What is the maximum value of P(a)?

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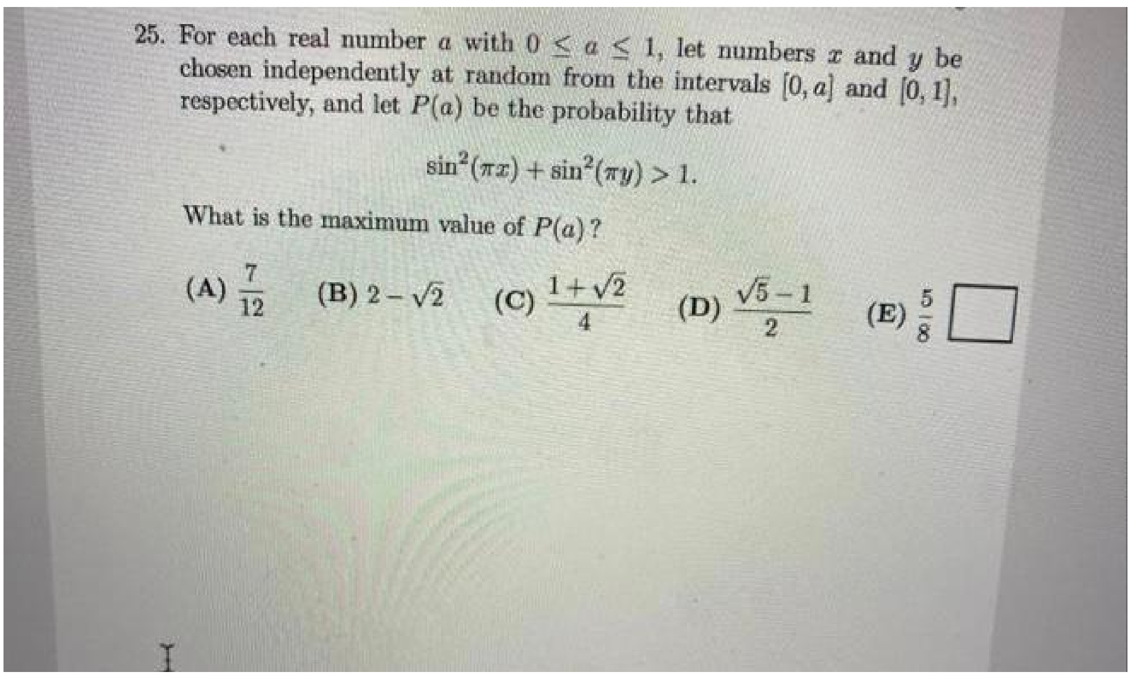
1 Problem



25. For each real number a with $0 \le a \le 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$$\sin^2(\pi x) + \sin^2(\pi y) > 1.$$

What is the maximum value of $P(a)$?

(A) $\frac{7}{12}$ (B) $2-\sqrt{2}$ (C) $\frac{1+\sqrt{2}}{4}$ (D) $\frac{\sqrt{5}-1}{2}$ (E) $\frac{5}{8}$

2 Solution

Let Prob{**e**} be the probability of an event **e** occurs. Thus $P(a)$ is:

$$P(a) = \text{Prob}\{\sin^2(\pi \text{x}) + \sin^2(\pi \text{y}) > 1\}, \tag{1}$$

where x and y be chosen independently at uniformly random from the interval $[0, a]$ and $[0, 1]$, respectively.

Because

$$
\begin{aligned}
\sin^2(\pi x) + \sin^2(\pi y) > 1 &\Rightarrow \frac{1-\cos(2\pi x)}{2} + \frac{1-\cos(2\pi y)}{2} > 1 \qquad \text{(pakai rumus } \cos(2\alpha)\text{)} \\
&\Rightarrow 2 - [\cos(2\pi x) + \cos(2\pi y)] > 2 \\
&\Rightarrow -[\cos(2\pi x) + \cos(2\pi y)] > 0 \\
&\Rightarrow \cos(2\pi x) + \cos(2\pi y) < 0 \\
&\Rightarrow 2\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0 \qquad \text{(pakai rumus } \cos\alpha + \cos\beta\text{)} \\
&\Rightarrow \cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0,
\end{aligned}
$$

then, the above event probabilty $P(a)$ in equation (1) is identical to:

$$P(a) = \text{Prob}\{\cos(\pi(\mathrm{x}+\mathrm{y})) \times \cos(\pi(\mathrm{x}-\mathrm{y})) < 0\}. \tag{2}$$

