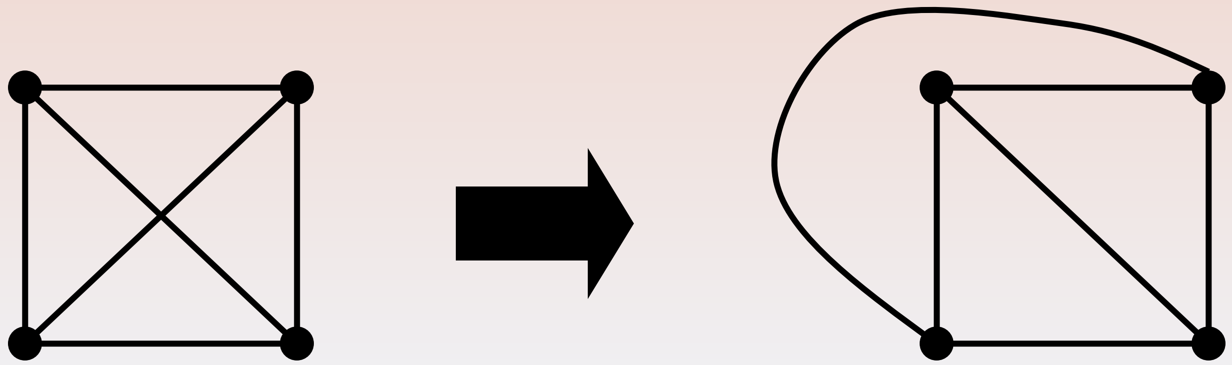
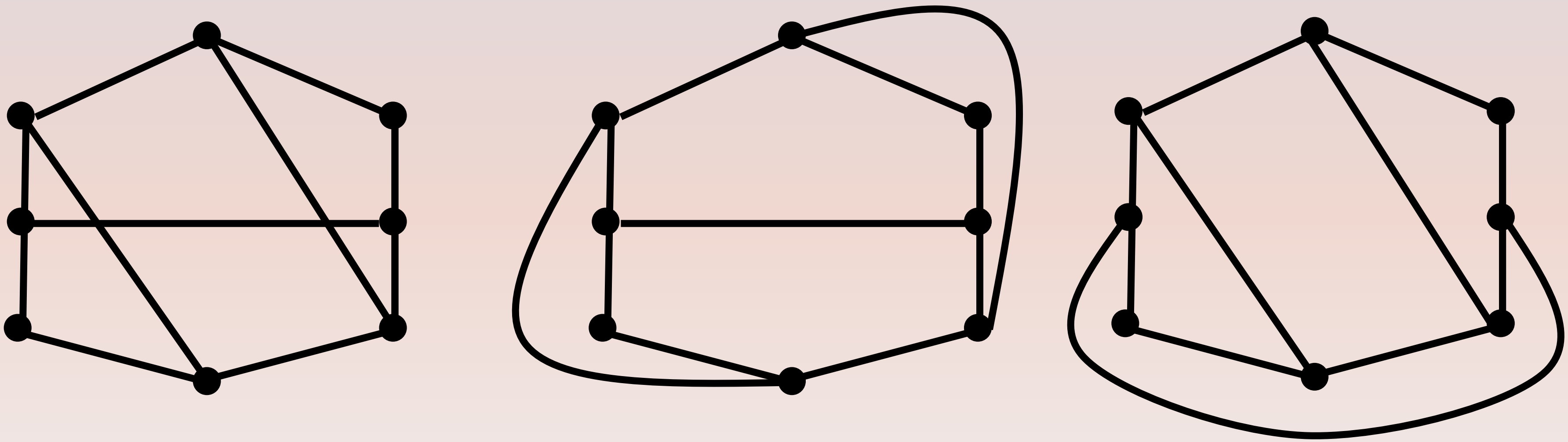
# Planar Graphs

# Planar Graphs

**Definition:** A graph that can be drawn in the plane without any of its edges intersecting is called a *planar graph*. A graph that is so drawn in the plane is also said to be embedded in the plane.

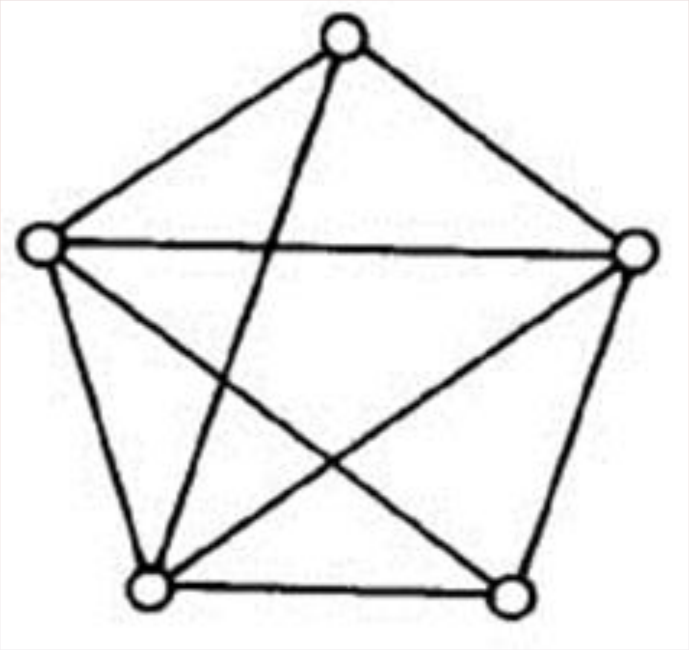


# Planar Graphs



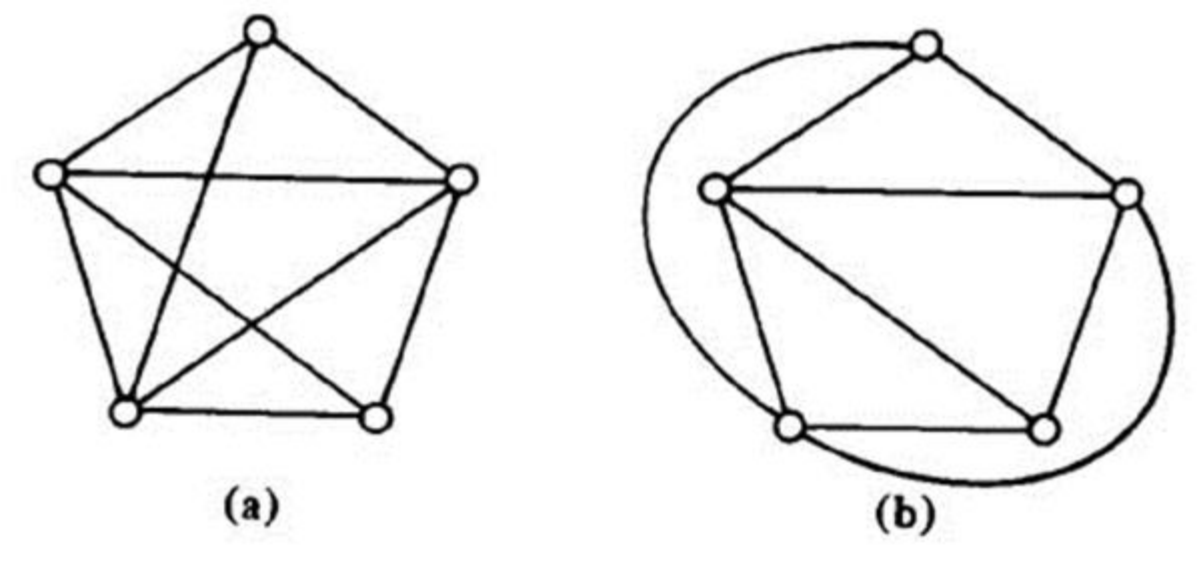
# Planar Graphs

Is the following graph planar?



# Planar Graphs

Yes this graph is planar.



