What is the maximum value of P(a)?

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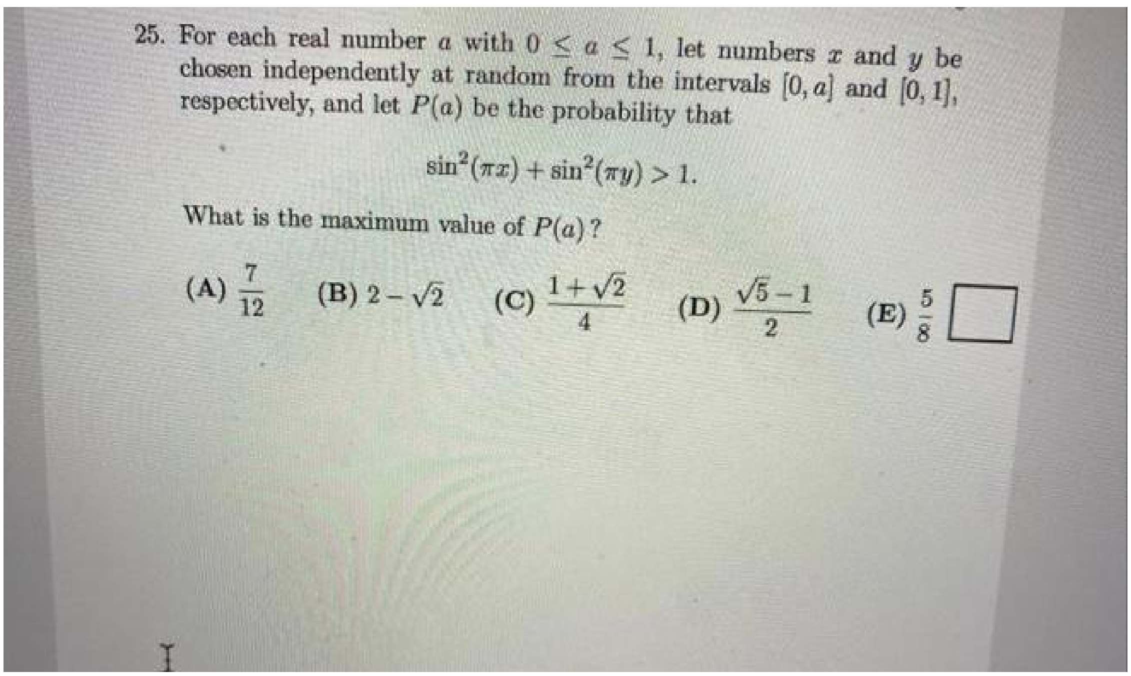
1 Problem



25. For each real number a with $0 \leq a \leq 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$$\sin^2(\pi x) + \sin^2(\pi y) > 1.$$

What is the maximum value of $P(a)$?

(A) $\frac{7}{12}$ **(B)** $2 - \sqrt{2}$ **(C)** $\frac{1+\sqrt{2}}{4}$ **(D)** $\frac{\sqrt{5}-1}{2}$ **(E)** $\frac{5}{8}$ $\square$

2 Solution

Let Prob{**e**} be the probability of an event **e** occurs. Thus $P(a)$ is:

$$P(a) = \text{Prob}\{\sin^2(\pi \text{x}) + \sin^2(\pi \text{y}) > 1\}, \tag{1}$$

where x and y be chosen independently at uniformly random from the interval $[0, a]$ and $[0, 1]$, respectively.

Because

$$\begin{aligned}
\sin^2(\pi x) + \sin^2(\pi y) > 1 &\Rightarrow \frac{1-\cos(2\pi x)}{2} + \frac{1-\cos(2\pi y)}{2} > 1 && \text{(pakai rumus } \cos(2\alpha)\text{)} \\
&\Rightarrow 2 - [\cos(2\pi x) + \cos(2\pi y)] > 2 \\
&\Rightarrow -[\cos(2\pi x) + \cos(2\pi y)] > 0 \\
&\Rightarrow \cos(2\pi x) + \cos(2\pi y) < 0 \\
&\Rightarrow 2\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0 && \text{(pakai rumus } \cos\alpha + \cos\beta\text{)} \\
&\Rightarrow \cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0,
\end{aligned}$$

then, the above event probabilty $P(a)$ in equation (1) is identical to:

$$P(a) = \mathrm{Prob}\{\cos(\pi(\mathrm{x}+\mathrm{y})) \times \cos(\pi(\mathrm{x}-\mathrm{y})) < 0\}. \tag{2}$$

