More on Fiedler vector

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Outline

- Graph model
- Partitioning functions
- Laplacian and normalized Laplacian
- Discrete formulation of a partitioning problem
- Relaxation of a discrete problem
- Bipartitioning algorithm
- Example

Model

$G = (V, B)$ is a simple, finite, undirected, weighted graph where:

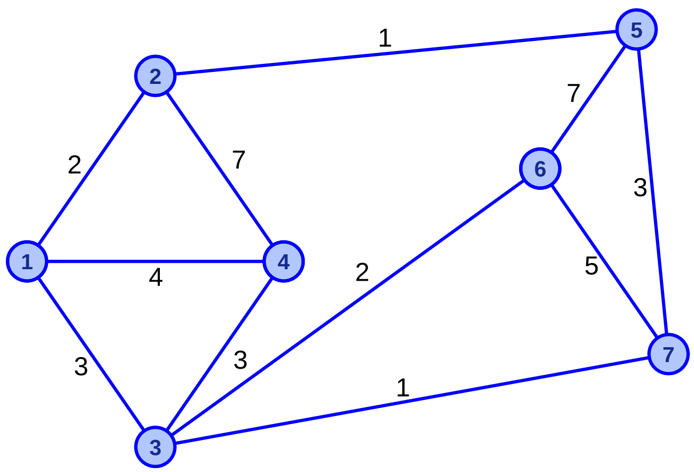
$V = \{1, 2, 3, ..., n\}$ is a set of nodes and

B is a set of edges $\{i, j\}$, $i, j \in V$, with weights $t(\{i, j\}) \in \mathbb{R}^+$.

The neighborhood matrix of G is a $n \times n$ matrix $W = [w_{ij}]$, s.t.

$$w_{ij} = \begin{cases} t(\{i, j\}), & \text{if } \{i, j\} \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Example (G_{small})



$$W = \begin{bmatrix} 0 & 2 & 3 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 7 & 1 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 & 2 & 1 \\ 4 & 7 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 7 & 3 \\ 0 & 0 & 2 & 0 & 7 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 & 5 & 0 \end{bmatrix}$$

Cut of the partition

Let $V_1, V_2 \subset V$, $V_1, V_2 \neq \emptyset$. We define

$$\mathrm{cut}(V_1, V_2) = \sum_{i \in V_1, j \in V_2} w_{ij},$$

$$t\,(i) = \sum_{j=1}^{n} w_{ij}$$

(weight of node i = weights of all edges incident to it)

$$t(V_l) = \sum_{i \in V_l} t(i) = \sum_{i \in V_l} \sum_{j \in V} w_{ij} = \mathrm{cut}(V_l, V \backslash V_l) + \mathrm{within}(V_l)$$

Partitioning functions

Proportional cut

$$R(V_1, V_2) = \frac{\mathrm{cut}(V_1, V_2)}{|V_1|} + \frac{\mathrm{cut}(V_1, V_2)}{|V_2|}$$

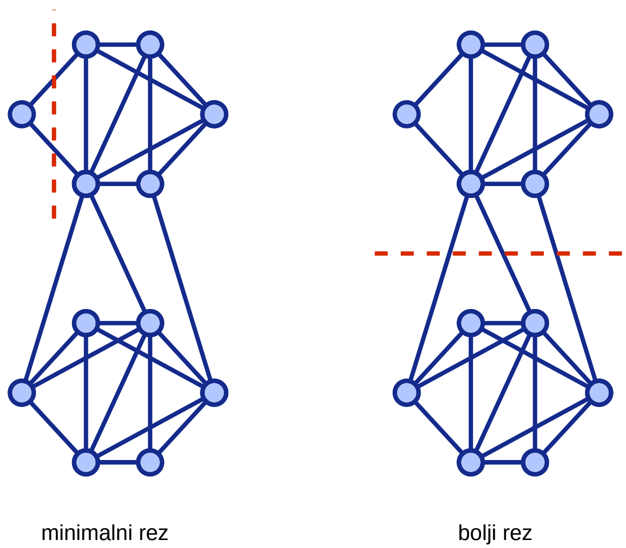
favors partitions into sets with equal number of nodes.

Normalized cut

$$N(V_1, V_2) = \frac{\mathrm{cut}(V_1, V_2)}{t(V_1)} + \frac{\mathrm{cut}(V_1, V_2)}{t(V_2)}$$

maximizes weights of edges within subsets.

Proportional v.s. normalized cut



Left partition:

$$\mathrm{cut}(V_1, V_2) = 2$$
$$R(V_1, V_2) = \frac{2}{1} + \frac{2}{11} = 2.18$$
$$N(V_1, V_2) = \frac{2}{2} + \frac{2}{50} = 1.04$$

Right partition:

$$\mathrm{cut}(V_1', V_2') = 3$$
$$R(V_1', V_2') = \frac{3}{6} + \frac{3}{6} = 1$$
$$N(V_1', V_2') = \frac{3}{27} + \frac{3}{25} = 0.23$$

NP-hard optimization problem

Theorem 1 (Papadimitrou, 1997) *Computing a normalized cut of a graph is NP-hard.*

Number of k-partitions of a set of n elements is given by Stirling number $S(n, k)$:

$$
\begin{aligned}
S(n, 2) &= 2^{n-1} - 1; \\
S(n, k) &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n .
\end{aligned}
$$

100 elements $\rightarrow 6.3383 \cdot 10^{29}$ bipartitions and $6.4176 \cdot 10^{80}$ 7-partitions.

