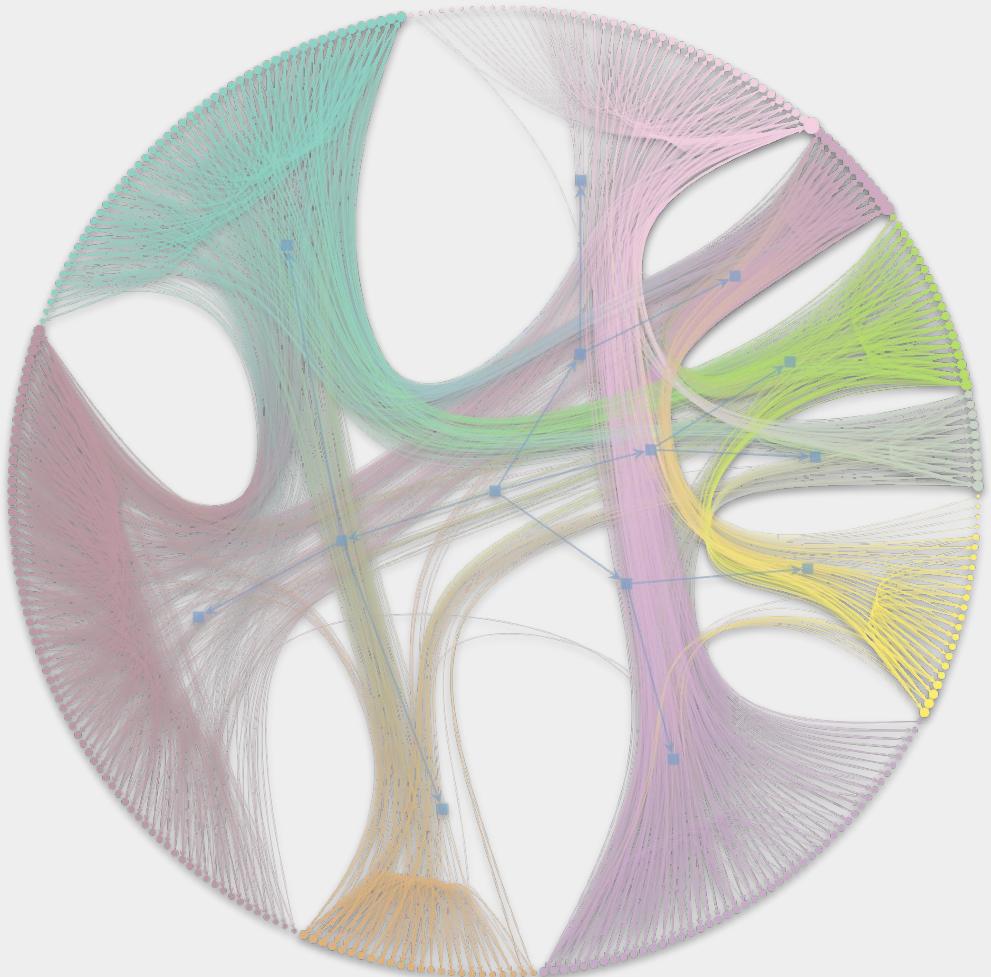


Network Elements

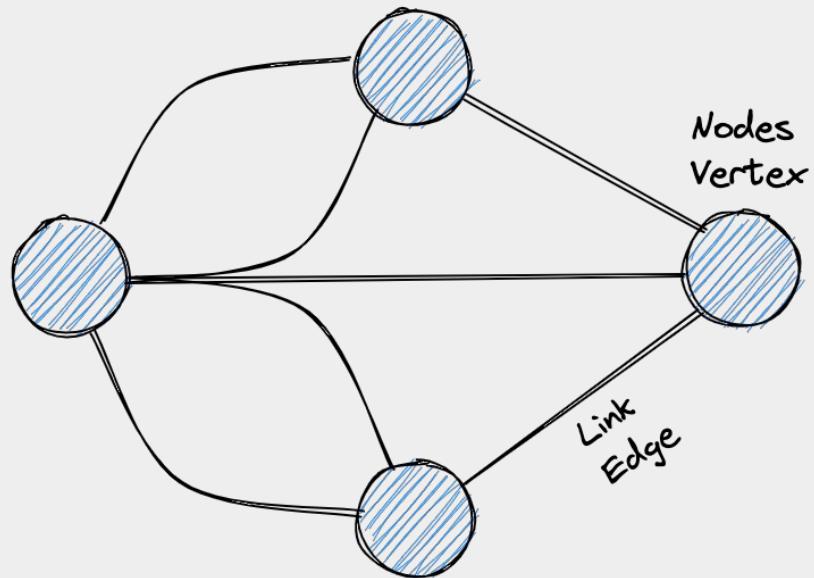
Part 02

ivanovitch.silva@ufrn.br
@ivanovitchm



Basic Definition

Network or Graph



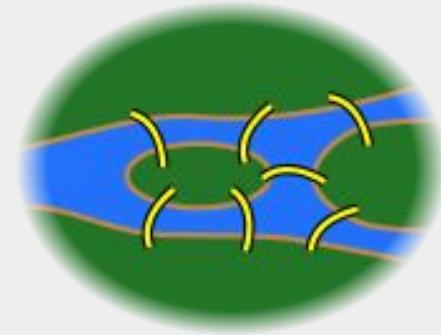
In very general terms a network, or graph, is a set of elements, which we call **nodes**, along with a set of connections between pairs of nodes, which we call **links**. The links represent the presence of a relation among the elements represented by the nodes.

Leonhard Euler, 1735



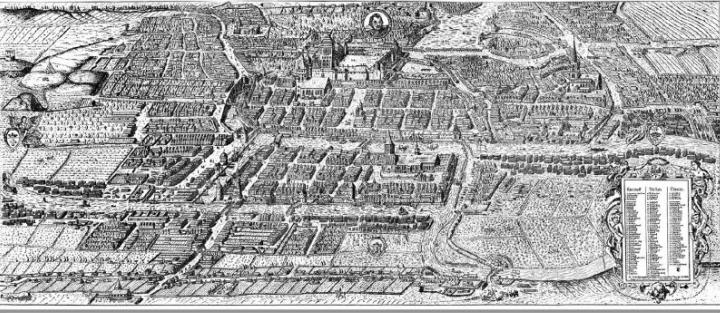
Seven Bridges of Königsberg

Can one walk across all seven bridges and never cross the same one twice?

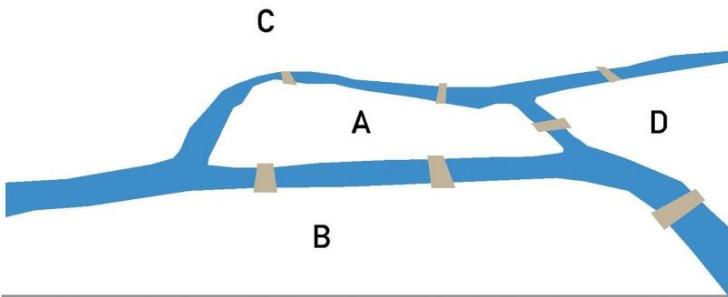


The rigorous language for the description of networks is found in graph theory, a field of mathematics that can be traced back to the pioneering work of Leonhard Euler in the eighteenth century.

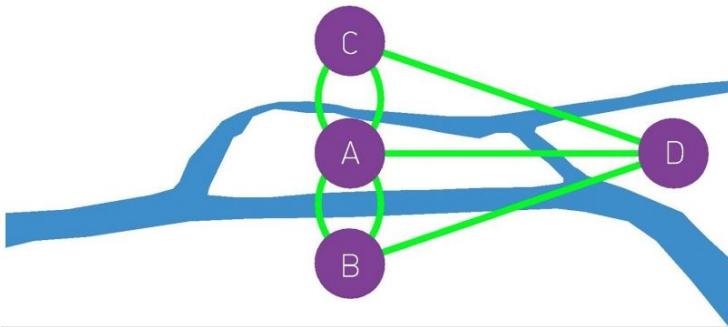
a.



b.



c.

