NA16 Primjene QR rastava

Ivan Slapničar

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1 Primjene QR rastava

1.1 Rješavanje sustava linearnih jednadžbi

QR rastav možemo koristiti za rješavanje sustava linearnih jednadžbi. U odnosu na rješenje pomoću Gaussove eliminacije vrijedi:

- broj računskih operacija se udvostruči,
- rješenje je nešto točnije, i
- nema rasta elemenata (pivotiranje nije potrebno).

In [1]:
```
import Random
Random.seed!(123)
using LinearAlgebra
n=10
A=rand(n,n)
b=rand(n)
Q,R=qr(A)
c=Q'*b
# Trokutasti sustav
x=R\c
```

Out[1]:
```
10-element Array{Float64,1}:
 -1.3513832062047446
 -2.803574341341646
 -6.212172382452086
 -0.4444920506128095
  2.120799184124583
  4.959355245144356
  1.7098940042037887
  1.1499707113735076
  1.533300587444642
  2.271279372899755
```

In [2]:
```
A*x-b
```

```
Out[2]: 10-element Array{Float64,1}:
          1.5543122344752192e-15
          1.3322676295501878e-15
          7.771561172376096e-16
         -3.1086244689504383e-15
         -4.440892098500626e-16
          1.3322676295501878e-15
         -8.881784197001252e-16
         -1.4432899320127035e-15
         -2.220446049250313e-16
          5.551115123125783e-16
```

1.2 Rješavanje problema najmanjih kvadrata

Programi za rješavanje problema najmanjih kvadrata uglavnom koriste QR rastav. Vrijedi

$$\|Ax-b\|_2^2 = \|QRx-b\|_2^2 = \|Q(Rx-Q^Tb)\|_2 = \|Rx-Q^Tb\|_2^2.$$

Neka je

$$R = \begin{bmatrix} R_0 \\ 0 \end{bmatrix}, \quad Q^Tb = \begin{bmatrix} c \\ d \end{bmatrix}.$$

Tada je

$$\|Rx-Q^Tb\|_2^2 = \|R_0x-c\|_2^2 + \|d\|_2^2$$

pa je rješenje trokutastog sustava

$$R_0x = c$$

rješenje problema najmanjih kvadrata.

```
In [3]: A=[1 2 3 6 7;1 1 1 1 1]'
```

```
Out[3]: 5×2 Adjoint{Int64,Array{Int64,2}}:
         1  1
         2  1
         3  1
         6  1
         7  1
```

```
In [4]: Q,R=qr(A)
```

```
Out[4]: LinearAlgebra.QRCompactWY{Float64,Array{Float64,2}}
        Q factor:
```

```
5×5 LinearAlgebra.QRCompactWYQ{Float64,Array{Float64,2}}:
 -0.100504  -0.694576  -0.36286   -0.423284  -0.443425
 -0.201008  -0.529614  -0.240632   0.433514   0.65823
 -0.301511  -0.364653   0.878914  -0.052476  -0.0296059
 -0.603023   0.130233  -0.135336   0.582037  -0.512172
 -0.703526   0.295195  -0.140086  -0.539791   0.326973
R factor:
2×2 Array{Float64,2}:
 -9.94987  -1.90957
  0.0      -1.16342
```

In [5]:
```
Q'*Q
```

Out[5]:
```
5×5 Array{Float64,2}:
  1.0           -8.32667e-17  -9.71445e-17  -1.66533e-16  -8.32667e-17
 -8.32667e-17   1.0           -6.93889e-17  -1.66533e-16  -1.52656e-16
 -9.71445e-17  -6.93889e-17   1.0           -5.55112e-17  -4.16334e-17
 -1.66533e-16  -1.66533e-16  -5.55112e-17   1.0           -2.77556e-17
 -8.32667e-17  -1.52656e-16  -4.16334e-17  -2.77556e-17   1.0
```

