

Imperial College London

**Multidimensional Utility Maximisation
in an Incomplete Market**

DEPARTMENT OF MATHEMATICS

Author:

Ivan Kirev
CID: 01738166

Supervisor:

Harry Zheng

Date: June 13, 2023

Acknowledgements

I wish to express my deepest gratitude to my supervisor, Prof. Harry Zheng, whose profound expertise, guidance, and unwavering support were pivotal throughout the course of this thesis. I would also like to extend my gratitude to Dr Kevin Webster, who generously provided me with support and guidance for the implementation of the Deep Learning algorithm.

Abstract

In this project, we study a general multidimensional linear quadratic stochastic control problem. We impose no constraints on the control, in which case the problem has an analytical solution. We formulate an equivalent dual problem and solve both the primal and dual problems using the Hamilton-Jacobi-Bellman equation approach and the Stochastic Maximum Principle approach. Each of the four solutions are shown to be equivalent. We then consider a similar problem, where the coefficients of the state process depend on a time-continuous finite state space Markov chain. With this modification, the four methods still give equivalent solutions, but the solutions are now in the form of systems of ODEs for each state i of the Markov chain, as opposed to a single system of ODEs in the non-Markov chain case. To complete the solutions, we solve numerically the system of ODEs for the 1-dimensional Brownian motion case. We employ the Runge-Kutta method to solve the coupled matrix ODEs, showing the equivalence of the solutions of the dual and primal problems. Finally, we want to consider a more general case where we impose constraints on the control and the cost is not necessarily quadratic, but convex. In this case, there are no longer analytical solutions, so we consider a deep learning algorithm with a focus on utility maximisation. We test this algorithm on the quadratic unconstrained case and compare the results with the already derived analytical results.

Contents

1	Unconstrained Optimisation with Quadratic Running Cost	6
1.1	Problem Description	6
1.1.1	Primal Problem	6
1.1.2	Dual Problem	7
1.2	Primal HJB Equation	10
1.2.1	Derivation of HJB equation	10
1.2.2	Finding the Optimal Control	11
1.2.3	Solving the Primal HJB Equation	13
1.3	Primal BSDE	14
1.3.1	Solution via the Primal BSDE	14
1.3.2	Equivalence of Primal HJB and Primal BSDE	17
1.4	Dual HJB Equation	18
1.4.1	The Dual HJB	18
1.4.2	Finding the Optimal Controls	18
1.4.3	Solving the Dual HJB	19
1.5	Dual BSDE	22
1.5.1	Solution via the Dual BSDE	22
1.5.2	Equivalence between Dual HJB and Dual BSDE	25
1.6	Primal and Dual Equivalence	26
2	Unconstrained Markovian Optimisation with Quadratic Running Cost	29
2.1	Problem Description	29
2.1.1	Primal Problem	29
2.1.2	Dual Problem	30
2.2	Primal HJB Equation	33
2.2.1	Deriving the Primal HJB	33
2.2.2	Finding the Optimal Control	33
2.2.3	Solving the HJB equation	35
2.3	Primal BSDE	37
2.3.1	Solution via the Primal BSDE	37
2.3.2	Equivalence of Primal HJB and Primal BSDE	40
2.4	Dual HJB Equation	42
2.4.1	The dual HJB	42
2.4.2	Finding the optimal control	42
2.4.3	Solving the dual HJB	42
2.5	Dual BSDE	46
2.5.1	Solution via the Dual BSDE	46
2.5.2	Equivalence of Dual HJB and Dual BSDE	50
2.6	Primal and Dual Equivalence	52

3	Numerical Solutions	55
3.1	Primal Problem for 1-Dimensional Brownian Motion Case	55
3.2	Dual Problem for for 1-Dimensional Brownian Motion Case	55
3.3	Runge-Kutta Method	56
4	Constrained Optimisation with Convex Running Cost	59
4.1	Problem Description	59
4.1.1	Primal Problem	59
4.1.2	Dual Problem	60
4.2	Primal and Dual HJB & BSDE Methods	61
4.2.1	Primal HJB Equation	61
4.2.2	Primal BSDE	61
4.2.3	Dual HJB Equation	62
4.2.4	Dual BSDE	63
4.3	Primal-Dual Relation	65
5	The Deep Controlled 2BSDE Method	67
5.1	Description of Algorithm	67
5.2	Applying the Algorithm to the Unconstrained Case	68
5.3	Verifying Results with Analytical Solution	70
5.4	Further Work	71

1 Unconstrained Optimisation with Quadratic Running Cost

In this chapter, we adopt the structural framework presented in [9]. This paper considers the Stochastic Maximum Principle method for solving our problem. We further consider the PDE method using the Hamilton-Jacobi-Bellman equation described in [8]. We modify the structure of the problem by implementing a quadratic running cost, which enables us to generate the analytical results, which we show in the following sections.

1.1 Problem Description

1.1.1 Primal Problem

We assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [t_0, T]}$ is the \mathbb{P} -augmentation of the natural filtration $\{F_t^W\}_{t \in [t_0, T]}$ generated by d -dimensional independent standard Brownian motions $\{(W_1(t), \dots, W_d(t))\}_{t \in [t_0, T]}$. Denote by $\mathcal{P}([t_0, T], \mathbb{R}^n)$ the set of all \mathbb{R}^n -valued progressively measurable processes on $[t_0, T] \times \Omega$, $\mathcal{H}([t_0, T], \mathbb{R}^n)$ the set of processes x in $\mathcal{P}([t_0, T], \mathbb{R}^n)$ such that $\mathbb{E}[\int_{t_0}^T |x(t)|^2 dt] < \infty$ and $\mathcal{S}([t_0, T], \mathbb{R}^n)$ the set of processes x in $\mathcal{P}([t_0, T], \mathbb{R}^n)$ such that $\mathbb{E}[\sup_{t_0 \leq t \leq T} |x(t)|^2] < \infty$.

In this section, we define the set of admissible controls by

$$\mathcal{A} := \{\pi(t) : \pi(t) \in \mathcal{H}([t_0, T], \mathbb{R}^m)\}$$

Given any $\pi(t) \in \mathcal{A}$, consider an n -dimensional state process $X(t)$ satisfying the following SDE:

$$\begin{cases} dX(t) &= [A(t)X(t) + B(t)\pi(t)]dt + \sum_{i=1}^d [C_i(t)X(t) + D_i(t)\pi(t)]dW_i(t) \\ X(t_0) &= x_0 \in \mathbb{R}^n \end{cases} \quad (1)$$

where $A(t), C(t) \in \mathbb{R}^{n \times n}$, and $B_i(t), D_i(t) \in \mathbb{R}^{n \times m}$, for $i = \{1, \dots, d\}$ are \mathbb{F} -progressively measurable and uniformly bounded. The pair (X, π) is *admissible* if X is a solution to SDE (1) with control $\pi \in \mathcal{A}$. For simplicity, we denote

$$\begin{aligned} b(t, X(t), \pi(t)) &:= A(t)X(t) + B(t)\pi(t) \in \mathbb{R}^n \\ \sigma(t, X(t), \pi(t)) &:= \left[\begin{pmatrix} C_1(t)X(t) + D_1(t)\pi(t) \\ \vdots \\ C_d(t)X(t) + D_d(t)\pi(t) \end{pmatrix} \right] \in \mathbb{R}^{n \times d} \end{aligned}$$

The system is then written as:

$$\begin{cases} dX(t) &= b(t, X(t), \pi(t))dt + \sigma(t, x(t), \pi(t))dW(t) \\ X(t_0) &= x_0 \in \mathbb{R}^n \end{cases} \quad (2)$$

Consider the functional $J : \mathcal{A} \rightarrow \mathbb{R}$, defined by

$$J(\pi) := \mathbb{E} \left[\int_{t_0}^T f(t, X(t), \pi(t))dt + g(X(T)) \right], \quad (3)$$

where $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are defined by

$$f(t, X(t), \pi(t)) = \frac{1}{2} X^T(t) Q(t) X(t) + X^T(t) S^T(t) \pi(t) + \frac{1}{2} \pi^T(t) R(t) \pi(t) \quad (4)$$

$$g(X(T)) = \frac{1}{2} X^T(T) G(T) X(T) + X^T(T) L(T). \quad (5)$$

We assume that the processes $Q(t) \in \mathcal{P}([t_0, T], \mathbb{R}^{n \times n})$, $S(t) \in \mathcal{P}([t_0, T], \mathbb{R}^{m \times n})$, $R(t) \in \mathcal{P}([t_0, T], \mathbb{R}^{m \times m})$ are uniformly bounded, $G(T) \in \mathbb{R}^{n \times n}$, $L(T) \in \mathbb{R}^n$, Q , R and G are symmetric matrices and

$$\begin{pmatrix} Q(t) & S^T(t) \\ S(t) & R(t) \end{pmatrix} \geq 0.$$

Under these assumptions, we know that J is a convex functional of π .

We consider the following optimisation problem:

$$\text{Minimise } J(\pi) \text{ subject to } (X, \pi) \text{ admissible.} \quad (6)$$

An admissible control $\hat{\pi}$ is *optimal* if $J(\hat{\pi}) \leq J(\pi)$ for all $\pi \in \mathcal{A}$.

1.1.2 Dual Problem

We now derive the dual problem. We follow the derivation in [6]. First, assume that the dual process Y is driven by an SDE of the form

$$dY(t) = \alpha_1(t) + \sum_{i=1}^d \beta_i(t) dW_i(t),$$

with initial condition $Y(t_0) = y$, where α_1 and β_i are stochastic processes to be determined. Ito's lemma gives

$$\begin{aligned} d(X^T Y) &= X^T dY + Y^T dX + dX^T dY \\ &= \underbrace{\left[X^T \left(\alpha_1 + A^T Y + \sum_{i=1}^d C_i^T \beta_i \right) \right]}_{:=\alpha} + \underbrace{\pi^T \left(B^T Y + \sum_{i=1}^d D_i^T \beta_i \right)}_{:=\beta} dt + \text{local martingale.} \end{aligned}$$

Denoting $\alpha = \alpha_1 + A^T Y + \sum_{i=1}^d C_i^T \beta_i$ and $\beta = B^T Y + \sum_{i=1}^d D_i^T \beta_i$, we have that the dual process $Y(t)$ satisfies

$$\begin{cases} dY(t) &= [\alpha(t) - A(t)^T Y(t) - \sum_{i=1}^d C_i(t)^T \beta_i(t)]dt + \sum_{i=1}^d \beta_i(t) dW_i(t) \\ Y(t_0) &= y. \end{cases} \quad (7)$$

There is a unique solution to the SDE for given $(y, \alpha, \beta_1, \dots, \beta_d)$. We call $(\alpha, \beta_1, \dots, \beta_d)$ the *admissible dual control* and $(Y, \alpha, \beta_1, \dots, \beta_d)$ the *admissible dual pair*. Returning to Ito's lemma to $X(t)^T Y(t)$, we have

$$d(X^T Y) = (X^T \alpha + \pi^T \beta)dt + \text{local martingale}.$$

The process $X^T(t)Y(t) - \int_{t_0}^t [X^T(s)\alpha(s) + \pi^T(s)\beta(s)]ds$ is a local martingale and a super-martingale if it is bounded below by an integrable process, which gives

$$\mathbb{E} \left[X^T(T)Y(T) - \int_{t_0}^T (X^T(s)\alpha(s) + \pi^T(s)\beta(s))ds \right] \leq X^T(t_0)Y(t_0) = x^T y. \quad (8)$$

The optimisation problem (6) can be written equivalently as

$$\sup_{\pi} \mathbb{E} \left[- \int_{t_0}^T f(t, X(t), \pi(t))dt - g(X(T)) \right].$$

Define the dual functions $\phi : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ by

$$\phi(t, \alpha, \beta) = \sup_{x, \pi} \{ x^T \alpha + \pi^T \beta - f(t, x, \pi) \} \quad (9)$$

and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$h(y) = \sup_x \{ -x^T y - g(x) \}. \quad (10)$$

Combining (8), (9), (10), we get the following inequality:

$$\sup_{\pi} \mathbb{E} \left[- \int_{t_0}^T f(t, X, \pi)dt - g(X(T)) \right] \leq \inf_{y, \alpha, \beta_1, \dots, \beta_d} \left[x^T y + \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha, \beta)dt + h(Y(T)) \right] \right]. \quad (11)$$

The dual control problem is defined by

$$\inf_{y, \alpha, \beta_1, \dots, \beta_d} \left[x^T y + \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha, \beta)dt + h(Y(T)) \right] \right], \quad (12)$$

where Y satisfies (7). Problem (12) can be solved in two steps: first, for fixed y , solve a stochastic control problem

$$-\tilde{v}(t, y) = \inf_{\alpha, \beta_1, \dots, \beta_d} \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha, \beta)dt + h(Y(T)) \right], \quad (13)$$

and second, solve a static optimisation problem

$$\inf_y (x^T y - \tilde{v}(t, y)).$$

To find ϕ and h , we substitute f and g from (4) and (5) into (9) and (10) and we can find the supremums by setting the derivatives to zero. We get

$$\phi(t, \alpha, \beta) = \sup_{x, \pi} \left\{ \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} \right\},$$

so setting the derivative to zero, we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0 \implies \begin{bmatrix} x^* \\ \pi^* \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Therefore

$$\begin{aligned} \pi^* &= [SQ^{-1}S^T - R]^{-1}(SQ^{-1}\alpha - \beta) \\ x^* &= Q^{-1}(\alpha - S^T\pi^*) \end{aligned}$$

Then ϕ is given by

$$\phi(t, \alpha, \beta) = \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Denoting

$$\begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1},$$

where

$$\tilde{Q} = Q^{-1} - Q^{-1}S^T(SQ^{-1}S^T - R)^{-1}SQ^{-1} \quad (14)$$

$$\tilde{R} = R^{-1} - R^{-1}S(S^TR^{-1}S - Q^{-1})^{-1}S^TR^{-1} \quad (15)$$

$$\tilde{S} = (SQ^{-1}S^T - R)^{-1}SQ^{-1} = R^{-1}S(S^TR^{-1}S - Q)^{-1} \quad (16)$$

we get

$$\phi(t, \alpha, \beta) = \frac{1}{2} \alpha^T \tilde{Q} \alpha + \alpha^T \tilde{S}^T \beta + \frac{1}{2} \beta^T \tilde{R} \beta, \quad (17)$$

where $\beta = B^T Y + \sum_{i=1}^d D_i^T \beta_i$. Similarly,

$$D_x \left[-x^T y - \frac{1}{2} x^T G x - x^T L \right] = -y - Gx - L \implies x^* = -G^{-1}(y + L).$$

Then $h(y)$ is given by

$$\begin{aligned} h(y) &= (y^T + L^T)G^{-1}y - \frac{1}{2}(y^T + L^T)G^{-1}(y + L) + (y^T + L^T)G^{-1}L \\ &= \frac{1}{2} [y^T G^{-1}y + L^T G^{-1}y + y^T G^{-1}L + L^T G^{-1}L] \\ &= \frac{1}{2} (y^T + L^T)G^{-1}(y + L) \end{aligned} \quad (18)$$

1.2 Primal HJB Equation

1.2.1 Derivation of HJB equation

We transform the minimisation problem to maximisation by noting that

$$\inf_{\pi(t) \in \mathbb{R}^m} \mathbb{E} \left[\int_{t_0}^T f(t, X(t), \pi(t)) dt + g(X(T)) \right] = - \sup_{\pi(t) \in \mathbb{R}^m} \mathbb{E} \left[\int_{t_0}^T -f(t, X(t), \pi(t)) dt - g(X(T)) \right],$$

and denote the value function

$$v(t, X(t)) = \sup_{\pi(t) \in \mathbb{R}^m} \mathbb{E} \left[\int_{t_0}^T -f(t, X(t), \pi(t)) dt - g(X(T)) \right] \quad (19)$$

Note that the optimal value of the optimisation problem is given by $-v(t, X(t))$. Consider the time interval $(t, t+h)$ and a constant control $\pi(t) = \pi$. According to the Dynamic programming principle,

$$v(t, X(t)) \geq \mathbb{E} \left[\int_t^{t+h} -f(s, X(s), \pi) ds + v(t+h, X(t+h)) \right], \quad (20)$$

where we denote $X(s)$ to be the solution of (2) given that we know the value of X at time t .

Applying Ito's formula between t and $t+h$ we get

$$\begin{aligned} v(t+h, X(t+h)) &= v(t, X(t)) + \int_t^{t+h} \left(\frac{\partial v(s, X(s))}{\partial t} + \mathcal{L}^\pi[v(s, X(s))] \right) ds \\ &\quad + \underbrace{\int_t^{t+h} [D_x v(s, X(s))]^T \sigma(s, X(s), \pi) dW(s)}_{\text{(local) martingale}}, \end{aligned}$$

where $\mathcal{L}^\pi[v(t, x)]$ is the generator given by

$$\mathcal{L}^\pi[v(t, X(t))] = b^T(t, X(t), \pi) D_x v(t, X(t)) + \frac{1}{2} \text{tr} \left[\sigma(t, X(t), \pi) \sigma^T(t, X(t), \pi) D_x^2 v(t, X(t)) \right] \quad (21)$$

Substituting into equation (20), we get

$$0 \geq \mathbb{E} \left[\int_t^{t+h} \frac{\partial v}{\partial t}(s, X(s)) + \mathcal{L}^\pi[v(s, X(s))] - f(s, X(s), \pi) ds \right]$$

Dividing by h and sending h to 0, this yields by the mean value theorem

$$0 \geq \frac{\partial v}{\partial t}(t, x) + \mathcal{L}^\pi[v(t, x)] - f(t, x, \pi).$$

Since this is true for any admissible π , we obtain the inequality

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\pi \in \mathbb{R}^m} [\mathcal{L}^\pi v(t, x) - f(t, x, \pi)] \leq 0. \quad (22)$$

On the other hand, suppose that π^* is an optimal control. Then by the dynamic programming principle,

$$v(t, x) = \mathbb{E} \left[\int_t^{t+h} -f(s, X^*(s), \pi^*(s)) ds + v(t+h, X^*(t+h)) \right], \quad (23)$$

where X^* is the solution to the initial SDE (2) with control π^* starting from x at time t . By similar reasoning, we get

$$\frac{\partial v}{\partial t}(t, x) + \mathcal{L}^{\pi^*}[v(t, x)] - f(t, x, \pi^*) = 0$$

which combined with (22) suggests that v should satisfy

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\pi \in \mathbb{R}^m} [\mathcal{L}^{\pi(t)}[v(t, x)] - f(t, x, \pi)] = 0, \quad \forall (t, x) \in [t_0, T] \times \mathbb{R}^n. \quad (24)$$

with the terminal condition:

$$v(T, x) = -g(x) = -\frac{1}{2}x^T G(T)x - x^T L(T), \quad \forall x \in \mathbb{R}^n.$$

Equation (24) is called the Hamilton-Jacobi-Bellman equation.

1.2.2 Finding the Optimal Control

The supremum in the HJB equation (24) can be found by setting the derivative with respect to π to zero. The derivative of the generator \mathcal{L}^π , $D_\pi[\mathcal{L}^\pi] \in \mathbb{R}^m$, is given by:

$$D_\pi[\mathcal{L}^\pi[v(t, x)]] = D_\pi[b(t, x, \pi)^T D_x[v(t, x)]] + D_\pi \left[\frac{1}{2} \text{tr}(\sigma(t, x, \pi) \sigma^T(t, x, \pi) D_x^2[v(t, x)]) \right]. \quad (25)$$

We have that

$$\begin{aligned} D_\pi[b^T(t, x, \pi) D_x[v(t, x)]] &= D_\pi[(x^T A^T(t) + \pi^T(t) B^T(t)) D_x[v(t, x)]] \\ &= B^T(t) D_x[v(t, x)] \end{aligned} \quad (26)$$

The latter derivative in (25) is given by:

$$D_\pi \left[\frac{1}{2} \text{tr}[\sigma(t, x, \pi) \sigma^T(t, x, \pi) D_x^2[v]] \right] = \frac{1}{2} D_\pi \left[\text{tr} \left[\sum_{i=1}^d (C_i x + D_i \pi) (C_i x + D_i \pi)^T D_x^2[v] \right] \right]$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=1}^d D_{\pi} \left[\text{tr}[(C_i x + D_i \pi)(C_i x + D_i \pi)^T D_x^2[v]] \right] \\
&= \frac{1}{2} \sum_{i=1}^d D_{\pi} \left[(C_i x + D_i \pi)^T D_x^2[v] (C_i x + D_i \pi) \right] \\
&= \sum_{i=1}^d D_i^T D_x^2[v(t, x)] (C_i x + D_i \pi) \tag{27}
\end{aligned}$$

The derivative of $f(t, x, \pi)$ with respect to π is

$$D_{\pi} f(t, x, \pi) = Sx + R\pi \tag{28}$$

Combining the three equations, (26), (27), (28), we get that

$$D_{\pi} [\mathcal{L}^{\pi}(t)[v(t, x)] - f(t, x, \pi)] = B^T D_x[v(t, x)] + \sum_{i=1}^d D_i^T D_x^2[v(t, x)] (C_i x + D_i \pi) - Sx - R\pi$$

Setting this to zero, we get

$$\pi^* = \left[\sum_{i=1}^d D_i^T D_x^2[v(t, x)] D_i - R \right]^{-1} \left[Sx - B^T D_x[v(t, x)] - \sum_{i=1}^d D_i^T D_x^2[v(t, x)] C_i x \right] \tag{29}$$

We now substitute (29) into (24) to get:

$$\frac{\partial v}{\partial t} + b(t, x, \pi^*)^T D_x[v] + \frac{1}{2} \text{tr}[\sigma(t, x, \pi^*) \sigma^T(t, x, \pi^*) D_x^2[v]] - \frac{1}{2} x^T Qx - \frac{1}{2} x^T S^T \pi^* - \frac{1}{2} \pi^{*T} Sx - \frac{1}{2} \pi^{*T} R \pi^* = 0$$

As $D_x^2[v]$ is a symmetric matrix, we can write

$$\begin{aligned}
\text{tr}[\sigma(t, x, \pi^*) \sigma^T(t, x, \pi^*) D_x^2[v]] &= \sum_{i=1}^d \text{tr}[(C_i x + D_i \pi^*)(C_i x + D_i \pi^*)^T D_x^2[v]] \\
&= \sum_{i=1}^d (C_i x + D_i \pi^*)^T D_x^2[v] (C_i x + D_i \pi^*),
\end{aligned}$$

we get the HJB equation

$$\begin{aligned}
\frac{\partial v}{\partial t} + (Ax + B\pi^*)^T D_x[v] + \frac{1}{2} \sum_{i=1}^d (C_i x + D_i \pi^*)^T D_x^2[v] (C_i x + D_i \pi^*) \\
- \frac{1}{2} x^T Qx - \frac{1}{2} x^T S^T \pi^* - \frac{1}{2} \pi^{*T} Sx - \frac{1}{2} \pi^{*T} R \pi^* = 0 \tag{30}
\end{aligned}$$

where π^* is as in (29) and the terminal condition is given by

$$v(T, x) = -g(x) = -\frac{1}{2} x^T G(T) x - x^T L(T).$$

1.2.3 Solving the Primal HJB Equation

We assume that $v(t, x)$ is a quadratic function in x and we use the ansatz

$$v(t, x) = \frac{1}{2}x^T P(t)x + x^T M(t) + N(t), \quad (31)$$

with terminal conditions:

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0. \quad (32)$$

Then

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) &= \frac{1}{2}x^T \dot{P}(t)x + x^T \dot{M}(t) + \dot{N}(t) \\ D_x[v(t, x)] &= P(t)x + M(t) \\ D_x^2[v(t, x)] &= P(t). \end{aligned}$$

Substituting in (29) we get the optimal control

$$\pi^* = \left[\sum_{i=1}^d D_i^T P D_i - R \right]^{-1} \left[Sx - B^T P x - B^T M - \sum_{i=1}^d D_i^T P C_i x \right] \quad (33)$$

We can write this as

$$\pi^* = \vartheta_1 x + \kappa_1,$$

where

$$\vartheta_1 = \left(\sum_{i=1}^d D_i^T P D_i - R \right)^{-1} \left(S - B^T P - \sum_{i=1}^d D_i^T P C_i \right), \quad \kappa_1 = - \left(\sum_{i=1}^d D_i^T P D_i + R \right)^{-1} B^T M \quad (34)$$

Substituting this into (30), we get

$$\begin{aligned} & \frac{\partial v}{\partial t} + (Ax + B(\vartheta_1 x + \kappa_1))^T D_x[v] + \frac{1}{2} \sum_{i=1}^d (C_i x + D_i(\vartheta_1 x + \kappa_1))^T D_x^2[v] (C_i x + D_i(\vartheta_1 x + \kappa_1)) \\ & - \frac{1}{2} x^T Q x - \frac{1}{2} x^T S^T (\vartheta_1 x + \kappa_1) - \frac{1}{2} (\vartheta_1 x + \kappa_1)^T S x - \frac{1}{2} (\vartheta_1 x + \kappa_1)^T R (\vartheta_1 x + \kappa_1) = 0 \implies \\ & \frac{1}{2} x^T \dot{P} x + x^T \dot{M} + \dot{N} + (x^T A^T + x^T \vartheta_1^T B^T + \kappa_1^T B^T) (P x + M) \\ & + \frac{1}{2} \sum_{i=1}^d (x^T C_i^T + x^T \vartheta_1^T D_i^T + \kappa_1^T D_i^T) P (C_i x + D_i \vartheta_1 x + D_i \kappa_1) \\ & - \frac{1}{2} x^T Q x - \frac{1}{2} x^T S^T (\vartheta_1 x + \kappa_1) - \frac{1}{2} (x^T \vartheta_1^T + \kappa_1^T) S x - \frac{1}{2} (x^T \vartheta_1^T + \kappa_1^T) R (\vartheta_1 x + \kappa_1) = 0 \end{aligned}$$

Rewriting this, we get

$$\begin{aligned}
& x^T \left[\frac{1}{2} \dot{P} + \frac{1}{2} A^T P + \frac{1}{2} P A + \frac{1}{2} \vartheta_1^T B^T P + \frac{1}{2} P B \vartheta_1 + \frac{1}{2} \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P (C_i + D_i \vartheta_1) \right. \\
& \left. - \frac{1}{2} Q - \frac{1}{2} \vartheta_1^T S - \frac{1}{2} S^T \vartheta_1 - \frac{1}{2} \vartheta_1^T R \vartheta_1 \right] x + x^T \left[\dot{M} + A^T M + P B \kappa_1 + \vartheta_1^T B^T M + \right. \\
& \left. \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P D_i \kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 \right] + \dot{N} + \kappa_1^T B^T M + \frac{1}{2} \sum_{i=1}^d \kappa_1^T D_i^T P D_i \kappa_1 - \frac{1}{2} \kappa_1^T R \kappa_1 = 0
\end{aligned}$$

This equation must equal zero for all x , hence the coefficients in front of the quadratic term, as well as x and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\begin{aligned}
& \frac{1}{2} \dot{P} + \frac{1}{2} A^T P + \frac{1}{2} P A + \frac{1}{2} \vartheta_1^T B^T P + \frac{1}{2} P B \vartheta_1 + \frac{1}{2} \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P (C_i + D_i \vartheta_1) \\
& - \frac{1}{2} Q - \frac{1}{2} \vartheta_1^T S - \frac{1}{2} S^T \vartheta_1 - \frac{1}{2} \vartheta_1^T R \vartheta_1 = 0
\end{aligned} \tag{35}$$

$$\dot{M} + A^T M + P B \kappa_1 + \vartheta_1^T B^T M + \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P D_i \kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 = 0 \tag{36}$$

$$\dot{N} + \kappa_1^T B^T M + \frac{1}{2} \sum_{i=1}^d \kappa_1^T D_i^T P D_i \kappa_1 - \frac{1}{2} \kappa_1^T R \kappa_1 = 0, \tag{37}$$

where ϑ_1 and κ_1 are as in (34) and the terminal conditions are as in (32).

