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## OPTIMAL STOPPING INVESTMENT WITH NON-SMOOTH UTILITY OVER AN INFINITE TIME HORIZON

XIAOSHAN CHEN

School of Mathematical Sciences, South China Normal University  
Guangzhou 510631, China

XUN LI

Department of Applied Mathematics  
The Hong Kong Polytechnic University, Hong Kong, China

FAHUAI YI

School of Finance, Guangdong University of Foreign Studies  
Guangzhou 510420, China

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**ABSTRACT.** This study addresses an investment problem facing a venture fund manager who has a non-smooth utility function. The theoretical model characterizes an absolute performance-based compensation package. Technically, the research methodology features stochastic control and optimal stopping by formulating a free-boundary problem with a nonlinear equation, which is transferred to a new one with a linear equation. Numerical results based on simulations are presented to better illustrate this practical investment decision mechanism.

**1. Introduction.** While the US economy is recovering as indicated by the first increase in interest rate announced by the Federal Reserve Board in December 2015, innovation-based new venture enterprises continued to serve as the engine of the economic growth (Carter-Mason-Tagg (2004)). However, financing their survival and growth is important yet challenging, especially at their early stages of development (Berger-Udell (1998)). Given the difficulties of securing debt financing due to their liability of newness and lack of collateral (Chua-Chrisman-Kellermanns-Wu (2011)), they have to rely on equity financing, most of which are from venture capitalists. In order to lower the level of risk they take, venture capitalists tend to establish venture funds and hire professional fund managers to make investment decisions. These venture fund managers are compensated by a fixed amount of salary plus an absolute performance-based incentive characterized by an American-style option,

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and their best interests are to optimize based on their utilities functions which take risk preferences into account.

Having said these, we formulate a venture fund manager's utility maximization-based investment decision problem using an optimal-stopping mechanism. It features the risk-taking behaviors in terms of the time point at which the manager quits an investment project. The exit options which the manager can choose from include, but are not limited to, leverage buy-out conducted by the entrepreneurs, management buy-out, initial public offerings, mergers and acquisitions, and sale to strategic investors (Cumming-Walz (2010)). By deciding to exit an investment project at a certain time point, the utility based on the manager's compensation package and risk preference may be maximized.

While optimal stopping theory has been extensively applied in pricing American options (Elliott-Kopp (1999); Yong-Zhou (1999); Ceci-Bassan (2004); Li-Zhou (2006); Shiryaev-Xu-Zhou (2008)), researchers usually run into more complicated situations when studying risk-taking behaviors of venture fund managers. This is mainly because of the existence of free-boundary problems with general controlled diffusion processes (Fleming-Soner (2006); Peskir-Shiryaev (2006)). To help fill these gaps in the literature, this study is rooted in Choi-Koo-Kwak (2004), Henderson-Hobson (2008) and Bensoussan-Cadenillas-Koo (2015), and it presents solutions to venture fund managers' utility maximization problems and interprets their risk-taking behaviors. In particular, Choi-Koo-Kwak (2004) and Bensoussan-Cadenillas-Koo (2015) proposed a general entrepreneurial/managerial decision problem which involves a broad class of nonconcave objective functions. Also, Bensoussan-Cadenillas-Koo (2015) proved that the optimization problem with a nonconcave objective function has the same solution as the optimization problem when the objective function is replaced by its concave hull, and thus the problems are equivalent to each other. However, it is technically difficult to present analytical or numerical solutions using the methods introduced in previous studies (Chang-Pang-Yong (2009); Dayanik-Karatzas (2003); Carpenter (2000)) because of the optimization process over the entire investment period and the simultaneous investment decisions on multiple projects. Thus, we develop a new approach by transforming the free-boundary problem with non-linear equations to a new one with linear equations in this work, and then optimize fund managers' investment strategies.

This study adds to the understanding of risk-taking behaviors and investment decision mechanisms of venture fund managers in at least three aspects below. First, it is one of the early studies addressing venture fund managers' decision-making process theoretically using a utility-maximization model and an American-style optimal stopping mechanism. Therefore, this study sheds light on the cutting-edge development of the entrepreneurial finance and entrepreneurship literature. Second, it contributes to the mathematical finance literature by formulating risk-sensitive problem in wealth management. Third, findings of this study provide important and timely implication practitioners and policy makers by helping them understand venture fund managers' risk-taking behaviors.

The rest of this paper is structured as the following: Section 2 formulates the utility-maximization model. Sections 3 and 4 solve the optimal stopping problem with and without constraints, respectively. Numerical results from simulations are presented in Section 5.

**2. Model formulation.** For technical simplicity, we assume that the continuous-time financial market is complete and arbitrage-free. A venture fund manager's

task is to develop an optimal investment strategy based on a risk-free asset with instantaneous interest rate  $r$  and a risky project whose assets  $S_t^i, i = 1, 2, \dots, n$  follow stochastic differential equations

$$\frac{dS_t^i}{S_t^i} = (r + \mu_i)dt + \sum_{j=1}^m \sigma_{ij} dW_t^j, \quad \text{for } i = 1, 2, \dots, n, \quad (2.1)$$

where  $\mu := (\mu_1, \mu_2, \dots, \mu_n)^\top$  is the excess appreciation rate,  $\sigma := (\sigma_{ij})_{n \times m}$  is the volatility, and  $W := (W^1, W^2, \dots, W^m)'$  is a standard  $n$ -dimensional Brownian motion defined on a complete probability  $(\Omega, \mathcal{F}, \mathcal{P})$ . In addition,  $\sigma\sigma' > \varepsilon I_{n \times n}$ , where  $\varepsilon > 0$ .

The process of the asset portfolio  $X_t$  follows

$$dX_t = (rX_t + \mu^\top \pi)dt + \pi^\top \sigma dW_t, \quad X_0 = x, \quad (2.2)$$

where an admissible investment strategy  $\pi \in \mathcal{U}$  is progressively measurable with respect to  $\{\mathcal{F}_t\}$  such that  $X_t \geq 0$  and satisfy  $\mathbb{E} \int_0^\infty |\pi|^2 dt < \infty$ . In particular,  $\mathcal{U}$  is a convex set in this paper, for example,  $\mathcal{U} \equiv \mathcal{L}_{\mathcal{F}}^2([0, \infty), \mathbb{R}^n)$  in Section 3, and  $\mathcal{U} \equiv \mathcal{L}_{\mathcal{F}}^2([0, \infty), \mathbb{R}_+^n)$  in Section 4, where each component is non-negative.