Laplacian

$L = [l_{ij}]$ is a real $n \times n$ matrix, s.t.

$$l_{ij} = \begin{cases} \sum_{k=1}^{n} w_{ik} & , i = j \\ -w_{ij} & , i \neq j \, , \{i, j\} \in B \\ 0 & , \text{ otherwise} \end{cases} .$$

Incidence matrix I_G of G is $|V| \times |B|$ matrix with one row/column for every node/edge.
The column corresponding to the edge $\{i, j\}$ is zero except in the i-th and j-th row, where the elements are $\sqrt{w_{ij}}$ and $-\sqrt{w_{ij}}$.

Laplace matrix and incidence matrix of G_{small}

$$L = \begin{bmatrix} \mathbf{9} & -2 & -3 & -4 & 0 & 0 & 0 \\ -2 & \mathbf{10} & 0 & -7 & -1 & 0 & 0 \\ -3 & 0 & \mathbf{9} & -3 & 0 & -2 & -1 \\ -4 & -7 & -3 & \mathbf{14} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \mathbf{11} & -7 & -3 \\ 0 & 0 & -2 & 0 & -7 & \mathbf{14} & -5 \\ 0 & 0 & -1 & 0 & -3 & -5 & \mathbf{9} \end{bmatrix},$$

$$I_G = \begin{bmatrix} \sqrt{2} & \sqrt{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & \sqrt{7} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & \sqrt{3} & 0 & \sqrt{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -\sqrt{7} & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \sqrt{7} & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & -\sqrt{7} & \sqrt{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -\sqrt{5} & -\sqrt{3} \end{bmatrix}$$

Properties of Laplacian (1)

(i) $L = D - W$, where D is diagonal with $d_{ii} = \sum_{j=1}^{n} w_{ij}$,

(ii) $L = I_G I_G^T$,

(iii) L is symmetric positive semi-definite,

Properties of Laplacian (2)

(iv) $L\mathbf{1} = 0$ for $\mathbf{1} = [1, ..., 1]^T$,

(v) If G has c components, then L has c zero eigenvalues,

(vi) For each $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{x}^T L\mathbf{x} = \sum_{i<j} w_{ij} \left(x_i - x_j\right)^2 ,$$

(vii) For each $\mathbf{x} \in \mathbb{R}^n$ and α,$\beta \in \mathbb{R}$

$$\left(\alpha\mathbf{x} + \beta\mathbf{1}\right)^T L \left(\alpha\mathbf{x} + \beta\mathbf{1}\right) = \alpha^2 \mathbf{x}^T L\mathbf{x}.$$

Normalized Laplacian

$L_n = [l_{n_{ij}}]$ is a $n \times n$ matrix, s.t.

$$l_{n_{ij}} = \begin{cases} 1 & , i = j \\ -\frac{w_{ij}}{\sqrt{d_{ii}}\sqrt{d_{jj}}} & , i \neq j \,, \{i, j\} \in B \\ 0 & , \text{ otherwise} \end{cases} .$$

In other words,

$$L_n = D^{-1/2}(D - W)D^{-1/2}.$$

Normalized Laplacian of G_{small}

$$L_n = \begin{bmatrix} 1 & -\frac{2}{\sqrt{9}\sqrt{10}} & -\frac{3}{\sqrt{9}\sqrt{9}} & -\frac{4}{\sqrt{9}\sqrt{14}} & 0 & 0 & 0 \\ -\frac{2}{\sqrt{9}\sqrt{10}} & 1 & 0 & -\frac{7}{\sqrt{10}\sqrt{14}} & -\frac{1}{\sqrt{10}\sqrt{11}} & 0 & 0 \\ -\frac{3}{\sqrt{9}\sqrt{9}} & 0 & 1 & -\frac{3}{\sqrt{9}\sqrt{14}} & 0 & -\frac{2}{\sqrt{9}\sqrt{14}} & -\frac{1}{\sqrt{9}\sqrt{9}} \\ -\frac{4}{\sqrt{9}\sqrt{14}} & -\frac{7}{\sqrt{10}\sqrt{14}} & -\frac{3}{\sqrt{9}\sqrt{14}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{10}\sqrt{11}} & 0 & 0 & 1 & -\frac{7}{\sqrt{11}\sqrt{14}} & -\frac{3}{\sqrt{11}\sqrt{9}} \\ 0 & 0 & -\frac{2}{\sqrt{9}\sqrt{14}} & 0 & -\frac{7}{\sqrt{11}\sqrt{14}} & 1 & -\frac{5}{\sqrt{14}\sqrt{9}} \\ 0 & 0 & -\frac{1}{\sqrt{9}\sqrt{9}} & 0 & -\frac{3}{\sqrt{11}\sqrt{9}} & -\frac{5}{\sqrt{14}\sqrt{9}} & 1 \end{bmatrix}$$

On spectra of L and L_n

The largest eigenvalue λ_n of L is bounded by

$$\lambda_n \leq 2 \max d_{ii},$$

Spectrum of the normalized Laplacian satisfies

$$\sigma \left(L_n \right) \subseteq [0, 2].$$

Discrete formulation

The partition $\pi = \{V_1, V_2\}$ of V is determined by a vector $\mathbf{y}$ s.t.

$$y_i = \begin{cases} \frac{1}{2}, & i \in V_1 \\ -\frac{1}{2}, & i \in V_2 \end{cases}$$

The proportional cut problem:

$$\min_{\substack{y_i \in \{-\frac{1}{2}, \frac{1}{2}\} \\ |\mathbf{y}^T \mathbf{1}| \leq \beta}} \frac{1}{2} \sum_{i,j} (y_i - y_j)^2 \, w_{ij}$$

Without balancing factor β, the trivial partition minimizes the problem. $2\mathbf{y}^T\mathbf{1}$ measures the difference between $|A|$ and $|B|$. $\beta = \frac{1}{2}$ requires the most even balancing. $\beta = \frac{n}{2}$ allows all bipartitions, including the trivial one.

Relaxation of the problem

The discrete problem:

$$\min_{\substack{y_i \in \{-\frac{1}{2}, \frac{1}{2}\} \\ |\mathbf{y}^T \mathbf{1}| \leq \beta}} \frac{1}{2} \sum_{i,j} (y_i - y_j)^2 w_{ij}$$

The relaxed problem

$$\min_{\substack{y \in \mathbb{R}^n \\ |\mathbf{y}^T \mathbf{1}| \leq \frac{2\beta}{\sqrt{n}} \\ \mathbf{y}^T \mathbf{y} = 1}} \quad \mathbf{y}^T L \mathbf{y}$$

$\mathbf{y}$ needs to be normalized – this is equivalent to $|\mathbf{y}^T \mathbf{1}| \leq \beta$ and $\mathbf{y}^T \mathbf{y} = n/4$.

β is irrelevant for the final partition!

For the normalized cut

The discrete problem is:

$$\min_{\substack{y_i \in \{-\frac{1}{2}, \frac{1}{2}\} \\ |\mathbf{y}^T D\mathbf{1}| \leq \beta}} \frac{1}{2} \sum_{i,j} (y_i - y_j)^2 \, w_{ij}$$

The condition $|\mathbf{y}^T D\mathbf{1}| \leq \beta$ controls the difference between the weights of the two sets. The relaxed problem is:

$$\min_{\substack{y \in \mathbb{R}^n \\ |\mathbf{y}^T D\mathbf{1}| \leq \frac{\beta}{\sqrt{\theta n}} \\ \mathbf{y}^T D\mathbf{y} = 1}} \quad \mathbf{y}^T L\mathbf{y}$$

This is equivalent to $|\mathbf{y}^T D\mathbf{1}| \leq \beta$ and $\mathbf{y}^T D\mathbf{y} = \theta n$, where $\theta > 0$ reduces the influence of nodes with very large or very small weights.

The Theorem

Theorem 2 *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix with eigenvalues* $\lambda_1 < \lambda_2 < \lambda_3 \leq \cdots \leq \lambda_n$ *and eigenvectors* $\mathbf{v}^{[1]}, \mathbf{v}^{[2]}, \ldots, \mathbf{v}^{[n]}$. *For a fixed* $0 \leq \alpha < 1$, *the problem*

$$\min_{\substack{\mathbf{y}\in\mathbb{R}^n \\ |\mathbf{y}^T\mathbf{v}^{[1]}|\leq\alpha \\ \mathbf{y}^T\mathbf{y}=1}} \quad \mathbf{y}^T A\mathbf{y}$$

has the solution $y = \pm\alpha\mathbf{v}^{[1]} \pm \sqrt{1-\alpha^2}\mathbf{v}^{[2]}$.

The Proof (1)

D. J. HIGHAM i M. KIBBLE, *A unified view of spectral clustering*, Mathematic Research Report 2, University of Strathclyde (2004)

Let $A = V\Lambda V^T$ and set $\mathbf{z} = V^T\mathbf{y}$. Then, the problem becomes

$$\min_{\substack{\mathbf{z}\in\mathbb{R}^n \\ |\mathbf{z}^T V^T \mathbf{v}^{[1]}|\le\alpha \\ \mathbf{z}^T\mathbf{z}=1}} \quad \mathbf{z}^T\Lambda\mathbf{z},$$

or

$$\min_{\substack{\mathbf{z}\in\mathbb{R}^n \\ |\mathbf{z}_1|\le\alpha \\ \mathbf{z}^T\mathbf{z}=1}} \sum_{i=1}^{n} \lambda_i z_i^2. \qquad (1)$$

The Proof (2)

From $\sum_{i=1}^{n} z_i^2 = 1$ and $\alpha^2 \geq z_1^2$ we have

$$\begin{aligned}
&\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + \cdots + \lambda_n z_n^2 \\
&= \lambda_1 z_1^2 + \lambda_2(1 - z_1^2 - z_3^2 - \cdots - z_n^2) + \lambda_3 z_3^2 + \cdots + \lambda_n z_n^2 \\
&= (\lambda_1 - \lambda_2) z_1^2 + (\lambda_3 - \lambda_2) z_3^2 + \cdots + (\lambda_n - \lambda_2) z_n^2 + \lambda_2 \geq \\
&\geq (\lambda_1 - \lambda_2)\alpha^2 + (\lambda_3 - \lambda_2) z_3^2 + \cdots + (\lambda_n - \lambda_2) z_n^2 + \lambda_2 \geq \\
&\geq (\lambda_1 - \lambda_2)\alpha^2 + \lambda_2 = \\
&= \alpha^2 \lambda_1 + (1 - \alpha^2)\lambda_2.
\end{aligned}$$

Thus, $z_1 = \pm\alpha$, $z_2 = \pm\sqrt{1 - \alpha^2}$ and $z_i = 0$ for $i > 2$, so

$$\mathbf{y} = V\mathbf{z} = \pm\alpha\mathbf{v}^{[1]} \pm \sqrt{1 - \alpha^2}\mathbf{v}^{[2]}.$$

The solution (1)

Corollary 1 *For* $0 \leq \beta < \frac{n}{2}$ *the relaxed proportional cut problem*

$$\min_{\substack{y \in \mathbb{R}^n \\ |\mathbf{y}^T \mathbf{1}| \leq \frac{2\beta}{\sqrt{n}} \\ \mathbf{y}^T \mathbf{y} = 1}} \quad \mathbf{y}^T L \mathbf{y}$$

has the solution

$$\mathbf{y} = \pm \frac{2\beta}{\sqrt{n}} \mathbf{1} \pm \sqrt{1 - 4\frac{\beta^2}{n^2}} \mathbf{v}^{[2]}.$$

$\mathbf{v}^{[2]}$ is the Fiedler vector of graph G.

The solution (2)

Corollary 2 *For* $0 \leq \beta < \sqrt{\theta n} \left\| D^{\frac{1}{2}} \mathbf{1} \right\|_2$ *the relaxed normalized cut problem*

$$\min_{\substack{y \in \mathbb{R}^n \\ |\mathbf{y}^T D \mathbf{1}| \leq \frac{\beta}{\sqrt{\theta n}} \\ \mathbf{y}^T D \mathbf{y} = 1}} \quad \mathbf{y}^T L \mathbf{y}$$

has the solution

$$\mathbf{y} = \pm \frac{\beta}{\sqrt{\theta n} \left\| D^{\frac{1}{2}} \mathbf{1} \right\|_2^2} \mathbf{1} \pm \sqrt{1 - \frac{\beta^2}{\theta n \left\| D^{\frac{1}{2}} \mathbf{1} \right\|_2^2}} D^{-\frac{1}{2}} \mathbf{w}^{[2]},$$

$D^{-\frac{1}{2}} \mathbf{w}^{[2]}$ is the normalized Fiedler vector (of a normalized Laplacian).

Constructing the partition

According to the definition, the sets V_1 and V_2 are determined by

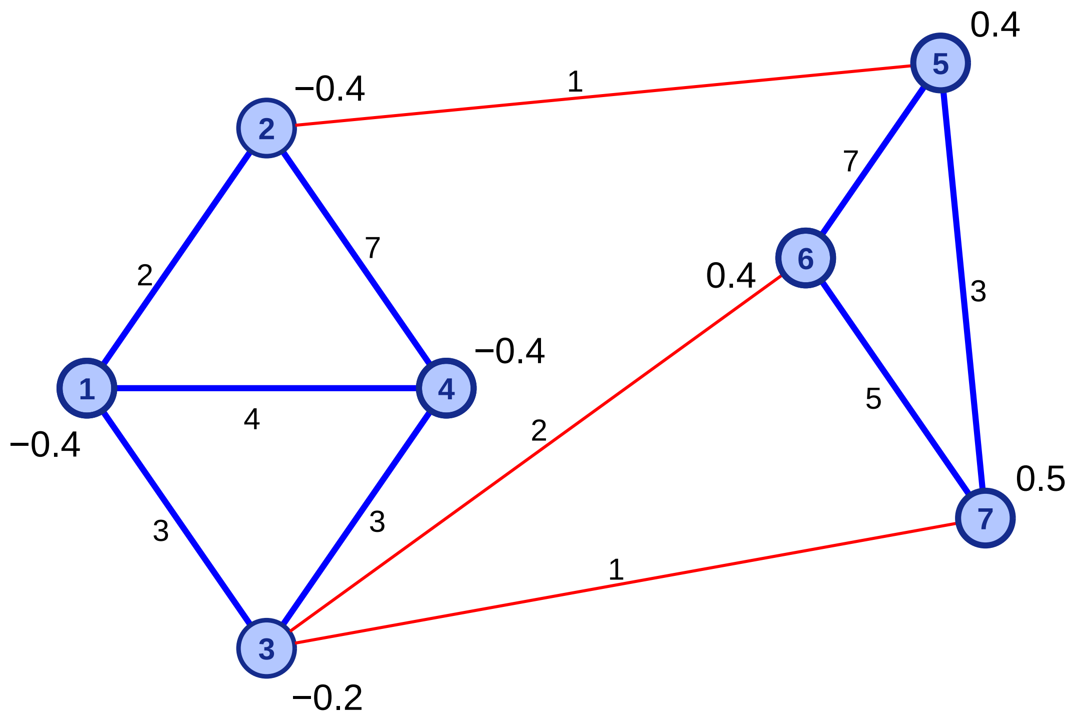
$$V_1 = \{i : \mathbf{v}^{[2]}(i) < 0\}, \quad V_2 = \{i : \mathbf{v}^{[2]}(i) \geq 0\},$$

for the proportional cut, and

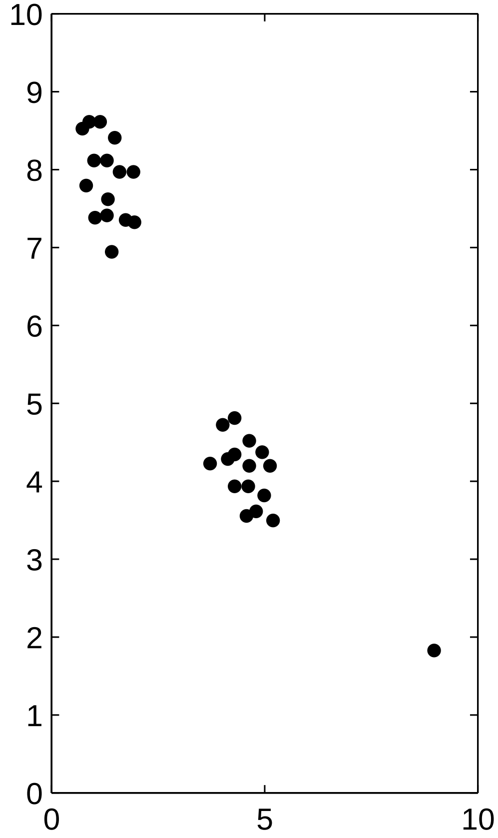
$$V_1 = \{i : D^{-\frac{1}{2}}\mathbf{w}^{[2]}(i) < 0\}, \quad V_2 = \{i : D^{-\frac{1}{2}}\mathbf{w}^{[2]}(i) \geq 0\}$$