1.3 Primal BSDE

1.3.1 Solution via the Primal BSDE

We define the Hamiltonian $\mathcal{H} : \Omega \times [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ by

$$\begin{aligned}
\mathcal{H}(t, x, \pi, p, q) &= b^T p + \text{tr}(\sigma^T q) - f(t, x, \pi) \\
&= x^T A^T p + \pi^T B^T p + \sum_{i=1}^d \left(x^T C_i^T q_i + \pi^T D_i^T q_i \right) - \frac{1}{2} x^T Q x - x^T S^T \pi - \frac{1}{2} \pi^T R \pi
\end{aligned} \tag{38}$$

where we denote by $q_i \in \mathbb{R}^n$ the i^{th} column of the matrix $q \in \mathbb{R}^{n \times d}$. The adjoint process is given by

$$\begin{cases} dp &= -D_x[\mathcal{H}(t, X(t), \pi(t), p(t), q(t))]dt + \sum_{i=1}^d q_i(t) dW_i(t) \\ p(T) &= -D_x[g(X(T))] = -G(T)X(T) - L(T) \end{cases} \tag{39}$$

According to the Stochastic Maximum Principle, the optimal control for the optimisation problem (6) satisfies the condition

$$D_\pi \mathcal{H}(t, X(t), \pi(t), p(t), q(t)) = 0.$$

We have

$$D_\pi \mathcal{H}(t, X(t), \pi(t), p(t), q(t)) = B^T p + \sum_{i=1}^d D_i^T q_i - SX - R\pi$$

so

$$B^T p + \sum_{i=1}^d D_i^T q_i - SX - R\pi = 0 \quad (40)$$

Solving this, we can find a linear solution for the control, which we denote by

$$\pi = \vartheta_2 X + \kappa_2. \quad (41)$$

Substituting the control in the Hamiltonian (38) we get

$$\begin{aligned} \mathcal{H} = & X^T A^T p + (\vartheta_2 X + \kappa_2)^T B^T p + \sum_{i=1}^d \left(X^T C_i^T q_i + (\vartheta_2 X + \kappa_2)^T D_i^T q_i \right) - \frac{1}{2} X^T Q X \\ & - \frac{1}{2} X^T S^T (\vartheta_2 X + \kappa_2) - \frac{1}{2} (\vartheta_2 X + \kappa_2)^T S X - \frac{1}{2} (\vartheta_2 X + \kappa_2)^T R (\vartheta_2 X + \kappa_2) \end{aligned} \quad (42)$$

The derivative of the Hamiltonian is then

$$D_x[\mathcal{H}] = A^T p + \vartheta_2^T B^T p + \sum_{i=1}^d C_i^T q_i + \sum_{i=1}^d \vartheta_2^T D_i^T q_i - QX - 2S^T \vartheta_2 X - S^T \kappa_2 - \vartheta_2^T R \vartheta_2 X - \vartheta_2^T R \kappa_2 \quad (43)$$

We try an ansatz for p of the form

$$p = \varphi(t)X(t) + \psi(t),$$

where $\varphi(t) \in \mathbb{R}^{n \times n}$ and $\psi(t) \in \mathbb{R}^n$. Applying Ito's formula, we get

$$\begin{aligned} dp &= \frac{\partial f}{\partial t} dt + (D_x[f])^T dX + \frac{1}{2} (dX)^T D_x^2[f] dX \\ &= (\dot{\varphi}X + \dot{\psi})dt + \varphi dX \\ &= (\dot{\varphi}X + \dot{\psi} + \varphi b(t, X, \pi))dt + \varphi \sigma(t, X, \pi) dW \\ &= \left[\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B\pi \right] dt + \varphi \sum_{i=1}^d (C_i X + D_i \pi) dW_i \\ &= \left[\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B\vartheta_2 X + \varphi B\kappa_2 \right] dt + \sum_{i=1}^d \varphi (C_i X + D_i \vartheta_2 X + D_i \kappa_2) dW_i \end{aligned} \quad (44)$$

Equating the coefficients of (44) with (39), we get the system

$$\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B \vartheta_2 X + \varphi B \kappa_2 = -D_x[\mathcal{H}(t, X(t), \pi(t), p(t), q(t))] \quad (45)$$

$$\varphi(C_i X + D_i \vartheta_2 X + D_i \kappa_2) = q_i \quad i \in \{1, \dots, d\} \quad (46)$$

$$B^T \varphi X + B^T \psi + \sum_{i=1}^d D_i^T q_i - S X - R(\vartheta_1 X + \kappa_1) = 0, \quad (47)$$

where the third equation is the Hamiltonian condition (40). We now substitute q_i from the second equation into (43) and (47), so that our system becomes

$$\begin{aligned} \dot{\varphi}(t)X(t) + \dot{\psi}(t) + \varphi(t)AX(t) + \varphi B \vartheta_2 X + \varphi B \kappa_2 = & -A^T \varphi X - A^T \psi - \vartheta_2^T B^T \varphi X - \vartheta_2^T B^T \psi \\ & - \sum_{i=1}^d C_i^T \varphi(C_i X + D_i \vartheta_2 X + D_i \kappa_2) - \sum_{i=1}^d \vartheta_2^T D_i^T \varphi(C_i X + D_i \vartheta_2 X + D_i \kappa_2) \\ & + QX + 2S^T \vartheta_2 X + S^T \kappa_2 + \vartheta_2^T R \vartheta_2 X + \vartheta_2^T R \kappa_2 \end{aligned} \quad (48)$$

$$B^T \varphi X + B^T \psi + \sum_{i=1}^d D_i^T [\varphi(C_i X + D_i \vartheta_2 X + D_i \kappa_2)] - S X - R \vartheta_2 X - R \kappa_2 = 0 \quad (49)$$

From (49), we get

$$\pi^* = \vartheta_2 X + \kappa_2 = \left[\sum_{i=1}^d D_i^T \varphi D_i - R \right]^{-1} \left[S X - B^T \varphi X - B^T \psi - \sum_{i=1}^d D_i^T \varphi C_i X \right], \quad (50)$$

i.e.,

$$\vartheta_2 = \left[\sum_{i=1}^d D_i^T \varphi D_i - 2R \right]^{-1} \left[S - B^T \varphi - \sum_{i=1}^d D_i^T \varphi C_i \right], \quad \kappa_2 = - \left[\sum_{i=1}^d D_i^T \varphi D_i - 2R \right]^{-1} B^T \psi. \quad (51)$$

We rewrite equation (48) as

$$\begin{aligned} & \left[\dot{\varphi} + \varphi A + A^T \varphi + \varphi B \vartheta_2 + \vartheta_2^T B^T \varphi + \sum_{i=1}^d (C_i^T + \vartheta_2^T D_i^T) \varphi(C_i + D_i \vartheta_2) - Q - S^T \vartheta_2 - \vartheta_2^T S - \vartheta_2^T R \vartheta_2 \right] X \\ & + \left[\dot{\psi} + \varphi B \kappa_2 + A^T \psi + \vartheta_2^T B^T \psi + \sum_{i=1}^d C_i^T \varphi D_i \kappa_2 + \sum_{i=1}^d \vartheta_2^T D_i^T \varphi D_i \kappa_2 - S^T \kappa_2 - \vartheta_2^T R \kappa_2 \right] = 0 \end{aligned}$$

Since this must be true for all X , the coefficient in front of X must be equal to zero, so we get the system of ODEs

$$\dot{\varphi} + \varphi A + A^T \varphi + \varphi B \vartheta_2 + \vartheta_2^T B^T \varphi + \sum_{i=1}^d (C_i^T + \vartheta_2^T D_i^T) \varphi(C_i + D_i \vartheta_2) - Q - S^T \vartheta_2 - \vartheta_2^T S - \vartheta_2^T R \vartheta_2 = 0 \quad (52)$$

$$\dot{\psi} + \varphi B \kappa_2 + A^T \psi + \vartheta_2^T B^T \psi + \sum_{i=1}^d C_i^T \varphi D_i \kappa_2 + \sum_{i=1}^d \vartheta_2^T D_i^T \varphi D_i \kappa_2 - S^T \kappa_2 - \vartheta_2^T R \kappa_2 = 0, \quad (53)$$

where ϑ_2 and κ_2 are as in (51) and the terminal conditions are given by

$$\varphi(T) = -G(T), \quad \psi(T) = -L(T). \quad (54)$$

1.3.2 Equivalence of Primal HJB and Primal BSDE

From the Primal HJB, the optimal control is given by (33):

$$\pi^* = \left[\sum_{i=1}^d D_i^T P D_i - R \right]^{-1} \left[Sx - B^T P x - B^T M - \sum_{i=1}^d D_i^T P C_i x \right] \quad (55)$$

and from the Primal BSDE the optimal control is (50):

$$\pi^* = \left[\sum_{i=1}^d D_i^T \varphi D_i - R \right]^{-1} \left[S X - B^T \varphi X - B^T \psi - \sum_{i=1}^d D_i^T \varphi C_i X \right], \quad (56)$$

Comparing, we see that the equations are identical and we get the relation

$$\varphi = P, \quad \psi = M.$$

The ODE from the Primal BSDE for φ is (52). Substituting $\varphi = P$ and $\vartheta_2 = \vartheta_1$ we get

$$\dot{P} + PA + A^T P + PB \vartheta_1 + \vartheta_1^T B^T P + \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P (C_i + D_i \vartheta_1) - Q - 2S^T \vartheta_1 - \vartheta_1^T R \vartheta_1 = 0,$$

which is equal to twice the ODE for P from the primal HJB (35), i.e.

$$\frac{1}{2} \left[\dot{P} + PA + A^T P + PB \vartheta_1 + \vartheta_1^T B^T P + \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P (C_i + D_i \vartheta_1) - Q - 2S^T \vartheta_1 - \vartheta_1^T R \vartheta_1 \right] = 0.$$

Similarly, substituting $\varphi = P, \psi = M, \vartheta_2 = \vartheta_1$ and $\kappa_2 = \kappa_1$ into (53) we get

$$\dot{M} + PB \kappa_1 + A^T M + \vartheta_1^T B^T M + \sum_{i=1}^d C_i^T P D_i \kappa_1 + \sum_{i=1}^d \vartheta_1^T D_i^T P D_i \kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 = 0$$

which is the same equation as (36):

$$\dot{M} + A^T M + PB \kappa_1 + \vartheta_1^T B^T M + \sum_{i=1}^d (C_i^T + \vartheta_1^T D_i^T) P D_i \kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 = 0$$

The terminal conditions for the primal BSDE are given by (54):

$$\varphi(T) = -G(T), \quad \psi(T) = -L(T),$$

and from the primal HJB (32):

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0.$$

So the ODEs for solving P, M and φ, ψ are identical, hence we have equivalence between the two methods.

1.4 Dual HJB Equation

1.4.1 The Dual HJB

Recall that the dual process Y satisfies (7):

$$\begin{cases} dY(t) &= [\alpha(t) - A(t)^T Y(t) - \sum_{i=1}^d C_i(t)^T \beta_i(t)] dt + \sum_{i=1}^d \beta_i(t) dW_i(t) \\ Y(t_0) &= y \end{cases}$$

and also recall from (13) that the dual value function is given by

$$\tilde{v}(t, Y(t)) = \sup_{\alpha, \beta_1, \dots, \beta_d} \mathbb{E} \left[- \int_{t_0}^T \phi(t, \alpha, \beta) dt - h(Y(T)) \right],$$

where $\beta = B^T Y + \sum_{i=1}^d D_i^T \beta_i$ and $\phi(t, \alpha, \beta)$ and $h(Y(T))$ are given in (17) and (18). Then, following the same steps as for the primal problem, the dual HJB equation is given by

$$\frac{\partial \tilde{v}}{\partial t}(t, y) + \sup_{\alpha, \beta_1, \dots, \beta_d} [\mathcal{L}^{\alpha, \beta_i}[\tilde{v}(t, y)] - \phi(t, \alpha, \beta)] = 0,$$

where the generator is given by

$$\mathcal{L}^{\alpha, \beta_i}[\tilde{v}(t, y)] = \left(\alpha^T - y^T A - \sum_{i=1}^d \beta_i^T C_i \right) D_y[\tilde{v}] + \frac{1}{2} \sum_{i=1}^d \beta_i^T D_y^2[\tilde{v}] \beta_i,$$

and the terminal condition is

$$\tilde{v}(T, y) = -h(y) = -\frac{1}{2}(y^T + L^T)G^{-1}(y + L).$$

1.4.2 Finding the Optimal Controls

To find the supremum, we set the derivatives with respect to $\alpha, \beta_1, \dots, \beta_d$ to zero. We have

$$D_\alpha[\mathcal{L}^{\alpha, \beta_i}[\tilde{v}(t, y)] - \phi] = D_y[\tilde{v}] - \tilde{Q}\alpha - \tilde{S}^T \left(B^T y + \sum_{i=1}^d D_i^T \beta_i \right) = 0 \quad (57)$$

$$D_{\beta_i}[\mathcal{L}^{\alpha, \beta_i}[\tilde{v}(t, y)] - \phi] = -C_i D_y[\tilde{v}] + D_y^2[\tilde{v}] \beta_i - D_i \left(\tilde{S} \alpha + \tilde{R} \left(B^T y + \sum_{i=1}^d D_i^T \beta_i \right) \right) = 0 \quad (58)$$

This is a system of $d+1$ equations in $d+1$ unknowns, so it can be solved and the optimal controls are linear functions of y , which we denote by α^* and β_i^* . The HJB equation then becomes

$$\frac{\partial \tilde{v}}{\partial t} + \left(\alpha^{*T} - Y^T A - \sum_{i=1}^d \beta_i^{*T} C_i \right) D_y[\tilde{v}] + \frac{1}{2} \sum_{i=1}^d \beta_i^{*T} D_y^2[\tilde{v}] \beta_i^* - \phi \left(t, \alpha^*, B^T y + \sum_{i=1}^d D_i^T \beta_i^* \right) = 0 \quad (59)$$

1.4.3 Solving the Dual HJB

Suppose that \tilde{v} is a quadratic function in y and use the ansatz

$$\tilde{v}(t, y) = \frac{1}{2} y^T \tilde{P}(t) y + y^T \tilde{M}(t) + \tilde{N}(t), \quad (60)$$

with terminal conditions

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = -\frac{1}{2} L^T(T)G^{-1}(T)L(T). \quad (61)$$

Then

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t, y) &= \frac{1}{2} y^T \frac{d\tilde{P}}{dt}(t) y + y^T \dot{\tilde{M}}(t) + \dot{\tilde{N}}(t) \\ D_y[v(\tilde{t}, y)] &= \tilde{P}(t)y + \tilde{M}(t) \\ D_y^2[v(\tilde{t}, y)] &= \tilde{P}(t) \end{aligned}$$

The system of equations (57) and (58) from which we derive the optimal controls α^* and β_i^* is now given by

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}\alpha - \tilde{S}^T \left(B^T y + \sum_{i=1}^d D_i^T \beta_i \right) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}\beta_i + D_i \left(\tilde{S}\alpha + \tilde{R} \left(B^T y + \sum_{i=1}^d D_i^T \beta_i \right) \right) = 0 \end{cases} \quad (62)$$

We do not solve this system explicitly, however, the solutions for α^* and β_i^* are linear in y , hence we simply denote by $\tilde{\vartheta}$ and $\tilde{\kappa}$ the coefficients before y and the free coefficient in α^* and similarly for β_i , i.e.

$$\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}, \quad \beta_i = \tilde{\vartheta}_i y + \tilde{\kappa}_i. \quad (63)$$

Substituting this into the HJB equation (59) we get

$$\frac{1}{2} y^T \frac{d\tilde{P}}{dt} y + y^T \frac{\tilde{M}}{dt} + \frac{\tilde{N}}{dt} + \left(y^T \tilde{\vartheta}^T + \tilde{\kappa}^T - y^T A - \sum_{i=1}^d (y^T \tilde{\vartheta}_i^T + \tilde{\kappa}_i^T) C_i \right) (\tilde{P}y + \tilde{M})$$

$$+\frac{1}{2}\sum_{i=1}^d(y^T\tilde{\mathfrak{s}}_i^T+\tilde{\kappa}_i^T)\tilde{P}(\tilde{\mathfrak{s}}_iy+\tilde{\kappa}_i)-\phi\left(t,\tilde{\mathfrak{s}}y+\tilde{\kappa},B^Ty+\sum_{i=1}^dD_i^T(\tilde{\mathfrak{s}}_iy+\tilde{\kappa}_i)\right)=0$$

Expanding $\phi\left(t,\tilde{\mathfrak{s}}y+\tilde{\kappa},B^Ty+\sum_{i=1}^dD_i^T(\tilde{\mathfrak{s}}_iy+\tilde{\kappa}_i)\right)$ we get

$$\begin{aligned}\phi(t,\alpha^*,\beta^*) &= \frac{1}{2}(\tilde{\mathfrak{s}}y+\tilde{\kappa})^T\tilde{Q}(\tilde{\mathfrak{s}}y+\tilde{\kappa})+(\tilde{\mathfrak{s}}y+\tilde{\kappa})^T\tilde{S}^T\left(B^Ty+\sum_{i=1}^dD_i^T(\tilde{\mathfrak{s}}_iy+\tilde{\kappa}_i)\right) \\ &\quad +\frac{1}{2}\left(B^Ty+\sum_{i=1}^dD_i^T(\tilde{\mathfrak{s}}_iy+\tilde{\kappa}_i)\right)^T\tilde{R}\left(B^Ty+\sum_{i=1}^dD_i^T(\tilde{\mathfrak{s}}_iy+\tilde{\kappa}_i)\right)\end{aligned}$$

Rearranging, we get

$$\begin{aligned}&\frac{1}{2}y^T\frac{d\tilde{P}}{dt}y+y^T\frac{d\tilde{M}}{dt}+\frac{d\tilde{N}}{dt}+y^T\tilde{\mathfrak{s}}^T\tilde{P}y+y^T\tilde{P}\tilde{\kappa}-y^TA\tilde{P}y-y^T\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TC_i\tilde{P}y-y^T\tilde{P}\sum_{i=1}^dC_i^T\tilde{\kappa}_i \\ &\quad +y^T\tilde{\mathfrak{s}}^T\tilde{M}+\tilde{\kappa}^T\tilde{M}-y^TA\tilde{M}-\sum_{i=1}^dy^T\tilde{\mathfrak{s}}_i^TC_i\tilde{M}-\sum_{i=1}^d\tilde{\kappa}_i^TC_i\tilde{M}+\frac{1}{2}y^T\sum_{i=1}^d\tilde{\mathfrak{s}}_i^T\tilde{P}\tilde{\mathfrak{s}}_iy \\ &\quad +y^T\sum_{i=1}^d\tilde{\mathfrak{s}}_i\tilde{P}\tilde{\kappa}_i+\frac{1}{2}\sum_{i=1}^d\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i-\frac{1}{2}y^T\tilde{\mathfrak{s}}^T\tilde{Q}\tilde{\mathfrak{s}}y-y^T\tilde{\mathfrak{s}}^T\tilde{Q}\tilde{\kappa}-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa}-y^T\tilde{\mathfrak{s}}^T\tilde{S}^T\left(B^T+\sum_{i=1}^dD_i^T\tilde{\mathfrak{s}}_i\right)y \\ &\quad -y^T\tilde{\mathfrak{s}}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i-y^T\left(B+\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TD_i\right)\tilde{S}\tilde{\kappa}-\tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i \\ &\quad -\frac{1}{2}y^T\left(B+\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TD_i\right)\tilde{R}\left(B^T+\sum_{i=1}^dD_i^T\tilde{\mathfrak{s}}_i\right)y-y^T\left(B+\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TD_i\right)\tilde{R}\sum_{i=1}^dD_i^T\tilde{\kappa}_i \\ &\quad -\frac{1}{2}\left(\sum_{i=1}^d\tilde{\kappa}_i^TD_i\right)\tilde{R}\left(\sum_{i=1}^dD_i^T\tilde{\kappa}_i\right)=0\end{aligned}$$

Grouping together the coefficients in front of y we get:

$$\begin{aligned}&y^T\left[\frac{1}{2}\frac{d\tilde{P}}{dt}+\tilde{\mathfrak{s}}^T\tilde{P}-A\tilde{P}-\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TC_i\tilde{P}+\frac{1}{2}\sum_{i=1}^d\tilde{\mathfrak{s}}_i^T\tilde{P}\tilde{\mathfrak{s}}_i-\frac{1}{2}\tilde{\mathfrak{s}}^T\tilde{Q}\tilde{\mathfrak{s}}-\tilde{\mathfrak{s}}^T\tilde{S}^T\left(B^T+\sum_{i=1}^dD_i^T\tilde{\mathfrak{s}}_i\right)\right. \\ &\quad \left.-\frac{1}{2}\left(B+\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TD_i\right)\tilde{R}\left(B^T+\sum_{i=1}^dD_i^T\tilde{\mathfrak{s}}_i\right)\right]y \\ &\quad +y^T\left[\frac{d\tilde{M}}{dt}+\tilde{P}\tilde{\kappa}-\tilde{P}\sum_{i=1}^dC_i^T\tilde{\kappa}_i+\tilde{\mathfrak{s}}^T\tilde{M}-A\tilde{M}-\sum_{i=1}^d\tilde{\mathfrak{s}}_i^TC_i\tilde{M}+\sum_{i=1}^d\tilde{\mathfrak{s}}_i\tilde{P}\tilde{\kappa}_i-\tilde{\mathfrak{s}}^T\tilde{Q}\tilde{\kappa}-\tilde{\mathfrak{s}}^T\tilde{S}^T\sum_{i=1}^dD_i^T\tilde{\kappa}_i\right.\end{aligned}$$

$$\begin{aligned}
& -\left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i\right) \tilde{S} \tilde{\kappa} - \left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i\right) \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i \Big] \\
& + \frac{d\tilde{N}}{dt} + \tilde{\kappa}^T \tilde{M} - \sum_{i=1}^d \tilde{\kappa}_i^T C_i \tilde{M} + \frac{1}{2} \sum_{i=1}^d \tilde{\kappa}_i^T \tilde{P} \tilde{\kappa}_i - \frac{1}{2} \tilde{\kappa}^T \tilde{Q} \tilde{\kappa} - \tilde{\kappa}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i \\
& - \frac{1}{2} \left(\sum_{i=1}^d \tilde{\kappa}_i^T D_i \right) \tilde{R} \left(\sum_{i=1}^d D_i^T \tilde{\kappa}_i \right) = 0
\end{aligned}$$

This equation must equal zero for all y , hence the coefficients in front of the quadratic term, as well as y^T and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\begin{aligned}
& \frac{1}{2} \frac{d\tilde{P}}{dt} + \tilde{\mathfrak{S}}^T \tilde{P} - A\tilde{P} - \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T C_i \tilde{P} + \frac{1}{2} \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T \tilde{P} \tilde{\mathfrak{S}}_i - \frac{1}{2} \tilde{\mathfrak{S}}^T \tilde{Q} \tilde{\mathfrak{S}} - \tilde{\mathfrak{S}}^T \tilde{S}^T \left(B^T + \sum_{i=1}^d D_i^T \tilde{\mathfrak{S}}_i \right) \\
& - \frac{1}{2} \left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i \right) \tilde{R} \left(B^T + \sum_{i=1}^d D_i^T \tilde{\mathfrak{S}}_i \right) = 0
\end{aligned} \tag{64}$$

$$\begin{aligned}
& \frac{d\tilde{M}}{dt} + \tilde{P} \tilde{\kappa} - \tilde{P} \sum_{i=1}^d C_i^T \tilde{\kappa}_i + \tilde{\mathfrak{S}}^T \tilde{M} - A\tilde{M} - \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T C_i \tilde{M} + \sum_{i=1}^d \tilde{\mathfrak{S}}_i \tilde{P} \tilde{\kappa}_i - \tilde{\mathfrak{S}}^T \tilde{Q} \tilde{\kappa} - \tilde{\mathfrak{S}}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i \\
& - \left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i \right) \tilde{S} \tilde{\kappa} - \left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i \right) \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i = 0
\end{aligned} \tag{65}$$

$$\begin{aligned}
& \frac{d\tilde{N}}{dt} + \tilde{\kappa}^T \tilde{M} - \sum_{i=1}^d \tilde{\kappa}_i^T C_i \tilde{M} + \frac{1}{2} \sum_{i=1}^d \tilde{\kappa}_i^T \tilde{P} \tilde{\kappa}_i - \frac{1}{2} \tilde{\kappa}^T \tilde{Q} \tilde{\kappa} - \tilde{\kappa}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i \\
& - \frac{1}{2} \left(\sum_{i=1}^d \tilde{\kappa}_i^T D_i \right) \tilde{R} \left(\sum_{i=1}^d D_i^T \tilde{\kappa}_i \right) = 0
\end{aligned} \tag{66}$$

where $\tilde{\mathfrak{S}}, \tilde{\kappa}, \tilde{\mathfrak{S}}_i$ and $\tilde{\kappa}_i$ satisfy the system (62):

