Linear Recurrence Relations

Linear Homogeneous Recurrence Relations

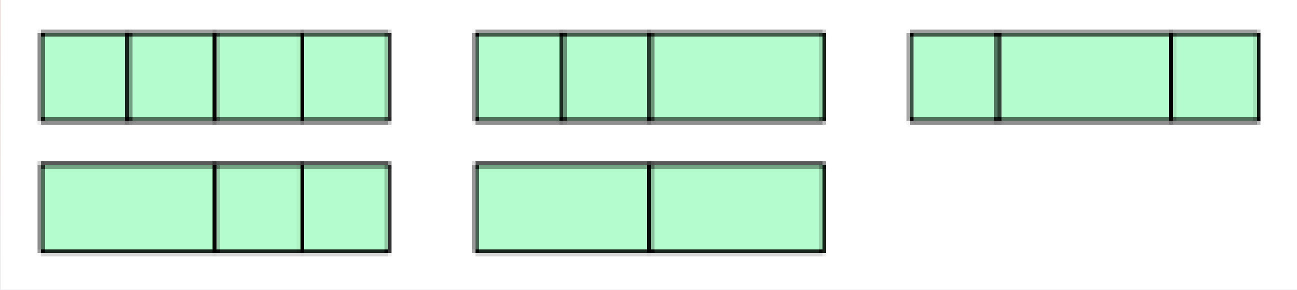
Tiles and Gold

Consider the following problem. We have two types of golden tiles, 1×1 tiles and 2×1 tiles. We want to use those tiles to cover an $n \times 1$ strip. Tiles cannot overlap and must cover exactly the strip. In how many ways this can be done for $n = 4$?

Linear Homogeneous Recurrence Relations

Tiles and Gold

Consider the following problem. We have two types of golden tiles, 1×1 tiles and 2×1 tiles. We want to use those tiles to cover an $n \times 1$ strip. Tiles cannot overlap and must cover exactly the strip. In how many ways this can be done? Figure shows all five tilings for $n = 4$.



Linear Homogeneous Recurrence Relations

Tiles and Gold

Denote by t_n the number of such tilings. For n = 0 we have $t_0 = 1$, because n = 0 represents an empty strip that requires no tiles, so there is one tiling, namely the tiling that has no tiles at all. For n = 1 we have $t_1 = 1$.

What about large n?

Linear Homogeneous Recurrence Relations

Tiles and Gold

Consider two types of such tilings:

Type-1 tilings ending with a 1 × 1 tile,

Type-2 tilings ending with a 2 × 1 tile.

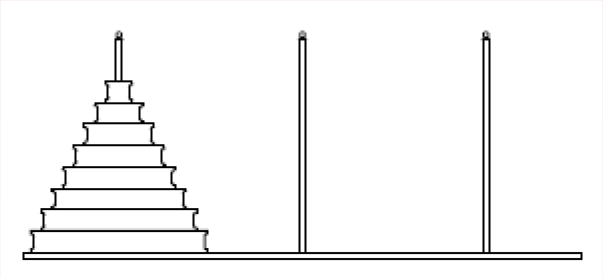
We have t_{n-1} tilings of Type 1, because in those tiles the last unit is occupied by a 1 × 1 tile but the remaining part, of length n − 1, can be tiled in t_{n-1} ways. Similarly, we have t_{n-2} tilings of Type 2. Putting it all together, we have

$$t_n = t_{n-1} + t_{n-2}, \text{ for } n \geq 2$$
$$t_0 = 1$$
$$t_1 = 1$$

Linear Non-Homogeneous Recurrence Relations
The Towers of Hanoi

According to legend, there is a temple in Hanoi with three posts and 64 gold disks of different sizes. Each disk has a hole through the center so that it fits on a post. In the misty past, all the disks were on the first post, with the largest on the bottom and the smallest on top.

Monks in the temple have labored through the years since to move all the disks to one of the other two posts according to the following rules:
The only permitted action is removing the top disk from one post and dropping it onto another post.
A larger disk can never lie above a smaller disk on any post.



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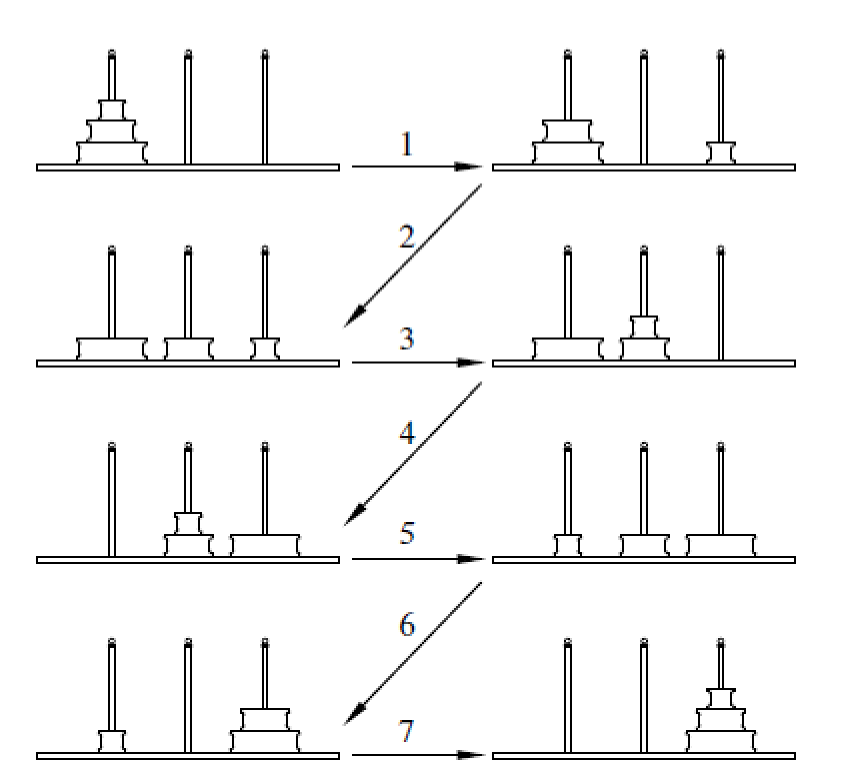
Linear Non-Homogeneous Recurrence Relations
The Towers of Hanoi



