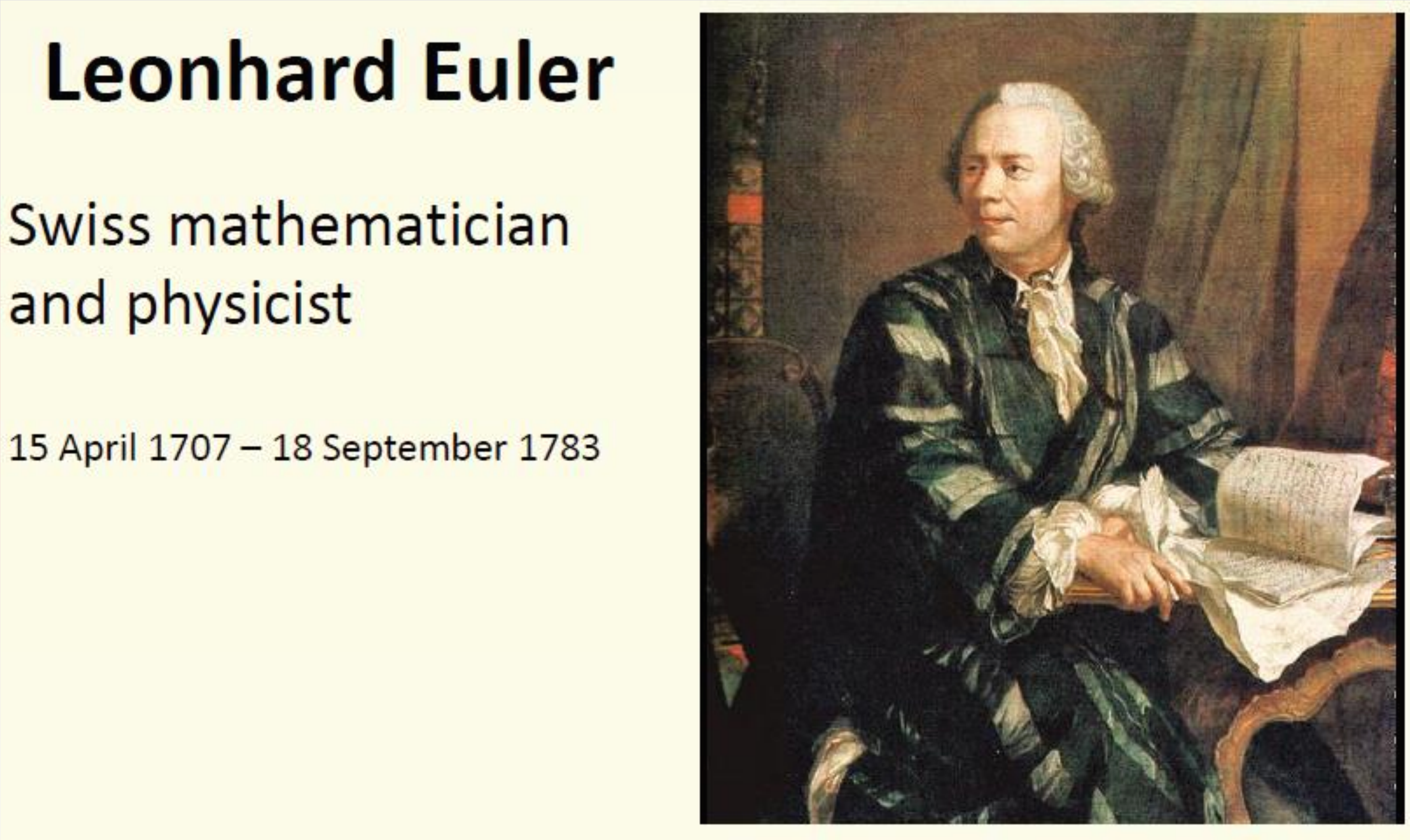
Graphs,
Euler Tour,
Hamiltonian Cycle,
Dirac’s Theorem,
Ore’s Theorem

Euler Tour

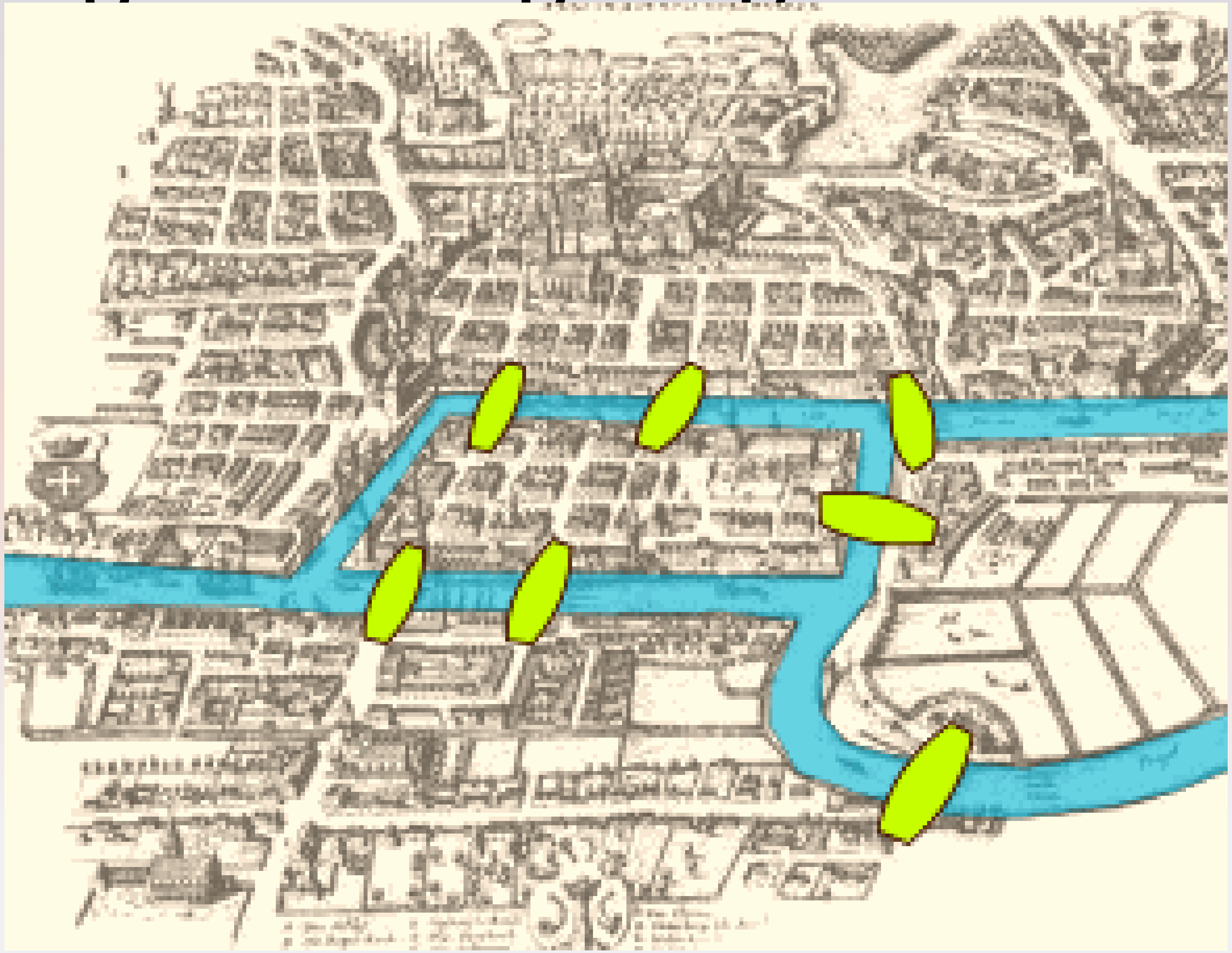

Leonhard Euler
Swiss mathematician and physicist
15 April 1707 – 18 September 1783

Original Problem

A resident of Konigsberg wrote to Leonard Euler saying that a popular pastime for couples was to try to cross each of the seven beautiful bridges in the city exactly once -- without crossing any bridge more than once.