As the first step, we need to find the location of points (x, y) inside the given area, that satisfy $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$, that is:

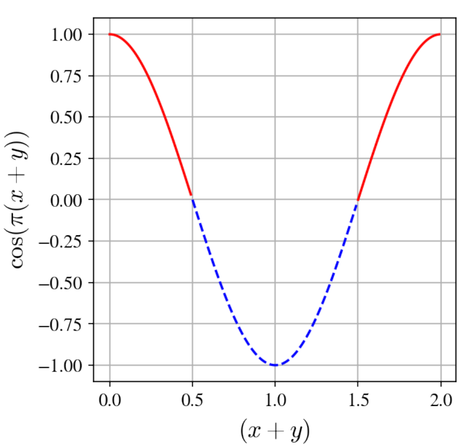
$$\{(x,y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0) \cap (0 \le x \le a) \cap (0 \le y \le 1)\}. \tag{3}$$

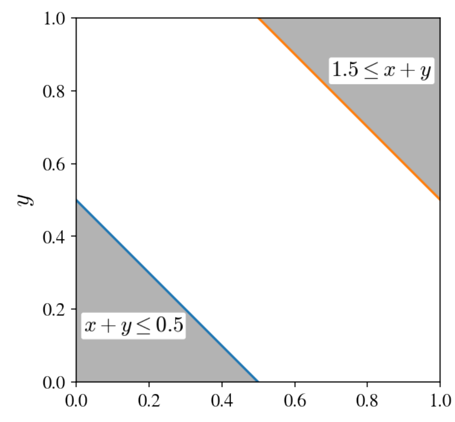
There are 2 cases that satisfy $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$, that is, $\cos(\pi(x+y)) > 0$ AND $\cos(\pi(x-y)) < 0$ as the first case, OR $\cos(\pi(x+y)) < 0$ AND $\cos(\pi(x-y)) > 0$ as the second case. So, equation (3) can be found as the union of the above 2 cases, that is (all inside $((0 \le x \le a) \cap (0 \le y \le 1))$):

$$
\begin{aligned}
\{(x,y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0)\} = & \qquad\qquad\qquad\qquad (4)\\
\{(x,y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\} & \qquad \text{(case 1)} \qquad (5)\\
\cup\{(x,y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\} & \qquad \text{(case 2).} \qquad (6)
\end{aligned}
$$

Keeping in mind that $x \in [0, a]$ (with $0 \le a \le 1$) and $y \in [0, 1]$, below we find the regions of (x, y) that satisfy case 1 and case 2.

2.1 Case 1: $\{(x,y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\}$

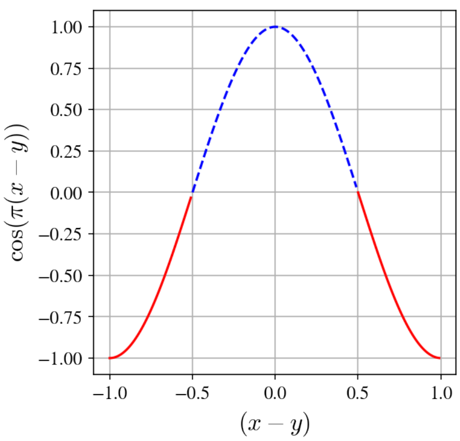


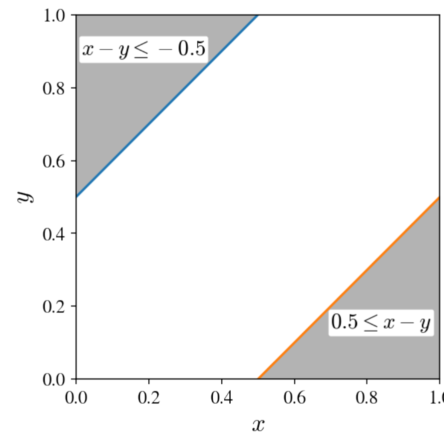


In case 1, the former term of equation (5), that is, $\cos(\pi(x+y))$, will satisfy $(\cos(\pi(x+y)) > 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(0 \leq (x+y) \leq 0.5) \cup (1.5 \leq (x+y) \leq 2.0), \tag{7}$$

which is shown in the above-right figure.





