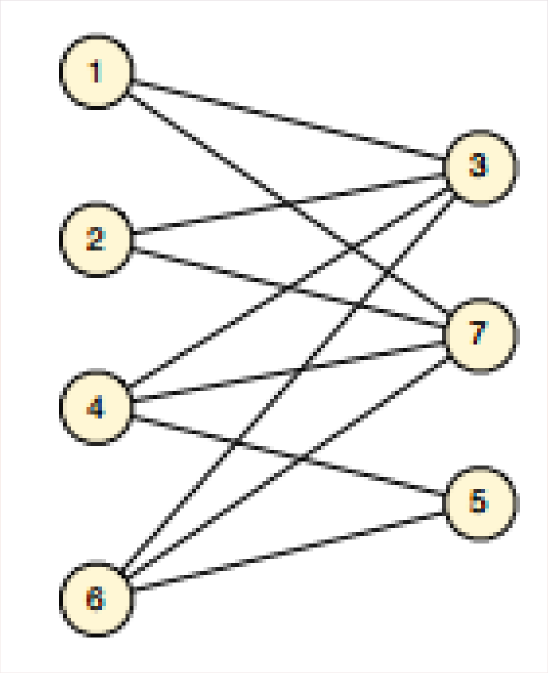
Bipartite graphs: matchings, Hall's Theorem

Bipartite graphs

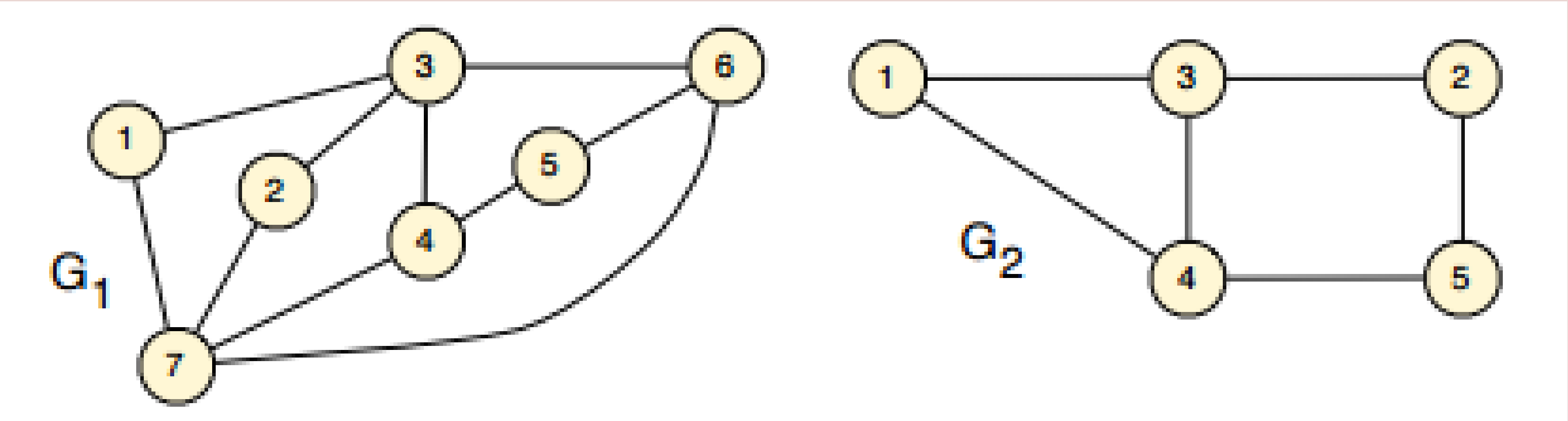


A graph G = (V, E) is bipartite if its vertex set V can be partitioned into two disjoint subsets L and R, such that every edge has one endpoint in L, and the other – in R.

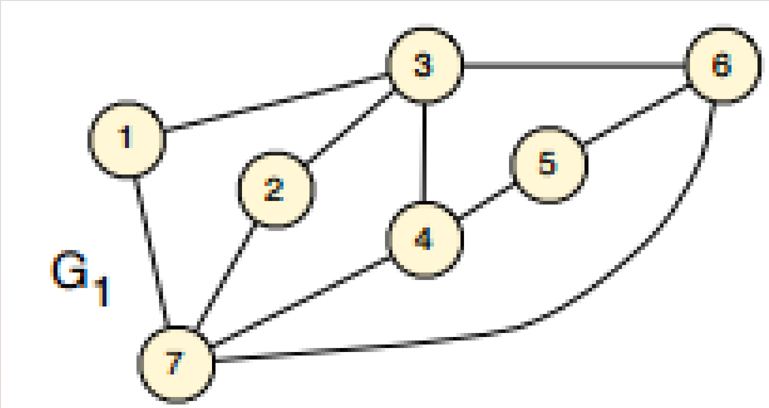
Bipartite graph notation: G=(L,R,E)

Bipartite graphs

Are these graphs bipartite?