As the first step, we need to find the location of points (x, y) inside the given area, that satisfy $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$, that is:

$$\{(x,y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0) \cap (0 \le x \le a) \cap (0 \le y \le 1)\}. \tag{3}$$

There are 2 cases that satisfy $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$, that is, $\cos(\pi(x+y)) > 0$ AND $\cos(\pi(x-y)) < 0$ as the first case, OR $\cos(\pi(x+y)) < 0$ AND $\cos(\pi(x-y)) > 0$ as the second case. So, equation (3) can be found as the union of the above 2 cases, that is (all inside $((0 \le x \le a) \cap (0 \le y \le 1))$):

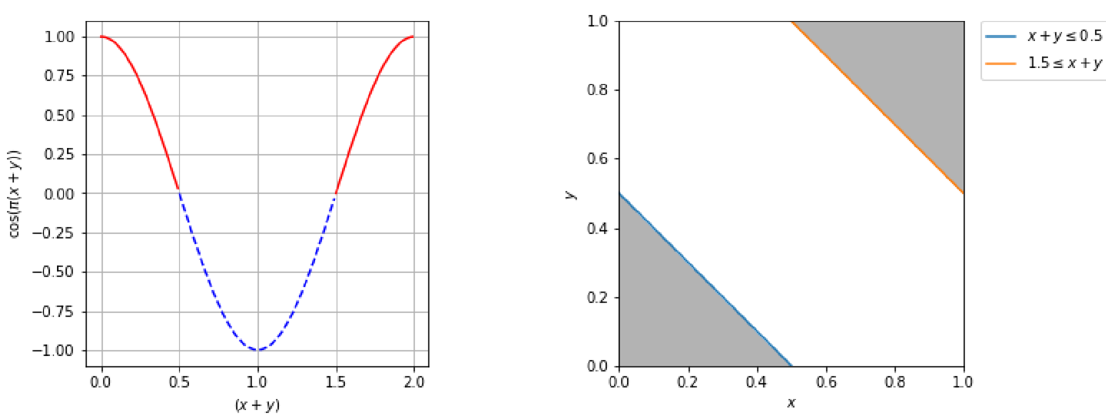
$$\{(x,y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0)\} = \tag{4}$$

$$\{(x,y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\} \qquad \text{(case 1)} \tag{5}$$

$$\cup\{(x,y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\} \qquad \text{(case 2).} \tag{6}$$

Keeping in mind that $x \in [0, a]$ (with $0 \le a \le 1$) and $y \in [0, 1]$, below we find the regions of (x, y) that satisfy case 1 and case 2.

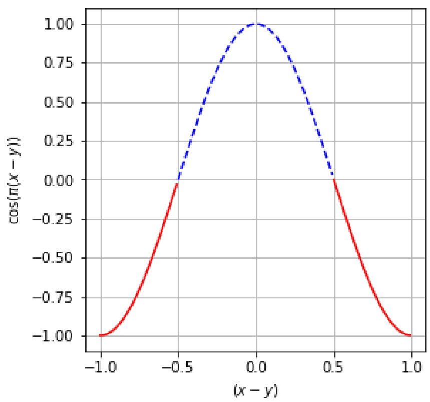
2.1 Case 1: $\{(x,y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\}$

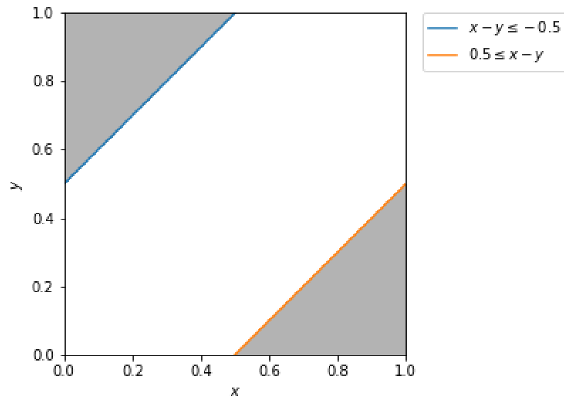


In case 1, the former term of equation (5), that is, $\cos(\pi(x+y))$, will satisfy $(\cos(\pi(x+y)) > 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(0 \leq (x+y) \leq 0.5) \cup (1.5 \leq (x+y) \leq 2.0), \tag{7}$$

which is shown in the above-right figure.