Similarly in case 1, the later term of equation (5), that is, $\cos(\pi(x-y))$, will satisfy $(\cos(\pi(x-y)) < 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(-1 \leq (x-y) \leq -0.5) \cup (0.5 \leq (x-y) \leq 1), \tag{8}$$

which is shown in the above-right figure.

We see from the figures that the region

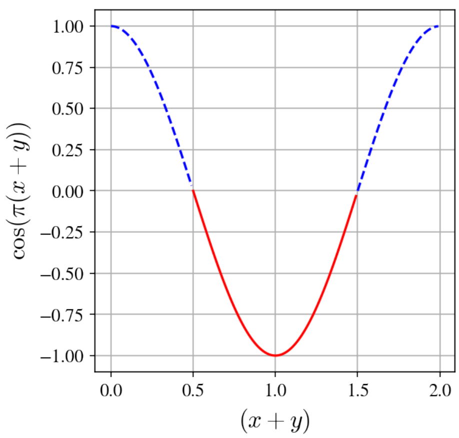
$$(0 \leq (x+y) \leq 0.5) \cup (1.5 \leq (x+y) \leq 2.0),$$

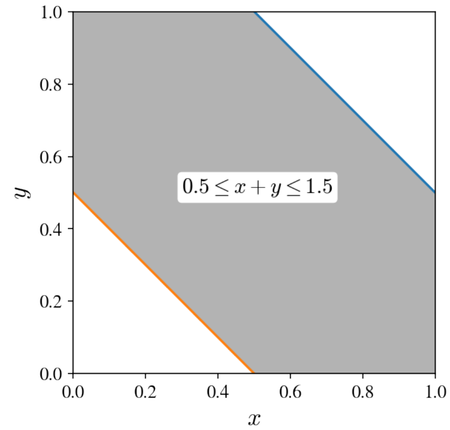
and the region

$$(-1 \leq (x-y) \leq -0.5) \cup (0.5 \leq (x-y) \leq 1),$$

do NOT overlap, so the region that satisfy case 1 does NOT exist.

2.2 Case 2: $\{(x, y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\}$

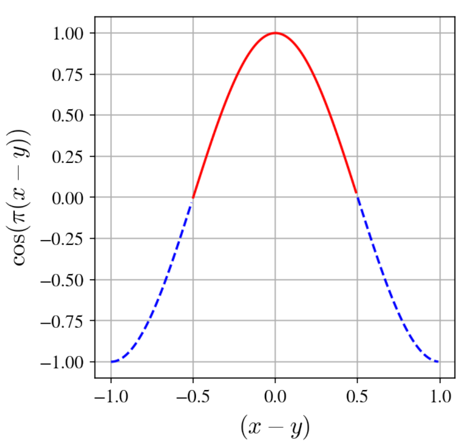


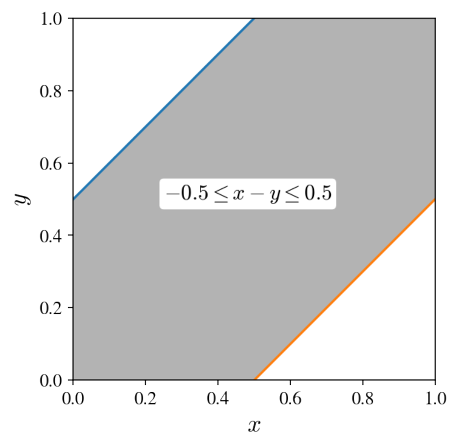


In case 2, the former term of equation (6), that is, $\cos(\pi(x+y))$, will satisfy $(\cos(\pi(x+y)) < 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(0.5 \le (x+y) \le 1.5), \tag{9}$$

which is shown in the above-right figure.





