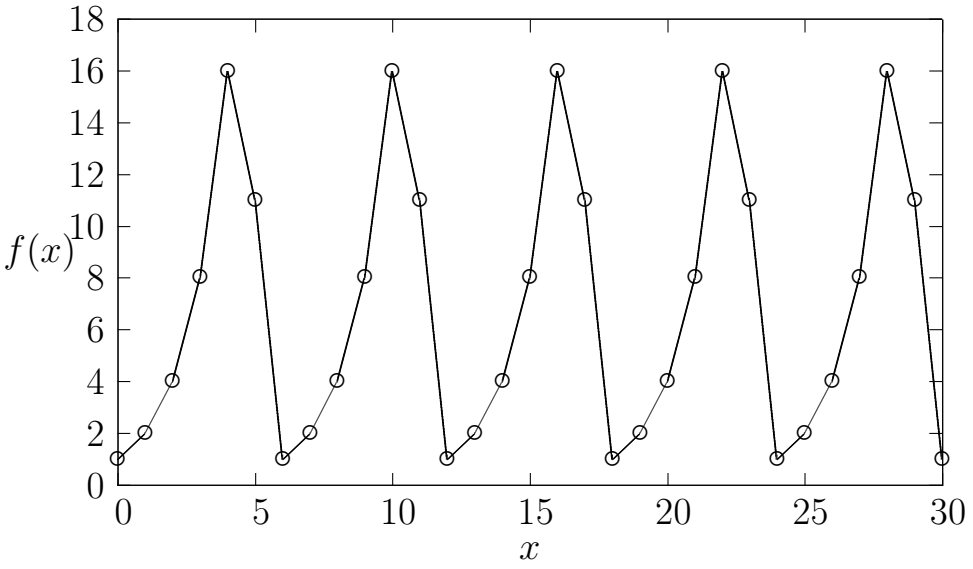


Figure 13.8: Plot of the function $f(x, a) = a^x \mod N$ for $a = 2$, $N = 21$. The period of the function is $r = 6$.

prime factors) of large numbers. Accordingly, algorithms that allow factorization are of particular interest. Below is a description of such an algorithm proposed by Shor [25].

13.3.1 Number Factorization and Period Finding of Functions

The problem of factoring a number N is closely related to finding the period of functions. Consider the following function, which is called modular exponentiation:

$$f(x, a) = a^x \mod N. \tag{13.5}$$

The function (13.5) depends on the number N under analysis and two arguments x and a. The argument a is chosen under the following conditions:

$$\begin{aligned} 0 < a < N, \\ \gcd(N, a) = 1. \end{aligned} \tag{13.6}$$

A typical graph of the function (13.5) is shown in Figure 13.8.

The conditions for choosing the coefficient a (13.6) ensure that a and N have no common divisors. If such divisors do exist, then they are the solution to the factorization problem and can be easily found using the Euclidean algorithm (see section 19.1).

The function (13.5) is periodic, i.e., there exists a number r such that $f(x + r, a) = f(x, a)$. The smallest such number r is called the period of the function (13.5).

To prove periodicity, note that $f(x,a)$ cannot be zero. Indeed, if $f(x,a)=0$ were true, then

$$\exists x \in \{0,1,\dots\} : a^x = k \cdot N,$$

where k is an integer, which is impossible due to the coprimeness of a and N (13.6) [4].

Thus, the range of the function (13.5) is limited to the set

$$f(x,a) \in \{1,\dots,N-1\},$$

from which it follows that

$$\exists k,j : k > j, k, j \in \{0,1,\dots,N\}, f(k,a) = f(j,a),$$

which proves the periodicity of the function (13.5).

Let $k = j + r$, then

$$a^k \mod N = a^{j+r} \mod N = a^j a^r \mod N = a^j \mod N,$$

since a and N are coprime, we can write

$$a^r \equiv 1 \mod N. \tag{13.7}$$

The period of the function (13.5) may be either even or odd. In Shor's algorithm, we are interested in the first case: the period is an even number. Otherwise, a new number a is chosen and the period finding is repeated. Thus, with $r = 2 \cdot l$, we can rewrite (13.7) as

$$a^{2 \cdot l} \equiv 1 \mod N,$$

and since r is the minimal number satisfying the periodicity condition,

$$a^l \not\equiv 1 \mod N.$$

If at the same time the number a is chosen such that

$$a^l \not\equiv -1 \mod N,$$

then we have

$$\left(a^l - 1\right)\left(a^l + 1\right) = k \cdot N, \tag{13.8}$$

where k is some integer. From (13.8), it follows that $a^l \pm 1$ have common nontrivial (different from 1) divisors with N.

[4] It is obviously assumed that $N > 1$.

Example 13.3.1. Finding the divisors of the number $N = 21$ *As an example, consider the problem of finding divisors of the number* $N = 21$. *Choosing* $a = 2$, *we get the period of the function* (13.5) $r = 6$ *(see Figure 13.8). Obviously,*

$$2^3 \equiv 8 \mod 21 \not\equiv -1 \mod 21.$$

Thus, by finding the corresponding greatest common divisors, we solve the problem

$$\begin{aligned} gcd\left(2^3 - 1, 21\right) &= gcd\left(7, 21\right) = 7, \\ gcd\left(2^3 + 1, 21\right) &= gcd\left(9, 21\right) = 3, \\ 21 &= 7 \cdot 3. \end{aligned}$$

Thus, the problem of factoring the number N can be reduced to the problem of finding the period of a certain function by means of the following algorithm:

Algorithm 1 Shor's Algorithm

$a \Leftarrow 0$
repeat
  Choose a new number a such that $0 < a < N$
  if GCD $(a, N) \neq 1$ **then**
    return a
  end if
  Find the period r of the function $f(x, a) = a^x \mod N$
until $(r \not\equiv 0 \mod 2)$ **or** $(a^{\frac{r}{2}} \equiv -1 \mod N)$
return GCD $\left(a^{\frac{r}{2}} \pm 1, N\right)$

13.4 Quantum Fourier Transform

For the analysis of periodic sequences (functions), the discrete Fourier transform (see section 18.2) can be used, which is defined by the following relation (18.4):

$$\tilde{X}_k = \sum_{m=0}^{M-1} x_m e^{-\frac{2\pi}{M} k \cdot m}, \tag{13.9}$$

where the original sequence of numbers $\{x_m\}$ has M terms.

13.4.1 Quantum Fourier Transform Circuit

The quantum Fourier transform [5] deals with states of the form

$$|x\rangle = \sum_{k=0}^{M-1} x_k |k\rangle , \tag{13.10}$$

where there is a sequence of amplitudes $\{x_k\}$, which defines the original sequence for the Fourier transform (13.9). In the basis vector $|k\rangle$ the index of this sequence element is recorded.

Obviously, the terms of the sequence (13.10) must satisfy the normalization condition

$$\sum_k |x_k|^2 = 1.$$

Suppose that some operator $\hat{F}^M$ (the quantum Fourier transform operator) transforms the basis vector $|k\rangle$ according to the rule given by relation (18.4):

$$\hat{F}^M |k\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{-i\omega kj} |j\rangle_{inv} \tag{13.11}$$

The systems of basis vectors $\{|k\rangle\}$ and $\{|k\rangle_{inv}\}$ represent the same set of vectors numbered in different ways.

From (13.10) and (13.11) we obtain

$$\begin{aligned}
\hat{F}^M |x\rangle = \sum_{j=0}^{M-1} x_k \hat{F}^M |k\rangle = \\
= \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \sum_{j=0}^{M-1} e^{-i\omega kj} x_k |j\rangle_{inv} = \\
= \sum_{j=0}^{M-1} \left\{ \frac{1}{\sqrt{M}} \left(\sum_{k=0}^{M-1} e^{-i\omega kj} x_k \right) \right\} |j\rangle_{inv} = \\
= \sum_{j=0}^{M-1} \tilde{X}_j |j\rangle_{inv} = \left| \tilde{X} \right\rangle_{inv},
\end{aligned}$$

where

$$\tilde{X}_j = \tilde{X}_j^M = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{-i\omega kj} x_k. \tag{13.12}$$

[5] For the analysis of the quantum Fourier transform circuit, the work [15] was used

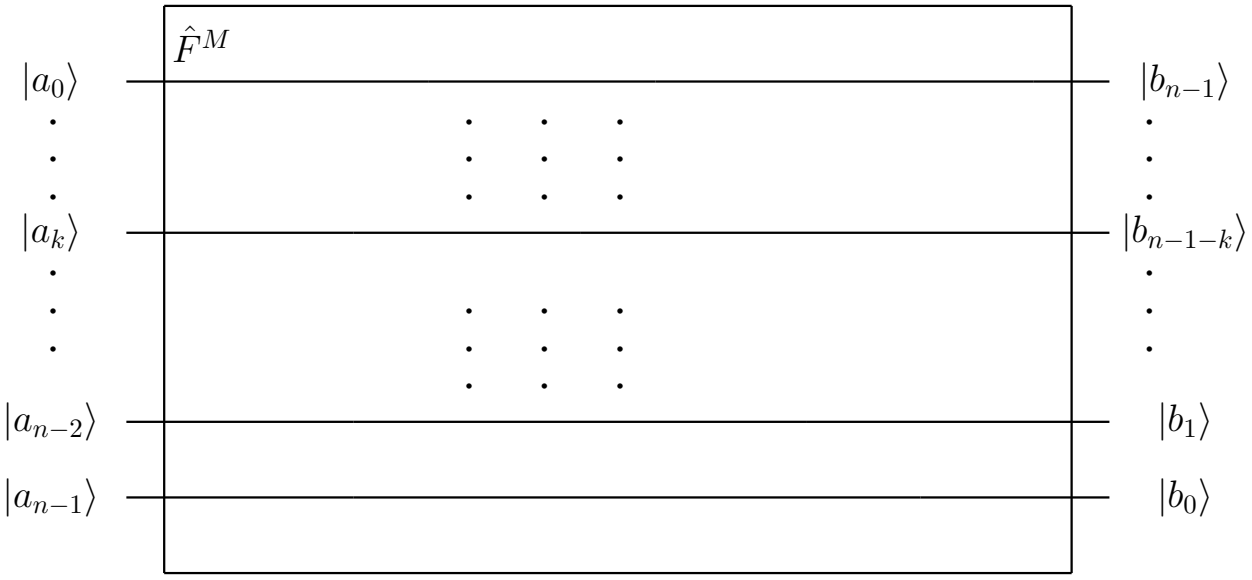


Figure 13.9: Diagram of the quantum Fourier transform based on the fast Fourier transform algorithm. Input and output data

The expression (13.12) repeats the classical analogue (18.4), i.e. we can write

$$|x\rangle \longleftrightarrow \left|\tilde{X}\right\rangle_{inv}.$$

Now suppose that the input to our system is a state of the form (13.10) which represents a superposition of M basis states $\{|k\rangle\}$ (see Figure 13.9). Assume that the number of basis states is a power of two, i.e. a basis state can be represented as a tensor product of $n = \log_2 M$ qubits:

$$|k\rangle = \left|a_0^{(k)}\right\rangle \otimes \left|a_1^{(k)}\right\rangle \otimes \cdots \otimes \left|a_{n-1}^{(k)}\right\rangle,$$

where

$$k = a_0^{(k)} + 2^1 a_1^{(k)} + \cdots + 2^{n-1} a_{n-1}^{(k)},$$
$$a_i^{(k)} \in \{0, 1\}.$$

At the output (see Figure 13.9) we have a superposition of M basis states $\{|j\rangle_{inv}\}$, where for the state $|j\rangle_{inv}$ we have

$$|j\rangle_{inv} = \left|b_{n-1}^{(j)}\right\rangle \otimes \left|b_{n-2}^{(j)}\right\rangle \otimes \cdots \otimes \left|b_0^{(j)}\right\rangle,$$

where

$$j = b_0^{(j)} + 2^1 b_1^{(j)} + \cdots + 2^{n-1} b_{n-1}^{(j)},$$
$$b_i^{(j)} \in \{0, 1\}.$$

From the formula (18.10) one can notice that if we have an input signal x consisting of $n = \log_2 M$ bits, then the bit $a_0^{(k)}$ can be used to select even (the first term of the sum (18.10)) or odd (the second term of the sum (18.10)) elements.

Indeed, excluding $a_0^{(k)}$, the state (13.10) can be represented as a sum of even and odd components:

$$\begin{aligned}
|x\rangle = \sum_{k=0}^{M-1} x_k |k\rangle = \sum_{k=0}^{M-1} x_k \left|a_0^{(k)}\right\rangle \otimes \left|a_1^{(k)}\right\rangle \otimes \cdots \otimes \left|a_{n-1}^{(k)}\right\rangle = \\
= \sum_{m=0}^{\frac{M}{2}-1} x_{k=2m} |0\rangle \otimes \left|a_1^{(k)}\right\rangle \otimes \cdots \otimes \left|a_{n-1}^{(k)}\right\rangle + \\
+ \sum_{m=0}^{\frac{M}{2}-1} x_{k=2m+1} |1\rangle \otimes \left|a_1^{(k)}\right\rangle \otimes \cdots \otimes \left|a_{n-1}^{(k)}\right\rangle = \\
= \sum_{m=0}^{\frac{M}{2}-1} x_{k=2m} |0\rangle \otimes |m\rangle + \sum_{m=0}^{\frac{M}{2}-1} x_{k=2m+1} |1\rangle \otimes |m\rangle = \\
= \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |2m\rangle + \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |2m+1\rangle ,
\end{aligned}$$

where

$$m = a_1^{(k)} + 2^1 a_2^{(k)} + \cdots + 2^{n-2} a_{n-1}^{(k)}.$$

Applying the Fourier transform only to the higher bits $\hat{F}^{\frac{M}{2}}$, i.e. excluding $a_0^{(k)}$, we get (see Figure 13.10):

$$\begin{aligned}
|x\rangle \to \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |2m\rangle + \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |2m+1\rangle = \\
= \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |0\rangle \otimes |m\rangle + \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |1\rangle \otimes |m\rangle = \\
= \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |0\rangle \otimes \hat{F}^{\frac{M}{2}} |m\rangle + \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |1\rangle \otimes \hat{F}^{\frac{M}{2}} |m\rangle . \qquad (13.13)
\end{aligned}$$

Taking into account expression (13.11) we get

$$\hat{F}^{\frac{M}{2}} |m\rangle = \sqrt{\frac{2}{M}} \sum_{j=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} |j\rangle_{inv} .$$

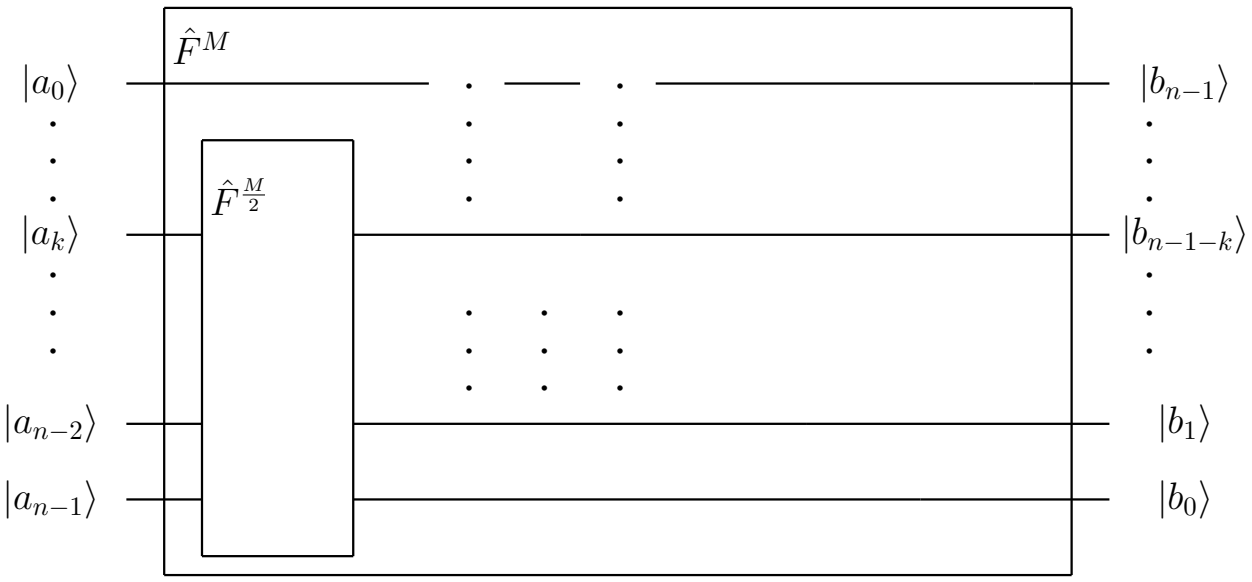


Figure 13.10: Diagram of the quantum Fourier transform based on the fast Fourier transform algorithm. Step 1: $|x\rangle \rightarrow \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |2m\rangle + \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |2m+1\rangle$

Thus, for (13.13) we have

$$
\begin{aligned}
|x\rangle \rightarrow \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |0\rangle \otimes \hat{F}^{\frac{M}{2}} |m\rangle + \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |1\rangle \otimes \hat{F}^{\frac{M}{2}} |m\rangle = \\
= \sqrt{\frac{2}{M}} \sum_{j=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |0\rangle \otimes |j\rangle_{inv} + \\
+ \sqrt{\frac{2}{M}} \sum_{j=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |1\rangle \otimes |j\rangle_{inv} = \\
= \sum_{j=0}^{\frac{M}{2}-1} \left(\sqrt{\frac{2}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} x_{2m} \right) |j\rangle_{inv} + \\
+ \sum_{j=0}^{\frac{M}{2}-1} \left(\sqrt{\frac{2}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} x_{2m+1} \right) \left| \frac{M}{2} + j \right\rangle_{inv} = \\
= \sum_{j=0}^{\frac{M}{2}-1} \tilde{A}_j |j\rangle_{inv} + \sum_{j=0}^{\frac{M}{2}-1} \tilde{B}_j \left| \frac{M}{2} + j \right\rangle_{inv},
\end{aligned}
$$

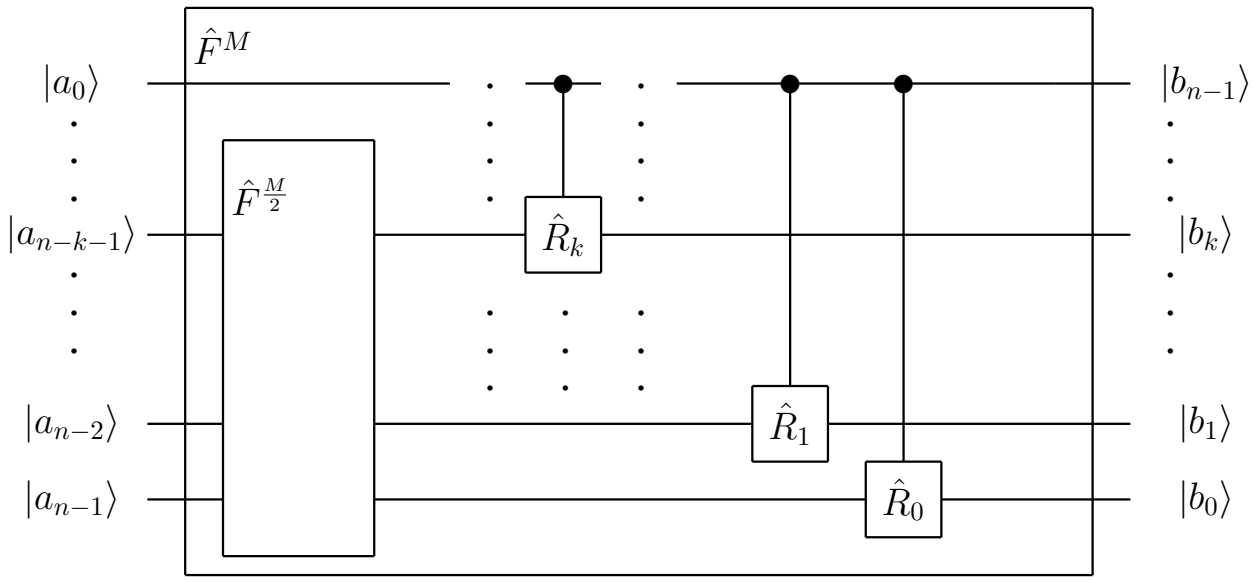


Figure 13.11: Quantum Fourier transform circuit based on the fast Fourier transform algorithm. Step 2: $|x\rangle \to \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |2m\rangle + \hat{R}\hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1}$

where

$$\tilde{A}_j = \sqrt{\frac{2}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} x_{2m}$$
$$\tilde{B}_j = \sqrt{\frac{2}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} x_{2m+1} \tag{13.14}$$

If we now add a phase shift for odd elements, i.e. for those where $a_0^k = 1$, then we obtain the circuit shown in Figure 13.11:

$$\begin{aligned} |x\rangle \to \hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m} |2m\rangle + \hat{R}\hat{F}^{\frac{M}{2}} \sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} |2m+1\rangle = \\ = \sum_{j=0}^{\frac{M}{2}-1} \tilde{A}_j |j\rangle_{inv} + \sum_{j=0}^{\frac{M}{2}-1} \tilde{B}_j \hat{R} \left|\frac{M}{2}+j\right\rangle_{inv}, \\ = \sum_{j=0}^{\frac{M}{2}-1} \tilde{A}_j |j\rangle_{inv} + \sum_{j=0}^{\frac{M}{2}-1} \tilde{C}_j \left|\frac{M}{2}+j\right\rangle_{inv}. \end{aligned} \tag{13.15}$$

Using the expression

$$\hat{R}_l \left| b_l^{(j)} \right\rangle = \exp\left\{ \left(-2\pi i \frac{b_l^{(j)}}{2^{n-l}} \right) \right\} \left| b_l^{(j)} \right\rangle$$

we get that the operator $\hat{R}$ acts on the state $\left| \frac{M}{2} + j \right\rangle_{inv}$ in the following way:

$$\begin{aligned} \hat{R} \left| \frac{M}{2} + j \right\rangle_{inv} &= \hat{R} \left| 1 \right\rangle \otimes \left| j \right\rangle_{inv} = \\ &= \left| 1 \right\rangle \otimes \hat{R}_0 \left| b_0^{(j)} \right\rangle \otimes \cdots \otimes \hat{R}_{n-2} \left| b_{n-2}^{(j)} \right\rangle = \\ &= \prod_{l=0}^{n-2} \exp\left\{ \left(-2\pi i \frac{2^l b_l^{(j)}}{2^n} \right) \right\} \left| 1 \right\rangle \otimes \left| j \right\rangle_{inv} = \\ &= \exp\left\{ \left(-2\pi i \frac{j}{M} \right) \right\} \left| \frac{M}{2} + j \right\rangle_{inv} \end{aligned} \tag{13.16}$$

In the derivation of (13.16), it was taken into account that $j = b_0^{(j)} + 2^1 b_1^{(j)} + \cdots + 2^{n-2} b_{n-2}^{(j)}$.

Thus, for $\tilde{C}_j$ in (13.15) we have

$$\begin{aligned} \tilde{C}_j &= \sqrt{\frac{2}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-2\pi i \frac{j}{M}} e^{-i\frac{4\pi}{M} m j} x_{2m+1} = \\ &= \sqrt{\frac{2}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{2\pi}{M}(2m+1)j} x_{2m+1} \end{aligned} \tag{13.17}$$

If we now apply the Hadamard transform to the qubit $|a_0\rangle$, we obtain the circuit shown in Figure 13.12. In this case the original state is transformed according

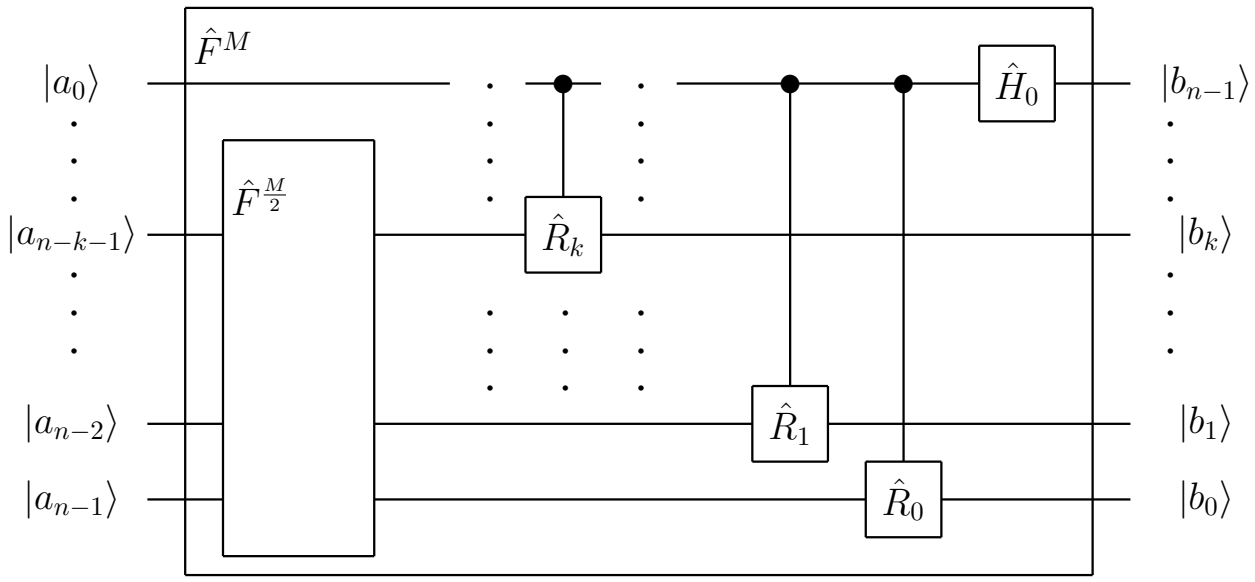


Figure 13.12: Quantum Fourier transform circuit based on the fast Fourier transform algorithm

to the following rule:

$$|x\rangle \to \hat{H}_0\hat{F}^{\frac{M}{2}}\sum_{m=0}^{\frac{M}{2}-1} x_{2m}\,|2m\rangle + \hat{H}_0\hat{R}\hat{F}^{\frac{M}{2}}\sum_{m=0}^{\frac{M}{2}-1} x_{2m+1} =$$

$$= \sum_{j=0}^{\frac{M}{2}-1} \tilde{A}_j\hat{H}\,|0\rangle\otimes|j\rangle_{inv} + \sum_{j=0}^{\frac{M}{2}-1} \tilde{C}_j\hat{H}\,|1\rangle\otimes|j\rangle_{inv} =$$

$$= \frac{1}{\sqrt{2}}\sum_{j=0}^{\frac{M}{2}-1} \tilde{A}_j\,(|0\rangle+|1\rangle)\otimes|j\rangle_{inv} + \frac{1}{\sqrt{2}}\sum_{j=0}^{\frac{M}{2}-1} \tilde{C}_j\,(|0\rangle-|1\rangle)\otimes|j\rangle_{inv} =$$

$$= \sum_{j=0}^{\frac{M}{2}-1} \frac{\tilde{A}_j+\tilde{C}_j}{\sqrt{2}}\,|0\rangle\otimes|j\rangle_{inv} + \sum_{j=0}^{\frac{M}{2}-1} \frac{\tilde{A}_j-\tilde{C}_j}{\sqrt{2}}\,|1\rangle\otimes|j\rangle_{inv} =$$

$$= \sum_{j=0}^{\frac{M}{2}-1} \frac{\tilde{A}_j+\tilde{C}_j}{\sqrt{2}}\,|j\rangle_{inv} + \sum_{j=0}^{\frac{M}{2}-1} \frac{\tilde{A}_j-\tilde{C}_j}{\sqrt{2}}\left|\frac{M}{2}+j\right\rangle_{inv}. \quad (13.18)$$

For the terms in (13.18) taking into account the equalities (13.14) and (13.17) we

have:

$$\frac{\tilde{A}_j + \tilde{C}_j}{\sqrt{2}} = \sqrt{\frac{1}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} x_{2m} + \sqrt{\frac{1}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{2\pi}{M}(2m+1)j} x_{2m+1} = \\ = \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}mj} x_m \quad (13.19)$$

and

$$\frac{\tilde{A}_j - \tilde{C}_j}{\sqrt{2}} = \sqrt{\frac{1}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{4\pi}{M}mj} x_{2m} - \sqrt{\frac{1}{M}} \sum_{m=0}^{\frac{M}{2}-1} e^{-i\frac{2\pi}{M}(2m+1)j} x_{2m+1} = \\ = \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}mj} x_m \frac{1+e^{-i\pi m}}{2} - \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}mj} x_m \frac{1-e^{-i\pi m}}{2} = \\ = \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}mj} e^{-i\pi m} x_m = \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}mj} e^{-i\frac{2\pi}{M}m\frac{M}{2}} x_m = \\ = \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}m\left(\frac{M}{2}+j\right)} x_m \quad (13.20)$$

Combining (13.18), (13.19) and (13.20) finally gives

$$|x\rangle \to \sum_{j=0}^{\frac{M}{2}-1} \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}mj} x_m |j\rangle_{inv} + \\ + \sum_{j=0}^{\frac{M}{2}-1} \sqrt{\frac{1}{M}} \sum_{m=0}^{M-1} e^{-i\frac{2\pi}{M}m\left(\frac{M}{2}+j\right)} x_m \left|\frac{M}{2}+j\right\rangle_{inv} = \\ = \sum_{j=0}^{M-1} \tilde{X}_j^M |j\rangle_{inv}$$

13.4.2 Finding the period of functions using quantum Fourier transform

To determine the period of the function (13.5), the circuit shown in Figure 13.13 is used.

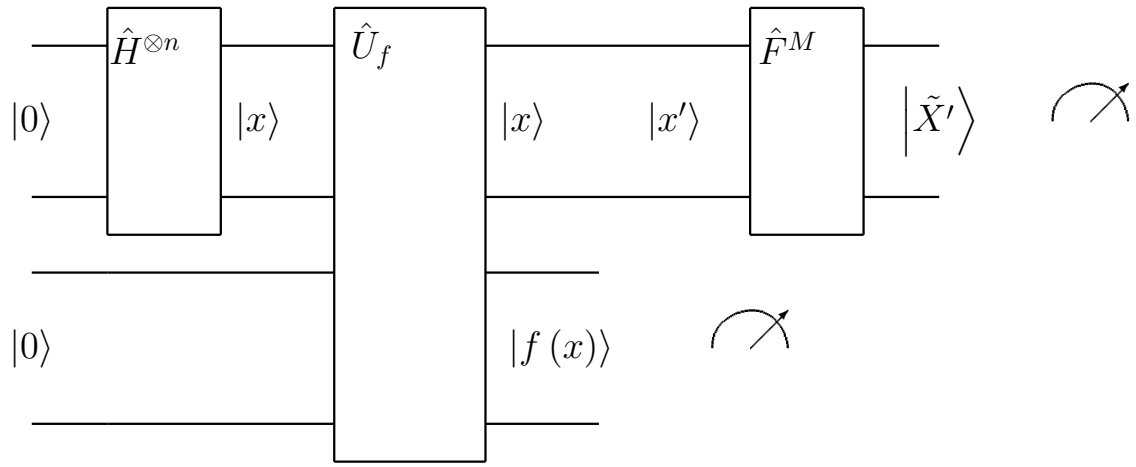


Figure 13.13: Period finding of functions using quantum Fourier transform

The first element is the Hadamard transform on n qubits, which prepares the initial state as:

$$|in\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes |0\rangle .$$

After the element computing the function $\hat{U}_f$, the state becomes

$$\hat{U}_f |in\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes |f(x)\rangle .$$

After measuring the value of the function, only those elements for which the function value equals the measured value remain in the coordinate list. As a result, the input to the Fourier transform measurement element is a state of the form

$$|in'\rangle = \sum_{x'} |x'\rangle ,$$

where all nonzero elements have the same amplitude and follow with a period equal to the period of the investigated function. The initial value will have a shift which depends on the experiment (different experiments will have different shifts). According to lemma 18.2.1, the Fourier image will be the same for different function measurements.

Furthermore, due to lemma 18.2.2 (on periodicity) (see also comment 18.2.1), it follows that the most probable measurements (maxima of probability) occur with a period related to the original period of the function. Thus, as a result of several experiments, the period of the desired function can be found with the required probability level (see Figure 13.14).

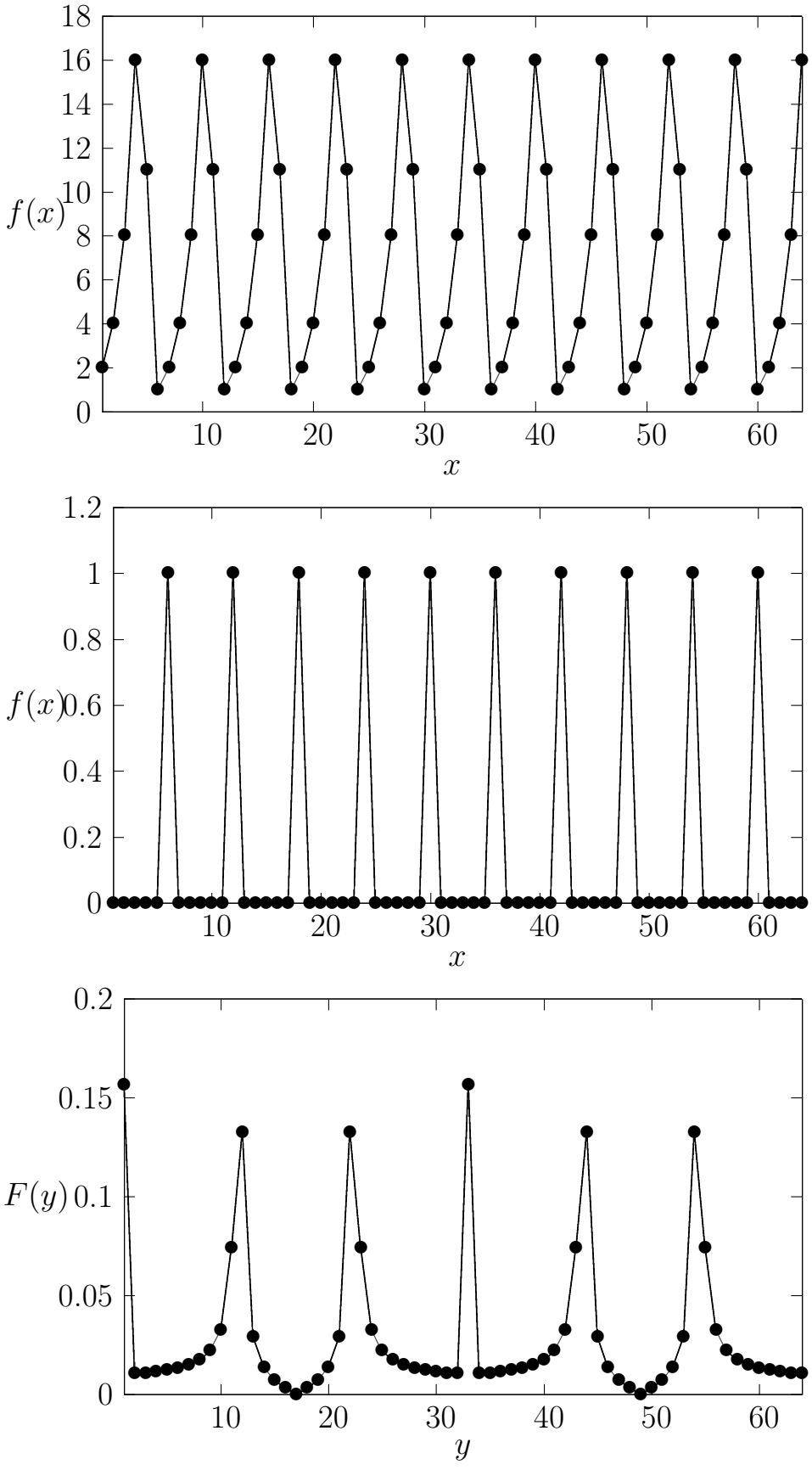


Figure 13.14: Shor's algorithm. Finding the period of the function $f(x, a) = a^x \mod N$ for $a = 2$, $N = 21$ (top graph). The function value 1 repeats with period $r = 6$ (middle graph). Local maxima of the Fourier transform of the middle graph occur with period $\frac{M}{r} \approx 10.67$ (bottom graph). See example 13.4.1

Example 13.4.1. Finding the period of the function $f(x) = 2^x \mod 21$ *As an example, consider the task of finding the period of the function* $f(x, a) = a^x \mod N$ *with* $a = 2$, $N = 21$ *see Figure 13.14*

The number of samples M *must be a power of two. In our example we choose* $M = 2^6 = 64$ *as the number of samples. Thus, 6 qubits are required for our example.*

The initial state after the Hadamard transform is:

$$|in\rangle = \frac{1}{8}\sum_{x=0}^{63} |x\rangle \otimes |0\rangle ,$$

where $|x\rangle$ *is the tensor product of 6 qubits which encode the binary representation of the argument of the investigated function. For example, for* $x = 5_{10} = 000101_2$ *we have*

$$|x\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$$

After calculating the function, the state is (see the upper graph in Figure 13.14)

$$\hat{U}_f |in\rangle = \frac{1}{8}\sum_{x=0}^{63} |x\rangle \otimes |f(x)\rangle =$$
$$= \frac{1}{8}\left(|0\rangle \otimes |2\rangle + |1\rangle \otimes |4\rangle + |2\rangle \otimes |8\rangle + \cdots + \right.$$
$$\left. + |62\rangle \otimes |8\rangle + |63\rangle \otimes |16\rangle\right). \qquad (13.21)$$

If the measurement result of the function was equal to 1*, then from the sum* (13.21) *only those terms remain for which the function value is equal to* 1 *(see the middle graph in Figure 13.14):*

$$|in'\rangle = \frac{1}{\sqrt{10}}\left(|5\rangle \otimes |1\rangle + |11\rangle \otimes |1\rangle + |17\rangle \otimes |1\rangle + \cdots + |60\rangle \otimes |1\rangle\right). \qquad (13.22)$$

Expression (13.22) *contains 10 terms of equal amplitude, so the normalization factor is* $\frac{1}{\sqrt{10}}$.