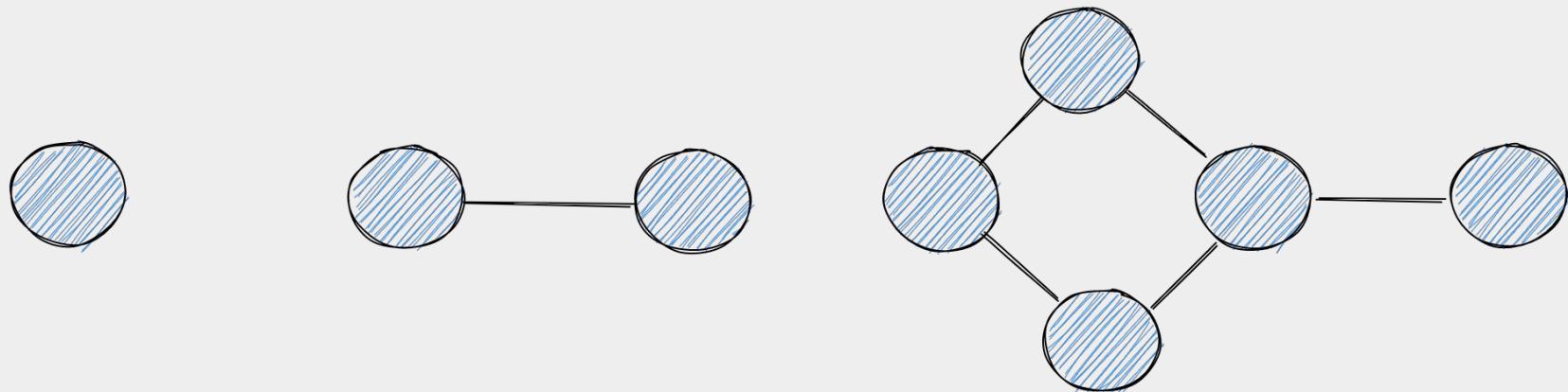


Despite many attempts, no one could find such path. The problem remained unsolved until 1735, when Leonhard Euler, offered a rigorous mathematical proof that such path does not exist.



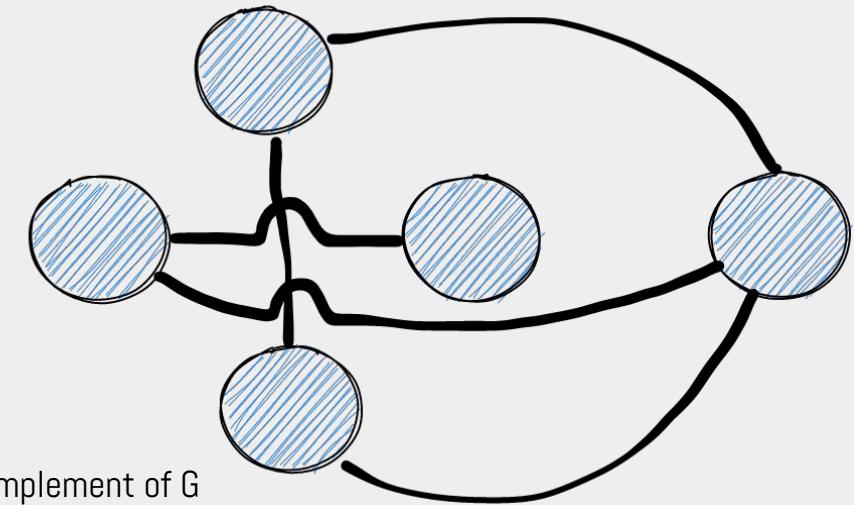
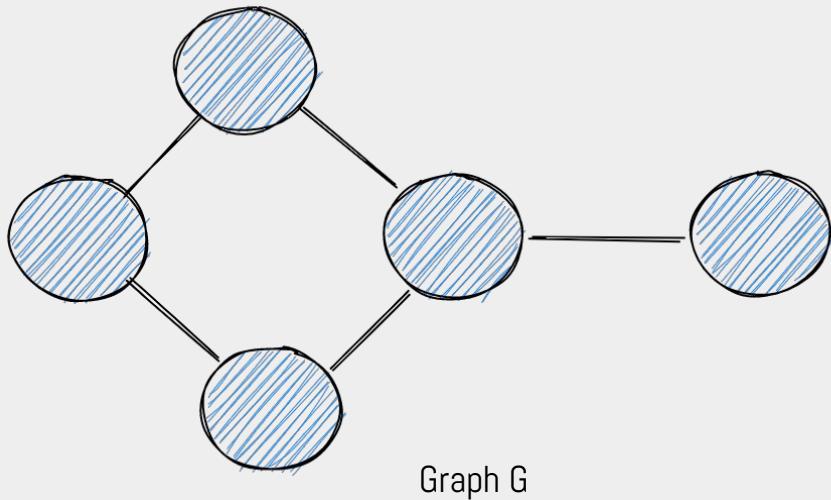
A more rigorous definition of a network

$$G = (V, E), \text{with } E \subseteq V \times V$$



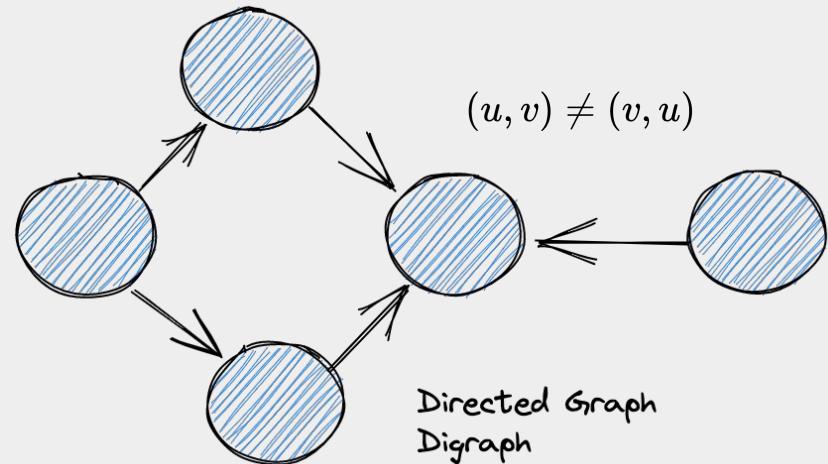
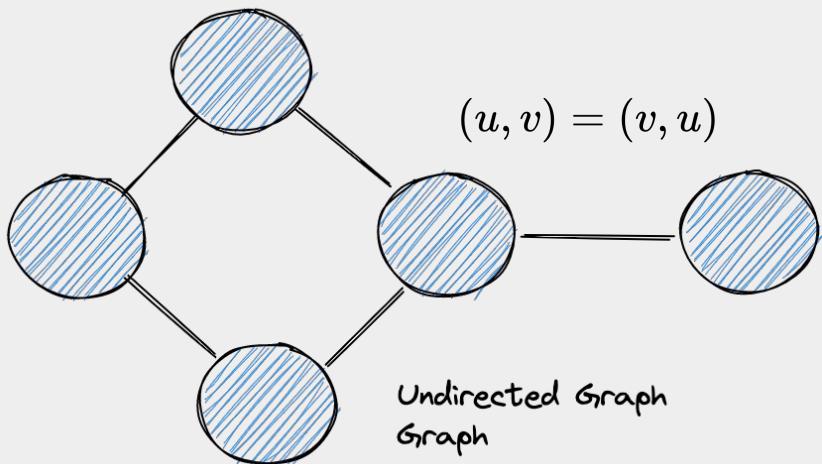
We can derive a series of special graphs

For instance, we can derive the complement of G . This is equivalent to remove all of the original edges of G , and then connect all the unconnected pairs of nodes in G .



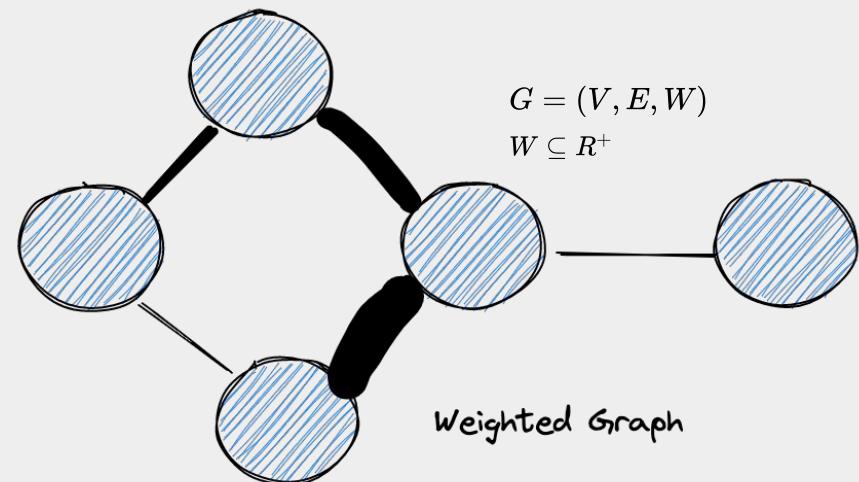
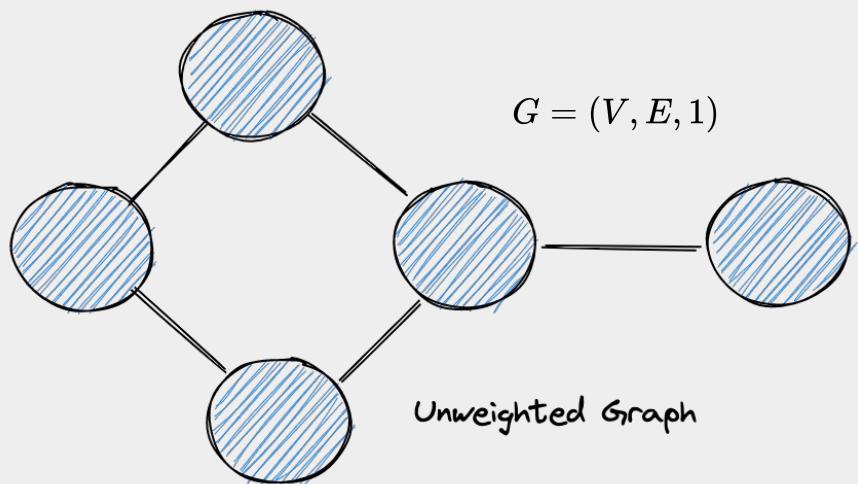
Sometimes you really need to complicate stuff

A first thing we will do is realizing that **not all relations are reciprocal**. We can introduce this asymmetry in the graph model.

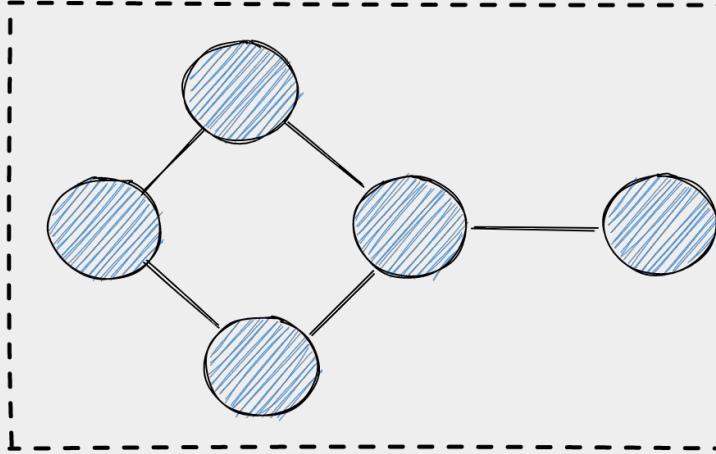


Sometimes you really need to complicate stuff

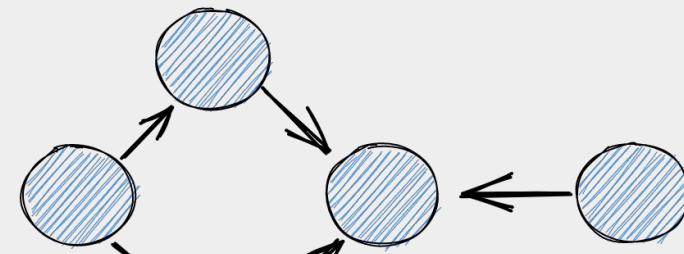
Another way to make edges more interesting is realizing that **two connections are not necessarily equally important** in the network.



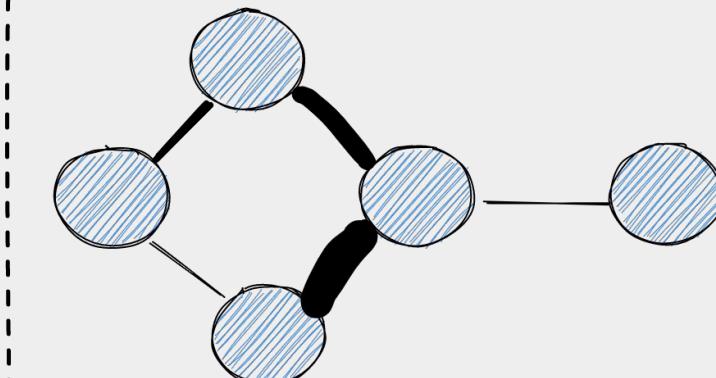
Undirected



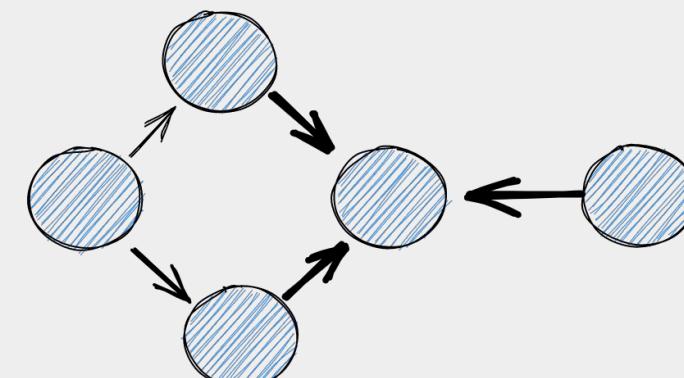
Directed



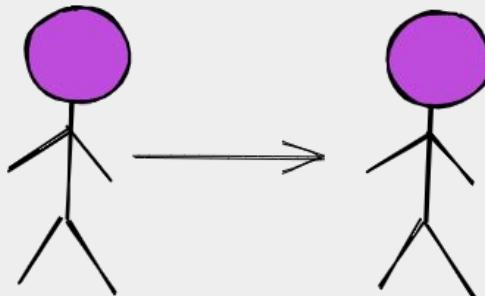
Unweighted



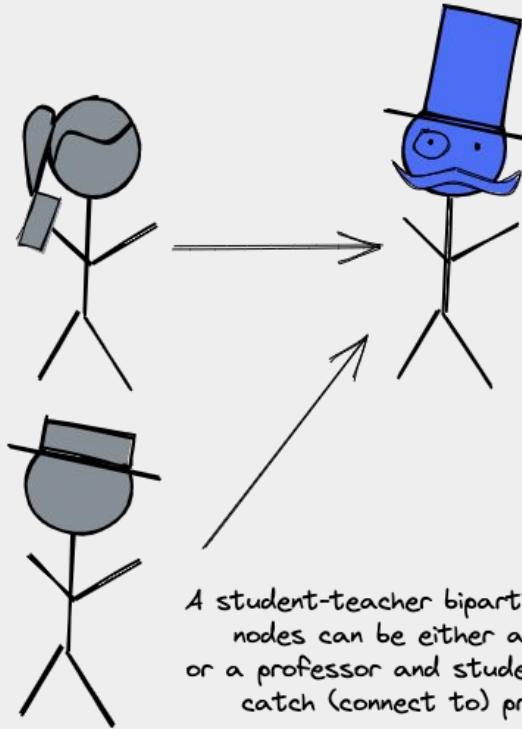
Weighted



Extended Graphs (bipartite network)

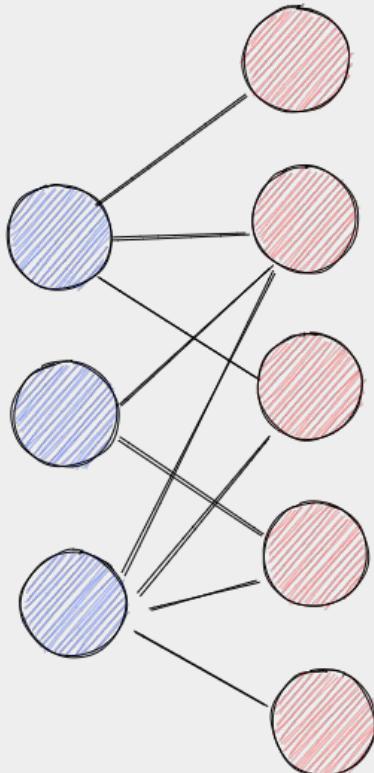


A simple graph representing
a social network with
no additional constraints.



A student-teacher bipartite network:
nodes can be either a student
or a professor and students can only
connect to professors.

Extended Graphs (bipartite network)



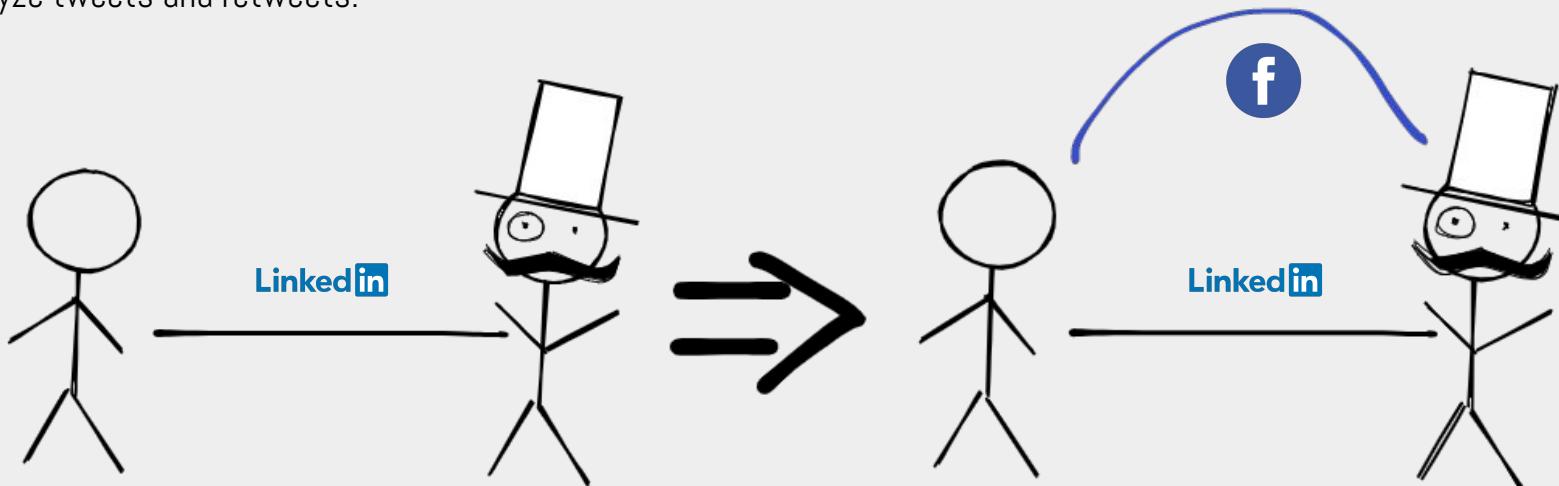
Bipartite networks are used for countless things, connecting:

1. countries to the products they export
2. hosts to guest in symbiotic relationships
3. users to the social media items they tag
4. bank-firm relationships in financial networks
5. players-bands in jazz
6. listener-band in music consumption
7. plant-pollinators in ecosystems

Extended Graphs (multilayer graph)

Traditionally, network scientists try to focus on one thing at a time. For instance, they will download a sample of the LinkedIn graph. Or they will analyze tweets and retweets.

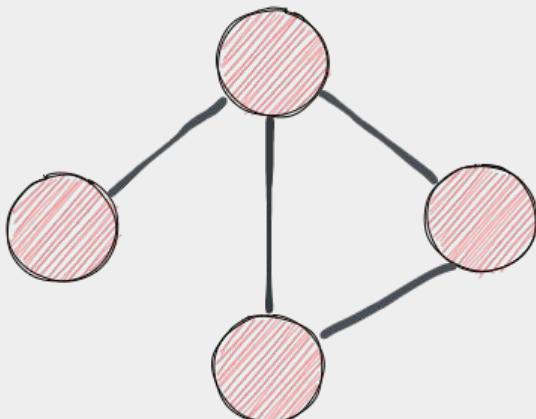
Two people might have started working in the same company and thus first connected on LinkedIn, and then became friends and connected on Facebook. Such scenario could not be captured by simply looking at one of the two networks.



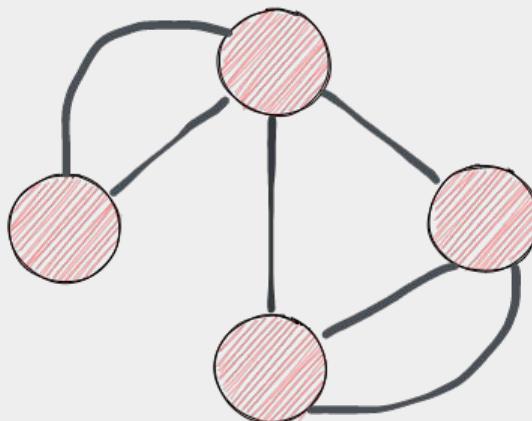
Extended Graphs (multilayer graph)

$$G = (V, E, L)$$

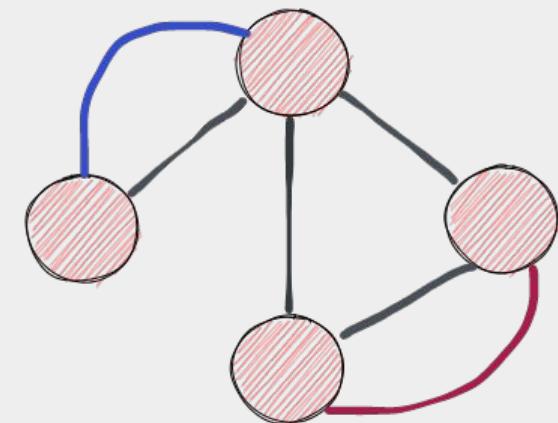
$(u, v, l) \in E$, with $u, v \in V$ and $l \in L$



A simple graph



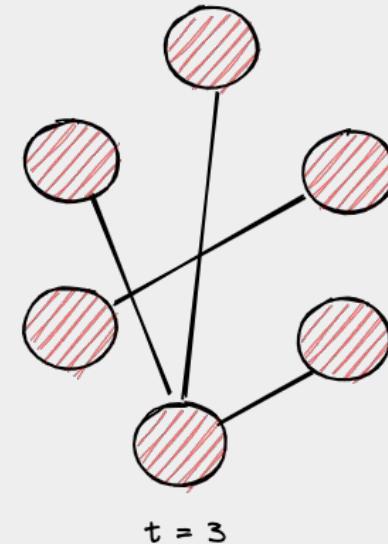
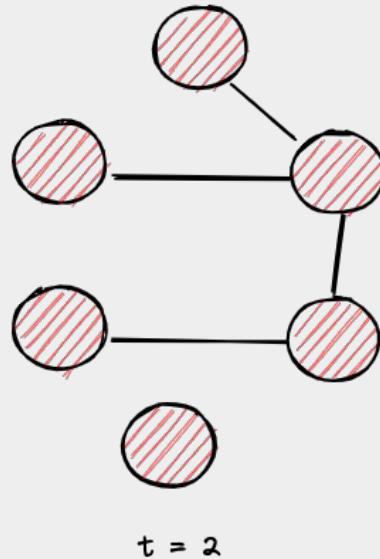
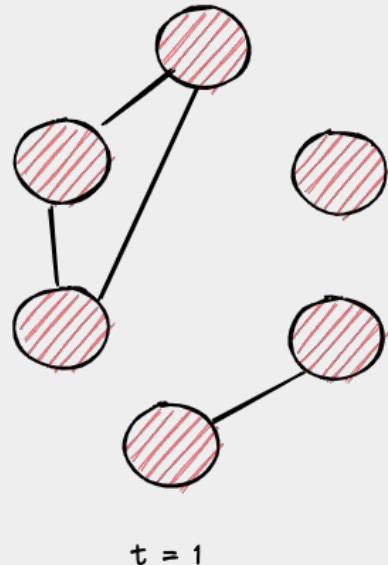
A multigraph, with multiple edges between the same node pairs



A multilayer network, where each edge has a type (represented by its color)

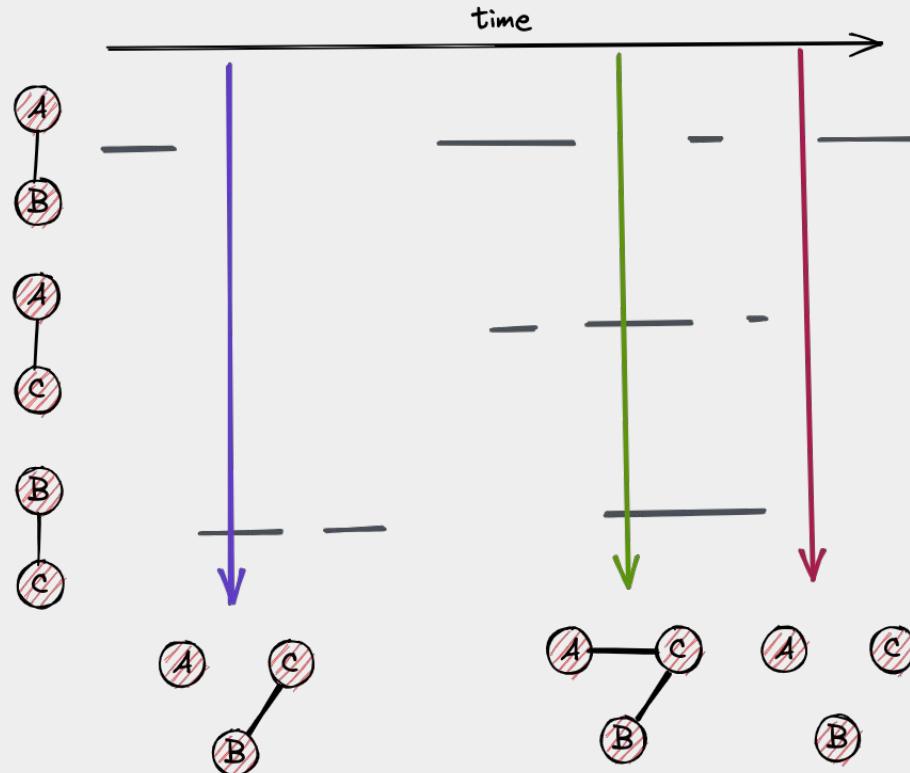
Extended Graphs (dynamic graph)

Imagine that your network represents a road graph. Nodes are intersections, and edges are stretches of the street connecting them. Roadworks might cut off a segment for a few days. If your network model cannot take this into account, you would end up telling drivers to use a road that is blocked, creating traffic jams and a lot of discomfort



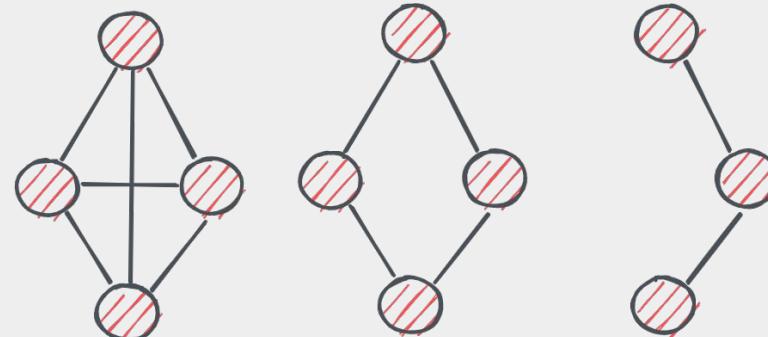
Extended Graphs (dynamic graph)

$$G = (G_1, G_2, \dots, G_n)$$
$$G_i = (V_i, E_i)$$



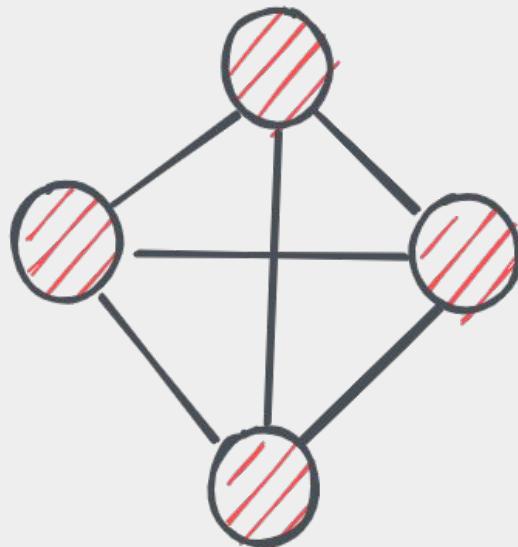


Density and Sparsity, Subnetworks



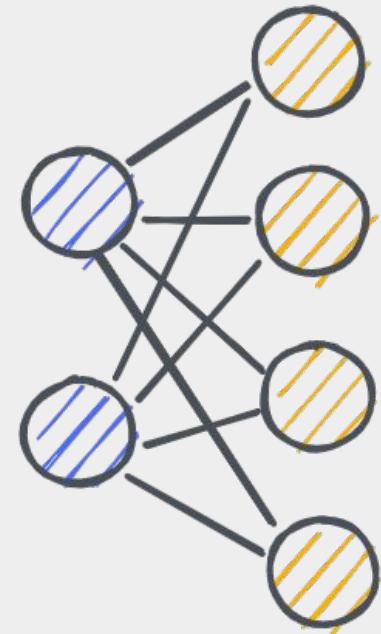
Density and Sparsity

The maximum number of links in a network is bounded by the possible number of distinct connections among the nodes of the system. The maximum number of links is therefore given by the number of pairs of nodes. A network with the maximum number of links, in which all possible pairs of nodes are connected by links, is called a **complete network**.



$$L_{max} = \binom{N}{2} = \frac{N(N - 1)}{2}$$

$$L_{max}^{directed} = N(N - 1)$$



$$L_{max} = N_1 \times N_2$$

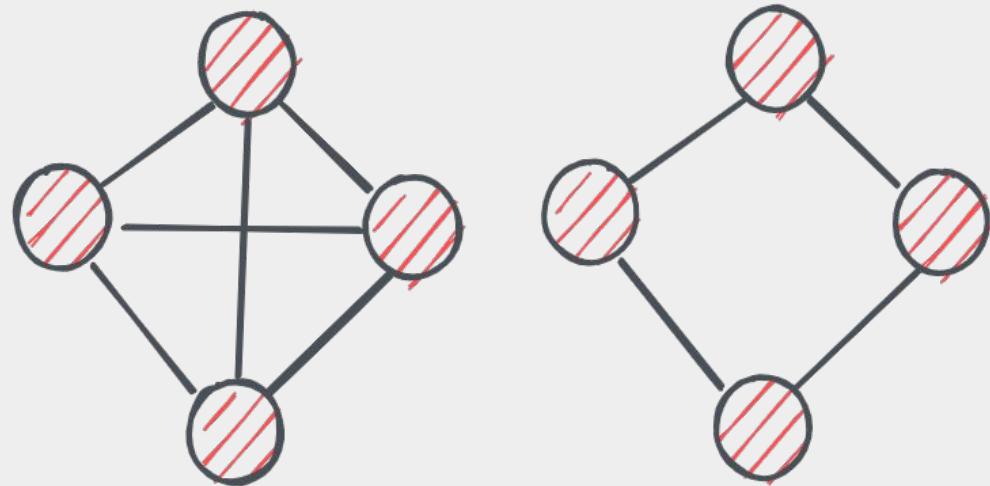
Density and Sparsity

$$d = \frac{L}{L_{max}} = \frac{2L}{N(N - 1)}$$

$$d = \frac{L}{L_{max}^{directed}} = \frac{L}{N(N - 1)}$$

The fraction of possible links that actually exist, which is the same as the fraction of pairs of nodes that are actually connected, is called the **density of the network**.

$$d = \frac{L}{L_{max}}$$

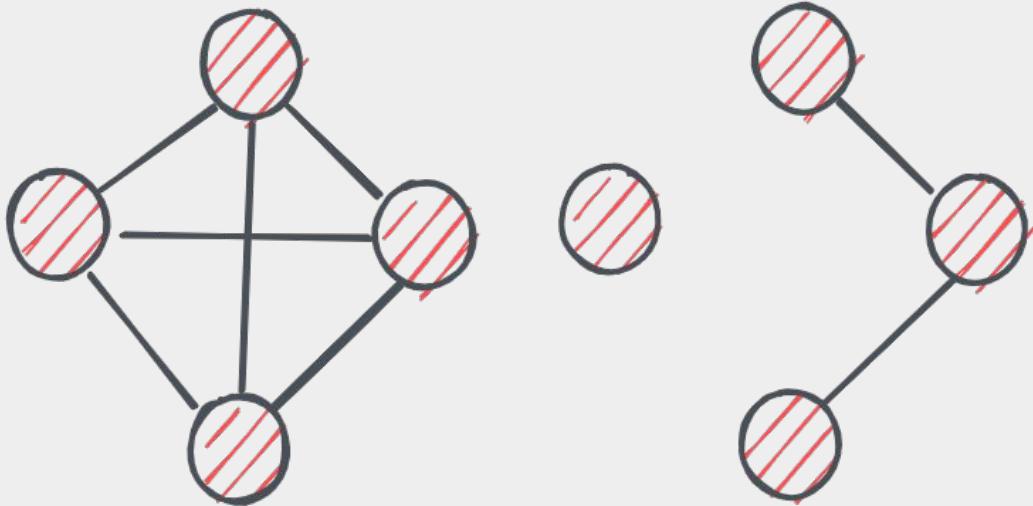


$$d = 1$$

$$d < 1$$

Density and Sparsity

We say that the **network is sparse** if the number of links grows proportionally to the number of nodes ($L \sim N$), or even slower. If instead the number of links grows faster, e.g. quadratically with network size ($L \sim N^2$), then we say that the **network is dense**.