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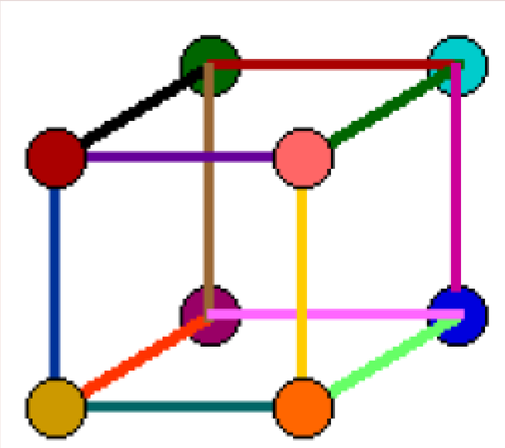
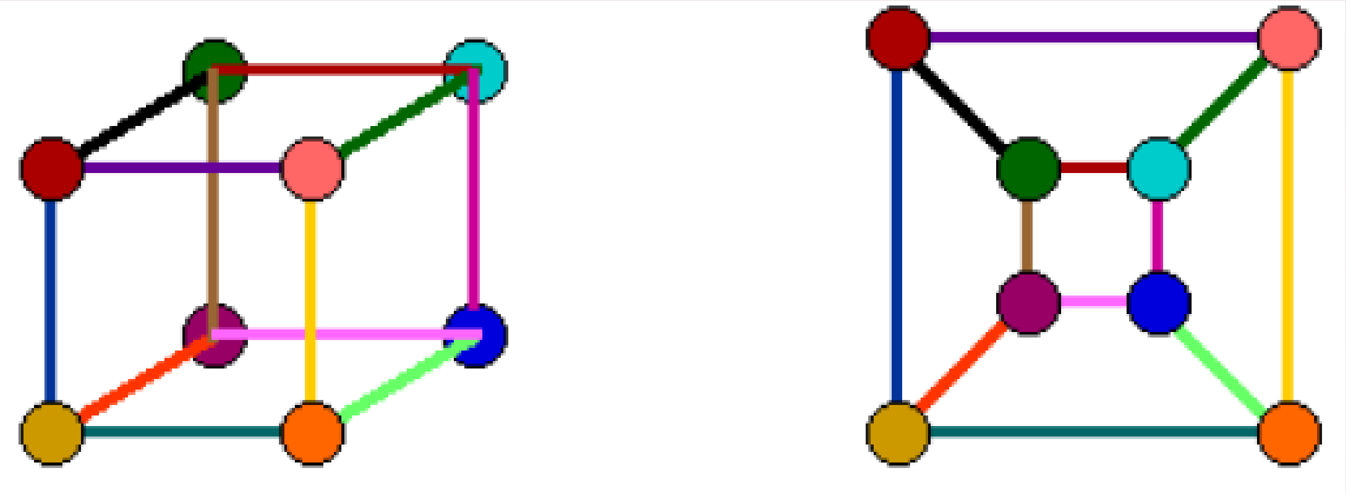
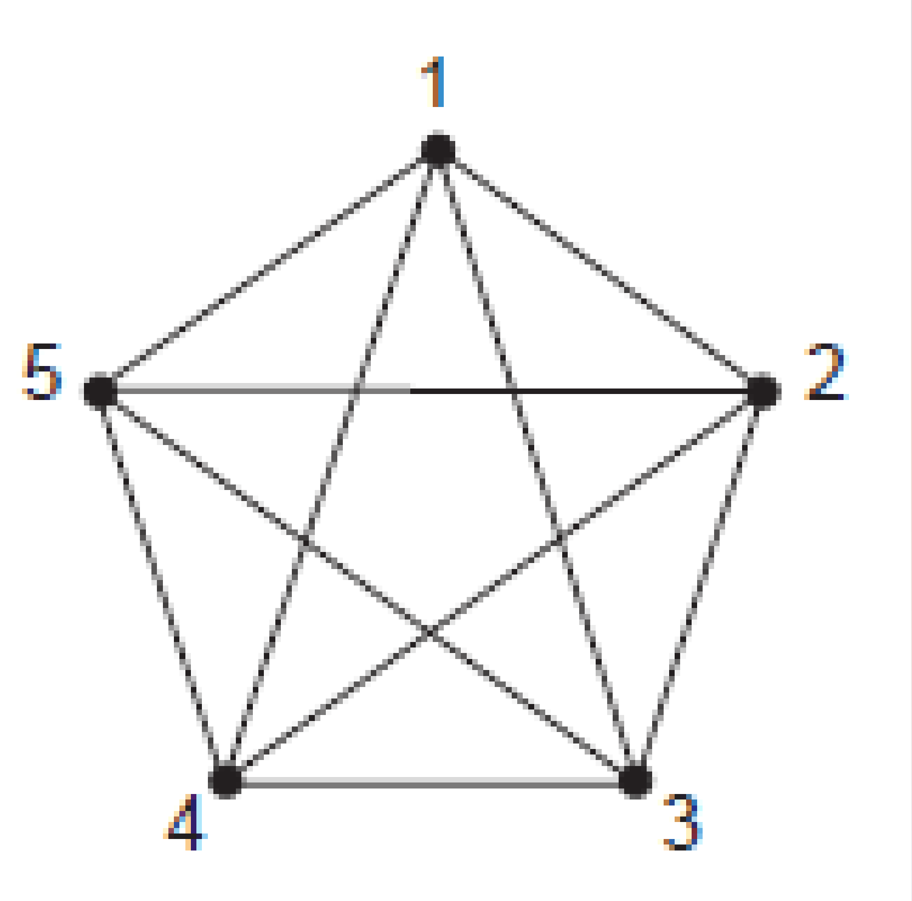


# Planar Graphs

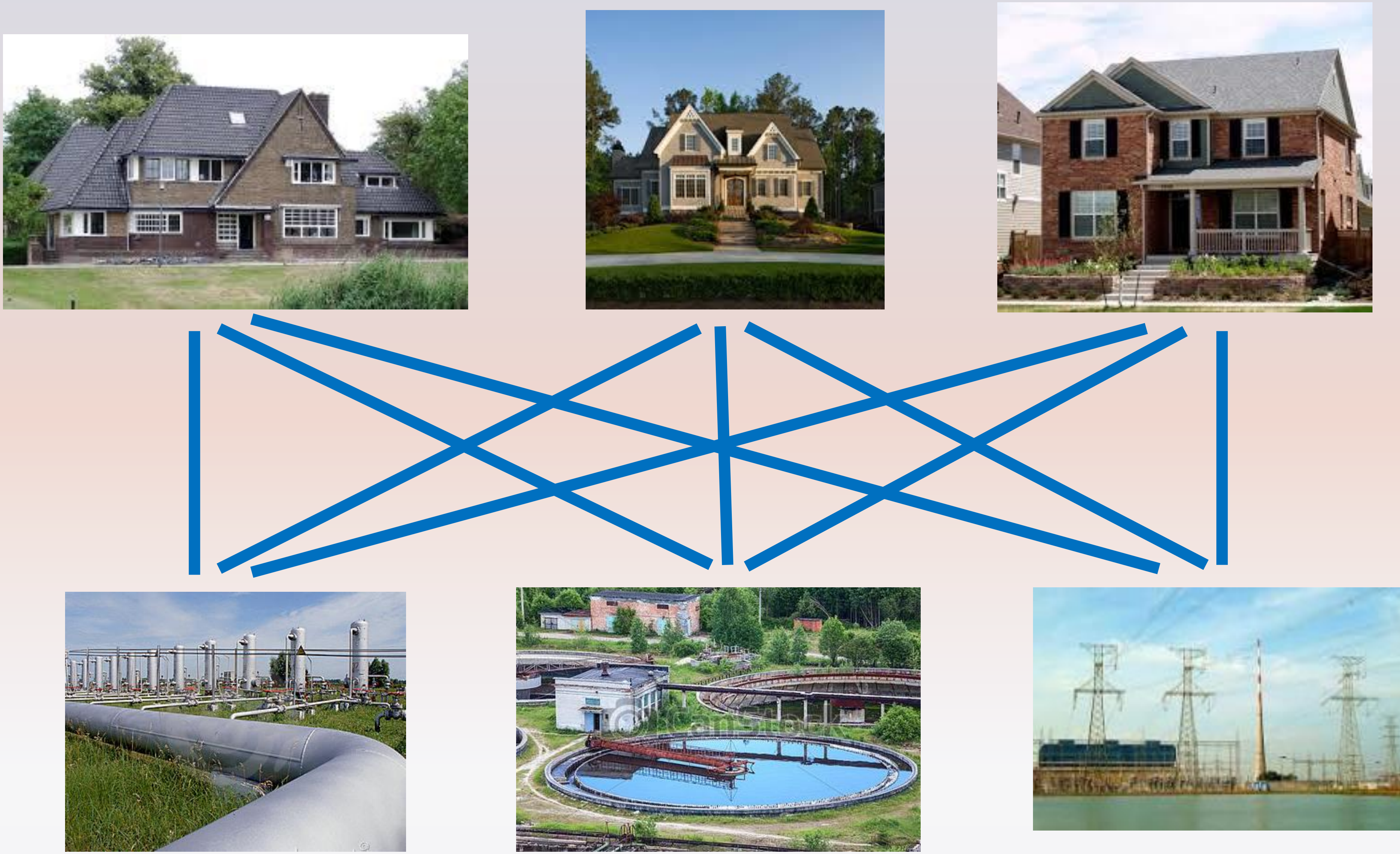
Yes this graph is planar



# $K_5$



# $K_{3,3}$



# Planar Graphs

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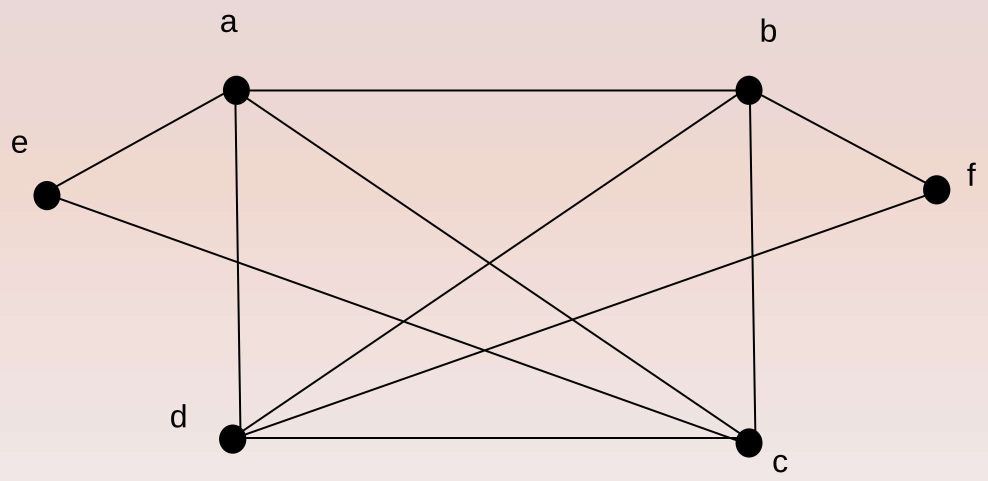
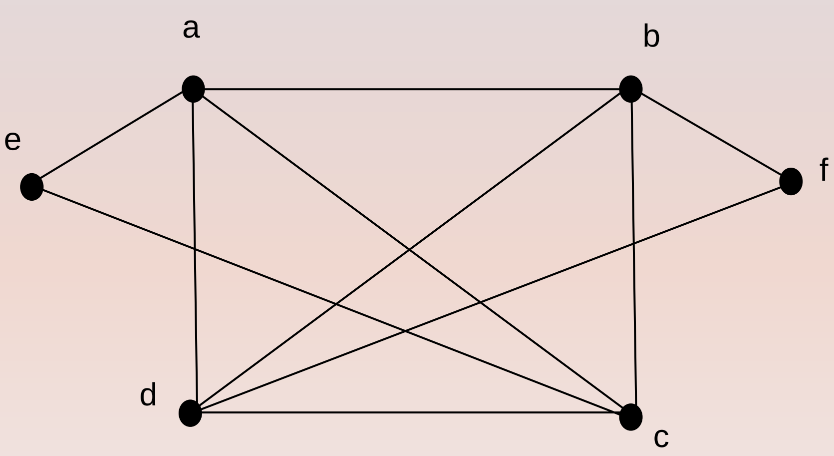
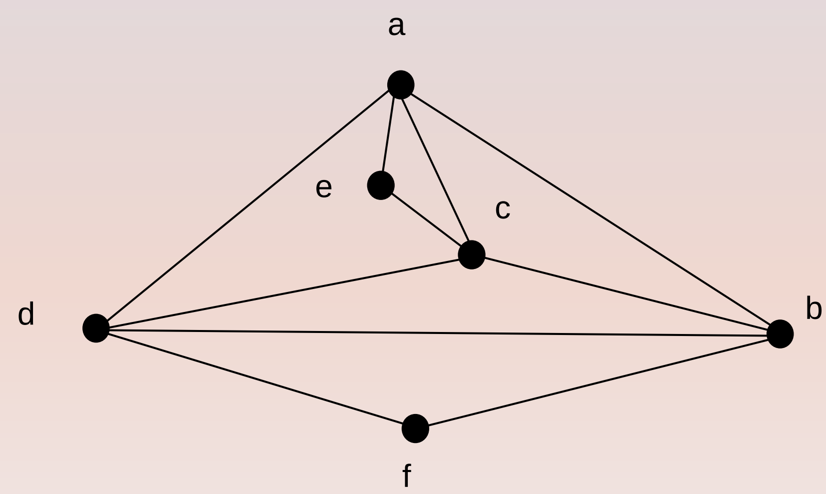



