Quantum Theory of Toroidal Solitons

Non-Perturbative Quantization, Internal Spectrum, and Soliton Statistics

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Abstract

This thirteenth document formulates the full quantum description of the toroidal solitons introduced in Documents 1–12. While previous works established classical stability, nonlinear dynamics, and EM/GR phenomenology, the present document constructs the non-perturbative quantum mechanics of solitons. We develop collective-coordinate quantization, analyze the internal vibrational spectrum, construct soliton creation operators, define a Hilbert space of multi-torus states, and derive the statistical and thermodynamic behavior of a gas of toroidal quanta. This provides the final layer needed to interpret the toroidal soliton as a genuine quantum object.

1 Introduction

Quantum solitons behave neither like point particles nor like classical extended objects. The toroidal soliton possesses:

- an internal phase degree of freedom,
- topologically quantized winding n,
- discrete vibrational modes,
- a conserved charge Q,
- a stable geometric profile.

These features yield a non-perturbative quantum structure analogous to:

- instantons (topology),
- Skyrmions (collective coordinates),

- Q-balls (charge quantization),
- flux tubes (vibrational spectra).

But the combination is unique to the unified motion theory.

2 Collective-Coordinate Quantization

The toroidal soliton solution depends on four key zero modes:

1. three translations,
2. one global phase rotation (U(1) symmetry).

Introduce collective coordinates:

$$\mathbf{X}(t), \qquad \Theta(t),$$

representing the soliton's center and phase.

Substitute into the action and expand:

$$\psi(\mathbf{x},t) = \rho_0(\mathbf{x}-\mathbf{X}(t))\, e^{i(\omega t+\Theta(t)+\delta\theta)}.$$

The effective Lagrangian becomes:

$$L_{\text{eff}} = \frac{1}{2} M_{\text{eff}}\, \dot{\mathbf{X}}^2 + Q\,\dot{\Theta} - E(Q,n),$$

with

$$M_{\text{eff}} = \int d^3x\, T_{00}.$$

Canonical momenta:

$$\mathbf{P} = M_{\text{eff}}\dot{\mathbf{X}}, \qquad P_\Theta = Q.$$

Quantization:

$$[\hat{X}_i, \hat{P}_j] = i\delta_{ij}, \qquad [\hat{\Theta}, \hat{Q}] = i.$$

Thus Q is the generator of phase rotations, consistent with Noether charge quantization.

3 Quantum Internal Spectrum

From the nonlinear stability analysis (Document 10), internal modes satisfy:

$$-\Omega_k^2 M\,\zeta_k = K\,\zeta_k.$$

Quantization gives creation/annihilation operators:

$$\hat{a}_k, \hat{a}_k^\dagger, \qquad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}.$$

Energy levels:

$$E_k = \Omega_k \left(n_k + \tfrac{1}{2}\right).$$

Key quantized modes:

- **Breathing mode:** Ω_{br},
- **Torsional mode:** $\Omega_{\text{tors}} \propto n/R_c$,
- **Kelvin-like bending modes:** $\Omega_m \propto m/R_c$,
- **Azimuthal phonons:** $\Omega_k \propto k/R_c$.

This predicts a discrete internal spectrum observable in GW/EM signals.

4 Soliton Creation and Annihilation Operators

Define a non-perturbative creation operator:

$$\hat{\mathcal{O}}^\dagger(Q,n) = \exp\left(\int d^3x\, \Phi_0(x;Q,n)\, \hat{\psi}^\dagger(x)\right).$$

Acting on the vacuum:

$$|\mathcal{T}_{Q,n}\rangle = \hat{\mathcal{O}}^\dagger(Q,n)\,|0\rangle.$$

This state is:

- not perturbative,
- not Gaussian,
- a coherent configuration of infinitely many quanta of $\hat{\psi}$,
- stabilized by nonlinearity and topology.

Multi-soliton states follow naturally:

$$|\mathcal{T}_{Q_1,n_1}\,\mathcal{T}_{Q_2,n_2}\cdots\rangle = \prod_i \hat{\mathcal{O}}^\dagger(Q_i,n_i)\,|0\rangle.$$

5 Quantum Interactions and Scattering

Soliton scattering is governed by:

$$S = \exp\left(i \int d^4x\, \mathcal{L}_{\rm int}(\psi, \rho_0, \theta_0) \right).$$

Collective-coordinate quantization yields:

$$\langle \mathcal{T}_f | S | \mathcal{T}_i \rangle = \int \mathcal{D}X(t)\, \mathcal{D}\Theta(t)\, e^{iS_{\rm eff}[X,\Theta]}.$$

Predictions:

- quantized scattering angles,
- resonant enhancement when Ω_k matches impact frequency,
- tunneling between different n-sectors is forbidden,
- fusion creates quantum superpositions:

$$|\mathcal{T}_{Q_1} \mathcal{T}_{Q_2}\rangle \to \sum_\alpha c_\alpha |\mathcal{T}_{Q_1+Q_2}, \alpha\rangle.$$

6 Quantum Statistics of Toroidal Solitons

The soliton carries:

- conserved charge Q,
- winding number n,
- internal phase Θ,
- geometric radius R_c,
- internal excitations.

Because it is an extended object with topological data:

- it is not a boson or fermion,
- it behaves as a **geometric anyon-like excitation**,
- exchange induces a Berry-phase-like rotation:

$$\phi_{\rm exch} = n_1 n_2 \Delta\theta.$$

Thus toroidal solitons obey **generalized quantum statistics**.

7 Thermodynamics of Soliton Gases

A dilute gas of toroidal quanta has partition function:

$$Z = \sum_{\{N_k\}} \exp\left[-\beta\left(\sum_k N_k E_k + \sum_{i<j} V_{\mathrm{int}}(i,j)\right)\right].$$

Phenomena:

- condensation into lowest torsional mode,
- ring-lattice formation,
- soliton crystals for large Q,
- topological phase transitions.

These phases have no analogue in standard scalar QFT.

8 Conclusion

The toroidal soliton is a fully consistent quantum object with:

- quantized collective coordinates,
- discrete internal spectrum,
- creation and annihilation operators,
- generalized anyonic statistics,
- well-defined thermodynamics,
- non-perturbative scattering amplitudes.

This completes the quantum layer of the unified motion theory and positions the toroidal soliton as a true *elementary quantum of geometry and motion.*