7 – step solution for n = 3:

$T_0 = 0$ $T_1 = 1$ $T_2 = 3$ $T_3 = 7$

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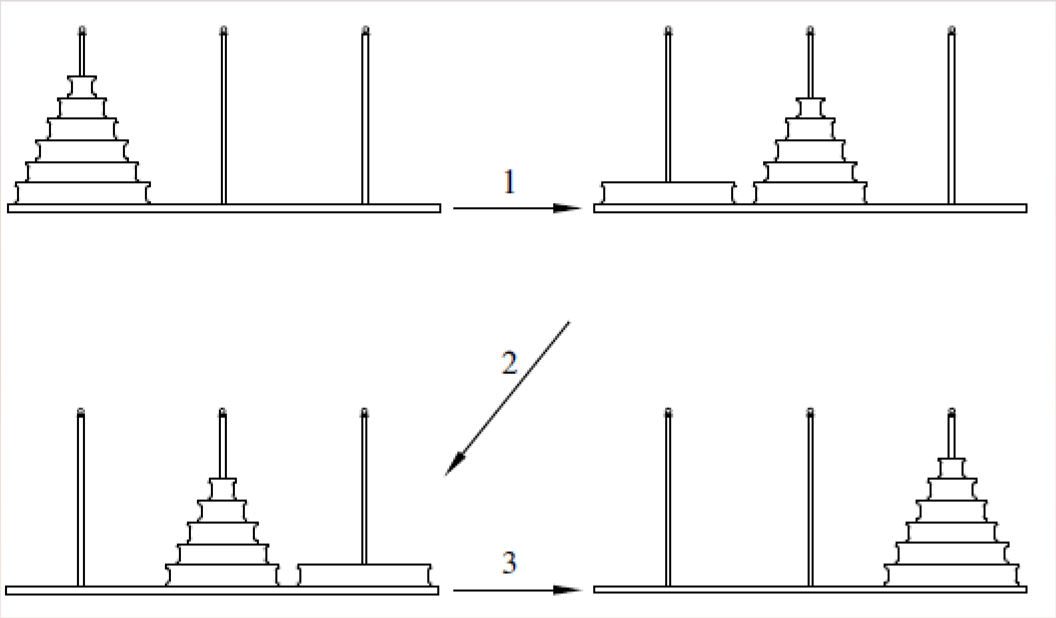
Linear Non-Homogeneous Recurrence Relations

The Towers of Hanoi

Recursive solution:



$$T_0 = 0, \quad \text{for } n \geq 1: \; T_n = 2T_{n-1} + 1$$

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Linear Homogeneous Recurrence Relations

Definition

A linear homogeneous recurrence relation of degree *k* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Where $c_1, c_2, \ldots, c_k$ are real numbers, $c_k \neq 0$.

Fibonacci numbers

$F_n = F_{n-1} + F_{n-2}$, $\quad$ *for* $n \geq 2$ $\qquad$ (1)

$F_0 = 1$

$F_1 = 1$

Since F_n grows exponentially, we will assume that F_n is proportional to x^n for some base x. Let $F_n = x^n$

$$x^n = x^{n-1} + x^{n-2}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$x_1 = \frac{1-\sqrt{5}}{2} \qquad x_2 = \frac{1+\sqrt{5}}{2}$$

Do they satisfy (1) ?

Linear Homogeneous Recurrence Relations

Theorem

Let c_1 and c_2 be real numbers. Suppose that

$r^2 + c_1 r + c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for n = 0, 1, 2,. . ., where α_1 and α_2 are constants.

Linear Homogeneous Recurrence Relations

Example

Solve the recurrence equation

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for} \quad a_0 = 1,\ a_1 = 4$$

Solution:

- The characteristic equation and its roots:

$$r^2 - 5r + 6 = 0, \qquad r_1=2,\ r_2=3$$

- General solution:

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

- General solution:

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

- System of linear equations for initial conditions and its solution:

$$a_0 = \alpha_1 2^0 + \alpha_2 3^0$$

$$a_1 = \alpha_1 2^1 + \alpha_2 3^1$$

$1 = \alpha_1 + \alpha_2$ $\qquad\qquad$ $\alpha_1 = -1$

$4 = 2\alpha_1 + 3\alpha_2$ $\qquad\qquad$ $\alpha_2 = 2$

- Final answer:

$$a_n = -1 \cdot 2^n + 2 \cdot 3^n$$

Linear Homogeneous Recurrence Relations

Example Solve the recurrence equation $Q_n = 2Q_{n-1} + 4Q_{n-2}$, for $Q_0 = 0$, $Q_1 = 2$. Follow the steps below.

(a) Characteristic polynomial and its roots:

$$x^2 - 2x - 4 = 0$$

The roots are $r_1 = 1 + \sqrt{5}$ and $r_2 = 1 - \sqrt{5}$.
(b) General solution:

$$Q_n = \alpha_1 \cdot \left(1 + \sqrt{5}\right)^n + \alpha_2 \cdot \left(1 - \sqrt{5}\right)^n$$

(c) Equations for initial conditions and its solution:

$$\alpha_1 + \alpha_2 = 0$$
$$\alpha_1(1 + \sqrt{5}) + \alpha_2(1 - \sqrt{5}) = 2$$

So $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$.

(d) Final answer:

$$Q_n = \frac{1}{\sqrt{5}} \cdot \left(1 + \sqrt{5}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(1 - \sqrt{5}\right)^n$$

Linear Homogeneous Recurrence

Example

Solve the recurrence relation $a_n = 4\,a_{n-1} - 4\,a_{n-2},$ with $a_0 = 1$ and $a_1 = 3$?

$$x^2 - 4x + 4 = 0$$

$$x_{1,2} = 2$$

?

Linear Homogeneous Recurrence Relations

Theorem

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 + c_1 r + c_2 = 0$ has a double root r_0.

A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

for n=0, 1, 2,. . .,where α_1 and α_2 are constants.