In [6]:
```
m=8
n=5
A=rand(m,n)
b=rand(m)
Q,R=qr(A)
```

Out[6]:
```
LinearAlgebra.QRCompactWY{Float64,Array{Float64,2}}
Q factor:
8×8 LinearAlgebra.QRCompactWYQ{Float64,Array{Float64,2}}:
 -0.0566351  -0.574505    -0.0230314  …  -0.428439    0.498927   -0.316609
 -0.208694   -0.212997     0.342354       0.408418   -0.169411   -0.646288
 -0.065727   -0.527101    -0.139024      -0.106073    0.0272808   0.257775
 -0.22419    -0.00607116   0.32014       -0.616738   -0.468554    0.212567
 -0.793791   -0.0286567   -0.467893       0.0531959  -0.248767   -0.0696813
 -0.378103    0.262243    -0.0385568  …   0.137668    0.611397    0.287264
 -0.342123    0.234304     0.644938      -0.15383     0.245325   -0.0412849
 -0.0924361  -0.471395     0.352264       0.461055   -0.078795    0.530431
R factor:
5×5 Array{Float64,2}:
 -1.23002  -0.827746  -1.40152   -0.968313  -0.828217
  0.0      -1.13251   -0.809317  -0.595303  -0.266157
  0.0       0.0        0.999553   0.588012   0.814772
  0.0       0.0        0.0       -0.661953   0.267255
  0.0       0.0        0.0        0.0        1.11871
```

In [7]:
```
c=Q[:,1:n]'*b
x=R\c
```

```
Out[7]: 5-element Array{Float64,1}:
         -0.1451430564241811
          0.6290700505109131
          0.6734603175535058
          0.1557491169211927
         -0.3955493171917202
```

```
In [8]: x1=A\b
```

```
Out[8]: 5-element Array{Float64,1}:
         -0.14514305642418174
          0.6290700505109127
          0.6734603175535074
          0.15574911692119206
         -0.3955493171917206
```

1.3 Numerička "ortogonalizacija" polinoma

Numerička ortogonalizacija potencija vektora daje ortogonalne polinome.

```
In [9]: # Standardna baza
        n=100
        x=range(-1,stop=1,length=n)
        # Kvazi Vandermondeova matrica
        V=[x.^0 x.^1 x.^2 x.^3 x.^4 x.^5]
```

```
Out[9]: 100×6 Array{Float64,2}:
         1.0  -1.0       1.0       -1.0       1.0       -1.0
         1.0  -0.979798  0.960004  -0.94061   0.921608  -0.902989
         1.0  -0.959596  0.920824  -0.883619  0.847918  -0.813658
         1.0  -0.939394  0.882461  -0.828978  0.778737  -0.731541
         1.0  -0.919192  0.844914  -0.776638  0.713879  -0.656192
         1.0  -0.89899   0.808183  -0.726548  0.65316   -0.587184
         1.0  -0.878788  0.772268  -0.67866   0.596398  -0.524107
         1.0  -0.858586  0.73717   -0.632923  0.543419  -0.466572
         1.0  -0.838384  0.702887  -0.589289  0.494051  -0.414204
         1.0  -0.818182  0.669421  -0.547708  0.448125  -0.366648
         1.0  -0.79798   0.636772  -0.508131  0.405478  -0.323563
         1.0  -0.777778  0.604938  -0.470508  0.36595   -0.284628
         1.0  -0.757576  0.573921  -0.434789  0.329385  -0.249534
         ⋮                                               ⋮
         1.0   0.777778  0.604938   0.470508  0.36595    0.284628
         1.0   0.79798   0.636772   0.508131  0.405478   0.323563
         1.0   0.818182  0.669421   0.547708  0.448125   0.366648
         1.0   0.838384  0.702887   0.589289  0.494051   0.414204
         1.0   0.858586  0.73717    0.632923  0.543419   0.466572
```

```
1.0   0.878788  0.772268   0.67866   0.596398   0.524107
1.0   0.89899   0.808183   0.726548  0.65316    0.587184
1.0   0.919192  0.844914   0.776638  0.713879   0.656192
1.0   0.939394  0.882461   0.828978  0.778737   0.731541
1.0   0.959596  0.920824   0.883619  0.847918   0.813658
1.0   0.979798  0.960004   0.94061   0.921608   0.902989
1.0   1.0       1.0        1.0       1.0        1.0
```

