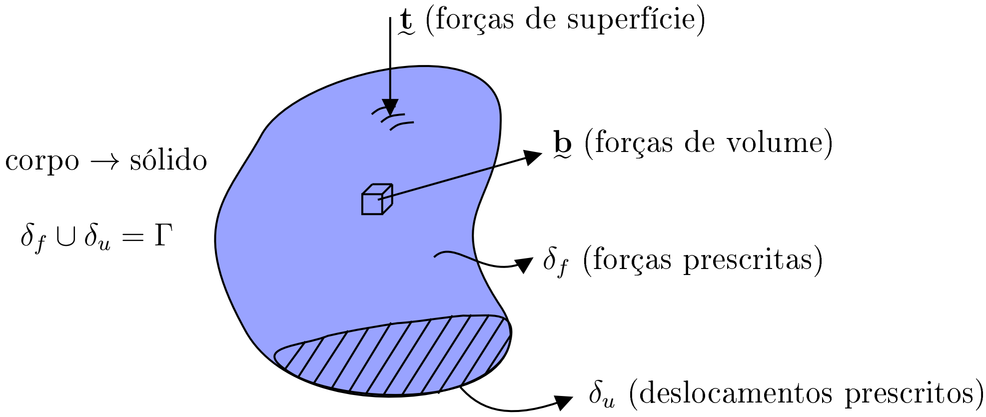
1 Estudos das deformações

1.1 Considerações iniciais

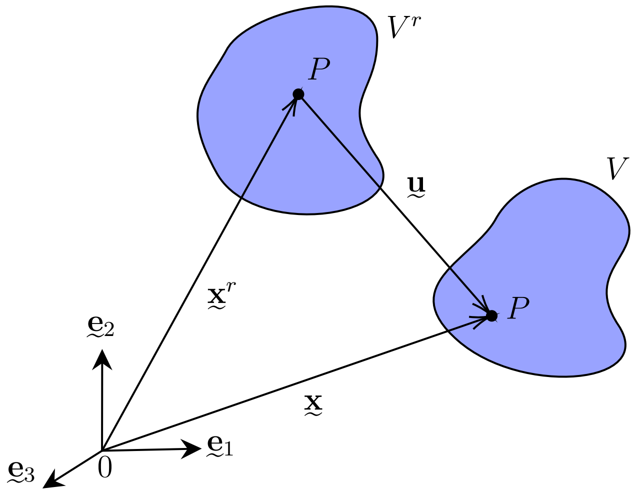
Figura 1: Considerações iniciais de corpo/sólido.



- Configuração do sólido: posição ocupada pelos pontos em um determinado instante t;
- Descrever a configuração deformada (V) a partir de uma configuração de referência (V^r);
- Considerando um conjunto de pontos da Geometria Euclidiana (E), e seja, $(\underset{\sim}{\mathbf{e}}_1,\ \underset{\sim}{\mathbf{e}}_2,\ \underset{\sim}{\mathbf{e}}_3)$ uma base do espaço vetorial da geometria clássica.

Podemos definir $(0,\ \underset{\sim}{\mathbf{e}}_1,\ \underset{\sim}{\mathbf{e}}_2,\ \underset{\sim}{\mathbf{e}}_3)$ como um sistema de referência:

Figura 2: Campo de deslocamentos.



Os vetores dos pontos de referência ($\underset{\sim}{\mathbf{x}}^r$) e na configuração deformada ($\underset{\sim}{\mathbf{x}}$) em relação aos versores da base podem ser expressos na notação de Einstein:

$$\underset{\sim}{\mathbf{x}}^r = \underset{\sim}{\mathbf{x}}^r - \underset{\sim}{\mathbf{0}} = \sum_{i=1}^{3} x^{ri} e_i = x^{r1}\underset{\sim}{\mathbf{e}}_1 + x^{r2}\underset{\sim}{\mathbf{e}}_2 + x^{r3}\underset{\sim}{\mathbf{e}}_3$$

$$\underset{\sim}{\mathbf{x}} = \underset{\sim}{\mathbf{x}} - \underset{\sim}{\mathbf{0}} = \sum_{i=1}^{3} x^{i} e_i = x^{1}\underset{\sim}{\mathbf{e}}_1 + x^{2}\underset{\sim}{\mathbf{e}}_2 + x^{3}\underset{\sim}{\mathbf{e}}_3$$

Seja f uma função que associa a posição de cada ponto na configuração V^r a sua posição na configuração V. Tal aplicação é uma transformação de V^r em V.

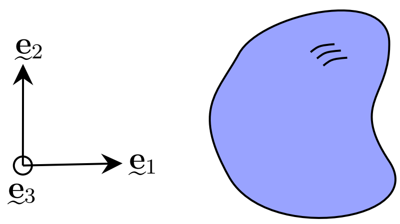
$$x^1 = \hat{f}_1(x^{r1},\ x^{r2},\ x^{r3})$$

$$x^2 = \hat{f}_2(x^{r1},\ x^{r2},\ x^{r3})$$

$$x^3 = \hat{f}_3(x^{r1},\ x^{r2},\ x^{r3})$$

Aqui, o circunflexo denota *em função de*, *i.e.*, a coordenada i da posição deformada está em função das coordenadas da posição de referência.

Exemplo: Considere um sólido cuja seção no plano $\underset{\sim}{\mathbf{e}}_1$, $\underset{\sim}{\mathbf{e}}_2$ é dada por:



Caracterize os seguintes campos de deslocamento:

a) Translação de corpo rígido de intensidade Δ na direção de $\underset{\sim}{\mathbf{e}}_1$;

b) Rotação de corpo rígido em torno de $\underset{\sim}{\mathbf{e}}_3$ de intensidade φ.

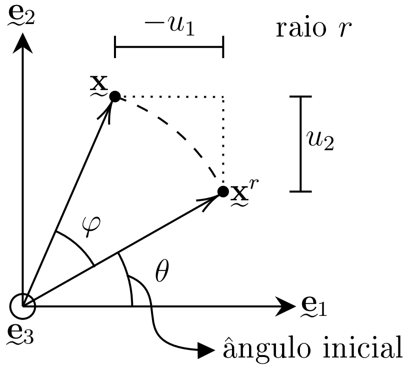
Resolução:

a)

$$\underset{\sim}{\mathbf{u}} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \Delta \\ 0 \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{x}} - \underset{\sim}{\mathbf{x}}^r \implies \underset{\sim}{\mathbf{x}} = \begin{cases} x^1 = x^{r1} + u_1 = x^{r1} + \Delta \\ x^2 = x^{r2} + u_2 = x^{r2} \\ x^3 = x^{r3} + u_3 = x^{r3} \end{cases}$$

b) A rotação de corpo rígido em torno de $\underset{\sim}{\mathbf{e}}_3$ de intensidade φ pode ser descrita como:



A configuração de referência:

$$x^{r1} = r \cdot \cos\theta$$

$$x^{r2} = r \cdot \sin\theta$$

$$x^{r3} = 0 \text{ (ou } x^{r3} \text{ para deixar genérico)}$$

A configuração deformada (a partir da imagem):

$$u^1 = r \cdot \cos(\varphi + \theta) - r \cdot \cos\theta$$

$$u^1 = r \cdot \cos\varphi \cdot \cos\theta - r \cdot \sin\varphi \cdot \sin\theta - r \cdot \cos\theta$$

$$u^2 = r \cdot \sin(\varphi + \theta) - r \cdot \sin\theta$$

$$u^2 = r \cdot \sin\varphi \cdot \cos\theta + r \cdot \sin\theta \cdot \cos\varphi - r \cdot \sin\theta$$

Substituindo as coordenadas de referência nas coordenadas deformadas:

$$u^1 = x^{r1} \cdot \cos\varphi - x^{r2} \cdot \sin\varphi - x^{r1}$$

$$u^2 = x^{r1} \cdot \sin\varphi + x^{r2} \cdot \cos\varphi - x^{r2}$$

$$u^3 = 0$$

Como sabemos que $\underset{\sim}{\mathbf{x}} = \underset{\sim}{\mathbf{x}}^r + \underset{\sim}{\mathbf{u}}$, temos:

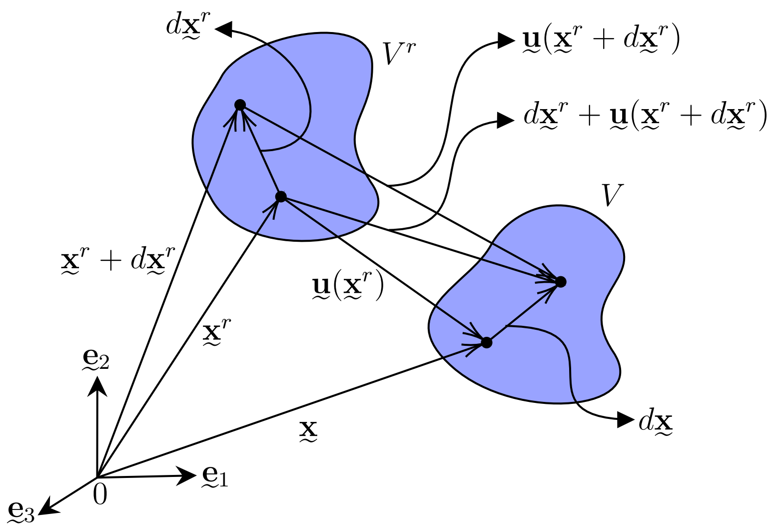
$$\begin{cases} x^1 = x^{r1} \cdot \cos\varphi - x^{r2} \cdot \sin\varphi \\ x^2 = x^{r1} \cdot \sin\varphi + x^{r2} \cdot \cos\varphi \\ x^3 = x^{r3} \end{cases}$$

Podemos escrever também na forma matricial:

$$\begin{Bmatrix} x^1 \\ x^2 \\ x^3 \end{Bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x^{r1} \\ x^{r2} \\ x^{r3} \end{Bmatrix}$$

1.2 Deformação Normal e por Cisalhamento

Figura 3: Deformação normal de uma fibra.



O assunto agora são fibras; como as fibras sofrem deformação. Seja $d\underset{\sim}{\mathbf{x}}^r$ o vetor infinitesimal que representa uma fibra a partir do ponto $\underset{\sim}{\mathbf{x}}^r$; $d\underset{\sim}{\mathbf{x}}$ o vetor infinitesimal da mesma fibra agora na configuração deformada, partindo do ponto $\underset{\sim}{\mathbf{x}}$. Para que o vetor $d\underset{\sim}{\mathbf{x}}^r$ deforme e se transforme em $d\underset{\sim}{\mathbf{x}}$, deve ocorrer uma transformação que depende de $\underset{\sim}{\mathbf{x}}^r + d\underset{\sim}{\mathbf{x}}^r$, *i.e.*, uma transformação $\underset{\sim}{\mathbf{u}}(\underset{\sim}{\mathbf{x}}^r + d\underset{\sim}{\mathbf{x}}^r)$.

Algumas relações podem ser estabelecidas a partir da imagem acima, sendo:

$$\underset{\sim}{\mathbf{u}}(\underset{\sim}{\mathbf{x}}^r) = d\underset{\sim}{\mathbf{x}}^r + \underset{\sim}{\mathbf{u}}(\underset{\sim}{\mathbf{x}}^r + d\underset{\sim}{\mathbf{x}}^r)$$

E a fibra na configuração deformada:

$$d\underset{\sim}{\mathbf{x}} = d\underset{\sim}{\mathbf{x}}^r + \underset{\sim}{\mathbf{u}}(\underset{\sim}{\mathbf{x}}^r + d\underset{\sim}{\mathbf{x}}^r) - \underset{\sim}{\mathbf{u}}(\underset{\sim}{\mathbf{x}}^r)$$

A equação acima na forma de componentes:

$$dx^i = dx^{ri} + u^i(x^{r1} + dx^{r1},\ x^{r2} + dx^{r2},\ x^{r3} + dx^{r3}) - u^i(x^{r1},\ x^{r2},\ x^{r3})$$

Lembrando de cálculo com múltiplas variáveis:

$$u^i(x^{r1} + dx^{r1},\ x^{r2} + dx^{r2},\ x^{r3} + dx^{r3}) - u^i(x^{r1},\ x^{r2},\ x^{r3}) = \frac{\partial u^i}{\partial x^{r1}}dx^{r1} + \frac{\partial u^i}{\partial x^{r2}}dx^{r2} + \frac{\partial u^i}{\partial x^{r3}}dx^{r3}$$

Para $i = 1$, 2 e 3. Essa iteração diz que cada coordenada i depende de acontecimentos nas 3 dimensões do espaço euclidiano.

Reescrevendo na forma matricial:

$$\underbrace{\begin{Bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{Bmatrix}}_{d\underset{\sim}{\mathbf{x}}} = \underbrace{\begin{Bmatrix} dx^{r1} \\ dx^{r2} \\ dx^{r3} \end{Bmatrix}}_{d\underset{\sim}{\mathbf{x}}^r} + \underbrace{\begin{bmatrix} \frac{\partial u^1}{\partial x^{r1}} & \frac{\partial u^1}{\partial x^{r2}} & \frac{\partial u^1}{\partial x^{r3}} \\ \frac{\partial u^2}{\partial x^{r1}} & \frac{\partial u^2}{\partial x^{r2}} & \frac{\partial u^2}{\partial x^{r3}} \\ \frac{\partial u^3}{\partial x^{r1}} & \frac{\partial u^3}{\partial x^{r2}} & \frac{\partial u^3}{\partial x^{r3}} \end{bmatrix}}_{\nabla \underset{\sim}{\mathbf{u}} = \underline{\mathbf{L}}} \underbrace{\begin{Bmatrix} dx^{r1} \\ dx^{r2} \\ dx^{r3} \end{Bmatrix}}_{d\underset{\sim}{\mathbf{x}}^r}$$

Onde $\nabla \underset{\sim}{\mathbf{u}}$ é o **gradiente dos deslocamentos**.

Logo,

$$d\underset{\sim}{\mathbf{x}} = d\underset{\sim}{\mathbf{x}}^r + \nabla \underset{\sim}{\mathbf{u}}\, d\underset{\sim}{\mathbf{x}}^r$$

Colocando $d\underset{\sim}{\mathbf{x}}^r$ em evidência, temos:

$$d\underset{\sim}{\mathbf{x}} = \underbrace{(\underline{\mathbf{I}} + \nabla \underset{\sim}{\mathbf{u}})}_{\underline{\mathbf{F}}}\, d\underset{\sim}{\mathbf{x}}^r$$

Onde $\underline{\mathbf{F}}$ é o **gradiente das deformações**.

Portanto, temos:

$$d\underset{\sim}{\mathbf{x}} = \underline{\mathbf{F}}\, d\underset{\sim}{\mathbf{x}}^r \tag{1}$$

Ou seja, a fibra deformada agora pode ser obtida a partir do gradiente dos deslocamentos ($\nabla \underset{\sim}{\mathbf{u}}$) e também a partir do gradiente das deformações ($\underline{\mathbf{F}}$).

O gradiente das deformações pode ser reescrito como:

$$\underline{\mathbf{F}} = \begin{bmatrix} 1 + \frac{\partial u^1}{\partial x^{r1}} & \frac{\partial u^1}{\partial x^{r2}} & \frac{\partial u^1}{\partial x^{r3}} \\ \frac{\partial u^2}{\partial x^{r1}} & 1 + \frac{\partial u^2}{\partial x^{r2}} & \frac{\partial u^2}{\partial x^{r3}} \\ \frac{\partial u^3}{\partial x^{r1}} & \frac{\partial u^3}{\partial x^{r2}} & 1 + \frac{\partial u^3}{\partial x^{r3}} \end{bmatrix}$$

Ou ainda, usando notação indicial de Einstein:

$$F_{ij} = \nabla u_{ij} + \delta_{ij}$$

Onde δ_{ij} é o delta de Kronecker, *i.e.*: $\delta_{ij} = \begin{cases} 1, & \text{se } i = j \\ 0, & \text{se } i \neq j \end{cases}$

E em notação indicial, temos:

$$\nabla u_{ij} = \frac{\partial u^i}{\partial x^{rj}}$$

Agora, expressemos:

$$\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{x}} - \underset{\sim}{\mathbf{x}}^r \implies \underset{\sim}{\mathbf{x}} = \underset{\sim}{\mathbf{u}} + \underset{\sim}{\mathbf{x}}^r$$

$$x^i = u^i + x^{ri}$$

Derivando ambos os lados da notação indicial acima em relação a $\underset{\sim}{\mathbf{x}}^r$, temos:

$$\underbrace{\frac{\partial x^i}{\partial x^{rj}}}_{\underline{\mathbf{F}}} = \underbrace{\frac{\partial u^i}{\partial x^{rj}}}_{\nabla u_{ij}} + \underbrace{\frac{\partial x^{ri}}{\partial x^{rj}}}_{\delta_{ij}}$$