Figure 1

# Planar Graphs

Is the following graph planar?



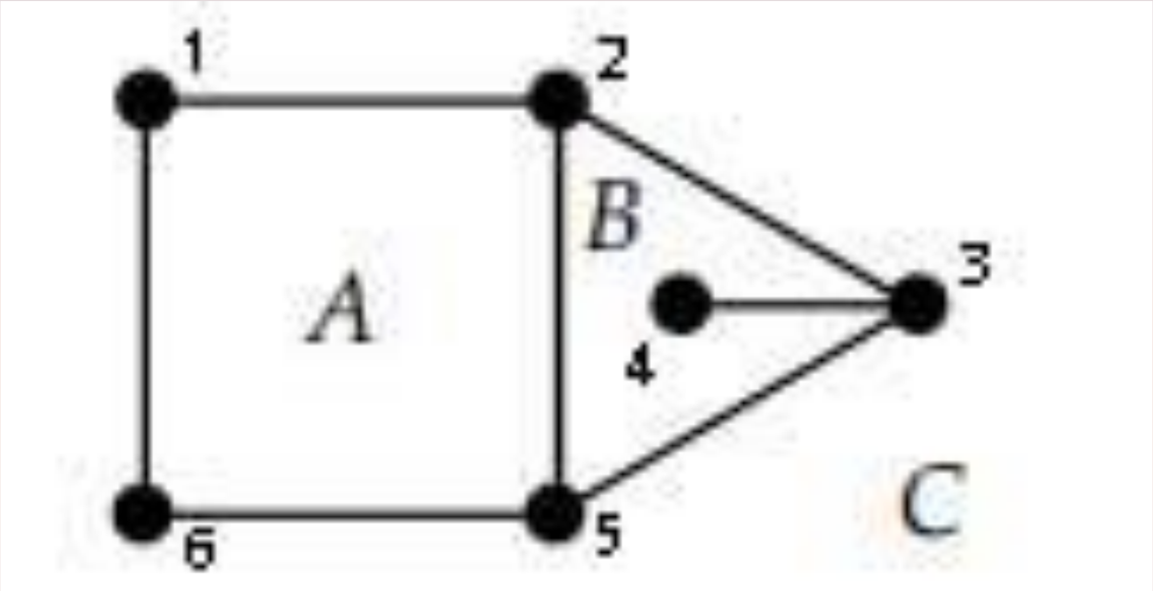
# Planar Graphs

- We can prove that a particular graph is planar by showing its planar embedding.
- However, not all graphs are planar.
- It may be difficult to show that a graph is nonplanar. We would have to show that there is *no way* to draw the graph without any edges crossing.

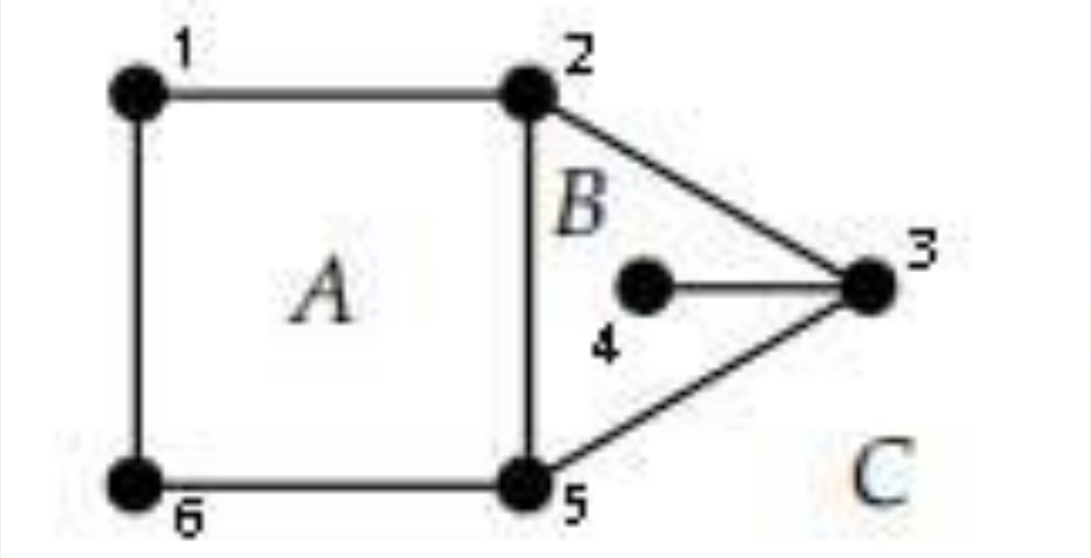
# Some Definitions

- When a planar graph is drawn with no crossing edges, it divides the plane into a set of regions, also called ***faces***.
- The unbounded area outside the whole graph is one face.
- The ***boundary*** of a face is the subgraph containing all the edges adjacent to that face, and a ***boundary walk*** is a closed walk containing all of those edges.
- The ***degree*** of a face is the number of edges on its boundary.

# Face, boundary



# Face, boundary



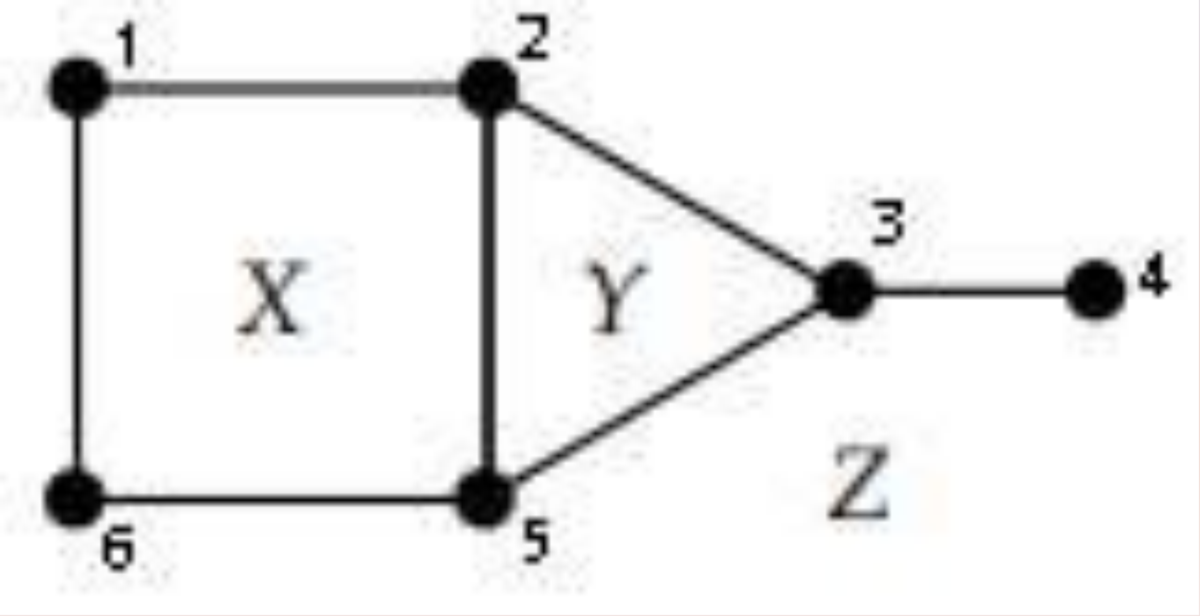

- 3 Faces A, B, C
- Degrees of all faces

  d(A) = 4 (*boundary walk* “12561”)

