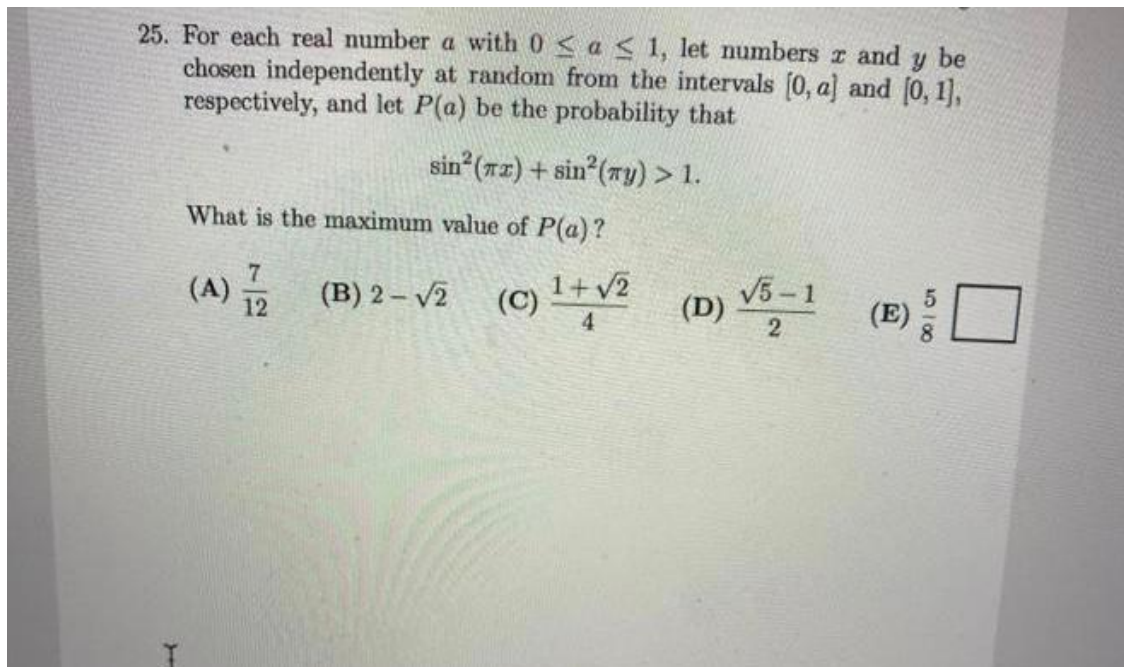


What is the maximum value of  $P(a)$ ?

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## 1 Problem



## 2 Solution

Let  $\text{Prob}\{\mathbf{e}\}$  be the probability of an event  $\mathbf{e}$  occurs. Thus  $P(a)$  is:

$$P(a) = \text{Prob}\{\sin^2(\pi x) + \sin^2(\pi y) > 1\}, \quad (1)$$

where  $x$  and  $y$  be chosen independently at uniformly random from the interval  $[0, a]$  and  $[0, 1]$ , respectively.

Because

$$\begin{aligned}
\sin^2(\pi x) + \sin^2(\pi y) > 1 &\Rightarrow \frac{1 - \cos(2\pi x)}{2} + \frac{1 - \cos(2\pi y)}{2} > 1 \quad (\text{pakai rumus } \cos(2\alpha)) \\
&\Rightarrow 2 - [\cos(2\pi x) + \cos(2\pi y)] > 2 \\
&\Rightarrow -[\cos(2\pi x) + \cos(2\pi y)] > 0 \\
&\Rightarrow \cos(2\pi x) + \cos(2\pi y) < 0 \\
&\Rightarrow 2\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0 \quad (\text{pakai rumus } \cos \alpha + \cos \beta) \\
&\Rightarrow \cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0,
\end{aligned}$$

then, the above event probabily  $P(a)$  in equation (1) is identical to:

$$P(a) = \text{Prob}\{\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0\}. \quad (2)$$

As the first step, we need to find the location of points  $(x, y)$  inside the given area, that satisfy  $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$ , that is:

$$\{(x, y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0) \cap (0 \leq x \leq a) \cap (0 \leq y \leq 1)\}. \quad (3)$$