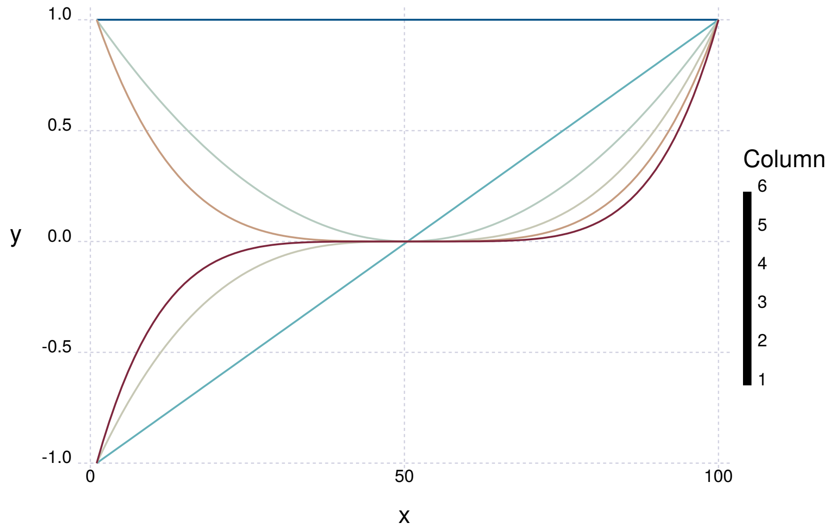
In [10]:
```
# Ortogonalizacija s težinskom funkcijom ω=1 daje normirane
# Legendreove polinome.
Q,R=qr(V)
Q=Q*sign.(diagm(0=>diag(R)))
```

Out[10]:
```
100×6 Array{Float64,2}:
 0.1  -0.171482  0.216998   -0.249164      0.271442   -0.285443
 0.1  -0.168017  0.203846   -0.218963      0.216605   -0.198945
 0.1  -0.164553  0.190963   -0.190302      0.166805   -0.124804
 0.1  -0.161089  0.178349   -0.16315       0.121798   -0.0620008
 0.1  -0.157625  0.166002   -0.137476      0.0813471  -0.00956382
 0.1  -0.15416   0.153925   -0.113247      0.0452209   0.0334314
 0.1  -0.150696  0.142115   -0.0904321     0.0131919   0.0678636
 0.1  -0.147232  0.130574   -0.0689992    -0.014962    0.0945669
 0.1  -0.143767  0.119302   -0.0489166    -0.0394579   0.114332
 0.1  -0.140303  0.108297   -0.0301525    -0.0605077   0.127905
 0.1  -0.136839  0.0975617  -0.0126751    -0.0783184   0.135994
 0.1  -0.133375  0.0870943   0.00354726   -0.0930919   0.139263
 0.1  -0.12991   0.0768952   0.0185465    -0.105025    0.138336
 ⋮                                                     ⋮
 0.1   0.133375  0.0870943  -0.00354726   -0.0930919  -0.139263
 0.1   0.136839  0.0975617   0.0126751    -0.0783184  -0.135994
 0.1   0.140303  0.108297    0.0301525    -0.0605077  -0.127905
 0.1   0.143767  0.119302    0.0489166    -0.0394579  -0.114332
 0.1   0.147232  0.130574    0.0689992    -0.014962   -0.0945669
 0.1   0.150696  0.142115    0.0904321     0.0131919  -0.0678636
 0.1   0.15416   0.153925    0.113247      0.0452209  -0.0334314
 0.1   0.157625  0.166002    0.137476      0.0813471   0.00956382
 0.1   0.161089  0.178349    0.16315       0.121798    0.0620008
 0.1   0.164553  0.190963    0.190302      0.166805    0.124804
 0.1   0.168017  0.203846    0.218963      0.216605    0.198945
 0.1   0.171482  0.216998    0.249164      0.271442    0.285443
```

