MPP03b Problem rubnih vrijednosti

Ivan Slapničar

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1 Problem rubnih vrijednosti

1.1 Klasifikacija

Neka je

$$a \cdot u_{xx} + b \cdot u_{xt} + c \cdot u_{tt} + d \cdot u_x + e \cdot u_t + f \cdot u + g = 0$$

i neka je

$$D = b^2 - 4ac.$$

Vrijedi sljedeća klasifikacija:

D	D=0	D<0	D>0
Vrsta	parabolička	eliptička	hiperbolička
Problem	difuzija	ravnoteža	valovi
Domena / Metoda	omeđena / SLP	omeđena / SLP	
	neomeđena / integr. trans.	neomeđena / integr. trans.	

Za neomeđeni interval $(0, \infty)$ koristi se Laplace-ova transformacija, a za interval (∞, ∞) koristi se Fourier-ova transformacija.

1.2 Jednadžba difuzije

Zadan je problem

$$\begin{aligned}
&u_t - u_{xx} = 0 \\
&u(x,0) = |x|, \quad -2 < x < 2 \\
&u_x(-2,t) = 0, \quad u_x(2,t) = 0, \quad t > 0
\end{aligned}$$

Pretpostavimo separaciju varijabli (rješenje je jedinstveno pa je svaka pretpostavka korektna ako daje rješenje):

$$u(x,t) = X(x)T(t).$$

Uvrštavanje u jednadžbu daje

$$XT' = X''T$$

odnosno (stavljamo $-\lambda$ po dogovoru)

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda,$$

za neki $\lambda \in \mathbb{R}$. Dakle,

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda,$$

za neki $\lambda \in \mathbb{R}$.

Dobili smo SLP i populacijsku jednadžbu:

1. SLP: $X'' + \lambda X = 0$ uz uvjete $X'(-2) = 0$ i $X'(2) = 0$
2. Populacijska jednadžba: $T' + \lambda T = 0$

Za $\lambda \geq 0$ SLP ima svojstvene vrijednosti (izračunajte!, vidi napomenu!)

$$\lambda_n = \frac{n^2\pi^2}{4}, \quad n \in \mathbb{N} \cup \{0\}.$$

i pripadne svojstvene funkcije

$$X_n(x) = A_n \cos\left(\frac{n\pi}{2}x\right).$$

Za svaki λ_n rješenje populacijske jednadžbe glasi

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{4}t}$$

što zajedno daje

$$u_n(x,t) = C_n \cos\left(\frac{n\pi}{2}x\right) e^{-\frac{n^2\pi^2}{4}t}.$$

Svaka funkcija u_n zadovoljava jednadžbu i rubne uvjete.

Prema *principu superpozicije* i funkcija

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\sum_{n=0}^{\infty}C_n\cos\big(\frac{n\pi}{2}x\big)e^{-\dfrac{n^2\pi^2}{4}t}$$

također zadovoljava jednadžbu i rubne uvjete. Treba još odabrati koeficijente C_n tako da se zadovolji i početni uvijet - radi se o razvoju u *(generalizirani) Fourierov red*:

$$u(x,0)=\sum_{n=0}^{\infty}C_n\cos\big(\frac{n\pi}{2}x\big)$$

$$C_n=\frac{\big(|x|,\cos\big(\frac{n\pi}{2}x\big)\big)}{\big(\cos\big(\frac{n\pi}{2}x\big),\cos\big(\frac{n\pi}{2}x\big)\big)} \qquad (*)$$

Napomena: Kod traženja svojstvenih vrijednosti za slučaj $\lambda>0$, zbog parnosti početnog uvjeta možemo odmah staviti da je koeficijent uz $\sin(\sqrt{\lambda}x)$ jednak nuli.

