Study Guide for CS111 Final

Graphs

1. Prove that in an undirected graph G the sum of all vertex degrees is equal to two times the number of edges.
2. Given a degree sequence, determine if there exists a graph with this degree sequence.
3. State the sufficient and necessary condition for an undirected graph to have an Euler cycle. Prove that if an undirected graph has an Euler cycle then all vertex degrees are even.
4. Can you generalize the characterization of Eulerian graphs to directed graphs?
5. State Dirac's theorem.
6. For graphs shown below ...., tell whether they have (a) an Euler tour, (b) a Hamiltonian cycle. Justify your answer.
7. For graphs shown below, determine the minimum number of colors needed to color them. Justify your answer. (Coloring of a graph with max vertex degree D.)
8. Bipartite graphs. State the sufficient and necessary condition for an undirected graph to be bipartite. Perfect matching. State Hall's theorem. For the bipartite graph below..., tell whether it has a perfect matching. Justify your answer.
9. Prove that if a bipartite graph G = (L, R, V) has a Hamiltonian cycle then it has a perfect matching.
10. Define a planar graph.
11. State Kuratowski's theorem (and define the terms used in this theorem).
12. Give Euler's formula for planar graphs (and define the terms involved in it.)
13. Using Euler's formula, prove that in a planar graph, $m \leq 3n-6$ (where m,n denote the numbers of edges and vertices.)
14. Use Euler's inequality ($m \leq 3n-6$) to prove that each planar graph has a vertex of degree at most 5.
    Stronger version: Each sufficiently large planar graph must have at least a certain number of vertices of degree at most 5. What is this number? Justify your answer.
15. Given the graphs below (...), determine whether they are planar or not. If a graph is planar, show a planar embedding. If a graph is not planar, prove it. (You can use Euler's inequality, Kuratowski's theorem, or a direct argument.)
16. How many colors are needed to color planar graphs? Give a proof that each planar graph can be colored with at most 6 colors. (Hint: induction. Use the fact that each planar graph has a vertex of degree no more than 5.)
17. Prove that a graph with the given degree sequence can be colored with k colors.

18. For a given directed graph, determine its strongly connected components.
19. Using the adjacency matrix of a given graph, compute its transitive closure.
20. For a given directed graph, prove that it is acyclic by showing its topological ordering.

Trees

1. Prove, by induction, that a tree with n nodes has exactly n-1 edges.
2. Binary trees
    - Prove (by induction) that a binary tree of height h has at most 2^h leaves.
    - Prove (by induction) that a binary tree of height h has at most $2^{h+1}-1$ nodes.
    - Draw a decision (comparison) tree for sorting 4 items.
    - Similarly as for sorting, we can consider decision trees for other problems that can be solved using comparisons. Draw a decision tree for selecting the second smallest element out of five.