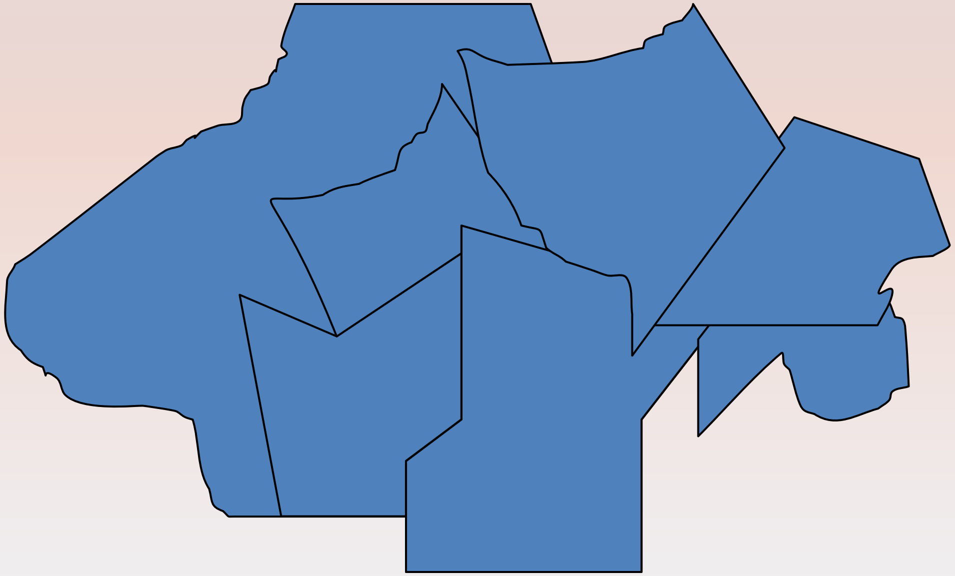
Graph Coloring

CS 111

Map Coloring

Map Coloring

Color the following map so that no two adjacent countries have the same color.



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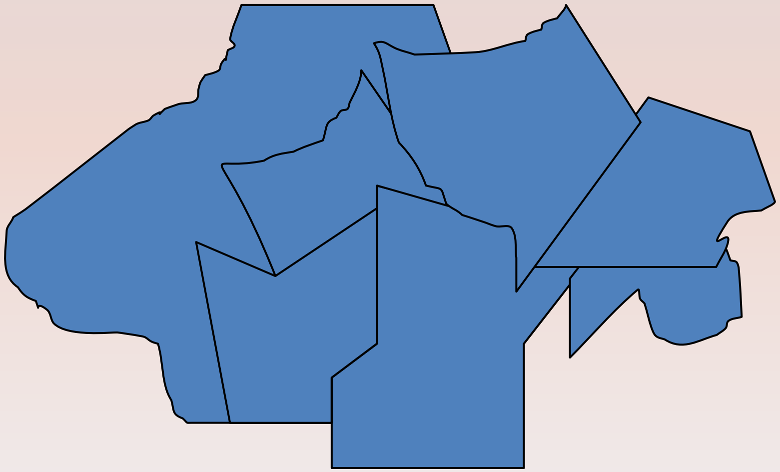

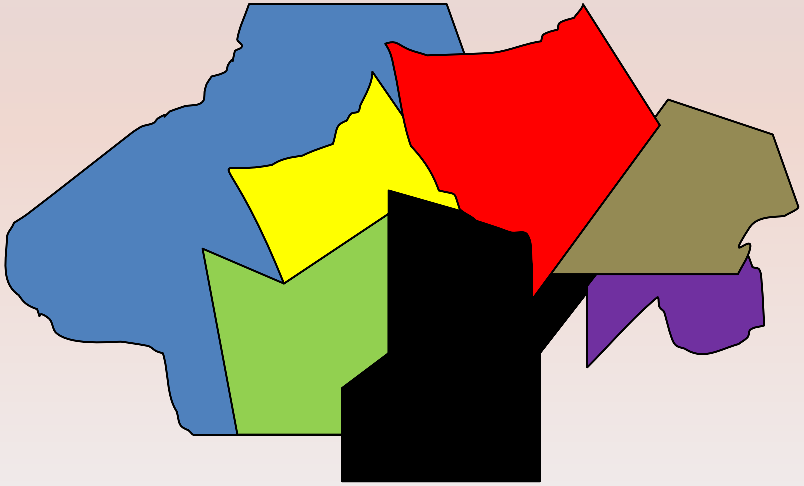

Do we have to use so many colors?