In [11]:
```
using Gadfly
```

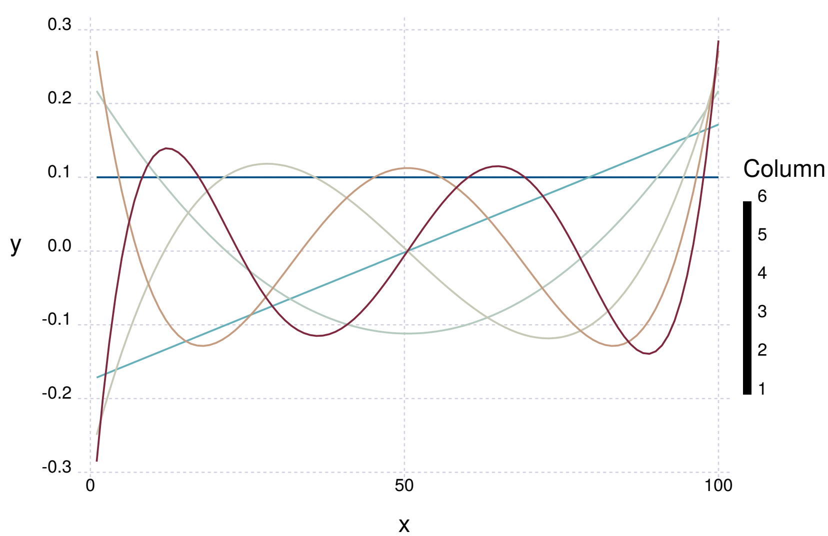
In [12]:
```
plot(V,x=Row.index, y=Col.value, color=Col.index, Geom.line)
```

Out[12]:



In [13]: plot(Q,x=Row.index, y=Col.value, color=Col.index, Geom.line)

Out[13]:



Dobiveni vektori su vrijednosti normiranih Legendreovih polinoma iz bilježnice NA12 Ortogonalni polinomi.ipynb.

Da bi dobili Čebiševljeve polinome, trebamo dodati težinsku funkciju ω i preraditi funkciju `myGramSchmidtQR()` iz bilježnice NA15 QR rastav.ipynb tako da računa *težinske skalarne produkte*. Dobiveni vektori su vrijednosti normiranih Čebiševljevih polinoma.

In [14]:
```
function myWeightedGramSchmidtQR(A::Array,ω::Vector)
    m,n=size(A)
    R=zeros(Float64,n,n)
    Q=Array{Float64,2}(undef,m,n)
    R[1,1]=norm(A[:,1])
    Q[:,1]=A[:,1]/R[1,1]
    for k=2:n
        for i=1:k-1
            R[i,k]=Q[:,i]⋅(A[:,k].*ω)/(Q[:,i]⋅(Q[:,i].*ω))
        end
        t=A[:,k]-sum([R[i,k]*Q[:,i] for i=1:k-1])
        R[k,k]=norm(t)
        Q[:,k]=t/R[k,k]
    end
    Q,R
end
```

Out[14]:
```
myWeightedGramSchmidtQR (generic function with 1 method)
```

In [15]:
```
n=100
x=range(-0.99,stop=0.99,length=n)
ω=1 ./(sqrt.(1.0.-x.^2))
# Kvazi Vandermonde-ova matrica
V=[x.^0 x.^1 x.^2 x.^3 x.^4 x.^5]
```

Out[15]:
```
100×6 Array{Float64,2}:
 1.0  -0.99  0.9801  -0.970299  0.960596  -0.95099
 1.0  -0.97  0.9409  -0.912673  0.885293  -0.858734
 1.0  -0.95  0.9025  -0.857375  0.814506  -0.773781
 1.0  -0.93  0.8649  -0.804357  0.748052  -0.695688
 1.0  -0.91  0.8281  -0.753571  0.68575   -0.624032
 1.0  -0.89  0.7921  -0.704969  0.627422  -0.558406
 1.0  -0.87  0.7569  -0.658503  0.572898  -0.498421
 1.0  -0.85  0.7225  -0.614125  0.522006  -0.443705
 1.0  -0.83  0.6889  -0.571787  0.474583  -0.393904
 1.0  -0.81  0.6561  -0.531441  0.430467  -0.348678
 1.0  -0.79  0.6241  -0.493039  0.389501  -0.307706
 1.0  -0.77  0.5929  -0.456533  0.35153   -0.270678
 1.0  -0.75  0.5625  -0.421875  0.316406  -0.237305
 ⋮                                         ⋮
 1.0   0.77  0.5929   0.456533  0.35153    0.270678
 1.0   0.79  0.6241   0.493039  0.389501   0.307706
 1.0   0.81  0.6561   0.531441  0.430467   0.348678
```

