Continuing the Lasso

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Question 1

Part a

Yes we can in fact use the lasso for classification problems. Say we are trying to find out whether an email is spam or not. The number of regressors may still in fact be large, so the lasso can help us distinguish which of these regressors contribute to classifying an email as spam or not

Part b

For large values of λ, the AL actually choses **less** variables than the lasso. In other words, if you penalize more in the lasso, the AL will penalize less. Will be more robust when your initial selection is large.

Part c

First, what is **Least Angle Regression**? Least Angle Regression (LARS) is a class of techniques for solving the lasso. Produce the entire path of solutions in a sequential (step-wise) manner as a piecewise linear path.

Delivers the entire solution path as a function of λ. It is efficient, but does not scale up to larger problems. Another way of looking at it is as a "democratic" forward stepwise regression.

LARS is a greedy algorithm that does not yield a provably consistent estimator. Conversely, the LASSO (and thus the LARS algorithm when used in LASSO mode) solves a convex data fitting problem.

Part d

Coordinate descent actually allows for upper and lower bounds on each coefficient and also allows for efficient computation. It is the workhorse algorithm in glmnet and is a simple extension from the lasso thus it can be used for the elastic net.

LARS is proposed in the paper by Zou and Hastie. If we fix the ridge regularizer, we essentially have a lasso on an augmented dataset and LARS works well.

The way this is done is by writing the EN as a lasso problem. This is shown in the slides. The same can be done for AL, where the weight is given in the penalty function. We essentially rescale all the variables by the weight. Dividing all by two means the parameter is multiplied by 2.

Question 2

Part a

$$y_i = \beta_1 x_{i,1} + \beta_2 x_{i,2} + \varepsilon_i, \quad i = 1, \dots, n$$

We look at the group lasso penalty and compare it to the lasso and ridge penalties

$$||\beta||_2 = \left(\sum_{j=1}^{k} |\beta_j|^2 \right)^{1/2}, \quad ||\beta||_1 = \sum_{j=1}^{k} |\beta_j|, \quad ||\beta||_2^2 = \sum_{j=1}^{k} \beta_j^2$$

For the group lasso, we consider j groups of covariates, can be denoted θ_j. The group lasso solves

$$\arg\min_{\theta_j} \left(||\mathbf{y} - \mathbf{X}\theta_j||_2^2/n + \lambda||\theta_j||_2 \right)$$

- Depending on λ, either the entire vector $\hat{\theta}_j$ is zero or all its elements will be nonzero
- If all groups are singletons, then this reduces to the normal lasso
- If we have one group, then this reduces to the ridge regression
- We consider only coefficients in group j, and all of these covariates are equally penalized
- The group penalty comes from the fact we square root **all** of the variables at once
- This applies only to the jˆth group

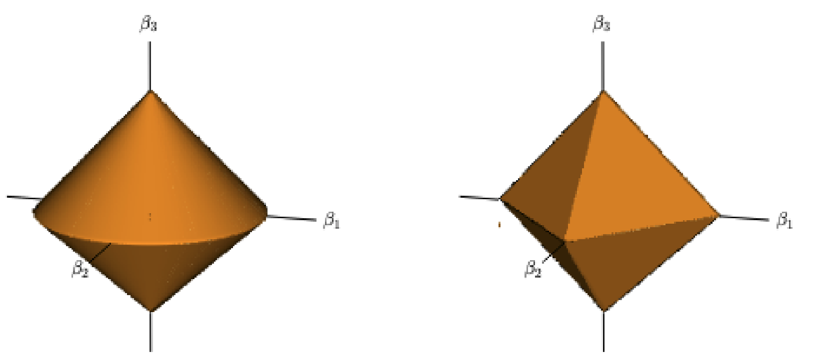


Figure 4.3 *The group lasso ball (left panel) in $\mathbb{R}^3$, compared to the ℓ_1 ball (right panel). In this case, there are two groups with coefficients $\theta_1 = (\beta_1, \beta_2) \in \mathbb{R}^2$ and $\theta_2 = \beta_3 \in \mathbb{R}^1$.*

Part b

$$\hat{\beta}_{GL} = \underset{\beta}{\arg\min}\left(\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta' \underline{x}_i)^2 + 2\gamma \|\beta\|_2\right)$$

take derivative wrt β

$$\frac{1}{n}\sum y_i^2 - 2y_i\beta'\underline{x}_i + \beta'^2 \underbrace{\underline{x}_i'\underline{x}_i}_{I} + 2\gamma\left(\sum \beta_i^2\right)^{\frac{1}{2}}$$

$$\frac{\partial}{\partial\beta} \to -2\sum y_i x_i/n + 2\beta + \boxed{\frac{\partial}{\partial\beta} 2\gamma\left(\sum \beta_i^2\right)^{\frac{1}{2}}} = 0$$

$$\frac{\partial}{\partial\beta} 2\gamma\left(\sum \beta_i^2\right)^{\frac{1}{2}} = \gamma\left(\sum \beta_i^2\right)^{-\frac{1}{2}} \cdot 2\beta_i = \frac{2\gamma\beta}{\left(\sum \beta_i^2\right)^{\frac{1}{2}}} = \frac{2\gamma\beta}{\|\beta_i\|_2}$$

we have to split to 2 conditions: $\beta_j = 0$ and $\beta_j \neq 0$ (can't divide by 0)

$$s(\beta) = \frac{\beta}{\|\beta_i\|_2} \to \text{if } \beta_i \neq 0, \qquad s(\beta) = (s_1(\beta), s_2(\beta))$$

If $\beta_i = 0 \to \mathbb{R}^{p=2}$ $\qquad \frac{f(x^*) - f(x)}{x^* - x} = \frac{\|x^*\|_2}{x^*}$ $\qquad$ collection of all subgradients is z. Take $x^* = c$

take $f(x) = \|x\|_2$, then

and $x = 0$

$$z \geq \frac{\|c\|_2}{c} \quad \text{for all } c < 0 \qquad\qquad z \leq \frac{\|c\|_2}{c} \quad \text{for all } c > 0$$

same condition if we multiply both sides

$$c\, z \leq \|c\|_2 \quad \text{for all } c \in \mathbb{R}^2$$

combining:

$$-2\sum_{i=1}^{n} y_i x_i + 2\beta + 2\gamma\, s(\beta) = 0$$

Part c

$$\hat{\beta}_{GL} = \hat{\beta}_{OLS} - \gamma s(\hat{\beta}_{GL}) \qquad s(\hat{\beta}_{GL}) = \begin{cases} \beta / \|\beta\|_2 & \text{if } \beta \neq 0 \\ z \text{ s.t. } \forall c \in \mathbb{R}_1^2 \; z'c \leq \|c\|_2 & \text{if } \beta = 0 \end{cases}$$

looking at $z'c \leq \|c\|_2 \rightarrow \|z\|_2 \leq 1$

$$\hat{\beta}_{GL} = 0 \Rightarrow \hat{\beta}_{OLS} = \gamma s(\hat{\beta}_{GL})$$

if $\hat{\beta}_{GL} = 0$

$$\hat{\beta}_{OLS} = \gamma z \quad \text{s.t. } \|z\|_2 \leq 1 \quad \Rightarrow \quad \frac{1}{\gamma}\hat{\beta}_{OLS} = z \Rightarrow \left\|\frac{1}{\gamma}\hat{\beta}_{OLS}\right\|_2 \leq 1$$

$$\left(\sum\left(\frac{1}{\gamma}\beta_{OLS}\right)^2\right)^{\frac{1}{2}} \leq 1 \quad \gamma \text{ is a constant} \Rightarrow \frac{1}{\gamma}\left(\sum(\hat{\beta}_{OLS})^2\right)^{\frac{1}{2}} \leq 1 \Rightarrow \left(\sum(\hat{\beta}_{OLS})^2\right)^{\frac{1}{2}} \leq \gamma$$

$$\therefore \boxed{\|\hat{\beta}_{OLS}\|_2 \leq \gamma}$$

Part d

$$\hat{\beta}_{GL} = \hat{\beta}_{OLS} - \gamma s(\hat{\beta}_{GL})$$

if $\hat{\beta}_{GL} \neq 0$

$$\hat{\beta} = \hat{\beta}_{OLS} - \gamma\left(\frac{\hat{\beta}}{\|\hat{\beta}\|_2}\right) \Rightarrow \hat{\beta} + \frac{\gamma\hat{\beta}}{\|\hat{\beta}\|_2} = \hat{\beta}_{OLS} \Rightarrow \hat{\beta}\left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right) = \hat{\beta}_{OLS}$$

$$\boxed{\hat{\beta} = \hat{\beta}_{OLS} / \left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)}$$

take ℓ_2-norm of above

$$\hat{\beta} = \left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)^{-1}\hat{\beta}_{OLS} \rightarrow \|\hat{\beta}\|_2 = \left\|\left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)^{-1}\hat{\beta}_{OLS}\right\|_2 \rightarrow \underbrace{\sqrt{\sum\left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)^{-2}\hat{\beta}_{OLS}^2}}_{\text{comes out}} = \left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)^{-1}\|\hat{\beta}_{OLS}\|_2$$

$$\|\hat{\beta}\|_2 \overbrace{\left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)}^{\text{moved to LHS}} = \|\hat{\beta}\|_2 + \gamma = \|\hat{\beta}_{OLS}\|_2$$

$$\|\hat{\beta}\|_2 = \|\hat{\beta}_{OLS}\|_2 - \gamma \Rightarrow \text{recall } \hat{\beta} = \left(1 + \frac{\gamma}{\|\hat{\beta}\|_2}\right)^{-1}\hat{\beta}_{OLS} \Rightarrow \left(1 + \frac{\gamma}{\|\hat{\beta}_{OLS}\|_2 - \gamma}\right)^{-1}\hat{\beta}_{OLS}$$

$$= \left(\frac{\|\hat{\beta}_{OLS}\|_2}{\|\hat{\beta}_{OLS}\|_2 - \gamma}\right)^{-1}\hat{\beta}_{OLS} = \left(\frac{\|\hat{\beta}_{OLS}\|_2 - \gamma}{\|\hat{\beta}_{OLS}\|_2}\right)\hat{\beta}_{OLS} = \boxed{\left(1 - \frac{\gamma}{\|\hat{\beta}_{OLS}\|_2}\right)\hat{\beta}_{OLS} = \hat{\beta}}$$

Part e

For Group Lasso: $\hat{\beta}_{GL} = \hat{\beta}_{OLS} - \gamma s(\beta)$ where $s(\beta) = (s_1(\beta), s_2(\beta))'$ and

$$s(\beta) = \begin{cases} \beta/\|\beta\|_2 & \text{if } \beta \neq 0 \\ z, \text{ st } \|z\|_2 \leq 1 & \text{if } \beta = 0 \end{cases}$$

if $\hat{\beta}_{GL} = 0 \Rightarrow \|\hat{\beta}_{OLS}\|_2 \leq \gamma$

if $\hat{\beta}_{GL} \neq 0$, $\hat{\beta}_{GL} = \hat{\beta}_{OLS}\left(1 - \frac{\gamma}{\|\hat{\beta}_{OLS}\|_2}\right)$

take $\hat{\beta}_{GL} \neq 0$

$$\hat{\beta}_{GL} = \hat{\beta}_{OLS}\left(1 - \underbrace{\frac{\gamma}{\|\hat{\beta}_{OLS}\|_2}}_{\geq 1}\right) \quad \text{st. } \|\hat{\beta}_{OLS}\|_2 \leq \gamma$$

Thus, $\hat{\beta}_{GL} = 0$ when our term inside parenthesis is ≤ 0. Can use max function.

$$\boxed{\hat{\beta}_{GL} = \hat{\beta}_{OLS} \max\left(1 - \frac{\gamma}{\|\hat{\beta}_{OLS}\|_2}, 0\right)}$$

Grouping effect occurs when we group vectors in order $\rightarrow \underline{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$. The ℓ_2-norm takes square root of entire group, and entire group will be set to zero if $1 - \frac{\gamma}{\|\hat{\beta}\|_2} \leq 0$. In other words: $\|\hat{\beta}_{OLS}\|_2 \leq \gamma$

Part f

$\mathbb{P}(\hat{\beta}_{GL} = 0) = \mathbb{P}(2\beta' z/\sqrt{n} + z'z/n \leq \gamma^2 - \beta'\beta)$ Goal!

we have $\hat{\beta}_{GL} = \hat{\beta}_{OLS} \max\left(1 - \frac{\gamma}{\|\hat{\beta}_{OLS}\|_2}, 0\right) \approx_d (\beta + z/\sqrt{n}) \max\left(1 - \frac{\gamma}{\|\beta + z/\sqrt{n}\|_2}, 0\right)$

where $z \sim N(0, I)$

$\mathbb{P}(\hat{\beta}_{GL} = 0) \approx \mathbb{P}(\|\beta + z/\sqrt{n}\|_2 \leq \gamma) = \mathbb{P}\left(\sum_{i=1}^{k} (\beta_i + z/\sqrt{n})^2 \leq \gamma^2\right) = \mathbb{P}\left(\sum(\beta\beta + 2\beta z/\sqrt{n} + z z/\sqrt{n}) \leq \gamma^2\right)$

$\mathbb{P}(\hat{\beta}_{GL} = 0) = \mathbb{P}\left(2\beta' z/\sqrt{n} + z'z/\sqrt{n} \leq \gamma^2 - \beta'\beta\right)$ Transpose comes with sum?

