Review of Prerequisite Topics

- Logic
- Sets and sequences
- Relations
- Basic combinatorics: counting, summation formulas
- Elementary number theory
- Proofs, proof techniques (mathematical induction)

Logic: propositional and predicate calculus

Propositional calculus:

- Deals with *propositions*, which are statements that can be assigned a boolean value of true or false (1 or 0)
- Establishes rules for:
  - combining propositions into more complex propositions using boolean operations
  - reasoning about validity of propositions

Example:

"if $\underbrace{\text{sun is yellow}}_{p}$ and $\underbrace{\text{cats bark}}_{q}$ then $\underbrace{\text{today is Monday}}_{r}$"

$$p \wedge q \Rightarrow r$$

$p\,,\, q\,,\, r$ are boolean variables (atomic propositions)

Logic: propositional and predicate calculus

Analyzing compound propositions using truth tables:

$p \Rightarrow q$

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

$\neg p \vee q$

p	q	$\neg p \vee q$
0	0	1
0	1	1
1	0	0
1	1	1

These propositions are equivalent!

$$p \Rightarrow q \equiv \neg p \vee q$$

Tautology: proposition that is true for all combination of values of its variables

$(p \wedge q) \Rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

Logic: propositional and predicate calculus

Basic laws

de Morgan laws:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

distributive laws:

$$r \vee (p \wedge q) \equiv (r \vee p) \wedge (r \vee q)$$

$$r \wedge (p \vee q) \equiv (r \wedge p) \vee (r \wedge q)$$

double negation:

$$\neg(\neg p) \equiv p$$

…

Logic: propositional and predicate calculus

Predicate calculus:

- Extension of propositional calculus, where propositions can involve *predicates*, which are properties of elements of some domain that we want to reason about
- We can form propositions from predicates by using quantifiers ∃ and ∀

Example:

"every bird flies"

Use predicates:

$B(x)$ = "x is a bird" $\qquad$ $\mathrm{F}(x)$ = "x flies"

Then "every bird flies" can be written as

$$\forall x \;\; B(x) \Rightarrow \; F(x)$$

Logic: propositional and predicate calculus

de Morgan laws extend to predicate calculus:

$$\neg \forall x P(x) \;\equiv\; \exists x \neg P(x)$$

$$\neg \exists x P(x) \;\equiv\; \forall x \neg P(x)$$

Question: Is the following "distributive law" true?

$$\forall x(\, P(x) \vee Q(x)\,) \;\equiv\; \forall x P(x) \vee \forall x Q(x)$$

No, only one implication is true

$$\forall x P(x) \vee \forall x Q(x) \longrightarrow \forall x(\, P(x) \vee Q(x)\,)$$

Logic: propositional and predicate calculus

Puzzle (zoom poll):

Which of the statements below is a negation of statement
"For each *X*, if *X* moos then *X* is a cow" ?

(a) "There is no X that does not moo and is not a cow"

(b) "For each X, X does not moo and X is not a cow"

(c) "For each X, if X does not moo then X is not a cow"

(d) "There exists an X that moos and is not a cow"

(e) None of the above

Logic: propositional and predicate calculus

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(e) None of the above

Solution:

$$\begin{aligned} \neg\forall x\,[M(x) \Rightarrow C(x)] &\equiv \neg\forall x\,[\,\neg M(x) \vee C(x)\,] \\ &\equiv \exists x\,\neg[\,\neg M(x) \vee C(x)\,] \\ &\equiv \exists x\, M(x) \wedge \neg C(x) \end{aligned}$$

So the answer is (d)

Sets: set notation, operations on sets

- Defining sets

$$
\begin{aligned}
A &= \{a, b, c\} \\
B &= \{1, 2, ..., 10\} \\
C &= \{x \in \mathbb{R} \,:\, x^3 - x^2 + x = 1\} \\
D &= \{p + q \,:\, p, q \in \mathbb{N} \text{ and } p, q \text{ are prime}\}
\end{aligned}
$$

- Relations involving sets

$$a \in \{a, b, c, d, e\}$$
$$\{a, b\} \subseteq \{a, b, c, d, e\}$$

Question: which of the following relations are true?

$1 \in \{0, \{1,2,3,4\}\}$	False
$\{1,2,3,4\} \subseteq \{0, \{1,2,3,4\}\}$	False
$\{1,2,3,4\} \in \{0, \{1,2,3,4\}\}$	True
$\{\{1,2,3,4\}\} \subseteq \{0, \{1,2,3,4\}\}$	True

Sets: set notation, operations on sets

- Basic operations on sets

$$X \cup Y \qquad X \cap Y \qquad \overline{Y}$$

- Power set of a set X : set of all subsets of X

$$\begin{aligned} X &= \{a, b, c\} \\ \mathcal{P}(X) &= \{\, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\,\} \end{aligned}$$

- Cartesian product of sets X and Y: set of all ordered pairs, one from X and one from Y

$$X = \{a, b\} \qquad Y = \{1, 2, 3\}$$

$$X \times Y = \{\,(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\,\}$$

- Cardinality of X : number of elements of X

$$\begin{aligned} X &= \{a, b, c, d\} \\ |X| &= 4 \end{aligned}$$

$$\begin{aligned} Y &= \{x \in \mathbf{N} \,:\, x^2 + 1 \text{ is prime}\} \\ |Y| &= ? \end{aligned}$$

It is an open problem in number theory whether this set is finite

Relations

- Let A be a set. Any subset $R \subseteq A \times A$ is called a *relation*.

in general, relations could be between different sets

Example: Some relations for $A = \mathbb{Z}$ (integers)

$$R = \{(1,3),(7,59),(2,17),(0,10)\}$$
$$Q = \{(a,b) \,:\, b = a^2\}$$
$$S = \{(a,b) \,:\, 3|a-b\}$$