Similarly in case 1, the later term of equation (5), that is, $\cos(\pi(x-y))$, will satisfy $(\cos(\pi(x-y)) < 0$ at the above-left red-solid-line cosine plot. So, the region of (x, y) is:

$$(-1 \leq (x-y) \leq -0.5) \cup (0.5 \leq (x-y) \leq 1), \tag{8}$$

which is shown in the above-right figure.

We see from the figures that the region

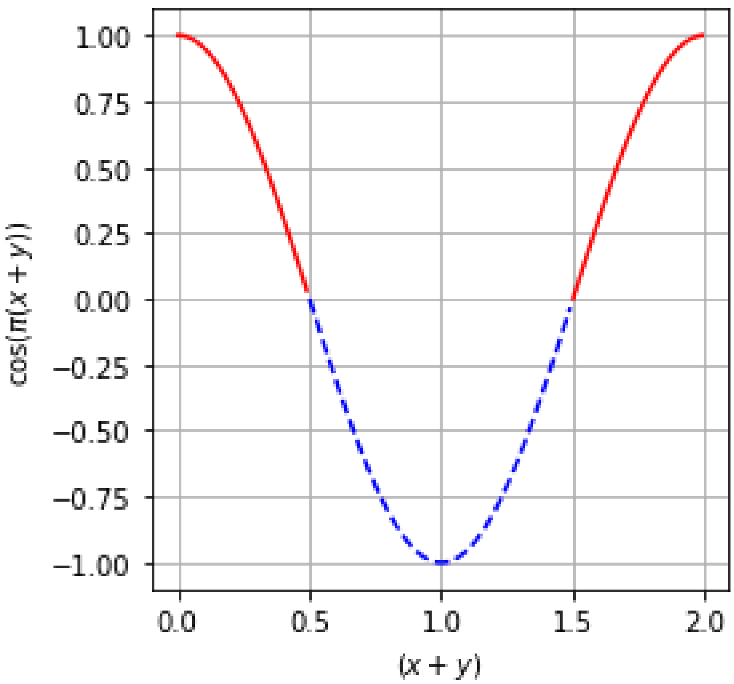
$$(0 \leq (x+y) \leq 0.5) \cup (1.5 \leq (x+y) \leq 2.0),$$