Probajmo simboličko računanje - treba nam paket `PyPlot.jl` za crtanje i paket `SymPy.jl` za simboličko računanje:

In [1]:
```
using PyPlot
using SymPy
```

In [2]:
```
# Definirajmo simbole
n=symbols("n",integer=true, nonnegative=true)
x=Sym("x")
```

Out[2]:

$$x$$

In [3]:
```
# Definirajmo skalarni produkt
import Base.⋅
⋅(f,g,a,b)=integrate(f*g,(x,a,b))
```

Out[3]:
```
dot (generic function with 12 methods)
```

In [4]:
```
g=abs(x)
```

Out[4]:

$$|x|$$

In [5]:
```
f(n)=cos(n*PI*x/2)
```

Out[5]:
```
f (generic function with 1 method)
```

```
In [6]: # Na primjer
        f(2)
```

Out[6]:

$$\cos(\pi x)$$

```
In [7]: f(0)
```

Out[7]:

$$1$$

Izračunajmo koeficijente C_n:

```
In [8]: C(n)=·(g,f(n),-2,2)/·(f(n),f(n),-2,2)
```

```
Out[8]: C (generic function with 1 method)
```

```
In [9]: C(0)
```

Out[9]:

$$1$$

```
In [10]: C(1)
```

Out[10]:

$$-\frac{8}{\pi^2}$$

```
In [11]: C(2)
```

Out[11]:

$$0$$

```
In [12]: C(3)
```

Out[12]:

$$-\frac{8}{9\pi^2}$$

```
In [13]: C(5)
```

Out[13]:

$$-\frac{8}{25\pi^2}$$

Vidimo da je

$$\begin{align} C_0 &= 1, \tag{1}\\ C_{2k} &= 0, \tag{2}\\ C_{2k-1} &= \frac{-8}{(2k-1)^2\pi^2}, \tag{3} \end{align}$$

odnosno

$$u(x,t) = 1 - \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2} \cos\left(\frac{(2k-1)\pi}{2}x\right) e^{-\dfrac{(2k-1)^2\pi^2}{4}t}.$$

Definirajmo sumu prvih n članova reda:

```
In [14]: k=symbols("k", integer=true)
         t=symbols("t", real=true, nonnegative=true)
```

Out[14]:

$$t$$

```
In [15]: u(n)=C(n)*f(n)*exp(-(n^2*PI^2*t/4))
```

```
Out[15]: u (generic function with 1 method)
```

```
In [16]: # Na primjer
         u(0)
```

Out[16]:

$$1$$

```
In [17]: u(3)
```

Out[17]:

$$-\frac{8e^{-\frac{9\pi^2 t}{4}}\cos\left(\frac{3\pi x}{2}\right)}{9\pi^2}$$

```
In [18]: # u(3) u nekoj točki (x,t)
         u(3)(0.5,0.5)
```

Out[18]:

$$-\frac{8\cos(0.75\pi)}{9\pi^2 e^{1.125\pi^2}}$$

```
In [19]: # Numerička vrijednost
         N(u(3)(0.5,0.5))
```

```
Out[19]: 9.59243054056247e-7
```

```
In [20]: # Suma prvih n članova reda
         U(n)=summation(u(k),(k,0,n))
```

```
Out[20]: U (generic function with 1 method)
```

```
In [21]: U(5)
```

Out[21]:

$$1-\frac{8e^{-\frac{\pi^2 t}{4}}\cos\left(\frac{\pi x}{2}\right)}{\pi^2}-\frac{8e^{-\frac{9\pi^2 t}{4}}\cos\left(\frac{3\pi x}{2}\right)}{9\pi^2}-\frac{8e^{-\frac{25\pi^2 t}{4}}\cos\left(\frac{5\pi x}{2}\right)}{25\pi^2}$$

```
In [22]: # Numerička vrijednost
         N(U(5)(0.5,0.5))
```

```
Out[22]: 0.8330895966582438
```

```
In [23]: # Za t=0 ovo mora konvergirati u |x|
         @time N(U(11)(0.5,0.0))
```

```
  2.427475 seconds (1.47 k allocations: 51.576 KiB)
```

```
Out[23]: 0.499415238140432
```

Napomena: Radi se o simboličkom računanju pa ne treba pretjerivati s n.