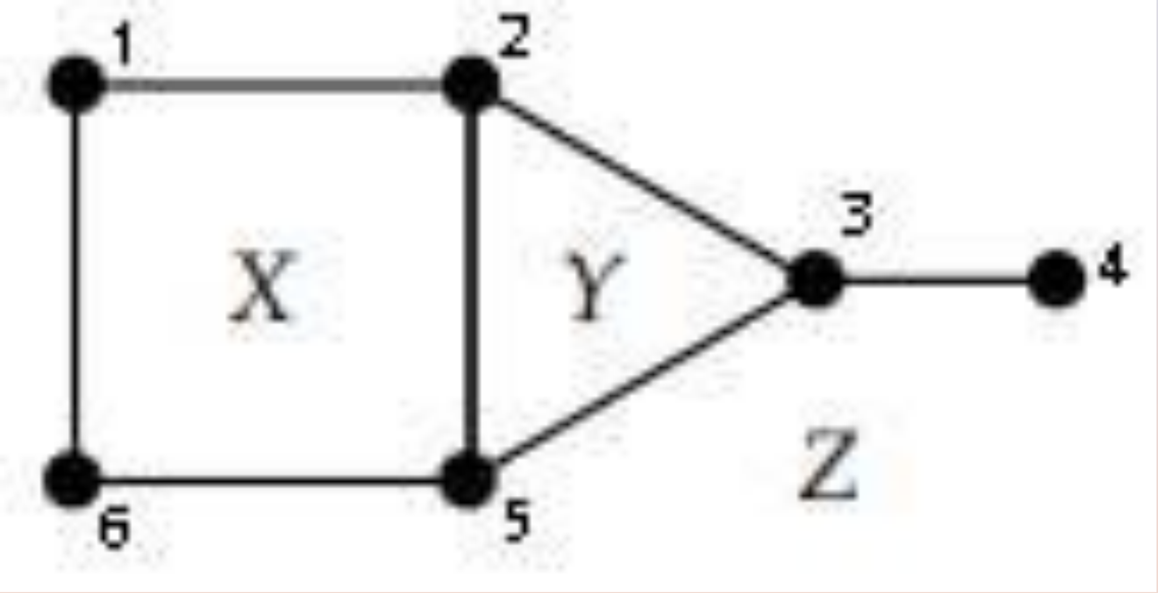
  d(B) = 5 (*boundary walk* “234352”)

  d(C) = 5 (*boundary walk* “123561”)

# Face, boundary



# Face, boundary




- 3 Faces X, Y, Z
- Degrees of all faces

  d(X) = 4 (*boundary walk* “12561”)

  d(Y) = 3 (*boundary walk* “2352”)

  d(Z) = 7 (*boundary walk* “12343561”)

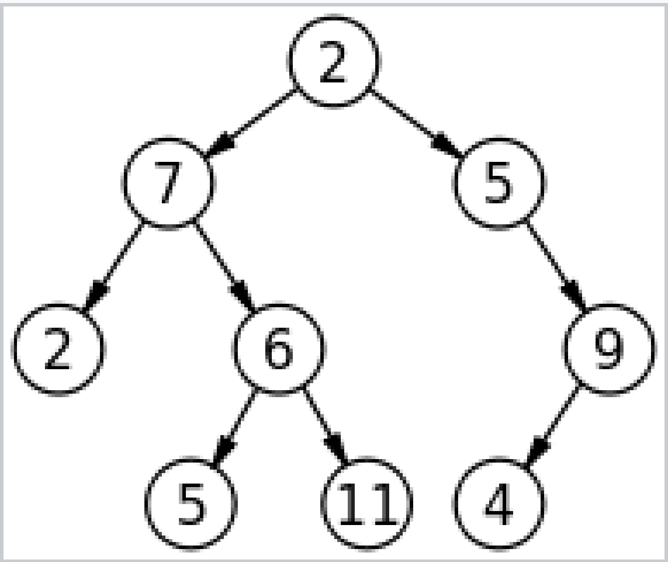
# Face, boundary

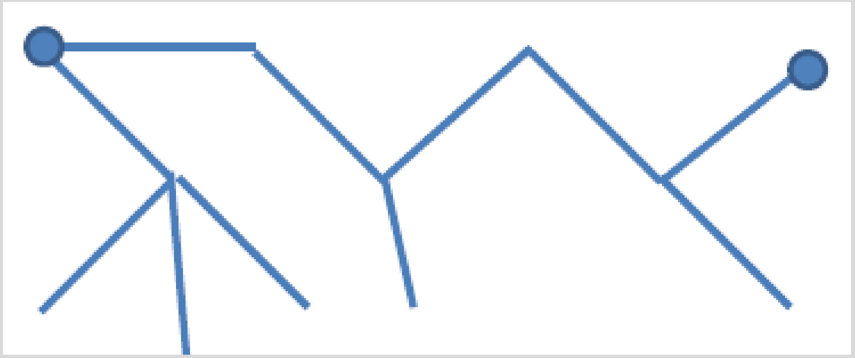
- Although the two drawings of the graph above had some different features, they both had the same number of faces, 3. In fact, this is always the case, as Euler’s formula shows.

- The number of faces does not depend on the (planar) drawing.

# Trees

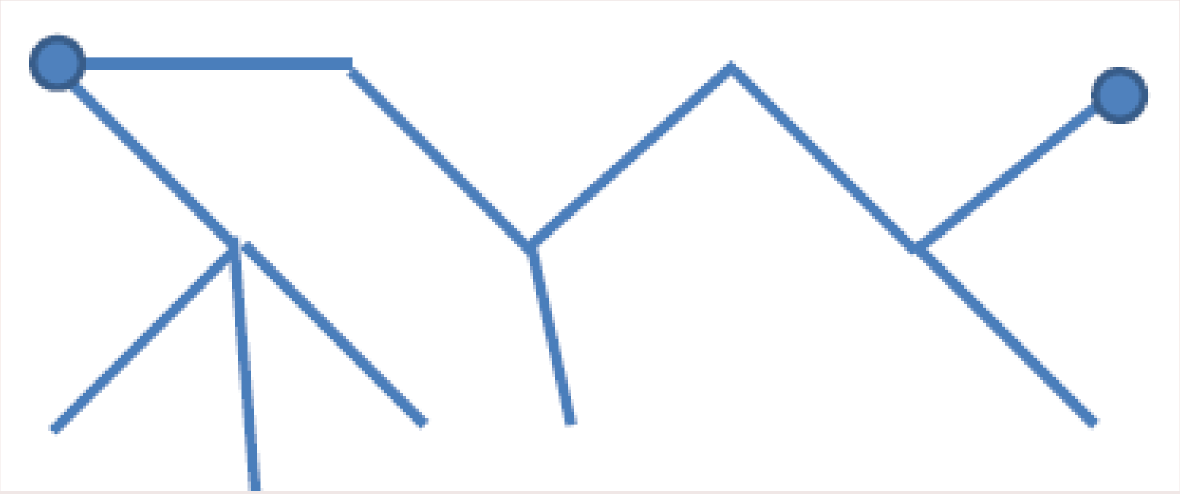
- A ***tree*** is any connected graph with no cycles.
- A ***free tree*** is an unrooted tree.



# Trees

- Let T be a free tree with $|V| = n$. What can you tell about m (the number of edges)?



Required

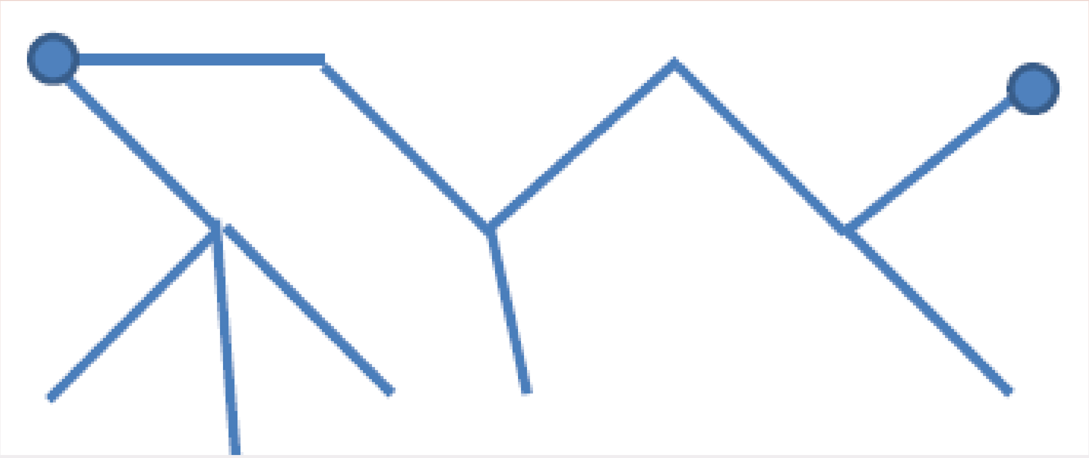
# Trees

**Lemma**

If T is a free tree, and has n vertices, then its number of edges is m = n – 1.

Proof: (by induction on the number of vertices or

on the number of edges ).



Required

# Euler's Formula

**Euler's Formula:** Let $G$ be a connected planar graph, and

$\boldsymbol{n}$ = number of vertices ($n > 0$),

$\boldsymbol{m}$ = number of edges,

$\boldsymbol{f}$ = number of regions (faces).

Then any planar embedding of $G$ has

$$\boldsymbol{f = m - n + 2} \text{ faces.}$$

Required for EC

# Euler's Formula

**Proof:**

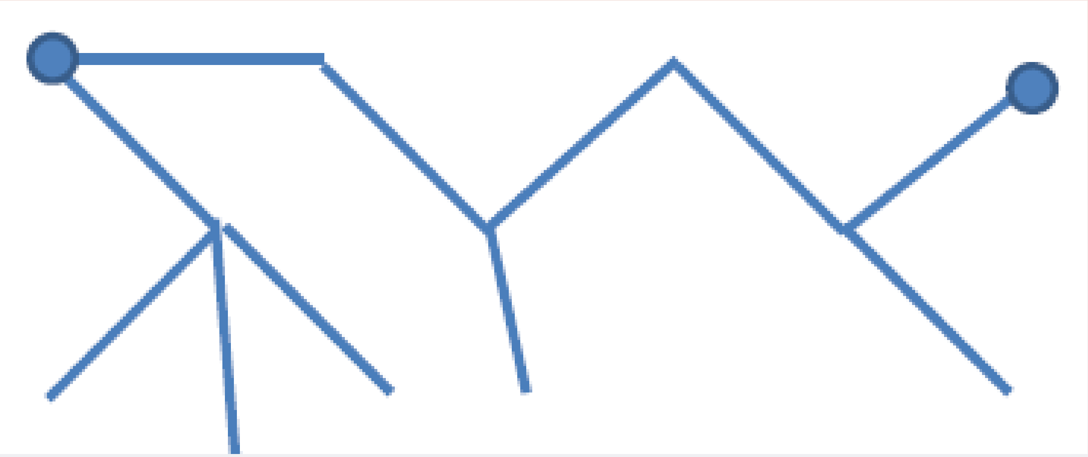
**<u>Case 1:</u>** G does not contain a cycle, so G is a free tree.

