String thermodynamics and black holes

From vibrating strings to the heart of gravitational darkness

What do we want to do

1. Apply results of thermodynamics to quantum violin string to count the possible states
2. Use them to find the string entropy and uncover the **Hagedorn temperature**
3. Look at partition function for the a single particle
4. Get partition function for a single string
5. BLACK HOLES!

Good old thermodynamics

- Recall familiar formulas:

Partition function:

$$Z \equiv \sum_{\alpha} e^{-\beta E_\alpha},$$

$$\beta = \frac{1}{kT}$$

The energy of a system:

$$E = -\frac{\partial \ln Z}{\partial \beta}$$

Entropy of a system:

$$S = k \ln \Omega(E)$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V$$

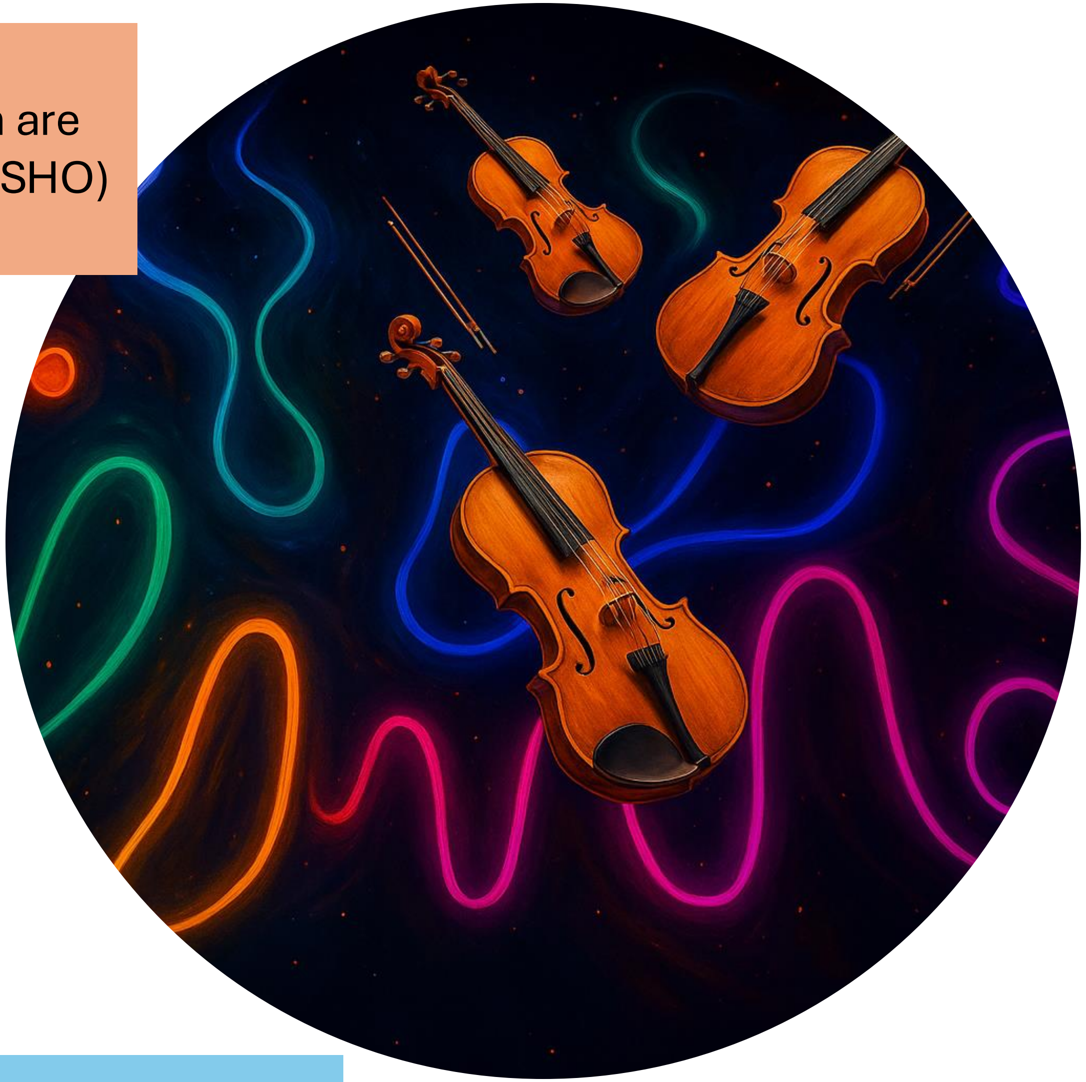
Temperature:

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

k – Boltzmann factor; T – Temperature; E_α – Energy of a state α;
$\Omega(E)$ – number of possible states of the system that have energy E;
$F = -kT\ln Z$ – free energy of a system

Quantum violin string

Quantum violin string:
A quantum string with zero spatial momentum, states of which are similar to those of a collection of simple harmonic oscillators (SHO) with frequencies $\omega_0, 2\omega_0, \dots n\omega_0$



If $a_l^\dagger$ is a creation operator of a SHO, then a state of a quantum violin string can be constructed:

$$|\psi\rangle = \left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2} \dots \left(a_l^\dagger\right)^{n_l} \dots |\Omega\rangle$$

Where $|\Omega\rangle$ is called a "vacuum state"- the lowest possible state.
(Similar to how we construct the states of a SHO in quantum mechanics)

Therefore, each state can be characterized by the set occupation numbers $\{n_1, n_2, \dots, n_l \dots\}$

Quick example: A state $|\psi\rangle = \left(a_1^\dagger\right)^2\left(a_3^\dagger\right)^4|\Omega\rangle$ is characterized by:
$\{2,0,4,0,0,\dots\}$

Still Quantum violin string

Introduce a number operator: $\hat{\mathrm{N}}|\psi\rangle = N|\psi\rangle, \quad N = n_1 + 2n_2 + \cdots \sum_l l\, n_l$

The energy is then given by: $\underline{\mathrm{E} = \hbar\omega_0 N}$

Big Question: For $N > 0$, how many states are there with $\hat{N}$ eigenvalue equal to N?

