

# MPP07c DAlembertovo rjesenje

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## 1 D'Alembertovo rješenje valne jednadžbe

### 1.1 Rubni uvjeti na konačnoj domeni

Promotrimo problem početnih vrijednosti

$$\begin{aligned}u_t + c u_x &= 0, & 0 < x < l, & \quad t > 0, \quad c > 0 \\u(x, 0) &= f(x).\end{aligned}$$

Rješenje je desni val

$$u(x, t) = f(x - ct),$$

a karakteristične krivulje su pravci

$$x(t) = ct + k,$$

odnosno

$$t(x) = \frac{1}{c}(x - k).$$

U ovom slučaju možemo zadati i *rubni uvjet* za  $x = 0$ :

$$u(0, t) = g(t).$$

U desnom rubu, za  $x = l$ , rješenje je određeno karakterističnim pravcima: za  $t \in [0, 1/c]$  rješenje je određeno početnim uvjetom (plave linije), a za  $t > 1/c$  rješenje je određeno rubnim uvjetom (crvene linije).

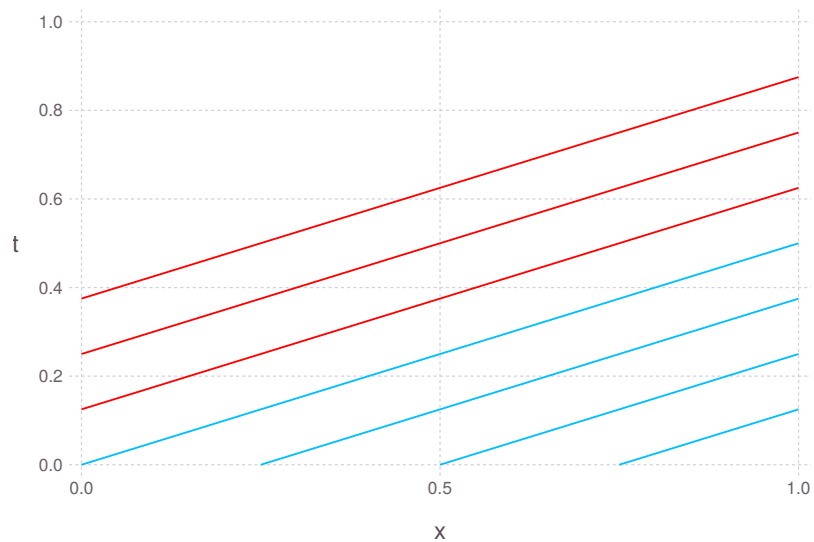
In [1]: `using Gadfly`

```

In [2]: # Za l=1
c=2
tx(x,k)=(x-k)/c
Gadfly.plot(layer(x->tx(x,0),0,1),
  layer(x->tx(x,0.25),0.25,1),
  layer(x->tx(x,0.5),0.5,1),
  layer(x->tx(x,0.75),0.75,1),
  layer(x->tx(x,-0.25),0,1,Theme(default_color=colorant"red")),
  layer(x->tx(x,-0.5),0,1,Theme(default_color=colorant"red")),
  layer(x->tx(x,-0.75),0,1,Theme(default_color=colorant"red")),
  Guide.xlabel("x"),Guide.ylabel("t"))

```

Out [2] :



### 1.1.1 Uvjeti kompatibilnosti

U prethodnom slučaju nema lijevih valova pa nema ni refleksije. U slučaju općenite valne jednadžbe možemo zadati rubne uvjete u oba ruba:

$$\begin{aligned}
 u_{tt} - c^2 u_{xx} &= 0, & 0 < x < l, & \quad t > 0, \\
 u(x,0) &= F(x), & u_t(x,0) &= G(x), \\
 u(0,t) &= a(t), & u(l,t) &= b(t).
 \end{aligned}$$

