Two-Subsystem Economic Dynamics: Coherence, Cycles, and Relaxation

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Abstract

This chapter develops the two-subsystem economic field model as a minimal laboratory for coherence, exchange cycles, and relaxation toward aligned configurations. Starting from the density–phase Lagrangian, we derive the reduced dynamical system for two interacting subsystems, analyze its conserved quantities, identify fixed points and their stability, and discuss the conditions under which persistent oscillations and phase-locked regimes emerge. The model serves both as a conceptual bridge between the field-theoretic formulation and numerical experiments, and as a template for understanding larger networks of economic subsystems.

Contents

1 Introduction

The full economic field is a many-component system in which densities and phases interact across a network of couplings. To gain intuition about its dynamical behavior, it is useful to consider the simplest non-trivial case: two interacting subsystems, labeled A and B.

In this reduced setting, the density–phase dynamics can be written as a four-dimensional ordinary differential equation (ODE) for $\rho_A(t)$, $\rho_B(t)$, $\theta_A(t)$, and $\theta_B(t)$. Despite its simplicity, this model already exhibits:

- conservation of total density,
- relaxation of phase differences,
- transient exchange cycles,
- the emergence of coherent aligned states.

The two-subsystem model thus plays the role of a *dynamical microscope* for the economic field, revealing core mechanisms that persist in more complex networks.

2 Two-Subsystem Lagrangian and Equations of Motion

We consider spatially uniform fields, so that all spatial gradients vanish and only temporal evolution remains. The Lagrangian for two subsystems A and B reduces to

$$L = \rho_A\dot{\theta}_A + \rho_B\dot{\theta}_B - V_A(\rho_A) - V_B(\rho_B) - J\,\rho_A\rho_B\big[1-\cos(\theta_A-\theta_B)\big], \tag{1}$$

where:

- $\rho_A(t)$, $\rho_B(t)$ are the densities of the two subsystems,
- $\theta_A(t)$, $\theta_B(t)$ are the phases,
- V_A and V_B are local potentials,
- $J \geq 0$ is the coupling strength.

Variation with respect to θ_A and θ_B yields:

$$\dot{\rho}_A = -J\,\rho_A\rho_B\,\sin(\theta_A-\theta_B), \tag{2}$$
$$\dot{\rho}_B = +J\,\rho_A\rho_B\,\sin(\theta_A-\theta_B). \tag{3}$$

Variation with respect to ρ_A and ρ_B gives:

$$\dot{\theta}_A = V_A'(\rho_A) + J\rho_B\big[1-\cos(\theta_A-\theta_B)\big], \tag{4}$$
$$\dot{\theta}_B = V_B'(\rho_B) + J\rho_A\big[1-\cos(\theta_A-\theta_B)\big]. \tag{5}$$

These four equations form the basic dynamical system.

3 Conserved Total Density

Adding Eqs. (2)–(3) we obtain:

$$\dot{\rho}_A + \dot{\rho}_B = -J\rho_A\rho_B\sin(\theta_A-\theta_B) + J\rho_A\rho_B\sin(\theta_A-\theta_B) = 0. \tag{6}$$

Hence

$$\rho_{\text{tot}} = \rho_A + \rho_B \tag{7}$$

is conserved:

$$\frac{\mathrm{d}\rho_{\text{tot}}}{\mathrm{d}t} = 0. \tag{8}$$

Economically, the two-subsystem dynamics reshuffle density between A and B, but do not create or annihilate it. The model describes *redistribution* under alignment forces, not exogenous growth or depletion.

4 Phase Difference and Interaction Energy

It is convenient to introduce the phase difference

$$\Delta\theta = \theta_A - \theta_B. \tag{9}$$

The interaction energy for the two subsystems is

$$E_{\text{int}} = J\rho_A\rho_B\big[1 - \cos(\Delta\theta)\big]. \tag{10}$$

This term is minimized when

$$\Delta\theta = 2\pi k, \qquad k \in \mathbb{Z}, \tag{11}$$

corresponding to perfectly aligned phases.

Differentiating $\Delta\theta$ with respect to time:

$$\begin{aligned}\dot{\Delta\theta} &= \dot{\theta}_A - \dot{\theta}_B \\ &= V_A'(\rho_A) - V_B'(\rho_B) + J\big[\rho_B - \rho_A\big]\big[1 - \cos(\Delta\theta)\big].\end{aligned} \tag{12}$$

This shows that both the asymmetry in local potentials and differences in density contribute to the evolution of the phase difference.