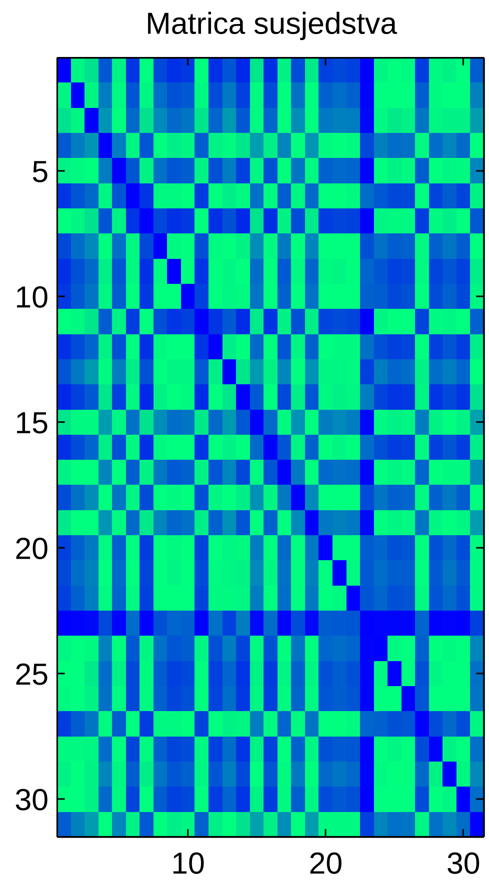
for the normalized cut.

Components of the Fiedler vector of G_{small}

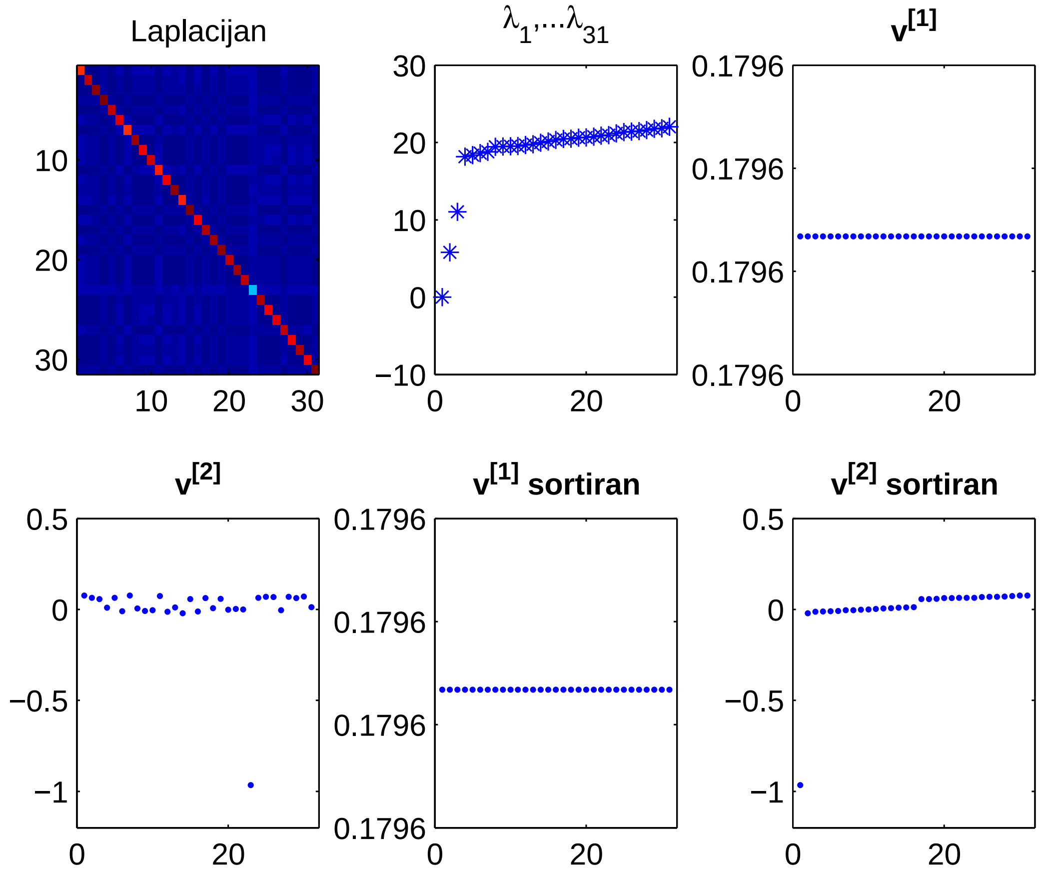


Proportional cut vs. normalized cut (1)

