CS/MATH 111 SPRING 2016
Final Test Version A
Solution Key

NAME: SID:

Problem 1: Amber needs to buy 30 bagels for a party. There are three flavors to choose from: poppyseed, blueberry, and garlic. She needs at least 3 poppyseed bagels, at most 12 blueberry bagels and at most 17 garlic bagels. How many possible combinations of bagels are there that satisfy these requirements? Show your work[1].

The problem is equivalent to computing the number of non-negative integer solutions to

$$\begin{aligned} p+b+g &= 33 \\ p &\geq 3 \\ b &\leq 11 \\ g &\leq 13 \end{aligned}$$

After the substitution for p, this reduces to computing the number of non-negative integer solutions to

$$\begin{aligned} p+b+g &= 30 \\ b &\leq 11 \\ g &\leq 13 \end{aligned}$$

As in class, let $S(P)$ be the number of solutions that satisfy condition P. So we need to compute $S(b \leq 11 \wedge g \leq 13)$. Denoting by S the number of all non-negative solutions, we have

$$S(b \leq 11 \wedge g \leq 13) = S - S(b \geq 12 \vee g \geq 14)$$

We now compute S:

$$S = \binom{30+2}{2} = \binom{32}{2} = 496.$$

To compute $S(b \geq 12 \vee g \geq 14)$, we use inclusion-exclusion:

$$\begin{aligned} S(b \geq 12 \vee g \geq 14) &= S(b \geq 12) + S(g \geq 14) - S(b \geq 12 \wedge g \geq 14) \\ &= \binom{30-12+2}{2} + \binom{30-14+2}{2} - \binom{30-26+2}{2} \\ &= \binom{20}{2} + \binom{18}{2} - \binom{6}{2} = 190 + 153 - 15 = 328. \end{aligned}$$

So

$$S(b \leq 11 \wedge g \leq 13) = 496 - 328 = \mathbf{168}.$$

[1] You must use the method for counting integer partitions that we covered in class. Brute force listing of all solutions will not be credited.

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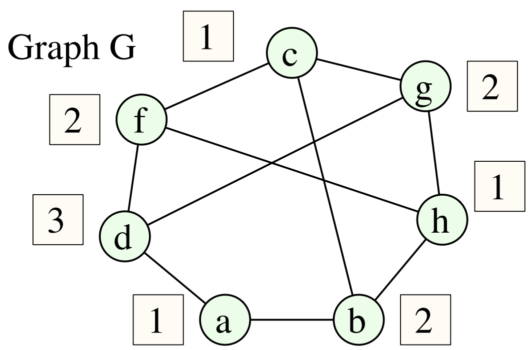
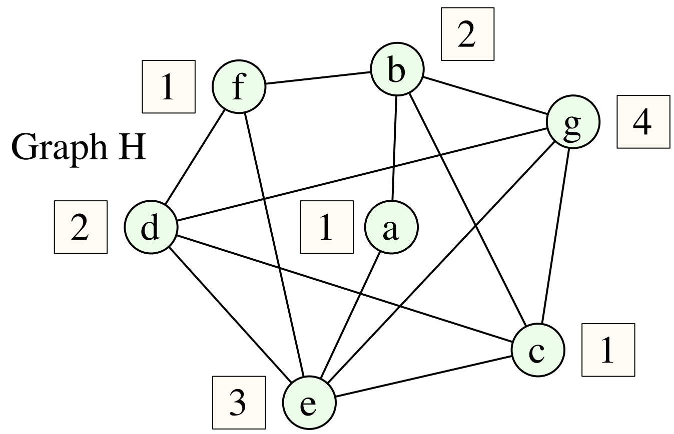
Problem 2: For each graph below determine the minimum number of colors necessary to color its vertices. Justify your answer, by giving a coloring and explaining why it is not possible to use fewer colors.

To give a coloring, use positive integers $1, 2, ...$ for colors and mark the color of each vertex in the box next to it. For ease of grading, assign color 1 to vertex **a** and color 2 to vertex **b**.





Graph G can be colored with 3 colors. Graph H can be colored with 4 colors. See the two colorings above.

Why the number of colors of G is minimized?	Why the number of colors of H is minimized?
G requires 3 colors because it contains an odd-length cycle, for example a, d, g, h, b, a.	H requires 4 colors because it contains a 4-vertex clique consising of vertices d, e, c, g.

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Problem 3: (a) Compute $12^{-1} \pmod{19}$. Show your work.

Listing multiples of 19 plus 1, we get 20, 39, 58, $77, 96$. Since $96 = 12 \cdot 8$, we have $12^{-1} \pmod{19} = 8$.