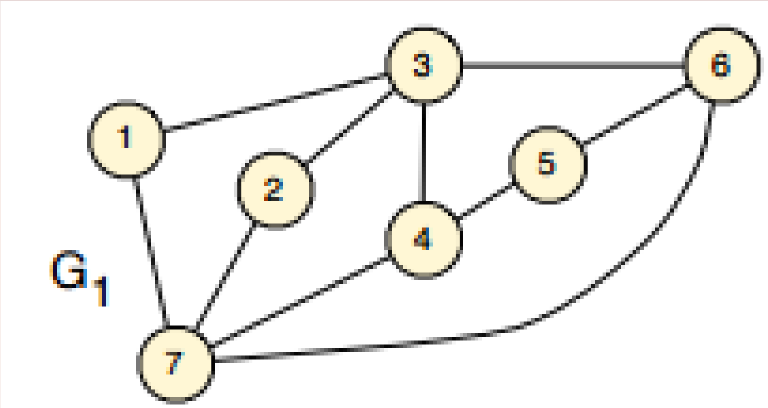


Bipartite graphs

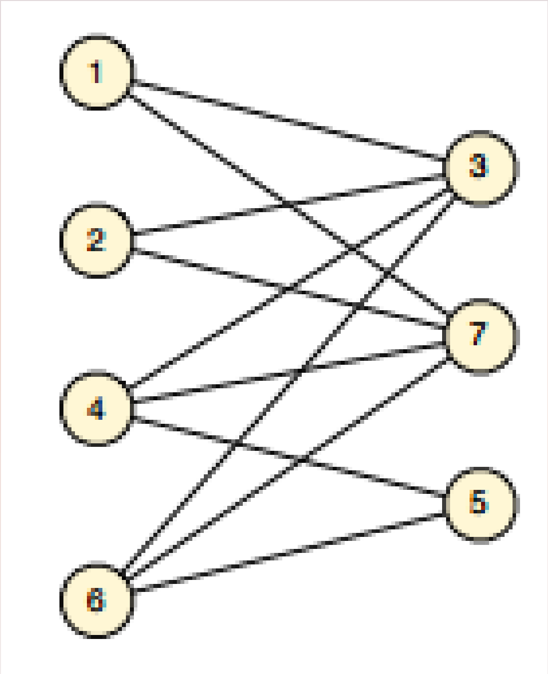


The graph G_1 is *bipartite*, because we can partition its vertex set into L = {1, 2, 4, 6} and R = {3, 5, 7}, and then each edge will have one endpoint in L and the other endpoint in R.

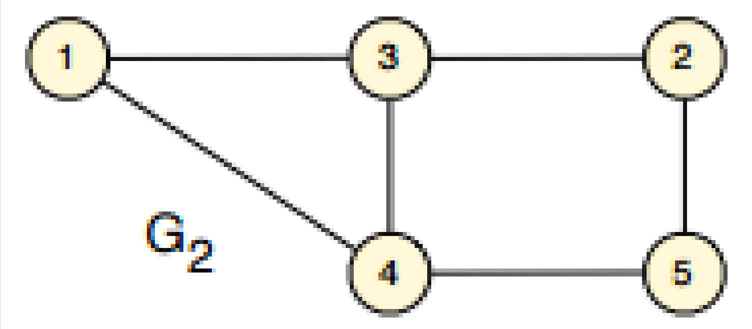
Bipartite graphs



G1



Bipartite graphs



The graph G_2 is *not bipartite*. Consider vertices 1, 3 and 4. No matter how we partition the vertices of this graph into two sets L and R, two of the three vertices 1, 3 and 4 will have to belong to the same set. But these vertices are connected by an edge, so we would have an edge inside L or inside R, which contradicts the definition of bipartite graphs.

Bipartite graphs

Theorem: A graph G is bipartite if and only if it does not contain any cycle of odd length.

Proof:

(⇒) Suppose G = (L, R, E) is bipartite, let's show that G does not contain any cycle of odd length.

Let $v_0, v_1, \ldots v_{k-1,} v_k = v_0$ be a cycle in G. Suppose $v_0 \in$ L. Then $v_1 \in$ R, since $\{v_0, v_1\} \in$ E. Then $v_2 \in$ L, since $\{v_1, v_2\}$ $\in$ E. Continuing this way, we see that if i is odd, then $v_i \in$ R, and if i is even then $v_i \in$ L. Thus, since $v_k = v_0 \in$ L, this implies that k is even, and thus the cycle is of even length.

Required EC

Bipartite graphs

Proof (cont.):

(⇐) Suppose G does not contain any cycle of odd length, let's prove that G is bipartite.

Pick any vertex u_0. For every vertex v ∈ V, let p_v be any path from u_0 to v, and let d_v be its length. Set L = {v ∈ V | d_v is even} and let R = {v ∈ V | d_v is odd}. Clearly V = L ∪ R is a partition of V . We now show that (L, R, E) is bipartite.

Bipartite graphs

Proof (cont.):

If (L, R, E) is not bipartite, then there is some edge{u, v} ∈ E such that both u, v ∈ L or both u, v ∈ R. In either case, there is a closed walk in G given by p_u,{u, v},p_v (from u_0 to u, then u to v, then v to u_0), whose total length is d_u + 1 + d_v, which is odd. Since G has a closed walk of odd length, then G also has a cycle of odd length. This is a contradiction.