The Fourier transform of the sequence (13.22) *is shown in the lower graph in Figure 13.14. The most probable values of the Fourier transform measurement result will be those corresponding to local maxima which repeat with a period* $\frac{M}{r} \approx 10.67$ *from which the period of the desired function* $r = 6$ *can be found.*

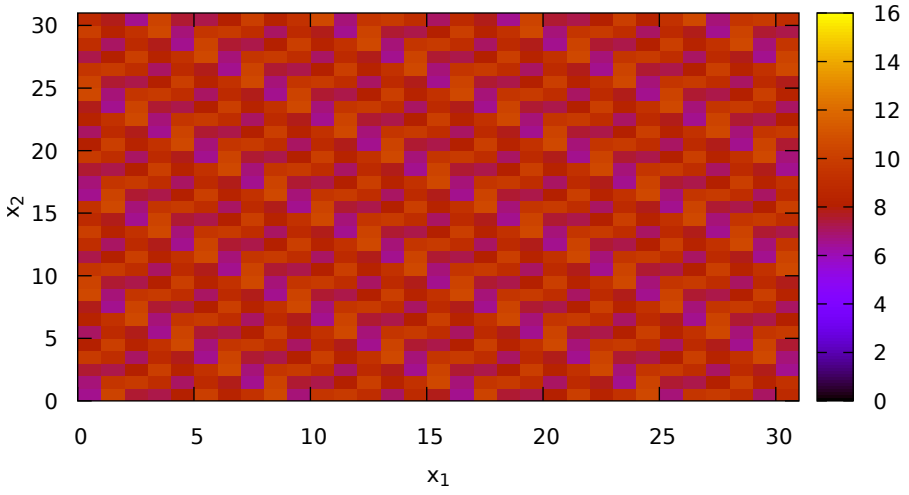


Figure 13.15: Investigated function $f(x_1, x_2) = 14^{x_1}3^{x_2}$

13.5 Quantum Fourier Transform and Discrete Logarithm

The discrete logarithm (see section 19.7) is the basis for a large number of modern cryptographic algorithms (see section 19.9, section 19.8). At the same time, the method proposed by Shor for factoring integers can also be applied to computing discrete logarithms, making it possible to break the corresponding cryptographic algorithms.

Let's set the problem as follows: there is an expression

$$b = a^x \mod p,$$

in which the numbers a, b, and p are given, and the number x is unknown and needs to be determined. By analogy with the application of the quantum Fourier transform for factoring numbers (see subsection 13.3.1), we must construct some periodic function, the period of which will allow us to determine the desired number x. We choose the analyzed function as

$$f(x_1, x_2) = b^{x_1}a^{x_2} = a^{x \cdot x_1}a^{x_2} \mod p \tag{13.23}$$

As an example, we will consider the quantum analog of solving the problem from example 19.7.1:

Example 13.5.1. ($ind_3 14 \mod 17$) *In our example,* $p = 17$, $b = 14$, *and* $a = 3$. *The function* (13.23) *has the form*

$$f(x_1, x_2) = b^{x_1}a^{x_2} = 14^{x_1}3^{x_2}.$$

and is shown in Figure 13.15.

Both $b = 14$ *and* $a = 3$ *are generators of* $(\mathbb{Z}/17\mathbb{Z})^{\times}$. *Moreover,* $3 \equiv 14^9 \mod 17$. *The periods of the depicted function, as seen in Figure 13.15, are the following numbers:*

$$\begin{aligned} t_1 &\equiv 1 \mod 16, \\ t_2 &\equiv 9 \mod 16 \end{aligned} \tag{13.24}$$

By analogy with the factorization problem solution, a measurement of this function is made. Suppose the result of the measurement is a number $c \in (\mathbb{Z}/p\mathbb{Z})^{\times}$. Since a is a generating element (see def. 16.1.4) of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^{\times}$ (see def. 16.1.5), $\exists x_0 : c = a^{x_0}$. Thus, taking into account Fermat's little theorem (Theorem 19.4.1) $a^{p-1} \equiv 1 \mod p$ and therefore

$$x_0 \equiv xx_1 + x_2 \mod q,$$

where $q = p - 1$. From this expression it follows that

$$x_2 \equiv x_0 - xx_1 \mod q.$$

That is, if the function is periodic in the first argument:

$$f(x_1 + t_1, x_2) = f(x_1, x_2),$$

then it will also be periodic in the second argument

$$f(x_1, x_2 + t_2) = f(x_1, x_2),$$

with

$$t_2 \equiv xt_1 \mod q. \tag{13.25}$$

13.5.1 Two-dimensional Fourier Transform and Period of the Function $f(x_1, x_2)$

Our function to analyze will be the following:

$$f'(x_1, x_2) = \begin{cases} 1, & xx_1 + x_2 \equiv x_0 \mod q \\ 0, & xx_1 + x_2 \not\equiv x_0 \mod q \end{cases} \tag{13.26}$$

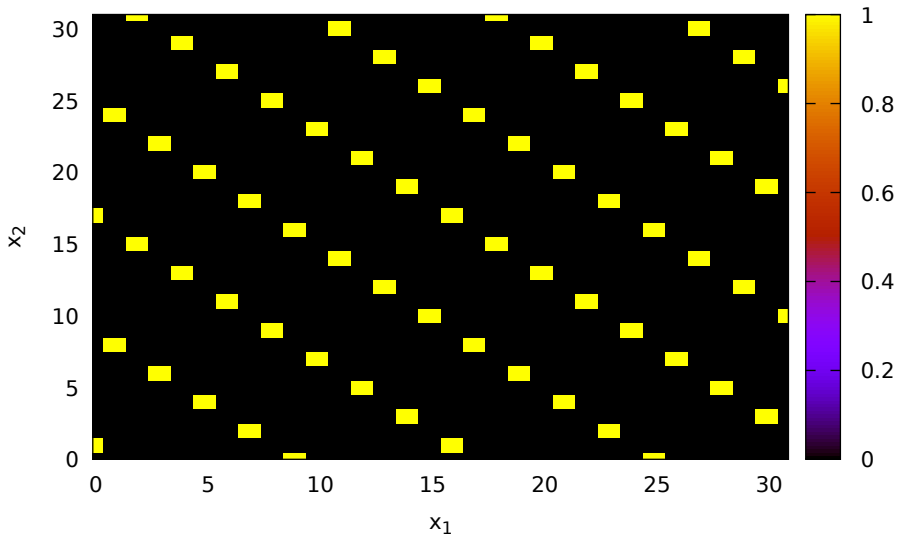


Figure 13.16: The investigated function $f(x_1, x_2) = 14^{x_1}3^{x_2}$. Only those pairs x_1, x_2 are marked for which $f(x_1, x_2) = 3, x_0 = 1$.

Example 13.5.2. ($ind_3 14 \mod 17$)

Continuing example 13.5.1 suppose that as a result of measuring the function f we obtained the value $f = 3$. That is, $f = a^{x_0} = 3^{x_0} \equiv 3 \mod 17$. As a result, only those values of x_1, x_2 remain that correspond to the observed function value (see Figure 13.16), i.e. $xx_1 + x_2 = x_0 \equiv 1 \mod 16$. Here, with fixed values x, x_1 and the number of samples $M = q = 16$.

For the Fourier image $\tilde{f}'$ we have

$$\tilde{f}'(j_1, j_2) = \frac{1}{M} \sum_{x_1=0}^{M-1} \sum_{x_2=0}^{M-1} f'(x_1, x_2)\, e^{-i\omega(x_1 j_1 + x_2 j_2)}, \tag{13.27}$$

where $\omega = \frac{2\pi}{M}$, M is the number of samples.

Consider first the case when $M = q$. In this case, there are two options for x_2:

1. $x_2 = x_0 - xx_1$, if $x_0 \geq xx_1$
2. $x_2 = x_0 + q - xx_1$, if $x_0 < xx_1$

Thus,

$$e^{-i\omega x_2 j_2} = e^{-i\omega(x_0 - xx_1 + q)j_2} =$$
$$= e^{-i\omega(x_0 - xx_1)j_2 - i\omega q j_2} = e^{-i\omega(x_0 - xx_1)j_2},$$

i.e., both cases coincide and can be reduced to the first: $x_2 = x_0 - xx_1$.

Therefore, continuing (13.27) we get

$$\tilde{f}'(j_1, j_2) = \frac{1}{M} \sum_{x_1=0}^{M-1} \sum_{x_2=0}^{M-1} f'(x_1, x_2)\, e^{-i\omega(x_1 j_1 + x_2 j_2)} = \\ = \frac{1}{M} \sum_{x_1=0}^{M-1} e^{-i\omega(x_1 j_1 + (x_0 - x x_1) j_2)} = \\ = e^{-i\omega x_0 j_2} \frac{1}{M} \sum_{x_1=0}^{M-1} e^{-i\omega x_1 (j_1 - x j_2)} = \frac{1}{M} e^{-i\omega x_0 j_2} \sum_{x_1=0}^{M-1} e^{-i\omega x_1 (j_1 - x j_2)}. \tag{13.28}$$

In the expression (13.28) $\tilde{f}'(j_1, j_2) = e^{-i\omega x_0 j_2} \neq 0$, if

$$j_1 \equiv x j_2 \mod M. \tag{13.29}$$

Otherwise, using the formula for a geometric series

$$\tilde{f}'(j_1 \neq x j_2, j_2) = e^{-i\omega x_0 j_2} \frac{1}{M} \sum_{x_1=0}^{M-1} e^{-i\omega x_1 (j_1 - x j_2)} = \\ = \frac{e^{-i\omega x_0 j_2}}{M} \frac{e^{-i\omega M (j_1 - x j_2)} - 1}{e^{-i\omega (j_1 - x j_2)} - 1} = \\ = \frac{e^{-i\omega x_0 j_2}}{M} \frac{e^{-i\frac{2\pi}{M} M (j_1 - x j_2)} - 1}{e^{-i\omega (j_1 - x j_2)} - 1} = 0.$$

Thus, to determine the period it is necessary to find the coordinates (j_1, j_2) of some maximum of the Fourier transform and use the expression

$$x \equiv j_1 j_2^{-1} \mod M, \tag{13.30}$$

which follows from (13.29).

Remark 13.5.1 (On zero divisors in $\mathbb{Z}/M\mathbb{Z}$)**.** *If there exists a number y such that*

$$j_2 y \equiv 0 \mod M,$$

then j_2 is called a zero divisor. It is also clear that

$$\gcd(j_2, M) \neq 1,$$

thus from subsection 19.2.2 it follows that j_2^{-1} does not exist. Therefore, for such j_2, the expression (13.30) is undefined. In this case, other coordinates (j_1, j_2) must be used.

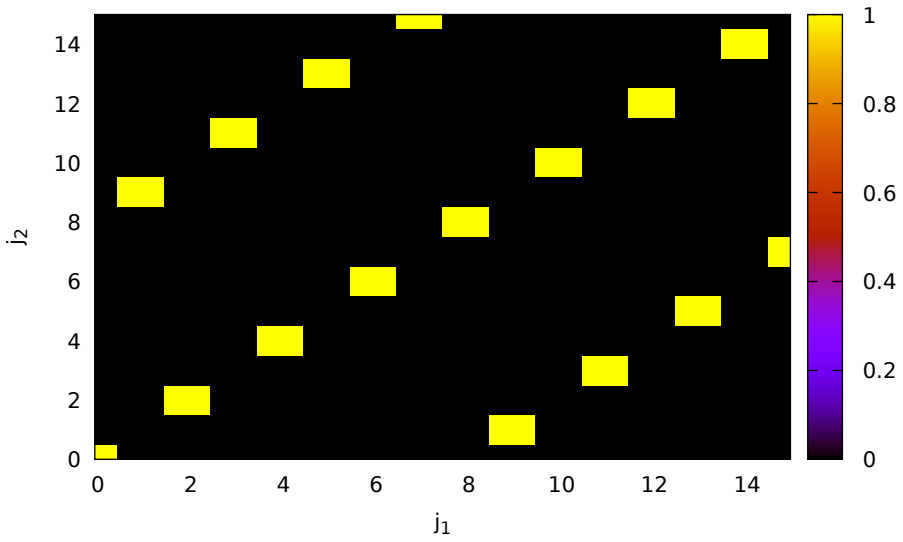


Figure 13.17: Fourier image of the function from Figure 13.16. Number of samples $M = 16$. Coordinates of the maximum $j_1 = 9$, $j_2 = 1$. The solution to the equation $3^x \equiv 14 \mod 17$ is $x \equiv 9 \cdot 1^{-1} \equiv 9 \mod 16$. Considering that the period is less than the number of samples, it can be concluded that $x = 9$

Example 13.5.3. ($ind_3 14 \mod 17$)

The Fourier image of the function from Figure 13.16 is shown in Figure 13.17, from which it is seen that with the highest probability, samples recorded follow with intervals $T_{j_1} = 9$ along coordinate j_1 and with interval $T_{j_2} = 1$ along coordinate j_2. Given that the number of samples $M = 16$, one can obtain the maximum coordinates of the Fourier transform $j_1 = 9$ and $j_2 = 1$. The solution of the equation $3^x \equiv 14 \mod 17$ is, according to (13.30), $x = 9 \cdot 1^{-1} = 9$, which corresponds to the result of example 19.7.1.

A similar result can be obtained by taking the point with coordinates $j_1 = 11, j_2 = 3$. Given that $3 \cdot 11 = 33 \equiv 1 \mod 16$, we have $j_2^{-1} \equiv 11 \mod 16$, i.e., $x \equiv 11 \cdot 11 \equiv 121 \equiv 9 \mod 16$, which again corresponds to the result of example 19.7.1.

It is worth noting that points lying on the diagonal, for example $j_1 = 6, j_2 = 6$ will not give a correct result because $\gcd(6, 16) = 2 \neq 1$

Note that the obtained result (13.30) directly corresponds to lemma 18.2.2 for the one-dimensional Fourier transform. At the same time, there is also an analog of commentary 18.2.1, which states that in case the number of samples in the Fourier transform is not equal to q: $M \neq q$, but $M \approx q$, we can still approximately consider the expression (13.30) [21].

Example 13.5.4. ($ind_2 14 \mod 59$)

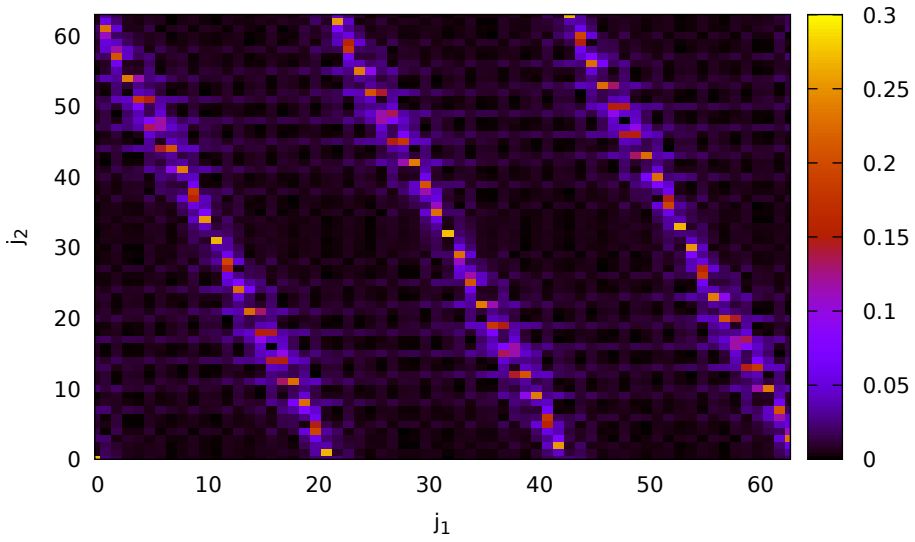


Figure 13.18: Fourier image of the samples of the function $f'(x_1, x_2)$ Number of samples $M = 64$. The three lowest maxima have coordinates approximately $(20, 1), (41, 2.2), (62, 3)$, which gives the following estimates for x: $x \approx 20, 18.6, 20.6$, which is close to the actual value $x = 19$

As an example, consider $p = 59$ *with number of samples* $M = 64 \approx q = p - 1 = 58$. *The group generator* $\mathbb{F}_{59}$ *(see subsection 19.2.3) is* $g = 2$, *i.e.* $\mathbb{F}_{59} = \langle 2 \rangle$. *This means that the equation* $2^x \equiv b \mod 59$ *has a solution for any* b, *in particular* $x = 19$ *is a solution of the equation*

$$2^x \equiv 14 \mod 59.$$

The function under study is

$$f(x_1, x_2) = 14^{x_1} 2^{x_2} \mod 59,$$

Suppose $x_0 = 50$, *i.e., the recorded function value is* $f(x_1, x_2) = 2^{x_0} = 2^{50} \equiv 3 \mod 59$.

As stated above, for the number of Fourier transform samples we have $M = 64$. *It should be noted that* $q = p - 1 = 58 \not\equiv 0 \mod 64$.

The Fourier image of the sampled function

$$f'(x_1, x_2) = \begin{cases} 1, & \text{if } 14^{x_1} 2^{x_2} \equiv 3 \mod 59 \\ 0, & \text{if } 14^{x_1} 2^{x_2} \not\equiv 3 \mod 59 \end{cases}$$

is shown in Figure 13.18. The three lowest maxima have coordinates

$$(j_1, j_2) \approx (20, 1), (41, 2.2), (62, 3),$$

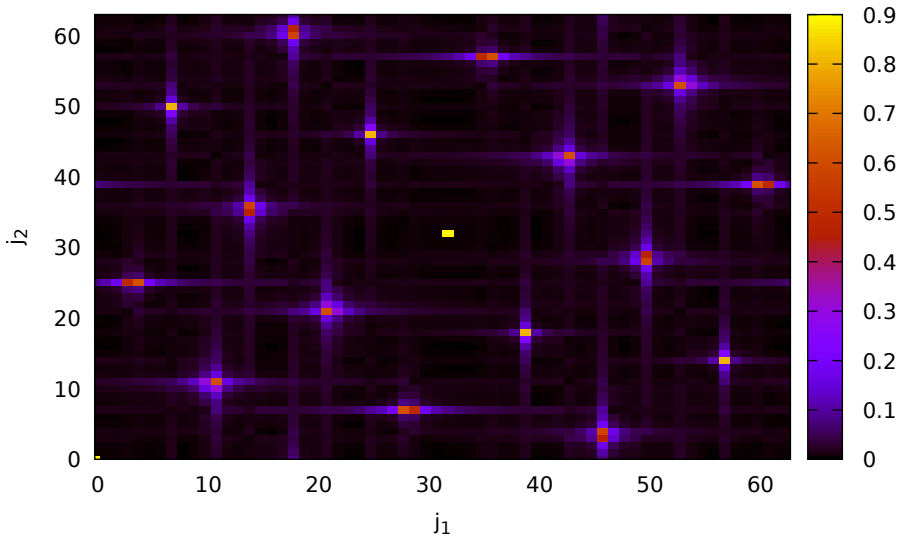


Figure 13.19: Fourier image of the samples of the function $f'(x_1, x_2)$ Number of samples $M = 64$. Coordinates of the maximum $j_1 \approx 46$, $j_2 \approx 3.5$. The solution of the equation $3^x \equiv 14 \mod 19$ is $x = 13 \approx \frac{46}{3.5} \approx 13.14$

which give the following estimates of x*:* $x \approx 20, 18.6, 20.6$*, which is close to the actual value* $x = 19$*.*

Example 13.5.5. ($ind_3 14 \mod 19$)

As an example, consider the problem of determining x *such that*

$$3^x \equiv 14 \mod 19.$$

The function under study is

$$f(x_1, x_2) = 14^{x_1} 3^{x_2} \mod 19,$$

Suppose $x_0 = 1$*, i.e., the recorded function value is* $f(x_1, x_2) = 3$*.*

We take the number of Fourier transform samples $M = 64$*. Note that,* $18 \not\equiv 0 \mod 64$*.*

The Fourier image of the sampled function

$$f'(x_1, x_2) = \begin{cases} 1, & \text{if } 14^{x_1} 3^{x_2} \equiv 3 \mod 19 \\ 0, & \text{if } 14^{x_1} 3^{x_2} \not\equiv 3 \mod 19 \end{cases}$$

is shown in Figure 13.19. The lowest maximum has coordinates $j_1 = 46, j_2 = 3.5$*, from which the estimate*

$$x \approx \frac{46}{3.5} \approx 13.14$$

follows. It is worth noting that the solution of the desired equation $x = 13$ *corresponds to the found approximate solution.*

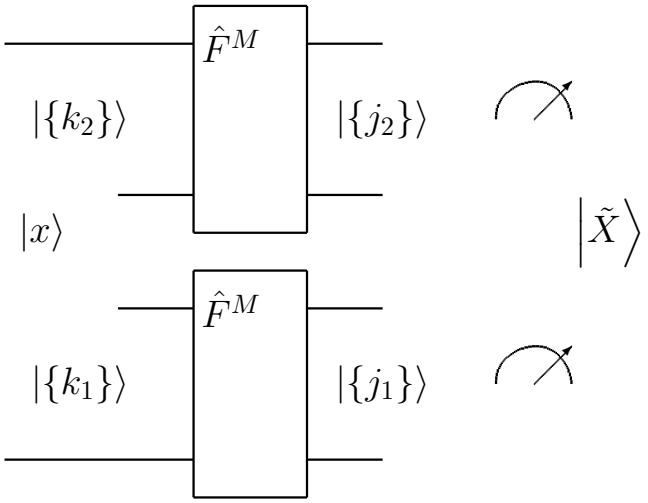


Figure 13.20: Scheme of the two-dimensional quantum Fourier transform. At the input, we have the signal $|x\rangle = \sum_{k_1,k_2=0}^{M-1} x\left(k_1,k_2\right)|k\rangle_1 \otimes |k\rangle_2$. The output signal $\left|\tilde{X}\right\rangle = \sum_{k_1,k_2=0}^{M-1} \tilde{X}\left(j_1,j_2\right)|j_1\rangle \otimes |j_2\rangle$ is the two-dimensional Fourier transform of the original $|x\rangle$.

13.5.2 Two-dimensional Quantum Fourier Transform

To determine the periods of a two-argument function one can use the two-dimensional Fourier transform, which can be constructed from blocks performing the one-dimensional Fourier transform, as shown in Figure 13.20. To analyze this scheme, consider the trivial case when at the input we have (see also (13.10))

$$|x\rangle = |x\rangle_1 \otimes |x\rangle_2 ,$$
$$|x\rangle_{1,2} = \sum_{k_{1,2}=0}^{M-1} x_{k_{1,2}}^{(1,2)} |k_{1,2}\rangle .$$

Considering that the output is

$$\left|\tilde{X}\right\rangle = \left|\tilde{X}_1\right\rangle \otimes \left|\tilde{X}_2\right\rangle ,$$

where

$$\left|\tilde{X}_{1,2}\right\rangle = \sum_{j_{1,2}=0}^{M-1} \tilde{X}_{j_{1,2}}^{(1,2)} |j_{1,2}\rangle$$

and, according to (13.12)

$$\tilde{X}_{j_{1,2}}^{(1,2)} = \frac{1}{\sqrt{M}} \sum_{k_{1,2}=0}^{M-1} e^{-i\omega_{1,2}k_{1,2}j_{1,2}} x_{k_{1,2}}^{(1,2)} .$$

we get

$$\left|\tilde{X}\right\rangle = \left|\tilde{X}_1\right\rangle \otimes \left|\tilde{X}_2\right\rangle =$$
$$= \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} \tilde{X}_{j_1}^{(1)} \tilde{X}_{j_2}^{(2)} \left|j_1\right\rangle \otimes \left|j_2\right\rangle =$$
$$= \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} \tilde{X}_{j_1,j_2} \left|j_1\right\rangle \otimes \left|j_2\right\rangle ,$$

where

$$\tilde{X}_{j_1,j_2} = \frac{1}{\left(\sqrt{M}\right)^2} \sum_{k_1=0}^{M-1}\sum_{k_2=0}^{M-1} e^{-i\omega(k_1 j_1 + k_2 j_2)} x_{k_1}^{(1)} x_{k_2}^{(2)} =$$
$$= \frac{1}{\left(\sqrt{M}\right)^2} \sum_{k_1=0}^{M-1}\sum_{k_2=0}^{M-1} e^{-i\omega(k_1 j_1 + k_2 j_2)} x_{k_1,k_2}$$

which, according to definition 18.3.1, is the two-dimensional Fourier transform of the original two-dimensional signal

$$\left|x\right\rangle = \sum_{k_1=0}^{M-1}\sum_{k_2=0}^{M-1} x_{k_1}^{(1)} x_{k_2}^{(2)} \left|k_1\right\rangle \otimes \left|k_2\right\rangle = \sum_{k_1=0}^{M-1}\sum_{k_2=0}^{M-1} x_{k_1,k_2} \left|k_1\right\rangle \otimes \left|k_2\right\rangle .$$

Therefore, using the scheme shown in Figure 13.21, one can determine the coordinates of the maxima of the two-dimensional Fourier transform j_1, j_2, and subsequently use (13.30) to find the desired x.

13.6 Shor's Algorithm and the Discrete Logarithm on Elliptic Curves

Consider an elliptic curve

$$E\left(\mathbb{F}\right) = \{(x,y) \in \mathbb{F}_p \times \mathbb{F}_p, y^2 = x^3 + ax + b, \}$$

with a given base point $g \in E\left(\mathbb{F}_p\right)$ such that:

$$ng = 0.$$

The task is to solve the following problem: given $q \in E\left(\mathbb{F}_p\right)$ find such x that

$$xg = q \mod n \tag{13.31}$$

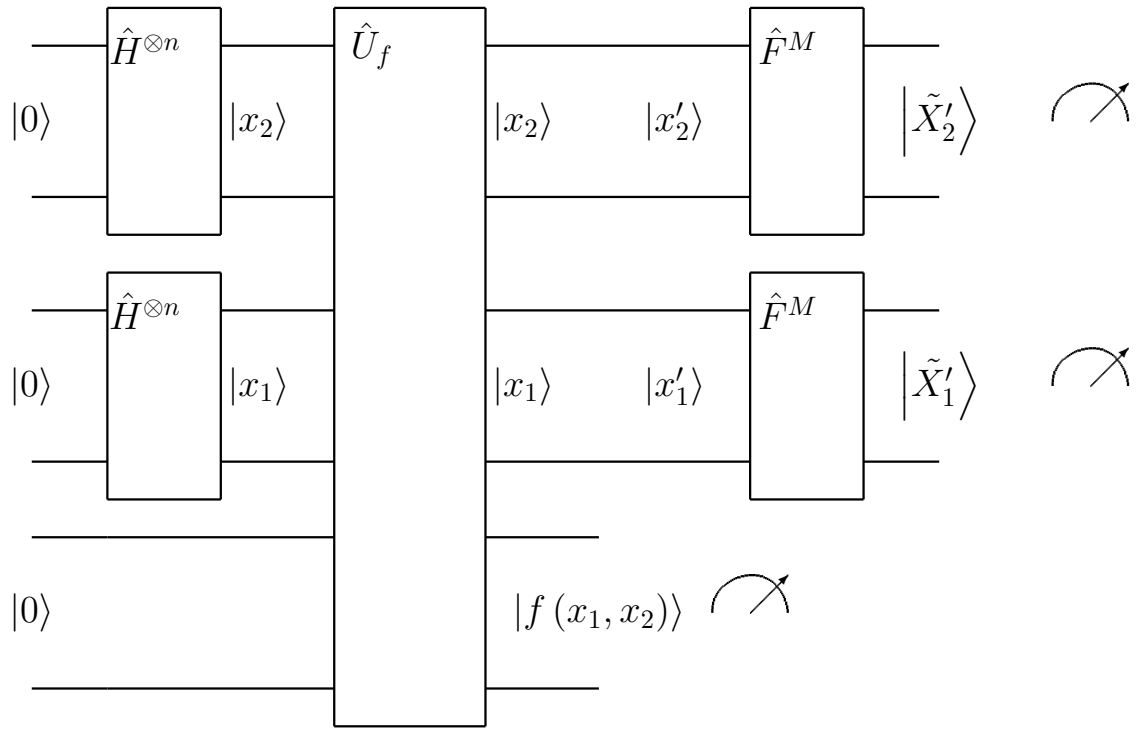


Figure 13.21: Determination of the period of two-argument functions using the quantum Fourier transform

Consider the following auxiliary function

$$f(x_1, x_2) = x_1 q + x_2 g = (x x_1 + x_2)\, g, \tag{13.32}$$

where $q, g \in E(\mathbb{F}_p)$ and are taken from the conditions of our problem (13.31). This function is analogous to (13.23) used in solving the discrete logarithm problem. Then a measurement is performed on this function. The result of this measurement is a certain point $c \in E(\mathbb{F}_p)$. At the same time, from (13.32) it follows that $c \in \langle g \rangle$, i.e. $\exists x_0$ such that $c = x_0 g$.

Thus, by analogy with (13.26), we compose the following function

$$f'(x_1, x_2) = \begin{cases} 1, x x_1 + x_2 \equiv x_0 \mod n \\ 0, x x_1 + x_2 \not\equiv x_0 \mod n \end{cases} \tag{13.33}$$

Coordinates (j_1, j_2) of the Fourier transform maximum $\tilde{f}'$ give, in accordance with formula (13.30), some value of the sought number x. In our case, practically always $n \neq M$ so we can only use an approximate estimate

$$x \approx \frac{j_1}{j_2}.$$

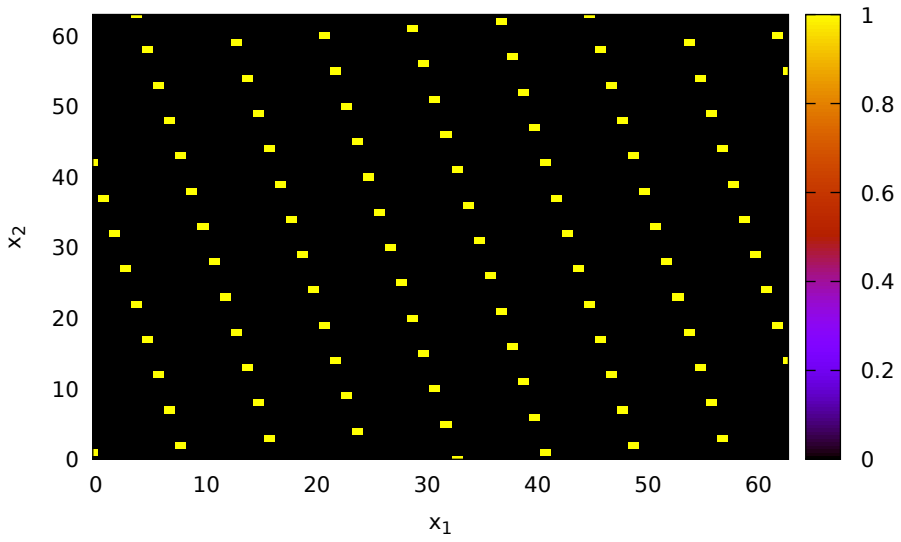


Figure 13.22: Graph of the function $f'(x_1, x_2)$ at $x_0 = 1$. Thus, the points x_1, x_2 corresponding to the relation $x_1 A + x_2 g = g$ are shown: $x_1(37, 35) + x_2(96, 93) = (96, 93)$, for example, $7(37, 35) + 7(96, 93) = (96, 93)$, $8(37, 35) + 2(96, 93) = (96, 93)$ or $16(37, 35) + 3(96, 93) = (96, 93)$. It is worth noting that the selected pairs of points satisfy the condition (13.33), indeed we have $x \cdot 7 + 7 \equiv x \cdot 8 + 2 \mod 41$. That is, if we subtract one from the other, we get an equation of the form $x\,(8 - 7) = -(2 - 7) = 5 \mod 41$. Or $x \equiv 5 \mod 41$.

Example 13.6.1 (Discrete logarithm on an elliptic curve)**.** *Consider the task from example 19.11.1 The curve and base point (see example 19.10.2) are given by*

$$(p, a, b, g, n, h) = (97, -7, 10, (96, 93), 41, 2)$$

Suppose we know Alice's public key

$$A = (37, 35)$$

and we want to find such $x \in \{0, 1, \ldots 40\}$ that $xg = A$, as follows from example 19.11.1 the answer is $x = d_a = 5$. Our investigated function will be

$$f\,(x_1, x_2) = x_1 A + x_2 g = x_1(37, 35) + x_2(96, 93)$$

As the measurement result, choose $c = g$, i.e. $x_0 = 1$. The graph of the function $f'(x_1, x_2)$ corresponding to this measurement is shown in Figure 13.22. It is worth noting that the function $f'(x_1, x_2)$ is periodic, and if we take any two close points, for example $(8, 2)$ and $(16, 3)$, we can notice that the period along the x_1 coordinate is $T_1 = 8$, and along the x_2 coordinate it is $T_2 = 1$. Solving the equation $x = T_2 T_1^{-1} \mod n$ gives $x = 8^{-1} \equiv 5 \mod 41$, which corresponds to the sought solution.