Map Coloring

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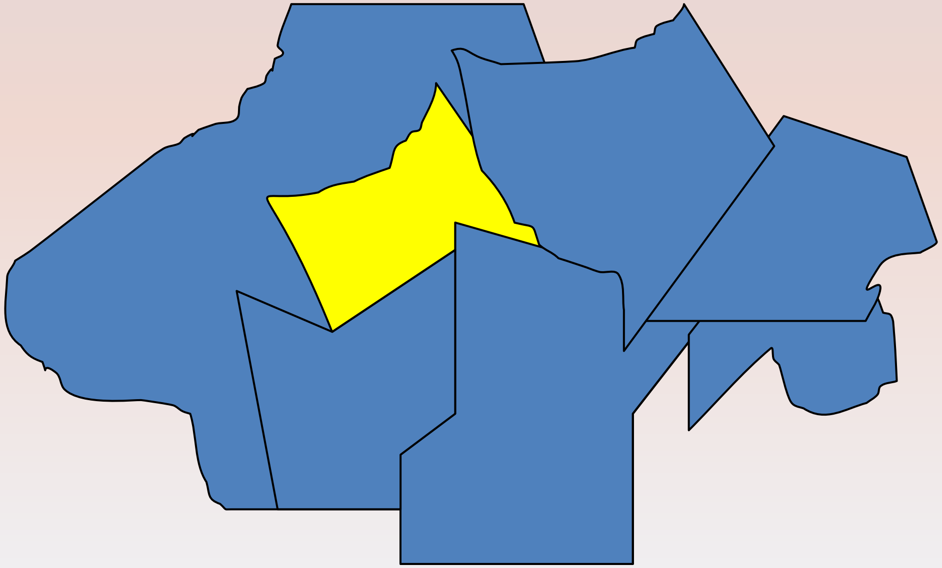

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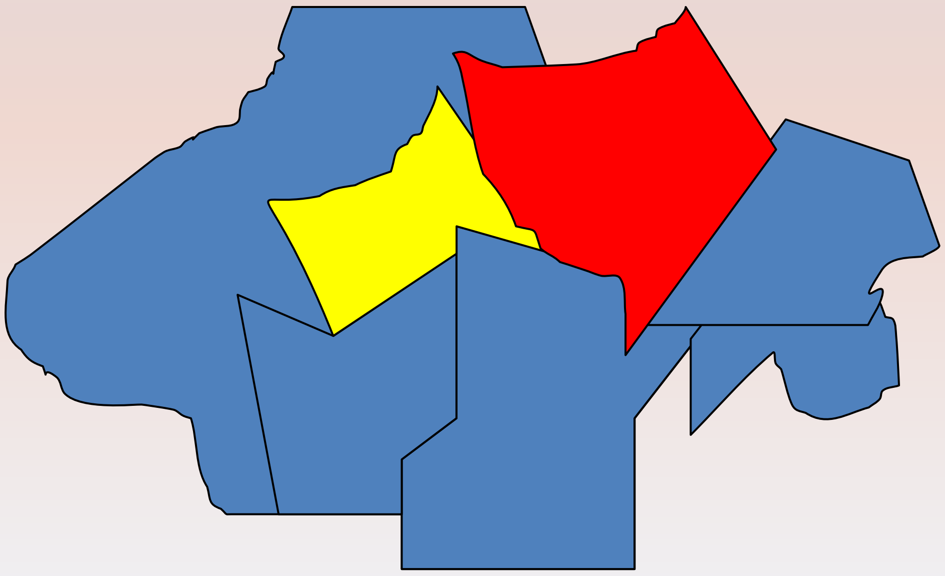