Part g

Consistent Selection

↳ All relevant groups are included if $n \to \infty$: $\mathbb{P}(S_0 \subseteq \hat{S}) \to 1$

↳ All irrelevant groups are excluded if $n \to \infty$: $\mathbb{P}(S_0^c \subseteq \hat{S}^c) \to 1$

$\Big\}\ \mathbb{P}(S_0 = \hat{S}) \to 1$

assume working with groups, and not single variables.

$$z_1 \sim N(0, I),\quad z_2 \sim N(0, I) \Rightarrow a z_1 + b z_2 \sim N(0, a^2 + b^2), \quad z_1^2 + z_2^2 \sim \chi^2(2)$$

$$\hat{\beta}_{GL} = \hat{\beta}_{OLS} \max\left(1 - \frac{\gamma}{\|\hat{\beta}_{OLS}\|_2}, 0\right) \Rightarrow \text{Suppose we have two parameters } \beta_1, \beta_2$$

where red parameters are one group

Case 1: If $\beta = 0$, $\mathbb{P}(\hat{\beta}_{GL} = 0) \to 1$ as $n \to \infty$

$$\mathbb{P}(\hat{\beta}_{GL} = 0) = \mathbb{P}(2\beta' z/\sqrt{n} + z'z/\sqrt{n} \le \gamma^2 - \beta'\beta)$$

$$= \mathbb{P}(z'z/\sqrt{n} \le \gamma^2) = \mathbb{P}(\underbrace{z_1^2 + z_2^2}_{\chi^2(2)} \le \sqrt{n}\gamma^2)$$

$\therefore$ we need $\sqrt{n}\gamma^2 \to \infty$ as $n \to \infty$

Case 2: If $\beta \ne 0$, $\mathbb{P}(\hat{\beta}_{GL} \ne 0) \to 1$ as $n \to \infty$ (or $\mathbb{P}(\hat{\beta}_{GL} = 0) \to 0$)

$\mathbb{P}(\hat{\beta}_{GL} = 0) = \mathbb{P}\left(2\beta' z/\sqrt{n} + z'z/\sqrt{n} \le \gamma^2 - \beta'\beta\right) \to 0$ if $\beta_1, \beta_2 \ne 0 \Rightarrow$ part of a group

β and z are vectors $\Rightarrow$ assume 2 elements

$$\mathbb{P}\Big(\underbrace{2(\beta_1 z_1 + \beta_2 z_2)}_{N(0, 4\beta_1^2 + 4\beta_2^2)}/\sqrt{n} + \underbrace{(z_1^2 + z_2^2)}_{\chi^2(2)}/\sqrt{n} \le \gamma^2 - \beta'\beta\Big)$$

$\mathbb{P}(2\beta_1 z_1 + 2\beta_2 z_2 + z_1 + z_2^2 \le \sqrt{n}(\gamma^2 - \beta'\beta)) \Rightarrow$ goes to zero when we go to ∞

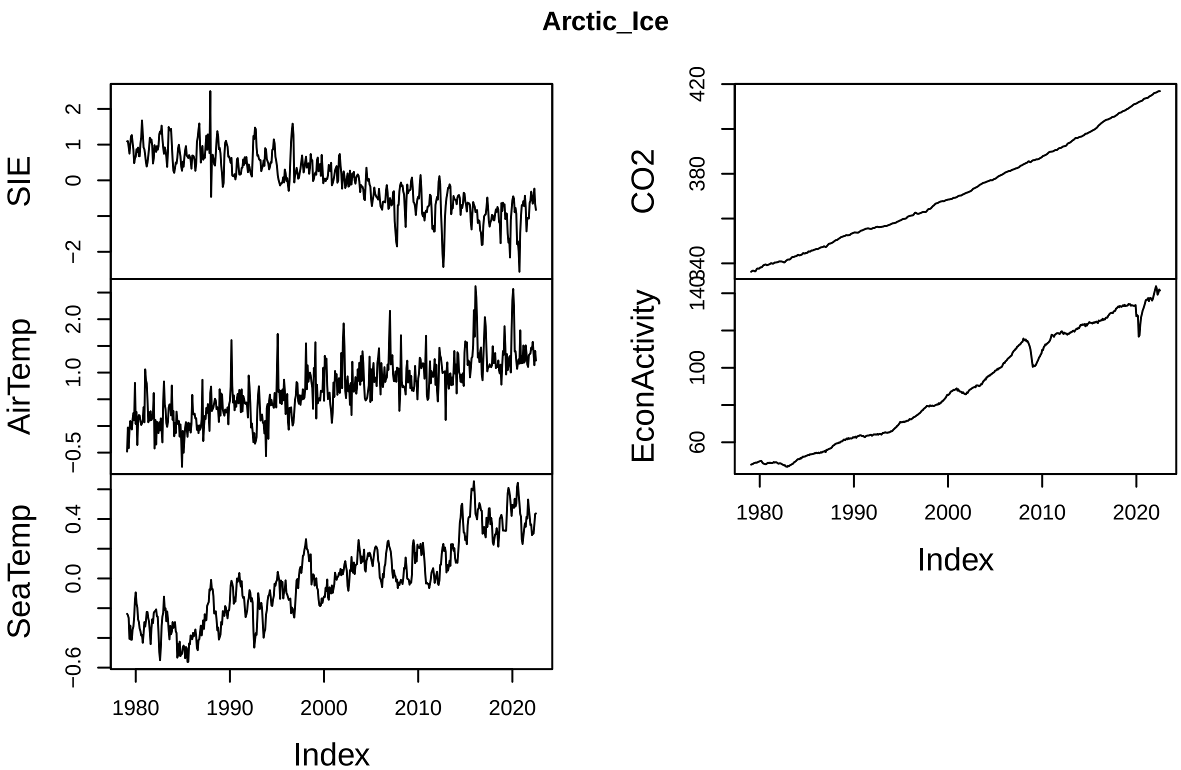
$\therefore$ we need $\sqrt{n}(\gamma^2 - \beta'\beta) \to -\infty$ as $n \to \infty$, or $\gamma^2 - \beta'\beta < 0 \Rightarrow \gamma^2 < \beta'\beta$

In this case, we need to note that the first case should be n and not $\sqrt{n}$. In the second case, we note that because we are dividing by $\sqrt{n}$ for the chi-square and by n for the normal distribution, the normal distribution takes over the chi square distribution. In this case, we simply need that it goes to negative infinity, which is what we have.

Question 3

```r
set.seed(20230215)
load("/Users/Ivan/Desktop/Areas/UM/CLASSES/YEAR_2/period_4/BIg_Data/Lasso_cont/Arctic_Ice.RData")
library(zoo)
```

```r
library(vars)
library(bigtime)
library(tseries)

plot(Arctic_Ice)
```



None of the time series look stationary, but we can perform unit tests to verify this.

```r
adf_test <- function(x) {
  return(adf.test(x)$p.value)
}
adf.results <- sapply(Arctic_Ice, adf_test)  # Applies the function adf_test to each column
adf.results
```

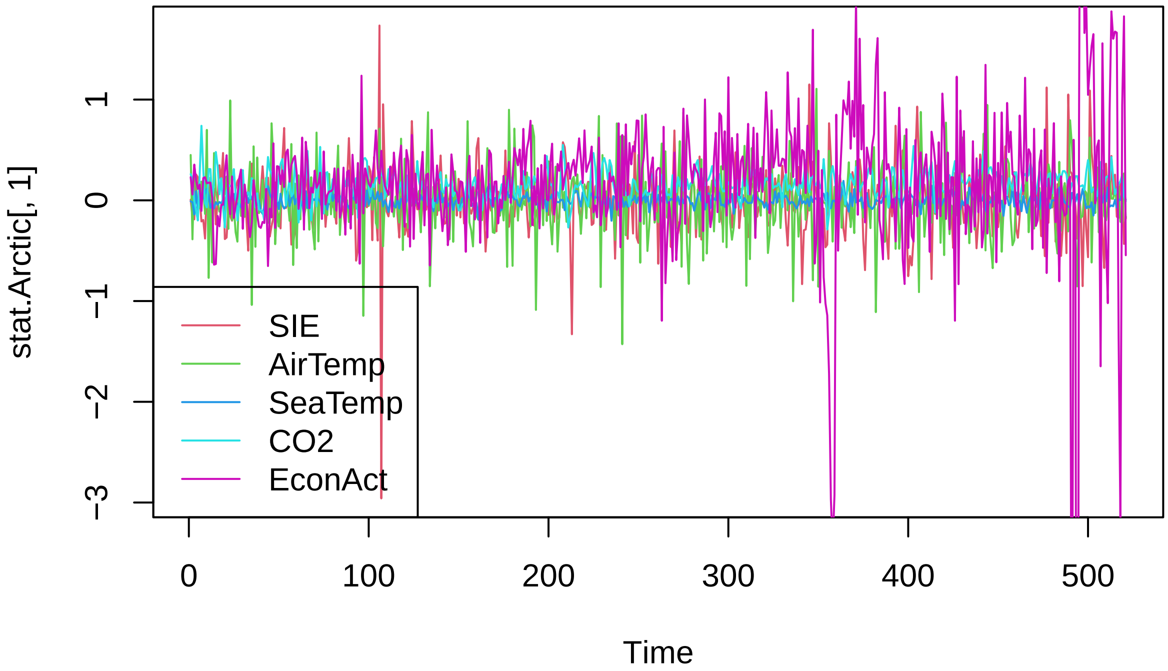
```
##          SIE      AirTemp      SeaTemp          CO2 EconActivity
##   0.01000000   0.01000000   0.01455506   0.98005701   0.01946908
```

```r
stat.Arctic <- as.zoo(sapply(Arctic_Ice,diff))
adf.results2 <- sapply(stat.Arctic, adf_test)
adf.results2
```

```
##          SIE      AirTemp      SeaTemp          CO2 EconActivity
##         0.01         0.01         0.01         0.01         0.01
```

```r
ts.plot(stat.Arctic[,1], col = 2)
lines(stat.Arctic[,2], col = 3)
lines(stat.Arctic[,3], col = 4)
lines(stat.Arctic[,4], col = 5)
lines(stat.Arctic[,5], col = 6)
legend("bottomleft",
       c("SIE", "AirTemp", "SeaTemp", "CO2", "EconAct"),
       lty = 1,
```

```r
      col = 2:6)
```



tests-1.pdf

Interesting that the null for CO2 is not rejected. Could be the case that it is trend stationary. But it seems (suspiciously enough) that all our series are stationary with the exception of `CO2`. Therefore, we use the "both" type option when estimating the VAR. We also select the number of lags using the `VARselect` function. In the end, this is not important because we are not performing inference for the VAR. Forecasting longer term is also not recommended with series in levels (But maybe more of a dimensionality problem).

```r
VARselect(Arctic_Ice,type = c("both"))$selection
```

```
## AIC(n)  HQ(n)  SC(n) FPE(n)
##      6      4      1      6
```

```r
VARselect(stat.Arctic,type = c("both"))$selection
```

```
## AIC(n)  HQ(n)  SC(n) FPE(n)
##      9      4      1      9
```

