MPP07c DAlembertovo rjesenje

Ivan Slapničar

11. siječnja 2019.

1 D'Alembertovo rješenje valne jednadžbe

1.1 Rubni uvjeti na konačnoj domeni

Promotrimo problem početnih vrijednosti

$$
\begin{aligned}
& u_t + c\,u_x = 0, \quad 0 < x < l, \quad t > 0, \quad c > 0 \\
& u(x,0) = f(x).
\end{aligned}
$$

Rješenje je desni val

$$u(x,t) = f(x-ct),$$

a karakteristične krivulje su pravci

$$x(t) = ct + k,$$

odnosno

$$t(x) = \frac{1}{c}(x-k).$$

U ovom slučaju možemo zadati i *rubni uvjet* za $x = 0$:

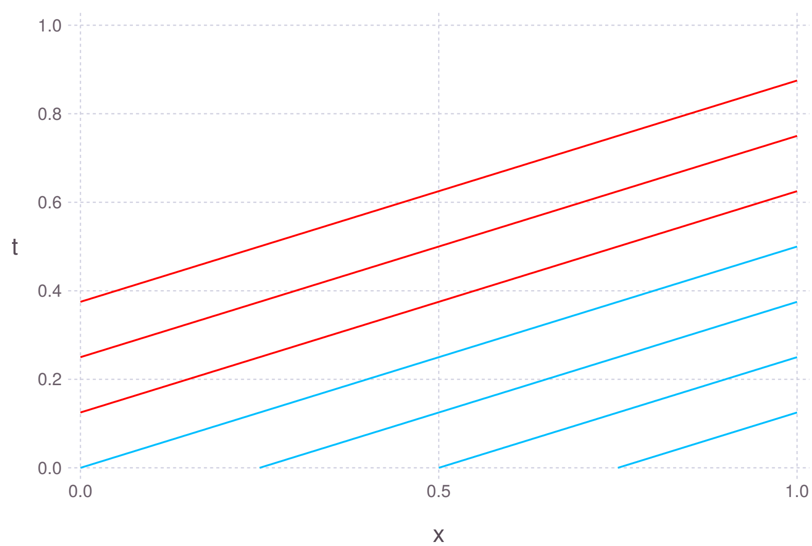
$$u(0,t) = g(t).$$

U desnom rubu, za $x = l$, rješenje je određeno karakterističnim pravcima: za $t \in [0, l/c]$ rješenje je određeno početnim uvjetom (plave linije), a za $t > l/c$ rješenje je određeno rubnim uvjetom (crvene linije).

```
In [1]: using Gadfly
```

```
In [2]: # Za l=1
        c=2
        tx(x,k)=(x-k)/c
        Gadfly.plot(layer(x->tx(x,0),0,1),
            layer(x->tx(x,0.25),0.25,1),
            layer(x->tx(x,0.5),0.5,1),
            layer(x->tx(x,0.75),0.75,1),
            layer(x->tx(x,-0.25),0,1,Theme(default_color=colorant"red")),
            layer(x->tx(x,-0.5),0,1,Theme(default_color=colorant"red")),
            layer(x->tx(x,-0.75),0,1,Theme(default_color=colorant"red")),
            Guide.xlabel("x"),Guide.ylabel("t"))
```

Out[2]:



1.1.1 Uvjeti kompatibilnosti

U prethodnom slučaju nema lijevih valova pa nema ni refleksije. U slučaju općenite valne jednadžbe možemo zadati rubne uvjete u oba ruba:

$$\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l, \quad t > 0, \\
&u(x,0) = F(x), \quad u_t(x,0) = G(x), \\
&u(0,t) = a(t), \quad u(l,t) = b(t).
\end{aligned}$$

U tom slučaju *uvjeti kompatibilnosti* trebaju biti ispunjeni u točkama $(0,0)$ i $(l,0)$:

$$
\begin{aligned}
F(0) &= u(0,0) = a(0) \\
F(l) &= u(l,0) = b(0) \\
G(0) &= u_t(0,0) = a'(0) \\
G(l) &= u_t(l,0) = b'(0).
\end{aligned}
$$