Linear Homogeneous Recurrence

Example

Solve the recurrence relation $a_n = 4\, a_{n-1} - 4\, a_{n-2},$ with $a_0 = 1$ and $a_1 = 3$?

$$x^2 - 4x + 4 = 0$$

$$x_{1,2} = 2$$

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n$$

$$a_n = 2^n + \frac{1}{2} n 2^n$$

Linear Homogeneous Recurrence Relations

Example

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

With the initial conditions

$$a_0 = 1 \text{ and } a_1 = -2, \text{ and } a_2 = -1.$$

$$x^3 + 3x^2 + 3x + 1 = 0$$

$$x_{1,\,2,\,3} = -1$$

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n + \alpha_3 n^2 r_0^n$$

Linear Homogeneous Recurrence Relations

Example

Find the solution to the recurrence relation

$$g_0 = 1, \quad g_1 = 2;$$
$$g_n = 2g_{n-1} - 2g_{n-2} \quad \text{for } n > 1$$

has the characteristic equation

$$\tau^2 - 2\tau + 2 = 0$$

with roots

$$\tau_1 = 1 + i \quad \text{and} \quad \tau_2 = 1 - i$$

Thus, the general solution is

$$g_n = b_1(1+i)^n + b_2(1-i)^n$$

Linear Homogeneous Recurrence Relations

The init conds yield the complex simultaneous equations

$$
\begin{aligned}
g_0 &= 1 = b_1(1+i)^0 + b_2(1-i)^0 = b_1 + b_2 \\
g_1 &= 2 = b_1(1+i)^1 + b_2(1-i)^1
\end{aligned}
$$

with solution

$$b_1 = \frac{i+1}{2i} \qquad b_2 = \frac{i-1}{2i}$$

Hence, the general solution is

$$g_n = \frac{1}{2i}(1+i)^{n+1} - \frac{1}{2i}(1-i)^{n+1}$$

Linear Homogeneous Recurrence Relations

Word problem (Extra credit)

Find a recurrence relation for the number of ternary strings (strings formed from 0, 1, 2) that do not contain three consecutive 2s.

Initial conditions: …

Recurrence:

Justification:

Linear Homogeneous Recurrence Relations

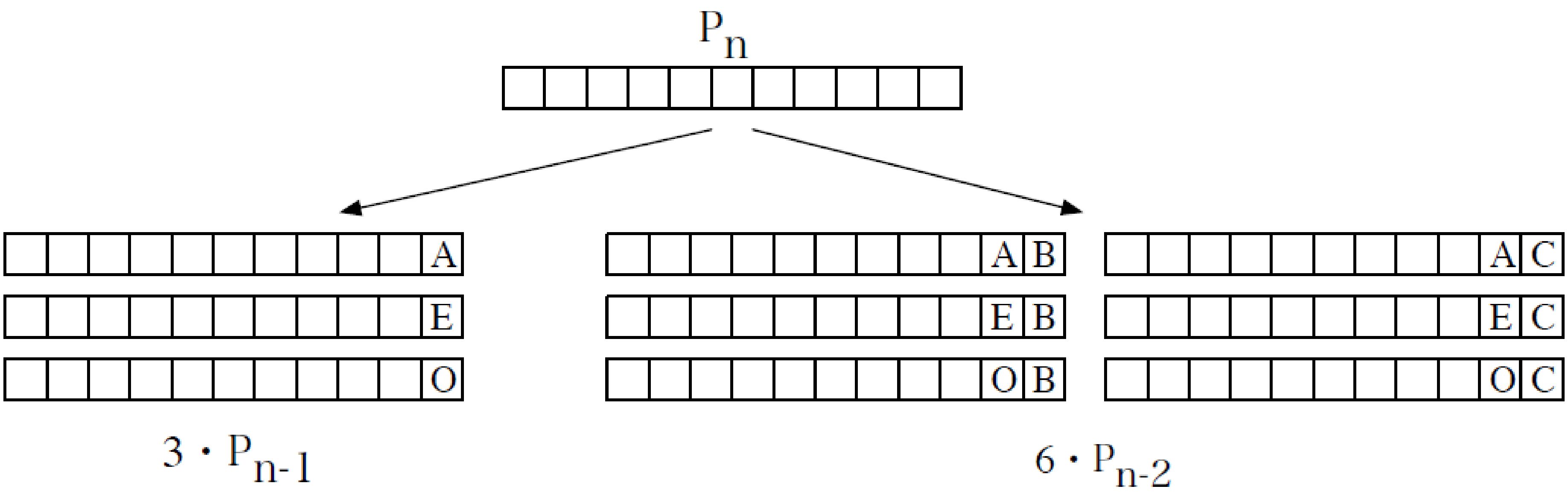
Word problem

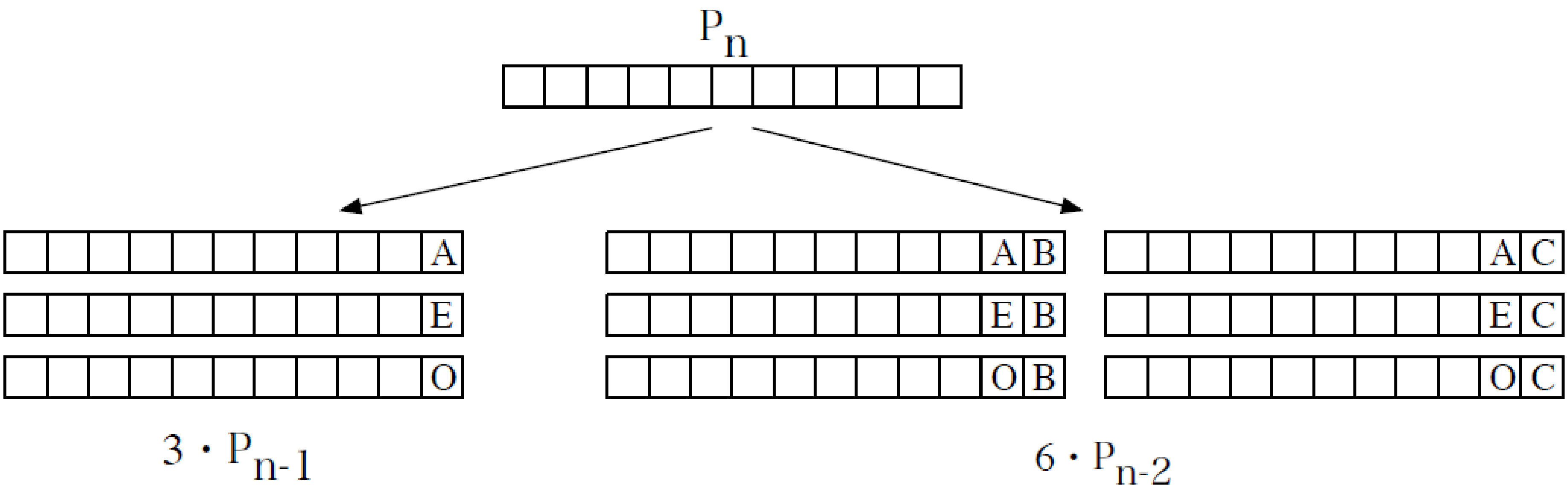
Problem 1: (a) Let P_n be the number of strings of length n formed from letters `A`, `B`, `C`, `E`, `O`, that do not contain two consecutive consonants (that is, `B` or `C`). For example, `AABOCA` and `BACOOEBO` satisfy this condition, while `AABCEC` does not. Derive a recurrence relation for the numbers P_n. You need to provide a complete justification for this recurrence. (But you *do not* need to solve it.)