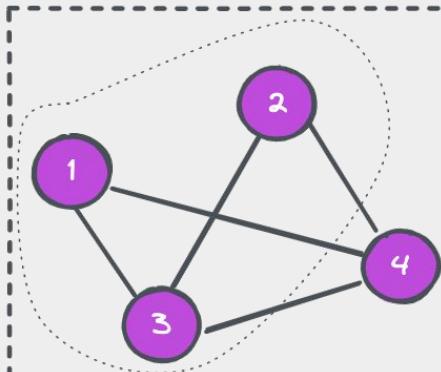


$$L \ll L_{max} \rightarrow d \ll 1$$

| Network | Type | Nodes (N) | Links (L) | Density (d) | Average degree ($\langle k \rangle$) |
|--------------------------------|------|--------------|--------------|----------------|---|
| Facebook Northwestern Univ. | | 10,567 | 488,337 | 0.009 | 92.4 |
| IMDB movies and stars | | 563,443 | 921,160 | 0.000006 | 3.3 |
| IMDB co-stars | W | 252,999 | 1,015,187 | 0.00003 | 8.0 |
| Twitter US politics | DW | 18,470 | 48,365 | 0.0001 | 2.6 |
| Enron email | DW | 87,273 | 321,918 | 0.00004 | 3.7 |
| Wikipedia math | D | 15,220 | 194,103 | 0.0008 | 12.8 |
| Internet routers | | 190,914 | 607,610 | 0.00003 | 6.4 |
| US air transportation | | 546 | 2,781 | 0.02 | 10.2 |
| World air transportation | | 3,179 | 18,617 | 0.004 | 11.7 |
| Yeast protein interactions | | 1,870 | 2,277 | 0.001 | 2.4 |
| <i>C. elegans</i> brain | DW | 297 | 2,345 | 0.03 | 7.9 |
| Everglades ecological food web | DW | 69 | 916 | 0.2 | 13.3 |

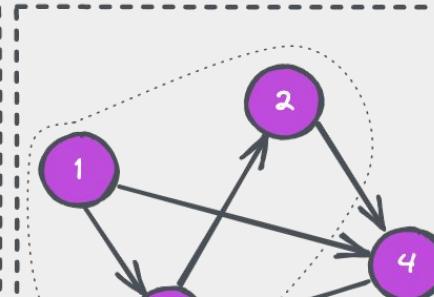
Subnetwork

Undirected



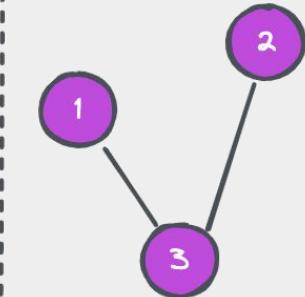
| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 |
| 2 | 0 | 0 | 1 | 1 |
| 3 | 1 | 1 | 0 | 1 |
| 4 | 1 | 1 | 1 | 0 |

Directed



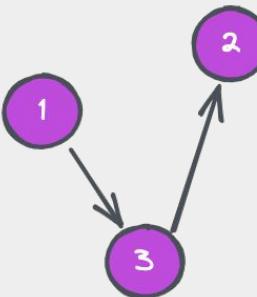
| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

Original



| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 2 | 0 | 0 | 1 |
| 3 | 1 | 1 | 0 |

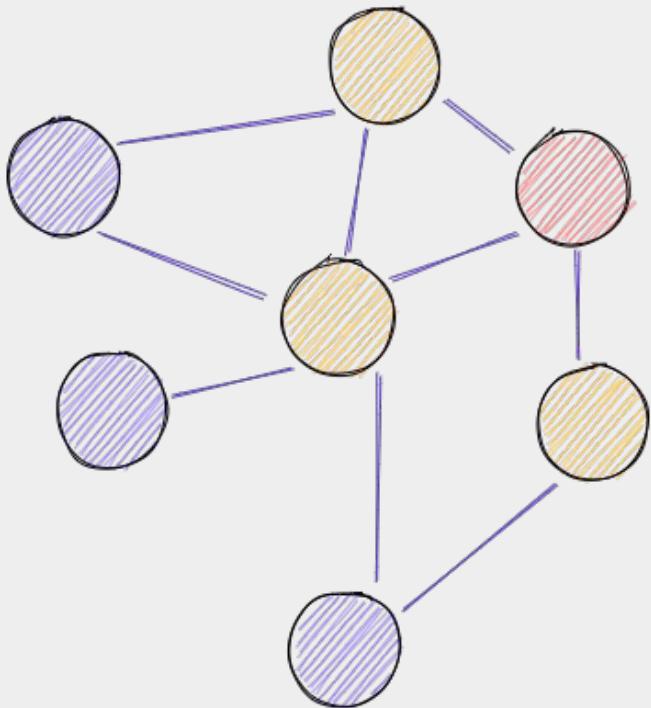
Subnetwork



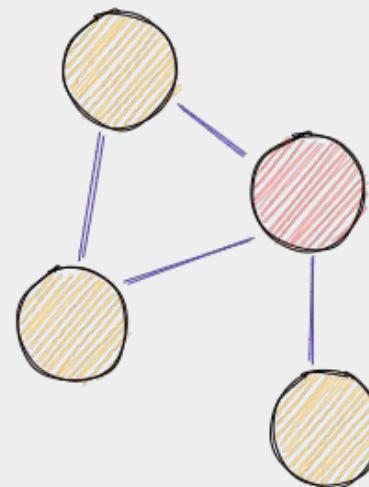
| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 |
| 3 | 0 | 1 | 0 |

Subnetwork

A special type of subnetwork is the ego network of a node, which is the subnetwork consisting of the chosen node — called the ego — and its neighbors. Ego networks are often studied in social network analysis.