1.2 D'Alembertovo rješenje

Teorem. Rješenje problema početnih vrijednosti

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
&u(x,0) = F(x), \quad u_t(x,0) = G(x)
\end{aligned}
$$

je

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} G(y)\, dy.$$

Dokaz: Karakteristične krivulje su pravci, a rješenje je kombinacija lijevog i desnog vala pa ima oblik

$$u(x,t) = f(x+ct) + g(x-ct).$$

Vrijedi

$$
\begin{aligned}
u(x,0) &= f(x) + g(x) = F(x), \qquad (1) \\
u_t(x,0) &= f'(x) \cdot c + g'(x) \cdot (-c) = G(x),
\end{aligned}
$$

odnosno

$$f'(x) - g'(x) = \frac{1}{c} G(x).$$

Integriranje daje

$$f(x) - g(x) = \frac{1}{c} \int G(x)\, dx = \frac{1}{c} \int_0^x G(y)\, dy + C. \qquad (2)$$

Zbrajanje (1) i (2) daje

$$f(x) = \frac{1}{2} F(x) + \frac{1}{2c} \int_0^x G(y)\, dy + \frac{1}{2} C,$$

a oduzimanje (2) od (1) daje

$$g(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int\limits_0^x G(y)\,dy - \frac{1}{2}C.$$

Dakle,

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c}\Bigg[\int\limits_0^{x+ct} G(y)\,dy - \int\limits_0^{x-ct} G(y)\,dy\Bigg] \end{aligned}$$

i teorem je dokazan.

Vidimo da vrijednost $u(x_0, y_0)$, ovisi o početnim uvjetima na intervalu $x \in [x_0 - c\,t_0, x_0 + c\,t_0]$.

Taj interval je *područje ovisnosti* (*domain of dependence*) točke (x_0, t_0).

Slično, početni uvjeti na intervalu $x \in [x_1, x_2]$ utječu na rješenje unutar područja omeđenog pravcima

$$x + c\,t = x_1, \quad x - c\,t = x_2, \quad t = 0.$$

To područje je *područje utjecaja* (*region of influence*).

Primjer. Riješenje problema

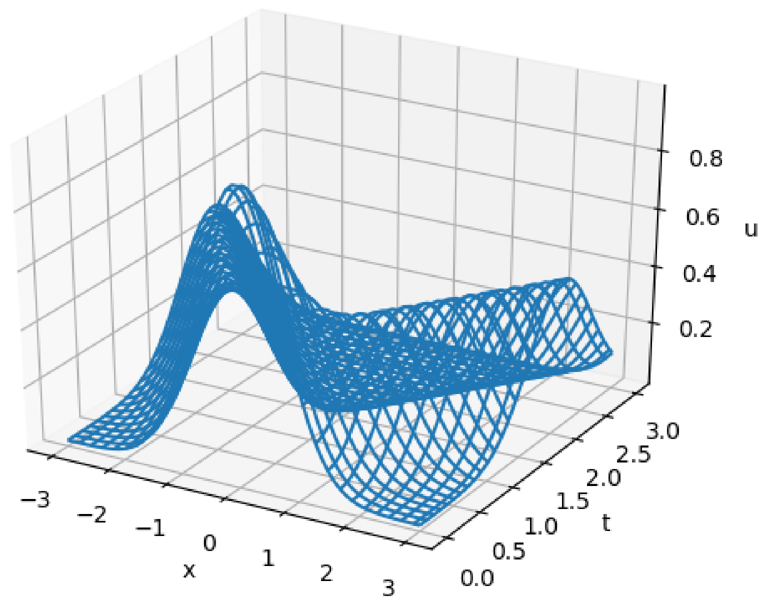
$$\begin{aligned} &u_{tt} - 2\,u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = e^{-x^2}, \quad u_t(x,0) = 0, \end{aligned}$$

