What is the value of tan(∠ACB)?

Ivan Setiawan* Takushi Itadani†

March 2, 2020

1 Problem

In Figure 1, the radius of the small circle (black) is the same as the edge length of a square. What is the value of tan(∠ACB)?

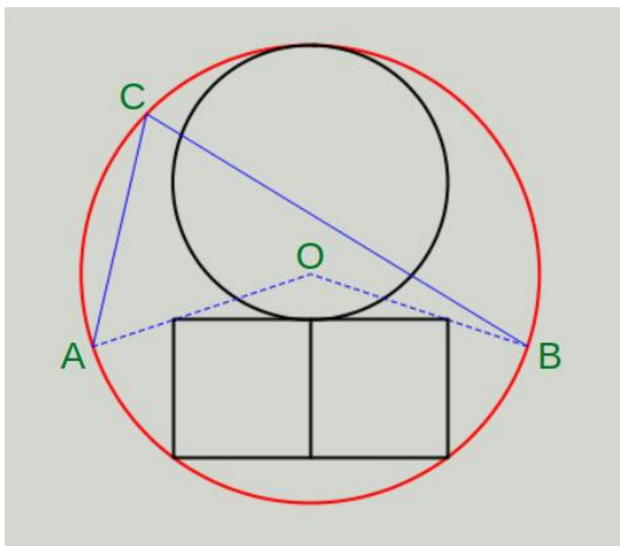

Figure 1: Problem's figure

2 Solution

From Figure 1, the inscribed angle ∠ACB has its intercepted arc $\overset{\frown}{\mathrm{AB}}$ (with angle ∠AOB). Because inscribed angle is half of its intercepted arc, we have:

$$\angle\mathrm{ACB} = \frac{1}{2} \times \angle\mathrm{AOB}. \tag{1}$$

Let point D (as shown in Figure 2: 'Problem's figure annotated with sizes' on page 2) be the bottom center (crossing point) of the 2 squares. Due to symmetry $\angle\mathrm{AOD} = \frac{1}{2} \times \angle\mathrm{AOB}$, and hence:

$$\tan(\angle\mathrm{ACB}) = \tan(\angle\mathrm{AOD}). \tag{2}$$

*The solver, Python-er, LaTeX-er, Markdown-er, https://github.com/ivansetiawantky/welovemath/blob/master/var-auth/20200225_geo_tan/20200225_geo_tan.pdf

†The creator

Figure 2 shows Figure 1 which is annotated with sizes relevant to the solution. Let point O be the center of the big red circle. Also, let the edge length of the square (which is also the radius of the small black circle) and the radius of the big red circle be a and R, respectively.

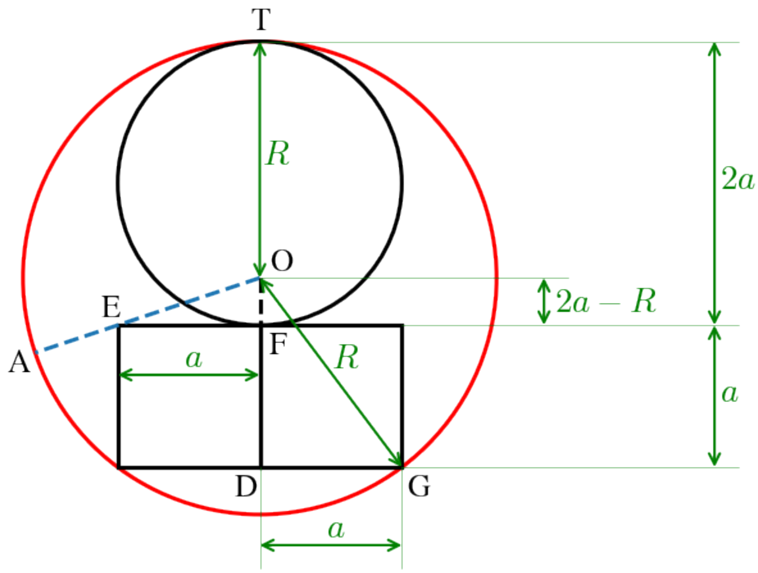

Figure 2: Problem's figure annotated with sizes

Let $\overline{\mathrm{XY}}$ and $|\overline{\mathrm{XY}}|$ be line segment XY and its length, respectively. Looking at right triangle $\triangle$ODG and applying Pythagoras' theorem:

$$\begin{aligned} |\overline{\mathrm{OD}}|^2 + |\overline{\mathrm{DG}}|^2 = |\overline{\mathrm{OG}}|^2 &\Rightarrow |\overline{\mathrm{OD}}|^2 + a^2 = R^2 \\ &\Rightarrow |\overline{\mathrm{OD}}| = \sqrt{R^2 - a^2}. \end{aligned} \tag{3}$$

Also from Figure 2, we have:

$$\begin{aligned} |\overline{\mathrm{OD}}| + |\overline{\mathrm{OT}}| = |\overline{\mathrm{DT}}| &\Rightarrow |\overline{\mathrm{OD}}| + R = |\overline{\mathrm{DF}}| + |\overline{\mathrm{FT}}| \\ &\Rightarrow |\overline{\mathrm{OD}}| + R = a + 2a \\ &\Rightarrow |\overline{\mathrm{OD}}| = 3a - R. \end{aligned} \tag{4}$$

From equation (3) and (4), we have:

$$\begin{aligned} \sqrt{R^2 - a^2} = 3a - R &\Rightarrow R^2 - a^2 = (3a - R)^2 \\ &\Rightarrow \cancel{R^2} - a^2 = 9a^2 - 6aR + \cancel{R^2} \\ &\Rightarrow 6aR = 9a^2 + a^2 = 10a^2 \\ &\Rightarrow 6\cancel{a}R = 10a^{\cancel{2}} \qquad (a \neq 0) \\ &\Rightarrow R = \frac{5}{3}a. \end{aligned} \tag{5}$$

Now look at right triangle $\triangle$EFO.

$$\begin{aligned}
\tan(\angle \mathrm{AOD}) = \tan(\angle \mathrm{EOF}) &= \frac{|\overline{\mathrm{EF}}|}{|\overline{\mathrm{FO}}|} \\
&= \frac{a}{2a - R} && \text{(from Figure 2)} \\
&= \frac{a}{2a - \frac{5}{3}a} && \text{(substituting } R \text{ from equation (5))} \\
&= \frac{\cancel{a}}{\frac{1}{3}\cancel{a}} \\
&= 3. && (6)
\end{aligned}$$

Hence, $\tan(\angle \mathrm{ACB}) = \tan(\angle \mathrm{AOD}) = 3$. ■