NA10 Interpolacija funkcija

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1 Interpolacija funkcija

Neka je zadana funkcija $f(x)$ na intervalu $[a,b]$.

Odaberimo $n+1$ točku $x_i, i=0,\ldots,n$, u intervalu $[a,b]$ tako da je $x_i\neq x_j$ te kroz točke $T_i=(x_i,f(x_i))$ provucimo interpolacijski polinom.

Za svaku točku $x\in[a,b]$ vrijedi *ocjena pogreške* (uz pretpostavku da funkcija f ima $n+1$ derivaciju)

$$
\begin{aligned}
f(x)-p_n(x)&=\frac{\omega(x)}{(n+1)!}f^{(n+1)}(\xi), \\
\omega(x)&=\prod_{k=0}^n (x-x_k)=(x-x_0)(x-x_1)\cdots(x-x_n), \quad \xi\in(a,b).
\end{aligned} \tag{1}
$$

Dokaz. Vidi Numerička matematika, str. 23.

1.1 Primjer

Promotrimo funkciju

$$
f(x)=\sin(x), \quad x\in[0,\pi].
$$

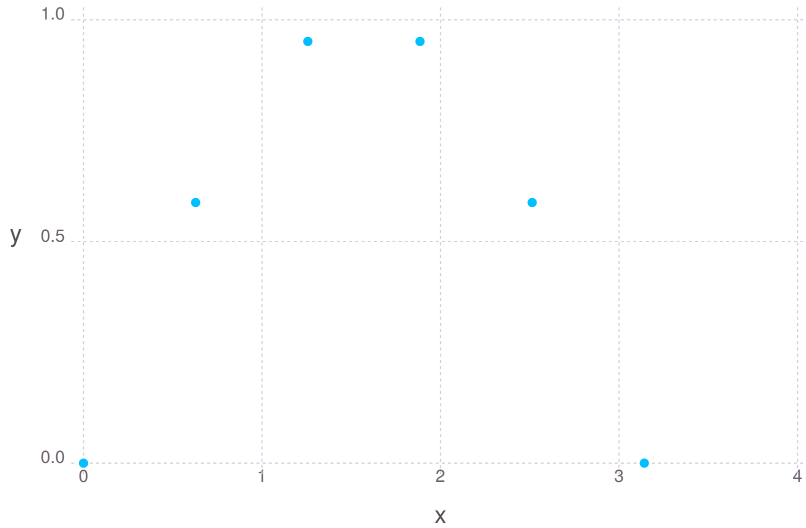
In [1]:
```
using Polynomials
using Gadfly
```

In [2]:
```
# Ova datoteka omogućuje manipulaciju s Vandermondeovim matricama
include("Vandermonde.jl")
```

Out[2]:
```
full (generic function with 1 method)
```

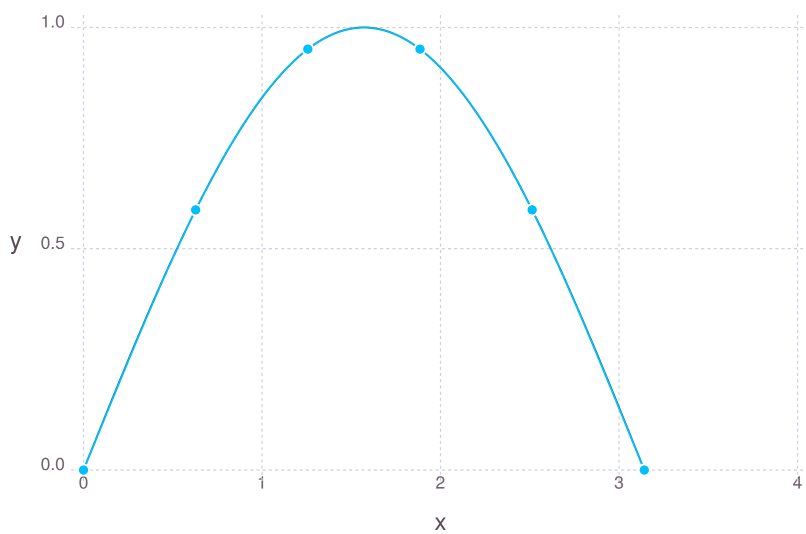
In [3]:
```
n=6
a=0
b=pi
x=collect(linspace(a,b,n))
y=sin.(x)
```

```
Out[3]: 6-element Array{Float64,1}:
         0.0
         0.587785
         0.951057
         0.951057
         0.587785
         1.22465e-16

In [4]: A=Vandermonde(x)

Out[4]: 6×6 Vandermonde{Float64}:
         1.0  0.0       0.0        0.0       0.0        0.0
         1.0  0.628319  0.394784   0.24805   0.155855   0.0979263
         1.0  1.25664   1.57914    1.9844    2.49367    3.13364
         1.0  1.88496   3.55306    6.69736  12.6242    23.7961
         1.0  2.51327   6.31655   15.8752   39.8988   100.277
         1.0  3.14159   9.8696    31.0063   97.4091   306.02

In [5]: c=A\y

Out[5]: 6-element Array{Float64,1}:
          0.0
          0.985329
          0.0524812
         -0.23308
          0.0370958
          6.25915e-16

In [6]: p=Poly(c)

Out[6]: Poly(0.9853290520718775*x + 0.05248115215994874 5*x^2
- 0.23307995080786054*x^3 + 0.03709582630668404 5*x^4 + 6.259152379042377e-16*x^5)

In [7]: plot(x=x,y=y)

Out[7]:
```



In [8]:
```
xx=linspace(a,b,100)
pS=polyval(p,xx)
sinus=sin.(xx)
plot(layer(x=x,y=y,Geom.point),layer(x=xx,y=pS,Geom.line),
    layer(x=xx,y=sinus,Theme(default_color=colorant"black"),
        Geom.line))
```

Out[8]:



```
In [9]: # maksimalne apsolutna i relativna pogreška
        norm(pS[2:end-1]-sinus[2:end-1],Inf),
        norm((pS[2:end-1]-sinus[2:end-1])./sinus[2:end-1],Inf)
```

```
Out[9]: (0.001311441310739292, 0.013073445174533032)
```

1.2 Čebiševljeve točke

Čebiševljevi polinomi su polinomi stupnja n dani formulom

$$T_n(x)=\cos(n\ \arccos x),\quad n=0,1,2,\ldots$$

Vrijedi rekurzivna formula:

$$\begin{aligned}T_0(x)&=1,\\ T_1(x)&=x,\\ T_{n+1}(x)&=2\,x\,T_n(x)-T_{n-1}(x),\quad n=1,2,\ldots\end{aligned}$$

Dakle,

$$T_2(x)=2x^2-1,\quad T_3(x)=4x^3-3x,\ldots$$

Nul-točke polinoma $T_n(x)$ su

$$x_k=\cos\left(\frac{2k-1}{2n}\pi\right),\quad k=1,2,\ldots,n.$$

Sve nul-točke leže unutar intervala $[-1,1]$.
Na intervalu $[-1,1]$ polinom $T_n(x)$ poprima vrijednosti u intervalu $[-1,1]$.
Napomena. Rekurzivna formula se dokazuje korištenjem adicione formule

$$\cos(n+1)\varphi+\cos(n-1)\varphi=2\cos\varphi\cos n\varphi$$

uz $\varphi=\arccos x$.

1.2.1 Primjer

```
In [10]: T(n,x)=cos.(n*acos.(x))
```

```
Out[10]: T (generic function with 1 method)
```

```
In [11]: x1=linspace(-1,1,100)
```

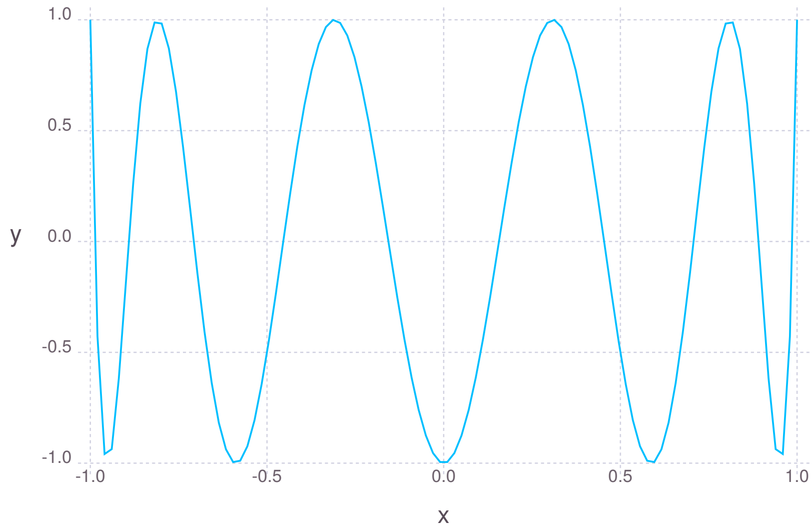
```
Out[11]: -1.0:0.020202020202020204:1.0
```

```
In [12]: y1=T(10,x1)
```

```
Out[12]: 100-element Array{Float64,1}:
           1.0
          -0.428362
          -0.958456
          -0.936674
          -0.616793
          -0.178048
           0.259411
           0.621398
           0.868645
           0.987601
           0.982859
           0.871005
           0.675712
           ⋮
           0.871005
           0.982859
           0.987601
           0.868645
           0.621398
           0.259411
          -0.178048
          -0.616793
          -0.936674
          -0.958456
          -0.428362
           1.0
```

```
In [13]: plot(x=x1,y=y1,Geom.line)
```

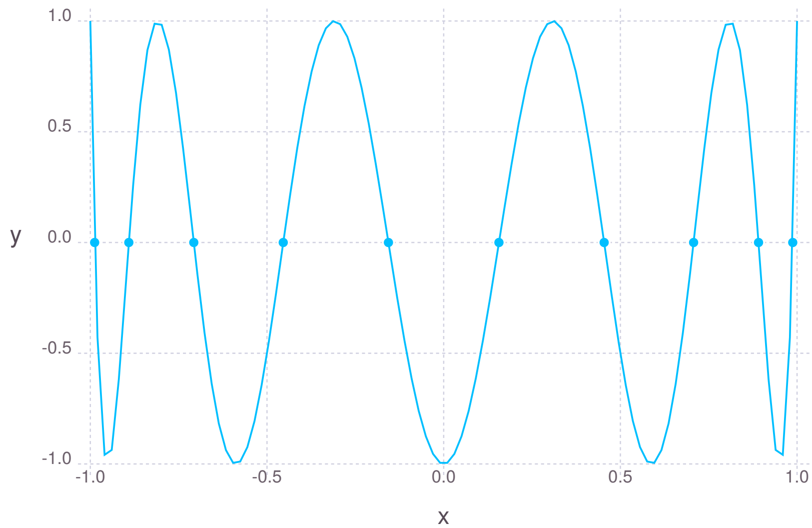
```
Out[13]:
```




```
In [14]: xn=[cos((2*k-1)*pi/(2*10)) for k=1:10]

Out[14]: 10-element Array{Float64,1}:
           0.987688
           0.891007
           0.707107
           0.45399
           0.156434
          -0.156434
          -0.45399
          -0.707107
          -0.891007
          -0.987688

In [15]: yn=T(10,xn)

Out[15]: 10-element Array{Float64,1}:
           2.72577e-15
          -1.07188e-15
           3.06162e-16
          -4.28626e-16
           5.51091e-16
          -2.44991e-15
          -9.80336e-16
          -2.69484e-15
          -7.35407e-16
          -6.49248e-15
```

```
In [16]: plot(layer(x=x1,y=y1,Geom.line),layer(x=xn,y=yn,Geom.point))
```

```
Out[16]:
```



1.2.2 Norme funkcija

Za funkcije

$$f, g : [a, b] \to \mathbb{R}$$

definiramo *skalarni produkt*

$$(f, g) = \int_a^b f(x)g(x)\,dx$$

i *težinski skalarni produkt* s *težinom* $\omega(x) > 0$

$$(f, g)_\omega = \int_a^b f(x)g(x)\omega(x)\,dx$$

Funkcije f i g su *ortogonalne* ako je $(f, g) = 0$ ili ako je $(f, g)_\omega = 0$.
Sljedeće tri *norme* su prirodna poopćenja odgovarajućih vektorskih normi:

$$
\begin{aligned}
\|f\|_2 &= \sqrt{(f,f)} = \sqrt{\int_a^b f^2(x)\,dx} \\
\|f\|_1 &= \int_a^b \big|f(x)\big|\,dx \\
\|f\|_\infty &= \max_{x\in[a,b]} \big|f(x)\big|
\end{aligned}
$$