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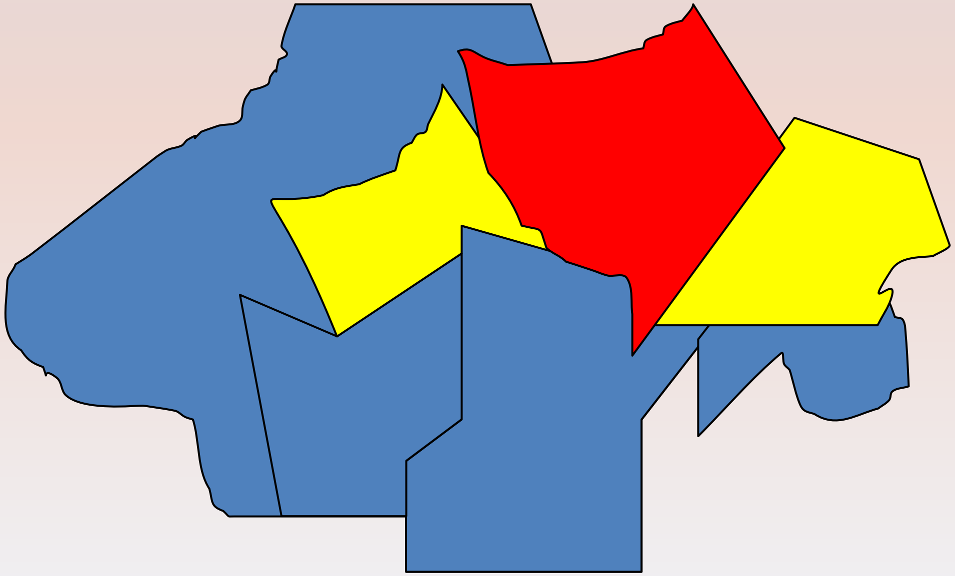

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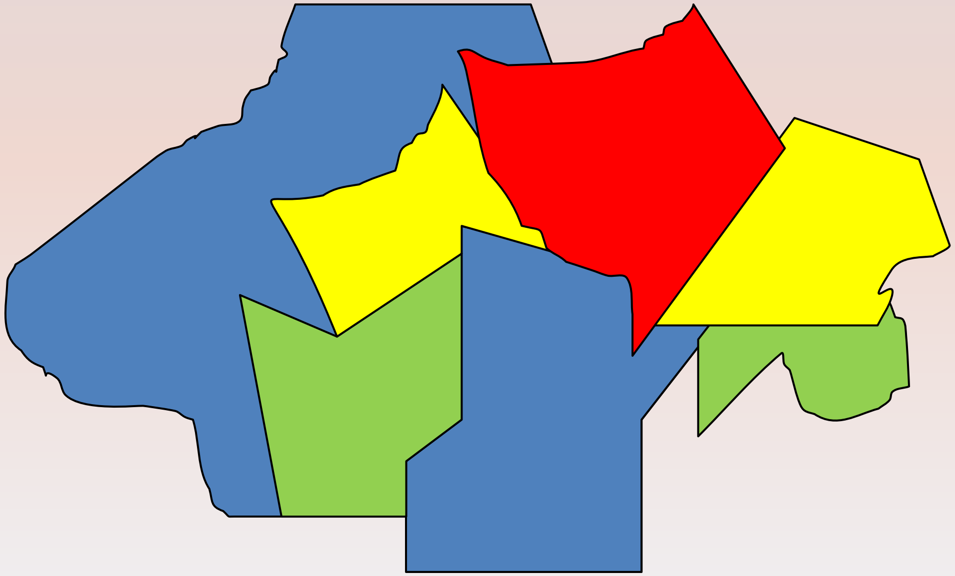
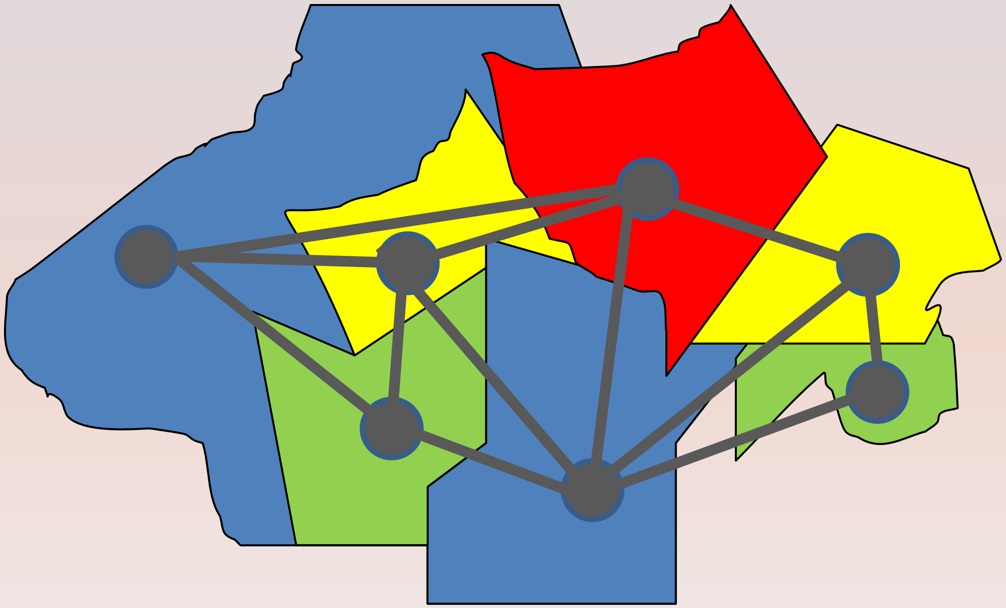
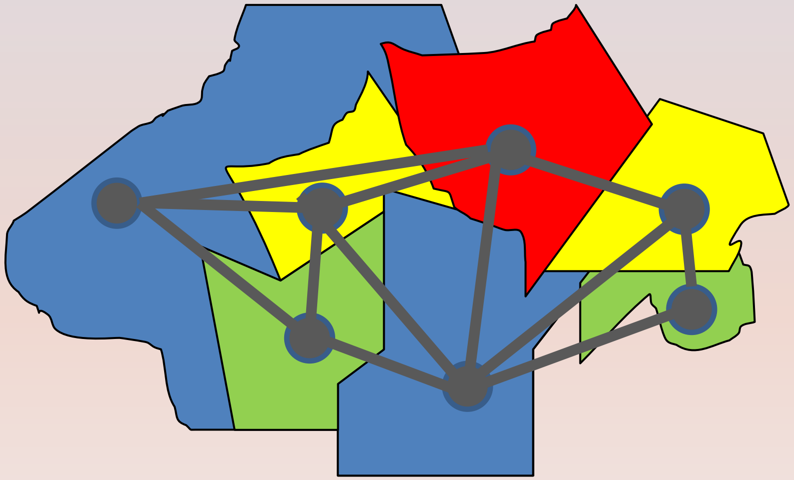