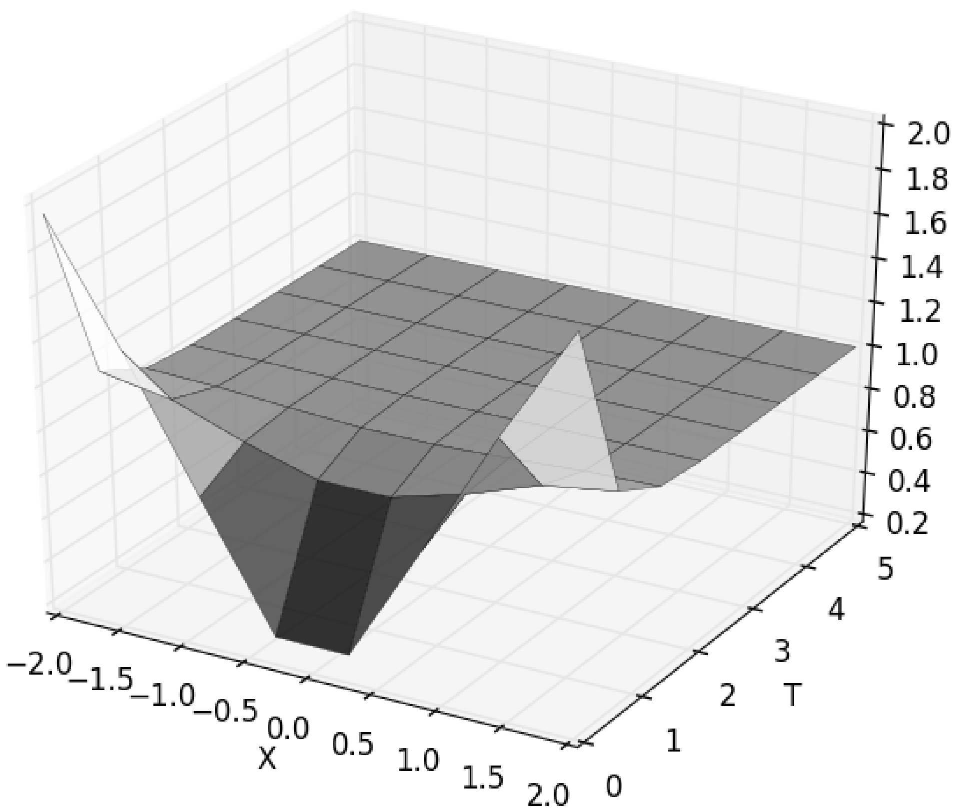
1.2.1 Crtanje

```
In [24]: m=8
         X=linspace(-2,2,m)
         T=linspace(0,5,m)
```

```
Out[24]: 0.0:0.7142857142857143:5.0
```

```
In [25]: # Ovo je sporo jer se radi o simboličkoj funkciji
         # (8*8*2.5=160 sekunda)
         z = zeros(m,m)
         for i in 1:m
             for j in 1:m
                 z[i,j] = N(U(9)(X[i],T[j]))
             end
         end
```

```
In [26]: plot_surface(X,T,z',rstride=1,edgecolors="k", cstride=1,
             cmap=ColorMap("gray"), alpha=0.8, linewidth=0.25)
         xlabel("X")
         ylabel("T")
```




```
Out[26]: PyObject <matplotlib.text.Text object at 0x7f426b8ca190>
```