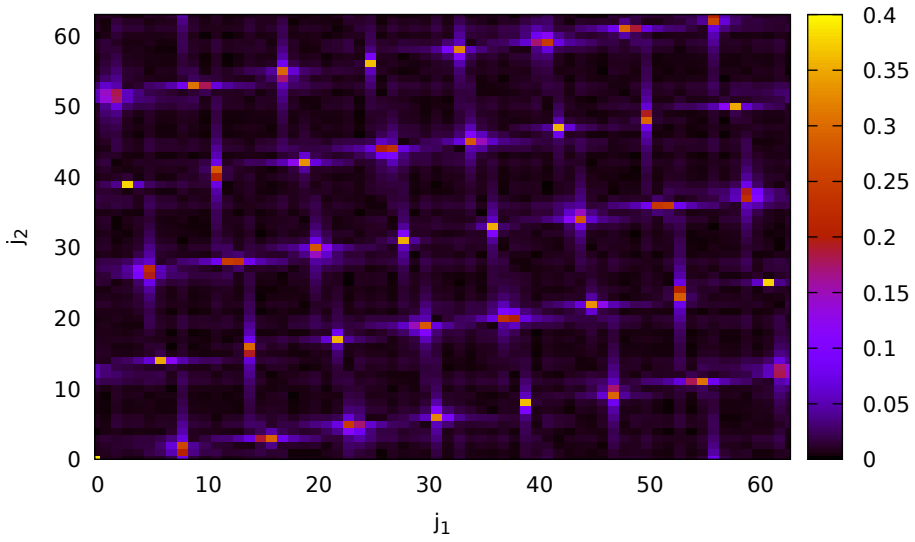


Figure 13.23: Fourier transform of the samples of the function $f'(x_1, x_2)$. Number of samples $M = 64$. The three bottom left maxima have coordinates approximately $(8, 2), (15, 3), (24, 5)$, which gives the following estimates for x: $x \approx 4, 5, 4.8$, which is close to the true value $x = 5$

The Fourier transform of the function $f'(x_1, x_2)$ is shown in Figure 13.23. From it, the sought $x = 5$ can be found.

13.7 Grover's Algorithm

Consider the following problem. Suppose there is a large data set consisting of N elements in which it is necessary to find an element satisfying certain conditions (see Figure 13.24). If the data are sorted, then using algorithms of the "divide and conquer" type, the desired element can be found in time on the order of $O(\log N)$ (see section 20.2). In some cases, the original data set cannot be prepared for fast search; in this case, a classical search takes time on the order of $O(N)$.

One example is symmetric encryption algorithms, where the task is to determine the key given known ciphertext and its corresponding original text. In this case, preprocessing the data seems impossible, and the straightforward solution to the problem is a simple brute force search over all possible values.

Grover's algorithm [11] solves the unstructured search problem in time on the order of $O\left(\sqrt{N}\right)$.

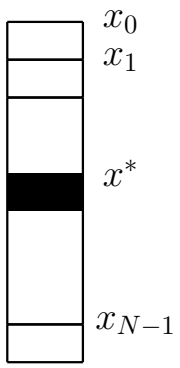


Figure 13.24: Search in an unstructured data volume (searching for a "needle in a haystack")

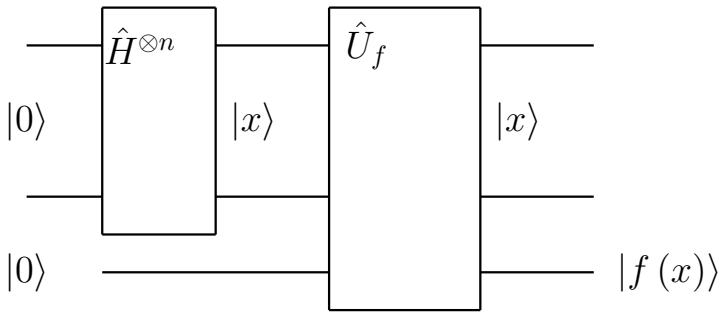


Figure 13.25: Calculation of the function $f(x)$. The output of the circuit is a superposition of states of the form $\frac{1}{\sqrt{N}}\left(\sum_{x\neq x^*}|x\rangle\otimes|0\rangle+|x^*\rangle\otimes|1\rangle\right)$

13.7.1 Algorithm Description

Suppose we have a quantum circuit that computes the value of a function $f(x)$ which can take only two values: 0 and 1. Here, the value 1 is true only for the desired element:

$$\begin{aligned} f(x)|_{x=x^*} &= 1, \\ f(x)|_{x\neq x^*} &= 0. \end{aligned} \tag{13.34}$$

Figure 13.25 shows a scheme for computing the desired function. At the output we have a state of the form

$$|out\rangle = \frac{1}{\sqrt{N}}\left(\sum_{x\neq x^*}|x\rangle\otimes|0\rangle+|x^*\rangle\otimes|1\rangle\right), \tag{13.35}$$

where N is the total number of elements in the sequence in which the search is performed.

Looking at expression (13.35), one can note that the proposed scheme, although it computes the function at the desired point, does not allow selecting the

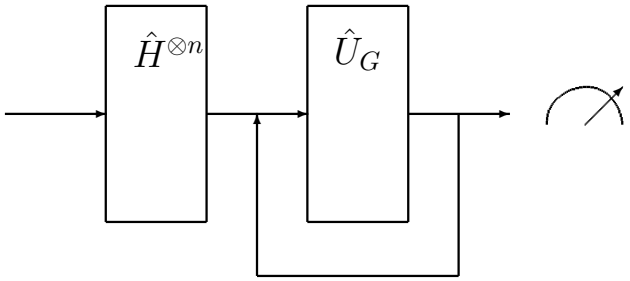


Figure 13.26: Grover's algorithm

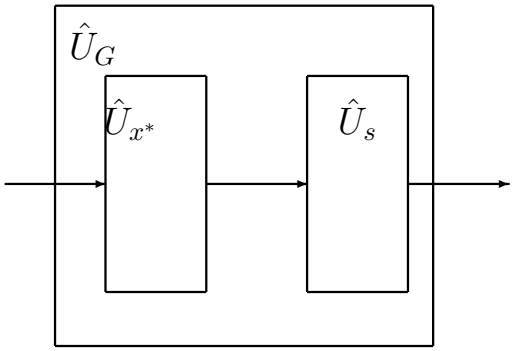


Figure 13.27: Grover's algorithm. Basic element

sought element, because all elements of the resulting sequence are equally probable, i.e., each element can be selected (as a result of measurement) with equal probability: $\frac{1}{N}$.

Grover proposed an algorithm that would increase the probability of detecting the desired element in the resulting superposition (13.35).

The scheme implementing Grover's algorithm is a certain block described by the operator $\hat{U}_G$, which is repeated a certain number of times (see Figure 13.26). At each step of iteration, the probability of detecting the desired element increases.

The basic element $\hat{U}_G$ represents a sequential application of two operators (see Figure 13.27):

$$\hat{U}_G = \hat{U}_s \hat{U}_{x^*},$$

where $\hat{U}_{x^*}$ is the phase inversion operator, and $\hat{U}_s$ is the inversion about the mean operator.

The action of the operator $\hat{U}_{x^*}$ is described by the following relation (see Figure 13.28):

$$\hat{U}_{x^*} \left(\sum_x \alpha_x |x\rangle \right) = \sum_x \alpha_x (-1)^{f(x)} |x\rangle . \tag{13.36}$$

The operator $\hat{U}_{x^*}$ can be rewritten in the form

$$\hat{U}_{x^*} = \hat{I} - 2 |x^*\rangle \langle x^*| .$$

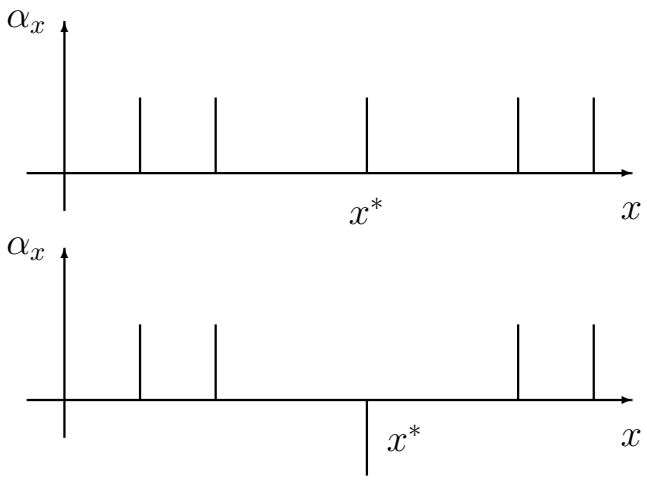


Figure 13.28: Grover's algorithm. Phase inversion. Described by the following relation $\hat{U}_{x^*}\left(\sum_x \alpha_x |x\rangle\right) = \sum_x \alpha_x (-1)^{f(x)} |x\rangle$

Indeed,

$$\left(\hat{I} - 2|x^*\rangle\langle x^*|\right)\left(\sum_x \alpha_x |x\rangle\right) =$$
$$= \sum_x \alpha_x |x\rangle - 2\alpha_{x^*}|x^*\rangle = \sum_{x\neq x^*} \alpha_x |x\rangle - \alpha_{x^*}|x^*\rangle =$$
$$= \sum_x \alpha_x (-1)^{f(x)} |x\rangle ,$$

which coincides with (13.36).

The action of the operator $\hat{U}_s$ is described by the following relation (see Figure 13.29):

$$\hat{U}_s\left(\sum_x \alpha_x |x\rangle\right) = \sum_x (2\mathcal{M} - \alpha_x)|x\rangle , \tag{13.37}$$

where $\mathcal{M} = \sum_x \frac{\alpha_x}{N}$.

The operator $\hat{U}_s$ can be rewritten in the following form

$$\hat{U}_s = 2|s\rangle\langle s| - \hat{I},$$

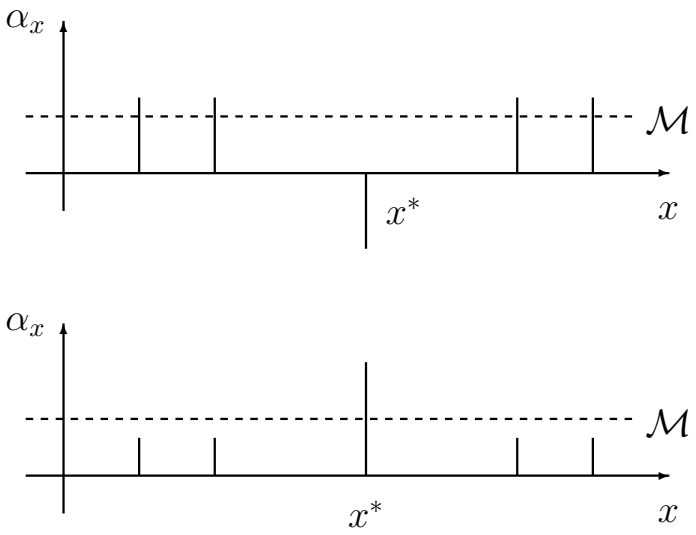


Figure 13.29: Grover's algorithm. Inversion about the average. Described by the following relation $\hat{U}_s \left(\sum_x \alpha_x |x\rangle\right) = \sum_x (2\mathcal{M} - \alpha_x) |x\rangle$

where $|s\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$ - the initial state in Grover's algorithm. Indeed,

$$\begin{aligned}
&\left(2 |s\rangle \langle s| - \hat{I}\right) \left(\sum_x \alpha_x |x\rangle\right) = \\
&= 2 \sum_x \alpha_x \langle s|x\rangle |s\rangle - \sum_x \alpha_x |x\rangle = \\
&= \frac{2}{N} \sum_x \alpha_x \sum_x |x\rangle - \sum_x \alpha_x |x\rangle = \\
&= \sum_x (2\mathcal{M} - \alpha_x) |x\rangle ,
\end{aligned}$$

which coincides with (13.37).

13.7.2 Analysis of Grover's Algorithm

The schematic form of Grover's algorithm is given in Alg. 2.

We will be interested in two questions: what is the algorithmic complexity of Grover's algorithm and whether there exist algorithms that can perform the search task in an unstructured data set more efficiently than Grover's algorithm.

The criterion of algorithm efficiency is the following fact: a good algorithm should find the desired value with the minimum number of calls to the function (13.34).

Algorithm 2 Grover's Algorithm

$|\psi\rangle_0 \Leftarrow \frac{1}{\sqrt{N}} \sum_x |x\rangle$
$t \Leftarrow 1$
repeat
  $|\psi\rangle_t \Leftarrow \hat{U}_s \hat{U}_{x^*} |\psi\rangle_{t-1}$
  $t \Leftarrow t+1$
until $(t < \frac{\pi}{4}\sqrt{N})$
return the measurement result of the state $|\psi\rangle_t$

Consider the very first iteration. The initial state $|\psi\rangle_0$ has the following form

$$|\psi\rangle_0 = \sum_x \alpha_x |x\rangle = |s\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle = \frac{1}{\sqrt{N}} \sum_{x \neq x^*} |x\rangle + \frac{1}{\sqrt{N}} |x^*\rangle .$$

Thus, the coefficient before the desired element is $\alpha_x^* = \frac{1}{\sqrt{N}}$.

After applying the phase inversion operator U_{x^*} from (13.36), we get

$$\hat{U}_{x^*} |\psi\rangle_0 = \frac{1}{\sqrt{N}} \sum_{x \neq x^*} |x\rangle - \frac{1}{\sqrt{N}} |x^*\rangle = \sum_x \beta_x |x\rangle ,$$

where $\beta_{x^*} = -\frac{1}{\sqrt{N}}$ and $\beta_{x \neq x^*} = \frac{1}{\sqrt{N}}$.

After applying the inversion about the mean operator $\hat{U}_s$ from (13.37), we obtain

$$\hat{U}_G |\psi\rangle_0 = \hat{U}_s \hat{U}_{x^*} |\psi\rangle_0 = \hat{U}_s \sum_x \beta_x |x\rangle =$$

$$= \sum_x (2M - \beta_x) |x\rangle \approx \sum_{x \neq x^*} \left(2\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N}} \right) |x\rangle +$$

$$+ \left(2\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} \right) |x^*\rangle = \frac{1}{\sqrt{N}} \sum_{x \neq x^*} |x\rangle + \frac{3}{\sqrt{N}} |x^*\rangle . \qquad (13.38)$$

In the derivation of (13.38) it was assumed that

$$\mathcal{M} = \frac{\sum_x \alpha_x}{N} \approx \frac{N}{N\sqrt{N}} = \frac{1}{\sqrt{N}}.$$

Thus, after the first iteration of Grover's algorithm, the amplitude α_{x^*} increased by $\frac{2}{\sqrt{N}}$. If this result is approximated to an arbitrary iteration, one can obtain that a 50% probability of finding $|x^*\rangle$ will be achievable after the following

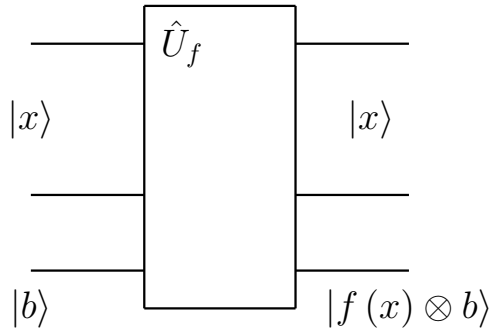


Figure 13.30: Grover's algorithm. Phase inversion implementation (operator $\hat{U}_{x^*}$). By taking $b = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$, we obtain for the depicted circuit $\hat{U}_f \left(\sum_x \alpha_x |x\rangle\right) \otimes |-\rangle = \sum_x \alpha_x (-1)^{f(x)} |x\rangle \otimes |-\rangle$

number of iterations:

$$\frac{1}{\sqrt{2}} \Big/ \frac{2}{\sqrt{N}} = \frac{\sqrt{N}}{2\sqrt{2}} = O\left(\sqrt{N}\right).$$

More precise calculations [19] give the number of iterations as $\frac{\pi}{4}\sqrt{N}$.

One can ask about the optimality of Grover's algorithm: does there exist a quantum algorithm that performs search in an unstructured data set faster than $O\left(\sqrt{N}\right)$ calls to the function (13.34)? In the article [6] it is shown that such an algorithm does not exist.

13.7.3 Implementation of basic elements of Grover's algorithm

Phase inversion

How can the phase inversion be implemented: what does the quantum logic gate look like that performs the transformation (13.36), i.e., how can $f(x)$ be "sent" into the phase?

Consider the circuit shown in Figure 13.30. The proposed circuit performs the following transformation:

$$|x\rangle \otimes |b\rangle \to |x\rangle \otimes |b \oplus f(x)\rangle,$$

where the following notation is introduced: $a \oplus b = a + b \mod 2$.

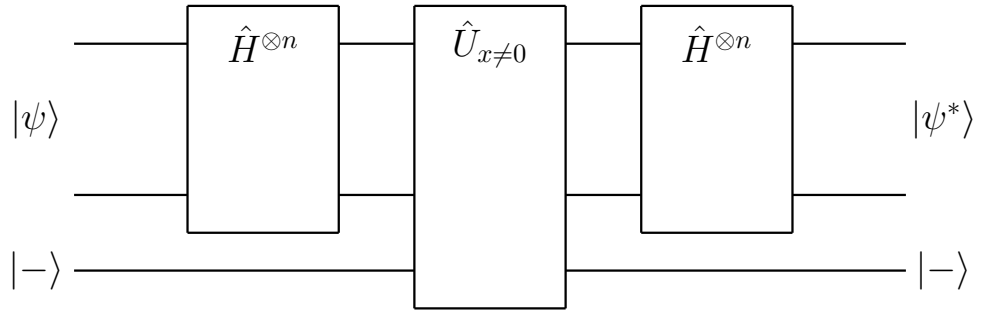


Figure 13.31: Grover's algorithm. Implementation of inversion about the mean (operator $\hat{U}_s$): $|\psi\rangle \otimes |-\rangle \rightarrow |\psi^*\rangle \otimes |-\rangle$, where $|\psi\rangle = \sum_x \alpha_x |x\rangle$, $|\psi^*\rangle = \sum_x (2\mathcal{M} - \alpha_x) |x\rangle$. The proposed circuit performs the following transformation: $\hat{H}^{\otimes n}\hat{U}_{x\neq 0}\hat{H}^{\otimes n} \sum_x \alpha_x |x\rangle \otimes |-\rangle = \sum_x (2\mathcal{M} - \alpha_x) |x\rangle \otimes |-\rangle$

For the case $|b\rangle = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ we have

$$
\begin{aligned}
|x\rangle \otimes |-\rangle \rightarrow |x\rangle \otimes \left(\frac{|0 \oplus 0\rangle - |1 \oplus 0\rangle}{\sqrt{2}} \right) = \\
= |x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = |x\rangle \otimes |-\rangle, x \neq x^*, \\
|x\rangle \otimes |-\rangle \rightarrow |x\rangle \otimes \left(\frac{|0 \oplus 1\rangle - |1 \oplus 1\rangle}{\sqrt{2}} \right) = \\
= |x\rangle \otimes \left(\frac{|1\rangle - |0\rangle}{\sqrt{2}} \right) = - |x\rangle \otimes |-\rangle, x = x^*,
\end{aligned}
$$

thus we have the following transformation

$$
|x\rangle \otimes |-\rangle \rightarrow (-1)^{f(x)} |x\rangle \otimes |-\rangle . \tag{13.39}
$$

Inversion about the mean

Consider the circuit shown in Figure 13.31. The element $\hat{U}_{x\neq 0}$ performs a transformation analogous to (13.39), but with the function $f(x = 0) = 0$ and $f(x \neq 0) = 1$, thus

$$
\begin{aligned}
\hat{U}_{x\neq 0} |x\rangle \otimes |-\rangle = |x\rangle \otimes |-\rangle, x = 0, \\
\hat{U}_{x\neq 0} |x\rangle \otimes |-\rangle = - |x\rangle \otimes |-\rangle, x \neq 0,
\end{aligned}
$$

i.e., the transformation matrix looks like:

$$\hat{U}_{x\neq 0} = \begin{pmatrix} 1\otimes|-\rangle & 0 & 0 & \cdots & 0 \\ 0 & -1\otimes|-\rangle & 0 & \cdots & 0 \\ 0 & 0 & -1\otimes|-\rangle & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1\otimes|-\rangle \end{pmatrix} =$$

$$= \left\{ \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \right\} \otimes |-\rangle \,.$$

Combining this result with two Hadamard transforms , and using (13.4), we obtain:

$$\hat{H}^{\otimes n} \left\{ \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \right\} \otimes |-\rangle \, \hat{H}^{\otimes n} =$$

$$= \left\{ \hat{H}^{\otimes n} \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \hat{H}^{\otimes n} - \hat{H}^{\otimes n} \hat{I} \hat{H}^{\otimes n} \right\} \otimes |-\rangle =$$

$$= \left\{ \begin{pmatrix} \frac{2}{N} & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \end{pmatrix} - \hat{I} \right\} \otimes |-\rangle =$$

$$= \left\{ \begin{pmatrix} \frac{2}{N}-1 & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N}-1 & \cdots & \frac{2}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N}-1 \end{pmatrix} \right\} \otimes |-\rangle \,. \qquad (13.40)$$

If we act with the operator $\hat{H}^{\otimes n}\hat{U}_{x\neq 0}\hat{H}^{\otimes n}$, then using the result (13.40) we

get:

$$\hat{H}^{\otimes n}\hat{U}_{x\neq 0}\hat{H}^{\otimes n}\sum_x \alpha_x \left|x\right\rangle =$$

$$= \begin{pmatrix} \frac{2}{N}-1 & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N}-1 & \cdots & \frac{2}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N}-1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{N}\sum_x \alpha_x - \alpha_0 \\ \frac{2}{N}\sum_x \alpha_x - \alpha_1 \\ \frac{2}{N}\sum_x \alpha_x - \alpha_2 \\ \vdots \\ \frac{2}{N}\sum_x \alpha_x - \alpha_{N-1} \end{pmatrix} = \sum_x \left(2\mathcal{M} - \alpha_x\right)\left|x\right\rangle .$$

Thus, the circuit proposed in Figure 13.31 indeed performs inversion about the mean.

13.8 Adiabatic Quantum Computers

Adiabatic computers are based on the so-called adiabatic theorem Theorem 14.7.1.
TBD

13.9 Algorithm Complexity Theory and Quantum Computing

13.9.1 Algorithm Complexity Classes P, NP, BQP

TBD

Part 5

Appendices

Chapter 14

Fundamentals of Quantum Mechanics

The necessity of this section is primarily caused by the specifics of teaching quantum mechanics in technical universities. The main emphasis is placed on practical applications, often overlooking the theoretical and philosophical foundations of quantum mechanics.

For example, practical techniques for processing the results of numerous similar measurements are described in detail, i.e., the rules for calculating the average values of physical quantities in a given quantum state are explained. At the same time, answers to questions arising from the analysis of single experiments are omitted, such as the relation between the instrument readings and the change of the wave function upon measurement, what decoherence is, and many others. The relevance of these questions has increased recently due to the necessity to analyze the results of single experiments, which has led to the ability to design new devices that rely on these properties of pure quantum states. In particular, the 2012 Nobel Prize in Physics was awarded to Serge Haroche and David J. Wineland for "for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems." [1]

14.1 Dirac Formulation of Quantum Mechanics

In the course of lectures on quantum optics, we will everywhere use the Dirac formalism [55]. In the usual formulation of quantum mechanics, we deal with wave functions, for example $\psi(q,t)$ - the wave function in the coordinate representation. The same state of the system can be described by wave functions in different

[1] "ground-breaking experimental methods that made it possible to measure and control individual quantum systems."

representations related by linear transformations. For instance, the wave function in the momentum representation is related to the wave function in the coordinate representation by the equality

$$\phi(p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \psi(q,t)\, e^{-i\frac{pq}{\hbar}} dq \tag{14.1}$$

The main point here is that the same state can be described by wave functions expressed through different variables. From this, it follows that one can introduce a more general entity that characterizes the state of the system independently of the representation. For such an entity, Dirac introduced the concept of the wave vector, or state vector, denoted as:

$$|\dots\rangle \tag{14.2}$$

and called a ket-vector.

14.1.1 Ket-vector

$|\dots\rangle$ is a general notation for a ket-vector; $|a\rangle$, $|x\rangle$, $|\psi\rangle$, etc. denote ket-vectors describing some particular states, whose symbols are written inside the brackets.

14.1.2 Bra-vectors

Each ket-vector has a conjugate bra-vector. The bra-vector is denoted:

$$\langle\dots|\,, \quad \langle a|\,, \quad \langle\psi|\,. \tag{14.3}$$

The names bra- and ket-vectors come from the first and second halves of the English word *bra-cket*.

Thus, to bra-vectors $\langle a|$, $\langle x|$, $\langle\psi|$ correspond conjugate ket-vectors $|a\rangle$, $|x\rangle$, $|\psi\rangle$ and vice versa. For state vectors, the same fundamental relations hold as for wave functions:

$$|u\rangle = |a\rangle + |b\rangle\,, \quad \langle u| = \langle a| + \langle b|\,, \quad |v\rangle = l\,|a\rangle\,, \quad \langle v| = l\,\langle a|\,. \tag{14.4}$$

Bra- and ket-vectors are connected by the operation of Hermitian conjugation:

$$|u\rangle = (\langle u|)^{\dagger}\,, \quad \langle u| = (|u\rangle)^{\dagger}\,. \tag{14.5}$$

In known cases, this reduces to the following relations:

$$(\psi(q))^{\dagger} = \psi^{*}(q)$$

for the wave function in the coordinate representation;

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^{\dagger} = (a_1^*, a_2^*, \cdots, a_n^*)$$

in the matrix representation.

Using bra- and ket-vectors, we can define the scalar product

$$\langle v|u\rangle = \langle u|v\rangle^* . \tag{14.6}$$

In specific cases, this means:

$$\langle \psi|\ \phi\rangle = \int \psi^* \phi dq$$

in the coordinate representation;

$$\langle a|b\rangle = (a_1^*, a_2^*, \cdots, a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n$$

in the matrix representation.

From relation (14.6) it follows that the norm of a vector is real. Additionally, we assume that the norm of the vector is positive or zero: $\langle a|a\rangle \geq 0$.

14.1.3 Operators

In quantum mechanics, linear operators are used. Operators connect one state vector with another:

$$|q\rangle = \hat{L}\,|p\rangle \tag{14.7}$$

The conjugate equality has the form

$$\langle q| = \langle p|\,\hat{L}^{\dagger} \tag{14.8}$$

where $\hat{L}^{\dagger}$ is the operator conjugate to $\hat{L}$.

Let us give some relations valid for linear operators:

$$\hat{L}^{++} = \hat{L}, \quad \left(l\hat{L}\,|a\rangle\right)^{\dagger} = l^{*}\,\langle a|\,\hat{L}^{\dagger},$$
$$\left(\left(\hat{L}_1 + \hat{L}_2\right)|a\rangle\right)^{\dagger} = \langle a|\left(\hat{L}_1{}^{\dagger} + \hat{L}_2{}^{\dagger}\right),$$
$$\left(\left(\hat{L}_1\hat{L}_2\right)|a\rangle\right)^{\dagger} = \langle a|\left(\hat{L}_2{}^{\dagger}\hat{L}_1{}^{\dagger}\right),$$
$$\left(\left(\hat{L}_1\hat{L}_2\hat{L}_3\right)|a\rangle\right)^{\dagger} = \langle a|\left(\hat{L}_3{}^{\dagger}\hat{L}_2{}^{\dagger}\hat{L}_1{}^{\dagger}\right), \text{ and so on} \tag{14.9}$$

Note that the algebra of operators coincides with the algebra of square matrices. The matrix elements of operators are denoted as follows:

$$\langle a|\,\hat{L}\,|b\rangle = L_{ab} \tag{14.10}$$

For matrix elements, the following equalities hold:

$$\langle a|\,\hat{L}\,|b\rangle^{*} = \langle b|\,\hat{L}^{\dagger}\,|a\rangle\,, \quad \langle a|\,\hat{L}_1\hat{L}_2\,|b\rangle^{*} = \langle b|\,\hat{L}_2{}^{\dagger}\hat{L}_1{}^{\dagger}\,|a\rangle \tag{14.11}$$

14.1.4 Eigenvalues and eigenvectors of operators

Eigenvalues and eigenvectors of operators are defined by the equality

$$\hat{L}\,|l_n\rangle = l_n\,|l_n\rangle\,, \tag{14.12}$$

where l_n is an eigenvalue; $|l_n\rangle$ an eigenvector.

For bra-vectors we have analogous equalities:

$$\langle d_n|\,\hat{D} = d_n\,\langle d_n|\,. \tag{14.13}$$

If operators correspond to observables, they must be self-adjoint:

$$\hat{L} = \hat{L}^{\dagger}. \tag{14.14}$$

The eigenvalues of a self-adjoint (Hermitian) operator are real. Indeed, from

$$\hat{L}\,|l\rangle = l\,|l\rangle$$

it follows that

$$\langle l|\,\hat{L}\,|l\rangle = l\,\langle l|l\rangle\,.$$

On the other hand, recalling (14.9): $\langle l|\,\hat{L}^{\dagger} = l^{*}\,\langle l|$, from (14.14) we have

$$\langle l|\,\hat{L}\,|l\rangle = l^{*}\,\langle l|l\rangle\,.$$

Thus $l \langle l|l\rangle = l^* \langle l|l\rangle$, i.e. $l = l^*$.

Eigenvectors of a self-adjoint operator are orthogonal. Indeed, consider two eigenvectors $|l_1\rangle$ and $|l_2\rangle$:

$$\hat{L}\,|l_1\rangle = l_1\,|l_1\rangle\,, \quad \hat{L}\,|l_2\rangle = l_2\,|l_2\rangle$$

From the second relation we get

$$\langle l_1|\,\hat{L}\,|l_2\rangle = l_2\,\langle l_1|l_2\rangle$$

Taking into account the reality of eigenvalues and relation (14.14) for the vector $|l_1\rangle$ we obtain:

$$\langle l_1|\,\hat{L} = l_1\,\langle l_1|\,.$$

Hence

$$\langle l_1|\,\hat{L}\,|l_2\rangle = l_1\,\langle l_1|l_2\rangle\,.$$

Thus

$$(l_1 - l_2)\,\langle l_1|l_2\rangle = 0, \quad \text{i.e. } \langle l_1|l_2\rangle = 0, \text{ since } l_1 \neq l_2.$$

14.1.5 Observables. Expansion in eigenvectors. Completeness of set of eigenvectors

Operators corresponding to physical observables are self-adjoint operators. This ensures the reality of the values of the observable physical quantity. We have a set of eigenstates of some Hermitian operator $|l_n\rangle$, $\hat{L}\,|l_n\rangle = l_n\,|l_n\rangle$. If the set of eigenstates is complete, according to the principles of quantum mechanics any state can be represented as a superposition of states $|l_n\rangle$:

$$|\psi\rangle = \sum_{(n)} c_n\,|l_n\rangle\,. \tag{14.15}$$

From here for the expansion coefficients we have: $c_n = \langle l_n|\,\psi\rangle$, and thus, the equality holds

$$|\psi\rangle = \sum_{(n)} \langle l_n|\,\psi\rangle\,|l_n\rangle = \sum_{(n)} |l_n\rangle\,\langle l_n|\,\psi\rangle\,. \tag{14.16}$$

From equality 14.16 follows an important relation:

$$\sum_{(n)} |l_n\rangle\,\langle l_n| = \hat{I}. \tag{14.17}$$

where $\hat{I}$ is the identity operator. This equality is the condition of completeness of the set of eigenvectors (the condition of decomposability).

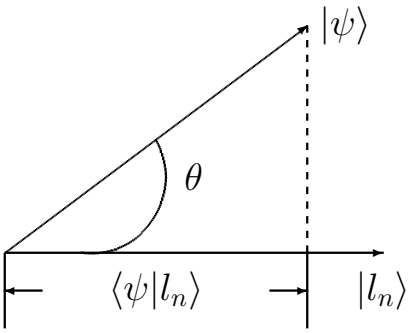


Figure 14.1: Projection operator. The action of the operator can be interpreted as the projection of the vector $|\psi\rangle$ onto the axis $|l_n\rangle$

14.1.6 Projection operator

Consider the operator $\hat{P}_n = |l_n\rangle \langle l_n|$. The result of action of this operator on the state $|\psi\rangle$ will be

$$\hat{P}_n |\psi\rangle = \sum_{(k)} |l_n\rangle \langle l_n| c_k |l_k\rangle = c_n |l_n\rangle . \tag{14.18}$$

The operator $\hat{P}_n = |l_n\rangle \langle l_n|$ is called the projection operator.

One can write the following properties of this operator

$$\sum_{(n)} \hat{P}_n = \hat{I}. \tag{14.19}$$

$$\hat{P}_n^2 = \hat{P}_n. \tag{14.20}$$

The action of the projection operator has a simple geometric interpretation (see Figure 14.1):

$$\hat{P}_n |\psi\rangle = \cos\theta |l_n\rangle ,$$

where $\cos\theta = \langle\psi|l_n\rangle = c_n$.

14.1.7 Trace of an operator

In an orthonormal basis $\{|l_n\rangle\}$, the quantity

$$Sp\hat{L} = \sum_n \langle l_n| \hat{L} |l_n\rangle \tag{14.21}$$

is called the trace of the operator $\hat{L}$. Under certain conditions [39], the series 14.21 converges absolutely and does not depend on the choice of basis.

If we use the matrix representation

$$L_{kn} = \langle l_k | \hat{L} | l_n \rangle ,$$

then the trace of the operator is the sum of the diagonal elements of the matrix representation

$$Sp\hat{L} = \sum_n L_{nn}$$

One can write the following properties of the trace:

$$Sp\left(l\hat{L} + m\hat{M}\right) = lSp\hat{L} + mSp\hat{M},$$
$$Sp\left(\hat{L}\hat{M}\right) = Sp\left(\hat{M}\hat{L}\right). \tag{14.22}$$

14.1.8 Mean values of operators

The mean value of operator $\hat{L}$ in the state $|\psi\rangle$ is given by the equality

$$\left\langle \hat{L} \right\rangle_\psi = \langle \psi | \hat{L} | \psi \rangle \tag{14.23}$$

under the condition

$$\langle \psi | \psi \rangle = 1.$$

Indeed, if one assumes that $|\psi\rangle$ is expanded in the series of eigenfunctions of the operator $\hat{L}$ as follows:

$$|\psi\rangle = \sum_n c_n |l_n\rangle ,$$

then $\hat{L}|\psi\rangle$ can be written as

$$\hat{L}|\psi\rangle = \sum_n l_n c_n |l_n\rangle ,$$

where l_n is the eigenvalue corresponding to the eigenstate $|l_n\rangle$. If now the last two expressions are substituted into (14.23) one obtains:

$$\langle \psi | \hat{L} | \psi \rangle = \sum_{n,m} l_n c_n c_m^* \langle l_m | l_n \rangle = \sum_n l_n c_n c_n^* = \sum_n l_n |c_n|^2 ,$$

which (under the condition $\langle \psi | \psi \rangle = 1$) proves that expression (14.23) indeed represents the expression for the mean value of the operator $\hat{L}$ in the state $|\psi\rangle$. [2]

[2] For this, it is enough to recall that $|c_n|^2$ gives the probability to find the system in the state $|l_n\rangle$, i.e., to obtain a reading of the measuring device equal to l_n

If one takes some orthonormal basis $\{|n\rangle\}$, forming a complete set, i.e., satisfying the condition (14.17): $\sum_n |n\rangle \langle n| = \hat{I}$, then expression (14.23) can be rewritten as follows:

$$\left\langle \hat{L} \right\rangle_\psi = \langle \psi | \hat{L} | \psi \rangle = \langle \psi | \hat{I} \hat{L} | \psi \rangle =$$
$$= \sum_n \langle \psi | n \rangle \langle n | \hat{L} | \psi \rangle = \sum_n \langle n | \hat{L} | \psi \rangle \langle \psi | n \rangle = Sp \left(\hat{L} \hat{\rho} \right),$$

where $\hat{\rho} = |\psi\rangle \langle \psi| = \hat{P}_\psi$ is the projection operator onto the state $|\psi\rangle$. Taking into account (14.22), one can write

$$\left\langle \hat{L} \right\rangle_\psi = Sp \left(\hat{\rho} \hat{L} \right). \tag{14.24}$$

14.1.9 Representation of operators through outer products of eigenvectors

Using the completeness condition (14.16) twice, we get

$$\hat{A} = \hat{I} \hat{A} \hat{I} = \sum_{(l)} \sum_{(l')} |l\rangle \langle l| \hat{A} |l'\rangle \langle l'| = \sum_{(l)} \sum_{(l')} |l\rangle \langle l'| A_{ll'}, \tag{14.25}$$

where $A_{ll'} = \langle l| \hat{A} |l'\rangle$ is the matrix element of the operator $\hat{A}$ in the $|l\rangle$ representation.