U tom slučaju *uvjeti kompatibilnosti* trebaju biti ispunjeni u točkama  $(0,0)$  i  $(l,0)$ :

$$\begin{aligned}
F(0) &= u(0,0) = a(0) \\
F(l) &= u(l,0) = b(0) \\
G(0) &= u_t(0,0) = a'(0) \\
G(l) &= u_t(l,0) = b'(0).
\end{aligned}$$

## 1.2 D'Alembertovo rješenje

**Teorem.** Rješenje problema početnih vrijednosti

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= F(x), \quad u_t(x,0) = G(x)
\end{aligned}$$

je

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} G(y) dy.$$

*Dokaz:* Karakteristične krivulje su pravci, a rješenje je kombinacija lijevog i desnog vala pa ima oblik

$$u(x,t) = f(x+ct) + g(x-ct).$$

Vrijedi

$$\begin{aligned}
u(x,0) &= f(x) + g(x) = F(x), \\
u_t(x,0) &= f'(x) \cdot c + g'(x) \cdot (-c) = G(x),
\end{aligned} \tag{1}$$

odnosno

$$f'(x) - g'(x) = \frac{1}{c} G(x).$$

Integriranje daje

$$f(x) - g(x) = \frac{1}{c} \int G(x) dx = \frac{1}{c} \int_0^x G(y) dy + C. \tag{2}$$

Zbrajanje (1) i (2) daje

$$f(x) = \frac{1}{2}F(x) + \frac{1}{2c} \int_0^x G(y) dy + \frac{1}{2}C,$$

a oduzimanje (2) od (1) daje

$$g(x) = \frac{1}{2}F(x) - \frac{1}{2c} \int_0^x G(y) dy - \frac{1}{2}C.$$

Dakle,

$$\begin{aligned} u(x, t) &= f(x + ct) + g(x - ct) \\ &= \frac{1}{2}[F(x + ct) + F(x - ct)] + \frac{1}{2c} \left[ \int_0^{x+ct} G(y) dy - \int_0^{x-ct} G(y) dy \right] \end{aligned}$$

i teorem je dokazan.

Vidimo da vrijednost  $u(x_0, y_0)$ , ovisi o početnim uvjetima na intervalu  $x \in [x_0 - c t_0, x_0 + c t_0]$ .

Taj interval je *područje ovisnosti (domain of dependence)* točke  $(x_0, t_0)$ .

Slično, početni uvjeti na intervalu  $x \in [x_1, x_2]$  utječu na rješenje unutar područja omeđenog pravcima

$$x + c t = x_1, \quad x - c t = x_2, \quad t = 0.$$

To područje je *područje utjecaja (region of influence)*.

**Primjer.** Rješenje problema

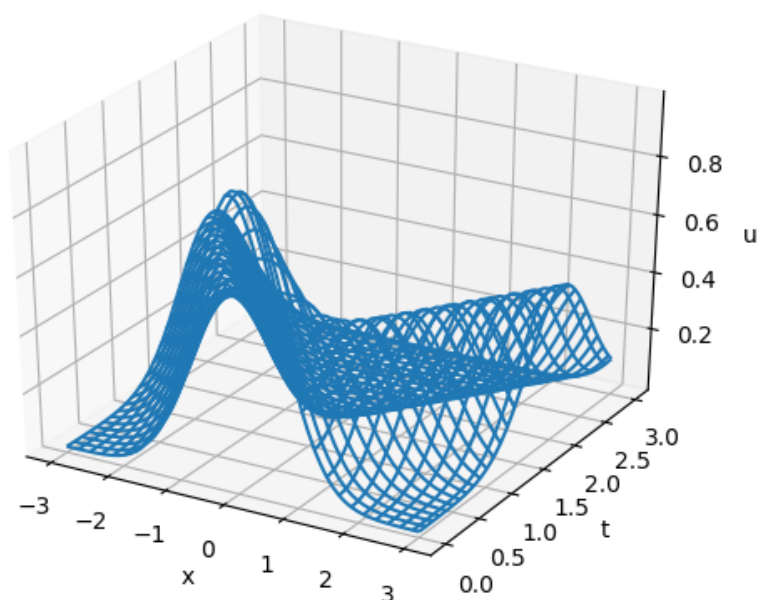
$$\begin{aligned} u_{tt} - 2 u_{xx} &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= e^{-x^2}, \quad u_t(x, 0) = 0, \end{aligned}$$

je

$$u(x, t) = \frac{1}{2} [e^{-(x+\sqrt{2}t)^2} + e^{-(x-\sqrt{2}t)^2}].$$

In [3]: `using` PyPlot

```
In [4]: # Rješenje
        gridsize=60
        X=range(-3,stop=3,length=gridsize)
        T=range(0,stop=3,length=gridsize)
        XT=collect(Iterators.product(X,T))
        u(x,t)=(exp(-(x+sqrt(2)*t)^2)+exp(-(x-sqrt(2)*t)^2))/2
        U=[u(XT[i,j][1],XT[i,j][2]) for i=1:gridsize,j=1:gridsize]
        PyPlot.mesh(X,T,Matrix(U'))
        xlabel("x")
        ylabel("t")
        zlabel("u")
```



Za  $k, h > 0$  definirajmo točke

$$\begin{aligned} A &= (x - ck, t - h), & B &= (x + ch, t + k), \\ C &= (x + ck, t + h), & D &= (x - ch, t - k). \end{aligned}$$

**Lema.** Funkcija  $u(x, t)$  zadovoljava jednadžbu

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (3)$$

ako i samo ako vrijedi

$$u(A) + u(C) = u(B) + u(D). \quad (4)$$

*Dokaz:* Pretpostavimo da je

$$u(x, t) = f(x + ct) + g(x - ct)$$

rješenje. Tada je

$$\begin{aligned}
u(A) &= f(x - ck + c(t - h)) + g(x - ck - c(t - h)) \\
&= f(x - ck + ct - ch) + g(x - ck - ct + ch) \\
u(B) &= f(x + ch + c(t + k)) + g(x + ch - c(t + k)) \\
&= f(x + ch + ct + ck) + g(x + ch - ct - ck) \\
u(C) &= f(x + ck + c(t + h)) + g(x + ck - c(t + h)) \\
&= f(x + ck + ct + ck) + g(x + ck - ct - ch) \\
u(D) &= f(x - ch + c(t - k)) + g(x - ch - c(t - k)) \\
&= f(x - ch + ct - ck) + g(x - ch - ct + ck)
\end{aligned}$$

pa (4) vrijedi.

Obrnuto, neka za  $u(x, t)$  vrijedi (4). Za  $h = 0$  vrijedi

$$u(x - ck, t) + u(x + ck, t) = u(x, t + k) + u(x, t - k). \quad (5)$$

Taylorova formula daje

$$\begin{aligned}
u(x - ck, t) &= u(x, t) + u_x(x, t) \cdot (-c)k + \frac{1}{2}u_{xx}(x, t) \cdot c^2k^2 + O(k^3) \\
u(x + ck, t) &= u(x, t) + u_x(x, t) \cdot ck + \frac{1}{2}u_{xx}(x, t) \cdot c^2k^2 + O(k^3) \\
u(x, t - k) &= u(x, t) + u_t(x, t) \cdot (-k) + \frac{1}{2}u_{tt}(x, t) \cdot k^2 + O(k^3) \\
u(x, t + k) &= u(x, t) + u_t(x, t) \cdot k + \frac{1}{2}u_{tt}(x, t) \cdot k^2 + O(k^3),
\end{aligned}$$

pa uvrštavanje u (5), dijeljenje s  $k^2$  i ignoriranje članova  $O(k)$  daje (3) i teorem je dokazan.

Lema je ilustrirana sljedećom slikom:

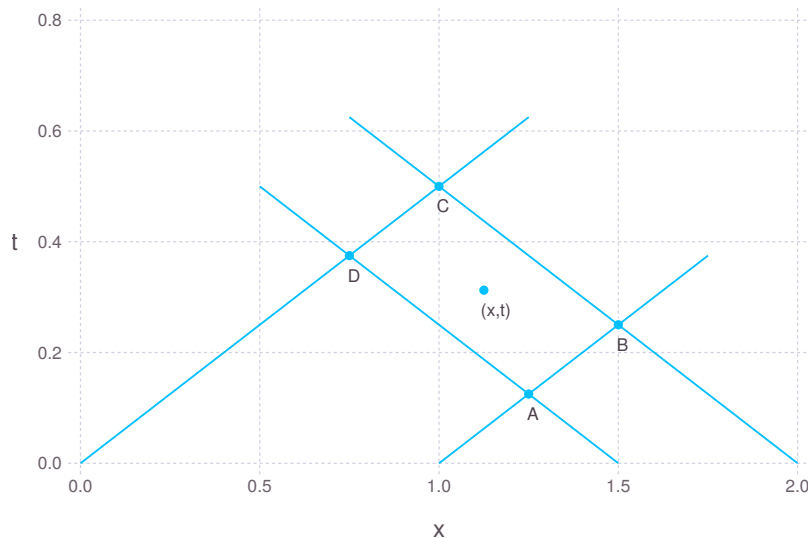
In [5]: c=2

```

X = [5/4, 3/2, 1, 3/4, 9/8]
T = [1/8, 1/4, 1/2, 3/8, 5/16]
Labels = ["A", "B", "C", "D", "(x, t)"]
Gadfly.plot(layer(x->(x-1)/c, 1, 1.75),
  layer(x->(x-0)/c, 0, 1.25),
  layer(x->(x-2)/(-c), 0.75, 2),
  layer(x->(x-1.5)/(-c), 0.5, 1.5),
  layer(x=X, y=T, label=Labels, Geom.point, Geom.label),
  Guide.xlabel("x"), Guide.ylabel("t"))

```

Out [5]:



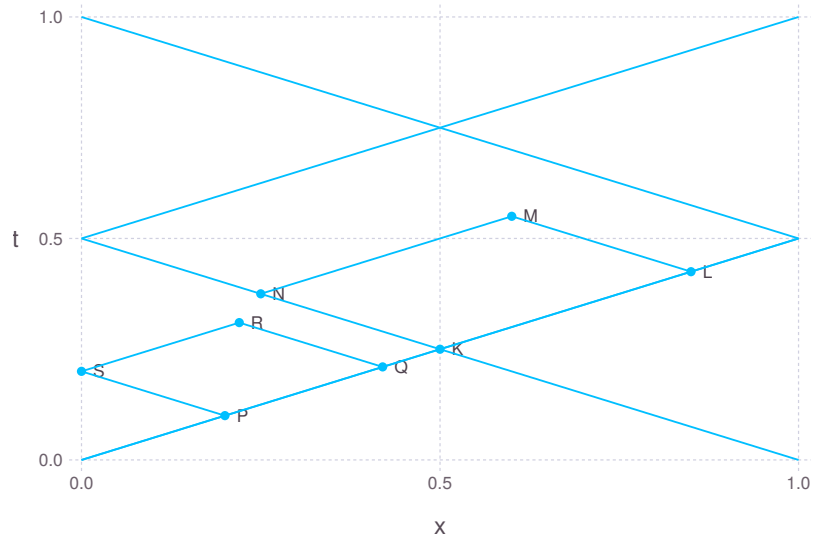
Na omeđenoj domeni rješenje se konstruira na sljedeći način:

- rješenje u donjem trokutu daje D'Alembert-ova formula,
- rješenje u lijevom trokutu se dobije pomoću leme:  $u(R) = u(S) + u(Q) - u(P)$ , pri čemu se  $u(P)$  i  $u(Q)$  izračunaju D'Alembert-ovom formulom (leže u donjem trokutu), a  $u(S)$  je rubni uvjet (rješenje u desnom trokutu se dobije slično),
- rješenje u prvom rombu se dobije pomoću leme:  $u(M) = u(L) + u(N) - u(K)$  pri čemu se  $u(K)$ ,  $u(L)$  i  $u(N)$  dobiju iz lijevog i desnog trokuta,
- u ostalim djelovima postupak se nastavlja analogno.