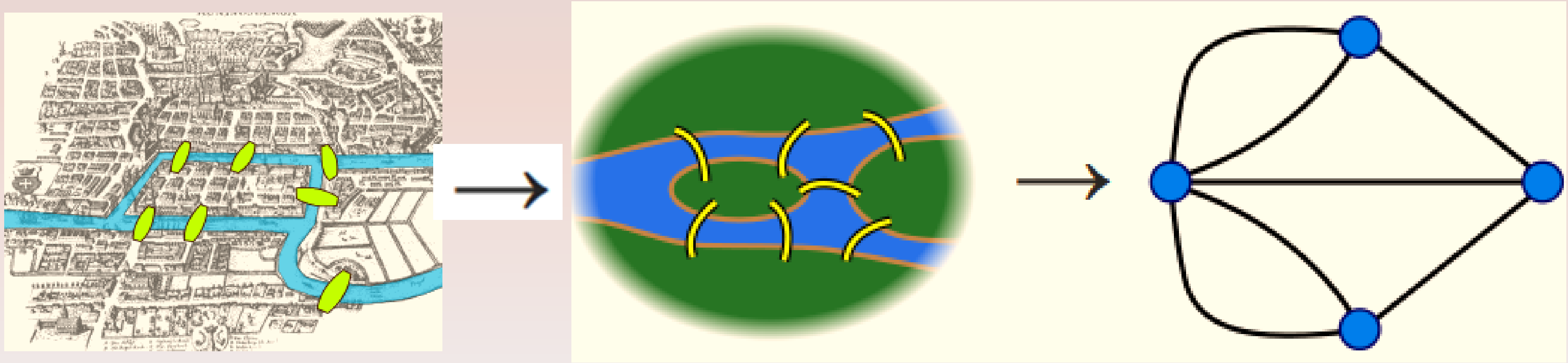
It was believed that it was impossible to do – but why? Could Euler explain the reason?

Euler Tour
Seven Bridges of Königsberg



Euler Invents Graph Theory

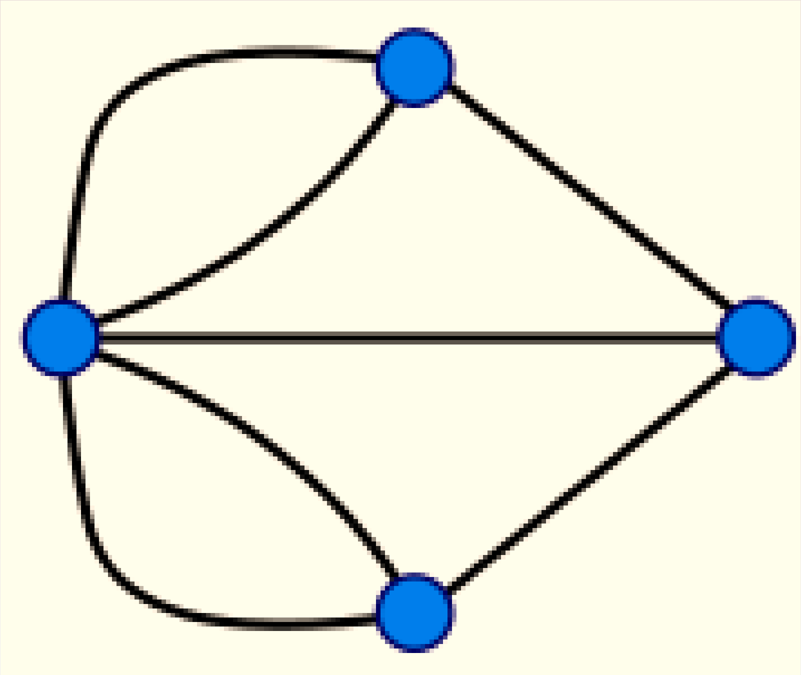
Euler realized that all problems of this form could be represented by replacing areas of land by points (what we call nodes), and the bridges to and from them by arcs (edges).



He formulated the solution to this type of problems as a theorem, which is considered to be the first theorem in graph theory.

Euler Path/Tour

The problem now becomes one of drawing this picture without retracing any line and without taking your pencil off the paper.



Graphs

Definition: A graph is an ordered pair $G = (V, E)$, where V is a set of nodes (vertices), and $E \subseteq V$xV is a set of edges.

If $e = (v_1, v_2) \in E$

- v_1 and v_2 are end-nodes of e
- e is *incident* to v_1 and v_2
- v_1 and v_2 are *adjacent*

Graphs

Definition: A **graph** is **connected** if there is a path between every pair of vertices. In a **connected graph**, there are no unreachable vertices.

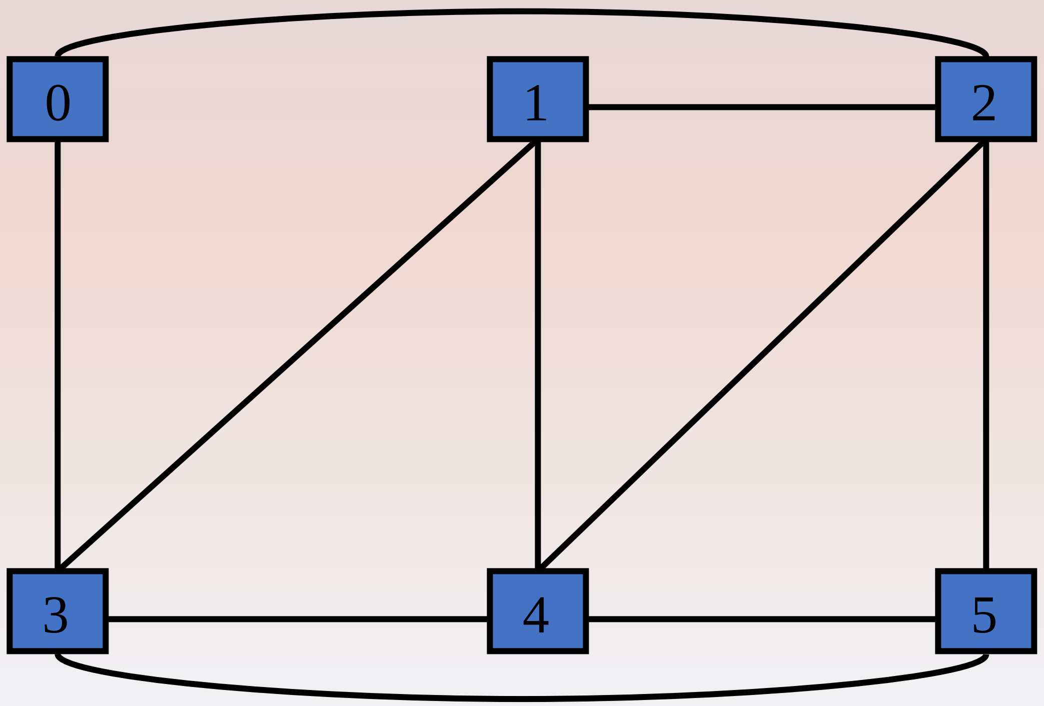
Give definitions of a:

- *Connected component of a graph*
- *Strongly connected component*
- *Weakly connected component*

Graphs

Example: Vertices = {0,1,2,3,4,5}

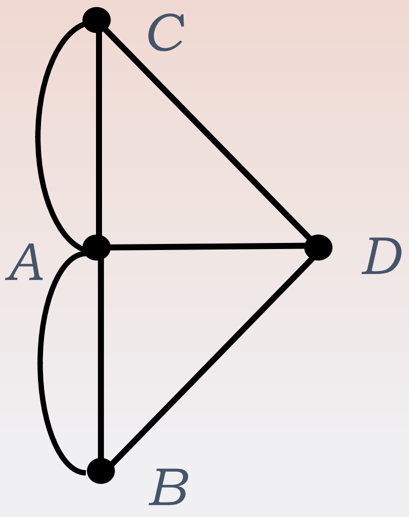
Edges = {{0,2}, {1,2}, {1,3}, {1,4}, {2,4}, {2,5}, …}



Euler Path and Tour

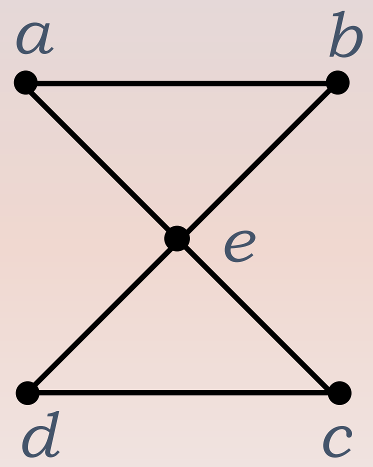
- An *Euler path* is a continuous path that passes through every edge exactly one time.
- An *Euler tour* is an Euler path that begins and ends at the same vertex: (tour that traverses each edge of the graph exactly once).