In simple terms, in how many ways can we arrange $\{n_1, n_2, \ldots, n_l \ldots\}$ to have a total of some N?

AhA!! We just need to find our familiar $\Omega(N)$

(we call it $p(N)$ – number of partitions of N)

Doctor's recipe:

1. Find Z for quantum violin string
2. Find free energy in the high-temperature regime
3. Find the entropy from the free energy
4. Use the relationship $\Omega \sim e^{\frac{S}{k}}$ to count the states

1. Partition function Z

$$Z = \sum_{\alpha} e^{-\beta E_\alpha}$$

- Start with the definition of the partition function:

Note:
Each term corresponds to a configuration of occupation numbers $\{n_l\}$ such that the total energy is distributed among oscillator modes.

$$Z = \sum_{n_1, n_2, \ldots} e^{-\frac{\hbar\omega_0}{kT}\sum_l l\, n_l}$$

- Use the expression for energy in terms of N and the expansion of N

$$Z = \prod_{l=1}^{\infty} \sum_{n_l=0}^{\infty} e^{-\frac{\hbar\omega_0}{kT} l n_l}$$

- Use the property of exponential functions ($e^{a+b} = e^a e^b$)

Note:
This partition function encodes all thermodynamic information of the string. It diverges at the Hagedorn temperature.

$$Z = \prod_{l=1}^{\infty} \left[1 - e_0^{-\frac{\hbar\omega}{kT} l}\right]^{-1}$$

- Use the geometric series: $\sum_{n=0}^{\infty} r^n = [1 - r]^{-1}$

2. Free energy $F = -kT \ln Z$

$$F = kT \sum_{l=1}^{\infty} \ln\left(1 - e_0^{-\frac{\hbar\omega}{kT}l}\right)$$

- Using the fact that the logarithm of the product is the sum of the logarithms

$$\frac{\hbar\omega_0}{kT} \ll 1$$

- We need this approximation to convert the sum into the integral

Note:
This implies that we are working with small level spacing compared to the thermal energy.
This is important because it defines the energy limit we work with.

$$F \cong kT \int_1^{\infty} dl \ln\left[1 - e_0^{-\frac{\hbar\omega l}{kT}}\right]$$

- $\int_0^{\infty} \ln(1 - e^{-x}) = \frac{\pi^2}{6}$

$$\boxed{F \cong -\frac{(kT)^2}{\hbar\omega_0}\frac{\pi^2}{6} = -\frac{1}{\hbar\omega_0}\frac{\pi^2}{6}\frac{1}{\beta^2}}$$

3. Entropy and Energy

$$F \cong -\frac{(kT)^2}{\hbar\omega_0}\frac{\pi^2}{6}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)$$

$$E = -\frac{\partial \ln Z}{\partial \beta}$$

$$S = k\frac{\pi^2}{3}\left(\frac{kT}{\hbar\omega_0}\right)$$

Note:
The primary goal is to find entropy as a function of N, since then we can simply find $\Omega(N)$

$$E = \frac{\pi^2}{6}\left(\frac{kT}{\hbar\omega_0}\right)^2 \hbar\omega$$

$$S = k\pi\sqrt{\frac{2}{3}\frac{E}{\hbar\omega_0}} = k2\pi\sqrt{\frac{N}{6}}$$

4. Counting the states

Finally, we can find the partition of N:

$$\ln\Omega(N) \cong 2\pi\sqrt{\frac{N}{6}}$$

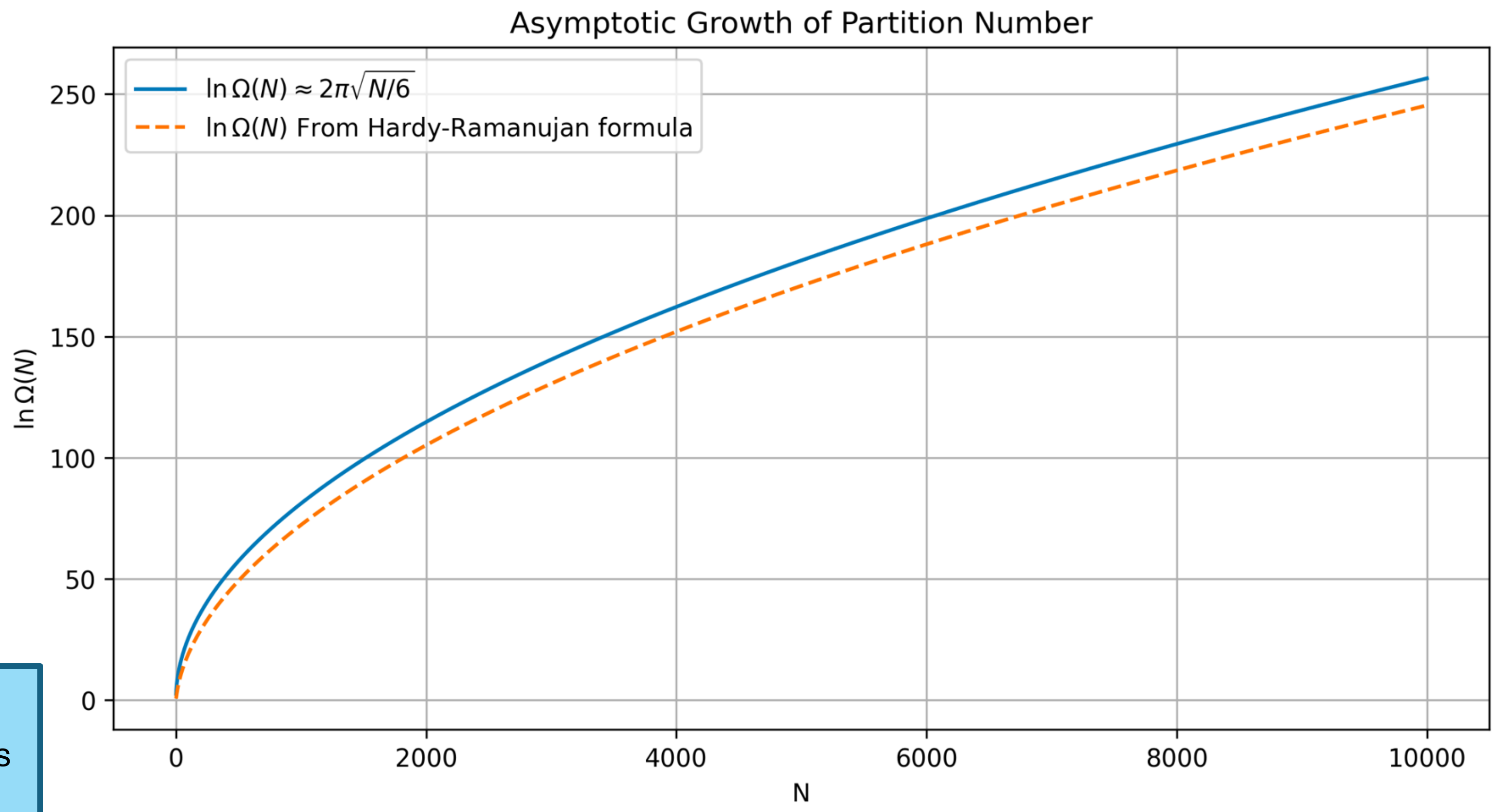
Note:
This is an approximate result, since

$$N = \frac{E}{\hbar\omega_0} \propto \left(\frac{kT}{\hbar\omega_0}\right)^2 \gg 1$$

This is a leading term approximation for the logarithm of the partition. A more accurate result is given by the Hardy-Ramanujan formula:

$$\Omega(N) \cong \frac{1}{4N\sqrt{3}} e^{2\pi\sqrt{\frac{N}{6}}}$$

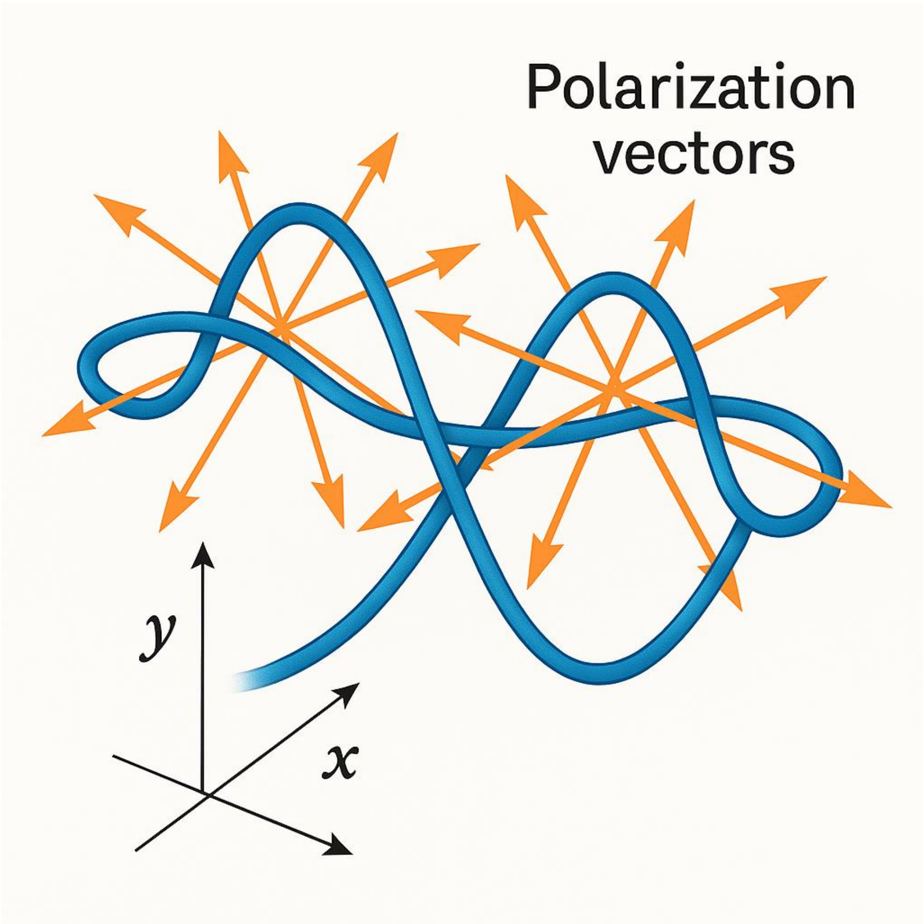
Note:
This exponential growth is exactly the reason why S scales as $S \propto \sqrt{E}$



* *Note that our approximation closely follows H-R result*

Generalization to vibration in b possible directions

Assume the string can vibrate in b transverse directions $\Longrightarrow$ $\forall\ l\omega_0\ \exists\ \ b$ harmonic oscillators that represent possible polarizations of the motion