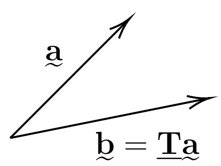
Ou seja, agora o gradiente das deformações está em função da transformação e sua forma matricial é:

$$\underline{\mathbf{F}} = \begin{bmatrix} \frac{\partial x^1}{\partial x^{r1}} & \frac{\partial x^1}{\partial x^{r2}} & \frac{\partial x^1}{\partial x^{r3}} \\ \frac{\partial x^2}{\partial x^{r1}} & \frac{\partial x^2}{\partial x^{r2}} & \frac{\partial x^2}{\partial x^{r3}} \\ \frac{\partial x^3}{\partial x^{r1}} & \frac{\partial x^3}{\partial x^{r2}} & \frac{\partial x^3}{\partial x^{r3}} \end{bmatrix}$$

Lembrete sobre operadores lineares:

$\underline{\mathbf{F}}$ e $\nabla\underset{\sim}{\mathbf{u}}$ são operadores lineares de $E \to E$. Seja $\underline{\mathbf{T}}$ um operador linear:

$$\underline{\mathbf{T}}(\alpha\underset{\sim}{\mathbf{x}} + \beta\underset{\sim}{\mathbf{y}}) = \alpha\underline{\mathbf{T}}\underset{\sim}{\mathbf{x}} + \beta\underline{\mathbf{T}}\underset{\sim}{\mathbf{y}}, \ \forall \ \underset{\sim}{\mathbf{x}} \text{ e } \underset{\sim}{\mathbf{y}} \in E$$

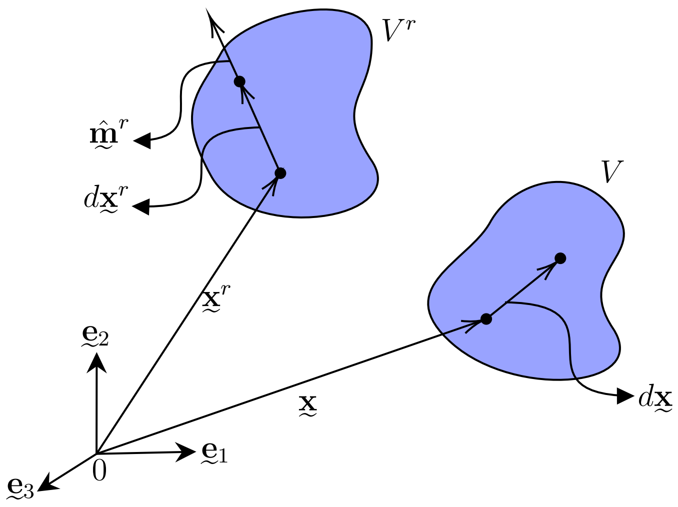


$$\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}}_{\underline{\mathbf{T}}} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Obs: Para o curso, um operador linear tem a mesma função de um tensor (simplificando).

1.3 Tensor das Deformações de Green-Lagrange

Figura 4: Alongamento de uma fibra.



Onde $\hat{\underset{\sim}{\mathbf{m}}}^r$ é um versor, $\dfrac{d\underset{\sim}{\mathbf{x}}^r}{||d\underset{\sim}{\mathbf{x}}^r||} = 1$, ds^r é o comprimento infinitesimal de uma fibra na configuração de referência na direção de $\hat{\underset{\sim}{\mathbf{m}}}^r$ e ds é o comprimento dessa mesma fibra na configuração deformada. Os comprimentos são definidos como:

$$ds^r = ||d\underset{\sim}{\mathbf{x}}^r|| = (d\underset{\sim}{\mathbf{x}}^r \cdot d\underset{\sim}{\mathbf{x}}^r)^{\frac{1}{2}}$$

$$ds = ||d\underset{\sim}{\mathbf{x}}|| = (d\underset{\sim}{\mathbf{x}} \cdot d\underset{\sim}{\mathbf{x}})^{\frac{1}{2}}$$

Lembrando sobre operadores lineares: Seja $\underline{\mathbf{A}}$ um operador linear ou tensor, temos

$$\underset{\sim}{\mathbf{a}} \cdot \underline{\mathbf{A}}\underset{\sim}{\mathbf{b}} = \underset{\sim}{\mathbf{b}} \cdot \underline{\mathbf{A}}^{\mathrm{T}}\underset{\sim}{\mathbf{a}},\ \forall\ \underset{\sim}{\mathbf{a}} \text{ e } \underset{\sim}{\mathbf{b}} \in E$$

Quando $\underline{\mathbf{A}} = \underline{\mathbf{A}}^{\mathrm{T}}$, $\underline{\mathbf{A}}$ é simétrico.

Exemplo: A *interpolação quadrática* ou *alongamento quadrático* é definida como:

$$\varepsilon_q = \frac{1}{2}\frac{(ds)^2 - (ds^r)^2}{(ds^r)^2}$$

Sabendo que,

$$ds^2 = d\underset{\sim}{\mathbf{x}} \cdot d\underset{\sim}{\mathbf{x}} = \underbrace{\underline{\mathbf{F}}d\underset{\sim}{\mathbf{x}}^r}_{\underset{\sim}{\mathbf{v}}} \cdot \underline{\mathbf{F}}d\underset{\sim}{\mathbf{x}}^r$$

$$ds^2 = \underset{\sim}{\mathbf{v}} \cdot \underline{\mathbf{F}}d\underset{\sim}{\mathbf{x}}^r$$

$$ds^2 = d\underset{\sim}{\mathbf{x}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underset{\sim}{\mathbf{v}}$$

$$ds^2 = d\underset{\sim}{\mathbf{x}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}d\underset{\sim}{\mathbf{x}}^r$$

Logo,

$$\varepsilon_q = \frac{1}{2}\frac{(d\underset{\sim}{\mathbf{x}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}d\underset{\sim}{\mathbf{x}}^r) - (d\underset{\sim}{\mathbf{x}}^r \cdot d\underset{\sim}{\mathbf{x}}^r)}{d\underset{\sim}{\mathbf{x}}^r \cdot d\underset{\sim}{\mathbf{x}}^r}$$

Lembrando que,

$$d\underset{\sim}{\mathbf{x}}^r = ||d\underset{\sim}{\mathbf{x}}^r||\underset{\sim}{\hat{\mathbf{m}}}^r = ds^r\underset{\sim}{\hat{\mathbf{m}}}^r$$

Então,

$$\varepsilon_q = \frac{1}{2}\frac{(ds^r\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}d\underset{\sim}{\mathbf{x}}^r) - (ds^r\underset{\sim}{\hat{\mathbf{m}}}^r \cdot ds^r\underset{\sim}{\hat{\mathbf{m}}}^r)}{ds^r\underset{\sim}{\hat{\mathbf{m}}}^r \cdot ds^r\underset{\sim}{\hat{\mathbf{m}}}^r}$$

$$\varepsilon_q = \frac{1}{2}\frac{ds^r\underset{\sim}{\hat{\mathbf{m}}}^r \cdot (\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} - \underline{\mathbf{I}})ds^r\underset{\sim}{\hat{\mathbf{m}}}^r}{ds^r\underset{\sim}{\hat{\mathbf{m}}}^r \cdot ds^r\underset{\sim}{\hat{\mathbf{m}}}^r}$$

$$\varepsilon_q = \frac{1}{2}[\underset{\sim}{\hat{\mathbf{m}}}^r \cdot (\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} - \underline{\mathbf{I}})\underset{\sim}{\hat{\mathbf{m}}}^r]$$

$$\varepsilon_q = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underbrace{\frac{1}{2}(\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} - \underline{\mathbf{I}})}_{\underline{\mathbf{E}}}\,\underset{\sim}{\hat{\mathbf{m}}}^r$$

$$\underline{\mathbf{E}} = \frac{1}{2}(\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} - \underline{\mathbf{I}})$$
$$\varepsilon_q(\underset{\sim}{\hat{\mathbf{m}}}^r) = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r$$

Onde $\underline{\mathbf{E}}$ é o Tensor das Deformações de Green-Lagrange e vale para qualquer magnitude de deslocamento. Algumas literaturas chamam de *finite displacement* os deslocamentos de qualquer magnitude.

