Measurements and Experimental Results:
Uncertainty Analysis

Introduction:

In science, no measurement is EVER perfect. In a well-designed experimental procedure, a value that is measured can at best only be said to be *near* the "true value". Scientists reporting their results usually specify a range of values that they expect this "true value" to fall within. The most common way to show this range of values is:

Measurement = best estimate ± uncertainty

Example: A measurement of 5.07 g ± 0.02 g means that the experimenter is confident that the actual value for the quantity being measured lies between 5.05 g and 5.09 g. The uncertainty is the experimenter's best *estimate* of how far an experimental quantity might be from the "true value".

The uncertainty in a measurement is limited by the precision and accuracy of the measuring instrument, human imperfection, environmental factors, *etc*. It is always an *estimated* value; it is up to you to determine an appropriate uncertainty for your set of particular measurements. If you estimate your uncertainty well, you are confident that the *majority* (*NOT all*) of the measurements you take will fall within the interval $y - \delta y$ to $y + \delta y$. Note that you can *never* estimate your uncertainty with 100% confidence (*i.e.* "all measurements ever attempted will fall in this range") – you would be saying that your estimated uncertainty is perfect, which is equally as impossible as saying that your measurements are perfect!

If a measured value lies far outside of the range of uncertainty, this may be due to errors in articulating the value of your uncertainty, and your measurement is deemed to be "inaccurate". Additionally, if your uncertainty is very large with respect to your measurement, your measurement is deemed to be "imprecise" (it would be easy enough to convey any quantity by "one plus or minus infinity", but that's quite meaningless in terms of extracting information).

Reference Acknowledgement: Rhett Allain, Southeastern Louisiana University

Significant Figures:

Experimental uncertainties should be rounded to one significant figure. Experimental uncertainties are, by nature, inexact. Uncertainties are almost always quoted to one significant digit (*eg.* 1.02 s ± 0.05 s). If the uncertainty starts with a one, some scientists quote the uncertainty to two significant digits (*eg*. 1.02 s ± 0.12 s).

In these introductory physics lab courses, always round the experimental measurement or result to the same decimal place as the uncertainty. It would be confusing (and perhaps dishonest) to suggest that you knew the digit in the hundredths place when you admit that you are unsure of the tenths place.

Wrong: 1.237 s ± 0.1 s
Right: 1.2 s ± 0.1 s

Types of Uncertainty:

Uncertainty can be divided into two broad categories:

1) **Random Uncertainty (δ_{ran}):** Random uncertainty in experimental measurements are caused by unknown and unpredictable changes in the measuring instruments or environmental conditions. This results in the uncertainty being unrepeatable and inconsistent.

2) **Systematic Uncertainty (δ_{sys})**: Systematic uncertainty in experimental measurements is caused by repeatable imperfections in the experiment. They may occur because of defects in measuring equipment such as calibration errors, loading errors, and spatial errors, or human errors such as reading instruments incorrectly, selecting improper measuring equipment, or improper data analysis. Systematic uncertainty is usually consistent and repeatable in a set of measurements (hence the name, "systematic").

Over time, with a lot of repetition and validation using different experimental methods, the uncertainty in measurements is reduced, which allows us to define "physical constants", or values that we accept to be true without making additional measurements (like g = 9.8 m/s, for example).

Comparing Experimentally Determined Numbers:

Uncertainty estimates are crucial for comparing experimental numbers. Are the measurements 9.86 m/s^2 and 9.73 m/s^2 the same or different? The answer depends on how *precise* those two measurements are. If the uncertainty is too large, it is impossible to determine whether the difference between the two numbers is real or just due to sloppy measurements. That's why properly estimating uncertainty is so important!

Measurements don't agree with Prediction	9.1 m/s^2 ± 0.1 m/s^2 and 9.8 m/s^2
Measurements do agree with Prediction	9.1 m/s^2 ± 0.8 m/s^2 and 9.8 m/s^2

If the ranges of the values don't overlap, your measurements are **discrepant** (the two numbers do not agree). If the ranges do overlap, the measurements are said to be **consistent**.

This does NOT mean that you can arbitrarily inflate your estimation of uncertainty to get agreement!! Your measurements are what you measure them to be! If your measurements **systematically** disagree with predicted values, then you need to consider what phenomena might be skewing your measured values away from what your physical model says they should be.

Sometimes this systematic uncertainty might result from the physical model not accounting for a significant phenomenon, like when we don't account for air resistance when we model kinematic motion like free-fall or projectile trajectory. Other times this systematic uncertainty may result from your apparatus, like in the event of a miscalibrated sensor that persistently measures times that are 2.3 seconds longer than they actually are.