The operator expressed through its own eigenvectors can be represented by the expansion [3]

$$\hat{L} = \sum_{(l)} l |l\rangle \langle l|. \tag{14.26}$$

The generalization of this equality for an operator function has the form

$$F \left(\hat{L} \right) = \sum_{(l)} F(l) |l\rangle \langle l|. \tag{14.27}$$

14.1.10 Wave functions in coordinate and momentum representations

Transition from the state vector to the wave function is carried out by scalar multiplying this state vector by the eigenvector of the corresponding observable. For example, for the wave function in the coordinate representation

$$\phi(q) = \langle q | \psi \rangle. \tag{14.28}$$

[3] Under the condition of normalization of eigenvectors: $\langle l|l\rangle = 1$

where $\langle q|$ is the eigenvector of the coordinate operator. In the momentum representation, we get

$$\phi(p) = \langle p|\, \psi\rangle\,. \tag{14.29}$$

where $\langle p|$ is the eigenvector of the momentum operator.

14.2 Dynamics of wave function change

The wave function $|\phi\rangle$ can change through two mechanisms:

- Collapse of the wave function during measurement
- Schrödinger equation in the intervals between two consecutive measurements

14.2.1 Schrödinger Equation

The change of the state of a pure quantum system between two consecutive measurements is described by the following equation (Schrödinger)

$$i\hbar\frac{\partial\,|\phi\rangle}{\partial t} = \hat{\mathcal{H}}\,|\phi\rangle\,. \tag{14.30}$$

Equation (14.30) is reversible and, accordingly, not applicable to describing the change of the wave function at the moment of measurement.

It is worth noting the connection of the Schrödinger equation with Stone's theorem (Theorem 17.3.1) TBD.

Schrödinger equation in the interaction picture

Suppose that the Hamiltonian can be separated into two parts:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{V}}.$$

We introduce the following transformation of the wave function:

$$|\phi\rangle_I = \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}|\phi\rangle$$

and examine the value of the following expression:

$$i\hbar\frac{\partial\left|\phi\right\rangle_I}{\partial t} = i\hbar\frac{i\hat{\mathcal{H}}_0}{\hbar}\exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\left|\phi\right\rangle + \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\} i\hbar\frac{\partial\left|\phi\right\rangle}{\partial t} =$$

$$= -\hat{\mathcal{H}}_0\exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\left|\phi\right\rangle + \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\left(\hat{\mathcal{H}}_0 + \hat{\mathcal{V}}\right)\left|\phi\right\rangle =$$

$$-\hat{\mathcal{H}}_0\exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\left|\phi\right\rangle + \hat{\mathcal{H}}_0\exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\left|\phi\right\rangle + \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\hat{\mathcal{V}}\left|\phi\right\rangle =$$

$$= \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\hat{\mathcal{V}}\left|\phi\right\rangle =$$

$$= \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\hat{\mathcal{V}}\exp\left\{\left(-\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\left|\phi\right\rangle =$$

$$= \hat{\mathcal{V}}_I\left|\phi\right\rangle_I,$$

where

$$\hat{\mathcal{V}}_I = \exp\left\{\left(\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\}\hat{\mathcal{V}}\exp\left\{\left(-\frac{i\hat{\mathcal{H}}_0 t}{\hbar}\right)\right\} \tag{14.31}$$

is the interaction Hamiltonian in the interaction picture.

Thus, we obtain the Schrödinger equation in the interaction picture:

$$i\hbar\frac{\partial\left|\phi\right\rangle_I}{\partial t} = \hat{\mathcal{V}}_I\left|\phi\right\rangle_I. \tag{14.32}$$

Equation of motion for the density matrix

From relation (14.30) we have

$$i\hbar\frac{\partial\left|\phi\right\rangle}{\partial t} = \hat{\mathcal{H}}\left|\phi\right\rangle,$$
$$-i\hbar\frac{\partial\left\langle\phi\right|}{\partial t} = \hat{\mathcal{H}}\left\langle\phi\right|,$$

thus, for the density matrix $\hat{\rho} = \left|\phi\right\rangle\left\langle\phi\right|$ we get

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = i\hbar\frac{\partial\left|\phi\right\rangle\left\langle\phi\right|}{\partial t} = i\hbar\left(\frac{\partial\left|\phi\right\rangle}{\partial t}\left\langle\phi\right| + \left|\phi\right\rangle\frac{\partial\left\langle\phi\right|}{\partial t}\right) =$$

$$= \hat{\mathcal{H}}\left|\phi\right\rangle\left\langle\phi\right| - \left|\phi\right\rangle\left\langle\phi\right|\hat{\mathcal{H}} = \left[\hat{\mathcal{H}},\hat{\rho}\right] \tag{14.33}$$

Equation (14.33) is often called the quantum Liouville equation and the von Neumann equation.

Evolution operator. Heisenberg and Schrödinger pictures

The change of the wave function according to the law (14.30) can also be described using a certain operator (evolution) $\hat{U}(t, t_0)$:

$$|\phi(t)\rangle = \hat{U}(t, t_0)\,|\phi(t_0)\rangle\,. \tag{14.34}$$

Equation (14.30) can be rewritten as

$$|\phi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{\mathcal{H}}(t - t_0)\right)|\phi(t_0)\rangle\,,$$

from which for the evolution operator we have

$$\hat{U}(t, t_0) = \exp\left(-\frac{i}{\hbar}\hat{\mathcal{H}}(t - t_0)\right) \tag{14.35}$$

The evolution operator is unitary. Indeed:

$$\hat{U}(t, t_0)\,\hat{U}^{\dagger}(t, t_0) = \\ = \exp\left(-\frac{i}{\hbar}\hat{\mathcal{H}}(t - t_0)\right)\exp\left(+\frac{i}{\hbar}\hat{\mathcal{H}}(t - t_0)\right) = \hat{I}$$

Alongside the Schrödinger picture where operators do not depend on time and wave functions evolve, there exists the Heisenberg picture where operators evolve with time.

Obviously, the expectation values of operators should not depend on the picture:

$$\langle\phi_H(t_0)|\,\hat{A}_H(t)\,|\phi_H(t_0)\rangle = \langle\phi_S(t)|\,\hat{A}_S\,|\phi_S(t)\rangle = \\ = \langle\phi_H(t_0)|\,\hat{U}^{\dagger}(t, t_0)\,\hat{A}_S\hat{U}(t, t_0)\,|\phi_H(t_0)\rangle\,,$$

from which, given $\hat{A}_H(t_0) = \hat{A}_S(t_0)$, we obtain the evolution law for operators in the Heisenberg picture:

$$\hat{A}_H(t) = \hat{U}^{\dagger}(t, t_0)\,\hat{A}_H(t_0)\,\hat{U}(t, t_0) \tag{14.36}$$

At the same time, the equation for the operator $\hat{A}_H$ looks like:

$$\frac{\partial\hat{A}_H}{\partial t} = \frac{i}{\hbar}\hat{\mathcal{H}}\hat{U}^{\dagger}(t, t_0)\,\hat{A}_H(t_0)\,\hat{U}(t, t_0) - \\ -\frac{i}{\hbar}\hat{U}^{\dagger}(t, t_0)\,\hat{A}_H(t_0)\,\hat{U}(t, t_0)\,\hat{\mathcal{H}} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{A}_H\right] \tag{14.37}$$

14.2.2 Differences between pure and mixed states. Decoherence

Definition 14.2.1 (Pure state)**.** If the state of a system is described by a density matrix $\hat{\rho}$ which can be written as

$$\hat{\rho} = |\psi\rangle \langle\psi| \tag{14.38}$$

then this state is called pure.

Definition 14.2.2 (Mixed state)**.** If the state of a system is described by a density matrix $\hat{\rho}$ which **cannot** be written in the form (14.38), i.e.

$$\hat{\rho} \neq |\psi\rangle \langle\psi|$$

then this state is called mixed.

Of particular interest is the difference between pure and mixed states, especially how the transition from pure to mixed states occurs.

Consider a two-level state (see Figure 14.2). In a pure state it is described by the following wave function:

$$|\phi\rangle = c_a |a\rangle + c_b |b\rangle ,$$

the corresponding density matrix is

$$\begin{aligned} \hat{\rho} = |\phi\rangle \langle\phi| = \\ = |c_a|^2 |a\rangle \langle a| + |c_b|^2 |b\rangle \langle b| + \\ + c_a c_b^* |a\rangle \langle b| + c_b c_a^* |b\rangle \langle a| , \end{aligned} \tag{14.39}$$

or in matrix form

$$\hat{\rho} = \begin{pmatrix} |c_a|^2 & c_a c_b^* \\ c_b c_a^* & |c_b|^2 \end{pmatrix} .$$

The density matrix for a mixed state has only diagonal elements:

$$\begin{aligned} \hat{\rho} = \begin{pmatrix} |c_a|^2 & 0 \\ 0 & |c_b|^2 \end{pmatrix} = \\ = |c_a|^2 |a\rangle \langle a| + |c_b|^2 |b\rangle \langle b| . \end{aligned} \tag{14.40}$$

The transition from (14.39) to (14.40) is called decoherence. In describing the decoherence process we will follow [53].

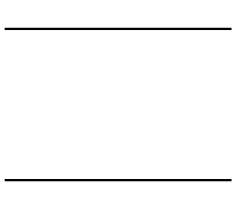


Figure 14.2: Model of a two-level atom used to describe decoherence.

The difference of mixed states from pure states manifests in the influence of the environment $\mathcal{E}$. In the case of pure states the considered system and its environment are independent, i.e.

$$|\phi\rangle_{pure} = |\phi\rangle_{at} \otimes |\mathcal{E}\rangle \,. \tag{14.41}$$

In the case of mixed states the atom and its environment form what is called an entangled state where the states $|a\rangle$ and $|b\rangle$ correspond to distinguishable environment states $|\mathcal{E}_a\rangle$ and $|\mathcal{E}_b\rangle$.

$$|\phi\rangle_{mix} = c_a \, |a\rangle \, |\mathcal{E}_a\rangle + c_b \, |b\rangle \, |\mathcal{E}_b\rangle \,. \tag{14.42}$$

The density matrix corresponding to (14.42) is

$$\begin{aligned} \hat{\rho}_{mix} = |\phi\rangle_{mix} \langle\phi|_{mix} = \\ = |c_a|^2 \, |a\rangle \langle a| \otimes |\mathcal{E}_a\rangle \langle\mathcal{E}_a| + |c_b|^2 \, |b\rangle \langle b| \otimes |\mathcal{E}_b\rangle \langle\mathcal{E}_b| + \\ + c_a c_b^* \, |a\rangle \langle b| \otimes |\mathcal{E}_a\rangle \langle\mathcal{E}_b| + c_b c_a^* \, |b\rangle \langle a| \otimes |\mathcal{E}_b\rangle \langle\mathcal{E}_a| \,. \end{aligned} \tag{14.43}$$

If we now apply averaging over the environment variables to expression (14.43), we obtain

$$\begin{aligned} \langle\hat{\rho}_{mix}\rangle_{\mathcal{E}} = Sp_{\mathcal{E}} \left(\hat{\rho}\right) = \\ = \langle\mathcal{E}_a| \, \hat{\rho}_{mix} \, |\mathcal{E}_a\rangle + \langle\mathcal{E}_b| \, \hat{\rho}_{mix} \, |\mathcal{E}_b\rangle = \\ = |c_a|^2 \, |a\rangle \langle a| + |c_b|^2 \, |b\rangle \langle b| \,. \end{aligned} \tag{14.44}$$

Expression (14.44) is obtained under the assumption of an orthonormal basis $\{|\mathcal{E}_a\rangle \,, |\mathcal{E}_b\rangle\}$:

$$\begin{aligned} \langle\mathcal{E}_a \, |\mathcal{E}_a\rangle = \langle\mathcal{E}_b \, |\mathcal{E}_b\rangle = 1, \\ \langle\mathcal{E}_a \, |\mathcal{E}_b\rangle = \langle\mathcal{E}_b \, |\mathcal{E}_a\rangle = 0. \end{aligned} \tag{14.45}$$

Conditions (14.45) are key to understanding why the considered basis of the atomic system is preferred and why, for instance, in the case of mixed states

one does not consider other bases such as the basis obtained by the Hadamard transform relative to the original :

$$|\mathcal{A}\rangle = \frac{|a\rangle + |b\rangle}{\sqrt{2}},$$
$$|\mathcal{B}\rangle = \frac{|a\rangle - |b\rangle}{\sqrt{2}}. \tag{14.46}$$

The environment states corresponding to the basis (14.46) are not orthogonal, hence it is impossible to use (14.46) as basis vectors for mixed states.

The decoherence process, i.e., the transition from (14.41) to (14.42), can be described using the Schrödinger equation and therefore theoretically reversible. The only requirement is the orthogonality of distinguishable environment states: $\langle \mathcal{E}_a \, | \mathcal{E}_b \rangle = 0$. This requirement is always fulfilled for macroscopic systems, where the state depends on a very large number of variables. Moreover, in the case of macroscopic systems there exist many possible final states $|\mathcal{E}_{a,b}\rangle$, which makes the reverse process practically unrealizable because it is necessary to control a large number of possible variables describing the environment state. In this sense, the decoherence process has the same nature as the second law of thermodynamics (increasing entropy), which describes irreversible processes. [4]

The decoherence process is very fast; in particular, [31] provides the following estimate: for a system of mass 1 g at a separation $\Delta x = 1$ cm and temperature $T = 300$ K, with relaxation time equal to the lifetime of the Universe $\tau_R = 10^{17}$ s, the decoherence process takes 10^{-23} s

14.2.3 Collapse of the wave function. Measurement in quantum mechanics

The process of selection (measurement result) is one of the most complex in quantum mechanics. Unlike deterministic change of the wave function described by the Schrödinger equation (14.30), the measurement process has a random character and different equations must be used to describe it.

Let us first consider pure states and suppose that a measurement is performed of a physical observable described by the operator $\hat{L}$. The eigenvalues and eigenfunctions of this operator are $\{l_k\}$ and $\{|l_k\rangle\}$ respectively. At the moment of measurement, the instrument readout can take values corresponding to the eigenvalues of the measured operator (see Figure 14.3). Suppose that the readout is l_n, in which case the wave function must be $|l_n\rangle$, i.e. the following change of the

[4] One should be somewhat careful here since the second law of thermodynamics applies to closed systems, while the decoherence processes happen in open systems.

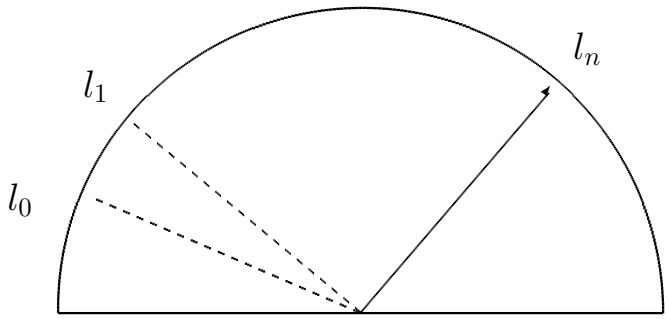
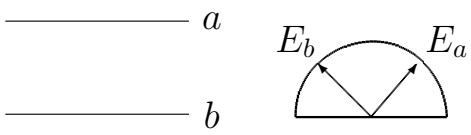


Figure 14.3: Measurement process. The instrument reading corresponds to one of the eigenvalues of the operator $\hat{L}$: $\{l_k\}$



$$|\psi\rangle = \tfrac{1}{\sqrt{2}}|a\rangle + \tfrac{1}{\sqrt{2}}|b\rangle$$

Figure 14.4: The process of measuring the energy of a two-level atom in a pure state $|\psi\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle$. The device registers the energy value E_a or E_b.

wave function occurred:

$$|\phi\rangle \to |l_n\rangle\,,$$

which can be described by the action of the projection operator $\hat{P}_n = |l_n\rangle\langle l_n|$ (14.18):

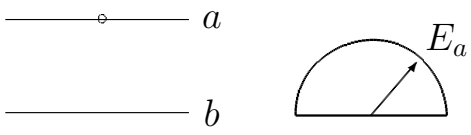
$$\hat{P}_n|\phi\rangle = c_n|l_n\rangle\,.$$

Example 14.2.1 (Measurement of energy of a two-level atom)**.** *Consider a two-level atom in a pure state (see Figure 14.4)* $|\psi\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle$.

Our instrument measures the energy of this atom and the Hamiltonian operator has 2 eigenfunctions $|a, b\rangle$*, corresponding to eigenvalues* E_a, E_b*. Thus, possible readings of the instrument belong to the set* $\{E_a, E_b\}$.

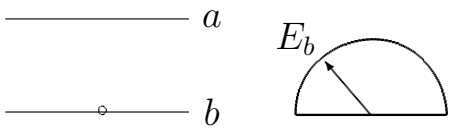
When the instrument pointer shows E_a*, the following collapse occurs (see Figure 14.5)*

$$|\psi\rangle \to |a\rangle\,.$$



$$|\psi\rangle \to |a\rangle$$

Figure 14.5: The process of measuring the energy of a two-level atom in a pure state $|\psi\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle$. The device registers the energy value E_a. During the measurement, the following reduction occurs $|\psi\rangle \to |a\rangle$



$$|\psi\rangle \to |b\rangle$$

Figure 14.6: The process of measuring the energy of a two-level atom in a pure state $|\psi\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle$. The device registers the energy value E_b. During the measurement, the following reduction occurs $|\psi\rangle \to |b\rangle$

Similarly, when the reading is E_b, the following collapse occurs (see Figure 14.6)

$$|\psi\rangle \to |b\rangle\,.$$

There is no way to predict the result that will be obtained from a single measurement. However, one can specify the probability of each possible result.

Indeed, in the case of a mixed state

$$\hat{\rho} = \sum_n |c_n|^2 |l_n\rangle \langle l_n|$$

the coefficients $P_n = |c_n|^2$ give the probabilities to find the system in state $|l_n\rangle$.

For a pure state

$$|\phi\rangle = \sum_n c_n |l_n\rangle$$

we also have that the probability to find the system in state $|l_n\rangle$ is given by the number $P_n = |c_n|^2$.

The main difference between pure and mixed states from the point of view of measurement is that in the former (pure states) the wave function and thus the state itself changes during measurement. In particular, if a certain final state $|l_i\rangle$ was obtained during measurement, one cannot say that it was the same before measurement. Mixed states behave like classical objects, i.e. if a state $|l_i\rangle$ was

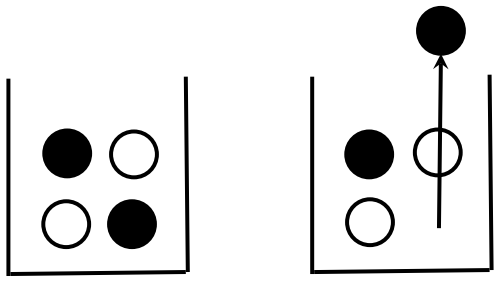


Figure 14.7: Example of a mixed state. The color of the sphere does not change as a result of the "measurement"

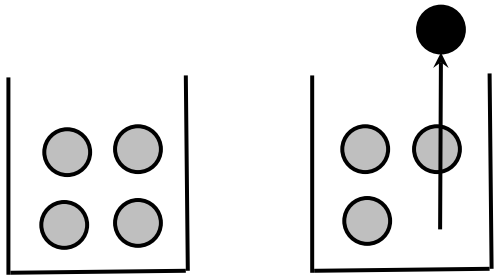


Figure 14.8: Example of a pure state. The color of the sphere changes as a result of "measurement"

obtained during measurement, it can be claimed that it was the same before measurement and the measurement itself represents choosing one state from many possible ones.

Example 14.2.2. Choosing from an urn with balls of two colors *Suppose we have an urn with 4 balls. With probability* $\frac{1}{2}$ *either a white or a black ball will be drawn. Suppose that as a result of the experiment a black ball was obtained. If the system under consideration is quantum and in a mixed state (see Figure 14.7), then the state of the drawn ball (color) did not change as a result of the experiment.*

If the system under consideration is pure (see Figure 14.8), then the state of each ball is described by a superposition of two colors - black and white. Thus, as a result of the experiment this superposition collapses and the ball acquires a definite color (black in our case), i.e., one can say that the color of the ball changes.

14.3 Measurability of Physical Quantities

14.3.1 Heisenberg Uncertainty Principle

Suppose two operators $\hat{A}$ and $\hat{B}$ do not commute with each other, i.e.

$$\left[\hat{A}\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C} \neq 0,$$

where $\hat{C}$ is some Hermitian operator.

Suppose the system is in the state $|\psi\rangle$. Then the average values of the operators are expressed by the following relations:

$$\langle\psi|\,\hat{A}\,|\psi\rangle = \left\langle \hat{A} \right\rangle,$$
$$\langle\psi|\,\hat{B}\,|\psi\rangle = \left\langle \hat{B} \right\rangle.$$

Define the uncertainty of measuring the quantities A and B as their variances as follows:

$$\Delta A = \sqrt{\langle\psi|\,\hat{\mathcal{A}}^2\,|\psi\rangle},$$
$$\Delta B = \sqrt{\langle\psi|\,\hat{\mathcal{B}}^2\,|\psi\rangle},$$

where

$$\hat{\mathcal{A}} = \hat{A} - \left\langle \hat{A} \right\rangle,$$
$$\hat{\mathcal{B}} = \hat{B} - \left\langle \hat{B} \right\rangle.$$

Introduce the operator $\hat{D}$ as follows:

$$\hat{D} = \hat{\mathcal{A}} + i\lambda\hat{\mathcal{B}}.$$

Consider the operator $\hat{D}^\dagger\hat{D}$, which is Hermitian. Its expectation in the state $|\psi\rangle$ is:

$$\langle\psi|\,\hat{D}^\dagger\hat{D}\,|\psi\rangle = \langle\phi|\,\phi\rangle \geq 0,$$

where $|\phi\rangle = \hat{D}\,|\psi\rangle$. On the other hand,

$$\langle\psi|\,\hat{D}^\dagger\hat{D}\,|\psi\rangle = \langle\psi|\left(\hat{\mathcal{A}} - i\lambda\hat{\mathcal{B}}\right)\left(\hat{\mathcal{A}} + i\lambda\hat{\mathcal{B}}\right)|\psi\rangle =$$
$$= \langle\psi|\,\hat{\mathcal{A}}^2\,|\psi\rangle + \lambda^2\,\langle\psi|\,\hat{\mathcal{B}}^2\,|\psi\rangle + i\lambda\,\langle\psi|\left[\hat{\mathcal{A}}, \hat{\mathcal{B}}\right]|\psi\rangle =$$
$$= (\Delta A)^2 + \lambda^2\,(\Delta B)^2 + i\lambda\,\langle\psi|\left[\hat{A}, \hat{B}\right]|\psi\rangle =$$
$$= \lambda^2\,(\Delta B)^2 - \lambda\,\langle C\rangle + (\Delta A)^2 \geq 0.$$

Consider the polynomial

$$f\,(\lambda) = \lambda^2\,(\Delta B)^2 - \lambda\,\langle C\rangle + (\Delta A)^2\,.$$

We have $f(\pm\infty) > 0$, so $f(\lambda) \geq 0$ if this polynomial has at most one real root, i.e.

$$\langle C\rangle^2 - 4(\Delta A)^2(\Delta B)^2 \leq 0$$

or

$$\Delta A \Delta B \geq \frac{|\langle C\rangle|}{2}, \tag{14.47}$$

which is the Heisenberg inequality.

14.3.2 Energy-Time Uncertainty Relation

Time in quantum mechanics does not have a corresponding operator, and therefore to estimate the time one should use some separate observable $\hat{O}$. Using (14.47), the following relation can be obtained:

$$\Delta E \Delta O \geq \frac{|\langle C\rangle|}{2},$$

where, taking into account (14.37),

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{O}\right]$$

and therefore

$$C = \frac{1}{i}\left[\hat{\mathcal{H}}, \hat{O}\right] = -\hbar\frac{d\hat{O}}{dt}.$$

Thus,

$$\Delta E \Delta O \geq \frac{|\langle C\rangle|}{2} = \frac{\hbar}{2}\left|\left\langle \frac{d\hat{O}}{dt}\right\rangle\right|.$$

Denoting $\Delta t = \frac{\Delta O}{\left|\left\langle \frac{d\hat{O}}{dt}\right\rangle\right|}$ [5] finally we get

$$\Delta E \Delta t \geq \frac{\hbar}{2}. \tag{14.48}$$

It is worth noting that states with definite energy do not contradict relation (14.48) because although $\Delta E = 0$, for any observable we have $\left\langle \frac{d\hat{O}}{dt}\right\rangle = 0$ and consequently, $\Delta t = \infty$.

[5] We can assume that at small times the observable $\hat{O}$ changes linearly, i.e. $\left\langle \frac{d\hat{O}}{dt}\right\rangle = \frac{\Delta O}{\Delta t}$. Thus, Δt is the time of significant change of the observable O.

14.3.3 Simultaneous Measurability of Physical Quantities

The question of simultaneous measurability of two physical quantities is very meaningful, especially when studying quantum mechanical paradoxes such as the EPR paradox (see section 10.1).

Suppose we have two observables $\hat{A}, \hat{B}$, each corresponding to a set of eigenfunctions $|a_i\rangle, |b_i\rangle$ and eigenvalues $\{a_i\}, \{b_i\}$, i.e.

$$\hat{A}|a_i\rangle = a_i|a_i\rangle,$$
$$\hat{B}|b_i\rangle = b_i|b_i\rangle.$$

If at some moment we can measure the values of these quantities simultaneously, then the obtained values can be described by numbers a and b, where obviously $a \in \{a_i\}, b \in \{b_i\}$ and

$$\hat{A}|a\rangle = a|a\rangle,$$
$$\hat{B}|b\rangle = b|b\rangle.$$

Since the measurement of the quantities was performed simultaneously, due to the collapse of the wavefunction (see subsection 14.2.3), we have

$$|a\rangle = |b\rangle,$$

that is, the operators $\hat{A}$ and $\hat{B}$ have common eigenfunctions. It is not necessary that all eigenvectors of these operators coincide, it is sufficient that a single vector matches.

Note that if the operators commute, then their sets of eigenvectors coincide, which is consistent with the Heisenberg inequality, which in this case takes the form

$$\Delta a \Delta b \geq 0.$$

Indeed, if

$$\hat{A}|a\rangle = a|a\rangle,$$

then, considering $\hat{A}\hat{B} = \hat{B}\hat{A}$,

$$\hat{A}\hat{B}|a\rangle = \hat{B}\hat{A}|a\rangle = a\hat{B}|a\rangle.$$

Denoting $|b\rangle = \hat{B}|a\rangle$, we have

$$\hat{A}|b\rangle = a|b\rangle,$$

i.e., $|b\rangle = |a\rangle$ and this vector is an eigenvector of both operators $\hat{A}$ and $\hat{B}$. Thus, if the operators commute, they have a common basis (set of eigenvectors). The converse is also true, see [36].

14.4 Quantization Procedure

As we have already found out, physical quantities in quantum mechanics correspond to self-adjoint operators. If we know the wave function (for example, by means of the Schrödinger equation), then knowing the expression for the operator of the physical quantity of interest, we can perform practical calculations, such as expectation values.

The procedure of deriving the relations that operators obey is called quantization.

Below is information about the connection between classical mechanics and the quantization procedure, as well as an example of quantizing the angular momentum operator.

14.4.1 Generalized Coordinates and Momenta

To fully describe a mechanical system, it is necessary to specify its coordinates and momenta. The coordinate is not always a classical coordinate, for example, positions in Cartesian coordinate system (x, y, z). Sometimes it is convenient to use special coordinates to describe the system under consideration, such as an angle in polar coordinates. In this case, the question arises of what should be considered the momentum corresponding to such a coordinate. To answer this question, one should consider the Lagrangian, which is defined as the difference between kinetic and potential energies:

$$\mathcal{L} = T - U.$$

If we have a set of generalized coordinates q_n, then the Lagrangian $\mathcal{L}$ can be presented as a function of these variables and their time derivatives:

$$\mathcal{L} = \mathcal{L}\left(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots\right).$$

We can define the generalized momentum p_i corresponding to the generalized coordinate q_i as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

In quantum mechanics, the equations of motion in Hamiltonian form play a special role:

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i}, \end{aligned} \tag{14.49}$$

where $\mathcal{H}$ is the Hamiltonian (total energy) of the system as a function of generalized coordinates and momenta.

Example 14.4.1 (Rectilinear Motion of a Material Point)**.** *Suppose we have a material point moving along the x-axis under the action of a force $F = -\frac{dU}{dx}$, where U is the potential energy. In this case:*

$$\mathcal{L} = T - U = \frac{m\dot{x}^2}{2} - U(x).$$

The generalized momentum p corresponding to coordinate x is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x},$$

which coincides with the classical definition of momentum.

The Hamiltonian is

$$\mathcal{H}(x, p) = T + U(x) = \frac{p^2}{2m} + U(x).$$

Equations of motion:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m},$$
$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{dU}{dx} = F,$$

where the first equation matches the definition of momentum, and the second represents the well-known Newton's law

$$F = m\ddot{x}.$$

14.4.2 Quantization of Angular Momentum

Classical Mechanics

The system under consideration consists of a material particle moving on a circle. The generalized coordinates describing the particle are the angle θ and the radius of the circle r, which is considered constant: $r = const|_t$ (see Figure 14.9).

The Hamiltonian of the system is given by

$$\mathcal{H} = T + U = \frac{mv^2}{2} + U(r) =$$
$$= \frac{mr^2\dot{\theta}^2}{2} + U(r),$$

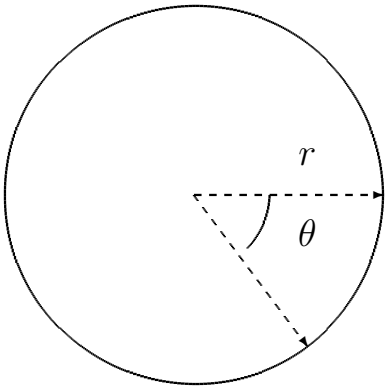


Figure 14.9: Quantization of angular momentum. The position of the particle is determined by two generalized coordinates: r and the angle θ.

where $T = \frac{mv^2}{2}$ is the kinetic energy of the particle, and U is the potential energy, which, due to the symmetry of the problem, does not depend on the angle θ and depends only on the distance r.

The Lagrangian of the system is

$$\mathcal{L} = T - U = \frac{mr^2\dot{\theta}^2}{2} - U\left(r\right).$$

The angular momentum (generalized momentum corresponding to the generalized coordinate θ) is defined as

$$l = \frac{d\mathcal{L}}{d\dot{\theta}} = mr^2\dot{\theta} = I\dot{\theta}, \tag{14.50}$$

where I denotes $I = mr^2$ — the moment of inertia.

Equations of motion for the coordinate r:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}},$$

from which

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} &= 0, \\ \frac{\partial T}{\partial r} - \frac{\partial U}{\partial r} &= 0, \\ \frac{\partial U}{\partial r} &= mr\dot{\theta}^2, \end{aligned}$$

or equivalently

$$U = \frac{mr^2\dot{\theta}^2}{2} = \frac{l^2}{2I}.$$

Thus, the Hamiltonian of the system under consideration is

$$\begin{aligned} \mathcal{H} &= \frac{mr^2\dot{\theta}^2}{2} + \frac{l^2}{2I} = \\ &= \frac{l^2}{2I} + \frac{l^2}{2I} = \frac{l^2}{I} \end{aligned} \tag{14.51}$$

Quantization

Let $|\psi\rangle$ be an eigenfunction of the angular momentum operator $\hat{L}$ corresponding to the eigenvalue l:

$$\hat{L}\,|\psi\rangle = l\,|\psi\rangle\,. \tag{14.52}$$

This same wave function must satisfy the Schrödinger equation

$$i\hbar\frac{\partial\,|\psi\rangle}{\partial t} = \hat{\mathcal{H}}\,|\psi\rangle\,.$$

From (14.51), we have

$$\hat{\mathcal{H}} = \frac{\hat{L}^2}{I},$$

and taking into account (14.52),

$$i\hbar\frac{\partial\,|\psi\rangle}{\partial t} = \frac{1}{I}\hat{L}\hat{L}\,|\psi\rangle = \frac{l^2}{I}\,|\psi\rangle\,,$$

or equivalently

$$\frac{\partial\,|\psi\rangle}{\partial t} = \frac{-il^2}{\hbar I}\,|\psi\rangle\,.$$

Thus,

$$|\psi\,(t)\rangle = C\cdot\exp\left\{\frac{-il^2}{\hbar I}t\right\}.$$

The wave function must satisfy the periodicity condition

$$|\psi(t)\rangle = |\psi(t+T)\rangle,$$

where T is the period of oscillations. It can be found from (14.50):

$$\begin{aligned}\dot{\theta} &= \frac{l}{I},\\ \theta &= \theta_0 + \frac{l\cdot t}{I},\\ 2\pi &= \frac{l\cdot T}{I},\\ T &= \frac{2\pi I}{l}.\end{aligned}$$

Thus,

$$2i\pi n = \frac{-il^2}{\hbar I}T = \frac{2\pi I}{l}\frac{-il^2}{\hbar I},$$

from which

$$l = -\hbar n.$$

14.5 Composite Systems

Systems consisting of several parts behave fundamentally differently in the case of classical and quantum systems.

As an example, consider a system of two particles. The position of the first is described by 3 coordinates, which represent a point in some linear space L_1: $(x_1, y_1, z_1) \in L_1$. For the second, suppose we have 2 coordinates in space L_2 fully defining the location: $(x_2, y_2) \in L_2$. Obviously, for a complete description of the positions of two particles, we need 5 coordinates: $(x_1, y_1, z_1, x_2, y_2) \in L_1 \times L_2$. Thus, composite systems in the classical case are described by points in the space $L^{classical}$, which is the Cartesian product of the originals: $L = L_1 \times L_2$. A distinctive feature of the Cartesian product is that the dimensions of the corresponding spaces add up:

$$\dim L^{classical} = \dim L_1 + \dim L_2.$$

In the quantum case, a composite system is described by vectors (points) in a space that is the tensor product of the originals:

$$L^{quantum} = L_1 \otimes L_2.$$

This space has 6 basis vectors, and accordingly, to describe the position of the system, 6 numbers are required: $(x_1 \cdot x_2, y_1 \cdot x_2, z_1 \cdot x_2, x_1 \cdot y_2, y_1 \cdot y_2, z_1 \cdot y_2) \in L^{quantum}$. Correspondingly, the dimensions multiply:

$$\dim L^{quantum} = \dim L_1 \cdot \dim L_2.$$

Thus, if we have two independent quantum systems with wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$, then the composite system will have the following wave function

$$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle ,$$

and obviously the following equalities hold

$$\begin{aligned} |\psi_1\rangle &= Sp_2 |\psi_{12}\rangle , \\ |\psi_2\rangle &= Sp_1 |\psi_{12}\rangle \end{aligned} \tag{14.53}$$

i.e., to obtain the state of subsystem 1, one must take the trace over the states of system 2 from the overall wave function.