Notations for a and b being related in R :

$$(a,b) \in R \qquad aRb \qquad R(a,b)$$

Types of relations:

- Functions
- Equivalence relations
- Partial orders
- …

Relations

- A *function* is a relation $R \subseteq A \times A$ in which the first argument is related only to one element:

$$aRb \wedge aRc \Rightarrow b = c \quad \forall a,b,c \in A$$

Examples:

- $f = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2 \}$
- $g= \{(x,y) \in \mathbb{N}\times\mathbb{N} : y \text{ is the smallest prime factor of } x \}$
- …

Notations for functions: $f(x) = y$ instead of xfy or $(x,y) \in f$

Relations

- An *equivalence relation* is a relation $R \subseteq A \times A$ that satisfies the following properties:
  - Reflexive: $aRa \quad \forall a \in A$
  - Symmetric: $aRb \Rightarrow bRa \quad \forall a,b \in A$
  - Transitive: $aRb \wedge bRc \Rightarrow aRc \quad \forall a,b,c \in A$

Examples:

- Isometry (in geometry)
- $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : |x| = |y| \}$
- Congruence relation for integers: $a \equiv b \pmod 5$ iff $5 \mid a - b$
- Parallel vectors:

$$D = \{((x, y), (u, v)) \in \mathbb{R}^2 \times \mathbb{R}^2 : xv = yu\}$$

Relations

Equivalence classes: $[a] = \{b \in A : aRb\}$, set of all elements in A related to a.

> Theorem: If $R \subseteq A \times A$ is an equivalence relation then its equivalence classes partition A into disjoint subsets.

Example: Congruence relation for integers: $a \equiv b \pmod 5$ iff $5 \mid a - b$.

Equivalence classes:

$$
\begin{aligned}
[0] &= \{..., -10, -5, 0, 5, 10, ...\} \\
[1] &= \{..., -9, -4, 1, 6, 11, ...\} \\
[2] &= \{..., -8, -3, 2, 7, 12, ...\} \\
[3] &= \{..., -7, -2, 3, 8, 13, ...\} \\
[4] &= \{..., -6, -1, 4, 9, 14, ...\}
\end{aligned}
$$

Relations

- A *partial order* is a relation $R \subseteq A \times A$ that satisfies the following properties:
  - Reflexive: $aRa \quad \forall a \in A$
  - Anti-symmetric: $aRb \wedge bRa \Rightarrow b = a \quad \forall a, b \in A$
  - Transitive: $aRb \wedge bRc \Rightarrow aRc \quad \forall a, b, c \in A$

Examples:

- "≤" relation on $\mathbb{R}$ or $\mathbb{Z}$
- divisibility relation for positive integers
- $\subseteq$ relation on subsets of a set

Question (zoom poll): Let $A = \mathscr{P}(B)$, the collection of all subsets of a set B. Is the element relation "$\in$" a partial order?

Relations

- A *partial order* is a relation $R \subseteq A \times A$ that satisfies the following properties:
  - Reflexive: $aRa \quad \forall a \in A$
  - Anti-symmetric: $aRb \wedge bRa \Rightarrow b=a \quad \forall a,b \in A$
  - Transitive: $aRb \wedge bRc \Rightarrow aRc \quad \forall a,b,c \in A$

Examples:

- "≤" relation on $\mathbb{R}$ or $\mathbb{Z}$
- divisibility relation for positive integers
- $\subseteq$ relation on subsets of a set

Question (zoom poll): Let $A = \mathscr{P}(B)$, the collection of all subsets of a set B. Is the element relation "$\in$" a partial order?

Answer: No. "$\in$" is a relation between *elements and sets*, not between two sets. So this is in fact an ill-posed question.

Relations

Sample Problem: You are given three relations $P,Q,R \subseteq \{a,b,c,d\} \times \{a,b,c,d\}$

P	a	b	c	d
a	Y	N	Y	N
b	N	Y	N	Y
c	Y	N	Y	N
d	N	Y	N	Y

Q	a	b	c	d
a	Y	Y	N	Y
b	N	Y	N	Y
c	N	N	Y	Y
d	N	N	N	Y

R	a	b	c	d
a	Y	N	N	N
b	N	N	N	Y
c	N	N	N	Y
d	N	N	Y	N

For each relation tell (write Y or N) whether it has the listed properties:

	reflexive	transitive	symmetric	anti-symmetric	partial order	equivalence
P						
Q						
R						

Combinatorics: counting and summation formulas

Counting basic combinatorial structures:

- Functions (also sequences, tuples, vectors)
- 1-1 Functions
- k-Permutations
- Permutations
- Subsets
- k-Subsets
- ….

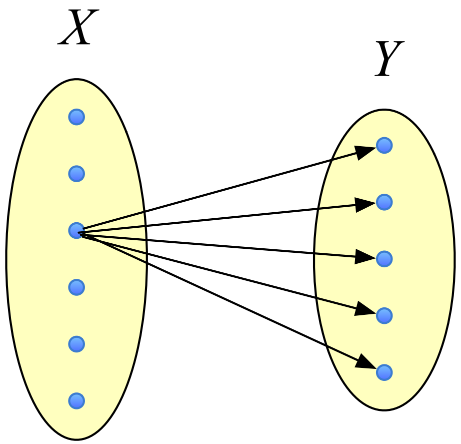
Combinatorics: counting and summation formulas

Principle of independent choices

- Simple form: $|X \times Y| = |X| \cdot |Y|$
- Generalized: If there are p choices to choose x, and for each x there are q choices to choose y, then there are pq choices of pairs (x, y)
- Extends naturally to more sets (or steps)

Combinatorics: counting and summation formulas

- Number of functions
  - $|X| = n$, $|Y| = m$
  - Compute number of functions $f : X \to Y$



Claim: There are m^n such functions

Proof: Use independence principle

- Assign a value to each $x \in X$ one by one
- We have n independent steps
- At each step there are m choices
- So the number of functions is

$$\underbrace{m \cdot m \cdot \ldots \cdot m}_{n \text{ times}} = m^n \qquad \blacksquare$$

Combinatorics: counting and summation formulas

- Number of binary strings of length n
- Number of subsets of $\{1,2, \dots ,n\}$

binary strings of length n $\xleftrightarrow{1:1}$ functions from $\{1,2, \dots ,n\}$ to $\{0,1\}$ $\xleftrightarrow{1:1}$ subsets of $\{1,2, \dots ,n\}$

So

- there are 2^n binary strings of length n
- there are 2^n subsets of $\{1,2, \dots ,n\}$

Example:

0 1 1 0 1

$\updownarrow$

x	1	2	3	4	5
$f(x)$	0	1	1	0	1

$\updownarrow$

$\{2,3,5\}$

Combinatorics: counting and summation formulas

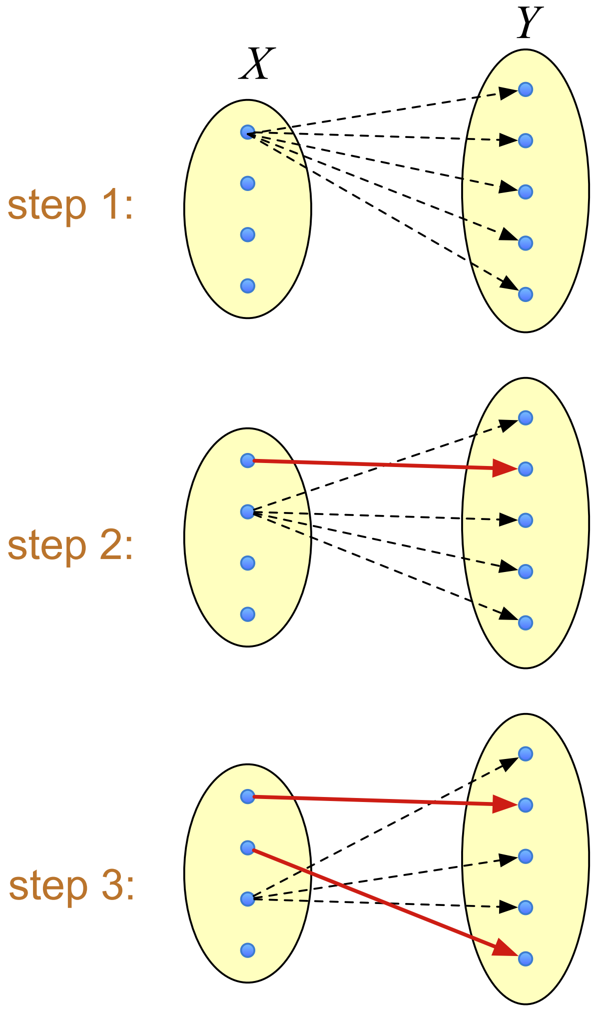
- Number of 1-1 functions
  - $|X| = n$, $|Y| = m \geq n$
  - Compute number of 1-1 functions $f : X \rightarrow Y$

Claim: The number of such 1-1 functions is $m!/(m - n)!$

Proof: Use independence principle

- Assign a value to each $x \in X$ one by one
- We have n steps
- At each step j there are $m - j + 1$ choices (independently of previous choices)
- So the number of 1-1 functions is

$$m \cdot (m-1) \cdot \ldots \cdot (m-n+1) = m!/(m-n)! \quad \blacksquare$$



Combinatorics: counting and summation formulas

Let $X = \{1,2, \ldots ,n\}$

A *permutation* of X is an ordering of elements of X

permutations of X $\xleftrightarrow{1:1}$ 1-1 functions from X to X

Corollary: The number of permutations of X is $n!$

A k-*permutation* of X is an ordered selection of k elements of X

k-permutations of X $\xleftrightarrow{1:1}$ 1-1 functions from $\{1,2, \ldots ,k\}$ to X

Corollary: The number of k-permutations of X is $n!/(n - k)!$

Example:

$X = \{1,2,3,4,5\}$

60 3-permutations:

$1, 2, 3$
$1, 2, 4$
$1, 2, 5$
$1, 3, 2$
$1, 3, 4$
$1, 3, 5$
$1, 4, 2$
$1, 4, 3$
$\ldots$

Combinatorics: counting and summation formulas

Let $X = \{1,2, \ldots ,n\}$

A *k-subset* of X is a subset of cardinality k

Claim: The number of k-subsets of X is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof:

- The number of k-permutations is $n!/(n - k)!$
- Each k-subset is counted $k!$ times in the list of k-permutations ■

Example: $X = \{1,2,3,4,5\}$

60 3-permutations:

$\boxed{1,2,3}$

$1,2,4$

$1,2,5$

$\boxed{1,3,2}$

$1,3,4$

$\ldots$

$\boxed{2,1,3}$

$\ldots$

$\boxed{2,3,1}$

$\ldots$

$\boxed{3,1,2}$

$\ldots$

$\boxed{3,2,1}$

$\ldots$

$\{1,2,3\}$ appears 6 times

Combinatorics: counting and summation formulas

Puzzle (zoom poll):

What is the number of binary strings of length 7 that have exactly 3 1's?

- 7
- 210
- 343
- 35
- none of the above

Combinatorics: counting and summation formulas

Puzzle (zoom poll):

What is the number of binary strings of length 7 that have exactly 3 1's?

- 7
- 210
- 343
- 35
- none of the above

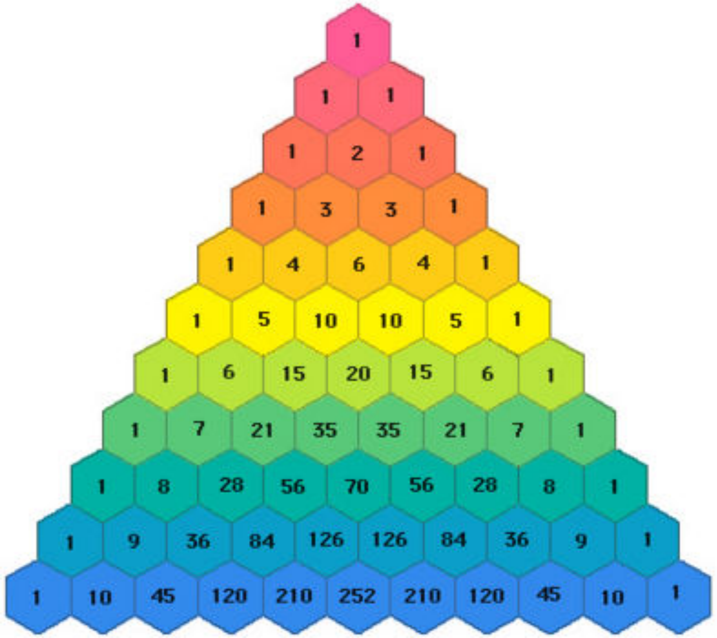
Solution: This is the same as the number of 3-subsets of $\{1,2,3,4,5,6,7\}$.

Answer: $$\binom{7}{3} = \frac{7!}{3!\cdot 4!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{(1\cdot 2\cdot 3)\cdot(1\cdot 2\cdot 3\cdot 4)} = 35$$

Combinatorics: counting and summation formulas

Let $1 \le k \le n-1$. Prove the following "Pascal triangle" equality

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

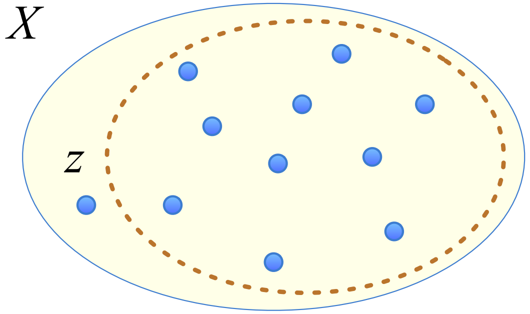


Proof: Let $X = \{1,2, \ldots ,n\}$.

The number of k-subsets of X is

Fix any $z \in X$. Consider two types of k-subsets of X:

- Those that do not contain z $\xleftrightarrow{1:1}$ k-subsets of $X \setminus \{z\}$
- Those that contain z $\xleftrightarrow{1:1}$ $(k-1)$-subsets of $X \setminus \{z\}$ ■



Combinatorics: counting and summation formulas

- Arithmetic sequence: $a_i = a + b \cdot i$ for $i = 0, 1, 2, \ldots$

notation for $a_0 + a_1 + \ldots + a_n$

Example:

$3, 10, 17, 24, 31, 38, \ldots$

Claim: $\sum_{i=0}^{n} a_i = \frac{1}{2}(n+1)(a_0 + a_n)$

Proof 1: Proof for sequence $0, 1, 2, \ldots, n$. We show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.

1	2	3	. . . .	n-1	n
n	n-1	n-2	. . . .	2	1
n+1	n+1	n+1	. . . .	n+1	n+1

total = $n(n+1)$

We double-count, so we need to divide $n(n+1)$ by 2 ∎

Combinatorics: counting and summation formulas

Proof 2: Proof for sequence $0, 1, 2, \ldots, n$. We use induction to show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.

Base case. For $n = 0$, we have $\sum_{i=1}^{0} i = 0 = \frac{1}{2}0(0+1)$

Inductive step. Assume that the claim holds for $n=k$, that is $\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$

Then for $n=k+1$, we have

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{1}{2} \cdot k(k+1) + (k+1) \\ &= (k+1)(\frac{1}{2} \cdot k + 1) \\ &= \frac{1}{2}(k+1)(k+2) \end{aligned}$$

Thus the claim holds for $n=k+1$. From the base case and the inductive step, the claim holds for all n. ■

Combinatorics: counting and summation formulas

- Geometric sequence: $a_i = c \cdot a^i$ for $i = 0, 1, 2, \ldots$ for $a \neq 1$.

Example:

$2, 6, 18, 54, 162, \ldots$

for simplicity, assume $c = 1$

Claim: $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$

Proof: Proof for sequence $1, 2, 4, \ldots, 2^n$. We show that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$1 \quad 1 \quad 2 \quad 4 \quad \ldots \quad 2^{i-1} \quad 2^i \quad \ldots \quad 2^n$

extra 1

2

4

8

$\cdots$

2^i

2^{i+1}

$\cdots$

2^{n+1}

$\blacksquare$

Combinatorics: counting and summation formulas

Claim: $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$

Proof 1: We can prove it by direct calculation:

$$\begin{aligned}(a-1)\cdot\sum_{i=0}^{n} a^i &= a\cdot\sum_{i=0}^{n} a^i - \sum_{i=0}^{n} a^i \\ &= \sum_{i=0}^{n} a^{i+1} - \sum_{i=0}^{n} a^i \\ &= \sum_{i=1}^{n+1} a^i - \sum_{i=0}^{n} a^i \\ &= a^{n+1} - 1 \qquad \blacksquare\end{aligned}$$

Combinatorics: counting and summation formulas

Claim: $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$

Proof 2: We now prove it using mathematical induction:

Base case. For $n=0$, we have $\text{LHS} = \sum_{i=0}^{0} a^i = a^0 = 1$ and $\text{RHS} = 1$

Inductive step. Assume that the claim holds for $n=k$, that is $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$

Then for $n=k+1$, we have

$$\begin{aligned}\sum_{i=0}^{k+1} a^i &= \sum_{i=0}^{k} a^i + a^{k+1} \\ &= \frac{a^{k+1}-1}{a-1} + a^{k+1} \\ &= \frac{a^{k+1}-1+(a-1)a^{k+1}}{a-1} \\ &= \frac{a^{k+2}-1}{a-1}\end{aligned}$$

Thus the claim holds for $n=k+1$. From the base case and the inductive step, the claim holds for all n. ■

Elementary number theory

- prime and composite numbers
- factorization
- greatest common divisor
- basic modular arithmetic

Elementary number theory

Integer numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

A natural number $p > 1$ is *prime* iff its only divisors are 1 and p. Otherwise it is called *composite*.

Example: first 15 primes 2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 …

Fundamental Theorem of Arithmetic: Every positive natural number has a unique representation as a product of prime numbers. (This product is called its factorization.)

Example:

$$\begin{aligned} 84 &= 2 \cdot 2 \cdot 3 \cdot 7 = 2^2 \cdot 3^1 \cdot 7^1 \\ 16335 &= 3 \cdot 3 \cdot 3 \cdot 5 \cdot 11 \cdot 11 = 3^3 \cdot 5^1 \cdot 11^2 \end{aligned}$$

Elementary number theory

Theorem: There are infinitely many prime numbers.

Proof: We give an argument by contradiction. Suppose that there are only finitely many prime numbers, say $p_1, p_2, \ldots, p_t$.

Consider $q = p_1 p_2 \cdots p_t + 1$. We have that $p_1 p_2 \cdots p_t$ is a multiple of each p_i and the next multiple of p_i is $p_1 p_2 \cdots p_t + p_i > q$. So q is not a multiple of any p_i.
Therefore either q is a prime itself or it has a prime divisor smaller than q that is not among p_1, $p_2, \ldots, p_t$. In either case we reach a contradiction with $p_1, p_2, \ldots, p_t$ being the list of all primes. ■

This proof was given by Euclid circa 300 BC !!

Elementary number theory

- Greatest common divisor $\gcd(a,b)$: Largest $c \in \mathbb{N}$ such that $c|a$ and $c|b$

Example: $\gcd(15, 27) = 3$ $\quad\gcd(16335, 693) = 99$

Theorem: Let the factorizations of a and b be

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t} \quad \text{and} \quad b = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t}.$$

Then $\gcd(a, b) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \dots p_t^{\min(\alpha_t,\beta_t)}$.

Example: $16335 = 3^3 \cdot 5^1 \cdot 7^0 \cdot 11^2$ and $693 = 3^2 \cdot 5^0 \cdot 7^1 \cdot 11^1$

So $\gcd(16335, 693) = 3^2 \cdot 11^1 = 99$

Numbers a, $b \in \mathbb{N}$ are called *relatively prime (a.k.a. co-prime)* iff $\gcd(a,b) = 1$

Example: $\gcd(15, 22) = 1$ $\quad\gcd(128, 81) = 1$

Combinatorics: counting and summation formulas

Puzzle (zoom poll): Are numbers 273 and 605 relatively prime? (True/False)

Combinatorics: counting and summation formulas

Puzzle (zoom poll): Are numbers 273 and 605 relatively prime? (True/False)

Solution: Factor these numbers: $253 \;=\; 3 \cdot 7 \cdot 13 \qquad 605 \;=\; 5 \cdot 11 \cdot 11$

Answer: Yes

Elementary number theory

- Modular arithmetic

Theorem: For any a, $b \in \mathbb{Z}$ there are $q \in \mathbb{N}$ and $r \in \{0,1,\ldots,q\text{-}1\}$ such that $a = b{\cdot}q + r$

$$q = \lfloor a/b \rfloor \qquad r = a \bmod b$$

remainder of a modulo b

Congruence relation: a and b are congruent modulo m , denoted $a \equiv b \pmod{m}$, iff $a \bmod m = b \bmod m$ (or, equivalently $m \mid a - b$).

caution: different meaning of "mod"

Example:

$$68 \equiv 12 \pmod{7}$$
$$57 \not\equiv 23 \pmod{11}$$

Elementary number theory

Theorem: For any fixed m, relation $a \equiv b \pmod{m}$ is an equivalence relation on $\mathbb{Z}$.

Proof: We just need to verify the conditions of equivalence relations:

- Reflexive: $a \equiv a \pmod{m}$ ✓
- Symmetric: $a \equiv b \pmod{m}$ implies $b \equiv a \pmod{m}$ ✓
- Transitive: $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ implies $a \equiv c \pmod{m}$ ✓

For transitivity, if $m \mid a-b$ and $m \mid b-c$ then $m \mid (a-b) + (b-c)$. So $m \mid a-c$. ∎

Theorem: Assume that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Then

$$a+c \equiv b+d \pmod{m} \quad \text{and} \quad a \cdot c \equiv b \cdot d \pmod{m}.$$

Proofs

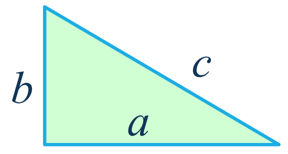
A *proof* is a rigorous argument justifying validity of a mathematical statement, showing that this statement logically follows from the assumptions.

Earlier in the lecture we have seen proofs of

- Formulas for the number of functions, 1-1 functions, permutations, subsets, ….
- Summation formulas for arithmetic and geometric sequences (including two proofs using induction)
- Pascal triangle equality
- That there are infinitely many primes (proof by contradiction)

Proofs

Pythagorean Theorem: Let a, b, and c be the width, height, and hypotenuse of a right triangle. Then $a^2 + b^2 = c^2$.



Proof 1: We use simple geometry and some calculation. Consider a $c \times c$ square $A_1A_2A_3A_4$ with four copies of our right triangle attached along its edges, as in the picture.

The angles at each A_i add up to 180 degrees each angle B_i is 90 degrees. So $B_1B_2B_3B_4$ is an $(a+b)\times(a+b)$ square.

Adding the area of four triangles and square $A_1A_2A_3A_4$ we have an equation

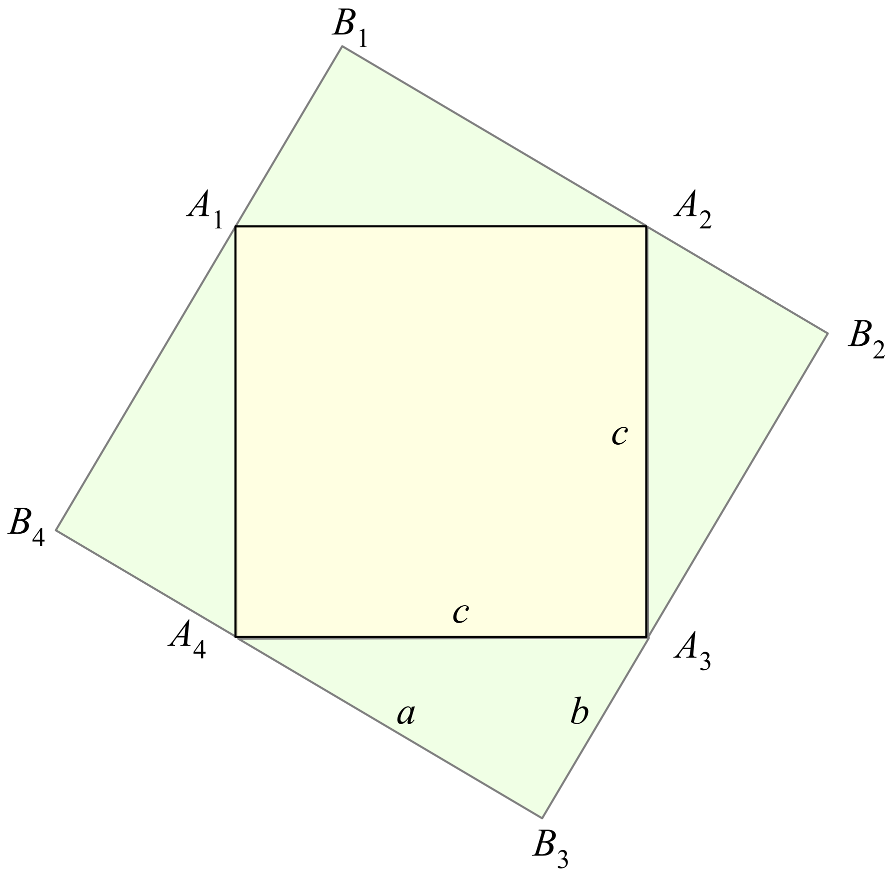
$$4 \cdot \frac{1}{2}ab + c^2 = (a+b)^2$$

This yields

$$2ab + c^2 = a^2 + 2ab + b^2$$

Therefore

$$c^2 = a^2 + b^2 \qquad \blacksquare$$



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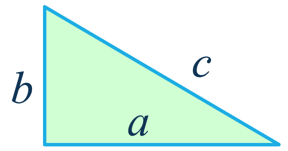
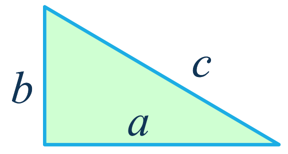


Intuition: The value c^2 represents the area of a $c \times c$ square. So there should be a way to slice a $c \times c$ square into pieces that can be then reassembled to form a $a \times a$ square and a $b \times b$ square.

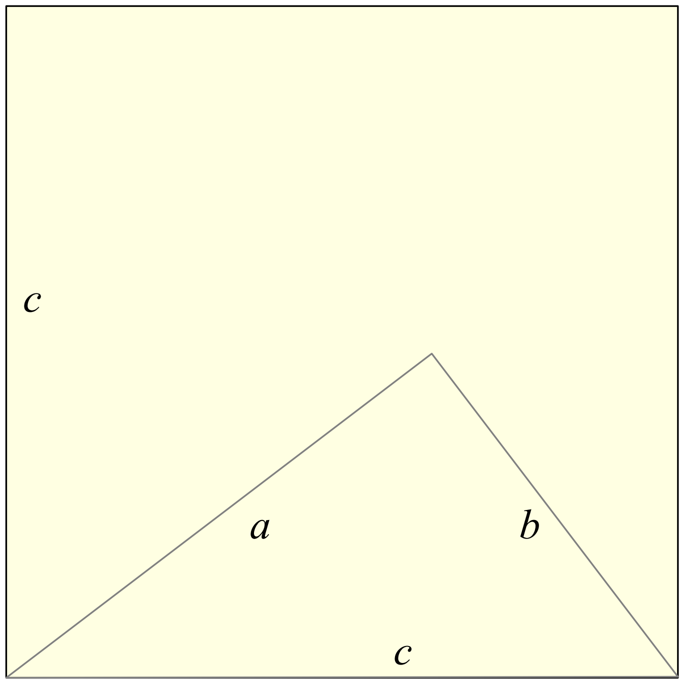
Proofs

Pythagorean Theorem: Let a, b, and c be the width, height, and hypotenuse of a right triangle. Then $a^2 + b^2 = c^2$.



Proof 2:

Draw a $c \times c$ square with one edge being the c edge of our triangle (in yellow).



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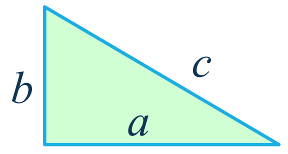
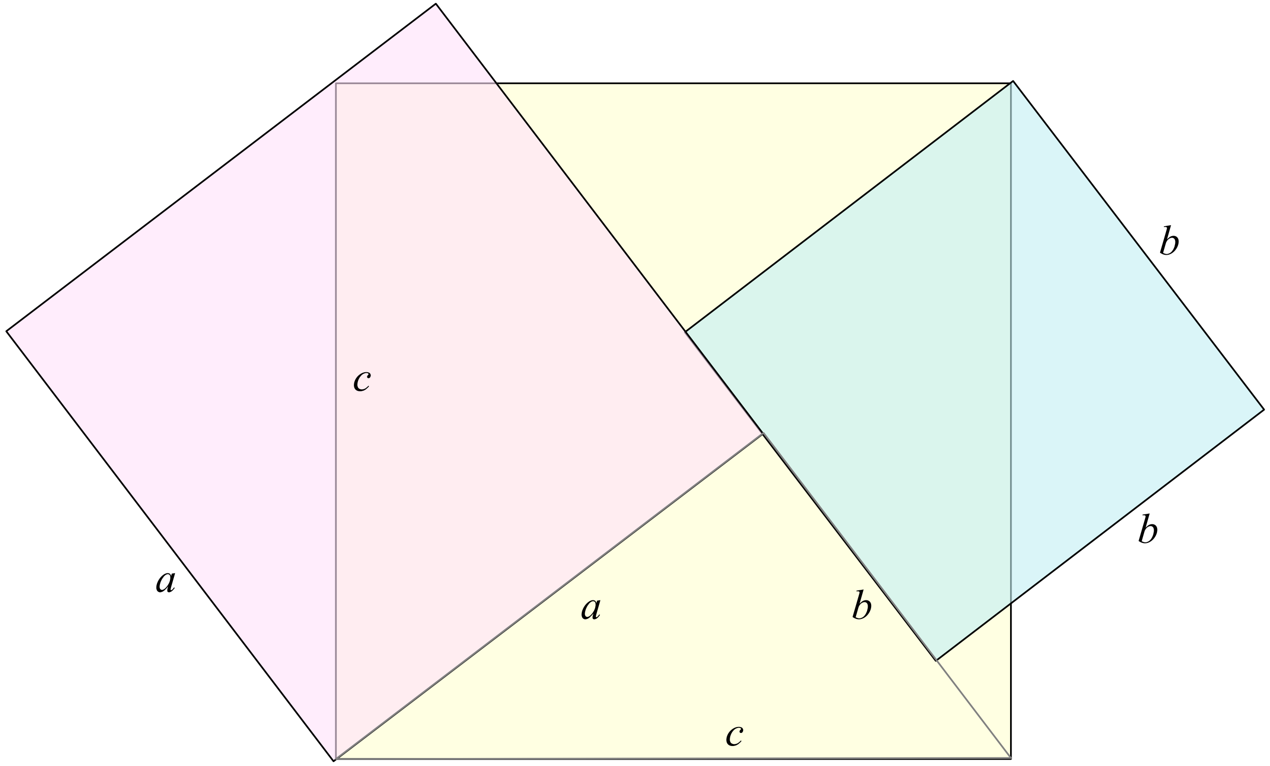


Proof 2:

Draw a $c \times c$ square with one edge being the c edge of our triangle (in yellow).

Draw an $a \times a$ square (pink) and a $b \times b$ square (blue) as in the picture.



Proofs

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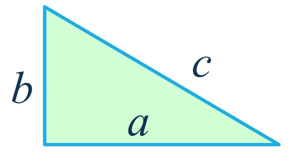
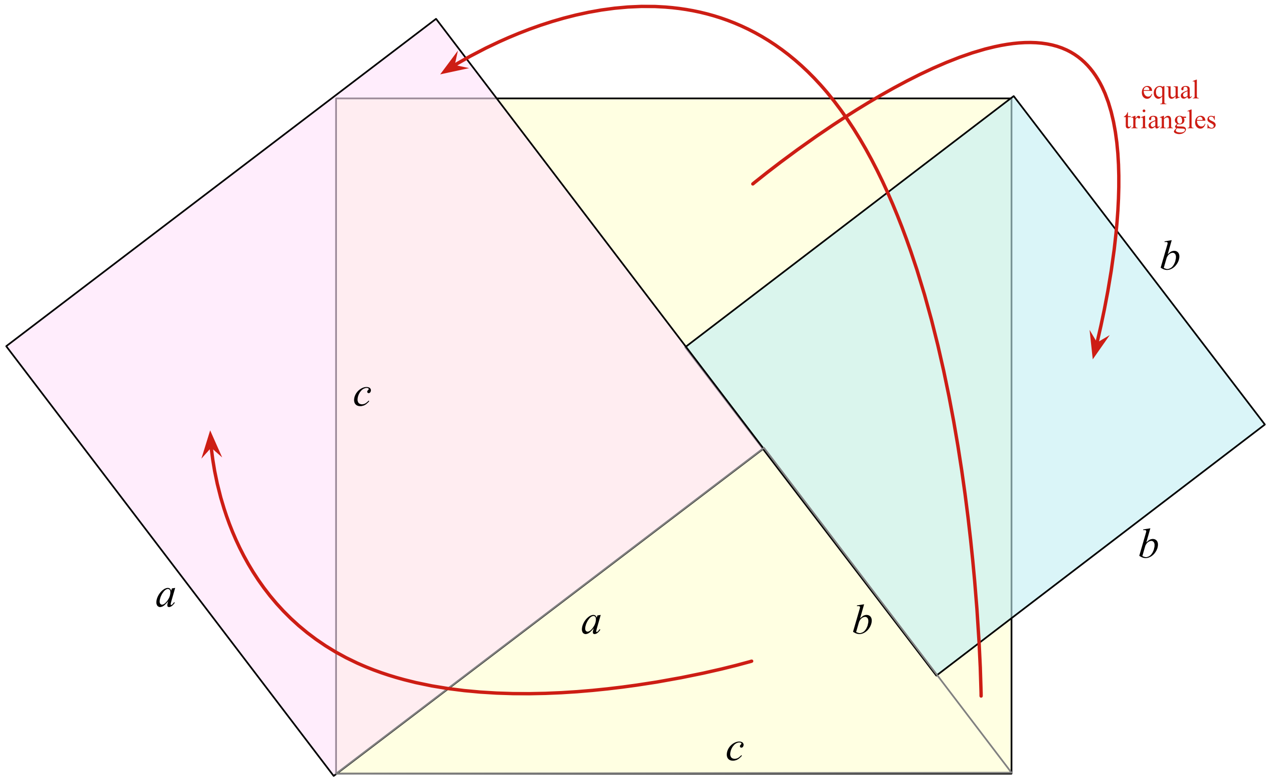


Proof 2:

Draw a $c \times c$ square with one edge being the c edge of our triangle (in yellow).

Draw an $a \times a$ square (pink) and a $b \times b$ square (blue) as in the picture.

This creates three pairs of identical triangles that can be rearranged following the arrows, convering the yellow square into the pink and the blue squares. ■



Proofs

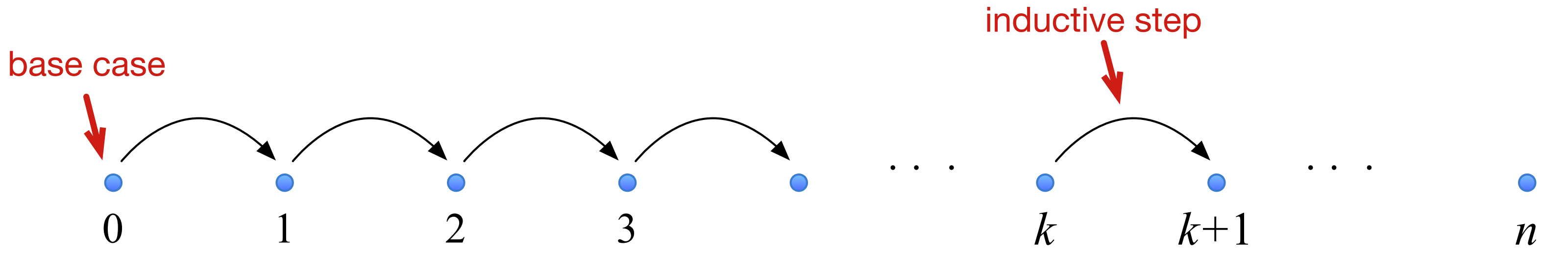
- *Mathematical induction*: technique for proving properties of integers

To prove that $\forall n\ \mathcal{P}(n)$ holds, show the following:

- Base case: $\mathcal{P}(0)$
- Inductive step: $\forall k\ \mathcal{P}(k) \Rightarrow \mathcal{P}(k+1)$

← (Note: $\forall n$ is shorthand for $\forall n \in \mathbb{N}$)

Intuition: boostraping property $\mathcal{P}(n)$:



Variants:

- Base case could be $\mathcal{P}(n_0)$, for some n_0. Then the proof shows that $\forall n \geq n_0\ \mathcal{P}(n)$.
- In strong induction, the inductive step is: $\forall k\ [\forall i \leq k\ \mathcal{P}(i)] \Rightarrow \mathcal{P}(k+1)$.

Proofs

Claim: $\forall n \;\; 5 | 7^n - 2^n$

Example:

$$\begin{aligned} 7^3 - 2^3 &= 343 - 8 \\ &= 335 \end{aligned}$$

Proof: We apply mathematical induction.

Base case. For $n = 0$, we have $7^0 - 2^0 = 0 = 5 \cdot 0$.

Inductive step. Consider $k \in \mathbb{N}$. Assume that the claim holds for $n = k$, that is $7^k - 2^k = 5 \cdot b$ for some $b \in \mathbb{N}$.

Then for $n = k+1$, we have

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2 \cdot (7^k - 2^k) \\ &= 5 \cdot 7^k + 2 \cdot (5b) \\ &= 5 \cdot (7^k + 2b) \end{aligned}$$

here we use inductive assumption

So $7^{k+1} - 2^{k+1}$ is a multiple of 5, completing the inductive step. ∎

Proofs

We will now prove that all horses have the same color! Formally:

Claim: $\forall n \geq 1$, if H is a set of n horses, then all horses in H have the same color.

Proof: We apply mathematical induction.

Base case. For $n = 1$, H has just one horse, so the claim is trivially true.

Inductive step. Consider $k \in \mathbb{N}$. Assume that the claim holds for any set of $n = k$ horses.

Let H be a set of $k+1$ horses, say $H = \{ h_1, h_2, \ldots, h_{k+1} \}$.

By the inductive assumption, all horses in these two sets:

$$\{ h_1, h_2, \ldots, h_k \} \qquad \{ h_2, \ldots, h_{k+1} \}.$$

have the same color.

Since these sets overlap, all horses in H also have the same color, completing the inductive step. ■

Question: *Where is the flaw???* These sets do not overlap when $k = 1$.

Proofs

Claim: $\forall n \quad \sum_{i=1}^{n} i^2 \; = \; \frac{1}{6}(2n+1)(n+1)n$

Proof: Class exercise.