(b) Compute $2^{5983207} \pmod{101}$. Show your work.

Computing modulo 101, using Fermat's theorem, we get

$$\begin{aligned} 2^{5983207} &= 2^{59832 \cdot 100 + 7} \\ &= (2^{100})^{59832} \cdot 2^7 \\ &= 1 \cdot 128 = 27. \end{aligned}$$

(c) Compute $7^{17} \pmod{23}$. Show your work.

Computing modulo 23, we get

$$\begin{aligned} 7^{17} &= 7 \cdot (7^2)^8 \\ &= 7 \cdot 49^8 \\ &= 7 \cdot 3^8 \\ &= 7 \cdot 9^4 \\ &= 7 \cdot 81^2 \\ &= 7 \cdot 12^2 \\ &= 7 \cdot 144 = 7 \cdot 6 = 42 = 19. \end{aligned}$$

Problem 4: Solve the following recurrence equation:

$$
\begin{aligned}
Z_n &= Z_{n-1} + 2Z_{n-2} + 3^n \\
Z_0 &= 3 \\
Z_1 &= 4
\end{aligned}
$$

To find a particular solution, we try $Z''_n = \beta 3^n$. After substituting, we get

$$\beta 3^n = \beta 3^{n-1} + 2\beta 3^{n-2} + 3^n$$

which simplifies to

$$9\beta = 3\beta + 2\beta + 9$$

so $\beta = \frac{9}{4}$. Thus $Z''_n = \frac{9}{4}3^n$.

Next, we compute the general solution of the homogeneous equation. The characteristic equation is

$$x^2 - x - 2 = 0$$

The roots are $-1, 2$. So the general solution for the homogeneous equation is

$$Z'_n = \alpha_1(-1)^n + \alpha_2 2^n.$$

We now combine it with the particular solution, getting the general solution of the inhomogeneous equation:

$$Z_n = \alpha_1(-1)^n + \alpha_2 2^n + \tfrac{9}{4}3^n.$$

Plugging into the initial condition, we get equations:

$$
\begin{aligned}
\alpha_1 + \alpha_2 + \tfrac{9}{4} &= 3 \\
-\alpha_1 + 2\alpha_2 + \tfrac{27}{4} &= 4
\end{aligned}
$$

From these equations, $\alpha_1 = \frac{17}{12}$ and $\alpha_2 = -\frac{2}{3}$. This gives us the final solution:

$$Z_n = \tfrac{17}{12}(-1)^n - \tfrac{2}{3}2^n + \tfrac{9}{4}3^n.$$

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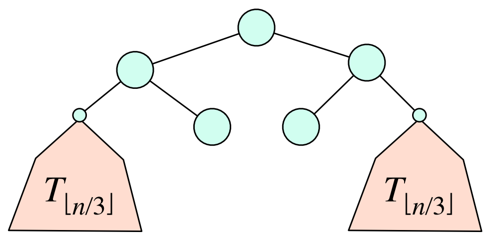
Problem 5: For each integer $n \geq 1$ we define a tree T_n, as follows: T_1 and T_2 consist of just a single node. For $n \geq 3$, T_n is formed by creating five new nodes and attaching to them two copies of subtree $T_{\lfloor n/3 \rfloor}$, as in the picture below:



Let $Q(n)$ be the number of nodes in T_n. For example, we have $Q(1) = Q(2) = 1$, $Q(3) = Q(4) = \ldots = Q(8) = 7$, and so on.

(a) Give a recurrence equation for $Q(n)$ and justify it. (b) Then determine the asymptotic value of $Q(n)$, expressing it using the Θ-notation.

(Reminder: $\lfloor x \rfloor$ is the largest integer not larger than x. For example, $\lfloor 2.7 \rfloor = 2$ and $\lfloor 23/3 \rfloor = 7$.)

T_n contains all nodes from both copies of $T_{\lfloor n/3 \rfloor}$, plus 5 additional nodes. Therefore the number of nodes $Q(n)$ satisfies the recurrence

$$Q(n) = 2 \cdot Q(\lfloor n/3 \rfloor) + 5.$$

To estimate $Q(n)$, we use Master Theorem. We have $a = 2$, $b = 3$ and $d = 0$, so $a > b^d$. So the solution is

$$Q(n) = \Theta(n^{\log_3 2}).$$

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Problem 6: Consider numbers B_n defined recursively as follows: $B_0 = B_1 = B_2 = 1$, and $B_n = B_{n-1} + B_{n-2} + B_{n-3}$ for all integers $n \geq 3$. Using mathematical induction, prove that $B_n \leq 2^n$ for all $n \geq 0$.

Base case: In the base case we verify that the inequality holds for $n = 0, 1, 2$. For $n = 0$, $B_0 = 1 \leq 2^0$, for $n = 1$, $B_1 = 1 \leq 2^1$, and for $n = 2$, $B_2 = 1 \leq 2^2$. So the inequality holds in the base case.

Inductive step: Now, let $k \geq 2$, and assume that $B_n \leq 2^n$ holds for all $n \leq k$. We show that it also holds for $k + 1$, that is $B_{k+1} \leq 2^{k+1}$. The derivation is as follows:

$$\begin{aligned}
B_{k+1} &= B_k + B_{k-1} + B_{k-2} \\
&\leq 2^k + 2^{k-1} + 2^{k-2} && \text{(from the inductive assumption)} \\
&= 2^{k-2}(4 + 2 + 1) \\
&= 7 \cdot 2^{k-2} \\
&\leq 8 \cdot 2^{k-2} \\
&= 2^{k+1}.
\end{aligned}$$

This implies that $B_{k+1} \leq 2^{k+1}$, completing the proof.

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Problem 7: Complete statements of the following theorems.

(a) Euler's Theorem: Let G be a connected graph. G has an Euler tour if and only if **each vertex in G has even degree**.

(b) Dirac's Theorem: Let G be a graph with n vertices. If **each vertex in G has degree at least $n/2$** then G has a hamiltonian cycle.

(c) Hall's Theorem: Let $G = (L, R, E)$ be a bipartite graph. G has a perfect matching if and only if $|L| = |R|$ **and for each $X \subseteq L$ we have $|N(X)| \geq |X|$.**

(d) Kuratowski's Theorem: Let G be a graph. G is planar if and only if G **does not contain a subgraph that is a sub-division of K_5 or $K_{3,3}$.**

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Problem 8: Give the formulas for the following quantities. Provide a justification for each.

(a) (2 points) The number of all strings of length n formed from letters $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}$.

For each n positions we have 5 choices, so the number of strings is 5^n.

(b) (2 points) The number of all strings of length n formed from letters $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}$ that contain exactly two $\mathtt{a}$'s and exactly two $\mathtt{b}$'s. (Here we assume $n \geq 4$.)

There are $\binom{n}{2}$ choices for the positions that have $\mathtt{a}$'s. Among the remaining $n-2$ positions, there are $\binom{n-2}{2}$ choices for the positions that have $\mathtt{b}$'s. The remaining $n-4$ positions can be filled in 3 ways each, for the total of 3^{n-4}. So the answer is

$$\binom{n}{2} \cdot \binom{n-2}{2} \cdot 3^{n-4} = \tfrac{1}{4} n(n-1)(n-2)(n-3)3^{n-4}.$$

(c) (6 points) The number of all strings of length n formed from letters $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}$ that contain at least two $\mathtt{a}$'s and at least two $\mathtt{b}$'s. (Here we assume $n \geq 4$.)

Some notation will be helpful. Let α be the number of $\mathtt{a}'s$ and β be the number of $\mathtt{b}$'s in the string. Let also $S(P)$ be the number of strings of length n with property P. So we want to compute $S(\alpha \geq 2 \wedge \beta \geq 2)$. Then

$$S(\alpha \geq 2 \wedge \beta \geq 2) = 5^n - S(\alpha < 2 \vee \beta < 2)$$

We now compute $S(\alpha < 2 \vee \beta < 2)$, using the inclusion-exclusion principle and breaking into cases:

$$\begin{aligned}
S(\alpha < 2 \vee \beta < 2) &= S(\alpha < 2) + S(\beta < 2) - S(\alpha < 2 \wedge \beta < 2) \\
&= S(\alpha = 0) + S(\alpha = 1) + S(\beta = 0) + S(\beta = 1) \\
&\qquad - [\, S(\alpha = 0 \wedge \beta = 0) + S(\alpha = 1 \wedge \beta = 0) \\
&\qquad\qquad + S(\alpha = 0 \wedge \beta = 1) + S(\alpha = 1 \wedge \beta = 1) \,] \\
&= 4^n + n4^{n-1} + 4^n + n4^{n-1} \\
&\qquad - [\, 3^n + n3^{n-1} + n3^{n-1} + n(n-1)3^{n-2} \,] \\
&= (2n+8)4^{n-1} - (n^2 + 5n + 9)3^{n-2}
\end{aligned}$$

So the answer is

$$S(\alpha \geq 2 \wedge \beta \geq 2) = 5^n - (2n+8)4^{n-1} + (n^2 + 5n + 9)3^{n-2}$$