Let A_b be the value of interest for the given configuration and A (without subscript) –– value we calculate earlier

$$Z_b = (Z)^b$$

(since we also sum over b in the exponent)

$$F_b = bF$$

$$E_b = bE$$

$$S_b = bS$$

$$E_b = bE = \hbar\omega_0 N$$
$$N = \sum_{l,q} \ln_l^{(q)}$$
(q) is like a polarization index

$$S_b = k2\pi\sqrt{\frac{Nb}{6}}$$

Since the number of transverse light-cone directions is 24, Hardy-Ramanujan formula becomes:

$$\Omega_{24}(N) \cong \frac{1}{\sqrt{2}} N^{-27/4} e^{4\pi\sqrt{N}}$$

Hagedorn Temperature

Consider open strings that carry no spatial momentum $\Longrightarrow$ energy levels are given by the rest mass of the states $E = M$

Recall:

The mass-squared of a given state is given by the number operator $N^{\perp}$:

$$M^2 = \frac{1}{\alpha'}(N^{\perp} - 1) \cong \frac{N^{\perp}}{\alpha'}$$

where $\alpha' \propto 1/T_0$ – slope parameter (inverseley proportional to the string tention)

$$\sqrt{N^{\perp}} = \sqrt{\alpha'}E \quad + \quad S_{24}(E) \longrightarrow \boxed{S(E) = k4\pi\sqrt{\alpha'}E}$$

$$\frac{1}{kT} = \frac{1}{k}\frac{\partial S}{\partial E}$$

$$\boxed{kT_H = \frac{1}{4\pi\sqrt{\alpha'}} \quad \text{– Hagedorn temperature}}$$

What is the physical meaning of Hagedorn temperature?

- Note that T_H is a constant and depends only on string tension.
- What happens if we heat the string up?

  Would imagine that it simply gains more thermal energy – up to infinity.
- In reality, once the string hits the Hagedorn temperature, no amount of energy increase can make it "warmer"; instead, it stays at that temperature.
- This is similar to a phase transition; the temperature stays constant and the new states appear.
- Entropy stops increasing!

Note:

Thermal energy associated with T_H is much smaller than rest energy of any particle:

Take a first massive particle $(N^{\perp} = 2 \implies E = M = 1/\sqrt{\alpha'})$

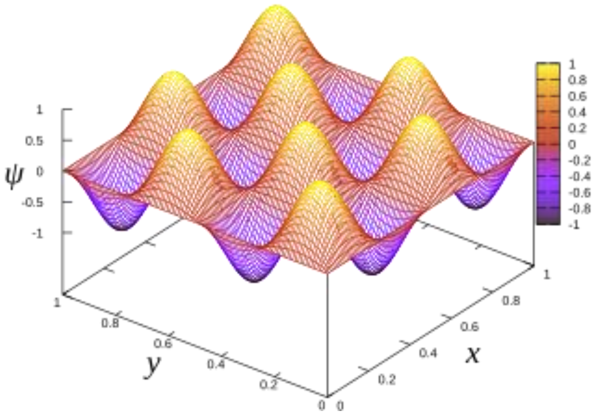
So $$\boxed{\frac{kT_H}{M} = \frac{1}{4\pi}}$$

Relativistic particle

Assume a particle that has mass m and momentum $\vec{p}$ that lives in D −dimensional spacetime ($d = D - 1$ dimensions of space). Assume that it is confined to a box of volume $V = L_1 L_2 \ldots L_d$ where L_a − size of the box in dimension a

$$Z(m^2) = \sum_{\vec{p}} e^{-\beta E(\vec{p})}$$

Particle in a box is quantized, and the periodicity of the wavefunctions requires:
$k_i L_i = 2\pi n_i, \quad i = 1,2,\ldots,d; \quad n_i \in \mathbb{Z}; \quad k_i = p_i/\hbar$
So $n_i = p_i \frac{L_i}{2\pi\hbar}$



The energy of a particle is given by:
$$E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$$

$$Z(m^2) \cong V \int_p \frac{d^d\vec{p}}{(2\pi\hbar)^d} e^{-\beta\sqrt{\vec{p}^2+m^2}}$$

This integral can be solved exactly, but we are particularly interested in the regime when the thermal energy is much smaller than the rest energy: ***$\beta m \gg 1$*, high mass limit (comparing to thermal energy)**

$$\boxed{Z(m^2) \cong V e^{-\beta m}\left(\frac{m}{2\pi\beta}\right)^{d/2}}$$

Need this for quantum relativistic strings

Single quantum relativistic string

What do we know about these strings?

- State:

$$|\lambda, p\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_m^{I\dagger}\right)^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle$$

$\lambda_{n,I}$ - occupation numbers, p – momentum of the string; (they label the states)

- d components $(p^+, \vec{p}_T)$ specify the light-cone energy via:

$$M^2(\{\lambda_{n,I}\}) = -p^2 = \frac{1}{\alpha'}(N^\perp - 1)$$

With $N^\perp = \sum_{n,I} n\lambda_{n,I}$

- $E(\{\lambda_{n,I}\}, \vec{p}) = \sqrt{M^2(\{\lambda_{n,I}\}) + \vec{p}_T^2}$

Want to find partition:

$$Z_{str} = \sum_{\lambda_{n,I}}\sum_{\vec{p}} e^{-\beta\sqrt{M^2(\{\lambda_{n,I}\})+\vec{p}^2}}$$

- We sum over $\lambda_{n,I}$ and $\vec{p}$ since they label the states
- Note that the second summation is simply the partition for the particle from the last slide

$$Z_{str} = \sum_{\lambda_{n,I}} Z\left(M^2(\{\lambda_{n,I}\})\right)$$

Since M^2 depends on $N^\perp$ only, we can change $\{\lambda_{n,I}\}$ to a sum over N, remembering that there are $\Omega_{24}(N)$ states with number eigenvalue N

$$Z_{str} = \sum_{N=0}^{\infty} \Omega_{24}(N) Z\left(M^2(N)\right)$$

Single string partition function ctnd.