These systematic sources of uncertainty are very difficult to come up with, but this is how scientific progress is made! Once we have decided that measured values are discrepant from predicted values, we know either that our data does not represent what we attempted to measure or that our physical model is imperfect. If we decide that our physical model is imperfect, the next step is to identify the physical phenomenon that our model does not account for, and then to construct that into the original model. This is how *any* science progresses!

Estimating Uncertainty from a Single Measurement:

In many circumstances, a single measurement of a quantity is sufficient for the purposes of the measurement being taken (*eg.* you usually don't need to weigh yourself more than once to collect your measurement). But if you only take one measurement, how can you estimate the uncertainty in that measurement? **Estimating the uncertainty in a single measurement requires *judgement* on the part of the experimenter.** The uncertainty of a single measurement is limited by the precision and accuracy of the measuring instrument, along with any other factors that might affect the ability of the experimenter to make the measurement and it is up to the experimenter to estimate the uncertainty.

Increasing Precision with Multiple Measurements:

One way to increase your confidence in measured experimental data is to repeat the same measurement many times. For example, you could estimate the time it takes something to happen by timing it once with a stopwatch. You could then decrease the uncertainty in this estimate by repeating it multiple times and taking the average. The more measurements you take, the better your estimate will be.

Taking multiple measurements helps you estimate the random uncertainty affecting precision. The precision of your estimate of the time depends on both the spread of the measurements (often measured using what is called the "standard deviation") and the number (N) of repeated measurements you take.

Propagation of Uncertainty:

In physics labs, you will often be asked to measure a quantity (and its associated uncertainty), then to use that measured quantity to calculate a different quantity through some physical hypothesis. For example, I might measure the length of one side of a cube to be $l = 1.2 \pm 0.1$ m, and I have the relation to find the cube's volume as $V = l^3$. The propagation of the uncertainty in your calculated value, δV is *not* found by plugging δl into the equation (*i.e.* $\delta V \neq \delta l^3$).

Let's say that we have two measured quantities, *x* and *y*, and that we have estimated uncertainties for both of those quantities, δ*x* and δy. If we want to calculate something (we'll call it *z*) based on the measured values of *x* and *y*, how to we determine the associated uncertainty in *z*, δ*z*? The method can be generalized by a simple formula.

We can say that *z* is calculated by some arbitrary function:

$$z = \mathrm{C}x^a y^b$$

where *z* is the calculated value, *x* and *y* are the measured values (with associated uncertainties δ*x* and δ*y*), and C, *a*, and *b* are constants. The uncertainty in *z* (δ*z*) can be determined from the general formula:

$$\frac{\delta z}{z} = \sqrt{\left(a\,\frac{\delta x}{x}\right)^2 + \left(b\,\frac{\delta y}{y}\right)^2}$$

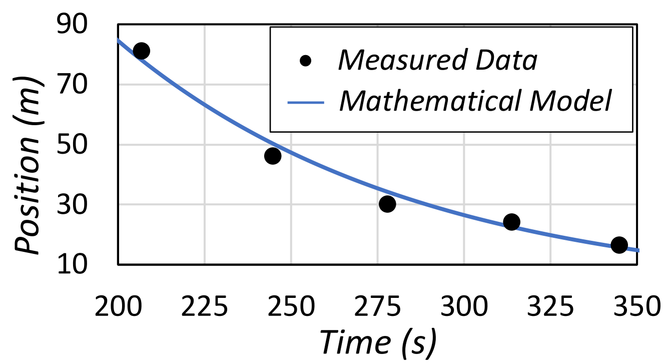
Taking the example given above where we need to calculate the uncertainty in *V* (δ*V*) based on the uncertainty in *l* (δ*l*), we would plug this into the propagation equation by taking *z* = *V*, C = 1, *x* = *l*, *a* = 3, *y* = 1, and *b* = 0. The resulting formula would look like:

$$\frac{\delta V}{V} = \sqrt{\left(3\,\frac{\delta l}{l}\right)^2}$$

This uncertainty propagation equation described here can be generalized for any polynomial calculation. For equations that cannot be expressed in polynomial form, the process of uncertainty propagation is slightly more complicated, but you will not need to worry about that for this course.

Comparing Graphed Data to Mathematical Models:

You know that all measured data has some degree of uncertainty associated with the measurement technique and the environment in which the measurements are acquired. These uncertainties are inherent to the measurements themselves. When we attempt to compare measured data with mathematical models, however, we produce a new kind of uncertainty, which we will call “deviation” or “graphical uncertainty”. These terms describe the “closeness” of the fit between mathematical models and physically measured data (example below).



The plot at right shows some measured position vs. time data (black data points) that have been fit with a mathematical model (smooth blue line) derived to describe the particular paradigm according to which the data were taken. The "deviation" or "graphical uncertainty" are the numerical values that tell us, on average, how far our measured values lie from the modeled behavior.

The deviation does NOT tell us anything about the uncertainty associated with any measurements! It is simply a value that tells us how closely our model matches our measured data! We can, however, use our estimated measurement uncertainty to more precisely compare measured data with the mathematical model, just like was described in the "Comparing Experimentally Determined Numbers" section above.

In the introductory physics lab curriculum, you will be asked many times to compare your measured and calculated data with some theoretical prediction or mathematical model. Be sure that you always explain what mathematical model you are using, the physics that is represented by that model, and the deviation between the model and your measured or calculated data with their associated uncertainties.

Statistical Uncertainty Analysis:

There are well-established statistical techniques that allow you to quantitatively analyze and propagate uncertainties associated with measured quantities. For the introductory physics lab courses, our objective is more to teach you how to identify the underlying physical causes of imperfections in your measurements rather than how to calculate the exact numbers. If you are interested, a good reference for these quantitative techniques can be found in *Data Reduction and Error Analysis for the Physical Sciences* (available at UCR's Orbach Library).

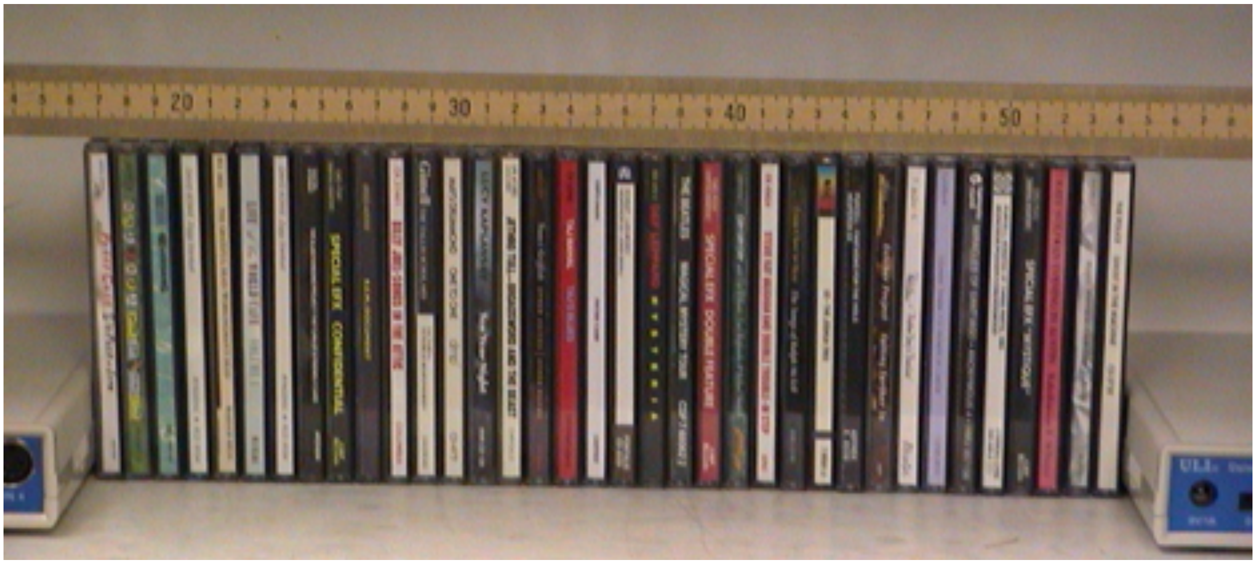
Some Examples:

Example #1: Measure the diameter of the tennis ball in the picture below using the meter stick. What is the uncertainty in your measurement?

Even though the meter stick can be read to the nearest 0.1 cm, you cannot determine the diameter of the ball to the nearest 0.1 cm from this picture. What factors limit your ability to determine the diameter of the ball? What is a more realistic estimate of the uncertainty in your measurement of the ball diameter?

Possible Answer: *It's hard to line up the edge of the ball with the marks on the ruler and the picture is blurry. Even though there are markings on the ruler for every 0.1 cm, only the markings at each 0.5 cm show up clearly. I figure I can reliably measure where the edge of the tennis ball is to within about half of one of these markings, or about 0.2 cm. The left edge is at about 50.2 cm and the right edge is at about 56.5 cm, so the diameter of the ball is about 6.3 cm ± 0.2 cm.*

Example #2: Determine the thickness of a CD case from the picture below.



How can you get the most precise measurement of the thickness of a single CD case from this picture? (Even though the ruler is blurry, you can determine the thickness of a single case to within less than 0.1 cm.). What assumptions are you making about the CD cases in your measurement and uncertainty estimation?

