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Problem 1: Find a general solution of the recurrence $A_n = 4A_{n-1} - 4A_{n-2} + 3n$. Show your work.

We consider the corresponding homogeneous equation first: $A'_n = 4A'_{n-1} - 4A'_{n-2}$. Its characteristic equation is $x^2 - 4x + 4 = 0$. This has root $x = 2$ with mulitplicity 2. So the general form of the homogeneous equation is:

$$A'_n = \alpha_1 2^n + \alpha_2 n 2^n.$$

Next, we look for a particular solution of the original inhomogeneous equation. We try solutions of the form $A''_n = \beta_1 n + \beta_2$. Plugging it into the equation, we get

$$\beta_1 n + \beta_2 = 4[\beta_1(n-1) + \beta_2] - 4[\beta_1(n-2) + \beta_2] + 3n$$

This simplifies to

$$(\beta_1 - 3)n - 4\beta_1 + \beta_2 = 0.$$

For this equation to be true for all n, we must have $\beta_1 - 3 = 0$ and $-4\beta_1 + \beta_2 = 0$. So $\beta_1 = 3$ and $\beta_2 = 12$, which gives us $A''_n = 3n + 12$.

Finally, adding the two solutions, the general solution of the original equation is

$$A_n = \alpha_1 2^n + \alpha_2 n 2^n + 3n + 12.$$

Problem 2: (a) Give the definition of Euler's totient function $\phi(n)$.

Definition: $\phi(n)$ is the number of integers in $\{1, 2, ..., n\}$ that are relatively prime to n.

(b) Give the formula for Euler's totient function.

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_k}\right)$$

where $p_1, p_2, ..., p_k$ are all different prime factors of n.

(c) Compute $\phi(6000)$.

The factorization of 6000 is $6000 = 2^4 \cdot 3 \cdot 5^3$. So

$$\begin{aligned}\phi(6000) &= 6000\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\\ &= 6000 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 1600.\end{aligned}$$

Problem 3: For each recurrence below, circle the correct solution (or "none of the above").

Recurrence	Solution
(a) $f(n) = 16f(n/4) + 2n^2$	$\Theta(n)$ $\Theta(\log n)$ $\Theta(n^{3/4})$ $\Theta(n^{\log_4 3})$ $\Theta(n^2)$ $\Theta(n^{\log_3 4})$ $\Theta(n \log n)$ none of the above
(b) $f(n) = 4f(n/3) + 2n^2$	$\Theta(n)$ $\Theta(\log n)$ $\Theta(n^{3/4})$ $\Theta(n^{\log_4 3})$ $\boxed{\Theta(n^2)}$ $\Theta(n^{\log_3 4})$ $\Theta(n \log n)$ none of the above
(c) $f(n) = 4f(n/3) + 3n$	$\Theta(n)$ $\Theta(\log n)$ $\Theta(n^{3/4})$ $\Theta(n^{\log_4 3})$ $\Theta(n^2)$ $\boxed{\Theta(n^{\log_3 4})}$ $\Theta(n \log n)$ none of the above