There are 2 cases that satisfy  $\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0$ , that is,  $\cos(\pi(x+y)) > 0$  AND  $\cos(\pi(x-y)) < 0$  as the first case, OR  $\cos(\pi(x+y)) < 0$  AND  $\cos(\pi(x-y)) > 0$  as the second case. So, equation (3) can be found as the union of the above 2 cases, that is (all inside  $((0 \leq x \leq a) \cap (0 \leq y \leq 1))$ ):

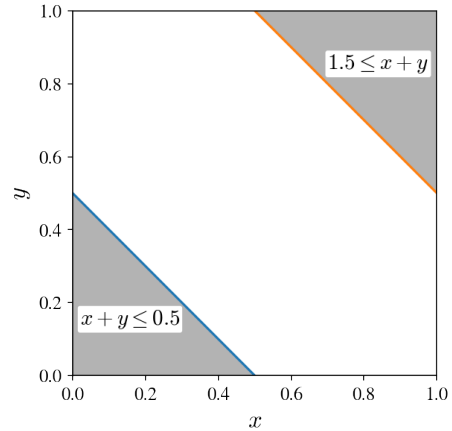
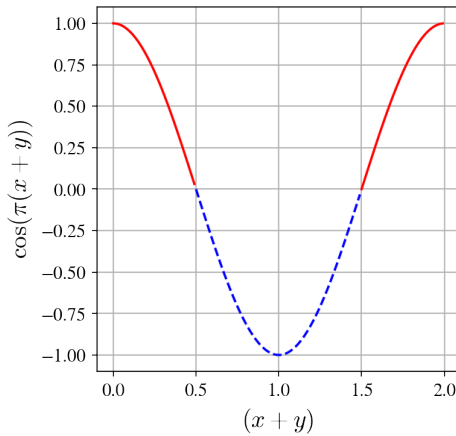
$$\{(x, y) \mid (\cos(\pi(x+y)) \times \cos(\pi(x-y)) < 0)\} = \quad (4)$$

$$\{(x, y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\} \quad (\text{case 1}) \quad (5)$$

$$\cup \{(x, y) \mid (\cos(\pi(x+y)) < 0) \cap (\cos(\pi(x-y)) > 0)\} \quad (\text{case 2}). \quad (6)$$

Keeping in mind that  $x \in [0, a]$  (with  $0 \leq a \leq 1$ ) and  $y \in [0, 1]$ , below we find the regions of  $(x, y)$  that satisfy case 1 and case 2.

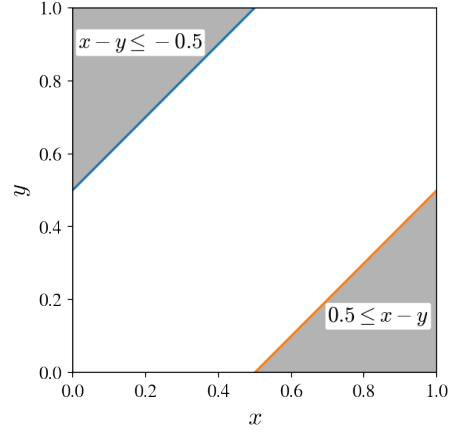
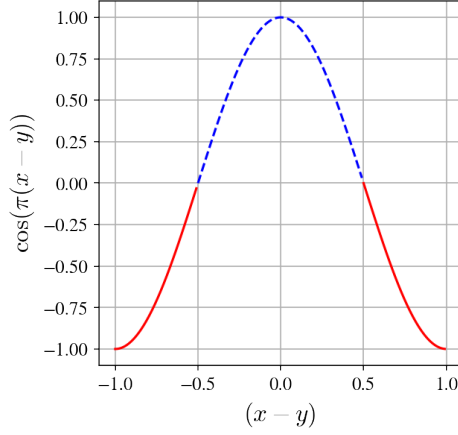
## 2.1 Case 1: $\{(x, y) \mid (\cos(\pi(x+y)) > 0) \cap (\cos(\pi(x-y)) < 0)\}$



In case 1, the former term of equation (5), that is,  $\cos(\pi(x+y))$ , will satisfy  $(\cos(\pi(x+y)) > 0$  at the above-left red-solid-line cosine plot. So, the region of  $(x, y)$  is:

$$(0 \leq (x+y) \leq 0.5) \cup (1.5 \leq (x+y) \leq 2.0), \quad (7)$$

which is shown in the above-right figure.