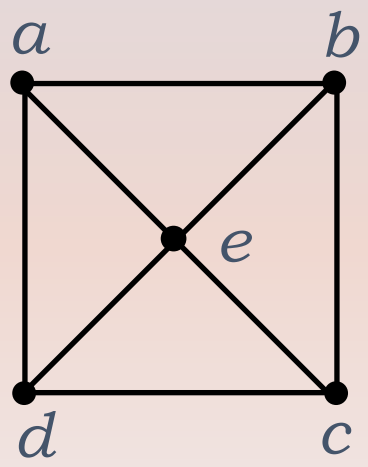
Does this graph have an Euler tour?

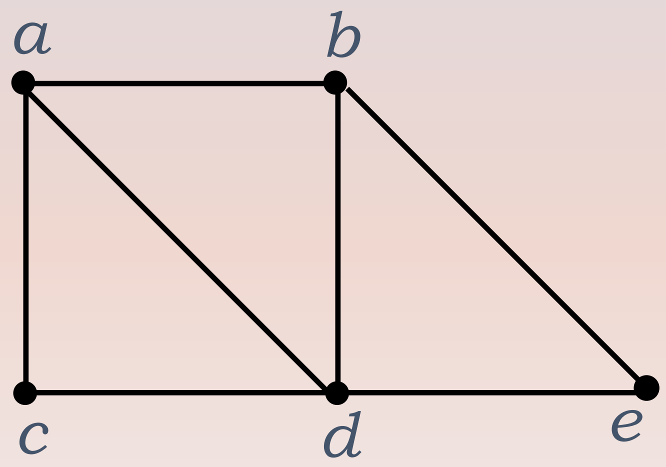


Example

- Which of the following graphs has an Euler *path*? Euler *tour*?

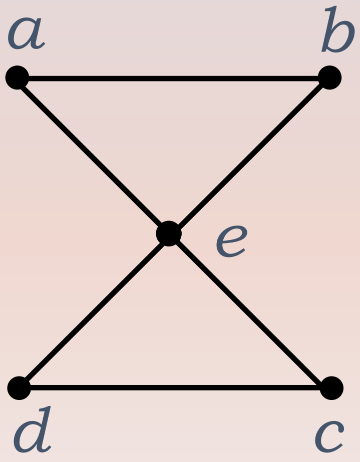




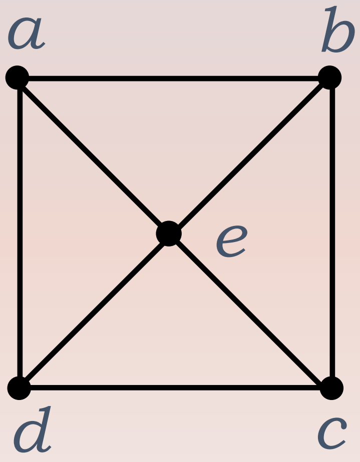


Example

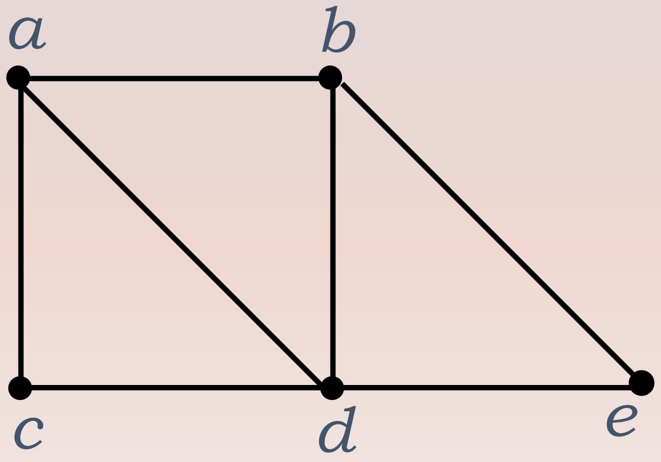
- Which of the following graphs has an *Euler path? Euler tour?*



yes

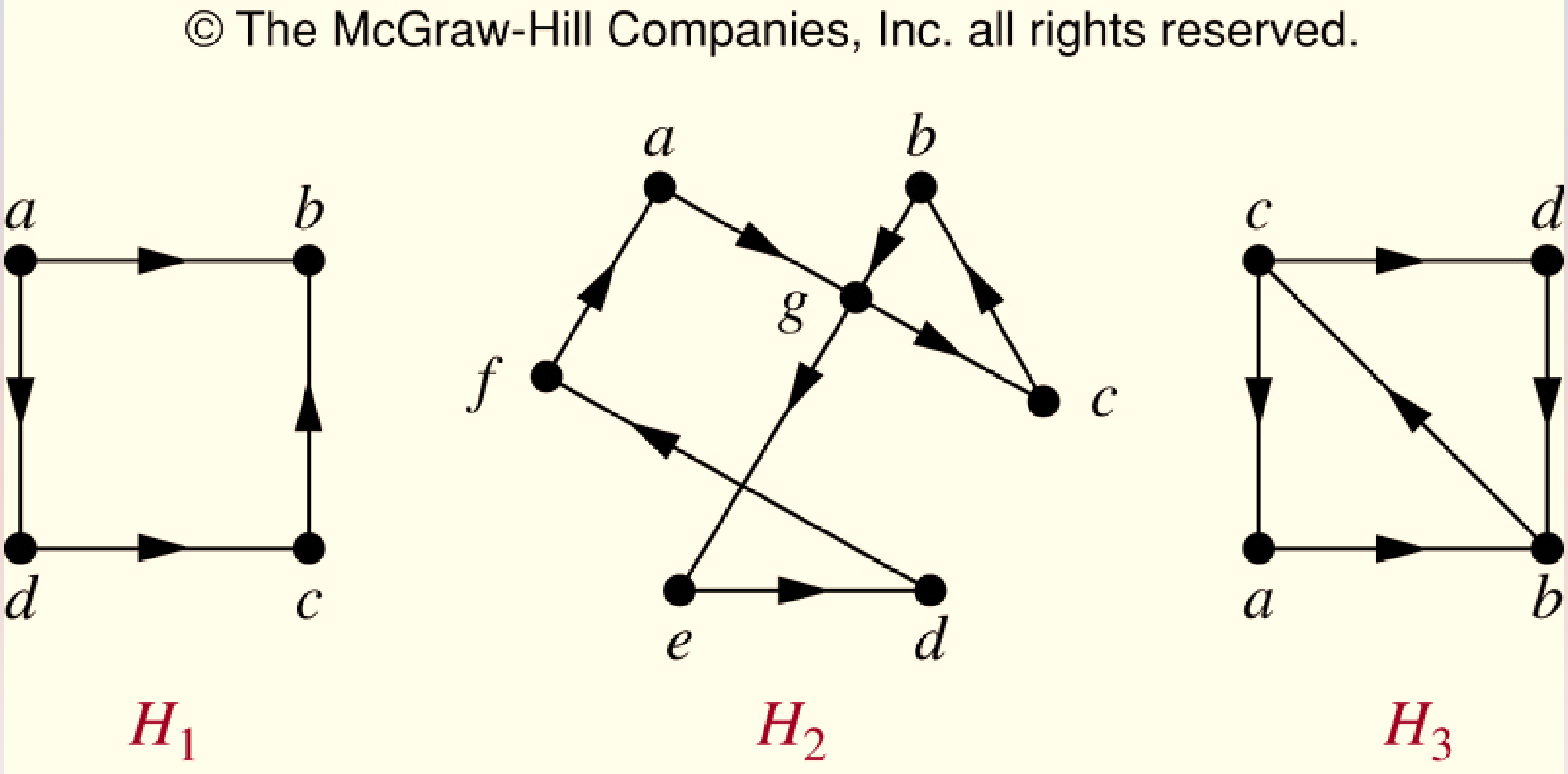


no

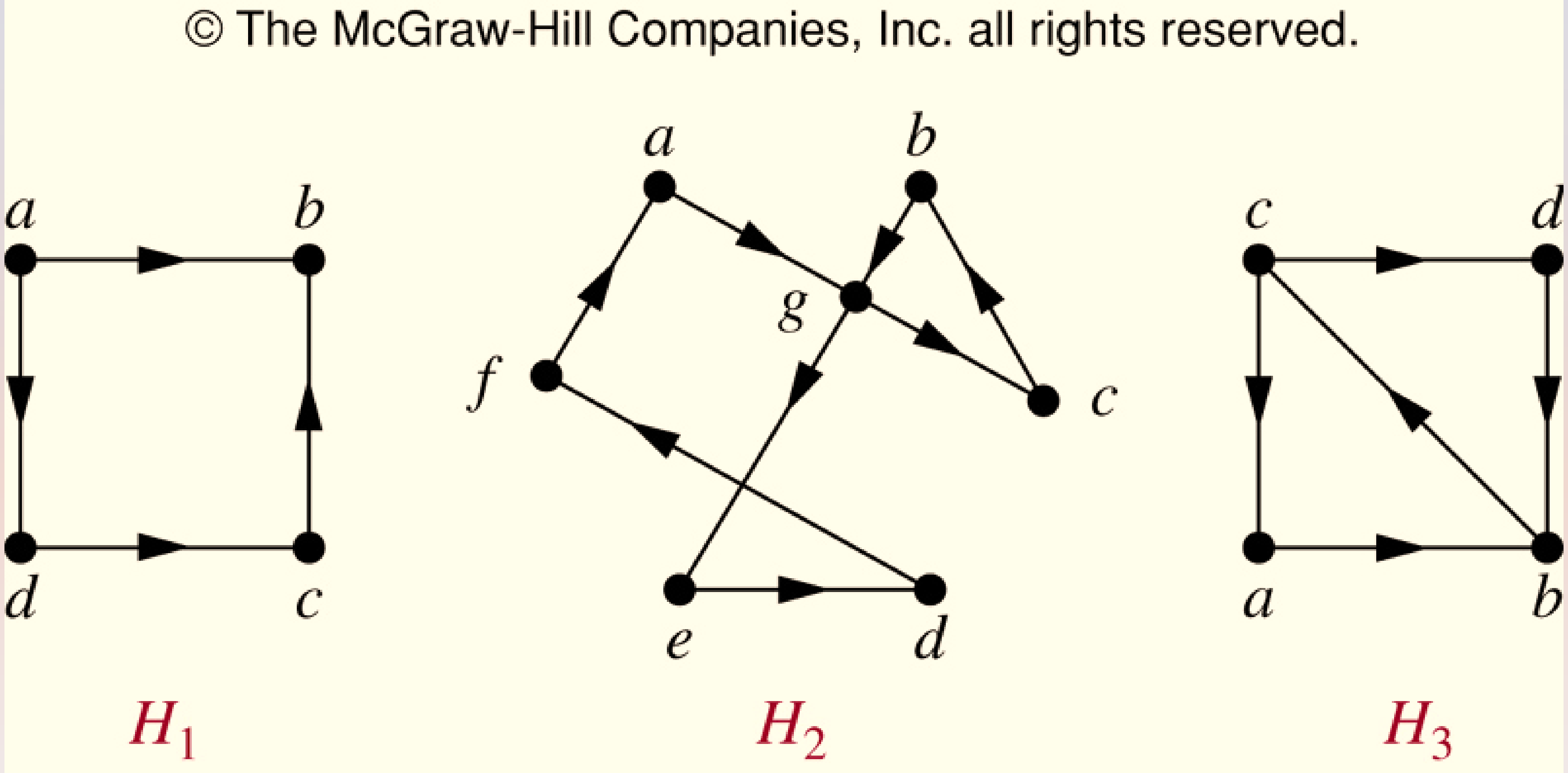


yes/no

Euler Path, Tour in Directed Graphs

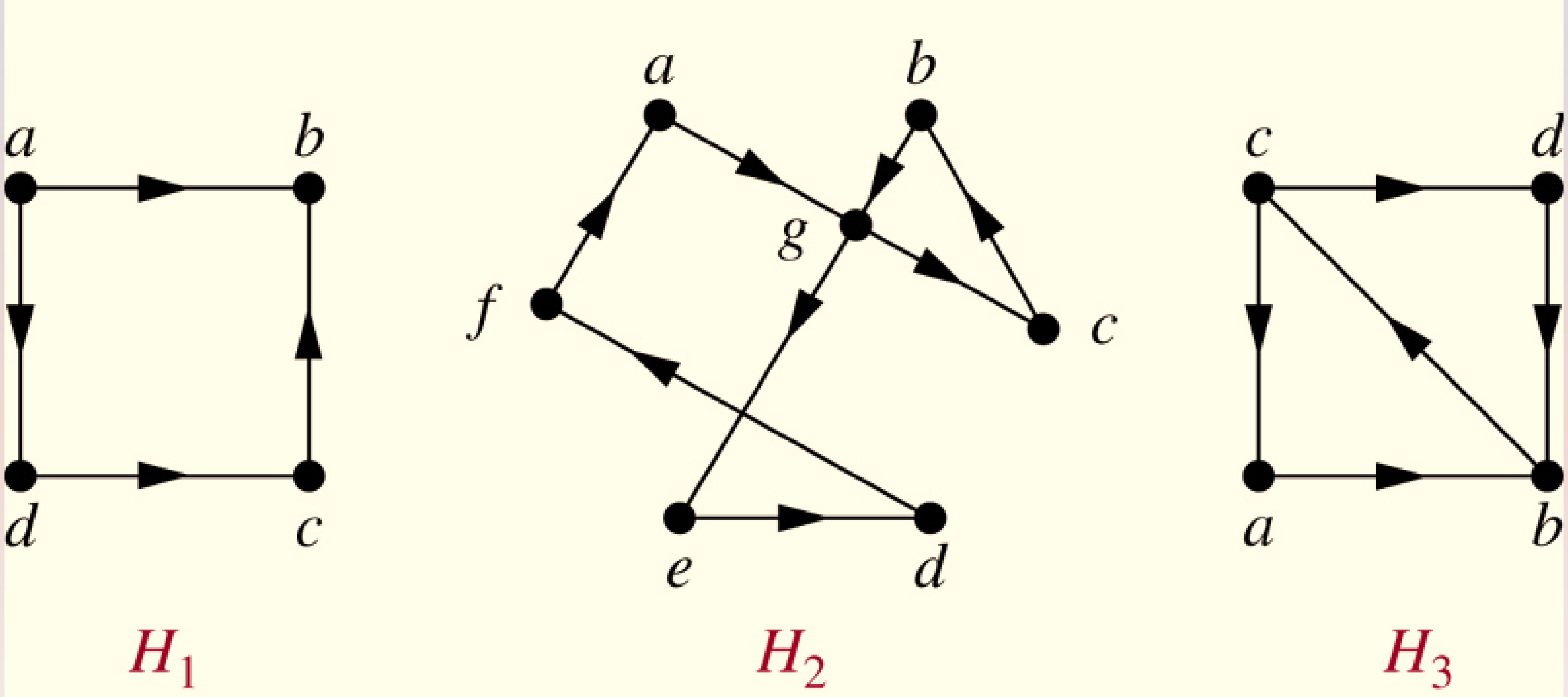


Euler Path in Directed Graphs



NO (a, g, c, b, g, e, d, f, a) (c, a, b, c, d, b)

Euler Tour in Directed Graphs

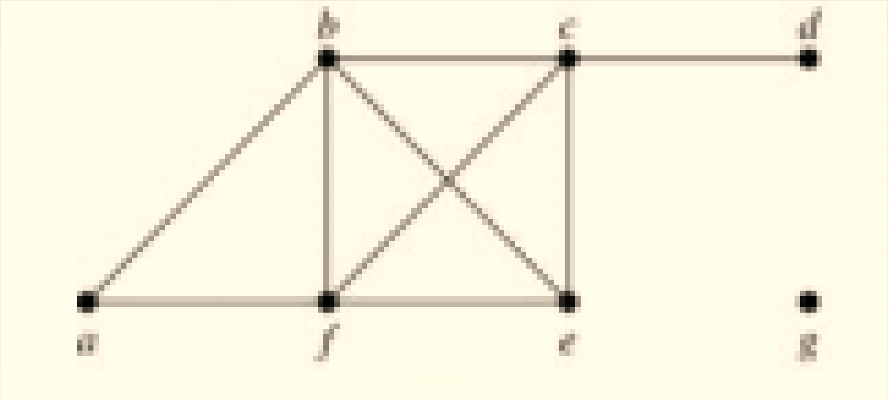


NO (a, g, c, b, g, e, d, f, a) NO

Graphs

Definition: The degree of a vertex in a undirected graph is the number of edges incident to the vertex. The degree of the vertex **v** is denoted by **deg(v)**.

Example: What are the degrees of the vertices in the graph?



deg(a) = 2,

deg(b) = deg(c) = deg(f) = 4,

deg(d) = 1, deg(e) = 3, deg(g) = 0.

Handshaking Lemma

Let G = (V, E) be a connected undirected graph.

Then

$\sum_{i=1}^{n}$ (deg(vi))= 2|E| (where |E| is the number of edges in E)

(In any graph the sum of the vertex degrees is equal to twice the number of edges.)

Required

Lemma

Let $G = (V, E)$ be a connected undirected graph,
s.t. $|V| = n$, and $|E| = m$.
Then $m \geq n - 1$.

Proof (by Math Induction on n).

Required

Lemma

Let $G = (V, E)$ be a connected undirected graph, s.t. $|V| = n$, and $|E| = m$. Then $m \geq n - 1$.

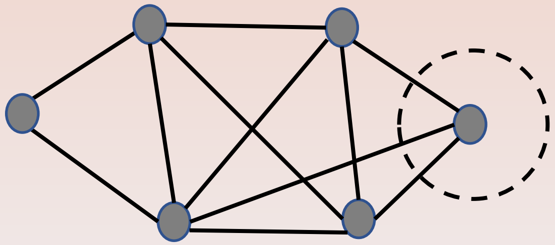
Proof (by Math Induction on n).

Base case: $n = 1$, then $m = 0$, $0 = 0$.

Inductive assumption: suppose, for all $n \leq k$ $(k \geq 1)$, $m \geq k - 1$.

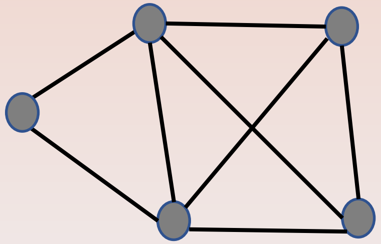
Need to prove: for $n = k + 1$: $m \geq k$.

G



$k + 1$ vertices, m edges

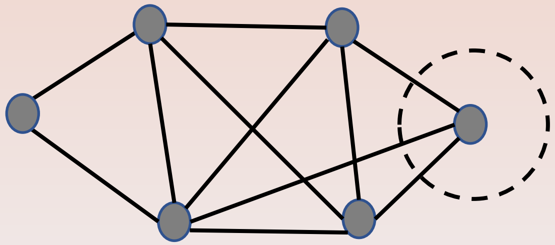
G`



k vertices, $m`$ edges

$m` \geq k - 1$ (assumption)

G



$k + 1$ vertices, m edges

$m \geq m` + 1 \geq k - 1 + 1$

$m \geq k$

Required

Which proves that the claim holds for $n = k + 1$, and thus proves the lemma.

Euler Tour

Theorem:

Let G = (V, E) be a connected undirected graph.

Then G has an Euler tour if and only if …

Euler Tour

Theorem:

Let G = (V, E) be a connected undirected graph.

Then G has an Euler tour if and only if all vertices in G have an even degree.

Required for EC

Euler Tour

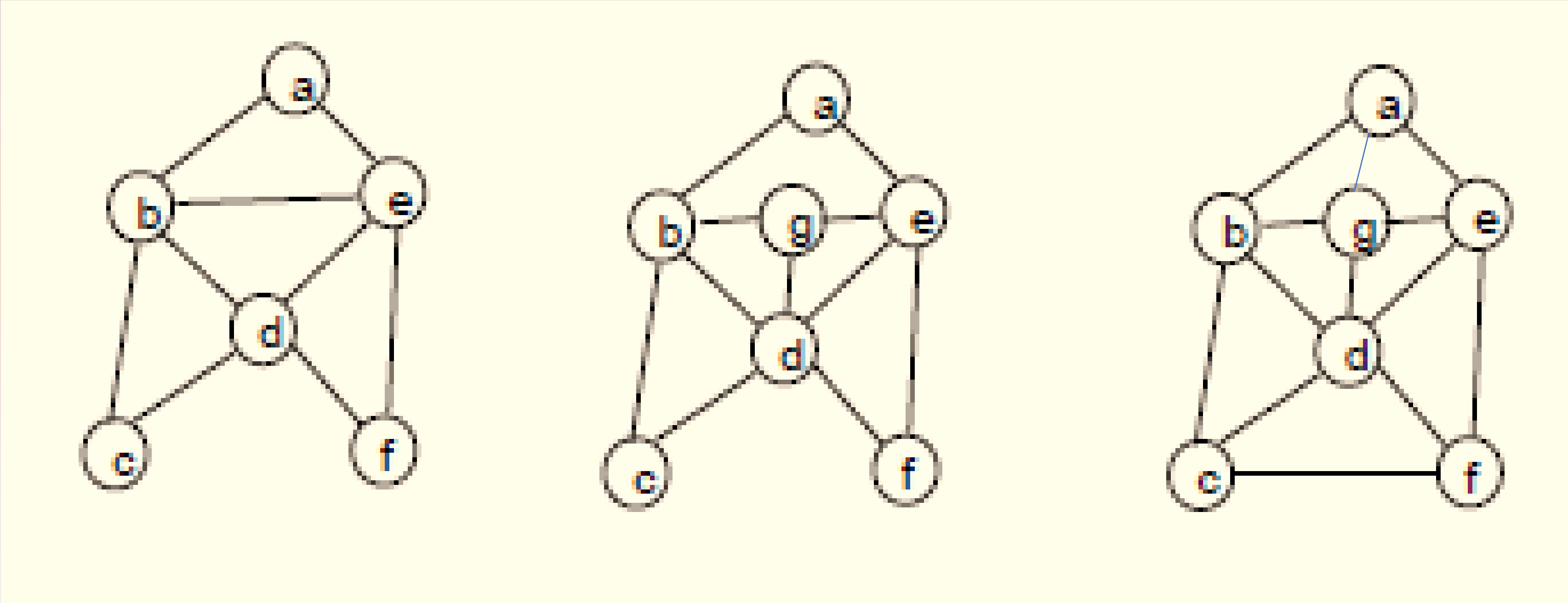
Necessary condition: …

Sufficient condition: Let's construct a tour. We begin at some initial node v_0 and draw a tour through G (thus eventually returning to v_0). Let this tour be denoted T_0. If T_0 happens to be an Euler tour, this is fine, we stop. If T_0 is not an Euler tour, then if we remove from G all edges used by tour T_0 (and call the new graph G'), there must be some edges left over. Moreover, at least two of these edges must be incident to some node v_1 through which tour T_0 has passed . This must be so since, by assumption, G is, first, connected and, second, all its nodes are of even degree (and T_0 has only used up an even number of edges which are incident to v_1). Thus, it is possible to draw another tour T_1, originating and terminating at v_1, which uses only edges of G', the graph left after we eliminated the edges of T_0 from G.

This procedure may now be continued until eventually, say after the n^{th} step, there will be no edges left uncovered. At that time, an Euler tour will also have been obtained which will be a combination of tours T_0, T_1, T_2, . . ., T_n.

Euler Tour

Which graph has an Euler Tour?

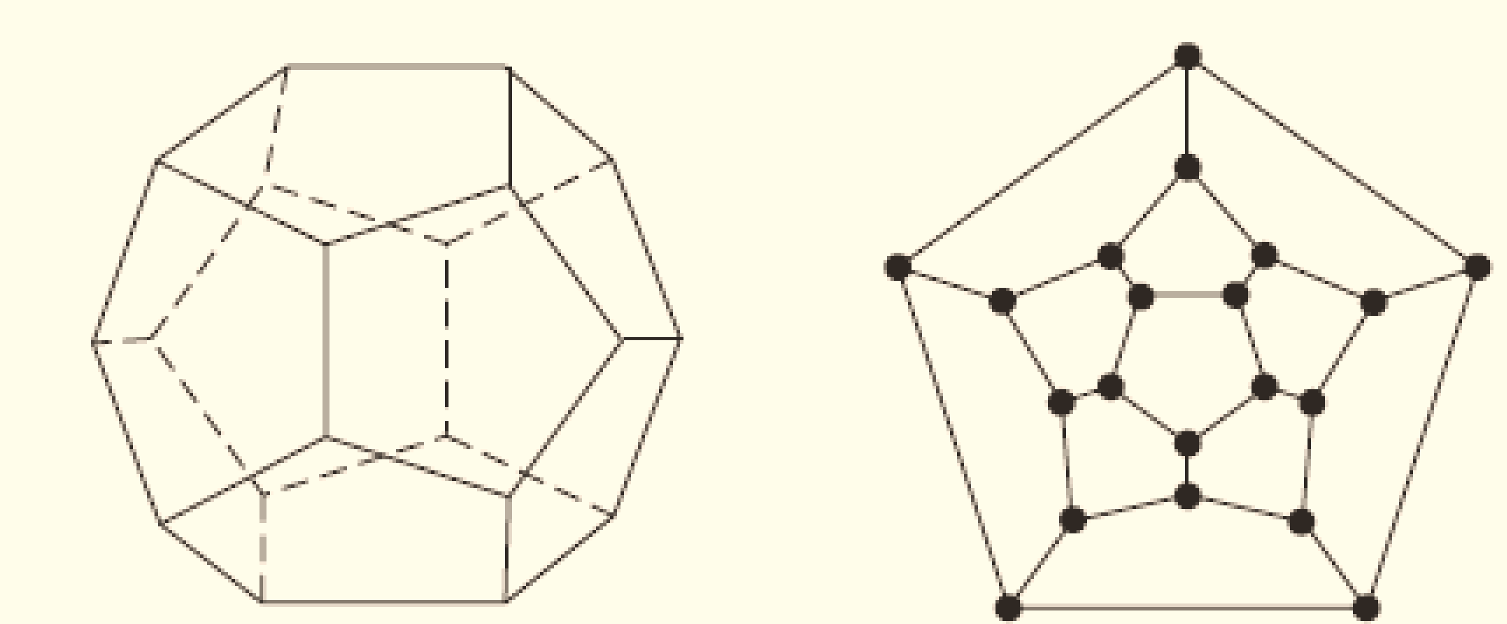


Hamiltonian Path and Cycle

- A *Hamiltonian path* in a graph G is a path which visits every vertex in G exactly once.

- A *Hamiltonian cycle* is a Hamilton path that begins and ends at the same node.

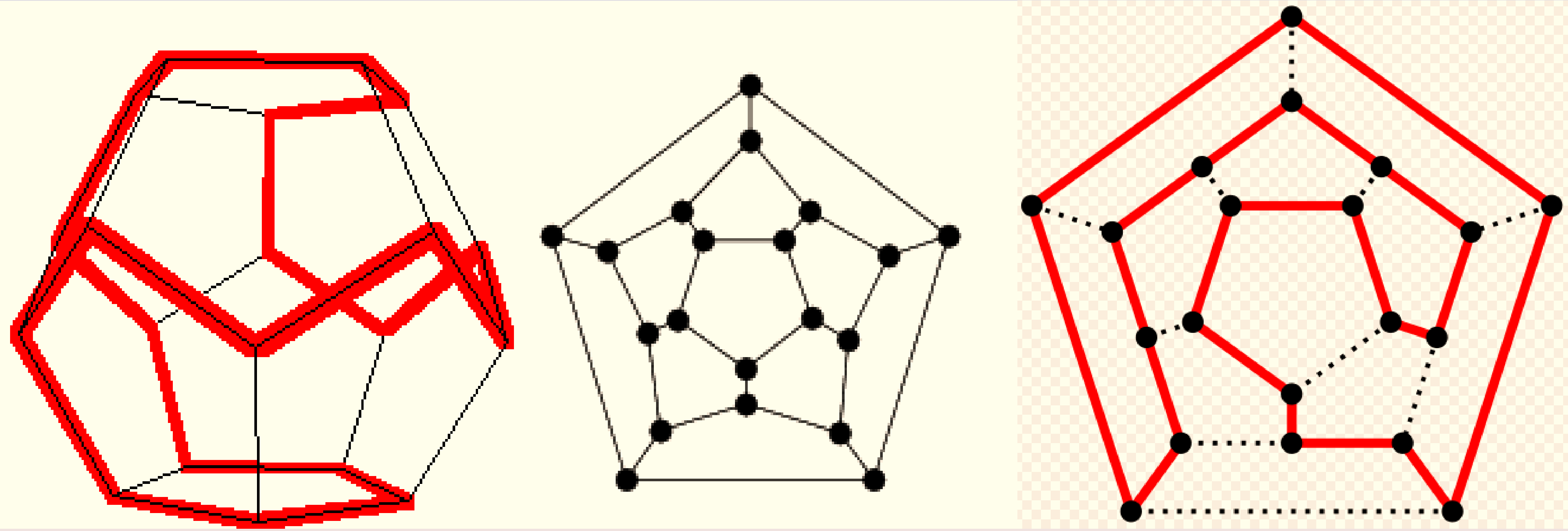
Hamiltonian Cycle



Dodecahedron puzzle and it’s equivalent graph

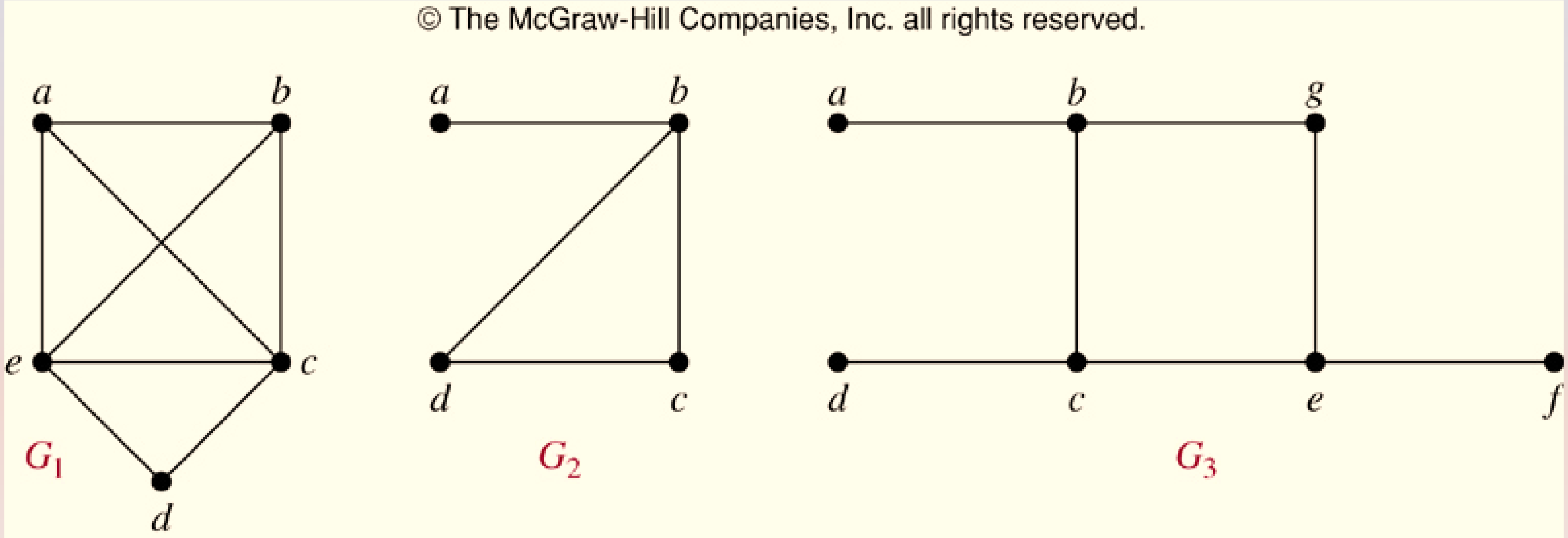
Is there a cycle in this graph that passes through each vertex exactly once?

Hamiltonian Cycle



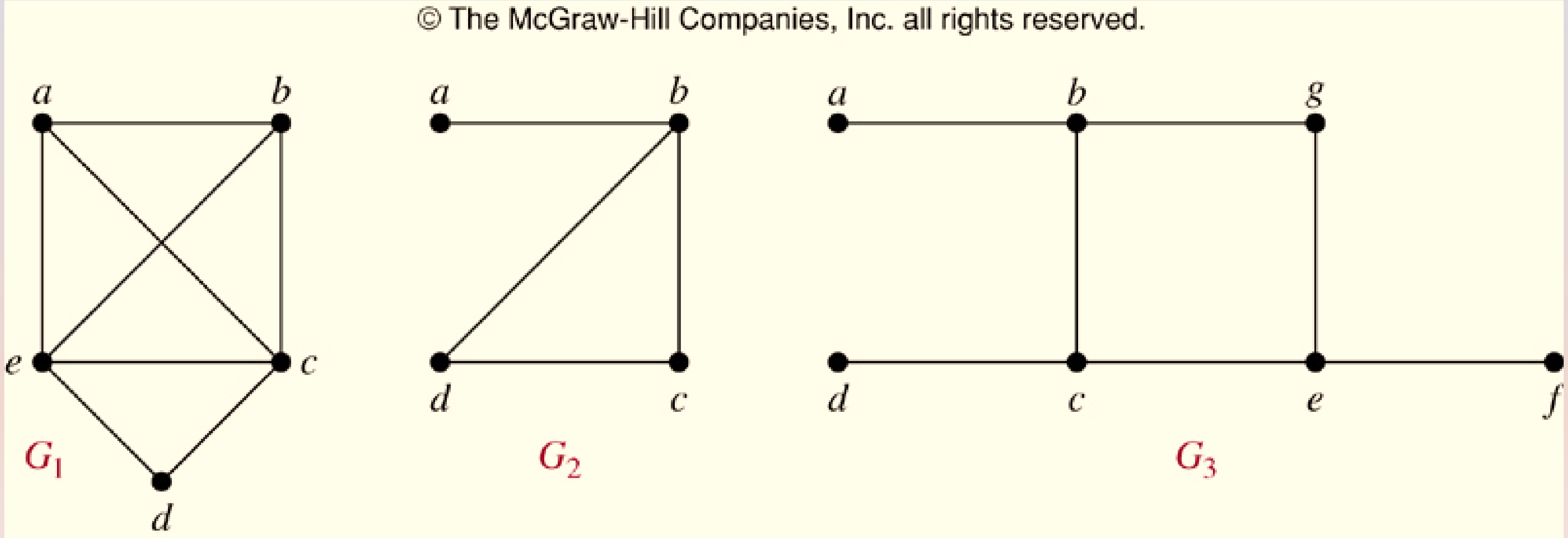
Yes; this is a cycle that passes through each vertex exactly once.

Finding Hamiltonian Cycle



Which graph has a Hamilton cycle?
Or, if no Hamilton cycle, a Hamilton path?

Finding Hamiltonian Cycle



- G_1 has a Hamilton cycle: a, b, c, d, e, a
- G_2 does not have a Hamilton cycle, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Dirac's Theorem

Theorem: Let G = (V, E) be a connected graph with **n** vertices in which each vertex has a degree at least **n/2**. Then G has a Hamiltonian cycle.

Proof: Let's show that if G satisfies the condition in the theorem then we can *construct* a Hamiltonian cycle in G.

Required for EC

Dirac's Theorem

Idea of the Proof: Pick some vertex v_1 arbitrarily, and gradually extend a path P starting from v_1, say P = $v_1 v_2 ... v_k$, where all vertices v_j are different. Eventually, if k = n, P will be a Hamiltonian path.

Initially, P = (v_1). Suppose that we have already constructed P = $v_1 v_2 ... v_k$. Let's show that as long as k < n we can always extend P.

Dirac's Theorem

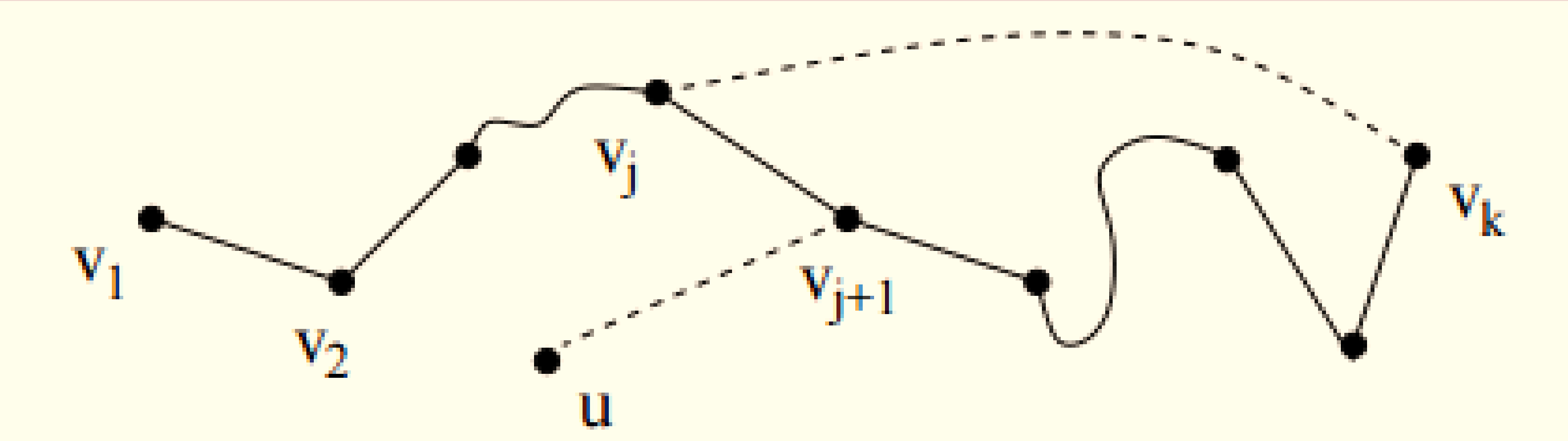
Case 1: v_k has a neighbor u ∈ V that is not on P.

If v_k has a neighbor u ∈ V that is not on P, then it is easy to extend P, for we can append u at the end of P. In other words, we can take v_{k+1} = u and the new extended path will be $v_1\ v_2 ...\ v_k\ v_{k+1}$.

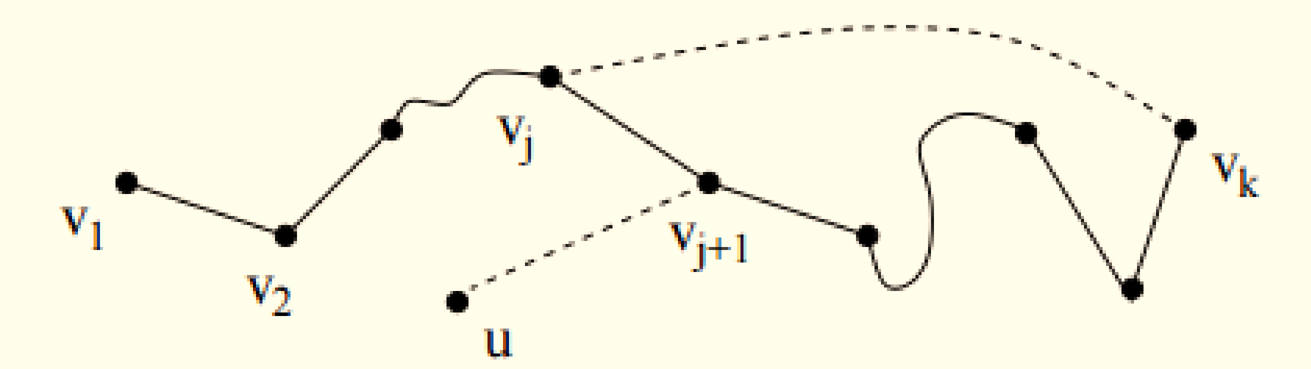
Dirac's Theorem

Case 2: All neighbors of v_k are on P.

Let's show that there is a neighbor v_j of v_k such that v_{j+1} has a neighbor outside P. Then we will perform a switch operation that transforms P into the following path: $v_1\, v_2 \ldots\, v_j\, v_k\, v_{k-1} \ldots\, v_{j+1}$ u, as in the figure below.



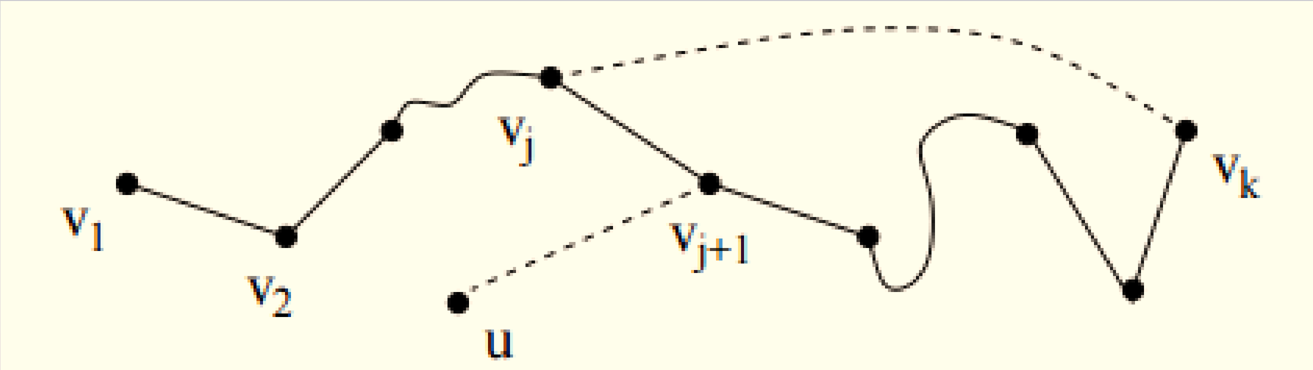
Dirac's Theorem



It is now sufficient to prove that such vertex v_j always exists. Since all neighbors of v_k are on P and are different than v_k, we have $k - 1 \geq deg(v_k) \geq n/2$, so $k \geq n/2 + 1$.

For each neighbor v_j of v_k, we mark the next vertex on P, that is v_{j+1}. Since all neighbors of v_k are on P, this way we will mark $deg(v_k)$ vertices.

Dirac's Theorem



Consider any vertex **u** not on **P**. Suppose, none of **u**'s neighbors were marked.

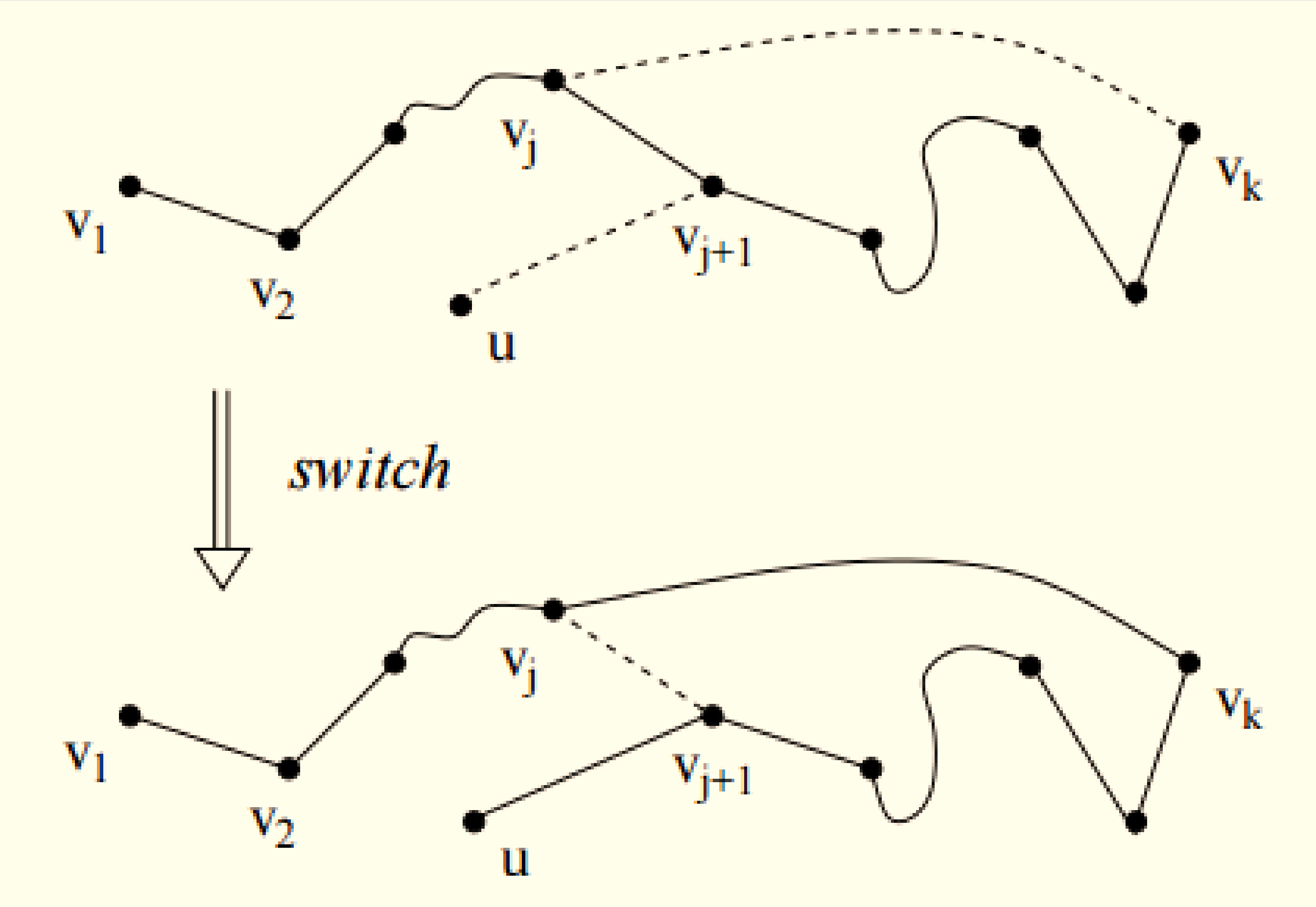
Then the total number of vertices in G is:

all of **u**'s neighbors, the marked vertices, and **u** itself:

$|V| \geq \deg(u) + k + 1 \geq n/2 + n/2 + 1 > n$ – a contradiction.

Therefore there must be a marked vertex that is a neighbor of **u**. Then there is a vertex v_j , a neighbor of v_k such that v_{j+1} has a neighbor outside P, and the switch operation can be applied.

Dirac's Theorem



Using the same arguments, we can continue until a Hamiltonian path P is constructed.

Dirac's Theorem

We have proved that G has a Hamiltonian path, but we need to prove that G has a Hamiltonian cycle.

Prove: you need to show how (under the assumptions from the theorem) you can convert P into a Hamiltonian cycle.

Ore’s Theorem

Ore’s Theorem: Let $G = (V, E)$ be a connected graph with $n \geq 3$ vertices. If G has the property that for each pair of non-adjacent vertices $u, v \in V$, **deg u + deg $v \geq$ n** then G contains a Hamiltonian cycle.

Required for EC