je

$$u(x,t) = \frac{1}{2}\big[e^{-(x+\sqrt{2}\,t)^2} + e^{-(x-\sqrt{2}\,t)^2}\big].$$

```
In [3]: using PyPlot
```

```
In [4]: # Rješenje
        gridsize=60
        X=range(-3,stop=3,length=gridsize)
        T=range(0,stop=3,length=gridsize)
        XT=collect(Iterators.product(X,T))
        u(x,t)=(exp(-(x+sqrt(2)*t)^2)+exp(-(x-sqrt(2)*t)^2))/2
        U=[u(XT[i,j][1],XT[i,j][2]) for i=1:gridsize,j=1:gridsize]
        PyPlot.mesh(X,T,Matrix(U'))
        xlabel("x")
        ylabel("t")
        zlabel("u")
```



Za $k, h \in \mathbb{R}$ definirajmo točke

$$\begin{aligned} A &= (x - c\,k, t - h), \quad & B &= (x + c\,h, t + k), \\ C &= (x + c\,k, t + h), \quad & D &= (x - c\,h, t - k). \end{aligned}$$

Lema. Funkcija $u(x,t)$ zadovoljava jednadžbu

$$u_{tt} - c^2\,u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3}$$

ako i samo ako vrijedi

$$u(A) + u(C) = u(B) + u(D). \tag{4}$$

Dokaz: Pretpostavimo da je

$$u(x,t) = f(x + c\,t) + g(x - c\,t)$$

rješenje. Tada je

$$
\begin{aligned}
u(A) &= f(x - ck + c(t-h)) + g(x - ck - c(t-h)) \\
&= f(x - ck + ct - ch) + g(x - ck - ct + ch) \\
u(B) &= f(x + ch + c(t+k)) + g(x + ch - c(t+k)) \\
&= f(x + ch + ct + ck) + g(x + ch - ct - ck) \\
u(C) &= f(x + ck + c(t+h)) + g(x + ck - c(t+h)) \\
&= f(x + ck + ct + ck) + g(x + ck - ct - ch) \\
u(D) &= f(x - ch + c(t-k)) + g(x - ch - c(t-k)) \\
&= f(x - ch + ct - ck) + g(x - ch - ct + ck)
\end{aligned}
$$

pa (4) vrijedi.

Obrnuto, neka za $u(x,t)$ vrijedi (4). Za $h=0$ vrijedi

$$
u(x-ck,t) + u(x+ck,t) = u(x,t+k) + u(x,t-k). \tag{5}
$$

Taylorova formula daje

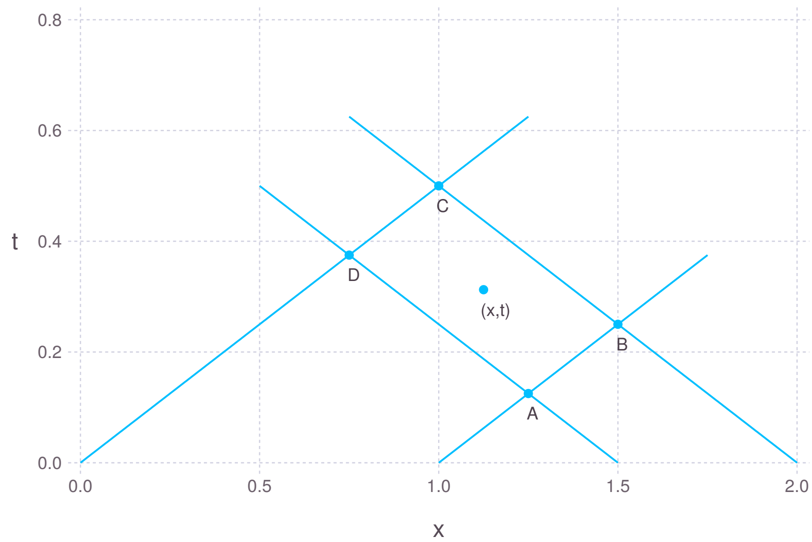
$$
\begin{aligned}
u(x-ck,t) &= u(x,t) + u_x(x,t)\cdot(-ck) + \frac{1}{2}u_{xx}(x,t)\cdot(-ck)^2 + O(k^3) \\
u(x+ck,t) &= u(x,t) + u_x(x,t)\cdot ck + \frac{1}{2}u_{xx}(x,t)\cdot(ck)^2 + O(k^3) \\
u(x,t-k) &= u(x,t) + u_t(x,t)\cdot(-k) + \frac{1}{2}u_{tt}(x,t)\cdot(-k)^2 + O(k^3) \\
u(x,t+k) &= u(x,t) + u_t(x,t)\cdot k + \frac{1}{2}u_{tt}(x,t)\cdot k^2 + O(k^3),
\end{aligned}
$$