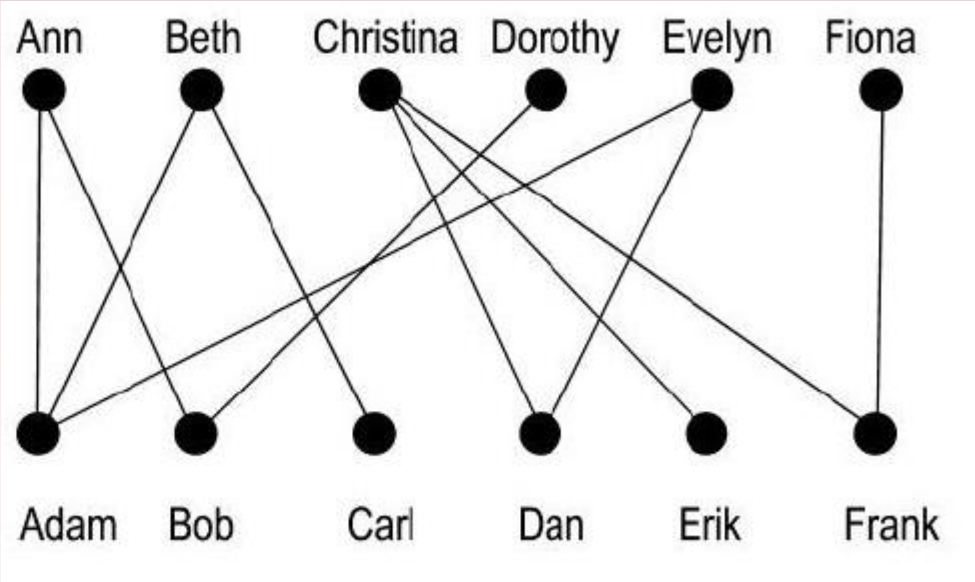
Thus G = (L, R, E) is bipartite.

(Note) Closed walk - sequence of vertices and edges where the first vertex is also the last

Cycle - closed walk where all vertices are different (except for first/last)

Perfect Matchings

Example: L - set of boys, R - set of girls, each edge (x, y) represents a pair x, y that like each other.



If G has a perfect matching, this means that we can marry all boys and girls happily, with everyone getting a partner they actually like.

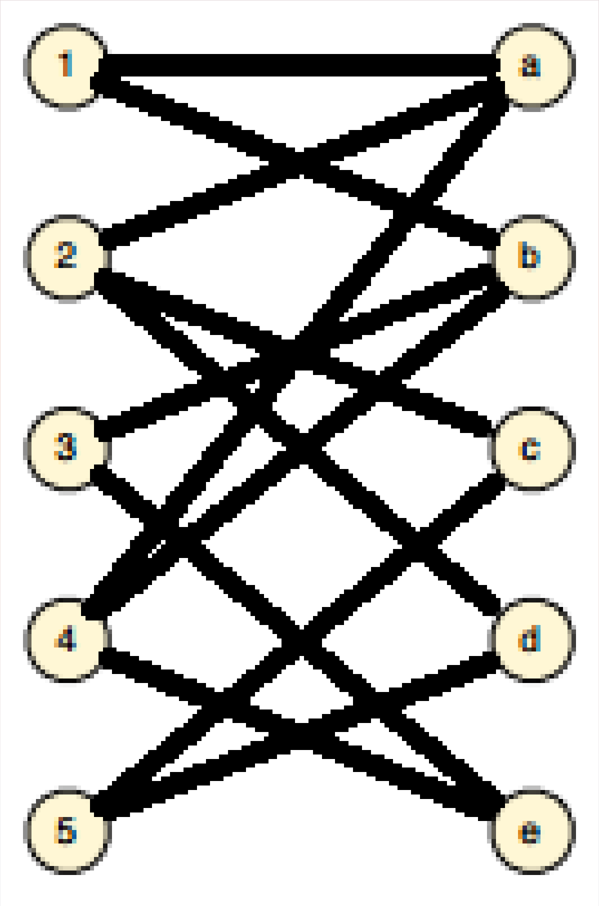
Perfect Matchings

$G = (L, R, E)$, where $|L| = |R|$

A ***perfect matching*** in such a graph is a set M of edges such that no two edges in M share an endpoint and every vertex has an edge that belongs to M.