Linear Homogeneous Recurrence Relations

Word problem

Problem 1: (a) Let P_n be the number of strings of length n formed from letters `A`, `B`, `C`, `E`, `O`, that do not contain two consecutive consonants (that is, `B` or `C`). For example, `AABOCA` and `BACOOEBO` satisfy this condition, while `AABCEC` does not. Derive a recurrence relation for the numbers P_n.





$$P_0 = 1$$
$$P_1 = 5$$
$$P_n = 3P_{n-1} + 6P_{n-2} \quad \text{for } n \geq 2$$

Linear Homogeneous Recurrence Relations

Word problem

(b) Let S_n be the number of strings of length n formed from letters $\mathtt{A}, \mathtt{B}, \mathtt{C}$, that do not contain substrings `AB`, `BA`, `AAA` or `BBB`. For example, for $n = 3$, all strings with this property are:

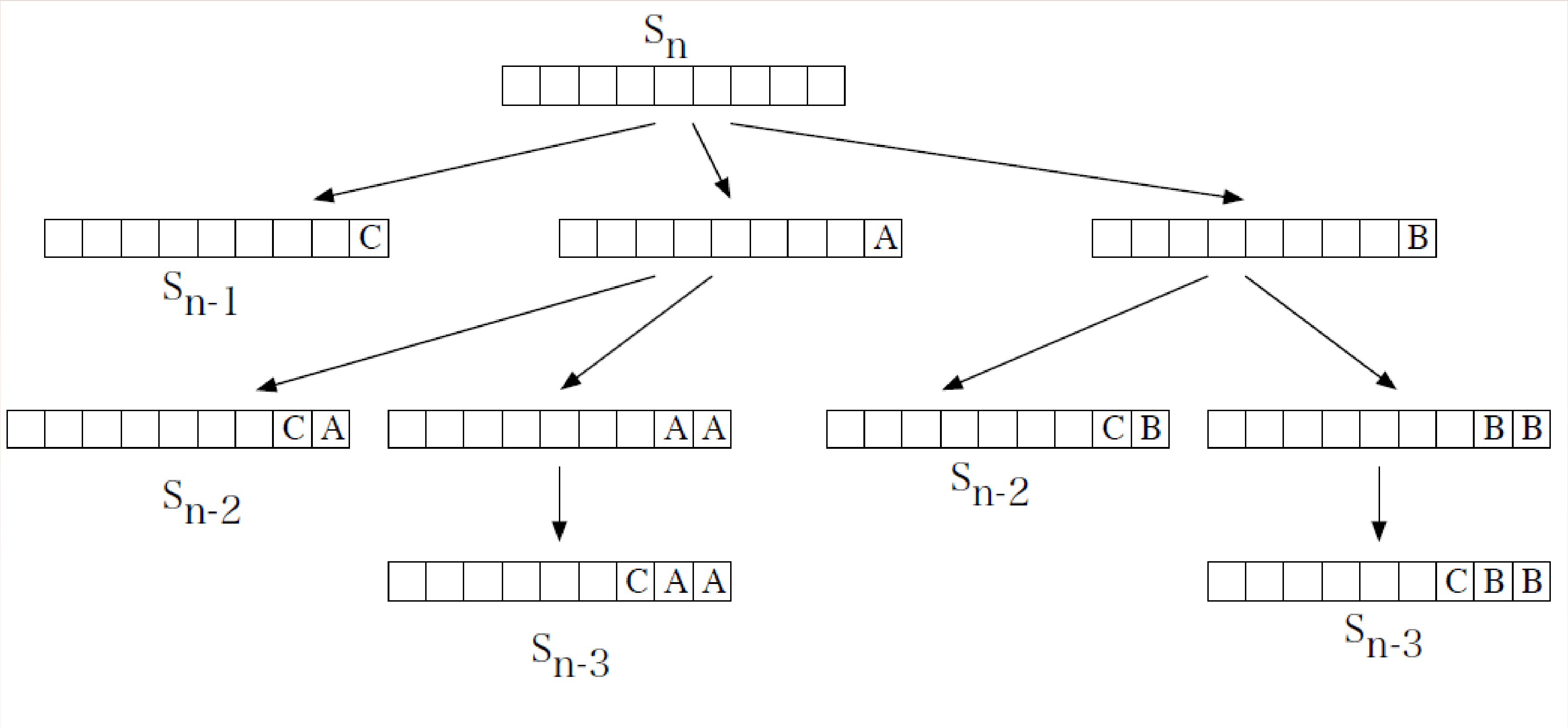
$$\mathtt{AAC}, \mathtt{ACA}, \mathtt{ACB}, \mathtt{ACC}, \mathtt{BBC}, \mathtt{BCA}, \mathtt{BCB}, \mathtt{BCC}, \mathtt{CAA}, \mathtt{CAC}, \mathtt{CBB}, \mathtt{CBC}, \mathtt{CCA}, \mathtt{CCB}, \mathtt{CCC},$$