Vrijedi sljedeći važan teorem:

Teorem. Od svih polinoma stupnja manje ili jednako n čiji je koeficijent uz najveću potenciju jednak 1, najmanju $\|\cdot\|_\infty$ na intervalu $[-1,1]$ ima upravo polinom $\frac{1}{2^{n-1}}T_n(x)$ i ta norma iznosi $\frac{1}{2^{n-1}}$.

Dokaz: Vidi Numerička matematika, str. 101.

Zaključujemo da će polinomna aproksimacija (1) biti najbolja ako na intervalu $[-1,1]$ odaberemo

$$
\omega(x) = \frac{1}{2^n}T_{n+1}(x),
$$

odnosno ako na intervalu $[a,b]$ za točke interpolacije $x_0, x_1, \ldots, x_n$ odaberemo upravo nul-točke polinoma $T_{n+1}(x)$ preslikane na interval $[a,b]$.

1.2.3 Promjena intervala

Sustav ortogonalnih funkcija Φ_i na intervalu $[-1,1]$ pomoću transformacije

$$
\gamma : [a,b] \to [-1,1], \quad \gamma(x) = \frac{2x}{b-a} - \frac{a+b}{b-a}
$$

prelazi u sustav ortogonalnih funkcija na intervalu $[a,b]$

$$
\Psi_i(x) = \Phi_i(\gamma(x)).
$$

```
In [17]: # Odaberimo za interpolaciju sinusa nultočke polinoma T(n,x)
         xc=(a+b)/2+(b-a)/2*map(Float64,[cos((2*k-1)*pi/(2*n)) for k=1:n])
```

```
Out[17]: 6-element Array{Float64,1}:
          3.08807
          2.68152
          1.97735
          1.16424
          0.460076
          0.0535236
```

```
In [18]: yc=sin.(xc)
```

Out[18]: 6-element Array{Float64,1}:

```
 0.053498
 0.444016
 0.91849
 0.91849
 0.444016
 0.053498
```

In [19]:

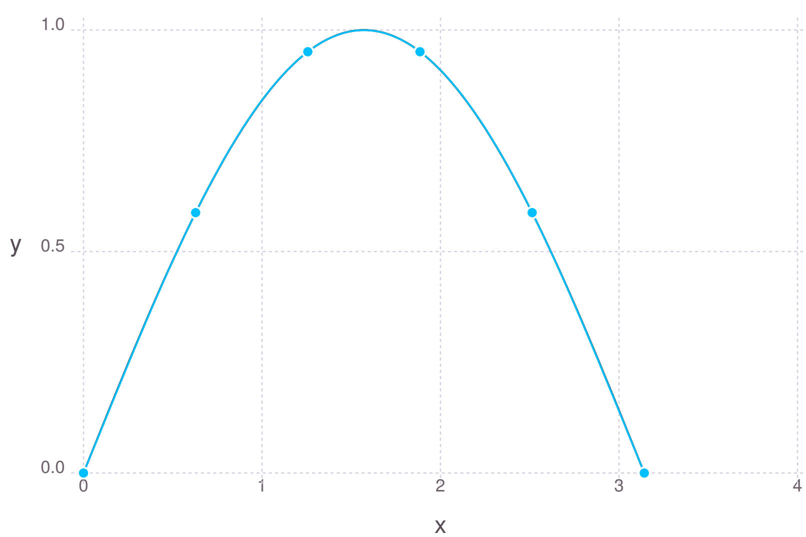
```
Ac=Vandermonde(xc)
cc=Ac\yc
pc=Poly(cc)
```

Out[19]: Poly(0.000583379034201581 + 0.9866631446414202*x + 0.04888081853370563*x^2 - 0.23105825

In [20]:

```
xx=linspace(a,b,100)
pC=polyval(pc,xx)
sinus=sin.(xx)
plot(layer(x=x,y=y,Geom.point),layer(x=xx,y=pS,Geom.line),
    layer(x=xx,y=sinus,Theme(default_color=colorant"Black"),
        Geom.line))
```

Out[20]:



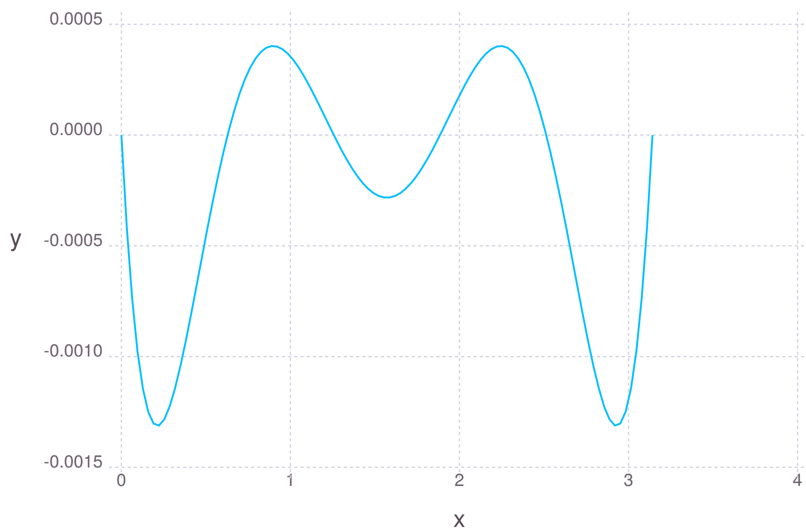
In [21]:

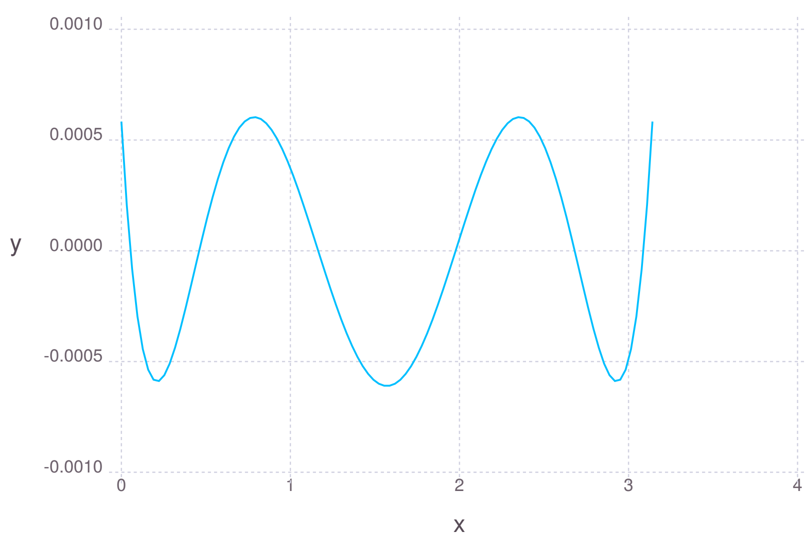
```
# maksimalne apsolutna i relativna pogreška
norm(pC[2:end-1]-sinus[2:end-1],Inf),
norm((pC[2:end-1]-sinus[2:end-1])./sinus[2:end-1],Inf)
```

Out[21]: (0.0006090743316018443, 0.006535549604570311)

Pogledajmo kako izgledaju stvarne pogreške u oba slučaja:

```
In [22]: p1=plot(x=xx,y=pS-sinus,Geom.line) # ravnomjerno raspoređene točke
         p2=plot(x=xx,y=pC-sinus,Geom.line) # Čebiševljeve točke
         display(p1),display(p2)
```





Out[22]: (nothing, nothing)

Vidimo da su za Čebiševljeve točke postignute manje pogreške.

Napomena: ovdje smo, radi jednostavnosti, koristili najmanje točnu varijantu računanja interpolacijskog polinoma.