We want to work in high N regime, since that is the range where $\Omega_{24}(N)$ works.

Choose $N \geq N_0$ such that $\Omega_{24}(N)$ works

$$Z_{str} = \underbrace{\sum_{N=0}^{N_0-1} \Omega_{24}(N) Z\big(M^2(N)\big)}_{Z_0} + \underbrace{\sum_{N=N_0}^{\infty} \Omega_{24}(N) Z\big(M^2(N)\big)}_{\cong \int_{N_0}^{\infty} dN \Omega_{24}(N) Z\left(M^2(N)\right)}$$

This integral can simply be evaluated if we change the variables to
$\Omega_{24}(N) dN = \rho(M) dM$
And use $N \cong \alpha' M^2$

$$Z_{str} \cong Z_0 + \frac{2^{11}}{\pi} V (kT\, kT_H)^{\frac{25}{2}} \left(\frac{T}{T_H - T}\right) e^{-4\pi\sqrt{N_0}\left[\frac{T_H}{T} - 1\right]}$$

In the limit $T \to T_H$ the second is much larger than Z_0

$$\boxed{Z_{str} \cong \frac{2^{11}}{\pi} V (kT_H)^{25} \left(\frac{T_H}{T_H - T}\right), \qquad T \to T_H}$$

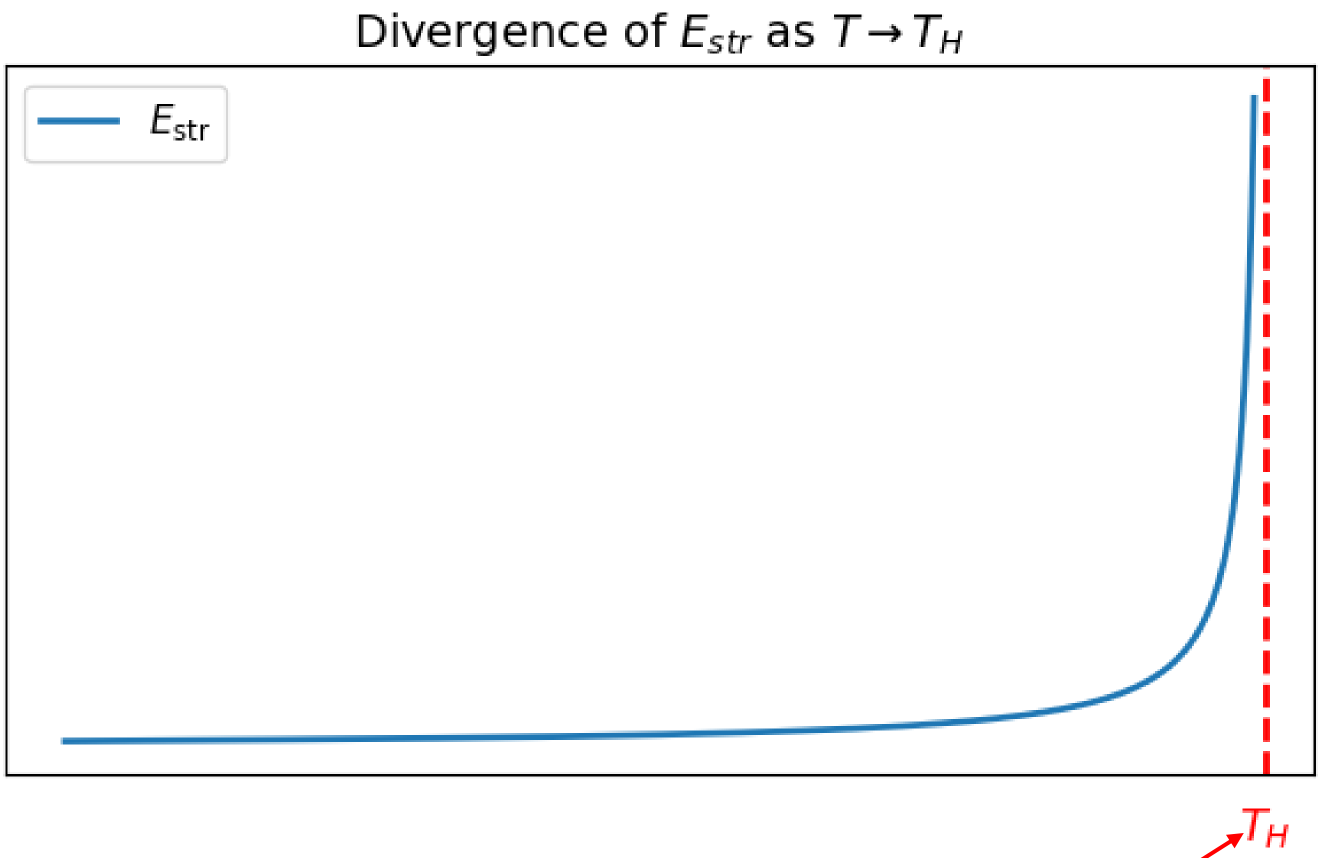
Energy of the string

We can find that

$$\boxed{E_{str} \cong kT_H\left(\frac{T_H}{T_H - T}\right)}$$

Note that the string’s energy grows without bound as $T \rightarrow T_H$:

- As more energy is pumped into the system, the string doesn’t get 'hotter' — it explores more and more excited vibrational states instead.
- Energy diverges → infinite excitation



Beyond this point, a string could collapse into a black hole.