Agora na notação de Einstein:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial x^k}{\partial x^{ri}}\frac{\partial x^k}{\partial x^{rj}} - \delta_{ij}\right)$$

Expressando os componentes de $\underline{\mathbf{E}}$ a partir do campo de deslocamentos:

$$\underline{\mathbf{E}} = \frac{1}{2}[(\nabla\underset{\sim}{\mathbf{u}}^{\mathrm{T}} + \underline{\mathbf{I}})(\nabla\underset{\sim}{\mathbf{u}} + \underline{\mathbf{I}}) - \underline{\mathbf{I}}]$$
$$\underline{\mathbf{E}} = \frac{1}{2}(\nabla\underset{\sim}{\mathbf{u}}^{\mathrm{T}}\nabla\underset{\sim}{\mathbf{u}} + \nabla\underset{\sim}{\mathbf{u}}^{\mathrm{T}} + \nabla\underset{\sim}{\mathbf{u}} + \underline{\mathbf{I}} - \underline{\mathbf{I}})$$
$$\underline{\mathbf{E}} = \frac{1}{2}(\nabla\underset{\sim}{\mathbf{u}} + \nabla\underset{\sim}{\mathbf{u}}^{\mathrm{T}} + \nabla\underset{\sim}{\mathbf{u}}^{\mathrm{T}}\nabla\underset{\sim}{\mathbf{u}})$$

Na notação de Einstein:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^{rj}} + \frac{\partial u^j}{\partial x^{ri}} + \underbrace{\frac{\partial u^k}{\partial x^{ri}}\frac{\partial u^k}{\partial x^{rj}}}_{*}\right)$$

Sendo $*$:

$$\frac{\partial u^k}{\partial x^{ri}}\frac{\partial u^k}{\partial x^{rj}} = \frac{\partial u^1}{\partial x^{ri}}\frac{\partial u^1}{\partial x^{rj}} + \frac{\partial u^2}{\partial x^{ri}}\frac{\partial u^2}{\partial x^{rj}} + \frac{\partial u^3}{\partial x^{ri}}\frac{\partial u^3}{\partial x^{rj}}$$

Acima, E_{ij} denota equações de compatibilidade; uma relação deslocamentos-deformações.

1.4 Medidas de deformação/alongamento

- Estiramento ou *stretch*:

$$\lambda = \frac{ds}{ds^r}$$

- Alongamento linear:

$$\varepsilon_l = \frac{ds - ds^r}{ds^r}$$

- Alongamento quadrático:

$$\varepsilon_q = \frac{1}{2}\frac{(ds)^2 - (ds^r)^2}{(ds^r)^2} = \frac{1}{2}\left[\left(\frac{ds}{ds^r}\right)^2 - 1\right] = \frac{1}{2}(\lambda^2 - 1)$$

- Alongamento logarítmico ou de Henry:

$$\varepsilon_n = \ln\left(\frac{ds}{ds^r}\right)$$

- Alongamento hiperbólico ou de Reiner:

$$\varepsilon_h = \frac{ds^r}{ds}$$

- Alongamento hiperbólico quadrático ou de Almansi:

$$\varepsilon_{hq} = \frac{1}{2}\frac{(ds^r)^2 - (ds)^2}{(ds)^2}$$

1.4.1 Relação entre alongamentos quadráticos e lineares

Para o alongamento quadrático:

$$\varepsilon_q = \frac{1}{2}(\lambda^2 - 1)$$

$$\lambda^2 = 1 + 2\varepsilon_q$$

$$\lambda = \sqrt{1 + 2\varepsilon_q}$$

$$\lambda(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{1 + 2(\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r)}$$

Para o alongamento linear:

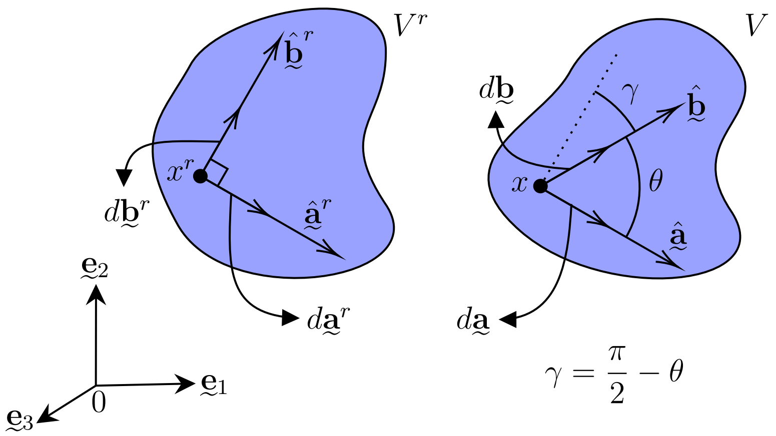
$$\varepsilon_l = \lambda - 1$$

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{1 + 2(\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r)} - 1$$

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{1 + 2\varepsilon_q} - 1$$

1.5 Variação do ângulo entre fibras (distorção)

Figura 5: Variação do ângulo entre fibras inicialmente ortogonais.



- $d\underset{\sim}{\mathbf{a}}^r$ e $d\underset{\sim}{\mathbf{b}}^r$ são inicialmente ortogonais;
- $\underline{\mathbf{F}} d\underset{\sim}{\mathbf{a}}^r$ tem a mesma direção de $d\underset{\sim}{\mathbf{a}}$ (hipótese);
- $\hat{\underset{\sim}{\mathbf{a}}}$ e $\hat{\underset{\sim}{\mathbf{b}}}$ são versores.

Lembrando que,

$$d\underset{\sim}{\mathbf{a}} = \underline{\mathbf{F}} d\underset{\sim}{\mathbf{a}}^r \text{ e } d\underset{\sim}{\mathbf{b}} = \underline{\mathbf{F}} d\underset{\sim}{\mathbf{b}}^r$$

A fim de encontrar novas relações, podemos definir o produto escalar,

$$d\underset{\sim}{\mathbf{a}} \cdot d\underset{\sim}{\mathbf{b}} = \underline{\mathbf{F}} d\underset{\sim}{\mathbf{a}}^r \cdot \underline{\mathbf{F}} d\underset{\sim}{\mathbf{b}}^r$$

$$d\underset{\sim}{\mathbf{a}} \cdot d\underset{\sim}{\mathbf{b}} = ||\underline{\mathbf{F}} d\underset{\sim}{\mathbf{a}}^r||\ ||\underline{\mathbf{F}} d\underset{\sim}{\mathbf{b}}^r|| \cos\theta$$

Logo,

$$\cos\theta = \sin\gamma = \frac{\underline{\mathbf{F}} d\underset{\sim}{\mathbf{a}}^r \cdot \underline{\mathbf{F}} d\underset{\sim}{\mathbf{b}}^r}{||\underline{\mathbf{F}} d\underset{\sim}{\mathbf{a}}^r||\ ||\underline{\mathbf{F}} d\underset{\sim}{\mathbf{b}}^r||} \tag{2}$$

E,

$$d\underset{\sim}{\mathbf{a}}^r = d\mathbf{a}^r \hat{\underset{\sim}{\mathbf{a}}}^r$$

$$d\underset{\sim}{\mathbf{b}}^r = d\mathbf{b}^r \hat{\underset{\sim}{\mathbf{b}}}^r$$

$$||\underline{\mathbf{F}}d\underset{\sim}{\mathbf{a}}^r|| = \sqrt{\underline{\mathbf{F}}d\underset{\sim}{\mathbf{a}}^r \cdot \underline{\mathbf{F}}d\underset{\sim}{\mathbf{a}}^r}$$

$$||\underline{\mathbf{F}}d\underset{\sim}{\mathbf{b}}^r|| = \sqrt{\underline{\mathbf{F}}d\underset{\sim}{\mathbf{b}}^r \cdot \underline{\mathbf{F}}d\underset{\sim}{\mathbf{b}}^r}$$