AIC chooses 6 lags which results in a high dimensional model with $5 \times 5 \times 6 = 150$ parameters to estimate. I also include both trend and intercept terms. Additionally, I estimate using the stationary values.

```r
VAR.estimate <- VAR(Arctic_Ice, p = 6, type = c("both"))
summary(VAR.estimate)
```

```
##
## VAR Estimation Results:
## =========================
## Endogenous variables: SIE, AirTemp, SeaTemp, CO2, EconActivity
## Deterministic variables: both
## Sample size: 516
## Log Likelihood: 494.184
## Roots of the characteristic polynomial:
## 0.9988 0.9348 0.8807 0.8595 0.8595 0.7561 0.7561 0.7416 0.7348 0.7348 0.7316 0.7316 0.7288 0.6903 0.
## Call:
## VAR(y = Arctic_Ice, p = 6, type = c("both"))
```

```
##
##
## Estimation results for equation SIE:
## ====================================
## SIE = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp.l2
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1           0.6617192  0.0456860  14.484  < 2e-16 ***
## AirTemp.l1      -0.0978275  0.0440580  -2.220  0.02685 *
## SeaTemp.l1      -0.2013697  0.2373861  -0.848  0.39670
## CO2.l1          -0.1325378  0.1233761  -1.074  0.28324
## EconActivity.l1 -0.0367817  0.0156493  -2.350  0.01916 *
## SIE.l2          -0.0149587  0.0549368  -0.272  0.78552
## AirTemp.l2      -0.0236538  0.0468644  -0.505  0.61398
## SeaTemp.l2      -0.3163881  0.3193253  -0.991  0.32228
## CO2.l2           0.3913520  0.2362478   1.657  0.09826 .
## EconActivity.l2  0.0117411  0.0238217   0.493  0.62233
## SIE.l3          -0.0042165  0.0541648  -0.078  0.93798
## AirTemp.l3      -0.0562523  0.0470476  -1.196  0.23242
## SeaTemp.l3       0.2021714  0.3194085   0.633  0.52706
## CO2.l3          -0.2329344  0.2763155  -0.843  0.39964
## EconActivity.l3  0.0329407  0.0238882   1.379  0.16855
## SIE.l4          -0.1568914  0.0539360  -2.909  0.00379 **
## AirTemp.l4      -0.0558023  0.0474211  -1.177  0.23988
## SeaTemp.l4       0.4776422  0.3173547   1.505  0.13296
## CO2.l4           0.0536381  0.2757944   0.194  0.84588
## EconActivity.l4 -0.0320199  0.0243258  -1.316  0.18870
## SIE.l5           0.0582774  0.0544640   1.070  0.28514
## AirTemp.l5      -0.0432329  0.0473292  -0.913  0.36146
## SeaTemp.l5       0.0100734  0.3172762   0.032  0.97468
## CO2.l5          -0.1676416  0.2372201  -0.707  0.48010
## EconActivity.l5  0.0382259  0.0243219   1.572  0.11668
## SIE.l6          -0.0279300  0.0456088  -0.612  0.54057
## AirTemp.l6      -0.0586526  0.0444309  -1.320  0.18743
## SeaTemp.l6       0.1073078  0.2339410   0.459  0.64666
## CO2.l6           0.0853028  0.1246639   0.684  0.49414
## EconActivity.l6 -0.0242619  0.0159021  -1.526  0.12774
## const            1.9759364  2.0969380   0.942  0.34651
## trend            0.0007024  0.0009140   0.769  0.44256
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.2842 on 484 degrees of freedom
## Multiple R-Squared: 0.8773,  Adjusted R-squared: 0.8694
## F-statistic: 111.6 on 31 and 484 DF,  p-value: < 2.2e-16
##
##
## Estimation results for equation AirTemp:
## ========================================
## AirTemp = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1           9.077e-03  4.687e-02   0.194  0.84650
```

```
## AirTemp.l1       3.287e-01  4.520e-02   7.274 1.42e-12 ***
## SeaTemp.l1       1.430e-01  2.435e-01   0.587  0.55732
## CO2.l1           1.014e-01  1.266e-01   0.801  0.42357
## EconActivity.l1 -1.932e-02  1.605e-02  -1.203  0.22942
## SIE.l2          -1.901e-02  5.636e-02  -0.337  0.73608
## AirTemp.l2       9.451e-02  4.808e-02   1.966  0.04989 *
## SeaTemp.l2       6.395e-01  3.276e-01   1.952  0.05148 .
## CO2.l2          -3.250e-01  2.424e-01  -1.341  0.18056
## EconActivity.l2  3.091e-02  2.444e-02   1.265  0.20657
## SIE.l3           1.323e-02  5.556e-02   0.238  0.81186
## AirTemp.l3      -6.303e-02  4.826e-02  -1.306  0.19216
## SeaTemp.l3      -4.363e-01  3.277e-01  -1.332  0.18359
## CO2.l3           5.070e-01  2.835e-01   1.789  0.07431 .
## EconActivity.l3  8.642e-03  2.451e-02   0.353  0.72451
## SIE.l4           9.920e-02  5.533e-02   1.793  0.07362 .
## AirTemp.l4      -1.525e-02  4.865e-02  -0.314  0.75398
## SeaTemp.l4       6.562e-01  3.256e-01   2.016  0.04438 *
## CO2.l4          -5.672e-01  2.829e-01  -2.005  0.04553 *
## EconActivity.l4 -6.618e-03  2.495e-02  -0.265  0.79097
## SIE.l5          -5.487e-02  5.587e-02  -0.982  0.32659
## AirTemp.l5      -5.739e-02  4.855e-02  -1.182  0.23780
## SeaTemp.l5      -6.978e-01  3.255e-01  -2.144  0.03254 *
## CO2.l5           3.589e-01  2.433e-01   1.475  0.14095
## EconActivity.l5 -6.006e-03  2.495e-02  -0.241  0.80987
## SIE.l6           4.470e-02  4.679e-02   0.955  0.33981
## AirTemp.l6      -1.621e-02  4.558e-02  -0.356  0.72226
## SeaTemp.l6       6.691e-01  2.400e-01   2.788  0.00551 **
## CO2.l6          -8.511e-02  1.279e-01  -0.666  0.50603
## EconActivity.l6  4.732e-03  1.631e-02   0.290  0.77188
## const            3.191e+00  2.151e+00   1.483  0.13865
## trend            2.344e-05  9.376e-04   0.025  0.98006
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.2915 on 484 degrees of freedom
## Multiple R-Squared: 0.723,   Adjusted R-squared: 0.7053
## F-statistic: 40.76 on 31 and 484 DF,  p-value: < 2.2e-16
##
##
## Estimation results for equation SeaTemp:
## ========================================
## SeaTemp = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1           1.225e-02  8.719e-03   1.405 0.160630
## AirTemp.l1       2.992e-02  8.408e-03   3.559 0.000409 ***
## SeaTemp.l1       9.244e-01  4.530e-02  20.404  < 2e-16 ***
## CO2.l1           3.119e-02  2.354e-02   1.325 0.185840
## EconActivity.l1  7.861e-04  2.986e-03   0.263 0.792494
## SIE.l2           3.463e-03  1.048e-02   0.330 0.741325
## AirTemp.l2      -2.013e-02  8.944e-03  -2.251 0.024826 *
## SeaTemp.l2      -1.764e-02  6.094e-02  -0.289 0.772406
## CO2.l2          -2.979e-02  4.509e-02  -0.661 0.509148
```

```
## EconActivity.l2  1.648e-03  4.546e-03   0.363 0.717070
## SIE.l3           6.604e-03  1.034e-02   0.639 0.523227
## AirTemp.l3       4.802e-03  8.978e-03   0.535 0.592980
## SeaTemp.l3      -3.537e-02  6.096e-02  -0.580 0.562040
## CO2.l3           2.024e-02  5.273e-02   0.384 0.701217
## EconActivity.l3 -6.631e-03  4.559e-03  -1.454 0.146457
## SIE.l4           1.570e-02  1.029e-02   1.525 0.127914
## AirTemp.l4       1.442e-02  9.050e-03   1.593 0.111809
## SeaTemp.l4       3.456e-02  6.056e-02   0.571 0.568555
## CO2.l4          -7.723e-02  5.263e-02  -1.467 0.142945
## EconActivity.l4  4.664e-03  4.642e-03   1.005 0.315563
## SIE.l5           3.203e-03  1.039e-02   0.308 0.758113
## AirTemp.l5      -1.047e-02  9.032e-03  -1.159 0.246969
## SeaTemp.l5      -1.957e-01  6.055e-02  -3.232 0.001311 **
## CO2.l5           5.703e-02  4.527e-02   1.260 0.208366
## EconActivity.l5 -1.618e-03  4.642e-03  -0.349 0.727591
## SIE.l6           6.935e-03  8.704e-03   0.797 0.425980
## AirTemp.l6       1.119e-02  8.479e-03   1.319 0.187692
## SeaTemp.l6       1.412e-01  4.464e-02   3.163 0.001660 **
## CO2.l6          -2.637e-04  2.379e-02  -0.011 0.991160
## EconActivity.l6  1.826e-03  3.035e-03   0.602 0.547753
## const           -5.379e-01  4.002e-01  -1.344 0.179494
## trend            6.333e-05  1.744e-04   0.363 0.716674
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.05424 on 484 degrees of freedom
## Multiple R-Squared: 0.9624,  Adjusted R-squared:  0.96
## F-statistic: 399.5 on 31 and 484 DF,  p-value: < 2.2e-16
##
##
## Estimation results for equation CO2:
## ====================================
## CO2 = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp.l2
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1          -0.0211825  0.0165250  -1.282  0.20051
## AirTemp.l1       0.0099050  0.0159361   0.622  0.53453
## SeaTemp.l1       0.0767154  0.0858646   0.893  0.37206
## CO2.l1           1.6920292  0.0446262  37.916  < 2e-16 ***
## EconActivity.l1 -0.0002413  0.0056605  -0.043  0.96601
## SIE.l2           0.0248266  0.0198711   1.249  0.21213
## AirTemp.l2       0.0094336  0.0169512   0.557  0.57812
## SeaTemp.l2       0.0708577  0.1155027   0.613  0.53985
## CO2.l2          -1.4168267  0.0854529 -16.580  < 2e-16 ***
## EconActivity.l2  0.0023641  0.0086165   0.274  0.78392
## SIE.l3          -0.0301779  0.0195919  -1.540  0.12413
## AirTemp.l3      -0.0506248  0.0170175  -2.975  0.00308 **
## SeaTemp.l3      -0.1158040  0.1155328  -1.002  0.31668
## CO2.l3           1.1896514  0.0999457  11.903  < 2e-16 ***
## EconActivity.l3  0.0068827  0.0086406   0.797  0.42610
## SIE.l4           0.0096869  0.0195091   0.497  0.61974
## AirTemp.l4       0.0526952  0.0171526   3.072  0.00225 **
```

```
## SeaTemp.l4       0.0631112  0.1147899   0.550  0.58271
## CO2.l4          -0.8452505  0.0997572  -8.473 2.89e-16 ***
## EconActivity.l4 -0.0060713  0.0087989  -0.690  0.49052
## SIE.l5          -0.0015165  0.0197001  -0.077  0.93867
## AirTemp.l5      -0.0341867  0.0171194  -1.997  0.04639 *
## SeaTemp.l5       0.0226370  0.1147615   0.197  0.84371
## CO2.l5           0.5256695  0.0858045   6.126 1.86e-09 ***
## EconActivity.l5 -0.0046957  0.0087975  -0.534  0.59376
## SIE.l6           0.0151634  0.0164971   0.919  0.35847
## AirTemp.l6      -0.0093980  0.0160710  -0.585  0.55897
## SeaTemp.l6       0.0934104  0.0846185   1.104  0.27018
## CO2.l6          -0.1489514  0.0450920  -3.303  0.00103 **
## EconActivity.l6  0.0048178  0.0057519   0.838  0.40267
## const            1.2864951  0.7584804   1.696  0.09050 .
## trend           -0.0001850  0.0003306  -0.560  0.57603
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.1028 on 484 degrees of freedom
## Multiple R-Squared:     1,   Adjusted R-squared:     1
## F-statistic: 8.288e+05 on 31 and 484 DF,  p-value: < 2.2e-16
##
##
## Estimation results for equation EconActivity:
## =============================================
## EconActivity = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + Se
##
##                  Estimate Std. Error t value Pr(>|t|)
## SIE.l1          -0.007381   0.131774  -0.056 0.955358
## AirTemp.l1      -0.127745   0.127078  -1.005 0.315282
## SeaTemp.l1       0.651594   0.684703   0.952 0.341752
## CO2.l1           0.049064   0.355859   0.138 0.890396
## EconActivity.l1  1.140061   0.045138  25.257  < 2e-16 ***
## SIE.l2          -0.074639   0.158457  -0.471 0.637827
## AirTemp.l2      -0.008479   0.135173  -0.063 0.950011
## SeaTemp.l2      -1.570234   0.921044  -1.705 0.088866 .
## CO2.l2           0.410897   0.681420   0.603 0.546790
## EconActivity.l2 -0.239561   0.068710  -3.487 0.000534 ***
## SIE.l3           0.140374   0.156230   0.899 0.369362
## AirTemp.l3      -0.141725   0.135701  -1.044 0.296826
## SeaTemp.l3       0.501417   0.921284   0.544 0.586514
## CO2.l3          -0.146557   0.796989  -0.184 0.854178
## EconActivity.l3  0.293888   0.068902   4.265  2.4e-05 ***
## SIE.l4          -0.125172   0.155570  -0.805 0.421445
## AirTemp.l4       0.101779   0.136779   0.744 0.457167
## SeaTemp.l4       0.014109   0.915360   0.015 0.987708
## CO2.l4          -0.113884   0.795486  -0.143 0.886221
## EconActivity.l4 -0.223996   0.070164  -3.192 0.001502 **
## SIE.l5          -0.104564   0.157093  -0.666 0.505971
## AirTemp.l5       0.314218   0.136514   2.302 0.021774 *
## SeaTemp.l5       1.062804   0.915134   1.161 0.246066
## CO2.l5          -0.364794   0.684224  -0.533 0.594174
## EconActivity.l5 -0.026787   0.070153  -0.382 0.702748
```

```
## SIE.l6            0.278844   0.131552   2.120 0.034543 *
## AirTemp.l6        0.112493   0.128154   0.878 0.380489
## SeaTemp.l6       -1.298586   0.674766  -1.924 0.054878 .
## CO2.l6            0.189712   0.359574   0.528 0.598016
## EconActivity.l6   0.014916   0.045867   0.325 0.745175
## const            -6.970628   6.048291  -1.152 0.249686
## trend             0.005213   0.002636   1.977 0.048562 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.8197 on 484 degrees of freedom
## Multiple R-Squared: 0.9993,  Adjusted R-squared: 0.9992
## F-statistic: 2.139e+04 on 31 and 484 DF,  p-value: < 2.2e-16
##
##
##
## Covariance matrix of residuals:
##                     SIE    AirTemp    SeaTemp        CO2 EconActivity
## SIE           0.0807718  0.0058268 -0.0013914 -0.0001395    -0.016671
## AirTemp       0.0058268  0.0849994  0.0001909 -0.0004837     0.005219
## SeaTemp      -0.0013914  0.0001909  0.0029416  0.0004362    -0.001049
## CO2          -0.0001395 -0.0004837  0.0004362  0.0105676    -0.002004
## EconActivity -0.0166707  0.0052185 -0.0010494 -0.0020036     0.671976
##
## Correlation matrix of residuals:
##                   SIE  AirTemp  SeaTemp       CO2 EconActivity
## SIE           1.000000  0.07032 -0.09027 -0.004776     -0.07156
## AirTemp       0.070322  1.00000  0.01207 -0.016139      0.02184
## SeaTemp      -0.090265  0.01207  1.00000  0.078232     -0.02360
## CO2          -0.004776 -0.01614  0.07823  1.000000     -0.02378
## EconActivity -0.071556  0.02184 -0.02360 -0.023777      1.00000
```

```r
VAR.stat.estimate <- VAR(stat.Arctic, p = 9, type = c("none"))
summary(VAR.stat.estimate)
```

```
##
## VAR Estimation Results:
## =========================
## Endogenous variables: SIE, AirTemp, SeaTemp, CO2, EconActivity
## Deterministic variables: none
## Sample size: 512
## Log Likelihood: 484.601
## Roots of the characteristic polynomial:
## 0.9859 0.9457 0.9457 0.8845 0.8845 0.8771 0.8771 0.8676 0.8676 0.8623 0.8623 0.8512 0.8512 0.8497 0.8
## Call:
## VAR(y = stat.Arctic, p = 9, type = c("none"))
##
##
## Estimation results for equation SIE:
## ====================================
## SIE = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp.l2
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1           -0.265118   0.045352  -5.846 9.48e-09 ***
```

```
## AirTemp.l1      -0.084520   0.043153  -1.959 0.050753 .
## SeaTemp.l1      -0.146612   0.242755  -0.604 0.546169
## CO2.l1          -0.148901   0.125539  -1.186 0.236189
## EconActivity.l1 -0.023086   0.015850  -1.457 0.145912
## SIE.l2          -0.227180   0.045980  -4.941 1.09e-06 ***
## AirTemp.l2      -0.055503   0.048358  -1.148 0.251657
## SeaTemp.l2      -0.467456   0.240791  -1.941 0.052819 .
## CO2.l2           0.289473   0.158968   1.821 0.069254 .
## EconActivity.l2 -0.024633   0.016161  -1.524 0.128138
## SIE.l3          -0.234975   0.045670  -5.145 3.95e-07 ***
## AirTemp.l3      -0.093041   0.050096  -1.857 0.063907 .
## SeaTemp.l3      -0.243785   0.243414  -1.002 0.317093
## CO2.l3          -0.018886   0.178182  -0.106 0.915633
## EconActivity.l3  0.008063   0.016248   0.496 0.619958
## SIE.l4          -0.353695   0.045958  -7.696 8.42e-14 ***
## AirTemp.l4      -0.120137   0.052000  -2.310 0.021305 *
## SeaTemp.l4       0.082172   0.242206   0.339 0.734564
## CO2.l4           0.024396   0.192143   0.127 0.899020
## EconActivity.l4 -0.020410   0.016937  -1.205 0.228804
## SIE.l5          -0.234942   0.047711  -4.924 1.18e-06 ***
## AirTemp.l5      -0.127985   0.052674  -2.430 0.015484 *
## SeaTemp.l5       0.026823   0.238097   0.113 0.910353
## CO2.l5          -0.038338   0.192997  -0.199 0.842628
## EconActivity.l5  0.018021   0.016801   1.073 0.284001
## SIE.l6          -0.179420   0.046189  -3.885 0.000117 ***
## AirTemp.l6      -0.160387   0.052041  -3.082 0.002178 **
## SeaTemp.l6      -0.050020   0.240549  -0.208 0.835364
## CO2.l6          -0.120362   0.192114  -0.627 0.531283
## EconActivity.l6  0.034087   0.016745   2.036 0.042354 *
## SIE.l7          -0.244714   0.045867  -5.335 1.49e-07 ***
## AirTemp.l7      -0.074888   0.050908  -1.471 0.141954
## SeaTemp.l7       0.099829   0.238352   0.419 0.675532
## CO2.l7           0.198193   0.177731   1.115 0.265367
## EconActivity.l7 -0.009927   0.016524  -0.601 0.548288
## SIE.l8          -0.220237   0.046054  -4.782 2.33e-06 ***
## AirTemp.l8      -0.042985   0.048216  -0.892 0.373120
## SeaTemp.l8       0.186960   0.240739   0.777 0.437782
## CO2.l8          -0.238834   0.156580  -1.525 0.127856
## EconActivity.l8 -0.024288   0.016599  -1.463 0.144096
## SIE.l9          -0.132843   0.045504  -2.919 0.003677 **
## AirTemp.l9      -0.135204   0.042755  -3.162 0.001667 **
## SeaTemp.l9       0.605331   0.240168   2.520 0.012053 *
## CO2.l9           0.036561   0.123719   0.296 0.767731
## EconActivity.l9  0.022207   0.016492   1.347 0.178775
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.2868 on 467 degrees of freedom
## Multiple R-Squared: 0.2929,  Adjusted R-squared: 0.2248
## F-statistic: 4.299 on 45 and 467 DF,  p-value: 2.963e-16
##
##
## Estimation results for equation AirTemp:
```

```
## ========================================
## AirTemp = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1           0.0458494  0.0480107   0.955 0.340080
## AirTemp.l1      -0.5489761  0.0456823 -12.017  < 2e-16 ***
## SeaTemp.l1       0.2832102  0.2569852   1.102 0.271008
## CO2.l1          -0.0201823  0.1328985  -0.152 0.879361
## EconActivity.l1 -0.0222142  0.0167786  -1.324 0.186165
## SIE.l2           0.0014723  0.0486754   0.030 0.975883
## AirTemp.l2      -0.3764569  0.0511925  -7.354 8.71e-13 ***
## SeaTemp.l2       0.8089421  0.2549058   3.173 0.001606 **
## CO2.l2          -0.1367614  0.1682869  -0.813 0.416822
## EconActivity.l2  0.0124648  0.0171086   0.729 0.466631
## SIE.l3           0.0114216  0.0483473   0.236 0.813350
## AirTemp.l3      -0.3821527  0.0530328  -7.206 2.33e-12 ***
## SeaTemp.l3       0.3289121  0.2576832   1.276 0.202441
## CO2.l3           0.1223421  0.1886268   0.649 0.516920
## EconActivity.l3  0.0185294  0.0172002   1.077 0.281913
## SIE.l4           0.0996311  0.0486521   2.048 0.041136 *
## AirTemp.l4      -0.3182798  0.0550483  -5.782 1.35e-08 ***
## SeaTemp.l4       0.9175557  0.2564040   3.579 0.000382 ***
## CO2.l4          -0.0338906  0.2034063  -0.167 0.867745
## EconActivity.l4  0.0118468  0.0179299   0.661 0.509112
## SIE.l5           0.0357512  0.0505078   0.708 0.479401
## AirTemp.l5      -0.3170983  0.0557612  -5.687 2.29e-08 ***
## SeaTemp.l5       0.0872622  0.2520540   0.346 0.729345
## CO2.l5          -0.2570339  0.2043105  -1.258 0.209000
## EconActivity.l5 -0.0083807  0.0177858  -0.471 0.637717
## SIE.l6           0.0766374  0.0488961   1.567 0.117709
## AirTemp.l6      -0.2795221  0.0550919  -5.074 5.64e-07 ***
## SeaTemp.l6       0.4735483  0.2546496   1.860 0.063570 .
## CO2.l6           0.3806543  0.2033753   1.872 0.061875 .
## EconActivity.l6  0.0056098  0.0177271   0.316 0.751798
## SIE.l7           0.0958301  0.0485553   1.974 0.049013 *
## AirTemp.l7      -0.2046985  0.0538927  -3.798 0.000165 ***
## SeaTemp.l7       0.4418259  0.2523242   1.751 0.080598 .
## CO2.l7          -0.4056170  0.1881490  -2.156 0.031607 *
## EconActivity.l7 -0.0295639  0.0174929  -1.690 0.091685 .
## SIE.l8          -0.0349020  0.0487533  -0.716 0.474417
## AirTemp.l8      -0.2206897  0.0510426  -4.324 1.88e-05 ***
## SeaTemp.l8       0.6278546  0.2548512   2.464 0.014114 *
## CO2.l8           0.4455830  0.1657584   2.688 0.007442 **
## EconActivity.l8  0.0364881  0.0175724   2.076 0.038400 *
## SIE.l9           0.0065118  0.0481715   0.135 0.892529
## AirTemp.l9      -0.1817860  0.0452611  -4.016 6.89e-05 ***
## SeaTemp.l9       0.3619627  0.2542461   1.424 0.155210
## CO2.l9          -0.0794382  0.1309708  -0.607 0.544455
## EconActivity.l9  0.0000898  0.0174587   0.005 0.995898
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.3036 on 467 degrees of freedom
```

```
## Multiple R-Squared: 0.342,   Adjusted R-squared: 0.2786
## F-statistic: 5.395 on 45 and 467 DF,  p-value: < 2.2e-16
##
##
## Estimation results for equation SeaTemp:
## ========================================
## SeaTemp = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTem
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1           0.0081863  0.0086052   0.951 0.341935
## AirTemp.l1       0.0327554  0.0081879   4.000 7.35e-05 ***
## SeaTemp.l1      -0.0376148  0.0460609  -0.817 0.414556
## CO2.l1           0.0431481  0.0238201   1.811 0.070719 .
## EconActivity.l1  0.0002831  0.0030073   0.094 0.925047
## SIE.l2           0.0072929  0.0087244   0.836 0.403626
## AirTemp.l2       0.0070286  0.0091755   0.766 0.444053
## SeaTemp.l2      -0.0762484  0.0456882  -1.669 0.095810 .
## CO2.l2           0.0121026  0.0301630   0.401 0.688426
## EconActivity.l2  0.0030779  0.0030665   1.004 0.316032
## SIE.l3           0.0126881  0.0086656   1.464 0.143813
## AirTemp.l3       0.0093171  0.0095054   0.980 0.327497
## SeaTemp.l3      -0.0967066  0.0461860  -2.094 0.036812 *
## CO2.l3           0.0232137  0.0338086   0.687 0.492662
## EconActivity.l3 -0.0029450  0.0030829  -0.955 0.339941
## SIE.l4           0.0229506  0.0087202   2.632 0.008772 **
## AirTemp.l4       0.0250396  0.0098666   2.538 0.011479 *
## SeaTemp.l4      -0.0283131  0.0459567  -0.616 0.538140
## CO2.l4          -0.0220690  0.0364576  -0.605 0.545252
## EconActivity.l4  0.0010894  0.0032137   0.339 0.734769
## SIE.l5           0.0191132  0.0090528   2.111 0.035277 *
## AirTemp.l5       0.0121090  0.0099944   1.212 0.226286
## SeaTemp.l5      -0.2258788  0.0451770  -5.000 8.13e-07 ***
## CO2.l5          -0.0450807  0.0366197  -1.231 0.218923
## EconActivity.l5 -0.0019047  0.0031879  -0.597 0.550482
## SIE.l6           0.0176584  0.0087639   2.015 0.044488 *
## AirTemp.l6       0.0232347  0.0098744   2.353 0.019036 *
## SeaTemp.l6      -0.0564790  0.0456423  -1.237 0.216550
## CO2.l6           0.0711563  0.0364521   1.952 0.051529 .
## EconActivity.l6 -0.0020842  0.0031773  -0.656 0.512165
## SIE.l7           0.0336012  0.0087028   3.861 0.000129 ***
## AirTemp.l7       0.0066187  0.0096595   0.685 0.493556
## SeaTemp.l7      -0.1055924  0.0452255  -2.335 0.019977 *
## CO2.l7          -0.0909137  0.0337230  -2.696 0.007273 **
## EconActivity.l7 -0.0019877  0.0031353  -0.634 0.526412
## SIE.l8           0.0208007  0.0087383   2.380 0.017694 *
## AirTemp.l8       0.0065666  0.0091487   0.718 0.473259
## SeaTemp.l8      -0.1081874  0.0456784  -2.368 0.018269 *
## CO2.l8           0.0812064  0.0297098   2.733 0.006508 **
## EconActivity.l8  0.0022586  0.0031496   0.717 0.473676
## SIE.l9           0.0077318  0.0086340   0.895 0.370983
## AirTemp.l9       0.0047161  0.0081124   0.581 0.561285
## SeaTemp.l9      -0.0538980  0.0455699  -1.183 0.237509
## CO2.l9          -0.0729540  0.0234746  -3.108 0.002000 **
## EconActivity.l9  0.0026324  0.0031292   0.841 0.400654
```