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\mathfrak{S}}y + \kappa) - \tilde{S}^T \left(B^T y + \sum_{i=1}^d D_i^T (\tilde{\mathfrak{S}}_i y + \kappa_i) \right) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\mathfrak{S}}_i y + \kappa_i) + D_i \left(\tilde{S}(\tilde{\mathfrak{S}}y + \kappa) + \tilde{R} \left(B^T y + \sum_{i=1}^d D_i^T (\tilde{\mathfrak{S}}_i y + \kappa_i) \right) \right) = 0 \end{cases} \tag{67}$$

and the terminal conditions are given by:

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = -\frac{1}{2}L^T(T)G^{-1}(T)L(T). \tag{68}$$

1.5 Dual BSDE

1.5.1 Solution via the Dual BSDE

The Hamiltonian $\tilde{\mathcal{H}} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^{nd} \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ for the dual problem is defined as

$$\begin{aligned} \tilde{\mathcal{H}}(t, Y, \alpha, \beta_1, \dots, \beta_d, p, q) &= -\phi\left(t, \alpha, B^T Y + \sum_{i=1}^d D_i^T \beta_i\right) + p^T (\alpha - A^T Y - \sum_{i=1}^d C_i^T \beta_i) + \sum_{i=1}^d \beta_i^T q_i \\ &= p^T \alpha - p^T A^T Y - p^T \sum_{i=1}^d C_i^T \beta_i + \sum_{i=1}^d \beta_i^T q_i - \phi\left(t, \alpha, B^T Y + \sum_{i=1}^d D_i^T \beta_i\right) \end{aligned} \quad (69)$$

The adjoint equation is given by the system

$$\begin{cases} dp(t) &= -D_y[\tilde{\mathcal{H}}]dt + \sum_{i=1}^d q_i dW_i \\ p(T) &= -D_y[h(Y(T))] = -G^{-1}Y(T) - G^{-1}L \end{cases} \quad (70)$$

Due to the Stochastic Maximum Principle, the optimal control can be found by setting $D_\alpha[\tilde{\mathcal{H}}] = 0$ and $D_{\beta_i}[\tilde{\mathcal{H}}] = 0$ for all $i = 1, \dots, d$, so we get the system

$$D_\alpha[\tilde{\mathcal{H}}] = p - \tilde{Q}\alpha - \tilde{S}^T\left(B^T Y + \sum_{i=1}^d D_i^T \beta_i\right) = 0 \quad (71)$$

$$D_{\beta_i}[\tilde{\mathcal{H}}] = q_i - C_i p - D_i \tilde{S} \alpha - D_i \tilde{R}\left(B^T Y + \sum_{i=1}^d D_i^T \beta_i\right) = 0 \quad (72)$$

There are $d+1$ equations in $d+1$ unknowns, so the system can be solved and the optimal controls are linear functions of Y , which we denote by

$$\alpha = \tilde{\vartheta}Y + \tilde{\kappa}, \quad \beta_i = \tilde{\vartheta}_i Y + \tilde{\kappa}_i, \quad i \in \{1, \dots, d\}$$

Substituting into the Hamiltonian (69) we get

$$\begin{aligned} \tilde{\mathcal{H}} &= p^T (\tilde{\vartheta}Y + \tilde{\kappa}) - p^T A^T Y - p^T \sum_{i=1}^d C_i^T (\tilde{\vartheta}_i Y + \tilde{\kappa}_i) + \sum_{i=1}^d (Y^T \tilde{\vartheta}_i^T + \tilde{\kappa}_i^T) q_i \\ &\quad - \phi\left(t, \tilde{\vartheta}Y + \tilde{\kappa}, B^T Y + \sum_{i=1}^d D_i^T (\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\right) \end{aligned}$$

The derivative of the dual Hamiltonian is then

$$D_y[\tilde{\mathcal{H}}] = \tilde{\vartheta}^T p - A p - \sum_{i=1}^d \tilde{\vartheta}_i^T C_i p + \sum_{i=1}^d \tilde{\vartheta}_i^T q_i - \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} Y - \tilde{\vartheta}^T \tilde{Q} \tilde{\kappa}$$

$$\begin{aligned}
& -2\tilde{\mathcal{S}}^T \tilde{\mathcal{S}} \left(B^T + \sum_{i=1}^d D_i^T \tilde{\mathcal{S}}_i \right) Y - \tilde{\mathcal{S}}^T \tilde{\mathcal{S}}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i - \left(B + \sum_{i=1}^d \tilde{\mathcal{S}}_i^T D_i \right) \tilde{\mathcal{S}} \tilde{\kappa} \\
& - \left(B + \sum_{i=1}^d \tilde{\mathcal{S}}_i^T D_i \right) \tilde{R} \left(B^T + \sum_{i=1}^d D_i^T \tilde{\mathcal{S}}_i \right) Y - \left(B^T + \sum_{i=1}^d D_i^T \tilde{\mathcal{S}}_i \right) \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i
\end{aligned} \tag{73}$$

We try the ansatz

$$p = \tilde{\varphi}(t)Y + \tilde{\psi}(t),$$

where $\varphi(t) \in \mathbb{R}^{n \times n}$ and $\psi(t) \in \mathbb{R}^n$. Applying Ito's formula to p , we get

$$\begin{aligned}
dp &= \left(\frac{d\tilde{\varphi}}{dt} Y + \frac{d\tilde{\psi}}{dt} \right) dt + \tilde{\varphi} dY \\
&= \left(\frac{d\tilde{\varphi}}{dt} Y + \frac{d\tilde{\psi}}{dt} \right) dt + \tilde{\varphi} \left[\alpha - A^T Y - \sum_{i=1}^d C_i^T \beta_i \right] dt + \tilde{\varphi} \sum_{i=1}^d \beta_i dW_i \\
&= \left[\frac{d\tilde{\varphi}}{dt} Y + \frac{d\tilde{\psi}}{dt} + \tilde{\varphi} \alpha - \tilde{\varphi} A^T Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \beta_i \right] dt + \tilde{\varphi} \sum_{i=1}^d \beta_i dW_i \\
&= \left[\frac{d\tilde{\varphi}}{dt} Y + \frac{d\tilde{\psi}}{dt} + \tilde{\varphi} \tilde{\mathcal{S}} Y + \tilde{\varphi} \tilde{\kappa} - \tilde{\varphi} A^T Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\mathcal{S}}_i Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\kappa}_i \right] dt + \tilde{\varphi} \sum_{i=1}^d (\tilde{\mathcal{S}}_i Y + \tilde{\kappa}_i) dW_i
\end{aligned} \tag{74}$$

Equating the coefficients of (74) and (70) we get

$$\frac{d\tilde{\varphi}}{dt} Y + \frac{d\tilde{\psi}}{dt} + \tilde{\varphi} \tilde{\mathcal{S}} Y + \tilde{\varphi} \tilde{\kappa} - \tilde{\varphi} A^T Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\mathcal{S}}_i Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\kappa}_i = -D_y[\tilde{\mathcal{H}}] \tag{75}$$

$$\tilde{\varphi} \tilde{\mathcal{S}}_i Y + \tilde{\varphi} \tilde{\kappa}_i = q_i \tag{76}$$

where the RHS of (75) is given by (73). We now substitute q_i from equation (76) into the system (71) and (72) we get

$$\begin{cases} \tilde{\varphi} Y + \tilde{\psi} - \tilde{Q}(\tilde{\mathcal{S}} Y + \tilde{\kappa}) - \tilde{\mathcal{S}}^T \left(B^T Y + \sum_{i=1}^d D_i^T (\tilde{\mathcal{S}}_i Y + \tilde{\kappa}_i) \right) = 0 \\ \tilde{\varphi} (\tilde{\mathcal{S}}_i Y + \tilde{\kappa}_i) - C_i (\tilde{\varphi} Y + \tilde{\psi}) - D_i \tilde{\mathcal{S}} (\tilde{\mathcal{S}} Y + \tilde{\kappa}) - D_i \tilde{R} \left(B^T Y + \sum_{i=1}^d D_i^T (\tilde{\mathcal{S}}_i Y + \tilde{\kappa}_i) \right) = 0 \end{cases} \tag{77}$$

which is the system we need to solve, to acquire the optimal controls $\alpha^* = \tilde{\mathcal{S}} Y + \tilde{\kappa}$ and $\beta_i^* = \tilde{\mathcal{S}}_i Y + \tilde{\kappa}_i$. Substituting q_i into (75) we get

$$\frac{d\tilde{\varphi}}{dt} Y + \frac{d\tilde{\psi}}{dt} + \tilde{\varphi} \tilde{\mathcal{S}} Y + \tilde{\varphi} \tilde{\kappa} - \tilde{\varphi} A^T Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\mathcal{S}}_i Y - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\kappa}_i = -\tilde{\mathcal{S}}^T p + A p + \sum_{i=1}^d \tilde{\mathcal{S}}_i^T C_i p$$

$$\begin{aligned}
& + \sum_{i=1}^d \tilde{\vartheta}_i^T (\tilde{\varphi} \tilde{\vartheta}_i Y + \tilde{\varphi} \tilde{\kappa}_i) + \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} Y + \tilde{\vartheta}^T \tilde{Q} \tilde{\kappa} + 2 \tilde{\vartheta}^T \tilde{S} \left(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i \right) Y + \tilde{\vartheta}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i \\
& + \left(B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i \right) \tilde{S} \tilde{\kappa} + \left(B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i \right) \tilde{R} \left(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i \right) Y + \left(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i \right) \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i
\end{aligned} \tag{78}$$

We rewrite equation (78) as

$$\left[\frac{d\tilde{\varphi}}{dt} + \tilde{\varphi} \tilde{\vartheta} - \tilde{\varphi} A^T - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\vartheta}_i + \tilde{\vartheta}^T \tilde{\varphi} - A \tilde{\varphi} - \sum_{i=1}^d \tilde{\vartheta}_i C_i \tilde{\varphi} - \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{\varphi} \tilde{\vartheta}_i - \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} \right. \tag{79}$$

$$\left. - 2 \tilde{\vartheta}^T \tilde{S} (B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i) - (B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i) \tilde{R} (B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i) \right] Y \tag{80}$$

$$+ \left[\frac{d\tilde{\psi}}{dt} + \tilde{\varphi} \tilde{\kappa} - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\kappa}_i + \tilde{\vartheta}^T \tilde{\psi} - A \tilde{\psi} - \sum_{i=1}^d \tilde{\vartheta}_i^T C_i \tilde{\psi} - \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{\varphi} \tilde{\kappa}_i \right. \tag{81}$$

$$\left. - \tilde{\vartheta}^T \tilde{Q} \tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i - (B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i) (\tilde{S} \tilde{\kappa} + \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i) \right] = 0 \tag{82}$$

Since this must be true for all Y , the coefficient in front of Y must be equal to zero, so we get

$$\begin{aligned}
& \frac{d\tilde{\varphi}}{dt} + \tilde{\varphi} \tilde{\vartheta} - \tilde{\varphi} A^T - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\vartheta}_i + \tilde{\vartheta}^T \tilde{\varphi} - A \tilde{\varphi} - \sum_{i=1}^d \tilde{\vartheta}_i C_i \tilde{\varphi} - \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{\varphi} \tilde{\vartheta}_i - \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} \\
& - 2 \tilde{\vartheta}^T \tilde{S} (B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i) - (B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i) \tilde{R} (B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i) = 0
\end{aligned} \tag{83}$$

$$\begin{aligned}
& \frac{d\tilde{\psi}}{dt} + \tilde{\varphi} \tilde{\kappa} - \tilde{\varphi} \sum_{i=1}^d C_i^T \tilde{\kappa}_i + \tilde{\vartheta}^T \tilde{\psi} - A \tilde{\psi} - \sum_{i=1}^d \tilde{\vartheta}_i^T C_i \tilde{\psi} - \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{\varphi} \tilde{\kappa}_i \\
& - \tilde{\vartheta}^T \tilde{Q} \tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i - (B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i) (\tilde{S} \tilde{\kappa} + \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i) = 0,
\end{aligned} \tag{84}$$

where $\tilde{\vartheta}$, $\tilde{\kappa}$, $\tilde{\vartheta}_i$, and $\tilde{\kappa}_i$ can be found from the system (77):

$$\begin{cases} \tilde{\varphi} Y + \tilde{\psi} - \tilde{Q} (\tilde{\vartheta} Y + \tilde{\kappa}) - \tilde{S}^T \left(B^T Y + \sum_{i=1}^d D_i^T (\tilde{\vartheta}_i Y + \tilde{\kappa}_i) \right) = 0 \\ \tilde{\varphi} (\tilde{\vartheta}_i Y + \tilde{\kappa}_i) - C_i (\tilde{\varphi} Y + \tilde{\psi}) - D_i \tilde{S} (\tilde{\vartheta} Y + \tilde{\kappa}) - D_i \tilde{R} \left(B^T Y + \sum_{i=1}^d D_i^T (\tilde{\vartheta}_i Y + \tilde{\kappa}_i) \right) = 0 \end{cases}$$

and the terminal conditions are given by

$$\tilde{\varphi}(T) = -G^{-1}(T), \quad \tilde{\psi}(T) = -G^{-1}(T)L(T). \tag{85}$$

1.5.2 Equivalence between Dual HJB and Dual BSDE

From the dual HJB we get that the optimal controls $\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}$ and $\beta^* = \tilde{\vartheta}_i y + \tilde{\kappa}_i$ are solution to the system of equations (62):

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\vartheta}y + \tilde{\kappa}) - \tilde{S}^T(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\vartheta}_i y + \tilde{\kappa}_i) + D_i \tilde{S}(\tilde{\vartheta}y + \tilde{\kappa}) + D_i \tilde{R}\left(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\right) = 0 \end{cases}$$

Similarly, the optimal controls from the dual BSDE method are found by solving the system (77):

$$\begin{cases} \tilde{\varphi}Y + \tilde{\psi} - \tilde{Q}(\tilde{\vartheta}Y + \tilde{\kappa}) - \tilde{S}^T\left(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\right) = 0 \\ \tilde{\varphi}(\tilde{\vartheta}_i Y + \tilde{\kappa}_i) - C_i(\tilde{\varphi}Y + \tilde{\psi}) - D_i \tilde{S}(\tilde{\vartheta}Y + \tilde{\kappa}) - D_i \tilde{R}\left(B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\right) = 0 \end{cases}$$

These systems are the same and therefore we get the relation

$$\tilde{\varphi} = \tilde{P}, \quad \tilde{\psi} = \tilde{M}.$$

The first ODE from the dual BSDE is (83), so substituting $\tilde{P} = \tilde{\varphi}$ in it we get

$$\begin{aligned} \frac{d\tilde{P}}{dt} + \tilde{P}\tilde{\vartheta} - \tilde{P}A^T - \tilde{P}\sum_{i=1}^d C_i^T \tilde{\vartheta}_i + \tilde{\vartheta}^T \tilde{P} - A\tilde{P} - \sum_{i=1}^d \tilde{\vartheta}_i C_i \tilde{P} - \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{P} \tilde{\vartheta}_i - \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} \\ - 2\tilde{\vartheta}^T \tilde{S}(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i) - (B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i) \tilde{R}(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i) = 0 \end{aligned}$$

The first ODE from the dual HJB equation is given by (64):

$$\begin{aligned} \frac{1}{2} \frac{d\tilde{P}}{dt} + \tilde{\vartheta}^T \tilde{P} - A\tilde{P} - \sum_{i=1}^d \tilde{\vartheta}_i^T C_i \tilde{P} + \frac{1}{2} \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{P} \tilde{\vartheta}_i - \frac{1}{2} \tilde{\vartheta}^T \tilde{Q} \tilde{\vartheta} - \tilde{\vartheta}^T \tilde{S}^T \left(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i\right) \\ - \frac{1}{2} \left(B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i\right) \tilde{R} \left(B^T + \sum_{i=1}^d D_i^T \tilde{\vartheta}_i\right) = 0 \end{aligned}$$

The two equations are equivalent with the second being the first one divided by 2. Similarly, plugging in $\tilde{P} = \tilde{\varphi}$ and $\tilde{M} = \tilde{\psi}$ in (84) we get

$$\begin{aligned} \frac{d\tilde{M}}{dt} + \tilde{P}\tilde{\kappa} - \tilde{P}\sum_{i=1}^d C_i^T \tilde{\kappa}_i + \tilde{\vartheta}^T \tilde{M} - A\tilde{M} - \sum_{i=1}^d \tilde{\vartheta}_i^T C_i \tilde{M} - \sum_{i=1}^d \tilde{\vartheta}_i^T \tilde{P} \tilde{\kappa}_i - \tilde{\vartheta}^T \tilde{Q} \tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i \\ - (B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i)(\tilde{S} \tilde{\kappa} + \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i) = 0 \end{aligned}$$

The respective ODE from the dual HJB is (65):

$$\begin{aligned} \frac{d\tilde{M}}{dt} + \tilde{P}\tilde{\kappa} - \tilde{P} \sum_{i=1}^d C_i^T \tilde{\kappa}_i + \tilde{\mathfrak{S}}^T \tilde{M} - A\tilde{M} - \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T C_i \tilde{M} + \sum_{i=1}^d \tilde{\mathfrak{S}}_i a \tilde{\kappa}_i - \tilde{\mathfrak{S}}^T \tilde{Q} \tilde{\kappa} - \tilde{\mathfrak{S}}^T \tilde{\mathfrak{S}}^T \sum_{i=1}^d D_i^T \tilde{\kappa}_i \\ - \left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i \right) \tilde{S} \tilde{\kappa} - \left(B + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i \right) \tilde{R} \sum_{i=1}^d D_i^T \tilde{\kappa}_i = 0 \end{aligned}$$

The terminal conditions from the dual HJB method are given by (68):

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = \frac{1}{2}L^T(T)G^{-1}(T)L(T)$$

and from the dual BSDE, we have (85):

$$\tilde{\varphi}(T) = -G^{-1}(T), \quad \tilde{\psi}(T) = -G^{-1}(T)L(T).$$

As we can see the equations are identical, and their terminal conditions are also the same, so the two methods are equivalent.

1.6 Primal and Dual Equivalence

Recall that we have the relation between the dual and primal value functions (11):

$$v(x) \leq \inf_y \{x^T y - \tilde{v}(y)\}$$

In this section, we impose no constraints on the control, and we will show that this leads to equality instead of inequality in the above equation. Substituting the respective ansatz, we have

$$\frac{1}{2}x^T P x + x^T M + N = \inf_y \left\{ x^T y - \frac{1}{2}y^T \tilde{P} y - y^T \tilde{M} - \tilde{N} \right\}$$

Setting the derivative of the RHS to zero, we get

$$y = \tilde{P}^{-1}(x - \tilde{M}),$$

so

$$\frac{1}{2}x^T P x + x^T M + N = -\frac{1}{2}(x^T - \tilde{M}^T)\tilde{P}^{-1}(x - \tilde{M}) - (x^T - \tilde{M}^T)\tilde{P}^{-1}\tilde{M} - \tilde{N} + x^T \tilde{P}^{-1}(x - \tilde{M})$$

Simplifying we get

$$\frac{1}{2}x^T P x + x^T M + N = \frac{1}{2}x^T \tilde{P}^{-1} x - x^T \tilde{P}^{-1} \tilde{M} + \frac{1}{2}\tilde{M}^T \tilde{P}^{-1} \tilde{M} - \tilde{N}.$$

Therefore, we get the relation

$$P = \tilde{P}^{-1}, \quad M = -\tilde{P}^{-1}\tilde{M}, \quad N = \frac{1}{2}\tilde{M}^T\tilde{P}^{-1}\tilde{M} - \tilde{N}. \quad (86)$$

To simplify computations, we consider a simpler case where $d = 1$ and $S = 0$. Then $\tilde{S} = 0, \tilde{Q} = Q^{-1}, \tilde{R} = R^{-1}$. The Riccati equation from the primal problem in this case is given by (35)

$$\dot{P} + 2PA + 2PB\vartheta_1 + (C_1^T + \vartheta_1^T D_1^T)P(C_1 + D_1\vartheta_1) - Q - \vartheta_1^T R\vartheta_1 = 0,$$

where

$$\vartheta_1 = (D_1^T P D_1 - R)^{-1}(-B^T P - D_1^T P C_1)$$

Therefore, our equation becomes

$$\begin{aligned} 0 &= \frac{dP}{dt} + 2PA - Q + C_1^T P C_1 + 2(PB + C_1^T P D_1)\vartheta_1 + \vartheta_1^T (D_1^T P D_1 - R)\vartheta_1 \\ &= \frac{dP}{dt} + 2PA - Q + C_1^T P C_1 - (PB + C_1^T P D_1)(D_1^T P D_1 - R)^{-1}(B^T P + D_1^T P C_1) \end{aligned}$$

Substituting $P = \tilde{P}^{-1}$, we get

$$\tilde{P}^{-1} \frac{d\tilde{P}}{dt} \tilde{P}^{-1} - 2\tilde{P}^{-1}A + Q - C_1^T \tilde{P}^{-1}C_1 + (\tilde{P}^{-1}B + C_1^T \tilde{P}^{-1}D_1)(D_1^T \tilde{P}^{-1}D_1 - R)^{-1}(B^T \tilde{P}^{-1} + D_1^T \tilde{P}^{-1}C_1) = 0$$

Multiplying on the left and on the right by \tilde{P} , we get

$$\frac{d\tilde{P}}{dt} - 2A\tilde{P} + \tilde{P}Q\tilde{P} - \tilde{P}C_1^T \tilde{P}^{-1}C_1\tilde{P} + \tilde{P}(\tilde{P}^{-1}B + C_1^T \tilde{P}^{-1}D_1)(D_1^T \tilde{P}^{-1}D_1 - R)^{-1}(B^T \tilde{P}^{-1} + D_1^T \tilde{P}^{-1}C_1)\tilde{P} = 0$$

Rewriting this, we get

$$\begin{aligned} \frac{d\tilde{P}}{dt} - 2A\tilde{P} + \tilde{P}Q\tilde{P} + \tilde{P}C_1^T (\tilde{P}^{-1}D_1(D_1^T \tilde{P}D_1 - R)^{-1}D_1^T \tilde{P}^{-1} - \tilde{P}^{-1})C_1\tilde{P} + B(D_1^T \tilde{P}D_1 - R)^{-1}B^T \\ + 2B(D_1^T \tilde{P}D_1 - R)^{-1}D_1^T \tilde{P}^{-1}C_1\tilde{P} = 0 \end{aligned} \quad (87)$$

On the other hand, for the dual problem we have:

$$\tilde{\vartheta} = Q\tilde{P}, \quad \tilde{\vartheta}_1 = (\tilde{P} - D_1 R^{-1} D_1^T)^{-1}(C_1 \tilde{P} + D_1 R^{-1} B^T)$$

The dual Riccati equation is (64):

$$\frac{d\tilde{P}}{dt} + 2\tilde{\vartheta}^T \tilde{P} - 2A\tilde{P} - 2\tilde{\vartheta}_1^T C_1 \tilde{P} + \tilde{\vartheta}_1^T \tilde{P} \tilde{\vartheta}_1 - \tilde{\vartheta}^T Q^{-1} \tilde{\vartheta} - (B + \tilde{\vartheta}_1^T D_1)R^{-1}(B^T + D_1^T \tilde{\vartheta}_1) = 0$$

Substituting for $\tilde{\mathfrak{s}}$ we get

$$\frac{d\tilde{P}}{dt} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T \tilde{\mathfrak{s}}_1 + \tilde{\mathfrak{s}}_1^T \tilde{P} \tilde{\mathfrak{s}}_1 - (B + \tilde{\mathfrak{s}}_1^T D_1)R^{-1}(B^T + D_1^T \tilde{\mathfrak{s}}_1) = 0$$

Substituting for $\tilde{\mathfrak{s}}_1$ we get

$$\begin{aligned} 0 &= \frac{d\tilde{P}}{dt} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T \tilde{\mathfrak{s}}_1 + \tilde{\mathfrak{s}}_1^T \tilde{P} \tilde{\mathfrak{s}}_1 - (B + \tilde{\mathfrak{s}}_1^T D_1)R^{-1}(B^T + D_1^T \tilde{\mathfrak{s}}_1) \\ &= \frac{d\tilde{P}}{dt} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T \tilde{\mathfrak{s}}_1 + \tilde{\mathfrak{s}}_1^T \tilde{P} \tilde{\mathfrak{s}}_1 - BR^{-1}B^T - 2BR^{-1}D_1^T \tilde{\mathfrak{s}}_1 - \tilde{\mathfrak{s}}_1^T D_1 R^{-1} D_1^T \tilde{\mathfrak{s}}_1 \\ &= \frac{d\tilde{P}}{dt} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - BR^{-1}B^T - 2(BR^{-1}D_1^T + \tilde{P}C_1^T)\tilde{\mathfrak{s}}_1 + \tilde{\mathfrak{s}}_1^T (\tilde{P} - D_1 R^{-1} D_1^T) \tilde{\mathfrak{s}}_1 \\ &= \frac{d\tilde{P}}{dt} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - BR^{-1}B^T - (BR^{-1}D_1^T + \tilde{P}C_1^T)(\tilde{P} - D_1 R^{-1} D_1^T)^{-1}(C_1 \tilde{P} + D_1 R^{-1} B^T) \end{aligned}$$

This is then rewritten as

$$\begin{aligned} \frac{d\tilde{P}}{dt} - 2A\tilde{P} + \tilde{P}Q\tilde{P} + \tilde{P}C_1^T (D_1 R^{-1} D_1^T - \tilde{P})^{-1} C_1 \tilde{P} + 2BR^{-1}D_1^T (D_1 R^{-1} D_1^T - \tilde{P})^{-1} C_1 \tilde{P} \\ + B(R^{-1}D_1^T (D_1 R^{-1} D_1^T - \tilde{P})^{-1} D_1 R^{-1} - R^{-1})B^T = 0 \quad (88) \end{aligned}$$

Now noting that

$$\begin{aligned} (D_1 R^{-1} D_1^T - \tilde{P})^{-1} &= \tilde{P}^{-1} D_1 (D_1^T \tilde{P} D_1 - R)^{-1} D_1^T \tilde{P}^{-1} - \tilde{P}^{-1} \\ R^{-1} D_1^T (D_1 R^{-1} D_1^T - \tilde{P})^{-1} D_1 R^{-1} - R^{-1} &= D_1^T \tilde{P} D_1 - R \\ R^{-1} D_1^T (D_1 R^{-1} D_1^T - \tilde{P})^{-1} &= (D_1^T \tilde{P} D_1 - R)^{-1} D_1^T \tilde{P}^{-1} \end{aligned}$$

we get exactly (87), so the dual and primal HJB methods are equivalent.