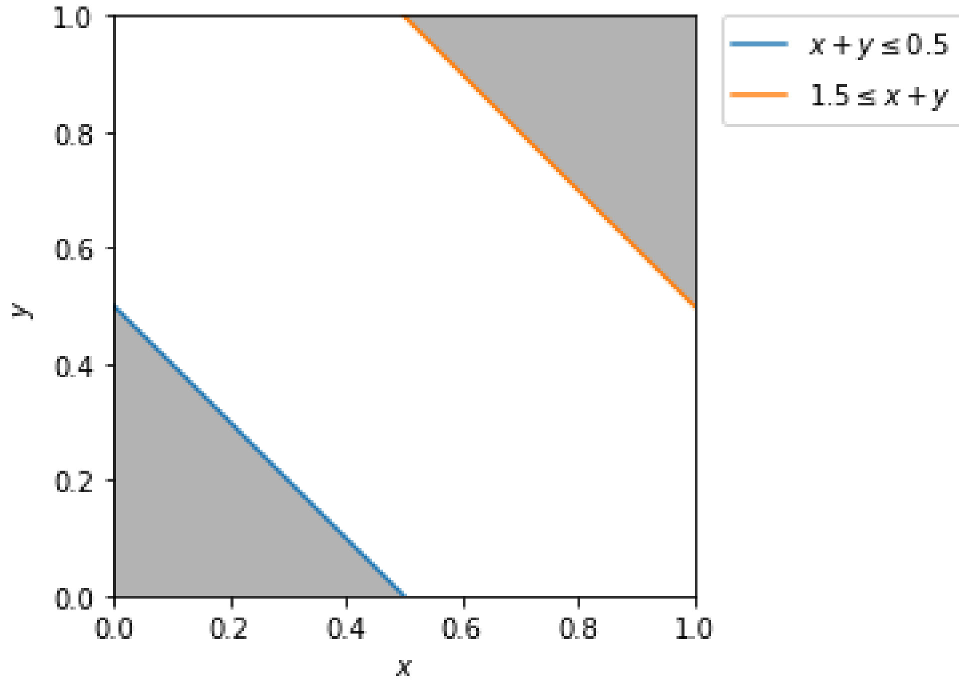
and the region

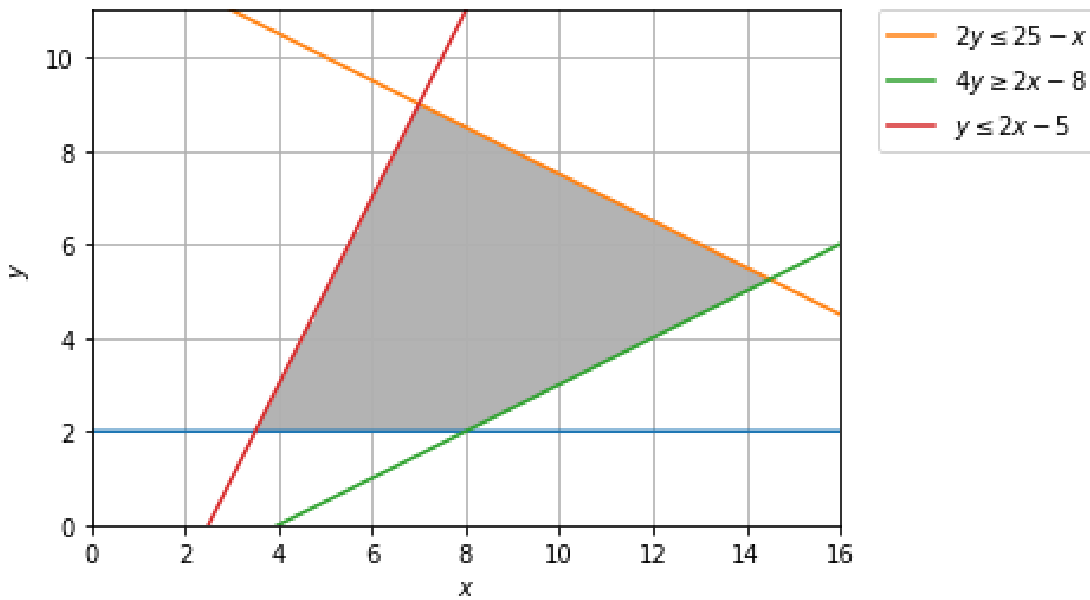
$$(-1 \leq (x-y) \leq -0.5) \cup (0.5 \leq (x-y) \leq 1),$$

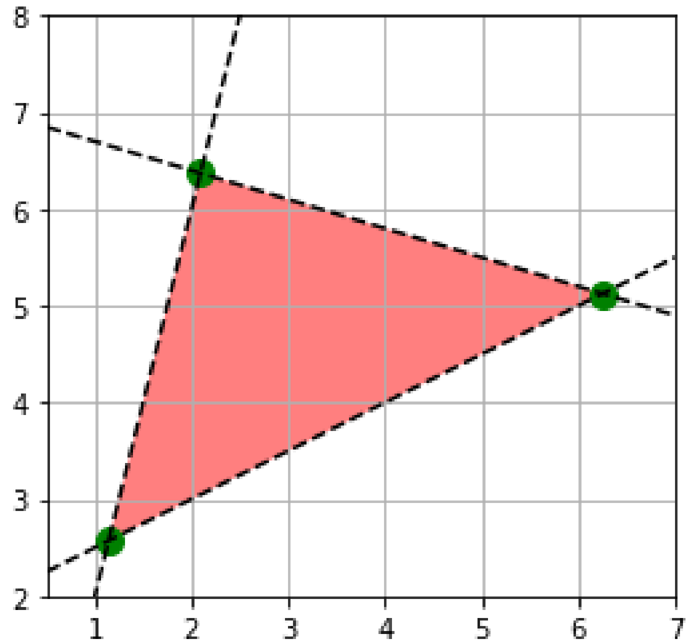
do NOT overlap, so the region that satisfy case 1 does NOT exist.

2.2 Case 2: $\{(x, y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\}$






$2y \leq 25 - x$
$4y \geq 2x - 8$
$y \leq 2x - 5$
x
y



$$c = \sqrt{a^2 + b^2} \tag{9}$$

$$\nabla \times \vec{\mathbf{B}} - \frac{1}{c}\frac{\partial \vec{\mathbf{E}}}{\partial t} = \frac{4\pi}{c}\vec{\mathbf{j}} \tag{10}$$

$$\nabla \cdot \vec{\mathbf{E}} = 4\pi\rho \tag{11}$$

$$\nabla \times \vec{\mathbf{E}} + \frac{1}{c}\frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\mathbf{0}} \tag{12}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{13}$$