pa uvrštavanje u (5), dijeljenje s k^2 i ignoriranje članova $O(k)$ daje (3) i teorem je dokazan.

Lema je ilustrirana sljedećom slikom:

In [5]:
```
c=2
X = [5/4,3/2,1,3/4,9/8]
T = [1/8,1/4,1/2,3/8,5/16]
Labels = ["A","B","C","D","(x,t)"]
Gadfly.plot(layer(x->(x-1)/c,1,1.75),
    layer(x->(x-0)/c,0,1.25),
    layer(x->(x-2)/(-c),0.75,2),
    layer(x->(x-1.5)/(-c),0.5,1.5),
    layer(x=X,y=T,label=Labels,Geom.point,Geom.label),
    Guide.xlabel("x"),Guide.ylabel("t"))
```

Out[5]:

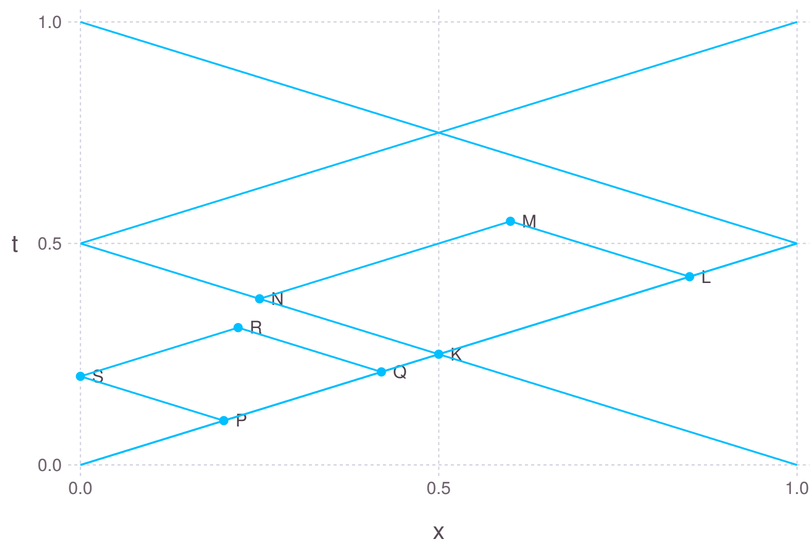


Na omeđenoj domeni rješenje se konstruira na sljedeći način:

- rješenje u donjem trokutu daje D'Alembert-ova formula,
- rješenje u lijevom trokutu dobije se pomoću leme: $u(R) = u(S) + u(Q) - u(P)$, pri čemu se $u(P)$ i $u(Q)$ izračunaju D'Alembert-ovom formulom (leže u donjem trokutu), a $u(S)$ je rubni uvjet (rješenje u desnom trokutu dobije se slično),
- rješenje u prvom rombu dobije se pomoću leme: $u(M) = u(L) + u(N) - u(K)$ pri čemu se $u(K)$, $u(L)$ i $u(N)$ dobiju iz lijevog i desnog trokuta,
- u ostalim djelovima postupak se nastavlja analogno.