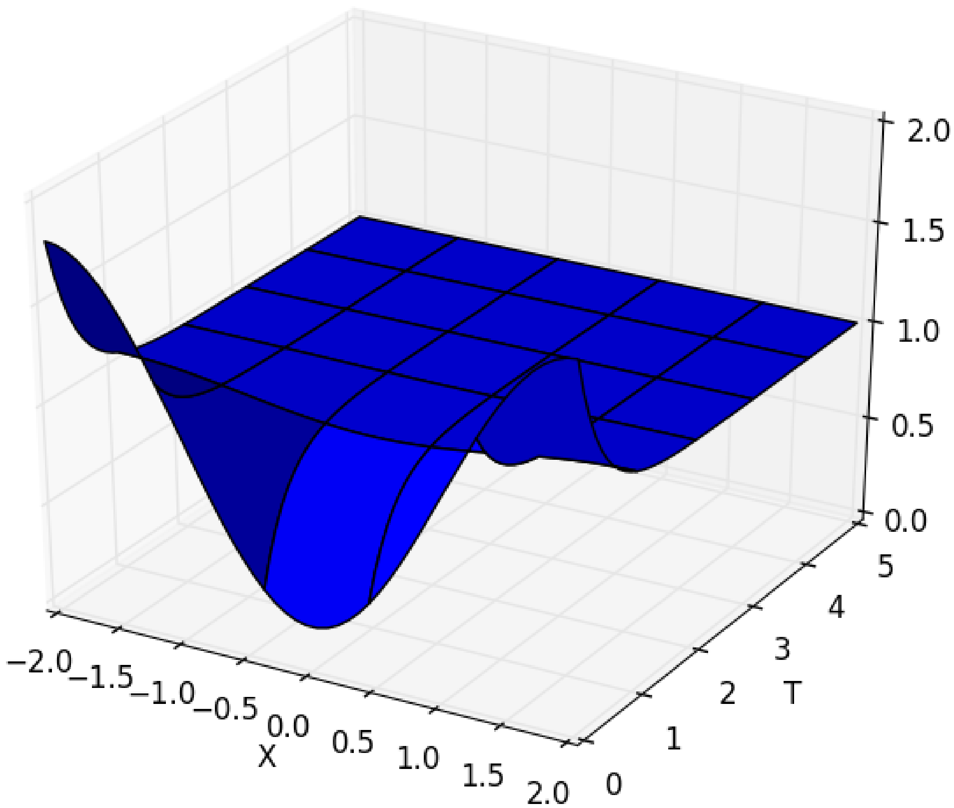
1.2.2 Numeričko računanje i crtanje

Pogledajmo interaktivno konvergenciju - treba nam paket `Interact.jl`:

```
In [27]: using Interact

INFO: Interact.jl: using new nbwidgetsextension protocol

In [28]: X=linspace(-2,2)
         T=linspace(0,5)
         XT=collect(Iterators.product(X,T))

Out[28]: 50×50 Array{Tuple{Float64,Float64},2}:
          (-2.0, 0.0)      (-2.0, 0.102041)      …  (-2.0, 5.0)
          (-1.91837, 0.0)  (-1.91837, 0.102041)     (-1.91837, 5.0)
          (-1.83673, 0.0)  (-1.83673, 0.102041)     (-1.83673, 5.0)
          (-1.7551, 0.0)   (-1.7551, 0.102041)      (-1.7551, 5.0)
          (-1.67347, 0.0)  (-1.67347, 0.102041)     (-1.67347, 5.0)
          (-1.59184, 0.0)  (-1.59184, 0.102041)  …  (-1.59184, 5.0)
          (-1.5102, 0.0)   (-1.5102, 0.102041)      (-1.5102, 5.0)
          (-1.42857, 0.0)  (-1.42857, 0.102041)     (-1.42857, 5.0)
          (-1.34694, 0.0)  (-1.34694, 0.102041)     (-1.34694, 5.0)
          (-1.26531, 0.0)  (-1.26531, 0.102041)     (-1.26531, 5.0)
          (-1.18367, 0.0)  (-1.18367, 0.102041)  …  (-1.18367, 5.0)
          (-1.10204, 0.0)  (-1.10204, 0.102041)     (-1.10204, 5.0)
          (-1.02041, 0.0)  (-1.02041, 0.102041)     (-1.02041, 5.0)
          ⋮                                      ⋱
          (1.10204, 0.0)   (1.10204, 0.102041)      (1.10204, 5.0)
          (1.18367, 0.0)   (1.18367, 0.102041)      (1.18367, 5.0)
          (1.26531, 0.0)   (1.26531, 0.102041)   …  (1.26531, 5.0)
          (1.34694, 0.0)   (1.34694, 0.102041)      (1.34694, 5.0)
          (1.42857, 0.0)   (1.42857, 0.102041)      (1.42857, 5.0)
          (1.5102, 0.0)    (1.5102, 0.102041)       (1.5102, 5.0)
          (1.59184, 0.0)   (1.59184, 0.102041)      (1.59184, 5.0)
          (1.67347, 0.0)   (1.67347, 0.102041)   …  (1.67347, 5.0)
          (1.7551, 0.0)    (1.7551, 0.102041)       (1.7551, 5.0)
          (1.83673, 0.0)   (1.83673, 0.102041)      (1.83673, 5.0)
          (1.91837, 0.0)   (1.91837, 0.102041)      (1.91837, 5.0)
          (2.0, 0.0)       (2.0, 0.102041)          (2.0, 5.0)

In [29]: g=figure()
         @manipulate for l in slider(1:10, value=1) ; withfig(g) do
                 h(xt)=1-8*sum([cos.((2*k-1)*pi*xt[1]/2).*
                        exp.(-(2*k-1)^2*pi^2*xt[2]/4)/((2*k-1)^2*π^2)
                         for k=1:l])
                 surf(X,T,map(h,XT)')
                 xlabel("X")
```

```
            ylabel("T")
        end
    end
```