This map can be colored with 4 colors.

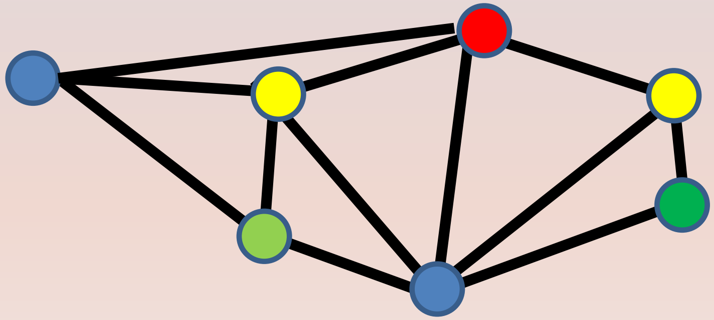
Converting a Map Coloring Problem into a Graph Coloring Problem



Converting a Map Coloring Problem into a Graph Coloring Problem

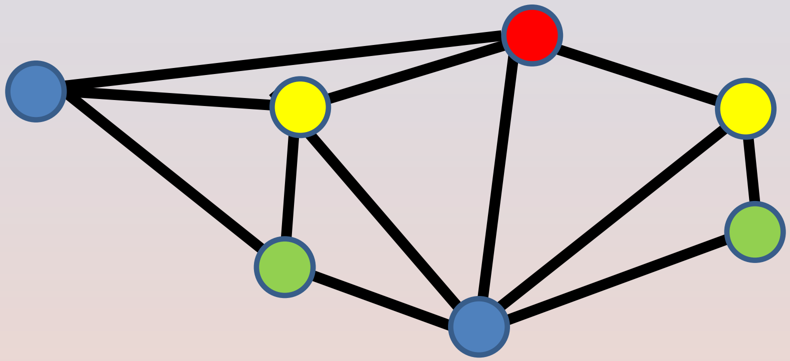


(a) Map coloring



(b) Graph coloring

Graph Coloring

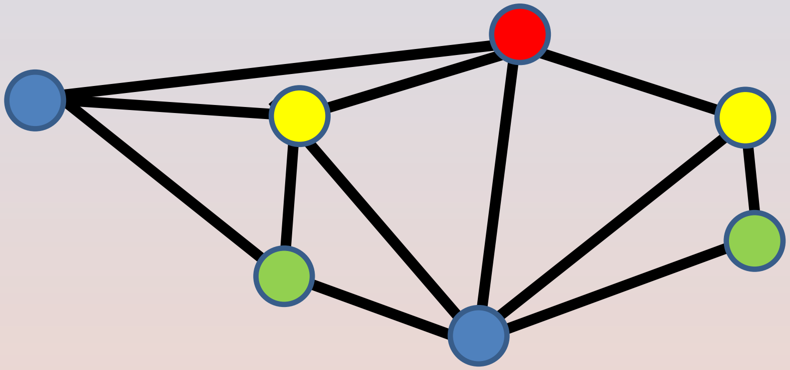


<u>Definition</u> A *coloring* of a simple graph is an assignment of a color to each vertex of the graph, so that no two adjacent vertices are assigned the same color.