In a free tree, **m = n – 1** (proof by induction).

The number of faces is always 1, so:

$\boldsymbol{f = 1,\ \ m - n + 2 = 1, \text{ and } \ f = m - n + 2}$

for any graph that has no cycles.



# Euler's Formula

**Case 2:** G has at least one cycle.

Proof by induction on the number of edges in the graph.

Base case: m = 0. Then $f = m - n + 2 = 1$ – true.

Inductive assumption: suppose, the formula works for any graph G with no more than k - 1 edges.

Need to Prove: the formula works for any graph with k edges.

# Euler's Formula

Let G be a graph with k edges.

Pick some edge *e* that belongs to a cycle. Remove *e* to get a graph G' with k - 1 edges.

Any cycle separates a plane into two faces. By removing *e*, we merged two faces into one. So, G' has one less face, than G.

Since G' has k-1 edges, the formula works for G' by assumption, and $f' = k' - n' + 2$, but we know that

$f' = f - 1$, $k' = k - 1$, and n' = n. Bringing back the removed edge, we get:

$f = f' + 1 = k' - n' + 2 + 1 = k - 1 - n + 2 + 1 = k - n + 2.$

Then $\boldsymbol{f = m - n + 2}$

# Euler's Formula: Corollary 1

**Corollary 1:** If G is a connected planar graph with $\boldsymbol{n} \geq 3$, then $\boldsymbol{m} \leq 3\boldsymbol{n} - 6$.

**Proof:** The sum of the degrees of the faces is equal to twice the number of edges. But each face must have degree $\geq 3$. So we have $2m \geq 3f$.

Euler's formula says that $\boldsymbol{f} = \boldsymbol{m} - \boldsymbol{n} + 2$, thus

$$3f = 3m - 3n + 6.$$

Combining this with $2m \geq 3f$, we get

$$2m \geq 3m - 3n + 6.$$

So

$$0 \geq m - 3n + 6, \text{ and}$$

$$\boldsymbol{m} \leq 3\boldsymbol{n} - 6.$$

# Euler's Formula

Corollary 1: If G is a connected planar graph with $n \geq 3$, then

$$m \leq 3n - 6.$$

<u>Question:</u> If a graph has more than $3n - 6$ edges ($m > 3n - 6$), what can we say? (Is it planar or not?)

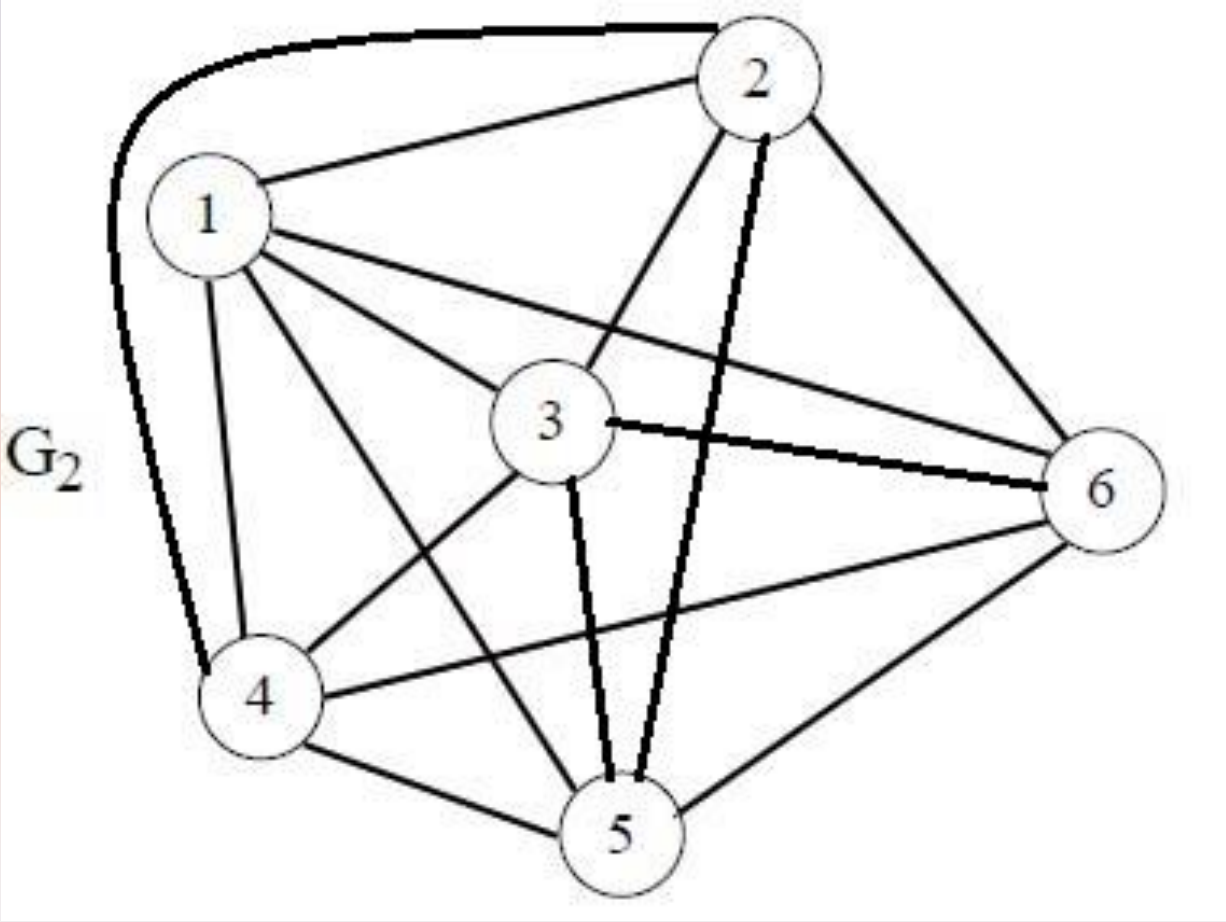
# Euler's Formula

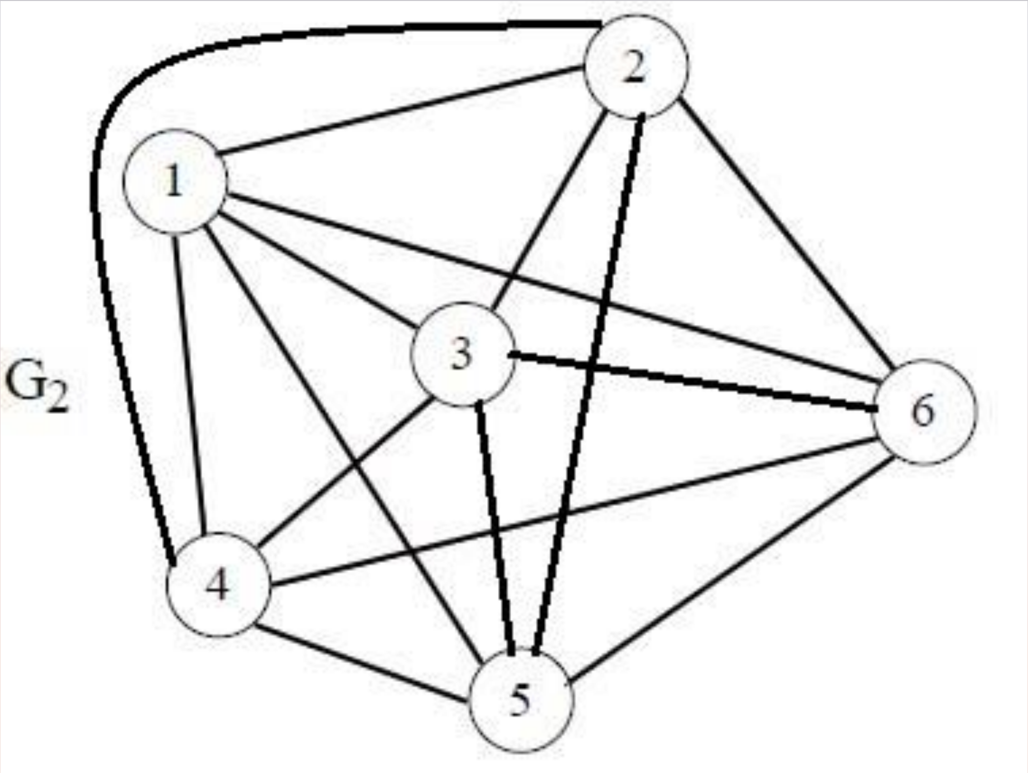
**Corollary 1:** If G is a connected planar graph with $n \geq 3$, then

$$m \leq 3n - 6.$$

<u>Answer:</u> If a graph has more than $3n - 6$ edges ($m > 3n - 6$) it must be **nonplanar.**

# Is this graph planar?





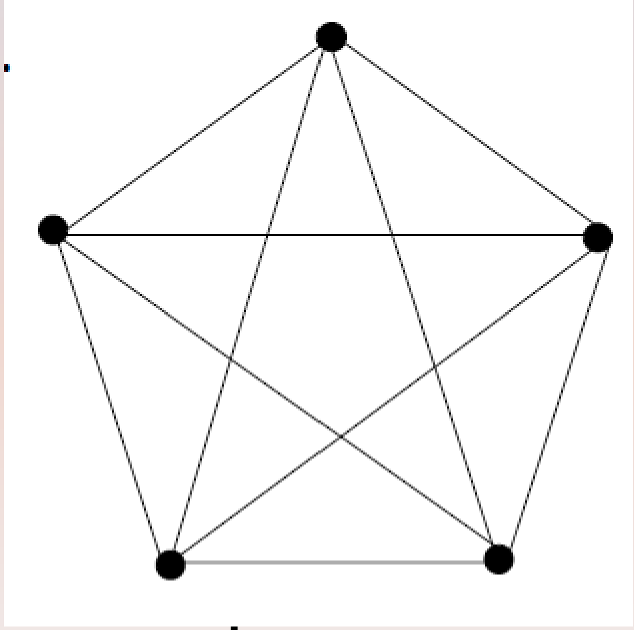

n=6 ≥ 3

m=15

15 > 3*6 – 6 = 12, so this graph is **nonplanar**

# Prove that $K_5$ is nonplanar.

Use Corollary 1: If G is a connected planar graph with **m** > 1, then **m** ≤ 3**n** – 6.