2 Unconstrained Markovian Optimisation with Quadratic Running Cost

The development of this section draws upon the study "Weak Necessary and Sufficient SMP for Markovian Regime-Switching Diffusion Models" [7]. This research elucidates the stochastic maximum principle for Markovian regime-switching diffusion models. We further consider the PDE method using the Hamilton-Jacobi-Bellman equation described in [8]. We integrate these concepts into the framework of Chapter 1, focusing specifically on the multidimensional linear quadratic unconstrained control problem. In this case, we are able to derive analytical solutions to our problem, as shown in the following sections.

2.1 Problem Description

2.1.1 Primal Problem

We now adapt the problem from the last section by introducing a Markov chain, independent of the d -dimensional Brownian motion. Let $\eta(t)$ be a continuous-time finite state observable Markov chain. Let the Markov chain take values in the state space $I = \{1, 2, \dots, k\}$ and start from an initial state $i_0 \in I$ with a $k \times k$ generator matrix $\mathcal{Q} = \{q_{ij}\}_{i,j=1}^k$. For each pair of distinct states (i, j) define the counting process $[\mathcal{Q}_{ij}] : \Omega \times [t_0, T] \rightarrow \mathbb{N}$ by

$$[\mathcal{Q}_{ij}](\omega, t) := \sum_{t_0 < s \leq t} \chi_{\{\eta(s-)=i\}}(\omega) \chi_{\{\eta(s)=j\}}(\omega), \quad \forall t \in [t_0, T],$$

and the compensator process $\langle \mathcal{Q}_{ij} \rangle : \Omega \times [t_0, T] \rightarrow [0, \infty)$ by

$$\langle \mathcal{Q}_{ij} \rangle(\omega, t) := q_{ij} \int_{t_0}^t \chi_{\{\eta(s-)=i\}}(\omega) ds, \quad \forall t \in [t_0, T].$$

The process

$$\mathcal{Q}_{ij}(\omega, t) := [\mathcal{Q}_{ij}](\omega, t) - \langle \mathcal{Q}_{ij} \rangle(\omega, t)$$

is a purely discontinuous square-integrable martingale with an initial value zero.

Let $W(t)$ be a d -dimensional Brownian motion and consider an n -dimensional process $X(t)$ described by

$$\begin{cases} dX(t) &= b(t, X(t), \pi(t), \eta(t-))dt + \sigma(t, x(t), \pi(t), \eta(t-))dW(t) \\ X(t_0) &= x_0 \in \mathbb{R}^n, \eta(0) = i_0 \in I \end{cases} \quad (89)$$

where $\pi(t) \in \mathbb{R}^m$ is the control and

$$b := A(t, \eta(t-))X(t) + B(t, \eta(t-))\pi(t) \in \mathbb{R}^n$$

$$\sigma := \left[\begin{pmatrix} C_1(t, \eta(t-))X(t) + D_1(t, \eta(t-))\pi(t) \end{pmatrix} \cdots \begin{pmatrix} C_d(t, \eta(t-))X(t) + D_d(t, \eta(t-))\pi(t) \end{pmatrix} \right] \in \mathbb{R}^{n \times d}$$

where $A, C_i \in \mathbb{R}^{n \times n}$, and $B, D_i \in \mathbb{R}^{n \times m}$, $i \in \{1, \dots, d\}$ are functions of both time and the Markov chain process.

The cost functional is given by

$$J(\pi) := \mathbb{E} \left[\int_{t_0}^T f(t, X(t), \pi(t), \eta(t)) dt + g(X(T), \eta(T)) \right], \quad (90)$$

where $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times I \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are given by

$$\begin{aligned} f(t, X(t), \pi(t), \eta(t)) &= \frac{1}{2} X^T(t) Q(t, \eta(t)) X(t) + X^T(t) S^T(t, \eta(t)) \pi(t) + \frac{1}{2} \pi^T(t) R(t, \eta(t)) \pi(t) \\ g(X(T), \eta(T)) &= \frac{1}{2} X^T(T) G(T, \eta(T)) X(T) + X^T(T) L(T, \eta(T)). \end{aligned}$$

The assumptions for this problem are the same to the ones in the previous section. We consider the following optimisation problem

$$\text{Minimise } J(\pi) \text{ subject to } (X, \pi) \text{ admissible.} \quad (91)$$

2.1.2 Dual Problem

We now derive the dual to the primal problem. First, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider the martingale $M : [t_0, T] \times \Omega \rightarrow \mathbb{R}^k$ generated by the Markov process $\eta(t)$:

$$M(t) := \eta(t) - \int_{t_0}^t Q^T \eta(s) ds,$$

where Q is the generating matrix of the Markov chain. Assume that the dual process $Y(t)$ follows

$$\begin{cases} dY(t) &= \left[\alpha(t) - A(t, \eta(t-))^T Y(t) - \sum_{j=1}^d C_j^T(t, \eta(t-)) \beta_j(t) \right] dt + \sum_{j=1}^d \beta_j(t) dW_j(t) + \sum_{j=1}^k \gamma_j(t) dM_j(t) \\ Y(t_0) &= y \end{cases} \quad (92)$$

where $\alpha(t) \in \mathcal{H}([t_0, T], \mathbb{R}^n)$, $\beta_j(t) \in \mathcal{H}([t_0, T], \mathbb{R}^n)$ and $\gamma_j(t) \in \mathcal{H}([t_0, T], \mathbb{R}^n)$ are the dual controls. Using Ito's lemma to $X(t)^T Y(t)$, we get

$$dX^T Y = \left[X^T \alpha + \pi^T \beta \right] dt + \sum_{j=1}^d \left(X^T \beta_j + Y^T (C_j X + D_j \pi) \right) dW_j,$$

where

$$\beta = B^T Y + \sum_{j=1}^d D_j^T \beta_j.$$

The process $X^T(t)Y(t) - \int_{t_0}^t [X^T(s)\alpha(s) + \pi^T(s)\beta(s)]ds$ is a local martingale and a super-martingale if it is bounded below by an integrable process, which gives

$$\mathbb{E} \left[X^T(T)Y(T) - \int_{t_0}^T (X^T \alpha + \pi^T \beta) ds \right] \leq x^T y. \quad (93)$$

The optimisation problem (91) can be written equivalently as

$$\sup_{\pi} \mathbb{E} \left[- \int_{t_0}^T f(t, X(t), \pi(t), \eta(t)) dt - g(X(T), \eta(T)) \right].$$

Define the dual functions $\phi : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times I \rightarrow \mathbb{R}$ by

$$\phi(t, \alpha, \beta, \eta(t)) = \sup_{x, \pi} \{ x^T \alpha + \pi^T \beta - f(t, x, \pi, \eta(t)) \} \quad (94)$$

and $h : \mathbb{R}^n \times I \rightarrow \mathbb{R}$ by

$$h(y, \eta(T)) = \sup_x \{ -x^T y - g(x, \eta(T)) \}. \quad (95)$$

Substituting f and g , we can find the supremums by setting the derivatives to zero. We get

$$\phi(t, \alpha, \beta, \eta(t)) = \sup_{x, \pi} \left\{ \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} \right\},$$

so setting the derivative to zero, we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0 \implies \begin{bmatrix} x^* \\ \pi^* \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Therefore

$$\begin{aligned} \pi^* &= [SQ^{-1}S^T - R]^{-1}(SQ^{-1}\alpha - \beta) \\ x^* &= Q^{-1}(\alpha - S^T \pi^*) \end{aligned}$$

Then ϕ is given by

$$\phi(t, \alpha, \beta, \eta(t)) = \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Denoting

$$\begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1},$$

where

$$\tilde{Q} = Q^{-1} - Q^{-1}S^T(SQ^{-1}S^T - R)^{-1}SQ^{-1} \quad (96)$$

$$\tilde{R} = R^{-1} - R^{-1}S(S^TR^{-1}S - Q^{-1})^{-1}S^TR^{-1} \quad (97)$$

$$\tilde{S} = (SQ^{-1}S^T - R)^{-1}SQ^{-1} = R^{-1}S(S^TR^{-1}S - Q)^{-1} \quad (98)$$

we get

$$\phi(t, \alpha, \beta, \eta(t)) = \frac{1}{2}\alpha^T \tilde{Q}(t, \eta(t))\alpha + \alpha^T \tilde{S}^T(t, \eta(t))\beta + \frac{1}{2}\beta^T \tilde{R}(t, \eta(t))\beta \quad (99)$$

Similarly,

$$D_x\left[-x^T y - \frac{1}{2}x^T Gx - x^T L\right] = -y - Gx - L \implies x^* = -G^{-1}(y + L).$$

Then $h(y, \eta(T))$ is given by

$$\begin{aligned} h(y, \eta(T)) &= (y^T + L^T)G^{-1}y - \frac{1}{2}(y^T + L^T)G^{-1}(y + L) + (y^T + L^T)G^{-1}L \\ &= \frac{1}{2}\left[y^T G^{-1}y + L^T G^{-1}y + y^T G^{-1}L + L^T G^{-1}L\right] \\ &= \frac{1}{2}(y^T + L^T)G^{-1}(y + L) \end{aligned} \quad (100)$$

Combining (93), (94), (10), we get the following inequality:

$$\sup_{\pi} \mathbb{E}\left[-\int_{t_0}^T f(t, X, \pi, \eta(t))dt - g(X(T), \eta(T))\right] \leq \inf_{y, \alpha, \beta_j} \left[x^T y + \mathbb{E}\left[\int_{t_0}^T \phi(t, \alpha, \beta, \eta(t))dt + h(Y(T), \eta(T))\right]\right] \quad (101)$$

The dual control problem is defined by

$$\inf_{y, \alpha, \beta_1, \dots, \beta_d} \left[x^T y + \mathbb{E}\left[\int_{t_0}^T \phi(t, \alpha, \beta, \eta(t))dt + h(Y(T), \eta(T))\right]\right], \quad (102)$$

This can be solved in two steps: first, for fixed y , solve a stochastic control problem

$$-\tilde{v}(t, y, \eta(t)) = \inf_{\alpha, \beta_1, \dots, \beta_d} \mathbb{E}\left[\int_{t_0}^T \phi(t, \alpha, \beta, \eta(t))dt + h(Y(T), \eta(T))\right], \quad (103)$$

and second, solve a static optimisation problem

$$\inf_y \{x^T y - \tilde{v}(t, y, \eta(t))\}.$$

2.2 Primal HJB Equation

2.2.1 Deriving the Primal HJB

We transform the minimisation problem (91) to maximisation by noting that

$$\inf_{\pi} \mathbb{E} \left[\int_{t_0}^T f(t, X(t), \pi(t), \eta(t)) dt + g(X(T), \eta(T)) \right] = - \sup_{\pi} \mathbb{E} \left[\int_{t_0}^T -f(t, X(t), \pi(t), \eta(t)) dt - g(X(T), \eta(T)) \right]$$

and denote the value function

$$v(t, X(t), \eta(t)) = \sup_{\pi} \mathbb{E} \left[\int_{t_0}^T -f(t, X(t), \pi(t), \eta(t)) dt - g(X(T), \eta(T)) \right] \quad (104)$$

The HJB equations is given by

$$\frac{\partial v}{\partial t}(t, x, i) + \sup_{\pi} \left[\mathcal{L}^{\pi}[v(t, x, i) - f(t, x, \pi, i)] \right] + \sum_{j \neq i}^k q_{ij}(v(t, x, j) - v(t, x, i)) = 0 \quad (105)$$

with terminal conditions

$$v(T, x, i) = -g(x, i) = -\frac{1}{2}x^T G(T, i)x + x^T L(T, i).$$

2.2.2 Finding the Optimal Control

The supremum can be found by setting the derivative with respect to π to zero. The derivative of the generator \mathcal{L}^{π} , $D_{\pi}[\mathcal{L}^{\pi}] \in \mathbb{R}^m$, is given by:

$$D_{\pi}[\mathcal{L}^{\pi}[v(t, x, i)]] = D_{\pi}[b(t, x, \pi, i)^T D_x[v(t, x, i)]] + D_{\pi}\left[\frac{1}{2}\text{tr}(\sigma(t, x, \pi, i)\sigma^T(t, x, \pi, i)D_x^2[v(t, x, i)])\right]. \quad (106)$$

We have that

$$\begin{aligned} D_{\pi}[b^T(t, x, \pi, i)D_x[v(t, x, i)]] &= D_{\pi}[(x^T A^T(t, i) + \pi^T(t)B^T(t, i))D_x[v(t, x, i)]] \\ &= B^T(t, i)D_x[v(t, x, i)] \end{aligned} \quad (107)$$

The latter derivative in (106) is given by:

$$\begin{aligned} D_{\pi}\left[\frac{1}{2}\text{tr}[\sigma(t, x, \pi, i)\sigma^T(t, x, \pi, i)D_x^2[v(t, x, i)]]\right] &= \frac{1}{2}D_{\pi}\left[\text{tr}\left[\sum_{j=1}^d (C_j x + D_j \pi)(C_j x + D_j \pi)^T D_x^2[v]\right]\right] \\ &= \frac{1}{2}\sum_{j=1}^d D_{\pi}\left[\text{tr}[(C_j x + D_j \pi)(C_j x + D_j \pi)^T D_x^2[v]]\right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{j=1}^d D_{\pi} \left[(C_j x + D_j \pi)^T D_x^2[v] (C_j x + D_j \pi) \right] \\
&= \sum_{j=1}^d D_j^T(t, i) D_x^2[v(t, x, i)] (C_j(t, i) x + D_j(t, i) \pi)
\end{aligned} \tag{108}$$

The derivative of $f(t, x, \pi, i)$ with respect to π is

$$D_{\pi} f(t, x, \pi, i) = S(t, i) x + R(t, i) \pi \tag{109}$$

Combining the three equations, (107), (108), (109), we get that

$$\begin{aligned}
D_{\pi} [\mathcal{L}^{\pi}(t)[v(t, x, i)] - f(t, x, \pi, i)] &= \sum_{j=1}^d D_j^T(t, i) D_x^2[v(t, x, i)] (C_j(t, i) x + D_j(t, i) \pi) \\
&\quad + B^T(t, i) D_x[v(t, x, i)] - S(t, i) x - R(t, i) \pi
\end{aligned}$$

The coefficients are all a function of time and the Markov chain process, however, for compactness, we write $S = S(t, i)$. Setting the derivative to zero, we get

$$\pi^* = \left[\sum_{j=1}^d D_j^T D_x^2[v(t, x, i)] D_j - R \right]^{-1} \left[Sx - B^T D_x[v(t, x, i)] - \sum_{j=1}^d D_j^T D_x^2[v(t, x, i)] C_j x \right] \tag{110}$$

We now substitute (110) into (105) to get:

$$\begin{aligned}
&\frac{\partial v}{\partial t}(t, x, i) + b(t, x, \pi^*, i)^T D_x[v(t, x, i)] + \frac{1}{2} \text{tr} \left[\sigma(t, x, \pi^*, i) \sigma^T(t, x, \pi^*, i) D_x^2[v(t, x, i)] \right] \\
&\quad - \frac{1}{2} x^T Q x - \frac{1}{2} x^T S^T \pi^* - \frac{1}{2} \pi^{*T} S x - \frac{1}{2} \pi^{*T} R \pi^* + \sum_{j \neq i} q_{ij} (v(t, x, j) - v(t, x, i)) = 0
\end{aligned}$$

As $D_x^2[v(t, x, i)]$ is a symmetric matrix, we can write

$$\begin{aligned}
\text{tr} \left[\sigma(t, x, \pi^*, i) \sigma^T(t, x, \pi^*, i) D_x^2[v(t, x, i)] \right] &= \sum_{j=1}^d \text{tr} \left[(C_j x + D_j \pi^*) (C_j x + D_j \pi^*)^T D_x^2[v(t, x, i)] \right] \\
&= \sum_{j=1}^d (C_j x + D_j \pi^*)^T D_x^2[v(t, x, i)] (C_j x + D_j \pi^*),
\end{aligned}$$

we get the Hamilton-Jacobi-Bellman equation

$$\frac{\partial v}{\partial t}(t, x, i) + (Ax + B\pi^*)^T D_x[v(t, x, i)] + \frac{1}{2} \sum_{j=1}^d (C_j x + D_j \pi^*)^T D_x^2[v(t, x, i)] (C_j x + D_j \pi^*)$$

$$-\frac{1}{2}x^T Qx - \frac{1}{2}x^T S^T \pi^* - \frac{1}{2}\pi^{*T} Sx - \frac{1}{2}\pi^{*T} R\pi^* + \sum_{j \neq i}^k q_{ij}(v(t, x, j) - v(t, x, i)) = 0 \quad (111)$$

where π^* is as in (110) and the terminal condition is given by

$$v(T, x, i) = -g(x, i) = -\frac{1}{2}x^T G(T, i)x - x^T L(T, i).$$

2.2.3 Solving the HJB equation

We solve (111) using the ansatz

$$v(t, x, i) = \frac{1}{2}x^T P(t, i)x + x^T M(t, i) + N(t, i) \quad (112)$$

with terminal conditions

$$P(T, i) = -G(T, i), \quad M(T, i) = -L(T, i), \quad N(T, i) = 0.$$

Then

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x, i) &= \frac{1}{2}x^T \frac{dP(t, i)}{dt}x + x^T \frac{dM(t, i)}{dt} + \frac{dN(t, i)}{dt} \\ D_x[v(t, x, i)] &= P(t, i)x + M(t, i) \\ D_x^2[v(t, x, i)] &= P(t, i) \end{aligned}$$

Substituting in (110) we get

$$\begin{aligned} \pi^* &= \left[\sum_{j=1}^d D_j^T(t, i)P(t, i)D_j(t, i) - R(t, i) \right]^{-1} \left[S(t, i)x - B^T(t, i)P(t, i)x - B^T(t, i)M(t, i) \right. \\ &\quad \left. - \sum_{j=1}^d D_j^T(t, i)P(t, i)C_j(t, i)x \right] \end{aligned} \quad (113)$$

We can write this as

$$\pi^* = \vartheta_1 x + \kappa_1,$$

where

$$\begin{aligned} \vartheta_1 &= \left(\sum_{j=1}^d D_j^T(t, i)P(t, i)D_j(t, i) - R(t, i) \right)^{-1} \left(S(t, i) - B^T(t, i)P(t, i) - \sum_{j=1}^d D_j^T(t, i)P(t, i)C_j(t, i) \right) \\ \kappa_1 &= - \left(\sum_{j=1}^d D_j^T(t, i)P(t, i)D_j(t, i) - R(t, i) \right)^{-1} B^T(t, i)M(t, i) \end{aligned}$$

Substituting into (111) we get

$$\begin{aligned}
& \frac{1}{2}x^T \frac{dP(t,i)}{dt}x + x^T \frac{dM(t,i)}{dt} + \frac{dN(t,i)}{dt} + (A(t,i)x + B(t,i)(\vartheta_1 x + \kappa_1))^T (P(t,i)x + M(t,i)) \\
& + \frac{1}{2} \sum_{j=1}^d (C_j(t,i)x + D_j(t,i)(\vartheta_1 x + \kappa_1))^T P(t,i) (C_j(t,i)x + D_j(t,i)(\vartheta_1 x + \kappa_1)) \\
& - \frac{1}{2}x^T Q(t,i)x - x^T S^T(t,i)(\vartheta_1 x + \kappa_1) - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T R(t,i)(\vartheta_1 x + \kappa_1) \\
& + \sum_{j \neq i}^k q_{ij} \left(\frac{1}{2}x^T (P(t,j) - P(t,i))x + x^T (M(t,j) - M(t,i)) + N(t,j) - N(t,i) \right) = 0
\end{aligned}$$

We rewrite this as

$$\begin{aligned}
& x^T \left[\frac{1}{2} \frac{dP(t,i)}{dt} + A^T(t,i)P(t,i) + \vartheta_1^T B^T(t,i)P(t,i) + \frac{1}{2} \sum_{j=1}^d (C_j(t,i) + D_j(t,i)\vartheta_1)^T P(t,i) (C_j(t,i) + D_j(t,i)\vartheta_1) \right. \\
& \quad \left. - \frac{1}{2}Q(t,i) - S^T(t,i)\vartheta_1 - \frac{1}{2}\vartheta_1^T R(t,i)\vartheta_1 + \frac{1}{2} \sum_{j \neq i}^k q_{ij}(P(t,j) - P(t,i)) \right] x \\
& + x^T \left[\frac{dM(t,i)}{dt} + A^T(t,i)M(t,i) + \vartheta_1^T B^T(t,i)M(t,i) + P(t,i)B(t,i)\kappa_1 \right. \\
& \quad \left. + \sum_{j=1}^d (C_j(t,i) + D_j(t,i)\vartheta_1)^T P(t,i)D_j\kappa_1 - S^T(t,i)\kappa_1 - \vartheta_1^T R(t,i)\kappa_1 + \sum_{j \neq i}^k q_{ij}(M(t,j) - M(t,i)) \right] \\
& + \frac{dN(t,i)}{dt} + \kappa_1^T M(t,i) + \frac{1}{2} \sum_{j=1}^d \kappa_1^T P(t,i)\kappa_1 - \frac{1}{2}\kappa_1^T R(t,i)\kappa_1 + \sum_{j \neq i}^k q_{ij}(N(t,j) - N(t,i)) = 0
\end{aligned}$$

This equation must equal zero for all x , hence the coefficients in front of the quadratic term, as well as x and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\begin{aligned}
& \frac{1}{2} \frac{dP(t,i)}{dt} + A^T(t,i)P(t,i) + \vartheta_1^T B^T(t,i)P(t,i) + \frac{1}{2} \sum_{j=1}^d (C_j(t,i) + D_j(t,i)\vartheta_1)^T P(t,i) (C_j(t,i) + D_j(t,i)\vartheta_1) \\
& - \frac{1}{2}Q(t,i) - S^T(t,i)\vartheta_1 - \frac{1}{2}\vartheta_1^T R(t,i)\vartheta_1 + \frac{1}{2} \sum_{j \neq i}^k q_{ij}(P(t,j) - P(t,i)) = 0
\end{aligned} \tag{114}$$

$$\frac{dM(t,i)}{dt} + A^T(t,i)M(t,i) + \vartheta_1^T B^T(t,i)M(t,i) + P(t,i)B(t,i)\kappa_1$$

$$+ \sum_{j=1}^d (C_j(t, i) + D_j(t, i) \vartheta_1)^T P(t, i) D_j \kappa_1 - S^T(t, i) \kappa_1 - \vartheta_1^T R(t, i) \kappa_1 + \sum_{j \neq i}^k q_{ij} (M(t, j) - M(t, i)) = 0 \quad (115)$$

$$+ \frac{dN(t, i)}{dt} + \kappa_1^T M(t, i) + \frac{1}{2} \sum_{j=1}^d \kappa_1^T P(t, i) \kappa_1 - \frac{1}{2} \kappa_1^T R(t, i) \kappa_1 + \sum_{j \neq i}^k q_{ij} (N(t, j) - N(t, i)) = 0 \quad (116)$$

with terminal conditions

$$P(T, i) = -G(T, i), \quad M(T, i) = -L(T, i), \quad N(T, i) = 0. \quad (117)$$

2.3 Primal BSDE

2.3.1 Solution via the Primal BSDE

Recall that the SDE (89) describing the state process is given by

$$\begin{cases} dX(t) &= b(t, X(t), \pi(t), \eta(t-))dt + \sigma(t, X(t), \pi(t), \eta(t-))dW(t) \\ X(t_0) &= x_0 \in \mathbb{R}^n, \quad \eta(0) = i_0 \in I \end{cases} \quad (118)$$

with cost functional (90)

$$J(\pi) := \mathbb{E} \left[\int_{t_0}^T f(t, X(t), \pi(t), \eta(t))dt + g(X(T), \eta(T)) \right]. \quad (119)$$

The Hamiltonian $\mathcal{H} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times I \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} \mathcal{H}(t, x, \pi, i, p, q) &:= -f(t, x, \pi, i) + b^T(t, x, \pi, i)p + \text{tr}(\sigma^T(t, x, \pi, i)q) \\ &= -\frac{1}{2}x^T Q(t, i)x - x^T S^T(t, i)\pi - \frac{1}{2}\pi^T R(t, i)\pi + (x^T A^T(t, i) + \pi^T B^T(t, i))p \\ &\quad + \sum_{j=1}^d (x^T C_j^T(t, i) + \pi^T D_j^T(t, i))q_j, \end{aligned} \quad (120)$$

where $q_j \in \mathbb{R}^n$ is the j^{th} column of $q \in \mathbb{R}^{n \times d}$. Given an admissible pair (x, π) , the adjoint equation in the unknown adapted processes $p(t), q(t)$ and $s(t) = (s^{(1)}(t), \dots, s^{(n)}(t))$, where $s^{(l)}(t) \in \mathbb{R}^{k \times k}$ for $l \in \{1, \dots, n\}$, is the following regime-switching BSDE:

$$\begin{cases} dp(t) &= -D_x[\mathcal{H}(t, X(t), \pi(t), \eta(t-), p(t), q(t))]dt + q(t)dW(t) + s(t) \cdot d\mathcal{Q}(t) \\ p(T) &= -D_x[g(X(T), \eta(T))] = -G(T, \eta(T))X(T) - L(T, \eta(T)) \end{cases} \quad (121)$$

where

$$s(t) \cdot d\mathcal{Q}(t) = \left(\sum_{j \neq i} s_{ij}^{(1)} d\mathcal{Q}_{ij}(t), \dots, s_{ij}^{(n)} d\mathcal{Q}_{ij}(t) \right)^T \quad (122)$$