Composite systems have an interesting property: although the parts of this system represent mixed states, the entire system as a whole is pure. As an example, consider the so-called entangled states :

$$|\psi\rangle = \frac{|0_1\rangle |1_2\rangle - |1_1\rangle |0_2\rangle}{\sqrt{2}}$$

which is pure. At the same time, for the first particle, we have

$$\begin{aligned} \rho_1 = Sp_2 |\psi\rangle \otimes \langle\psi| = \\ = \langle 0_2|\psi\rangle \langle\psi|0_2\rangle + \langle 1_2|\psi\rangle \langle\psi|1_2\rangle = \\ = \frac{|1_1\rangle \langle 1_1| + |0_1\rangle \langle 0_1|}{2} \end{aligned}$$

i.e., the state of the first particle is mixed.

14.6 Coordinate representation of a coherent state

Let's start with the vacuum state, which satisfies the equation

$$\hat{a} |0\rangle = 0. \tag{14.54}$$

In the coordinate representation $\hat{q} = q$, $\hat{p} = -i\frac{\partial}{\partial q}$, then

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega q + \hbar \frac{\partial}{\partial q} \right) , \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega q - \hbar \frac{\partial}{\partial q} \right) . \end{aligned} \tag{14.55}$$

Substituting (14.55) into (14.54), we obtain the equation for the wave function of the vacuum state of the mode in the coherent representation:

$$\left(\omega q + \hbar\frac{\partial}{\partial q}\right)\Phi_0(q) = 0, \tag{14.56}$$

where $\Phi_0(q) = \langle q|0\rangle$. The normalized solution of this equation is

$$\Phi_0(q) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\omega q^2}{2\hbar}}.$$

The uncertainty of this solution $\Delta q \Delta p = \frac{\hbar}{2}$ is minimal. The probability density to detect a certain coordinate q (in our case a certain electric field strength) is:

$$\Phi_0(q)\,\Phi_0^*(q) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega q^2}{\hbar}}. \tag{14.57}$$

For the coherent state, the following equation holds

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,,$$

which in the coordinate representation has the form

$$\left(\omega q + \hbar\frac{\partial}{\partial q}\right)\Phi_\alpha(q) = \alpha\Phi_\alpha(q)\,, \tag{14.58}$$

where

$$\Phi_\alpha(q) = \langle q|\,\alpha\rangle\,.$$

Comparing (14.56) and (14.58), it is easy to see that the solution of equation (14.58) differs from (14.57) by a shift of $\alpha\sqrt{\frac{2\hbar}{\omega}}$. Indeed, equation (14.58) can be rewritten as

$$\left(\omega\left(q - \frac{\alpha}{\omega}\right) + \hbar\frac{\partial}{\partial q}\right)\Phi_\alpha(q) = 0.$$

The solution of this equation is:

$$\Phi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\omega}{2\hbar}\left(q-\sqrt{\frac{2\hbar}{\omega}}\alpha\right)^2}. \tag{14.59}$$

The coherent state is not a stationary state. It must satisfy the non-stationary Schrödinger equation. The parameter $\alpha = \alpha(t)$ depends on time. Substituting (14.59) into the Schrödinger equation, this dependence can be determined, but even without that it is clear that $\alpha(t)$ must oscillate harmonically at the mode frequency. We assume

$$\alpha(t) = |\alpha|\, e^{-i\omega t}.$$

Then the probability density takes the form

$$\Phi_\alpha(q)\,\Phi_\alpha^*(q) = Ne^{-\frac{\omega}{2\hbar}\left[\left(q-\sqrt{\frac{2\hbar}{\omega}}\alpha\right)^2+\left(q-\sqrt{\frac{2\hbar}{\omega}}\alpha^*\right)^2\right]}.$$

Let's transform the exponent's argument

$$\left[\left(q-\sqrt{\frac{2\hbar}{\omega}}\alpha\right)^2+\left(q-\sqrt{\frac{2\hbar}{\omega}}\alpha^*\right)^2\right] =$$

$$= q^2+\frac{2\hbar}{\omega}\alpha^2-2q\alpha\sqrt{\frac{2\hbar}{\omega}}+q^2+\frac{2\hbar}{\omega}\alpha^{2*}-2q\alpha^*\sqrt{\frac{2\hbar}{\omega}} =$$

$$= 2\left[q^2+\frac{2\hbar}{\omega}\frac{\alpha^2+\alpha^{*2}}{2}-2q\sqrt{\frac{2\hbar}{\omega}}\frac{\alpha+\alpha^*}{2}+\frac{2\hbar}{\omega}(\alpha\alpha^*)-\frac{2\hbar}{\omega}(\alpha\alpha^*)\right] =$$

$$= 2\left[q^2+\frac{4\hbar}{\omega}\left(\frac{\alpha+\alpha^*}{2}\right)^2-2q\sqrt{\frac{2\hbar}{\omega}}\frac{\alpha+\alpha^*}{2}-\frac{2\hbar}{\omega}(\alpha\alpha^*)\right] =$$

$$= 2\left[q-\sqrt{\frac{2\hbar}{\omega}}\frac{\alpha+\alpha^*}{2}\right]^2+const =$$

$$= 2\left[q-\sqrt{\frac{2\hbar}{\omega}}|\alpha|\cos\omega t\right]^2+const.$$

The constant will go into the normalization factor, so we obtain:

$$\Phi_\alpha(q,t)\,\Phi_\alpha^*(q,t) = Ne^{-\frac{\omega}{\hbar}\left[q-\sqrt{\frac{2\hbar}{\omega}}|\alpha|\cos\omega t\right]^2}. \tag{14.60}$$

From expression (14.60) it is seen that the probability density for q (in our case for the electric field strength) at a fixed time is the same as for the vacuum state, but shifted by $\sqrt{\frac{2\hbar}{\omega}}\alpha(t)$. Over time, the center of the distribution moves harmonically, as depicted in Figure 14.10

Since the width of the distribution does not depend on the amplitude, the relative uncertainty decreases with increasing amplitude. Therefore, the state will tend to become classical.

14.7 Adiabatic Theorem

Theorem 14.7.1 (Adiabatic Theorem)**.** *A physical system remains in its instantaneous eigenstate if the perturbation acts sufficiently slowly and if this state is separated by an energy gap from the rest of the Hamiltonian spectrum.*

Proof. TBD □

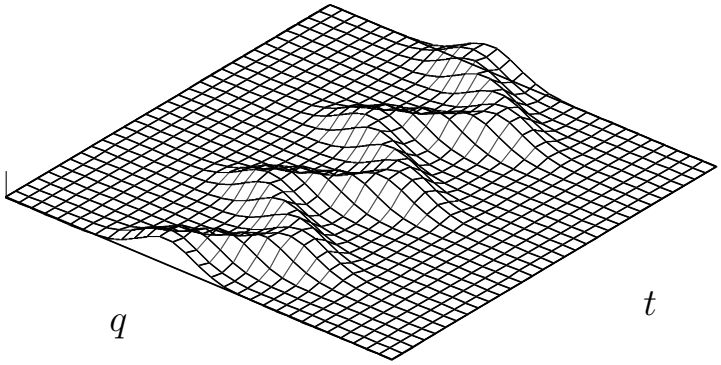


Figure 14.10: Evolution of the distribution function of the coherent state $|\Phi_\alpha(q,t)|^2$ over time.

14.8 Bosons and Fermions

Consider an ensemble of identical particles. The state of a system consisting of n identical particles can be written in the following form

$$|\psi\rangle = |x_1, x_2, \ldots, x_k, \ldots, x_j, \ldots, x_n\rangle . \tag{14.61}$$

For each of the particles in expression (14.61) we are interested in some physical characteristic (for example, coordinate) which is denoted by x.

Obviously, from the classical point of view, there should be no difference if we swap two particles. At the same time, the wave function, which is defined up to a phase, may change. Thus, denoting by $\hat{S}_{k,j}$ the operator that swaps two particles with indices k and j, we get

$$\begin{gathered}\hat{S}_{k,j}|\psi\rangle = |x_1, x_2, \ldots, x_j, \ldots, x_k, \ldots, x_n\rangle = \\ = e^{i\phi}|x_1, x_2, \ldots, x_k, \ldots, x_j, \ldots, x_n\rangle = e^{i\phi}|\psi\rangle\end{gathered}$$

If the operator $\hat{S}_{k,j}$ is applied twice, it is obvious that the system will be brought completely back to the original state, i.e.,

$$\hat{S}^2_{k,j}|\psi\rangle = e^{2i\phi}|\psi\rangle = |\psi\rangle$$

from which it follows that ϕ can take two values 0 and π. Thus, we can divide all particles into two classes. For the first of them the wave function does not change when swapping two particles ($\psi = 0$), and for the second it changes sign ($\psi = \pi$).

Example 14.8.1. *Consider a system consisting of two particles. Suppose the first particle can be in one of two states* $\left|\psi_1^{(1)}\right\rangle$ *or* $\left|\psi_2^{(1)}\right\rangle$. *Similarly, the second particle can be in one of two states* $\left|\psi_1^{(2)}\right\rangle$ *or* $\left|\psi_2^{(2)}\right\rangle$.

Assume the total wave function $\left|\psi^{(12)}\right\rangle$ *of the two particles can be represented as a superposition of the single-particle wave functions, i.e.*

$$\left|\psi^{(12)}\right\rangle = \left|\psi_{1,2}^{(1)}\right\rangle \pm \left|\psi_{1,2}^{(2)}\right\rangle$$

The case of a symmetric (under particle exchange) wave function will have the form

$$\left|\psi^{(12)}\right\rangle = \left|\psi_{1,2}^{(1)}\right\rangle + \left|\psi_{1,2}^{(2)}\right\rangle$$

and both particles can be in the same state, i.e., have the same wave function

$$|\psi\rangle = \left|\psi_1^{(1)}\right\rangle = \left|\psi_1^{(2)}\right\rangle. \tag{14.62}$$

In this case

$$\left|\psi^{(12)}\right\rangle = 2\,|\psi\rangle\,,$$

i.e., the particles can simultaneously be in the same state.

For the antisymmetric wave function case we have

$$\left|\psi^{(12)}\right\rangle = \left|\psi_{1,2}^{(1)}\right\rangle - \left|\psi_{1,2}^{(2)}\right\rangle$$

and if both particles are in the same state (14.62)*, then for* $\psi^{(12)}$ *we get*

$$\left|\psi^{(12)}\right\rangle = |\psi\rangle - |\psi\rangle = 0,$$

i.e., in this case particles cannot be in the same state simultaneously.

Bearing in mind Example 14.8.1, we can consider the general case of an ensemble of identical particles, each of which is in the same state or equivalently has the same wave function. In this case, under particle exchange, the actual wave function of the ensemble does not change:

$$\left|\psi^{(12)}\right\rangle = \left|\psi^{(21)}\right\rangle.$$

That is, in the antisymmetric case

$$\left|\psi^{(12)}\right\rangle = -\left|\psi^{(21)}\right\rangle = -\left|\psi^{(12)}\right\rangle = 0.$$

Thus, we can say that particles whose ensemble states are symmetric can be simultaneously in the same state. Such particles are called bosons.

For the antisymmetric case, we assume that such particles cannot simultaneously be in the same state. Such particles are called fermions.

Consider a state with a definite number of particles n $|\psi\rangle = |n\rangle$. In the case of bosons we will have

$$|\psi_b\rangle = |n\rangle_b . \tag{14.63}$$

The creation operators $\hat{a}_b^\dagger$ and annihilation operators $\hat{a}_b$ act on the state (14.63) as follows:

$$\hat{a}_b^\dagger |n\rangle_b = \sqrt{n+1}\,|n+1\rangle_b ,$$
$$\hat{a}_b |n\rangle_b = \sqrt{n}\,|n-1\rangle_b$$

For fermions there are two possible states: $|0\rangle_f$ and $|1\rangle_f$, with the corresponding creation operators $\hat{a}_f^\dagger$ and annihilation operators $\hat{a}_f$ acting as follows:

$$\hat{a}_f^\dagger |0\rangle_f = |1\rangle_f ,$$
$$\hat{a}_f^\dagger |1\rangle_f = 0,$$
$$\hat{a}_f |0\rangle_f = 0,$$
$$\hat{a}_f |1\rangle_f = |0\rangle_f .$$

14.9 Fermi's Golden Rule

Fermi's golden rule allows one to use time-dependent perturbation theory to calculate the transition probability between two states of a quantum system.

Theorem 14.9.1 (Fermi's Golden Rule)**.** *Suppose that a quantum system is initially in the state $|i\rangle$ and we want to estimate the probability of transition to the state $|f\rangle$. The Hamiltonian of the considered system consists of two parts: a stationary Hamiltonian $\hat{H}_0$ and a weak perturbation $\hat{V}$:*

$$\hat{H} = \hat{H}_0 + \hat{V}$$

In this case, the desired probability is given by

$$W_{i\to f} = \frac{2\pi}{\hbar} \left| \langle f| \hat{V} |i\rangle \right|^2 \rho,$$

where ρ is the density of final states.

Proof. Suppose there exists a set of states $\{|n\rangle\}$ forming a complete set of eigenvectors of the Hamiltonian $\hat{H}_0$, i.e.,

$$\hat{H}_0 |n\rangle = E_n |n\rangle ,$$

while the considered initial state $|i\rangle$ and final state $|f\rangle$ belong to this vector system, i.e.,

$$|i\rangle , |f\rangle \in \{|n\rangle\} .$$

At some moment in time, the system is in the state $|\psi\rangle$, which satisfies the Schrödinger equation:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle . \tag{14.64}$$

Our task is to calculate the probability that upon observation the system will be found in the state $|f\rangle$, given that the initial condition for (14.64) is

$$|\psi\rangle|_{t=0} = |i\rangle . \tag{14.65}$$

Due to the completeness of the system $\{|n\rangle\}$, the state $|\psi\rangle$ can be represented in the form

$$|\psi\rangle = \sum_n a_n (t) |n\rangle e^{\frac{-iE_n t}{\hbar}} \tag{14.66}$$

Substituting (14.66) into the Schrödinger equation (14.64) yields

$$i\hbar \sum_n \frac{\partial a_n (t)}{\partial t} |n\rangle e^{\frac{-iE_n t}{\hbar}} + \sum_n a_n (t) |n\rangle i\hbar \frac{\partial e^{\frac{-iE_n t}{\hbar}}}{\partial t} = \tag{14.67}$$

$$= i\hbar \sum_n e^{\frac{-iE_n t}{\hbar}} |n\rangle \left(\frac{\partial a_n (t)}{\partial t} - \frac{iE_n}{\hbar} \right) =$$

$$= i\hbar \sum_n e^{\frac{-iE_n t}{\hbar}} \frac{\partial a_n (t)}{\partial t} |n\rangle + \sum_n E_n a_n (t) |n\rangle e^{\frac{-iE_n t}{\hbar}} =$$

$$= i\hbar \sum_n e^{\frac{-iE_n t}{\hbar}} \frac{\partial a_n (t)}{\partial t} |n\rangle + \sum_n E_n a_n (t) |n\rangle e^{\frac{-iE_n t}{\hbar}} . \tag{14.68}$$

On the other hand, expression (14.68) must equal

$$\hat{H} |\psi\rangle = \sum_n E_n a_n (t) |n\rangle e^{\frac{-iE_n t}{\hbar}} + \sum_n a_n (t) e^{\frac{-iE_n t}{\hbar}} \hat{V} |n\rangle ,$$

i.e.,

$$i\hbar \sum_n e^{\frac{-iE_n t}{\hbar}} \frac{\partial a_n (t)}{\partial t} |n\rangle = \sum_n a_n (t) e^{\frac{-iE_n t}{\hbar}} \hat{V} |n\rangle . \tag{14.69}$$

We will solve expression (14.69) by perturbation theory, i.e.,

$$a_n(t) = a_n^{(0)}(t) + a_n^{(1)}(t) + \ldots,$$

and suppose that

$$a_n^{(0)}(t) = const|_t = a_n^{(0)}(0),$$

from which, given the initial condition (14.65),

$$a_n^{(0)}(t) = \delta_{ni}.$$

Thus, for the first order of perturbation theory from (14.69) we get

$$i\hbar \sum_n e^{\frac{-iE_n t}{\hbar}} \frac{\partial a_n^{(1)}(t)}{\partial t} |n\rangle = \sum_n \delta_{ni} e^{\frac{-iE_n t}{\hbar}} \hat{V} |n\rangle = e^{\frac{-iE_i t}{\hbar}} \hat{V} |i\rangle . \tag{14.70}$$

Therefore, from (14.70) we have

$$\frac{\partial a_n^{(1)}(t)}{\partial t} = -\frac{i}{\hbar} e^{\frac{-i(E_i - E_n)t}{\hbar}} \langle n| \hat{V} |i\rangle ,$$

denoting $E_i - E_n = \hbar\omega_{in}$, where ω_{in} is the transition frequency between the states $|i\rangle$ and $|n\rangle$, after integrating we get

$$a_n^{(1)}(t) = -\frac{i}{\hbar} \int_0^t e^{-i\omega_{in}t'} \langle n| \hat{V} |i\rangle \, dt',$$

i.e.,

$$\begin{aligned} a_n^{(1)}(t) &= \frac{1}{\hbar\omega_{in}} \langle n| \hat{V} |i\rangle \left(e^{-i\omega_{in}t} - 1\right) = \\ &= \frac{1}{\hbar\omega_{in}} \langle n| \hat{V} |i\rangle e^{\frac{-i\omega_{in}t}{2}} \left(e^{\frac{-i\omega_{in}t}{2}} - e^{\frac{i\omega_{in}t}{2}}\right) = \\ &= -\frac{2i}{\hbar\omega_{in}} \langle n| \hat{V} |i\rangle e^{\frac{-i\omega_{in}t}{2}} \sin\left(\frac{\omega_{in}t}{2}\right) \end{aligned} \tag{14.71}$$

We are interested in the rate $W_{i\to f}$ of transition to the state $|f\rangle$, which is given by the expression

$$W_{i\to f} = \frac{\left|a_f^{(1)}(t)\right|^2}{t}.$$

Or

$$W_{i\to f} = \frac{t}{\hbar^2} \left|\langle f| \hat{V} |i\rangle\right|^2 \frac{\sin^2\left(\frac{\omega_{if}t}{2}\right)}{\left(\frac{\omega_{if}t}{2}\right)^2}. \tag{14.72}$$

In the case where the final state $|f\rangle$ represents some continuous spectrum near the energy E_f, the expression (14.72) must be summed over all final states for which $\omega_{nf} \approx 0$. We introduce the concept of the density of states $\rho(E) = \frac{dE}{dn}$, so that the number of states dn corresponding to an infinitesimal energy interval dE can be represented as $dn = \rho dE$. Therefore, the transition rate can be rearranged as:

$$W_{i\to f} = \sum_{n:E_n \approx E_f} \rho\left(E_n\right) W_{i\to f}\left(E_n\right),$$

i.e., assuming for the considered energy interval $\rho\left(E_n\right) \approx \rho\left(E_f\right)$,

$$\left|\langle n|\,\hat{V}\,|i\rangle\right|^2 \approx \left|\langle f|\,\hat{V}\,|i\rangle\right|^2$$

and replacing summation by integration, we get

$$\begin{aligned} W_{i\to f} &\approx \int_{E_n \approx E_f} \rho\left(E_n\right) dE_n \frac{t}{\hbar^2} \left|\langle n|\,\hat{V}\,|i\rangle\right|^2 \frac{\sin^2\left(\frac{\omega_{in}t}{2}\right)}{\left(\frac{\omega_{in}t}{2}\right)^2} = \\ &= \int_{\omega_{in} \approx \omega_{if}} \rho\left(E_n\right) dE_n \frac{t}{\hbar^2} \left|\langle n|\,\hat{V}\,|i\rangle\right|^2 \frac{\sin^2\left(\frac{\omega_{in}t}{2}\right)}{\left(\frac{\omega_{in}t}{2}\right)^2} \approx \\ &\approx \frac{2}{\hbar}\left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \rho \int_{-\infty}^{\infty} d\left(\frac{\omega t}{2}\right) \frac{\sin^2\left(\frac{\omega t}{2}\right)}{\left(\frac{\omega t}{2}\right)^2} = \frac{2\pi}{\hbar}\left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \rho, \end{aligned} \tag{14.73}$$

which coincides with the expression to be proved.

□

Chapter 15

Probability Theory

When talking about quantum mechanics, it is often considered an applied part of classical probability theory. Such an assumption is only partly true; the probabilities underlying effects such as entangled states and nonclassical light have a fundamentally nonclassical nature, i.e., they are not described by classical Kolmogorov probability theory. The material in this section is mainly based on the monographs [35] and [37], which provide a comparison between classical and quantum probability theory.

15.1 Set algebra and basic concepts of probability theory

Definition 15.1.1 (σ-algebra)**.** Suppose that Ω is some set, then a family $\mathcal{F}$ of subsets of the set Ω is called a σ-algebra if the following conditions are satisfied:

- $\mathcal{F}$ contains Ω: $\Omega \in \mathcal{F}$
- if $\Upsilon \in \mathcal{F}$ then $\Omega \setminus \Upsilon \in \mathcal{F}$
- the union or intersection of a countable subfamily of $\mathcal{F}$ belongs to $\mathcal{F}$

Definition 15.1.2 (Measure)**.** If $\mathcal{F}$ is some σ-algebra then the following function

$$\mu : \mathcal{F} \to [0, \infty]$$

is called a measure if

- $\mu(\emptyset) = 0$
- $\forall A, B \in \mathcal{F}, A \cap B = \emptyset$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$

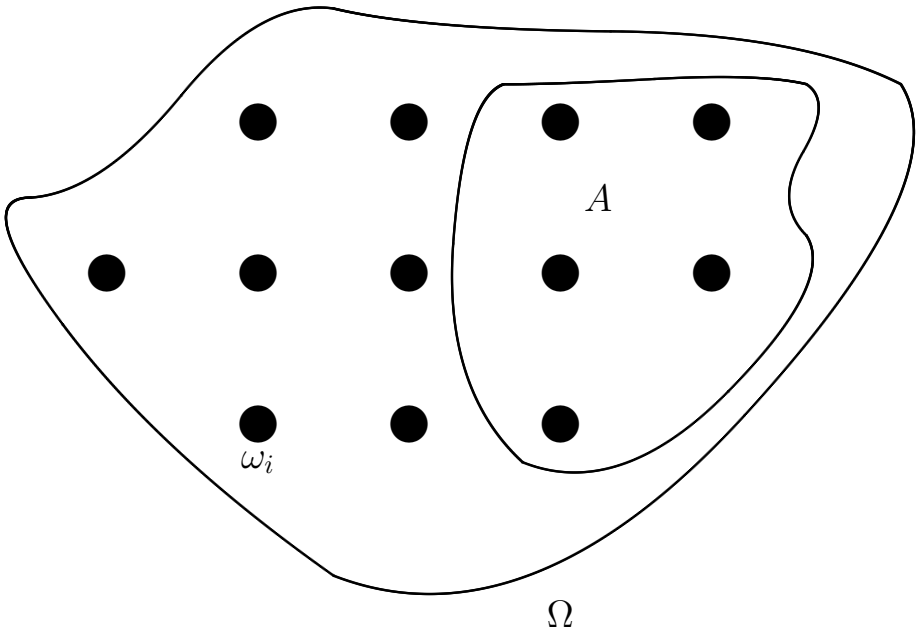


Figure 15.1: The sample space Ω is a set consisting of 12 elements (elementary events) $\omega_i \in \Omega$. The subset $A \subset \Omega$ (event) contains 5 elements (elementary events) of the set Ω

15.2 Classical Kolmogorov Probability Theory

15.2.1 Basic Concepts and Axioms of Probability Theory

Classical probability theory deals with set theory and is based on several simple axioms. This axiomatics was proposed in the 1930s by Kolmogorov A. N. [33].

First of all, a few definitions of the objects with which we will be dealing.

Definition 15.2.1 (Event Space)**.** $\mathcal{F}$ is some σ-algebra defined on some set Ω and is called an event space.

Remark 15.2.1 (Event Space)**.** *By virtue of the properties of a σ-algebra, we have the following facts: if $\Omega \subset \Omega$, $\emptyset \subset \Omega$ and for any $A, B \subset \Omega$ it holds that $A \cup B \subset \Omega$, $A \cap B \subset \Omega$ and $A \setminus B \subset \Omega$. An element of the set $\omega \in \Omega$ is called an elementary event. A subset $A \subset \Omega$ or $A \in \mathcal{F}$ is called an event.*

Definition 15.2.2 (Probability P)**.** A finite measure defined on $\mathcal{F}$ over the set Ω is called a probability if $P(\Omega) = 1$.

Example 15.2.1 (Event Space)**.** *In Figure 15.1, the set Ω is depicted, which we call the event space. Elements of the set $\omega_i \in \Omega$ (black dots in Figure 15.1) are elementary events. A subset $A \subset \Omega$ is an event.*

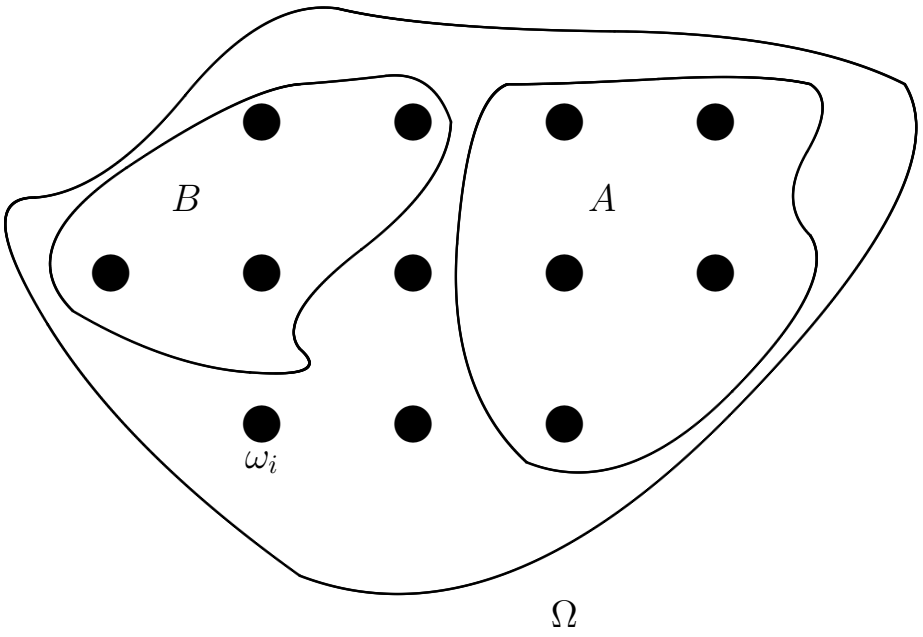


Figure 15.2: Sample space Ω, elementary event $\omega \in \Omega$, events $A \subset \Omega$ and $B \subset \Omega$. $P(\omega) = \frac{1}{12}$, $P(\Omega) = \frac{12}{12} = 1$, $P(A) = \frac{5}{13}$, $P(B) = \frac{4}{13}$, $P(A \cup B) = P(A) + P(B) = \frac{5}{12} + \frac{4}{12} = \frac{3}{4}$

Now, strictly speaking, the axioms.

Axiom 15.2.1 (Non-negativity). *The probability of an event $A \subset \Omega$ is a non-negative real number, i.e. $P(A) \geq 0$*

Axiom 15.2.2 (Normalization). *The probability of the event space Ω is 1, i.e. $P(\Omega) = 1$*

Axiom 15.2.3 (Additivity). *If $A_i \cap A_j = \emptyset$, then $P(A_i \cup A_j) = P(A_i) + P(A_j)$*

Example 15.2.2 (Axioms of Kolmogorov's Probability Theory). *Assign to each elementary event $\omega_i \in \Omega$ in Figure 15.2 a non-negative number $P(\omega_i) = \frac{1}{12}$. The probability of event A is $P(A) = \frac{5}{12}$ and for event B we have $P(B) = \frac{4}{12}$. Thus, since $A \cap B = \emptyset$, using axiom 15.2.3, we get:*

$$P(A \cup B) = P(A) + P(B) = \frac{5}{12} + \frac{4}{12} = \frac{3}{4}.$$

For the event Ω, by axiom 15.2.2 we have $P(\Omega) = 1$. On the other hand, $\Omega = \cup_i \omega_i$, i.e.

$$P(\Omega) = \sum_i P(\omega_i) = 12 \cdot \frac{1}{12} = 1.$$

All known facts of probability theory are derived from these three axioms. TBD

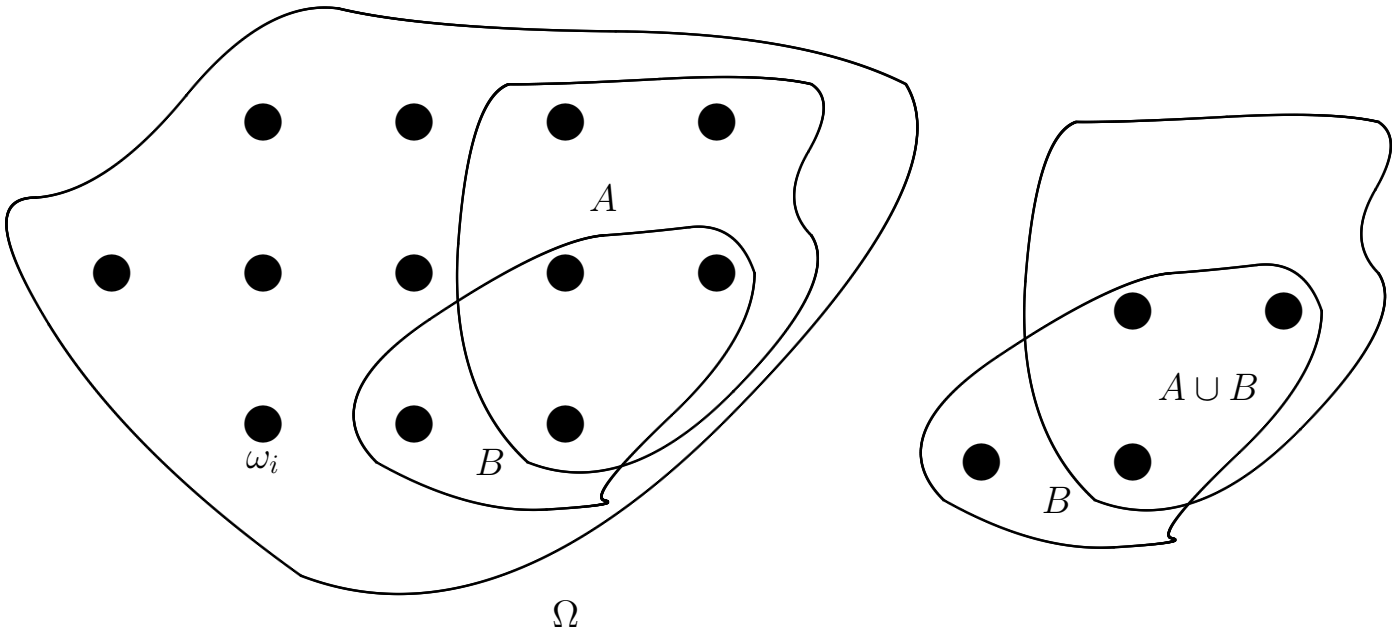


Figure 15.3: Conditional probability $P(A|B)$

15.2.2 Random Variable. Distribution Functions

Definition 15.2.3 (Random Variable)**.** If Ω is the sample space (see definition 15.2.1), then a mapping $X : \Omega \to \mathbb{R}$ is called a random variable.

It is worth noting that if a random variable is denoted by an uppercase letter X, then a specific realization is denoted by a lowercase letter x, i.e. if $\omega \in \Omega$, then $x = X(\omega)$. In particular, for discrete random variables, to each realization $x = X(\omega)$ one can assign some probability $P(x) = P(\omega)$. Otherwise, distribution functions are used.

Definition 15.2.4 (Distribution Function of a Random Variable)**.** The distribution function F_X of a random variable X is a function $F_X : \mathbb{R} \to [0, 1]$ defined by the formula

$$F_X(x) = P(X \leq x),$$

i.e. the distribution function at argument x equals the probability of those events $\omega \in \Omega$ for which $X(\omega) \leq x$.

Definition 15.2.5 (Probability Density Function of a Random Variable)**.** In the case of continuous random variables, it makes sense to speak about the probability density function $f_X(x)$, which is defined by the following expression

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Definition 15.2.6 (Expectation of a Random Variable)**.** For a discrete random variable, i.e. if the set of elementary outcomes is countable, the expectation $E(X)$ of the random variable X is defined by the following relation

$$E(X) = \sum_{\omega} p_{\omega} \cdot X(\omega) = \sum_{x} p_x \cdot x,$$

where the summation is over all possible elementary outcomes ω, $x = X(\omega)$, $p_{\omega} = p_{X(\omega)} = p_x$. For continuous random variables, this relation transforms into

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx.$$

TBD

15.2.3 Poisson Distribution

The Poisson distribution plays a very important role in quantum optics and therefore deserves a more detailed consideration.

Definition 15.2.7 (Poisson Distribution)**.**

TBD

15.2.4 Random Processes

To fully describe a random process $x(t)$, it is necessary to know all the joint probability density functions

$$p_n\left(\{x\}_n, \{t\}_n\right) = p_n\left(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_n\right).$$

For example, using $p_1 = p_1(x,t)$, one can calculate the mean value of the random variable

$$\langle x(t) \rangle = \int x p_1(x,t)\, dx, \tag{15.1}$$

and using $p_2 = p_2(x_1, x_2, t_1, t_2)$ — the so-called two-time autocorrelation function

$$r(t_1, t_2) = \langle x^*(t_1)\, x(t_2) \rangle = \int\int x_1 x_2 p_2(x_2, t_2, x_1, t_1)\, dx_1 dx_2. \tag{15.2}$$

In the case where the mean of the random variable (15.1) does not depend on t:

$$\langle x(t) \rangle = const_t, \tag{15.3}$$

and the autocorrelation function (15.2) depends only on the difference $t_1 - t_2 = \tau$:

$$r(t_1, t_2) = r(t_1 - t_2) = r(\tau), \tag{15.4}$$

then the random process is called wide-sense stationary. Obviously, for such processes, the following identity is valid:

$$r(-\tau) = r^*(\tau), \tag{15.5}$$

indeed, from (15.4) we have

$$r(-\tau) = r(t_2 - t_1) = r(t_2, t_1) = r^*(t_1, t_2) = r^*(\tau).$$

For wide-sense stationary random processes, the following theorem holds:

Theorem 15.2.1 (Khinchin-Wiener)**.** *The autocorrelation function of wide-sense stationary random processes and their spectral density are linked by the direct and inverse Fourier transforms.*

Proof. The proof will be informal and serves as an attempt to explain the meaning of this theorem. Consider the following integral

$$\tilde{x}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt, \tag{15.6}$$

which can be interpreted as the Fourier transform (see section 18.1) of the random process $x(t)$. For stationary random processes, it is obvious that the integral (15.6) generally does not exist. Indeed, if this integral existed, the Parseval relation would hold:

$$\int_{-\infty}^{\infty} |\tilde{x}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |x(t)|^2 dt, \tag{15.7}$$

which obviously diverges in the case of a stationary random process. At the same time, by virtue of (15.7), $S(\omega) = |\tilde{x}(\omega)|^2$ represents the distribution of energy along the frequency axis for a particular realization of the process, and the averaged value $\langle S(\omega) \rangle$ represents a certain characteristic of the random process that describes the energy distribution over frequencies.