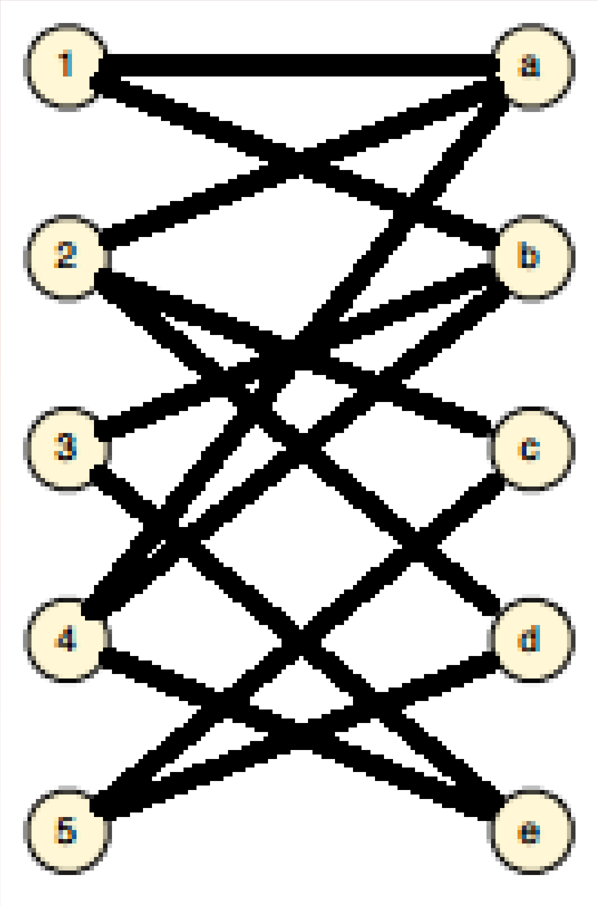
Perfect Matchings

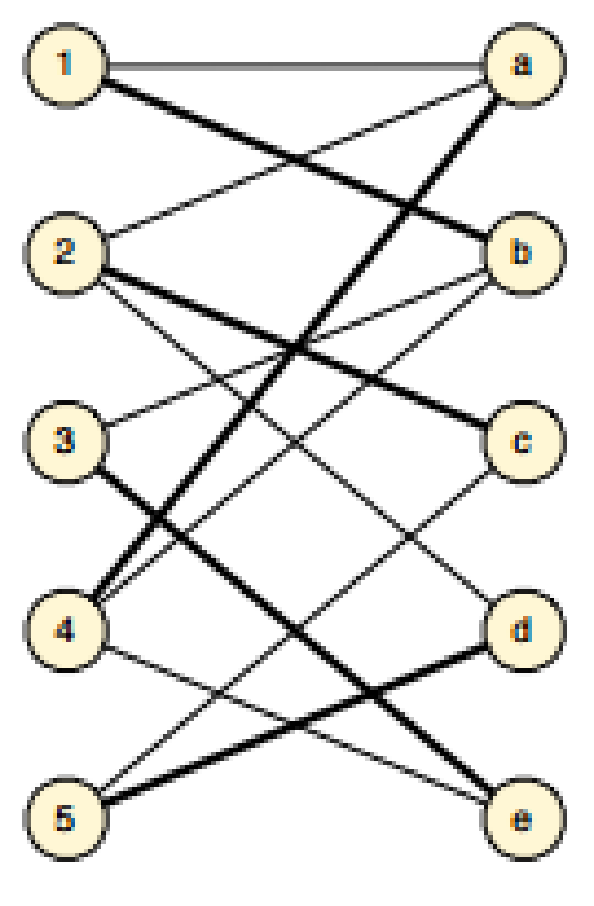
Does this graph have a perfect matching?



Perfect Matchings

This graph has a perfect matching. M={(1,b), (2,c), (3,e), (4,a), (5,d)}





Perfect Matchings

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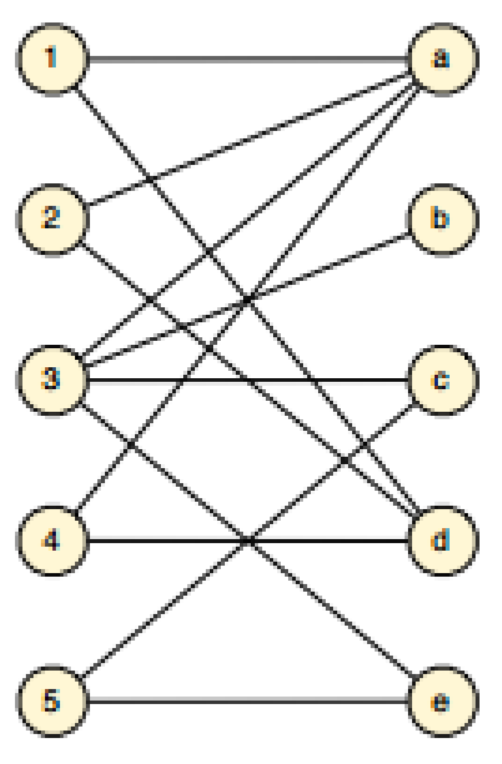
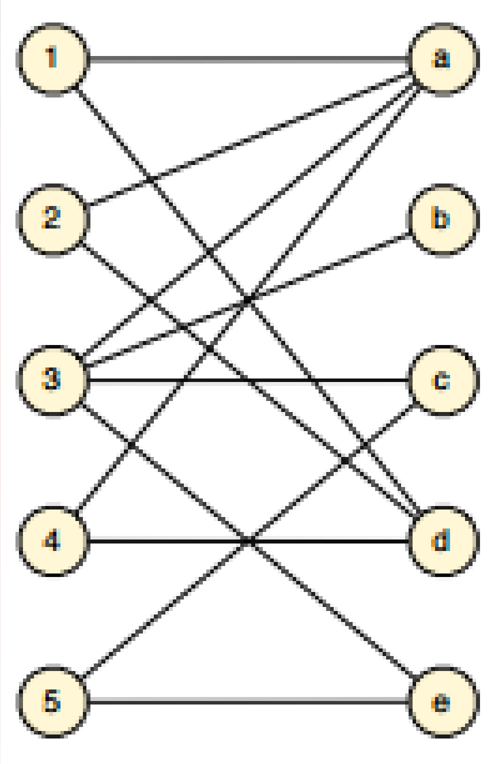


Perfect Matchings

This graph does not have a perfect matching.

Explain why



Hall’s Theorem

Hall’s Theorem:

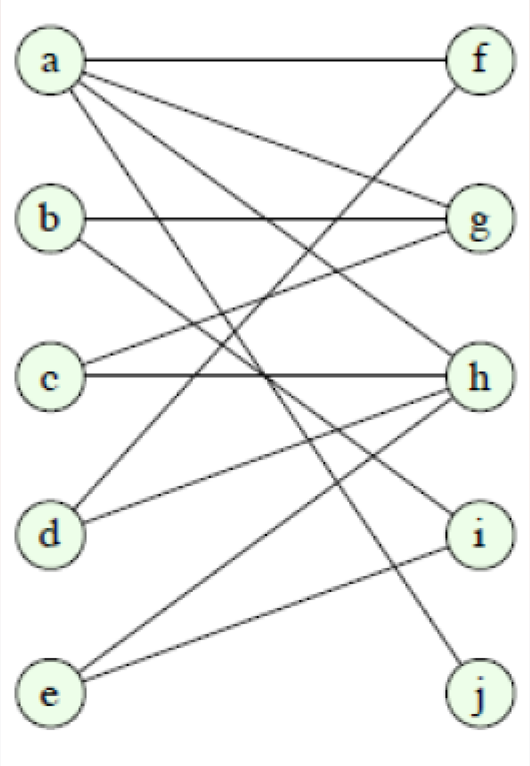
A bipartite graph $G = (L, R, E)$ with $|L| = |R|$ has a perfect matching if and only if each set $X \subseteq L$ satisfies $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbors of X.

Philip Hall was an English mathematician

(11 April 1904 – 30 December 1982).

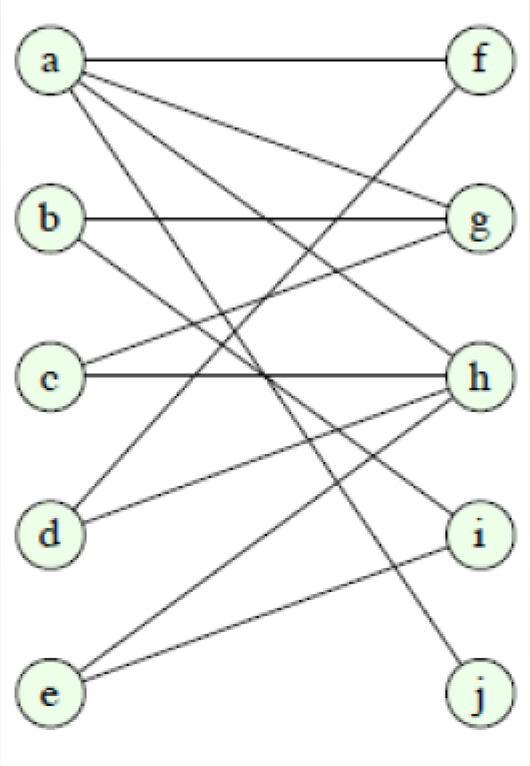
Perfect Matchings

Tell whether the following graph has a perfect matching. Justify your answer



Perfect Matchings

Perfect matching (a,j) (d,f) (b,g) (c,h) (e,i)



Perfect Matchings

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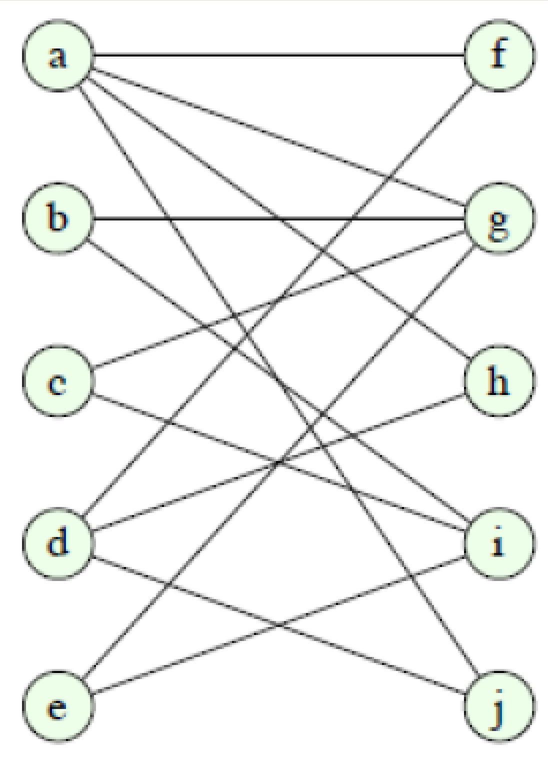


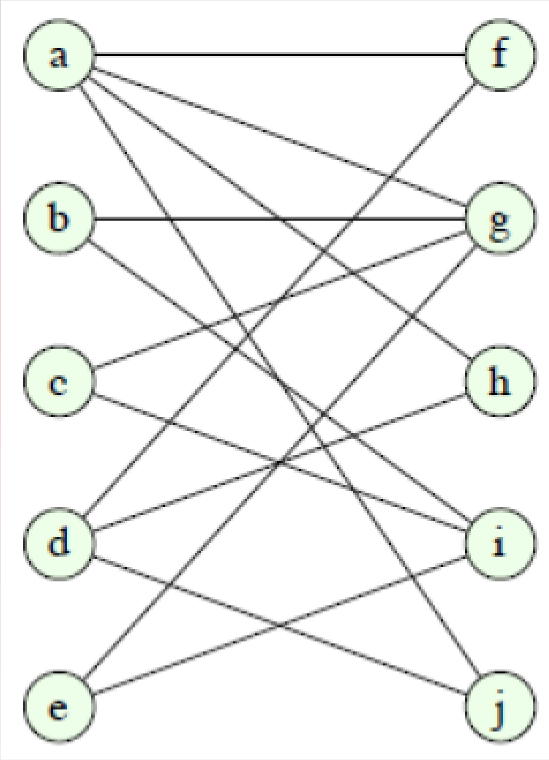
Perfect Matchings

No perfect matching.

$X=\{b,c,e\} \ngeq N(X)=\{g,i\}$

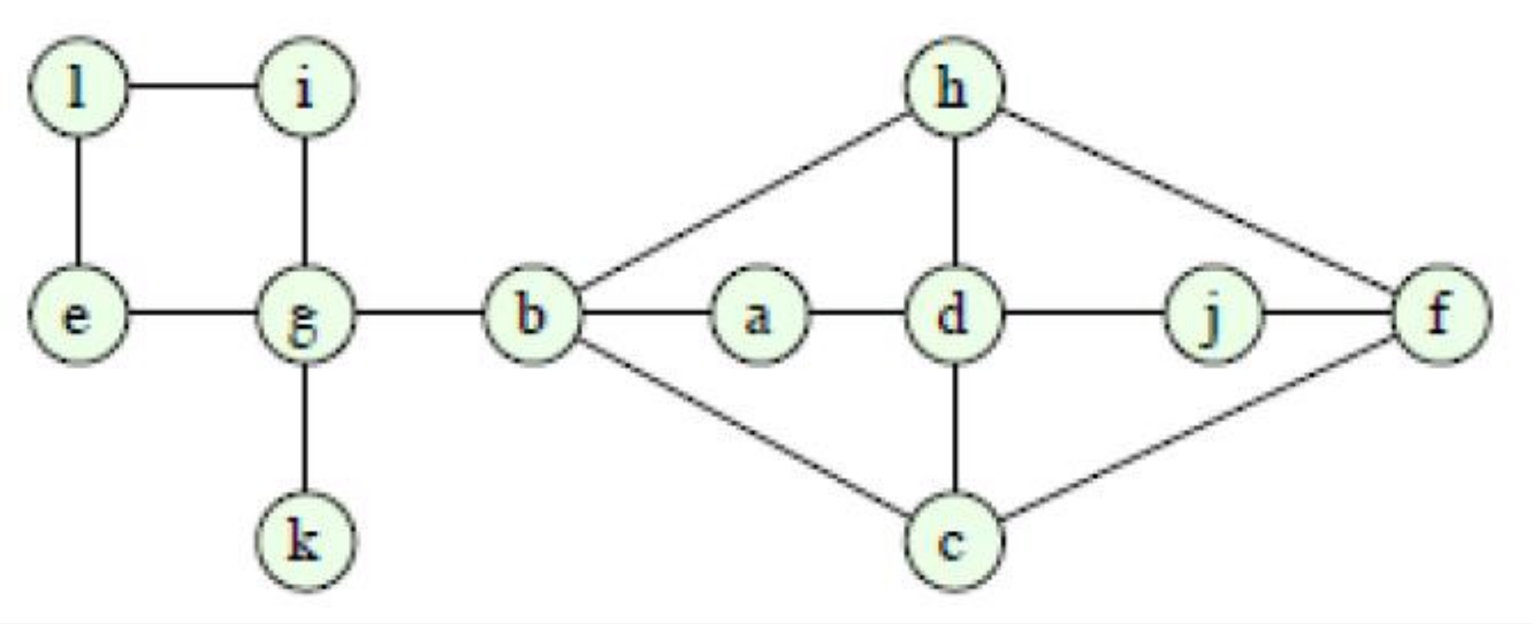
$|X|=3, |N(X)|=2$

$2=|N(X)| \quad |X|=3$



Example

Determine, if the graph below is bipartite, and if yes, whether it has a perfect matching.



Example

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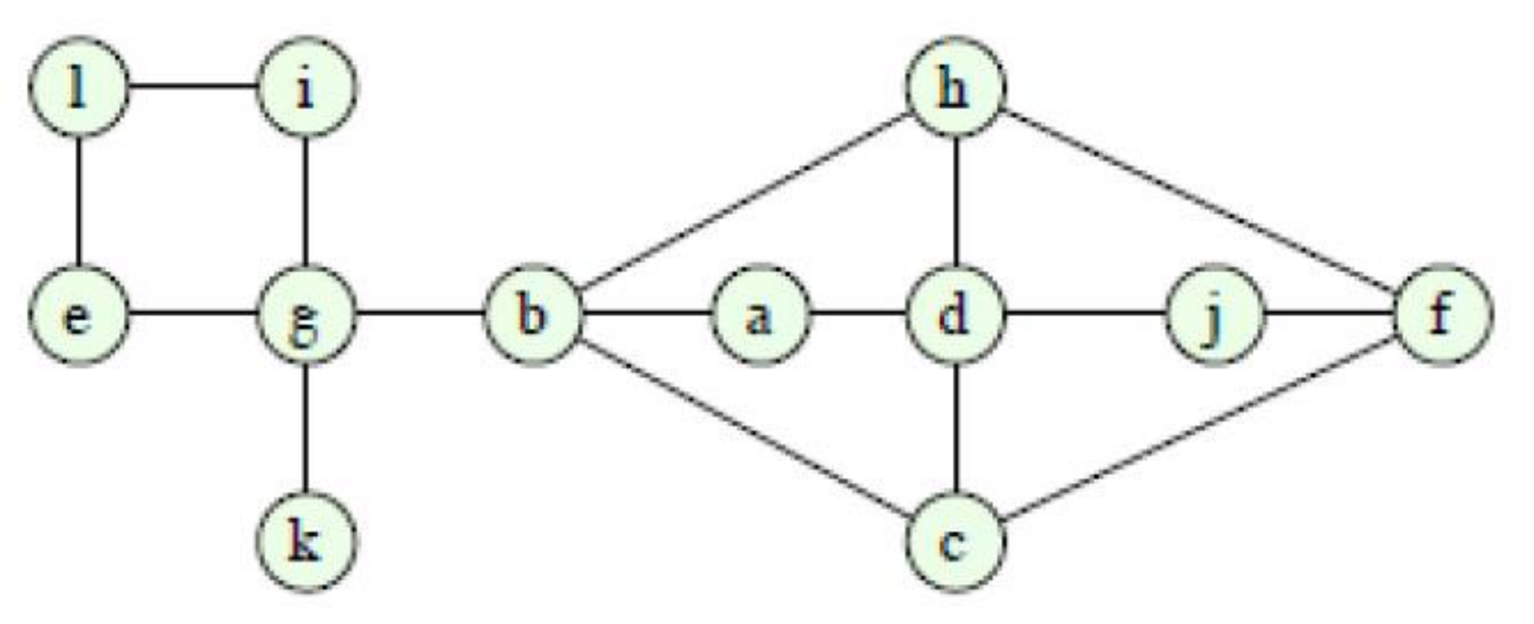
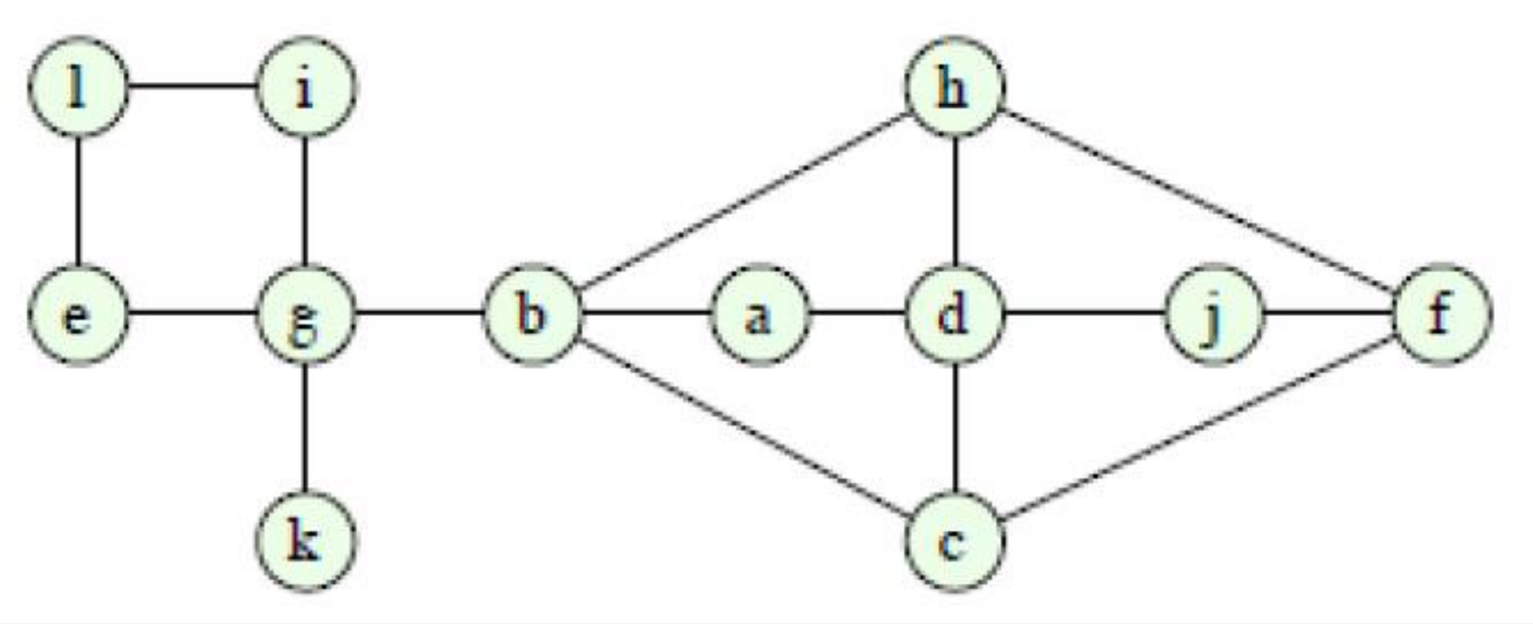


This graph is bipartite

L = {a, c, h, j, g, l}; R = {b, d, f, e, i, k}

Example

Determine, if the graph below is bipartite, and if yes, whether it has a perfect matching.



No perfect matching.

Let X = {a, c, j, h}. Then N(X) = {b, d, f}. So |N(X)| < |X|.

Hall’s Theorem

Hall’s Theorem: A bipartite graph $G = (L, R, E)$ with $|L| = |R|$ has a perfect matching if and only if each set $X \subseteq L$ satisfies $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbors of X.

Proof: ($\Rightarrow$) Suppose that G has a perfect matching M and pick any set $X \subseteq L$. Denote by $M(X)$ the set of vertices matched to those in X via the edges in M. Recall that we match only neighbors, so

$|N(X)| \geq |M(X)| = |X|$, and $|N(X)| \geq |X|$.

Hall's Theorem

($\Leftarrow$) Suppose that $G = (L, R, E)$ is a bipartite graph with $|L| = |R|$ such that each set $X \subseteq L$ satisfies $|N(X)| \geq |X|$ $(*)$. Then prove that G has a perfect matching.

Prove by induction on $n = |L| = |R|$.