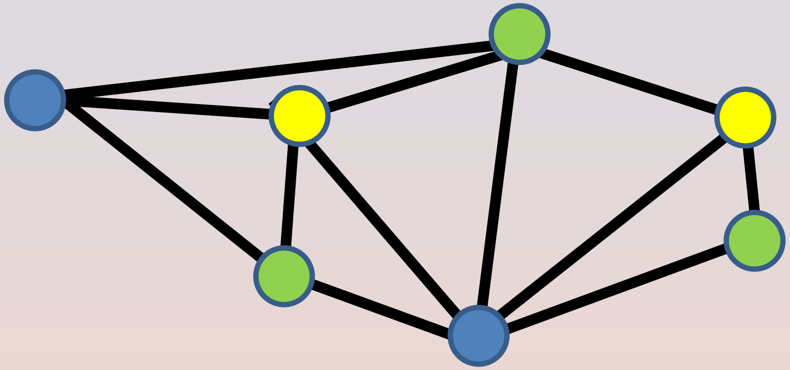
Graph coloring can be used to model different problems: scheduling, resource allocation, etc.

Can we model the game of Sudoku as a graph coloring problem?

Chromatic Number of a Graph



(a) 4-coloring of G



(b) 3-coloring of G

Definition The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph.

Example The chromatic number of graph G is 3.

In general, the chromatic number problem is NP-complete.

Coloring of a Graph with the Maximum Vertex Degree d.

Theorem. Let d be the maximum vertex degree of a graph G. Then G can be colored with at most $d + 1$ colors.
(The chromatic number of G is at most $d + 1$.)

Proof 1. (By induction on the number of vertices n in G).
Let $G = (V, E)$ be a simple graph with $n = |V|$ and max degree d.
We claim that G can be colored with at most $d + 1$ colors.

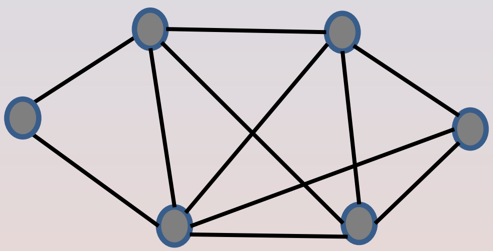
<u>Base case:</u> If $n = 1$, then $d = 0$, and G **can** be colored with $d + 1 = 1$ color.

<u>Induction hypothesis:</u> Assume that the claim holds for all values $n \leq k$ (where k is a positive integer, $k \geq 1$).

<u>Inductive step</u>: Need to prove that the claim holds for $n = k + 1$.

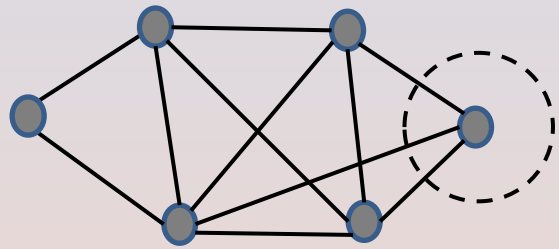
Required

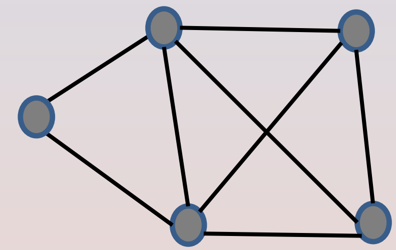
Coloring of a Graph with the Maximum Vertex Degree d.