Similarly in case 1, the later term of equation (5), that is,  $\cos(\pi(x-y))$ , will satisfy  $(\cos(\pi(x-y)) < 0$  at the above-left red-solid-line cosine plot. So, the region of  $(x, y)$  is:

$$(-1 \leq (x-y) \leq -0.5) \cup (0.5 \leq (x-y) \leq 1), \quad (8)$$

which is shown in the above-right figure.

We see from the figures that the region

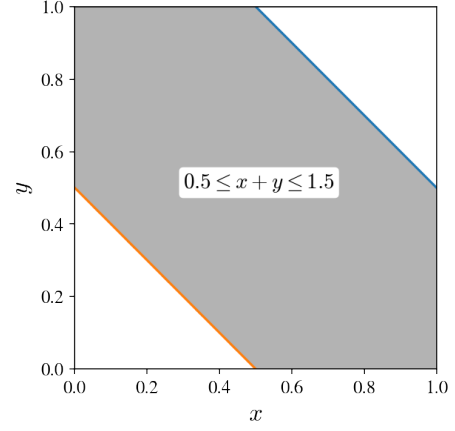
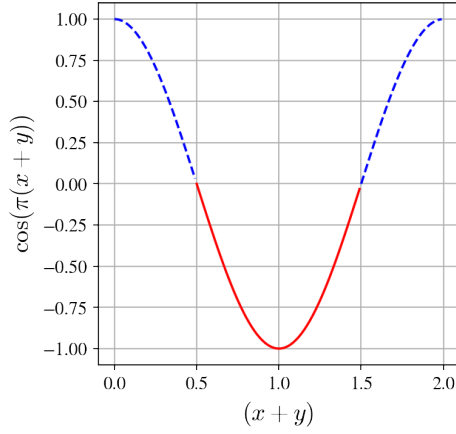
$$(0 \leq (x+y) \leq 0.5) \cup (1.5 \leq (x+y) \leq 2.0),$$

and the region

$$(-1 \leq (x-y) \leq -0.5) \cup (0.5 \leq (x-y) \leq 1),$$

do NOT overlap, so the region that satisfy case 1 does NOT exist.

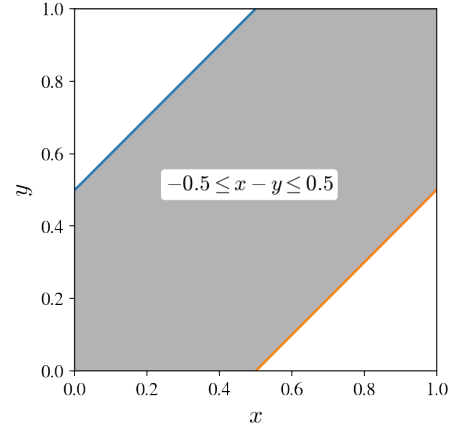
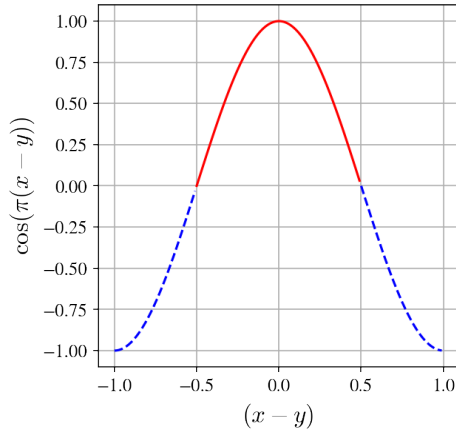
**2.2 Case 2:**  $\{(x, y) \mid (\cos(\pi(x + y)) < 0) \cap (\cos(\pi(x - y)) > 0)\}$



In case 2, the former term of equation (6), that is,  $\cos(\pi(x + y))$ , will satisfy  $(\cos(\pi(x + y)) < 0)$  at the above-left red-solid-line cosine plot. So, the region of  $(x, y)$  is:

$$(0.5 \leq (x + y) \leq 1.5), \quad (9)$$

which is shown in the above-right figure.