Possible Answer: *The best way to do the measurement is to measure the thickness of the stack and divide by the number of cases in the stack. That way, the uncertainty in the measurement is spread out over all 36 CD cases. It's hard to read the ruler in the picture any closer than within about 0.2 cm. The stack starts at about the 16.5 cm mark and ends at about the 54.5 cm mark, so the stack is about 38.0 ± 0.2 cm long. Divide the length of the stack by the number of CD cases in the stack (36) to get the thickness of a single case: 1.056 cm ± 0.006 cm. By "spreading out" the uncertainty over the entire stack of cases, you can get a measurement that is more precise than what can be determined by measuring just one of the cases with the same ruler. We are assuming that all the cases are the same thickness and that there is no space between any of the cases.*

Example #3: Maria timed how long it takes for a steel ball to fall from the top of a table to the floor using the same stopwatch five times:

0.42 s, 0.54 s, 0.44 s, 0.58 s, 0.48 s

Estimate the random uncertainty that you would assign to these measurements.

Possible Answer: *By taking five measurements, Maria has a crude estimate of the random uncertainty in her time measurements. It is very likely that the "true" time it takes for the ball to fall is somewhere between 0.42 s and 0.58 s, so we might say that the measured value is the average of those five values with the uncertainty equal to the apparent spread: 0.49 s ± 0.08 s. Statistical methods like a standard deviation calculating function are required to get a more sophisticated estimate of the uncertainty.*

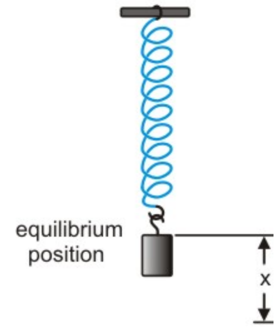
Example #3.5: Given Maria's answer from above, in which she determined her measurement of the time it takes for a steel ball to fall from the top of a table to the floor, calculate the height of the table, *y*, and its associated uncertainty using the model: (use g = 9.8 m/s^2)

$$y = \frac{1}{2} g t^2$$

Possible Answer: *Maria's value for the time of fall is t = 0.49 s ± 0.08 s. First Maria must calculate a value for the table height: y = 1/2*(9.8 m/s^2)*0.49 s → y = 2.4 m. Then she must propagate the uncertainty from her time measurement, δt, to her calculated value, δy. The equation above can be made to look like the general formula from page 5 by taking: y = z, C = 1/2*g, x = t, a = 2, y = 1, and b = 0. So the propagation calculation should look like:*

$$\frac{\delta y}{y} = \sqrt{\left(2\,\frac{\delta t}{t}\right)^2} \quad \text{-- or --} \quad \delta y = (2.4\ m)\sqrt{\left(2\,\frac{(0.08\ s)}{(0.49\ s)}\right)^2} = 0.78\ m$$

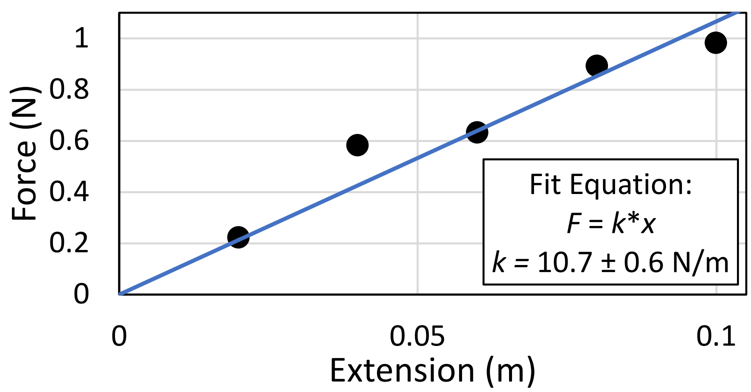
So Maria has determined that the table height is *y* = 2.4 m ± 0.8 m.



Example #4: Lucas did an experiment in which they were attempting to verify Hooke's Law, which describes the amount of force (*F*) a spring applies to a suspended mass depending on the extension of the spring (*x*) and the stiffness of the spring (*k*): $F = kx$. Lucas measured the force applied by a particular spring when the hanging mass was extended to different positions, and they generated the data in the table below.

Force (N)	Extension (m)
0.22 ± 0.02	0.020
0.58 ± 0.05	0.040
0.63 ± 0.04	0.060
0.89 ± 0.05	0.080
0.98 ± 0.05	0.10

Lucas plotted the data and fit it with a proportional model ($y = A*x$) as shown below. Estimate the deviation between the mathematical model and the measured data.



Possible Answer: *Since it is already provided to us as a fit parameter, the deviation between the mathematical model and the measured data is 0.6 N/m, so we would write that our graphical analysis yields a measured spring stiffness, k, equal to 10.7 N/m ± 0.6 N/m. Scientifically, this comparison is incomplete, since it does not account for the estimated uncertainty in our measurements, but this is the extent to which you will be asked to analyze graphical uncertainty the introductory physics curriculum.*