Out[29]:



1.3 Primjer 1

$$
\begin{aligned}
&u_t - u_{xx} = -u \\
&u(x,0) = f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases} \\
&u(-1,t) = 0, \quad u(1,t) = 0
\end{aligned}
$$

Za detalje o simboličkom računanju pogledajte SymPy Tutorial.

Uvrštavanjem

$$u(x,t) = X(x)T(t)$$

jednadžba prelazi u jednadžbu

$$T'X - TX'' = -TX,$$

što daje dvije jednadžbe:

$$\frac{X''}{X} = \frac{T' + T}{T} = -\lambda.$$

Jednadžba po T je populacijska jednadžba koja glasi

$$T' = -(\lambda + 1)T$$

i čije rješenje je

$$T = Ce^{-(\lambda+1)t}.$$

Riješimo SLP po X:

$$X'' = -\lambda X, \quad X(-1) = 0, \quad X(1) = 0.$$

```
In [30]: F = SymFunction("F")
```

```
Out[30]: F
```

```
In [31]: l=symbols("l",real=true,positive=true)
         diffeq = Eq(diff(F(x), x, x) +l*F(x), 0)
```

```
Out[31]:
```

$$lF(x) + \frac{d^2}{dx^2}F(x) = 0$$

```
In [32]: ex = dsolve(diffeq)
```

```
Out[32]:
```

$$F(x) = C_1 \sin\left(\sqrt{l}x\right) + C_2 \cos\left(\sqrt{l}x\right)$$

```
In [33]: ex1 = rhs(ex)
```

```
Out[33]:
```

$$C_1 \sin\left(\sqrt{l}x\right) + C_2 \cos\left(\sqrt{l}x\right)$$

Uvrstimo rubne uvjete:

```
In [34]: ex1a=subs(ex1,x,-1)
```

Out[34]:

$$-C_1 \sin\left(\sqrt{l}\right) + C_2 \cos\left(\sqrt{l}\right)$$

```
In [35]: ex1b=subs(ex1,x,1)
```

Out[35]:

$$C_1 \sin\left(\sqrt{l}\right) + C_2 \cos\left(\sqrt{l}\right)$$

```
In [36]: solve(cos(sqrt(l)),l)
```

Out[36]:

$$\begin{bmatrix} \frac{\pi^2}{4} \\ \frac{9\pi^2}{4} \end{bmatrix}$$

Sustav jednadžbi je homogen i glasi

$$\begin{bmatrix} -C_1 & C_2 \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} \sin\sqrt{\lambda} \\ \cos\sqrt{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Trivijalno rješenje je u ovom slučaju očito nemoguće, a netrivijalna rješenje postoje kada je matrica sustava singularna, odnosno kada je $C_1 = 0$ ili $C_2 = 0$.
Kada je $C_1 = 0$ onda je $\cos\sqrt{\lambda} = 0$ pa je

$$\sqrt{\lambda} = \frac{2n+1}{2}\pi, \quad n = 0, 1, 2, 3, \ldots$$

Kada je $C_2 = 0$ onda je $\sin\sqrt{\lambda} = 0$ pa je

$$\sqrt{\lambda} = n\pi, \quad n = 0, 1, 2, 3, \ldots$$

Dakle, rješenje problema koje zadovoljava jednadžbu i rubne uvjete ima oblik:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2n+1}{2}\pi x\right) e^{-\left(\left[\frac{2n+1}{2}\pi\right]^2+1\right)t} + b_n \sin(n\pi x)\, e^{-([n\pi]^2+1)t}.$$

Potrebno je zadovoljiti još početni uvjet:

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2n+1}{2}\pi x\right) + b_n \sin(n\pi x) = f(x).$$

Radi se o razvoju u generalizirani Fourierov red funkcije f(x):

In [37]:
```
p=piecewise((0,Lt(x,0)),(x,Ge(x,0)))
```

Out[37]:

$$\begin{cases} 0 & \text{for } x < 0 \\ x & \text{otherwise} \end{cases}$$

In [39]:
```
xx=linspace(-1,1)
y=[p(xx[i]) for i=1:length(xx)];
```

Provjerimo ortonormiranost sustava funkcija.