Similarly in case 2, the later term of equation (6), that is, $\cos(\pi(x-y))$, will satisfy $(\cos(\pi(x-y)) > 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(-0.5 \le (x-y) \le 0.5), \tag{10}$$

which is shown in the above-right figure.

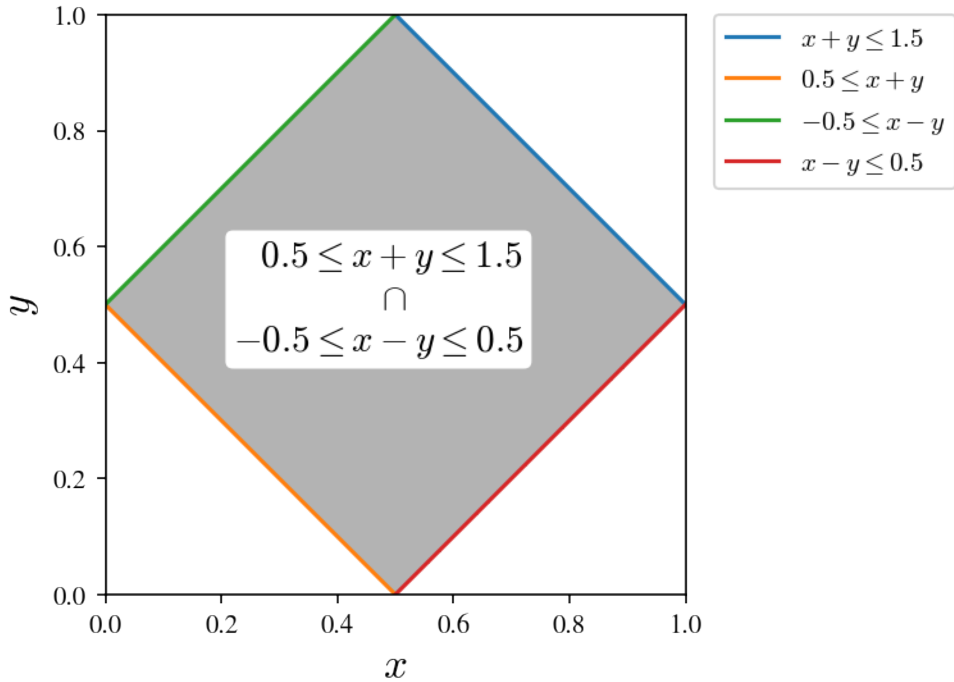
We see from the figures that the region

$$(0.5 \leq (x+y) \leq 1.5),$$

and the region

$$(-0.5 \leq (x-y) \leq 0.5),$$

do overlap, so the region that satisfy case 2 is:



Thus, the above region is the region of (x, y) which satisfy $\{\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0\}$:

$$\{(x, y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0) \cap (0 \leq x \leq 1) \cap (0 \leq y \leq 1)\}, \tag{11}$$

which is also the region of (x, y) which satisfy $\{\sin^2(\pi x) + \sin^2(\pi y) > 1\}$:

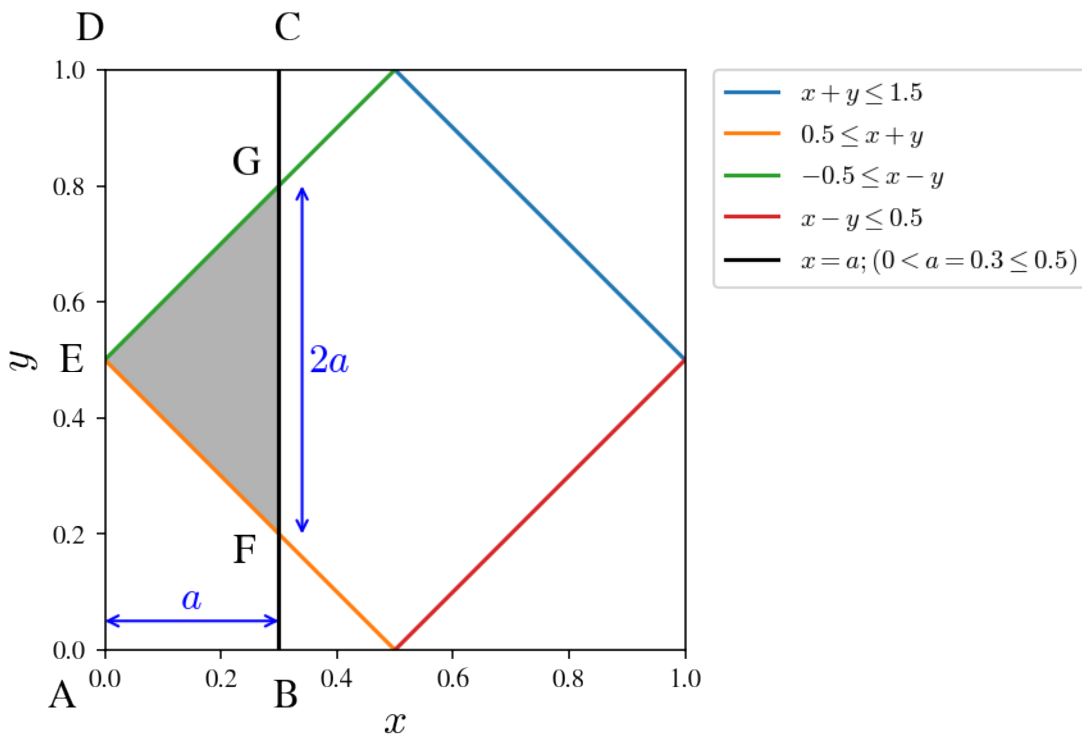
$$\{(x, y) \mid (\sin^2(\pi x) + \sin^2(\pi y) > 1) \cap (0 \leq x \leq 1) \cap (0 \leq y \leq 1)\}. \tag{12}$$

2.3 Formulation of function $P(a)$

$P(a)$ is the probability of point (x, y) lies inside the above shown region with additional constraint $\{x \in [0, a]\}$, when x and y is chosen uniform-randomly independently from region $\{(x, y) \mid x \in [0, a], y \in [0, 1]\}$. This probability is the ratio of "the area of the above shown region with additional constraint $\{x \in [0, a]\}$" to "the area of the region $\{(x, y) \mid x \in [0, a], y \in [0, 1]\}$." Hence,

$$\begin{aligned} P(a) &= \frac{\text{Area of the above region } \cap \{0 \leq x \leq a\}}{\text{Area of the region } \{(x, y) \mid x \in [0, a], y \in [0, 1]\}} \\ &= \frac{\text{Area of the above region } \cap \{0 \leq x \leq a\}}{a}. \end{aligned}$$

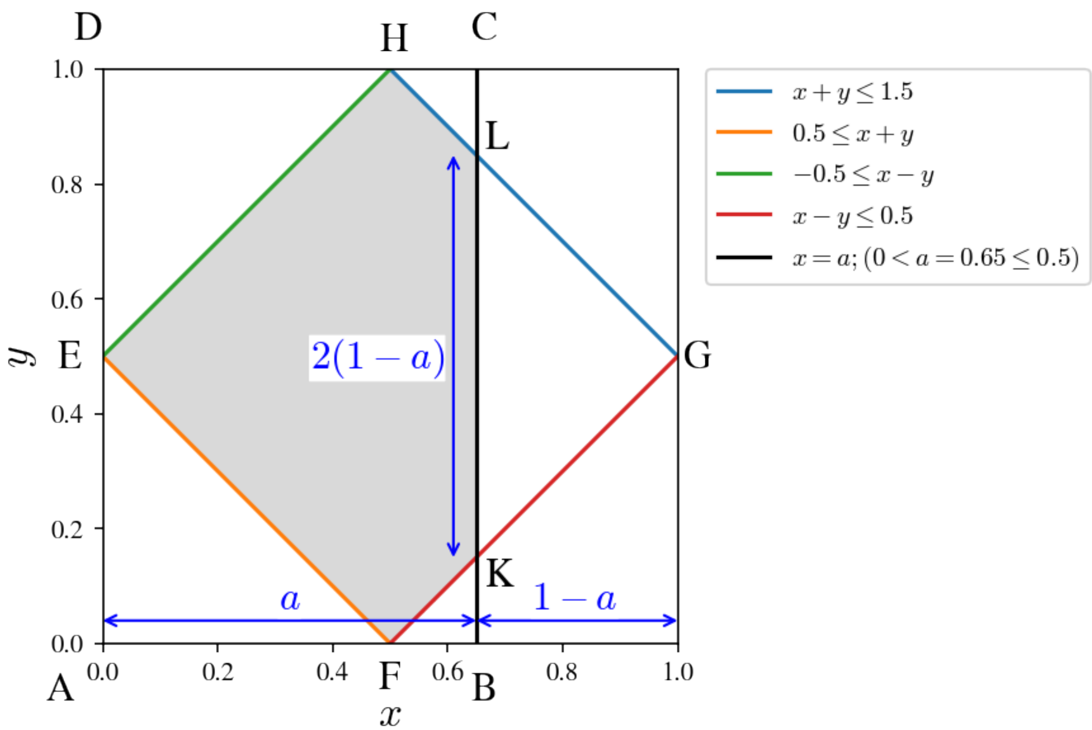
2.3.1 Function $P(a)$ when $0 < a \leq 0.5$



From the above figure, when $0 < a \leq 0.5$, then $P(a)$ is:

$$\begin{aligned} P(a) &= \frac{\text{Area of the shaded region, that is, area of }\triangle\text{EFG}}{\text{Area of rectangle ABCD}} \\ &= \frac{\frac{1}{2} \times 2a \times a}{a \times 1.0} \\ &= a. \end{aligned} \tag{13}$$

2.3.2 Function $P(a)$ when $0.5 < a \leq 1.0$



From the above figure, when $0.5 < a \leq 1$, then $P(a)$ is:

$$
\begin{aligned}
P(a) &= \frac{\text{Area of the shaded region, that is, area of } \square\text{EFGH} - \text{area of } \triangle\text{GLK}}{\text{Area of rectangle ABCD}} \\
&= \frac{2 \times (\text{area of } \triangle\text{EGH}) - \text{area of } \triangle\text{GLK}}{\text{Area of rectangle ABCD}} \\
&= \frac{2 \times (\frac{1}{2} \times 1.0 \times 0.5) - \frac{1}{2} \times 2(1-a) \times (1-a)}{a \times 1.0} \\
&= \frac{\frac{1}{2} - (1 - 2a + a^2)}{a} \\
&= \frac{-\frac{1}{2} + 2a - a^2}{a} \\
&= -a + 2 - \frac{1}{2a}.
\end{aligned} \tag{14}
$$

2.3.3 Final form of function $P(a)$

Noticing that $P(a=0) = 0$ is satisfied in equation (13), and noticing that $P(a=0.5) = 0.5$ is satisfied in both equation (13) and (14), then $P(a)$ can be expressed as a piecewise-defined function as follows:

$$
P(a) = \begin{cases} a & 0 \leq a \leq 0.5 \\ -a + 2 - \frac{1}{2a} & 0.5 \leq a \leq 1 \end{cases} \tag{15}
$$

2.4 Maximum value of function $P(a)$ when $a \in [0, 1]$

Function $P(a)$ has following properties:

1. In the domain $\{a \mid 0 \leq a \leq 0.5\}$, $P(a)$ is linearly increasing with maximum value of 0.5 at $a = 0.5$.
2. At $a = 0.5$, the first derivative values of $P(a)$ from left and right are the same and take a positive value.
$$\lim_{a \to 0.5^-} P'(a) = \lim_{a \to 0.5^+} P'(a) = 1 > 0$$
3. In the domain $\{a \mid 0.5 \leq a \leq 1.0\}$, $P(0.5) = P(1.0) = 0.5$.

So, piecewise-defined function $P(a)$ takes maximum at the second piece (domain), i.e., $\{a \mid 0.5 \leq a \leq 1.0\}$.