The manager controls assets with initial value  $x_0$ , and her wealth at exercise time  $\tau$  is the payoff of a call option on the asset with strike price (or benchmark payoff)  $B$  plus a constant  $K > 0$ , that includes fixed compensation and personal wealth. By choosing an optimal investment strategy  $\pi^*$  and an optimal stopping time  $\tau^*$ , the manager intends to maximize her expected utility such that

$$V(x) = \sup_{\pi, \tau} \mathbb{E}[e^{-\beta\tau} U(\alpha(X_\tau - B)^+ + K)], \quad (2.3)$$

where  $\alpha$  represents the number of options or the percentage of positive profits,  $\beta > 0$  is the discounted factor,  $U(x) = \frac{1-\gamma}{\gamma} \left( \frac{A(x-w)}{1-\gamma} \right)^\gamma$ ,  $x > w$ ,  $0 < \gamma < 1$ ,  $w < K$ ,  $A > 0$  is the utility function.

**3. Stopping problem with unconstrained portfolio.** Applying the principle of dynamic programming, we may obtain the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \min \left\{ -\max_{\pi} \left[ \frac{1}{2} \pi^\top \sigma \sigma^\top \pi V''(x) + \mu^\top \pi V'(x) \right] - rxV'(x) + \beta V(x), \right. \\ \left. V(x) - U(\alpha(x - B)^+ + K) \right\} = 0, \quad x > 0, \\ V(0) = U(K). \end{cases} \quad (3.1)$$

We conjecture that there exists a free boundary point  $x^* > B$  such that

$$\begin{cases} V(x) > U(\alpha(x - B)^+ + K), & 0 < x < x^*; \\ V(x) = U(\alpha(x - B)^+ + K) = U(\alpha(x - B) + K), & x \geq x^* > B. \end{cases}$$

Furthermore, problem (3.1) can be reduced to

$$\begin{cases} \max_{\pi} \left[ \frac{1}{2} \pi^\top \sigma \sigma^\top \pi V''(x) + \mu^\top \pi V'(x) \right] + rxV'(x) - \beta V(x) = 0, & 0 < x < x_*, \\ V(x) = U(\alpha(x - B) + K), & x \geq x_* > B, \\ V(0) = U(K). \end{cases} \quad (3.2)$$

Suppose that  $V(x)$  is increasing and concave, i.e.,  $V'(x) > 0$ ,  $V''(x) < 0$ . Then

$$\pi^* = -(\sigma\sigma^\top)^{-1}\mu \frac{V'(x)}{V''(x)}.$$

Let  $\theta = \sigma^{-1}\mu$ . Therefore, problem (3.2) is equivalent to

$$\begin{cases} \frac{1}{2}\|\theta\|^2 \frac{(V'(x))^2}{V''(x)} - rxV'(x) + \beta V(x) = 0, & 0 < x < x_*, \\ V(x) = U(\alpha(x - B) + K), & x \geq x_* > B, \\ V(0) = U(K). \end{cases} \quad (3.3)$$

**Remark 3.1. Discussion of the special cases of  $K = w = 0, B > 0$  or  $K = w = B = 0$ .**

These special case are simple. We need not adopt dual methods to derive their strategies. Denote

$$F(x) := U(\alpha(x - B)^+ + K) = \frac{1-\gamma}{\gamma} \left( \frac{A(\alpha(x - B)^+ + K - w)}{1-\gamma} \right)^\gamma$$

**Case 1.**  $K = w = 0, B > 0$ .

When  $K = w = 0$ , then  $F(x) = 0$  if  $x \leq B$ ,  $F(0) = F'(0) = 0$ , by the equation in (3.3) and  $V(0) = 0$ , we have the general solution is

$$V(x) = cx^\gamma,$$

where  $c$  will be defined later by the second condition in (3.3). Substituting the expression of  $V(x)$  into the equation in (3.3) yields

$$\left( \frac{1}{2}\|\theta\|^2 \frac{\gamma}{\gamma-1} - r\gamma + \beta \right) \cdot c = 0.$$

(i) If  $\frac{1}{2}\|\theta\|^2 \frac{\gamma}{\gamma-1} - r\gamma + \beta = 0$ , then we have

$$\begin{cases} V(x_*) = U(\alpha(x_* - B)), \\ V'(x_*) = \alpha U'(\alpha(x_* - B)), \end{cases}$$

i.e.,

$$\begin{cases} cx_*^\gamma = \frac{1-\gamma}{\gamma} \left( \frac{A\alpha(x_* - B)}{1-\gamma} \right)^\gamma, \\ c\gamma x_*^{\gamma-1} = A\alpha \left( \frac{A\alpha(x_* - B)}{1-\gamma} \right)^{\gamma-1}. \end{cases}$$

Hence,

$$\begin{cases} x_* = +\infty, \\ c = \frac{(A\alpha)^\gamma}{\gamma(1-\gamma)^{\gamma-1}}, \end{cases}$$

which implies that problem (3.3) has no free boundary. The value function is always larger than payoff function. This means that managers will not stop the investment, i.e.,  $\tau^* = \infty$ .

(ii) If  $\frac{1}{2}\|\theta\|^2 \frac{\gamma}{\gamma-1} - r\gamma + \beta \neq 0$ , then  $c = 0$ , but  $v(x) = 0$  is not the solution to problem (3.3), which implies that the original problem has no solution in this case.

**Case 2.**  $K = w = B = 0$ . Denote

$$F(x) := U(\alpha x) = \frac{1-\gamma}{\gamma} \left( \frac{A\alpha x}{1-\gamma} \right)^\gamma = \frac{(\alpha A)^\gamma}{\gamma(1-\gamma)^{\gamma-1}} x^\gamma.$$

In this case  $F(0) = 0$  and  $F'(0) = +\infty$ . Moreover,  $F(x)$  is a concave function in  $x$ . Note that

$$F'(x) = \frac{(\alpha A)^\gamma}{(1-\gamma)^{\gamma-1}} x^{\gamma-1}, \quad F''(x) = \frac{(\alpha A)^\gamma}{(1-\gamma)^{\gamma-2}} x^{\gamma-2},$$

Substituting the expression of  $F(x)$  into the equation in (3.3) leads to

$$\begin{aligned} & \frac{1}{2} \|\theta\|^2 \frac{(F'(x))^2}{F''(x)} - rxF'(x) + \beta F(x) \\ &= -\frac{1}{2} \|\theta\|^2 \frac{(\alpha A)^\gamma}{(1-\gamma)^\gamma} x^\gamma - \frac{r(\alpha A)^\gamma}{(1-\gamma)^{\gamma-1}} x^\gamma + \frac{\beta(\alpha A)^\gamma}{\gamma(1-\gamma)^{\gamma-1}} x^\gamma \\ &= \frac{(\alpha A)^\gamma}{(1-\gamma)^\gamma} \left( -\frac{1}{2} \|\theta\|^2 - r(1-\gamma) + \frac{\beta(1-\gamma)}{\gamma} \right) x^\gamma. \end{aligned}$$

(i) If  $-\frac{1}{2} \|\theta\|^2 - r(1-\gamma) + \frac{\beta(1-\gamma)}{\gamma} \geq 0$ , then the solution to problem (3.1) is  $F(x) = U(\alpha x)$ . The value function is the same as payoff function and optimal stopping  $\tau^* = 0$ .