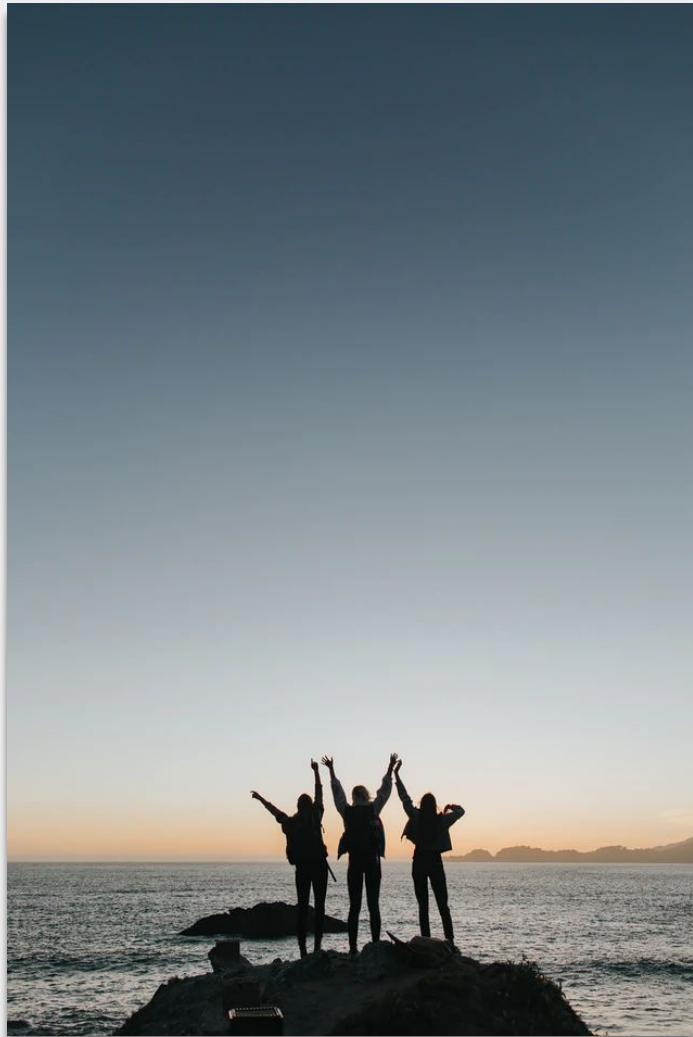


Original Network

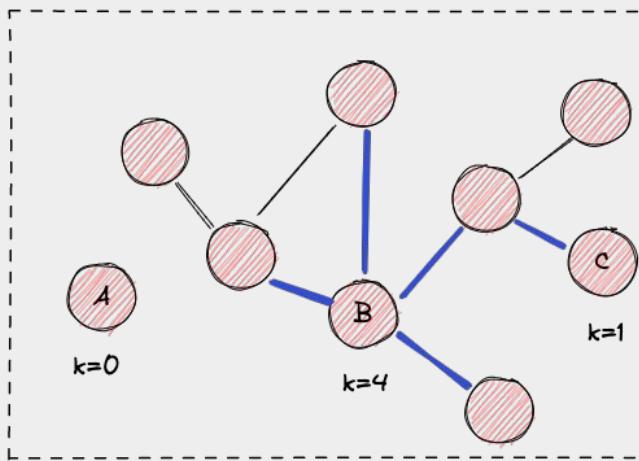


Ego Network

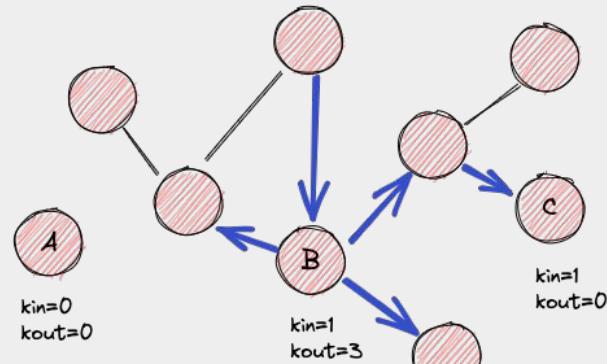
Degree and Network Representation



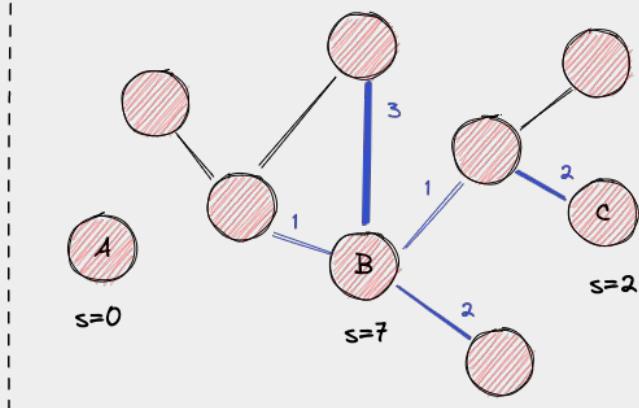
Undirected



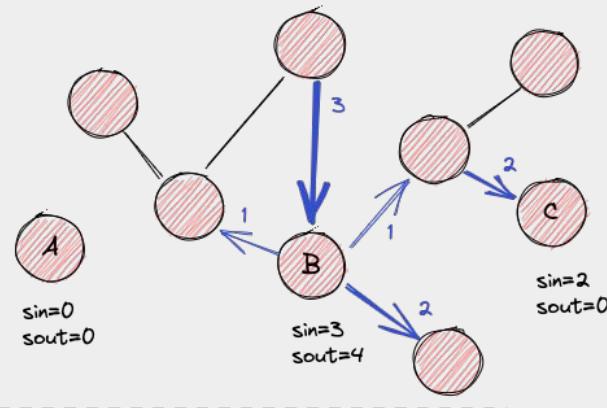
Directed



Unweighted

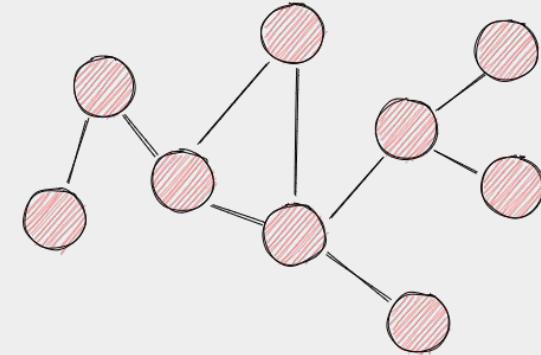


Weighted



The average degree of a network is defined as:

$$\langle k \rangle = \frac{\sum_i k_i}{N}$$



Since each link contributes to the degree of two nodes in an undirected network:

$$\langle k \rangle = \frac{2L}{N} = d(N - 1)$$

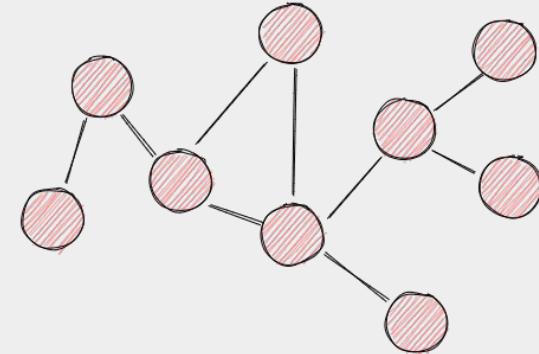
$$d = \frac{2L}{N(N - 1)}$$

The average degree of a network is defined as:

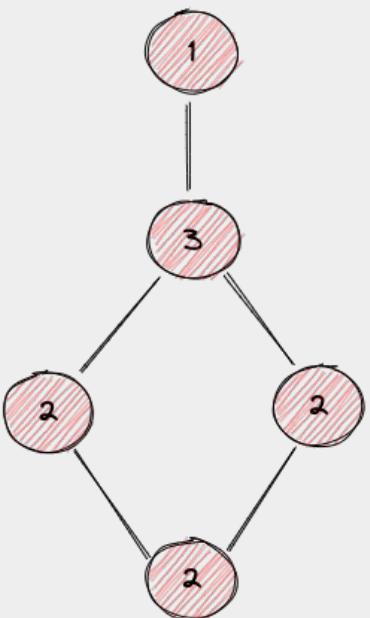
$$\langle k \rangle = \frac{2L}{N} = d(N - 1)$$

$$d = \frac{\langle k \rangle}{N - 1}$$

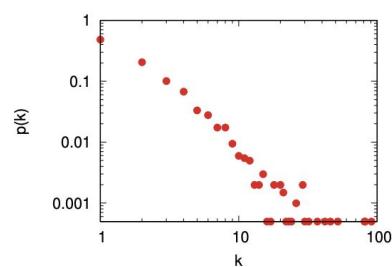
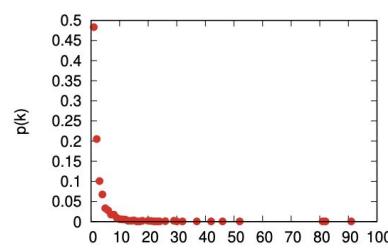
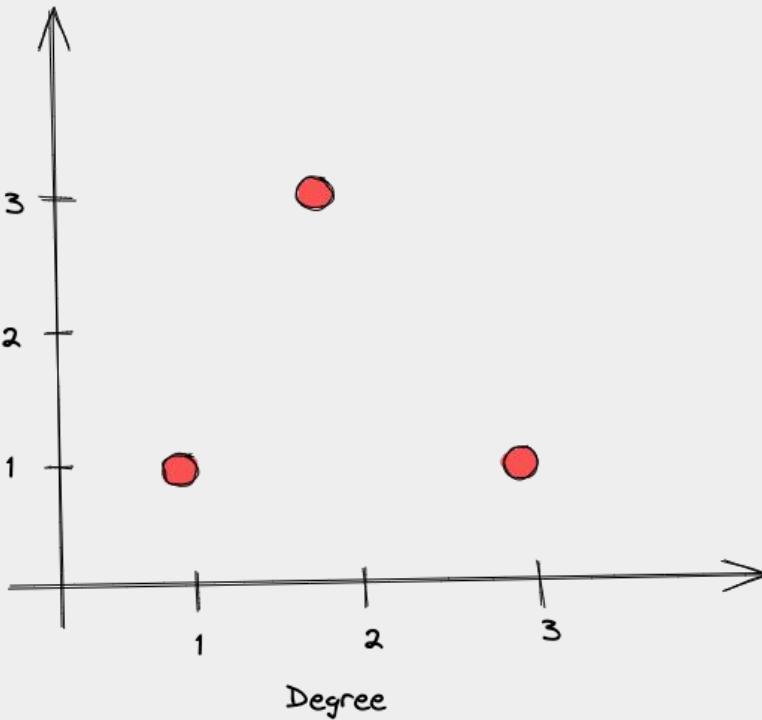
This makes sense because the **maximum possible degree** of a node is $k_{\max} = N - 1$, obtained when the node is connected to every other node. Intuitively, the **density is the ratio between the average and maximum degree**.



Degree Distribution

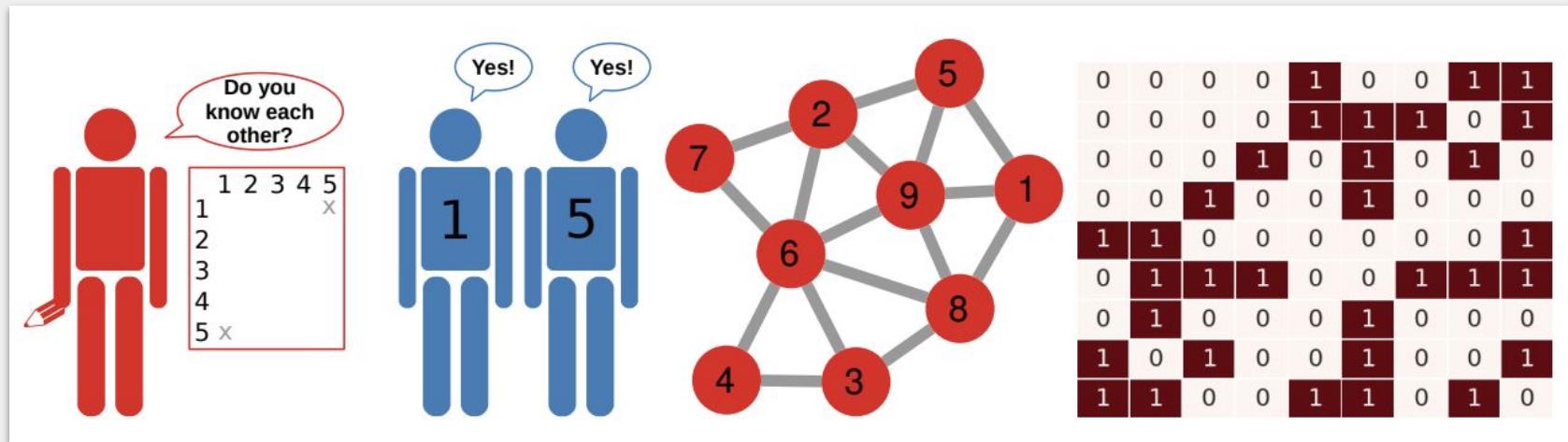


Nodes



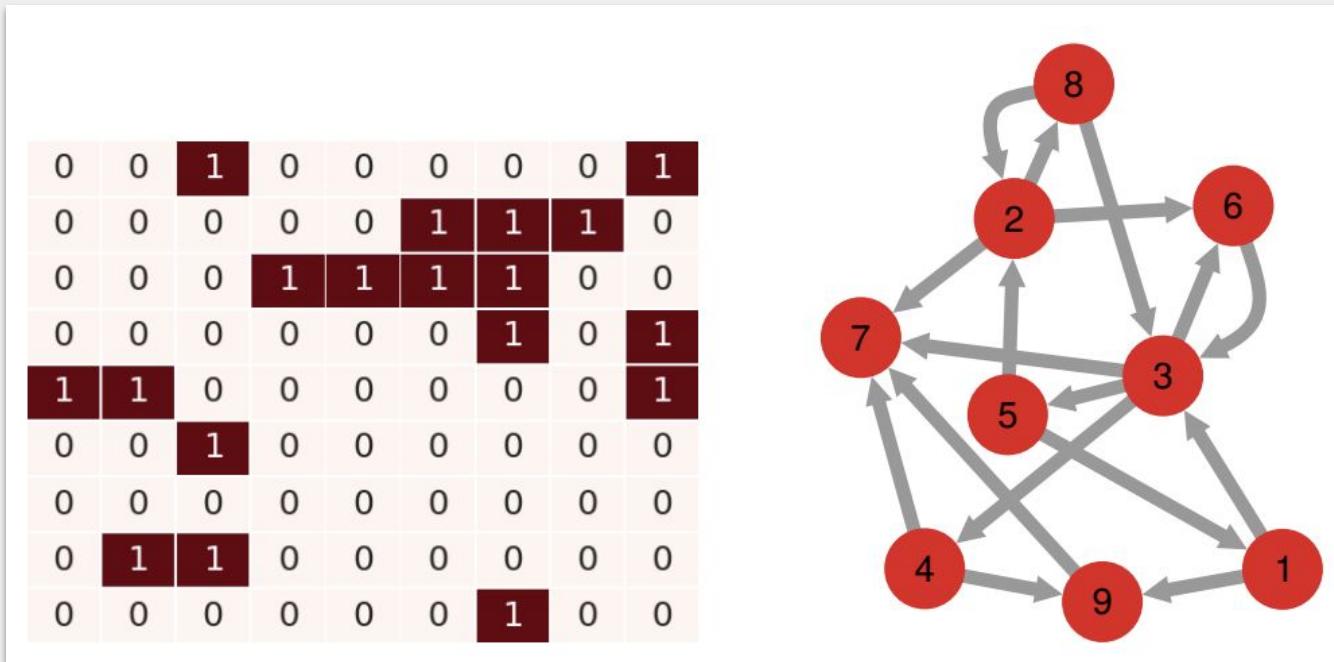
Network Representation

The **adjacency matrix** is the basic representation of a graph as a **matrix**. Each row/column corresponds to a node. Each cell represents an edge, set to one if the edge exists, and zero otherwise.



Network Representation

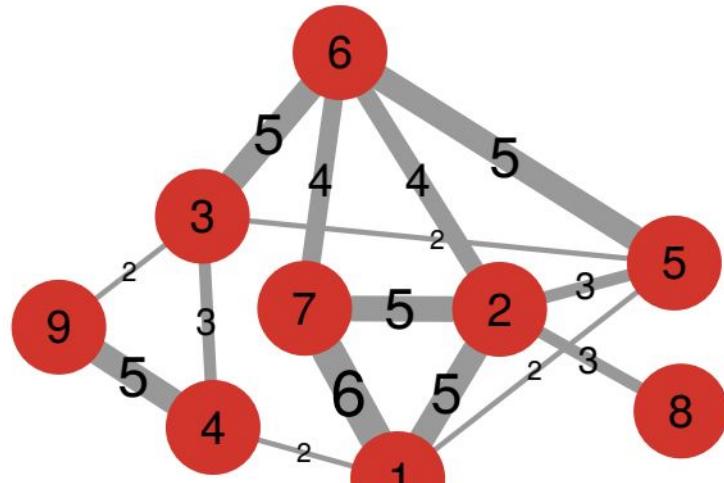
A non-symmetric adjacency matrix corresponding to a directed graph



Network Representation

A non-binary adjacency matrix corresponding a weighted graph

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 0 | 2 | 2 | 0 | 6 | 0 | 0 |
| 5 | 0 | 0 | 0 | 3 | 4 | 5 | 3 | 0 |
| 0 | 0 | 0 | 3 | 2 | 5 | 0 | 0 | 2 |
| 2 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 5 |
| 2 | 3 | 2 | 0 | 0 | 5 | 0 | 0 | 0 |
| 0 | 4 | 5 | 0 | 5 | 0 | 4 | 0 | 0 |
| 6 | 5 | 0 | 0 | 0 | 4 | 0 | 0 | 0 |
| 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 2 | 5 | 0 | 0 | 0 | 0 | 0 |



Network Representation

A non-square adjacency matrix corresponding a bipartite graph

$$\begin{bmatrix} |V_1| & |V_2| \\ |V_1| & 0 & A \\ |V_2| & A^T & 0 \end{bmatrix}$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |

