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A First Course in String Theory by Barton Zwiebach

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1 Introduction

Throughout the semester, we covered Chapters 4 through 15 and Chapter 17 of Barton Zwiebach's "*A First Course in String Theory*", excluding the introductory Chapters 1–3 and Chapter 16.
The main focus of the course was to build up the formalism of string theory. We began by studying classical nonrelativistic strings, drawing on the concepts of action and Lagrangian mechanics. From there, we moved on to relativistic particles and relativistic strings, using the principle of reparameterization invariance to derive the appropriate equations of motion. This included boundary conditions, parametrization constraints, and the wave equation that governs string dynamics.
Next, we selected suitable world-sheet parametrizations and applied Noether's theorem to identify conserved currents and charges. An important advancement came with the introduction of light-cone coordinates, which provided a powerful tool for solving the wave equation and simplifying the analysis.
We then extended the light-cone formalism to scalar fields, laying the groundwork for the quantization of scalar and later gauge fields. With this foundation, we proceeded to quantize relativistic point particles and strings (both open and closed), and explored the emergence of particles such as the photon and graviton in the quantum string spectrum. We also introduced the concept of superstrings, incorporating fermions through the Ramond and Neveu–Schwarz sectors.
Finally, we applied this theoretical toolkit to study D-branes and gauge fields, culminating in a discussion of T-duality for closed strings, a profound symmetry of string theory.
In this report, Chapters 1 through 3 of Zwiebach's book are omitted, as they primarily provide introductory context and background in special relativity, electromagnetism, and early motivation for string theory. We thus begin directly with the core technical development of this fascinating theory.

2 Nonrelativistic Strings

Let us start by looking at classical nonrelativistic strings. What are these beasts, and what do we know about them?
Classical waves support two types of oscillation: transverse and longitudinal. We are interested only in transverse modes, since fundamental strings have no internal structure and cannot oscillate longitudinally.
A classical string is characterized by its tension T_0 and mass per unit length μ_0. The wave velocity along the string is given by:

$$v_0 = \sqrt{T_0/\mu_0}$$

To describe its dynamics, we construct an action based on kinetic and potential energy. The kinetic energy of an infinitesimal segment of mass $\mu_0 dx$ combines with the potential energy from stretching, $T_0 \Delta l$. Minimizing the action yields the expected wave equation.
To find exact solutions, we impose boundary conditions. For open strings, endpoints can either move freely or be fixed. The former gives **Neumann conditions**, where the spatial derivative vanishes at the endpoint; the latter gives **Dirichlet conditions**, where the endpoint is fixed and its time derivative vanishes.
Dirichlet conditions appear problematic because a fixed endpoint breaks momentum conservation—the wall must absorb force. This issue is resolved in string theory by introducing **D-branes**: dynamical extended objects to which strings can attach. When a string ends on a D-brane, mo-

mentum is conserved via transfer to the brane itself.

3 The relativistic point particle

To describe a free relativistic point particle, we require an action that is **Lorentz invariant** that is, one that gives the same value for all inertial observers. The natural choice is to make the action proportional to the proper time s, since it is an invariant quantity:

$$S = -mc \int_{\mathcal{P}} ds$$

where $ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$ is the spacetime interval along the world-line $\mathcal{P}$. This ensures that the action is a Lorentz scalar, preserving the symmetry of spacetime under Lorentz transformations.
The action is also reparameterization invariant—its value does not depend on how the path is parameterized (e.g., by coordinate time t, or by some arbitrary parameter τ). That is, the action depends only on the geometry of the world-line, not on the specific labels assigned to points on it.
When written in terms of an arbitrary parameter τ, the action becomes:

$$S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau$$

This form is manifestly Lorentz invariant and reparameterization invariant under $\tau \to \tau'$, setting the template for the action of the relativistic string.
Varying the action yields the equations of motion:

$$\frac{d^2 x^\mu}{ds^2} = 0$$

which describes uniform motion: the particle traces a straight world-line in spacetime.
This construction generalizes naturally to relativistic strings, where instead of tracing a one-dimensional path (a world-line), the string sweeps out a two-dimensional surface in spacetime—a world-sheet.