Black holes and entropy

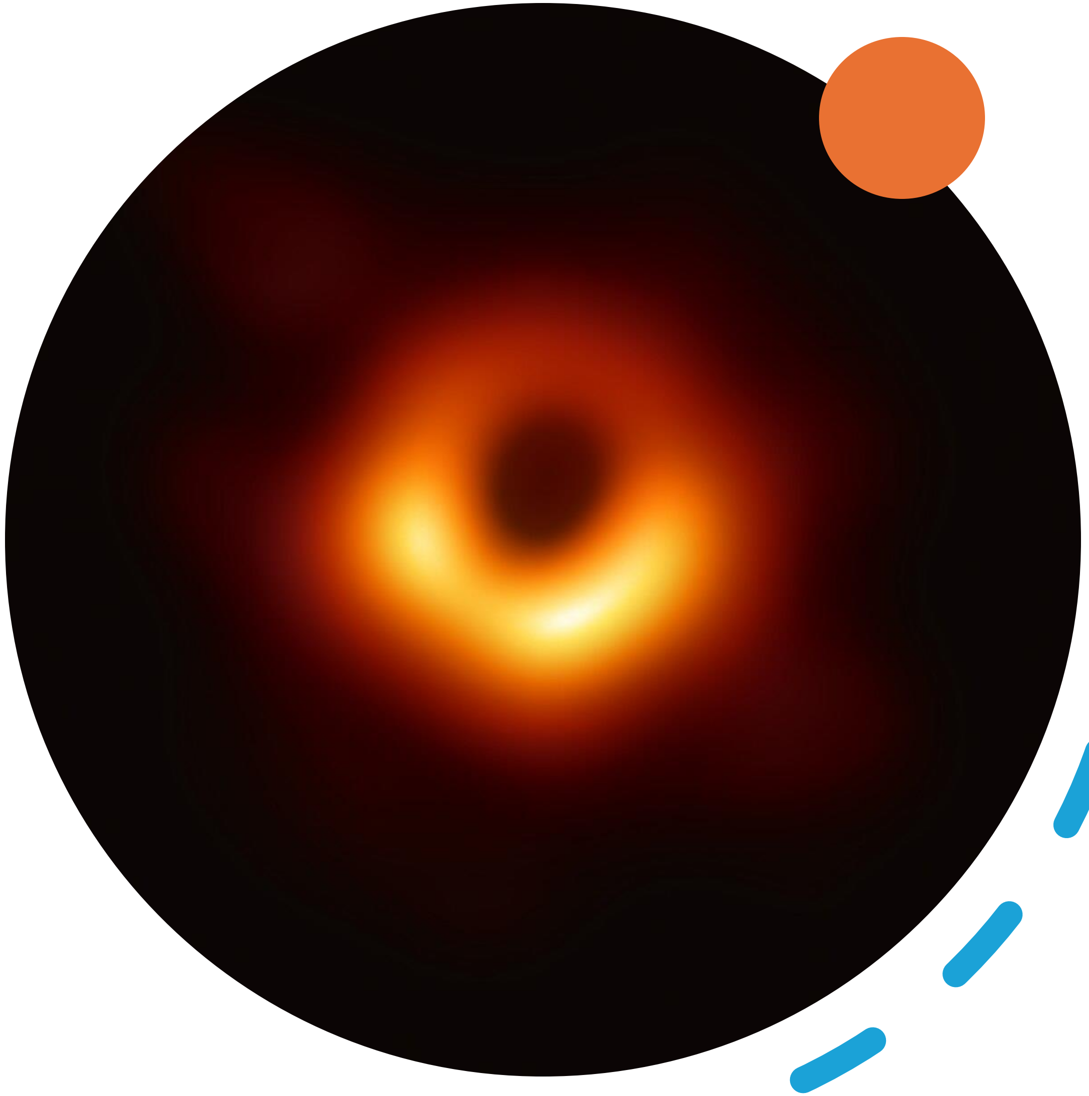
- What are black holes?
- Simplest case – Schwarzschild black hole
- Radius of an event horizon:

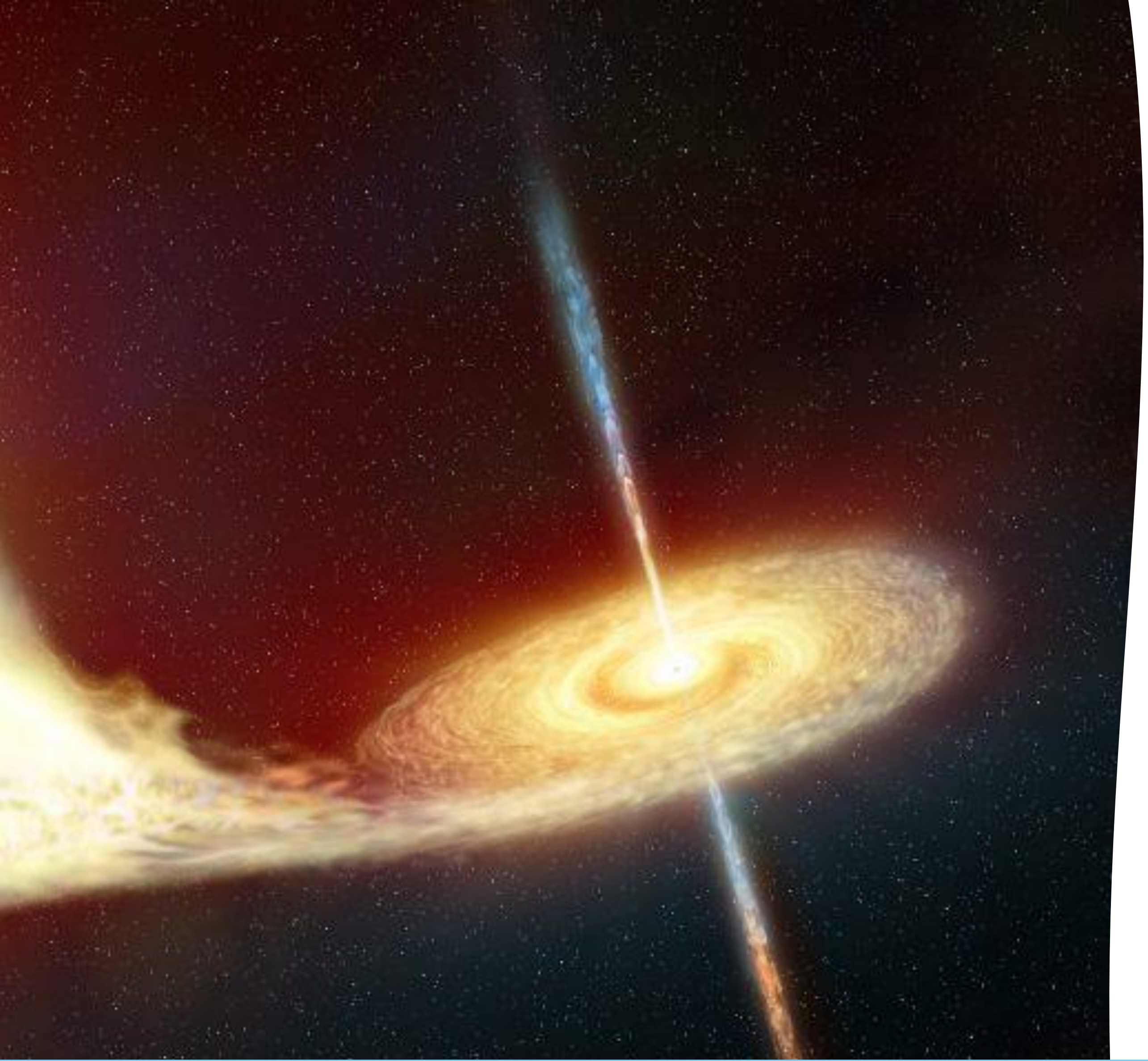
$$R = \frac{2GM}{c^2}$$

G- gravitational constant

M- black hole's mass

c-speed of light





Where does entropy come from?

- If a black hole consumes some gas, its mass should increase, and the entropy of a BH-gas system should increase.
- Black Hole is practically a point mass singularity – what are the states?
- However, Black holes emit radiation at a well-defined temperature: Hawking temperature $\bar{T}_H$ which emerges from the near horizon limit and controlled by the intensity of gravity:

$$\boxed{k\bar{T}_H = \frac{\hbar c^3}{8\pi GM}}$$

Quick example: If a photon had an energy kT_H then its wavelength would be 80 times the size of a radius a Black Hole:

$$h\nu = \frac{hc}{\lambda} = k\bar{T}_H = \frac{\hbar c^3}{8\pi GM} = \left\{R = \frac{2GM}{c^2}\right\} = \frac{hc^3\,2}{2\pi 8\pi R c^2} \Rightarrow \frac{hc}{8\pi^2 R} = \frac{hc}{\lambda} \Rightarrow \frac{\lambda}{R} = 8\pi^2 \approx 80$$

Finding the entropy

Denote the entropy of a Black hole by S_B – Bekenstein entropy
Energy of a BH is: $E = Mc^2$

$$dE = c^2 dM = (\text{Definition of temperature}) = \bar{T}_H dS_B$$

$$\boxed{\frac{S_B}{k} = \frac{4\pi G}{\hbar c} M^2}$$

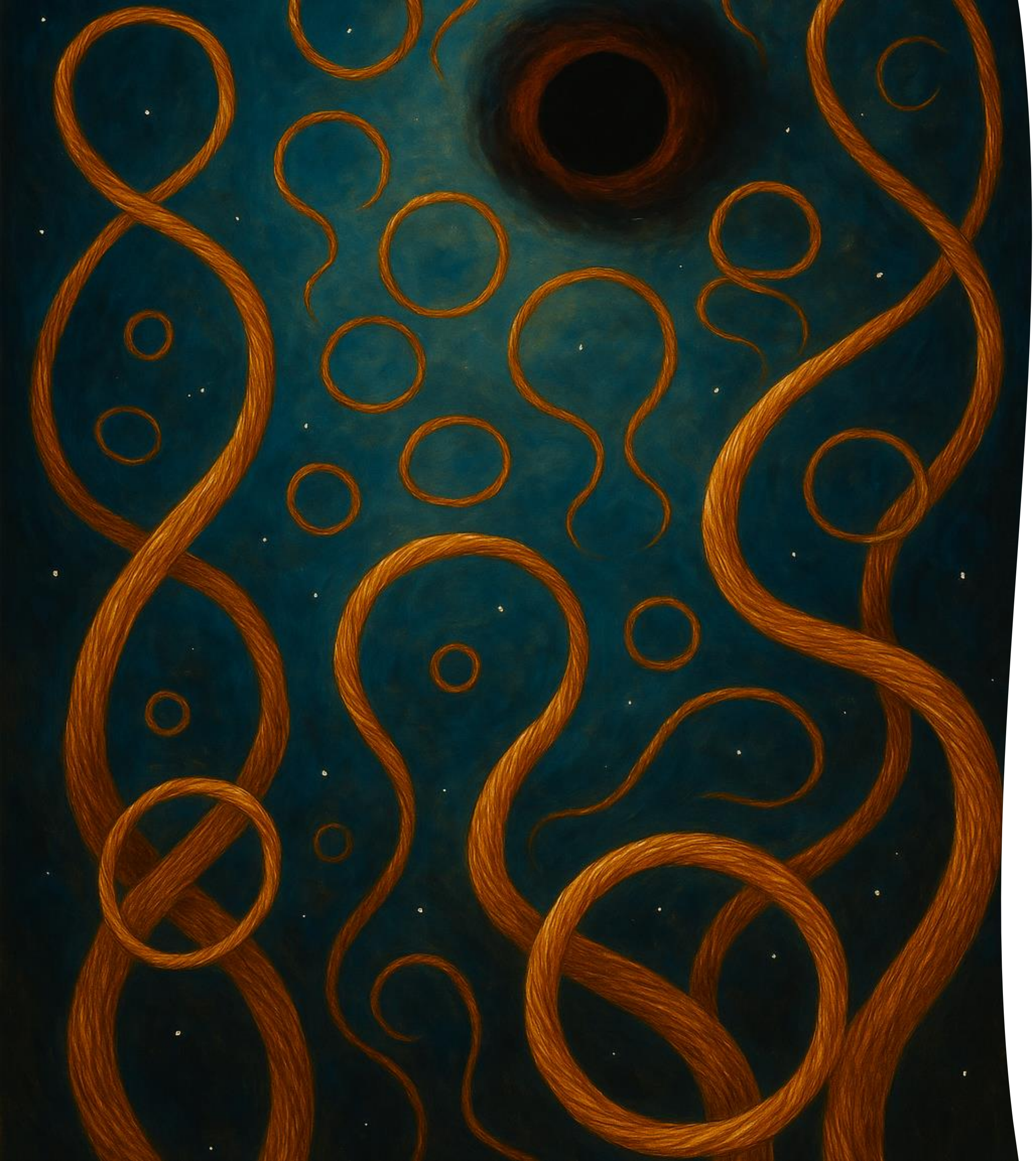
If we take the surface area of the event horizon: $A = 4\pi R^2$ → Use Schwarzschild radius →