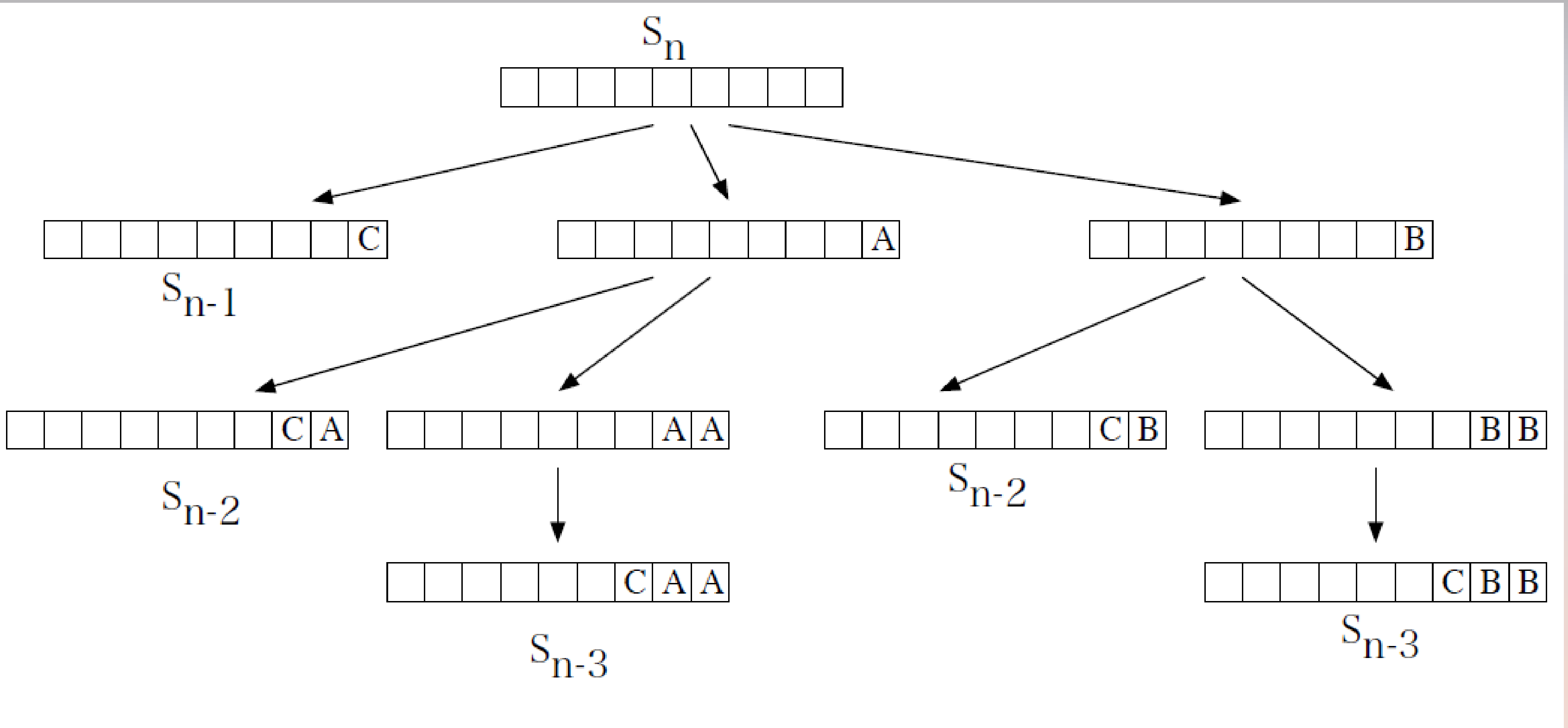
and thus $S_3 = 15$. (Note that $S_0 = 1$, because the empty string satisfies the condition.) Derive a recurrence relation for the numbers S_n. You need to provide a complete justification for this recurrence. (But you *do not need* to solve it.)

(b) Let S_n be the number of strings of length n formed from letters A, B, C, that do not contain substrings AB, BA, AAA or BBB. For example, for $n = 3$, all strings with this property are:

AAC, ACA, ACB, ACC, BBC, BCA, BCB, BCC, CAA, CAC, CBB, CBC, CCA, CCB, CCC,

and thus $S_3 = 15$. (Note that $S_0 = 1$, because the empty string satisfies the condition.) Derive a recurrence relation for the numbers S_n. You need to provide a complete justification for this recurrence. (But you *do not need* to solve it.)





$$
\begin{aligned}
S_0 &= 1 \\
S_1 &= 3 \\
S_2 &= 7 \\
S_n &= S_{n-1} + 2S_{n-2} + 2S_{n-3} \quad \text{for } n \geq 3
\end{aligned}
$$

Linear Non-Homogeneous Recurrence Relations with Constant Coefficients

Linear homogeneous recurrence relations of degree *k* with constant coefficients:

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$a_n = 4\, a_{n-1} - 4\, a_{n-2},$$

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

Linear non-homogeneous recurrence relation of degree *k* with constant coefficients:

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$$

$$a_n = 4\, a_{n-1} - 4\, a_{n-2} + n^2 + 1$$

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} + 2^n(n+1)$$

Linear Non-Homogeneous Recurrence Relations with Constant Coefficients

A linear homogeneous recurrence relation of degree *k* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, $c_k \neq 0$

A linear non-homogeneous recurrence relation of degree *k* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + g(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, $c_k \neq 0$, and g(n) a function not identically zero depending only on n.

Linear Non-Homogeneous Recurrence Relations with Constant Coefficients

If non-homogeneous recurrence is

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \ldots + c_k f_{n-k} + g(n)$$

the associated homogeneous recurrence is

$$f_{nc} = c_1 f_{n-1} + c_2 f_{n-2} + \ldots + c_k f_{n-k}$$

Let f'_n *be the* ***general solution*** *of* f_{nc},

and f''_n *– a* ***particular (arbitrary) solution*** *of* f_n *. Then*

$$f_n = f'_n + f''_n$$

Linear Non-Homogeneous Recurrence Relations

Theorem (particular solution)

Suppose $\{f_n\}$ satisfies the linear nonhomogeneous recurrence relation $f_n = c_1 f_{n-1} + c_2 f_{n-2} + \ldots + c_k f_{n-k} + g(n)$, *and*

$$g(n) = (b_t n^t + b_{t-1} n^{t-1} + .. + b_1 n + b_0)\, s^n$$

where $b_0, b_1, .., b_n, s \in R$.