In [6]: `c=2`

```
X = [0,0.2,0.42,0.22,0.5,0.25,0.6,0.85]
T = [0.2,0.1,0.21,0.31,0.25,0.375,0.55,0.425]
Labels = ["S", "P", "Q", "R", "K", "N", "M", "L"]
Gadfly.plot(layer(x->x/c,0,1),
  layer(x->(x-0)/c,0,1),
  layer(x->(x-1)/(-c),0,1),
  layer(x->(x+1)/c,0,1),
  layer(x->(x-2)/(-c),0,1),
  layer(x->(x+0.4)/c,0,0.22),
  layer(x->(x-0.4)/(-c),0,0.2),
  layer(x->(x-0.84)/(-c),0.22,0.42),
  layer(x->(x+0.5)/c,0.25,0.6),
  layer(x->(x-1.7)/(-c),0.6,0.85),
  layer(x=X,y=T,label=Labels,Geom.point,Geom.label(position=:right)),
  Guide.xlabel("x"),Guide.ylabel("t"),
)
```

Out [6] :



**Primjer.** Primijenimo postupak na poluomeđenu domenu:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad x > 0, \\ u(0, t) &= h(t), \quad t > 0. \end{aligned}$$

Uvjeti kompatibilnosti su  $f(0) = h(0)$  i  $g(0) = h'(0)$ . Karakteristične krivulje su pravci

$$t(x) = \frac{1}{c}(x - k).$$

Za  $0 < t < \frac{x}{c}$  rješenje je dano D'Alembert-ovom formulom:

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy. \quad (6)$$

Za  $0 < \frac{x}{c} < t$  rješenje slijedi primjenom leme (lijevi trokut na gornjoj slici), odnosno

$$u(x, t) = u(R) = u(S) + u(Q) - u(P). \quad (7)$$

Točka  $S$  leži na pravcu



$$y(\xi) = \frac{\xi}{c} + \left(t - \frac{x}{c}\right)$$

pa je

$$u(S) = h\left(t - \frac{x}{c}\right).$$

Točka  $P$  je presjek pravca  $y(\xi) = \xi/c$  s pravcem

$$y(\xi) = -\frac{\xi}{c} + \left(t - \frac{x}{c}\right),$$

odnosno

$$P = \left(\frac{ct - x}{2}, \frac{ct - x}{2c}\right),$$

pa (6) daje

$$u(P) = \frac{1}{2}[f(ct - x) + f(0)] + \frac{1}{2c} \int_0^{ct-x} g(y) dy.$$

Slično, točka  $Q$  je presjek pravca  $y(\xi) = \xi/c$  s pravcem

$$y(\xi) = -\frac{\xi}{c} + \left(t + \frac{x}{c}\right),$$

pa je

$$u(Q) = \frac{1}{2}[f(ct + x) + f(0)] + \frac{1}{2c} \int_0^{ct+x} g(y) dy.$$

Uvrštavanje u (7) konačno daje

$$u(x, t) = h\left(t - \frac{x}{c}\right) + \frac{1}{2}[f(ct + x) - f(ct - x)] + \frac{1}{2c} \int_{ct-x}^{ct+x} g(y) dy.$$

Nacrtajmo rješenje problema:

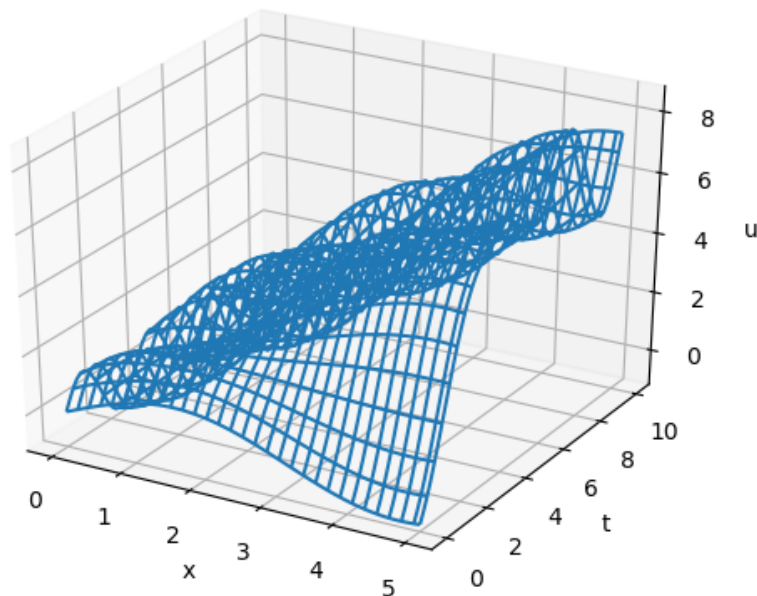
$$\begin{aligned} u_{tt} - 2u_{xx} &= 0, & x > 0, & \quad t > 0, \\ u(x, 0) &= \sin x, & u_t(x, 0) &= 2, & x > 0, \\ u(0, t) &= \sin 2t, & t > 0. \end{aligned}$$

In [7]: `using QuadGK`

```

In [8]: c=sqrt(2)
        f(x)=sin(x)
        g(x)=2*x^0
        h(t)=sin(2*t)
        gridsize=60
        ζ=range(0,stop=5,length=gridsize)
        τ=range(0,stop=10,length=gridsize)
        X=repeat(ζ,1,gridsize)
        T=repeat(τ',gridsize,1)
        U=Array{Float64}(undef,gridsize,gridsize)
        for i=1:gridsize,j=1:gridsize
            x=X[i,j]
            t=T[i,j]
            if t<=x/c
                U[i,j]=(f(x+c*t)+f(x-c*t))/2+quadgk(g,x-c*t,x+c*t)[1]/(2*c)
            else
                U[i,j]=(f(x+c*t)-f(c*t-x))/2+quadgk(g,c*t-x,x+c*t)[1]/(2*c)+h(t-x/c)
            end
        end
        mesh(X,T,U)
        xlabel("x")
        ylabel("t")
        zlabel("u")

```

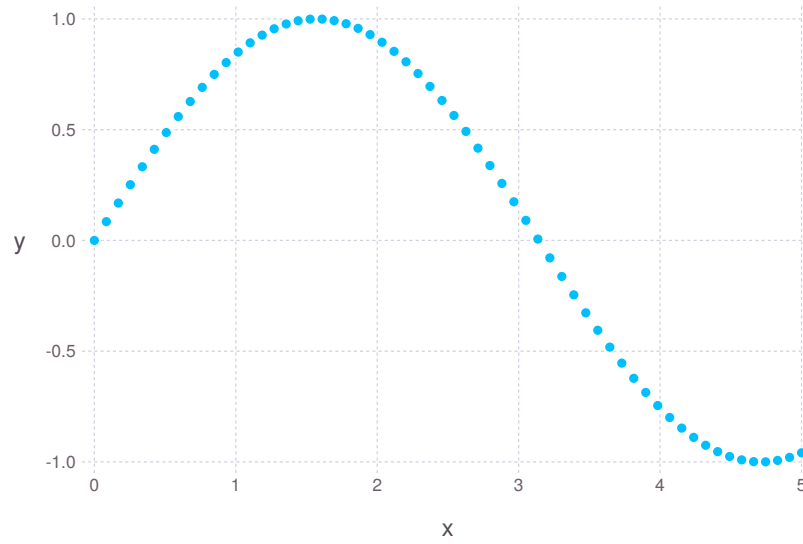


```

In [9]: # Prouverimo početni uvjet
        Gadfly.plot(x=ζ,y=U[:,1])

```

Out [9] :



In [10]: *# Provjerimo rubni uvjet*  
`Gadfly.plot(x= $\tau$ , y=U[1,:])`

Out [10] :