$K_5$ has 5 vertices and 10 edges:

n = 5, m = 10, and

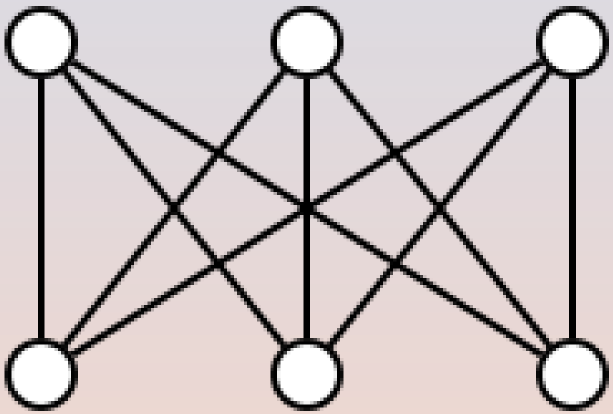
10 > 3*5 – 6 = 9, so

**m ≤** 3**n** – 6 is not true, thus, $K_5$ is nonplanar.

$K_5$ is complete graph with 5 vertices.

*Complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

# What about $K_{3,3}$?



If G is a connected planar graph with **m** > 1, then **m** ≤ 3**n** – 6.

$K_{3,3}$ has n = 6 vertices and m = 9 edges:

9 < 3*6 – 6 = 12, but $K_{3,3}$ is nonplanar!

$K_{3,3}$ is complete bipartite graph with 3 vertices in both L and R sets.

# Euler's Formula: Corollary 2

**Corollary 2:** If G is a connected planar graph with $n \geq 4$, and no cycles of length 3, then $m \leq 2n - 4$.

**Proof:** The sum of the degrees of the faces is equal to twice the number of edges. But each face must have degree $\geq 4$. So we have $2m \geq 4f$.

Euler's formula says that $f = m - n + 2$, thus

$$4f = 4m - 4n + 8.$$

Combining this with $2m \geq 4f$, we get

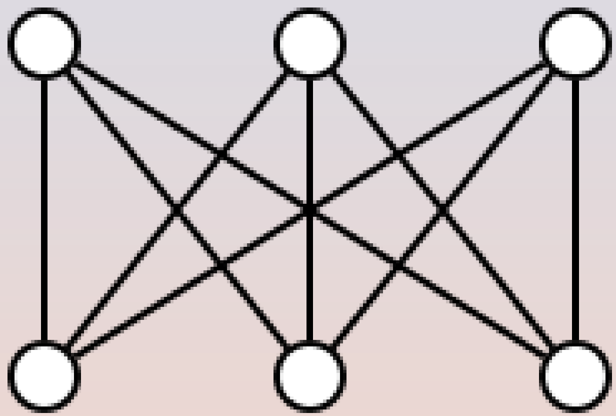
$$2m \geq 4m - 4n + 8.$$

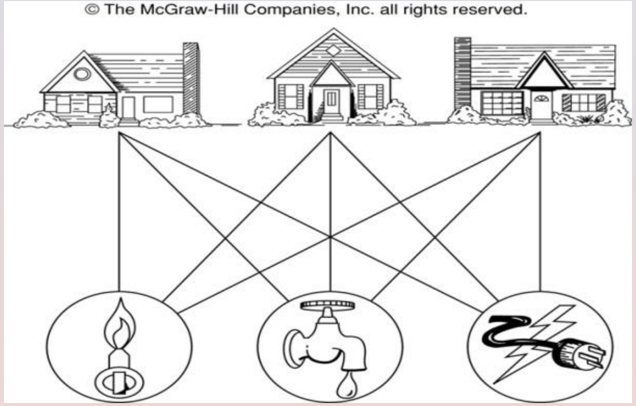
So

$$0 \geq 2m - 4n + 8, \text{ and}$$

$$m \leq 2n - 4$$

# $K_{3,3}$ is nonplanar.






Corollary 2: If G is a connected planar graph with n ≥ 3, and no cycles of length 3, then m ≤ 2n – 4.

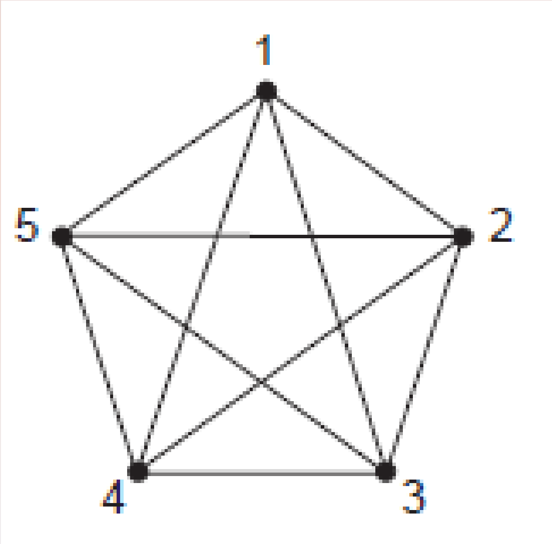
$K_{3,3}$ has n = 6 vertices and m = 9 edges:

9 > 2*6 – 4 = 8, so m ≤ 2n – 4 is not true, thus, $K_{3,3}$ is nonplanar.

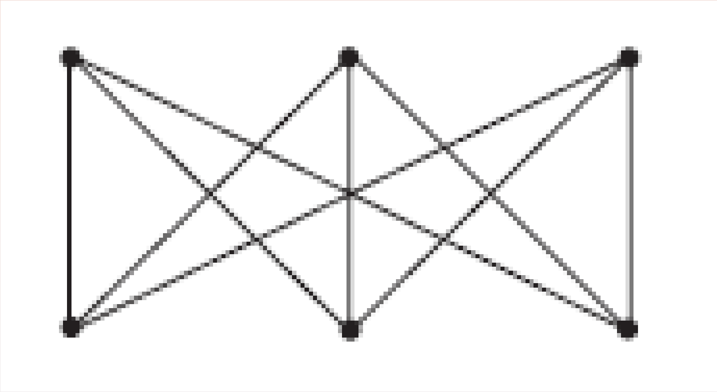
$K_{3,3}$ is complete bipartite graph with 3 vertices in both L and R sets.

# Planar Graphs

**Kuratowski's Theorem:** A finite graph G is planar if and only if it has no subgraph that is homeomorphic to $K_5$ or $K_{3,3}$.




$K_5$

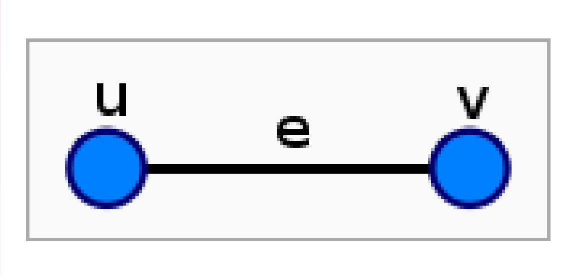


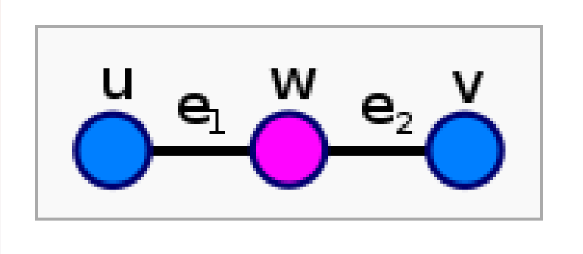
$K_{3,3}$

# Planar Graphs

Two graphs, G and G' are homeomorphic if there is a graph isomorphism from some **subdivision** of G to some **subdivision** of G'.