In [40]:
```
⋅(cos((2*n+1)*pi*x/2),cos((2*n+1)*pi*x/2),-1,1)
```

Out[40]:

$$1$$

In [41]:
```
⋅(sin(n*pi*x),sin(n*pi*x),-1,1)
```

Out[41]:

$$\begin{cases} 1 & \text{for } n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Norme svih funkcija su jednake 1 pa ne trebanmo računati nazivnike.

In [42]:
```
a(n)=⋅(p(x),cos((2*n+1)*PI*x/2),-1,1)
```

Out[42]: `a (generic function with 1 method)`

In [43]:
```
a(0)
```

Out[43]:

$$-\frac{4}{\pi^2} + \frac{2}{\pi}$$

In [44]:
```
N(a(0))
```

Out[44]: `0.23133503779823025`

In [45]:
```
b(n)=⋅(p(x),sin(n*PI*x),-1,1)
```

Out[45]: `b (generic function with 1 method)`

```
In [46]: b(0)
```

Out[46]:

$$0$$

```
In [47]: b(1)
```

Out[47]:

$$\frac{1}{\pi}$$

Pripremimo se za brže računanje tako da ćemo unaprijed izračunati numeričke vrijednosti koeficijenata a_n i b_n.

```
In [48]: A=[N(a(n)) for n=0:20]
```

```
Out[48]: 21-element Array{Float64,1}:
           0.231335
          -0.257238
           0.111113
          -0.0992168
           0.065732
          -0.061224
           0.0465726
          -0.0442426
           0.0360459
          -0.034629
           0.0293962
          -0.0284453
           0.0248163
          -0.0241345
           0.0214705
          -0.0209579
           0.0189193
          -0.01852
           0.0169099
          -0.01659
           0.0152862
```

```
In [49]: B=[N(b(n)) for n=0:20]
```

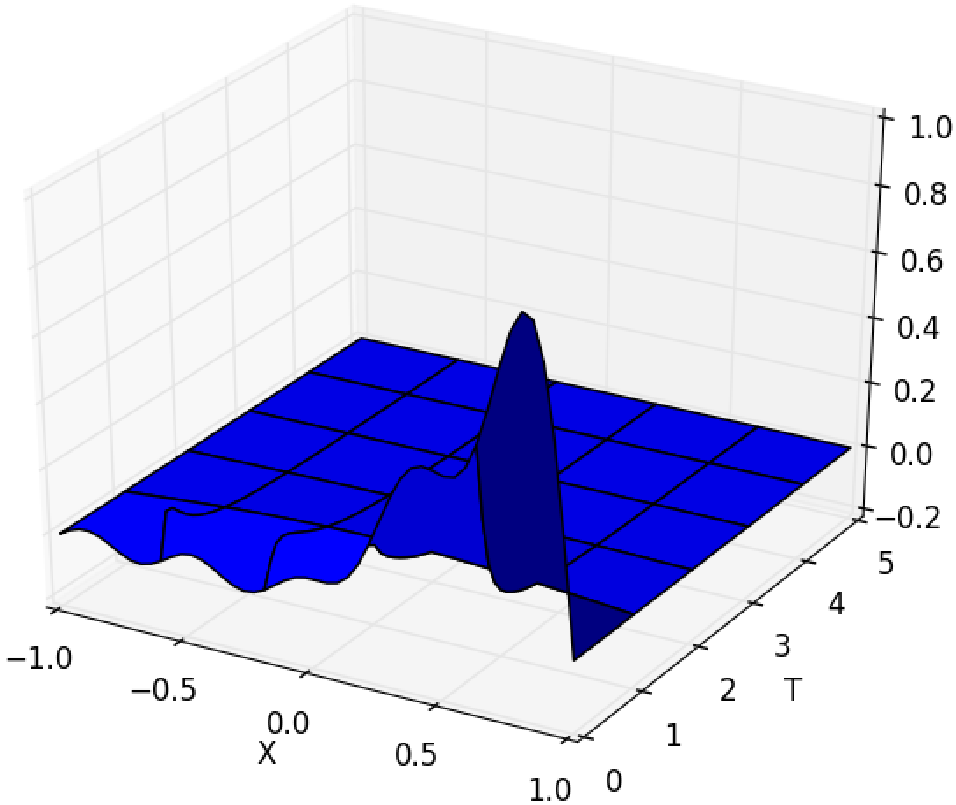
```
Out[49]: 21-element Array{Real,1}:
           0
           0.31831
          -0.159155
```

```
 0.106103
-0.0795775
 0.063662
-0.0530516
 0.0454728
-0.0397887
 0.0353678
-0.031831
 0.0289373
-0.0265258
 0.0244854
-0.0227364
 0.0212207
-0.0198944
 0.0187241
-0.0176839
 0.0167532
-0.0159155
```

In [50]:
```
X=linspace(-1,1)
T=linspace(0,5)
XT=collect(Iterators.product(X,T));
```

In [52]:
```
g=figure()
@manipulate for l in slider(1:20, value=5); withfig(g) do
        h(xt)=sum([A[k]*cos.((2*k-1)*pi*xt[1]/2).*
                exp.(-(((2*k-1)*pi/2)^2/4+1)*xt[2])+
                B[k]*sin.((k-1)*pi*xt[1]).*exp.(-(((k-1)*pi)^2+1)*xt[2])
                for k=1:l])
        surf(X,T,map(h,XT)')
        xlabel("X")
        ylabel("T")
    end
end
```

Out[52]:



1.4 Homogenizacija

U oba prethodna primjera zadani su homogeni rubni uvjeti. Ukoliko rubni uvjeti nisu homogeni, zadani problem je potrebno *homogenizirati* kako bi mogli dobiti regularni SLP.

Navedimo primjer. Neka je zadan problem

$$
\begin{aligned}
&u_t - u_{xx} = 0, \quad 0 < x < l, \quad t > 0 \\
&u(x,0) = f(x), \quad 0 < x < l \\
&u(0,t) = g(t), \quad u(l,t) = h(t), \quad t > 0.
\end{aligned}
$$

Nađimo rješenje u obliku

$$u(x,t) = v(x,t) + U(x,t),$$

gdje je v rješenje problema sa homogenim rubnim uvjetima. Vrijedi

$$
\begin{aligned}
u &= v + U \\
u_t &= v_t + U_t \\
u_{xx} &= v_{xx} + U_{xx}
\end{aligned}
$$

pa zadana PDJ prelazi u

$$v_t + U_t = v_{xx} + U_{xx}.$$

Početni uvjet za v glasi

$$v(x,0) = u(x,0) - U(x,0) = f(x) - U(x,0),$$

a rubni uvjeti glase

$$
\begin{aligned}
v(0,t) &= u(0,t) - U(0,t) = g(t) - U(0,t) = 0 \quad \text{(želimo homogeni uvjet)} \\
v(l,t) &= u(l,t) - U(l,t) = h(t) - U(l,t) = 0 \quad \text{(želimo homogeni uvjet)}
\end{aligned}
$$

Zaključujemo da će v zadovoljavati homogene rubne uvjete ako je

$$U(x,t) = g(t) + \frac{x}{l}[h(t) - g(t)], \quad 0 < x < l.$$

Za ovako definiranu funkciju U vrijedi

$$
\begin{aligned}
U_t &= g'(t) + \frac{x}{l}[h'(t) - g'(t)] \\
U_{xx} &= 0.
\end{aligned}
$$

Uvrštavanjem slijedi da je v rješenje *homogenog* reakcijsko-difuzijskog problema

$$
\begin{aligned}
v_t &= v_{xx} - g'(t) - \frac{x}{l}[h'(t) - g'(t)], \quad 0 < x < l, \quad t > 0 \\
v(x,0) &= f(x) - g(0) - \frac{x}{l}[h(0) - g(0)], \quad 0 < x < l \\
v(0,t) &= 0, \quad v(l,t) = 0, \quad t > 0,
\end{aligned}
$$

dok je rješenje polaznog problema

$$u(x,t) = v(x,t) + g(t) + \frac{x}{l}[h(t) - g(t)].$$