Although the integral (15.7) diverges for the considered random processes, as shown by Khinchin and Wiener, the integral

$$\langle \tilde{x}^*(\omega) \tilde{x}(\omega') \rangle = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle x^*(t) x(t') \rangle e^{i(\omega' t' - \omega t)} dt dt', \tag{15.8}$$

exists and can be interpreted as the power spectral density $S(\omega, \omega')$ of the considered random process. Thus, taking into account stationarity,

$$\langle x^*(t) x(t') \rangle = r(t' - t) = r(\tau),$$

we obtain

$$\begin{aligned} S(\omega,\omega') &= \langle \tilde{x}^*(\omega)\,\tilde{x}(\omega')\rangle = \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle x^*(t)\,x(t')\rangle\, e^{i(\omega' t'-\omega t)}dtdt' = \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(\tau)\, e^{i(\omega'-\omega)t}e^{i\omega'(t'-t)}dtd\tau = \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{i(\omega'-\omega)t}dt\int_{-\infty}^{\infty} r(\tau)\, e^{i\omega'\tau}d\tau = \\ &= \tilde{r}(\omega')\,\delta(\omega'-\omega), \end{aligned} \tag{15.9}$$

where the integral representation of the delta function (18.3) has been used.

Thus, from (15.9) it follows that the Fourier transform of the autocorrelation function $\tilde{r}(\omega)$ can be regarded as the spectral density of the random process (15.8). □

15.3 Quantum Probability Theory

When speaking about quantum probability theory, we will rely on the classical (Kolmogorov) theory, while trying to choose a formulation that is applicable in both cases. As a model, we will consider a system with n possible outcomes, i.e. one where the classical probability distribution is given by the following distribution:

$$p = \{p_1, p_2, \ldots, p_n\} \tag{15.10}$$

A quantum state is fully described in the general case (both pure and mixed states) by the density matrix

$$\hat{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{bmatrix}.$$

In the classical case, this object corresponds to the probability distribution (15.10), which can also be represented as a certain matrix

$$\hat{p} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix}.$$

It is worth noting that for both cases the equality holds

$$Sp\left(\hat{\rho}\right) = Sp\left(\hat{p}\right) = 1,$$

which is a rewritten version of axiom 15.2.2.

An observable $\hat{x}$ corresponds to a certain matrix

$$\hat{x} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}, \tag{15.11}$$

which in the classical case is diagonal

$$\hat{x} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}. \tag{15.12}$$

It is also worth noting that the calculation of the mean then takes the familiar classical form

$$\bar{x} = Sp\left(\hat{p}\hat{x}\right) = \sum_{i=1}^{n} p_i x_i.$$

Remark 15.3.1. *It is worth noting that for any observable $\hat{X}$, due to self-adjointness $\hat{X} = \hat{X}^{\dagger}$, one can choose a basis in which the corresponding matrix will be diagonal, with eigenvalues on the diagonal. Moreover, if two observables $\hat{X}$ and $\hat{Y}$ commute:*

$$\hat{X}\hat{Y} = \hat{Y}\hat{X},$$

then they can be diagonalized in a common basis.

The difference between quantum probability theory and classical theory can be shown in several ways [35, 37], one of which is based on the noncommutativity of the quantum description. Indeed, it can be noted that in the general case quantum observables, represented by matrices of the form (15.11), do not commute with each other. At the same time, classical observables, written as diagonal matrices (15.12) always commute with each other. From this, many facts follow TBD

Chapter 16

Abstract Algebra

Here you can find definitions and basic properties of groups, rings, and fields.

16.1 Introduction to Group Theory

Definition 16.1.1. A group $(\mathcal{G}, \circ)$ is called a set of elements $g \in \mathcal{G}$ for which a certain binary operation $\circ$ (often called multiplication or addition) is defined:

$$\begin{aligned} &\forall g_1, g_2 \in \mathcal{G}, \\ &g_1 \circ g_2 \in \mathcal{G}. \end{aligned} \tag{16.1}$$

The operation defined by (16.1) satisfies the property of associativity:

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3.$$

The considered set must contain an element $e_{\mathcal{G}}$ that has the following property, valid for any element of the set g:

$$g \circ e_{\mathcal{G}} = e_{\mathcal{G}} \circ g = g.$$

For each element of the group g, there must exist an inverse element $g^{-1} \in \mathcal{G}$, possessing the following property:

$$g \circ g^{-1} = g^{-1} \circ g = e_{\mathcal{G}}$$

Definition 16.1.2 (Monoid)**.** If for some set of elements $\mathcal{G}$ the last property of groups (existence of inverse element) is not fulfilled, then this set is called a monoid or a semigroup.

Definition 16.1.3 (Abelian Group)**.** A group $(\mathcal{A}, \circ)$ is called abelian, or commutative, if $\forall a_1, a_2 \in \mathcal{A}$: $a_1 \circ a_2 = a_2 \circ a_1$.

Example 16.1.1. Group $(\mathbb{Z}, +)$ *The set of integers* $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ *forms a group under the operation of addition.*

Definition 16.1.4 (Cyclic Group)**.** A cyclic group G is a group generated by a single element $g : G =< g >$, i.e. all its elements are powers of g. The element g is called the generating element or the generator of the group G.

Definition 16.1.5 (Multiplicative Group of the Ring of Residues)**.** Consider the set of integers coprime with n and less than n, denoted by $(\mathbb{Z}/n\mathbb{Z})^{\times}$. As the operation of multiplication of two elements $a, b \in (\mathbb{Z}/n\mathbb{Z})^{\times}$ we take

$$a \circ b = a \cdot b \mod n.$$

The identity element $e_{(\mathbb{Z}/n\mathbb{Z})^{\times}}$ is 1. Moreover, it can be shown that for each $a \in (\mathbb{Z}/n\mathbb{Z})^{\times}$ there exists $a^{-1} \in (\mathbb{Z}/n\mathbb{Z})^{\times}$ such that $a \circ a^{-1} = 1$. Thus, $(\mathbb{Z}/n\mathbb{Z})^{\times}$ is a group.

Theorem 16.1.1 (On the order of $(\mathbb{Z}/n\mathbb{Z})^{\times}$)**.** *The order of the group* $(\mathbb{Z}/n\mathbb{Z})^{\times}$ *is given by the formula*

$$\left|(\mathbb{Z}/n\mathbb{Z})^{\times}\right| = \phi(n),$$

where $\phi(n)$ *is the Euler's totient function (19.3.1) . Moreover, if* $n = p$ *is a prime number, then*

$$\left|(\mathbb{Z}/p\mathbb{Z})^{\times}\right| = \phi(p),$$

and if $a \in (\mathbb{Z}/p\mathbb{Z})^{\times}, a \neq 1$*, then* a *is a generator of the considered group, i.e.,*

$$a^{p-1} = 1.$$

Proof. TBD □

Theorem 16.1.2 (Lagrange)**.** *For any finite group* $\mathcal{G}$*, the order (number of elements) of any subgroup* $\mathcal{H}$ *divides the order of* $\mathcal{G}$*:*

$$|\mathcal{G}| = h\,|\mathcal{H}|\,,$$

where the integer h *is called the index of the subgroup.*

16.2 Fields

Definition 16.2.1 (Field (algebra))**.** Let an Abelian group be given (see definition 16.1.3) $(\mathcal{F}, +)$. The identity element of this group is $e_{\mathcal{F}}$ - 0. Also let $(\mathcal{F} \setminus \{0\}, \cdot)$

be another group (also Abelian) with the identity element 1. Furthermore, the operations $+, \cdot$ satisfy the distributive property, i.e. $\forall a, b, c \in \mathcal{F}$:

$$c \cdot (a + b) = c \cdot a + c \cdot b,$$
$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

In this case $(\mathcal{F}, +, \cdot)$ is called a field.

Example 16.2.1 (Field $\mathbb{Q}$)**.** *Note that $\mathbb{Z}$ is not a field because not every integer has a multiplicative inverse. However, the following set will form a field: $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\}$. The inverse of $a/b \in (\mathbb{Q} \setminus \{0\}, \cdot)$ is b/a.*

Example 16.2.2 (Field $\mathbb{R}$)**.** *The real numbers form a field.*

Example 16.2.3 (Field $\mathbb{C}$)**.** *The complex numbers form a field.*

Chapter 17

Linear Algebra

17.1 Linear Space

Definition 17.1.1 (Linear Space)**.** TBD

17.1.1 Direct sum of linear spaces and tensor product

Suppose we have two linear spaces $\mathbb{A}$ and $\mathbb{B}$ with bases $\{a_i\} \subset \mathbb{A}$ and $\{b_j\} \subset \mathbb{B}$ respectively.

Definition 17.1.2 (Tensor product of linear spaces)**.** The linear space $\mathbb{L} = \mathbb{A} \otimes \mathbb{B}$ with basis $l_{ij} = a_i \otimes b_j$ is called the tensor product of linear spaces $\mathbb{A}$ and $\mathbb{B}$.

Definition 17.1.3 (Tensor (external) product of linear operators)**.** Consider two linear operators $\hat{A} : \mathbb{A} \to \mathbb{A}$ and $\hat{B} : \mathbb{B} \to \mathbb{B}$ then the operator $\hat{L} = \hat{A} \otimes \hat{B} : \mathbb{L} \to \mathbb{L}$ acting on the basis by the rule

$$\hat{L} l_{ij} = \hat{L} a_i \otimes b_j = \hat{A} a_i \otimes \hat{B} b_j$$

is called the tensor product of operators.

17.1.2 Theorem on Operator Decomposition

Theorem 17.1.1 (On the Operator Identity)**.**

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \dots \tag{17.1}$$

Proof. To prove this, we introduce the following auxiliary function:

$$f(\alpha) = e^{\alpha \hat{A}} \hat{B} e^{-\alpha \hat{A}}, \tag{17.2}$$

which coincides with the desired expression at $\alpha = 1$. We expand (17.2) into a Taylor series:

$$f(\alpha) = f(0) + \frac{\alpha}{1!} f'(0) + \frac{\alpha^2}{2!} f''(0) + \cdots + \frac{\alpha^k}{k!} f^{(k)}(0) + \ldots. \tag{17.3}$$

The following recursive relation holds:

$$\begin{aligned} f^{(k)}(\alpha) &= \left[\hat{A}, f^{(k-1)}(\alpha)\right] = \\ &= \underbrace{[\hat{A}[\ldots[\hat{A}}_{\text{k times}}, f^{(0)}(\alpha) \underbrace{]\ldots]]}_{\text{k times}}, \end{aligned} \tag{17.4}$$

where

$$f^{(0)}(\alpha) = f(\alpha).$$

Indeed, by induction for the first derivative we have:

$$\begin{aligned} f^{(1)}(\alpha) &= \hat{A} e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}} - e^{\alpha\hat{A}} \hat{B} \hat{A} e^{-\alpha\hat{A}} = \\ &= \hat{A} e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}} - e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}} \hat{A} = \\ &= \left[\hat{A}, e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}}\right] = \left[\hat{A}, f(\alpha)\right] = \left[\hat{A}, f^{(0)}(\alpha)\right], \end{aligned} \tag{17.5}$$

i.e. (17.4) holds for $k = 1$. In deriving (17.5) we used the fact that

$$\hat{A} e^{-\alpha\hat{A}} = e^{-\alpha\hat{A}} \hat{A}.$$

For the $(k+1)$-th derivative we get:

$$\begin{aligned} f^{(k+1)}(\alpha) &= \frac{\partial f^{(k)}(\alpha)}{\partial\alpha} = \\ &= \frac{\partial}{\partial\alpha}\left(\underbrace{[\hat{A}[\ldots[\hat{A}}_{k}, f^{(0)}(\alpha) \underbrace{]\ldots]]}_{k}\right) = \underbrace{[\hat{A}[\ldots[\hat{A}}_{k}, \frac{\partial}{\partial\alpha} f^{(0)}(\alpha) \underbrace{]\ldots]]}_{k} = \\ &= \underbrace{[\hat{A}[\ldots[\hat{A}}_{k}, f^{(1)}(\alpha) \underbrace{]\ldots]]}_{k} = \underbrace{[\hat{A}[\ldots[\hat{A}}_{k}, \left[\hat{A}, f^{(0)}(\alpha)\right] \underbrace{]\ldots]]}_{k} = \\ &= \underbrace{[\hat{A}[\ldots[\hat{A}}_{k+1}, f^{(0)}(\alpha) \underbrace{]\ldots]]}_{k+1}, \end{aligned}$$

which together with (17.5) proves the validity of (17.4).

Considering that

$$f^{(0)}(\alpha) = \hat{B},$$

we can obtain from (17.3) the desired identity (17.1) at $\alpha = 1$. □

17.2 Hilbert Space

A Hilbert space is a generalization of Euclidean space allowing for infinite dimensionality.

Definition 17.2.1 (Hilbert Space)**.** TBD

17.2.1 Stone's Theorem

Theorem 17.2.1 (Stone)**.** *TBD*

17.3 Hilbert Space

A Hilbert space is a generalization of Euclidean space allowing for infinite dimensionality.

Definition 17.3.1 (Hilbert Space)**.** TBD

17.3.1 Stone's Theorem

Theorem 17.3.1 (Stone)**.** *TBD*

Chapter 18

Fourier Transform

18.1 Fourier Transform

In this course, we will use the following form of the Fourier transform [1]

$$\tilde{x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\, e^{i\omega t} dt, \tag{18.1}$$

while the inverse Fourier transform takes the following form

$$x(t) = \int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{-i\omega t} dt, \tag{18.2}$$

The following table contains important transform pairs

Function $x(t)$	Transform $\tilde{x}(\omega)$
1	$\delta(\omega)$
$\delta(t)$	$\frac{1}{2\pi}$

Table 18.1: Fourier Transform

From Table 18.1 follows an important formula (integral representation of the delta function) [1]

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} dt \tag{18.3}$$

[1] Most commonly, the Fourier transform is defined as

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)\, e^{-i\omega t} dt.$$

However, according to [2], there are alternative definitions differing by the sign before the frequency and the coefficient before the integral.

18.2 Discrete Fourier Transform

The Fourier transform plays an important role in digital signal processing, particularly for the analysis of periodic sequences.

18.2.1 Definition

Definition 18.2.1. Suppose there are M samples $x_0, x_1, \ldots, x_{M-1}$ then the discrete Fourier transform is given by the relation

$$\tilde{X}_k = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} x_m e^{-\frac{2\pi i}{M} k \cdot m}, \tag{18.4}$$

which is also written as

$$\{x_m\} \longleftrightarrow \left\{\tilde{X}_k\right\}.$$

The inverse Fourier transform can be obtained by the analogous relation

$$x_k = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \tilde{X}_m e^{\frac{2\pi i}{M} k \cdot m},$$

Figure Figure 18.1 shows the graph of some periodic function and its Fourier transform.

Expression (18.4) can also be rewritten in the matrix form

$$\vec{\tilde{X}} = \hat{F}\vec{x},$$

where

$$\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{M-1} \end{pmatrix}, \vec{\tilde{X}} = \begin{pmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \vdots \\ \tilde{X}_{M-1} \end{pmatrix},$$

and the matrix $\hat{F}$ has the form

$$\hat{F} = \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\omega} & e^{-2i\omega} & \cdots & e^{-(M-1)i\omega} \\ 1 & e^{-2i\omega} & e^{-4i\omega} & \cdots & e^{-2(M-1)i\omega} \\ 1 & e^{-3i\omega} & e^{-6i\omega} & \cdots & e^{-3(M-1)i\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-(M-1)i\omega} & e^{-2(M-1)i\omega} & \cdots & e^{-(M-1)(M-1)i\omega} \end{pmatrix}, \tag{18.5}$$

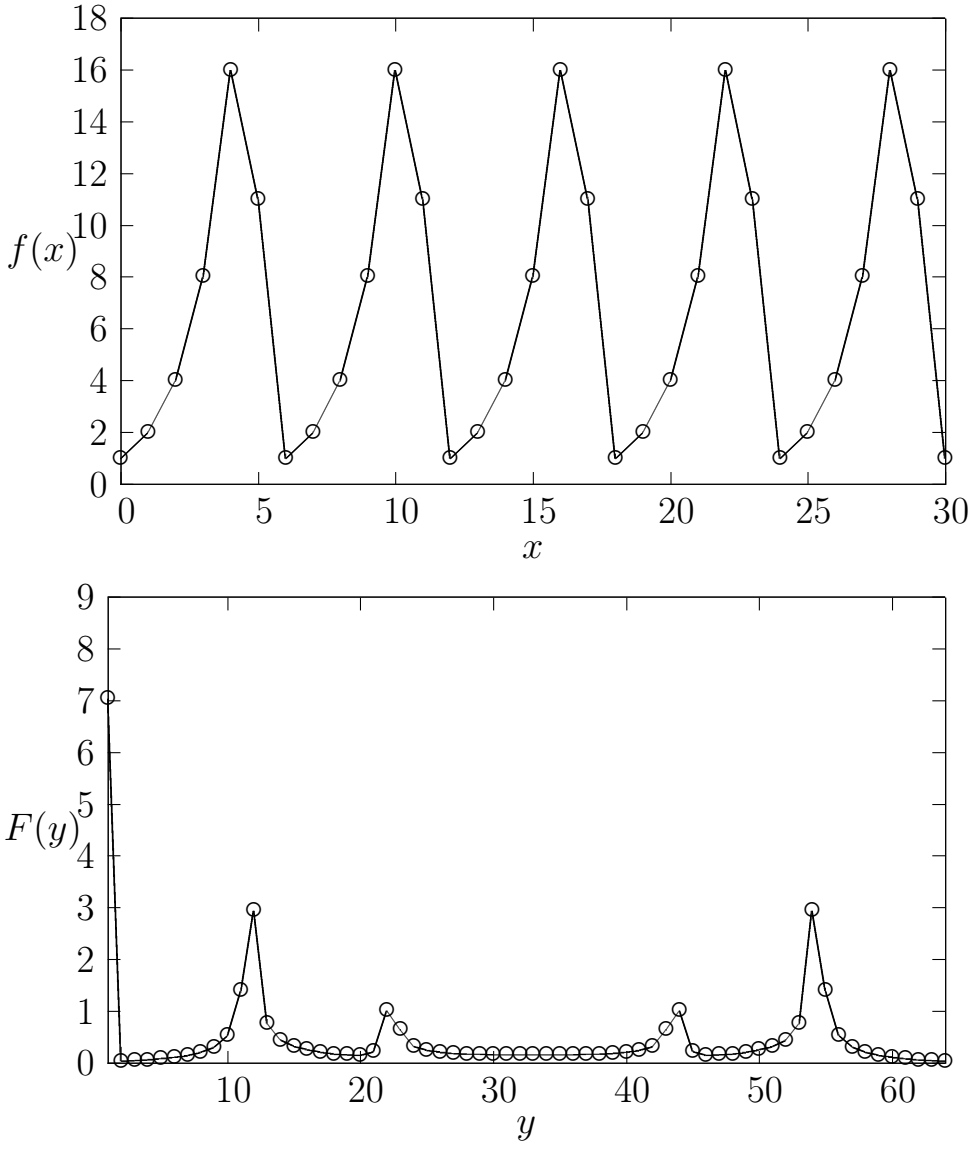


Figure 18.1: Periodic function $f\,(x, a) = a^x \mod N$ at $a = 2$, $N = 21$ (upper graph) and its discrete Fourier transform (lower graph). The period of the original function is $r = 6$. The number of samples is $M = 64$, local maxima with period $T \approx \frac{M}{r} \approx 10.67$ are visible

where

$$\omega = \frac{2\pi}{M}.$$

For a matrix element of the matrix (18.5), one can write

$$F_{nm} = \frac{1}{\sqrt{M}} e^{-i\omega nm}, \tag{18.6}$$

where $n, m \in \{0, 1, \ldots, M-1\}$.

18.2.2 Properties of the Discrete Fourier Transform

The following properties of the Fourier transform should be noted, which play a fundamental role in Shor's algorithm:

Lemma 18.2.1. (Shift) *If* $\{x_n\} \longleftrightarrow \left\{\tilde{X}_k\right\}$ *then* $\left\{x_{(n-m) \mod M}\right\} \longleftrightarrow \left\{e^{-i\omega mk}\tilde{X}_k\right\}$

Proof. For the sequence $\left\{x_{(n-m) \mod M}\right\}$ one can write

$$\left\{x_{(n-m) \mod M}\right\} = \begin{pmatrix} x_{-m \mod M} \\ x_{-m+1 \mod M} \\ \vdots \\ x_{-1 \mod M} \\ x_0 \\ x_1 \\ \vdots \\ x_{M-m-1} \end{pmatrix} = \begin{pmatrix} x_{M-m} \\ x_{M-m+1} \\ \vdots \\ x_{M-1} \\ x_0 \\ x_1 \\ \vdots \\ x_{M-m-1} \end{pmatrix},$$

and

$$\hat{F}\left\{x_{(n-m) \mod M}\right\} =$$

$$= \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\omega} & e^{-2i\omega} & \cdots & e^{-(M-1)i\omega} \\ 1 & e^{-2i\omega} & e^{-4i\omega} & \cdots & e^{-2(M-1)i\omega} \\ 1 & e^{-3i\omega} & e^{-6i\omega} & \cdots & e^{-3(M-1)i\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-(M-1)i\omega} & e^{-2(M-1)i\omega} & \cdots & e^{-(M-1)(M-1)i\omega} \end{pmatrix} \begin{pmatrix} x_{M-m} \\ x_{M-m+1} \\ \vdots \\ x_0 \\ \vdots \\ x_{M-m-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{x_{M-m}}{\sqrt{M}} + \frac{x_{M-m+1}}{\sqrt{M}} + \cdots + \frac{x_0}{\sqrt{M}} + \cdots + \frac{x_{M-m-1}}{\sqrt{M}} \\ \frac{x_{M-m}}{\sqrt{M}} + \frac{e^{-i\omega}x_{M-m+1}}{\sqrt{M}} + \cdots + \frac{e^{-i\omega m}x_0}{\sqrt{M}} + \cdots + \frac{e^{-i\omega(M-1)}x_{M-m-1}}{\sqrt{M}} \\ \frac{x_{M-m}}{\sqrt{M}} + \frac{e^{-2i\omega}x_{M-m+1}}{\sqrt{M}} + \cdots + \frac{e^{-2i\omega m}x_0}{\sqrt{M}} + \cdots + \frac{e^{-2i\omega(M-1)}x_{M-m-1}}{\sqrt{M}} \\ \vdots \\ \frac{x_{M-m}}{\sqrt{M}} + \frac{e^{-mi\omega}x_{M-m+1}}{\sqrt{M}} + \cdots + \frac{e^{-mi\omega m}x_0}{\sqrt{M}} + \cdots + \frac{e^{-mi\omega(M-1)}x_{M-m-1}}{\sqrt{M}} \\ \vdots \end{pmatrix}. \quad (18.7)$$

Taking into account the relation

$$e^{-i\omega kM} = 1, k \in \{0, 1, \dots\},$$

the expression (18.7) can be rewritten in the following form

$$\hat{F}\left\{x_{(n-m) \mod M}\right\} =$$

$$= \frac{1}{\sqrt{M}} \begin{pmatrix} x_{M-m} + \cdots + x_{M-m-1} \\ e^{-i\omega m}e^{-i\omega(M-m)}x_{M-m} + \cdots + e^{-i\omega 2m}e^{-i2\omega(M-m-1)} \\ e^{-i\omega 2m}e^{-i2\omega(M-m)}x_{M-m} + \cdots + e^{-i\omega 2m}e^{-i2\omega(M-1)} \\ \vdots \end{pmatrix} =$$

$$= \begin{pmatrix} \tilde{X}_0 \\ e^{-i\omega m}\tilde{X}_1 \\ e^{-i\omega 2m}\tilde{X}_2 \\ \vdots \end{pmatrix}.$$

□

Lemma 18.2.2. (Periodicity) *If the sequence* $\{x_n\}$ *has period* r*:* $x_n = x_{n+r}$*, and the number of samples* M *is divisible by* r*, then the nonzero terms of the Fourier transform appear with period* $\frac{M}{r}$*.*

Proof. Indeed, if $M \mod r = 0$ and $kr \mod M \neq 0$, then

$$1 - e^{-i\omega kr} \neq 0,$$

whence

$$\tilde{X}_k = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} e^{-i\omega kn} x_n =$$

$$= \frac{1}{\sqrt{M}} \left(\sum_{n=0}^{r-1} e^{-i\omega kn} x_n + \sum_{n=0}^{r-1} e^{-i\omega k(n+r)} x_{n+r} + \right.$$

$$\left. + \sum_{n=0}^{r-1} e^{-i\omega k(n+2r)} x_{n+2r} + \dots \right) =$$

$$= \frac{1}{\sqrt{M}} \left(\sum_{n=0}^{r-1} e^{-i\omega kn} x_n + \sum_{n=0}^{r-1} e^{-i\omega k(n+r)} x_n + \sum_{n=0}^{r-1} e^{-i\omega k(n+2r)} x_n + \dots \right) =$$

$$= \frac{1}{\sqrt{M}} \left(\sum_{n=0}^{r-1} x_n e^{-i\omega kn} \sum_{l=0}^{\frac{M}{r}-1} e^{-i\omega klr} = \sum_{n=0}^{r-1} x_n e^{-i\omega kn} \frac{1 - e^{-i\omega k \frac{M}{r} r}}{1 - e^{-i\omega kr}} \right) =$$

$$= \frac{1}{\sqrt{M}} \frac{1 - e^{-i\omega kM}}{1 - e^{-i\omega kr}} \sum_{n=0}^{r-1} x_n e^{-i\omega kn} = 0. \quad (18.8)$$

If $M \mod r = 0$ and $kr \mod M = 0$, then

$$e^{-i\omega kr} = e^{-i\frac{2\pi}{M} kr} = 1,$$

whence

$$\tilde{X}_k = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} e^{-i\omega kn} x_n =$$

$$= \frac{1}{\sqrt{M}} \left(\sum_{n=0}^{r-1} e^{-i\omega kn} x_n + \sum_{n=0}^{r-1} e^{-i\omega kn} x_{n+r} + \sum_{n=0}^{r-1} e^{-i\omega kn} x_{n+2r} + \dots \right) =$$

$$= \frac{1}{\sqrt{M}} \left(\sum_{n=0}^{r-1} e^{-i\omega kn} x_n + \sum_{n=0}^{r-1} e^{-i\omega kn} x_n + \sum_{n=0}^{r-1} e^{-i\omega kn} x_n + \dots \right) =$$

$$= \frac{1}{\sqrt{M}} \frac{M}{r} \sum_{n=0}^{r-1} e^{-i\omega kn} x_n \neq 0. \quad (18.9)$$

Thus, from expressions (18.8) and (18.9) it follows that the nonzero terms appear with a period $T = \frac{M}{r}$. □

Remark 18.2.1. (Periodicity of maxima) *It is worth noting that expression* (18.8) *(in the case when the period is not a divisor of the number of samples:* $M \bmod r \neq 0$*) will be approximately equal to 0 for those values of* k *that are far from multiples of* $\frac{M}{r}$*, i.e. local maxima of the Fourier transform will repeat with period* $\frac{M}{r}$*.*

18.2.3 Fast Fourier Transform

Calculations using formula (18.4) have complexity of order $O\left(M^2\right)$, where M is the number of elements (samples) [2].

There exists an algorithm for fast calculation using formula (18.4) that has complexity $O\left(M \log M\right)$.

Using the "divide and conquer" paradigm (see section 20.2) one can pay attention to the form of the expression (18.6) and notice that the expression (18.4) can be rewritten as

$$\tilde{X}_k = \sum_{m=0}^{M-1} F_{km}^M x_m,$$

where the notation F_{km}^M indicates that the matrix (18.6) of size $M \times M$ is used. If M is even, then

$$\tilde{X}_k = \sum_{m=0}^{M-1} F_{k,m}^M x_m = \sum_{m=0}^{\frac{M}{2}-1} F_{k,2m}^M x_{2m} + \sum_{m=0}^{\frac{M}{2}-1} F_{k,2m+1}^M x_{2m+1},$$

where

$$F_{k,2m}^M = e^{-i\omega k 2m} = e^{-ikm\frac{2\pi}{\frac{M}{2}}} = F_{k,m}^{\frac{M}{2}},$$
$$F_{k,2m+1}^M = e^{-i\omega k(2m+1)} = e^{-i\omega k} e^{-ikm\frac{2\pi}{\frac{M}{2}}} = e^{-2\pi i\frac{k}{M}} F_{k,m}^{\frac{M}{2}},$$

i.e.

$$\tilde{X}_k = \sum_{m=0}^{\frac{M}{2}-1} F_{k,m}^{\frac{M}{2}} x_{2m} + \exp\left\{\left(-2\pi i\frac{k}{M}\right)\right\} \sum_{m=0}^{\frac{M}{2}-1} F_{k,m}^{\frac{M}{2}} x_{2m+1}. \tag{18.10}$$

The complexity of computations using formula (18.10) is determined by the following relation

$$T\left(M\right) = 2T\left(\frac{M}{2}\right) + O\left(M\right). \tag{18.11}$$

[2] Indeed, it is necessary to obtain M elements, for the computation of each of which requires M multiplication operations

The truth of (18.11) can be verified by noting that the calculations with complexity $T(M)$ in (18.10) split into two subproblems each with complexity $T\left(\frac{M}{2}\right)$.

Using the Master Theorem for recurrence relations (case 2) (Theorem 20.1.1) we obtain

$$T(M) = O(M \log M).$$

18.3 Two-Dimensional Fourier Transform

18.3.1 Definition

Definition 18.3.1 (Two-Dimensional Fourier Transform)**.** Suppose we have a two-dimensional signal $x(k_1, k_2)$, where $k_1, k_2 \in \{0, \ldots, M-1\}$. The two-dimensional Fourier transform is called a two-dimensional function

$$\tilde{X}(j_1, j_2), j_1, j_2 \in \{0, \ldots, M-1\}$$

such that

$$\tilde{X}(j_1, j_2) = \frac{1}{M} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{M-1} x(k_1, k_2)\, e^{-i\omega(k_1 j_1 + k_2 j_2)},$$

where

$$\omega = \frac{2\pi}{M}.$$

Chapter 19

Discrete Mathematics

19.1 Greatest Common Divisor. Euclidean Algorithm

Definition 19.1.1. The greatest common divisor of numbers a and b (GCD (a, b)) is called the largest of their common divisors.

Theorem 19.1.1 (Theorem on common divisors)**.** *Suppose the following inequalities hold $a > b > 0$, and the number r is the remainder of the division of a by b. Thus, we can write*

$$a = x \cdot b + r, \tag{19.1}$$

where $x \geq 1$, $b > r \geq 0$. If $r = 0$, then b is the largest number that divides both a and b without remainder. In the case when $r > 0$, then

$$GCD\,(a, b) = GCD\,(b, r)\,. \tag{19.2}$$

Proof. To prove (19.2), we show that any divisor of the pair of numbers (a, b) is a divisor of the pair of numbers (b, r). Let d be a common divisor of numbers a and b, i.e. $a = d \cdot x_1$, $b = d \cdot x_2$. Thus, from (19.1) it follows

$$r = a - x \cdot b = d \cdot (x_1 - x \cdot x_2)\,,$$

i.e. d is a divisor of the number r.

Now we prove that any common divisor of numbers b and r is also a common divisor of numbers a and b: indeed, let d be a common divisor of numbers b and r, i.e. $b = y_1 \cdot d$ and $r = y_2 \cdot d$, thus (19.1) can be rewritten as

$$a = x \cdot y_1 \cdot d + y_2 \cdot d = d \cdot (x \cdot y_1 + y_2)\,,$$

i.e. d is a divisor of the number a.

Thus, the pairs of numbers (a, b) and (b, r) have common divisors, including the greatest divisor for one pair is also such for the other. □

Relation (19.2) leads to the following algorithm for computing the greatest common divisor

Algorithm 3 Euclidean Algorithm

$a > b$
if $b = 0$ **then**
  return a
end if
$a \Leftarrow 0$
$r \Leftarrow b$
$b \Leftarrow a$
repeat
  $a \Leftarrow b$
  $b \Leftarrow r$
  $r \Leftarrow$ remainder of division of a by b
until $(r \neq 0)$
return b

To estimate the complexity of algorithm 3, we write it in the following form

$$\begin{aligned} f_k &= x_k \cdot f_{k+1} + f_{k+2}, \\ f_0 &= a,\ f_1 = b, \\ x_k \geq 1,\ f_k &> f_{k+1} > f_{k+2}, \end{aligned}$$

thus $f_k > 2 \cdot f_{k+2}$, or else $f_0 > 2 \cdot f_2 > \cdots > 2^n f_{2n}$ i.e. the algorithm stops at $n = \log_2(f_0) = \log_2(a)$ [1]. The number of steps of the algorithm is obviously equal to $2n$ or $2 \cdot \log_2(a)$. Thus the algorithmic complexity of the Euclidean algorithm can be written as $O(\log(a))$.

Example 19.1.1. (GCD $(2345, 1456)$)

$$\begin{aligned} 2345 &= 1456 + 889, \\ 1456 &= 889 + 567, \\ 889 &= 567 + 322, \\ 567 &= 322 + 245, \\ 322 &= 245 + 77, \\ 245 &= 3 \cdot 77 + 14, \\ 77 &= 5 \cdot 14 + 7, \\ 14 &= 2 \cdot 7. \end{aligned}$$

[1] The following derivations are made assuming that $\log_2(a)$ is an integer

thus $GCD\,(2345, 1456) = 7$. *The number of algorithm steps is* $8 < 2 \cdot \log_2 2345 \approx 2 \cdot 11.2 = 22.4$.

19.1.1 Bezout's Identity

Theorem 19.1.2 (Bezout)**.** *If numbers* a *and* b *are coprime, then the equation*

$$ax + by = 1$$

has integer solutions.

Proof. We use the Euclidean algorithm 3 to find $\mathrm{GCD}\,(a, b)$. Assuming $a > b$, then

$$\begin{aligned} r_1 &= a - bq_0, \\ r_2 = b - r_1 q_1 = b - (a - bq_0)q_1 &= b(1 + q_1 q_0) - aq_1, \\ r_3 = r_1 - r_2 q_2 = a - bq_0 - (b(1 + q_1 q_0) - aq_1)\, q_2 &= \\ = a(1 + q_1 q_2) - b(q_0 + q_2 + q_0 q_1 q_2), & \\ \dots & \\ \mathrm{GCD}\,(a, b) = r_n = r_{n-2} - r_{n-1} q_{n-1} = \dots = ax + by, & \end{aligned} \tag{19.3}$$

which proves our statement. □

Remark 19.1.1 (On the complexity of computing Bezout's identity)**.** *Calculations by* (19.3) *are equivalent to the steps in the algorithm 3. The number of these steps is* $O\,(\log_2(a))$*, thus the algorithm described by equations* (19.3) *is sufficiently efficient and has complexity* $O\,(\log_2(a))$.

Example 19.1.2 (Bezout's Identity)**.** *Suppose* $a = 25, b = 14$. *The formulas of the Euclidean algorithm are*

$$\begin{aligned} 11 &= 25 - 14 \cdot 1, \\ 3 &= 14 - 11 \cdot 1, \\ 2 &= 11 - 3 \cdot 3, \\ 1 &= 3 - 2 \cdot 1. \end{aligned}$$

From these formulas we get

$$\begin{aligned} 11 &= 25 - 14 \cdot 1, \\ 3 = 14 - 25 + 14 &= 2 \cdot 14 - 25, \\ 2 = 11 - 3 \cdot 3 = 25 - 14 - 3 \cdot (2 \cdot 14 - 25) &= 4 \cdot 25 - 7 \cdot 14, \\ 1 = 3 - 2 \cdot 1 &= 9 \cdot 14 - 5 \cdot 25. \end{aligned}$$

Thus,

$$25 \cdot (-5) + 14 \cdot 9 = 1.$$

19.2 Modular Arithmetic

Definition 19.2.1. The notation

$$a \equiv b \mod c \tag{19.4}$$

means that a and b have the same remainders when divided by c or that a and b are congruent modulo the natural number c. Here, the number c is called the modulus of the congruence.

