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Multidimensional Utility Maximisation in Incomplete Markets

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Abstract

In this project, we study a general multidimensional linear quadratic stochastic control problem. We impose no constraints on the control, in which case the problem has an analytical solution. We formulate an equivalent dual problem and solve both the primal and dual problems using the Hamilton-Jacobi-Bellman equation approach and the Stochastic Maximum Principle approach. Each of the four solutions are shown to be equivalent. We then consider a similar problem, where the coefficients of the state process depend on a time-continuous finite state space Markov chain. With this modification, the four methods still give equivalent solutions, but the solutions are now in the form of systems of ODEs for each state of the Markov chain, as opposed to a single system of ODEs in the non-Markov chain case. To complete the solutions, we solve numerically the system of ODEs for the 1-dimensional Brownian motion case. We employ the Runge-Kutta method to solve the coupled matrix ODEs, showing the equivalence of the solutions of the dual and primal problems. Finally, we want to consider a more general case where we impose constraints on the control and the cost is not necessarily quadratic, but convex. In this case, there are no longer analytical solutions, so we consider a deep learning algorithm with a focus on utility maximisation. We test this algorithm on the quadratic unconstrained case and compare the results with the already derived analytical results.

Contents

Notation

- $t \in [0, T]$: Time variable.
- $W(t)$: A d-dimensional Brownian motion, i.e., $W(t) = (W_1(t), W_2(t), \ldots, W_d(t))^\top$, where each $W_i(t)$ is an independent standard Brownian motion.
- $\mathbb{R}^d$: The d-dimensional real vector space.
- $\mathbb{E}[\cdot]$: Expectation operator.
- $\mathcal{F}_t$: Filtration up to time t, representing the information available at time t.
- $\mathcal{P}([t_0, T], \mathbb{R}^n)$: progressively measurable $\mathbb{R}^n$-valued processes;
- $\mathcal{H}([t_0, T], \mathbb{R}^n)$: processes in $\mathcal{P}$ with finite L^2-norm in time, i.e., $\mathbb{E}\big[\int_{t_0}^{T} \|x(t)\|^2 \, \mathrm{d}t\big] < \infty$;
- $\mathcal{S}([t_0, T], \mathbb{R}^n)$: processes in $\mathcal{P}$ with bounded second moment in supremum norm, i.e., $\mathbb{E}\big[\sup_{t\in[t_0,T]} \|x(t)\|^2\big] < \infty$.

1 Framework

1.1 Unconstrained Case

We consider a control model where the state of the system is governed by a stochastic differential equation (SDE) valued in $\mathbb{R}^n$:

$$\mathrm{d}X^\pi(t) = b(t, X^\pi(t), \pi(t))\,\mathrm{d}t + \sigma(t, X^\pi(t), \pi(t))\ \mathrm{d}W(t), \tag{1.1}$$

where $W(t)$ is a $d-$dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. The control $\pi(t)$ is a progressively measurable process with respect to $\mathbb{F}$, taking values in $\mathbb{R}^m$.

We fix a finite time horizon $0 < T < \infty$ and define the set of admissible controls:

$$\mathcal{A} := \left\{ \pi : [0,T] \times \Omega \to \mathbb{R}^m \;\middle|\; \mathbb{E} \int_0^T |\pi(t)|^2\,\mathrm{d}t < \infty \right\}.$$

In our framework the coefficients of the state system are linear functions of the control and the underlying process:

$$\begin{aligned} b(t,x,\pi) &:= A(t)x + B(t)\pi \in \mathbb{R}^n, \\ \sigma(t,x,\pi) &:= \begin{bmatrix} (C_1(t)x + D_1(t)\pi) & \cdots & (C_d(t)x + D_d(t)\pi) \end{bmatrix} \in \mathbb{R}^{n\times d}, \end{aligned}$$

where $A(t), C_i(t) \in \mathbb{R}^{n\times n}$, and $B(t), D_i(t) \in \mathbb{R}^{n\times m}$, for $i = 1, \ldots, d$, are deterministic coefficient processes. We assume that these coefficient processes are uniformly bounded and measurable functions.

1.1.1 Primal Problem

For the state and control processes $X^\pi(t)$ and $\pi(t)$, we define the gain functional as:

$$J(\pi) := \mathbb{E}\left[\int_0^T f(t, X^\pi(t), \pi(t))\,\mathrm{d}t + g(X^\pi(T)) \right],$$

where the *running cost* $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the *terminal cost* $g : \mathbb{R}^n \to \mathbb{R}$ are quadratic functions of the form:

$$f(t,x,\pi) = \tfrac{1}{2}x^\top Q(t)x + x^\top S^\top(t)\pi + \tfrac{1}{2}\pi^\top R(t)\pi, \quad g(x) = \tfrac{1}{2}x^\top G(T)x + x^\top L(T). \tag{1.2}$$

The coefficients are assumed to satisfy the following conditions:

Assumptions 1.1.1. We assume that the coefficients $Q(t) \in \mathbb{R}^{n\times n}$ and $R(t) \in \mathbb{R}^{m\times m}$ are symmetric and positive semidefinite for all $t \in [0,T]$, and that $S(t) \in \mathbb{R}^{m\times n}$, $G(T) \in \mathbb{R}^{n\times n}$ (symmetric), and $L(T) \in \mathbb{R}^n$ are all bounded and measurable. Furthermore, the extended running cost matrix

$$\begin{bmatrix} Q(t) & S^\top(t) \\ S(t) & R(t) \end{bmatrix} \geq 0.$$

Definition 1.1.1 (Primal Problem)**.** *Suppose that $X(t)$ is the solution to the SDE Eq.* (1.1)*. The* ***primal problem*** *is the following optimisation task*

$$J(\pi^\star) = \inf_{\pi \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(t, X^\pi(t), \pi(t))\,\mathrm{d}t + g(X^\pi(T))\right].$$

Ofter we work with a transformed version of the primal problem where the optimisation is via maximisation instead of minimisation:

$$J(\pi^\star) = -\sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\int_0^T -f(t, X^\pi(t), \pi(t))\,\mathrm{d}t - g(X^\pi(T))\right].$$

We denote the ***value function****, $v(t,x)$, of the primal problem as*

$$v(t, X(t)) := \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\int_0^T -f(t, X^\pi(t), \pi(t))\,\mathrm{d}t - g(X^\pi(T))\right].$$

1.1.2 Dual Problem

We now formulate the dual problem corresponding to the transformed primal maximisation problem. To this end, define the auxiliary process $Y(t) \in \mathbb{R}^n$ satisfying the stochastic differential equation:

$$\mathrm{d}Y(t) = \left[\alpha(t) - A(t)^\top Y(t) - \sum_{i=1}^d C_i(t)^\top \beta_i(t)\right]\mathrm{d}t + \sum_{i=1}^d \beta_i(t)\,\mathrm{d}W_i(t), \quad Y(0) = y, \tag{1.3}$$

where $\alpha(t) \in \mathbb{R}^n$, $\beta_i(t) \in \mathbb{R}^n$ for $i = 1, \ldots, d$, are progressively measurable processes.

There is a unique solution to the SDE for a given $(y, \alpha, \beta_1, \ldots, \beta_d)$. We call $(\alpha, \beta_1, \ldots, \beta_d)$ the *admissible dual control* and $(Y, \alpha, \beta_1, \ldots, \beta_d)$ the *admissible dual pair.*

Definition 1.1.2 (Dual Problem)**.** *Suppose $Y(t)$ is the solution to the SDE* (1.3)*. The dual problem is given by*

$$\inf_{y \in \mathbb{R}^n}\left\{x^\top y + \inf_{(\alpha,\beta_1,\ldots,\beta_d)} \mathbb{E}\left[\int_0^T \phi(t, \alpha(t), \beta(t))\,\mathrm{d}t + h(Y(T))\right]\right\},$$

where $\beta(t) = B(t)^\top Y(t) + \sum_{i=1}^d D_i(t)^\top \beta_i(t)$ and the dual functions are given by

$$\phi(t, \alpha, \beta) := \frac{1}{2}\alpha^T \tilde{Q}\alpha + \alpha^T \tilde{S}^T \beta + \frac{1}{2}\beta^T \tilde{R}\beta, \quad h(y) := \frac{1}{2}(y^T + L^T)G^{-1}(y + L)$$

where

$$\begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} := \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1}.$$

This optimisation problem can be solved in two steps: first, for fixed y, solve a stochastic control problem

$$-\tilde{v}(t, y) := \inf_{(\alpha,\beta_1,\ldots,\beta_d)} \mathbb{E}\left[\int_0^T \phi(t, \alpha(t), \beta(t))\,\mathrm{d}t + h(Y(T))\right],$$

where we denote by $\tilde{v}(t,y)$ the ***dual value function****. Second, solve the following static optimisation problem*

$$\inf_y \{x^T y - \tilde{v}(t, y)\}.$$

Remark 1.1.1. Note that explicit forms for the dual coefficients $\tilde{Q}, \tilde{S}$ and $\tilde{R}$ can be computed as follows:

$$\begin{aligned}
\tilde{Q} &= Q^{-1} - Q^{-1}S^T(SQ^{-1}S^T - R)^{-1}SQ^{-1} \\
\tilde{R} &= R^{-1} - R^{-1}S(S^TR^{-1}S - Q^{-1})^{-1}S^TR^{-1} \\
\tilde{S} &= (SQ^{-1}S^T - R)^{-1}SQ^{-1} = R^{-1}S(S^TR^{-1}S - Q)^{-1}
\end{aligned}$$

Theorem 1.1.1 (Primal-Dual Relationship)**.** *For the primal problem defined as in Definition 1.1.1 and the corresponding dual problem from Definition 1.1.2, the following inequality holds:*

$$\sup_{\pi} \mathbb{E}\Big[-\int_0^T f(t, X, \pi)\,\mathrm{d}t - g(X(T))\Big] \leq \inf_{y, \alpha, \beta_1, \ldots, \beta_d} \Big[x^T y + \mathbb{E}\Big[\int_0^T \phi(t, \alpha, \beta)\,\mathrm{d}t + h(Y(T))\Big]\Big].$$

Remark 1.1.2. Specifically, the primal problem acts as a maximisation task, while the dual problem minimises the objective. Both problems optimise similar controls, but one does so from below while the other from above. In some cases equality can be achieved, removing the gap between the two tasks.

Proof. The proof closely follows ideas from [6]. First, assume that the dual process Y is driven by an SDE of the form

$$\mathrm{d}Y(t) = \alpha_1(t) + \sum_{i=1}^{d} \beta_i(t)\,\mathrm{d}W_i(t),$$

with initial condition $Y(0) = y$, where α_1 and β_i are stochastic processes to be determined. Ito's lemma gives

$$\begin{aligned}
\mathrm{d}(X^TY) &= X^T\,\mathrm{d}Y + Y^T\,\mathrm{d}X + \mathrm{d}X^T\,\mathrm{d}Y \\
&= \Bigg[X^T \underbrace{\Big(\alpha_1 + A^TY + \sum_{i=1}^{d} C_i^T\beta_i\Big)}_{:=\alpha} + \pi^T \underbrace{\Big(B^TY + \sum_{i=1}^{d} D_i^T\beta_i\Big)}_{:=\beta}\Bigg]\,\mathrm{d}t + \text{local martingale.}
\end{aligned}$$

Denoting $\alpha = \alpha_1 + A^TY + \sum_{i=1}^{d} C_i^T\beta_i$ and $\beta = B^TY + \sum_{i=1}^{d} D_i^T\beta_i$, we have that the dual process $Y(t)$ satisfies

$$\begin{cases}
\mathrm{d}Y(t) &= \big[\alpha(t) - A^T(t)Y(t) - \sum_{i=1}^{d} C_i^T(t)\beta_i(t)\big]\,\mathrm{d}t + \sum_{i=1}^{d} \beta_i(t)\,\mathrm{d}W_i(t) \\
Y(0) &= y.
\end{cases} \tag{1.4}$$

Returning to Ito's lemma to $X^T(t)Y(t)$, we have

$$\mathrm{d}(X^TY) = (X^T\alpha + \pi^T\beta)\,\mathrm{d}t + \text{local martingale.}$$

The process $X^T(t)Y(t) - \int_0^t [X^T(s)\alpha(s) + \pi^T(s)\beta(s)]\,\mathrm{d}s$ is a local martingale and a supermartingale if it is bounded below by an integrable process, which gives

$$\mathbb{E}\Big[X^T(T)Y(T) - \int_0^T \big(X^T(s)\alpha(s) + \pi^T(s)\beta(s)\big)\,\mathrm{d}s\Big] \leq X^T(0)Y(0) = x^Ty. \tag{1.5}$$

Using this, and the following dual functions $\phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$\phi(t, \alpha, \beta) = \sup_{x, \pi} \big\{x^T\alpha + \pi^T\beta - f(t, x, \pi)\big\} \tag{1.6}$$

and $h : \mathbb{R}^n \to \mathbb{R}$

$$h(y) = \sup_{x} \big\{-x^Ty - g(x)\big\}, \tag{1.7}$$

we arrive at the desired inequality

$$\sup_{\pi} \mathbb{E}\Big[-\int_0^T f(t,X,\pi)\,\mathrm{d}t - g(X(T))\Big] \le \inf_{y,\alpha,\beta_1,\dots,\beta_d} \Big[x^T y + \mathbb{E}\Big[\int_0^T \phi(t,\alpha,\beta)\,\mathrm{d}t + h(Y(T))\Big]\Big], \tag{1.8}$$

where the RHS is the dual control problem.

Now to find explicit forms for ϕ and h, recall that f, g take the quadratic forms Eq. (1.2), so finding the supremums in Eq. (1.6) and Eq. (1.7) is done by setting the derivatives to zero. We have

$$\phi(t,\alpha,\beta) = \sup_{x,\pi} \left\{ \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x \\ \pi \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} \right\},$$

so setting the derivative to zero, we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0 \implies \begin{bmatrix} x^* \\ \pi^* \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Therefore

$$\begin{aligned} \pi^* &= \big[SQ^{-1}S^T - R\big]^{-1}(SQ^{-1}\alpha - \beta) \\ x^* &= Q^{-1}(\alpha - S^T\pi^*) \end{aligned}$$

Then ϕ is given by

$$\phi(t,\alpha,\beta) = \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Denoting

$$\begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}^{-1},$$

we get

$$\phi(t,\alpha,\beta) = \frac{1}{2}\alpha^T \tilde{Q}\alpha + \alpha^T \tilde{S}^T \beta + \frac{1}{2}\beta^T \tilde{R}\beta,$$

Similarly,

$$D_x\big[-x^T y - \frac{1}{2}x^T G x - x^T L\big] = -y - Gx - L \implies x^* = -G^{-1}(y+L).$$

Then $h(y)$ is given by

$$\begin{aligned} h(y) &= (y^T + L^T)G^{-1}y - \frac{1}{2}(y^T + L^T)G^{-1}(y+L) + (y^T + L^T)G^{-1}L \\ &= \frac{1}{2}\big[y^T G^{-1} y + L^T G^{-1} y + y^T G^{-1} L + L^T G^{-1} L\big] \\ &= \frac{1}{2}(y^T + L^T)G^{-1}(y+L) \end{aligned}$$

□

1.2 Unconstrained Markovian Case

1.3 Constrained Case

2 Unconstrained Problem

In this chapter, we adopt the structural framework introduced in [8], which employs the Stochastic Maximum Principle for solving optimal control problems. Additionally, we explore the partial differential equation (PDE) approach via the Hamilton-Jacobi-Bellman (HJB) equation, as presented in [7]. We modify the problem structure by incorporating a quadratic running cost, which enables analytical tractability and facilitates explicit computations, as demonstrated in the following sections.

2.1 Solving the Primal Problem

2.1.1 HJB Method

Derivation of HJB equation

We transform the minimisation problem to maximisation by noting that

$$\inf_{\pi(t)\in\mathbb{R}^m} \mathbb{E}\left[\int_{t_0}^{T} f(t,X(t),\pi(t))\,\mathrm{d}t + g(X(T))\right] = -\sup_{\pi(t)\in\mathbb{R}^m} \mathbb{E}\left[\int_{t_0}^{T} -f(t,X(t),\pi(t))\,\mathrm{d}t - g(X(T))\right],$$

and denote the value function

$$v(t,X(t)) = \sup_{\pi(t)\in\mathbb{R}^m} \mathbb{E}\left[\int_{t_0}^{T} -f(t,X(t),\pi(t))\,\mathrm{d}t - g(X(T))\right] \tag{2.1}$$

Note that the optimal value of the optimisation problem is given by $-v(t,X(t))$. Consider the time interval $(t,t+h)$ and a constant control $\pi(t)=\pi$. According to the Dynamic programming principle,

$$v(t,X(t)) \geq \mathbb{E}\left[\int_{t}^{t+h} -f(s,X(s),\pi)\,\mathrm{d}s + v(t+h,X(t+h))\right], \tag{2.2}$$

where we denote $X(s)$ to be the solution of (**??**) given that we know the value of X at time t.

Applying Ito's formula between t and $t+h$ we get

$$\begin{aligned} v(t+h,X(t+h)) = v(t,X(t)) &+ \int_{t}^{t+h}\left(\frac{\partial v(s,X(s))}{\partial t} + \mathcal{L}^{\pi}[v(s,X(s))]\right)\mathrm{d}s \\ &+ \underbrace{\int_{t}^{t+h}\big[D_x v(s,X(s))\big]^T \sigma(s,X(s),\pi)\,\mathrm{d}W(s)}_{\text{(local) martingale}}, \end{aligned}$$

where $\mathcal{L}^{\pi}[v(t,x)]$ is the generator given by

$$\mathcal{L}^{\pi}[v(t,X(t))] = b^T(t,X(t),\pi)D_x v(t,X(t)) + \frac{1}{2}\mathrm{tr}\big[\sigma(t,X(t),\pi)\sigma^T(t,X(t),\pi)D_x^2 v(t,X(t))\big] \tag{2.3}$$

Substituting into equation (2.2), we get

$$0 \geq \mathbb{E}\bigg[\int_t^{t+h} \frac{\partial v}{\partial t}(s, X(s)) + \mathcal{L}^\pi[v(s, X(s))] - f(s, X(s), \pi)\,\mathrm{d}s\bigg]$$

Dividing by h and sending h to 0, this yields by the mean value theorem

$$0 \geq \frac{\partial v}{\partial t}(t, x) + \mathcal{L}^\pi[v(t, x)] - f(t, x, \pi).$$

Since this is true for any admissible π, we obtain the inequality

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\pi \in \mathbb{R}^m}[\mathcal{L}^\pi v(t, x) - f(t, x, \pi)] \leq 0. \tag{2.4}$$

On the other hand, suppose that π^* is an optimal control. Then by the dynamic programming principle,

$$v(t, x) = \mathbb{E}\bigg[\int_t^{t+h} -f(s, X^*(s), \pi^*(s))\,\mathrm{d}s + v(t+h, X^*(t+h))\bigg], \tag{2.5}$$

where X^* is the solution to the initial SDE (**??**) with control π^* starting from x at time t. By similar reasoning, we get

$$\frac{\partial v}{\partial t}(t, x) + \mathcal{L}^{\pi^*}[v(t, x)] - f(t, x, \pi^*) = 0$$

which combined with (2.4) suggests that v should satisfy

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\pi \in \mathbb{R}^m}[\mathcal{L}^{\pi(t)}[v(t, x)] - f(t, x, \pi)] = 0, \quad \forall (t, x) \in [t_0, T) \times \mathbb{R}^n. \tag{2.6}$$

with the terminal condition:

$$v(T, x) = -g(x) = -\frac{1}{2}x^T G(T)x - x^T L(T), \quad \forall x \in \mathbb{R}^n.$$

Equation (2.6) is called the Hamilton-Jacobi-Bellman equation.

