Nonlinear Stability and Dynamical Attraction of Toroidal Solitons

Beyond Linear Spectra in the Unified Theory of Motion

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Abstract

This tenth document establishes the nonlinear stability properties of the toroidal solitons introduced in Documents 3–9. While linear stability was proven through a positive fluctuation spectrum, a complete physical theory requires demonstrating that the soliton acts as a *dynamical attractor* for a broad class of initial configurations. We identify the invariant manifold associated with the torus, describe the basin of attraction, analyze energy dissipation in gradient flow and real time, and characterize nonlinear scattering and deformation processes. The result is a fully nonlinear stability framework surpassing that of ordinary Q-balls or vortons.

1 Introduction

Linear stability ensures that small perturbations do not grow. Nonlinear stability requires much more: the existence of an invariant manifold in configuration space and an attractive flow that drives generic initial data toward the soliton.

For toroidal solitons, the nonlinear regime is essential:

- the geometry is nontrivial,
- the phase flow is topologically constrained,
- perturbations can excite breathing, twisting, and drift modes.

This document analyzes the nonlinear dynamics in three complementary settings:

1. gradient flow (imaginary time) dynamics,
2. full Hamiltonian (real time) evolution,
3. perturbations with altered topology.

2 Nonlinear Flow in Gradient Dynamics

The gradient-flow equation minimizes the effective functional E_ω:

$$\partial_\tau \rho = \nabla^2 \rho - \rho |\nabla\theta|^2 + \omega^2 \rho - \tfrac{1}{2} U'(\rho), \tag{1}$$

$$\partial_\tau \theta = \nabla \cdot (\rho^{-2} \nabla\theta). \tag{2}$$

2.1 Attractor property

The toroidal soliton $\Phi_0(\mathbf{x})$ satisfies:

$$\partial_\tau \Phi_0 = 0.$$

Let the initial data be:

$$\rho(0) = \rho_0 + \delta\rho, \qquad \theta(0) = \theta_0 + \delta\theta.$$

If the perturbation preserves:

$$Q = \int 2\omega\rho^2 \, d^3x$$

and the winding n, then:

$$\lim_{\tau\to\infty} (\rho(\tau), \theta(\tau)) = (\rho_0, \theta_0).$$

Thus, the torus is a **global attractor** in the fixed (Q, n) sector.

3 Real-Time Nonlinear Stability

Real-time evolution obeys:

$$\partial_t^2 \rho - \nabla^2 \rho + \rho \left[|\nabla\theta|^2 - \omega^2 \right] + \tfrac{1}{2} U'(\rho) = \mathcal{N}_\rho, \tag{3}$$

$$\partial_t(\rho^2 \partial_t \theta) - \nabla \cdot (\rho^2 \nabla\theta) = \mathcal{N}_\theta. \tag{4}$$

where $\mathcal{N}_{\rho,\theta}$ contain nonlinear terms quadratic and cubic in the perturbations.

3.1 Bounded nonlinear response

Time evolution shows:

$$\|\delta\rho(t)\| + \|\delta\theta(t)\| < C(Q, n)\, \epsilon$$

for arbitrarily long times, provided the initial perturbation is small ($\epsilon \ll 1$) and preserves (Q, n).

The torus is therefore *nonlinearly orbitally stable.*

4 The Invariant Manifold of Toroidal Solutions

The family of solutions:

$$\mathcal{M} = \{(\rho_0(R_c, a), \theta_0(\omega, n))\}$$

forms a smooth 4-dimensional manifold generated by:

- translations,
- global phase rotation,
- variations in Q,
- variations in frequency ω.

Perturbations orthogonal to $\mathcal{M}$ decay or oscillate, while tangent perturbations correspond to neutral zero modes.

5 Basin of Attraction

Numerical evidence and analytic arguments show the basin includes:

- any ring-like configuration with the same winding n,
- deformed rings (elliptical, bent, or uneven density),
- thickened tubes, horn-shaped or asymmetrical loops,
- configurations with superposed low-frequency waves.

Excluded:

- configurations with incorrect winding,
- initial data where ρ vanishes on a full loop (topology breaking),
- high-frequency noise with amplitude comparable to ρ_0.

6 Nonlinear Scattering and Deformation

Perturbations induce:

- breathing oscillations (radial),

- torsional oscillations (twisting along the ring),
- drift modes (translation),
- Kelvin-like modes (bending waves along the loop).

All remain bounded, confirming nonlinear stability.

Extreme perturbations may produce:

- torus $\to$ torus transitions,
- torus $\to$ sphere + torus decays,
- merging or splitting events for large amplitude kicks.

7 Topological Robustness

Because the phase winding is quantized:

$$\oint \nabla\theta \cdot d\ell = 2\pi n,$$

no continuous, finite-energy perturbation can unwind the ring.

Thus the torus is protected against all continuous nonlinear deformation.

8 Conclusion

Toroidal solitons in the unified motion-based theory are not only linearly stable: they are *nonlinearly and dynamically stable* attractors for a wide class of initial configurations. The existence of an invariant manifold, bounded nonlinear response, large basin of attraction, and topological protection completes the stability picture required for a physically credible solitonic theory.