We know that the optimal control maximises the Hamiltonian (120), that is, the derivative with respect to the control vanishes:

$$D_\pi[\mathcal{H}] = B^T(t, i)p + \sum_{i=1}^d D_i^T(t, i)q_i - S(t, i)X - R(t, i)\pi = 0 \quad (123)$$

From this, we know that the control π is linear in X , so it is of the form

$$\pi = \vartheta_2 X + \kappa_2, \quad (124)$$

where $\vartheta_2 \in \mathbb{R}^{m \times n}$ and $\kappa_2 \in \mathbb{R}^m$. Substituting the control in the Hamiltonian (120) we get

$$\begin{aligned} \mathcal{H} = & X^T A^T(t, i)p + (\vartheta_2 X + \kappa_2)^T B^T(t, i)p + \sum_{j=1}^d \left(X^T C_j^T(t, i)q_j + (\vartheta_2 X + \kappa_2)^T D_j^T(t, i)q_j \right) \\ & - \frac{1}{2} X^T Q(t, i)X - X^T S^T(t, i)(\vartheta_2 X + \kappa_2) - \frac{1}{2} (\vartheta_2 X + \kappa_2)^T R(t, i)(\vartheta_2 X + \kappa_2) \end{aligned} \quad (125)$$

The derivative of the Hamiltonian is then

$$\begin{aligned} D_x[\mathcal{H}] = & A^T(t, i)p + \vartheta_2^T B^T(t, i)p + \sum_{j=1}^d (C_j^T(t, i) + \vartheta_2^T D_j^T(t, i))q_j - Q(t, i)X \\ & - 2S^T(t, i)\vartheta_2 X - S^T(t, i)\kappa_2 - \vartheta_2^T R(t, i)\vartheta_2 X - \vartheta_2^T R(t, i)\kappa_2 \end{aligned} \quad (126)$$

We try an ansatz for p of the form:

$$p = \varphi(t, \eta(t))X(t) + \psi(t, \eta(t))$$

where $\varphi(t, \eta(t)) \in \mathbb{R}^{n \times n}$ and $\psi(t, \eta) \in \mathbb{R}^n$. Applying Ito's formula to $p = \varphi(t, \eta(t))X(t) + \psi(t, \eta(t))$, we have

$$\begin{aligned} dp = & \sum_{i=1}^k \chi_{\{\eta(t-)=i\}} \left[\left(\varphi(t, i)A(t, i) + \Delta\varphi(t, i) \right) X + \Delta\psi(t, i) + \varphi(t, i)B(t, i)(\vartheta_2 X + \kappa_2) \right] dt \\ & + \varphi(t, \eta(t-))\sigma(t, \eta(t-))dW \\ & + \sum_{i \neq j} \left[\left(\varphi(t, j) - \varphi(t, i) \right) X + \psi(t, j) - \psi(t, i) \right] dQ_{ij} \end{aligned}$$

where

$$\Delta\varphi(t, i) = \frac{\partial\varphi}{\partial t}(t, i) + \sum_{j=1}^k q_{ij}(\varphi(t, j) - \varphi(t, i))$$

$$\Delta\psi(t, i) = \frac{\partial\psi}{\partial t}(t, i) + \sum_{j=1}^k q_{ij}(\psi(t, j) - \psi(t, i))$$

Equating coefficients with (121) and setting $\eta(t-) = i$, we get

$$(\varphi(t, i)A(t, i) + \Delta\varphi(t, i))X + \Delta\psi(t, i) + \varphi(t, i)B(t, i)(\vartheta_2 X + \kappa_2) = -D_x[\mathcal{H}] \quad (127)$$

$$\varphi(t, i)\sigma(t, X(t), \pi(t), i) = q(t) \quad (128)$$

$$(\varphi(t, j) - \varphi(t, i))X + \psi(t, i) - \psi(t, j) = s_{ij}(t) \quad (129)$$

$$B^T(t, i)(\varphi X + \psi) + \sum_{j=1}^d D_j^T(t, i)q_j - S(t, i)X - R(t, i)(\vartheta_2 X + \kappa_2) = 0 \quad (130)$$

where the last equation is the Hamiltonian condition (123). We now substitute $q(t)$ from the second equation into the rest, and our system becomes

$$\begin{aligned} &(\varphi(t, i)A(t, i) + \Delta\varphi(t, i))X + \Delta\psi(t, i) + \varphi(t, i)B(t, i)(\vartheta_2 X + \kappa_2) = -A^T(t, i)(\varphi X + \psi) \\ & -\vartheta_2^T B^T(t, i)(\varphi X + \psi) - \sum_{j=1}^d (C_j^T(t, i) + \vartheta_2^T D_j^T(t, i))\varphi(C_j(t, i)X + D_j(t, i)(\vartheta_2 X + \kappa_2)) \\ & + Q(t, i)X + 2S^T(t, i)\vartheta_2 X + S^T(t, i)\kappa_2 + \vartheta_2^T R(t, i)\vartheta_2 X + \vartheta_2^T R(t, i)\kappa_2 \end{aligned} \quad (131)$$

$$(\varphi(t, j) - \varphi(t, i))X + \psi(t, i) - \psi(t, j) = s_{ij}(t) \quad (132)$$

$$\begin{aligned} &B^T(t, i)(\varphi X + \psi) + \sum_{j=1}^d D_j^T(t, i)\varphi(C_j(t, i)X + D_j(t, i)(\vartheta_2 X + \kappa_2)) \\ & - S(t, i)X - R(t, i)(\vartheta_2 X + \kappa_2) = 0 \end{aligned} \quad (133)$$

From (133) we find the optimal control $\pi^* = \vartheta_2 X + \kappa_2$:

$$\pi^* = \left[\sum_{j=1}^d D_j^T(t, i)\varphi D_j(t, i) - R(t, i) \right]^{-1} \left[S(t, i)X - \sum_{j=1}^d D_j^T(t, i)\varphi C_j(t, i)X - B^T(t, i)\varphi X - B^T(t, i)\psi \right] \quad (134)$$

i.e.,

$$\begin{aligned} \vartheta_2 &= \left[\sum_{j=1}^d D_j^T(t, i)\varphi D_j(t, i) - R(t, i) \right]^{-1} \left(S(t, i) - B^T(t, i)\varphi - \sum_{j=1}^d D_j^T(t, i)\varphi C_j(t, i) \right) \\ \kappa_2 &= - \left[\sum_{j=1}^d D_j^T(t, i)\varphi D_j(t, i) - R(t, i) \right]^{-1} B^T(t, i)\psi. \end{aligned}$$

We rewrite equation (131) as

$$\left[\frac{\partial\varphi}{\partial t}(t, i) + \sum_{j=1}^k q_{ij}(\varphi(t, j) - \varphi(t, i)) + \varphi(t, i)A(t, i) + A^T(t, i)\varphi + 2\vartheta_2^T B^T(t, i)\varphi \right.$$

$$\begin{aligned}
& + \sum_{j=1}^d (C_j^T(t, i) + \vartheta_2^T D_j^T(t, i)) \varphi(C_j(t, i) + D_j(t, i) \vartheta_2) - Q(t, i) - 2S^T(t, i) \vartheta_2 - \vartheta_2^T R(t, i) \vartheta_2 \Big] X \\
& + \left[\frac{\partial \psi}{\partial t}(t, i) + \sum_{j=1}^k q_{ij}(\psi(t, j) - \psi(t, i)) + \varphi(t, i) B(t, i) \kappa_2 + A^T(t, i) \psi + \vartheta_2^T B^T(t, i) \psi \right. \\
& \quad \left. + \sum_{j=1}^d (C_j^T(t, i) + \vartheta_2^T D_j^T(t, i)) \varphi D_j(t, i) \kappa_2 - S^T(t, i) \kappa_2 - \vartheta_2^T R(t, i) \kappa_2 \right] = 0
\end{aligned}$$

Since this must be true for all X , the coefficient in front of X must be equal to zero, so we get

$$\begin{aligned}
& \frac{\partial \varphi}{\partial t}(t, i) + 2A^T(t, i) \varphi(t, i) + 2\vartheta_2^T B^T(t, i) \varphi(t, i) \\
& + \sum_{j=1}^d (C_j^T(t, i) + \vartheta_2^T D_j^T(t, i)) \varphi(t, i) (C_j(t, i) + D_j(t, i) \vartheta_2) \\
& - Q(t, i) - 2S^T(t, i) \vartheta_2 - \vartheta_2^T R(t, i) \vartheta_2 + \sum_{j=1}^k q_{ij}(\varphi(t, j) - \varphi(t, i)) = 0 \tag{135}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i) B(t, i) \kappa_2 + A^T(t, i) \psi(t, i) + \vartheta_2^T B^T(t, i) \psi(t, i) \\
& + \sum_{j=1}^d (C_j^T(t, i) + \vartheta_2^T D_j^T(t, i)) \varphi(t, i) D_j(t, i) \kappa_2 \\
& - S^T(t, i) \kappa_2 - \vartheta_2^T R(t, i) \kappa_2 + \sum_{j=1}^k q_{ij}(\psi(t, j) - \psi(t, i)) = 0 \tag{136}
\end{aligned}$$

with terminal conditions

$$\varphi(T, i) = -G(T, i), \quad \psi(T, i) = -L(T, i). \tag{137}$$

2.3.2 Equivalence of Primal HJB and Primal BSDE

The optimal control from the primal HJB was given by (113):

$$\pi^* = \left[\sum_{j=1}^d D_j^T(t, i) P D_j(t, i) - R(t, i) \right]^{-1} \left[S(t, i) x - \sum_{j=1}^d D_j^T(t, i) P C_j(t, i) x - B^T(t, i) P x - B^T(t, i) M \right]$$

and from the primal BSDE (134)

$$\pi^* = \left[\sum_{j=1}^d D_j(t, i) \varphi D_j(t, i) - R(t, i) \right]^{-1} \left[S(t, i) X - \sum_{j=1}^d D_j^T(t, i) \varphi C_j(t, i) X - B^T(t, i) \varphi X - B^T(t, i) \psi \right]$$

Comparing, we see that the equations are the same and we get the relation

$$\varphi(t, i) = P(t, i), \quad \psi(t, i) = M(t, i),$$

so $\vartheta_1 = \vartheta_2$ and $\kappa_1 = \kappa_2$. The ODE from the primal HJB for $P(t, i)$ is given by (114):

$$\begin{aligned} \frac{1}{2} \frac{dP(t, i)}{dt} + A^T(t, i)P(t, i) + \vartheta_1^T B^T(t, i)P(t, i) + \frac{1}{2} \sum_{j=1}^d (C_j(t, i) + D_j(t, i)\vartheta_1)^T P(t, i)(C_j(t, i) + D_j(t, i)\vartheta_1) \\ - \frac{1}{2} Q(t, i) - S^T(t, i)\vartheta_1 - \frac{1}{2} \vartheta_1^T R(t, i)\vartheta_1 + \frac{1}{2} \sum_{j \neq i}^k q_{ij}(P(t, j) - P(t, i)) = 0 \end{aligned}$$

Letting $P(t, i) = \varphi(t, i)$ and $\vartheta_1 = \vartheta_2, \kappa_1 = \kappa_2$, and multiplying by 2 on both sides we get

$$\begin{aligned} \frac{d\varphi}{dt}(t, i) + 2A^T(t, i)\varphi(t, i) + 2\vartheta_1^T B^T(t, i)\varphi(t, i) + \sum_{j=1}^d (C_j(t, i) + D_j(t, i)\vartheta_1)^T \varphi(t, i)(C_j(t, i) + D_j(t, i)\vartheta_1) \\ - Q(t, i) - 2S^T(t, i)\vartheta_1 - \vartheta_1^T R(t, i)\vartheta_1 + \sum_{j \neq i}^k q_{ij}(\varphi(t, j) - \varphi(t, i)) = 0 \end{aligned}$$

which is the same ODE as the one from the primal BSDE (135).

Similarly, the ODE from the primal HJB for $M(t, i)$ is given by (115):

$$\begin{aligned} \frac{dM}{dt}(t, i) + A^T M(t, i) + \vartheta_1^T B^T M(t, i) + P(t, i)B\kappa_1 + \sum_{j=1}^d (C_j + D_j\vartheta_1)^T P(t, i)D_j\kappa_1 \\ - S^T \kappa_1 - \vartheta_1^T R\kappa_1 + \sum_{j \neq i}^k q_{ij}(M(t, j) - M(t, i)) = 0 \end{aligned}$$

Substituting $P(t, i) = \varphi(t, i), M(t, i) = \psi(t, i)$ we get

$$\begin{aligned} \frac{d\psi}{dt}(t, i) + A^T(t, i)\psi(t, i) + \vartheta_1^T B^T(t, i)\psi(t, i) + \varphi(t, i)B(t, i)\kappa_1 + \sum_{j=1}^d (C_j(t, i) + D_j(t, i)\vartheta_1)^T \varphi(t, i)D_j\kappa_1 \\ - S^T(t, i)\kappa_1 - \vartheta_1^T R(t, i)\kappa_1 + \sum_{j \neq i}^k q_{ij}(\psi(t, j) - \psi(t, i)) = 0 \end{aligned}$$

the same equations as the one from the primal BSDE (136). The terminal conditions from the primal HJB are (117)

$$P(T, i) = -G(T, i), \quad M(T, i) = -L(T, i), \quad N(T, i) = 0$$

and the terminal conditions from the primal BSDE are (137)

$$\varphi(T, i) = -G(T, i), \quad \psi(T, i) = -L(T, i). \quad (138)$$

As we can see, the terminal conditions are the same, so we can conclude that the two methods are equivalent.

2.4 Dual HJB Equation

2.4.1 The dual HJB

The dual HJB is given by

$$0 = \frac{\partial v}{\partial t}(t, y, i) + \sup_{\alpha, \beta, \gamma} \left\{ \left(\alpha - A^T y - \sum_{j=1}^d C_j^T \beta_j - \sum_{j=1}^k q_{ij}(\gamma_j - \gamma_i)^T \right) D_y[v(t, y, i)] + \frac{1}{2} \sum_{j=1}^d \beta_j^T D_y^2[v(t, y, i)] \beta_j \right. \\ \left. - \phi\left(t, \alpha, B^T y + \sum_{j=1}^d D_j^T \beta_j\right) + \sum_{j \neq i}^k q_{ij} v(t, y + \gamma_j - \gamma_i, j) \right\}$$

with terminal condition

$$v(T, y, i) = -h(y, i) = -\frac{1}{2}(y^T + L^T(T, i))G^{-1}(T, i)(y + L(T, i)).$$

2.4.2 Finding the optimal control

We find the optimal controls $\alpha, \beta_1, \dots, \beta_d$ by setting the derivatives with respect to α and β_j to zero:

$$D_y[\tilde{v}] - \tilde{Q}\alpha - \tilde{S}^T(B^T y + \sum_{i=1}^d D_i^T \beta_i) = 0 \quad (139)$$

$$-C_j D_y[\tilde{v}] + D_y^2[\tilde{v}] \beta_j - D_j(\tilde{S}\alpha + \tilde{R}(B^T y + \sum_{j=1}^d D_j^T \beta_j)) = 0 \quad (140)$$

This is a system of $d+1$ equations in $d+1$ unknowns, so it can be solved and the optimal controls are linear functions of y , which we denote by α^* and β_j^* . The HJB equation is then

$$0 = \frac{\partial v}{\partial t}(t, y, i) + \sup_{\gamma} \left\{ \left(\alpha^* - A^T y - \sum_{j=1}^d C_j^T \beta_j^* - \sum_{j=1}^k q_{ij}(\gamma_j - \gamma_i)^T \right) D_y[v(t, y, i)] \right. \\ \left. + \frac{1}{2} \sum_{j=1}^d \beta_j^{*T} D_y^2[v(t, y, i)] \beta_j^* - \phi\left(t, \alpha^*, B^T y + \sum_{j=1}^d D_j^T \beta_j^*\right) + \sum_{j \neq i}^k q_{ij} v(t, y + \gamma_j - \gamma_i, j) \right\} \quad (141)$$

2.4.3 Solving the dual HJB

Suppose that \tilde{v} is a quadratic function in y and use the ansatz

$$\tilde{v}(t, y, i) = \frac{1}{2} y^T \tilde{P}(t, i) y + y^T \tilde{M}(t, i) + \tilde{N}(t, i), \quad (142)$$

where $\tilde{P}(t, i) \in \mathbb{R}^{n \times n}$, $\tilde{M}(t, i) \in \mathbb{R}^n$, $\tilde{N}(t, i) \in \mathbb{R}$, with terminal conditions

$$\tilde{P}(T, i) = -G^{-1}(T, i), \tilde{M}(T, i) = -G^{-1}(T, i)L(T, i), \tilde{N}(T, i) = \frac{1}{2}L^T(T, i)G^{-1}(T, i)L(T, i). \quad (143)$$

Then

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t, y, i) &= \frac{1}{2}y^T \frac{d\tilde{P}}{dt}(t, i)y + y^T \frac{d\tilde{M}}{dt}(t, i) + \frac{d\tilde{N}}{dt}(t, i) \\ D_y[\tilde{v}(t, y, i)] &= \tilde{P}(t, i)y + \tilde{M}(t, i) \\ D_y^2[\tilde{v}(t, y, i)] &= \tilde{P}(t, i) \end{aligned}$$

The system of equations from which we derive the optimal controls α^* and β_i^* is now given by

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}\alpha - \tilde{S}^T(B^T Y + \sum_{i=1}^d D_i^T \beta_i) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}\beta_i + D_i(\tilde{S}\alpha + \tilde{R}(B^T y + \sum_{j=1}^d D_j^T \beta_j)) = 0 \end{cases} \quad (144)$$

We do not solve this system explicitly, however, the solutions for α^* and β^* are linear in y , hence we denote by $\tilde{\vartheta}$ and $\tilde{\kappa}$ the coefficients before y and the free coefficient in α^* and similarly for β_j , i.e.

$$\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}, \quad \beta_j = \tilde{\vartheta}_j y + \tilde{\kappa}_j. \quad (145)$$

Substituting this into the HJB (141) equation we get

$$\begin{aligned} & \frac{1}{2}y^T \frac{d\tilde{P}}{dt}(t, i)y + y^T \frac{d\tilde{M}}{dt}(t, i) + \frac{d\tilde{N}}{dt}(t, i) \\ & + \sup_{\gamma} \left\{ \left(y^T \tilde{\vartheta}^T + \tilde{\kappa}^T - y^T A - \sum_{j=1}^d (\tilde{\vartheta}_j y + \tilde{\kappa}_j)^T C_j^T - \sum_{j \neq i}^k q_{ij}(\gamma_j - \gamma_i)^T \right) (\tilde{P}(t, i)y + \tilde{M}(t, i)) \right. \\ & + \frac{1}{2} \sum_{j=1}^d (\tilde{\vartheta}_j y + \tilde{\kappa}_j)^T \tilde{P}(t, i) (\tilde{\vartheta}_j y + \tilde{\kappa}_j) - \phi\left(t, \tilde{\vartheta}y + \tilde{\kappa}, B^T y + \sum_{j=1}^d D_j^T (\tilde{\vartheta}_j y + \tilde{\kappa}_j)\right) \\ & + \sum_{j=1}^k q_{ij} \left(\frac{1}{2}y^T (\tilde{P}(t, j) - \tilde{P}(t, i))y + y^T (\tilde{M}(t, j) - \tilde{M}(t, i)) + \tilde{N}(t, j) - \tilde{N}(t, i) \right) \\ & \left. + \sum_{j=1}^k q_{ij} \left(\frac{1}{2}(\gamma_j - \gamma_i)^T \tilde{P}(t, j) (\gamma_j - \gamma_i) + (\gamma_j - \gamma_i)^T (\tilde{P}(t, j)y + \tilde{M}(t, j)) \right) \right\} = 0 \end{aligned}$$

Setting the derivative w.r.t. γ_j , $j \neq i$ to zero we get

$$(\gamma_j - \gamma_i) = -\tilde{P}^{-1}(t, j) \left[(\tilde{P}(t, j) - \tilde{P}(t, i))y + \tilde{M}(t, j) - \tilde{M}(t, i) \right]$$

The HJB is then

$$\frac{1}{2}y^T \frac{d\tilde{P}}{dt}(t, i)y + y^T \frac{d\tilde{M}}{dt}(t, i) + \frac{d\tilde{N}}{dt}(t, i) + \left(y^T \tilde{\vartheta}^T + \tilde{\kappa}^T - y^T A - \sum_{j=1}^d (\tilde{\vartheta}_j y + \tilde{\kappa}_j)^T C_j^T \right) (\tilde{P}(t, i)y + \tilde{M}(t, i))$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{j=1}^d (\tilde{\mathfrak{S}}_j y + \tilde{\kappa}_j)^T \tilde{P}(t, i) (\tilde{\mathfrak{S}}_j y + \tilde{\kappa}_j) - \phi(t, \tilde{\mathfrak{S}} y + \tilde{\kappa}, B^T y + \sum_{j=1}^d D_j^T (\tilde{\mathfrak{S}}_j y + \tilde{\kappa}_j), i) \\
& + \sum_{j \neq i}^k q_{ij} \left\{ \frac{1}{2} y^T (\tilde{P}(t, j) - \tilde{P}(t, i)) y + y^T (\tilde{M}(t, j) - \tilde{M}(t, i)) + \tilde{N}(t, j) - \tilde{N}(t, i) \right. \\
& \left. - \frac{1}{2} \left[y^T (\tilde{P}(t, j) - \tilde{P}(t, i)) + \tilde{M}^T(t, j) - \tilde{M}^T(t, i) \right] \tilde{P}^{-1}(t, j) \left[(\tilde{P}(t, j) - \tilde{P}(t, i)) y + \tilde{M}(t, j) - \tilde{M}(t, i) \right] \right\}
\end{aligned}$$

Rearranging, we get

$$\begin{aligned}
& y^T \left[\frac{1}{2} \frac{d\tilde{P}}{dt}(t, i) + (\tilde{\mathfrak{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathfrak{S}}_j^T C_j^T(t, i)) \tilde{P}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\mathfrak{S}}_j^T \tilde{P}(t, i) \tilde{\mathfrak{S}}_j - \frac{1}{2} \tilde{\mathfrak{S}}^T \tilde{Q}(t, i) \tilde{\mathfrak{S}} \right. \\
& - \tilde{\mathfrak{S}}^T \tilde{S}^T(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\mathfrak{S}}_i \right) - \frac{1}{2} \left(B(t, i) + \sum_{i=1}^d \tilde{\mathfrak{S}}_i^T D_i(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\mathfrak{S}}_i \right) \\
& \left. + \sum_{j \neq i}^k q_{ij} \left[\frac{1}{2} \tilde{P}(t, i) - \frac{1}{2} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) \tilde{P}(t, i) \right] y \right] \\
& + y^T \left[\frac{d\tilde{M}}{dt}(t, i) + \tilde{\mathfrak{S}}^T \tilde{M}(t, i) - A(t, i) \tilde{M}(t, i) + \tilde{P}(t, i) \tilde{\kappa} - \sum_{j=1}^d \tilde{\mathfrak{S}}_j^T C_j^T(t, i) \tilde{M}(t, i) + \sum_{j=1}^d \tilde{P}(t, i) C_j(t, i) \tilde{\kappa}_j \right. \\
& \tilde{\mathfrak{S}}_j^T \tilde{P}(t, i) \tilde{\kappa}_j - \tilde{\mathfrak{S}}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\mathfrak{S}}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathfrak{S}}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} \\
& \left. - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathfrak{S}}_j^T D_j(t, i) \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j + \sum_{j \neq i}^k q_{ij} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) \right] \\
& \frac{d\tilde{N}}{dt}(t, i) + \left(\tilde{\kappa}^T - \sum_{j=1}^d \tilde{\kappa}_j^T C_j^T(t, i) \right) \tilde{M}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\kappa}_j^T \tilde{P}(t, i) \tilde{\kappa}_j \\
& - \frac{1}{2} \tilde{\kappa}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\kappa}^T \tilde{S}^T \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \frac{1}{2} \left(\sum_{j=1}^d \tilde{\kappa}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(\sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j \right) \\
& + \sum_{j \neq i}^k q_{ij} \left[\tilde{N}(t, j) - \tilde{N}(t, i) - \frac{1}{2} (\tilde{M}^T(t, j) - \tilde{M}^T(t, i)) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) \right] = 0
\end{aligned}$$

This equation must equal zero for all y , hence the coefficients in front of the quadratic term, as well as y^T and the free coefficient must be zero. Setting the coefficients to

zero, we get the system

$$\begin{aligned}
& \frac{1}{2} \frac{d\tilde{P}}{dt}(t, i) + (\tilde{\mathcal{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j^T(t, i)) \tilde{P}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \tilde{P}(t, i) \tilde{\mathcal{S}}_j - \frac{1}{2} \tilde{\mathcal{S}}^T \tilde{Q}(t, i) \tilde{\mathcal{S}} \\
& - \tilde{\mathcal{S}}^T \tilde{S}^T(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\mathcal{S}}_i \right) - \frac{1}{2} \left(B(t, i) + \sum_{i=1}^d \tilde{\mathcal{S}}_i^T D_i(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\mathcal{S}}_i \right) \\
& + \sum_{j \neq i}^k q_{ij} \left[\frac{1}{2} \tilde{P}(t, i) - \frac{1}{2} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) \tilde{P}(t, i) \right] = 0
\end{aligned} \tag{146}$$

$$\begin{aligned}
& \frac{d\tilde{M}}{dt}(t, i) + \tilde{\mathcal{S}}^T \tilde{M}(t, i) - A(t, i) \tilde{M}(t, i) + \tilde{P}(t, i) \tilde{\kappa} - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j^T(t, i) \tilde{M}(t, i) + \sum_{j=1}^d \tilde{P}(t, i) C_j(t, i) \tilde{\kappa}_j \\
& \tilde{\mathcal{S}}_j^T \tilde{P}(t, i) \tilde{\kappa}_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\mathcal{S}}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} \\
& - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j + \sum_{j \neq i}^k q_{ij} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) = 0
\end{aligned} \tag{147}$$

$$\begin{aligned}
& \frac{d\tilde{N}}{dt}(t, i) + \left(\tilde{\kappa}^T - \sum_{j=1}^d \tilde{\kappa}_j^T C_j^T(t, i) \right) \tilde{M}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\kappa}_j^T \tilde{P}(t, i) \tilde{\kappa}_j \\
& - \frac{1}{2} \tilde{\kappa}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\kappa}^T \tilde{S}^T \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \frac{1}{2} \left(\sum_{j=1}^d \tilde{\kappa}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(\sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j \right) \\
& + \sum_{j \neq i}^k q_{ij} \left[\tilde{N}(t, j) - \tilde{N}(t, i) - \frac{1}{2} (\tilde{M}^T(t, j) - \tilde{M}^T(t, i)) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) \right] = 0
\end{aligned} \tag{148}$$

where $\tilde{\mathcal{S}}, \tilde{\kappa}, \tilde{\mathcal{S}}_j$ and $\tilde{\kappa}_j$ satisfy the system of equations (144):