Similarly in case 2, the later term of equation (6), that is,  $\cos(\pi(x - y))$ , will satisfy  $(\cos(\pi(x - y)) > 0)$  at the above-left red-solid-line cosine plot. So, the region of  $(x, y)$  is:

$$(-0.5 \leq (x - y) \leq 0.5), \quad (10)$$

which is shown in the above-right figure.

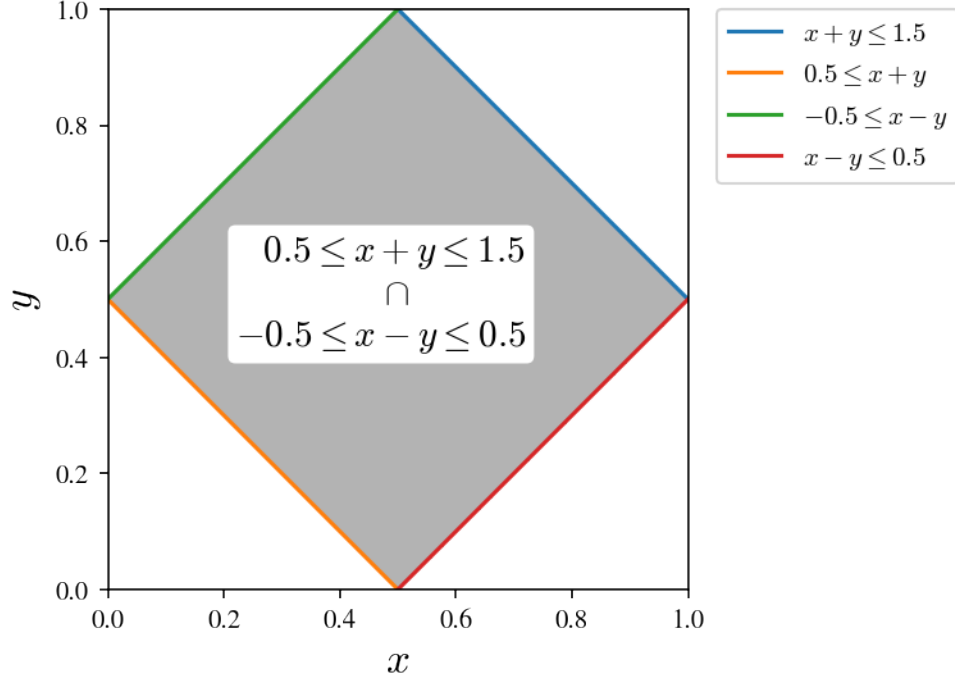
We see from the figures that the region

$$(0.5 \leq (x + y) \leq 1.5),$$

and the region

$$(-0.5 \leq (x - y) \leq 0.5),$$

do overlap, so the region that satisfy case 2 is:



Thus, the above region is the region of  $(x, y)$  which satisfy  $\{\cos(\pi(x + y)) \times \cos(\pi(x - y)) < 0\}$ :

$$\{(x, y) \mid (\cos(\pi(x + y)) \times \cos(\pi(x - y)) < 0) \cap (0 \leq x \leq 1) \cap (0 \leq y \leq 1)\}, \quad (11)$$

which is also the region of  $(x, y)$  which satisfy  $\{\sin^2(\pi x) + \sin^2(\pi y) > 1\}$ :