$$\boxed{\frac{S_B}{k} = \frac{1}{4}\frac{c^3}{\hbar G} A = \frac{A}{4\ell_p^2}}$$

ℓ_p^2 – Plank's length

Note:
The entropy of the black hole depends on the area only.
This then implies that S_B is not an <u>extensive</u> quantity
(quantity which linearly depends on the size, so we would expect $S_B \propto V$)
This failure to be extensive is a feature of gravitational physics

So the entropy is one-fourth of the area of a horizon expressed in units of Plank-length squared



Strings+BH = <3?

- In string theory, we want to relate a Schwarzschild black hole to a highly energetic string with zero momentum ($E = M$)
- Previously found (let $\hbar = c = 1$):

$$\frac{S_{str}}{k} = 4\pi\sqrt{\alpha'}E = 4\pi\sqrt{\alpha'}M$$

$$\frac{S_B}{k} = 4\pi G M^2$$

- Problem: They disagree on the powers of
- Why this happens:

– When calculating the string entropy, we assumed a <u>single</u> string → neglected string interactions.

– Gravity arises from string interactions

New string entropy result:

We are not striving to rederive the expressions; we want to take a look at them from a different perspective:

Consider a string of mass M, length L and tension T_0 ($T_0 \propto 1/\alpha'$), such that $M \sim T_0 L \sim \frac{1}{\alpha'} L$

We can imagine that the string is built out of small pieces with length $\ell_s \sim \sqrt{\alpha'}$, each of which can point in n directions

There are L/ℓ_s pieces so we can build our assumed string in Ω different ways:

$$\Omega \sim n^{\frac{L}{\alpha'}} \sim e^{M\sqrt{\alpha'}\ln(n)}$$

Taking log

$$\frac{S_{str}}{k} \sim M\sqrt{a'} \sim M\ell_s$$

We'll need this later

Note:
Similar to how Plank's length can be derived out of the constants $G, c,$ and $\hbar$ using the dimensional analysis, we can derive a fundamental length $\ell_s = \hbar c\sqrt{\alpha'}$

Gravitational coupling constant and the gravity

$$G \sim g^2\alpha' \sim M\ell_s^2$$

g – string coupling constant, which we set to zero by neglecting all interactions

$$\frac{S_B}{k} \sim GM^2 \sim g^2\ell_s^2 M^2$$

$$R \sim GM \sim g^2\ell_s^2 M$$

Assume large BH with $S_0, M_0, R_0 \gg \ell_s$ (since radius is much larger than the string's length → can neglect string corrections to GR

And fix $g \to g_0$

$$\frac{S_0}{k} \sim g_0^2\ell_s^2 M_0^2$$

$$R_0 \sim g_0^2\ell_s^2 M_0$$

Let's decrease $g_0 \to g_*$ and let M_*, R_* – final values of mass and radius

The entropy should remain fixed at S_0 → $M \sim 1/g$

$$\frac{S_0}{k} \sim g_0^2\ell_s^2 M_0^2 = g_*^2\ell_s^2 M_*^2$$

$$R_* \sim g_*^2\ell_s^2 M_*$$

The min value of R_* is ℓ_s

$$\ell_s \sim g_*^2\ell_s^2 M_* \to M_* \sim \frac{1}{g_*^2\ell_s}$$

$$\boxed{\frac{S_0}{k} \sim \frac{1}{g_*^2}}$$

Uniting the strings and the Gravity

We have found:

$$\boxed{\frac{S_{str}}{k} \sim M\sqrt{a'} \sim M\ell_s}$$

Plug in M_* from before

$$\boxed{\frac{S_{str}}{k} \sim \frac{1}{g_*^2}}$$

$$\boxed{\frac{S_0}{k} \sim \frac{1}{g_*^2}}$$

So S_0 and S_{str} match!

This shows that a Schwarzschild black hole is the strong coupling version of a string with a very high degree of excitation!
It is possible to show that for any g there is a mass of a string beyond which any excited string state is smaller than its Schwarzschild radius. So, very massive string states will form black holes!