(ii) If  $-\frac{1}{2} \|\theta\|^2 - r(1-\gamma) + \frac{\beta(1-\gamma)}{\gamma} < 0$ , then the solution to problem (3.1) satisfies

$$\begin{cases} \frac{1}{2} \|\theta\|^2 \frac{(V'(x))^2}{V''(x)} - rxV'(x) + \beta V(x) = 0, & x > 0, \\ V(0) = 0. \end{cases}$$

Similar to (ii) in the case 1, the original problem has no solution.  $\square$

In general case  $K > 0$ , we employ dual method to problem (3.3), we define

$$v(y) = \max_{x>0} [V(x) - xy], \quad y \in \left[ \lim_{x \rightarrow +\infty} V'(x), V'(0) \right]. \quad (3.4)$$

Since  $V''(x) < 0$  and the critical value  $x_y$  satisfies  $V'(x_y) = y$ , then there exists an inverse function  $I(y)$  of  $V'(x)$  such that  $x_y = I(y)$ . Hence

$$v(y) = V(x_y) - x_y y = V(I(y)) - yI(y), \quad y \in \left[ \lim_{x \rightarrow +\infty} V'(x), V'(0) \right]. \quad (3.5)$$

Furthermore, we have

$$v'(y) = V'(I(y))I'(y) - I(y) - yI'(y) = -I(y) < 0, \quad (3.6)$$

$$v''(y) = -I'(y) = \frac{-1}{V''(I(y))} > 0, \quad (3.7)$$

which implies that  $I(y)$  is strictly decreasing, and  $v(y)$  is strictly convex and decreasing on  $(\lim_{x \rightarrow +\infty} V'(x), V'(0))$ . Moreover, in view of (3.4), we have

$$V(x) = \min_{y \in \left[ \lim_{x \rightarrow +\infty} V'(x), V'(0) \right]} [v(y) + xy], \quad x > 0. \quad (3.8)$$

Then for any  $x > 0$ , let  $y_x = V'(x)$ , in view of (3.6), we have

$$v'(y_x) = v'(V'(x)) = -I(V'(x)) = -x.$$

In terms of (3.6), we obtain  $I(y_x) = x$ , which implies

$$y_x = I^{-1}(x) = V'(x). \quad (3.9)$$

It follows from (3.8) that we have

$$v(y_x) + xy_x = V(x).$$

Set  $x = I(y)$ . Then (3.3) reads

$$\begin{cases} \frac{1}{2}\|\theta\|^2 \frac{y^2}{V''(I(y))} - ryI(y) + \beta V(I(y)) = 0, & y_* < y < y_0, \\ V(I(y)) = U(\alpha(I(y) - B) + K), & \lim_{x \rightarrow +\infty} V'(x) < y \leq y_*, \\ V(I(y_0)) = U(K), \end{cases} \quad (3.10)$$

where  $y_0$  is defined by  $y_0 = V'(0)$  and  $y_*$  is defined by  $y_* = V'(x_*)$ .

By (3.5), (3.6) and (3.7), we deduce the first two equations of (3.10) into

$$\begin{cases} \frac{1}{2}\|\theta\|^2 y^2 v''(y) + (\beta - r)yv'(y) - \beta v(y) = 0, & y_* < y < y_0, \\ v(y) - yv'(y) = U(\alpha(-v'(y) - B) + K), & \lim_{x \rightarrow +\infty} V'(x) < y \leq y_*. \end{cases} \quad (3.11)$$

The definition of  $y_0$  implies that  $I(y_0) = 0$ . Then  $v'(y_0) = -I(y_0) = 0$ . Combining with (3.5) yields

$$V(0) = V(I(y_0)) = v(y_0) + y_0 I(y_0) = v(y_0).$$

Since the boundary condition of  $V(x)$  at  $x = 0$  is  $U(K)$ , thus we obtain the boundary conditions at  $y = y_0$  ( $y_0$  is unknown)

$$\begin{cases} v(y_0) = U(K), \\ v'(y_0) = 0. \end{cases}$$

By the smooth-fit condition, we know  $V'(x)$  continuously goes through  $x = x_*$ , then

$$V'(x_*) = \alpha A \left( \frac{A(\alpha(x_* - B) + K - w)}{1 - \gamma} \right)^{\gamma-1}. \quad (3.12)$$

Substituting  $y_*$  in the second equality of (3.11) and combining with the definition of  $y_*$ , we obtain  $-v'(y_*) = I(y_*) = x_* > B$ . Hence, the boundary conditions at  $y = y_*$  become

$$\begin{cases} v(y_*) - y_* v'(y_*) = U(\alpha(-v'(y_*) - B) + K), \\ y_* = \alpha A \left( \frac{A(\alpha(-v'(y_*) - B) + K - w)}{1 - \gamma} \right)^{\gamma-1}. \end{cases}$$

In view of the above analysis, we have the following lemma

**Lemma 3.2.** *The function  $v(y)$  satisfies the following differential equation*

$$\frac{1}{2}\|\theta\|^2 y^2 v''(y) + (\beta - r)yv'(y) - \beta v(y) = 0, \quad y_* < y < y_0, \quad (3.13)$$

with the boundary condition

$$v(y_0) = U(K), \quad (3.14)$$

$$v'(y_0) = 0, \quad (3.15)$$

$$v(y_*) - y_* v'(y_*) = U(\alpha(-v'(y_*) - B) + K), \quad (3.16)$$

$$y_* = \alpha A \left( \frac{A(\alpha(-v'(y_*) - B) + K - w)}{1 - \gamma} \right)^{\gamma-1}. \quad (3.17)$$

**Lemma 3.3.** *When  $K - \omega - \alpha B < 0$ , there exists a unique solution of (3.13)-(3.17).*

*Proof.* Obviously, the general solution of the ordinary differential equation (3.13) can be expressed by

$$v(y) = D_1 y^{n_1} + D_2 y^{n_2}, \quad y \in [y_*, y_0], \quad (3.18)$$

where

$$\begin{cases} n_1 = \frac{-(\beta - r - \frac{1}{2}\|\theta\|^2) + \sqrt{(\beta - r - \frac{1}{2}\|\theta\|^2)^2 + 2\|\theta\|^2\beta}}{\|\theta\|^2} > 1, \\ n_2 = \frac{-(\beta - r - \frac{1}{2}\|\theta\|^2) - \sqrt{(\beta - r - \frac{1}{2}\|\theta\|^2)^2 + 2\|\theta\|^2\beta}}{\|\theta\|^2} < 0. \end{cases} \quad (3.19)$$