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}\alpha - \tilde{S}^T(B^T Y + \sum_{i=1}^d D_i^T \beta_i) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}\beta_i + D_i(\tilde{S}\alpha + \tilde{R}(B^T y + \sum_{j=1}^d D_j^T \beta_j)) = 0 \end{cases}$$

and the terminal conditions are given by (143):

$$\tilde{P}(T, i) = -G^{-1}(T, i), \tilde{M}(T, i) = -G^{-1}(T, i)L(T, i), \tilde{N}(T, i) = \frac{1}{2}L^T(T, i)G^{-1}(T, i)L(T, i). \tag{149}$$

2.5 Dual BSDE

2.5.1 Solution via the Dual BSDE

Recall that the SDE (92) describing the state process is given by

$$dY = \left[\alpha(t) - A(t, \eta(t-))^T Y(t) - \sum_{j=1}^d C_j^T(t, \eta(t-)) \beta_j(t) \right] dt + \sum_{j=1}^d \beta_j(t) dW_j(t) + \sum_{j=1}^k \gamma_j(t) dM_j(t)$$

$$Y(t_0) = y$$

and the cost functional is

$$\inf_{\alpha, \beta_1, \dots, \beta_d} \mathbb{E} \left[\int_{t_0}^T \phi \left(t, \alpha, B^T(t, i)Y + \sum_{j=1}^d D_j^T(t, i) \beta_j, \eta(t) \right) dt + h(Y(T), \eta(T)) \right]$$

where ϕ is given in (99):

$$\phi(t, \alpha, \beta, i) = \frac{1}{2} \alpha^T \tilde{Q}(t, i) \alpha + \alpha^T \tilde{S}^T(t, i) \beta + \frac{1}{2} \beta^T \tilde{R}(t, i) \beta$$

and h is given in (100). The Hamiltonian $\tilde{\mathcal{H}} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^{n(d+1)} \times I \times \mathbb{R}^n \times \mathbb{R}^{nd} \times \mathbb{R}^{k \times nk} \rightarrow \mathbb{R}$ for the dual problem is defined as

$$\begin{aligned} \tilde{\mathcal{H}}(t, Y, \alpha, \beta_1, \dots, \beta_d, i, p, q, s) = & -\phi \left(t, \alpha, B^T(t, i)Y + \sum_{j=1}^d D_j^T(t, i) \beta_j \right) \\ & + p^T \left(\alpha - A^T(t, i)Y - \sum_{j=1}^d C_j^T(t, i) \beta_j \right) + \sum_{j=1}^d \beta_j^T q_j + \sum_{j=1}^k s_{ij} \gamma_j \end{aligned} \quad (150)$$

where $p(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n \times d}$ and $s(t) = (s^{(1)}(t), \dots, s^{(n)}(t))$, where $s^{(l)}(t) \in \mathbb{R}^{k \times k}$ for $l = 1, \dots, n$. The adjoint equations are given by the system

$$\begin{cases} dp &= -D_y[\tilde{\mathcal{H}}(t, Y, \alpha, \beta_1, \dots, \beta_d, \eta(t-), p, q, s)] dt + \sum_{j=1}^d q_j(t) dW_j(t) + s(t) \cdot dQ(t) \\ p(T) &= -D_y[h(Y(T), \eta(T))] = -G^{-1}(T, \eta(T))Y(T) - G^{-1}(T, \eta(T))L(T, \eta(T)) \end{cases} \quad (151)$$

where

$$s(t) \cdot dQ(t) = \left(\sum_{j \neq i} s_{ij}^{(1)}(t) dQ_{ij}(t), \dots, \sum_{j \neq i} s_{ij}^{(n)}(t) dQ_{ij}(t) \right)^T.$$

Due to the Stochastic Maximum Principle, the optimal controls are found by setting $D_\alpha[\tilde{\mathcal{H}}]$, $D_{\beta_j}[\tilde{\mathcal{H}}]$ and $D_{\gamma_j}[\tilde{\mathcal{H}}]$ to zero, so we get the system

$$D_\alpha[\tilde{\mathcal{H}}] = p - \tilde{Q}(t, i) \alpha - \tilde{S}^T(t, i) \left(B^T(t, i)Y + \sum_{j=1}^d D_j^T(t, i) \beta_j \right) = 0 \quad (152)$$

$$D_{\beta_j}[\tilde{\mathcal{H}}] = q_j - C_j(t, i)p - D_j(t, i)\tilde{S}(t, i)\alpha - D_j(t, i)\tilde{R}(t, i)\left(B^T(t, i)Y + \sum_{k=1}^d D_k^T(t, i)\beta_k\right) = 0 \quad (153)$$

$$D_{\gamma_j}[\tilde{\mathcal{H}}] = s_{ij} = 0 \quad (154)$$

Without the last condition, there are $d + 1$ equations in $d + 1$ unknowns, so we know that we can find a linear solution for the controls, which we denote as

$$\alpha^* = \tilde{\mathcal{S}}Y + \tilde{\kappa}, \quad \beta_j^* = \tilde{\mathcal{S}}_j Y + \tilde{\kappa}_j, \quad j \in \{1, \dots, d\}$$

Substituting into the Hamiltonian (150) we get

$$\begin{aligned} \tilde{\mathcal{H}} = & p^T(\tilde{\mathcal{S}}Y + \tilde{\kappa}) - p^T A^T(t, i)Y - p^T \sum_{j=1}^d C_j^T(t, i)(\tilde{\mathcal{S}}_j Y + \tilde{\kappa}_j) + \sum_{j=1}^d (Y^T \tilde{\mathcal{S}}_j^T + \tilde{\kappa}_j^T)q_j + \sum_{j=1}^k s_{ij}\gamma_j \\ & - \phi\left(t, \tilde{\mathcal{S}}Y + \tilde{\kappa}, B^T(t, i)Y + \sum_{j=1}^d D_j^T(t, i)(\tilde{\mathcal{S}}_j Y + \tilde{\kappa}_j)\right) \end{aligned}$$

The derivative of the Hamiltonian w.r.t. Y is then

$$\begin{aligned} D_Y[\tilde{\mathcal{H}}] = & \tilde{\mathcal{S}}^T p - A(t, i)p - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i)p + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T q_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i)\tilde{\mathcal{S}}Y - \tilde{\mathcal{S}}^T \tilde{Q}(t, i)\tilde{\kappa} \\ & - 2\tilde{\mathcal{S}}^T \tilde{S}(t, i)\left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j\right)Y - \tilde{\mathcal{S}}^T \tilde{S}^T(t, i)\sum_{j=1}^d D_j^T(t, i)\tilde{\kappa}_j \\ & - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i)\right)\tilde{S}(t, i)\tilde{\kappa} - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i)\right)\tilde{R}(t, i)\left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j\right)Y \\ & - \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j\right)\tilde{R}(t, i)\sum_{j=1}^d D_j^T(t, i)\tilde{\kappa}_j \end{aligned} \quad (155)$$

We try an ansatz for p of the form

$$p = \varphi(t, \eta(t))Y + \psi(t, \eta(t)).$$

where $\varphi(t, \eta(t)) \in \mathbb{R}^{n \times n}$ and $\psi(t, \eta(t)) \in \mathbb{R}^n$. Applying Ito's formula to $p = \varphi(t, \eta(t))Y(t) + \psi(t, \eta(t))$, we have

$$dp = \sum_{i=1}^k \chi_{\{\eta(t-) = i\}} \left[\frac{\partial \varphi}{\partial t}(t, i)Y - \varphi(t, i)A(t, i)^T Y + \frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i)\alpha - \varphi(t, i) \sum_{j=1}^d C_j^T(t, i)\beta_j \right]$$

$$\begin{aligned}
& -\varphi(t, i) \sum_{j=1}^k q_{ij}(\gamma_j - \gamma_i) \Big] dt + \sum_{j=1}^d \varphi(t, i) \beta_j dW_j \\
& + \sum_{i \neq j}^k \left[(\varphi(t, j) - \varphi(t, i))Y + \psi(t, i) - \psi(t, j) + \varphi(t, j)(\gamma_j - \gamma_i) \right] dQ_{ij}
\end{aligned} \tag{156}$$

Equating the coefficients of (156) and (151) and setting $\eta(t-) = i$ we get

$$\begin{aligned}
& \frac{\partial \varphi}{\partial t}(t, i)Y - \varphi(t, i)A(t, i)^T Y + \frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i)(\tilde{\mathcal{S}}Y + \tilde{\kappa}) - \varphi(t, i) \sum_{j=1}^d C_j^T(t, i)(\tilde{\mathcal{S}}_j Y + \tilde{\kappa}_j) \\
& - \varphi(t, i) \sum_{j=1}^k q_{ij}(\gamma_j - \gamma_i) = -D_y[\tilde{\mathcal{H}}]
\end{aligned} \tag{157}$$

$$\varphi(t, i)\tilde{\mathcal{S}}_j Y + \varphi(t, i)\tilde{\kappa}_j = q_j, \quad \forall j \in \{1, \dots, d\} \tag{158}$$

$$(\varphi(t, j) - \varphi(t, i))Y + \psi(t, i) - \psi(t, j) + \varphi(t, j)(\gamma_j - \gamma_i) = s_{ij}(t) \tag{159}$$

where the RHS of (157) is given by (155). From (159) and (154) we get

$$\gamma_j - \gamma_i = -\varphi(t, j)^{-1} \left((\varphi(t, j) - \varphi(t, i))Y + \psi(t, j) - \psi(t, i) \right) \tag{160}$$

We now substitute q_j from equation (158) and $\gamma_j - \gamma_i$ from (160) into (157) and we get

$$\begin{aligned}
& \frac{\partial \varphi}{\partial t}(t, i)Y - \varphi(t, i)A(t, i)^T Y + \frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i)(\tilde{\mathcal{S}}Y + \tilde{\kappa}) - \varphi(t, i) \sum_{j=1}^d C_j^T(t, i)(\tilde{\mathcal{S}}_j Y + \tilde{\kappa}_j) \\
& + \varphi(t, i) \sum_{j=1}^k q_{ij} \varphi(t, j)^{-1} \left((\varphi(t, j) - \varphi(t, i))Y + \psi(t, j) - \psi(t, i) \right) = \\
& (-\tilde{\mathcal{S}}^T + A(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i))(\varphi(t, i)Y + \psi(t, i)) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \varphi(t, i)(\tilde{\mathcal{S}}_j Y + \tilde{\kappa}_j) \\
& + \tilde{\mathcal{S}}^T \tilde{Q}(t, i)\tilde{\mathcal{S}}Y + \tilde{\mathcal{S}}^T \tilde{Q}(t, i)\tilde{\kappa} + 2\tilde{\mathcal{S}}^T \tilde{S}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j \right) Y + \tilde{\mathcal{S}}^T \tilde{S}(t, i) \sum_{j=1}^d D_j^T(t, i)\tilde{\kappa}_j \\
& + \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{S}(t, i)\tilde{\kappa} + \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j \right) Y \\
& + \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T(t, i)\tilde{\kappa}_j
\end{aligned} \tag{161}$$

We rewrite equation (161) as

$$\begin{aligned}
& \left[\frac{\partial \varphi}{\partial t}(t, i) + \varphi(t, i) \left(\tilde{\mathcal{S}} - A^T(t, i) - \sum_{j=1}^d C_j^T(t, i) \tilde{\mathcal{S}}_j \right) + \left(\tilde{\mathcal{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i) \right) \varphi(t, i) \right. \\
& \quad \left. + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \varphi(t, i) \tilde{\mathcal{S}}_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i) \tilde{\mathcal{S}} - 2 \tilde{\mathcal{S}}^T \tilde{S}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\mathcal{S}}_j \right) \right. \\
& \quad \left. - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\mathcal{S}}_j \right) + \sum_{j=1}^k q_{ij} (\varphi(t, i) - \varphi(t, i) \varphi^{-1}(t, j) \varphi(t, i)) \right] Y \\
& \quad + \left[\frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i) \tilde{\kappa} - \varphi(t, i) \sum_{j=1}^d C_j^T(t, i) \tilde{\kappa}_j + \left(\tilde{\mathcal{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i) \right) \psi(t, i) \right. \\
& \quad \left. + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \varphi(t, i) \tilde{\kappa}_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\mathcal{S}}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} \right. \\
& \quad \left. - \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\mathcal{S}}_j \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j + \sum_{j=1}^k q_{ij} \varphi(t, i) \varphi^{-1}(t, j) (\psi(t, j) - \psi(t, i)) \right] = 0
\end{aligned}$$

Since this must be true for all Y , the coefficient in front of Y must be equal to zero, so we get the ODEs

$$\begin{aligned}
& \frac{\partial \varphi}{\partial t}(t, i) + 2 \left(\tilde{\mathcal{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i) \right) \varphi(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \varphi(t, i) \tilde{\mathcal{S}}_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i) \tilde{\mathcal{S}} \\
& - 2 \tilde{\mathcal{S}}^T \tilde{S}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\mathcal{S}}_j \right) - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\mathcal{S}}_j \right) \\
& \quad + \sum_{j=1}^k q_{ij} (\varphi(t, i) - \varphi(t, i) \varphi^{-1}(t, j) \varphi(t, i)) = 0
\end{aligned} \tag{162}$$

$$\begin{aligned}
& \frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i) \tilde{\kappa} - \varphi(t, i) \sum_{j=1}^d C_j^T(t, i) \tilde{\kappa}_j + \left(\tilde{\mathcal{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i) \right) \psi(t, i) \\
& \quad + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \varphi(t, i) \tilde{\kappa}_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\mathcal{S}}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j \\
& \quad - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} - \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\mathcal{S}}_j \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j
\end{aligned}$$

$$+ \sum_{i=1}^k q_{ij} \varphi(t, i) \varphi^{-1}(t, j) (\psi(t, j) - \psi(t, i)) = 0, \quad (163)$$

with terminal conditions given by

$$\varphi(T, i) = -G^{-1}(T, i), \quad \psi(T, i) = -G^{-1}(T, i)L(T, i). \quad (164)$$

2.5.2 Equivalence of Dual HJB and Dual BSDE

From the dual HJB we get that the controls are given by $\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}$ and $\beta_j^* = \tilde{\vartheta}_j y + \tilde{\kappa}_j$, which can be computed from the system of equations (139) and (140):

$$\begin{aligned} \tilde{P}(t, i)y + \tilde{M}(t, i) - \tilde{Q}(t, i)\alpha - \tilde{S}^T(t, i) \left(B^T(t, i)y + \sum_{j=1}^d D_j^T(t, i)\beta_j \right) &= 0 \\ C_j(t, i)(\tilde{P}(t, i)y + \tilde{M}(t, i)) - \tilde{P}(t, i)\beta_j + D_j(t, i) \left(\tilde{S}(t, i)\alpha + \tilde{R}(t, i) \left(B^T(t, i)y + \sum_{j=1}^d D_j^T(t, i)\beta_j \right) \right) &= 0 \end{aligned}$$

Similarly, the system of equations from the dual BSDE are given by (152) and (153)

$$\begin{aligned} \varphi(t, i)Y + \psi(t, i) - \tilde{Q}(t, i)\alpha - \tilde{S}^T(t, i) \left(B^T(t, i)Y + \sum_{j=1}^d D_j^T(t, i)\beta_j \right) &= 0 \\ C_j(t, i)(\varphi(t, i)Y + \psi(t, i)) - \varphi(t, i)\beta_j + D_j(t, i) \left(\tilde{S}(t, i)\alpha + \tilde{R}(t, i) \left(B^T(t, i)Y + \sum_{k=1}^d D_k^T(t, i)\beta_k \right) \right) &= 0 \end{aligned}$$

These equations are the same, therefore we get the relation

$$\varphi(t, i) = \tilde{P}(t, i), \quad \psi(t, i) = \tilde{M}(t, i)$$

The first ODE from the dual HJB is (146):

$$\begin{aligned} \frac{1}{2} \frac{d\tilde{P}}{dt}(t, i) + (\tilde{\vartheta}^T - A(t, i) - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t, i))\tilde{P}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\vartheta}_j^T \tilde{P}(t, i) \tilde{\vartheta}_j - \frac{1}{2} \tilde{\vartheta}^T \tilde{Q}(t, i) \tilde{\vartheta} \\ - \tilde{\vartheta}^T \tilde{S}^T(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\vartheta}_i \right) - \frac{1}{2} \left(B(t, i) + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\vartheta}_i \right) \\ + \sum_{j \neq i}^k q_{ij} \left[\frac{1}{2} \tilde{P}(t, i) - \frac{1}{2} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) \tilde{P}(t, i) \right] = 0 \end{aligned}$$

Letting $\tilde{P}(t, i) = \varphi(t, i)$ and multiplying by 2 we get

$$\begin{aligned} & \frac{d\varphi}{dt}(t, i) + 2(\tilde{\vartheta}^T - A(t, i) - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t, i))\varphi(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T \varphi(t, i) \tilde{\vartheta}_j - \tilde{\vartheta}^T \tilde{Q}(t, i) \tilde{\vartheta} \\ & - 2\tilde{\vartheta}^T \tilde{S}^T(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\vartheta}_i \right) - \left(B(t, i) + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i) \tilde{\vartheta}_i \right) \\ & + \sum_{j \neq i}^k q_{ij} [\varphi(t, i) - \varphi(t, i) \varphi^{-1}(t, j) \varphi(t, i)] = 0 \end{aligned}$$

which is the same ODE as the first one from the dual BSDE (162):

$$\begin{aligned} & \frac{\partial \varphi}{\partial t}(t, i) + 2 \left(\tilde{\vartheta}^T - A(t, i) - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t, i) \right) \varphi(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T \varphi(t, i) \tilde{\vartheta}_j - \tilde{\vartheta}^T \tilde{Q}(t, i) \tilde{\vartheta} \\ & - 2\tilde{\vartheta}^T \tilde{S}^T(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\vartheta}_j \right) - \left(B(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i) \tilde{\vartheta}_j \right) \\ & + \sum_{j=1}^k q_{ij} (\varphi(t, j) - \varphi(t, i) \varphi^{-1}(t, j) \varphi(t, i)) = 0 \end{aligned}$$

The ODE for $\tilde{M}(t, i)$ from the dual HJB is given by (147):

$$\begin{aligned} & \frac{d\tilde{M}}{dt}(t, i) + \tilde{\vartheta}^T \tilde{M}(t, i) - A(t, i) \tilde{M}(t, i) + \tilde{P}(t, i) \tilde{\kappa} - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t, i) \tilde{M}(t, i) + \sum_{j=1}^d \tilde{P}(t, i) C_j(t, i) \tilde{\kappa}_j \\ & \sum_{j=1}^d \tilde{\vartheta}_j^T \tilde{P}(t, i) \tilde{\kappa}_j - \tilde{\vartheta}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \left(B(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} \\ & - \left(B(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T D_j(t, i) \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T \tilde{\kappa}_j + \sum_{j \neq i}^k q_{ij} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) = 0 \end{aligned}$$

Letting $\tilde{P} = \varphi$ and $\tilde{M} = \psi$ we get

$$\begin{aligned} & \frac{d\psi}{dt}(t, i) + \tilde{\vartheta}^T \psi(t, i) - A(t, i) \psi(t, i) + \varphi(t, i) \tilde{\kappa} - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t, i) \psi(t, i) + \sum_{j=1}^d \varphi(t, i) C_j(t, i) \tilde{\kappa}_j \\ & \sum_{j=1}^d \tilde{\vartheta}_j^T \varphi(t, i) \tilde{\kappa}_j - \tilde{\vartheta}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \left(B(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} \\ & - \left(B(t, i) + \sum_{j=1}^d \tilde{\vartheta}_j^T D_j(t, i) \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T \tilde{\kappa}_j + \sum_{j \neq i}^k q_{ij} \varphi(t, i) \varphi^{-1}(t, j) (\psi(t, j) - \psi(t, i)) = 0 \end{aligned}$$

which is the same ODE as the one from the dual BSDE (163):

$$\begin{aligned}
& \frac{\partial \psi}{\partial t}(t, i) + \varphi(t, i)\tilde{\kappa} - \varphi(t, i) \sum_{j=1}^d C_j^T(t, i)\tilde{\kappa}_j + \left(\tilde{\mathcal{S}}^T - A(t, i) - \sum_{j=1}^d \tilde{\mathcal{S}}_j^T C_j(t, i) \right) \psi(t, i) \\
& + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T \varphi(t, i)\tilde{\kappa}_j - \tilde{\mathcal{S}}^T \tilde{Q}(t, i)\tilde{\kappa} - \tilde{\mathcal{S}}^T \tilde{\mathcal{S}}^T(t, i) \sum_{j=1}^d D_j^T(t, i)\tilde{\kappa}_j \\
& - \left(B(t, i) + \sum_{j=1}^d \tilde{\mathcal{S}}_j^T D_j(t, i) \right) \tilde{\mathcal{S}}(t, i)\tilde{\kappa} - \left(B^T(t, i) + \sum_{j=1}^d D_j^T(t, i)\tilde{\mathcal{S}}_j \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T(t, i)\tilde{\kappa}_j \\
& + \sum_{i=1}^k q_{ij} \varphi(t, i) \varphi^{-1}(t, j) (\psi(t, j) - \psi(t, i)) = 0
\end{aligned}$$

The terminal conditions from the dual HJB are given by (149):

$$\tilde{P}(T, i) = -G^{-1}(T, i), \quad \tilde{M}(T, i) = -G^{-1}(T, i)L(T, i)$$

and from the dual BSDE, we have (164):

$$\varphi(T, i) = -G^{-1}(T, i), \quad \psi(T, i) = -G^{-1}(T, i)L(T, i).$$

As we can see the equations are identical, and their terminal conditions are also the same, so the two methods are equivalent.

2.6 Primal and Dual Equivalence

This section shows equivalence between the primal HJB and dual HJB solutions. Recall that we have the relation between the dual and primal value functions (101):

$$v(t, x, i) \leq \inf_y \{x^T y - \tilde{v}(t, y, i)\}$$

In this section, we impose no constraints on the control, and we will show that this leads to equality instead of inequality in the above equation. Substituting the respective ansatz, we have

$$\frac{1}{2}x^T P(t, i)x + x^T M(t, i) + N(t, i) = \inf_y \left\{ x^T y - \frac{1}{2}y^T \tilde{P}(t, i)y - y^T \tilde{M}(t, i) - \tilde{N}(t, i) \right\}$$

Setting the derivative of the RHS to zero, we get

$$y = \tilde{P}^{-1}(t, i)(x - \tilde{M}(t, i)),$$

so

$$\frac{1}{2}x^T P(t, i)x + x^T M(t, i) + N(t, i) = \frac{1}{2}x^T \tilde{P}^{-1}(t, i)x - x^T \tilde{P}^{-1}(t, i)\tilde{M}(t, i) + \frac{1}{2}\tilde{M}^T(t, i)\tilde{P}^{-1}(t, i)\tilde{M}(t, i) - \tilde{N}(t, i).$$

Therefore, we get the relation

$$P(t, i) = \tilde{P}^{-1}(t, i), \quad M(t, i) = -\tilde{P}^{-1}(t, i)\tilde{M}(t, i), \quad N(t, i) = \frac{1}{2}\tilde{M}^T(t, i)\tilde{P}^{-1}(t, i)\tilde{M}(t, i) - \tilde{N}(t, i). \quad (165)$$

In section 1.6 we have shown equivalence between the primal and dual HJB in the non-Markovian case. Now we work in a Markov setting, and the resulting ODEs from the primal and dual HJB methods are the same as the ones from the non-Markovian case, apart from the fact that the parameters now depend also on the Markov chain (not only on time) and there are also additional terms concerning the Markov chain. Therefore, we will only show here the equivalence between the additional terms, as the equivalence of the rest of the ODEs is already shown.

The system of ODEs from the primal HJB is given by (114), (115) and (116)

$$\begin{aligned} \frac{1}{2} \frac{dP(t, i)}{dt} + A^T(t, i)P(t, i) + \vartheta_1^T B^T(t, i)P(t, i) + \frac{1}{2} \sum_{j=1}^d (C_j(t, i) + D_j(t, i)\vartheta_1)^T P(t, i)(C_j(t, i) + D_j(t, i)\vartheta_1) \\ - \frac{1}{2} Q(t, i) - S^T(t, i)\vartheta_1 - \frac{1}{2} \vartheta_1^T R(t, i)\vartheta_1 + \frac{1}{2} \sum_{j \neq i}^k q_{ij}(P(t, j) - P(t, i)) = 0 \\ \frac{dM(t, i)}{dt} + A^T(t, i)M(t, i) + \vartheta_1^T B^T(t, i)M(t, i) + P(t, i)B(t, i)\kappa_1 \\ + \sum_{j=1}^d (C_j(t, i) + D_j(t, i)\vartheta_1)^T P(t, i)D_j\kappa_1 - S^T(t, i)\kappa_1 - \vartheta_1^T R(t, i)\kappa_1 + \sum_{j \neq i}^k q_{ij}(M(t, j) - M(t, i)) = 0 \\ + \frac{dN(t, i)}{dt} + \kappa_1^T M(t, i) + \frac{1}{2} \sum_{j=1}^d \kappa_1^T P(t, i)\kappa_1 - \frac{1}{2} \kappa_1^T R(t, i)\kappa_1 + \sum_{j \neq i}^k q_{ij}(N(t, j) - N(t, i)) = 0 \end{aligned}$$

with terminal conditions (117):

$$P(T, i) = -G(T, i), \quad M(T, i) = -L(T, i), \quad N(T, i) = 0.$$

Similarly, the ODEs from the dual HJB are given by (146), (147) and (148):

$$\begin{aligned} \frac{1}{2} \frac{d\tilde{P}}{dt}(t, i) + (\tilde{\vartheta}^T - A(t, i) - \sum_{j=1}^d \tilde{\vartheta}_j^T C_j^T(t, i))\tilde{P}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\vartheta}_j^T \tilde{P}(t, i)\tilde{\vartheta}_j - \frac{1}{2} \tilde{\vartheta}^T \tilde{Q}(t, i)\tilde{\vartheta} \\ - \tilde{\vartheta}^T \tilde{S}^T(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i)\tilde{\vartheta}_i \right) - \frac{1}{2} \left(B(t, i) + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i(t, i) \right) \tilde{R}(t, i) \left(B^T(t, i) + \sum_{i=1}^d D_i^T(t, i)\tilde{\vartheta}_i \right) \\ + \sum_{j \neq i}^k q_{ij} \left[\frac{1}{2} \tilde{P}(t, i) - \frac{1}{2} \tilde{P}(t, i)\tilde{P}^{-1}(t, j)\tilde{P}(t, i) \right] = 0 \end{aligned}$$

$$\begin{aligned}
& \frac{d\tilde{M}}{dt}(t, i) + \tilde{S}^T \tilde{M}(t, i) - A(t, i) \tilde{M}(t, i) + \tilde{P}(t, i) \tilde{\kappa} - \sum_{j=1}^d \tilde{S}_j^T C_j^T(t, i) \tilde{M}(t, i) + \sum_{j=1}^d \tilde{P}(t, i) C_j(t, i) \tilde{\kappa}_j \\
& \quad \tilde{S}_j^T \tilde{P}(t, i) \tilde{\kappa}_j - \tilde{S}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{S}^T \tilde{S}^T(t, i) \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \left(B(t, i) + \sum_{j=1}^d \tilde{S}_j^T D_j(t, i) \right) \tilde{S}(t, i) \tilde{\kappa} \\
& \quad - \left(B(t, i) + \sum_{j=1}^d \tilde{S}_j^T D_j(t, i) \right) \tilde{R}(t, i) \sum_{j=1}^d D_j^T \tilde{\kappa}_j + \sum_{j \neq i}^k q_{ij} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) = 0 \\
& \quad \frac{d\tilde{N}}{dt}(t, i) + \left(\tilde{\kappa}^T - \sum_{j=1}^d \tilde{\kappa}_j^T C_j^T(t, i) \right) \tilde{M}(t, i) + \frac{1}{2} \sum_{j=1}^d \tilde{\kappa}_j^T \tilde{P}(t, i) \tilde{\kappa}_j \\
& \quad - \frac{1}{2} \tilde{\kappa}^T \tilde{Q}(t, i) \tilde{\kappa} - \tilde{\kappa}^T \tilde{S}^T \sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j - \frac{1}{2} \left(\sum_{j=1}^d \tilde{\kappa}_j^T D_j(t, i) \right) \tilde{R}(t, i) \left(\sum_{j=1}^d D_j^T(t, i) \tilde{\kappa}_j \right) \\
& \quad + \sum_{j \neq i}^k q_{ij} \left[\tilde{N}(t, j) - \tilde{N}(t, i) - \frac{1}{2} (\tilde{M}^T(t, j) - \tilde{M}^T(t, i)) \tilde{P}^{-1}(t, j) (\tilde{M}(t, j) - \tilde{M}(t, i)) \right] = 0
\end{aligned}$$

where the terminal conditions are given by (149):

$$\tilde{P}(T, i) = -G^{-1}(T, i), \tilde{M}(T, i) = -G^{-1}(T, i) L(T, i), \tilde{N}(T, i) = \frac{1}{2} L^T(T, i) G^{-1}(T, i) L(T, i).$$

Using the relation (165) it is easy to check that the terminal conditions are equivalent. Now, for the first ODE of the primal HJB, the additional terms are

$$\frac{1}{2} \sum_{j \neq i}^k q_{ij} (P(t, j) - P(t, i))$$

When showing equivalence in section 1.6 we transformed the ODE from the primal by multiplying on both sides by $\tilde{P}(t, i)$. Doing so to the additional term becomes

$$\begin{aligned}
\frac{1}{2} \sum_{j \neq i}^k q_{ij} \tilde{P}(t, i) (P(t, j) - P(t, i)) \tilde{P}(t, i) &= \frac{1}{2} \sum_{j \neq i}^k q_{ij} \tilde{P}(t, i) (\tilde{P}^{-1}(t, j) - \tilde{P}^{-1}(t, i)) \tilde{P}(t, i) \\
&= \sum_{j \neq i}^k q_{ij} \left[\frac{1}{2} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) \tilde{P}(t, i) - \frac{1}{2} \tilde{P}(t, i) \right]
\end{aligned}$$

which is exactly the same as the Markov term from the dual HJB:

$$\sum_{j \neq i}^k q_{ij} \left[\frac{1}{2} \tilde{P}(t, i) \tilde{P}^{-1}(t, j) \tilde{P}(t, i) - \frac{1}{2} \tilde{P}(t, i) \right].$$

Similarly, one can check for the additional Markov terms in the ODE for $\tilde{M}(t, i)$.

3 Numerical Solutions

In this chapter, we solve the system of coupled ODEs which we derived as a solution to the non-Markov problem in Chapter 1, using the Runge-Kutta numerical scheme. Since the built-in libraries in Python using this method do not allow for matrix ODEs as in our case (as far as I saw), I have written the code to solve them from scratch, which can be found here [5].

So far, we have derived an analytical solution for both the Primal and Dual problems in the case of quadratic cost and no constraints on the control. These solutions are in the form of a system of three ODEs, which we will now solve numerically.

For simplicity, we will consider the case where $d = 1$, i.e. we work with one-dimensional Brownian motion. We first present some of the results we have derived in previous sections for the $d = 1$ case:

3.1 Primal Problem for 1-Dimensional Brownian Motion Case

Recall that the value function for the primal problem is given by

$$v(t, x) = \frac{1}{2}x^T Px + x^T M + N,$$

where $P(t), M(t)$ and $N(t)$ are solutions to the system of ODEs (35), (36), (37):

$$\begin{aligned} \frac{dP}{dt} + 2A^T P + 2PB\vartheta_1 + (C^T + \vartheta_1^T D^T)P(C + D\vartheta_1) - Q - 2S^T \vartheta_1 - \vartheta_1^T R \vartheta_1 &= 0 \\ \frac{dM}{dt} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M + (C^T + \vartheta_1^T D^T)PD\kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 &= 0 \\ \frac{dN}{dt} + \kappa_1^T B^T M + \frac{1}{2}\kappa_1^T D^T PD\kappa_1 - \frac{1}{2}\kappa_1^T R \kappa_1 &= 0, \end{aligned}$$

where ϑ_1 and κ_1 are given by (34):

$$\vartheta_1 = (D^T PD - R)^{-1} (S - B^T P - D^T PC), \quad \kappa_1 = -(D^T PD + R)^{-1} B^T M$$

and the terminal conditions are given by (32):

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0. \quad (166)$$

3.2 Dual Problem for 1-Dimensional Brownian Motion Case

The value function for the dual problem is given by

$$\tilde{v}(t, y) = \frac{1}{2}y^T \tilde{P}y + y^T \tilde{M} + \tilde{N},$$

where $\tilde{P}(t), \tilde{M}(t)$ and $\tilde{N}(t)$ are solutions to the system of ODEs (64), (65), (66) (in the $d = 1$ case):

$$\begin{aligned} \frac{d\tilde{P}}{dt} + 2\tilde{\mathfrak{S}}^T \tilde{P} - 2A\tilde{P} - 2\tilde{\mathfrak{S}}_1^T C\tilde{P} + \tilde{\mathfrak{S}}_1^T \tilde{P} \tilde{\mathfrak{S}}_1 - \tilde{\mathfrak{S}}^T \tilde{Q} \tilde{\mathfrak{S}} - 2\tilde{\mathfrak{S}}^T \tilde{S}^T (B^T + D^T \tilde{\mathfrak{S}}_1) - (B + \tilde{\mathfrak{S}}_1^T D) \tilde{R} (B^T + D^T \tilde{\mathfrak{S}}_1) &= 0 \\ \frac{d\tilde{M}}{dt} + \tilde{P} \tilde{\kappa} - \tilde{P} C^T \tilde{\kappa}_1 + \tilde{\mathfrak{S}}^T \tilde{M} - A\tilde{M} - \tilde{\mathfrak{S}}_1^T C\tilde{M} + \tilde{\mathfrak{S}}_1 \tilde{P} \tilde{\kappa}_1 - \tilde{\mathfrak{S}}^T \tilde{Q} \tilde{\kappa} - \tilde{\mathfrak{S}}^T \tilde{S}^T D^T \tilde{\kappa}_1 \\ - (B + \tilde{\mathfrak{S}}_1^T D) \tilde{S} \tilde{\kappa} - (B + \tilde{\mathfrak{S}}_1^T D) \tilde{R} D^T \tilde{\kappa}_1 &= 0 \\ \frac{d\tilde{N}}{dt} + \tilde{\kappa}^T \tilde{M} - \tilde{\kappa}_1^T C\tilde{M} + \frac{1}{2} \tilde{\kappa}_1^T \tilde{P} \tilde{\kappa}_1 - \frac{1}{2} \tilde{\kappa}^T \tilde{Q} \tilde{\kappa} - \tilde{\kappa}^T \tilde{S}^T D^T \tilde{\kappa}_1 - \frac{1}{2} \tilde{\kappa}_1^T D \tilde{R} D^T \tilde{\kappa}_1 &= 0 \end{aligned}$$

where $\tilde{\mathfrak{S}}, \tilde{\kappa}$ and $\tilde{\mathfrak{S}}_1, \tilde{\kappa}_1$ are solutions to the system (67)

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\mathfrak{S}}y + \kappa) - \tilde{S}^T B^T y - \tilde{S}^T D^T (\tilde{\mathfrak{S}}_1 y + \kappa_1) = 0 \\ C(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\mathfrak{S}}_1 y + \kappa_1) + D\tilde{S}(\tilde{\mathfrak{S}}y + \kappa) + D\tilde{R}B^T y + D\tilde{R}D^T (\tilde{\mathfrak{S}}_1 y + \kappa_1) = 0 \end{cases}$$

and the terminal conditions are given by (68):

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = -\frac{1}{2}L^T(T)G^{-1}(T)L(T).$$

3.3 Runge-Kutta Method

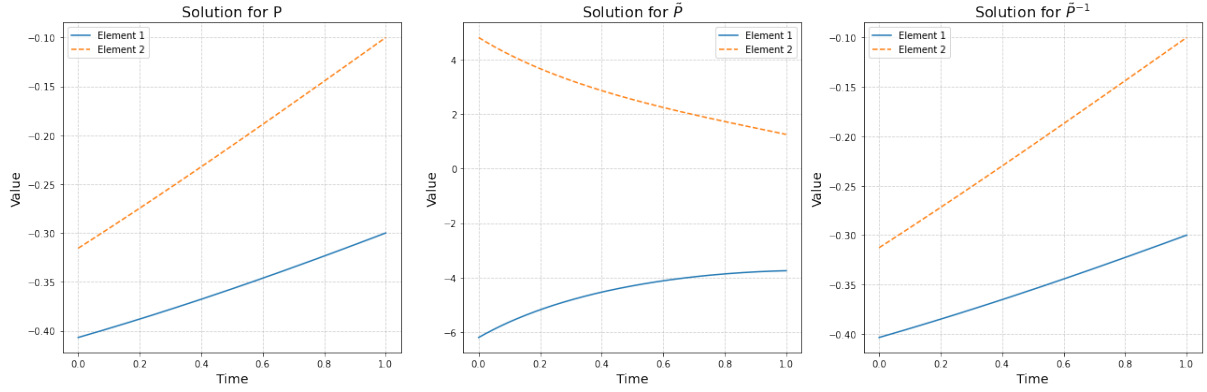
Next, we will describe the method employed to resolve the two systems of ordinary differential equations (ODEs). For ease of understanding, let's set $n = 2$, which implies that X exists within a two-dimensional space. We also assume the coefficients $A(t), B(t), C(t), D(t)$ to be time-independent, meaning they're constant, and we further take them to be symmetric.

Our initial focus lies on the equations for P and \tilde{P} . The solution to these ODEs is achieved via the Runge-Kutta method. It's important to highlight that in our situation, both P and \tilde{P} are represented as 2×2 matrices. We omit the details of the method, as it is a widely-used approach for solving such ODEs.

From (86) we know that the relationship between P and \tilde{P} is given by

$$P = \tilde{P}^{-1}.$$

As the matrices P and \tilde{P} are symmetric 2×2 matrices, we only show the solution for the 2 distinct elements (in our case we have simplified the coefficients so that we only get two distinct values in P and \tilde{P}). The results are shown below:

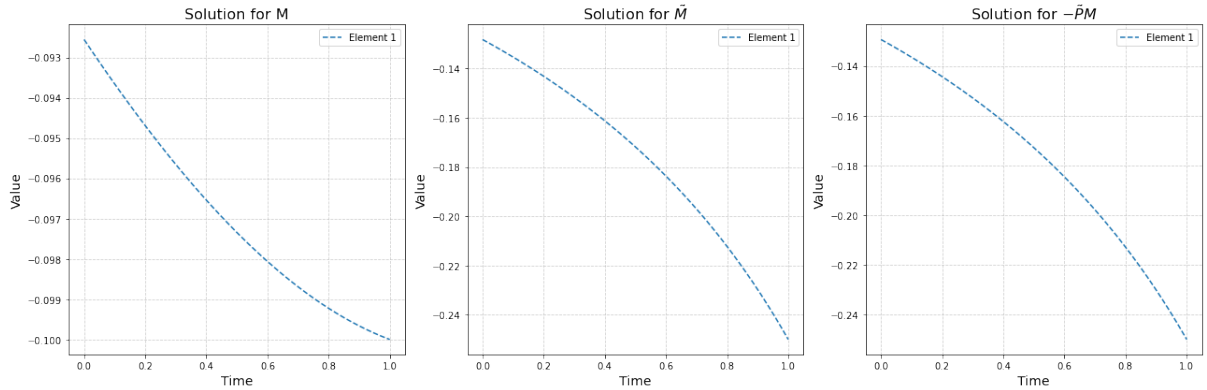


We are considering the time interval from 0 to 1 which we have divided into 200 time-steps. As we can see from the plots, the solutions for P and \tilde{P}^{-1} are identical, as expected from the analytically derived results.

After having obtained the solutions for P and \tilde{P} , we can then compute the solutions for M and \tilde{M} using the same method. Note that we needed to compute the ODEs for P and \tilde{P} first, as their solutions are part of the ODEs for the rest of the parameters. From (86) we know that the relationship between M and \tilde{M} is given by

$$\tilde{M} = -\tilde{P}M.$$

Our numerical solutions give us the following



As we can see, the last two plots are identical, confirming our analytical results. Very similar results can be shown for the ODEs for N and \tilde{N} .

While these results are a mere confirmation for the 1-dimensional Brownian motion case of our analytical results, we will make use of this method when we employ the BSDE Deep learning algorithm to this problem in chapter 5, which is essentially an alternative way to reach the optimal control. The Runge-Kutta solution will help us compare these analytical results with the BSDE method results.

If we wish to work in the setting of Chapter 2, i.e. the case where the coefficients depend on a continuous-time finite state space Markov chain, we simply need to consider multiple systems of ODEs corresponding to each state of the chain, which can be done using the same methods as here.

4 Constrained Optimisation with Convex Running Cost

In this section, we return to the framework of Chapter 1, however, now we impose constraints on the control and no longer consider quadratic cost functions. This general framework is as described in [9]. In this case, we no longer have an analytical solution to the problem, however, we can derive a useful dual-primal relationship, which will help us later with implementing the Deep BSDE algorithm.

4.1 Problem Description

4.1.1 Primal Problem

For simplicity, we consider the case with one-dimensional Brownian motion, or $d = 1$. The state process is then given by the following SDE:

$$\begin{cases} dX(t) &= [A(t)X(t) + B(t)\pi(t)]dt + [C(t)X(t) + D(t)\pi(t)]dW(t) \\ X(t_0) &= x_0 \in \mathbb{R}^n. \end{cases} \quad (167)$$

So far, we have not imposed any constraints on the control $\pi(t)$, i.e. we had $\pi(t) \in \mathbb{R}^m$. Now, we consider the case where $\pi(t) \in K \subset \mathbb{R}^m$, where K is a convex set.

In the previous sections, we also had quadratic functions f and g . Now we relax these conditions and consider a more general case where $f : \Omega \times [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable functions, f being \mathbb{F} -progressively measurable for fixed (x, π) , convex in (x, π) , C^1 in x , continuous in π , and g being \mathcal{F}_T -measurable for fixed x , convex and C^1 in x .

The set of admissible controls in this section is defined as

$$\mathcal{A} := \{\pi \in \mathcal{H}([t_0, T], \mathbb{R}^m) : \pi(t) \in K \text{ for } t \in [t_0, T], a.e.\}.$$

The optimisation problem is defined as

$$\inf_{\pi \in K} \mathbb{E} \left[\int_{t_0}^T f(t, X(t), \pi(t)) dt + g(X(T)) \right]. \quad (168)$$

If we denote by $\tilde{f}(t, x, \pi)$ the function

$$\tilde{f}(t, x, \pi) = f(t, x, \pi) + \Psi_K(\pi),$$

where

$$\Psi_K(\pi) = \begin{cases} 0 & \text{if } \pi \in K \\ +\infty & \text{if } \pi \notin K. \end{cases}$$

then the optimisation problem can be formulated as

$$\sup_{\pi \in \mathbb{R}^m} \mathbb{E} \left[- \int_{t_0}^T \tilde{f}(t, X, \pi) dt - g(X(T)) \right]. \quad (169)$$

4.1.2 Dual Problem

We now derive the dual problem in a similar way as before (adapted from [6]). Assume that the dual process Y is driven by an SDE

$$dY(t) = \alpha_1(t)dt + \beta_1(t)dW(t),$$

with initial condition $Y(t_0) = y$, where $\alpha_1(t) \in \mathbb{R}^n$ and $\beta_1(t) \in \mathbb{R}^n$ are two stochastic processes to be determined. Ito's lemma gives

$$d(X^T Y) = \left[X^T \underbrace{(\alpha_1 + A^T Y + C^T \beta_1)}_{:=\alpha} + \pi^T \underbrace{(B^T Y + D^T \beta_1)}_{:=\beta} \right] dt + \text{local martingale}.$$

Denoting $\alpha = \alpha_1 + A^T Y + C^T \beta_1$ and $\beta = B^T Y + D^T \beta_1$, we have that the dual process $Y(t)$ satisfies:

$$\begin{cases} dY(t) &= [\alpha(t) - A(t)^T Y(t) - C^T(t) \beta_1(t)] dt + \beta_1(t) dW(t) \\ Y(t_0) &= y, \end{cases} \quad (170)$$

Returning to Ito's lemma to $X(t)^T Y(t)$, we get

$$dX^T Y = (X^T \alpha + \pi^T \beta) dt + \text{local martingale}$$

where

$$\beta = B^T Y + D^T \beta_1. \quad (171)$$

The process $X^T(t)Y(t) - \int_{t_0}^t [X^T(s)\alpha(s) + \pi^T(s)\beta(s)] ds$ is a local martingale and a super-martingale if it is bounded below by an integrable process, which gives

$$\mathbb{E} \left[X^T(T)Y(T) - \int_{t_0}^T (X^T \alpha + \pi^T \beta) ds \right] \leq x^T y. \quad (172)$$

Define the dual functions $\phi : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ by

$$\phi(t, \alpha, \beta) = \sup_{x, \pi} \{ x^T \alpha + \pi^T \beta - \tilde{f}(t, x, \pi) \} \quad (173)$$

and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$h(y) = \sup_x \{ -x^T y - g(x) \}. \quad (174)$$

We have ϕ and h are proper closed convex functions [2, Proposition 1.1.6, Proposition 1.6.1].

Combining (172), (173) and (174) we get the following inequality:

$$\sup_{\pi} \mathbb{E} \left[- \int_{t_0}^T \tilde{f}(t, X, \pi) dt - g(X(T)) \right] \leq \inf_{y, \alpha, \beta_1} \left\{ x^T y + \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha, \beta) dt + h(Y(T)) \right] \right\}.$$

The dual control problem is defined by

$$\inf_{y, \alpha, \beta_1} \left\{ x^T y + \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha, \beta) dt + h(Y(T)) \right] \right\}, \quad (175)$$

where Y satisfies (170). Problem (175) can be solved in two steps: first, for fixed y solve a stochastic control problem

$$-\tilde{v}(y) = \inf_{\alpha, \beta_1} \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha, \beta) dt + h(Y(T)) \right] \quad (176)$$

and second, solve a static optimisation problem:

$$\inf_y \{ x^T y + V(y) \}. \quad (177)$$

4.2 Primal and Dual HJB & BSDE Methods

4.2.1 Primal HJB Equation

The HJB equation is the same as in section 1.2.1 (derivation taken from [8]), except that the supremum is taken over the set K instead of the whole space \mathbb{R}^m . Similar to the unconstrained case, we define the value function $v : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$v(t, X(t)) = \sup_{\pi \in K} \mathbb{E} \left[\int_{t_0}^T -f(t, X(t), \pi(t)) dt - g(X(T)) \right]$$

The HJB is then given by:

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\pi \in K} \{ F(t, x, \pi, D_x[v], D_x^2[v]) \} = 0 \quad (178)$$

where the Hamiltonian F is defined as

$$F(t, x, \pi, D_x[v], D_x^2[v]) = (AX + B\pi)^T D_x[v(t, x(t))] + \frac{1}{2} (Cx + D\pi)^T D_x^2[v(t, x)] (Cx + D\pi) - f(t, x, \pi)$$

and the terminal condition is given by:

$$v(T, x) = -g(x).$$

4.2.2 Primal BSDE

This section is adapted from [9]. Proofs and details of the Stochastic Maximum Principle can be found in [3]. To state the SMP, we need the following assumption:

Let $(\hat{X}, \hat{\pi})$ be admissible pair satisfying

$$\mathbb{E} \left[\int_{t_0}^T |D_x[f(t, \hat{X}(t), \hat{\pi}(t))]|^2 dt \right] < \infty \quad \text{and} \quad \mathbb{E} \left[\int_{t_0}^T |D_x[g(\hat{X}(T))]|^2 dt \right] < \infty.$$

For any admissible pair (X, π) , there exists $Z \in \mathcal{P}([t_0, T], \mathbb{R})$ and an \mathcal{F}_T -measurable random variable \tilde{Z} satisfying $\mathbb{E} \int_{t_0}^T |Z(t)| dt < \infty$, $\mathbb{E}|\tilde{Z}| < \infty$ such that

$$\begin{aligned} Z(t) &\geq \frac{f(t, \hat{X}(t) + \varepsilon(X(t) - \hat{X}(t)), \hat{\pi}(t) + \varepsilon(\pi(t) - \hat{\pi}(t))) - f(t, \hat{X}(t), \hat{\pi}(t))}{\varepsilon} \\ \tilde{Z} &\geq \frac{g(\hat{X}(T) + \varepsilon(X(T) - \hat{X}(T))) - g(\hat{X}(T))}{\varepsilon} \end{aligned}$$

for $(\mathbb{P} \otimes Leb) - a.e. (\omega, t) \in \Omega \times [t_0, T]$ and $\varepsilon \in (0, 1]$.