Finding the Optimal Control

The supremum in the HJB equation (2.6) can be found by setting the derivative with respect to π to zero. The derivative of the generator $\mathcal{L}^\pi$, $D_\pi[\mathcal{L}^\pi] \in \mathbb{R}^m$, is given by:

$$D_\pi\big[\mathcal{L}^\pi[v(t, x)]\big] = D_\pi\big[b(t, x, \pi)^T D_x[v(t, x)]\big] + D_\pi\bigg[\frac{1}{2}\mathrm{tr}(\sigma(t, x, \pi)\sigma^T(t, x, \pi)D_x^2[v(t, x)])\bigg]. \tag{2.7}$$

We have that

$$\begin{aligned} D_\pi\big[b^T(t, x, \pi)D_x[v(t, x)]\big] &= D_\pi\big[(x^T A^T(t) + \pi^T(t)B^T(t))D_x[v(t, x)]\big] \\ &= B^T(t)D_x[v(t, x)] \end{aligned} \tag{2.8}$$

The latter derivative in (2.7) is given by:

$$\begin{aligned} D_\pi\bigg[\frac{1}{2}\mathrm{tr}[\sigma(t, x, \pi)\sigma^T(t, x, \pi)D_x^2[v]]\bigg] &= \frac{1}{2}D_\pi\bigg[\mathrm{tr}\bigg[\sum_{i=1}^d (C_i x + D_i\pi)(C_i x + D_i\pi)^T D_x^2[v]\bigg]\bigg] \\ &= \frac{1}{2}\sum_{i=1}^d D_\pi\big[\mathrm{tr}[(C_i x + D_i\pi)(C_i x + D_i\pi)^T D_x^2[v]]\big] \\ &= \frac{1}{2}\sum_{i=1}^d D_\pi\big[(C_i x + D_i\pi)^T D_x^2[v](C_i x + D_i\pi)\big] \\ &= \sum_{i=1}^d D_i^T D_x^2[v(t, x)](C_i x + D_i\pi) \end{aligned} \tag{2.9}$$

The derivative of $f(t,x,\pi)$ with respect to π is

$$D_\pi f(t,x,\pi) = Sx + R\pi \tag{2.10}$$

Combining the three equations,(2.8), (2.9), (2.10), we get that

$$D_\pi[\mathcal{L}^\pi(t)[v(t,x)] - f(t,x,\pi)] = B^T D_x[v(t,x)] + \sum_{i=1}^{d} D_i^T D_x^2[v(t,x)](C_i x + D_i \pi) - Sx - R\pi$$

Setting this to zero, we get

$$\pi^* = \left[\sum_{i=1}^{d} D_i^T D_x^2[v(t,x)] D_i - R\right]^{-1} \left[Sx - B^T D_x[v(t,x)] - \sum_{i=1}^{d} D_i^T D_x^2[v(t,x)] C_i x\right] \tag{2.11}$$

We now substitute (2.11) into (2.6) to get:

$$\frac{\partial v}{\partial t} + b(t,x,\pi^*)^T D_x[v] + \frac{1}{2}\mathrm{tr}\big[\sigma(t,x,\pi^*)\sigma^T(t,x,\pi^*)D_x^2[v]\big] - \frac{1}{2}x^T Q x - \frac{1}{2}x^T S^T \pi^* - \frac{1}{2}\pi^{*T} S x - \frac{1}{2}\pi^{*T} R \pi^* = 0$$

As $D_x^2[v]$ is a symmetric matrix, we can write

$$\begin{aligned}
\mathrm{tr}\big[\sigma(t,x,\pi^*)\sigma^T(t,x,\pi^*)D_x^2[v]\big] &= \sum_{i=1}^{d} \mathrm{tr}\big[(C_i x + D_i \pi^*)(C_i x + D_i \pi^*)^T D_x^2[v]\big] \\
&= \sum_{i=1}^{d} (C_i x + D_i \pi^*)^T D_x^2[v] (C_i x + D_i \pi^*),
\end{aligned}$$

we get the HJB equation

$$\begin{aligned}
\frac{\partial v}{\partial t} + (Ax + B\pi^*)^T D_x[v] + \frac{1}{2}\sum_{i=1}^{d} (C_i x + D_i \pi^*)^T D_x^2[v] (C_i x + D_i \pi^*) \\
- \frac{1}{2}x^T Q x - \frac{1}{2}x^T S^T \pi^* - \frac{1}{2}\pi^{*T} S x - \frac{1}{2}\pi^{*T} R \pi^* = 0
\end{aligned} \tag{2.12}$$

where π^* is as in (2.11) and the terminal condition is given by

$$v(T,x) = -g(x) = -\frac{1}{2}x^T G(T) x - x^T L(T).$$

Solving the Primal HJB Equation

We assume that $v(t,x)$ is a quadratic function in x and we use the ansatz

$$v(t,x) = \frac{1}{2}x^T P(t) x + x^T M(t) + N(t), \tag{2.13}$$

with terminal conditions:

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0. \tag{2.14}$$

Then

$$\begin{aligned}
\frac{\partial v}{\partial t}(t,x) &= \frac{1}{2}x^T \dot{P}(t) x + x^T \dot{M}(t) + \dot{N}(t) \\
D_x[v(t,x)] &= P(t)x + M(t) \\
D_x^2[v(t,x)] &= P(t).
\end{aligned}$$

Substituting in (2.11) we get the optimal control

$$\pi^* = \Big[\sum_{i=1}^{d} D_i^T P D_i - R\Big]^{-1}\Big[Sx - B^T P x - B^T M - \sum_{i=1}^{d} D_i^T P C_i x\Big] \tag{2.15}$$

We can write this as

$$\pi^* = \vartheta_1 x + \kappa_1,$$

where

$$\vartheta_1 = \Big(\sum_{i=1}^{d} D_i^T P D_i - R\Big)^{-1}\Big(S - B^T P - \sum_{i=1}^{d} D_i^T P C_i\Big), \quad \kappa_1 = -\Big(\sum_{i=1}^{d} D_i^T P D_i + R\Big)^{-1} B^T M \tag{2.16}$$

Substituting this into (2.12), we get

$$\begin{aligned}
&\frac{\partial v}{\partial t} + (Ax + B(\vartheta_1 x + \kappa_1))^T D_x[v] + \frac{1}{2}\sum_{i=1}^{d}(C_i x + D_i(\vartheta_1 x + \kappa_1))^T D_x^2[v](C_i x + D_i(\vartheta_1 x + \kappa_1))\\
&-\frac{1}{2}x^T Q x - \frac{1}{2}x^T S^T(\vartheta_1 x + \kappa_1) - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T S x - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T R(\vartheta_1 x + \kappa_1) = 0 \implies\\
&\frac{1}{2}x^T \dot{P} x + x^T \dot{M} + \dot{N} + (x^T A^T + x^T \vartheta_1^T B^T + \kappa_1^T B^T)(Px + M)\\
&+\frac{1}{2}\sum_{i=1}^{d}(x^T C_i^T + x^T \vartheta_1^T D_i^T + \kappa_1^T D_i^T)P(C_i x + D_i \vartheta_1 x + D_i \kappa_1)\\
&-\frac{1}{2}x^T Q x - \frac{1}{2}x^T S^T(\vartheta_1 x + \kappa_1) - \frac{1}{2}(x^T \vartheta_1^T + \kappa_1^T) S x - \frac{1}{2}(x^T \vartheta_1^T + \kappa_1^T) R(\vartheta_1 x + \kappa_1) = 0
\end{aligned}$$

Rewriting this, we get

$$\begin{aligned}
&x^T\Big[\frac{1}{2}\dot{P} + \frac{1}{2}A^T P + \frac{1}{2}PA + \frac{1}{2}\vartheta_1^T B^T P + \frac{1}{2}PB\vartheta_1 + \frac{1}{2}\sum_{i=1}^{d}(C_i^T + \vartheta_1^T D_i^T)P(C_i + D_i\vartheta_1)\\
&-\frac{1}{2}Q - \frac{1}{2}\vartheta_1^T S - \frac{1}{2}S^T\vartheta_1 - \frac{1}{2}\vartheta_1^T R\vartheta_1\Big]x + x^T\Big[\dot{M} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M +\\
&\sum_{i=1}^{d}(C_i^T + \vartheta_1^T D_i^T)PD_i\kappa_1 - S^T\kappa_1 - \vartheta_1^T R\kappa_1\Big] + \dot{N} + \kappa_1^T B^T M + \frac{1}{2}\sum_{i=1}^{d}\kappa_1^T D_i^T P D_i \kappa_1 - \frac{1}{2}\kappa_1^T R \kappa_1 = 0
\end{aligned}$$

This equation must equal zero for all x, hence the coefficients in front of the quadratic term, as well as x and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\begin{aligned}
\frac{1}{2}\dot{P} + \frac{1}{2}A^T P + \frac{1}{2}PA + \frac{1}{2}\vartheta_1^T B^T P + \frac{1}{2}PB\vartheta_1 + \frac{1}{2}\sum_{i=1}^{d}(C_i^T + \vartheta_1^T D_i^T)P(C_i + D_i\vartheta_1)&\\
-\frac{1}{2}Q - \frac{1}{2}\vartheta_1^T S - \frac{1}{2}S^T\vartheta_1 - \frac{1}{2}\vartheta_1^T R\vartheta_1 = 0&
\end{aligned} \tag{2.17}$$

$$\dot{M} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M + \sum_{i=1}^{d}(C_i^T + \vartheta_1^T D_i^T)PD_i\kappa_1 - S^T\kappa_1 - \vartheta_1^T R\kappa_1 = 0 \tag{2.18}$$

$$\dot{N} + \kappa_1^T B^T M + \frac{1}{2}\sum_{i=1}^{d}\kappa_1^T D_i^T P D_i\kappa_1 - \frac{1}{2}\kappa_1^T R\kappa_1 = 0, \tag{2.19}$$

where ϑ_1 and κ_1 are as in (2.16) and the terminal conditions are as in (2.14).

2.1.2 BSDE Method

Solution via the Primal BSDE

We define the Hamiltonian $\mathcal{H} : \Omega \times [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times R^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{H}(t,x,\pi,p,q) &= b^T p + \operatorname{tr}(\sigma^T q) - f(t,x,\pi)\\ &= x^T A^T p + \pi^T B^T p + \sum_{i=1}^{d}\left(x^T C_i^T q_i + \pi^T D_i^T q_i\right) - \frac{1}{2}x^T Q x - x^T S^T \pi - \frac{1}{2}\pi^T R\pi \end{aligned} \tag{2.20}$$

where we denote by $q_i \in \mathbb{R}^n$ the i^{th} column of the matrix $q \in \mathbb{R}^{n\times d}$. The adjoint process is given by

$$\begin{cases} \mathrm{d}p &= -D_x[\mathcal{H}(t, X(t), \pi(t), p(t), q(t))]\,\mathrm{d}t + \sum_{i=1}^{d} q_i(t)\,\mathrm{d}W_i(t)\\ p(T) &= -D_x[g(X(T))] = -G(T)X(T) - L(T)\end{cases} \tag{2.21}$$

According to the Stochastic Maximum Principle, the optimal control for the optimisation problem (**??**) satisfies the condition

$$D_\pi \mathcal{H}(t, X(t), \pi(t), p(t), q(t)) = 0.$$

We have

$$D_\pi \mathcal{H}(t, X(t), \pi(t), p(t), q(t)) = B^T p + \sum_{i=1}^{d} D_i^T q_i - SX - R\pi$$

so

$$B^T p + \sum_{i=1}^{d} D_i^T q_i - SX - R\pi = 0 \tag{2.22}$$

Solving this, we can find a linear solution for the control, which we denote by

$$\pi = \vartheta_2 X + \kappa_2. \tag{2.23}$$

Substituting the control in the Hamiltonian (2.20) we get

$$\begin{aligned}\mathcal{H} = X^T A^T p + (\vartheta_2 X + \kappa_2)^T B^T p + \sum_{i=1}^{d}\left(X^T C_i^T q_i + (\vartheta_2 X + \kappa_2)^T D_i^T q_i\right) - \frac{1}{2}X^T Q X\\ -\frac{1}{2}X^T S^T(\vartheta_2 X + \kappa_2) - \frac{1}{2}(\vartheta_2 X + \kappa_2)^T S X - \frac{1}{2}(\vartheta_2 X + \kappa_2)^T R(\vartheta_2 X + \kappa_2)\end{aligned} \tag{2.24}$$

The derivative of the Hamiltonian is then

$$D_x[\mathcal{H}] = A^T p + \vartheta_2^T B^T p + \sum_{i=1}^{d} C_i^T q_i + \sum_{i=1}^{d} \vartheta_2^T D_i^T q_i - QX - 2S^T\vartheta_2 X - S^T\kappa_2 - \vartheta_2^T R\vartheta_2 X - \vartheta_2^T R\kappa_2 \tag{2.25}$$

We try an ansatz for p of the form

$$p = \varphi(t)X(t) + \psi(t),$$

where $\varphi(t) \in \mathbb{R}^{n\times n}$ and $\psi(t) \in R^n$. Applying Ito's formula, we get

$$\begin{aligned}\mathrm{d}p &= \frac{\partial f}{\partial t}\,\mathrm{d}t + (D_x[f])^T\,\mathrm{d}X + \frac{1}{2}(\mathrm{d}X)^T D_x^2[f]\,\mathrm{d}X\\ &= (\dot\varphi X + \dot\psi)\,\mathrm{d}t + \varphi\,\mathrm{d}X\\ &= (\dot\varphi X + \dot\psi + \varphi b(t,X,\pi))\,\mathrm{d}t + \varphi\sigma(t,X,\pi)\,\mathrm{d}W\\ &= \left[\dot\varphi X + \dot\psi + \varphi A X + \varphi B\pi\right]\mathrm{d}t + \varphi\sum_{i=1}^{d}(C_i X + D_i\pi)\,\mathrm{d}W_i\\ &= \left[\dot\varphi X + \dot\psi + \varphi A X + \varphi B\vartheta_2 X + \varphi B\kappa_2\right]\mathrm{d}t + \sum_{i=1}^{d}\varphi(C_i X + D_i\vartheta_2 X + D_i\kappa_2)\,\mathrm{d}W_i\end{aligned} \tag{2.26}$$

Equating the coefficients of (2.26) with (2.21), we get the system

$$\dot{\varphi}X + \dot{\psi} + \varphi AX + \varphi B\vartheta_2 X + \varphi B\kappa_2 = -D_x[\mathcal{H}(t, X(t), \pi(t), p(t), q(t))] \tag{2.27}$$

$$\varphi(C_i X + D_i\vartheta_2 X + D_i\kappa_2) = q_i \quad i \in \{1, \dots, d\} \tag{2.28}$$

$$B^T\varphi X + B^T\psi + \sum_{i=1}^{d} D_i^T q_i - SX - R(\vartheta_1 X + \kappa_1) = 0, \tag{2.29}$$

where the third equation is the Hamiltonian condition (2.22). We now substitute q_i from the second equation into (2.25) and (2.29), so that our system becomes

$$\begin{aligned}\dot{\varphi}(t)X(t) + \dot{\psi}(t) + \varphi(t)AX(t) + \varphi B\vartheta_2 X + \varphi B\kappa_2 = &-A^T\varphi X - A^T\psi - \vartheta_2^T B^T\varphi X - \vartheta_2^T B^T\psi \\ &- \sum_{i=1}^{d} C_i^T\varphi(C_i X + D_i\vartheta_2 X + D_i\kappa_2) - \sum_{i=1}^{d}\vartheta_2^T D_i^T\varphi(C_i X + D_i\vartheta_2 X + D_i\kappa_2) \\ &+ QX + 2S^T\vartheta_2 X + S^T\kappa_2 + \vartheta_2^T R\vartheta_2 X + \vartheta_2^T R\kappa_2 \end{aligned} \tag{2.30}$$

$$B^T\varphi X + B^T\psi + \sum_{i=1}^{d} D_i^T\left[\varphi(C_i X + D_i\vartheta_2 X + D_i\kappa_2)\right] - SX - R\vartheta_2 x - R\kappa_2 = 0 \tag{2.31}$$

From (2.31), we get

$$\pi^* = \vartheta_2 X + \kappa_2 = \left[\sum_{i=1}^{d} D_i^T\varphi D_i - R\right]^{-1}\left[SX - B^T\varphi X - B^T\psi - \sum_{i=1}^{d} D_i^T\varphi C_i X\right], \tag{2.32}$$

i.e.,

$$\vartheta_2 = \left[\sum_{i=1}^{d} D_i^T\varphi D_i - 2R\right]^{-1}\left[S - B^T\varphi - \sum_{i=1}^{d} D_i^T\varphi C_i\right], \quad \kappa_2 = -\left[\sum_{i=1}^{d} D_i^T\varphi D_i - 2R\right]^{-1} B^T\psi. \tag{2.33}$$

We rewrite equation (2.30) as

$$\begin{aligned}&\left[\dot{\varphi} + \varphi A + A^T\varphi + \varphi B\vartheta_2 + \vartheta_2^T B^T\varphi + \sum_{i=1}^{d}(C_i^T + \vartheta_2^T D_i^T)\varphi(C_i + D_i\vartheta_2) - Q - S^T\vartheta_2 - \vartheta_2^T S - \vartheta_2^T R\vartheta_2\right]X \\ &\quad + \left[\dot{\psi} + \varphi B\kappa_2 + A^T\psi + \vartheta_2^T B^T\psi + \sum_{i=1}^{d} C_i^T\varphi D_i\kappa_2 + \sum_{i=1}^{d}\vartheta_2^T D_i^T\varphi D_i\kappa_2 - S^T\kappa_2 - \vartheta_2^T R\kappa_2\right] = 0\end{aligned}$$

Since this must be true for all X, the coefficient in front of X must be equal to zero, so we get the system of ODEs

$$\dot{\varphi} + \varphi A + A^T\varphi + \varphi B\vartheta_2 + \vartheta_2^T B^T\varphi + \sum_{i=1}^{d}(C_i^T + \vartheta_2^T D_i^T)\varphi(C_i + D_i\vartheta_2) - Q - S^T\vartheta_2 - \vartheta_2^T S - \vartheta_2^T R\vartheta_2 = 0 \tag{2.34}$$

$$\dot{\psi} + \varphi B\kappa_2 + A^T\psi + \vartheta_2^T B^T\psi + \sum_{i=1}^{d} C_i^T\varphi D_i\kappa_2 + \sum_{i=1}^{d}\vartheta_2^T D_i^T\varphi D_i\kappa_2 - S^T\kappa_2 - \vartheta_2^T R\kappa_2 = 0, \tag{2.35}$$

where ϑ_2 and κ_2 are as in (2.33) and the terminal conditions are given by

$$\varphi(T) = -G(T), \quad \psi(T) = -L(T). \tag{2.36}$$

Equivalence of Primal HJB and Primal BSDE

From the Primal HJB, the optimal control is given by (2.15):

$$\pi^* = \left[\sum_{i=1}^{d} D_i^T P D_i - R\right]^{-1}\left[Sx - B^T Px - B^T M - \sum_{i=1}^{d} D_i^T PC_i x\right] \tag{2.37}$$

and from the Primal BSDE the optimal control is (2.32):

$$\pi^* = \left[\sum_{i=1}^{d} D_i^T \varphi D_i - R\right]^{-1} \left[SX - B^T\varphi X - B^T\psi - \sum_{i=1}^{d} D_i^T \varphi C_i X\right], \tag{2.38}$$

Comparing, we see that the equations are identical and we get the relation

$$\varphi = P, \quad \psi = M.$$

The ODE from the Primal BSDE for φ is (2.34). Substituting $\varphi = P$ and $\vartheta_2 = \vartheta_1$ we get

$$\dot{P} + PA + A^T P + PB\vartheta_1 + \vartheta_1^T B^T P + \sum_{i=1}^{d} (C_i^T + \vartheta_1^T D_i^T) P (C_i + D_i \vartheta_1) - Q - 2S^T \vartheta_1 - \vartheta_1^T R \vartheta_1 = 0,$$

which is equal to twice the ODE for P from the primal HJB (2.17), i.e.

$$\frac{1}{2}\left[\dot{P} + PA + A^T P + PB\vartheta_1 + \vartheta_1^T B^T P + \sum_{i=1}^{d} (C_i^T + \vartheta_1^T D_i^T) P (C_i + D_i \vartheta_1) - Q - 2S^T \vartheta_1 - \vartheta_1^T R \vartheta_1\right] = 0.$$

Similarly, substituting $\varphi = P, \psi = M$, $\vartheta_2 = \vartheta_1$ and $\kappa_2 = \kappa_1$ into (2.35) we get

$$\dot{M} + PB\kappa_1 + A^T M + \vartheta_1^T B^T M + \sum_{i=1}^{d} C_i^T P D_i \kappa_1 + \sum_{i=1}^{d} \vartheta_1^T D_i^T P D_i \kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 = 0$$

which is the same equation as (2.18):

$$\dot{M} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M + \sum_{i=1}^{d} (C_i^T + \vartheta_1^T D_i^T) P D_i \kappa_1 - S^T \kappa_1 - \vartheta_1^T R \kappa_1 = 0$$

The terminal conditions for the primal BSDE are given by (2.36):

$$\varphi(T) = -G(T), \quad \psi(T) = -L(T),$$

and from the primal HJB (2.14):

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0.$$

So the ODEs for solving P, M and φ, ψ are identical, hence we have equivalence between the two methods.

2.2 Solving the Dual Problem

2.2.1 HJB Method

The Dual HJB

Recall that the dual process Y satisfies (1.4):

$$\begin{cases} \mathrm{d}Y(t) &= \left[\alpha(t) - A(t)^T Y(t) - \sum_{i=1}^d C_i(t)^T \beta_i(t)\right] \mathrm{d}t + \sum_{i=1}^d \beta_i(t)\, \mathrm{d}W_i(t) \\ Y(t_0) &= y \end{cases}$$

and also recall from (**??**) that the dual value function is given by

$$\tilde{v}(t, Y(t)) = \sup_{\alpha,\beta_1,\dots,\beta_d} \mathbb{E}\left[- \int_{t_0}^T \phi(t,\alpha,\beta)\,\mathrm{d}t - h(Y(T))\right],$$

where $\beta = B^T Y + \sum_{i=1}^d D_i^T \beta_i$ and $\phi(t,\alpha,\beta)$ and $h(Y(T))$ are given in (**??**) and (**??**). Then, following the same steps as for the primal problem, the dual HJB equation is given by

$$\frac{\partial \tilde{v}}{\partial t}(t,y) + \sup_{\alpha,\beta_1,\dots,\beta_d} \left[\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] - \phi(t,\alpha,\beta)\right] = 0,$$

where the generator is given by

$$\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] = \left(\alpha^T - y^T A - \sum_{i=1}^d \beta_i^T C_i\right) D_y[\tilde{v}] + \frac{1}{2}\sum_{i=1}^d \beta_i^T D_y^2[\tilde{v}]\beta_i,$$

and the terminal condition is

$$\tilde{v}(T,y) = -h(y) = -\frac{1}{2}(y^T + L^T)G^{-1}(y+L).$$

Finding the Optimal Controls

To find the supremum, we set the derivatives with respect to $\alpha, \beta_1, \dots, \beta_d$ to zero. We have

$$D_\alpha\left[\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] - \phi\right] = D_y[\tilde{v}] - \tilde{Q}\alpha - \tilde{S}^T\left(B^T y + \sum_{i=1}^d D_i^T \beta_i\right) = 0 \tag{2.39}$$

$$D_{\beta_i}\left[\mathcal{L}^{\alpha,\beta_i}[\tilde{v}(t,y)] - \phi\right] = -C_i D_y[\tilde{v}] + D_y^2[\tilde{v}]\beta_i - D_i\left(\tilde{S}\alpha + \tilde{R}\left(B^T y + \sum_{i=1}^d D_i^T \beta_i\right)\right) = 0 \tag{2.40}$$

This is a system of $d+1$ equations in $d+1$ unknowns, so it can be solved and the optimal controls are linear functions of y, which we denote by α^* and β_i^*. The HJB equation then becomes

$$\frac{\partial \tilde{v}}{\partial t} + \left(\alpha^{*T} - Y^T A - \sum_{i=1}^d \beta_i^{*T} C_i\right) D_y[\tilde{v}] + \frac{1}{2}\sum_{i=1}^d \beta_i^{*T} D_y^2[\tilde{v}]\beta_i^* - \phi\left(t, \alpha^*, B^T y + \sum_{i=1}^d D_i^T \beta_i^*\right) = 0 \tag{2.41}$$

Solving the Dual HJB

Suppose that $\tilde{v}$ is a quadratic function in y and use the ansatz

$$\tilde{v}(t,y) = \frac{1}{2} y^T \tilde{P}(t) y + y^T \tilde{M}(t) + \tilde{N}(t), \tag{2.42}$$

with terminal conditions

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = -\frac{1}{2}L^T(T)G^{-1}(T)L(T). \tag{2.43}$$

Then

$$\frac{\partial \tilde{v}}{\partial t}(t,y) = \frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t)y + y^T\dot{\tilde{M}}(t) + \dot{\tilde{N}}(t)$$
$$D_y[v(\tilde{t,y})] = \tilde{P}(t)y + \tilde{M}(t)$$
$$D_y^2[v(\tilde{t,y})] = \tilde{P}(t)$$

The system of equations (2.39) and (2.40) from which we derive the optimal controls α^* and β_i^* is now given by

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}\alpha - \tilde{S}^T\Big(B^Ty + \sum_{i=1}^d D_i^T\beta_i\Big) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}\beta_i + D_i\Big(\tilde{S}\alpha + \tilde{R}\Big(B^Ty + \sum_{i=1}^d D_i^T\beta_i\Big)\Big) = 0 \end{cases} \tag{2.44}$$

We do not solve this system explicitly, however, the solutions for α^* and β_i^* are linear in y, hence we simply denote by $\tilde{\vartheta}$ and $\tilde{\kappa}$ the coefficients before y and the free coefficient in α^* and similarly for β_i, i.e.

$$\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}, \quad \beta_i = \tilde{\vartheta}_i y + \tilde{\kappa}_i. \tag{2.45}$$

Substituting this into the HJB equation (2.41) we get

$$\frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}y + y^T\frac{\tilde{M}}{\mathrm{d}t} + \frac{\tilde{N}}{\mathrm{d}t} + \Big(y^T\tilde{\vartheta}^T + \tilde{\kappa}^T - y^TA - \sum_{i=1}^d(y^T\tilde{\vartheta}_i^T + \tilde{\kappa}_i^T)C_i\Big)(\tilde{P}y + \tilde{M})$$
$$+\frac{1}{2}\sum_{i=1}^d(y^T\tilde{\vartheta}_i^T + \tilde{\kappa}_i^T)\tilde{P}(\tilde{\vartheta}_i y + \tilde{\kappa}_i) - \phi\Big(t, \tilde{\vartheta}y + \tilde{\kappa}, B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\Big) = 0$$

Expanding $\phi\Big(t, \tilde{\vartheta}y + \tilde{\kappa}, B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\Big)$ we get

$$\phi(t,\alpha^*,\beta^*) = \frac{1}{2}(\tilde{\vartheta}y + \tilde{\kappa})^T\tilde{Q}(\tilde{\vartheta}y + \tilde{\kappa}) + (\tilde{\vartheta}y + \tilde{\kappa})^T\tilde{S}^T\big(B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\big)$$
$$+\frac{1}{2}\big(B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\big)^T\tilde{R}(B^Ty + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i))$$

Rearranging, we get

$$\frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}y + y^T\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \frac{\mathrm{d}\tilde{N}}{\mathrm{d}t} + y^T\tilde{\vartheta}^T\tilde{P}y + y^T\tilde{P}\tilde{\kappa} - y^TA\tilde{P}y - y^T\sum_{i=1}^d\tilde{\vartheta}_i^TC_i\tilde{P}y - y^T\tilde{P}\sum_{i=1}^dC_i^T\tilde{\kappa}_i$$
$$+y^T\tilde{\vartheta}^T\tilde{M} + \tilde{\kappa}^T\tilde{M} - y^TA\tilde{M} - \sum_{i=1}^d y^T\tilde{\vartheta}_i^TC_i\tilde{M} - \sum_{i=1}^d\tilde{\kappa}_i^TC_i\tilde{M} + \frac{1}{2}y^T\sum_{i=1}^d\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i y$$
$$+y^T\sum_{i=1}^d\tilde{\vartheta}_i\tilde{P}\tilde{\kappa}_i + \frac{1}{2}\sum_{i=1}^d\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i - \frac{1}{2}y^T\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}y - y^T\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa} - y^T\tilde{\vartheta}^T\tilde{S}^T(B^T + \sum_{i=1}^d D_i^T\tilde{\vartheta}_i)y$$
$$-y^T\tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^d D_i^T\tilde{\kappa}_i - y^T(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i)\tilde{S}\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^d D_i^T\tilde{\kappa}_i$$
$$-\frac{1}{2}y^T\Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^d D_i^T\tilde{\vartheta}_i\Big)y - y^T\Big(B + \sum_{i=1}^d\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\sum_{i=1}^d D_i^T\tilde{\kappa}_i$$
$$-\frac{1}{2}\Big(\sum_{i=1}^d\tilde{\kappa}_i^TD_i\Big)\tilde{R}\Big(\sum_{i=1}^d D_i^T\tilde{\kappa}_i\Big) = 0$$

Grouping together the coefficients in front of y we get:

$$
\begin{aligned}
y^T\Bigg[\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{\vartheta}^T\tilde{P} - A\tilde{P} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{P} + \frac{1}{2}\sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} - \tilde{\vartheta}^T\tilde{S}^T\Big(B^T + \sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)& \\
-\frac{1}{2}\Big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big)\Bigg]y& \\
+y^T\Bigg[\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \tilde{P}\tilde{\kappa} - \tilde{P}\sum_{i=1}^{d}C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{M} - A\tilde{M} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{M} + \sum_{i=1}^{d}\tilde{\vartheta}_i\tilde{P}\tilde{\kappa}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i& \\
-\big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\big)\tilde{S}\tilde{\kappa} - \big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\big)\tilde{R}\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i\Bigg]& \\
+\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t} + \tilde{\kappa}^T\tilde{M} - \sum_{i=1}^{d}\tilde{\kappa}_i^T C_i\tilde{M} + \frac{1}{2}\sum_{i=1}^{d}\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i - \frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i& \\
-\frac{1}{2}\Big(\sum_{i=1}^{d}\tilde{\kappa}_i^T D_i\Big)\tilde{R}\Big(\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i\Big) = 0&
\end{aligned}
$$

This equation must equal zero for all y, hence the coefficients in front of the quadratic term, as well as y^T and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{\vartheta}^T\tilde{P} - A\tilde{P} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{P} + \frac{1}{2}\sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{P}\tilde{\vartheta}_i - \frac{1}{2}\tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} - \tilde{\vartheta}^T\tilde{S}^T\big(B^T + \sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\big)& \\
-\frac{1}{2}\Big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^{d}D_i^T\tilde{\vartheta}_i\Big) = 0& \qquad (2.46)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \tilde{P}\tilde{\kappa} - \tilde{P}\sum_{i=1}^{d}C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{M} - A\tilde{M} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{M} + \sum_{i=1}^{d}\tilde{\vartheta}_i\tilde{P}\tilde{\kappa}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i& \\
-\big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\big)\tilde{S}\tilde{\kappa} - \big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\big)\tilde{R}\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i = 0& \qquad (2.47)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t} + \tilde{\kappa}^T\tilde{M} - \sum_{i=1}^{d}\tilde{\kappa}_i^T C_i\tilde{M} + \frac{1}{2}\sum_{i=1}^{d}\tilde{\kappa}_i^T\tilde{P}\tilde{\kappa}_i - \frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^T\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i& \\
-\frac{1}{2}\Big(\sum_{i=1}^{d}\tilde{\kappa}_i^T D_i\Big)\tilde{R}\Big(\sum_{i=1}^{d}D_i^T\tilde{\kappa}_i\Big) = 0& \qquad (2.48)
\end{aligned}
$$

where $\tilde{\vartheta}, \tilde{\kappa}, \tilde{\vartheta}_i$ and $\tilde{\kappa}_i$ satisfy the system (2.44):

$$
\begin{cases}
\tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\vartheta}y + \kappa) - \tilde{S}^T\Big(B^T y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i y + \kappa_i)\Big) = 0 \\
C_i(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\vartheta}_i y + \kappa_i) + D_i\Big(\tilde{S}(\tilde{\vartheta}y + \kappa) + \tilde{R}\Big(B^T y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i y + \kappa_i)\Big)\Big) = 0
\end{cases}
\qquad (2.49)
$$

and the terminal conditions are given by:

$$
\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = -\frac{1}{2}L^T(T)G^{-1}(T)L(T). \qquad (2.50)
$$

2.2.2 BSDE Method

Solution via the Dual BSDE

The Hamiltonian $\tilde{\mathcal{H}} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^{nd} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ for the dual problem is defined as

$$\begin{aligned}\tilde{\mathcal{H}}(t, Y, \alpha, \beta_1, \dots, \beta_d, p, q) &= -\phi\Big(t, \alpha, B^T Y + \sum_{i=1}^d D_i^T \beta_i\Big) + p^T(\alpha - A^T Y - \sum_{i=1}^d C_i^T \beta_i) + \sum_{i=1}^d \beta_i^T q_i \\ &= p^T\alpha - p^T A^T Y - p^T \sum_{i=1}^d C_i^T \beta_i + \sum_{i=1}^d \beta_i^T q_i - \phi\Big(t, \alpha, B^T Y + \sum_{i=1}^d D_i^T \beta_i\Big)\end{aligned} \tag{2.51}$$

The adjoin equation is given by the system

$$\begin{cases} \mathrm{d}p(t) &= -D_y[\tilde{\mathcal{H}}]\,\mathrm{d}t + \sum_{i=1}^d q_i\,\mathrm{d}W_i \\ p(T) &= -D_y[h(Y(T))] = -G^{-1}Y(T) - G^{-1}L \end{cases} \tag{2.52}$$

Due to the Stochastic Maximum Principle, the optimal control can be found by setting $D_\alpha[\tilde{\mathcal{H}}] = 0$ and $D_{\beta_i}[\tilde{\mathcal{H}}] = 0$ for all $i = 1, \dots, d$, so we get the system

$$D_\alpha[\tilde{\mathcal{H}}] = p - \tilde{Q}\alpha - \tilde{S}^T\Big(B^T Y + \sum_{i=1} D_i^T \beta_i\Big) = 0 \tag{2.53}$$

$$D_{\beta_i}[\tilde{\mathcal{H}}] = q_i - C_i p - D_i\tilde{S}\alpha - D_i\tilde{R}\Big(B^T Y + \sum_{i=1} D_i^T \beta_i\Big) = 0 \tag{2.54}$$

There are $d+1$ equations in $d+1$ unknowns, so the system can be solved and the optimal controls are linear functions of Y, which we denote by

$$\alpha = \tilde{\vartheta}Y + \tilde{\kappa}, \quad \beta_i = \tilde{\vartheta}_i Y + \tilde{\kappa}_i, \quad i \in \{1, \dots, d\}$$

Substituting into the Hamiltonian (2.51) we get

$$\begin{aligned}\tilde{\mathcal{H}} = p^T(\tilde{\vartheta}Y + \tilde{\kappa}) - p^T A^T Y - p^T \sum_{i=1}^d C_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i) + \sum_{i=1}^d (Y^T\tilde{\vartheta}_i^T + \tilde{\kappa}_i^T)q_i \\ -\phi\Big(t, \tilde{\vartheta}Y + \tilde{\kappa}, B^T Y + \sum_{i=1}^d D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big)\end{aligned}$$

The derivative of the dual Hamiltonian is then

$$\begin{aligned}D_y[\tilde{\mathcal{H}}] = \tilde{\vartheta}^T p - Ap - \sum_{i=1}^d \tilde{\vartheta}_i^T C_i p + \sum_{i=1}^d \tilde{\vartheta}_i^T q_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}Y - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} \\ -2\tilde{\vartheta}^T\tilde{S}\Big(B^T + \sum_{i=1}^d D_i^T\tilde{\vartheta}_i\Big)Y - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^d D_i^T\tilde{\kappa}_i - \Big(B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i\Big)\tilde{S}\tilde{\kappa} \\ -\Big(B + \sum_{i=1}^d \tilde{\vartheta}_i^T D_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^d D_i^T\tilde{\vartheta}_i\Big)Y - \Big(B^T + \sum_{i=1}^d D_i^T\tilde{\vartheta}_i\Big)\tilde{R}\sum_{i=1}^d D_i^T\tilde{\kappa}_i\end{aligned} \tag{2.55}$$

We try the ansatz

$$p = \tilde{\varphi}(t)Y + \tilde{\psi}(t),$$

where $\varphi(t) \in \mathbb{R}^{n\times n}$ and $\psi(t) \in R^n$. Applying Ito's formula to p, we get

$$
\begin{aligned}
\mathrm{d}p &= \left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y + \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\right)\mathrm{d}t + \tilde{\varphi}\,\mathrm{d}Y \\
&= \left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y + \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\right)\mathrm{d}t + \tilde{\varphi}\big[\alpha - A^TY - \sum_{i=1}^{d} C_i^T\beta_i\big]\,\mathrm{d}t + \tilde{\varphi}\sum_{i=1}^{d}\beta_i\,\mathrm{d}W_i \\
&= \left[\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y + \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\alpha - \tilde{\varphi}A^TY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\beta_i\right]\mathrm{d}t + \tilde{\varphi}\sum_{i=1}^{d}\beta_i\,\mathrm{d}W_i \\
&= \left[\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y + \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta}Y + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}A^TY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\vartheta}_iY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i\right]\mathrm{d}t + \tilde{\varphi}\sum_{i=1}^{d}(\tilde{\vartheta}_iY + \tilde{\kappa}_i)\,\mathrm{d}W_i
\end{aligned}
\tag{2.56}
$$

Equating the coefficients of (2.56) and (2.52) we get

$$
\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y + \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta}Y + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}A^TY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\vartheta}_iY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i = -D_y[\tilde{\mathcal{H}}] \tag{2.57}
$$

$$
\tilde{\varphi}\tilde{\vartheta}_iY + \tilde{\varphi}\tilde{\kappa}_i = q_i \tag{2.58}
$$

where the RHS of (2.57) is given by (2.55). We now substitute q_i from equation (2.58) into the system (2.53) and (2.54) we get

$$
\begin{cases}
\tilde{\varphi}Y + \tilde{\psi} - \tilde{Q}(\tilde{\vartheta}Y + \tilde{\kappa}) - \tilde{S}^T\Big(B^TY + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_iY + \tilde{\kappa}_i)\Big) = 0 \\
\tilde{\varphi}(\tilde{\vartheta}_iY + \tilde{\kappa}_i) - C_i(\tilde{\varphi}Y + \tilde{\psi}) - D_i\tilde{S}(\tilde{\vartheta}Y + \tilde{\kappa}) - D_i\tilde{R}\Big(B^TY + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_iY + \tilde{\kappa}_i)\Big) = 0
\end{cases}
\tag{2.59}
$$

which is the system we need to solve, to acquire the optimal controls $\alpha^* = \tilde{\vartheta}Y + \tilde{\kappa}$ and $\beta_i^* = \tilde{\vartheta}_i + \tilde{\kappa}_i$. Substituting q_i into (2.57) we get

$$
\begin{aligned}
&\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}Y + \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta}Y + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}A^TY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\vartheta}_iY - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i = -\tilde{\vartheta}^Tp + Ap + \sum_{i=1}^{d}\tilde{\vartheta}_i^TC_ip \\
&+ \sum_{i=1}^{d}\tilde{\vartheta}_i^T(\tilde{\varphi}\tilde{\vartheta}_iY + \tilde{\varphi}\tilde{\kappa}_i) + \tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta}Y + \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} + 2\tilde{\vartheta}^T\tilde{S}\Big(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i\Big)Y + \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i \\
&+ \Big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i\Big)\tilde{S}\tilde{\kappa} + \Big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i\Big)Y + \Big(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i\Big)\tilde{R}\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i
\end{aligned}
\tag{2.60}
$$

We rewrite equation (2.60) as

$$
\Big[\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta} - \tilde{\varphi}A^T - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\vartheta}_i + \tilde{\vartheta}^T\tilde{\varphi} - A\tilde{\varphi} - \sum_{i=1}^{d}\tilde{\vartheta}_iC_i\tilde{\varphi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{\varphi}\tilde{\vartheta}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} \tag{2.61}
$$

$$
-2\tilde{\vartheta}^T\tilde{S}(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i) - (B + \sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i)\tilde{R}(B^T + \sum_{i=1}^{d} D_i^T\tilde{\vartheta}_i)\Big]Y \tag{2.62}
$$

$$
+\Big[\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{\psi} - A\tilde{\psi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^TC_i\tilde{\psi} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{\varphi}\tilde{\kappa}_i \tag{2.63}
$$

$$
-\tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i - (B + \sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i)(\tilde{S}\tilde{\kappa} + \tilde{R}\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i)\Big] = 0 \tag{2.64}
$$

Since this must be true for all Y, the coefficient in front of Y must be equal to zero, so we get

$$\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\vartheta} - \tilde{\varphi}A^T - \tilde{\varphi}\sum_{i=1}^{d} C_i^T \tilde{\vartheta}_i + \tilde{\vartheta}^T\tilde{\varphi} - A\tilde{\varphi} - \sum_{i=1}^{d} \tilde{\vartheta}_i C_i \tilde{\varphi} - \sum_{i=1}^{d} \tilde{\vartheta}_i^T \tilde{\varphi}\tilde{\vartheta}_i - \tilde{\vartheta}^T \tilde{Q}\tilde{\vartheta}$$
$$-2\tilde{\vartheta}^T \tilde{S}(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i) - (B + \sum_{i=1}^{d} \tilde{\vartheta}_i^T D_i)\tilde{R}(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i) = 0 \tag{2.65}$$

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \tilde{\varphi}\tilde{\kappa} - \tilde{\varphi}\sum_{i=1}^{d} C_i^T \tilde{\kappa}_i + \tilde{\vartheta}^T \tilde{\psi} - A\tilde{\psi} - \sum_{i=1}^{d} \tilde{\vartheta}_i^T C_i \tilde{\psi} - \sum_{i=1}^{d} \tilde{\vartheta}_i^T \tilde{\varphi}\tilde{\kappa}_i$$
$$-\tilde{\vartheta}^T \tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T \sum_{i=1}^{d} D_i^T \tilde{\kappa}_i - (B + \sum_{i=1}^{d} \tilde{\vartheta}_i^T D_i)(\tilde{S}\tilde{\kappa} + \tilde{R}\sum_{i=1}^{d} D_i^T \tilde{\kappa}_i) = 0, \tag{2.66}$$

where $\tilde{\vartheta}, \tilde{\kappa}, \tilde{\vartheta}_i$, and $\tilde{\kappa}_i$ can be found from the system (2.59):

$$\begin{cases} \tilde{\varphi}Y + \tilde{\psi} - \tilde{Q}(\tilde{\vartheta}Y + \tilde{\kappa}) - \tilde{S}^T\Big(B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big) = 0 \\ \tilde{\varphi}(\tilde{\vartheta}_i Y + \tilde{\kappa}_i) - C_i(\tilde{\varphi}Y + \tilde{\psi}) - D_i\tilde{S}(\tilde{\vartheta}Y + \tilde{\kappa}) - D_i\tilde{R}\Big(B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big) = 0 \end{cases}$$

and the terminal conditions are given by

$$\tilde{\varphi}(T) = -G^{-1}(T), \quad \tilde{\psi}(T) = -G^{-1}(T)L(T). \tag{2.67}$$

Equivalence between Dual HJB and Dual BSDE

From the dual HJB we get that the optimal controls $\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}$ and $\beta^* = \tilde{\vartheta}_i y + \tilde{\kappa}_i$ are solution to the system of equations (2.44):

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\vartheta}y + \tilde{\kappa}) - \tilde{S}^T(B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\vartheta}_i y + \tilde{\kappa}_i) + D_i\tilde{S}(\tilde{\vartheta}y + \tilde{\kappa}) + D_i\tilde{R}\Big(B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i y + \tilde{\kappa}_i)\Big) = 0 \end{cases}$$

Similarly, the optimal controls from the dual BSDE method are found by solving the system (2.59):

$$\begin{cases} \tilde{\varphi}Y + \tilde{\psi} - \tilde{Q}(\tilde{\vartheta}Y + \tilde{\kappa}) - \tilde{S}^T\Big(B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big) = 0 \\ \tilde{\varphi}(\tilde{\vartheta}_i Y + \tilde{\kappa}_i) - C_i(\tilde{\varphi}Y + \tilde{\psi}) - D_i\tilde{S}(\tilde{\vartheta}Y + \tilde{\kappa}) - D_i\tilde{R}\Big(B^T Y + \sum_{i=1}^{d} D_i^T(\tilde{\vartheta}_i Y + \tilde{\kappa}_i)\Big) = 0 \end{cases}$$

These systems are the same and therefore we get the relation

$$\tilde{\varphi} = \tilde{P}, \quad \tilde{\psi} = \tilde{M}.$$

The first ODE from the dual BSDE is (2.65), so substituting $\tilde{P} = \tilde{\varphi}$ in it we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}\tilde{\vartheta} - \tilde{P}A^T - \tilde{P}\sum_{i=1}^{d} C_i^T \tilde{\vartheta}_i + \tilde{\vartheta}^T\tilde{P} - A\tilde{P} - \sum_{i=1}^{d} \tilde{\vartheta}_i C_i \tilde{P} - \sum_{i=1}^{d} \tilde{\vartheta}_i^T \tilde{P}\tilde{\vartheta}_i - \tilde{\vartheta}^T \tilde{Q}\tilde{\vartheta}$$
$$-2\tilde{\vartheta}^T \tilde{S}(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i) - (B + \sum_{i=1}^{d} \tilde{\vartheta}_i^T D_i)\tilde{R}(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i) = 0$$

The first ODE from the dual HJB equation is given by (2.46):

$$\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{\vartheta}^T\tilde{P} - A\tilde{P} - \sum_{i=1}^{d} \tilde{\vartheta}_i^T C_i \tilde{P} + \frac{1}{2}\sum_{i=1}^{d} \tilde{\vartheta}_i^T \tilde{P}\tilde{\vartheta}_i - \frac{1}{2}\tilde{\vartheta}^T \tilde{Q}\tilde{\vartheta} - \tilde{\vartheta}^T \tilde{S}^T(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i)$$
$$-\frac{1}{2}\Big(B + \sum_{i=1}^{d} \tilde{\vartheta}_i^T D_i\Big)\tilde{R}\Big(B^T + \sum_{i=1}^{d} D_i^T \tilde{\vartheta}_i\Big) = 0$$

The two equations are equivalent with the second being the first one divided by 2. Similarly, plugging in $\tilde{P} = \tilde{\varphi}$ and $\tilde{M} = \tilde{\psi}$ in (2.66) we get

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \tilde{P}\tilde{\kappa} - \tilde{P}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{M} - A\tilde{M} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{M} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T\tilde{P}\tilde{\kappa}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i$$
$$-(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i)(\tilde{S}\tilde{\kappa} + \tilde{R}\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i) = 0$$

The respective ODE from the dual HJB is (2.47):

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \tilde{P}\tilde{\kappa} - \tilde{P}\sum_{i=1}^{d} C_i^T\tilde{\kappa}_i + \tilde{\vartheta}^T\tilde{M} - A\tilde{M} - \sum_{i=1}^{d}\tilde{\vartheta}_i^T C_i\tilde{M} + \sum_{i=1}^{d}\tilde{\vartheta}_i a\tilde{\kappa}_i - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i$$
$$-\big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\big)\tilde{S}\tilde{\kappa} - \big(B + \sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i\big)\tilde{R}\sum_{i=1}^{d} D_i^T\tilde{\kappa}_i = 0$$

The terminal conditions from the dual HJB method are given by (2.50):

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = \frac{1}{2}L^T(T)G^{-1}(T)L(T)$$

and from the dual BSDE, we have (2.67):

$$\tilde{\varphi}(T) = -G^{-1}(T), \quad \tilde{\psi}(T) = -G^{-1}(T)L(T).$$

As we can see the equations are identical, and their terminal conditions are also the same, so the two methods are equivalent.

2.3 Primal and Dual Equivalence

Recall that we have the relationship between the dual and primal value functions (**??**):

$$v(x) \leq \inf_y \{x^T y - \tilde{v}(y)\}$$

In this section, we impose no constraints on the control, and we will show that this leads to equality instead of inequality in the above equation. Substituting the respective ansatz, we have

$$\frac{1}{2}x^T P x + x^T M + N = \inf_y \left\{ x^T y - \frac{1}{2} y^T \tilde{P} y - y^T \tilde{M} - \tilde{N} \right\}$$

Setting the derivative of the RHS to zero, we get

$$y = \tilde{P}^{-1}(x - \tilde{M}),$$

so

$$\frac{1}{2}x^T P x + x^T M + N = -\frac{1}{2}(x^T - \tilde{M}^T)\tilde{P}^{-1}(x - \tilde{M}) - (x^T - \tilde{M}^T)\tilde{P}^{-1}\tilde{M} - \tilde{N} + x^T \tilde{P}^{-1}(x - \tilde{M})$$

Simplifying we get

$$\frac{1}{2}x^T P x + x^T M + N = \frac{1}{2}x^T \tilde{P}^{-1} x - x^T \tilde{P}^{-1}\tilde{M} + \frac{1}{2}\tilde{M}^T \tilde{P}^{-1}\tilde{M} - \tilde{N}.$$

Therefore, we get the relation

$$P = \tilde{P}^{-1}, \quad M = -\tilde{P}^{-1}\tilde{M}, \quad N = \frac{1}{2}\tilde{M}^T \tilde{P}^{-1}\tilde{M} - \tilde{N}. \tag{2.68}$$

To simplify computations, we consider a simpler case where $d = 1$ and $S = 0$. Then $\tilde{S} = 0, \tilde{Q} = Q^{-1}, \tilde{R} = R^{-1}$. The Riccati equation from the primal problem in this case is given by (2.17)

$$\dot{P} + 2PA + 2PB\vartheta_1 + (C_1^T + \vartheta_1^T D_1^T)P(C_1 + D_1\vartheta_1) - Q - \vartheta_1^T R \vartheta_1 = 0,$$

where

$$\vartheta_1 = (D_1^T P D_1 - R)^{-1}(-B^T P - D_1^T P C_1)$$

Therefore, our equation becomes

$$\begin{aligned} 0 &= \frac{\mathrm{d}P}{\mathrm{d}t} + 2PA - Q + C_1^T P C_1 + 2(PB + C_1^T P D_1)\vartheta_1 + \vartheta_1^T (D_1^T P D_1 - R)\vartheta_1 \\ &= \frac{\mathrm{d}P}{\mathrm{d}t} + 2PA - Q + C_1^T P C_1 - (PB + C_1^T P D_1)(D_1^T P D_1 - R)^{-1}(B^T P + D_1^T P C_1) \end{aligned}$$

Substituting $P = \tilde{P}^{-1}$, we get

$$\tilde{P}^{-1}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}\tilde{P}^{-1} - 2\tilde{P}^{-1}A + Q - C_1^T \tilde{P}^{-1}C_1 + (\tilde{P}^{-1}B + C_1^T \tilde{P}^{-1}D_1)(D_1^T \tilde{P}^{-1}D_1 - R)^{-1}(B^T \tilde{P}^{-1} + D_1^T \tilde{P}^{-1}C_1) = 0$$

Multiplying on the left and on the right by $\tilde{P}$, we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} - 2A\tilde{P} + \tilde{P}Q\tilde{P} - \tilde{P}C_1^T \tilde{P}^{-1}C_1\tilde{P} + \tilde{P}(\tilde{P}^{-1}B + C_1^T \tilde{P}^{-1}D_1)(D_1^T \tilde{P}^{-1}D_1 - R)^{-1}(B^T \tilde{P}^{-1} + D_1^T \tilde{P}^{-1}C_1)\tilde{P} = 0$$

Rewriting this, we get

$$\begin{aligned} &\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} - 2A\tilde{P} + \tilde{P}Q\tilde{P} + \tilde{P}C_1^T(\tilde{P}^{-1}D_1(D_1^T \tilde{P}D_1 - R)^{-1}D_1^T \tilde{P}^{-1} - \tilde{P}^{-1})C_1\tilde{P} + B(D_1^T \tilde{P}D_1 - R)^{-1}B^T \\ &\qquad + 2B(D_1^T \tilde{P}D_1 - R)^{-1}D_1^T \tilde{P}^{-1}C_1\tilde{P} = 0 \end{aligned} \tag{2.69}$$

On the other hand, for the dual problem we have:

$$\tilde{\vartheta} = Q\tilde{P}, \quad \tilde{\vartheta}_1 = (\tilde{P} - D_1 R^{-1} D_1^T)^{-1}(C_1\tilde{P} + D_1 R^{-1} B^T)$$

The dual Riccati equation is (2.46):

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + 2\tilde{\vartheta}^T\tilde{P} - 2A\tilde{P} - 2\tilde{\vartheta}_1^T C_1\tilde{P} + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - \tilde{\vartheta}^T Q^{-1}\tilde{\vartheta} - (B + \tilde{\vartheta}_1^T D_1)R^{-1}(B^T + D_1^T\tilde{\vartheta}_1) = 0$$

Substituting for $\tilde{\vartheta}$ we get

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - (B + \tilde{\vartheta}_1^T D_1)R^{-1}(B^T + D_1^T\tilde{\vartheta}_1) = 0$$

Substituting for $\tilde{\vartheta}_1$ we get

$$\begin{aligned} 0 &= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - (B + \tilde{\vartheta}_1^T D_1)R^{-1}(B^T + D_1^T\tilde{\vartheta}_1) \\ &= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - 2\tilde{P}C_1^T\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - BR^{-1}B^T - 2BR^{-1}D_1^T\tilde{\vartheta}_1 - \tilde{\vartheta}_1^T D_1 R^{-1} D_1^T\tilde{\vartheta}_1 \\ &= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - BR^{-1}B^T - 2(BR^{-1}D_1^T + \tilde{P}C_1^T)\tilde{\vartheta}_1 + \tilde{\vartheta}_1^T(\tilde{P} - D_1R^{-1}D_1^T)\tilde{\vartheta}_1 \\ &= \frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + \tilde{P}Q\tilde{P} - 2A\tilde{P} - BR^{-1}B^T - (BR^{-1}D_1^T + \tilde{P}C_1^T)(\tilde{P} - D_1R^{-1}D_1^T)^{-1}(C_1\tilde{P} + D_1R^{-1}B^T) \end{aligned}$$

This is then rewritten as

$$\begin{aligned} &\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} - 2A\tilde{P} + \tilde{P}Q\tilde{P} + \tilde{P}C_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}C_1\tilde{P} + 2BR^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}C_1\tilde{P} \\ &\qquad + B(R^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}D_1R^{-1} - R^{-1})B^T = 0 \end{aligned} \tag{2.70}$$

Now noting that

$$\begin{aligned} (D_1R^{-1}D_1^T - \tilde{P})^{-1} &= \tilde{P}^{-1}D_1(D_1^T\tilde{P}D_1 - R)^{-1}D_1^T\tilde{P}^{-1} - \tilde{P}^{-1} \\ R^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1}D_1R^{-1} - R^{-1} &= D_1^T\tilde{P}D_1 - R \\ R^{-1}D_1^T(D_1R^{-1}D_1^T - \tilde{P})^{-1} &= (D_1^T\tilde{P}D_1 - R)^{-1}D_1^T\tilde{P}^{-1} \end{aligned}$$

we get exactly (2.69), so the dual and primal HJB methods are equivalent.

3 Unconstrained Markovian Optimisation with Quadratic Running Cost

The development of this section draws upon the study "Weak Necessary and Sufficient SMP for Markovian Regime-Switching Diffusion Models" [5]. This research elucidates the stochastic maximum principle for Markovian regime-switching diffusion models. We further consider the PDE method using the Hamilton-Jacobi-Bellman equation described in [7]. We integrate these concepts into the framework of Chapter 1, focusing specifically on the multidimensional linear quadratic unconstrained control problem. In this case, we are able to derive analytical solutions to our problem, as shown in the following sections.

3.1 Primal HJB Equation

3.1.1 Deriving the Primal HJB

We transform the minimisation problem (**??**) to maximisation by noting that

$$\inf_{\pi} \mathbb{E}\left[\int_{t_0}^{T} f(t, X(t), \pi(t), \eta(t))\,\mathrm{d}t + g(X(T), \eta(T))\right] = -\sup_{\pi} \mathbb{E}\left[\int_{t_0}^{T} -f(t, X(t), \pi(t), \eta(t))\,\mathrm{d}t - g(X(T), \eta(T))\right]$$

and denote the value function

$$v(t, X(t), \eta(t)) = \sup_{\pi} \mathbb{E}\left[\int_{t_0}^{T} -f(t, X(t), \pi(t), \eta(t))\,\mathrm{d}t - g(X(T), \eta(T))\right] \tag{3.1}$$

The HJB equations is given by

$$\frac{\partial v}{\partial t}(t, x, i) + \sup_{\pi} \left[\mathcal{L}^{\pi}[v(t, x, i) - f(t, x, \pi, i)]\right] + \sum_{j \neq i}^{k} q_{ij}(v(t, x, j) - v(t, x, i)) = 0 \tag{3.2}$$

with terminal conditions

$$v(T, x, i) = -g(x, i) = -\frac{1}{2} x^T G(T, i) x + x^T L(T, i).$$

3.1.2 Finding the Optimal Control

The supremum can be found by setting the derivative with respect to π to zero. The derivative of the generator $\mathcal{L}^{\pi}$, $D_{\pi}[\mathcal{L}^{\pi}] \in \mathbb{R}^m$, is given by:

$$D_{\pi}\left[\mathcal{L}^{\pi}[v(t, x, i)]\right] = D_{\pi}\left[b(t, x, \pi, i)^T D_x[v(t, x, i)]\right] + D_{\pi}\left[\frac{1}{2}\mathrm{tr}(\sigma(t, x, \pi, i)\sigma^T(t, x, \pi, i) D_x^2[v(t, x, i)])\right]. \tag{3.3}$$

We have that

$$\begin{aligned} D_{\pi}\left[b^T(t, x, \pi, i) D_x[v(t, x, i)]\right] &= D_{\pi}\left[(x^T A^T(t, i) + \pi^T(t) B^T(t, i)) D_x[v(t, x, i)]\right] \\ &= B^T(t, i) D_x[v(t, x, i)] \end{aligned} \tag{3.4}$$

The latter derivative in (3.3) is given by:

$$
\begin{aligned}
D_\pi\left[\frac{1}{2}\mathrm{tr}[\sigma(t,x,\pi,i)\sigma^T(t,x,\pi,i)D_x^2[v(t,x,i)]]\right] &= \frac{1}{2}D_\pi\left[\mathrm{tr}\left[\sum_{j=1}^{d}(C_jx+D_j\pi)(C_jx+D_j\pi)^T D_x^2[v]\right]\right] \\
&= \frac{1}{2}\sum_{j=1}^{d} D_\pi\big[\mathrm{tr}[(C_jx+D_j\pi)(C_jx+D_j\pi)^T D_x^2[v]]\big] \\
&= \frac{1}{2}\sum_{j=1}^{d} D_\pi\big[(C_jx+D_j\pi)^T D_x^2[v](C_jx+D_j\pi)\big] \\
&= \sum_{j=1}^{d} D_j^T(t,i)D_x^2[v(t,x,i)](C_j(t,i)x+D_j(t,i)\pi) \qquad (3.5)
\end{aligned}
$$

The derivative of $f(t,x,\pi,i)$ with respect to π is

$$
D_\pi f(t,x,\pi,i) = S(t,i)x + R(t,i)\pi \tag{3.6}
$$

Combining the three equations,(3.4), (3.5), (3.6), we get that

$$
\begin{aligned}
D_\pi[\mathcal{L}^\pi(t)[v(t,x,i)] - f(t,x,\pi,i)] = &\sum_{i=j}^{d} D_j^T(t,i)D_x^2[v(t,x,i)](C_j(t,i)x+D_j(t,i)\pi) \\
&+B^T(t,i)D_x[v(t,x,i)] - S(t,i)x - R(t,i)\pi
\end{aligned}
$$

The coefficients are all a function of time and the Markov chain process, however, for compactness, we write $S = S(t,i)$. Setting the derivative to zero, we get

$$
\pi^* = \left[\sum_{j=1}^{d} D_j^T D_x^2[v(t,x,i)]D_j - R\right]^{-1}\left[Sx - B^T D_x[v(t,x,i)] - \sum_{j=1}^{d} D_j^T D_x^2[v(t,x,i)]C_jx\right] \tag{3.7}
$$

We now substitute (3.7) into (3.2) to get:

$$
\begin{aligned}
&\frac{\partial v}{\partial t}(t,x,i) + b(t,x,\pi^*,i)^T D_x[v(t,x,i)] + \frac{1}{2}\mathrm{tr}\big[\sigma(t,x,\pi^*,i)\sigma^T(t,x,\pi^*,i)D_x^2[v(t,x,i)]\big] \\
&-\frac{1}{2}x^TQx - \frac{1}{2}x^TS^T\pi^* - \frac{1}{2}\pi^{*T}Sx - \frac{1}{2}\pi^{*T}R\pi^* + \sum_{j\neq i} q_{ij}(v(t,x,j)-v(t,x,i)) = 0
\end{aligned}
$$

As $D_x^2[v(t,x,i)]$ is a symmetric matrix, we can write

$$
\begin{aligned}
\mathrm{tr}\big[\sigma(t,x,\pi^*,i)\sigma^T(t,x,\pi^*,i)D_x^2[v(t,x,i)]\big] &= \sum_{j=1}^{d}\mathrm{tr}\big[(C_jx+D_j\pi^*)(C_jx+D_j\pi^*)^T D_x^2[v(t,x,i)]\big] \\
&= \sum_{j=1}^{d}(C_jx+D_j\pi^*)^T D_x^2[v(t,x,i)](C_jx+D_j\pi^*),
\end{aligned}
$$

we get the Hamilton-Jacobi-Bellman equation

$$
\begin{aligned}
&\frac{\partial v}{\partial t}(t,x,i) + (Ax+B\pi^*)^T D_x[v(t,x,i)] + \frac{1}{2}\sum_{j=1}^{d}(C_jx+D_j\pi^*)^T D_x^2[v(t,x,i)](C_jx+D_j\pi^*) \\
&\quad -\frac{1}{2}x^TQx - \frac{1}{2}x^TS^T\pi^* - \frac{1}{2}\pi^{*T}Sx - \frac{1}{2}\pi^{*T}R\pi^* + \sum_{j\neq i}^{k} q_{ij}(v(t,x,j)-v(t,x,i)) = 0 \qquad (3.8)
\end{aligned}
$$

where π^* is as in (3.7) and the terminal condition is given by

$$
v(T,x,i) = -g(x,i) = -\frac{1}{2}x^TG(T,i)x - x^TL(T,i).
$$

3.1.3 Solving the HJB equation

We solve (3.8) using the ansatz

$$v(t,x,i) = \frac{1}{2}x^T P(t,i)x + x^T M(t,i) + N(t,i) \tag{3.9}$$

with terminal conditions

$$P(T,i) = -G(T,i), \quad M(T,i) = -L(T,i), \quad N(T,i) = 0.$$

Then

$$\begin{aligned}
\frac{\partial v}{\partial t}(t,x,i) &= \frac{1}{2}x^T \frac{\mathrm{d}P(t,i)}{\mathrm{d}t}x + x^T \frac{\mathrm{d}M(t,i)}{\mathrm{d}t} + \frac{\mathrm{d}N(t,i)}{\mathrm{d}t} \\
D_x[v(t,x,i)] &= P(t,i)x + M(t,i) \\
D_x^2[v(t,x,i)] &= P(t,i)
\end{aligned}$$

Substituting in (3.7) we get

$$\begin{aligned}
\pi^* = \left[\sum_{j=1}^d D_j^T(t,i)P(t,i)D_j(t,i) - R(t,i)\right]^{-1} \Bigg[& S(t,i)x - B^T(t,i)P(t,i)x - B^T(t,i)M(t,i) \\
& - \sum_{j=1}^d D_j^T(t,i)P(t,i)C_j(t,i)x\Bigg]
\end{aligned} \tag{3.10}$$

We can write this as

$$\pi^* = \vartheta_1 x + \kappa_1,$$

where

$$\begin{aligned}
\vartheta_1 &= \left(\sum_{j=1}^d D_j^T(t,i)P(t,i)D_j(t,i) - R(t,i)\right)^{-1}\left(S(t,i) - B^T(t,i)P(t,i) - \sum_{j=1}^d D_j(t,i)P(t,i)C_j(t,i)\right) \\
\kappa_1 &= -\left(\sum_{j=1}^d D_j^T(t,i)P(t,i)D_j(t,i) - R(t,i)\right)^{-1} B^T(t,i)M(t,i)
\end{aligned}$$

Substituting into (3.8) we get

$$\begin{aligned}
\frac{1}{2}x^T \frac{\mathrm{d}P(t,i)}{\mathrm{d}t}x + x^T \frac{\mathrm{d}M(t,i)}{\mathrm{d}t} + \frac{\mathrm{d}N(t,i)}{\mathrm{d}t} + (A(t,i)x + B(t,i)(\vartheta_1 x + \kappa_1))^T (P(t,i)x + M(t,i)) \\
+\frac{1}{2}\sum_{j=1}^d (C_j(t,i)x + D_j(t,i)(\vartheta_1 x + \kappa_1))^T P(t,i)(C_j(t,i)x + D_j(t,i)(\vartheta_1 x + \kappa_1)) \\
-\frac{1}{2}x^T Q(t,i)x - x^T S^T(t,i)(\vartheta_1 x + \kappa_1) - \frac{1}{2}(\vartheta_1 x + \kappa_1)^T R(t,i)(\vartheta_1 x + \kappa_1) \\
+\sum_{j\neq i}^k q_{ij}\left(\frac{1}{2}x^T(P(t,j) - P(t,i))x + x^T(M(t,j) - M(t,i)) + N(t,j) - N(t,i)\right) = 0
\end{aligned}$$

We rewrite this as

$$x^T\left[\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t}+A^T(t,i)P(t,i)+\vartheta_1^TB^T(t,i)P(t,i)+\frac{1}{2}\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)\right.$$
$$\left.-\frac{1}{2}Q(t,i)-S^T(t,i)\vartheta_1-\frac{1}{2}\vartheta_1^TR(t,i)\vartheta_1+\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))\right]x$$
$$+x^T\left[\frac{\mathrm{d}M(t,i)}{\mathrm{d}t}+A^T(t,i)M(t,i)+\vartheta_1^TB^T(t,i)M(t,i)+P(t,i)B(t,i)\kappa_1\right.$$
$$\left.+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)D_j\kappa_1-S^T(t,i)\kappa_1-\vartheta_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(M(t,j)-M(t,i))\right]$$
$$+\frac{\mathrm{d}N(t,i)}{\mathrm{d}t}+\kappa_1^TM(t,i)+\frac{1}{2}\sum_{j=1}^{d}\kappa_1^TP(t,i)\kappa_1-\frac{1}{2}\kappa_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(N(t,j)-N(t,i))=0$$

This equation must equal zero for all x, hence the coefficients in front of the quadratic term, as well as x and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t}&+A^T(t,i)P(t,i)+\vartheta_1^TB^T(t,i)P(t,i)+\frac{1}{2}\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)\\&-\frac{1}{2}Q(t,i)-S^T(t,i)\vartheta_1-\frac{1}{2}\vartheta_1^TR(t,i)\vartheta_1+\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))=0\end{aligned}\tag{3.11}$$

$$\begin{aligned}&\frac{\mathrm{d}M(t,i)}{\mathrm{d}t}+A^T(t,i)M(t,i)+\vartheta_1^TB^T(t,i)M(t,i)+P(t,i)B(t,i)\kappa_1\\&+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)D_j\kappa_1-S^T(t,i)\kappa_1-\vartheta_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(M(t,j)-M(t,i))=0\end{aligned}\tag{3.12}$$

$$+\frac{\mathrm{d}N(t,i)}{\mathrm{d}t}+\kappa_1^TM(t,i)+\frac{1}{2}\sum_{j=1}^{d}\kappa_1^TP(t,i)\kappa_1-\frac{1}{2}\kappa_1^TR(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(N(t,j)-N(t,i))=0\tag{3.13}$$

with terminal conditions

$$P(T,i)=-G(T,i),\quad M(T,i)=-L(T,i),\quad N(T,i)=0.\tag{3.14}$$

3.2 Primal BSDE

3.2.1 Solution via the Primal BSDE

Recall that the SDE (**??**) describing the state process is given by

$$\begin{cases} \mathrm{d}X(t) &= b(t,X(t),\pi(t),\eta(t-))\,\mathrm{d}t + \sigma(t,X(t),\pi(t),\eta(t-))\,\mathrm{d}W(t) \\ X(t_0) &= x_0 \in \mathbb{R}^n,\ \eta(0) = i_0 \in I \end{cases} \tag{3.15}$$

with cost functional (**??**)

$$J(\pi) := \mathbb{E}\bigg[\int_{t_0}^{T} f(t,X(t),\pi(t),\eta(t))\,\mathrm{d}t + g(X(T),\eta(T))\bigg]. \tag{3.16}$$

The Hamiltonian $\mathcal{H} : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times I\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}$ is given by

$$\begin{aligned} \mathcal{H}(t,x,\pi,i,p,q) :&= -f(t,x,\pi,i) + b^T(t,x,\pi,i)p + \mathrm{tr}(\sigma^T(t,x,\pi,i)q) \\ &= -\frac{1}{2}x^TQ(t,i)x - x^TS^T(t,i)\pi - \frac{1}{2}\pi^TR(t,i)\pi + (x^TA^T(t,i)+\pi^TB^T(t,i))p \\ &+ \sum_{j=1}^{d}(x^TC_j^T(t,i)+\pi^TD_j^T(t,i))q_j, \end{aligned} \tag{3.17}$$

where $q_j \in \mathbb{R}^n$ is the j^{th} column of $q \in \mathbb{R}^{n\times d}$. Given an admissible pair (x,π), the adjoint equation in the unknown adapted processes $p(t), q(t)$ and $s(t) = (s^{(1)}(t),\dots,s^{(n)}(t))$, where $s^{(l)}(t)\in\mathbb{R}^{k\times k}$ for $l\in\{1,\cdots,n\}$, is the following regime-switching BSDE:

$$\begin{cases} \mathrm{d}p(t) &= -D_x[\mathcal{H}(t,X(t),\pi(t),\eta(t-),p(t),q(t))]\,\mathrm{d}t + q(t)\,\mathrm{d}W(t) + s(t)\cdot\mathrm{d}\mathcal{Q}(t) \\ p(T) &= -D_x[g(X(T),\eta(T))] = -G(T,\eta(t))X(T) - L(T,\eta(t)) \end{cases} \tag{3.18}$$

where

$$s(t)\cdot\mathrm{d}\mathcal{Q}(t) = \bigg(\sum_{j\neq i} s_{ij}^{(1)}\,\mathrm{d}\mathcal{Q}_{ij}(t),\cdots,s_{ij}^{(n)}\,\mathrm{d}\mathcal{Q}_{ij}(t)\bigg)^T \tag{3.19}$$

We know that the optimal control maximises the Hamiltonian (3.17), that is, the derivative with respect to the control vanishes:

$$D_\pi[\mathcal{H}] = B^T(t,i)p + \sum_{i=1}^{d} D_i^T(t,i)q_i - S(t,i)X - R(t,i)\pi = 0 \tag{3.20}$$

From this, we know that the control π is linear in X, so it is of the form

$$\pi = \vartheta_2 X + \kappa_2, \tag{3.21}$$

where $\vartheta_2\in\mathbb{R}^{m\times n}$ and $\kappa_2\in\mathbb{R}^m$. Substituting the control in the Hamiltonian (3.17) we get

$$\begin{aligned} \mathcal{H} = X^TA^T(t,i)p &+ (\vartheta_2X+\kappa_2)^TB^T(t,i)p + \sum_{j=1}^{d}\Big(X^TC_j^T(t,i)q_j + (\vartheta_2X+\kappa_2)^TD_j^T(t,i)q_j\Big) \\ &-\frac{1}{2}X^TQ(t,i)X - X^TS^T(t,i)(\vartheta_2X+\kappa_2) - \frac{1}{2}(\vartheta_2X+\kappa_2)^TR(t,i)(\vartheta_2X+\kappa_2) \end{aligned} \tag{3.22}$$

The derivative of the Hamiltonian is then

$$\begin{aligned} D_x[\mathcal{H}] = A^T(t,i)p &+ \vartheta_2^TB^T(t,i)p + \sum_{j=1}^{d}(C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))q_j - Q(t,i)X \\ &-2S^T(t,i)\vartheta_2X - S^T(t,i)\kappa_2 - \vartheta_2^TR(t,i)\vartheta_2X - \vartheta_2^TR(t,i)\kappa_2 \end{aligned} \tag{3.23}$$

We try an ansatz for p of the form:

$$p = \varphi(t, \eta(t))X(t) + \psi(t, \eta(t))$$

where $\varphi(t, \eta(t)) \in \mathbb{R}^{n\times n}$ and $\psi(t, \eta) \in \mathbb{R}^n$. Applying Ito's formula to $p = \varphi(t, \eta(t))X(t) + \psi(t, \eta(t))$, we have

$$\begin{aligned}
\mathrm{d}p = \sum_{i=1}^{k} \chi_{\{\eta(t-)=i\}} \Big[\big(\varphi(t,i)A(t,i) + \Delta\varphi(t,i)\big)X + \Delta\psi(t,i) + \varphi(t,i)B(t,i)(\vartheta_2 X + \kappa_2) \Big] \,\mathrm{d}t \\
+\varphi(t, \eta(t-))\sigma(t, \eta(t-)) \,\mathrm{d}W \\
+ \sum_{i\neq j} \big[\big(\varphi(t,j) - \varphi(t,i)\big)X + \psi(t,i) - \psi(t,i) \big] \,\mathrm{d}\mathcal{Q}_{ij}
\end{aligned}$$

where

$$\begin{aligned}
\Delta\varphi(t,i) &= \frac{\partial\varphi}{\partial t}(t,i) + \sum_{j=1}^{k} q_{ij}(\varphi(t,j) - \varphi(t,i)) \\
\Delta\psi(t,i) &= \frac{\partial\psi}{\partial t}(t,i) + \sum_{j=1}^{k} q_{ij}(\psi(t,j) - \psi(t,i))
\end{aligned}$$

Equating coefficients with (3.18) and setting $\eta(t-) = i$, we get

$$\big(\varphi(t,i)A(t,i) + \Delta\varphi(t,i)\big)X + \Delta\psi(t,i) + \varphi(t,i)B(t,i)(\vartheta_2 X + \kappa_2) = -D_x[\mathcal{H}] \tag{3.24}$$

$$\varphi(t,i)\sigma(t, X(t), \pi(t), i) = q(t) \tag{3.25}$$

$$\big(\varphi(t,j) - \varphi(t,i)\big)X + \psi(t,i) - \psi(t,i) = s_{ij}(t) \tag{3.26}$$

$$B^T(t,i)(\varphi X + \psi) + \sum_{j=1}^{d} D_j^T(t,i)q_j - S(t,i)X - R(t,i)(\vartheta_2 X + \kappa_2) = 0 \tag{3.27}$$

where the last equation is the Hamiltonian condition (3.20). We now substitute $q(t)$ from the second equation into the rest, and our system becomes

$$\begin{aligned}
\big(\varphi(t,i)A(t,i) + \Delta\varphi(t,i)\big)X + \Delta\psi(t,i) + \varphi(t,i)B(t,i)(\vartheta_2 X + \kappa_2) = -A^T(t,i)(\varphi X + \psi) \\
-\vartheta_2^T B^T(t,i)(\varphi X + \psi) - \sum_{j=1}^{d}(C_j^T(t,i) + \vartheta_2^T D_j^T(t,i))\varphi(C_j(t,i)X + D_j(t,i)(\vartheta_2 X + \kappa_2)) \\
+Q(t,i)X + 2S^T(t,i)\vartheta_2 X + S^T(t,i)\kappa_2 + \vartheta_2^T R(t,i)\vartheta_2 X + \vartheta_2^T R(t,i)\kappa_2
\end{aligned} \tag{3.28}$$

$$\big(\varphi(t,j) - \varphi(t,i)\big)X + \psi(t,i) - \psi(t,i) = s_{ij}(t) \tag{3.29}$$

$$\begin{aligned}
B^T(t,i)(\varphi X + \psi) + \sum_{j=1}^{d} D_j^T(t,i)\varphi(C_j(t,i)X + D_j(t,i)(\vartheta_2 X + \kappa_2)) \\
-S(t,i)X - R(t,i)(\vartheta_2 X + \kappa_2) = 0
\end{aligned} \tag{3.30}$$

From (3.30) we find the optimal control $\pi^* = \vartheta_2 X + \kappa_2$:

$$\pi^* = \Big[\sum_{j=1}^{d} D_j(t,i)\varphi D_j(t,i) - R(t,i) \Big]^{-1} \Big[S(t,i)X - \sum_{j=1}^{d} D_j^T(t,i)\varphi C_j(t,i)X - B^T(t,i)\varphi X - B^T(t,i)\psi \Big] \tag{3.31}$$

i.e.,

$$\begin{aligned}
\vartheta_2 &= \Big[\sum_{j=1}^{d} D_j^T(t,i)\varphi D_j(t,i) - R(t,i) \Big]^{-1} \Big(S(t,i) - B^T(t,i)\varphi - \sum_{j=1}^{d} D_j^T(t,i)\varphi C_j(t,i) \Big) \\
\kappa_2 &= -\Big[\sum_{j=1}^{d} D_j^T(t,i)\varphi D_j(t,i) - R(t,i) \Big]^{-1} B^T(t,i)\psi.
\end{aligned}$$

We rewrite equation (3.28) as

$$\Big[\frac{\partial\varphi}{\partial t}(t,i)+\sum_{j=1}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i))+\varphi(t,i)A(t,i)+A^T(t,i)\varphi+2\vartheta_2^TB^T(t,i)\varphi$$

$$+\sum_{j=1}^{d}(C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))\varphi(C_j(t,i)+D_j(t,i)\vartheta_2)-Q(t,i)-2S^T(t,i)\vartheta_2-\vartheta_2^TR(t,i)\vartheta_2\Big]X$$

$$+\Big[\frac{\partial\psi}{\partial t}(t,i)+\sum_{j=1}^{k}q_{ij}(\psi(t,j)-\psi(t,i))+\varphi(t,i)B(t,i)\kappa_2+A^T(t,i)\psi+\vartheta_2^TB^T(t,i)\psi$$

$$+\sum_{j=1}^{d}(C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))\varphi D_j(t,i)\kappa_2-S^T(t,i)\kappa_2-\vartheta_2^TR(t,i)\kappa_2\Big]=0$$

Since this must be true for all X, the coefficient in front of X must be equal to zero, so we get

$$\frac{\partial\varphi}{\partial t}(t,i)+2A^T(t,i)\varphi(t,i)+2\vartheta_2^TB^T(t,i)\varphi(t,i)$$

$$+\sum_{j=1}^{d}(C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))\varphi(t,i)(C_j(t,i)+D_j(t,i)\vartheta_2)$$

$$-Q(t,i)-2S^T(t,i)\vartheta_2-\vartheta_2^TR(t,i)\vartheta_2+\sum_{j=1}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i))=0 \tag{3.32}$$

$$\frac{\partial\psi}{\partial t}(t,i)+\varphi(t,i)B(t,i)\kappa_2+A^T(t,i)\psi(t,i)+\vartheta_2^TB^T(t,i)\psi(t,i)$$

$$+\sum_{j=1}^{d}(C_j^T(t,i)+\vartheta_2^TD_j^T(t,i))\varphi(t,i)D_j(t,i)\kappa_2$$

$$-S^T(t,i)\kappa_2-\vartheta_2^TR(t,i)\kappa_2+\sum_{j=1}^{k}q_{ij}(\psi(t,j)-\psi(t,i))=0 \tag{3.33}$$

with terminal conditions

$$\varphi(T,i)=-G(T,i),\quad \psi(T,i)=-L(T,i). \tag{3.34}$$

3.2.2 Equivalence of Primal HJB and Primal BSDE

The optimal control from the primal HJB was given by (3.10):

$$\pi^*=\Big[\sum_{j=1}^{d}D_j^T(t,i)PD_j(t,i)-R(t,i)\Big]^{-1}\Big[S(t,i)x-\sum_{j=1}^{d}D_j^T(t,i)PC_j(t,i)x-B^T(t,i)Px-B^T(t,i)M\Big]$$

and from the primal BSDE (3.31)

$$\pi^*=\Big[\sum_{j=1}^{d}D_j(t,i)\varphi D_j(t,i)-R(t,i)\Big]^{-1}\Big[S(t,i)X-\sum_{j=1}^{d}D_j^T(t,i)\varphi C_j(t,i)X-B^T(t,i)\varphi X-B^T(t,i)\psi\Big]$$

Comparing, we see that the equations are the same and we get the relation

$$\varphi(t,i)=P(t,i),\quad \psi(t,i)=M(t,i),$$

so $\vartheta_1=\vartheta_2$ and $\kappa_1=\kappa_2$. The ODE from the primal HJB for $P(t,i)$ is given by (3.11):

$$\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t}+A^T(t,i)P(t,i)+\vartheta_1^TB^T(t,i)P(t,i)+\frac{1}{2}\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^TP(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)$$

$$-\frac{1}{2}Q(t,i)-S^T(t,i)\vartheta_1-\frac{1}{2}\vartheta_1^TR(t,i)\vartheta_1+\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))=0$$

Letting $P(t,i)=\varphi(t,i)$ and $\vartheta_1=\vartheta_2, \kappa_1=\kappa_2$, and multiplying by 2 on both sides we get

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t,i)+2A^T(t,i)\varphi(t,i)+2\vartheta_1^T B^T(t,i)\varphi(t,i)+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^T\varphi(t,i)(C_j(t,i)+D_j(t,i)\vartheta_1)$$
$$-Q(t,i)-2S^T(t,i)\vartheta_1-\vartheta_1^T R(t,i)\vartheta_1+\sum_{j\neq i}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i))=0$$

which is the same ODE as the one from the primal BSDE (3.32).

Similarly, the ODE from the primal HJB for $M(t,i)$ is given by (3.12):

$$\frac{\mathrm{d}M}{\mathrm{d}t}(t,i)+A^T M(t,i)+\vartheta_1^T B^T M(t,i)+P(t,i)B\kappa_1+\sum_{j=1}^{d}(C_j+D_j\vartheta_1)^T P(t,i)D_j\kappa_1$$
$$-S^T\kappa_1-\vartheta_1^T R\kappa_1+\sum_{j\neq i}^{k}q_{ij}(M(t,j)-M(t,i))=0$$

Substituting $P(t,i)=\varphi(t,i), M(t,i)=\psi(t,i)$ we get

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t,i)+A^T(t,i)\psi(t,i)+\vartheta_1^T B^T(t,i)\psi(t,i)+\varphi(t,i)B(t,i)\kappa_1+\sum_{j=1}^{d}(C_j(t,i)+D_j(t,i)\vartheta_1)^T\varphi(t,i)D_j\kappa_1$$
$$-S^T(t,i)\kappa_1-\vartheta_1^T R(t,i)\kappa_1+\sum_{j\neq i}^{k}q_{ij}(\psi(t,j)-\psi(t,i))=0$$

the same equations as the one from the primal BSDE (3.33). The terminal conditions from the primal HJB are (3.14)

$$P(T,i)=-G(T,i), \quad M(T,i)=-L(T,i), \quad N(T,i)=0$$

and the terminal conditions from the primal BSDE are (3.34)

$$\varphi(T,i)=-G(T,i), \quad \psi(T,i)=-L(T,i). \tag{3.35}$$

As we can see, the terminal conditions are the same, so we can conclude that the two methods are equivalent.

3.3 Dual HJB Equation

3.3.1 The dual HJB

The dual HJB is given by

$$0 = \frac{\partial v}{\partial t}(t,y,i) + \sup_{\alpha,\beta,\gamma}\Bigg\{\Big(\alpha - A^T y - \sum_{j=1}^{d} C_j^T\beta_j - \sum_{j=1}^{k} q_{ij}(\gamma_j-\gamma_i)^T\Big)D_y[v(t,y,i)] + \frac{1}{2}\sum_{j=1}^{d}\beta_j^T D_y^2[v(t,y,i)]\beta_j$$
$$-\phi\big(t,\alpha,B^T y + \sum_{j=1}^{d} D_j^T\beta_j\big) + \sum_{j\neq i}^{k} q_{ij} v(t,y+\gamma_j-\gamma_i,j)\Bigg\}$$

with terminal condition

$$v(T,y,i) = -h(y,i) = -\frac{1}{2}(y^T + L^T(T,i))G^{-1}(T,i)(y+L(T,i)).$$

3.3.2 Finding the optimal control

We find the optimal controls $\alpha,\beta_1,\ldots,\beta_d$ by setting the derivatives with respect to α and β_j to zero:

$$D_y[\tilde{v}] - \tilde{Q}\alpha - \tilde{S}^T(B^T y + \sum_{i=1}^{d} D_i^T\beta_i) = 0 \tag{3.36}$$

$$-C_j D_y[\tilde{v}] + D_y^2[\tilde{v}]\beta_j - D_j(\tilde{S}\alpha + \tilde{R}(B^T y + \sum_{j=1}^{d} D_j^T\beta_j)) = 0 \tag{3.37}$$

This is a system of $d+1$ equations in $d+1$ unknowns, so it can be solved and the optimal controls are linear functions of y, which we denote by α^* and β_j^*. The HJB equation is then

$$0 = \frac{\partial v}{\partial t}(t,y,i) + \sup_{\gamma}\Bigg\{\Big(\alpha^* - A^T y - \sum_{j=1}^{d} C_j^T\beta_j^* - \sum_{j=1}^{k} q_{ij}(\gamma_j-\gamma_i)^T\Big)D_y[v(t,y,i)]$$
$$+\frac{1}{2}\sum_{j=1}^{d}\beta_j^{*T} D_y^2[v(t,y,i)]\beta_j^* - \phi\big(t,\alpha^*,B^T y + \sum_{j=1}^{d} D_j^T\beta_j^*\big) + \sum_{j\neq i}^{k} q_{ij} v(t,y+\gamma_j-\gamma_i,j)\Bigg\} \tag{3.38}$$

3.3.3 Solving the dual HJB

Suppose that $\tilde{v}$ is a quadratic function in y and use the ansatz

$$\tilde{v}(t,y,i) = \frac{1}{2}y^T\tilde{P}(t,i)y + y^T\tilde{M}(t,i) + \tilde{N}(t,i), \tag{3.39}$$

where $\tilde{P}(t,i)\in\mathbb{R}^{n\times n}$, $\tilde{M}(t,i)\in\mathbb{R}^n$, $\tilde{N}(t,i)\in\mathbb{R}$, with terminal conditions

$$\tilde{P}(T,i) = -G^{-1}(T,i), \tilde{M}(T,i) = -G^{-1}(T,i)L(T,i), \tilde{N}(T,i) = \frac{1}{2}L^T(T,i)G^{-1}(T,i)L(T,i). \tag{3.40}$$

Then

$$\begin{aligned}
\frac{\partial\tilde{v}}{\partial t}(t,y,i) &= \frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)y + y^T\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i) + \frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i)\\
D_y[\tilde{v}(t,y,i)] &= \tilde{P}(t,i)y + \tilde{M}(t,i)\\
D_y^2[\tilde{v}(t,y,i)] &= \tilde{P}(t,i)
\end{aligned}$$

The system of equations from which we derive the optimal controls α^* and β_i^* is now given by

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}\alpha - \tilde{S}^T(B^TY + \sum_{i=1}^d D_i^T\beta_i) = 0 \\ C_i(\tilde{P}y + \tilde{M}) - \tilde{P}\beta_i + D_i(\tilde{S}\alpha + \tilde{R}(B^Ty + \sum_{j=1}^d D_j^T\beta_j)) = 0 \end{cases} \tag{3.41}$$

We do not solve this system explicitly, however, the solutions for α^* and β^* are linear in y, hence we denote by $\tilde{\vartheta}$ and $\tilde{\kappa}$ the coefficients before y and the free coefficient in α^* and similarly for β_j, i.e.

$$\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}, \quad \beta_j = \tilde{\vartheta}_jy + \tilde{\kappa}_j. \tag{3.42}$$

Substituting this into the HJB (3.38) equation we get

$$\begin{aligned} &\frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)y + y^T\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i) + \frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i) \\ &+ \sup_\gamma \Bigg\{ \Big(y^T\tilde{\vartheta}^T + \tilde{\kappa}^T - y^TA - \sum_{j=1}^d(\tilde{\vartheta}_jy + \tilde{\kappa}_j)^TC_j^T - \sum_{j\neq i}^k q_{ij}(\gamma_j - \gamma_i)^T\Big)(\tilde{P}(t,i)y + \tilde{M}(t,i)) \\ &+ \frac{1}{2}\sum_{j=1}^d(\tilde{\vartheta}_jy + \tilde{\kappa}_j)^T\tilde{P}(t,i)(\tilde{\vartheta}_jy + \tilde{\kappa}_j) - \phi\big(t, \tilde{\vartheta}y + \tilde{\kappa}, B^Ty + \sum_{j=1}^d D_j^T(\tilde{\vartheta}_jy + \tilde{\kappa}_j)\big) \\ &+ \sum_{j=1}^k q_{ij}\Big(\frac{1}{2}y^T(\tilde{P}(t,j) - \tilde{P}(t,i))y + y^T(\tilde{M}(t,j) - \tilde{M}(t,i)) + \tilde{N}(t,j) - \tilde{N}(t,i)\Big) \\ &+ \sum_{j=1}^k q_{ij}\big(\frac{1}{2}(\gamma_j - \gamma_i)^T\tilde{P}(t,j)(\gamma_j - \gamma_i) + (\gamma_j - \gamma_i)^T(\tilde{P}(t,j)y + \tilde{M}(t,j))\big)\Bigg\} = 0 \end{aligned}$$

Setting the derivative w.r.t. γ_j, $j \neq i$ to zero we get

$$(\gamma_j - \gamma_i) = -\tilde{P}^{-1}(t,j)\big[(\tilde{P}(t,j) - \tilde{P}(t,i))y + \tilde{M}(t,j) - \tilde{M}(t,i)\big]$$

The HJB is then

$$\begin{aligned} &\frac{1}{2}y^T\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)y + y^T\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i) + \frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i) + \Big(y^T\tilde{\vartheta}^T + \tilde{\kappa}^T - y^TA - \sum_{j=1}^d(\tilde{\vartheta}_jy + \tilde{\kappa}_j)^TC_j^T\Big)(\tilde{P}(t,i)y + \tilde{M}(t,i)) \\ &+ \frac{1}{2}\sum_{j=1}^d(\tilde{\vartheta}_jy + \tilde{\kappa}_j)^T\tilde{P}(t,i)(\tilde{\vartheta}_jy + \tilde{\kappa}_j) - \phi\big(t, \tilde{\vartheta}y + \tilde{\kappa}, B^Ty + \sum_{j=1}^d D_j^T(\tilde{\vartheta}_jy + \tilde{\kappa}_j), i\big) \\ &+ \sum_{j\neq i}^k q_{ij}\Big\{\frac{1}{2}y^T(\tilde{P}(t,j) - \tilde{P}(t,i))y + y^T(\tilde{M}(t,j) - \tilde{M}(t,i)) + \tilde{N}(t,j) - \tilde{N}(t,i) \\ &- \frac{1}{2}\big[y^T(\tilde{P}(t,j) - \tilde{P}(t,i)) + \tilde{M}^T(t,j) - \tilde{M}^T(t,i)\big]\tilde{P}^{-1}(t,j)\big[(\tilde{P}(t,j) - \tilde{P}(t,i))y + \tilde{M}(t,j) - \tilde{M}(t,i)\big]\Big\} \end{aligned}$$

Rearranging, we get

$$
\begin{aligned}
y^T\Bigg[&\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)+(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i))\tilde{P}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\vartheta}_j-\frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}\\
&-\tilde{\vartheta}^T\tilde{S}^T(t,i)\big(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\big)-\frac{1}{2}\Bigg(B(t,i)+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i(t,i)\Bigg)\tilde{R}(t,i)\Bigg(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\Bigg)\\
&+\sum_{j\neq i}^{k}q_{ij}\big[\frac{1}{2}\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\big]\Bigg]y\\
+y^T\Bigg[&\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i)+\tilde{\vartheta}^T\tilde{M}(t,i)-A(t,i)\tilde{M}(t,i)+\tilde{P}(t,i)\tilde{\kappa}-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i)\tilde{M}(t,i)+\sum_{j=1}^{d}\tilde{P}(t,i)C_j(t,i)\tilde{\kappa}_j\\
&\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\Bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Bigg)\tilde{S}(t,i)\tilde{\kappa}\\
&-\Bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Bigg)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j+\sum_{j\neq i}^{k}q_{ij}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))\Bigg]\\
&\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i)+\Bigg(\tilde{\kappa}^T-\sum_{j=1}^{d}\tilde{\kappa}_j^T C_j^T(t,i)\Bigg)\tilde{M}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\kappa}_j^T\tilde{P}(t,i)\tilde{\kappa}_j\\
&-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\kappa}^T\tilde{S}^T\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\frac{1}{2}\Bigg(\sum_{j=1}^{d}\tilde{\kappa}_j^T D_j(t,i)\Bigg)\tilde{R}(t,i)\Bigg(\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\Bigg)\\
&+\sum_{j\neq i}^{k}q_{ij}\big[\tilde{N}(t,j)-\tilde{N}(t,i)-\frac{1}{2}(\tilde{M}^T(t,j)-\tilde{M}^T(t,i))\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))\big]=0
\end{aligned}
$$

This equation must equal zero for all y, hence the coefficients in front of the quadratic term, as well as

y^T and the free coefficient must be zero. Setting the coefficients to zero, we get the system

$$\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)+(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i))\tilde{P}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\vartheta}_j-\frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$
$$-\tilde{\vartheta}^T\tilde{S}^T(t,i)\big(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\big)-\frac{1}{2}\bigg(B(t,i)+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i(t,i)\bigg)\tilde{R}(t,i)\bigg(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\bigg)$$
$$+\sum_{j\neq i}^{k}q_{ij}\big[\frac{1}{2}\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\big]=0 \tag{3.43}$$

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i)+\tilde{\vartheta}^T\tilde{M}(t,i)-A(t,i)\tilde{M}(t,i)+\tilde{P}(t,i)\tilde{\kappa}-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i)\tilde{M}(t,i)+\sum_{j=1}^{d}\tilde{P}(t,i)C_j(t,i)\tilde{\kappa}_j$$
$$\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{S}(t,i)\tilde{\kappa}$$
$$-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j+\sum_{j\neq i}^{k}q_{ij}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))=0 \tag{3.44}$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i)+\bigg(\tilde{\kappa}^T-\sum_{j=1}^{d}\tilde{\kappa}_j^T C_j^T(t,i)\bigg)\tilde{M}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\kappa}_j^T\tilde{P}(t,i)\tilde{\kappa}_j$$
$$-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\kappa}^T\tilde{S}^T\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\frac{1}{2}\bigg(\sum_{j=1}^{d}\tilde{\kappa}_j^T D_j(t,i)\bigg)\tilde{R}(t,i)\bigg(\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\bigg)$$
$$+\sum_{j\neq i}^{k}q_{ij}\big[\tilde{N}(t,j)-\tilde{N}(t,i)-\frac{1}{2}(\tilde{M}^T(t,j)-\tilde{M}^T(t,i))\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))\big]=0 \tag{3.45}$$

where $\tilde{\vartheta},\tilde{\kappa},\tilde{\vartheta}_j$ and $\tilde{\kappa}_j$ satisfy the system of equations (3.41):

$$\begin{cases}\tilde{P}y+\tilde{M}-\tilde{Q}\alpha-\tilde{S}^T(B^TY+\sum_{i=1}^{d}D_i^T\beta_i)=0\\ C_i(\tilde{P}y+\tilde{M})-\tilde{P}\beta_i+D_i(\tilde{S}\alpha+\tilde{R}(B^Ty+\sum_{j=1}^{d}D_j^T\beta_j))=0\end{cases}$$

and the terminal conditions are given by (3.40):

$$\tilde{P}(T,i)=-G^{-1}(T,i),\tilde{M}(T,i)=-G^{-1}(T,i)L(T,i),\tilde{N}(T,i)=\frac{1}{2}L^T(T,i)G^{-1}(T,i)L(T,i). \tag{3.46}$$

3.4 Dual BSDE

3.4.1 Solution via the Dual BSDE

Recall that the SDE (**??**) describing the state process is given by

$$\mathrm{d}Y = \left[\alpha(t) - A(t,\eta(t-))^T Y(t) - \sum_{j=1}^{d} C_j^T(t,\eta(t-))\beta_j(t)\right]\mathrm{d}t + \sum_{j=1}^{d}\beta_j(t)\,\mathrm{d}W_j(t) + \sum_{j=1}^{k}\gamma_j(t)\,\mathrm{d}M_j(t)$$
$$Y(t_0) = y$$

and the cost functional is

$$\inf_{\alpha,\beta_1,\ldots,\beta_d}\mathbb{E}\left[\int_{t_0}^{T}\phi\left(t,\alpha,B^T(t,i)Y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j,\eta(t)\right)\mathrm{d}t + h(Y(T),\eta(t)))\right]$$

where ϕ is given in (**??**):

$$\phi(t,\alpha,\beta,i) = \frac{1}{2}\alpha^T\tilde{Q}(t,i)\alpha + \alpha^T\tilde{S}^T(t,i)\beta + \frac{1}{2}\beta^T\tilde{R}(t,i)\beta$$

and h is given in (**??**). The Hamiltonian $\tilde{\mathcal{H}} : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^{n(d+1)}\times I\times\mathbb{R}^n\times\mathbb{R}^{nd}\times\mathbb{R}^{k\times nk}\to\mathbb{R}$ for the dual problem is defined as

$$\begin{aligned}\tilde{\mathcal{H}}(t,Y,\alpha,\beta_1,\ldots,\beta_d,i,p,q,s) = -\phi&\left(t,\alpha,B^T(t,i)Y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j\right)\\ &+p^T\left(\alpha - A^T(t,i)Y - \sum_{j=1}^{d} C_j^T(t,i)\beta_j\right) + \sum_{j=1}^{d}\beta_j^T q_j + \sum_{j=1}^{k} s_{ij}\gamma_j\end{aligned} \tag{3.47}$$

where $p(t)\in\mathbb{R}^n, q(t)\in\mathbb{R}^{n\times d}$ and $s(t) = (s^{(1)}(t),\ldots,s^{(n)}(t))$, where $s^{(l)}(t)\in\mathbb{R}^{k\times k}$ for $l = 1,\ldots,n$. The adjoint equations are given by the system

$$\begin{cases}\mathrm{d}p &= -D_y[\tilde{\mathcal{H}}(t,Y,\alpha,\beta_1,\ldots,\beta_d,\eta(t-),p,q,s]\,\mathrm{d}t + \sum_{j=1}^{d} q_j(t)\,\mathrm{d}W_j(t) + s(t)\cdot\mathrm{d}\mathcal{Q}(t)\\ p(T) &= -D_y[h(Y(T),\eta(T))] = -G^{-1}(T,\eta(T))Y(T) - G^{-1}(T,\eta(T))L(T,\eta(T))\end{cases} \tag{3.48}$$

where

$$s(t)\cdot\mathrm{d}\mathcal{Q}(t) = \left(\sum_{j\neq i} s_{ij}^{(1)}(t)\,\mathrm{d}\mathcal{Q}_{ij}(t),\ldots,\sum_{j\neq i} s_{ij}^{(n)}(t)\,\mathrm{d}\mathcal{Q}_{ij}(t)\right)^T.$$

Due to the Stochastic Maximum Principle, the optimal controls are found by setting $D_\alpha[\tilde{\mathcal{H}}]$, $D_{\beta_j}[\tilde{\mathcal{H}}]$ and $D_{\gamma_j}[\tilde{\mathcal{H}}]$ to zero, so we get the system

$$D_\alpha[\tilde{\mathcal{H}}] = p - \tilde{Q}(t,i)\alpha - \tilde{S}^T(t,i)\left(B^T(t,i)Y + \sum_{j=1} D_j^T(t,i)\beta_j\right) = 0 \tag{3.49}$$

$$D_{\beta_j}[\tilde{\mathcal{H}}] = q_j - C_j(t,i)p - D_j(t,i)\tilde{S}(t,i)\alpha - D_j(t,i)\tilde{R}(t,i)\left(B^T(t,i)Y + \sum_{k=1}^{d} D_k^T(t,i)\beta_k\right) = 0 \tag{3.50}$$

$$D_{\gamma_j}[\tilde{\mathcal{H}}] = s_{ij} = 0 \tag{3.51}$$

Without the last condition, there are $d+1$ equations in $d+1$ unknowns, so we know that we can find a linear solution for the controls, which we denote as

$$\alpha^* = \tilde{\vartheta}Y + \tilde{\kappa},\quad \beta_j^* = \tilde{\vartheta}_j Y + \tilde{\kappa}_j,\quad j\in\{1,\ldots,d\}$$

Substituting into the Hamiltonian (3.47) we get

$$\tilde{\mathcal{H}} = p^T(\tilde{\vartheta}Y+\tilde{\kappa}) - p^T A^T(t,i)Y - p^T\sum_{j=1}^{d} C_j^T(t,i)(\tilde{\vartheta}_jY+\tilde{\kappa}_j) + \sum_{j=1}^{d}(Y^T\tilde{\vartheta}_j^T+\tilde{\kappa}_j^T)q_j + \sum_{j=1}^{k} s_{ij}\gamma_j$$
$$-\phi\Big(t, \tilde{\vartheta}Y+\tilde{\kappa}, B^T(t,i)Y + \sum_{j=1}^{d} D_j^T(t,i)(\tilde{\vartheta}_jY+\tilde{\kappa}_j)\Big)$$

The derivative of the Hamiltonian w.r.t. Y is then

$$\begin{aligned}D_y[\tilde{\mathcal{H}}] &= \tilde{\vartheta}^T p - A(t,i)p - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)p + \sum_{j=1}^{d}\tilde{\vartheta}_j^T q_j - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}Y - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}\\
&\quad -2\tilde{\vartheta}^T\tilde{S}(t,i)\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)Y - \tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\\
&\quad -\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa} - \Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)Y\\
&\quad -\Big(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\Big)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\end{aligned} \tag{3.52}$$

We try an ansatz for p of the form

$$p = \varphi(t,\eta(t))Y + \psi(t,\eta(t)).$$

where $\varphi(t,\eta(t)) \in \mathbb{R}^{n\times n}$ and $\psi(t,\eta(t)) \in \mathbb{R}^n$. Applying Ito's formula to $p = \varphi(t,\eta(t))Y(t) + \psi(t,\eta(t))$, we have

$$\begin{aligned}\mathrm{d}p &= \sum_{i=1}^{k}\chi_{\{\eta(t-)=i\}}\Big[\frac{\partial\varphi}{\partial t}(t,i)Y - \varphi(t,i)A(t,i)^TY + \frac{\partial\psi}{\partial t}(t,i) + \varphi(t,i)\alpha - \varphi(t,i)\sum_{j=1}^{d}C_j^T(t,i)\beta_j\\
&\quad -\varphi(t,i)\sum_{j=1}^{k}q_{ij}(\gamma_j-\gamma_i)\Big]\,\mathrm{d}t + \sum_{j=1}^{d}\varphi(t,i)\beta_j\,\mathrm{d}W_j\\
&\quad +\sum_{i\neq j}^{k}\big[\big(\varphi(t,j)-\varphi(t,i)\big)Y + \psi(t,i) - \psi(t,i) + \varphi(t,j)(\gamma_j-\gamma_i)\big]\,\mathrm{d}\mathcal{Q}_{ij}\end{aligned} \tag{3.53}$$

Equating the coefficients of (3.53) and (3.48) and setting $\eta(t-) = i$ we get

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(t,i)Y - \varphi(t,i)A(t,i)^TY + \frac{\partial\psi}{\partial t}(t,i) + \varphi(t,i)(\tilde{\vartheta}Y+\tilde{\kappa}) - \varphi(t,i)\sum_{j=1}^{d}C_j^T(t,i)(\tilde{\vartheta}_jY+\tilde{\kappa}_j)\\
&-\varphi(t,i)\sum_{j=1}^{k}q_{ij}(\gamma_j-\gamma_i) = -D_y[\tilde{\mathcal{H}}]\end{aligned} \tag{3.54}$$

$$\varphi(t,i)\tilde{\vartheta}_jY + \varphi(t,i)\tilde{\kappa}_j = q_j, \quad \forall j\in\{1,\dots,d\} \tag{3.55}$$

$$\big(\varphi(t,j)-\varphi(t,i)\big)Y + \psi(t,i) - \psi(t,i) + \varphi(t,j)(\gamma_j-\gamma_i) = s_{ij}(t) \tag{3.56}$$

where the RHS of (3.54) is given by (3.52). From (3.56) and (3.51) we get

$$\gamma_j - \gamma_i = -\varphi(t,j)^{-1}\Big(\big(\varphi(t,j)-\varphi(t,i)\big)Y + \psi(t,j) - \psi(t,i)\Big) \tag{3.57}$$

We now substitute q_j from equation (3.55) and $\gamma_j - \gamma_i$ from (3.57) into (3.54) and we get

$$
\begin{aligned}
&\frac{\partial \varphi}{\partial t}(t,i)Y - \varphi(t,i)A(t,i)^T Y + \frac{\partial \psi}{\partial t}(t,i) + \varphi(t,i)(\tilde{\vartheta}Y + \tilde{\kappa}) - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)(\tilde{\vartheta}_j Y + \tilde{\kappa}_j) \\
&+\varphi(t,i)\sum_{j=1}^{k} q_{ij}\varphi(t,j)^{-1}\Big(\big(\varphi(t,j)-\varphi(t,i)\big)Y + \psi(t,j) - \psi(t,i)\Big) = \\
&(-\tilde{\vartheta}^T + A(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i))(\varphi(t,i)Y + \psi(t,i)) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)(\tilde{\vartheta}_j Y + \tilde{\kappa}_j) \\
&+\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}Y + \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa} + 2\tilde{\vartheta}^T\tilde{S}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big)Y + \tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j \\
&+\Big(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa} + \Big(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big)Y \\
&+\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big)\tilde{R}(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j
\end{aligned}
\tag{3.58}
$$

We rewrite equation (3.58) as

$$
\begin{aligned}
&\Big[\frac{\partial \varphi}{\partial t}(t,i) + \varphi(t,i)\Big(\tilde{\vartheta} - A^T(t,i) - \sum_{j=1}^{d} C_j^T(t,i)\tilde{\vartheta}_j\Big) + \Big(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\Big)\varphi(t,i) \\
&+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\vartheta}_j - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta} - 2\tilde{\vartheta}^T\tilde{S}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big) \\
&-\Big(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big) + \sum_{j=1}^{k} q_{ij}(\varphi(t,i) - \varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i))\Big]Y \\
&+\Big[\frac{\partial \psi}{\partial t}(t,i) + \varphi(t,i)\tilde{\kappa} - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)\tilde{\kappa}_j + \Big(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\Big)\psi(t,i) \\
&+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\kappa}_j - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j - \Big(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa} \\
&-\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big)\tilde{R}(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j + \sum_{j=1}^{k} q_{ij}\varphi(t,i)\varphi^{-1}(t,j)(\psi(t,j) - \psi(t,i))\Big] = 0
\end{aligned}
$$

Since this must be true for all Y, the coefficient in front of Y must be equal to zero, so we get the ODEs

$$\frac{\partial \varphi}{\partial t}(t,i) + 2\Big(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d} \tilde{\vartheta}_j^T C_j(t,i)\Big)\varphi(t,i) + \sum_{j=1}^{d} \tilde{\vartheta}_j^T \varphi(t,i)\tilde{\vartheta}_j - \tilde{\vartheta}^T \tilde{Q}(t,i)\tilde{\vartheta}$$
$$-2\tilde{\vartheta}^T \tilde{S}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big) - \Big(B(t,i) + \sum_{j=1}^{d} \tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big)$$
$$+\sum_{j=1}^{k} q_{ij}(\varphi(t,i) - \varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i)) = 0 \tag{3.59}$$

$$\frac{\partial \psi}{\partial t}(t,i) + \varphi(t,i)\tilde{\kappa} - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)\tilde{\kappa}_j + \Big(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d} \tilde{\vartheta}_j^T C_j(t,i)\Big)\psi(t,i)$$
$$+\sum_{j=1}^{d} \tilde{\vartheta}_j^T \varphi(t,i)\tilde{\kappa}_j - \tilde{\vartheta}^T \tilde{Q}(t,i)\tilde{\kappa} - \tilde{\vartheta}^T \tilde{S}^T(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j$$
$$-\Big(B(t,i) + \sum_{j=1}^{d} \tilde{\vartheta}_j^T D_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa} - \Big(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\Big)\tilde{R}(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j$$
$$+\sum_{i=1}^{k} q_{ij}\varphi(t,i)\varphi^{-1}(t,j)(\psi(t,j) - \psi(t,i)) = 0, \tag{3.60}$$

with terminal conditions given by

$$\varphi(T,i) = -G^{-1}(T,i), \quad \psi(T,i) = -G^{-1}(T,i)L(T,i). \tag{3.61}$$

3.4.2 Equivalence of Dual HJB and Dual BSDE

From the dual HJB we get that the controls are given by $\alpha^* = \tilde{\vartheta}y + \tilde{\kappa}$ and $\beta_j^* = \tilde{\vartheta}_j y + \tilde{\kappa}_j$, which can be computed from the system of equations (3.36) and (3.37):

$$\tilde{P}(t,i)y + \tilde{M}(t,i) - \tilde{Q}(t,i)\alpha - \tilde{S}^T(t,i)\Big(B^T(t,i)y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j\Big) = 0$$
$$C_j(t,i)(\tilde{P}(t,i)y + \tilde{M}(t,i)) - \tilde{P}(t,i)\beta_j + D_j(t,i)\Big(\tilde{S}(t,i)\alpha + \tilde{R}(t,i)\big(B^T(t,i)y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j\big)\Big) = 0$$

Similarly, the system of equations from the dual BSDE are given by (3.49) and (3.50)

$$\varphi(t,i)Y + \psi(t,i) - \tilde{Q}(t,i)\alpha - \tilde{S}^T(t,i)\Big(B^T(t,i)Y + \sum_{j=1}^{d} D_j^T(t,i)\beta_j\Big) = 0$$
$$C_j(t,i)(\varphi(t,i)Y + \psi(t,i)) - \varphi(t,i)\beta_j + D_j(t,i)\Big(\tilde{S}(t,i)\alpha + \tilde{R}(t,i)\big(B^T(t,i)Y + \sum_{k=1}^{d} D_k^T(t,i)\beta_k\big)\Big) = 0$$

These equations are the same, therefore we get the relation

$$\varphi(t,i) = \tilde{P}(t,i), \quad \psi(t,i) = \tilde{M}(t,i)$$

The first ODE from the dual HJB is (3.43):

$$\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)+(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i))\tilde{P}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\vartheta}_j-\frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$

$$-\tilde{\vartheta}^T\tilde{S}^T(t,i)\big(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\big)-\frac{1}{2}\bigg(B(t,i)+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i(t,i)\bigg)\tilde{R}(t,i)\bigg(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\bigg)$$

$$+\sum_{j\neq i}^{k}q_{ij}\big[\frac{1}{2}\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\big]=0$$

Letting $\tilde{P}(t,i)=\varphi(t,i)$ and multiplying by 2 we get

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t,i)+2(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i))\varphi(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\vartheta}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$

$$-2\tilde{\vartheta}^T\tilde{S}^T(t,i)\big(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\big)-\bigg(B(t,i)+\sum_{i=1}^{d}\tilde{\vartheta}_i^T D_i(t,i)\bigg)\tilde{R}(t,i)\bigg(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\bigg)$$

$$+\sum_{j\neq i}^{k}q_{ij}\big[\varphi(t,i)-\varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i)\big]=0$$

which is the same ODE as the first one from the dual BSDE (3.59):

$$\frac{\partial\varphi}{\partial t}(t,i)+2\bigg(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\bigg)\varphi(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\vartheta}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$

$$-2\tilde{\vartheta}^T\tilde{S}(t,i)\bigg(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\bigg)-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{R}(t,i)\bigg(B^T(t,i)+\sum_{j=1}^{d}D_j^T(t,i)\tilde{\vartheta}_j\bigg)$$

$$+\sum_{j=1}^{k}q_{ij}(\varphi(t,j)-\varphi(t,i)\varphi^{-1}(t,j)\varphi(t,i))=0$$

The ODE for $\tilde{M}(t,i)$ from the dual HJB is given by (3.44):

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i)+\tilde{\vartheta}^T\tilde{M}(t,i)-A(t,i)\tilde{M}(t,i)+\tilde{P}(t,i)\tilde{\kappa}-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i)\tilde{M}(t,i)+\sum_{j=1}^{d}\tilde{P}(t,i)C_j(t,i)\tilde{\kappa}_j$$

$$\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{S}(t,i)\tilde{\kappa}$$

$$-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j+\sum_{j\neq i}^{k}q_{ij}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))=0$$

Letting $\tilde{P}=\varphi$ and $\tilde{M}=\psi$ we get

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t,i)+\tilde{\vartheta}^T\psi(t,i)-A(t,i)\psi(t,i)+\varphi(t,i)\tilde{\kappa}-\sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j^T(t,i)\psi(t,i)+\sum_{j=1}^{d}\varphi(t,i)C_j(t,i)\tilde{\kappa}_j$$

$$\sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{S}(t,i)\tilde{\kappa}$$

$$-\bigg(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\bigg)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j+\sum_{j\neq i}^{k}q_{ij}\varphi(t,i)\varphi^{-1}(t,j)(\psi(t,j)-\psi(t,i))=0$$

which is the same ODE as the one from the dual BSDE (3.60):

$$\frac{\partial\psi}{\partial t}(t,i) + \varphi(t,i)\tilde{\kappa} - \varphi(t,i)\sum_{j=1}^{d} C_j^T(t,i)\tilde{\kappa}_j + \left(\tilde{\vartheta}^T - A(t,i) - \sum_{j=1}^{d}\tilde{\vartheta}_j^T C_j(t,i)\right)\psi(t,i)$$

$$+ \sum_{j=1}^{d}\tilde{\vartheta}_j^T\varphi(t,i)\tilde{\kappa}_j - \tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j$$

$$-\left(B(t,i) + \sum_{j=1}^{d}\tilde{\vartheta}_j^T D_j(t,i)\right)\tilde{S}(t,i)\tilde{\kappa} - \left(B^T(t,i) + \sum_{j=1}^{d} D_j^T(t,i)\tilde{\vartheta}_j\right)\tilde{R}(t,i)\sum_{j=1}^{d} D_j^T(t,i)\tilde{\kappa}_j$$

$$+ \sum_{i=1}^{k} q_{ij}\varphi(t,i)\varphi^{-1}(t,j)(\psi(t,j) - \psi(t,i)) = 0$$

The terminal conditions from the dual HJB are given by (3.46):

$$\tilde{P}(T,i) = -G^{-1}(T,i), \quad \tilde{M}(T,i) = -G^{-1}(T,i)L(T,i)$$

and from the dual BSDE, we have (3.61):

$$\varphi(T,i) = -G^{-1}(T,i), \quad \psi(T,i) = -G^{-1}(T,i)L(T,i).$$

As we can see the equations are identical, and their terminal conditions are also the same, so the two methods are equivalent.

3.5 Primal and Dual Equivalence

This section shows equivalence between the primal HJB and dual HJB solutions. Recall that we have the relation between the dual and primal value functions (**??**):

$$v(t,x,i) \leq \inf_y \{x^T y - \tilde{v}(t,y,i)\}$$

In this section, we impose no constraints on the control, and we will show that this leads to equality instead of inequality in the above equation. Substituting the respective ansatz, we have

$$\frac{1}{2}x^T P(t,i)x + x^T M(t,i) + N(t,i) = \inf_y \left\{ x^T y - \frac{1}{2} y^T \tilde{P}(t,i) y - y^T \tilde{M}(t,i) - \tilde{N}(t,i) \right\}$$

Setting the derivative of the RHS to zero, we get

$$y = \tilde{P}^{-1}(t,i)(x - \tilde{M}(t,i)),$$

so

$$\frac{1}{2}x^T P(t,i)x + x^T M(t,i) + N(t,i) = \frac{1}{2}x^T \tilde{P}^{-1}(t,i)x - x^T \tilde{P}^{-1}(t,i)\tilde{M}(t,i) + \frac{1}{2}\tilde{M}^T(t,i)\tilde{P}^{-1}(t,i)\tilde{M}(t,i) - \tilde{N}(t,i).$$

Therefore, we get the relation

$$P(t,i) = \tilde{P}^{-1}(t,i), \quad M(t,i) = -\tilde{P}^{-1}(t,i)\tilde{M}(t,i), \quad N(t,i) = \frac{1}{2}\tilde{M}^T(t,i)\tilde{P}^{-1}(t,i)\tilde{M}(t,i) - \tilde{N}(t,i). \quad (3.62)$$

In section 2.3 we have shown equivalence between the primal and dual HJB in the non-Markovian case. Now we work in a Markov setting, and the resulting ODEs from the primal and dual HJB methods are the same as the ones from the non-Markovian case, apart from the fact that the parameters now depend also on the Markov chain (not only on time) and there are also additional terms concerning the Markov chain. Therefore, we will only show here the equivalence between the additional terms, as the equivalence of the rest of the ODEs is already shown.

The system of ODEs from the primal HJB is given by (3.11), (3.12) and (3.13)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}P(t,i)}{\mathrm{d}t} + A^T(t,i)P(t,i) + \vartheta_1^T B^T(t,i)P(t,i) + \frac{1}{2}\sum_{j=1}^{d}(C_j(t,i) + D_j(t,i)\vartheta_1)^T P(t,i)(C_j(t,i) + D_j(t,i)\vartheta_1)&\\ -\frac{1}{2}Q(t,i) - S^T(t,i)\vartheta_1 - \frac{1}{2}\vartheta_1^T R(t,i)\vartheta_1 + \frac{1}{2}\sum_{j\neq i}^{k} q_{ij}(P(t,j) - P(t,i)) = 0&\\ \frac{\mathrm{d}M(t,i)}{\mathrm{d}t} + A^T(t,i)M(t,i) + \vartheta_1^T B^T(t,i)M(t,i) + P(t,i)B(t,i)\kappa_1&\\ + \sum_{j=1}^{d}(C_j(t,i) + D_j(t,i)\vartheta_1)^T P(t,i)D_j\kappa_1 - S^T(t,i)\kappa_1 - \vartheta_1^T R(t,i)\kappa_1 + \sum_{j\neq i}^{k} q_{ij}(M(t,j) - M(t,i)) = 0&\\ + \frac{\mathrm{d}N(t,i)}{\mathrm{d}t} + \kappa_1^T M(t,i) + \frac{1}{2}\sum_{j=1}^{d}\kappa_1^T P(t,i)\kappa_1 - \frac{1}{2}\kappa_1^T R(t,i)\kappa_1 + \sum_{j\neq i}^{k} q_{ij}(N(t,j) - N(t,i)) = 0&\end{aligned}$$

with terminal conditions (3.14):

$$P(T,i) = -G(T,i), \quad M(T,i) = -L(T,i), \quad N(T,i) = 0.$$

Similarly, the ODEs from the dual HJB are given by (3.43), (3.44) and (3.45):

$$\frac{1}{2}\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t,i)+(\tilde{\vartheta}^T-A(t,i)-\sum_{j=1}^{d}\tilde{\vartheta}_j^TC_j^T(t,i))\tilde{P}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\vartheta}_j-\frac{1}{2}\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\vartheta}$$

$$-\tilde{\vartheta}^T\tilde{S}^T(t,i)\big(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\big)-\frac{1}{2}\Big(B(t,i)+\sum_{i=1}^{d}\tilde{\vartheta}_i^TD_i(t,i)\Big)\tilde{R}(t,i)\Big(B^T(t,i)+\sum_{i=1}^{d}D_i^T(t,i)\tilde{\vartheta}_i\Big)$$

$$+\sum_{j\neq i}^{k}q_{ij}\big[\frac{1}{2}\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)\big]=0$$

$$\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t}(t,i)+\tilde{\vartheta}^T\tilde{M}(t,i)-A(t,i)\tilde{M}(t,i)+\tilde{P}(t,i)\tilde{\kappa}-\sum_{j=1}^{d}\tilde{\vartheta}_j^TC_j^T(t,i)\tilde{M}(t,i)+\sum_{j=1}^{d}\tilde{P}(t,i)C_j(t,i)\tilde{\kappa}_j$$

$$\tilde{\vartheta}_j^T\tilde{P}(t,i)\tilde{\kappa}_j-\tilde{\vartheta}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\vartheta}^T\tilde{S}^T(t,i)\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^TD_j(t,i)\Big)\tilde{S}(t,i)\tilde{\kappa}$$

$$-\Big(B(t,i)+\sum_{j=1}^{d}\tilde{\vartheta}_j^TD_j(t,i)\Big)\tilde{R}(t,i)\sum_{j=1}^{d}D_j^T\tilde{\kappa}_j+\sum_{j\neq i}^{k}q_{ij}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))=0$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t}(t,i)+\Big(\tilde{\kappa}^T-\sum_{j=1}^{d}\tilde{\kappa}_j^TC_j^T(t,i)\Big)\tilde{M}(t,i)+\frac{1}{2}\sum_{j=1}^{d}\tilde{\kappa}_j^T\tilde{P}(t,i)\tilde{\kappa}_j$$

$$-\frac{1}{2}\tilde{\kappa}^T\tilde{Q}(t,i)\tilde{\kappa}-\tilde{\kappa}^T\tilde{S}^T\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j-\frac{1}{2}\Big(\sum_{j=1}^{d}\tilde{\kappa}_j^TD_j(t,i)\Big)\tilde{R}(t,i)\Big(\sum_{j=1}^{d}D_j^T(t,i)\tilde{\kappa}_j\Big)$$

$$+\sum_{j\neq i}^{k}q_{ij}\big[\tilde{N}(t,j)-\tilde{N}(t,i)-\frac{1}{2}(\tilde{M}^T(t,j)-\tilde{M}^T(t,i))\tilde{P}^{-1}(t,j)(\tilde{M}(t,j)-\tilde{M}(t,i))\big]=0$$

where the terminal conditions are given by (3.46):

$$\tilde{P}(T,i)=-G^{-1}(T,i),\tilde{M}(T,i)=-G^{-1}(T,i)L(T,i),\tilde{N}(T,i)=\frac{1}{2}L^T(T,i)G^{-1}(T,i)L(T,i).$$

Using the relation (3.62) it is easy to check that the terminal conditions are equivalent. Now, for the first ODE of the primal HJB, the additional terms are

$$\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}(P(t,j)-P(t,i))$$

When showing equivalence in section 2.3 we transformed the ODE from the primal by multiplying on both sides by $\tilde{P}(t,i)$. Doing so to the additional term becomes

$$\begin{aligned}\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}\tilde{P}(t,i)(P(t,j)-P(t,i))\tilde{P}(t,i)&=\frac{1}{2}\sum_{j\neq i}^{k}q_{ij}\tilde{P}(t,i)(\tilde{P}^{-1}(t,j)-\tilde{P}^{-1}(t,i))\tilde{P}(t,i)\\&=\sum_{j\neq i}^{k}q_{ij}\left[\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\right]\end{aligned}$$

which is exactly the same as the Markov term from the dual HJB:

$$\sum_{j\neq i}^{k}q_{ij}\left[\frac{1}{2}\tilde{P}(t,i)\tilde{P}^{-1}(t,j)\tilde{P}(t,i)-\frac{1}{2}\tilde{P}(t,i)\right].$$

Similarly, one can check for the additional Markov terms in the ODE for $\tilde{M}(t,i)$.

4 Numerical Solutions

In this chapter, we solve the system of coupled ODEs which we derived as a solution to the non-Markov problem in Chapter 1, using the Runge-Kutta numerical scheme. Since the built-in libraries in Python using this method do not allow for matrix ODEs as in our case (as far as I saw), I have written the code to solve them from scratch, which can be found here [4].

So far, we have derived an analytical solution for both the Primal and Dual problems in the case of quadratic cost and no constraints on the control. These solutions are in the form of a system of three ODEs, which we will now solve numerically.

For simplicity, we will consider the case where $d = 1$, i.e. we work with one-dimensional Brownian motion. We first present some of the results we have derived in previous sections for the $d = 1$ case:

4.1 Primal Problem for 1-Dimensional Brownian Motion Case

Recall that the value function for the primal problem is given by

$$v(t,x) = \frac{1}{2}x^T Px + x^T M + N,$$

where $P(t), M(t)$ and $N(t)$ are solutions to the system of ODEs (2.17), (2.18), (2.19):

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} + 2A^T P + 2PB\vartheta_1 + (C^T + \vartheta_1^T D^T)P(C + D\vartheta_1) - Q - 2S^T\vartheta_1 - \vartheta_1^T R\vartheta_1 = 0 \\
\frac{\mathrm{d}M}{\mathrm{d}t} + A^T M + PB\kappa_1 + \vartheta_1^T B^T M + (C^T + \vartheta_1^T D^T)PD\kappa_1 - S^T\kappa_1 - \vartheta_1^T R\kappa_1 = 0 \\
\frac{\mathrm{d}N}{\mathrm{d}t} + \kappa_1^T B^T M + \frac{1}{2}\kappa_1^T D^T PD\kappa_1 - \frac{1}{2}\kappa_1^T R\kappa_1 = 0,
\end{aligned}$$

where ϑ_1 and κ_1 are given by (2.16):

$$\vartheta_1 = \big(D^T PD - R\big)^{-1}\big(S - B^T P - D^T PC\big), \quad \kappa_1 = -\big(D^T PD + R\big)^{-1}B^T M$$

and the terminal conditions are given by (2.14):

$$P(T) = -G(T), \quad M(T) = -L(T), \quad N(T) = 0. \tag{4.1}$$

4.2 Dual Problem for for 1-Dimensional Brownian Motion Case

The value function for the dual problem is given by

$$\tilde{v}(t,y) = \frac{1}{2}y^T\tilde{P}y + y^T\tilde{M} + \tilde{N},$$

where $\tilde{P}(t), \tilde{M}(t)$ and $\tilde{N}(t)$ are solutions to the system of ODEs (2.46), (2.47), (2.48) (in the $d = 1$ case):

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} + 2\tilde{\vartheta}^T\tilde{P} - 2A\tilde{P} - 2\tilde{\vartheta}_1^TC\tilde{P} + \tilde{\vartheta}_1^T\tilde{P}\tilde{\vartheta}_1 - \tilde{\vartheta}^T\tilde{Q}\tilde{\vartheta} - 2\tilde{\vartheta}^T\tilde{S}^T\left(B^T + D^T\tilde{\vartheta}_1\right) - \ (B + \tilde{\vartheta}_1^TD)\tilde{R}(B^T + D^T\tilde{\vartheta}_1) = 0$$

$$\begin{aligned}\frac{\mathrm{d}\tilde{M}}{\mathrm{d}t} + \tilde{P}\tilde{\kappa} - \tilde{P}C^T\tilde{\kappa}_1 + \tilde{\vartheta}^T\tilde{M} - A\tilde{M} - \tilde{\vartheta}_1^TC\tilde{M} + \tilde{\vartheta}_1\tilde{P}\tilde{\kappa}_1 - \tilde{\vartheta}^T\tilde{Q}\tilde{\kappa} - \tilde{\vartheta}^T\tilde{S}^TD^T\tilde{\kappa}_1 \\ -\left(B + \tilde{\vartheta}_1^TD\right)\tilde{S}\tilde{\kappa} - \left(B + \tilde{\vartheta}_1^TD\right)\tilde{R}D^T\tilde{\kappa}_1 = 0\end{aligned}$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{d}t} + \tilde{\kappa}^T\tilde{M} - \tilde{\kappa}_1^TC\tilde{M} + \frac{1}{2}\tilde{\kappa}_1^T\tilde{P}\tilde{\kappa}_1 - \frac{1}{2}\tilde{\kappa}^T\tilde{Q}\tilde{\kappa} - \tilde{\kappa}^T\tilde{S}^TD^T\tilde{\kappa}_1 - \frac{1}{2}\tilde{\kappa}_1^TD\tilde{R}D^T\tilde{\kappa}_1 = 0$$

where $\tilde{\vartheta}, \tilde{\kappa}$ and $\tilde{\vartheta}_1, \tilde{\kappa}_1$ are solutions to the system (2.49)

$$\begin{cases} \tilde{P}y + \tilde{M} - \tilde{Q}(\tilde{\vartheta}y + \kappa) - \tilde{S}^TB^Ty - \tilde{S}^TD^T(\tilde{\vartheta}_1y + \kappa_1) = 0 \\ C(\tilde{P}y + \tilde{M}) - \tilde{P}(\tilde{\vartheta}_1y + \kappa_1) + D\tilde{S}(\tilde{\vartheta}y + \kappa) + D\tilde{R}B^Ty + D\tilde{R}D^T(\tilde{\vartheta}_1y + \kappa_1) = 0 \end{cases}$$

and the terminal conditions are given by (2.50):

$$\tilde{P}(T) = -G^{-1}(T), \quad \tilde{M}(T) = -G^{-1}(T)L(T), \quad \tilde{N}(T) = -\frac{1}{2}L^T(T)G^{-1}(T)L(T).$$

4.3 Runge-Kutta Method

Next, we will describe the method employed to resolve the two systems of ordinary differential equations (ODEs). For ease of understanding, let's set $n = 2$, which implies that X exists within a two-dimensional space. We also assume the coefficients $A(t), B(t), C(t), D(t)$ to be time-independent, meaning they're constant, and we further take them to be symmetric.

Our initial focus lies on the equations for P and $\tilde{P}$. The solution to these ODEs is achieved via the Runge-Kutta method. It's important to highlight that in our situation, both P and $\tilde{P}$ are represented as 2×2 matrices. We omit the details of the method, as it is a widely-used approach for solving such ODEs.

From (2.68) we know that the relationship between P and $\tilde{P}$ is given by

$$P = \tilde{P}^{-1}.$$

As the matrices P and $\tilde{P}$ are symmetric 2×2 matrices, we only show the solution for the 2 distinct elements (in our case we have simplified the coefficients so that we only get two distinct values in P and $\tilde{P}$). The results are shown below: We are considering the time interval from 0 to 1 which we have divided into 200 time-steps. As we can see from the plots, the solutions for P and $\tilde{P}^{-1}$ are identical, as expected from the analytically derived results.

After having obtained the solutions for P and $\tilde{P}$, we can then compute the solutions for M and $\tilde{M}$ using the same method. Note that we needed to compute the ODEs for P and $\tilde{P}$ first, as their solutions are part of the ODEs for the rest of the parameters. From (2.68) we know that the relationship between M and $\tilde{M}$ is given by

$$\tilde{M} = -\tilde{P}M.$$

Our numerical solutions give us the following As we can see, the last two plots are identical, confirming our analytical results. Very similar results can be shown for the ODEs for N and $\tilde{N}$.

While these results are a mere confirmation for the 1-dimensional Brownian motion case of our analytical results, we will make use of this method when we employ the BSDE Deep learning algorithm to this problem in chapter 6, which is essentially an alternative way to reach the optimal control. The Runge-Kutta solution will help us compare these analytical results with the BSDE method results.

If we wish to work in the setting of Chapter 2, i.e. the case where the coefficients depend on a continuous-time finite state space Markov chain, we simply need to consider multiple systems of ODEs corresponding to each state of the chain, which can be done using the same methods as here.

5 Constrained Optimisation with Convex Running Cost

In this section, we return to the framework of Chapter **??**, however, now we impose constraints on the control and no longer consider quadratic cost functions. This general framework is as described in [8]. In this case, we no longer have an analytical solution to the problem, however, we can derive a useful dual-primal relationship, which will help us later with implementing the Deep BSDE algorithm.

5.1 Primal and Dual HJB & BSDE Methods

5.1.1 Primal HJB Equation

The HJB equation is the same as in section 2.1.1 (derivation taken from [7]), except that the supremum is taken over the set K instead of the whole space $\mathbb{R}^m$. Similar to the unconstrained case, we define the value function $v : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$ as

$$v(t, X(t)) = \sup_{\pi \in K} \mathbb{E}\left[\int_{t_0}^{T} -f(t, X(t), \pi(t))\,\mathrm{d}t - g(X(T)) \right]$$

The HJB is then given by:

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\pi \in K} \left\{ F(t, x, \pi, D_x[v], D_x^2[v]) \right\} = 0 \tag{5.1}$$

where the Hamiltonian F is defined as

$$F(t, x, \pi, D_x[v], D_x^2[v]) = (AX + B\pi)^T D_x[v(t, x(t))] + \frac{1}{2}(Cx + D\pi)^T D_x^2[v(t, x)](Cx + D\pi) - f(t, x, \pi)$$

and the terminal condition is given by:

$$v(T, x) = -g(x).$$

5.1.2 Primal BSDE

This section is adapted from [8]. Proofs and details of the Stochastic Maximum Principle can be found in [2]. To state the SMP, we need the following assumption:

Let $(\hat{X}, \hat{\pi})$ be admissible pair satisfying

$$\mathbb{E}\left[\int_{t_0}^{T} |D_x[f(t, \hat{X}(t), \hat{\pi}(t))]|^2\,\mathrm{d}t \right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_{t_0}^{T} |D_x[g(\hat{X}(T))]|^2\,\mathrm{d}t \right] < \infty.$$

For any admissible pair (X, π), there exists $Z \in \mathcal{P}([t_0, T], \mathbb{R})$ and an $\mathcal{F}_T$-measurable random variable $\tilde{Z}$ satisfying $\mathbb{E} \int_{t_0}^{T} |Z(t)| \, \mathrm{d}t < \infty$, $\mathbb{E}|\tilde{Z}| < \infty$ such that

$$\begin{aligned} Z(t) &\geq \frac{f(t, \hat{X}(t) + \varepsilon(X(t) - \hat{X}(t)), \hat{\pi}(t) + \varepsilon(\pi(t) - \hat{\pi}(t))) - f(t, \hat{X}(t), \hat{\pi}(t))}{\varepsilon} \\ \tilde{Z} &\geq \frac{g(\hat{X}(T) + \varepsilon(X(T) - \hat{X}(T))) - g(\hat{X}(T))}{\varepsilon} \end{aligned}$$

for $(\mathbb{P} \otimes \mathit{Leb}) - a.e.$ $(\omega, t) \in \Omega \times [t_0, T]$ and $\varepsilon \in (0, 1]$.

We define the generalised Hamiltonian $\mathcal{H} : \Omega \times [t_0, T] \times \mathbb{R}^n \times K \times R^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{H}(t, x, \pi, p, q) = x^T A^T p + \pi^T B^T p + \Big(x^T C^T + \pi^T D^T\Big) q - f(t, x, \pi).$$

The adjoint process is given by

$$\begin{cases} \mathrm{d}p &= -D_x[\mathcal{H}(t, X(t), \pi(t), p(t), q(t))] \, \mathrm{d}t + q(t) \, \mathrm{d}W(t) \\ &= -\big[A^T p + C^T q - D_x[f(t, X, \pi)]\big] \, \mathrm{d}t + q(t) \, \mathrm{d}W(t) \\ p(T) &= -D_x[g(X(T))] \end{cases} \tag{5.2}$$

The Stochastic Maximum Principle for problem (**??**) states that if X is the solution to (**??**) and p, q are the solution to the adjoint process (5.2), then $\hat{\pi} \in \mathcal{A}$, such that the assumption above holds, is optimal if and only if it satisfies the condition

$$\mathcal{H}(t, X(t), \hat{\pi}(t), p(t), q(t)) = \sup_{\pi \in K} \mathcal{H}(t, X(t), \pi(t), p(t), q(t))$$

for $(\mathbb{P} \otimes \mathit{Leb}) - a.e.$ $(\omega, t) \in \Omega \times [t_0, T]$. Moreover, if $D_\pi[f]$ exists, then this conditions is equivalent to

$$[\hat{\pi} - \pi]^T \Big[B^T p + D^T q - D_\pi[f(t, X, \hat{\pi})]\Big] \geq 0, \quad \forall \pi \in K.$$

5.1.3 Dual HJB Equation

Recall that the dual process $Y(t)$ satisfies the SDE (**??**):

$$\begin{cases} \mathrm{d}Y(t) &= \big[\alpha(t) - A(t)^T Y(t) - C^T(t)\beta_1(t)\big] \, \mathrm{d}t + \beta_1(t) \, \mathrm{d}W(t) \\ Y(t_0) &= y, \end{cases}$$

and also recall from (**??**) that the dual value function is given by

$$\tilde{v}(t, Y(t)) = \sup_{\alpha, \beta_1} \mathbb{E}\Big[\int_{t_0}^{T} -\phi(t, \alpha, \beta) \, \mathrm{d}t - h(Y(T))\Big]$$

where $\beta = B^T Y + D^T \beta_1$ and ϕ and h are given by (**??**) and (**??**):

$$\begin{aligned} \phi(t, \alpha, \beta) &= \sup_{x, \pi} \big\{x^T \alpha + \pi^T \beta - \tilde{f}(t, x, \pi)\big\} \\ h(y) &= \sup_{x} \big\{ - x^T y - g(x)\big\}. \end{aligned}$$

Then, using the same reasoning as in section 2.2.1, the dual HJB equation is given by

$$\frac{\partial \tilde{v}}{\partial t}(t, y) + \sup_{\alpha, \beta_1} \big[\mathcal{L}^{\alpha, \beta_1}[\tilde{v}(t, y)] - \phi(t, \alpha, \beta)\big] = 0,$$

where the generator is given by

$$\mathcal{L}^{\alpha,\beta_1}[\tilde{v}(t,y)] = \big(\alpha^T - y^T A - \beta_1^T C\big) D_y[\tilde{v}] + \frac{1}{2}\beta_1^T D_y^2[\tilde{v}]\beta_1,$$

and the terminal condition is

$$\tilde{v}(T,y) = -h(y).$$

If we assume that $n < m$ and $rank(D(t)) = n$. Then $D^\dagger(t) := D^T(t)(D(t)D^T(t))^{-1} \in \mathbb{R}^{m\times n}$ is the Moore-Penrose inverse of D and satisfied $DD^\dagger = I_n$. From (**??**) we can write

$$\beta_1(t) = (D^\dagger)^T(t)\big(\beta(t) - B^T(t)Y(t)\big)$$

Then the HJB equation is given by

$$\begin{aligned}\frac{\partial \tilde{v}}{\partial t}(t,y) + \sup_{\alpha,\beta}\Big\{ &-\phi(t,\alpha,\beta) + \big(\alpha^T - y^T A - \big(\beta(t) - B^T(t)Y(t)\big)^T D^\dagger(t)C\big)D_y[\tilde{v}] \\ &+\frac{1}{2}\big(\beta(t) - B^T(t)Y(t)\big)^T D^\dagger(t)D_y^2[\tilde{v}](D^\dagger)^T(t)\big(\beta(t) - B^T(t)Y(t)\big)\Big\} = 0.\end{aligned}$$

5.1.4 Dual BSDE

This section is once again adapted from [8]. To state the Stochastic Maximum Principle for the dual problem, we need a similar assumption to that of the primal problem:

Let $(\hat{Y},\hat{\alpha},\hat{\beta}_1)$ be a given admissible dual pair. For any admissible pair (Y,α,β_1), there exists $Z \in \mathcal{P}([t_0,T],\mathbb{R})$ and an $\mathcal{F}_T$-measurable random variable $\tilde{Z}$ satisfying $\mathbb{E}\int_{t_0}^T |Z(t)|\,\mathrm{d}t < \infty$, $\mathbb{E}|\tilde{Z}| < \infty$ such that

$$\begin{aligned} Z(t) &\geq \frac{\phi(t,\hat{\alpha}+\varepsilon\alpha,\hat{\beta}+\varepsilon\beta) - \phi(t,\hat{\alpha},\hat{\beta})}{\varepsilon} \\ \tilde{Z} &\geq \frac{h(\hat{Y}(T)+\varepsilon Y(T)) - h(\hat{Y}(T))}{\varepsilon}\end{aligned}$$

for $(\mathbb{P}\otimes \mathit{Leb}) - a.e.\ (\omega,t) \in \Omega\times[t_0,T]$ and $\varepsilon\in(0,1]$. Furthermore, h is C^1 in y and satisfies $\mathbb{E}[D_y[h^2(Y(T))]] < \infty$.

The Hamiltonian $\tilde{\mathcal{H}} : [t_0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ for the dual problem is defined as

$$\begin{aligned}\tilde{\mathcal{H}}(t,Y,\alpha,\beta_1,p,q) &= p^T(\alpha - A^T Y - C^T\beta_1) + \beta_1^T q - \phi\Big(t,\alpha,B^T Y + D^T\beta_1\Big) \\ &= p^T\alpha - p^T A^T Y - p^T C^T\beta_1 + \beta_1^T q - \phi\Big(t,\alpha,B^T Y + D^T\beta_1\Big)\end{aligned}$$

If ϕ is C^1 in β, and under the assumption above, then the adjoint equation is given by the system

$$\begin{cases}\mathrm{d}p(t) &= -D_y[\tilde{\mathcal{H}}(t,Y(t),\alpha(t),\beta_1(t),p(t),q(t))]\,\mathrm{d}t + q(t)\,\mathrm{d}W(t) \\ &= \big[A(t)p(t) + B(t)D_\beta[\phi(t,\alpha(t),B^T(t)Y(t)+D^T(t)\beta_1(t))]\big]\,\mathrm{d}t + q(t)\,\mathrm{d}W(t) \\ p(T) &= -D_y[h(Y(T))].\end{cases} \tag{5.3}$$

According to the Stochastic Maximum Principle, the optimal controls $\hat{\alpha},\hat{\beta}_1$ satisfy the condition

$$\tilde{\mathcal{H}}(t,Y(t),\hat{\alpha}(t),\hat{\beta}_1(t),p(t),q(t)) = \sup_{\alpha,\beta_1}\tilde{\mathcal{H}}(t,Y(t),\alpha(t),\beta_1(t),p(t),q(t)).$$

If, however, the condition that ϕ is differential in β is not satisfied, then the adjoint equation is not well defined. We consider the following additional assumption:

Assume that $n < m$ and $rank(D(t)) = n$. Then $D^{\dagger}(t) := D^T(t)(D(t)D^T(t))^{-1} \in \mathbb{R}^{m\times n}$ is the Moore-Penrose inverse of D and satisfied $DD^{\dagger} = I_n$. From (**??**) we can write

$$\beta_1(t) = (D^{\dagger})^T(t)\big(\beta(t) - B^T(t)Y(t)\big)$$

Using this and (**??**), the dual process Y satisfies the following SDE:

$$\begin{cases} \mathrm{d}Y(t) = & \big[\alpha(t) - A(t)^T Y(t) - C^T(t)(D^{\dagger})^T(t)\big(\beta(t) - B^T(t)Y(t)\big)\big]\,\mathrm{d}t \\ & \qquad\qquad\qquad\qquad +(D^{\dagger})^T(t)\big(\beta(t) - B^T(t)Y(t)\big)\,\mathrm{d}W(t) \\ Y(t_0) = & y, \end{cases} \tag{5.4}$$

The dual optimisation problem (**??**) is now equivalent to

$$\inf_{y,\alpha,\beta}\left\{x^T y + \mathbb{E}\bigg[\int_{t_0}^{T} \phi(t,\alpha(t),\beta(t)\,\mathrm{d}t + h(Y(T))\bigg]\right\}. \tag{5.5}$$

The adjoint equation associated with (t,α,β) and Y in (5.4) is given by

$$\begin{cases} \mathrm{d}p & = \big[(A(t) - B(t)D^{\dagger}(t)C(t))p(t) + B(t)D^{\dagger}(t)q(t)\big]\,\mathrm{d}t + q(t)\,\mathrm{d}W(t) \\ p(T) & = -D_y[h(Y(T))] \end{cases} \tag{5.6}$$

The Stochastic Maximum Principle for the process (5.4) with the adjoint process (5.6) states that $(\hat{y},\hat{\alpha},\hat{\beta})$ are optimal for the dual problem (5.5) if and only if the following conditions are satisfied:

$$\begin{cases} p(t_0) = x \\ (p, D^{\dagger}q - D^{\dagger}Cp) \in \partial\phi(\hat{\alpha},\hat{\beta}) \\ D^{\dagger}q - D^{\dagger}Cp \in K \end{cases} \tag{5.7}$$

for $(\mathbb{P}\otimes Leb)-a.e.$ $(\omega,t) \in \Omega\times[t_0,T]$, where $\partial\phi(\hat{\alpha}(t),\hat{\beta}(t))$ is the sub-differential of ϕ at $(\hat{\alpha}(t),\hat{\beta}(t))$.

5.2 Primal-Dual Relation

In this section, we state some results on the dual-primal relation. We need to assume that the function $D_x[g](\omega,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is a bijection for any ω such that $z=-D_x[g(x)]$ if and only if $x=-D_y[h(z)]$, that is, the inverse function of $-D_x[g]$ is $-D_y[h]$.

We first show how we can recover the primal optimal solution from that of the dual problem. Proof of this statement can be found in [8].

Suppose that (y,α,β) is optimal for the dual problem (5.5), and let $(Y,p_{\text{dual}},q_{\text{dual}})$ be the associated state and adjoint processes in (5.4) and (5.6). Define

$$\pi(t):=D^{\dagger}(t)q_{\text{dual}}(t)-D^{\dagger}(t)C(t)p_{\text{dual}},\quad t\in[t_0,T]. \tag{5.8}$$

Then $\pi(t)$ is the optimal control for the primal problem (**??**). Further, for $t\in[t_0,T]$, the optimal state and associated adjoint processes satisfy

$$\begin{cases} X(t)=p_{\text{dual}}(t)\\ p_{\text{primal}}(t)=Y(t)\\ q_{\text{primal}}(t)=(D^{\dagger})^T(t)\big(\beta(t)-B^TY(t)\big),\end{cases}$$

where p_{primal} and q_{primal} denote the adjoint processes associated with the primal problem.

We can also recover the dual optimal solution from the primal. Suppose that $\pi(t)\in\mathcal{A}$ is optimal for the primal problem (**??**) and let $(X,p_{primal},q_{primal})$ be the associated state and adjoint processes in (**??**) and (5.2). Define

$$\begin{cases} y=p_{primal}(0)\\ \alpha(t)=D_x[f(t,X(t),\pi(t))]\\ \beta(t)=B^T(t)p_{primal}(t)+D^T(t)q_{primal}(t)\end{cases}$$

Then $(y,\alpha(t),\beta(t))$ is the optimal control of the dual problem (5.5). For $y\in[t_0,T]$, the optimal dual state process and associated adjoint processes satisfy

$$\begin{cases} Y(t)=p_{primal}(t)\\ p_{dual}(t)=X(t)\\ D^{\dagger}(t)q_{dual}(t)=\pi(t)+D^{\dagger}(t)C(t)X(t).\end{cases}$$

6 The Deep Controlled 2BSDE Method

In this section, we adapt the results from [3]. We first state the main results and then apply the algorithm for the unconstrained case described in chapter **??**, in which we know the theoretical optimal control, with the purpose of verifying the results from the deep controlled 2BSDE method.

The algorithm was developed mainly following the idea from [3] as well as [1]. The code was written from scratch, as the first paper only provides a pseudo-code and the second one utilises an older version of `tensorflow`, but the ideas are similar, with adaptations to our specific problem. The code for this project can be found at [4].

6.1 Description of Algorithm

We try to solve the problem using machine learning techniques. The idea is to simulate all processes in the forward direction, introducing the two new variables v_0, z_0 and moving forward through a discretisation of the time interval $[t_0, T]$. We set Γ to be a neural network that at each time t_i takes the ith state position denoted by X_i as input since we are approximating $D_x^2[v]$ with this process. The control π is also a neural network, taking in the same state X_i as input at each time step.

Each of the neural networks has L number of layers with l hidden nodes. We use a nonlinear activation function between each of the hidden layers. For our networks, we choose the `ReLu` activation function, although this choice is rather arbitrary (provided it is nonlinear). No activation function is applied to the output. We denote by θ_i the parameters of the neural network for the controls π_i at each time t_i of the discretisation of the time interval. We further denote by $N_{\theta_i}(X_i)$ the output of the neural network. Similarly, for Γ we denote the parameters by λ_i and the output of the neural network at time t_i by $N_{\lambda_i}(X_i)$.

We then use the following Euler-Maruyama Scheme. Set $X_0 = x_0$, and initialise $V_0 = v_0, Z_0 = z_0$. For $i = 0, \ldots, N-1$ let $\pi_i = N_{\theta_i^0}(X_i)$, $\Gamma_i = N_{\lambda_i^0}(X_i)$ and

$$\begin{aligned}
X_{i+1} &= X_i + (t_{i+1} - t_i)(AX_i + B\pi_i) + (CX_i + D\pi_i)\,\mathrm{d}W_i \\
V_{i+1} &= V_i + Z_i^T(CX_i + D\pi_i)\,\mathrm{d}W_i \\
Z_{i+1} &= Z_i - (t_{i+1} - t_i)D_x[H(t_i, X_i, \pi_i, Z_i, \Gamma_i(CX_i + D\pi_i))] + \Gamma_i(CX_i + D\pi_i)\,\mathrm{d}W_i,
\end{aligned}$$

where $\mathrm{d}W_i$ is a multivariate normal $N(0, (t_{i+1} - t_i)$.

Once we have repeated this iteration $N-1$ times we arrive at time T, at which point we have not satisfied the terminal conditions. We, therefore, define the loss function

$$L_1(\{\lambda_i\}_{i=0}^{N-1} \cup \{V_0, Z_0\}) := \mathbb{E}\Bigg[|V_N + g(X_N)|^2 + |Z_N + D_x[g(X_N)]|^2\Bigg].$$

This loss function is for the parameters of the neural network for Γ and the initial values v_0, z_0. In addition, we need a second loss function for the parameters of the neural network for π. We define for each time step

$$L_2(\theta_i, i) := \mathbb{E}\big[\big|D_\pi[F(t_i, X_i, \pi_i, Z_i, \Gamma_i)]\big|^2\big].$$

Since we want to maximise the Hamiltonian F w.r.t. π, we minimise the square of its derivative. For each of these loss functions, we update the parameters using the Adam algorithm. We repeat this procedure until the parameters converge. The algorithm can be described as follows:

Algorithm 1 Primal Deep BSDE Method

```
Initialise V_0 ~ Unif([−0.1, 0.1]) and Z_0 ~ Unif([−0.1, 0.1])
for i=0, 1, ..., N-1 do
    Initialise θ_i ~ Unif([−0.1, 0.1]) and λ_i ~ Unif([−0.1, 0.1])
end for
for i=0, 1, ..., NumEpochs do
    Generate W_k for k = 1, ..., B
    Generate X^k, V^k, Z^k, π^k, Γ^k using {θ_i, λ_i}_{i=0}^{N−1} and V_0, Z_0
    Set L_1 := (1/B) Σ_{k=1}^{B} [ |V_N^k + g(X_N^k)|^2 + |Z_N^k + D_x[g(X_N^k)]|^2 ]
    Update {λ_i}_{i=0}^{N−1} ∪ {V_0, Z_0} using the ADAM algorithm
    Regenerate X^k, V^k, Z^k, π^k, Γ^k using {θ_i, }_{i=0}^{N−1} and V_0, Z_0 and the updated {λ_i}_{i=0}^{N−1}
    for i=0, 1, 2, ..., N-1 do
        Set L_2(θ_i, i) := −(1/B) Σ_{k=1}^{B} [F(t_i, X_i^k, π_i^k, Z_i^k, Γ_i^k)]
        Update θ_i using the ADAM algorithm
    end for
end for
```

6.2 Applying the Algorithm to the Unconstrained Case

We perform the Deep Controlled 2BSDE algorithm using `tensorflow` for the case where the control is unconstrained and the problem is quadratic, i.e. the framework of chapter **??**. We do this because we know the analytical solution for this problem and want to compare the two methods to verify the Deep Controlled 2BSDE method.

We iterate the steps described above until the loss functions have converged. We use constant values for the parameters of the model, so that $A, B, C, D, Q, R, S, G, L$ are not dependent on time (for simplicity). We take the learning rate for the BSDE step to be larger than the learning rate for the control step, ensuring that the loss functions converge.

The figure below shows the loss functions of the primal problem. We have used a batch size of 1024 and the time interval $[0, 1]$ is divided into 20 time-steps. The image on the left represents the loss L_1, corresponding to the terminal conditions and on the right we have shown the loss L_2, which ensures the optimality of the control. In this method, we have used an initial learning rate of 10^{-2} for minimising L_1 and 10^{-3} for minimising L_2. After a certain number of iterations, we divide this learning rates by 10.

Extending the Algorithm to the Dual Problem

As described in [3], this algorithm can be extended to the dual problem. We perform the algorithm for our problem, and show the loss functions for the dual problem: For the control loss, we have plotted the derivative of the loss function L_3 with respect to y. We have used learning rates of 10^{-2} and 10^{-3} for the respective losses.

6.3 Verifying Results with Analytical Solution

Now that we have both the primal and dual solutions from the Deep BSDE algorithm, we compare the results with the analytical solution. In chapter 4 we described how to solve the systems of ODEs for the primal and dual problem using the Runge-Kutta numerical scheme, giving us the solutions to the unconstrained quadratic case.

For a given simulation of the process X_t, we can compute the analytical solution to the value function for this problem using the results form 4, and plot them together with the results from the Deep BSDE method. We get the following results: We can see that both the primal and dual solutions are very close to the solution.

Next, we recall that we can obtain the optimal control from (5.8) given that we have obtained the dual processes. From the primal processes, we can also get the optimal control π, so we can compare the results, together with the analytical solution given by the Runge-Kutta scheme. Below we show the results for a given simulation of X, (we only show the first element of the control π): Once again, all three algorithms show relatively close results.

6.4 Further Work

With the successful demonstration that the Deep BSDE algorithm is capable of producing results congruent with those derived analytically, we find ourselves in a position to extend its application to scenarios of increased complexity. In these scenarios, achieving analytical results following conventional methods may not be feasible.

Our primary focus of interest is instances where the application of control is subject to limitations. In such situations, it's generally infeasible to use traditional methods to derive the optimal control analytically. This limitation presents us with the opportunity to exploit the capabilities of our Deep BSDE algorithm to its full potential, providing a practical alternative to conventional analytical methods.

Another area of application we are considering is the exploration of more varied types of cost functions. Until now, we've been predominantly working with quadratic cost functions. However, the real-world scenarios can be more diverse and might require dealing with cost functions of a more general convex nature. In this context, we expect our algorithm to prove its versatility and adaptability, thus expanding the range of problems it can tackle.

The algorithm is applied to both primal and dual problems in these scenarios. The primal problem focuses on the task of maximisation, whereas the dual problem centres around minimisation. While it's important to acknowledge that the solutions to these two problems might not be perfectly identical, they can, however, establish upper and lower bounds for the optimal value function. As long as the gap between these boundaries remains relatively small, they can provide invaluable insights.

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