Substituindo as 4 relações acima na Equação (2),

$$\sin\gamma = \frac{d\underset{\sim}{\mathbf{b}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}d\underset{\sim}{\mathbf{a}}^r}{\sqrt{d\underset{\sim}{\mathbf{a}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}d\underset{\sim}{\mathbf{a}}^r}\sqrt{d\underset{\sim}{\mathbf{b}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}d\underset{\sim}{\mathbf{b}}^r}}$$

$$\sin\gamma = \frac{(d\mathbf{b}^r\underset{\sim}{\hat{\mathbf{b}}}^r) \cdot [\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}(d\mathbf{a}^r\underset{\sim}{\hat{\mathbf{a}}}^r)]}{\sqrt{(d\mathbf{a}^r\underset{\sim}{\hat{\mathbf{a}}}^r) \cdot [\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}(d\mathbf{a}^r\underset{\sim}{\hat{\mathbf{a}}}^r)]}\sqrt{(d\mathbf{b}^r\underset{\sim}{\hat{\mathbf{b}}}^r) \cdot [\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}(d\mathbf{b}^r\underset{\sim}{\hat{\mathbf{b}}}^r)]}}$$

$$\sin\gamma = \frac{(d\mathbf{b}^r d\mathbf{a}^r)[\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{a}}}^r]}{(d\mathbf{b}^r d\mathbf{a}^r)\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{a}}}^r}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{b}}}^r}}$$

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{a}}}^r}{\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{a}}}^r}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{b}}}^r}} \tag{3}$$

Lembrando que,

$$\underline{\mathbf{E}} = \frac{1}{2}(\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} - \underline{\mathbf{I}})$$

$$\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} = 2\underline{\mathbf{E}} + \underline{\mathbf{I}}$$

Substituindo a expressão acima na Equação (3),

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot (2\underline{\mathbf{E}} + \underline{\mathbf{I}})\underset{\sim}{\hat{\mathbf{a}}}^r}{\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot (2\underline{\mathbf{E}} + \underline{\mathbf{I}})\underset{\sim}{\hat{\mathbf{a}}}^r}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot (2\underline{\mathbf{E}} + \underline{\mathbf{I}})\underset{\sim}{\hat{\mathbf{b}}}^r}}$$

$$(2\underline{\mathbf{E}} + \underline{\mathbf{I}})\underset{\sim}{\hat{\mathbf{a}}}^r = 2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r + \underline{\mathbf{I}}\underset{\sim}{\hat{\mathbf{a}}}^r = 2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r + \underset{\sim}{\hat{\mathbf{a}}}^r$$

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot (2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r + \underset{\sim}{\hat{\mathbf{a}}}^r)}{\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot (2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r + \underset{\sim}{\hat{\mathbf{a}}}^r)}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot (2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{b}}}^r + \underset{\sim}{\hat{\mathbf{b}}}^r)}}$$

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot 2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r + \underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underset{\sim}{\hat{\mathbf{a}}}^r}{\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot 2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r + \underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underset{\sim}{\hat{\mathbf{a}}}^r}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot 2\underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{b}}}^r + \underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underset{\sim}{\hat{\mathbf{b}}}^r}}$$

Por fim,

$$\sin\gamma(\underset{\sim}{\hat{\mathbf{a}}}^r, \underset{\sim}{\hat{\mathbf{b}}}^r) = \frac{2(\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r)}{\sqrt{2(\underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r) + 1}\sqrt{2(\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{b}}}^r) + 1}} \tag{4}$$

Onde γ é a distorção, *i.e.*, a variação de ângulo entre duas fibras inicialmente ortogonais.

1.6 Interpretação de $\underline{\mathbf{E}}$

Partindo-se de uma base ortonormal ($\underset{\sim}{e_1}$, $\underset{\sim}{e_2}$, $\underset{\sim}{e_3}$), o Tensor das Deformações de Green-Lagrange é definido como,

$$\underline{\mathbf{E}} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

Lembrando do alongamento quadrático, por exemplo, de uma fibra alinhada na direção de $\underset{\sim}{\hat{\mathbf{e}}_1}$,

$$\varepsilon_q(\underset{\sim}{\hat{\mathbf{e}}_1}) = \underset{\sim}{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{e}}_1}$$

$$\varepsilon_q(\underset{\sim}{\hat{\mathbf{e}}_1}) = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix} \cdot \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon_q(\underset{\sim}{\hat{\mathbf{e}}_1}) = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix} \cdot \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{Bmatrix} = E_{11}$$

Da mesma forma, uma fibra na direção de $\underset{\sim}{\hat{\mathbf{e}}_2}$ ou $\underset{\sim}{\hat{\mathbf{e}}_3}$,

$$\varepsilon_q(\underset{\sim}{\hat{\mathbf{e}}_2}) = E_{22}$$

$$\varepsilon_q(\underset{\sim}{\hat{\mathbf{e}}_3}) = E_{33}$$

Ou seja, a diagonal de $\underline{\mathbf{E}}$ guarda o alongamento quadrático. E quanto à distorção? Escolhendo dois vetores da base ortonormal quaisqueres, por exemplo, $\utilde{\hat{\mathbf{e}}_1}$ e $\utilde{\hat{\mathbf{e}}_2}$ e lembrando da Equação (4),

$$\sin\gamma(\utilde{\hat{\mathbf{e}}_1}, \utilde{\hat{\mathbf{e}}_2}) = \frac{2(\utilde{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{E}}\utilde{\hat{\mathbf{e}}_1})}{\sqrt{2(\utilde{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{E}}\utilde{\hat{\mathbf{e}}_1}) + 1}\sqrt{2(\utilde{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{E}}\utilde{\hat{\mathbf{e}}_2}) + 1}} \tag{5}$$

Desenvolvendo o produto escalar no numerador da expressão acima na equação do alongamento quadrático,

$$\varepsilon_q = \utilde{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{E}}\utilde{\hat{\mathbf{e}}_1}$$

$$\varepsilon_q = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \cdot \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon_q = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \cdot \begin{Bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{Bmatrix} = E_{21}$$

Agora, invertendo os versores escolhidos para a Equação (4) e como o denominador continuará o mesmo, só o numerador sofrerá alteração. Desenvolvendo-o da mesma forma na equação do alongamento quadrático,

$$\varepsilon_q = \utilde{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{E}}\utilde{\hat{\mathbf{e}}_2}$$

$$\varepsilon_q = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix} \cdot \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\varepsilon_q = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix} \cdot \begin{Bmatrix} E_{12} \\ E_{22} \\ E_{32} \end{Bmatrix} = E_{12}$$

Ou seja, como a distorção entre esses versores deve ser igual, apenas invertendo os versores utilizados pode-se chegar a conclusão de que $E_{21} = E_{12}$. Isso significa que $\underline{\mathbf{E}}$ é simétrico.

Pode-se reescrever a Equação (5) como,

$$\sin\gamma(\underset{\sim}{\hat{\mathbf{e}}}_{\mathbf{1}}, \underset{\sim}{\hat{\mathbf{e}}}_{\mathbf{2}}) = \frac{2E_{21}}{\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}}$$

1.7 Tensor das Deformações de Cauchy-Green (da direita)

O Tensor das Deformações de Cauchy-Green ($\underline{\mathbf{C}}$) serve para grandes deformações e é definido como,

$$\underline{\mathbf{C}} = \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} \tag{6}$$

Podendo ser reescrito na notação de Einstein como,

$$C_{ij} = F^{\mathrm{T}}_{ik}F_{kj}$$

$$C_{ij} = F_{ki}F_{kj}$$

Na forma diferencial,

$$C_{ij} = \frac{\partial \underset{\sim}{\mathbf{x}}^k}{\partial x^{ri}}\frac{\partial \underset{\sim}{\mathbf{x}}^k}{\partial x^{rj}}$$

1.7.1 Relações entre alongamentos quadráticos e lineares

O alongamento quadrático,

$$\lambda = \sqrt{2\varepsilon_q + 1}$$

Elevando ambos os lados ao quadrado,

$$\lambda^2 = 2\varepsilon_q + 1 = 2(\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r) + 1$$

Como $\underline{\mathbf{C}} = \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}$, $\underline{\mathbf{E}}$ pode ser reescrito como,

$$\underline{\mathbf{E}} = \frac{1}{2}(\underline{\mathbf{C}} - \underline{\mathbf{I}}) \tag{7}$$

Pode-se reescrever λ^2 como,

$$\lambda^2 = 2\left\{\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \left[\frac{1}{2}(\underline{\mathbf{C}} - \underline{\mathbf{I}})\right] \underset{\sim}{\hat{\mathbf{m}}}^r\right\} + 1$$

$$\lambda^2 = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r - \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{I}}\underset{\sim}{\hat{\mathbf{m}}}^r + 1$$

$$\lambda^2 = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r - \underbrace{\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underset{\sim}{\hat{\mathbf{m}}}^r}_{1} + 1$$

$$\lambda^2 = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r$$

Logo,

$$\lambda(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r} \tag{8}$$

Para o alongamento linear,

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \lambda - 1$$

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r} - 1 \tag{9}$$

Para a distorção, a partir da Equação (3),

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{a}}}^r}{\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{a}}}^r}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}}\underset{\sim}{\hat{\mathbf{b}}}^r}}$$

Como $\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} = \underline{\mathbf{C}}$,

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{a}}}^r}{\sqrt{\underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{a}}}^r}\sqrt{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{b}}}^r}}$$

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{a}}}^r}{\lambda(\underset{\sim}{\hat{\mathbf{a}}}^r)\lambda(\underset{\sim}{\hat{\mathbf{b}}}^r)}$$

Ou seja, pode-se concluir que $\underline{\mathbf{C}}$ é simétrico.

1.7.2 Interpretação de $\underline{\mathbf{C}}$ na base ortonormal

$$\lambda(\underset{\sim}{\hat{\mathbf{e}}_1}) = \sqrt{\underset{\sim}{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_1}} = \sqrt{C_{11}} \therefore \lambda^2(\underset{\sim}{\hat{\mathbf{e}}_1}) = C_{11}$$

$$\lambda(\underset{\sim}{\hat{\mathbf{e}}_2}) = \sqrt{\underset{\sim}{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_2}} = \sqrt{C_{22}} \therefore \lambda^2(\underset{\sim}{\hat{\mathbf{e}}_2}) = C_{22}$$

$$\lambda(\underset{\sim}{\hat{\mathbf{e}}_3}) = \sqrt{\underset{\sim}{\hat{\mathbf{e}}_3} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_3}} = \sqrt{C_{33}} \therefore \lambda^2(\underset{\sim}{\hat{\mathbf{e}}_3}) = C_{33}$$

Ou seja, a diagonal de $\underline{\mathbf{C}}$ guarda o alongamento quadrático.

$$\sin(\underset{\sim}{\hat{\mathbf{e}}_1}, \underset{\sim}{\hat{\mathbf{e}}_2}) = \frac{\underset{\sim}{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_2}}{\lambda(\underset{\sim}{\hat{\mathbf{e}}_1})\lambda(\underset{\sim}{\hat{\mathbf{e}}_2})} = \frac{C_{12}}{\lambda(\underset{\sim}{\hat{\mathbf{e}}_1})\lambda(\underset{\sim}{\hat{\mathbf{e}}_2})}$$

1.8 Deslocamentos infinitesimais

Relembrando de $\underline{\mathbf{E}}$ na forma diferencial,

$$\underline{\mathbf{E}} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^{rj}} + \frac{\partial u^j}{\partial x^{ri}} + \frac{\partial u^k}{\partial x^{ri}}\frac{\partial u^k}{\partial x^{rj}}\right)$$

Para deslocamentos infinitesimais, pode-se desprezar o produto contido na expressão acima. Ficando como,

$$E^l_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^{rj}} + \frac{\partial u^j}{\partial x^{ri}}\right)$$

Ou seja, obtém-se o Tensor das Deformações infinitesimais linearizado, denotado por $\underline{\mathbf{E}}^l$ ou E^l_{ij}. Sendo,

$$\underline{\mathbf{E}}^l = \frac{1}{2}(\nabla u^{\mathrm{T}} + \nabla u)$$

Agora, lembrando do alongamento linear em função do vetor unitário $\underset{\sim}{\hat{\mathbf{m}}}^r$ (direção de uma fibra na configuração de referência),

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{1 + 2(\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r)} - 1$$

Elevando ambos os lados ao quadrado,

$$(\varepsilon_l + 1)^2 = 1 + 2(\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r)$$

$$\varepsilon_l^2 + 2\varepsilon_l + 1 = 1 + 2(\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r)$$

Desprezando a ordem superior,

$$\varepsilon_l = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{m}}}^r$$

Ou seja, $\underline{\mathbf{E}}$ tende para $\underline{\mathbf{E}}^l$ para deslocamentos lineares, portanto,

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{E}}^l\underset{\sim}{\hat{\mathbf{m}}}^r$$

E para a distorção,

$$\sin\gamma(\underset{\sim}{\hat{\mathbf{a}}}^r, \underset{\sim}{\hat{\mathbf{b}}}^r) = \frac{2(\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r)}{\sqrt{2(\underset{\sim}{\hat{\mathbf{a}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{a}}}^r) + 1}\sqrt{2(\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{E}}\underset{\sim}{\hat{\mathbf{b}}}^r) + 1}}$$

Admitindo que o denominador acima será muito pequeno em relação a 1,

$$\sin\gamma(\underset{\sim}{\hat{\mathbf{a}}}^r, \underset{\sim}{\hat{\mathbf{b}}}^r) = 2(\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{E}}^l\underset{\sim}{\hat{\mathbf{a}}}^r)$$

Exercício - Capítulo 3/página 108 do livro *The Mechanics of Solids and Structures - Hierarchical Modeling and The Finite Element Process of Solution (Bucalem, Bathe):*

Configuração deformada para um bloco de aresta a,

$$\begin{cases} x^1 = x^{r1} + \tan\beta x^{r2} \\ x^2 = x^{r2} \\ x^3 = x^{r3} \end{cases}$$

//Inserir desenho

1. Calcular o campo de deslocamentos e deformações normais (ε_l) nas direções ($\underset{\sim}{\hat{\mathbf{e}}_1}$), ($\underset{\sim}{\hat{\mathbf{e}}_2}$), ($\underset{\sim}{\mathbf{m}_1} = \frac{\sqrt{2}}{2}\underset{\sim}{\hat{\mathbf{e}}_1} + \frac{\sqrt{2}}{2}\underset{\sim}{\hat{\mathbf{e}}_2}$) e ($\underset{\sim}{\mathbf{m}_2} = -\frac{\sqrt{2}}{2}\underset{\sim}{\hat{\mathbf{e}}_1} + \frac{\sqrt{2}}{2}\underset{\sim}{\hat{\mathbf{e}}_2}$).

2. Calcular as deformações por cisalhamento para os pares de fibras nas direções ($\underset{\sim}{\hat{\mathbf{e}}_1}$, $\underset{\sim}{\hat{\mathbf{e}}_2}$) e ($\underset{\sim}{\mathbf{m}_1}$, $\underset{\sim}{\mathbf{m}_2}$).

3. Repetir os itens 1 e 2 assumindo um β pequeno, obter os resultados usando a teoria dos pequenos deslocamentos, e mostrar que estes mesmos resultados são também obtidos de 1 e 2.

Solução:

1. O campo de deslocamentos é dado por,

$$u^i = x^i - x^{ri}$$

Aplicando para as coordenadas,

$$u^1 = x^1 - x^{r1} = (x^{r1} + \tan\beta x^{r2}) - x^{r1} = \tan\beta x^{r2}$$

$$u^2 = x^2 - x^{r2} = x^{r2} - x^{r2} = 0$$

$$u^3 = x^2 - x^{r2} = x^{r3} - x^{r3} = 0$$

Considerando grandes deslocamentos, precisamos calcular o gradiente das deformações ($\underline{\mathbf{F}}$)

$$\underline{\mathbf{F}} = \begin{bmatrix} 1 & \tan\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Nota-se aqui que através do bloco, os elementos de F são constantes (independentes de x^{r1}, x^{r2} e x^{r3}).

Então,

$$\underline{\mathbf{F}}^{\mathrm{T}}\underline{\mathbf{F}} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Usando a Equação (9) (alongamento linear) para as fibras do item 1:

Para $\utilde{\hat{\mathbf{e}}_1}$,

$$\varepsilon_l(\utilde{\hat{\mathbf{e}}_1}) = \sqrt{\utilde{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{C}}\utilde{\hat{\mathbf{e}}_1}} - 1$$

$$\utilde{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{C}}\utilde{\hat{\mathbf{e}}_1} = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix} \cdot \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\utilde{\hat{\mathbf{e}}_1} \cdot \underline{\mathbf{C}}\utilde{\hat{\mathbf{e}}_1} = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ \tan\beta \\ 0 \end{Bmatrix} = 1$$

$$\varepsilon_l(\utilde{\hat{\mathbf{e}}_1}) = \sqrt{1} - 1 = 0$$

Para $\utilde{\hat{\mathbf{e}}_2}$,

$$\varepsilon_l(\utilde{\hat{\mathbf{e}}_2}) = \sqrt{\utilde{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\utilde{\hat{\mathbf{e}}_2}} - 1$$

$$\utilde{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\utilde{\hat{\mathbf{e}}_2} = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \cdot \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\utilde{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\utilde{\hat{\mathbf{e}}_2} = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \cdot \begin{Bmatrix} \tan\beta \\ \tan^2\beta + 1 \\ 0 \end{Bmatrix} = \tan^2\beta + 1$$

$$\varepsilon_l(\utilde{\hat{\mathbf{e}}_2}) = \sqrt{\tan^2\beta + 1} - 1$$

Para $\utilde{\mathbf{m}_1}$ (sendo $\utilde{\mathbf{m}_1}$ unitário, obrigatoriamente),

$$\varepsilon_l(\utilde{\mathbf{m}_1}) = \sqrt{\utilde{\mathbf{m}_1} \cdot \underline{\mathbf{C}}\utilde{\mathbf{m}_1}} - 1$$

$$\utilde{\mathbf{m}_1} \cdot \underline{\mathbf{C}}\utilde{\mathbf{m}_1} = \begin{Bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{Bmatrix} \cdot \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{Bmatrix}$$

$$\utilde{\mathbf{m}_1} \cdot \underline{\mathbf{C}}\utilde{\mathbf{m}_1} = \begin{Bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{Bmatrix} \cdot \begin{Bmatrix} \frac{\sqrt{2}}{2}(\tan\beta + 1) \\ \frac{\sqrt{2}}{2}(\tan^2\beta + 1 + \tan\beta) \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\mathbf{m_1}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_1}} = \frac{1}{2}(\tan\beta + 1) + \frac{1}{2}(\tan^2\beta + 1 + \tan\beta)$$

$$\underset{\sim}{\mathbf{m_1}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_1}} = \tan\beta + 1 + \frac{\tan^2\beta}{2}$$

$$\varepsilon_l(\underset{\sim}{\mathbf{m_1}}) = \sqrt{\tan\beta + 1 + \frac{\tan^2\beta}{2}} - 1$$

Para $\underset{\sim}{\mathbf{m_2}}$ (sendo $\underset{\sim}{\mathbf{m_2}}$ unitário, obrigatoriamente),

$$\varepsilon_l(\underset{\sim}{\mathbf{m_2}}) = \sqrt{\underset{\sim}{\mathbf{m_2}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_2}}} - 1$$

$$\underset{\sim}{\mathbf{m_2}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_2}} = \left\{-\tfrac{\sqrt{2}}{2} \quad \tfrac{\sqrt{2}}{2} \quad 0\right\} \cdot \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\mathbf{m_2}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_2}} = \left\{-\tfrac{\sqrt{2}}{2} \quad \tfrac{\sqrt{2}}{2} \quad 0\right\} \cdot \begin{Bmatrix} \frac{\sqrt{2}}{2}(\tan\beta - 1) \\ \frac{\sqrt{2}}{2}(\tan^2\beta + 1 - \tan\beta) \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\mathbf{m_2}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_2}} = -\frac{1}{2}(\tan\beta - 1) + \frac{1}{2}(\tan^2\beta + 1 - \tan\beta)$$

$$\underset{\sim}{\mathbf{m_2}} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m_2}} = -\tan\beta + 1 + \frac{\tan^2\beta}{2}$$

$$\varepsilon_l(\underset{\sim}{\mathbf{m_2}}) = \sqrt{1 + \frac{\tan^2\beta}{2} - \tan\beta} - 1$$

2. As tensões de cisalhamento podem ser calculadas como segue,

   Sabendo que,

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r} - 1$$

$$1 + \varepsilon_l(\underset{\sim}{\hat{\mathbf{m}}}^r) = \sqrt{\underset{\sim}{\hat{\mathbf{m}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{m}}}^r}$$

   Substituindo na Equação (3),

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{b}}}^r \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{a}}}^r}{\sqrt{1 + \varepsilon_l(\underset{\sim}{\hat{\mathbf{a}}}^r)}\sqrt{1 + \varepsilon_l(\underset{\sim}{\hat{\mathbf{b}}}^r)}}$$

   Para o par de fibras $(\underset{\sim}{\hat{\mathbf{e}}_1}, \underset{\sim}{\hat{\mathbf{e}}_2})$,

$$\sin\gamma = \frac{\underset{\sim}{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_1}}{\sqrt{1 + \varepsilon_l(\underset{\sim}{\hat{\mathbf{e}}_1})}\sqrt{1 + \varepsilon_l(\underset{\sim}{\hat{\mathbf{e}}_2})}} = \frac{\underset{\sim}{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_1}}{\sqrt{1 + \tan^2\beta}}$$

$$\underset{\sim}{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_1} = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \cdot \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\hat{\mathbf{e}}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\hat{\mathbf{e}}_1} = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ \tan\beta \\ 0 \end{Bmatrix} = \tan\beta$$

$$\sin\gamma = \frac{\tan\beta}{\sqrt{1 + \tan^2\beta}}$$

Usando a identidade $\sec^2\beta - \tan^2\beta = 1$,

$$\sin\gamma = \frac{\tan\beta}{\sqrt{\sec^2\beta}} = \frac{\tan\beta}{\sec\beta} = \frac{\dfrac{\sin\beta}{\cos\beta}}{\dfrac{1}{\cos\beta}} = \sin\beta$$

O que implica que $\gamma = \beta$.

Para o par de fibras $(\underset{\sim}{\mathbf{m}_1}, \underset{\sim}{\mathbf{m}_2})$,

$$\sin\gamma = \frac{\underset{\sim}{\mathbf{m}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m}_1}}{\sqrt{1 + \varepsilon_l(\underset{\sim}{\mathbf{m}_1})}\sqrt{1 + \varepsilon_l(\underset{\sim}{\mathbf{m}_2})}}$$

$$\underset{\sim}{\mathbf{m}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m}_1} = \begin{Bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{Bmatrix} \cdot \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & \tan^2\beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\mathbf{m}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m}_1} = \begin{Bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{Bmatrix} \cdot \begin{Bmatrix} \frac{\sqrt{2}}{2}(1 + \tan\beta) \\ \frac{\sqrt{2}}{2}(\tan\beta + \tan^2\beta + 1) \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\mathbf{m}_2} \cdot \underline{\mathbf{C}}\underset{\sim}{\mathbf{m}_1} = -\frac{1}{2}(1 + \tan\beta) + \frac{1}{2}(\tan\beta + \tan^2\beta + 1) = \tan^2\beta$$

$$\sin\gamma = \frac{\tan^2\beta}{\sqrt{\tan\beta + 1 + \dfrac{\tan^2\beta}{2}}\sqrt{1 + \dfrac{\tan^2\beta}{2} - \tan\beta}}$$

3. Quando os deslocamentos são infinitesimais, podemos calcular o alongamento normal e deformação cisalhante usando o tensor infinitesimal das deformações, sendo definido como,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^{rj}} + \frac{\partial u^j}{\partial x^{ri}}\right)$$

Então,

$$E_{11} = \frac{\partial u^1}{\partial x^{r1}} = 0,\ E_{22} = \frac{\partial u^2}{\partial x^{r2}} = 0,\ E_{33} = \frac{\partial u^3}{\partial x^{r3}} = 0$$

E como para deslocamentos infinitesimais $\tan\beta = \beta$,

$$E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial u^1}{\partial x^{r2}} + \frac{\partial u^2}{\partial x^{r1}}\right) = \frac{\beta}{2}$$

$$E_{13} = E_{31} = \frac{1}{2}\left(\frac{\partial u^1}{\partial x^{r3}} + \frac{\partial u^3}{\partial x^{r1}}\right) = 0$$

$$E_{23} = E_{32} = \frac{1}{2}\left(\frac{\partial u^2}{\partial x^{r3}} + \frac{\partial u^3}{\partial x^{r2}}\right) = 0$$

Na forma matricial,

$$\underline{\mathbf{E}} = \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Obtendo-se,

$$\varepsilon_l(\underset{\sim}{\hat{\mathbf{e}}_1}) = E_{11} = 0,\ \varepsilon_l(\underset{\sim}{\hat{\mathbf{e}}_2}) = E_{22} = 0,\ \sin(\underset{\sim}{\hat{\mathbf{e}}_1},\ \underset{\sim}{\hat{\mathbf{e}}_2}) = 2E_{12} = \beta$$

A deformação normal das fibras nas direções $\underset{\sim}{\mathbf{m_1}}$ e $\underset{\sim}{\mathbf{m_2}}$ podem ser calculadas por,

$$\varepsilon_l(\underset{\sim}{\mathbf{m_1}}) = \underset{\sim}{\mathbf{m_1}} \cdot \underline{\mathbf{E}}\underset{\sim}{\mathbf{m_1}}$$

$$\varepsilon_l(\underset{\sim}{\mathbf{m_1}}) = \left\{\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0\right\} \cdot \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{Bmatrix} = \frac{\beta}{2}$$

$$\varepsilon_l(\underset{\sim}{\mathbf{m_2}}) = \underset{\sim}{\mathbf{m_2}} \cdot \underline{\mathbf{E}}\underset{\sim}{\mathbf{m_2}}$$

$$\varepsilon_l(\underset{\sim}{\mathbf{m_2}}) = \left\{-\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0\right\} \cdot \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{Bmatrix} = -\frac{\beta}{2}$$

A deformação entre as fibras de direção $\underset{\sim}{\mathbf{m_1}}$, $\underset{\sim}{\mathbf{m_1}}$ é dada por,

$$\gamma(\underset{\sim}{\mathbf{m_1}},\ \underset{\sim}{\mathbf{m_2}}) = 2(\underset{\sim}{\mathbf{m_1}} \cdot \underline{\mathbf{E}}^l\underset{\sim}{\mathbf{m_2}})$$

$$\gamma(\underset{\sim}{\mathbf{m_1}},\ \underset{\sim}{\mathbf{m_2}}) = 2\left\{\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0\right\} \cdot \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{Bmatrix} = 0$$

A fim de mostrar que obtivemos os resultados acima dos valores calculados em 1 e 2, precisamos considerar β infinitesimal nas expressões em 1 e 2.

1.9 Movimentos rígidos: Rotações

//Inserir imagem

Para a figura acima em uma base ortonormal e o eixo de rotação $\underset{\sim}{\hat{\mathbf{e}}} = \underset{\sim}{\hat{\mathbf{e}}_3}$, θ é a magnitude de rotação, $\underset{\sim}{\theta} = \theta\underset{\sim}{\hat{\mathbf{e}}}$ ($\underset{\sim}{\theta}$ define totalmente a rotação) e $||\underset{\sim}{\hat{\mathbf{e}}}|| = 1$.

Lembrando do exercício de rotação da primeira aula,

$$\underbrace{\begin{Bmatrix} x^1 \\ x^2 \\ x^3 \end{Bmatrix}}_{\underset{\approx}{\mathbf{x}}} = \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{\mathbf{Q}}} \underbrace{\begin{Bmatrix} x^{r1} \\ x^{r2} \\ x^{r3} \end{Bmatrix}}_{\underset{\approx}{\mathbf{x}}^r}$$

Onde $\underline{\mathbf{Q}}$ é o operador (tensor) ortogonal.

Pode-se definir um operador ortogonal de forma equivalente por uma das relações abaixo,

- $||\underline{\mathbf{Q}}\underset{\sim}{\mathbf{w}}|| = ||\underset{\sim}{\mathbf{w}}||\ \forall\ \underset{\sim}{\mathbf{w}}$ (Não muda o comprimento).
- $\underline{\mathbf{Q}\mathbf{Q}}^{\mathrm{T}} = \underline{\mathbf{Q}}^{\mathrm{T}}\underline{\mathbf{Q}} = \underline{\mathbf{I}} \implies \underline{\mathbf{Q}}^{-1} = \underline{\mathbf{Q}}^{\mathrm{T}}$
- A partir de item acima,

$$\det(\underline{\mathbf{Q}\mathbf{Q}}^{\mathrm{T}}) = \det(\underline{\mathbf{I}})$$

$$\det(\underline{\mathbf{Q}})\det(\underline{\mathbf{Q}}^{\mathrm{T}}) = 1$$

$$(\det\underline{\mathbf{Q}})^2 = 1$$

$$\det(\underline{\mathbf{Q}}) = \pm 1$$

Pode-se demonstrar que para qualquer tensor ortogonal com determinante positivo ($\det(\underline{\mathbf{Q}}) = 1$), a rotação é descrita através da equação,

$$\underset{\sim}{\mathbf{x}} = \underline{\mathbf{Q}}\underset{\sim}{\mathbf{x}}^{r}$$

Exercício - Capítulo 3/página 144 do livro *The Mechanics of Solids and Structures - Hierarchical Modeling and The Finite Element Process of Solution (Bucalem, Bathe):*

Considere o tensor $\underline{\mathbf{Q}}$ na base ortogonal,

$$\underline{\mathbf{Q}} = \begin{bmatrix} \frac{4}{9} + \frac{5\sqrt{3}}{18} & \frac{5}{18} + \frac{2\sqrt{3}}{9} & \frac{5}{9} - \frac{\sqrt{3}}{9} \\ \frac{11}{18} - \frac{2\sqrt{3}}{9} & \frac{4}{9} + \frac{5\sqrt{3}}{18} & -\frac{1}{9} - \frac{\sqrt{3}}{9} \\ -\frac{1}{9} - \frac{\sqrt{3}}{9} & \frac{5}{9} - \frac{\sqrt{3}}{9} & \frac{1}{9} + \frac{4\sqrt{3}}{9} \end{bmatrix}$$

1. Verificar se $\underline{\mathbf{Q}}$ é um tensor ortogonal;
2. Obter o eixo e magnitude dada por $\underline{\mathbf{Q}}$.

Solução:

1. Deve-se verificar se $\underline{\mathbf{Q}\mathbf{Q}}^{\mathrm{T}} = 1$
2. Considerando que $\underset{\sim}{\hat{\mathbf{e}}}$ é um vetor unitário ($||\underset{\sim}{\hat{\mathbf{e}}}|| = 1$) na direção do eixo de rotação, então:

$$\underline{\mathbf{Q}}\underset{\sim}{\hat{\mathbf{e}}} = \underset{\sim}{\hat{\mathbf{e}}} \text{ (Não muda a magnitude)}$$

$$\underline{\mathbf{Q}}\underset{\sim}{\hat{\mathbf{e}}} - \underset{\sim}{\hat{\mathbf{e}}} = 0$$

$$(\underline{\mathbf{Q}} - \underline{\mathbf{I}})\underset{\sim}{\hat{\mathbf{e}}} = 0$$

Na forma matricial,

$$\begin{bmatrix} Q_{11} - 1 & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} - 1 & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} - 1 \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Resolvendo o sistema

$$\underset{\sim}{\hat{\mathbf{e}}} = \begin{Bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{Bmatrix}$$

Verificando se não houve falha no processo de cálculo,

$$||\underset{\sim}{\hat{\mathbf{e}}}|| = \sqrt{(2/3)^2 + (2/3)^2 + (1/3)^2} = 1$$

Adotando $\underset{\sim}{\mathbf{g}}$ como um vetor unitário ortogonal ao eixo de rotação e escolhendo qualquer um dos vetores da base,

$$\underset{\sim}{\mathbf{g}} = \frac{\underset{\sim}{\hat{\mathbf{e}}} \times \underset{\sim}{\hat{\mathbf{e}}_{\mathbf{2}}}}{||\underset{\sim}{\hat{\mathbf{e}}} \times \underset{\sim}{\hat{\mathbf{e}}_{\mathbf{2}}}||} = \begin{Bmatrix} -\sqrt{5}/5 \\ 0 \\ \frac{2\sqrt{5}}{5} \end{Bmatrix}$$

Aplicando a rotação em $\underset{\sim}{\mathbf{g}}$ para obter um vetor $\underset{\sim}{\mathbf{f}}$,

$$\underset{\sim}{\mathbf{f}} = \underline{\mathbf{Q}}\underset{\sim}{\mathbf{g}} = \begin{Bmatrix} -\frac{\sqrt{15}}{10} + \frac{2\sqrt{5}}{15} \\ -\frac{5}{6} \\ \frac{\sqrt{15}}{5} + \frac{\sqrt{5}}{15} \end{Bmatrix}$$

Encontrando o ângulo de rotação pelo produto escalar entre $\underset{\sim}{\mathbf{g}}$ e $\underset{\sim}{\mathbf{f}}$,

$$\underset{\sim}{\mathbf{g}} \cdot \underset{\sim}{\mathbf{f}} = ||\underset{\sim}{\mathbf{g}}||\ ||\underset{\sim}{\mathbf{f}}|| \cos\theta$$

$$\cos\theta = \frac{\sqrt{3}}{2} \therefore \theta = 30°$$