5 Local Potentials and Structural Baselines

To obtain a more explicit form, we consider quadratic potentials:

$$V_A(\rho_A) = \frac{1}{2}k_A(\rho_A - \rho_{0A})^2, \tag{13}$$

$$V_B(\rho_B) = \frac{1}{2}k_B(\rho_B - \rho_{0B})^2, \tag{14}$$

so that

$$V_A'(\rho_A) = k_A(\rho_A - \rho_{0A}), \tag{15}$$

$$V_B'(\rho_B) = k_B(\rho_B - \rho_{0B}). \tag{16}$$

These forms stabilize ρ_A and ρ_B around structural baselines ρ_{0A} and ρ_{0B}, with stiffness controlled by k_A and k_B.

Substituting into (12) yields:

$$\dot{\Delta\theta} = k_A(\rho_A - \rho_{0A}) - k_B(\rho_B - \rho_{0B}) + J(\rho_B - \rho_A)\,[1 - \cos(\Delta\theta)]. \tag{17}$$

6 Fixed Points and Coherent Alignment

Fixed points of the dynamics satisfy:

$$\dot{\rho}_A = 0, \qquad \dot{\rho}_B = 0, \tag{18}$$

$$\dot{\theta}_A = 0, \qquad \dot{\theta}_B = 0. \tag{19}$$

From (2)–(3), either:

- $\rho_A = 0$ or $\rho_B = 0$, or

- $\sin(\Delta\theta) = 0$.

We are interested in non-degenerate configurations with both densities strictly positive, $\rho_A > 0$ and $\rho_B > 0$. Then

$$\sin(\Delta\theta^*) = 0 \quad \Rightarrow \quad \Delta\theta^* = n\pi, \qquad n \in \mathbb{Z}. \tag{20}$$

For the interaction energy,

$$E^*_{\text{int}} = J\rho_A\rho_B\big[1 - \cos(\Delta\theta^*)\big], \tag{21}$$

this yields:

- if $\Delta\theta^* = 2\pi k$, then $E^*_{\text{int}} = 0$ (aligned phases, minimal interaction energy),
- if $\Delta\theta^* = (2k+1)\pi$, then $E^*_{\text{int}} = 2J\rho_A\rho_B$ (anti-aligned, maximal interaction energy).

Thus, the *coherent fixed points* are those with:

$$\Delta\theta^* = 2\pi k, \qquad k \in \mathbb{Z}, \tag{22}$$

and densities satisfying the stationary conditions from Eqs. (4)–(5).

7 Linear Stability of the Coherent Fixed Point

We focus on a coherent fixed point with $\Delta\theta^* = 0$ and (ρ^*_A, ρ^*_B) such that:

$$0 = V'_A(\rho^*_A), \tag{23}$$
$$0 = V'_B(\rho^*_B). \tag{24}$$

For quadratic potentials, this implies $\rho^*_A = \rho_{0A}$ and $\rho^*_B = \rho_{0B}$.

We introduce small perturbations:

$$\rho_A(t) = \rho^*_A + \delta\rho_A(t), \tag{25}$$
$$\rho_B(t) = \rho^*_B + \delta\rho_B(t), \tag{26}$$
$$\Delta\theta(t) = 0 + \delta\theta(t), \tag{27}$$

with $|\delta\rho_A|, |\delta\rho_B|, |\delta\theta| \ll 1$.

Expanding Eqs. (2), (3), and (17) to linear order in the perturbations, we use:

$$\sin(\delta\theta) \approx \delta\theta, \tag{28}$$
$$1 - \cos(\delta\theta) \approx \tfrac{1}{2}(\delta\theta)^2 \approx 0 \quad \text{at leading order.} \tag{29}$$

Thus, to first order,

$$\dot{\delta\rho}_A \approx -J\rho^*_A\rho^*_B\,\delta\theta, \tag{30}$$
$$\dot{\delta\rho}_B \approx +J\rho^*_A\rho^*_B\,\delta\theta, \tag{31}$$

and, since $V'_A(\rho^*_A) = V'_B(\rho^*_B) = 0$, we have:

$$\dot{\delta\theta} \approx k_A\delta\rho_A - k_B\delta\rho_B. \tag{32}$$

It is convenient to define:

$$\delta\rho_{\text{tot}} = \delta\rho_A + \delta\rho_B, \tag{33}$$

$$\delta\rho_{\text{rel}} = \delta\rho_A - \delta\rho_B. \tag{34}$$

From Eqs. (30)–(31):

$$\dot{\delta\rho}_{\text{tot}} = 0, \tag{35}$$

$$\dot{\delta\rho}_{\text{rel}} = -2J\rho_A^*\rho_B^*\,\delta\theta. \tag{36}$$

The total density perturbation is thus conserved at linear order.

Using (32) and $\delta\rho_A = (\delta\rho_{\text{tot}} + \delta\rho_{\text{rel}})/2$, $\delta\rho_B = (\delta\rho_{\text{tot}} - \delta\rho_{\text{rel}})/2$, we obtain:

$$\dot{\delta\theta} = \frac{1}{2}(k_A - k_B)\,\delta\rho_{\text{tot}} + \frac{1}{2}(k_A + k_B)\,\delta\rho_{\text{rel}}. \tag{37}$$

If we focus on perturbations that preserve total density at the linear level (i.e. $\delta\rho_{\text{tot}} = 0$), the coupled system reduces to:

$$\dot{\delta\rho}_{\text{rel}} = -2J\rho_A^*\rho_B^*\,\delta\theta, \tag{38}$$

$$\dot{\delta\theta} = \frac{1}{2}(k_A + k_B)\,\delta\rho_{\text{rel}}. \tag{39}$$

Differentiating (39) with respect to time and using (38) gives:

$$\ddot{\delta\theta} = \frac{1}{2}(k_A + k_B)\,\dot{\delta\rho}_{\text{rel}} = -(k_A + k_B)\,J\rho_A^*\rho_B^*\,\delta\theta. \tag{40}$$

Therefore,

$$\ddot{\delta\theta} + \omega^2\delta\theta = 0, \qquad \omega^2 = (k_A + k_B)\,J\rho_A^*\rho_B^* > 0. \tag{41}$$

This is a harmonic oscillator equation: small perturbations of the phase difference around the coherent fixed point oscillate with frequency ω. In the presence of additional damping mechanisms (not included here), these oscillations would decay, leading to exponential relaxation toward $\Delta\theta = 0$.

7.1 Qualitative picture

The coherent fixed point with aligned phases and densities at their structural baselines is linearly stable: small deviations in phase and relative density lead to bounded oscillations. In realistic settings, frictional or adaptation effects would damp these oscillations, making the coherent state an attractor.

Economically, this corresponds to two strongly interdependent subsystems that settle into a stable, mutually aligned configuration, with exchange flows organizing into recurrent and predictable patterns.

8 Exchange Cycles and Oscillatory Regimes

The linear analysis shows that small perturbations behave like a harmonic oscillator. Beyond the linear regime, the full nonlinear dynamics can generate:

- larger-amplitude oscillations of $\Delta\theta(t)$,

- periodic or quasi-periodic cycles in $\rho_A(t)$ and $\rho_B(t)$,
- transitions between different phase-locked regimes.

These oscillatory regimes can be interpreted as *exchange cycles*: the two subsystems repeatedly transfer density back and forth, driven by phase misalignment and then pulled back toward coherence by the alignment force.

Depending on the shape of the potentials V_A and V_B, and on the coupling strength J, the system may exhibit:

- fast or slow relaxation to coherence,
- persistent oscillations around the coherent state,
- metastable patterns in which one subsystem temporarily dominates.

9 Connection to Numerical Experiments

The two-subsystem model can be implemented numerically using standard ODE integrators. For example, one may choose:

- quadratic potentials with given $(k_A, k_B, \rho_{0A}, \rho_{0B})$,
- initial densities $\rho_A(0)$, $\rho_B(0)$ with $\rho_A(0) + \rho_B(0) = \rho_{\text{tot}}$,
- a non-zero initial phase difference $\Delta\theta(0)$.

The resulting trajectories show:

- that $\rho_A(t) + \rho_B(t)$ remains approximately constant,
- that $\Delta\theta(t)$ tends to fluctuate around a coherent value,
- how the characteristic frequency of oscillations depends on J, k_A, k_B, and the structural baselines.

Such numerical experiments provide a concrete, visual representation of the abstract analysis carried out in this chapter, and form a bridge between the analytic theory and applications to more complex multi-agent networks.

10 Conclusions

The two-subsystem economic model encapsulates, in minimal form, the essence of density–phase dynamics:

- conservation of total density as a consequence of phase symmetry,
- coherent fixed points with aligned phases and structural densities,
- oscillatory regimes around these coherent states,
- exchange cycles driven by misalignment and constrained by coupling.

This model serves as a foundational building block for the study of larger economic networks, where many subsystems interact. The mechanisms of coherence, oscillation, and relaxation observed here will reappear in richer forms when we consider clusters, modular structures, and global economic fields.