n = k + 1

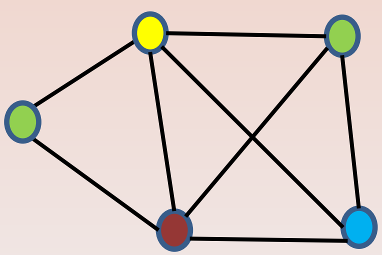
max degree $=$ d





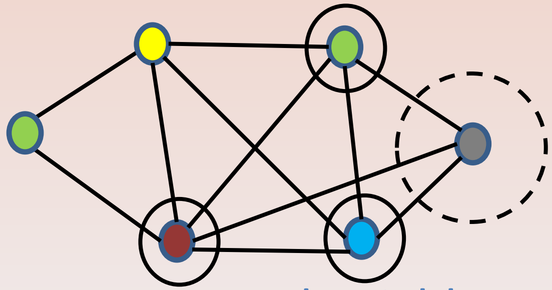
n = k

max degree $\leq$ d

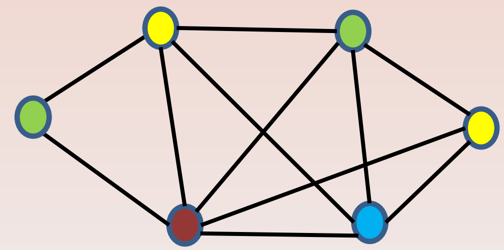


n = k

max degree $\leq$ d



at most d neighbors



n = k + 1

max degree $=$ d

Which proves that the claim holds for $n = k + 1$, and thus proves the theorem.

Graph Coloring

Theorem. Let d be the maximum vertex degree of a graph G.
Then G can be colored with at most $d + 1$ colors.
(The chromatic number of G is at most $d + 1$.)

Proof 2. We can provide a greedy algorithm:

Input: Graph $G = (V, E)$ (with max vertex degree $\leq d$),

available colors $C = \{c_1, c_2, \dots c_{d+1}\}$.

Output: a coloring of G.

Arrange the vertices of G in linear order: $v_1, v_2, \dots, v_n$.

for i = 1 to n do

assign c_j to v_i, where j is the smallest available color index, different

from all the colors, already used by the neighbors of v_i.

Proof of correctness:

easy

Coloring of Planar Graphs

Lemma. The average degree of any planar graph **G** is less then **6**.

Proof. The average degree of a graph is $\mathbf{D = \dfrac{2m}{n}}$.

It follows from Euler's Formula, that for $\mathbf{n \geq 3}$,

$$\mathbf{m \leq 3n - 6;}$$

Then $$\mathbf{D \leq \frac{2\,(3n-6)}{n} = 6 - \frac{12}{n} = 6\,\left(1 - \frac{2}{n}\right) < 6}.$$

For $\mathbf{n < 3}$, we can check directly.

Coloring of Planar Graphs
The 6 Color Theorem

Theorem. Every planar graph is 6-colorable.

Proof. (By induction on the number of vertices n in G).
Let $G = (V, E)$ be a simple planar graph with $n = |V|$.
We claim that G can be colored with at most 6 colors.

Base case**:** A planar graphs with $n \leq 6$ vertices can be colored with at most 6 colors.

Induction hypothesis: Assume that the claim holds for all values $n \leq m$ (where m is a positive integer).

Inductive step: Prove, that any G on $n = m + 1$ vertices can be colored with 6 colors.
From the Lemma, G must have at least 1 vertex with degree **at most** 5. Remove this vertex ... . (The remaining prove is similar to the prove of the theorem about coloring graphs with the max degree d.)

Coloring of Planar Graphs
5 and 4 Color Theorems

Theorem. Every planar graph is 5-colorable.
The proof is more complicated than that of the 6 Color Theorem, but still not difficult.

Theorem. Every planar graph is 4-colorable.

Four Color Map Theorem. A map can be colored with at most 4 colors.
Complicated proof.
The first major theorem that was proved using a computer (1972, K.Appel and W.Haken).

The last proof was given in 2005 by Georges Gonthier (using general-purpose theorem-proving software).

Graph Coloring
True or False

- A graph on 10 vertices with the maximum vertex degree 5 cannot be colored with 7 colors.
- There is no graph on 10 vertices with the maximum vertex degree 5 that can be colored with 3 colors.
- Every graph can be colored with at most 6 colors.
- The chromatic number of a cycle graph is less than 3.
- A bipartite graph is 2 – colorable.

Summary

- A *coloring* of a simple graph is an assignment of a color to each vertex of the graph, so that no two adjacent vertices are assigned the same color.

- Graph coloring can be used to model different problems.

- The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph.

- A graph with the maximum vertex degree d can be colored with at most $d + 1$ colors.

- Every planar graph is 4-colorable (later).