The first derivative function (w.r.t a) of $P(a)$ in the domain $\{a \mid 0.5 \leq a \leq 1.0\}$ is:

$$P'(a) = -1 + \frac{1}{2a^2}. \tag{16}$$

$P(a)$ takes extreme value when $P'(a) = 0$:

$$\begin{aligned} P'(a) = 0 &\Rightarrow -1 + \frac{1}{2a^2} = 0 \\ &\Rightarrow \frac{1}{2a^2} = 1 \Rightarrow a^2 = \frac{1}{2} \\ &\Rightarrow a = \frac{1}{\sqrt{2}}. \end{aligned}$$

We confirm that $a = \frac{1}{\sqrt{2}} \approx 0.71 \in [0.5, 1.0]$, and also:

$$P''(a) = -\frac{1}{a^3} \Rightarrow P''(a)\big|_{a=\frac{1}{\sqrt{2}}} = -2\sqrt{2} < 0,$$

so $P(a = \frac{1}{\sqrt{2}})$ is a maximum extreme value, with the maximum value of $P(a)$ is (from equation (14)):

$$P(a = \frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} + 2 - \frac{1}{2 \times \frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{2} + 2 - \frac{\sqrt{2}}{2} = 2 - \sqrt{2}, \tag{17}$$

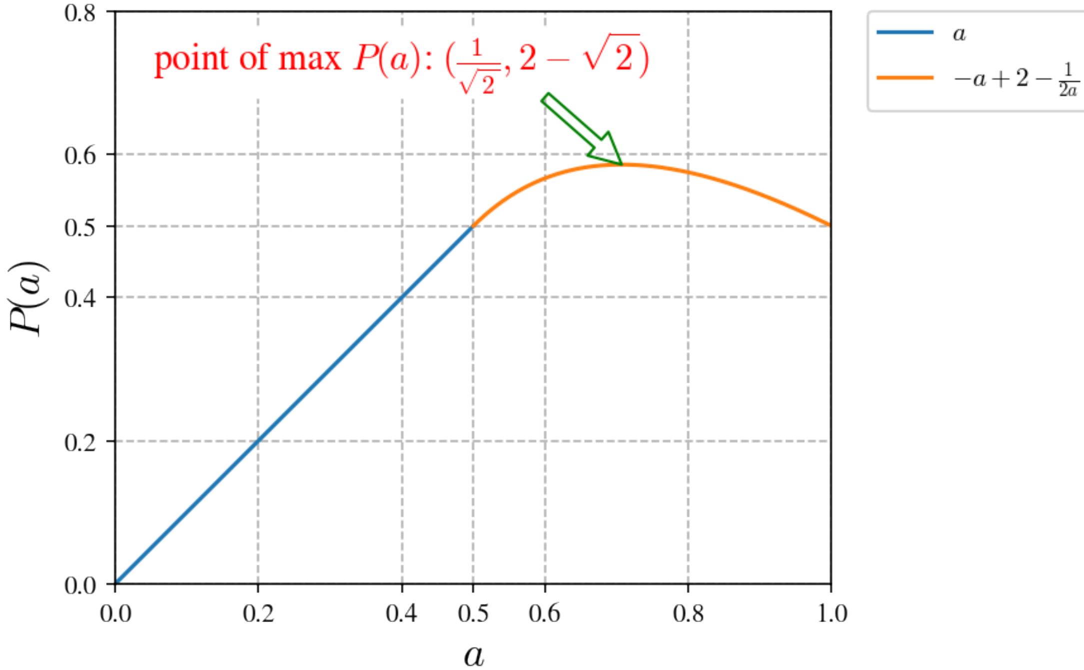
which is B. ■

3 Bonus

3.1 $P(a)$ function plot

Plot of function $P(a)$ as in equation (15):

$$P(a) = \begin{cases} a & 0 \leq a \leq 0.5 \\ -a + 2 - \frac{1}{2a} & 0.5 \leq a \leq 1 \end{cases}$$


point of max $P(a)$: $(\frac{1}{\sqrt{2}}, 2-\sqrt{2})$
a
$-a+2-\frac{1}{2a}$
0.8
0.6
0.5
0.4
0.2
0.0
$P(a)$
0.0
0.2
0.4
0.5
0.6
0.8
1.0
a