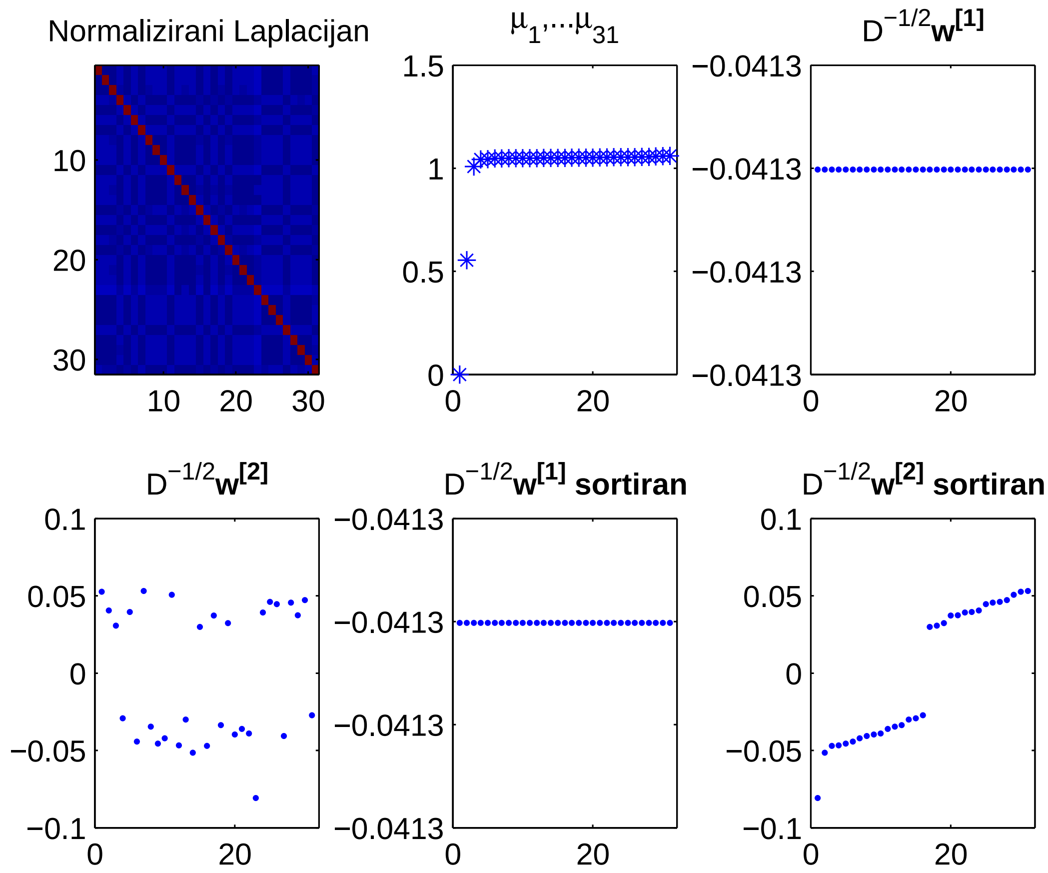




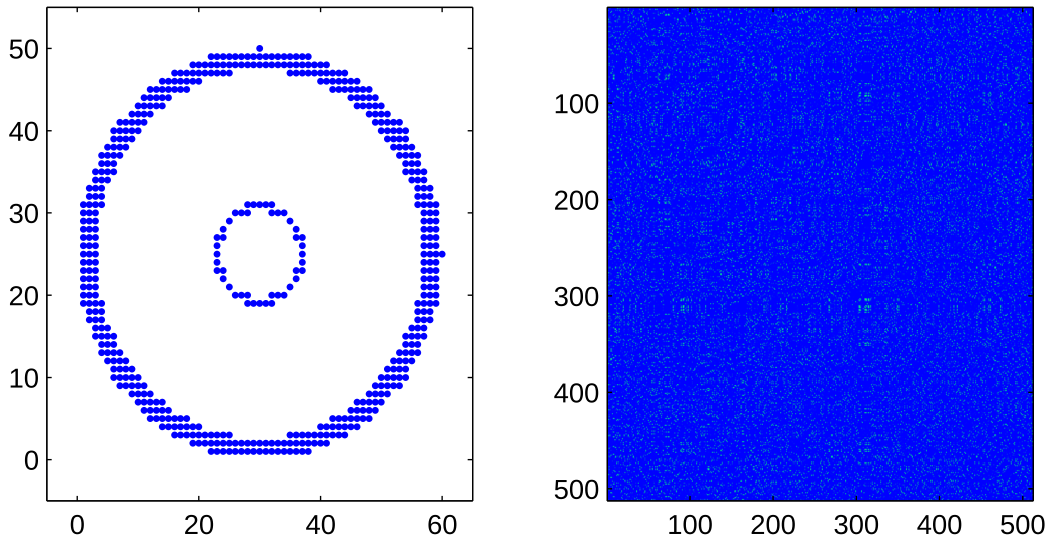
Proportional cut vs. normalized cut (2)