In [6]:
```
c=2
X = [0,0.2,0.42,0.22,0.5,0.25,0.6,0.85]
T = [0.2,0.1,0.21,0.31,0.25,0.375,0.55,0.425]
Labels = ["S","P","Q","R","K","N","M","L"]
Gadfly.plot(layer(x->x/c,0,1),
    layer(x->(x-0)/c,0,1),
    layer(x->(x-1)/(-c),0,1),
    layer(x->(x+1)/c,0,1),
    layer(x->(x-2)/(-c),0,1),
    layer(x->(x+0.4)/c,0,0.22),
    layer(x->(x-0.4)/(-c),0,0.2),
    layer(x->(x-0.84)/(-c),0.22,0.42),
    layer(x->(x+0.5)/c,0.25,0.6),
    layer(x->(x-1.7)/(-c),0.6,0.85),
    layer(x=X,y=T,label=Labels,Geom.point,Geom.label(position=:right)),
    Guide.xlabel("x"),Guide.ylabel("t"),
)
```

Out[6]:



Primjer. Primijenimo postupak na poluomeđenu domenu:

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t > 0,\\
&u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x > 0,\\
&u(0,t) = h(t), \quad t > 0.
\end{aligned}
$$

Uvjeti kompatibilnosti su $f(0) = h(0)$ i $g(0) = h'(0)$. Karakteristične krivulje su pravci

$$t(x) = \frac{1}{c}(x - k).$$

Za $0 < t < \frac{x}{c}$ rješenje je dano D'Alembert-ovom formulom:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(y)\,dy. \tag{6}$$

Za $0 < \frac{x}{c} < t$ rješenje slijedi primjenom leme (lijevi trokut na gornjoj slici), odnosno

$$u(x,t) = u(R) = u(S) + u(Q) - u(P). \tag{7}$$

Točka S leži na pravcu

$$y(\xi) = \frac{\xi}{c} + \big(t - \frac{x}{c}\big)$$

pa je

$$u(S) = h\big(t - \frac{x}{c}\big).$$

Točka P je presjek pravca $y(\xi) = \xi/c$ s pravcem

$$y(\xi) = -\frac{\xi}{c} + \big(t - \frac{x}{c}\big),$$

odnosno

$$P = \Big(\frac{ct-x}{2}, \frac{ct-x}{2c}\Big),$$

pa (6) daje

$$u(P) = \frac{1}{2}[f(ct-x) + f(0)] + \frac{1}{2c}\int\limits_0^{ct-x} g(y)\, dy.$$

Slično, točka Q je presjek pravca $y(\xi) = \xi/c$ s pravcem

$$y(\xi) = -\frac{\xi}{c} + \big(t + \frac{x}{c}\big),$$

pa je

$$u(Q) = \frac{1}{2}[f(ct+x) + f(0)] + \frac{1}{2c}\int\limits_0^{ct+x} g(y)\, dy.$$

Uvrštavanje u (7) konačno daje

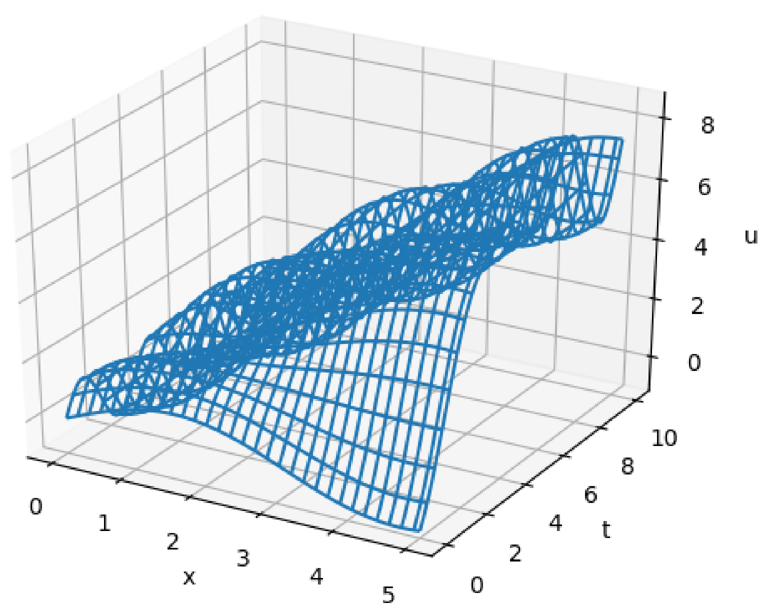
$$u(x,t) = h\big(t - \frac{x}{c}\big) + \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int\limits_{ct-x}^{ct+x} g(y)\, dy.$$

Nacrtajmo rješenje problema:

$$\begin{aligned}
&u_{tt} - 2\,u_{xx} = 0, \quad x > 0, \quad t > 0,\\
&u(x,0) = \sin x, \quad u_t(x,0) = 2, \quad x > 0,\\
&u(0,t) = \sin 2t, \quad t > 0.
\end{aligned}$$

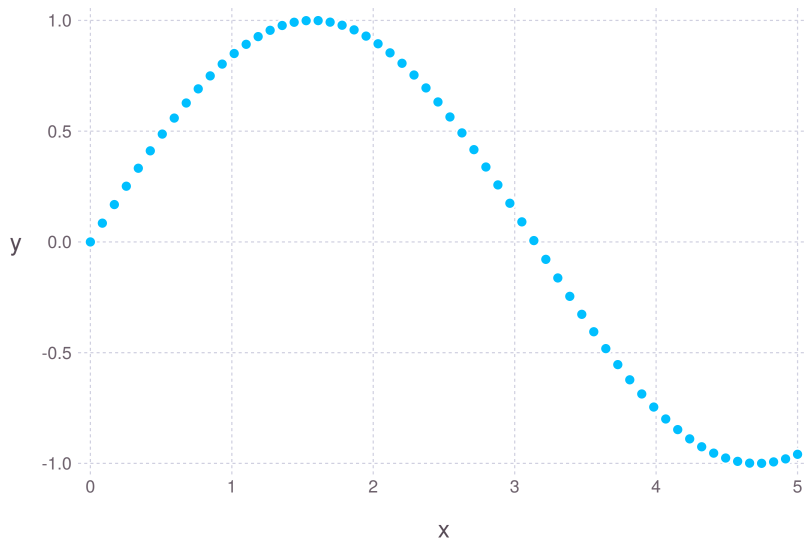
In [7]:
```
using QuadGK
```

In [8]:
```
c=sqrt(2)
f(x)=sin(x)
g(x)=2*x^0
h(t)=sin(2*t)
gridsize=60
ξ=range(0,stop=5,length=gridsize)
τ=range(0,stop=10,length=gridsize)
X=repeat(ξ,1,gridsize)
T=repeat(τ',gridsize,1)
U=Array{Float64}(undef,gridsize,gridsize)
for i=1:gridsize,j=1:gridsize
    x=X[i,j]
    t=T[i,j]
   if t<=x/c
        U[i,j]=(f(x+c*t)+f(x-c*t))/2+quadgk(g,x-c*t,x+c*t)[1]/(2*c)
    else
        U[i,j]=(f(x+c*t)-f(c*t-x))/2+quadgk(g,c*t-x,x+c*t)[1]/(2*c)+h(t-x/c)
    end
end
mesh(X,T,U)
xlabel("x")
ylabel("t")
zlabel("u")
```



In [9]:
```
# Provjerimo početni uvjet
Gadfly.plot(x=ξ,y=U[:,1])
```

Out[9]:



In [10]:
```
# Provjerimo rubni uvjet
Gadfly.plot(x=τ,y=U[1,:])
```

Out[10]: