Toroidal Q–Ball Configurations as Geometric Quanta of Motion: A Field-Theoretic and Numerical Study

Ivan Salines

Independent Researcher

November 15, 2025

Abstract

We investigate a family of non–topological solitons of toroidal geometry arising in a complex scalar field theory with global $U(1)$ symmetry. By imposing cylindrical symmetry and a spatial winding number n, and minimizing the energy under fixed charge Q, we obtain stable ring–shaped configurations whose major and minor radii are asymptotically independent of Q. We interpret such objects as geometric "quanta of motion", where the distribution of energy is determined by dynamical constraints rather than by particle–like localization. The analysis combines analytic approximations (thin–torus limit) and fully nonlinear numerical gradient–flow evolution, yielding a coherent theoretical framework with internal consistency and predictive power.

1 Introduction

Non–topological solitons supported by a conserved charge, known as Q–balls, arise in scalar field theories with global $U(1)$ symmetry when the self–interaction potential satisfies suitable conditions. Their existence is tied to a dynamical balance among gradient, potential and charge contributions.

In the present work we show that when the internal phase of the field carries a spatial winding number n, the lowest–energy configuration at fixed charge Q becomes *toroidal*, forming a ring–like soliton (Q–torus) rather than a spherical lump.

Our main results are:

- the toroidal geometry is dynamically preferred and stable for a range of parameters,
- the major radius R_c and tube radius a are nearly independent of the total charge Q,
- the energy per unit charge satisfies $E/Q < m$, ensuring absolute stability,
- the thin–torus analytic model predicts the geometry, which is confirmed by full numerical simulations.

These findings support the interpretation of toroidal Q–balls as *geometric quanta of motion*, where curvature and internal phase flow determine the spatial structure.

2 Field Theory Model

We consider a relativistic complex scalar field $\Phi(t, \mathbf{x}) \in \mathbb{C}$ with global $U(1)$ invariance. The Lagrangian is:

$$\mathcal{L} = \partial_\mu \Phi^* \, \partial^\mu \Phi - U(|\Phi|), \tag{1}$$

with potential

$$U(|\Phi|) = \frac{1}{2}m^2|\Phi|^2 - \frac{g}{3}|\Phi|^3 + \frac{h}{4}|\Phi|^4, \qquad m^2 > 0,\ g > 0,\ h > 0. \tag{2}$$

The Noether charge associated with the global symmetry is:

$$Q = i\int d^3x\,\left(\Phi^*\dot{\Phi} - \dot{\Phi}^*\Phi\right). \tag{3}$$

Stationary Ansatz

We use the standard Q–ball ansatz

$$\Phi(t, \mathbf{x}) = e^{i\omega t}\psi(\mathbf{x}),$$

which yields:

$$Q = 2\omega\int d^3x\,|\psi|^2, \qquad E = \int d^3x\big(|\nabla\psi|^2 + \omega^2|\psi|^2 + U(|\psi|)\big).$$

3 Axially Symmetric Reduction

In cylindrical coordinates (R, φ, z), we impose a spatial winding number $n \in \mathbb{Z}$:

$$\psi(R, \varphi, z) = \rho(R, z)\,e^{in\varphi}.$$

Then:

$$|\nabla\psi|^2 = (\partial_R\rho)^2 + (\partial_z\rho)^2 + \frac{n^2}{R^2}\rho^2.$$

Integrating over φ, we obtain the reduced energy:

$$E[\rho] = 2\pi\int_0^\infty dR\int_{-\infty}^{+\infty} dz\ R\left[(\partial_R\rho)^2 + (\partial_z\rho)^2 + \frac{n^2}{R^2}\rho^2 + \omega^2\rho^2 + U(\rho)\right]. \tag{4}$$

Charge:

$$Q[\rho] = 4\pi\omega\int_0^\infty dR\int_{-\infty}^{+\infty} dz\ R\rho^2. \tag{5}$$

4 Thin–Torus Approximation

Gradient–flow minimization of (4) produces toroidal configurations peaked around a circle in the plane $z = 0$, with major radius R_c and minor radius a.

In the limit $a \ll R_c$, the configuration behaves as a straight Q–ball filament bent into a circle. The effective energy is:

$$E(R, a) \simeq C_1\frac{Q}{a^2} + C_2\frac{n^2Q}{R^2} + C_3Q + \cdots. \tag{6}$$

Minimization yields:

$$R = R_*(n, m^2, g, h), \qquad a = a_*(n, m^2, g, h),$$

independent of Q for sufficiently large charge. This prediction is borne out by full numerical simulations.

5 Numerical Method

We implement gradient–flow relaxation:

$$\rho_{t+\Delta t} = \rho_t - \Delta t \, \frac{\delta E}{\delta \rho_t},$$

with rescaling at each step to enforce $Q[\rho] = Q_{\text{target}}$.

A regulator $\alpha_{\text{core}} e^{-(R/R_{\text{core}})^2} \rho^2$ prevents spurious concentration near $R = 0$.

We use:

$$n = 3, \qquad m^2 = 1, \qquad g = 1, \qquad h = 0.5,$$

with $Q = 100, 200, 400$.

6 Results

Stable toroidal solutions were obtained for all charges. The radii $R_{\text{in}}, R_{\text{out}}$ are extracted from the half–maximum condition of $\rho(R, 0)$.

Q	E	E/Q	R_{in}	R_{out}	$R_c \simeq (R_{\text{in}} + R_{\text{out}})/2$
100	89.95	0.899	6.015	9.925	7.970
200	174.64	0.873	6.015	9.925	7.970
400	336.23	0.841	6.015	9.925	7.970

The minor radius is:

$$a = \frac{R_{\text{out}} - R_{\text{in}}}{2} \simeq 1.955,$$

identical (within numerical precision) for all charges tested.

Thus the geometry of the toroidal Q–ball is effectively independent of Q, in agreement with the thin–torus prediction.

7 Physical Interpretation: Geometric Quanta of Motion

The charge–constrained energy minimization selects a ring–shaped configuration where geometry is dictated by dynamical constraints:

- the winding term n^2/R^2 stabilizes the circular shape,
- radial compression scales as $1/a^2$,
- potential energy fixes the internal density scale.

The insensitivity of (R_c, a) to Q shows that geometry is a *dynamical invariant*. These toroidal structures represent geometric quanta of motion: coherent, localized flows of the field where motion, energy and geometry are inseparably linked.

8 Conclusion

We demonstrated the existence of stable toroidal Q–ball solutions in a $U(1)$ scalar field theory with spatial winding. The geometry is stable, nearly independent of charge, and consistent with analytic predictions from the thin–torus model. These ring–like solitons provide a natural example of organized internal motion giving rise to emergent geometric structure.

References

[1] S. Coleman, *Q-balls*, Nucl. Phys. B 262 (1985) 263.

[2] T. D. Lee and Y. Pang, *Nontopological solitons*, Phys. Rept. 221 (1992) 251.

[3] R. Friedberg, T. D. Lee, and A. Sirlin, *Class of scalar-field soliton solutions*, Phys. Rev. D 13 (1976) 2739.

[4] Y. Shnir, *Topological and Non-Topological Solitons in Scalar Field Theories*, Cambridge Univ. Press (2018).