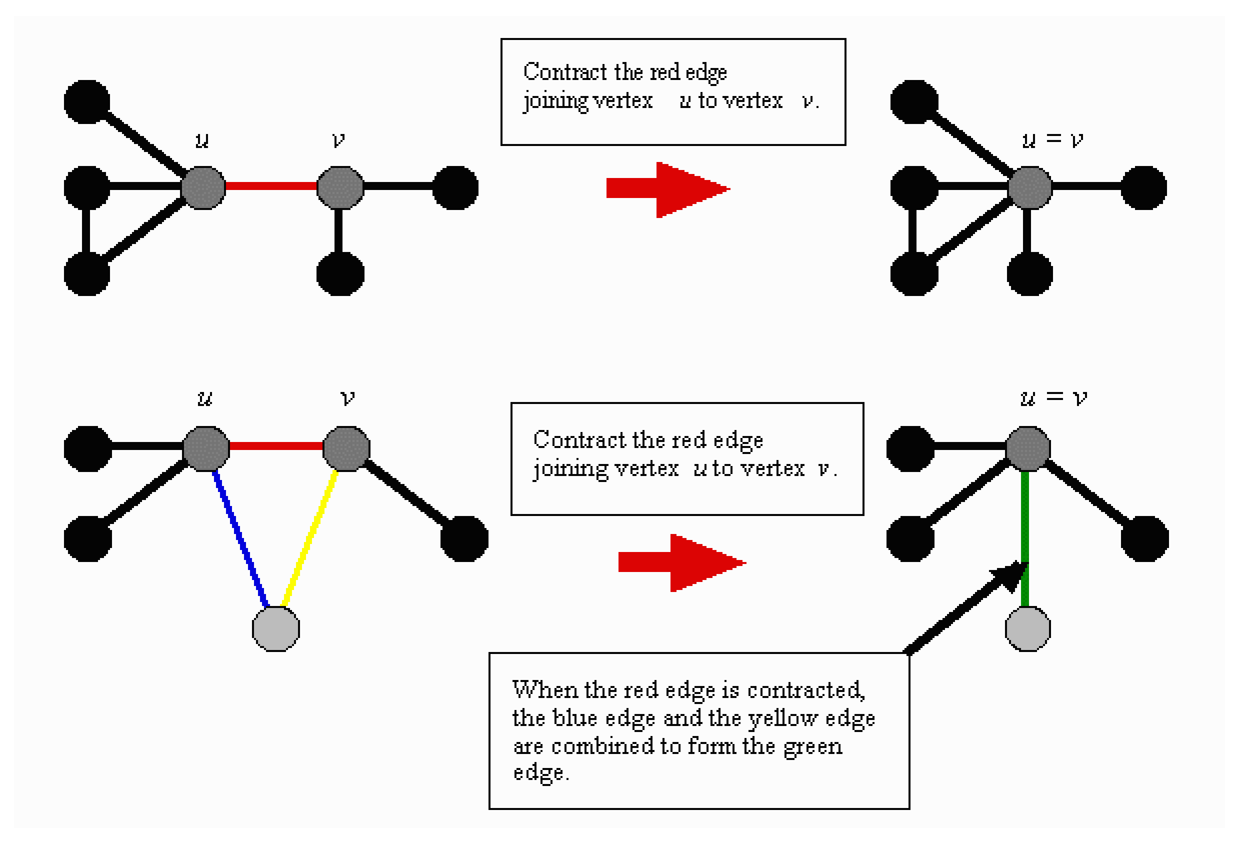
***Edge subdivision:***



# Planar Graphs

***Edge contraction:***



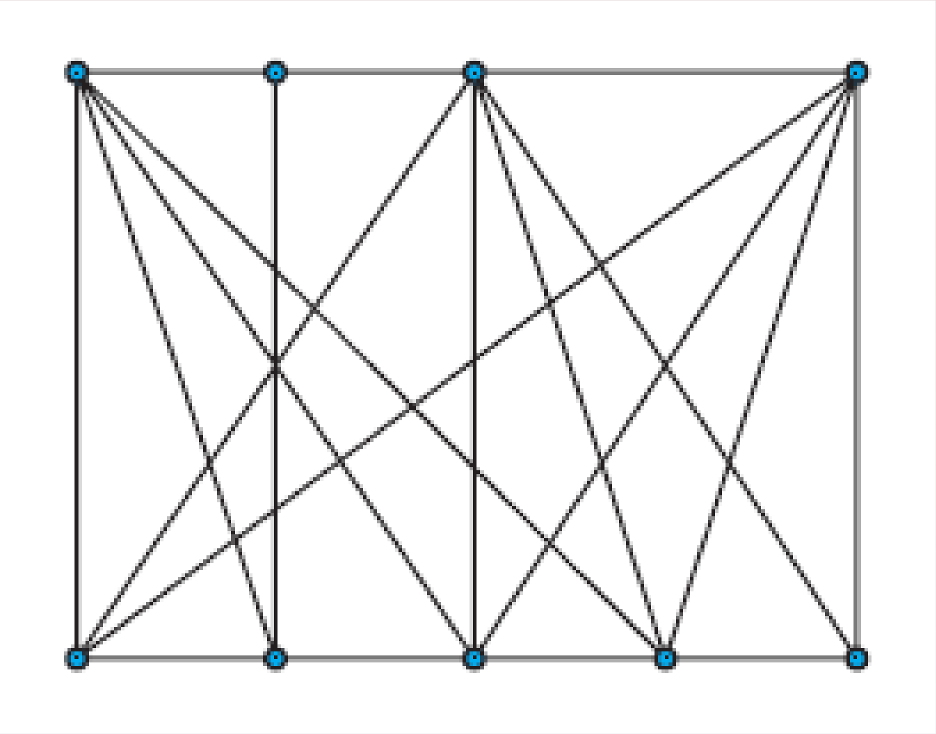
# Planar Graphs

**Kuratowski's Theorem:** A finite graph G is planar if and only if it has no subgraph that is homeomorphic to $K_5$ or $K_{3,3}$.

Kuratowski's theorem tells us that ***any nonplanar graph has either $K_5$ or $K_{3,3}$ hidden inside of it***.

# Planar Graphs

Can you use Kuratowski's Theorem to prove that the graph below is nonplanar?



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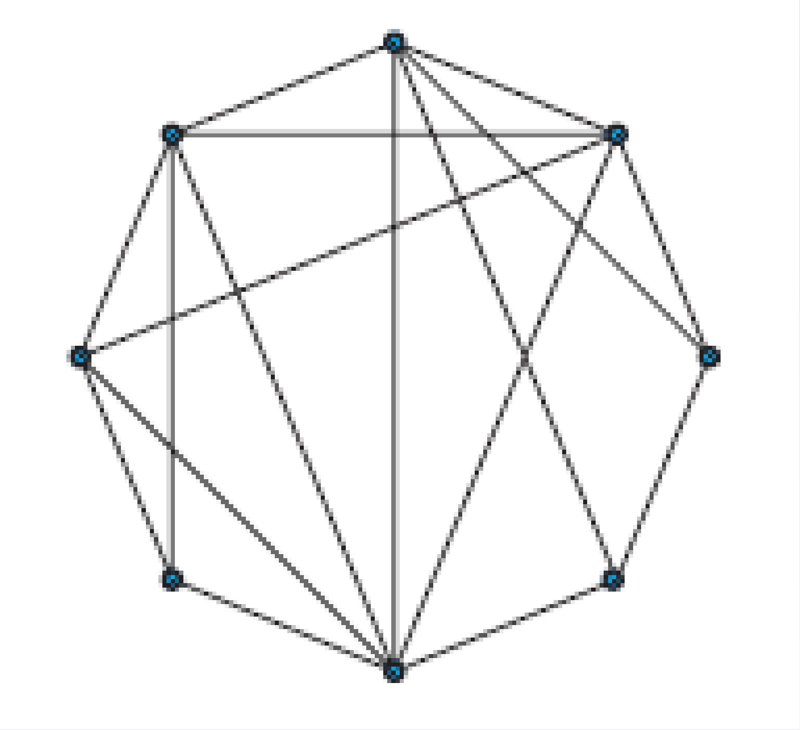
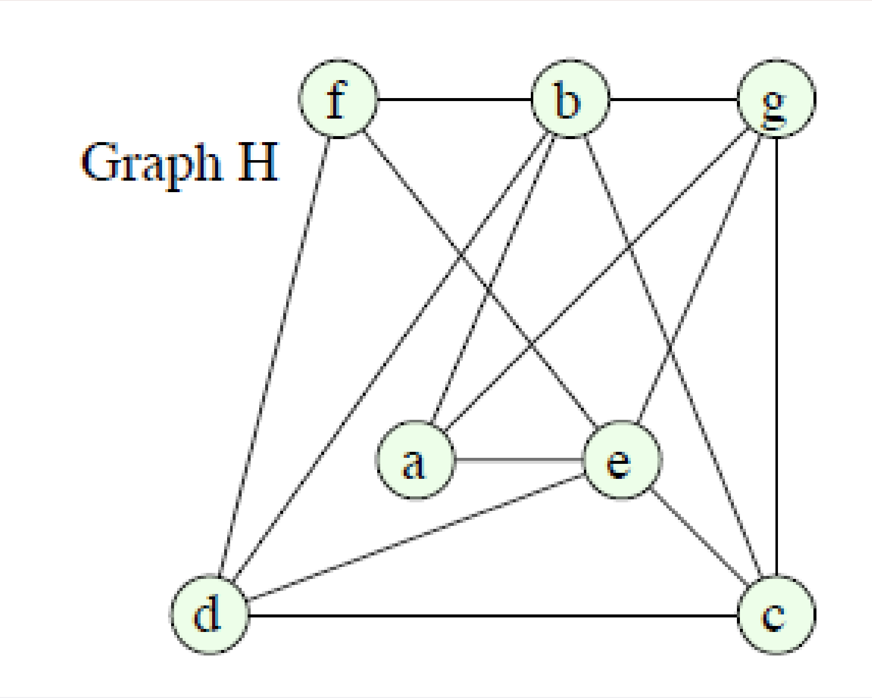


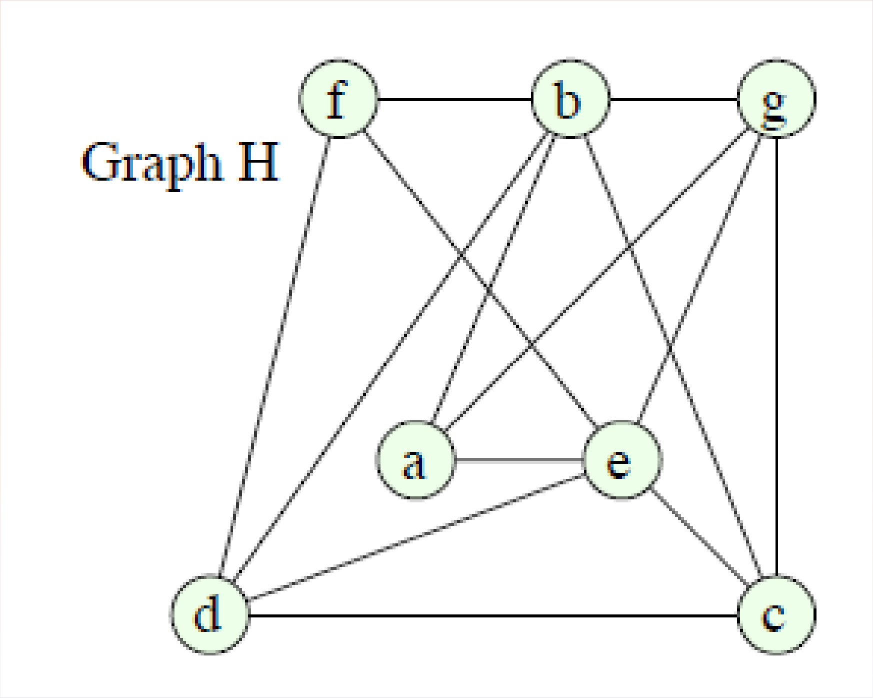
# Planar Graphs

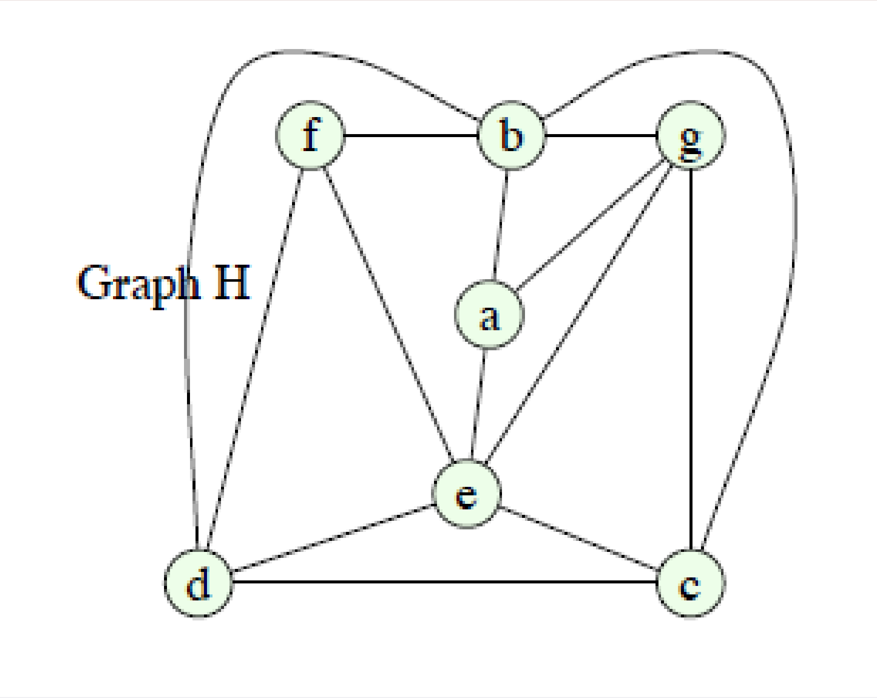
Use Kuratowski's Theorem to prove that graph H is nonplanar or show its planar embedding



# Planar Graphs

Planar embedding of graph H



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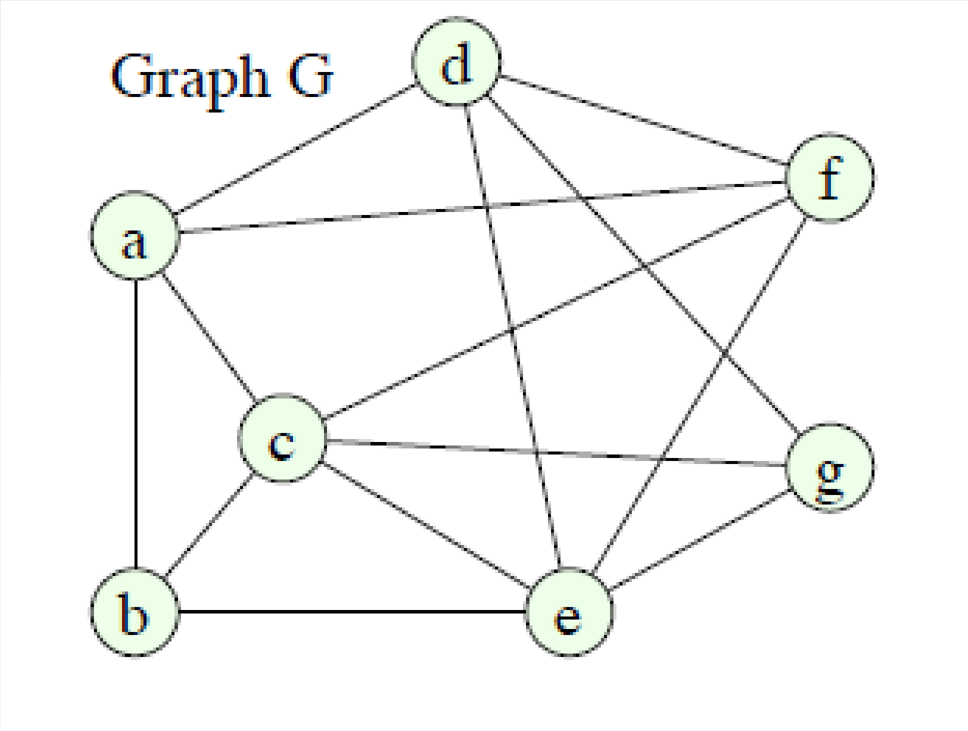
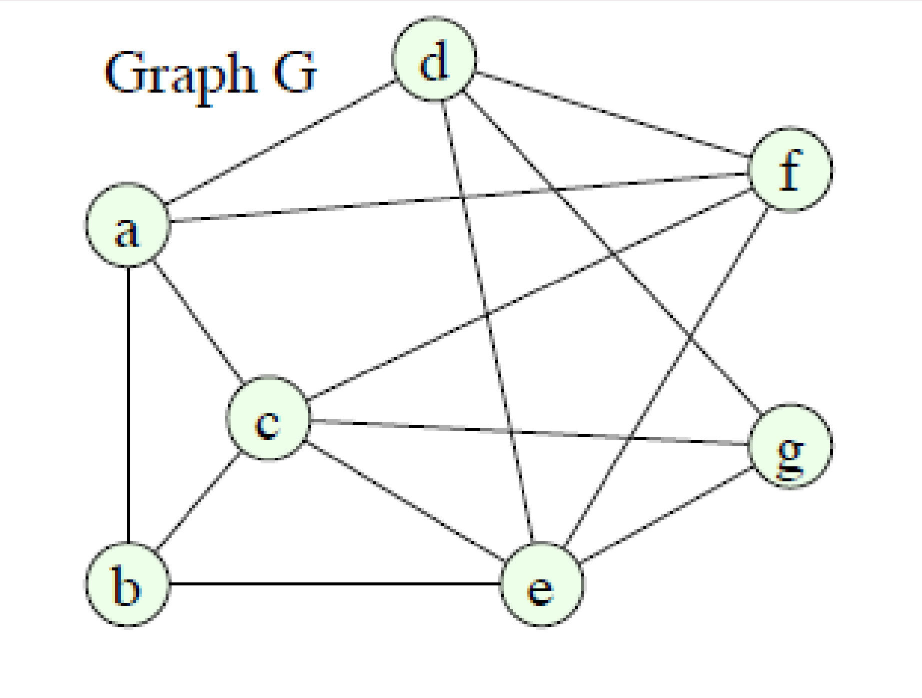
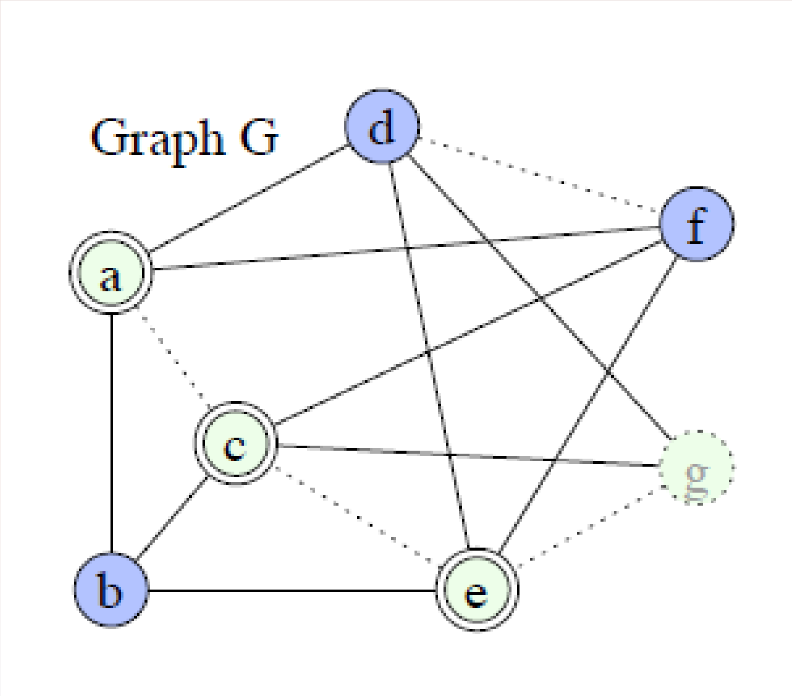


# Planar Graphs

Graph G contains $K_{3,3}$



# 6 – Coloring Theorem

**Lemma.** For any planar graph G, the average degree of G is strictly less than 6.

**Proof.** The average degree of a graph is D = 2m/n. Using $\boldsymbol{m} \leq 3\boldsymbol{n} - 6$ (for $n \geq 3$), we get

$$D \leq 2(3\boldsymbol{n} - 6)/n \text{ or } D \leq 6 - 12/n = 6(1 - 2/n).$$

For n $\geq 3$, $D < 6$

For n $\leq 3$, we can check directly.

# 6 – Coloring Theorem

**Theorem.** Any planar graph is 6-colorable.

**Proof:** Proof by induction on the number of vertices.

**Base case:** Suppose we have a graph with $n \leq 6$.

**Inductive assumption:** assume, that any planar graph on $n = k$ vertices can be colored with 6 colors.

**Prove**, that any G on $n = k + 1$ vertices can be colored with 6 colors.

From lemma above, G must have at least 1 vertex with degree at most 5. Remove it ...

# 5 – Color Theorem, 4 – Color Theorem

**Theorem.** Any planar graph is 5 - colorable.

**Theorem.** Any planar graph is 4 - colorable.