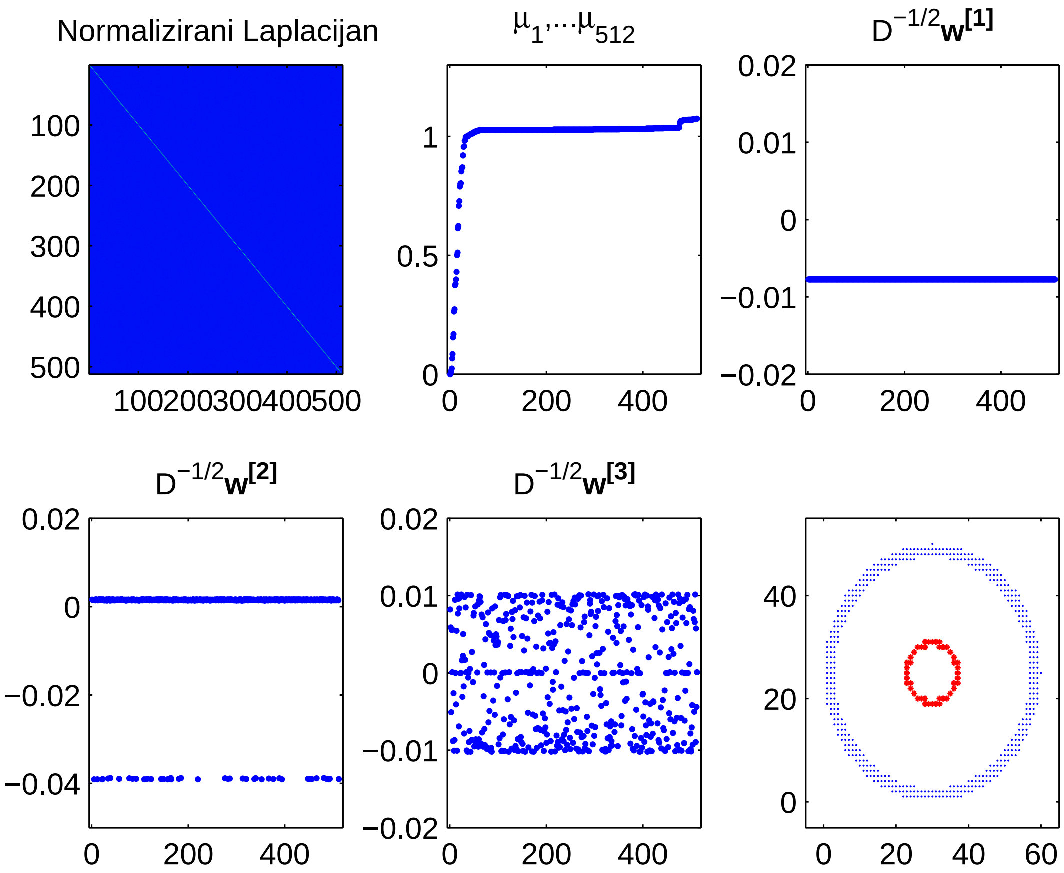
Proportional cut vs. normalized cut (3)



Concentric circles (1)



Concentric circles (2)



k-partitioning

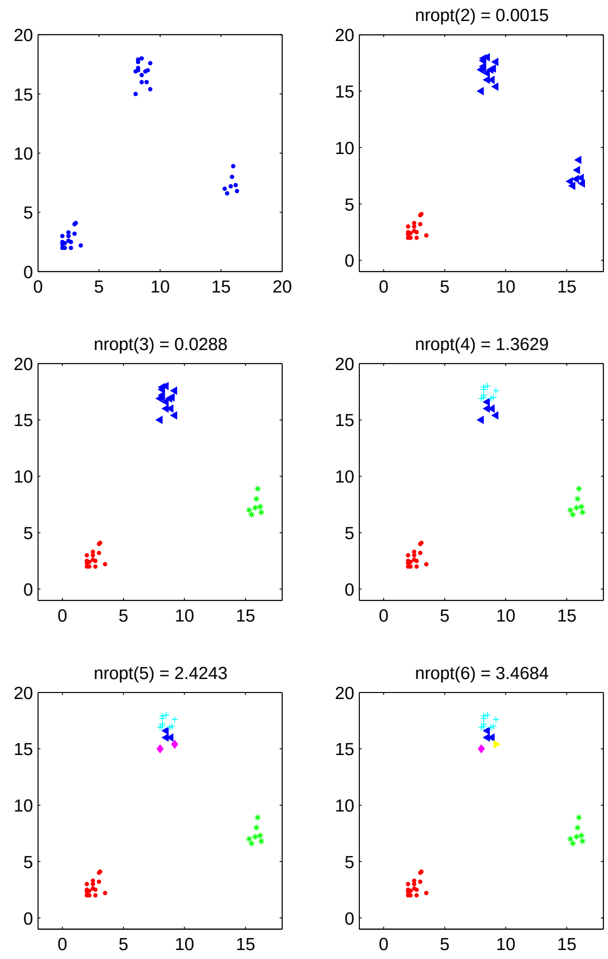
1. Bipartition V; Set counter $k_c = 2$;

2. If $k_c < k$,

   - for each subset of V compute the optimal bipartition;
   - within all $(k_c + 1)$-partitions, choose one with the smallest value of the partitioning function;
   - set $k_c = k_c + 1$ and repeat step 2.

3. Stop.

k-partitioning example



Bipartite graph

Undirected bipartite graph G is a triplet

$$G = (R, D, B).$$

$R = \{r_1, \cdots, r_m\}$ and $D = \{d_1, ..., d_n\}$ are two sets of nodes and

$$B = \{\{r_i, d_j\} : r_i \in R, d_j \in D\}$$

is a set of edges.
For example, D is a set of documents, R is a set of words and edge $\{r_i, d_j\}$ exists if document d_j contains word r_i.

Laplacian

Let e.g. $R = \{r_1, \cdots, r_5\}$ and $D = \{d_1, d_2\}$. Then,

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 & 3 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

where $A \in \mathbb{R}^{m \times n}$ is the terms $\times$ documents matrix.