Note that,  $y_0 > 0$  is finite, otherwise problem (3.13)–(3.17) has no solution. Substituting the expression (3.18) of  $v(y)$  into (3.14) and (3.15) yields

$$D_1 = \frac{-n_2 U(K)}{(n_1 - n_2)y_0^{n_1}}, \quad D_2 = \frac{n_1 U(K)}{(n_1 - n_2)y_0^{n_2}}. \quad (3.20)$$

Thus,

$$v(y) = \frac{-n_2 U(K)}{n_1 - n_2} \left(\frac{y}{y_0}\right)^{n_1} + \frac{n_1 U(K)}{n_1 - n_2} \left(\frac{y}{y_0}\right)^{n_2}, \quad y \in [y_*, y_0], \quad (3.21)$$

where  $(y_0, y_*)$  satisfy (3.16)–(3.17). Substituting (3.21) into (3.16), we have

$$\frac{(n_1 - 1)n_2 U(K)}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_1} + \frac{(1 - n_2)n_1 U(K)}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_2} = U(\alpha(-v'(y_*) - B) + K). \quad (3.22)$$

It follows from (3.22) and the definition of  $U(x)$  that we obtain

$$\begin{aligned} & \frac{(n_1 - 1)n_2(K - w)^\gamma}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_1} + \frac{(1 - n_2)n_1(K - w)^\gamma}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_2} \\ &= \left\{ \alpha \left[ \frac{n_1 n_2 U(K)}{(n_1 - n_2)y_0} \left(\frac{y_*}{y_0}\right)^{n_1-1} - \frac{n_1 n_2 U(K)}{(n_1 - n_2)y_0} \left(\frac{y_*}{y_0}\right)^{n_2-1} - B \right] + K - w \right\}^\gamma \\ &= \left[ \frac{\alpha n_1 n_2 U(K)}{(n_1 - n_2)y_0} \left( \left(\frac{y_*}{y_0}\right)^{n_1-1} - \left(\frac{y_*}{y_0}\right)^{n_2-1} \right) + K - w - \alpha B \right]^\gamma. \end{aligned}$$

Re-arranging the above equation leads to

$$\begin{aligned} & \frac{\alpha n_1 n_2 U(K)}{(n_1 - n_2)y_0} \left( \left(\frac{y_*}{y_0}\right)^{n_1-1} - \left(\frac{y_*}{y_0}\right)^{n_2-1} \right) \\ &= \left[ \frac{(n_1 - 1)n_2(K - w)^\gamma}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_1} + \frac{(1 - n_2)n_1(K - w)^\gamma}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_2} \right]^{\frac{1}{\gamma}} - K + w + \alpha B. \end{aligned}$$

This implies that  $y_0$  can be expressed by  $\frac{y_*}{y_0}$ , i.e.,

$$y_0 = \frac{\alpha n_1 n_2 U(K) \left( \left(\frac{y_*}{y_0}\right)^{n_1-1} - \left(\frac{y_*}{y_0}\right)^{n_2-1} \right)}{(n_1 - n_2) \left\{ \left[ \frac{(n_1 - 1)n_2}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_1} + \frac{(1 - n_2)n_1}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_2} \right]^{\frac{1}{\gamma}} (K - w) - K + w + \alpha B \right\}}. \quad (3.23)$$

From (3.16), (3.17) and the definition of  $U(x)$ , we obtain

$$\frac{v(y_*) - y_* v'(y_*)}{y_*} = \frac{\frac{1-\gamma}{\gamma} \frac{A^\gamma}{(1-\gamma)^\gamma} [\alpha(-v'(y_*) - B) + K - w]^\gamma}{\alpha \frac{A^\gamma}{(1-\gamma)^{\gamma-1}} [\alpha(-v'(y_*) - B) + K - w]^{\gamma-1}},$$

which implies

$$v(y_*) + \left(\frac{1}{\gamma} - 1\right) y_* v'(y_*) = \frac{K - w - \alpha B}{\alpha \gamma} y_*.$$

Similarly,  $y_*$  can be expressed by  $\frac{y_*}{y_0}$ , i.e.,

$$y_* = \frac{\alpha \gamma U(K)}{K - w - \alpha B} \left[ \frac{-(1 + n_1(\frac{1}{\gamma} - 1))n_2}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_1} + \frac{(1 + n_2(\frac{1}{\gamma} - 1))n_1}{n_1 - n_2} \left(\frac{y_*}{y_0}\right)^{n_2} \right]. \quad (3.24)$$

Let  $\delta = \frac{y_*}{y_0}$ . It follows from (3.23) and (3.24) that we obtain

$$y_0 = \frac{\alpha n_1 n_2 U(K) (\delta^{n_1-1} - \delta^{n_2-1})}{(n_1 - n_2) \left\{ \left[ \frac{(n_1-1)n_2}{n_1-n_2} \delta^{n_1} + \frac{(1-n_2)n_1}{n_1-n_2} \delta^{n_2} \right]^{\frac{1}{\gamma}} (K - w) - K + w + \alpha B \right\}} \quad (3.25)$$

and

$$y_* = \frac{\alpha \gamma U(K)}{K - w - \alpha B} \left[ \frac{-(1 + n_1(\frac{1}{\gamma} - 1))n_2}{n_1 - n_2} \delta^{n_1} + \frac{(1 + n_2(\frac{1}{\gamma} - 1))n_1}{n_1 - n_2} \delta^{n_2} \right]. \quad (3.26)$$

Since  $0 < y_* < y_0$ , we have  $0 < \delta < 1$ . Also, since  $n_1 > 1$  and  $n_2 < 0$ , we have  $0 < \delta^{n_1-n_2} < 1$ . Furthermore, we get  $0 < \delta^{n_1-1} < \delta^{n_2-1}$ . From (3.25), we know

$$\left[ \frac{(n_1-1)n_2}{n_1-n_2} \delta^{n_1} + \frac{(1-n_2)n_1}{n_1-n_2} \delta^{n_2} \right]^{\frac{1}{\gamma}} (K - w) - K + w + \alpha B > 0. \quad (3.27)$$

It then follows from (3.25) and (3.26) that we obtain

$$\begin{aligned} & n_1 n_2 [\delta^{n_1} - \delta^{n_2}] \\ &= \frac{\gamma}{K - w - \alpha B} \left[ -\left(1 + n_1\left(\frac{1}{\gamma} - 1\right)\right) n_2 \delta^{n_1} + \left(1 + n_2\left(\frac{1}{\gamma} - 1\right)\right) n_1 \delta^{n_2} \right] \\ & \cdot \left\{ \left[ \frac{(n_1-1)n_2}{n_1-n_2} \delta^{n_1} + \frac{(1-n_2)n_1}{n_1-n_2} \delta^{n_2} \right]^{\frac{1}{\gamma}} (K - w) - K + w + \alpha B \right\}. \end{aligned} \quad (3.28)$$