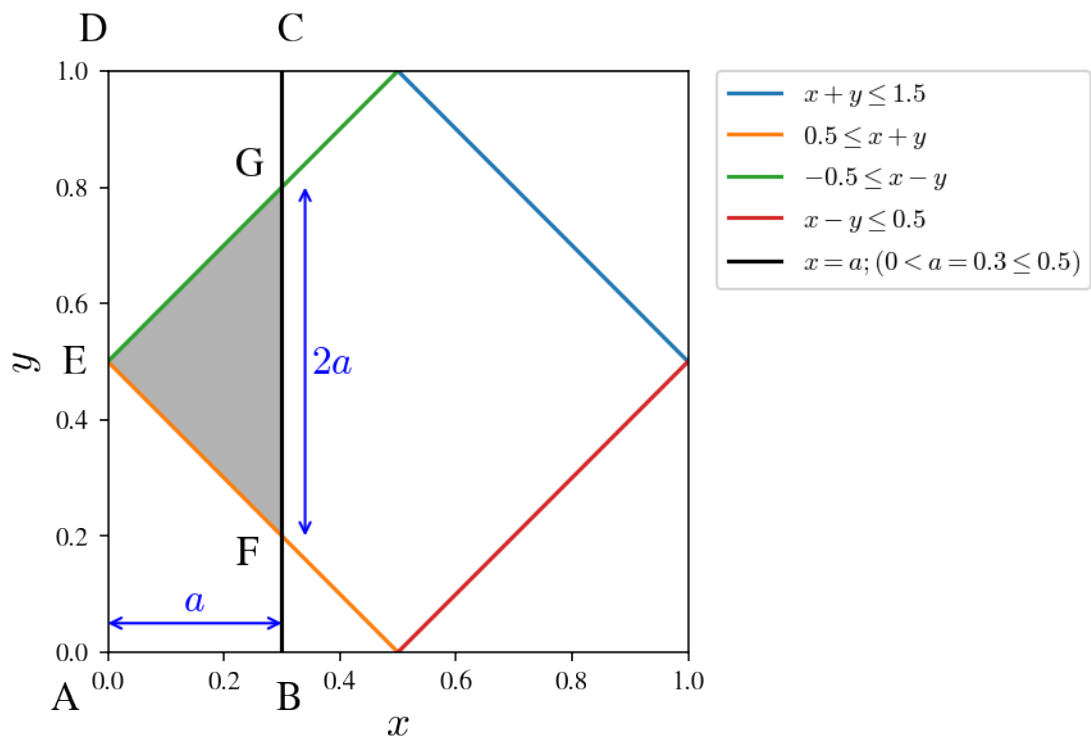
$$\{(x, y) \mid (\sin^2(\pi x) + \sin^2(\pi y) > 1) \cap (0 \leq x \leq 1) \cap (0 \leq y \leq 1)\}. \quad (12)$$

### 2.3 Formulation of function $P(a)$

$P(a)$  is the probability of point  $(x, y)$  lies inside the above shown region with additional constraint  $\{x \in [0, a]\}$ , when  $x$  and  $y$  is chosen uniform-randomly independently from region  $\{(x, y) \mid x \in [0, a], y \in [0, 1]\}$ . This probability is the ratio of "the area of the above shown region with additional constraint  $\{x \in [0, a]\}$ " to "the area of the region  $\{(x, y) \mid x \in [0, a], y \in [0, 1]\}$ ." Hence,

$$\begin{aligned} P(a) &= \frac{\text{Area of the above region} \cap \{0 \leq x \leq a\}}{\text{Area of the region } \{(x, y) \mid x \in [0, a], y \in [0, 1]\}} \\ &= \frac{\text{Area of the above region} \cap \{0 \leq x \leq a\}}{a}. \end{aligned}$$