Connection to SVD (1)

Let

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad L = \begin{bmatrix} D_1 & -A \\ -A^T & D_2 \end{bmatrix}$$

Then

$$L_n = D^{-\frac{1}{2}} \begin{bmatrix} D_1 & -A \\ -A^T & D_2 \end{bmatrix} D^{-\frac{1}{2}} = \begin{bmatrix} I & -D_1^{-\frac{1}{2}} A D_2^{-\frac{1}{2}} \\ -D_2^{-\frac{1}{2}} A D_1^{-\frac{1}{2}} & I \end{bmatrix}$$

Connection to SVD (2)

Let

$$\mathbf{w} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \quad \mathbf{u} \in \mathbb{R}^m, \quad \mathbf{v} \in \mathbb{R}^n,$$

be an eigenvector of the normalized Laplacian,

$$D^{-\frac{1}{2}} L D^{-\frac{1}{2}} \mathbf{w} = \lambda \mathbf{w}.$$

Then

$$\begin{aligned} D_1^{-\frac{1}{2}} A D_2^{-\frac{1}{2}} \mathbf{v} &= (1-\lambda)\mathbf{u}, \\ D_2^{-\frac{1}{2}} A^T D_1^{-\frac{1}{2}} \mathbf{u} &= (1-\lambda)\mathbf{v}. \end{aligned}$$

Connection to SVD

Instead of computing the Fiedler vector of L_n, we compute the left and right singular vector of the normalized matrix $A_n = D_1^{-\frac{1}{2}} A D_2$ which correspond to the second largest singular value,

$$A_n \mathbf{v}^{[2]} \;=\; \sigma_2 \mathbf{u}^{[2]},$$

where $\sigma_2 = 1 - \lambda_2$

This is more stable!

$\mathbf{u}^{[2]}$ partitions terms and $\mathbf{v}^{[2]}$ partitions documents!

Multipartitioning algorithm

1. For given matrix A compute $A_n = D_1^{-\frac{1}{2}} A D_2^{-\frac{1}{2}}$;
2. Compute k singular vectors of A_n, $\mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[k]}$ and $\mathbf{v}^{[1]}, \ldots, \mathbf{v}^{[k]}$, and form the matrix

$$Z = \begin{bmatrix} D_1^{-\frac{1}{2}} U \\ D_2^{-\frac{1}{2}} V \end{bmatrix},$$

where

$$U = \left[\mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[k]}\right] \quad \text{i} \quad V = \left[\mathbf{v}^{[1]}, \ldots, \mathbf{v}^{[k]}\right].$$

3. Use k-means algorithm on the rows of the matrix Z.

Multipartitioning algorithm

k-means algorithm outputs means of the partitions

$$\mathbf{c}_1, \ldots, \mathbf{c}_k$$

and the vector $[\sigma_1, \sigma_2, \cdots, \sigma_{m+n}]$, where

$$\sigma_i \in \{1, 2, \cdots, k\}, \quad i = 1, \cdots, m+n,$$

denotes the number of the partition to which $Z(i)$ belongs, that is, the i-th word belongs to the partition σ_i, $i = 1, \cdots, m$, and the j-th document belongs to the partition σ_{j+m}, $j = 1, \cdots, n$,

Example

(Digital) textbook *Mathematics 1* (http://lavica.fesb.hr/mat1) consists of 146 documents divided in six chapters:

Basics, Linear algebra, Vector algebra and analytic geometry, Functions of a real variable, Derivatives and applications, Sequences and series.

Results of the spectral partitioning of documents and words:

testni data	algorithm	norm. cut	time (s)
(a)	mp	3.4254	0.672
($>$ 2 letters in a word)	rbp	3.582	3.75
3522×146	km	5.3561	1.766
(b)	mp	3.0196	0.484
($>$ 4 letters in word)	rbp	3.115	3.875
3213×146	km	5.1435	2.797

Example

Result of the partitioning with multipartitioning algorithm:

Basics (21) - $[2\ 6\ 1\ 3\ 3\ 3\ 3\ 4\ 1\ 1\ 1\ 1\ 3\ 1\ 3\ 3\ 1\ 3\ 2\ 2\ 3]^T$

Linear algebra (25) - $[4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 2\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4]^T$

Vector algebra and analytic geometry (20) - $[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2]^T$

Functions of a real variable (31) - $[6\ 6\ 2\ 6\ 6\ 6\ 3\ 3\ 3\ 6\ 6\ 6\ 6\ 6\ 3\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6]^T$

Derivatives and applications (27) - $[6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6]^T$

Sequences and series (21) - $[6\ 3\ 3\ 3\ 3\ 3\ 3\ 6\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 6\ 6\ 2\ 5\ 5]^T$

Example

Result of partitioning with recursive bipartitioning algorithm:

Basics (21) - $[2\ 1\ 1\ 3\ 3\ 3\ 3\ 5\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 2\ 3\ 2\ 2]^T$

Linear algebra (25) - $[5\ 3\ 5\ 3\ 5\ 5\ 5\ 3\ 5\ 5\ 5\ 5\ 5\ 5\ 4\ 5\ 5\ 5\ 5\ 5\ 3\ 5\ 5\ 5\ 5]^T$

Vector algebra and analytic geometry (20) - $[4\ 4\ 4\ 3\ 3\ 3\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 3\ ]^T$

Functions of a real variable (31) - $[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 2\ 3\ 3\ 2\ 3\ 2\ 3\ 2\ 2\ 2\ 2\ 2]^T$

Derivatives and applications (27) - $[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 6]^T$

Sequences and series (21) - $[2\ 2\ 3\ 3\ 2\ 2\ 3\ 2\ 1\ 3\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 6\ 6]^T$

IWASEP X

10th International Workshop on Accurate Solution of Eigenvalue Problems

June 2-5, 2014, Dubrovnik, Croatia

- I Split, 1996
- II Penn State, 1998 (LAA 309/1-3, 2000)
- III Hagen, 2000 (LAA 358, 2003)
- IV Split, 2002 (part of LAA 417/2-3, 2006)
- V Hagen, 2004 (SIMAX 28/4, 2006)
- VI Penn State, 2006 (SIMAX 31/1, 2009)
- VII Dubrovnik, 2008
- VIII Berlin, 2010
- IX Berkeley (Napa Valley), 2012 (LAA, deadline Feb 28, 2013)