What is the maximum value of P(a)?

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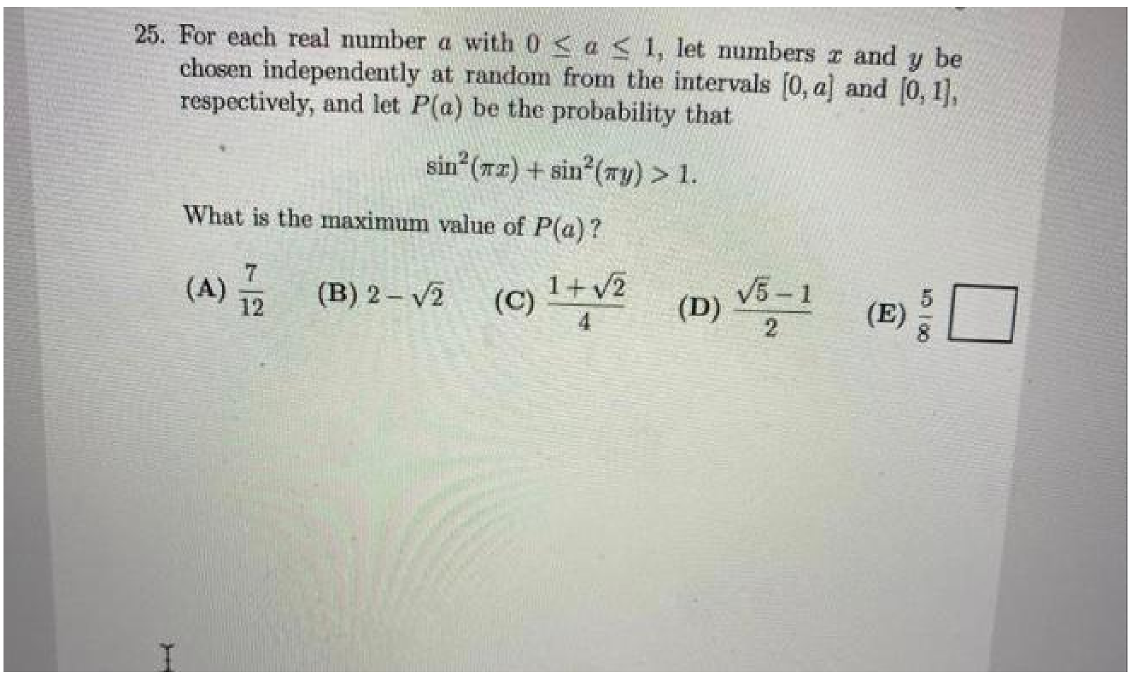
February 14, 2020

1 Problem



25. For each real number a with $0 \le a \le 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$$\sin^2(\pi x) + \sin^2(\pi y) > 1.$$

What is the maximum value of $P(a)$?

(A) $\frac{7}{12}$ **(B)** $2 - \sqrt{2}$ **(C)** $\frac{1+\sqrt{2}}{4}$ **(D)** $\frac{\sqrt{5}-1}{2}$ **(E)** $\frac{5}{8}$

2 Solution

Let Prob{**e**} be the probability of an event **e** occurs. Thus $P(a)$ is:

$$P(a) = \text{Prob}\{\sin^2(\pi \text{x}) + \sin^2(\pi \text{y}) > 1\}, \tag{1}$$

where x and y be chosen independently at uniformly random from the interval $[0, a]$ and $[0, 1]$, respectively.

Because

$$\begin{aligned}
\sin^2(\pi x) + \sin^2(\pi y) > 1 &\Rightarrow \frac{1-\cos(2\pi x)}{2} + \frac{1-\cos(2\pi y)}{2} > 1 \qquad \text{(pakai rumus } \cos(2\alpha)\text{)} \\
&\Rightarrow 2 - [\cos(2\pi x) + \cos(2\pi y)] > 2 \\
&\Rightarrow -[\cos(2\pi x) + \cos(2\pi y)] > 0 \\
&\Rightarrow \cos(2\pi x) + \cos(2\pi y) < 0 \\
&\Rightarrow 2\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0 \quad \text{(pakai rumus } \cos\alpha + \cos\beta\text{)} \\
&\Rightarrow \cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0,
\end{aligned}$$

then, the above event probabilty $P(a)$ in equation (1) is identical to:

$$P(a) = \text{Prob}\{\cos(\pi(\mathrm{x}+\mathrm{y})) \times \cos(\pi(\mathrm{x}-\mathrm{y})) < 0\}. \tag{2}$$

As the first step, we need to find the location of points (x, y) inside the given area, that satisfy $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$, that is:

$$\{(x,y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0) \cap (0 \le x \le a) \cap (0 \le y \le 1)\}. \tag{3}$$

There are 2 cases that satisfy $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$, that is, $\cos(\pi(x+y)) > 0$ AND $\cos(\pi(x-y)) < 0$ as the first case, OR $\cos(\pi(x+y)) < 0$ AND $\cos(\pi(x-y)) > 0$ as the second case. So, equation (3) can be found as the union of the above 2 cases, that is (all inside $((0 \le x \le a) \cap (0 \le y \le 1))$):

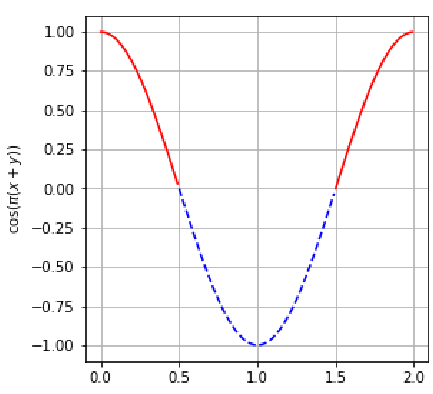
$$\{(x,y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0)\} = \tag{4}$$

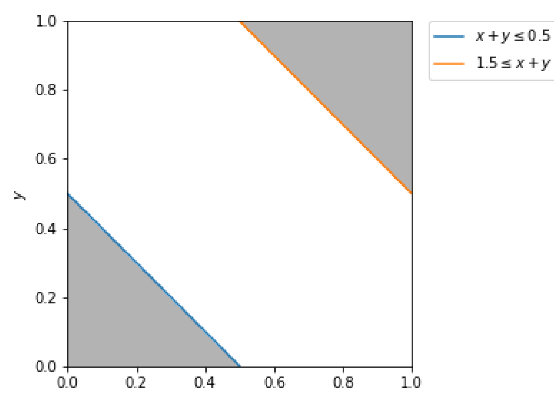
$$\{(x,y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\} \qquad \text{(case 1)} \tag{5}$$

$$\cup\{(x,y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\} \qquad \text{(case 2).} \tag{6}$$

Keeping in mind that $x \in [0, a]$ (with $0 \le a \le 1$) and $y \in [0, 1]$, below we find the regions of (x, y) that satisfy case 1 and case 2.

2.1 Case 1: $\{(x,y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\}$

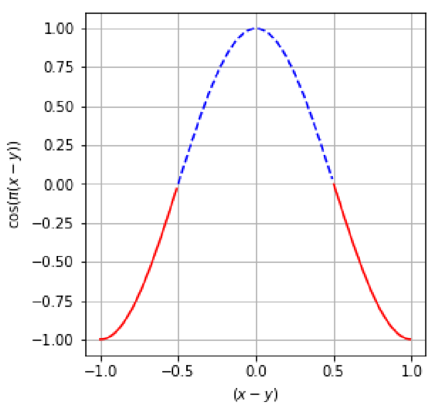


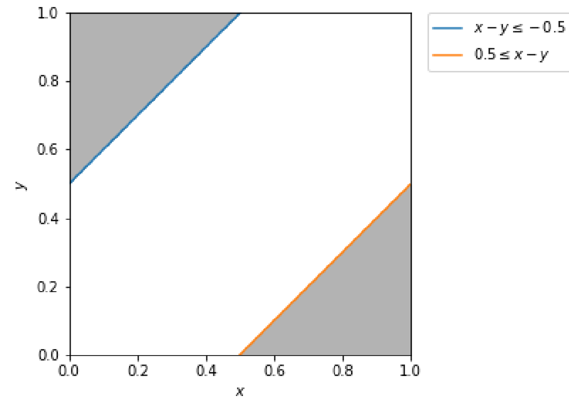


In case 1, the former term of equation (5), that is, $\cos(\pi(x+y))$, will satisfy $(\cos(\pi(x+y)) > 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(0 \le (x+y) \le 0.5) \cup (1.5 \le (x+y) \le 2.0), \tag{7}$$

which is shown in the above-right figure.





Similarly in case 1, the later term of equation (5), that is, $\cos(\pi(x-y))$, will satisfy $(\cos(\pi(x-y)) < 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(-1 \le (x-y) \le -0.5) \cup (0.5 \le (x-y) \le 1), \tag{8}$$

which is shown in the above-right figure.

We see from the figures that the region

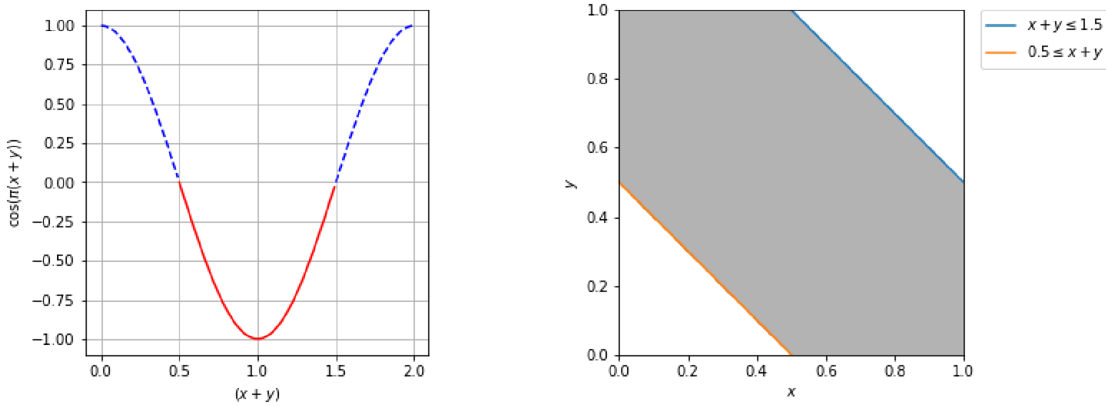
$$(0 \le (x+y) \le 0.5) \cup (1.5 \le (x+y) \le 2.0),$$

and the region

$$(-1 \le (x-y) \le -0.5) \cup (0.5 \le (x-y) \le 1),$$

do NOT overlap, so the region that satisfy case 1 does NOT exist.

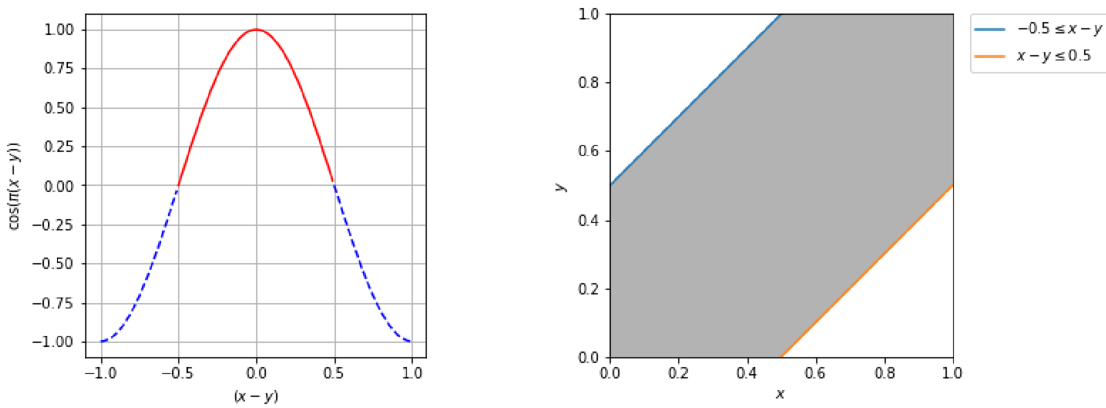
2.2 Case 2: $\{(x,y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\}$



In case 2, the former term of equation (6), that is, $\cos(\pi(x+y))$, will satisfy $(\cos(\pi(x+y)) < 0$ at the above-left red-solid-line cosine plot. So, the region of (x,y) is:

$$(0.5 \leq (x+y) \leq 1.5), \tag{9}$$

which is shown in the above-right figure.



Similarly in case 2, the later term of equation (6), that is, $\cos(\pi(x-y))$, will satisfy $(\cos(\pi(x-y)) > 0$ at the above-left red-solid-line cosine plot. So, the region of (x,y) is:

$$(-0.5 \leq (x-y) \leq 0.5), \tag{10}$$

which is shown in the above-right figure.

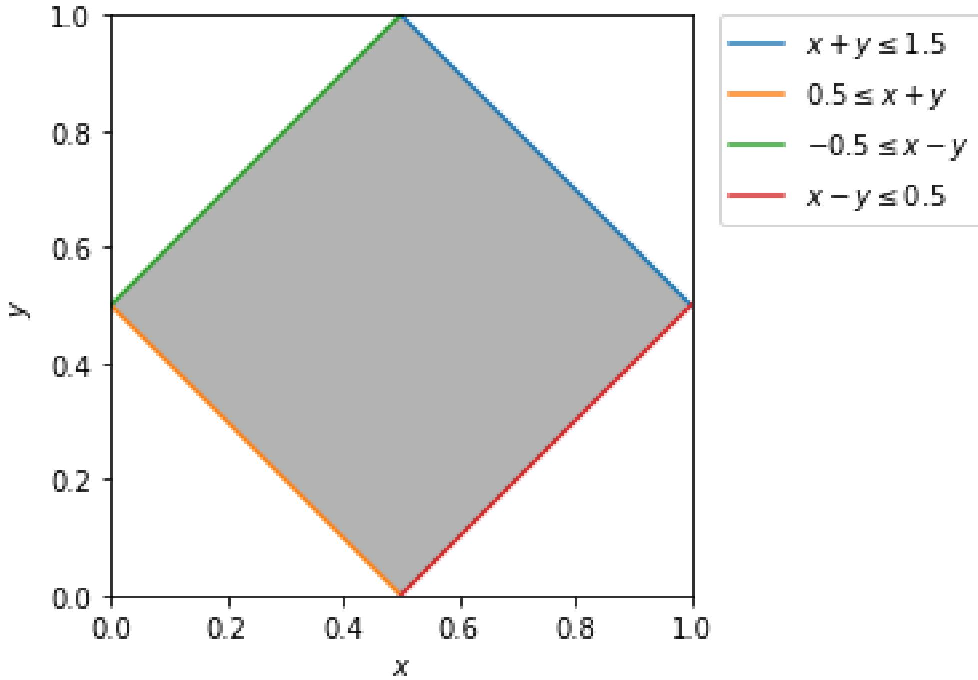
We see from the figures that the region

$$(0.5 \leq (x+y) \leq 1.5),$$

and the region

$$(-0.5 \leq (x-y) \leq 0.5),$$

do overlap, so the region that satisfy case 2 is:



Thus, the above region is the region of (x, y) which satisfy $\{\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0\}$:

$$\{(x, y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0) \cap (0 \leq x \leq 1) \cap (0 \leq y \leq 1)\}, \tag{11}$$

which is also the region of (x, y) which satisfy $\{\sin^2(\pi x) + \sin^2(\pi y) > 1\}$:

$$\{(x, y) \mid (\sin^2(\pi x) + \sin^2(\pi y) > 1) \cap (0 \leq x \leq 1) \cap (0 \leq y \leq 1)\}. \tag{12}$$

2.3 Formulation of function $P(a)$

$P(a)$ is the probability of point (x, y) lies inside the above shown region with additional constraint $\{x \in [0, a]\}$, when x and y is chosen uniform-randomly independently from region $\{(x, y) \mid x \in [0, a], y \in [0, 1]\}$. This probability is the ratio of "the area of the above shown region with additional constraint $\{x \in [0, a]\}$" to "the area of the region $\{(x, y) \mid x \in [0, a], y \in [0, 1]\}$." Hence,

$$P(a) = \frac{\text{Area of the above region } \cap \{0 \leq x \leq a\}}{\text{Area of the region } \{(x, y) \mid x \in [0, a], y \in [0, 1]\}} \tag{13}$$

$$= \frac{\text{Area of the above region } \cap \{0 \leq x \leq a\}}{a}. \tag{14}$$

$$c = \sqrt{a^2 + b^2} \tag{15}$$

$$\nabla \times \vec{\mathbf{B}} - \frac{1}{c} \frac{\partial \vec{\mathbf{E}}}{\partial t} = \frac{4\pi}{c} \vec{\mathbf{j}} \tag{16}$$

$$\nabla \cdot \vec{\mathbf{E}} = 4\pi\rho \tag{17}$$

$$\nabla \times \vec{\mathbf{E}} + \frac{1}{c} \frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\mathbf{0}} \tag{18}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{19}$$