We define the generalised Hamiltonian $\mathcal{H} : \Omega \times [t_0, T] \times \mathbb{R}^n \times K \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\mathcal{H}(t, x, \pi, p, q) = x^T A^T p + \pi^T B^T p + \left(x^T C^T + \pi^T D^T \right) q - f(t, x, \pi).$$

The adjoint process is given by

$$\begin{cases} dp &= -D_x[\mathcal{H}(t, X(t), \pi(t), p(t), q(t))]dt + q(t)dW(t) \\ &= -[A^T p + C^T q - D_x[f(t, X, \pi)]]dt + q(t)dW(t) \\ p(T) &= -D_x[g(X(T))] \end{cases} \quad (179)$$

The Stochastic Maximum Principle for problem (168) states that if X is the solution to (167) and p, q are the solution to the adjoint process (179), then $\hat{\pi} \in \mathcal{A}$, such that the assumption above holds, is optimal if and only if it satisfies the condition

$$\mathcal{H}(t, X(t), \hat{\pi}(t), p(t), q(t)) = \sup_{\pi \in K} \mathcal{H}(t, X(t), \pi(t), p(t), q(t))$$

for $(\mathbb{P} \otimes Leb) - a.e. (\omega, t) \in \Omega \times [t_0, T]$. Moreover, if $D_\pi[f]$ exists, then this conditions is equivalent to

$$[\hat{\pi} - \pi]^T \left[B^T p + D^T q - D_\pi[f(t, X, \hat{\pi})] \right] \geq 0, \quad \forall \pi \in K.$$

4.2.3 Dual HJB Equation

Recall that the dual process $Y(t)$ satisfies the SDE (170):

$$\begin{cases} dY(t) &= [\alpha(t) - A(t)^T Y(t) - C^T(t)\beta_1(t)]dt + \beta_1(t)dW(t) \\ Y(t_0) &= y, \end{cases}$$

and also recall from (176) that the dual value function is given by

$$\tilde{v}(t, Y(t)) = \sup_{\alpha, \beta_1} \mathbb{E} \left[\int_{t_0}^T -\phi(t, \alpha, \beta) dt - h(Y(T)) \right]$$

where $\beta = B^T Y + D^T \beta_1$ and ϕ and h are given by (173) and (174):

$$\begin{aligned} \phi(t, \alpha, \beta) &= \sup_{x, \pi} \{x^T \alpha + \pi^T \beta - \tilde{f}(t, x, \pi)\} \\ h(y) &= \sup_x \{-x^T y - g(x)\}. \end{aligned}$$

Then, using the same reasoning as in section 1.4, the dual HJB equation is given by

$$\frac{\partial \tilde{v}}{\partial t}(t, y) + \sup_{\alpha, \beta_1} [\mathcal{L}^{\alpha, \beta_1}[\tilde{v}(t, y)] - \phi(t, \alpha, \beta)] = 0,$$

where the generator is given by

$$\mathcal{L}^{\alpha, \beta_1}[\tilde{v}(t, y)] = (\alpha^T - y^T A - \beta_1^T C) D_y[\tilde{v}] + \frac{1}{2} \beta_1^T D_y^2[\tilde{v}] \beta_1,$$

and the terminal condition is

$$\tilde{v}(T, y) = -h(y).$$

If we assume that $n < m$ and $\text{rank}(D(t)) = n$. Then $D^\dagger(t) := D^T(t)(D(t)D^T(t))^{-1} \in \mathbb{R}^{m \times n}$ is the Moore-Penrose inverse of D and satisfied $DD^\dagger = I_n$. From (171) we can write

$$\beta_1(t) = (D^\dagger)^T(t) (\beta(t) - B^T(t)Y(t))$$

Then the HJB equation is given by

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t, y) + \sup_{\alpha, \beta} \left\{ -\phi(t, \alpha, \beta) + (\alpha^T - y^T A - (\beta(t) - B^T(t)Y(t))^T D^\dagger(t) C) D_y[\tilde{v}] \right. \\ \left. + \frac{1}{2} (\beta(t) - B^T(t)Y(t))^T D^\dagger(t) D_y^2[\tilde{v}] (D^\dagger)^T(t) (\beta(t) - B^T(t)Y(t)) \right\} = 0. \end{aligned}$$

4.2.4 Dual BSDE

This section is once again adapted from [9]. To state the Stochastic Maximum Principle for the dual problem, we need a similar assumption to that of the primal problem:

Let $(\hat{Y}, \hat{\alpha}, \hat{\beta}_1)$ be a given admissible dual pair. For any admissible pair (Y, α, β_1) , there exists $Z \in \mathcal{P}([t_0, T], \mathbb{R})$ and an \mathcal{F}_T -measurable random variable \tilde{Z} satisfying $\mathbb{E} \int_{t_0}^T |Z(t)| dt < \infty$, $\mathbb{E}|\tilde{Z}| < \infty$ such that

$$Z(t) \geq \frac{\phi(t, \hat{\alpha} + \varepsilon \alpha, \hat{\beta} + \varepsilon \beta) - \phi(t, \hat{\alpha}, \hat{\beta})}{\varepsilon}$$

$$\tilde{Z} \geq \frac{h(\hat{Y}(T) + \varepsilon Y(T)) - h(\hat{Y}(T))}{\varepsilon}$$

for $(\mathbb{P} \otimes \text{Leb})$ -a.e. $(\omega, t) \in \Omega \times [t_0, T]$ and $\varepsilon \in (0, 1]$. Furthermore, h is C^1 in y and satisfies $\mathbb{E}[D_y[h^2(Y(T))]] < \infty$.

The Hamiltonian $\tilde{\mathcal{H}} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ for the dual problem is defined as

$$\begin{aligned} \tilde{\mathcal{H}}(t, Y, \alpha, \beta_1, p, q) &= p^T(\alpha - A^T Y - C^T \beta_1) + \beta_1^T q - \phi\left(t, \alpha, B^T Y + D^T \beta_1\right) \\ &= p^T \alpha - p^T A^T Y - p^T C^T \beta_1 + \beta_1^T q - \phi\left(t, \alpha, B^T Y + D^T \beta_1\right) \end{aligned}$$

If ϕ is C^1 in β , and under the assumption above, then the adjoint equation is given by the system

$$\begin{cases} dp(t) &= -D_y[\tilde{\mathcal{H}}(t, Y(t), \alpha(t), \beta_1(t), p(t), q(t))]dt + q(t)dW(t) \\ &= [A(t)p(t) + B(t)D_\beta[\phi(t, \alpha(t), B^T(t)Y(t) + D^T(t)\beta_1(t))]]dt + q(t)dW(t) \\ p(T) &= -D_y[h(Y(T))]. \end{cases} \quad (180)$$

According to the Stochastic Maximum Principle, the optimal controls $\hat{\alpha}, \hat{\beta}_1$ satisfy the condition

$$\tilde{\mathcal{H}}(t, Y(t), \hat{\alpha}(t), \hat{\beta}_1(t), p(t), q(t)) = \sup_{\alpha, \beta_1} \tilde{\mathcal{H}}(t, Y(t), \alpha(t), \beta_1(t), p(t), q(t)).$$

If, however, the condition that ϕ is differential in β is not satisfied, then the adjoint equation is not well defined. We consider the following additional assumption:

Assume that $n < m$ and $\text{rank}(D(t)) = n$. Then $D^\dagger(t) := D^T(t)(D(t)D^T(t))^{-1} \in \mathbb{R}^{m \times n}$ is the Moore-Penrose inverse of D and satisfied $DD^\dagger = I_n$. From (171) we can write

$$\beta_1(t) = (D^\dagger)^T(t)(\beta(t) - B^T(t)Y(t))$$

Using this and (170), the dual process Y satisfies the following SDE:

$$\begin{cases} dY(t) = [\alpha(t) - A(t)^T Y(t) - C^T(t)(D^\dagger)^T(t)(\beta(t) - B^T(t)Y(t))]dt \\ \quad + (D^\dagger)^T(t)(\beta(t) - B^T(t)Y(t))dW(t) \\ Y(t_0) = y, \end{cases} \quad (181)$$

The dual optimisation problem (175) is now equivalent to

$$\inf_{y, \alpha, \beta} \left\{ x^T y + \mathbb{E} \left[\int_{t_0}^T \phi(t, \alpha(t), \beta(t))dt + h(Y(T)) \right] \right\}. \quad (182)$$

The adjoint equation associated with (t, α, β) and Y in (181) is given by

$$\begin{cases} dp &= [(A(t) - B(t)D^\dagger(t)C(t))p(t) + B(t)D^\dagger(t)q(t)]dt + q(t)dW(t) \\ p(T) &= -D_y[h(Y(T))] \end{cases} \quad (183)$$

The Stochastic Maximum Principle for the process (181) with the adjoint process (183) states that $(\hat{y}, \hat{\alpha}, \hat{\beta})$ are optimal for the dual problem (182) if and only if the following conditions are satisfied:

$$\begin{cases} p(t_0) = x \\ (p, D^\dagger q - D^\dagger C p) \in \partial\phi(\hat{\alpha}, \hat{\beta}) \\ D^\dagger q - D^\dagger C p \in K \end{cases} \quad (184)$$

for $(\mathbb{P} \otimes Leb) - a.e. (\omega, t) \in \Omega \times [t_0, T]$, where $\partial\phi(\hat{\alpha}(t), \hat{\beta}(t))$ is the subdifferential of ϕ at $(\hat{\alpha}(t), \hat{\beta}(t))$.

4.3 Primal-Dual Relation

In this section, we state some results on the dual-primal relation. We need to assume that the function $D_x[g](\omega, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijection for any ω such that $z = -D_x[g(x)]$ if and only if $x = -D_y[h(z)]$, that is, the inverse function of $-D_x[g]$ is $-D_y[h]$.

We first show how we can recover the primal optimal solution from that of the dual problem. Proof of this statement can be found in [9].

Suppose that (y, α, β) is optimal for the dual problem (182), and let $(Y, p_{\text{dual}}, q_{\text{dual}})$ be the associated state and adjoint processes in (181) and (183). Define

$$\pi(t) := D^\dagger(t)q_{\text{dual}}(t) - D^\dagger(t)C(t)p_{\text{dual}}, \quad t \in [t_0, T]. \quad (185)$$

Then $\pi(t)$ is the optimal control for the primal problem (168). Further, for $t \in [t_0, T]$, the optimal state and associated adjoint processes satisfy

$$\begin{cases} X(t) = p_{\text{dual}}(t) \\ p_{\text{primal}}(t) = Y(t) \\ q_{\text{primal}}(t) = (D^\dagger)^T(t)(\beta(t) - B^T Y(t)), \end{cases}$$

where p_{primal} and q_{primal} denote the adjoint processes associated with the primal problem.

We can also recover the dual optimal solution from the primal. Suppose that $\pi(t) \in \mathcal{A}$ is optimal for the primal problem (168) and let $(X, p_{\text{primal}}, q_{\text{primal}})$ be the associated state and adjoint processes in (167) and (179). Define

$$\begin{cases} y = p_{\text{primal}}(0) \\ \alpha(t) = D_x[f(t, X(t), \pi(t))] \\ \beta(t) = B^T(t)p_{\text{primal}}(t) + D^T(t)q_{\text{primal}}(t) \end{cases}$$

Then $(y, \alpha(t), \beta(t))$ is the optimal control of the dual problem (182). For $y \in [t_0, T]$, the optimal dual state process and associated adjoint processes satisfy

$$\begin{cases} Y(t) = p_{\text{primal}}(t) \\ p_{\text{dual}}(t) = X(t) \\ D^\dagger(t)q_{\text{dual}}(t) = \pi(t) + D^\dagger(t)C(t)X(t). \end{cases}$$

5 The Deep Controlled 2BSDE Method

In this section, we adapt the results from [4]. We first state the main results and then apply the algorithm for the unconstrained case described in chapter 1, in which we know the theoretical optimal control, with the purpose of verifying the results from the deep controlled 2BSDE method.

The algorithm was developed mainly following the idea from [4] as well as [1]. The code was written from scratch, as the first paper only provides a pseudo-code and the second one utilises an older version of tensorflow, but the ideas are similar, with adaptations to our specific problem. The code for this project can be found at [5].

5.1 Description of Algorithm

We try to solve the problem using machine learning techniques. The idea is to simulate all processes in the forward direction, introducing the two new variables v_0, z_0 and moving forward through a discretisation of the time interval $[t_0, T]$. We set Γ to be a neural network that at each time t_i takes the i th state position denoted by X_i as input since we are approximating $D_x^2[v]$ with this process. The control π is also a neural network, taking in the same state X_i as input at each time step.

Each of the neural networks has L number of layers with l hidden nodes. We use a nonlinear activation function between each of the hidden layers. For our networks, we choose the ReLu activation function, although this choice is rather arbitrary (provided it is nonlinear). No activation function is applied to the output. We denote by θ_i the parameters of the neural network for the controls π_i at each time t_i of the discretisation of the time interval. We further denote by $N_{\theta_i}(X_i)$ the output of the neural network. Similarly, for Γ we denote the parameters by λ_i and the output of the neural network at time t_i by $N_{\lambda_i}(X_i)$.

We then use the following Euler-Maruyama Scheme. Set $X_0 = x_0$, and initialise $V_0 = v_0, Z_0 = z_0$. For $i = 0, \dots, N-1$ let $\pi_i = N_{\theta_i^0}(X_i)$, $\Gamma_i = N_{\lambda_i^0}(X_i)$ and

$$\begin{aligned} X_{i+1} &= X_i + (t_{i+1} - t_i)(AX_i + B\pi_i) + (CX_i + D\pi_i)dW_i \\ V_{i+1} &= V_i + Z_i^T(CX_i + D\pi_i)dW_i \\ Z_{i+1} &= Z_i - (t_{i+1} - t_i)D_x[H(t_i, X_i, \pi_i, Z_i, \Gamma_i(CX_i + D\pi_i))] + \Gamma_i(CX_i + D\pi_i)dW_i, \end{aligned}$$

where dW_i is a multivariate normal $N(0, (t_{i+1} - t_i))$.

Once we have repeated this iteration $N-1$ times we arrive at time T , at which point we have not satisfied the terminal conditions. We, therefore, define the loss function

$$L_1(\{\lambda_i\}_{i=0}^{N-1} \cup \{V_0, Z_0\}) := \mathbb{E} \left[|V_N + g(X_N)|^2 + |Z_N + D_x[g(X_N)]|^2 \right].$$

This loss function is for the parameters of the neural network for Γ and the initial values v_0, z_0 . In addition, we need a second loss function for the parameters of the neural network for π . We define for each time step

$$L_2(\theta_i, i) := \mathbb{E} \left[\left| D_\pi[F(t_i, X_i, \pi_i, Z_i, \Gamma_i)] \right|^2 \right].$$

Since we want to maximise the Hamiltonian F w.r.t. π , we minimise the square of its derivative. For each of these loss functions, we update the parameters using the Adam algorithm. We repeat this procedure until the parameters converge. The algorithm can be described as follows:

Algorithm 1 Primal Deep BSDE Method

```

Initialise  $V_0 \sim Unif([-0.1, 0.1])$  and  $Z_0 \sim Unif([-0.1, 0.1])$ 
for  $i=0, 1, \dots, N-1$  do
    Initialise  $\theta_i \sim Unif([-0.1, 0.1])$  and  $\lambda_i \sim Unif([-0.1, 0.1])$ 
end for
for  $i=0, 1, \dots, \text{NumEpochs}$  do
    Generate  $W_k$  for  $k = 1, \dots, B$ 
    Generate  $X^k, V^k, Z^k, \pi^k, \Gamma^k$  using  $\{\theta_i, \lambda_i\}_{i=0}^{N-1}$  and  $V_0, Z_0$ 
    Set  $L_1 := \frac{1}{B} \sum_{k=1}^B \left[ |V_N^k + g(X_N^k)|^2 + |Z_N^k + D_x[g(X_N^k)]|^2 \right]$ 
    Update  $\{\lambda_i\}_{i=0}^{N-1} \cup \{V_0, Z_0\}$  using the ADAM algorithm
    Regenerate  $X^k, V^k, Z^k, \pi^k, \Gamma^k$  using  $\{\theta_i, \lambda_i\}_{i=0}^{N-1}$  and  $V_0, Z_0$  and the updated  $\{\lambda_i\}_{i=0}^{N-1}$ 
    for  $i=0, 1, 2, \dots, N-1$  do
        Set  $L_2(\theta_i, i) := -\frac{1}{B} \sum_{k=1}^B [F(t_i, X_i^k, \pi_i^k, Z_i^k, \Gamma_i^k)]$ 
        Update  $\theta_i$  using the ADAM algorithm
    end for
end for

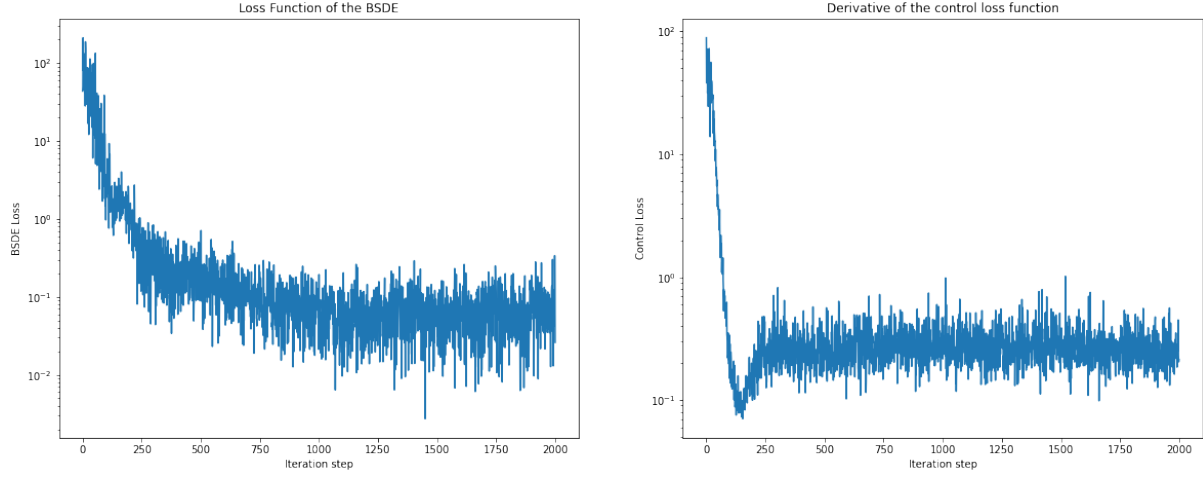
```

5.2 Applying the Algorithm to the Unconstrained Case

We perform the Deep Controlled 2BSDE algorithm using tensorflow for the case where the control is unconstrained and the problem is quadratic, i.e. the framework of chapter 1. We do this because we know the analytical solution for this problem and want to compare the two methods to verify the Deep Controlled 2BSDE method.

We iterate the steps described above until the loss functions have converged. We use constant values for the parameters of the model, so that $A, B, C, D, Q, R, S, G, L$ are not dependent on time (for simplicity). We take the learning rate for the BSDE step to be larger than the learning rate for the control step, ensuring that the loss functions converge.

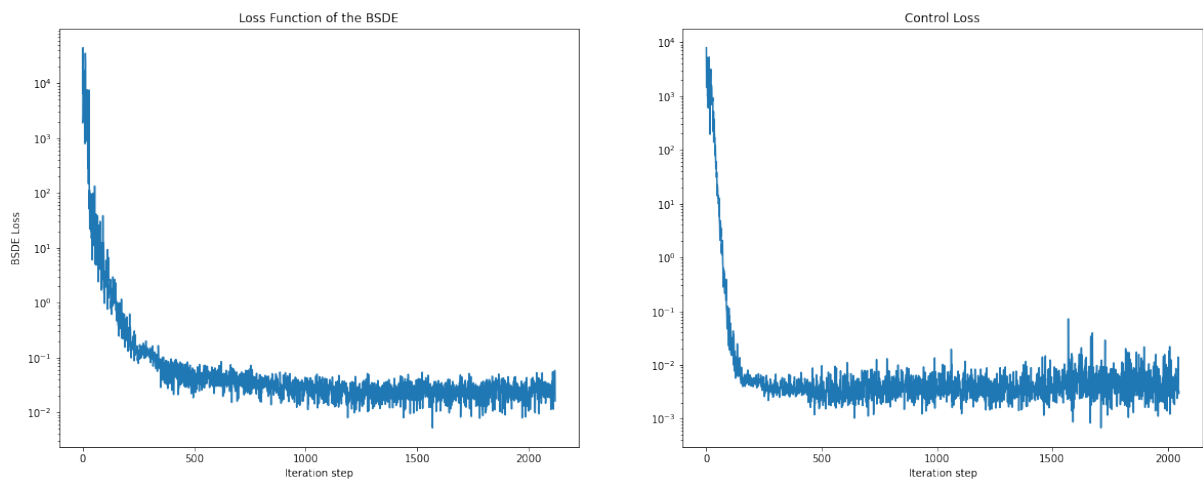
The figure below shows the loss functions of the primal problem. We have used a batch size of 1024 and the time interval $[0, 1]$ is divided into 20 time-steps.



The image on the left represents the loss L_1 , corresponding to the terminal conditions and on the right we have shown the loss L_2 , which ensures the optimality of the control. In this method, we have used an initial learning rate of 10^{-2} for minimising L_1 and 10^{-3} for minimising L_2 . After a certain number of iterations, we divide this learning rates by 10.

Extending the Algorithm to the Dual Problem

As described in [4], this algorithm can be extended to the dual problem. We perform the algorithm for our problem, and show the loss functions for the dual problem:

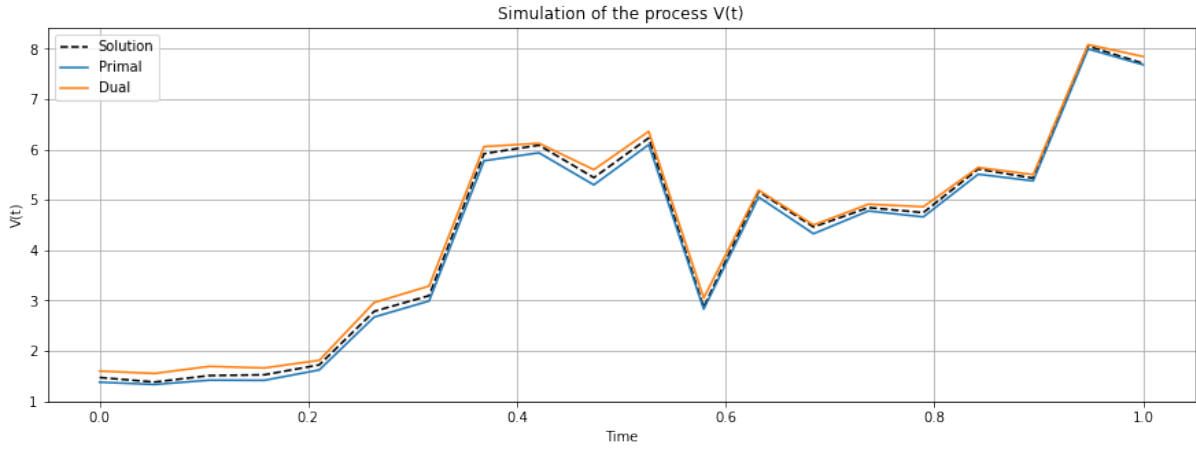


For the control loss, we have plotted the derivative of the loss function L_3 with respect to y . We have used learning rates of 10^{-2} and 10^{-3} for the respective losses.

5.3 Verifying Results with Analytical Solution

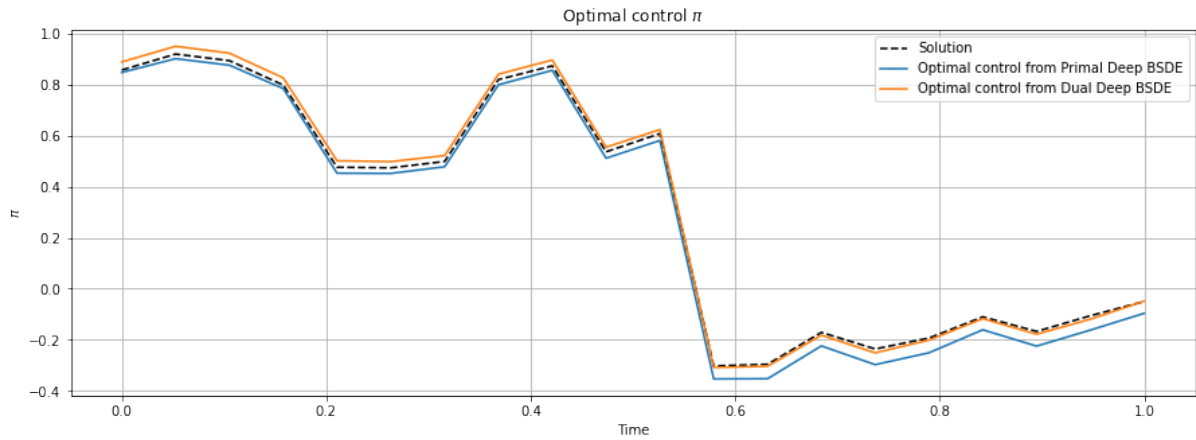
Now that we have both the primal and dual solutions from the Deep BSDE algorithm, we compare the results with the analytical solution. In chapter 3 we described how to solve the systems of ODEs for the primal and dual problem using the Runge-Kutta numerical scheme, giving us the solutions to the unconstrained quadratic case.

For a given simulation of the process X_t , we can compute the analytical solution to the value function for this problem using the results from 3, and plot them together with the results from the Deep BSDE method. We get the following results:



We can see that both the primal and dual solutions are very close to the solution.

Next, we recall that we can obtain the optimal control from (185) given that we have obtained the dual processes. From the primal processes, we can also get the optimal control π , so we can compare the results, together with the analytical solution given by the Runge-Kutta scheme. Below we show the results for a given simulation of X , (we only show the first element of the control π):



Once again, all three algorithms show relatively close results.

5.4 Further Work

With the successful demonstration that the Deep BSDE algorithm is capable of producing results congruent with those derived analytically, we find ourselves in a position to extend its application to scenarios of increased complexity. In these scenarios, achieving analytical results following conventional methods may not be feasible.

Our primary focus of interest is instances where the application of control is subject to limitations. In such situations, it's generally infeasible to use traditional methods to derive the optimal control analytically. This limitation presents us with the opportunity to exploit the capabilities of our Deep BSDE algorithm to its full potential, providing a practical alternative to conventional analytical methods.

Another area of application we are considering is the exploration of more varied types of cost functions. Until now, we've been predominantly working with quadratic cost functions. However, the real-world scenarios can be more diverse and might require dealing with cost functions of a more general convex nature. In this context, we expect our algorithm to prove its versatility and adaptability, thus expanding the range of problems it can tackle.

The algorithm is applied to both primal and dual problems in these scenarios. The primal problem focuses on the task of maximisation, whereas the dual problem centres around minimisation. While it's important to acknowledge that the solutions to these two problems might not be perfectly identical, they can, however, establish upper and lower bounds for the optimal value function. As long as the gap between these boundaries remains relatively small, they can provide invaluable insights.

References

- [1] Christian Beck, Weinan E, and Arnulf Jentzen. **Machine Learning Approximation Algorithms for High-Dimensional Fully Nonlinear Partial Differential Equations and Second-order Backward Stochastic Differential Equations**. 2019. URL: <https://doi.org/10.1007%2Fs00332-018-9525-3>.
- [2] D. Bertsekas. **Convex Optimization Theory**. Athena Scientific, 2009. URL: <https://books.google.co.uk/books?id=0H1iQwAACAAJ>.
- [3] Abel Cadenillas and Ioannis Karatzas. **The Stochastic Maximum Principle for Linear, Convex Optimal Control with Random Coefficients**. 1995. URL: <https://doi.org/10.1137/S0363012992240722>.
- [4] Ashley Davey and Harry Zheng. **Deep Learning for Constrained Utility Maximisation**. 2021. URL: <https://doi.org/10.1007%2Fs11009-021-09912-3>.
- [5] Ivan Kirev. **Stochastic Control in Mathematical Finance**. <https://github.com/ivankirev13/Stochastic-Control-in-Mathematical-Finance>. 2023.
- [6] Yusong Li and Harry Zheng. **Constrained Quadratic Risk Minimization via Forward and Backward Stochastic Differential Equations**. 2018. URL: <https://doi.org/10.1137/15M1052457>.
- [7] Yusong Li and Harry Zheng. **Weak Necessary and Sufficient Stochastic Maximum Principle for Markovian Regime-Switching Diffusion Models**. 2012. URL: <https://arxiv.org/abs/1210.0371>.
- [8] H. Pham. **Continuous-time Stochastic Control and Optimization with Financial Applications**. Springer, 2009.
- [9] Engel John Dela Vega and Harry Zheng. **Duality Method for Multidimensional Nonsmooth Constrained Linear Convex Stochastic Control**. 2022.