If ***s*** *is* ***not*** *a root of* $\boldsymbol{f_{nc}}$, *there is a particular solution of the form*

$$f''_n = (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

If ***s*** *is a root of* $\boldsymbol{f_{nc}}$, *and its multiplicity is m, there is a particular solution of the form*

$$f''_n = n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

Linear Non-Homogeneous Recurrence Relations

What form does a particular solution of the linear non-homogeneous recurrence relation

$f_n = 6f_{n-1} - 9f_{n-2} + g(n)$ have when

$g(n) = 5$

$g(n) = 5n + 1$

$g(n) = 5n^2 + 1$

$g(n) = 5n^2 + n + 1$

$g(n) = n2^n$

$g(n) = 2^n (5n^2 + n + 1)$

$g(n) = n3^n$

$g(n) = 3^n (5n^2 + n + 1)$

Linear Non-Homogeneous Recurrence Relations

$f_n = 6f_{n-1} - 9f_{n-2} + g(n)$

$f'_{nc} = \alpha_1 3^n + \alpha_2 n3^n$

$f''_n = n^m(p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)\, s^n$

$g(n) = 5$	$f''_n =$
$g(n) = 5n + 1$	$f''_n =$
$g(n) = 5n^2 + 1$	$f''_n = p_2 n^2 + p_1 n + p_0$
$g(n) = 5n^2 + n + 1$	$f''_n =$
$g(n) = n2^n$	$f''_n = (p_1 n + p_0)\, 2^n$
$g(n) = 2^n (5n^2 + n + 1)$	$f''_n =$

Linear Non-Homogeneous Recurrence Relations

$f_n = 6f_{n-1} - 9f_{n-2} + g(n)$

$f'_{nc} = \alpha_1 3^n + \alpha_2 n 3^n$

$f''_n = n^m(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$

$g(n) = 5$	$f''_n = p_0$
$g(n) = 5n + 1$	$f''_n = p_1 n + p_0$
$g(n) = 5n^2 + 1$	$f''_n = p_2 n^2 + p_1 n + p_0$
$g(n) = 5n^2 + n + 1$	$f''_n = p_2 n^2 + p_1 n + p_0$
$g(n) = n2^n$	$f''_n = (p_1 n + p_0)\, 2^n$
$g(n) = 2^n (5n^2 + n + 1)$	$f''_n = (p_2 n^2 + p_1 n + p_0)\, 2^n$

Linear Non-Homogeneous Recurrence Relations

$f_n = 6f_{n-1} - 9f_{n-2} + g(n)$

$f'_{nc} = \alpha_1 3^n + \alpha_2 n3^n$

$f''_n = n^m(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$

If s is not a root of the characteristic equation, its multiplicity m = 0:

$g(n) = n2^n$ $\qquad$ $f''_n = (p_1 n + p_0)\, 2^n$

$g(n) = 2^n (5n^2 + n + 1)$ $\qquad$ $f''_n = (p_2 n^2 + p_1 n + p_0)\, 2^n$

If s is a root of the characteristic equation, its multiplicity m > 0:

$g(n) = n3^n$ $\qquad$ $f''_n = n^2(p_1 n + p_0)\, 3^n$

$g(n) = 3^n (5n^2 + n + 1)$ $\qquad$ $f''_n = n^2(p_2 n^2 + p_1 n + p_0)\, 3^n$

Example

Solve

$$\begin{aligned} f_n &= f_{n-1} + 6f_{n-2} + 2 \\ f_0 &= 5 \\ f_1 &= 7 \end{aligned}$$

The associated homogeneous equation is $f_n = f_{n-1} + 6f_{n-2}$.

its general solution $f'_n = \alpha_1 3^n + \alpha_2(-2)^n$.

$$q(n) = 2$$

a particular solution of the form $f''(n) = \beta$,

$$\beta = \beta + 6\beta + 2$$

$$\beta = -\frac{1}{3}$$

Example (cont.)

$$f_n = \alpha_1 3^n + \alpha_2(-2)^n - \tfrac{1}{3}.$$

Using the initial conditions, we have

$$\begin{aligned} \alpha_1 3^0 + \alpha_2(-2)^0 - \tfrac{1}{3} &= 5 \\ \alpha_1 3^1 + \alpha_2(-2)^1 - \tfrac{1}{3} &= 7 \end{aligned}$$

which simplifies to

$$\begin{aligned} \alpha_1 + \alpha_2 &= \tfrac{16}{3} \\ 3\alpha_1 - 2\alpha_2 &= \tfrac{22}{3} \end{aligned}$$

We solve it, getting $\alpha_1 = \frac{18}{5}$ and $\alpha_2 = \frac{26}{15}$. So the solution is

$$f_n = \tfrac{18}{5} \cdot 3^n + \tfrac{26}{15} \cdot (-2)^n - \tfrac{1}{3}$$

Example

- Consider the recurrence $a_n = 3a_{n-1} + 2n$
- Here, $s = 1$ and characteristic root is 3
- Hence, there exists a particular solution of the form $p_1 n + p_0$
- Now, solve for p_0, p_1:

$$p_1 n + p_0 = 3(p_1(n-1) + p_0) + 2n$$

- Rearrange: $2n(p_1 + 1) + (2p_0 - 3p_1) = 0$
- A solution $p_1 = -1, p_0 = -\frac{3}{2}$
- A particular solution: $-n - \frac{3}{2}$

Example

Problem 3: Find a particular solution of the recurrence $V_n = 3V_{n-1} - 4V_{n-2} + 3 \cdot 4^n$.

Solution: We guess the solution, $U_n = c \cdot 4^n$.