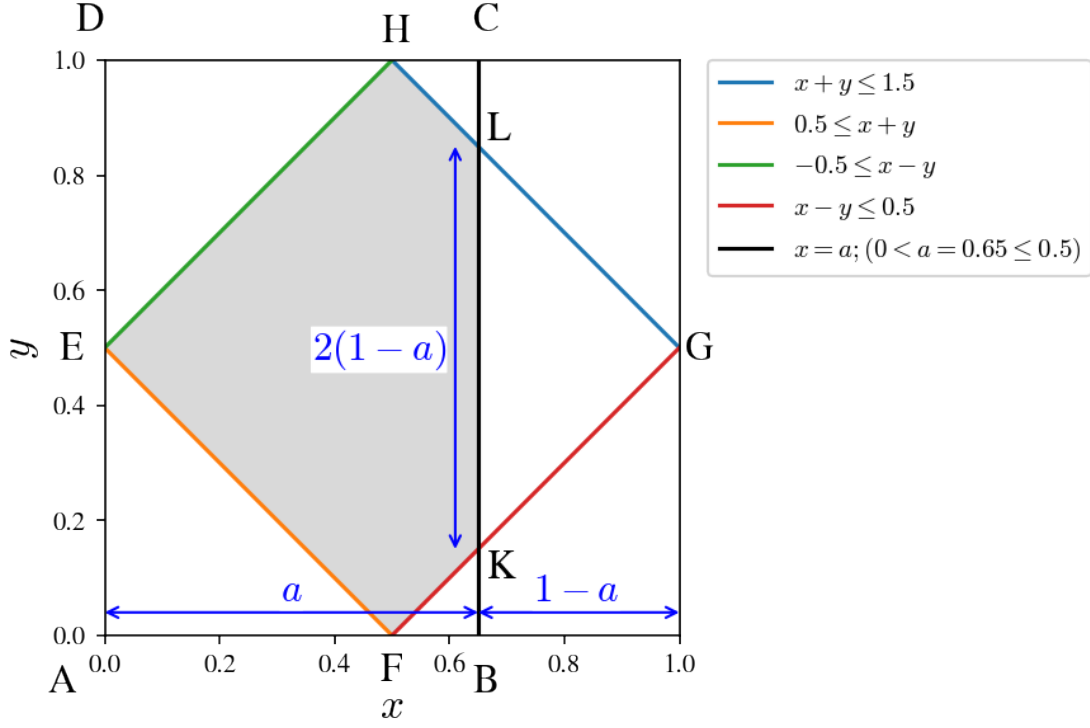
### 2.3.1 Function $P(a)$ when $0 < a \leq 0.5$



From the above figure, when  $0 < a \leq 0.5$ , then  $P(a)$  is:

$$\begin{aligned}
 P(a) &= \frac{\text{Area of the shaded region, that is, area of } \triangle EFG}{\text{Area of rectangle } ABCD} \\
 &= \frac{\frac{1}{2} \times 2a \times a}{a \times 1.0} \\
 &= a.
 \end{aligned} \tag{13}$$

### 2.3.2 Function $P(a)$ when $0.5 < a \leq 1.0$



From the above figure, when  $0.5 < a \leq 1$ , then  $P(a)$  is:

$$\begin{aligned}
 P(a) &= \frac{\text{Area of the shaded region, that is, area of } \square EFGH - \text{area of } \triangle GLK}{\text{Area of rectangle } ABCD} \\
 &= \frac{2 \times (\text{area of } \triangle EGH) - \text{area of } \triangle GLK}{\text{Area of rectangle } ABCD} \\
 &= \frac{2 \times (\frac{1}{2} \times 1.0 \times 0.5) - \frac{1}{2} \times 2(1-a) \times (1-a)}{a \times 1.0} \\
 &= \frac{\frac{1}{2} - (1 - 2a + a^2)}{a} \\
 &= \frac{-\frac{1}{2} + 2a - a^2}{a} \\
 &= -a + 2 - \frac{1}{2a}.
 \end{aligned} \tag{14}$$

### 2.3.3 Final form of function $P(a)$

Noticing that  $P(a = 0) = 0$  is satisfied in equation (13), and noticing that  $P(a = 0.5) = 0.5$  is satisfied in both equation (13) and (14), then  $P(a)$  can be expressed as a piecewise-defined function as follows:

$$P(a) = \begin{cases} a & 0 \leq a \leq 0.5 \\ -a + 2 - \frac{1}{2a} & 0.5 \leq a \leq 1 \end{cases} \tag{15}$$

## 2.4 Maximum value of function $P(a)$ when $a \in [0, 1]$

Function  $P(a)$  has following properties:

1. In the domain  $\{a \mid 0 \leq a \leq 0.5\}$ ,  $P(a)$  is linearly increasing with maximum value of 0.5 at  $a = 0.5$ .
2. At  $a = 0.5$ , the first derivative values of  $P(a)$  from left and right are the same and take a positive value.

$$\lim_{a \rightarrow 0.5^-} P'(a) = \lim_{a \rightarrow 0.5^+} P'(a) = 1 > 0$$

3. In the domain  $\{a \mid 0.5 \leq a \leq 1.0\}$ ,  $P(a)$  is a continuous function and  $P(0.5) = P(1.0) = 0.5$ .

So, piecewise-defined function  $P(a)$  takes maximum at the second piece (domain), i.e.,  $\{a \mid 0.5 \leq a \leq 1.0\}$ .

The first derivative function (w.r.t  $a$ ) of  $P(a)$  in the domain  $\{a \mid 0.5 \leq a \leq 1.0\}$  is:

$$P'(a) = -1 + \frac{1}{2a^2}. \quad (16)$$

$P(a)$  takes extreme value when  $P'(a) = 0$ :

$$\begin{aligned} P'(a) = 0 &\Rightarrow -1 + \frac{1}{2a^2} = 0 \\ &\Rightarrow \frac{1}{2a^2} = 1 \Rightarrow a^2 = \frac{1}{2} \\ &\Rightarrow a = \frac{1}{\sqrt{2}}. \end{aligned}$$

We confirm that  $a = \frac{1}{\sqrt{2}} \approx 0.71 \in [0.5, 1.0]$ , and also:

$$P''(a) = -\frac{1}{a^3} \Rightarrow P''(a)|_{a=\frac{1}{\sqrt{2}}} = -2\sqrt{2} < 0,$$

so  $P(a = \frac{1}{\sqrt{2}})$  is a maximum extreme value, with the maximum value of  $P(a)$  is (from equation (14)):

$$P(a = \frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} + 2 - \frac{1}{2 \times \frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{2} + 2 - \frac{\sqrt{2}}{2} = 2 - \sqrt{2}, \quad (17)$$

which is B. ■

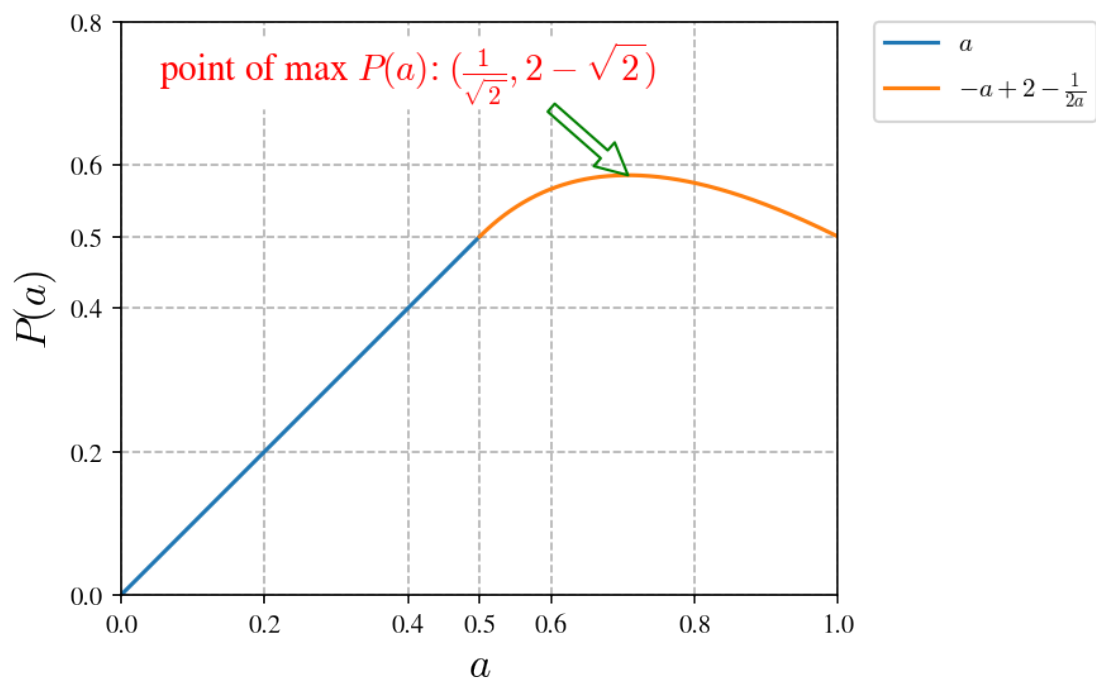
## 3 Bonus

### 3.1 $P(a)$ function plot

Plot of function  $P(a)$  as in equation (15):

$$P(a) = \begin{cases} a & 0 \leq a \leq 0.5 \\ -a + 2 - \frac{1}{2a} & 0.5 \leq a \leq 1 \end{cases}$$





### 3.2 Function $P(a)$ animated plot

[29]: <IPython.core.display.HTML object>