4 Relativistic strings

In relativistic string theory, the string sweeps out a two-dimensional surface in spacetime called the **world-sheet**, which generalizes the world-line of a point particle. The natural invariant quantity for such a surface is its **proper area**, and the string action must be both **Lorentz invariant** and **reparameterization invariant**.

We describe the world-sheet using coordinates τ (time-like) and σ (space-like), and define the embedding of the string in spacetime by the functions $X^\mu(\tau, \sigma)$. The infinitesimal area element is constructed using the **induced metric**

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \, \partial_\beta X_\mu, \quad \alpha, \beta \in \{\tau, \sigma\},$$

and the action is given by the **Nambu–Goto action**:

$$S = -\frac{T_0}{c} \int d\tau \, d\sigma \sqrt{-\det(\gamma_{\alpha\beta})},$$

where T_0 is the string tension. This action reduces to that of a point particle in the limit where the string collapses to a single point.

Varying this action leads to the equations of motion and determines the boundary conditions.

The equations of motion for the relativistic string, open or closed take the form of a conservation law:

$$\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0.$$

Where the **canonical momenta** conjugate to X^μ are given by:

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{6.49}$$

$$\mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{6.50}$$

For **open strings**, the possible boundary conditions, described before, become:

- **Neumann boundary condition:** $\mathcal{P}^\sigma_\mu = 0$ (free endpoint),
- **Dirichlet boundary condition:** $\partial_\tau X^\mu = 0$ (fixed endpoint),

To simplify calculations, we often use the **static gauge** in which $\tau = t$. For **closed strings**, σ is periodic and parameterizes a circle, making the world-sheet cylindrical.

Using the static gauge, the energy of a stretched string can be calculated. The **mass per unit length** μ_0 is related to the tension by:

$$\mu_0 = \frac{T_0}{c^2},$$

and the total energy of a string of length a is $E = T_0 a$, leading to a rest mass $m = \mu_0 a$. Thus, mass in string theory arises entirely due to tension.

We also define the **transverse velocity** $\vec{v}_\perp$ of the string, which is the component of the string's motion perpendicular to itself. This leads to an alternate form of the action:

$$S = -T_0 \int dt \int d\sigma \, \frac{ds}{d\sigma} \sqrt{1 - \frac{v_\perp^2}{c^2}},$$

where $ds = \left| \partial_\sigma \vec{X} \right| d\sigma$ is the physical length element along the string.

Finally, a striking result from the dynamics of open strings is that their endpoints must move at the speed of light:

$$v^2 = c^2.$$

To go further, we now seek explicit solutions to the equations of motion derived from the Nambu–Goto action. This leads us to examine the general motion of classical strings, which is the focus of the next chapter.

5 String parametrization and classical motion

To study classical string motion, we begin by fixing a convenient parametrization in which lines of constant σ are orthogonal to lines of constant time t. Furthermore, we require that each string segment of equal parameter length carries the same amount of energy. This choice simplifies the analysis and leads to a natural form of the equations of motion. By varyig the Nambu-Goto action we can find the equations of motion, which, subject to the described parametrization produce the wave equation:

$$\frac{\partial^2 \vec{X}}{\partial \sigma^2} - \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = 0,$$

subject to two key constraints that can simply be written as:

$$\boxed{\left(\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \vec{X}}{\partial t}\right)^2 = 1}$$

These ensure that energy and momentum are distributed consistently along the string.

For **open strings**, the boundary conditions require the spatial derivative to vanish at the endpoints:

$$\frac{\partial \vec{X}}{\partial \sigma}\Big|_{\sigma=0} = \frac{\partial \vec{X}}{\partial \sigma}\Big|_{\sigma=\sigma_1} = 0.$$

The general solution is a superposition of left- and right-moving waves:

$$\vec{X}(t,\sigma) = \frac{1}{2}\left(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma)\right),$$

with $|\vec{F}'(u)| = 1$, meaning $\vec{F}$ traces out a curve at the speed of light.

For **closed strings**, the solution is:

$$\vec{X}(t,\sigma) = \frac{1}{2}\left(\vec{F}(ct+\sigma) + \vec{G}(ct-\sigma)\right),$$

with periodicity in both $\vec{F}(u)$ and $\vec{G}(v)$. These functions represent waves moving in opposite directions around the closed string.

A notable feature of closed string motion is the possible appearance of **cusps**—points where the left- and right-moving tangent vectors add to zero. At a cusp, the string momentarily doubles back on itself and moves sharply, which has implications for radiation and high-energy phenomena.

These classical results are the foundation for understanding the quantized string and will reappear in the mode expansions of the next chapters.

6 World-sheet currents

Symmetries in a physical theory lead to conserved quantities. Just as in electromagnetism the conservation of electric charge follows from the continuity equation for the current, in string theory we define conserved **world-sheet currents** by analyzing the symmetries of the action.

A general principle from Lagrangian mechanics states that if a system's Lagrangian is invariant under a continuous transformation, then a conserved quantity must exist. For particles, this results in conserved charges like momentum. For strings, which are described by a Lagrangian density over a two-dimensional world-sheet, such symmetries lead to conserved **currents** rather than point-like charges.

In the case of the Nambu–Goto action, translation invariance in spacetime implies the conservation of the **spacetime momentum** carried by the string. Each spacetime coordinate X^μ behaves as a dynamical field on the world-sheet, and a shift symmetry in X^μ leads to a conserved current density associated with momentum flow $\mathcal{P}^\tau_\mu$.

These world-sheet currents have two components: one in the time direction and one in the spatial direction of the string. Conservation of the current means that momentum cannot be created or destroyed on the world-sheet—only moved around. For closed strings and open strings with free endpoints, the total momentum is conserved. In the case of open strings with Dirichlet boundary conditions, the momentum of the string may change, but only by transferring it to or from the D-brane that the string is attached to.

Beyond translations, the relativistic string action is also Lorentz invariant. Infinitesimal Lorentz transformations give rise to additional conserved currents $\mathcal{M}^\alpha_{\mu\nu}$. These correspond to the **angular momentum** of the string in spacetime. The associated conserved charges $M_{\mu\nu}$, obtained by integrating the current over the world-sheet, yield the total angular momentum carried by the string:

$$M_{\mu\nu} = \int_\gamma (\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau)$$

Finally, we define the **slope parameter** α', which relates the angular momentum and energy of a rotating string. In the classical theory, it acts as a proportionality constant between angular momentum (measured in units of $\hbar$) and the square of the energy. In the quantum theory, it will later determine the spacing between string mass levels and plays a central role in the spectrum of string excitations. We define α' as:

$$\boxed{\alpha' = \frac{1}{2\pi T_0 \hbar c} \quad \text{and} \quad T_0 = \frac{1}{2\pi\alpha' \hbar c}}$$

7 Light-Cone Relativistic Strings

To simplify the dynamics of relativistic strings, we adopt the **light-cone gauge**, identifying the world-sheet time parameter τ with a fixed linear combination of spacetime coordinates:

$$n_\mu X^\mu(\tau, \sigma) = \lambda\tau.$$

This choice explicitly resolves the constraints and isolates the physical degrees of freedom.

We also fix the spatial parametrization σ so that the momentum density along n^μ is constant:

$$\boxed{n \cdot \mathcal{P}^\tau = \frac{n \cdot p}{\pi} \quad \text{is an open string world-sheet constant}}$$

This leads to a simplified momentum conservation condition:

$$\boxed{n \cdot \mathcal{P}^\sigma = 0},$$

which holds for both open and closed strings.

Using these choices and equations (6.49) and (6.50), the constraints reduce to:

$$\dot{X} \cdot X' = 0, \quad \dot{X}^2 + X'^2 = 0.$$

In this gauge, the coordinate X^+ evolves linearly in τ:

$$X^+(\tau, \sigma) = \beta\alpha' p^+ \tau, \quad \text{where} \quad p^+ = \frac{1}{2\pi\alpha'} \int_0^{\sigma_1} \mathcal{P}^+ d\sigma.$$

The general solution for the open string is given by the mode expansion:

$$\boxed{X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma} \tag{9.56}$$

Only the transverse components X^I are independent; the longitudinal component X^- is fixed by the constraints:

$$\dot{X}^- \pm X'^- = \frac{1}{2p^+} (\dot{X}^I \pm X'^I)^2.$$

We define the **transverse Virasoro modes** $L_n^\perp$ as:

$$\boxed{\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+} L_n^\perp, \quad L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I} \tag{9.77}$$

The zero mode relates directly to the energy:

$$\alpha' p^+ p^- = L_0^\perp.$$

Finally, we arrive at the **mass formula** for the string:

$$\boxed{M^2 = -p^2 = 2p^+p^- - p^I p^I = \frac{1}{\alpha'} L_0^\perp - p^I p^I = \frac{1}{\alpha'} \sum_{n=1} n\, \alpha_n^{I*} \alpha_n^I} \tag{9.83}$$

This relation connects classical string vibrations to quantum particle states and underlies the physical spectrum of string theory.

8 Light-cone Fields and Particles

Next, we shift our focus from strings to fields and particles, using scalar fields as the simplest starting point. Scalar fields are functions of spacetime coordinates that transform trivially under Lorentz transformations, making them ideal for illustrating field quantization clearly and intuitively.

We start by constructing a consistent Lagrangian density for a free scalar field. Requiring Lorentz invariance naturally leads us to the Klein–Gordon equation:

$$(\partial^2 - m^2)\phi = 0,$$

which describes free relativistic scalar particles. Solutions to this equation are combinations of plane waves, and reality conditions require both positive and negative frequency solutions. Thus, the general scalar field solution involves an integral over momenta, known as a Fourier integral.

When quantizing the scalar field, classical amplitudes become quantum operators, expressed in terms of creation and annihilation operators. These operators arise naturally as quantum analogs of harmonic oscillator ladder operators, adding (creation) or removing (annihilation) particle excitations from the vacuum state: $a_p^\dagger|\Omega\rangle$, where $|\Omega\rangle$ is the vacuum state. Physically, applying a creation operator to the vacuum is like plucking a string or exciting a harmonic oscillator, thus generating a particle state.

We then extend this discussion to electromagnetic (Maxwell) fields. Using the light-cone gauge simplifies Maxwell's equations significantly, leaving only transverse polarization states as physically meaningful degrees of freedom. Specifically, in D-dimensional spacetime, photons have $D-2$ physical polarization states, each represented by a creation operator analogous to those introduced for scalar particles:

$$a^{\dagger I}_{p^+,\vec{p}_T}|\Omega\rangle, \quad I = 1, \ldots, D-2.$$

Finally, the same logic applies to gravitational fields. In the light-cone formalism, graviton states arise as quantum excitations of transverse-traceless metric perturbations around flat spacetime. After gauge fixing and applying constraints, we find that gravitons have $\frac{(D-2)(D-3)}{2}$ polarization states. Each graviton state is similarly generated by applying symmetric, traceless creation operators to the vacuum, conceptually resembling the construction of photon states.

Thus, this chapter provides a clear bridge between classical field theories, their quantum counterparts, and the resulting particle spectra—essential concepts for interpreting excitations in string theory.

9 Relativistic Quantum Point Particle

Now, we extend the relativistic point particle into the quantum domain, introducing a framework that mirrors the structure we'll soon apply to strings. The classical action leads to the constraint $p^2 + m^2 = 0$, which becomes a condition on physical states in the quantum theory:

$$(p^2 + m^2)|\psi\rangle = 0.$$

This is the Klein–Gordon equation in operator form.

Quantization promotes x^μ and p^μ to operators obeying $[x^\mu, p^\nu] = i\eta^{\mu\nu}$. Physical one-particle states are created by acting on the vacuum with a creation operator:

$$|p^+, \vec{p}_T\rangle = a^\dagger_{p^+,\vec{p}_T}|\Omega\rangle.$$

These states are labeled by the light-cone momentum p^+ and transverse momentum $\vec{p}_T$, and satisfy the mass-shell condition.

There is a direct correspondence between the quantum wavefunction of a relativistic particle and the classical scalar field. Both are governed by the same equation, and a classical field can be seen as a superposition of quantum particle states.

We also define the Lorentz generators $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, which represent orbital angular momentum and encode the symmetry of Minkowski spacetime. This connects directly to Chapter 8, where similar charges arose as conserved currents on the world-sheet. The point particle thus serves as a simplified but structurally complete model for understanding constrained quantization and spacetime symmetries—ideas we are now ready to apply to strings.

10 Relativistic quantum open strings

Finally, it is possible to quantize the relativistic string in the light-cone gauge, using the tools found in Chapters 9 and 11, and extending the point-particle framework into an infinite set of quantum harmonic oscillators.
The light-cone gauge drastically simplifies the dynamics, reducing the degrees of freedom to transverse components $X^I(\tau, \sigma)$, with their conjugate momenta $\mathcal{P}^{\tau I}(\tau, \sigma)$, which, together with x_0^- and p^+ make up a full set of Schrödinger operators. The fundamental commutation relation then is:

$$\left[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')\right] = i\eta^{IJ}\delta(\sigma - \sigma') \tag{12.10}$$

From 9.56 we know that we can write the mode expansion of the string in terms of creation and annihilation operators α_n^I and α_{-n}^I which satisfy

$$\left[\alpha_m^I, \alpha_n^J\right] = m\eta^{IJ}\delta_{m+n,0} \tag{12.45}$$

These operators build the string Hilbert space, similar to how it was done in quantum field theory. Each transverse direction I contributes a copy of this oscillator algebra.
After constructing the mode expansion for the transverse coordinates, we repeat the procedure for X^+ and X^-. To understand the dynamics of the quantized strings, we use the transverse Virasoro modes from 9.77, which, upon quantization become operators. Particularly, note that the zeros mode $L_0^\perp$ is of a great importance, as it connects directly to the energy of the string. We find that:

$$H = L_0^\perp + a, \quad L_0^\perp = \alpha' p^+ p^-$$

The transverse Virasoro operators satisfy a commutation relation forming the Virasoro algebra, which encodes the symmetry structure of the string world-sheet under reparametrizations. Ensuring that these operators obey the correct algebra leads to an important consistency condition: the requirement of Lorentz invariance. In the quantum theory, this imposes constraints on the spacetime dimension. By evaluating the commutator of the Lorentz generators, we find that the algebra closes only in $D = 26$ dimensions. This striking result shows that the requirement of quantum consistency alone determines the dimension of spacetime. Moreover, it finds determines the constant a in the Hamiltonian of the theory, fixing it to the value of -1. So we obtain

$$H = L_0^\perp - 1$$

Further, we can construct the quantum states and the state space of the theory by acting with the creation operators on the ground states $|p^+, \vec{p}_t\rangle$ and so the general basis state $|\lambda\rangle$ of the state space can be written as:

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_n^{I\dagger}\right)^{\lambda_{n,I}}|p^+, \vec{p}_T\rangle \tag{12.162}$$

where the non-negative integer $\lambda_{n,I}$ denotes the number of times that the creation operator appears.
Moreover, we can find the mass squared operator, similar to how it was done in 9.83, which we can express in terms of the number operator:

$$M^2 = \frac{1}{a'}(-1 + N^\perp), \quad N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I$$

And so the first excited states have $N^{\perp} = 1$, built by acting once with the raising operator on the ground state, or $\lambda_{1,I} = 1$, which gives $D - 2 = 24$ such **massless** states, forming a vector representation under the transverse rotation group. These states have the same labels and properties as photon states from Chapter 10, confirming that open string excitations include gauge bosons - photons, supporting the idea that gauge interactions emerge naturally in the string spectrum. However, there also exist states called **tachyons** which correspond to $N^{\perp} = 0$ and result in a complex mass. These states pose a serious threat to a bosonic string theory as they create instability.

11 Quantum Closed Strings

Closed strings differ from open strings by being periodic in σ, leading to two sets of independent oscillations—left- and right-movers, similar to what we saw in Chapter 7. The general mode expansion reflects this structure:

$$X^{\mu}(\tau, \sigma) = X_L^{\mu}(\tau + \sigma) + X_R^{\mu}(\tau - \sigma),$$

and introduces two oscillator sets: α_n^{μ} and $\tilde{\alpha}_n^{\mu}$, satisfying their own commutation relations.

As in Chapter 12, we impose the light-cone gauge, which reduces dynamics to the transverse coordinates X^I. The physical states are built from transverse oscillators α_{-n}^I and $\tilde{\alpha}_{-n}^I$, and the total energy depends on both left and right Virasoro operators:

$$\alpha' M^2 = 2\left(L_0^{\perp} + \tilde{L}_0^{\perp} - 2\right).$$

Consistency imposes the *level-matching condition* $L_0^{\perp} = \tilde{L}_0^{\perp}$, which ensures coherence of left/right excitations.

The Hilbert space is now spanned by states of the form $\alpha_{-n}^I \tilde{\alpha}_{-m}^J |p^+, \vec{p}_T\rangle$, whose first excited level contains the **graviton**, **dilaton**, and **antisymmetric tensor**, matching the gauge bosons introduced in Chapter 10.

12 A look at relativistic superstrings

To address the shortcomings of bosonic string theory—most notably, the absence of fermions and the presence of a tachyon—Chapter 14 introduces **superstring theory**, which incorporates **supersymmetry** on the world-sheet. This brings fermionic degrees of freedom into the theory and results in a more physically viable string spectrum.

The key new ingredient is the **world-sheet fermion** $\psi^{\mu}(\tau, \sigma)$, which is a 2D spinor on the world-sheet and a vector under spacetime Lorentz transformations. These fermions are introduced alongside the bosonic fields X^{μ}, and together they form a supersymmetric world-sheet theory.

Fermionic fields satisfy one of two boundary conditions:

- **Neveu–Schwarz (NS)**: antiperiodic in σ, leading to half-integer moded oscillators.
- **Ramond (R)**: periodic in σ, producing integer-moded oscillators.

The **NS sector** gives rise to spacetime bosons, while the **R sector** produces spacetime fermions. After applying the **GSO projection**, which removes unphysical and tachyonic states, the spectrum becomes supersymmetric.

This construction applies to both open and closed strings. In the closed case, combining left- and right-moving sectors leads to four possibilities: NS–NS, R–R, NS–R, and R–NS. Depending on how these sectors are paired and projected, different consistent superstring theories emerge.

Consistency of the quantized theory, particularly the closure of the supersymmetry and Lorentz algebras, requires the spacetime dimension to be $D = 10$—a reduction from the 26 dimensions of the bosonic string. This is the critical dimension where anomalies cancel and the theory is well-defined.

Superstring theory thus provides a unified, tachyon-free framework incorporating both fermions and bosons, and forms the basis for realistic high-energy models of nature.

13 D-branes and Gauge Fields

Having introduced open strings and their quantization, we now explore the extended objects to which their endpoints can attach: **D-branes** (Dirichlet-branes). These arise naturally when imposing Dirichlet boundary conditions on some string coordinates, allowing momentum conservation to be maintained. A **Dp-brane** is a p-dimensional hypersurface in spacetime, where the string satisfies Neumann boundary conditions along the brane directions (NN), Dirichlet conditions in transverse directions (DD), and mixed ND/DN conditions when stretching between branes of different dimensions.

These boundary conditions determine the string mode content. NN sectors yield massless vector fields, which become gauge fields on the brane. DD sectors correspond to scalar fields representing the brane's motion in transverse directions. ND and DN sectors are important for brane intersections and breaking of supersymmetry.

When multiple D-branes coincide, open strings stretching between them lead to **non-Abelian gauge fields**, enhancing the gauge symmetry from $U(1)$ to $U(N)$. The resulting low-energy theory becomes a supersymmetric gauge theory, tightly linking D-branes to the emergence of gauge interactions in string theory.

14 T-duality of Closed Strings

When one spatial dimension is compactified on a circle of radius R, closed strings can carry quantized momentum n/R and wind around the circle w times, contributing energy proportional to wR/α'. The string spectrum depends on both n and w, but remains invariant under the transformation:

$$R \to \frac{\alpha'}{R}, \quad n \leftrightarrow w.$$

This symmetry, known as **T-duality**, has no analogue in point particle theories and is a uniquely stringy feature. It implies that physics at radius R is equivalent to that at radius α'/R, revealing a minimal length scale and a breakdown of classical geometry at small distances.

On the world-sheet, T-duality acts by asymmetrically transforming left- and right-moving modes, flipping the sign of one component. While preserving the physical content of the theory, it alters the geometric interpretation.

T-duality also exchanges Dp-branes with D$(p \pm 1)$-branes and maps Type IIA to Type IIB string theory and vice versa, playing a key role in unifying the various string theories within a larger duality framework.

References

[1] Zwiebach, B. (2009) *A First Course in String Theory* (2nd ed.) Cambridge University Press.