Definition 19.4 can also be interpreted as saying that the difference $a - b$ is divisible by c.

Example 19.2.1. Modular Congruence: $30 \equiv 8 \mod 11$, *because* $30 = 2 \cdot 11 + 8$.

Definition 19.2.2 (Negative Element)**.** If $a < n$, then $n - a$ will be called the negative element relative to a and denoted by $-a \mod n$.

Example 19.2.2. Negative Element:

$$-5 \equiv 6 \mod 11,$$

since $5 < 11$ *and* $6 = 11 - 5$.

19.2.1 Arithmetic Operations

Lemma 19.2.1 (Addition modulo)**.** *If* $a_1 \equiv a_2 \mod n, b_1 \equiv b_2 \mod n$, *then*

$$a_1 + b_1 \equiv a_2 + b_2 \mod n$$

Proof. We can write $a_1 = k_1 n + r_a, a_2 = k_2 n + r_a, b_1 = l_1 n + r_b, b_2 = l_2 n + r_b$ from which

$$a_1 + b_1 = (k_1 + l_1)n + r_a + r_b \equiv r_a + r_b \mod n$$

and

$$a_2 + b_2 = (k_2 + l_2)n + r_a + r_b \equiv r_a + r_b \mod n$$

hence

$$a_1 + b_1 \equiv a_2 + b_2 \equiv r_a + r_b \mod n$$

□

Lemma 19.2.2 (Multiplication modulo)**.** *If* $a_1 \equiv a_2 \mod n, b_1 \equiv b_2 \mod n$, *then*

$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \mod n$$

Proof. If $a_1 \equiv a_2 \mod n, b_1 \equiv b_2 \mod n$, then

$$a_1 + b_1 \equiv a_2 + b_2 \mod n$$

We can write $a_1 = k_1 n + r_a, a_2 = k_2 n + r_a, b_1 = l_1 n + r_b, b_2 = l_2 n + r_b$ from which

$$a_1 \cdot b_1 = k_1 l_1 n + l_1 n r_a + k_1 n r_b + r_a r_b \equiv r_a r_b \mod n$$

and

$$a_2 \cdot b_2 = k_2 l_2 n + l_2 n r_a + k_2 n r_b + r_a r_b \equiv r_a r_b \mod n$$

hence

$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \equiv r_a r_b \mod n$$

□

19.2.2 Solving Equations

In cryptography, one often deals with equations of the form

$$ax \equiv b \mod n, \tag{19.5}$$

where a, b, n are known integers, and x is an unknown parameter to be determined.

Obviously, if we find an integer a^{-1} such that

$$aa^{-1} \equiv 1 \mod n,$$

then

$$x \equiv ba^{-1} \mod n.$$

If $\gcd(a, n) = 1$, then according to Bézout's identity (see Bézout's theorem (Theorem 19.1.2)) there exist integers x, y such that $ax + ny = 1$, i.e.

$$x \equiv a^{-1} \mod n.$$

Moreover, according to remark 19.1.1, a^{-1} and the solution of equation (19.5) can be found quite efficiently.

19.2.3 The Field $\mathbb{F}_p$

As we have seen, in modular arithmetic one can add, subtract, multiply, and even divide if the modulo is a prime number. Moreover, addition and multiplication are commutative and satisfy the distributive property. Thus, the remainders modulo a prime form a field (see definition 16.2.1) which is called the Galois field and denoted by $\mathbb{F}_p$.

19.3 Euler's Function

19.3.1 Definition

Definition 19.3.1 (Euler's Function)**.** Euler's function $\phi(n)$ shows how many numbers $k \in \{1, ...n-1\}$ are coprime with n, i.e. $\text{GCD}(k, n) = 1$.

Example 19.3.1 (Euler's Function)**.** *If we take the number $n = 15$, there are 8 numbers coprime with 15: $1, 2, 4, 7, 8, 11, 13, 14$. The remaining 7 numbers are not coprime with n since they have a greatest common divisor different from 1, for example $GCD(6, 15) = 3$. Thus, $\phi(15) = 8$.*

19.3.2 Properties

Property 19.3.1 (Euler's Function of a Prime Number)**.** *If p is a prime number, then $\phi(p) = p - 1$*

Proof. Follows from definition 19.3.1. □

Property 19.3.2 (Euler's Function of a Product (Generalized Multiplicativity))**.** *If $GCD(n, m) = 1$, then $\phi(n \cdot m) = \phi(n)\phi(m)$*

Proof. TBD □

Remark 19.3.1 (On the Complexity of Computing Euler's Function)**.** *Computing Euler's function for large numbers is a very complex task. Most often, property 19.3.1 is used in combination with 19.3.2 for computation. Applying these properties to an arbitrary number requires its factorization, so the complexity of computing Euler's function is comparable to the complexity of the factorization problem.*

19.4 Fermat's Little Theorem

Theorem 19.4.1 (Fermat's Little Theorem)**.** *If p is a prime number, and a is not divisible by p, then*

$$a^{p-1} \equiv 1 \mod p \tag{19.6}$$

Proof. Consider the following relation

$$a \cdot k_i \mod p,$$

where $k_i \subset \{1, \ldots, p-1\}$.

Obviously,

$$a \cdot k_i \equiv k_j \mod p. \tag{19.7}$$

Indeed,

$$a \cdot k_i \mod p \subset \{1, \ldots, p-1\},$$

since any remainder from division by p takes values $0, 1, \ldots, p-1$. A zero remainder is impossible because a and p are coprime.

Moreover, each of the remainders $a \cdot k_i \mod p$ occurs only once. Indeed, suppose that $a \cdot k_i \mod p = a \cdot k_j \mod p$ or $a \cdot (k_i - k_j) \equiv 0 \mod p$, i.e. a is divisible by p, which contradicts the condition that they are coprime.

Multiplying all expressions Equation 19.7 we get

$$a \cdot 2a \cdot 3a \cdot \cdots \cdot a\,(p-1) \equiv 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (p-1) \mod p.$$

or

$$a^{p-1}\,(p-1)! \equiv (p-1)! \mod p,$$

from which we obtain the required equality due to the coprimality of p and $(p-1)!$:

$$a^{p-1} \equiv 1 \mod p$$

□

19.4.1 Pseudoprime numbers

The generalization of Fermat's Little Theorem is not true, i.e., if a and p are coprime numbers satisfying the relation Equation 19.6, then p may not be prime. For example,

$$2^{341-1} \equiv 1 \mod 341,$$

even though $341 = 11 \cdot 31$.

Numbers p satisfying the relation Equation 19.6, but not being prime, are called pseudoprime numbers to the base a. For example, 341 is the first pseudoprime number to base 2.

19.5 Chinese Remainder Theorem

Theorem 19.5.1. *If there are pairwise coprime integers $n_1, n_2, \ldots, n_k$, then for any set of integers $a_1, a_2, \ldots a_k$ there exists an x such that*

$$\begin{aligned} x &\equiv a_1 \mod n_1, \\ x &\equiv a_2 \mod n_2, \\ &\vdots \\ x &\equiv a_k \mod n_k, \end{aligned} \tag{19.8}$$

Moreover, for any x_1, x_2 satisfying this relation the following equality holds

$$x_1 \equiv x_2 \mod N,$$

where $N = n_1 \cdot n_2 \cdot \dots \cdot n_k$.

19.6 RSA Algorithm

The RSA algorithm (an acronym for the surnames Rivest, Shamir, and Adleman) is a public-key encryption algorithm [2], based on the difficulty of factoring a number into prime factors.

19.6.1 Key Generation

Consists of several steps

- Two prime numbers p and q are chosen
- The product of the chosen prime numbers is calculated $n = p \cdot q$
- The Euler function (19.3.1) (is calculated) (see properties 19.3.1 and 19.3.2) $\phi(n) = (p-1)(q-1)$
- An integer e is chosen such that $1 < e < \phi(n)$ and e and $\phi(n)$ are coprime, i.e. $\gcd(e, \phi(n)) = 1$.
- Compute $d \equiv e^{-1} \mod \phi(n)$, i.e. $d \cdot e \equiv 1 \mod \phi(n)$.

The public key consists of two numbers: the modulus n and the public exponent e. These two numbers are used to encrypt the original message.

The private key also consists of two numbers: the modulus n and the private exponent d.

The original numbers p and q are kept secret, since they make computing $\phi(n)$ trivial.

It should be noted that to obtain the private key from the public key, one must compute $\phi(n)$ given n. This is a difficult problem (if p and q are unknown), as noted in remark 19.3.1.

Example 19.6.1. (RSA. Key Generation) *We choose two prime numbers $p = 3$ and $q = 7$. The product of these numbers is $n = 21$. Euler's function (19.3.1) $\phi(n) = (p-1)(q-1) = 2 \cdot 6 = 12$.*

[2] A public-key (asymmetric) encryption algorithm is one in which two different keys are used: one for encryption and the other for decryption

We select a number e *(public exponent) such that* $1 < e < 12$ *and* $\gcd(e, 12) = 1$. *Obviously,* $e = 5$ *satisfies these conditions.*

We compute the private exponent $d \equiv 5^{-1} \mod 12$, *i.e.* $d = 5$. *Indeed,* $5 \cdot 5 = 25 = 2 \cdot 12 + 1$, *so* $5 \cdot 5 \equiv 1 \mod 12$.

Thus, we get

- *Public key* $(n = 21, e = 5)$
- *Private key* $(n = 21, d = 5)$

19.6.2 Encryption

Suppose we need to encrypt a certain message M. First, it is converted into an integer (or integers) m such that $0 < m < \phi(n)$. Then the ciphertext c is computed as:

$$c \equiv m^e \mod n \tag{19.9}$$

Example 19.6.2. (RSA. Encryption) *Suppose we have a public key* $(n = 21, e = 5)$ *(see example 19.6.1) and we want to encrypt the following message* $m = 1101_2 = 11_{10}$. *The ciphertext is computed using formula (19.9) as* $c \equiv 11^5 \mod 21 = 2$.

19.6.3 Decryption

m can be recovered from c using the following formula:

$$m \equiv c^d \mod n. \tag{19.10}$$

Having m, the original message M can be restored.

Example 19.6.3. (RSA. Decryption) *Suppose we have a private key* $(n = 21, d = 5)$ *(see example 19.6.1) and ciphertext* $c = 2$ *from example 19.6.2.*

The original text is calculated by formula (19.10) $m \equiv 2^5 \mod 21 = 11 = 1101_2$.

19.6.4 Proof

We want to prove that

$$(m^e)^d \equiv m \mod p \cdot q$$

for any positive number m when p and q are prime numbers, and e and d satisfy the relation

$$d \cdot e \equiv 1 \mod \phi(p \cdot q),$$

which can be rewritten as

$$d \cdot e - 1 = h\,(p-1)\,(q-1)\,.$$

Thus,

$$m^{e \cdot d} = m \cdot m^{h(p-1)(q-1)}.$$

Next, two cases are possible: when m is divisible by p, and when m and p are coprime.

In the first case,

$$m^{e \cdot d} \equiv m \equiv 0 \mod p.$$

In the second case, we use Fermat's little theorem (Theorem 19.4.1) :

$$m \cdot m^{h(p-1)(q-1)} = m\left(m^{p-1}\right)^{h(q-1)} \equiv m \cdot 1^{h(q-1)} \equiv m \mod p.$$

Similarly, we have either

$$m^{e \cdot d} \equiv m \equiv 0 \mod q$$

or, by Fermat's little theorem,

$$m \cdot m^{h(p-1)(q-1)} = m\left(m^{q-1}\right)^{h(p-1)} \equiv m \cdot 1^{h(p-1)} \equiv m \mod q.$$

Thus, we have the following two types of congruences: the trivial equalities

$$\begin{aligned} x_1 = m \equiv m \mod p, \\ x_1 = m \equiv m \mod q, \end{aligned}$$

and the just obtained congruences

$$\begin{aligned} x_2 = m^{ed} \equiv m \mod p, \\ x_2 = m^{ed} \equiv m \mod q, \end{aligned}$$

from which, by the Chinese remainder theorem (Theorem 19.5.1) , we get

$$x_1 \equiv x_2 \mod p \cdot q,$$

i.e.

$$m^{e \cdot d} \equiv m \mod n.$$

19.7 Discrete Logarithm

Definition 19.7.1 (Discrete Logarithm)**.** Consider some abelian multiplicative group G and the equation

$$g^x = a \tag{19.11}$$

A solution of this equation, i.e., a non-negative integer x satisfying the equality (19.11) is called a discrete logarithm.

The equation (19.11) does not, in general, have a solution for any value of a. But in the case when g is a generating element of G, i.e., $G = \langle g \rangle$, then the solution always exists and it is unique. Further, when speaking about the discrete logarithm, we will assume that g is chosen such that $G = \langle g \rangle$.

In applied cryptography, a special kind of discrete logarithm in the ring of residues modulo p is often used:

Definition 19.7.2 (Discrete Logarithm in the Ring of Residues modulo p)**.** The discrete logarithm $ind_g(a) \mod p$ [3] is called the minimal number x that satisfies the following equation (if such a number exists):

$$g^x \equiv a \mod p \tag{19.12}$$

Example 19.7.1. ($ind_3 14 \mod 17$) *Let us solve the problem by brute force [28]:*

$$\begin{array}{lllll} 3^1 \equiv 3 \mod 17, & 3^2 \equiv 9 \mod 17, & 3^3 \equiv 10 \mod 17, & 3^4 \equiv 13 \mod 17, & \\ 3^5 \equiv 5 \mod 17, & 3^6 \equiv 15 \mod 17, & 3^7 \equiv 11 \mod 17, & 3^8 \equiv 16 \mod 17, & \\ 3^9 \equiv 14 \mod 17, & 3^{10} \equiv 8 \mod 17, & 3^{11} \equiv 7 \mod 17, & 3^{12} \equiv 4 \mod 17, & \\ 3^{13} \equiv 12 \mod 17, & 3^{14} \equiv 2 \mod 17, & 3^{15} \equiv 6 \mod 17, & 3^{16} \equiv 1 \mod 17, & \end{array}$$

thus, one can see that $ind_3 14 \mod 17 = 9$, *since* $3^9 \equiv 14 \mod 17$.

The problem of finding the discrete logarithm is a hard problem. The fastest known algorithm [10] solves it in time approximately $O\left(c \cdot \exp\left(\log p^{\frac{1}{3}} \log\log p^{\frac{2}{3}}\right)\right)$, where c is some constant, which explains the widespread use of algorithms that utilize the discrete logarithm in cryptography.

19.8 Diffie-Hellman Protocol (Diffie-Hellman, DH)

Suppose there are two participants Alice and Bob. They know two numbers g and p, which are not secret.

[3] From the word **ind**ex - an alternative name for the discrete logarithm

Alice chooses a random number a and sends Bob the following value

$$A \equiv g^a \mod p. \tag{19.13}$$

Bob computes the following number (using a secret random value b)

$$B \equiv g^b \mod p. \tag{19.14}$$

Alice, using only the number a known to her, computes the key

$$K \equiv B^a \mod p \equiv g^{ab} \mod p. \tag{19.15}$$

Bob can obtain the same key value using his secret number b:

$$K \equiv A^b \mod p \equiv g^{ab} \mod p. \tag{19.16}$$

Thus Alice and Bob obtain the same key, which can later be used to transmit messages using symmetric encryption algorithms (for example AES).

Example 19.8.1. (Diffie-Hellman) *Initial data (public information):* $g = 2$, $p = 23$. *Alice chooses a random number* $a = 6$ *and computes by formula* (19.13) *the number* $A = 18$ *and sends it to Bob. Bob chooses a random number* $b = 9$ *and, using formula* (19.14), *obtains* $B = 6$ *and sends this number to Alice.*

Alice calculates the key $K = 12$ *by formula* (19.15). *Bob can also obtain the same key value* $K = 12$ *using* (19.16)

An adversary Eve knows the numbers g, p, A and B. To obtain the key K, Eve needs to obtain one of the secret numbers a or b:

$$a \equiv ind_g(A) \mod p,$$

from which, using (19.15), the sought value K is obtained.

19.9 Elgamal Scheme

One of the variations of the Diffie-Hellman protocol is the Elgamal scheme. It is important to distinguish between the Elgamal encryption algorithm and the Elgamal digital signature algorithm. The Elgamal digital signature underlies the US Digital Signature Algorithm (DSA) and the Russian standard (GOST R 34.10-94).

Below we will consider the algorithm in encryption mode.

19.9.1 Key Generation

- Generate a prime number p.
- Choose an integer g.
- Choose a random integer $x : 1 < x < p$.
- Compute $y = g^x \mod p$.

The public key is the triple p, g, y. The private key is the number x.

Example 19.9.1 (Key Generation (Elgamal)). *Choose* $p = 21, g = 10, x = 3$. $y = 10^3 \mod 21 = 13$.

19.9.2 Encryption

The message to be encrypted M must satisfy $0 < M < p$.

- Choose a session key — a random number $k : 1 < k < p - 1$.
- Compute $a = g^k \mod p$.
- Compute $b = y^k M \mod p$.

The pair of numbers (a, b) is considered the ciphertext.

Example 19.9.2 (Encryption (Elgamal)). *Suppose we want to encrypt* $M = 6$. *Choose* $k = 7$. *Then* $a = 10^7 \mod 21 = 10$ *and* $b = 13^7 \cdot 6 \mod 21 = 15$.

19.9.3 Decryption

Knowing the private key x, we can recover the original message by

$$M = b \cdot (a^x)^{-1} \mod p, \tag{19.17}$$

indeed, since

$$(a^x)^{-1} = g^{-kx} \mod p,$$

we have

$$b \cdot (a^x)^{-1} = y^k M g^{-kx} = g^{kx} M g^{-kx} = M \mod p.$$

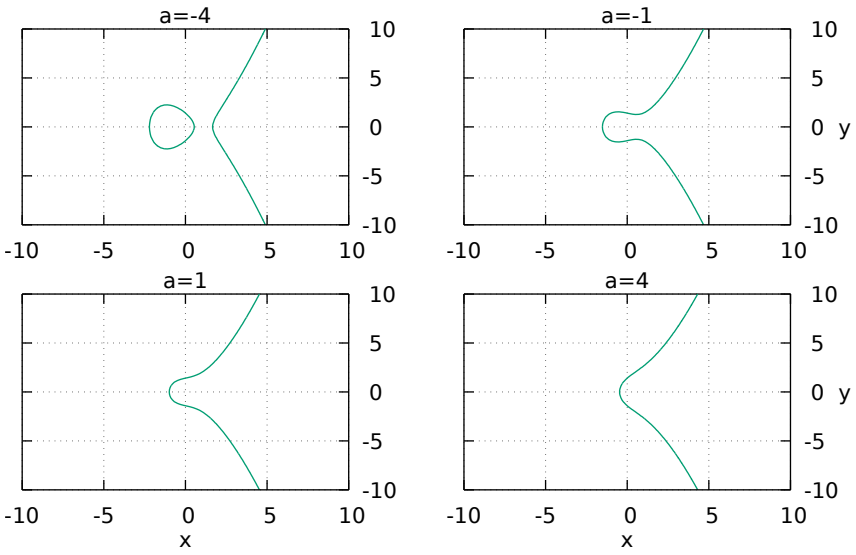


Figure 19.1: Elliptic curve $y^2 = x^3 + ax + 2$ over the field of real numbers $\mathbb{R}$ for different values of the parameter a

Example 19.9.3 (Decryption (Elgamal))**.** *The encrypted message from example 19.9.3 is* $C = (a = 10, b = 15)$*. Using* (19.17) *we have*

$$M = 15 \cdot 13 \mod 21 = 6,$$

where

$$\left(10^3\right)^{-1} \equiv 13 \mod 21,$$

since $13 \cdot 10^3 = 1 \mod 21$*. Thus, we have recovered the message* $M = 6$ *encrypted in example 19.9.3.*

19.10 Elliptic Cryptography

19.10.1 Elliptic Curves over the Field $\mathbb{R}$

In elliptic cryptography, certain sets of objects that form a group are considered (see section 16.1). As such a set, we will consider points belonging to a certain curve (see Figure 19.1):

$$E : y^2 = x^3 + ax + b,$$

where the coefficients a, b must satisfy the following relation

$$4a^3 + 27b^2 \neq 0,$$

in this case the cubic equation $x^3 + ax + b = 0$ will have 3 distinct real roots [26].

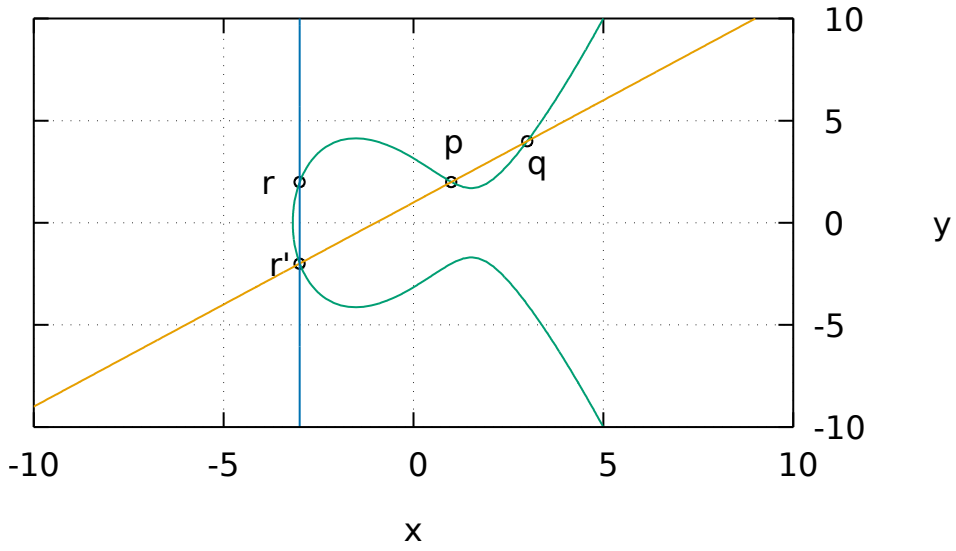


Figure 19.2: Elliptic curve $y^2 = x^3 - 7x + 10$ over the field of real numbers $\mathbb{R}$. Addition of two points $p(1, 2)$ and $q(3, 4)$. The line passing through these points intersects the curve at a third point $r'(-3, -2)$. The point $r(-3, 2)$ symmetric to r' with respect to the curve is the sum of the original two: $p + q = r$

According to definition 16.1.1, for points on the curve there is a certain binary operation which assigns a third point r with coordinates (r_x, r_y) to two points p, q with coordinates (p_x, p_y) and (q_x, q_y), respectively. We will call this operation addition:

$$p + q = r.$$

There is a simple geometric interpretation of the addition operation (see Figure 19.2). Suppose there are 2 points on the curve that we want to add: p and q with coordinates $(x_p, y_p), (x_q, y_q)$ respectively. If $x_p \neq x_q$, then through these points one can draw a straight line with slope

$$m = \frac{y_p - y_q}{x_p - x_q}.$$

and intersects the curve at a point r'. If this point has coordinates $(x_{r'}, y_{r'})$, then since it lies on the line with slope m:

$$m = \frac{y_{r'} - y_p}{x_{r'} - x_p},$$

therefore

$$y_{r'} = y_p + m\left(x_{r'} - x_p\right).$$

This point must belong to the curve, i.e.

$$y_{r'}^2 = \left(y_p + m\left(x_{r'} - x_p\right)\right)^2 = x_{r'}^3 + a x_{r'} + b$$

which can be rewritten as

$$x_{r'}^3 - m^2 x_{r'}^2 + \cdots = 0.$$

The equation $x^3 - m^2 x^2 + \cdots = 0$ has 3 roots: $x_p, x_q, x_{r'}$, i.e., it can also be rewritten as

$$(x - x_{r'})(x - x_p)(x - x_q) = x^3 - (x_{r'} + x_p + x_q)x^2 + \cdots = 0.$$

Thus,

$$x_{r'} + x_p + x_q = m^2.$$

Therefore

$$\begin{aligned} x_{r'} &= m^2 - x_p - x_q, \\ y_{r'} &= y_p + m\left(x_{r'} - x_p\right), \end{aligned}$$

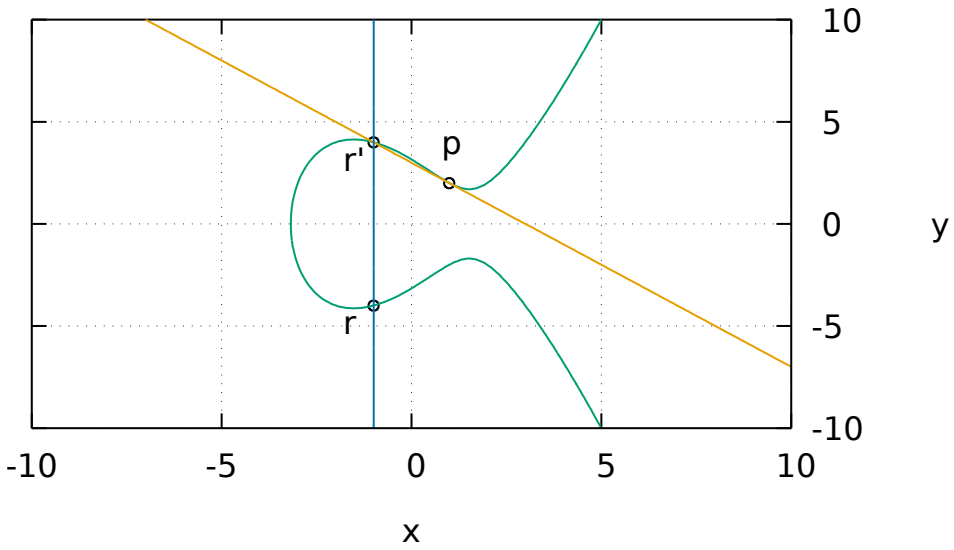


Figure 19.3: Elliptic curve $y^2 = x^3 - 7x + 10$ over the field of real numbers $\mathbb{R}$. Addition of two points with the same coordinates $p(1, 2)$: $2p = r(-1, -4)$

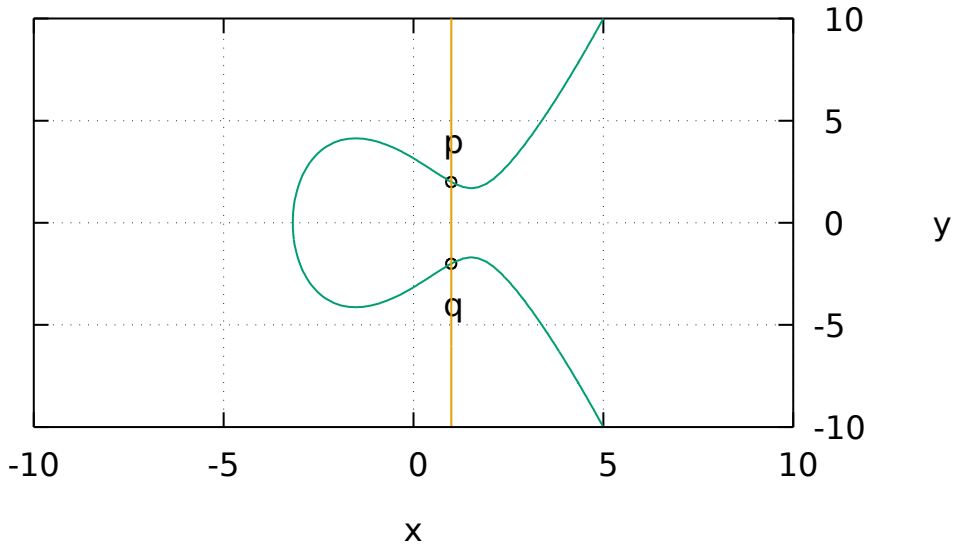


Figure 19.4: Elliptic curve $y^2 = x^3 - 7x + 10$ over the field of real numbers $\mathbb{R}$. Addition of two points $p(1, 2)$ and $q(1, -2)$. The line passing through these points does not intersect the curve. The result of the addition is the zero element: $p + q = 0$, i.e., $q = -p$

Reflecting this point relative to the X-axis we get the final point r which we will call the sum of the original points (see Figure 19.2). The coordinates of this point x_r, y_r can be obtained by the following formulas

$$\begin{aligned} x_r &= m^2 - x_p - x_q, \\ y_r &= -y_p + m\left(x_p - x_r\right). \end{aligned} \tag{19.18}$$

In the case $x_p = x_q$ there are two possibilities:

1. $y_p = y_q$ (see Figure 19.3). When the points on the curve approach each other, the straight line drawn through them tends to the tangent line. The coefficient m can be found by the following formula: $m = \frac{dy}{dx}$. Taking into account $2ydy = 3x^2dx + adx$ we have $m = \frac{dy}{dx} = \frac{3x^2+a}{2y}$. Further calculation is carried out using formulas (19.18).

2. $y_p \neq y_q$ (see Figure 19.4). In this case, due to the symmetry of the curve with respect to the X-axis, only one possibility exists: $y_p = y_q$ and the line passing through these two points does not intersect the curve at a third point. For this case, an additional point 0 is introduced, where the line passing through the two points goes. Thus, in this case we have $p + q = 0, q = -p$.

Thus, we can define an elliptic curve over the field of real numbers $\mathbb{R}$ as the following set of points

$$E\left(\mathbb{R}\right) = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : y^2 = x^3 + ax + b, \right\} \cup \{0\}. \tag{19.19}$$

where $a, b \in \mathbb{R}$.

For these points, a binary operation is defined, which we called addition. There is a zero element on the set, and for each element an inverse element can be defined. It can be proven that the introduced operation is associative: $(a + b) + c = a + (b + c)$ [26]. Thus, the set $E\left(\mathbb{R}\right)$ forms a group with respect to addition. Considering the obvious equality $p + q = q + p$, this group will be commutative, i.e., Abelian (see definition 16.1.3).

Remark 19.10.1 (On the Addition Operation)**.** *The introduced addition operation for points on an elliptic curve initially looks unnatural; why not use more obvious operations of addition of points on the plane, for example, vector addition rules. In this case, if $p, q \in E\left(\mathbb{R}\right)$, then it is quite possible that $p + q \notin E\left(\mathbb{R}\right)$ and, consequently, the main property of the group — closure — would be violated. Meanwhile, for the binary operation (19.18) defined by us, all group properties hold, and accordingly, all properties following from this, such as Lagrange's theorem (see theorem 16.1.2), which can be applied to the introduced objects.*

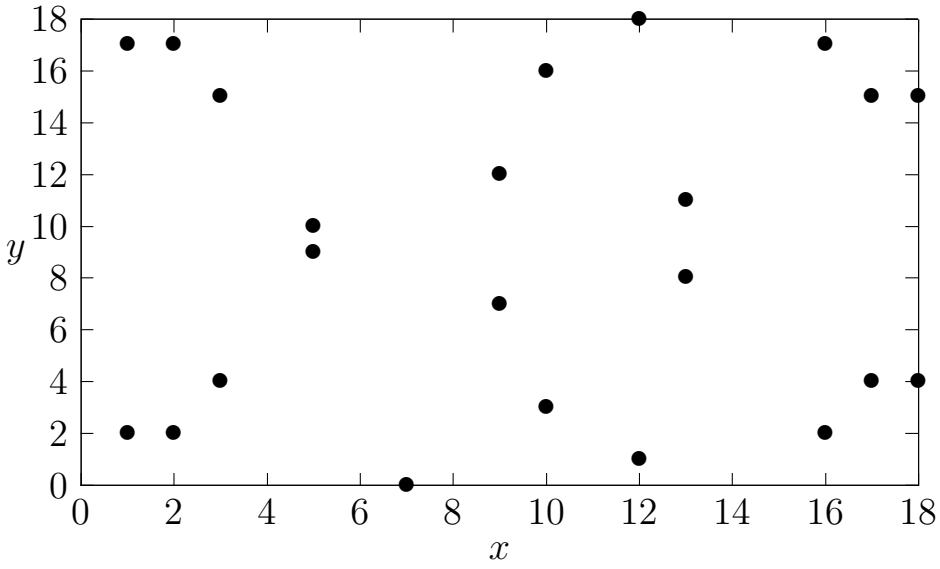


Figure 19.5: Elliptic curve $y^2 = x^3 - 7x + 10$ over the field $\mathbb{F}_{19}$

19.10.2 Elliptic Curves over the Field $\mathbb{F}_p$

The set (19.19) together with the addition operation (19.18) can be defined over an arbitrary field (see definition 16.2.1), , i.e., not only over $\mathbb{R}$. From the cryptographic point of view, special interest lies in the field $\mathbb{F}_p$ (see subsection 19.2.3). It is possible to define the set of elements of an elliptic curve over the field $\mathbb{F}_p$ similarly to expression (19.2.3):

$$E\left(\mathbb{F}_p\right) = \left\{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : y^2 \equiv x^3 + ax + b, \right\} \cup \{0\}. \tag{19.20}$$

where $a, b \in \mathbb{F}_p$.

Definition 19.10.1 (Order of an Elliptic Curve)**.** The number of points of an elliptic curve E is called its order and is denoted as $|E|$

Figure 19.5 shows such a set for the field $\mathbb{F}_{19}$, i.e., $p = 19$. The curve equation is $y^2 \equiv x^3 - 7x + 10 \mod 19$.

For each point a with coordinates x_a, y_a, the inverse element $-a$ is defined with coordinates $x_{-a} = x_a, y_{-a} \equiv -y_a \mod p$.

For points on this curve, the following addition law is defined $a+b = c, b \neq -a$

$$\begin{aligned} x_c &\equiv m^2 - x_a - x_b \mod p, \\ y_c &\equiv -y_a + m\left(x_a - x_c\right) \mod p, \end{aligned} \tag{19.21}$$

where $(x_{a,b,c}, y_{a,b,c})$ are the coordinates of points a, b and c respectively. For the coefficient m, the following relations are used:

$$\begin{aligned} m &= \left(y_a - y_b\right)\left(x_a - x_b\right)^{-1} \mod p, \text{ if } x_a \neq x_b \\ m &= \left(3x^2 + a\right)\left(2y\right)^{-1} \mod p, \text{ if } x_a = x_b. \end{aligned}$$

Obviously, if $b = -a$, then $a + b = a + (-a) = 0$.

19.10.3 Scalar Multiplication and Discrete Logarithm

Assume n is some natural number $n \in \mathbb{N}$ and $a \in E(\mathbb{F}_p)$. Define scalar multiplication as follows:

$$n \cdot a = a + a + \cdots + a = \sum_{k=1}^{n} a$$

A naive implementation requires $O(n)$ addition operations, but using the divide and conquer paradigm (see section 20.2) it can be reduced to $O(\log n)$ addition operations.

Example 19.10.1 (Scalar Multiplication)**.** *Consider the elliptic curve*

$$E(\mathbb{F}_{19}) = \{(x, y) \in \mathbb{F}_{19} \times \mathbb{F}_{19} : y^2 \equiv x^3 - 7x + 10\} \cup \{0\},$$

shown in Figure 19.5. Choose the point $p = (13, 8)$*, then*

$$\begin{aligned}
0 \cdot p &= 0, \\
1 \cdot p = p &= (13, 8), \\
2 \cdot p = p + p &= (16, 7), \\
3 \cdot p = 2 \cdot p + p &= (18, 5), \\
4 \cdot p = 3 \cdot p + p &= (12, 1), \\
5 \cdot p = 4 \cdot p + p &= (5, 10), \\
6 \cdot p = 5 \cdot p + p &= (7, 0), \\
7 \cdot p = 6 \cdot p + p &= (5, 9), \\
8 \cdot p = 7 \cdot p + p &= (12, 18), \\
9 \cdot p = 8 \cdot p + p &= (18, 4), \\
10 \cdot p = 9 \cdot p + p &= (16, 2), \\
11 \cdot p = 10 \cdot p + p &= (13, 11), \\
12 \cdot p = 11 \cdot p + p &= 0
\end{aligned}$$

As seen from example 19.10.1, every element of an elliptic curve is a generator of some cyclic subgroup. At the same time, the entire group of points on the elliptic curve is not necessarily cyclic (TBD). On the other hand, for formulating the discrete logarithm problem, we need exactly a cyclic group. Thus, for a given

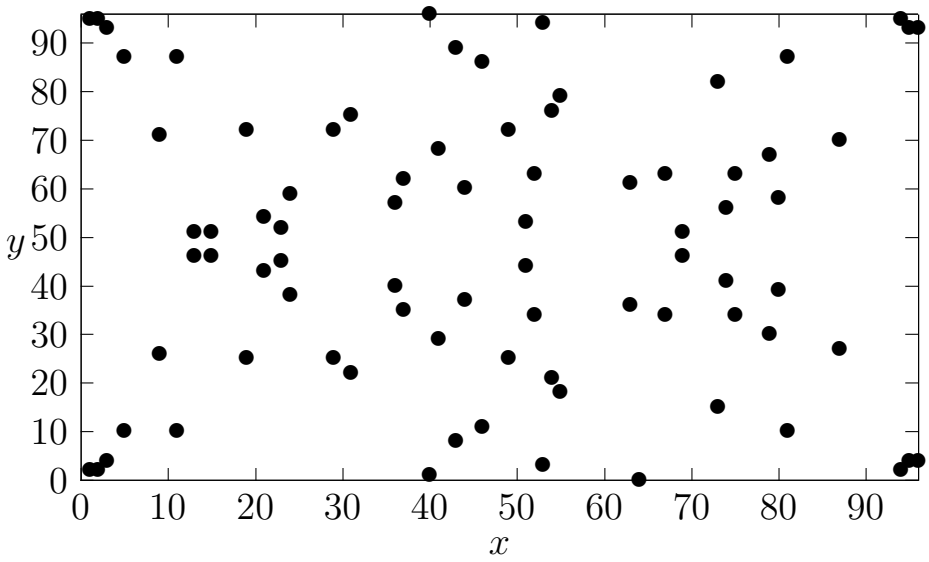


Figure 19.6: Elliptic curve $y^2 = x^3 - 7x + 10$ over the field $\mathbb{F}_{97}$

elliptic curve, first its order (see def. 19.10.1) is calculated, for which there exists an efficient Schoof's algorithm [22]. Then a prime divisor of the found order is located and a point is sought which is a generator of a subgroup of the chosen order. To do this, the following fact is used. For any point $g \in E$, the relation holds

$$Ng = 0,$$

where $N = |E|$ is the order (number of points) of the elliptic curve. Assume p is a prime divisor of N:

$$N = hp$$

then

$$Ng = p\,(hg) = 0.$$

Thus, if $hg \neq 0$, then the point $g' = hg$ will be a generator of a cyclic subgroup of order p.

Remark 19.10.2. *It makes sense to choose the order of the subgroup to be a prime number, since if the order of the group is prime then by Lagrange's theorem (see theorem 16.1.2) it has only trivial subgroups — the group itself and the subgroup consisting of the identity element. Thus, due to $hg \neq 0$, only the group itself remains, which is cyclic with order p.*

Example 19.10.2 (Choosing a Base Point)**.** *Consider the elliptic curve*

$$E = E\,(\mathbb{F}_{97}) = \{(x, y) \in \mathbb{F}_{97} \times \mathbb{F}_{97} : y^2 \equiv x^3 - 7x + 10\} \cup \{0\},$$

shown in Figure 19.6. The order of this curve is:

$$N = |E| = 82.$$

The number 82 has 2 divisors: 41 and 2. Thus, there exists a cyclic subgroup of order 41, i.e., $h = 2$.

Take the point $g = (1, 2) \in E$*, its order:* $|\langle g \rangle| = 82$*, i.e., this point is not suitable. Calculate*

$$g' = hg = 2g = (96, 93) \neq 0.$$

At the same time, $|\langle g' \rangle| = 41$*, i.e., we have found the required base point.*

If there are two points on the curve $a, b \in E(\mathbb{F}_p)$, the question of the existence of such $x \in \mathbb{N}$ makes sense:

$$x \cdot a = b$$

this problem is called the discrete logarithm problem on elliptic curves.

19.11 ECDH Algorithm

The ECDH algorithm is a modification of the Diffie-Hellman algorithm (see section 19.8) for elliptic curves. The Diffie-Hellman protocol is a key exchange protocol. In our case, the following elliptic curve parameters are published: (p, a, b, g, n, h), where p, a, b define the curve

$$E(\mathbb{F}_p) = \{(x, y) : y^2 \equiv x^3 + ax + b \mod p\} \cup \{0\},$$

g is the base point of order n: $|\langle g \rangle| = n$, h is the cofactor of the group $\langle g \rangle$, i.e. the order of the curve (see def. 19.10.1) $|E| = nh$.

Alice chooses a private key $d_a \in \{1, \ldots, n-1\}$ and forms the public key $A = d_a g$. Bob also forms private key $d_b \in \{1, \ldots, n-1\}$ and public key $B = d_b g$. Alice and Bob exchange these keys. Then each of them computes the shared key according to the rule $K = d_a B = d_b A$.

Example 19.11.1 (ECDH Algorithm)**.** *Let's take the curve and base point from example 19.10.2. Thus*

$$(p, a, b, g, n, h) = (97, -7, 10, (96, 93), 41, 2)$$

Alice chooses $d_a = 5$*, i.e.* $A = (37, 35)$*. Bob chooses* $d_b = 15$*, so* $B = (15, 51)$*. Alice's key* $K = d_a B = (46, 11)$ *and Bob's key* $K = d_b A = (46, 11)$ *match.*

Chapter 20

Algorithm Theory

20.1 Master Theorem for Recurrence Relations

Theorem 20.1.1 (Master Theorem for Recurrence Relations)**.** *If the following recurrence relation holds for the complexity of some algorithm*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n),$$

then it is possible to determine the asymptotic behavior of the function $T(n)$ *in the following cases*

1. *If* $f(n) = O\left(n^{\log_b a - \epsilon}\right)$*, for some* $\epsilon > 0$*, then* $T(n) = \Theta\left(n^{\log_b a}\right)$
2. *If* $f(n) = \Theta\left(n^{\log_b a} \log^k n\right)$*, then* $T(n) = \Theta\left(n^{\log_b a} \log^{k+1} n\right)$
3. *If* $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$*, for some* $\epsilon > 0$ *and* $af\left(\frac{n}{b}\right) \leq cf(n)$ *for some constant* $c < 1$ *and large* n*, then* $T(n) = \Theta(f(n))$

20.2 Divide and Conquer

"Divide and Conquer" is an important paradigm in solving algorithmic problems, which consists in dividing the original problem into simpler ones.

20.3 Boolean Logic. CNF Boolean Formulas

Definition 20.3.1 (Conjunction (logical "AND"))**.** For the logical operation "AND" $(a\&b)$, the following truth table 20.1 holds.

Definition 20.3.2 (Disjunction (logical "OR"))**.** For the logical operation "OR" $(a\|b)$, the following truth table 20.2 holds.

a	b	$a\&b$
0	0	0
0	1	0
1	0	0
1	1	1

Table 20.1: Conjunction $a\&b$

a	b	$a\|\|b$
0	0	0
0	1	1
1	0	1
1	1	1

Table 20.2: Disjunction $a\|\|b$

Definition 20.3.3 (Boolean Formula)**.** A Boolean formula is defined as a set of Boolean literals combined by logical operations.

Definition 20.3.4 (CNF - Conjunctive Normal Form (SAT))**.** A conjunctive normal form in Boolean logic is called a Boolean formula that has the form of a conjunction of disjunctions of literals.

Example 20.3.1. SAT Problem *The SAT problem is the problem of satisfiability of CNF. Suppose the following Boolean formula in CNF form is given:*

$$(x_1 \| x_2 \| \bar{x_3}) \,\&\, (x_1 \| \bar{x_2} \| x_3)$$

It is satisfiable; it suffices to assign $x_1 = 1$*, and the values of* $x_{2,3}$ *can be arbitrary.*

At the same time, the following formula is unsatisfiable:

$$(x_1 \| x_2 \| \bar{x_3}) \,\&\, (\bar{x_1} \| \bar{x_2} \| x_3)\,.$$

Theorem 20.3.1 (Reduction to CNF)**.** *Any Boolean formula can be reduced to CNF.*

Proof. For the proof, one can use: the law of double negation, De Morgan's law, and distributivity. □

Theorem 20.3.2 (Reduction to 3-CNF (3-SAT))**.** *Any Boolean formula in conjunctive normal form can be reduced to 3-CNF.*

Proof. TBD □

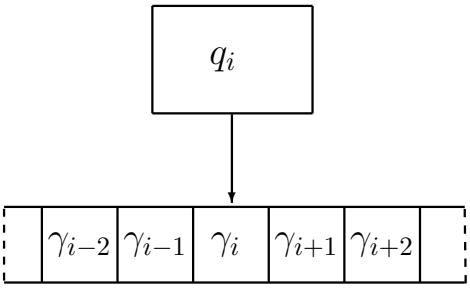


Figure 20.1: Turing machine. The current state of the machine is described by two numbers (q_i, γ_i), where $q_i \in Q$ is the state of the control device, which points to the tape element $\gamma_i \in \Gamma$.

20.4 Turing Machine

A Turing machine TM is a mathematical object consisting of a control device (head) and an infinite tape divided into cells, each of which can contain one symbol.

The control device of TM can be in one of the states $q_i \in Q$, where Q is the set of different states. The state in which TM was at the initial moment in time $q_0 \in Q$ is called the initial state.

Among the elements of the set Q a subset $F \subset Q$ of final states is chosen. If TM ends up in a state $f \in F \subset Q$, it is said that the Turing machine has finished (terminated) its work.

The symbols written on the tape form some set (alphabet) $\Gamma = \{\gamma_i\}$.

The input alphabet Σ of the Turing machine is called a subset of elements of Γ with which the control device TM operates, i.e. those symbols that it can recognize on the tape. Those symbols that the control device cannot recognize will be denoted by the symbol β (blank).

During operation, TM performs the following actions:

- reads the element pointed to by the control device (see Figure 20.1)
- changes (or leaves unchanged) this symbol to an element $\sigma \in \Sigma \subset \Gamma$
- changes the position of the control device by one cell to the right (R), to the left (L), or does not change it (S)

The change of state of TM is described by the transition table Δ. An element of the transition table is written as: $\delta_j\,(q_{old}, \gamma_{old})$ where $q_{old} \in Q$ is the current state, $\gamma_{old} \in \Gamma$ is the selected element on the tape. The transition value may be undefined or written as $\{q_{new}, \gamma_{new}, d\}$ (see Figure 20.2), where $q_{new} \in Q$ is the new state of the Turing machine, $\gamma_{new} \in \Gamma$ is the new value of the element on the tape pointed to by the control device, i.e. γ_{old} is replaced by γ_{new}. $d \in D = \{L, R, S\}$

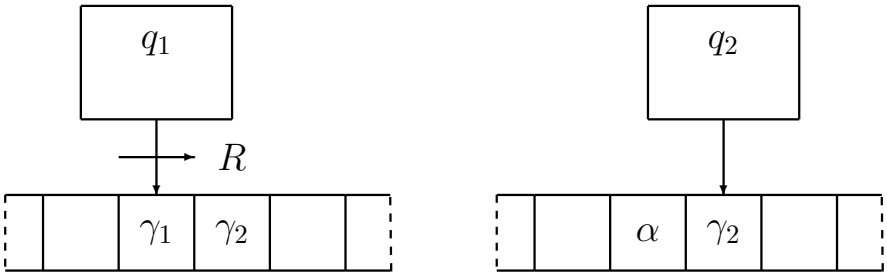


Figure 20.2: Transition of a Turing machine from state (q_1, γ_1) to state (q_2, γ_2). The corresponding element of the transition table $\delta : \{q_1, \gamma_1\} \to \{q_2, \alpha, R\}$. That is, the tape element initially pointed to by the control unit (γ_1) changes its value to α, and the control unit shifts its position one cell to the right (R).

is the direction of movement of the control device (L left, R right, S stay in the same place). In the case of an undefined transition, it is said that the Turing machine has halted in the state q_{old}.

Definition 20.4.1 (Language of a Turing Machine $L(M)$)**.** The initial symbols on the tape can be interpreted as some string x which is input to the Turing machine M. All those strings which bring the machine M to some final state $f \in F$ are called the language $L(M)$ of the machine M

Definition 20.4.2 (Deterministic Turing Machine)**.** If in the transition table the mapping $(q_{old}, \gamma_{old}) \to \{q_{new}, \gamma_{new}, d\}$ is a bijection [1], then the corresponding Turing machine is called a deterministic Turing machine

Definition 20.4.3 (Non-deterministic Turing Machine)**.** If in the transition table the mapping $(q_{old}, \gamma_{old}) \to \{q_{new}, \gamma_{new}, d\}$ is not a bijection, i.e., the state (q_{old}, γ_{old}) may correspond to several elements of the transition table $\{q_{new}, \gamma_{new}, d\}$, then the corresponding Turing machine is called a non-deterministic Turing machine

20.5 Computability Theory

A Turing machine M (see Figure 20.3) can be interpreted as a certain function m whose argument is a set of symbols representing the initial state of the tape x. As a result of processing x, the machine M can finish its work, i.e., end up in some final state $f \in F \subset Q$. In this case, it is said that M accepts x. Then the corresponding function value is $m(x) = 1$.

At the same time, the computation of the function $m(x)$ can be interpreted as a certain problem.

[1] A bijection is a one-to-one mapping, i.e., in our case, for each state (q_{old}, γ_{old}) there corresponds only one element of the transition table $\{q_{new}, \gamma_{new}, d\}$

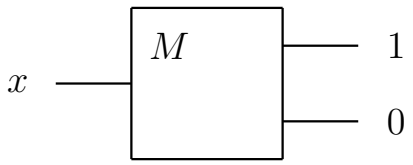


Figure 20.3: Turing machine M as a function $m(x)$. The input consists of some sequence of initial data x (initial symbols on the tape). At the output, M may be in some final state (1 - accept x), hang (0 - reject x) – i.e., be in a state from which there are no further transitions. The machine may also enter an infinite loop and never finish its work

Definition 20.5.1 (Problem)**.** We call a problem with respect to a Turing machine a yes/no answer to a question about some set of possible objects.

Example 20.5.1 (Hamiltonian Graph)**.** *The Hamiltonian graph problem - for a given graph, whether there exists a path (chain) containing each vertex of this graph exactly once.*

In the case of halting (halt) of M, i.e., when M reaches a state from which no further transition is defined, it is said that M rejects x. In this case, the corresponding function value is $m(x) = 0$.

Also, the machine M on input x may never terminate its operation. In this case, the value of the function $m(x)$ is undefined.

Definition 20.5.2 (Recursive Language)**.** The set of strings L [2] is called a recursive language if there exists a Turing machine M such that $\forall x \in L$ the corresponding function $m(x) = 1$, and $\forall x \notin L$ the corresponding function $m(x) = 0$.

Thus, M corresponding to a recursive language $L(M)$ halts on any input. The set of recursive languages is denoted R.

Definition 20.5.3 (Recursively Enumerable Language)**.** The set of strings L is called a recursively enumerable language if there exists a Turing machine M such that $\forall x \in L$, M halts in some final state, i.e., the corresponding function $m(x) = 1$.

Thus, M corresponding to a recursively enumerable language $L(M)$ halts on every input belonging to the language, but may never halt on input not belonging to the language $L(M)$. The set of recursively enumerable languages is denoted RE. Obviously $R \subset RE$.

A Turing machine defines a class of functions for which the notion of computability can be defined.

[2] the set of initial tape states of some Turing machine

Definition 20.5.4 (Computable Function)**.** [3] A function m is called computable, and the corresponding problem - decidable if there exists some Turing machine M for it, and the language corresponding to this machine is recursive, $L(M) \in R$. Otherwise, the corresponding problem is called undecidable.

Theorem 20.5.1 (On the cardinality of the set of Turing Machines)**.** *The set of Turing machines is countable.*

Proof. A set is called countable [54] if there exists a natural number assigned to each element. A Turing machine is uniquely determined by the following group of elements:

- Set of states Q
- Start state q_0
- Set of final states $F \subset Q$
- Alphabet Σ
- Transition table Δ

Since each of the listed elements of a Turing machine takes on a finite number of values, a unique (for the corresponding machine) number can be assigned to each element, and therefore to the entire Turing machine, [4] which proves the countability of the set of all Turing machines. □

After introducing the notion of a computable function, the question arises about the computability of an arbitrary function. Suppose we have some input data (function argument) x. Does there exist a Turing machine M that accepts this data, i.e., M halts in some final state? The following theorem answers this question.

Theorem 20.5.2 (On Undecidability)**.** *There exists a language L_d that is not accepted by any Turing machine, i.e., the corresponding problem L_d is fundamentally undecidable (not computable).*

Language L_d is often called the diagonalization language because if one constructs a matrix [13] whose row indices denote the numbers of input data of Turing machines, and the column numbers correspond to the Turing machine number, then the value of this matrix a_{ij} is equal to 1 if M_j accepts x_i and 0 otherwise. The inverted [5] numbers on the diagonal define the language L_d.

[3] In fact, the notion of computability cannot be defined strictly mathematically and is given here in the formulation of the Church-Turing thesis.

[4] For example, one can encode each machine by a unique sequence of numbers 0 and 1. The resulting sequence is finite due to the finiteness of the set of elements comprising the Turing machine. Any finite binary sequence corresponds to a unique natural number for which the considered sequence is the binary representation.

[5] 0 replaced by 1 and 1 by 0

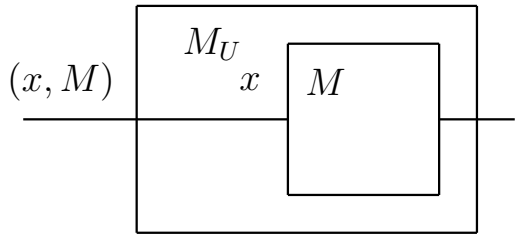


Figure 20.4: Universal Turing machine M_U. M_U receives as input a pair of strings (x, M), where M represents the code of a certain Turing machine on which the input data x should be executed

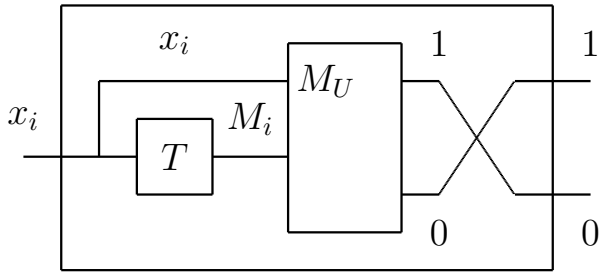


Figure 20.5: Proof that $L_U \notin R$. If $L_U \in R$, then a universal Turing machine M_U can be constructed for the Turing machine M_d, which accepts all strings of the language L_d, which is impossible according to the theorem 20.5.2. T is a special device that, given a string x_i, creates a code for the Turing machine M_i, which does not accept x_i.

Proof. By theorem On the cardinality of the set of Turing Machines (Theorem 20.5.1) , one can talk about the i-th Turing machine M_i, where i is some natural number. Suppose x_i is a string not recognized by M_i, i.e., $x_i \notin L(M_i)$. The strings x_i form some language L_d. By the definition of L_d, no Turing machine exists that accepts all strings of this language. □

Definition 20.5.5 (Universal Turing Machine)**.** A universal Turing machine M_u is a machine that receives on input a string x and a code for some Turing machine M and computes the result of applying x to M.

The set of pairs of strings (x, M) form some language L_U of the machine M_U.

Theorem 20.5.3 (On the Language of the Universal Turing Machine)**.** *The language of the universal Turing machine is recursively enumerable but not recursive, i.e., $L_U \in RE$ and $L_U \notin R$. Thus, the problem of recognizing the language of the universal Turing machine is undecidable.*

Proof. The statement $L_U \in RE$ follows directly from the definition of the universal Turing machine: if M accepts x, then M_U accepts the pair of strings (x, M).

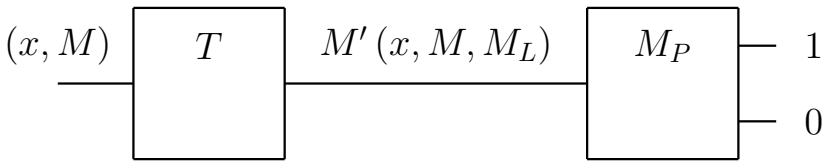


Figure 20.6: Rice's theorem. The Turing machine M_P recognizing a nontrivial property of the language P is used to construct a universal Turing machine M_U which can both accept and reject pairs of strings (x, M). The adapter T is used to transform the pair of strings (x, M) into a code for the Turing machine M' which satisfies the property P if and only if M accepts x.

We prove $L_U \notin R$ by contradiction. Suppose $L_U \in R$. Using such a machine M_U, construct a Turing machine M_d that will accept all strings of the language L_d. For each string $x_i \in L_d$ there is a corresponding Turing machine M_i which does not accept this string, i.e., $x_i \notin L(M_i)$. The pair (x_i, M_i) can be input to the universal Turing machine as shown in Figure 20.5. If $L_U \in R$, then machine M_U can decide whether M_i accepts input x_i or not. If M_i does not accept x_i, then machine M_d should accept this data. Thus, we have constructed a Turing machine M_d that accepts all strings $x_i \in L_d$ and rejects all strings $x_i \notin L_d$, which is impossible due to theorem On Undecidability (Theorem 20.5.2). □

Every Turing machine M has some code which can be interpreted as input data for another Turing machine M_P. The machine can either accept or reject this code. Due to the one-to-one correspondence between a Turing machine M and its language $L(M)$, one can talk about a Turing machine that accepts or rejects languages L. If machine M_P accepts some set of languages, it is said that these languages possess some property P.

Definition 20.5.6 (Nontrivial Property of a Turing Machine Language)**.** A property of a language is called nontrivial if among all languages one can distinguish both those which satisfy the property and those which do not.

Theorem 20.5.4 (Rice's Theorem)**.** *The problem of determining any nontrivial property of the language $L(M)$ of a Turing machine M is undecidable.*

Proof. Suppose we have a machine M_P that can determine some nontrivial language property P, i.e., it can classify the languages of all Turing machines into two sets (each of which is nonempty). The first set contains all languages satisfying P (the machine M_P accepts languages from this set), and the second set contains all languages that do not satisfy property P.

Let us try to construct on the basis of M_P a universal Turing machine M_U that accepts (or rejects) pairs (x, M) (see Figure 20.6).

Suppose we have a machine M_L that accepts strings l from some language L [6].

As the first step, construct a machine M' that accepts strings l. At the same time, the pair (x, M) is fixed and embedded in the code of M'. The key property of M' is the following: M' accepts strings $l \in L$ if and only if M accepts x. Thus, two cases are possible:

- $L(M') = L(M_L)$ if M' accepts l as a result of accepting the pair (x, M)
- $L(M') = \emptyset$ if M' rejects any input l, i.e., in the case where M never halts on input x.

Next, recall the property P and suppose that the language L (accepted by machine M') has property P, that is, $L \in P$. Also assume that $\emptyset \notin P$, [7]. Thus, if the code for M' is given as input to machine M_p, then M_p accepts the input (code M') if $L(M') \in P$ and rejects it if $L(M') = \emptyset$.

On the other hand, the pair (x, M) is embedded in the original code of M', so we can assume that M_P receives exactly the pair (x, M) as input, accepting it if M accepts x ($L(M') \in P$) and rejecting this pair if M does not accept x ($L(M') = \emptyset \notin P$). Thus, using M_P we have constructed a universal Turing machine and shown that its language is recursive, which is impossible by theorem On the Language of the Universal Turing Machine (Theorem 20.5.3) . □

20.6 Complexity Classes P, NP

When solving a particular problem, two questions arise primarily. Does the given problem have a solution in principle, i.e. can an algorithm be constructed that solves the given problem? Once a positive answer to the first question is found, i.e. an algorithm solving the problem is constructed, the next issue is the practical feasibility of the proposed solution. Here, the theory of algorithm complexity helps us address the feasibility of various algorithms.

In the following table [38] some large numbers useful for analyzing algorithm complexity are given

Example 20.6.1. Prime factorization of integers by trial division *Suppose we have the problem of factorizing an integer x (see example 13.0.2). In binary representation, x consists of N digits, i.e. $2^N \leq x < 2^{N+1}$. In the simplest approach (trial division), we can try all integers from 2 up to $2^{\frac{N}{2}}$, and for each such integer $2 < y \leq 2^{\frac{N}{2}}$ find the remainder of division of x by y. If x is divisible*

[6] There is no relation between language L and property P at this step

[7] If $\emptyset \in P$, then one can consider the property $\bar{P} = \{\forall L : L \notin P\}$. In this case, $\emptyset \notin \bar{P}$.

Physical analogy	Number
Probability of being struck by lightning (in one day)	1 in 9 billion (2^{33})
Probability of winning the grand prize in the US state lottery	1 in 4,000,000 (2^{22})
Probability of winning the grand prize in the US state lottery and being struck by lightning on the same day	1 in 2^{61}
Probability of drowning (in the USA within a year)	1 in 59,000 (2^{16})
Probability of dying in a car accident (in the USA in a year)	1 in 6,100 (2^{13})
Probability of dying in a car accident (in the USA during a lifetime)	1 in 88 (2^{7})
Time until next glaciation	14,000 (2^{14}) years
Time until the Sun becomes a supernova	10^{9} (2^{30}) years
Age of the planet	10^{9} (2^{30}) years
Age of the Universe	10^{10} (2^{34}) years
Number of atoms in the planet	10^{51} (2^{170})
Number of atoms in the Sun	10^{57} (2^{190})
Number of atoms in the galaxy	10^{67} (2^{223})
Number of atoms in the Universe	10^{77} (2^{265})
Volume of the Universe	10^{84} (2^{280}) cm^3
If the Universe is finite:	
Total lifetime of the universe	10^{11} (2^{37}) years 10^{18} (2^{61}) seconds
If the Universe is infinite:	
Time until cooling of low-mass stars	10^{14} (2^{47}) years
Time until planets detach from stars	10^{15} (2^{50}) years
Time until stars detach from galaxies	10^{19} (2^{64}) years
Time until orbital decay by gravitational radiation	10^{20} (2^{67}) years
Time until black holes evaporate by Hawking processes	10^{64} (2^{213}) years
Time until matter becomes liquid at zero temperature	10^{65} (2^{216}) years

Table 20.3: Large numbers [38]

by y, the problem is solved and y is declared a divisor of x. If none of the numbers y divide x, then x is declared prime. Thus, the original problem is divided into $2^{\frac{N}{2}}$ subproblems involving computing remainders.

Assume x has 20 digits, i.e. $N = 20$. In that case, $2^{10} = 1024$ division operations need to be performed to find a solution to the original problem, which can be easily done on simple computers.

However, if the number of digits increases 30-fold to $N = 600$, then to solve the given problem, one must perform 2^{300} division operations. If you look at Table 20.3, you can see that this number is much larger than the number of atoms in the universe, which is on the order of 2^{265}. That is, even if each atom in the universe represents a very simple computational device that computes the remainder of division of two integers, using all atoms in the universe would be insufficient to solve the problem.

Thus, despite the existence of a simple algorithm, the problem of prime factorization, using this algorithm, becomes practically infeasible for numbers of length 600 bits, even though the number 600 does not seem very large compared to 20 (where the task is practically feasible).

20.6.1 Hierarchy of Complexity Classes

Below are formal definitions for problems that are practically feasible, P. The class of problems that are practically infeasible (until it is proven that $P = NP$) is also defined.

Definition 20.6.1 (Problem size)**.** The size of the problem N is defined as the number of symbols from the alphabet of the Turing machine contained on the tape in the initial state, that is, in other words, the size of the problem is considered to be the length of the string input to the Turing machine.

For example, in example 20.6.1, the problem size is the length of number x (whose divisors are being sought) in binary representation.

Definition 20.6.2 (Class P)**.** A problem (algorithm) belongs to class P (Polynomial) if it can be solved in $O\left(N^k\right)$ steps on a deterministic Turing machine. Here N is the size of the original problem, k is any integer independent of N.

The class P defines the class of problems that are practically feasible. For example, the problem of sorting unordered data (a list) can be solved in $O\left(N \log N\right) = O\left(N^2\right)$ steps, where N is the number of elements in the list to be sorted (the problem size).

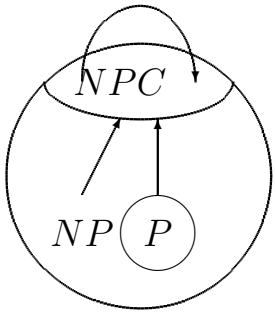


Figure 20.7: The relationship between the complexity classes P, NP, and NPC. The presented picture corresponds to the assumption $P \neq NP$. In this case, $P \subset NP$, $NPC \subset NP$. Any problem from P and NP can be reduced to NPC in polynomial time.

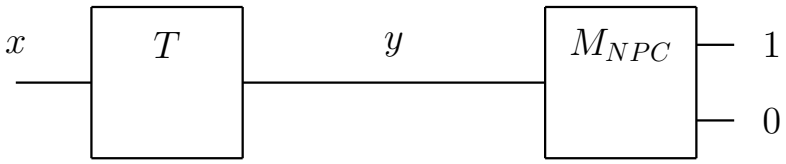


Figure 20.8: Class NPC. The input for any NP problem x can be transformed into an input y for some problem from the class NPC. The transformation $x \to y$ can be performed in $O\left(N^k\right)$ steps, where $N = |x|$ is the length of x, and k is some integer independent of N.

Definition 20.6.3 (Class NP)**.** A problem (algorithm) belongs to class NP (Nondeterministic Polynomial) if it can be solved in $O\left(N^k\right)$ steps on a nondeterministic Turing machine. Here N is the size of the original problem, k is any integer independent of N.

One example of an NP problem is integer factorization. A nondeterministic Turing machine can guess a solution (in one step) and verify it (multiply the two guessed numbers and compare the product to the original number) in polynomial time. At the same time, the question of whether the factorization problem belongs to class P remains open.

The problem of equality of classes P and NP is one of the most important unsolved problems in mathematics and probably the most important unsolved problem in algorithm theory. To show how an NP problem can be reduced to a P problem, it is necessary to introduce the class of NP-complete problems, which are, in some sense, the hardest NP problems.

Definition 20.6.4 (Class NPC (NP-complete))**.** A problem M belongs to class NPC if any NP problem can be reduced to M (see Figure 20.8).

Thus, if a polynomial algorithm is found for at least one NPC problem, then polynomial algorithms will be found for all NP problems (see Figure 20.7).

20.6.2 Cook-Levin Theorem

As soon as we introduced the complexity class NPC, the question arises about the problems that belong to it. Below it will be shown that the SAT problem (the satisfiability of a Boolean formula in conjunctive normal form) is NP-complete.

Theorem 20.6.1 (Cook-Levin)**.** *The problem of satisfiability of a Boolean formula in CNF is NP-complete, i.e., it belongs to the class NPC.*

Proof. The proof consists of two parts. First, we must prove that the SAT problem (see example 20.3.1) belongs to the class NP. Second, we must show that any problem from the class NP can be reduced to the SAT problem in polynomial time.

Membership in the class NP is proven in an obvious way, since it is possible to verify the correctness of a solution in $O(N)$ steps, where N is the number of Boolean operations used in the considered Boolean formula.

TBD □

Part 6
Answers to Questions

Chapter 21

Quantum Electrodynamics

21.1 Quantization of the Electromagnetic Field

21.1.1 5: Number of modes

How many electromagnetic field modes with wavelength $\lambda \geq 500\text{nm}$ are in the quantization cube with side $L = 1\text{mm}$?

The wavelength range $\lambda \geq \lambda_0 = 500\text{nm}$ corresponds to frequency range $\omega \leq \omega_0 = \frac{2\pi c}{\lambda_0}$.

Using formula 1.47

$$dN = 2\left(\frac{L}{2\pi c}\right)^3 \omega^2 d\omega d\Omega$$

hence

$$N = \int_0^{\omega_0} 2\left(\frac{L}{2\pi c}\right)^3 \omega^2 d\omega d \int_{4\pi} d\Omega =$$
$$= 8\pi \int_0^{\omega_0} \left(\frac{L}{2\pi c}\right)^3 d\omega = 8\pi \left(\frac{L}{2\pi c}\right)^3 \frac{\omega_0^3}{3} =$$
$$= \frac{8\pi}{3}\left(\frac{L}{\lambda_0}\right)^3 \approx 67 \cdot 10^9$$

21.2 Interaction of Light with an Atom

21.2.1 13: Determination of lithium atom transition frequency

Let us denote by $\Omega_R^{(1,2)}$ the effective Rabi frequencies for measurements 1 and 2, respectively. The frequency detunings for the two experiments are $\delta_{1,2}$. Thus,

we have

$$\Omega_R^{(1)} = \sqrt{\omega_R^2 + \delta_1^2},$$
$$\Omega_R^{(2)} = \sqrt{\omega_R^2 + \delta_2^2} \tag{21.1}$$

On the other hand, from the graphs Figure 2.19 it is seen that the maximum probability to detect lithium atoms in state $|2\rangle$ for frequency $\omega_2 = 2\pi \cdot 228.4\text{MHz}$ will be $P_2 = 0.49$, while from (2.27) we have

$$P_2 = \left(\frac{\omega_R}{\Omega_R^{(2)}}\right)^2 = \frac{1}{1 + \frac{\delta_2^2}{\omega_R^2}},$$

from which

$$\omega_R = 0.98 \cdot \delta_2. \tag{21.2}$$

On the other hand, one can conclude that

$$\frac{\Omega_R^{(1)} \cdot 6\mu\text{s}}{2} = \pi$$

and

$$\frac{\Omega_R^{(2)} \cdot 12.5\mu\text{s}}{2} = \pi,$$

thus

$$\Omega_R^{(1)} = 2\pi \cdot 0.167\text{MHz},$$

and

$$\Omega_R^{(2)} = 2\pi \cdot 0.08\text{MHz},$$

From (21.1) and (21.2) one can find

$$\delta_2 = 2\pi \cdot 0.057\text{MHz},$$

i.e. two candidates

$$f_0 = 228.4 \pm 0.057\text{MHz} = 228.343\text{MHz},\ 228.457\text{MHz} \tag{21.3}$$

For δ_1 we have

$$\delta_1 = \sqrt{\left(\Omega_R^{(1)}\right)^2 - \omega_R^2} = 2\pi\sqrt{0.167^2 - (0.98 \cdot 0.057)^2}\text{MHz} = 2\pi \cdot 0.157\text{MHz}$$

thus we get the next two candidates

$$f_0 = 228.5 \pm 0.157\text{MHz} = 228.343\text{MHz},\ 228.657\text{MHz} \tag{21.4}$$

From (21.3) and (21.4) we finally have

$$f_0 = 228.343\text{MHz}.$$

It is worth noting that the obtained value differs from the real $f_0 = 228.205$MHz. This deviation is caused by the influence of the Zeeman effect from the magnetic field, which is usually present in such experiments.

Chapter 22

Quantum Optics

22.1 Quantum Theory of the Laser

22.1.1 6: Natural line width

From (3.24) we have

$$D = \frac{A}{2\bar{n}_{st}} = \frac{10^6}{2 \cdot 10^6} = 0.5\text{Hz}.$$

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