1.2.4 Primjer

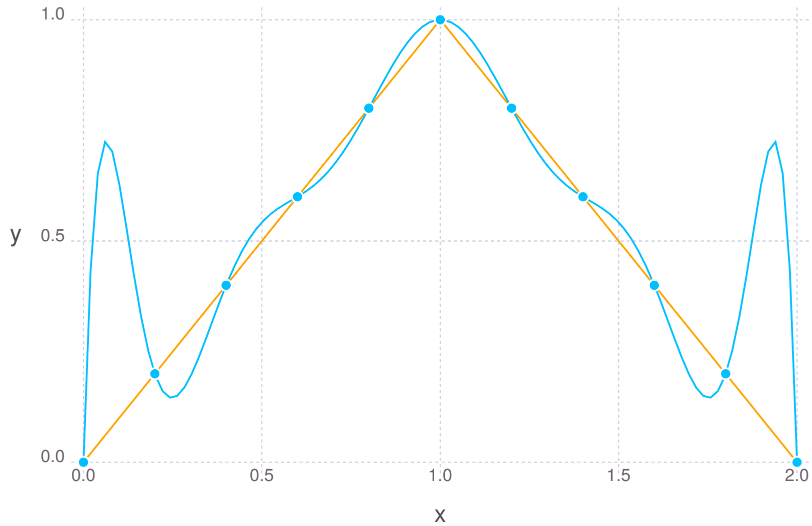
Napravimo još jedan zanimljiv primjer (vidi Numerička matematika, str. 24): interpolirajmo funkciju

$$f(x) = 1 - |x-1|, \quad x \in [0,2]$$

polinomima stupnja 10.

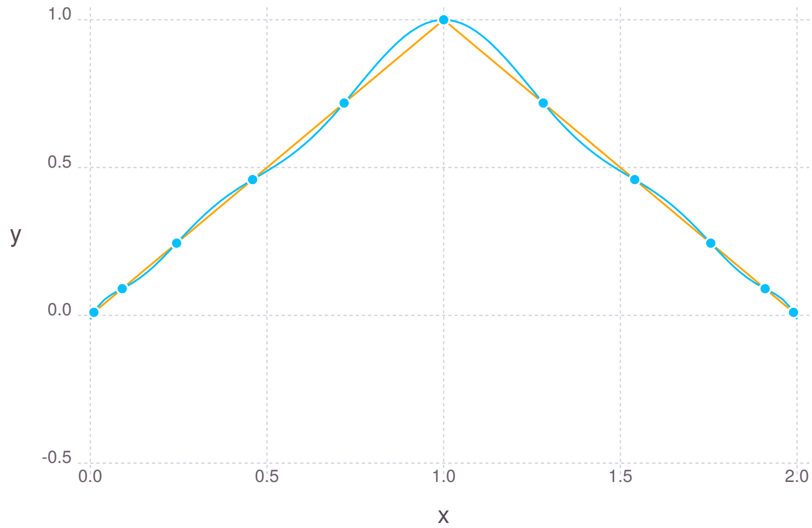
In [23]:
```
n=11
a=0
b=2
f(x)=1-abs.(x-1)

# Ravnomjerno raspoređene točke
x=collect(linspace(a,b,n))
y=f(x)
A=Vandermonde(x)
c=A\y
p=Poly(c)

xx=linspace(a,b,100)
pS=polyval(p,xx)
F=f(xx)
plot(layer(x=x,y=y,Geom.point),layer(x=xx,y=pS,Geom.line),
    layer(x=xx,y=F,Theme(default_color=colorant"orange"),
        Geom.line))
```

Out[23]:



In [24]:
```
# Čebiševljeve točke
xc=(a+b)/2+(b-a)/2*map(Float64,[cos((2*k-1)*pi/(2*n)) for k=1:n])
yc=f(xc)
Ac=Vandermonde(xc)
cc=Ac\yc
pc=Poly(cc)
pCheb=polyval(pc,xx)
plot(layer(x=xc,y=yc,Geom.point),layer(x=xx,y=pCheb,Geom.line),
    layer(x=xx,y=F,Theme(default_color=colorant"orange"),
        Geom.line))
```

Out[24]:



1.0
0.5
0.0
-0.5
y
0.0
0.5
1.0
1.5
2.0
x