Base case: $n = 1$ - then L and R have each just one vertex and $N(L) \geq |L| = 1$, so these vertices must be connected by an edge. Thus this graph has a perfect matching, consisting of just this one edge.

Hall's Theorem

Inductive step: fix some $n > 1$ and suppose that $(*)$ holds for all bipartite graphs with fewer than n vertices in each partition (so $(*)$ holds for all bipartite graphs with fewer than n vertices both in L and V).

Consider any graph $G = (L, R, E)$ with $|L| = |R| = n$ that satisfies $|N(X)| \geq |X|$ for each $X \subseteq L$. Using the inductive assumption, we want to show that there is a perfect matching in G.

Hall’s Theorem

Case 1: $|N(X)| \geq |X| + 1$ for all $X \subset L$.

Chose an arbitrary edge $(x, y) \in E$. Let G’ = (L’, R’, E’) be obtained from G by removing x and y. Then, for any $X \subseteq L'$, we have

$$|N_{G'}(X)| \geq |N_G(X)| - 1 \geq (|X| + 1) - 1 = |X|,$$

Since L’ has fewer than n vertices and $|N_{G'}(X)| \geq |X|$, the inductive assumption implies that G’ has a perfect matching.

Let’s call this matching M’. By adding (x, y) to M’ we obtain a perfect matching in M, completing the proof for this case.

Hall's Theorem

Case 2: $|N(X)| = |X|$ for some $X \subset L$. Introduce the following notations:

$$Y = N_G(X)$$

$$H = (X, Y, F)$$

$$X' = L - X$$

$$Y' = R - Y$$

$$H' = (X', Y', F')$$

where F is the set of edges between X and Y and F' is the set of edges between X' and Y' .

Note that graphs H and H' have equal size partitions, namely $|X| = |Y|$ and $|X'| = |Y'|$.

We will argue that each of them must have a perfect matching and then we will combine these matchings into one.

Hall's Theorem

Case 2(cont.): Let's start with H. The key property of H is that it inherits all edges of G that have an endpoint in X, because of the way Y is defined.

For any Z ⊆ X, by the definition of H we have

$$N_H(\mathrm{Z}) = N_G(\mathrm{Z}), \text{ so } |N_H(\mathrm{Z})| = |N_G(\mathrm{Z})| \geq |\mathrm{Z}|.$$

So H has fewer vertices than G and $|N_H(\mathrm{Z})| \geq |\mathrm{Z}|$, so from the inductive assumption we obtain that H has a perfect matching. Let's call this matching Q.

Hall’s Theorem

What about H’?

Let Z ⊆ X’ . We consider the set Z ∪ X. We have that

$$|N_G(Z \cup X)| \geq |Z \cup X|.$$

Further, $N_G(Z \cup X) = Y \cup N_{H'}(Z)$, and these two sets are actually disjoint. Putting this together, we get

$$|Y| + |N_{H'}(Z)| = |N_G(Z \cup X)| \geq |Z \cup X| = |Z| + |X|.$$

Hall’s Theorem

$$|Y| + |N_{H'}(Z)| \geq |Z| + |X|.$$

Since we have $|Y| = |X|$, this gives us $|N_{H'}(Z)| \geq |Z|$.

So we showed that $|N_{H'}(Z)| \geq |Z|$ holds for all $Z \subseteq X'$, which, using the inductive assumption implies that H’ has a perfect matching, say Q’. Joining these two matchings together, that is letting M = Q ∪ Q’ , we obtain that M is a perfect matching in G, proving the inductive claim.

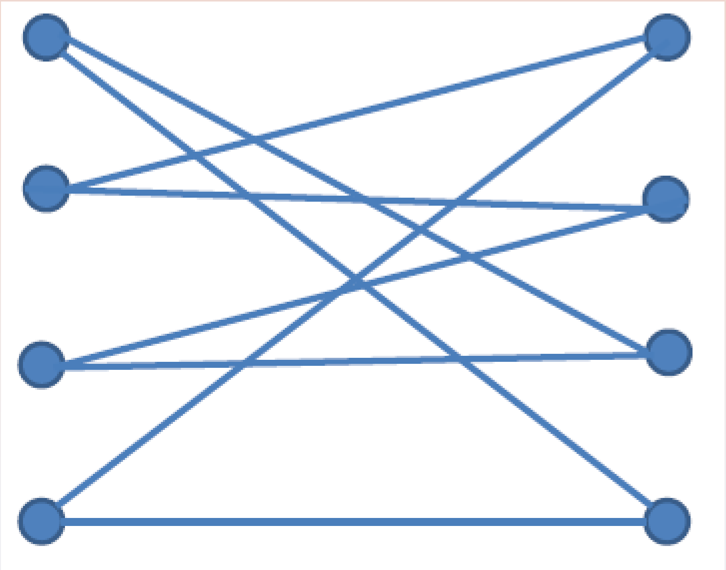
Perfect Matchings

Prove or disprove: “If a bipartite graph G has a Hamiltonian cycle, then it has a perfect matching.”

Perfect Matchings

Prove or disprove: "If a bipartite graph G has a Hamiltonian cycle, then it has a perfect matching."

Proof: Let $H = v_1 v_2 \ldots v_n v_1$ be a Hamiltonian cycle. Consider the set of edges *M*, that consists of every second edge from *H*: $\{v_1, v_2, v_3, v_4, \ldots, v_{n-1}, v_n\}$. *Every vertex of G is covered by M*, and no two edges of *M* share an endpoint.



Perfect Matchings