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Problem 1: (a) Find a particular solution of the recurrence $G_n = 3G_{n-1} + 2G_{n-2} + n + 4$. Show your work.

The inhomogeneous term is a linear function, so we try a particular solution of the form $G''_n = \beta_1 n + \beta_2$. Plugging this in, we get an equation

$$\beta_1 n + \beta_2 = 3[\beta_1(n-1) + \beta_2] + 2[\beta_1(n-2) + \beta_2] + n + 4$$

that simplifies to

$$(4\beta_1 + 1)n + (-7\beta_1 + 4\beta_2 + 4) = 0.$$

Since this equation must hold for all n, we must have $4\beta_1 + 1 = 0$ and $-7\beta_1 + 4\beta_2 + 4 = 0$. So $\beta_1 = -\frac{1}{4}$ and $\beta_2 = -\frac{23}{16}$. This gives us the following particular solution:

$$G''_n = -\tfrac{1}{4}n - \tfrac{23}{16}$$

(b) Find a particular solution of the recurrence $Q_n = 3Q_{n-1} + 2Q_{n-2} + 2 \cdot 3^n$. Show your work.

The inhomogeneous term is an exponential function (times a constant), so we try a particular solution of the form $Q''_n = \beta \cdot 3^n$. Plugging this in, we get an equation

$$\beta \cdot 3^n = 3\beta \cdot 3^{n-1} + 2\beta \cdot 3^{n-2} + 2 \cdot 3^n$$

that simplifies (after dividing by 3^{n-2}) to

$$9\beta = 9\beta + 2\beta + 18.$$

So $\beta = -9$, giving us the following particular solution:

$$Q''_n = -9 \cdot 3^n$$

Problem 2: (a) Give the definition of Euler's totient function $\phi(n)$.

For a positive integer n, $\phi(n)$ is the number of integers in the range $\{1, 2, ..., n\}$ that are relatively prime to n.

(b) Give the formula for Euler's totient function.

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot ... \cdot \left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, ..., p_k$ are all the different prime factors of n.

(c) Compute $\phi(18000)$.

The factorization of 18000 is $18000 = 3^2 \cdot 2^4 \cdot 5^3$. So

$$\phi(18000) = 18000 \cdot (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(1 - \tfrac{1}{5}) = 18000 \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{4}{5} = 4800.$$

Problem 3: For each recurrence equation below, mark (circle) the correct solution.

Recurrence	Solution
(a) $f(n) = 4f(n/3) + 3n$	$\boxed{\Theta(n^{\log_3 4})}$
	$\Theta(n)$
	$\Theta(n^{3/4})$
	$\Theta(n^{\log_4 3})$
	$\Theta(n^2)$
	$\Theta(n \log n)$
	none of the above
(b) $f(n) = 16f(n/4) + 2n^2$	$\Theta(n^{\log_3 4})$
	$\Theta(n)$
	$\Theta(n^{3/4})$
	$\Theta(n^{\log_4 3})$
	$\Theta(n^2)$
	$\Theta(n \log n)$
	$\boxed{\text{none of the above}}$
(c) $f(n) = 4f(n/3) + 2n^2$	$\Theta(n^{\log_3 4})$
	$\Theta(n)$
	$\Theta(n^{3/4})$
	$\Theta(n^{\log_4 3})$
	$\boxed{\Theta(n^2)}$
	$\Theta(n \log n)$
	none of the above