Set

$$\begin{aligned} f(\delta) &\stackrel{\Delta}{=} n_1 n_2 (\delta^{n_1} - \delta^{n_2}) \\ &+ \frac{\gamma}{K - w - \alpha B} \left[ \left(1 + n_1\left(\frac{1}{\gamma} - 1\right)\right) n_2 \delta^{n_1} - \left(1 + n_2\left(\frac{1}{\gamma} - 1\right)\right) n_1 \delta^{n_2} \right] \\ &\cdot \left\{ \left[ \frac{(n_1-1)n_2}{n_1-n_2} \delta^{n_1} + \frac{(1-n_2)n_1}{n_1-n_2} \delta^{n_2} \right]^{\frac{1}{\gamma}} (K - w) - K + w + \alpha B \right\} \\ &= \gamma g(\delta) + \frac{\gamma(K - w)}{(K - w - \alpha B)(n_1 - n_2)^{\frac{1}{\gamma}}} \left[ \frac{n_1 n_2}{\gamma} (\delta^{n_1} - \delta^{n_2}) - g(\delta) \right] [g(\delta)]^{\frac{1}{\gamma}}, \end{aligned} \quad (3.29)$$

where

$$g(\delta) \stackrel{\Delta}{=} (n_1 - 1) n_2 \delta^{n_1} + (1 - n_2) n_1 \delta^{n_2}.$$

By the definition of  $g(\delta)$ , we obtain

$$\begin{aligned} g(0) &= +\infty, \quad g(1) = n_1 - n_2 > 0, \\ g'(\delta) &= n_1 n_2 [(n_1 - 1)\delta^{n_1-1} + (1 - n_2)\delta^{n_2-1}] < 0, \quad \delta \in (0, 1], \end{aligned}$$

hence

$$g(\delta) > 0, \quad \delta \in (0, 1].$$

Define  $h(\delta) = \frac{1}{\gamma}f(\delta)[g(\delta)]^{-\frac{1}{\gamma}}$ , we will show that  $h(\delta) = 0$  has a unique solution in  $[0, 1]$ . Since

$$\begin{aligned} h(\delta) &= \frac{1}{\gamma}f(\delta)[g(\delta)]^{-\frac{1}{\gamma}} \\ &= g(\delta)^{1-\frac{1}{\gamma}} + \frac{K-w}{(K-w-\alpha B)(n_1-n_2)^{\frac{1}{\gamma}}} \left[ \frac{n_1 n_2}{\gamma} (\delta^{n_1} - \delta^{n_2}) - g(\delta) \right]. \end{aligned}$$

Note that  $\beta$  is large enough such that  $1+n_2\left(\frac{1}{\gamma}-1\right) < 0$ . When  $K-w-\alpha B < 0$ , we have

$$\begin{aligned} h(0) &= 0 + \frac{K-w}{(K-w-\alpha B)(n_1-n_2)^{\frac{1}{\gamma}}} \lim_{\delta \rightarrow 0} \left[ n_2 \left( \frac{n_1}{\gamma} - n_1 + 1 \right) \delta^{n_1} - n_1 \left( \frac{n_2}{\gamma} - n_2 + 1 \right) \delta^{n_2} \right] \\ &= -\infty, \\ h(1) &= (n_1 - n_2)^{1-\frac{1}{\gamma}} \left( 1 - \frac{K-w}{K-w-\alpha B} \right) = (n_1 - n_2)^{1-\frac{1}{\gamma}} \frac{-\alpha B}{K-w-\alpha B} > 0, \\ h'(\delta) &= \frac{\gamma-1}{\gamma} g(\delta)^{-\frac{1}{\gamma}} g'(\delta) \\ &\quad + \frac{n_1 n_2 (K-w)}{(K-w-\alpha B)(n_1-n_2)^{\frac{1}{\gamma}}} \left[ \left( \frac{n_1}{\gamma} - n_1 + 1 \right) \delta^{n_1-1} - \left( \frac{n_2}{\gamma} - n_2 + 1 \right) \delta^{n_2-1} \right] > 0, \end{aligned}$$

the last inequality holds with the facts  $0 < \gamma < 1$  and  $g'(\delta) < 0$ . Hence, there exists a unique solution of  $h(\delta) = 0$  in  $[0, 1]$ .

Thus, there exists a unique  $\delta_0 \in (0, 1)$  such that

$$f(\delta_0) = 0.$$

Substituting  $\delta_0$  into (3.25) and (3.26), we obtain

$$y_0 = \frac{\alpha n_1 n_2 U(K) (\delta_0^{n_1-1} - \delta_0^{n_2-1})}{(n_1 - n_2) \left\{ \left[ \frac{(n_1-1)n_2}{n_1-n_2} \delta_0^{n_1} + \frac{(1-n_2)n_1}{n_1-n_2} \delta_0^{n_2} \right]^{\frac{1}{\gamma}} (K-w) - K + w + \alpha B \right\}}, \quad (3.30)$$

$$y_* = \frac{\alpha \gamma U(K)}{K-w-\alpha B} \left[ \frac{-(1+n_1(\frac{1}{\gamma}-1))n_2}{n_1-n_2} \delta_0^{n_1} + \frac{(1+n_2(\frac{1}{\gamma}-1))n_1}{n_1-n_2} \delta_0^{n_2} \right]. \quad (3.31)$$

Then we substitute the above  $y_0$  into (3.21) to get the expression of  $v(y)$  for  $y \in [y_*, y_0]$ .  $\square$

Furthermore, we can derive  $V(x)$  by

$$V(x) = \min_{y \in [y_*, y_0]} [v(y) + xy] = \min_{y \in [y_*, y_0]} [D_1 y^{n_1} + D_2 y^{n_2} + xy], \quad x \in (0, x_*), \quad (3.32)$$

where

$$x_* = I(y_*) = -v'(y_*) = \frac{n_1 n_2 U(K)}{(n_1 - n_2) y_0} [\delta_0^{n_1-1} - \delta_0^{n_2-1}], \quad (3.33)$$

with  $y_0$  given in (3.30). Hence, the minimum  $y_x$  satisfies

$$D_1 n_1 y_x^{n_1-1} + D_2 n_2 y_x^{n_2-1} + x = 0, \quad x \in (0, x_*). \quad (3.34)$$

Therefore, for any  $x \in (0, x_*)$ , solving (3.34) to obtain the minimum  $y_x$  and substituting it into (3.32), we obtain

$$V(x) = \begin{cases} D_1 y_x^{n_1} + D_2 y_x^{n_2} + xy_x, & x \in (0, x_*), \\ U(\alpha(x - B) + K), & x \in [x_*, +\infty). \end{cases} \quad (3.35)$$

In terms of (3.18) with  $n_1 > 1$ ,  $n_2 < 0$ ,  $D_1, D_2 > 0$ , we have

$$v''(y) = D_1 n_1(n_1 - 1)y^{n_1-2} + D_2 n_2(n_2 - 1)y^{n_2-2} > 0, \quad y \in [y_*, y_0].$$

Thus, we obtain the first and second orders of  $V(x)$  as below

$$\begin{aligned} V'(x) &= V'(I(y_x)) = y_x > 0, \\ V''(x) &= \begin{cases} V''(I(y)) = -\frac{1}{v''(y)} < 0, & x \in (0, x_*), \\ -(\alpha A)^2 \left( \frac{A(\alpha(x-B)+K-w)}{1-\gamma} \right)^{\gamma-2} < 0, & x \in [x_*, +\infty), \end{cases} \end{aligned}$$

which imply that  $V(x)$  is increasing and concave.

Moreover, the optimal free boundary can be presented by

$$\begin{aligned} x_* &= I(y_*) = -v'(y_*) \\ &= \frac{n_1 n_2 U(K)}{(n_1 - n_2) y_0} [\delta_0^{n_1-1} - \delta_0^{n_2-1}] \\ &= \frac{1}{\alpha} \left\{ \left[ \frac{(n_1 - 1)n_2(K - w)^\gamma}{n_1 - n_2} \delta_0^{n_1} + \frac{(1 - n_2)n_1(K - w)^\gamma}{n_1 - n_2} \delta_0^{n_2} \right]^{\frac{1}{\gamma}} - K + w + \alpha B \right\}. \end{aligned}$$

The third equality is due to  $\delta_0 = \frac{y_*}{y_0}$  and the last one is due to (3.30).

Suppose  $V(x)$  is given by (3.35), with  $n_1, n_2, D_1, D_2$  are given in (3.19) and (3.20), respectively,  $x_*$  which can be interpreted as the optimal exercise boundary is shown in (3.33), then  $V(x)$  is the solution to problem (3.1).

We now prove the following verification theorem.

**Theorem 3.4.** Suppose  $V(x)$  is the solution to problem (3.1), then for any admissible  $\pi$  and  $\tau$ , we have

$$V(x) \geq J_{\pi, \tau}(x). \quad (3.36)$$

Moreover, the optimal strategy pair  $(\pi^*, \tau^*)$  is

$$\begin{cases} \pi^* = -(\sigma\sigma^\top)^{-1}\mu \frac{V'(x)}{V''(x)}, \\ \tau^* = \inf\{t > 0 : X_t \geq x_*\}. \end{cases} \quad (3.37)$$

and

$$V(x) = J_{\pi^*, \tau^*}(x). \quad (3.38)$$

*Proof.* For any admissible  $\pi$ , by Itô formula,

$$\begin{aligned} d[e^{-\beta t} V(X_t)] &= e^{-\beta t} \left[ -\beta V(X_t) + (rX_t + \mu^\top \pi)V'(X_t) \right. \\ &\quad \left. + \frac{1}{2}\pi^\top (\sigma\sigma^\top)\pi V''(X_t) \right] ds + e^{-\beta t} \pi^\top \sigma dW_t. \end{aligned}$$

Thus, for any stopping time  $\tau \geq 0$ ,

$$\begin{aligned} V(x) &= \mathbb{E}[e^{-\beta(\tau \wedge T)} V(X_{\tau \wedge T})] + \mathbb{E} \int_0^{\tau \wedge T} e^{-\beta t} \left[ \beta V(X_t) - (rX_t + \mu^\top \pi) V'(X_t) \right. \\ &\quad \left. - \frac{1}{2} \pi^\top (\sigma \sigma^\top) \pi V''(X_t) \right] dt \\ &\geq \mathbb{E}[e^{-\beta(\tau \wedge T)} V(X_{\tau \wedge T})] \\ &\quad + \mathbb{E} \int_0^{\tau \wedge T} e^{-\beta t} \left[ - \sup_\pi \left( \frac{1}{2} \pi^\top (\sigma \sigma^\top) \pi V''(X_t) + \mu^\top \pi V'(X_t) \right) \right. \\ &\quad \left. - rX_t V'(X_t) + \beta V(X_t) \right] dt \\ &\geq \mathbb{E}[e^{-\beta(\tau \wedge T)} U(\alpha(X_{\tau \wedge T} - B)^+ + K)]. \end{aligned}$$

Taking  $\underline{\lim}_{T \rightarrow \infty}$  in both sides, then applying Fatou Lemma we obtain

$$V(x) \geq \underline{\lim}_{T \rightarrow \infty} \mathbb{E}[e^{-\beta(\tau \wedge T)} U(\alpha(X_{\tau \wedge T} - B)^+ + K)] \geq J_{\pi, \tau}(x).$$

We obtain (3.36).

On the other hand, define  $\pi(x) := -(\sigma \sigma^\top)^{-1} \mu \frac{V'(x)}{V''(x)}$ . Let  $X_t^*$  be the solution of the following SDE,

$$\begin{cases} dX_t = (rX_t + \mu^\top \pi(X_t))ds + \pi(X_t)^\top \sigma dW_t, \\ X_0 = x, \end{cases}$$

and let

$$\pi^* = \pi(X_t^*), \quad \tau^* = \inf\{t \geq 0 : V(X_t^*) = U(\alpha(X_t^* - B)^+ + K)\}.$$

Since  $V(X_t^*) > U(\alpha(X_t^* - B)^+ + K)$  when  $t < \tau^*$ , we have

$$\left[ -\left( \frac{1}{2} \pi^{*\top} (\sigma \sigma^\top) \pi^* V'' + \mu^\top \pi^* V' \right) - rX_t^* V' + \beta V \right](X_t^*) = 0, \quad t < \tau^*.$$

Applying Itô formula yields

$$\begin{aligned} V(x) &= \mathbb{E}[e^{-\beta(\tau^* \wedge T)} V(X_{\tau^* \wedge T}^*)] + \mathbb{E} \int_0^{\tau^* \wedge T} e^{-\beta t} \left[ \beta V - (rX_t^* + \mu^\top \pi^*) V' \right. \\ &\quad \left. - \frac{1}{2} \pi^{*\top} (\sigma \sigma^\top) \pi^* V'' \right] (X_t^*) dt \\ &= \mathbb{E}[e^{-\beta(\tau^* \wedge T)} V(X_{\tau^* \wedge T}^*)] \\ &= \mathbb{E}\{e^{-\beta(\tau^* \wedge T)} [U(\alpha(X_{\tau^* \wedge T}^* - B)^+ + K) 1_{\{\tau^* \leq T\}} + V(X_T^*) 1_{\{\tau^* > T\}}]\} \\ &= \mathbb{E}\{e^{-\beta(\tau^* \wedge T)} U(\alpha(X_{\tau^* \wedge T}^* - B)^+ + K) 1_{\{\tau^* \leq T\}}\} \\ &\quad + \mathbb{E}\{e^{-\beta(\tau^* \wedge T)} V(X_T^*) 1_{\{\tau^* > T\}}\} \\ &= \mathbb{E}\{e^{-\beta\tau^*} U(\alpha(X_{\tau^*}^* - B)^+ + K) 1_{\{\tau^* \leq T\}}\} + \mathbb{E}\{e^{-\beta T} V(X_T^*) 1_{\{\tau^* > T\}}\} \\ &\leq \mathbb{E}\{e^{-\beta\tau^*} U(\alpha(X_{\tau^*}^* - B)^+ + K)\} + \mathbb{E}\{e^{-\beta T} V(X_T^*) 1_{\{\tau^* > T\}}\}. \end{aligned}$$

Thus

$$V(x) \leq J_{\pi^*, \tau^*}(x) + e^{-\beta T} \mathbb{E}\{V(X_T^*) 1_{\{\tau^* > T\}}\}. \quad (3.39)$$

Recalling (3.3), if  $\tau^* > T$ , then  $X_T^* \leq x^*$  where  $x^*$  is a finite number, hence  $V(X_T^*) \leq V(x^*)$  by  $V'(x) \geq 0$ , so

$$V(x) \leq J_{\pi^*, \tau^*}(x) + e^{-\beta T} V(x^*).$$

Letting  $T \rightarrow +\infty$ , we obtain (3.38).  $\square$

**4. Stopping problem with no-shorting constraint.** If there is a no-shorting constraint, the manager's utility maximization problem is

$$\bar{V}(x) = \sup_{\pi \in \mathbb{R}_+^m, \tau} \mathbb{E}[e^{-\beta\tau} U(\alpha(X_\tau - B)^+ + K)]. \quad (4.1)$$

where  $X_\cdot$  is the solution to the stochastic differential equation (2.2), and again, we have

$$\begin{cases} \min \left\{ - \max_{\pi \in \mathbb{R}_+^m} \left[ \frac{1}{2} \pi^\top \sigma \sigma^\top \pi \bar{V}''(x) + \mu^\top \pi \bar{V}'(x) \right] - rx \bar{V}'(x) + \beta \bar{V}(x), \right. \\ \left. \bar{V}(x) - U(\alpha(x - B)^+ + K) \right\} = 0, \quad x > 0, \\ \bar{V}(0) = U(K). \end{cases} \quad (4.2)$$

We conjecture that there exists a free boundary point  $\bar{x}^* > B$  such that

$$\begin{cases} \bar{V}(x) > U(\alpha(x - B)^+ + K), & 0 < x < \bar{x}^*, \\ \bar{V}(x) = U(\alpha(x - B)^+ + K) = U(\alpha(x - B) + K), & x \geq \bar{x}^* > B. \end{cases}$$

Furthermore, problem (4.2) can be reduced to

$$\begin{cases} \max_{\pi \in \mathbb{R}_+^m} \left[ \frac{1}{2} \pi^\top \sigma \sigma^\top \pi \bar{V}''(x) + \mu^\top \pi \bar{V}'(x) \right] + rx \bar{V}'(x) - \beta \bar{V}(x) = 0, & 0 < x < \bar{x}_*, \\ \bar{V}(x) = U(\alpha(x - B)^+ + K) = U(\alpha(x - B) + K), & x \geq \bar{x}_* > B, \\ \bar{V}(0) = U(K). \end{cases} \quad (4.3)$$

Let

$$\bar{z} := \operatorname{argmin}_{z \in \mathbb{R}_+^m} \frac{1}{2} \|\sigma^{-1} z + \sigma^{-1} \mu\|^2, \quad (4.4)$$

and

$$\bar{\xi} := \sigma^{-1} \bar{z} + \sigma^{-1} \mu. \quad (4.5)$$

Since  $\bar{V}'(x) \geq 0$  and  $\bar{V}''(x) < 0$ , we have  $-\frac{\bar{V}'(x)}{\bar{V}''(x)} \geq 0$ . By Lemma 3.2 (Xu-Shreve (1992)), we obtain the supremum in the HJB equation (4.3) by

$$\bar{\pi} = -(\sigma \sigma^\top)^{-1} (\bar{z} + \mu) \frac{\bar{V}'(x)}{\bar{V}''(x)}. \quad (4.6)$$

Substituting (4.6) into the HJB equation (4.3) results in the equation

$$\begin{cases} \frac{1}{2} \|\bar{\theta}\|^2 \frac{(\bar{V}'(x))^2}{\bar{V}''(x)} - rx \bar{V}'(x) + \beta \bar{V}(x) = 0, & 0 < x < \bar{x}_*, \\ \bar{V}(x) = U(\alpha(x - B) + K), & x \geq \bar{x}_* > B, \\ \bar{V}(0) = U(K), \end{cases} \quad (4.7)$$

where  $\bar{\theta} = \sigma^{-1}(\bar{z} + \mu)$ .

By Lemma 3.2 (Li-Zhou-Lim (2002)), we also can get the above equation (4.7) from the following unconstrained HJB equation (4.8)

$$\begin{cases} \max_{\pi} \left\{ \frac{1}{2} \pi^\top \sigma \sigma^\top \pi \bar{V}''(x) + (\bar{z} + \mu)^\top \pi \bar{V}'(x) \right\} + rx \bar{V}'(x) - \beta \bar{V}(x) = 0, & 0 < x < \bar{x}_*, \\ \bar{V}(x) = U(\alpha(x - B)^+ + K) = U(\alpha(x - B) + K), & x \geq \bar{x}_* > B, \\ \bar{V}(0) = U(K) \end{cases} \quad (4.8)$$

using the same portfolio form

$$\bar{\pi} = -(\sigma \sigma')^{-1} (\bar{z} + \mu) \frac{\bar{V}'(x)}{\bar{V}''(x)}. \quad (4.9)$$

Based on the above optimization analysis, the constrained portfolio problem (4.1) can be transformed into the equivalent unconstrained problem

$$\sup_{\pi, \tau} \mathbb{E}[e^{-\beta \tau} U(\alpha(\bar{X}_\tau - B)^+ + K)], \quad (4.10)$$

where the wealth process follows

$$d\bar{X}_t = [r\bar{X}_t + (\bar{z} + \mu)^\top \pi] dt + \pi^\top \sigma dW_t, \quad \bar{X}_0 = x. \quad (4.11)$$

In view of the above analysis and section 3, we have the following theorem.

**Theorem 4.1.** Suppose  $V(x)$  is the value function (4.1) of the no-shorting model, then  $V(x)$  is given by (3.35) with  $\theta$  replaced by  $\bar{\theta} = \sigma^{-1}(\bar{z} + \mu)$ , the optimal exercised boundary  $\bar{x}_*$  to the no-shorting model is also given by (3.33) with  $\theta$  replaced by  $\bar{\theta}$ . Moreover, the optimal strategy pair  $(\bar{\pi}, \bar{\tau})$

$$\begin{cases} \bar{\pi} = -(\sigma \sigma^\top)^{-1} (\bar{z} + \mu) \frac{V'(\bar{X}_t)}{V''(\bar{X}_t)}, \\ \bar{\tau} = \inf\{t > 0 : \bar{X}_t \geq \bar{x}_*\}, \end{cases} \quad (4.12)$$

where  $\bar{X}_t$  is the solution to the stochastic differential equation (4.11) and  $\bar{z}$  is shown in (4.4).

Also, we present the following verification theorem.

**Theorem 4.2.** Suppose  $V(x)$  is the solution to problem (4.2), then for any admissible  $\pi$  and  $\tau$ , we have

$$V(x) \geq J_{\pi, \tau}(x). \quad (4.13)$$

Moreover, there exist  $\bar{\pi}$  and  $\bar{\tau}$  such that

$$V(x) = J_{\bar{\pi}, \bar{\tau}}(x). \quad (4.14)$$

The proof of this theorem is similar to the proving procedure of Theorem 3.4.

**5. Numerical results.** In this section, a numerical example with constant coefficients is presented to demonstrate the results in the previous section. Let  $\gamma = 0.5$ ,  $A = 1$ ,  $w = 0$ ,  $\beta = 0.25$ ,  $m = 3$ . The interest rate of the bond and the appreciation rate of the  $m$  stocks are  $r = 0.03$  and  $\mu = (\mu_1, \mu_2, \mu_3)^\top = (0.09, 0.12, 0.15)^\top$ , respectively, and the volatility matrix is

$$\sigma = \begin{bmatrix} 0.2500 & 0 & 0 \\ 0.1500 & 0.2598 & 0 \\ -0.2500 & 0.2887 & 0.3227 \end{bmatrix}.$$

Then we have

$$\sigma^{-1} = \begin{bmatrix} 4.0000 & 0 & 0 \\ -2.3094 & 3.8490 & 0 \\ 5.1640 & -3.4427 & 3.0984 \end{bmatrix}$$

and

$$\begin{cases} \theta = \sigma^{-1}\mu = (0.3600, 0.2540, 0.5164)^\top, \\ (\sigma\sigma^\top)^{-1}\mu = (3.52, -0.8, 1.6)^\top. \end{cases}$$

We see that there exists a shorting case in policy (4.9). Using (4.4), we obtain the following  $\bar{z}$  to re-construct the no-shorting policy

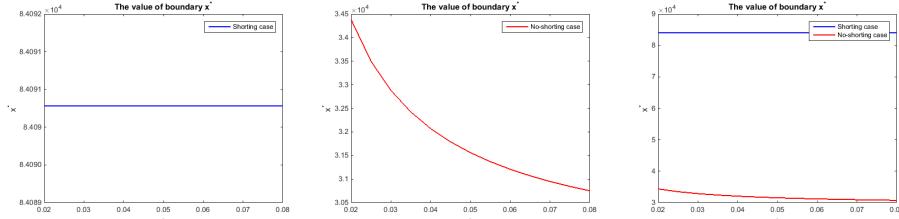
$$\bar{z} := \underset{z \in \mathbb{R}_+^m}{\operatorname{argmin}} \frac{1}{2} \|\sigma^{-1}z + \sigma^{-1}\mu\|^2 = (0, 0.03, 0)^\top.$$

Hence,

$$\begin{cases} \bar{\theta} = \sigma^{-1}(\bar{z} + \mu) = (0.3600, 0.3695, 0.4131)^\top, \\ (\sigma\sigma^\top)^{-1}(\bar{z} + \mu) = (2.72, 0, 1.28)^\top. \end{cases}$$

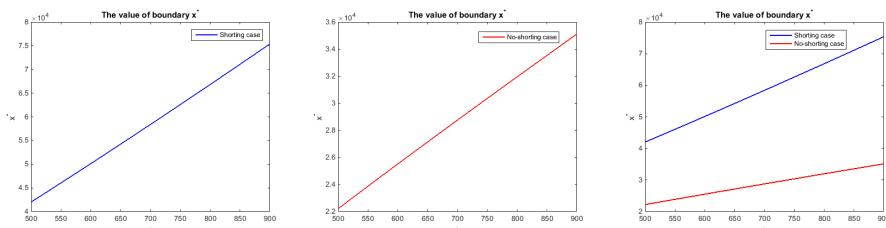
We now study how  $x^*$  and  $\bar{x}$  change when  $\alpha$ ,  $B$  and  $K$  run in the different intervals.

**Case 1.** Let  $0.02 \leq \alpha \leq 0.08$ ,  $B = 1000$  and  $K = 100$ .



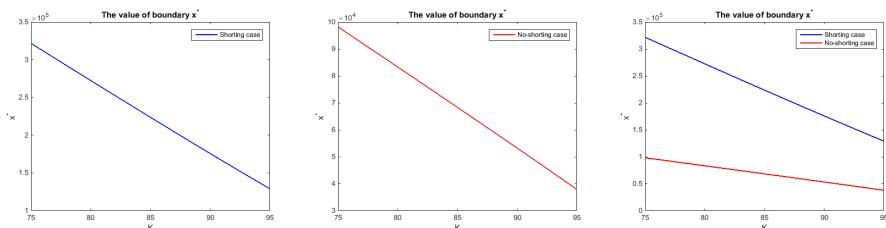
The free boundaries  $x^*$  and  $\bar{x}$  change when  $\alpha$  changes.

**Case 2.** Let  $\alpha = 0.01$ ,  $500 \leq B \leq 900$  and  $K = 100$ .



The free boundaries  $x^*$  and  $\bar{x}$  change when  $\alpha$  changes.

**Case 3.** Let  $\alpha = 0.01$ ,  $B = 1000$  and  $75 \leq K \leq 95$ .



The free boundaries  $x^*$  and  $\bar{x}$  change when  $K$  changes.

**6. Conclusions and implications.** Formulating a non-smooth utility maximization problem with optimal-stopping features, we study investment decision-making mechanisms and risk-taking behaviors of venture fund managers. Due to the challenging nature on the technical side of this free-boundary problem with a nonlinear equation, we have developed the methodology to convert it to a new one with a linear equation in order to optimize venture fund managers' investment strategies and risk-taking behaviors. Findings of this study add to our knowledge in the fields of risk management, financial investment, venture fund management, and entrepreneurial finance. More importantly, it provides critical implications for fund managers to optimize their investment strategies and for policy makers and entrepreneurs to improve risk management (Sparrow and Bentley (2000)). Future research may extend the findings of this study by empirical testing the propositions developed in this study, so that more convincing results may be obtained and presented.

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*E-mail address:* [xschen@m.scnu.edu.cn](mailto:xschen@m.scnu.edu.cn) (Xiaoshan Chen)

*E-mail address:* [malixun@polyu.edu.hk](mailto:malixun@polyu.edu.hk) (Xun Li)

*E-mail address:* [fhyi@scnu.edu.cn](mailto:fhyi@scnu.edu.cn) (Fahuai Yi)