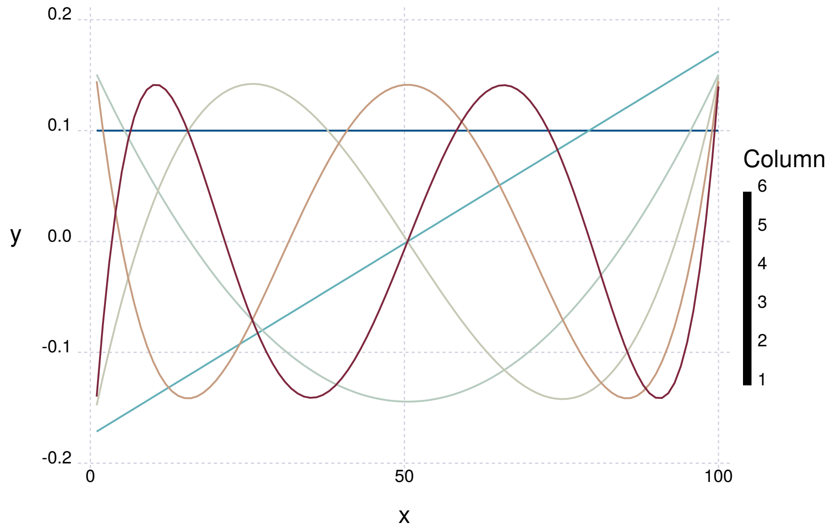
```
1.0   0.83  0.6889   0.571787  0.474583   0.393904
1.0   0.85  0.7225   0.614125  0.522006   0.443705
1.0   0.87  0.7569   0.658503  0.572898   0.498421
1.0   0.89  0.7921   0.704969  0.627422   0.558406
1.0   0.91  0.8281   0.753571  0.68575    0.624032
1.0   0.93  0.8649   0.804357  0.748052   0.695688
1.0   0.95  0.9025   0.857375  0.814506   0.773781
1.0   0.97  0.9409   0.912673  0.885293   0.858734
1.0   0.99  0.9801   0.970299  0.960596   0.95099
```

In [16]:
```
Q,R=myWeightedGramSchmidtQR(V,ω)
Q=Q*sign.(diagm(0=>diag(R)))
```

Out[16]:
```
100×6 Array{Float64,2}:
 0.1  -0.171482  0.150624   -0.147774     0.144547    -0.139908
 0.1  -0.168017  0.138822   -0.12221      0.100715    -0.0743526
 0.1  -0.164553  0.12726    -0.0980281    0.0613082   -0.0193541
 0.1  -0.161089  0.11594    -0.0752001    0.0261088    0.0259973
 0.1  -0.157625  0.10486    -0.0536975   -0.00509746   0.0625685
 0.1  -0.15416   0.0940209  -0.0334917   -0.0325198    0.091184
 0.1  -0.150696  0.0834228  -0.0145544   -0.056363     0.112627
 0.1  -0.147232  0.0730655   0.00314315  -0.0768273    0.127639
 0.1  -0.143767  0.0629492   0.0196293   -0.0941083    0.136924
 0.1  -0.140303  0.0530737   0.0349326   -0.108397     0.141146
 0.1  -0.136839  0.043439    0.0490815   -0.11988      0.14093
 0.1  -0.133375  0.0340453   0.0621045   -0.12874      0.136867
 0.1  -0.12991   0.0248924   0.0740302   -0.135153     0.12951
 ⋮                                                     ⋮
 0.1   0.133375  0.0340453  -0.0621045   -0.12874     -0.136867
 0.1   0.136839  0.043439   -0.0490815   -0.11988     -0.14093
 0.1   0.140303  0.0530737  -0.0349326   -0.108397    -0.141146
 0.1   0.143767  0.0629492  -0.0196293   -0.0941083   -0.136924
 0.1   0.147232  0.0730655  -0.00314315  -0.0768273   -0.127639
 0.1   0.150696  0.0834228   0.0145544   -0.056363    -0.112627
 0.1   0.15416   0.0940209   0.0334917   -0.0325198   -0.091184
 0.1   0.157625  0.10486     0.0536975   -0.00509746  -0.0625685
 0.1   0.161089  0.11594     0.0752001    0.0261088   -0.0259973
 0.1   0.164553  0.12726     0.0980281    0.0613082    0.0193541
 0.1   0.168017  0.138822    0.12221      0.100715     0.0743526
 0.1   0.171482  0.150624    0.147774     0.144547     0.139908
```

In [17]:
```
plot(Q,x=Row.index, y=Col.value, color=Col.index, Geom.line)
```

Out[17]:



0.2
0.1
0.0
-0.1
-0.2
y
0
50
100
x
Column
6
5
4
3
2
1