It is not a root of the chacteristic equation. Substitute into the original recurrence to get:

$$c \cdot 4^n = 3 \cdot c \cdot 4^{n-1} - 4 \cdot c \cdot 4^{n-2} + 3 \cdot 4^n$$

or, $16c = 12c - 4c + 48$

or, $c = 6$

Particular solution: $V_n = 6 \cdot 4^n$

Particular Solution

f''_n = $(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$

What if s = 1 ?

$$f_n = f'_n + f''_n$$

Compare:

If $\quad f_n = 2f_{n-1} + n, \quad f''_n = p_1 n + p_0$

If $\quad f_n = f_{n-1} + n, \quad f''_n = n(p_1 n + p_0)$

Example

- Find a particular solution for $a_n = 6a_{n-1} - 9a_{n-2} + 2^n$
- Characteristic equation:
- Particular solution of the form:
- Find p_0 such that $p_0 \cdot 2^n = 6(p_0 \cdot 2^{n-1}) - 9(p_0 \cdot 2^{n-2}) + 2^n$
- Solve for p_0:
- Particular solution:

Linear Non-Homogeneous Recurrence Relations

Word Problems

<u>Example</u> Let h_n be the number of "hello"s printed by the program below on input n, where $n \geq 0$. Compute h_n.

```
procedure Hello(n)
    print("hello")
    if n ≥ 2 then
        Hello(n − 1) ; Hello(n − 1)
        Hello(n − 2) ; Hello(n − 2) ; Hello(n − 2)
```

Linear Non-Homogeneous Recurrence Relations
Word Problems

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```

Recurrence

$$h_n = 2h_{n-1} + 3h_{n-2} + 1$$
$$h_0 = 1$$
$$h_1 = 1$$

Word Problems

Recurrence

$$h_n = 2h_{n-1} + 3h_{n-2} + 1$$
$$h_0 = 1$$
$$h_1 = 1$$

Characteristic equation:

$$x^2 - 2x - 3 = 0$$

The roots are -1 and 3. So the general solution of the homogeneous equation is:

$$h'_n = \alpha_1(-1)^n + \alpha_2 3^n.$$

To find a particular solution, substitute $h''_n = \beta$, which gives us the equation

$$\beta = 2\beta + 3\beta + 1$$

so $\beta = -\frac{1}{4}$. Thus the general solution is:

$$h_n = \alpha_1(-1)^n + \alpha_2 3^n - \frac{1}{4}.$$

Word Problems

Recurrence

$$h_n = 2h_{n-1} + 3h_{n-2} + 1$$
$$h_0 = 1$$
$$h_1 = 1$$

general solution $\quad h_n = \alpha_1(-1)^n + \alpha_2 3^n - \frac{1}{4}.$

Initial condition equations:

$$\alpha_1 + \alpha_2 - \tfrac{1}{4} = 1$$
$$-\alpha_1 + 3\alpha_2 - \tfrac{1}{4} = 1$$

This gives $\alpha_1 = \frac{5}{8}$ and $\alpha_2 = \frac{5}{8}$. So the solution is

$$h_n = \tfrac{5}{8}(-1)^n + \tfrac{5}{8}3^n - \tfrac{1}{4}$$

Grabbits

Problem 2: Grabbits are genetically modified rabbits that live forever and reproduce asexually on a precise schedule: each grabbit gives birth to three grabbits every Wednesday starting two weeks after birth. So if you start with 1 newly born grabbit, after one week you will still only have 1 grabbit. After two weeks you will have 4 grabbits, namely your first grabbit plus its 3 offspring. In general, how many grabbits will you have after n weeks if you start with one newly born grabbit? Set up a recurrence relation for this problem and solve it.

Grabbits

Denote by G_n the number of grabbits after n weeks. So $G_0 = 1$ and $G_1 = 1$, as explained above. To write a recurrence for G_n we reason as follows. In week n we will have some old rabbits (that were around the previous week) and newly born rabbits. The number of old rabbits is G_{n-1}. How many new rabbits do we have? New rabbits are offspring of the rabbits that were around two weeks ago. Two weeks ago we had G_{n-2} grabbits and each has 3 offspring this week, so there are $3G_{n-2}$ new grabbits. Thus the recurrence is

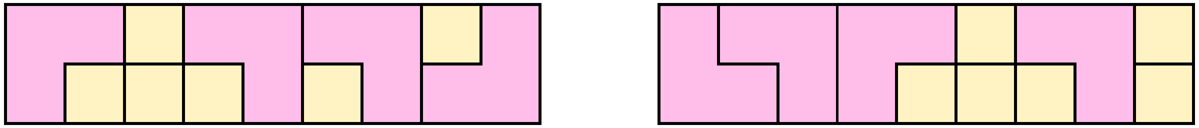
$$G_n = G_{n-1} + 3G_{n-2}$$

Final answer:

$$\begin{aligned} G_n &= \frac{1+\sqrt{13}}{2\sqrt{13}} \cdot \left(\frac{1+\sqrt{13}}{2}\right)^n + \frac{-1+\sqrt{13}}{2\sqrt{13}} \cdot \left(\frac{1-\sqrt{13}}{2}\right)^n \\ &= \frac{1}{\sqrt{13}}\left[\left(\frac{1+\sqrt{13}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{13}}{2}\right)^{n+1}\right] \end{aligned}$$

One More Problem

Problem 8: We want to tile a $2 \times n$ strip with 1×1 tiles and L-shaped tiles of width and height 2. Here are two examples of such a tiling of a 2×9 strip:



Let $A(n)$ be the number of such tilings. (a) Give a recurrence relation for $A(n)$ and justify it. (b) Solve the recurrence to compute $A(n)$.

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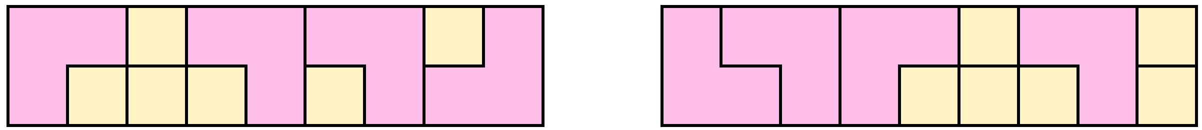

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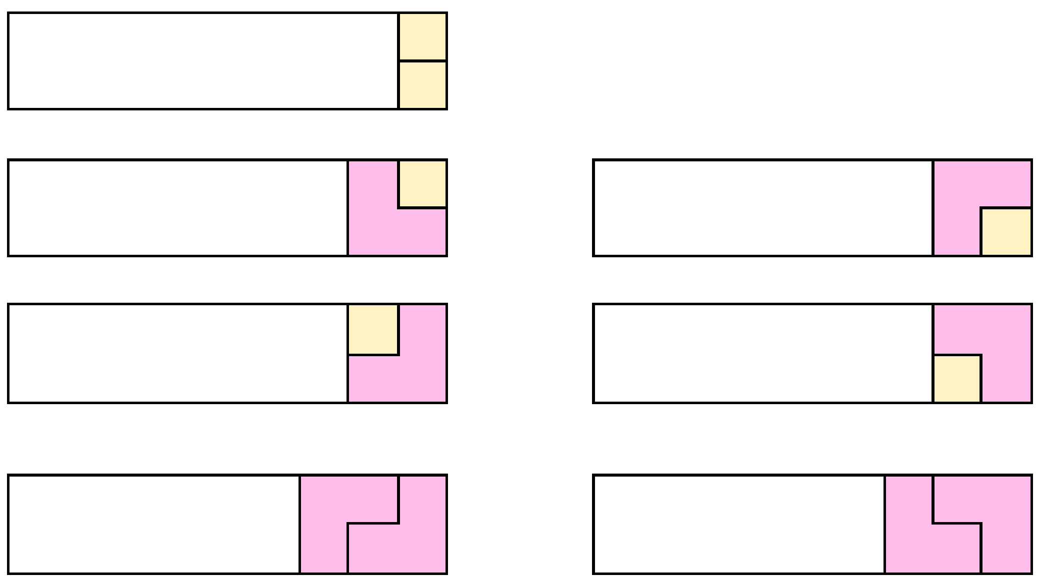

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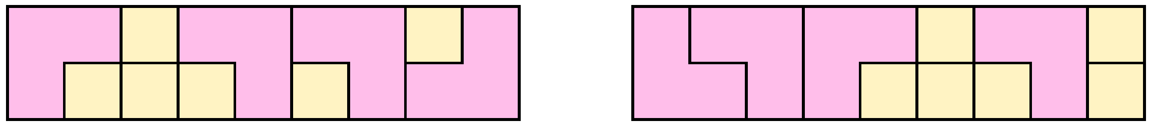

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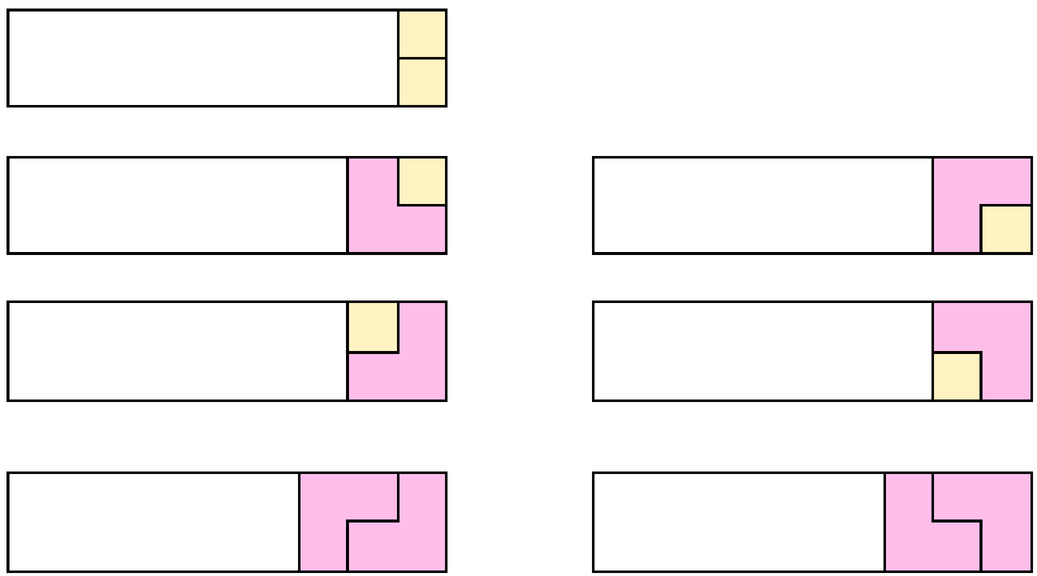

$$A(n) = A(n-1) + 4A(n-2) + 2A(n-3)$$

with $A(0) = 1$, $A(1) = 1$, and $A(2) = 5$.

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with $A(0) = 1$, $A(1) = 1$, and $A(2) = 5$.

The characteristic equation is $x^3 - x^2 - 4x - 2 = 0$ and its roots are -1, $1 - \sqrt{3}$, and $1 + \sqrt{3}$. So the general form of the solution is

$$A(n) = \alpha_1(-1)^n + \alpha_2(1-\sqrt{3})^n + \alpha_3(1+\sqrt{3})^n.$$

The initial conditions give us

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ \alpha_1(-1) + \alpha_2(1-\sqrt{3}) + \alpha_3(1+\sqrt{3}) &= 1 \\ \alpha_1 + \alpha_2(4-2\sqrt{3}) + \alpha_3(4+2\sqrt{3}) &= 5 \end{aligned}$$

Solving, we get $\alpha_1 = 1$, $\alpha_2 = -1/\sqrt{3}$ and $\alpha_3 = 1/\sqrt{3}$. So

$$A(n) = (-1)^n - \frac{1}{\sqrt{3}}(1-\sqrt{3})^n + \frac{1}{\sqrt{3}}(1+\sqrt{3})^n.$$

Solve the following recurrence equation:

$$\begin{aligned} f_n &= 13f_{n-2} + 12f_{n-3} + 2n + 1 \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 1 \end{aligned}$$

$$f'_n = \alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3 4^n$$

$$\beta_1 n + \beta_2 = 13[\beta_1(n-2) + \beta_2] + 12[\beta_1(n-3) + \beta_2] + 2n + 1$$

$$(-24\beta_1 - 2)n + (62\beta_1 - 24\beta_2 - 1) = 0.$$

$$f_n = \alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3 4^n - \frac{1}{12}n - \frac{37}{144}$$

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= \frac{37}{144} \\ -3\alpha_1 - \alpha_2 + 4\alpha_3 &= \frac{193}{144} \\ 9\alpha_1 + \alpha_2 + 16\alpha_3 &= \frac{205}{144} \end{aligned}$$

$$f_n = -\frac{29}{112}(-3)^n + \frac{3}{10}(-1)^n + \frac{68}{315}4^n - \frac{1}{12}n - \frac{37}{144}.$$