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.05442 on 467 degrees of freedom
## Multiple R-Squared: 0.2179,  Adjusted R-squared: 0.1425
## F-statistic: 2.891 on 45 and 467 DF,  p-value: 9.902e-09
##
##
## Estimation results for equation CO2:
## ====================================
## CO2 = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + SeaTemp.l2
##
##                   Estimate Std. Error t value Pr(>|t|)
## SIE.l1          -0.0195758  0.0164698  -1.189 0.235206
## AirTemp.l1       0.0067412  0.0156710   0.430 0.667270
## SeaTemp.l1       0.0912173  0.0881571   1.035 0.301339
## CO2.l1           0.7748670  0.0455900  16.996  < 2e-16 ***
## EconActivity.l1  0.0007733  0.0057558   0.134 0.893179
## SIE.l2           0.0041304  0.0166978   0.247 0.804738
## AirTemp.l2       0.0239460  0.0175613   1.364 0.173361
## SeaTemp.l2       0.2010407  0.0874438   2.299 0.021941 *
## CO2.l2          -0.6774256  0.0577298 -11.734  < 2e-16 ***
## EconActivity.l2  0.0032559  0.0058690   0.555 0.579327
## SIE.l3          -0.0259300  0.0165852  -1.563 0.118627
## AirTemp.l3      -0.0248698  0.0181926  -1.367 0.172273
## SeaTemp.l3       0.0226383  0.0883966   0.256 0.797987
## CO2.l3           0.5664602  0.0647072   8.754  < 2e-16 ***
## EconActivity.l3  0.0093135  0.0059004   1.578 0.115142
## SIE.l4          -0.0102133  0.0166898  -0.612 0.540871
## AirTemp.l4       0.0324989  0.0188840   1.721 0.085917 .
## SeaTemp.l4       0.1129498  0.0879578   1.284 0.199731
## CO2.l4          -0.3629851  0.0697772  -5.202 2.96e-07 ***
## EconActivity.l4  0.0003429  0.0061507   0.056 0.955568
## SIE.l5          -0.0149594  0.0173264  -0.863 0.388367
## AirTemp.l5       0.0001164  0.0191285   0.006 0.995148
## SeaTemp.l5       0.0720375  0.0864655   0.833 0.405194
## CO2.l5           0.3016068  0.0700874   4.303 2.05e-05 ***
## EconActivity.l5 -0.0021460  0.0061013  -0.352 0.725197
## SIE.l6           0.0167153  0.0167735   0.997 0.319507
## AirTemp.l6      -0.0067259  0.0188989  -0.356 0.722083
## SeaTemp.l6       0.0353562  0.0873559   0.405 0.685856
## CO2.l6          -0.0326556  0.0697666  -0.468 0.639954
## EconActivity.l6 -0.0067350  0.0060812  -1.108 0.268641
## SIE.l7          -0.0086036  0.0166566  -0.517 0.605731
## AirTemp.l7       0.0030096  0.0184875   0.163 0.870755
## SeaTemp.l7       0.2022137  0.0865582   2.336 0.019905 *
## CO2.l7           0.2198398  0.0645433   3.406 0.000716 ***
## EconActivity.l7  0.0064815  0.0060008   1.080 0.280657
## SIE.l8          -0.0018897  0.0167245  -0.113 0.910086
## AirTemp.l8      -0.0007801  0.0175098  -0.045 0.964485
## SeaTemp.l8       0.1939987  0.0874251   2.219 0.026965 *
## CO2.l8          -0.0579655  0.0568624  -1.019 0.308540
## EconActivity.l8  0.0073992  0.0060281   1.227 0.220271
```

```
## SIE.l9          -0.0097826  0.0165249  -0.592 0.554145
## AirTemp.l9       0.0040034  0.0155265   0.258 0.796642
## SeaTemp.l9       0.0243482  0.0872175   0.279 0.780241
## CO2.l9           0.1921303  0.0449287   4.276 2.31e-05 ***
## EconActivity.l9 -0.0005729  0.0059891  -0.096 0.923834
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.1041 on 467 degrees of freedom
## Multiple R-Squared: 0.7598,  Adjusted R-squared: 0.7367
## F-statistic: 32.83 on 45 and 467 DF,  p-value: < 2.2e-16
##
##
## Estimation results for equation EconActivity:
## ==============================================
## EconActivity = SIE.l1 + AirTemp.l1 + SeaTemp.l1 + CO2.l1 + EconActivity.l1 + SIE.l2 + AirTemp.l2 + Se
##
##                  Estimate Std. Error t value Pr(>|t|)
## SIE.l1          -0.004598   0.132177  -0.035 0.972265
## AirTemp.l1      -0.170106   0.125766  -1.353 0.176852
## SeaTemp.l1       0.472052   0.707497   0.667 0.504965
## CO2.l1           0.194735   0.365878   0.532 0.594813
## EconActivity.l1  0.170086   0.046193   3.682 0.000258 ***
## SIE.l2          -0.060839   0.134007  -0.454 0.650038
## AirTemp.l2      -0.236028   0.140936  -1.675 0.094661 .
## SeaTemp.l2      -1.230406   0.701773  -1.753 0.080209 .
## CO2.l2           0.558606   0.463305   1.206 0.228544
## EconActivity.l2 -0.068986   0.047101  -1.465 0.143693
## SIE.l3           0.120091   0.133103   0.902 0.367393
## AirTemp.l3      -0.406334   0.146003  -2.783 0.005603 **
## SeaTemp.l3      -0.451891   0.709419  -0.637 0.524445
## CO2.l3           0.228773   0.519302   0.441 0.659750
## EconActivity.l3  0.230045   0.047353   4.858 1.62e-06 ***
## SIE.l4          -0.035722   0.133942  -0.267 0.789818
## AirTemp.l4      -0.299906   0.151552  -1.979 0.048414 *
## SeaTemp.l4      -0.271158   0.705897  -0.384 0.701055
## CO2.l4           0.320018   0.559991   0.571 0.567956
## EconActivity.l4 -0.014611   0.049362  -0.296 0.767364
## SIE.l5          -0.143096   0.139051  -1.029 0.303972
## AirTemp.l5       0.007919   0.153514   0.052 0.958884
## SeaTemp.l5       1.150846   0.693921   1.658 0.097894 .
## CO2.l5          -0.349630   0.562480  -0.622 0.534518
## EconActivity.l5 -0.020108   0.048966  -0.411 0.681518
## SIE.l6           0.174263   0.134614   1.295 0.196120
## AirTemp.l6       0.093940   0.151672   0.619 0.535979
## SeaTemp.l6      -0.236810   0.701067  -0.338 0.735677
## CO2.l6           0.216939   0.559906   0.387 0.698595
## EconActivity.l6 -0.042682   0.048804  -0.875 0.382259
## SIE.l7           0.140929   0.133676   1.054 0.292309
## AirTemp.l7       0.054232   0.148370   0.366 0.714891
## SeaTemp.l7      -0.572818   0.694665  -0.825 0.410022
## CO2.l7          -0.265756   0.517987  -0.513 0.608154
## EconActivity.l7  0.057684   0.048159   1.198 0.231608
```

```
## SIE.l8           0.151223   0.134221   1.127 0.260460
## AirTemp.l8       0.137116   0.140524   0.976 0.329694
## SeaTemp.l8      -0.985460   0.701622  -1.405 0.160821
## CO2.l8          -0.016228   0.456344  -0.036 0.971647
## EconActivity.l8 -0.037597   0.048378  -0.777 0.437467
## SIE.l9           0.141294   0.132619   1.065 0.287242
## AirTemp.l9       0.035827   0.124607   0.288 0.773841
## SeaTemp.l9      -0.560816   0.699956  -0.801 0.423414
## CO2.l9           0.023114   0.360571   0.064 0.948915
## EconActivity.l9 -0.075897   0.048065  -1.579 0.115000
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.8358 on 467 degrees of freedom
## Multiple R-Squared: 0.1911,  Adjusted R-squared: 0.1131
## F-statistic: 2.451 on 45 and 467 DF,  p-value: 1.684e-06
##
##
##
## Covariance matrix of residuals:
##                    SIE    AirTemp    SeaTemp        CO2 EconActivity
## SIE           8.225e-02  7.789e-03 -1.181e-03  9.082e-05    -0.010764
## AirTemp       7.789e-03  9.217e-02  2.256e-05 -4.885e-04     0.004944
## SeaTemp      -1.181e-03  2.256e-05  2.954e-03  7.110e-04    -0.001736
## CO2           9.082e-05 -4.885e-04  7.110e-04  1.080e-02    -0.001862
## EconActivity -1.076e-02  4.944e-03 -1.736e-03 -1.862e-03     0.698506
##
## Correlation matrix of residuals:
##                   SIE   AirTemp   SeaTemp       CO2 EconActivity
## SIE           1.000000  0.089456 -0.075774  0.003048     -0.04491
## AirTemp       0.089456  1.000000  0.001367 -0.015487      0.01949
## SeaTemp      -0.075774  0.001367  1.000000  0.125899     -0.03821
## CO2           0.003048 -0.015487  0.125899  1.000000     -0.02144
## EconActivity -0.044908  0.019486 -0.038211 -0.021442      1.00000
```

In Levels

It seems the most important variable to explain SIE is the lag term, but other variables make little to no impact. Perhaps air temp and economic activity help explain the current SIE slightly, but it is a marginal impact.

Other interesting thing to note is that CO2 emissions can be predicted quite well by the lag terms. This is because of the strong linear trend it exhibits from the plot above.

Additionally, we take 6 lags, as recommended by AIC.

First Differences

In this case, we find more relevance in the lag terms to explain SIE. Additionally, Air temp, Sea temp, and CO2 emission seem strong candidates for helping predict SIE.

Other variables look to be better explained other than econ activity, which is not affected by the other response variables.

We take 9 lags in this case, as recommended by AIC.

Penalized Regressions

The problem with this approach is we will almost certainly overfit the data and with an R^2 of 0.9993, We seem to explain a lot of the variance of the dataset given the amount of parameters we have. The solution here would then be penalized regression. We use the `bigtime` package with the `L1` penalty option to impose a lasso penalty to the VAR estimates of the time series in levels and differenced.

```
sparse.VAR.estimate <- sparseVAR(scale(Arctic_Ice), VARpen = "L1", selection = "cv")
sparse.VAR.estimate$Phihat
```

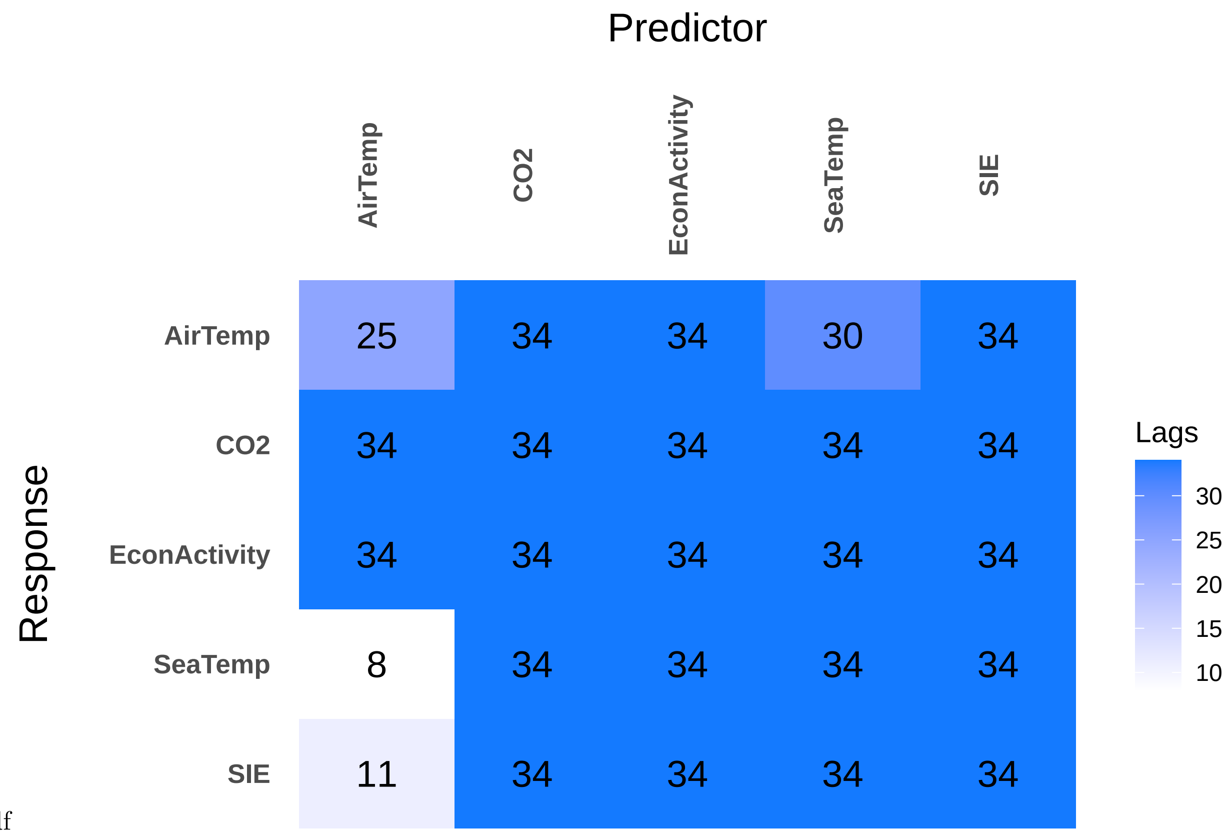
```
##              [,1]         [,2]         [,3]         [,4]         [,5]
## [1,]  0.060433469 -0.006825680 -0.011456998 -0.004318293 -0.008839745
## [2,] -0.003404310  0.034321381  0.022024554  0.006267798  0.007030615
## [3,] -0.004244145  0.025232998  0.056548106  0.007134901  0.003274967
## [4,] -0.005516516  0.005004217  0.005950470  0.006609076  0.006608718
## [5,] -0.005775285  0.005039321  0.005774196  0.006600327  0.006865662
##              [,6]         [,7]         [,8]         [,9]        [,10]
## [1,]  0.035148078 -0.009964037 -0.009632488 -0.004204211 -0.008192351
## [2,] -0.003311153  0.023404695  0.022729209  0.006184153  0.007270151
## [3,]  0.000000000  0.019351604  0.047674915  0.006963856  0.003189857
## [4,] -0.005528354  0.004992324  0.005929913  0.006598014  0.006600009
## [5,] -0.005794854  0.005006766  0.005754519  0.006589497  0.006847184
##             [,11]        [,12]        [,13]        [,14]        [,15]
## [1,]  0.014529547 -0.014012858 -0.004403776 -0.004192058 -0.007665841
## [2,] -0.000380447  0.013070590  0.020973787  0.006109009  0.007307221
## [3,]  0.000000000  0.016531507  0.038948682  0.006788293  0.003162363
## [4,] -0.005566658  0.004996822  0.005923200  0.006586751  0.006593146
## [5,] -0.005828181  0.004996752  0.005750291  0.006578253  0.006830552
##             [,16]       [,17]       [,18]        [,19]        [,20]
## [1,]  0.000000000 -0.01602179 0.000000000 -0.004227330 -0.007516843
## [2,]  0.000000000  0.00788035 0.020258353  0.006023251  0.007098057
## [3,]  0.000000000  0.01589571 0.030956242  0.006652237  0.003312055
## [4,] -0.005593471  0.00498206 0.005935385  0.006575476  0.006587098
## [5,] -0.005849386  0.00498966 0.005766289  0.006566761  0.006812224
##             [,21]        [,22]       [,23]        [,24]        [,25]
## [1,]  0.000000000 -0.014728132 0.000000000 -0.004212165 -0.007282465
## [2,]  0.000000000  0.003582262 0.017145988  0.005967229  0.006961558
## [3,]  0.000000000  0.011870187 0.023847508  0.006577469  0.003488040
## [4,] -0.005618194  0.004961201 0.005940852  0.006564478  0.006576625
## [5,] -0.005863095  0.005001050 0.005773436  0.006555468  0.006789657
##             [,26]        [,27]       [,28]        [,29]        [,30]
## [1,]  0.000000000 -0.013955754 0.000000000 -0.004147049 -0.007383024
## [2,]  0.000000000  0.001729472 0.016717321  0.005911556  0.006858095
## [3,]  0.000000000  0.009797490 0.019951524  0.006529200  0.003685479
## [4,] -0.005641374  0.004937801 0.005941605  0.006553596  0.006564571
## [5,] -0.005855127  0.005007041 0.005767520  0.006544381  0.006766490
##             [,31]        [,32]       [,33]        [,34]        [,35]
## [1,]  0.000000000 -0.007982202 0.000000000 -0.004083344 -0.007922337
## [2,]  0.000000000  0.001335112 0.015194055  0.005821700  0.006744631
## [3,]  0.000000000  0.004676234 0.017482743  0.006480332  0.003936251
## [4,] -0.005667372  0.004924228 0.005931565  0.006542408  0.006554450
## [5,] -0.005856751  0.005010026 0.005754239  0.006533123  0.006745187
```

```
##             [,36]         [,37]       [,38]        [,39]        [,40]
## [1,]  0.000000000 -0.0072694341 0.000000000 -0.004041891 -0.008065441
## [2,] -0.001915099  0.0006952204 0.013545071  0.005769479  0.006906344
## [3,]  0.000000000  0.0020988078 0.016844782  0.006448757  0.004225417
## [4,] -0.005678523  0.0048993869 0.005918130  0.006530628  0.006545822
## [5,] -0.005849685  0.0050057972 0.005742045  0.006521440  0.006724783
##             [,41]        [,42]       [,43]        [,44]        [,45]
## [1,]  0.004927364 -0.011360202 0.000000000 -0.003997139 -0.007861580
## [2,]  0.000000000  0.000000000 0.010988350  0.005680842  0.006812478
## [3,]  0.000000000  0.000000000 0.017228597  0.006415203  0.004430805
## [4,] -0.005683655  0.004861092 0.005895846  0.006518837  0.006538402
## [5,] -0.005844327  0.004973041 0.005729346  0.006509820  0.006706277
##              [,46]        [,47]        [,48]        [,49]        [,50]
## [1,]  0.0160828249 -0.002531950 -0.003761293 -0.003956729 -0.007808702
## [2,]  0.0000000000  0.007496839  0.008624573  0.005586840  0.006768304
## [3,] -0.0003148893  0.000000000  0.018012590  0.006407743  0.004463754
## [4,] -0.0056747196  0.004820299  0.005873171  0.006507632  0.006532694
## [5,] -0.0058413166  0.004933667  0.005723645  0.006498756  0.006690444
##             [,51]        [,52]        [,53]        [,54]        [,55]
## [1,]  0.024987787 -0.002744663 -0.011176433 -0.003897033 -0.007890616
## [2,]  0.000000000  0.007233923  0.007474612  0.005531987  0.006795867
## [3,] -0.004278718  0.000000000  0.017397136  0.006411350  0.004406963
## [4,] -0.005671153  0.004780469  0.005845734  0.006496466  0.006527867
## [5,] -0.005840749  0.004912187  0.005721693  0.006487839  0.006675800
##             [,56]       [,57]        [,58]        [,59]        [,60]
## [1,]  0.030140478 0.000000000 -0.013895777 -0.003780707 -0.007717778
## [2,]  0.000000000 0.005068363  0.006742395  0.005510821  0.006620584
## [3,] -0.002903331 0.000000000  0.015889588  0.006389814  0.004436624
## [4,] -0.005665684 0.004789947  0.005829483  0.006485456  0.006521259
## [5,] -0.005829617 0.004918731  0.005728251  0.006477032  0.006660315
##             [,61]       [,62]        [,63]        [,64]        [,65]
## [1,]  0.023743081 0.000000000 -0.013848061 -0.003692022 -0.007563192
## [2,] -0.002644458 0.004886634  0.005138352  0.005472697  0.006475468
## [3,]  0.000000000 0.000000000  0.012911631  0.006357623  0.004469689
## [4,] -0.005642967 0.004769836  0.005818391  0.006474661  0.006513904
## [5,] -0.005807681 0.004882166  0.005734489  0.006466354  0.006645089
##             [,66]       [,67]        [,68]        [,69]        [,70]
## [1,]  0.014632180 0.000000000 -0.009857291 -0.003635607 -0.007207138
## [2,]  0.000000000 0.001141163  0.005547910  0.005405314  0.006343963
## [3,]  0.000000000 0.000000000  0.008299852  0.006328876  0.004606326
## [4,] -0.005658580 0.004750969  0.005821401  0.006463814  0.006506988
## [5,] -0.005829817 0.004866752  0.005756567  0.006455540  0.006630152
##             [,71]       [,72]        [,73]        [,74]        [,75]
## [1,]  0.001665857 0.000000000 -0.002720282 -0.003534197 -0.006908935
## [2,]  0.000000000 0.000000000  0.005792354  0.005337898  0.006434902
## [3,]  0.000000000 0.000000000  0.002997966  0.006303778  0.004761604
## [4,] -0.005701563 0.004757425  0.005814793  0.006452637  0.006501882
## [5,] -0.005864081 0.004881997  0.005767569  0.006444283  0.006617859
##              [,76]       [,77]       [,78]        [,79]        [,80]
## [1,]  0.0000000000 0.000000000 0.000000000 -0.003420996 -0.006985920
## [2,] -0.0002587865 0.000000000 0.005199884  0.005305372  0.006395034
## [3,]  0.0000000000 0.000000000 0.000000000  0.006289764  0.005043881
## [4,] -0.0057194799 0.004765799 0.005813067  0.006441577  0.006494451
## [5,] -0.0058735668 0.004887782 0.005780152  0.006433077  0.006604621
```

```
##             [,81]       [,82]       [,83]        [,84]        [,85]
## [1,]  0.000000000 0.000000000 0.000000000 -0.003337727 -0.006803883
## [2,]  0.000000000 0.000000000 0.003347778  0.005315363  0.006347802
## [3,]  0.000000000 0.000000000 0.000000000  0.006267676  0.005269136
## [4,] -0.005740672 0.004750906 0.005814803  0.006430556  0.006487325
## [5,] -0.005887789 0.004877638 0.005793509  0.006421903  0.006590938
##             [,86]       [,87]       [,88]        [,89]        [,90]
## [1,]  0.000000000 0.000000000 0.000000000 -0.003283223 -0.006868384
## [2,]  0.000000000 0.000000000 0.002100752  0.005330851  0.006250418
## [3,]  0.000000000 0.000000000 0.000000000  0.006254182  0.005653443
## [4,] -0.005747376 0.004755358 0.005822929  0.006419254  0.006480287
## [5,] -0.005867653 0.004876664 0.005812781  0.006410448  0.006576747
##             [,91]       [,92]       [,93]        [,94]        [,95]
## [1,]  0.000000000 0.000000000 0.000000000 -0.003271405 -0.007016672
## [2,]  0.000000000 0.000000000 0.001555077  0.005319797  0.006255622
## [3,]  0.000000000 0.000000000 0.000000000  0.006257169  0.005916694
## [4,] -0.005757954 0.004749938 0.005824035  0.006408181  0.006472706
## [5,] -0.005860285 0.004861236 0.005819419  0.006399155  0.006562414
##             [,96]       [,97]       [,98]        [,99]       [,100]
## [1,]  0.000000000 0.000000000 0.000000000 -0.003271555 -0.006803471
## [2,]  0.000000000 0.000000000 0.002437153  0.005284600  0.006279510
## [3,]  0.000000000 0.000000000 0.000000000  0.006253668  0.006095272
## [4,] -0.005766236 0.004754608 0.005805169  0.006397474  0.006466279
## [5,] -0.005854518 0.004871889 0.005807826  0.006388059  0.006549322
##            [,101]      [,102]      [,103]       [,104]       [,105]
## [1,]  0.001207063 0.000000000 0.000000000 -0.003263974 -0.006629551
## [2,]  0.000000000 0.000000000 0.001436871  0.005268162  0.006253393
## [3,] -0.001387190 0.000000000 0.000000000  0.006245500  0.006281204
## [4,] -0.005747170 0.004745157 0.005776026  0.006386693  0.006460807
## [5,] -0.005830068 0.004863063 0.005790157  0.006376742  0.006537987
##            [,106]      [,107]      [,108]       [,109]       [,110]
## [1,]  0.010209404 0.000000000 0.000000000 -0.003185064 -0.006301964
## [2,]  0.000000000 0.002394482 0.001476568  0.005277248  0.006195239
## [3,] -0.007587162 0.000000000 0.001137613  0.006218538  0.006395168
## [4,] -0.005680967 0.004747397 0.005734983  0.006375908  0.006456542
## [5,] -0.005767941 0.004873077 0.005754836  0.006365399  0.006529407
##            [,111]      [,112]        [,113]       [,114]       [,115]
## [1,]  0.017002927 0.000000000 -0.0037926353 -0.003099165 -0.006155325
## [2,]  0.000000000 0.006797626  0.0004613283  0.005247176  0.006287461
## [3,] -0.010283521 0.000000000  0.0028260607  0.006174014  0.006343564
## [4,] -0.005629930 0.004755626  0.0056772188  0.006365673  0.006453444
## [5,] -0.005720411 0.004869837  0.0057000578  0.006354553  0.006523934
##            [,116]      [,117]       [,118]       [,119]       [,120]
## [1,]  0.021637949 0.000000000 -0.008996673 -0.003063672 -0.006111336
## [2,]  0.000000000 0.009672357  0.000000000  0.005203907  0.006415608
## [3,] -0.008995321 0.000000000  0.002745137  0.006130983  0.006281049
## [4,] -0.005587993 0.004727098  0.005614163  0.006355854  0.006452429
## [5,] -0.005674552 0.004835169  0.005642789  0.006343999  0.006519589
##            [,121]      [,122]        [,123]       [,124]       [,125]
## [1,]  0.014699717 0.000000000 -0.0108406330 -0.003002960 -0.005854878
## [2,]  0.000000000 0.004714376  0.0000000000  0.005182540  0.006406404
## [3,] -0.004428837 0.000000000  0.0008031119  0.006100398  0.006115406
## [4,] -0.005538284 0.004741939  0.0055658095  0.006345764  0.006453111
## [5,] -0.005625926 0.004826044  0.0056002469  0.006333077  0.006516652
```

```
##            [,126]      [,127]       [,128]       [,129]       [,130]
## [1,]  0.003200890 0.000000000 -0.008435128 -0.002939948 -0.005572147
## [2,]  0.000000000 0.000000000  0.000000000  0.005173873  0.006224604
## [3,]  0.000000000 0.000000000  0.000000000  0.006066876  0.005990554
## [4,] -0.005538020 0.004739873  0.005538132  0.006335477  0.006458313
## [5,] -0.005623669 0.004812049  0.005576236  0.006322051  0.006518581
##            [,131]      [,132]       [,133]       [,134]       [,135]
## [1,]  0.000000000 0.000000000 -0.002436151 -0.002928305 -0.005460148
## [2,]  0.000000000 0.000000000  0.000000000  0.005179429  0.005992832
## [3,]  0.000000000 0.000000000  0.000000000  0.006024426  0.005949467
## [4,] -0.005550069 0.004731818  0.005513829  0.006325334  0.006470058
## [5,] -0.005641873 0.004794594  0.005559220  0.006311361  0.006527317
##            [,136]      [,137]       [,138]       [,139]       [,140]
## [1,]  0.000000000 0.000000000 0.0000000000 -0.002909460 -0.005400096
## [2,]  0.000000000 0.000000000 0.0003210325  0.005184328  0.005739371
## [3,]  0.000000000 0.000000000 0.0000000000  0.005966374  0.005887381
## [4,] -0.005564714 0.004733134 0.0054894080  0.006315848  0.006482290
## [5,] -0.005668553 0.004793635 0.0055465452  0.006301402  0.006536109
##            [,141]      [,142]      [,143]       [,144]       [,145]
## [1,]  0.000000000 0.000000000 0.000000000 -0.002855442 -0.005349395
## [2,]  0.000000000 0.000000000 0.000220876  0.005177717  0.005539854
## [3,]  0.000000000 0.000000000 0.000000000  0.005884756  0.005753587
## [4,] -0.005597010 0.004677341 0.005474223  0.006306579  0.006481588
## [5,] -0.005694359 0.004751765 0.005528590  0.006291612  0.006533703
##            [,146]      [,147]       [,148]       [,149]       [,150]
## [1,]  0.000000000 0.000000000 0.0000000000 -0.002813473 -0.005288953
## [2,]  0.000000000 0.000000000 0.0001882431  0.005174368  0.005416906
## [3,]  0.000000000 0.000000000 0.0000000000  0.005789737  0.005654671
## [4,] -0.005626451 0.004596435 0.0054503859  0.006296985  0.006480203
## [5,] -0.005707613 0.004687775 0.0055010356  0.006281408  0.006532085
##            [,151]      [,152]      [,153]       [,154]       [,155]
## [1,]  0.000000000 0.000000000 0.000000000 -0.002763639 -0.005134518
## [2,] -0.001939415 0.000000000 0.000000000  0.005134509  0.005338088
## [3,]  0.000000000 0.000000000 0.000000000  0.005720836  0.005511189
## [4,] -0.005643402 0.004540616 0.005438587  0.006287072  0.006479294
## [5,] -0.005705976 0.004643600 0.005488878  0.006270896  0.006530781
##            [,156]      [,157]      [,158]       [,159]       [,160]
## [1,]  0.000000000 0.000000000 0.000000000 -0.002687490 -0.005077753
## [2,] -0.003726876 0.000000000 0.000000000  0.005085027  0.005300699
## [3,]  0.000000000 0.000000000 0.000000000  0.005667234  0.005262899
## [4,] -0.005657130 0.004541058 0.005425061  0.006276947  0.006471390
## [5,] -0.005701561 0.004653369 0.005465378  0.006260145  0.006523502
##            [,161]      [,162]       [,163]       [,164]       [,165]
## [1,]  0.000000000 0.000000000 0.0000000000 -0.002634729 -0.004987017
## [2,] -0.003411475 0.000000000 0.0000000000  0.005053002  0.005266089
## [3,] -0.006707291 0.000000000 0.0006053221  0.005585955  0.005019338
## [4,] -0.005667007 0.004575808 0.0054026240  0.006266740  0.006463477
## [5,] -0.005693570 0.004687231 0.0054336390  0.006249369  0.006516504
##            [,166]      [,167]        [,168]       [,169]       [,170]
## [1,]  0.007605807 0.000000000 -0.0008120345 -0.002629462 -0.004924101
## [2,] -0.002392740 0.000000000  0.0000000000  0.005005014  0.005291784
## [3,] -0.010294572 0.000000000  0.0055826637  0.005502837  0.004789418
## [4,] -0.005631747 0.004615105  0.0053719283  0.006256529  0.006455677
## [5,] -0.005656203 0.004714243  0.0054071838  0.006238706  0.006509318
```

```r
Phihat.viz <- lagmatrix(fit=sparse.VAR.estimate, returnplot=TRUE)
```



VAR levels-1.pdf

Here, the 5 rows each represent the response variables. We can see we can actually account for much longer time series here, with 34 lags being selected for each response. The heatmap shows that SIE is affected for 34 months from all regressors except AirTemp. We can do the same to the differenced dataset.

```r
sparse.diff.VAR.estimate <- sparseVAR(scale(stat.Arctic), VARpen = "L1", selection = "bic")
```

```
##
##
## #### Selected the following lambda ####
##
##      AIC      BIC       HQ
## 14.72934 68.36755 40.98527
##
##
## #### Details ####
##
##      lambda         AIC         BIC          HQ
## 1  190.2368     -0.0611     -0.0611     -0.0611
## 2  114.0439     -0.2688     -0.2524     -0.2624
## 3   68.3675     -0.4505 ==> -0.2953     -0.3897
## 4   40.9853     -0.6783     -0.1147 ==> -0.4575
## 5     24.57     -0.8279      0.7241       -0.22
## 6   14.7293 ==> -0.8556      2.0197      0.2706
## 7      8.83      -0.809      3.3078      0.8036
## 8    5.2935     -0.7334      4.3637      1.2632
## 9    3.1733     -0.6511      5.1076      1.6046
```

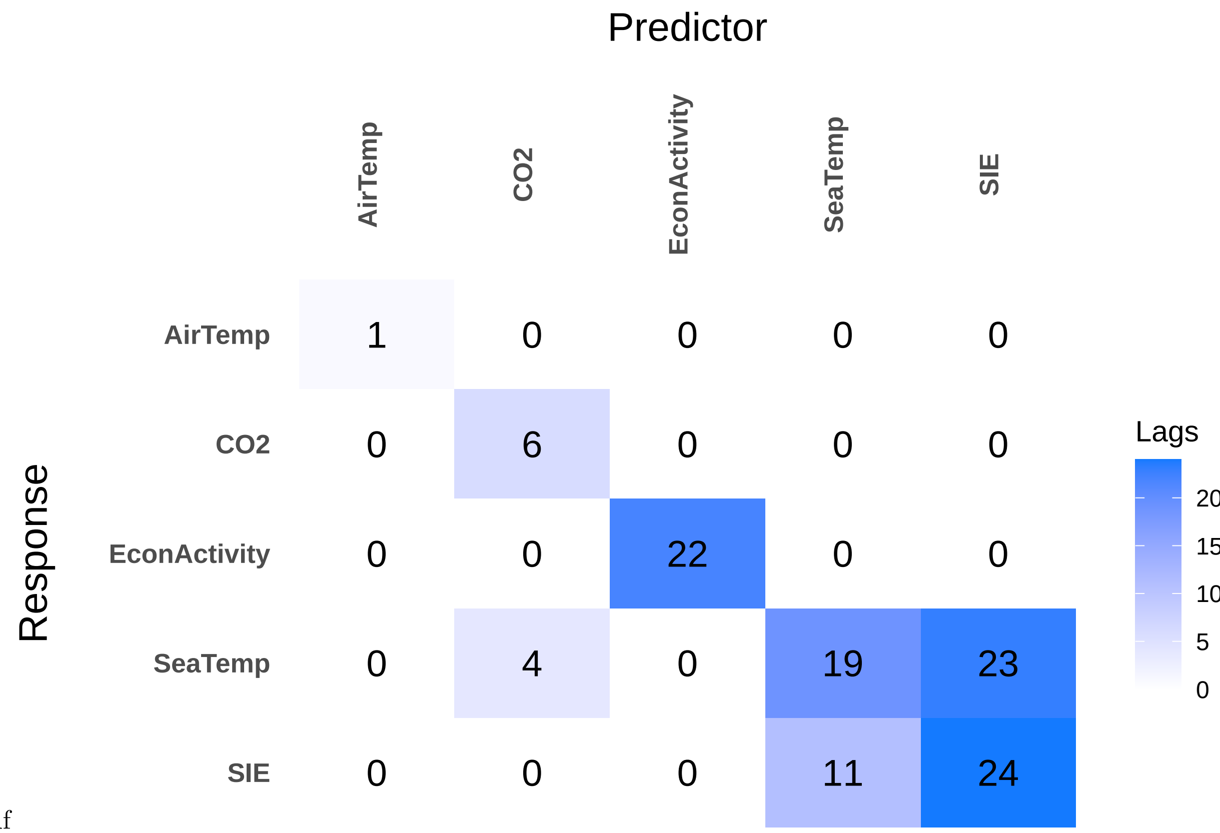
```
## 10   1.9024     -0.4663      5.9459      2.0454
```

```
sparse.diff.VAR.estimate$Phihat
```

```
##      [,1]       [,2] [,3]      [,4]       [,5] [,6] [,7] [,8]        [,9] [,10]
## [1,]    0  0.0000000    0 0.0000000 0.00000000    0    0    0  0.00000000     0
## [2,]    0 -0.2373358    0 0.0000000 0.00000000    0    0    0  0.00000000     0
## [3,]    0  0.0000000    0 0.0000000 0.00000000    0    0    0  0.00000000     0
## [4,]    0  0.0000000    0 0.3039376 0.00000000    0    0    0 -0.05831355     0
## [5,]    0  0.0000000    0 0.0000000 0.03320218    0    0    0  0.00000000     0
##      [,11] [,12] [,13] [,14]      [,15]       [,16] [,17] [,18]        [,19]
## [1,]     0     0     0     0 0.00000000 -0.04419163     0     0  0.000000000
## [2,]     0     0     0     0 0.00000000  0.00000000     0     0  0.000000000
## [3,]     0     0     0     0 0.00000000  0.00000000     0     0 -0.002228516
## [4,]     0     0     0     0 0.00000000  0.00000000     0     0  0.000000000
## [5,]     0     0     0     0 0.05270757  0.00000000     0     0  0.000000000
##      [,20] [,21] [,22]       [,23] [,24] [,25] [,26] [,27] [,28]      [,29]
## [1,]     0     0     0  0.00000000     0     0     0     0     0 0.00000000
## [2,]     0     0     0  0.00000000     0     0     0     0     0 0.00000000
## [3,]     0     0     0 -0.05654296     0     0     0     0     0 0.00000000
## [4,]     0     0     0  0.00000000     0     0     0     0     0 0.01153829
## [5,]     0     0     0  0.00000000     0     0     0     0     0 0.00000000
##      [,30]        [,31] [,32] [,33] [,34] [,35] [,36] [,37] [,38] [,39] [,40]
## [1,]     0 -0.003641395     0     0     0     0     0     0     0     0     0
## [2,]     0  0.000000000     0     0     0     0     0     0     0     0     0
## [3,]     0  0.003062701     0     0     0     0     0     0     0     0     0
## [4,]     0  0.000000000     0     0     0     0     0     0     0     0     0
## [5,]     0  0.000000000     0     0     0     0     0     0     0     0     0
##      [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48] [,49] [,50]       [,51]
## [1,]     0     0     0     0     0     0     0     0     0     0  0.00000000
## [2,]     0     0     0     0     0     0     0     0     0     0  0.00000000
## [3,]     0     0     0     0     0     0     0     0     0     0 -0.04407686
## [4,]     0     0     0     0     0     0     0     0     0     0  0.00000000
## [5,]     0     0     0     0     0     0     0     0     0     0  0.00000000
##      [,52]        [,53] [,54] [,55]      [,56] [,57] [,58] [,59] [,60] [,61]
## [1,]     0 -0.003647723     0     0 0.05345415     0     0     0     0     0
## [2,]     0  0.000000000     0     0 0.00000000     0     0     0     0     0
## [3,]     0  0.000000000     0     0 0.00000000     0     0     0     0     0
## [4,]     0  0.000000000     0     0 0.00000000     0     0     0     0     0
## [5,]     0  0.000000000     0     0 0.00000000     0     0     0     0     0
##      [,62] [,63] [,64] [,65] [,66] [,67] [,68] [,69] [,70] [,71] [,72] [,73]
## [1,]     0     0     0     0     0     0     0     0     0     0     0     0
## [2,]     0     0     0     0     0     0     0     0     0     0     0     0
## [3,]     0     0     0     0     0     0     0     0     0     0     0     0
## [4,]     0     0     0     0     0     0     0     0     0     0     0     0
## [5,]     0     0     0     0     0     0     0     0     0     0     0     0
##      [,74] [,75]       [,76] [,77] [,78] [,79] [,80] [,81] [,82] [,83] [,84]
## [1,]     0     0 0.000000000     0     0     0     0     0     0     0     0
## [2,]     0     0 0.000000000     0     0     0     0     0     0     0     0
## [3,]     0     0 0.007557284     0     0     0     0     0     0     0     0
## [4,]     0     0 0.000000000     0     0     0     0     0     0     0     0
## [5,]     0     0 0.000000000     0     0     0     0     0     0     0     0
##      [,85] [,86] [,87] [,88] [,89] [,90] [,91] [,92]        [,93] [,94] [,95]
## [1,]     0     0     0     0     0     0     0     0  0.000000000     0     0
## [2,]     0     0     0     0     0     0     0     0  0.000000000     0     0
```

```
## [3,]     0     0     0     0     0     0     0     0 -0.000521853     0     0
## [4,]     0     0     0     0     0     0     0     0  0.000000000     0     0
## [5,]     0     0     0     0     0     0     0     0  0.000000000     0     0
##      [,96] [,97] [,98] [,99] [,100] [,101] [,102] [,103] [,104] [,105] [,106]
## [1,]     0     0     0     0      0      0      0      0      0      0      0
## [2,]     0     0     0     0      0      0      0      0      0      0      0
## [3,]     0     0     0     0      0      0      0      0      0      0      0
## [4,]     0     0     0     0      0      0      0      0      0      0      0
## [5,]     0     0     0     0      0      0      0      0      0      0      0
##      [,107] [,108] [,109]      [,110]        [,111] [,112] [,113] [,114] [,115]
## [1,]      0      0      0  0.00000000  0.000000e+00      0      0      0      0
## [2,]      0      0      0  0.00000000  0.000000e+00      0      0      0      0
## [3,]      0      0      0  0.00000000 -9.194534e-05      0      0      0      0
## [4,]      0      0      0  0.00000000  0.000000e+00      0      0      0      0
## [5,]      0      0      0 -0.01453844  0.000000e+00      0      0      0      0
##          [,116] [,117] [,118] [,119] [,120] [,121] [,122] [,123] [,124] [,125]
## [1,] 0.05186137      0      0      0      0      0      0      0      0      0
## [2,] 0.00000000      0      0      0      0      0      0      0      0      0
## [3,] 0.00000000      0      0      0      0      0      0      0      0      0
## [4,] 0.00000000      0      0      0      0      0      0      0      0      0
## [5,] 0.00000000      0      0      0      0      0      0      0      0      0
##      [,126] [,127] [,128] [,129] [,130] [,131] [,132] [,133] [,134] [,135]
## [1,]      0      0      0      0      0      0      0      0      0      0
## [2,]      0      0      0      0      0      0      0      0      0      0
## [3,]      0      0      0      0      0      0      0      0      0      0
## [4,]      0      0      0      0      0      0      0      0      0      0
## [5,]      0      0      0      0      0      0      0      0      0      0
##      [,136] [,137] [,138] [,139] [,140] [,141] [,142] [,143] [,144] [,145]
## [1,]      0      0      0      0      0      0      0      0      0      0
## [2,]      0      0      0      0      0      0      0      0      0      0
## [3,]      0      0      0      0      0      0      0      0      0      0
## [4,]      0      0      0      0      0      0      0      0      0      0
## [5,]      0      0      0      0      0      0      0      0      0      0
##      [,146] [,147] [,148] [,149] [,150] [,151] [,152] [,153] [,154] [,155]
## [1,]      0      0      0      0      0      0      0      0      0      0
## [2,]      0      0      0      0      0      0      0      0      0      0
## [3,]      0      0      0      0      0      0      0      0      0      0
## [4,]      0      0      0      0      0      0      0      0      0      0
## [5,]      0      0      0      0      0      0      0      0      0      0
##      [,156] [,157] [,158] [,159] [,160] [,161] [,162] [,163] [,164] [,165]
## [1,]      0      0      0      0      0      0      0      0      0      0
## [2,]      0      0      0      0      0      0      0      0      0      0
## [3,]      0      0      0      0      0      0      0      0      0      0
## [4,]      0      0      0      0      0      0      0      0      0      0
## [5,]      0      0      0      0      0      0      0      0      0      0
##      [,166] [,167] [,168] [,169] [,170]
## [1,]      0      0      0      0      0
## [2,]      0      0      0      0      0
## [3,]      0      0      0      0      0
## [4,]      0      0      0      0      0
## [5,]      0      0      0      0      0
```

```
Phihat.diff.viz <- lagmatrix(fit=sparse.diff.VAR.estimate, returnplot=TRUE)
```



diff VAR levels-1.pdf

Here, we see that we get much different results, and actually see that SIE is affected by 24 lags of itself, and also by the SeaTemp for 11 lags. Note for cv we actually obtain zeros for all coefficient lags. Any reason why?

Forecasting

We can use our model in levels for forecasting to avoid any complications with returning a differenced series back to levels. Although the SIE level has already reached 0, we can see it is going down once again with the SIE reaching a level of 1.406 in June of 2022 and 1.226 in July 2022. We will compare this to forecast from a normal VAR with p = 6 lags.

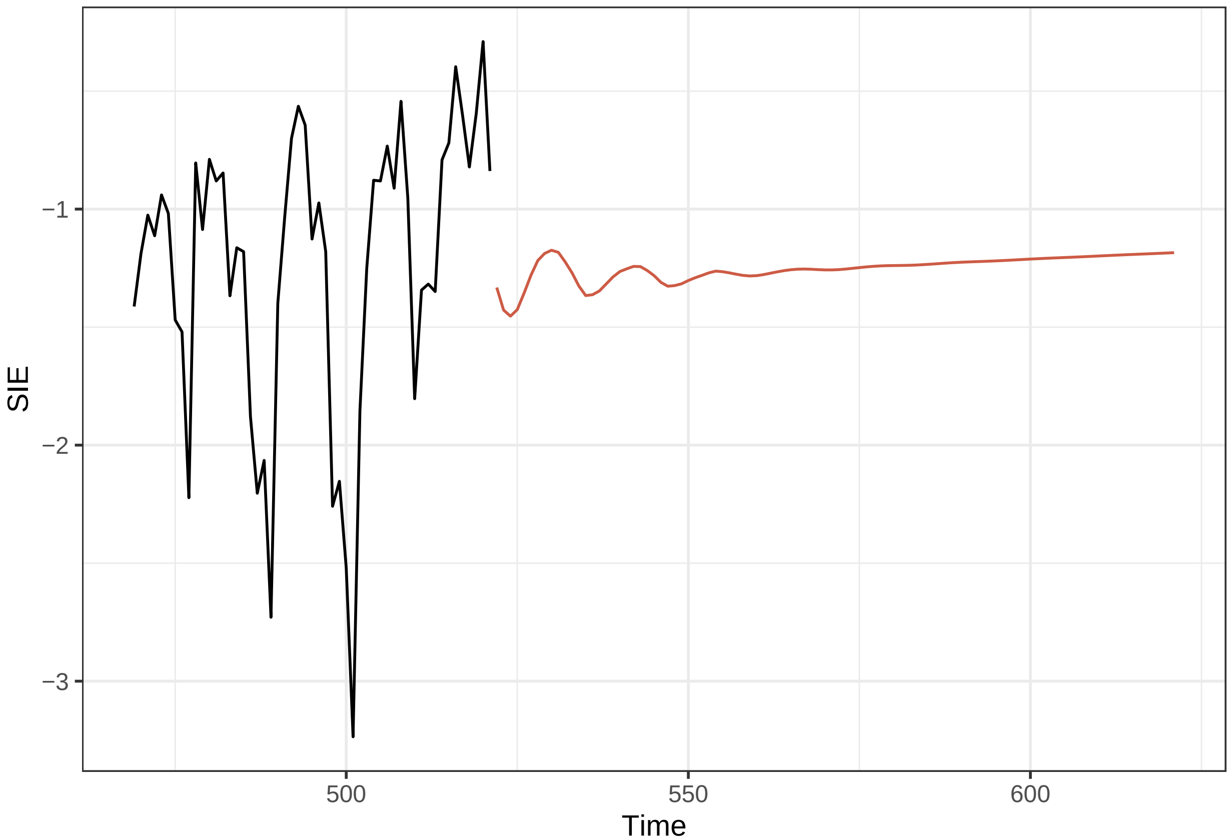
```
Y <- Arctic_Ice[-nrow(Arctic_Ice),]  # Leave the last 10 observations out for comparison
Ytest <- Arctic_Ice[nrow(Arctic_Ice),]

VARcv <- sparseVAR(Y = scale(Y), VARpen = "L1", selection = "cv", h = 1)

VARf <- directforecast(VARcv, h = 1)
mean((VARf-Ytest)^2)
```

```
## [1] 38400.27
```

```
# Longer horizon forecasting
is.stable(VARcv)
```

```
## [1] TRUE
```

```
rec_fcst <- recursiveforecast(VARcv, h = 100)
plot(rec_fcst, series = "SIE")
```

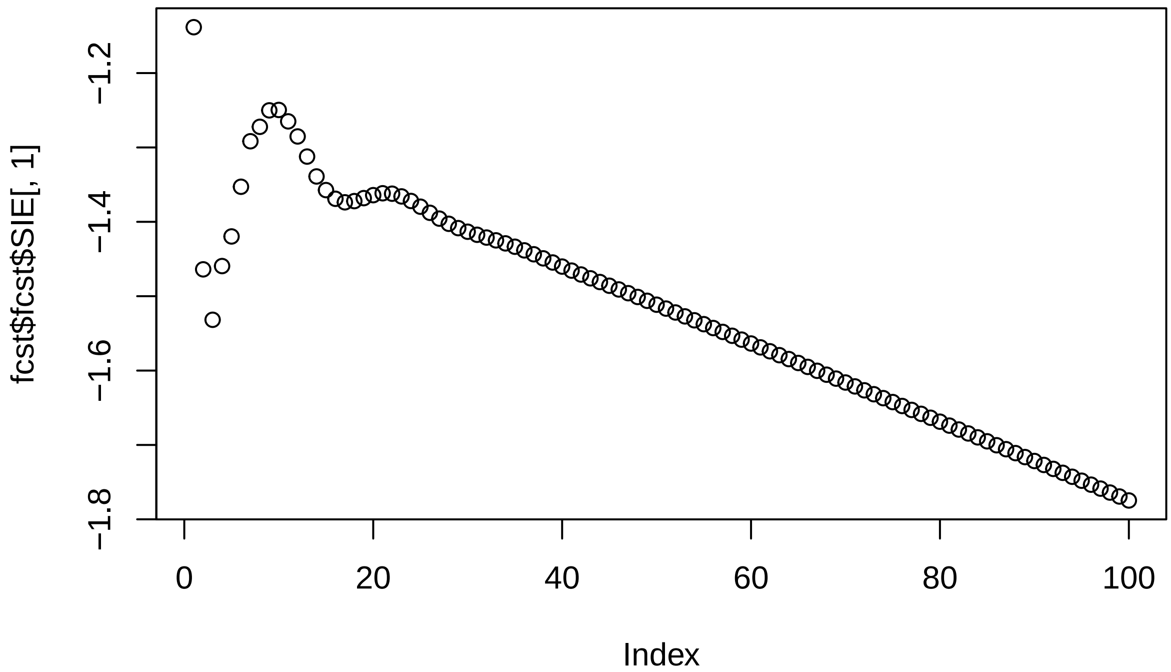


Here, we note that SIE is on the rise, and this is captured by the forecast. We can compare this to the forecast for the regular VAR.

```
library(forecast)
```

```
## Warning: package 'forecast' was built under R version 4.1.2
```

```
VARnormf <- VAR(Arctic_Ice, p = 6, type = c("both"))
fcst <- predict(VARnormf, n.ahead = 100)
plot(fcst$fcst$SIE[,1])
```



forecasting-1.pdf

Here, we actually go down in forecast interestingly enough. We can compare the two models using cross validation

Penalized VAR Implementation

We create a function that implements a penalized VAR. Note we assume that the data is all standardized and centered around zero

```r
# Soft Thresholding
soft.thresh <- function(x, lambda){
  sign(x) * pmax(abs(x) - lambda, 0)
}

# Stopping Criteria: verify that average loss hasn't changed more than tol
# k is the iteration
# window used to find window length
# patience is further down, used to see how many time points to consider for loss
should.stop <- function(loss, window, k, tol = 1e-10) {
  window_length = floor(window / 2) - 1
  prev_loss <- mean(loss[(k-(window-1)):(k-(window-1)+window_length)])
  curr_loss <- mean(loss[(k-(window-1)+window_length):k])

  abs(curr_loss - prev_loss) < tol
}

# Coordinate Descent
coord.desc <- function(y, X, lambda, max.iter = 500, tol = 1e-4, patience = 10){
  X <- scale(X)
  y <- scale(y)
  beta.hat <- rep(0,ncol(X))

  # Value of loss function at each iteration
```

```r
  loss <- rep(0, max.iter)

  for (k in 1:max.iter){
    # The full residual used to check the value of the loss function
    residual <- y - X %*% beta.hat
    loss[k] <- mean(residual*residual)

    for (j in 1:ncol(X)){
      # Partial residual (effect on all other covariates)
      residual <- residual + X[,j] * beta.hat[j]

      # Single variable OLS estimate
      beta.ols.j <- mean(residual * X[,j])

      # Soft thresholding
      beta.hat[j] <- soft.thresh(beta.ols.j, lambda)
    }

    # Early Stopping Criteria
    if (k>patience){
      if (should.stop(loss, k = k, window = patience, tol = tol)) {
        break
      }
    }
  }
  beta.hat
}

penal.VAR <- function(Y, VARp = 1, lambda = 0, max.iter = 500, tol = 1e-4){
  k <- ncol(Y)
  lagged.Y <- embed(Y, VARp+1)[,-(1:k)]  # Long parameter matrix of all lags from t through VARp
  original.Y <- embed(Y,VARp+1)[,1:k]

  # Best way of doing this is row by row?
  # Perform coordinate descent on each output variable independently

  beta.hat <- matrix(data = NA, nrow = k, ncol = k*VARp)
  for (series in 1:k){
    beta.hat[series,] <- coord.desc(original.Y[,series], lagged.Y, lambda = lambda)
  }
  array(beta.hat, dim = c(k,k,VARp))
}

penal.VAR(Arctic_Ice, VARp = 9, lambda = 0.05)
```

```
## , , 1
##
##            [,1]       [,2]       [,3]          [,4]         [,5]
## [1,] 0.07730636 0.00000000 0.00000000  0.000000e+00  0.000000000
## [2,] 0.03849500 0.07464964 0.00000000 -7.777491e-05 -0.011956432
## [3,] 0.01952927 0.00000000 0.06275992  0.000000e+00 -0.019258608
## [4,] 0.00000000 0.00000000 0.00000000  0.000000e+00  0.000000000
## [5,] 0.00000000 0.00000000 0.00000000  0.000000e+00  0.001275843
##
```

```
## , , 2
##
##            [,1]       [,2]        [,3] [,4]        [,5]
## [1,] 0.05143979 0.00000000 0.000000000    0 0.000000000
## [2,] 0.01074057 0.02867807 0.009137487    0 0.000000000
## [3,] 0.02261985 0.00000000 0.048409267    0 0.000000000
## [4,] 0.00000000 0.00000000 0.000000000    0 0.000000000
## [5,] 0.00000000 0.00000000 0.000000000    0 0.001239299
##
## , , 3
##
##            [,1] [,2]       [,3] [,4]        [,5]
## [1,] 0.00000000    0 0.00000000    0 0.000000000
## [2,] 0.00000000    0 0.01659203    0 0.000000000
## [3,] 0.02536523    0 0.05136880    0 0.000000000
## [4,] 0.00000000    0 0.00000000    0 0.000000000
## [5,] 0.00000000    0 0.00000000    0 0.001425646
##
## , , 4
##
##            [,1]         [,2]       [,3] [,4]        [,5]
## [1,] 0.00000000 -0.006344609 0.00000000    0 0.000000000
## [2,] 0.01588261  0.000000000 0.01876364    0 0.000000000
## [3,] 0.03115406  0.000000000 0.04889036    0 0.000000000
## [4,] 0.00000000  0.000000000 0.00000000    0 0.000000000
## [5,] 0.00000000  0.000000000 0.00000000    0 0.001071204
##
## , , 5
##
##             [,1]         [,2]       [,3]         [,4] [,5]
## [1,] 0.000000000 -0.007813545 0.00000000 0.000000e+00    0
## [2,] 0.003232268  0.000000000 0.01046169 0.000000e+00    0
## [3,] 0.031540325  0.000000000 0.03896605 0.000000e+00    0
## [4,] 0.000000000  0.000000000 0.00000000 0.000000e+00    0
## [5,] 0.000000000  0.000000000 0.00000000 2.220623e-05    0
##
## , , 6
##
##            [,1]        [,2]         [,3]         [,4] [,5]
## [1,] 0.00000000 -0.01813985 0.000000e+00 0.000000e+00    0
## [2,] 0.01003684  0.00000000 9.369036e-03 0.000000e+00    0
## [3,] 0.03192738  0.00000000 3.328114e-02 0.000000e+00    0
## [4,] 0.00000000  0.00000000 1.937851e-05 0.000000e+00    0
## [5,] 0.00000000  0.00000000 0.000000e+00 1.054117e-05    0
##
## , , 7
##
##            [,1] [,2]         [,3]         [,4] [,5]
## [1,] 0.00000000    0 0.0000000000 0.000000e+00    0
## [2,] 0.00000000    0 0.0070795879 0.000000e+00    0
## [3,] 0.03068736    0 0.0173565837 0.000000e+00    0
## [4,] 0.00000000    0 0.0002084364 0.000000e+00    0
## [5,] 0.00000000    0 0.0000000000 4.783547e-06    0
##
```

```
## , , 8
##
##            [,1] [,2]         [,3]         [,4] [,5]
## [1,] 0.00000000    0 0.0000000000 0.000000e+00    0
## [2,] 0.00000000    0 0.0136515049 0.000000e+00    0
## [3,] 0.01987252    0 0.0000000000 0.000000e+00    0
## [4,] 0.00000000    0 0.0004753935 0.000000e+00    0
## [5,] 0.00000000    0 0.0000000000 1.573915e-05    0
##
## , , 9
##
##      [,1]        [,2]       [,3]         [,4]       [,5]
## [1,]    0 -0.03364101 0.00000000 -0.001128428 -0.8058947
## [2,]    0  0.00000000 0.05607771  0.006174460  0.7182466
## [3,]    0  0.00000000 0.06847915  0.020227830  0.8256062
## [4,]    0  0.00000000 0.00620189  0.019322506  0.9389976
## [5,]    0  0.00000000 0.00000000  0.007354537  0.9420944
```

```
penal.VAR(stat.Arctic, VARp = 9, lambda = 0.05)
```

```
## , , 1
##
##             [,1]        [,2]        [,3]      [,4]      [,5]
## [1,] -0.05800071 -0.01648397 0.000000000 0.0000000 0.0000000
## [2,]  0.00000000 -0.32403154 0.000000000 0.0000000 0.0000000
## [3,]  0.00000000  0.08524363 0.000000000 0.0000000 0.0000000
## [4,]  0.00000000  0.00000000 0.000000000 0.3636441 0.0000000
## [5,]  0.00000000  0.00000000 0.009477648 0.0000000 0.1114224
##
## , , 2
##
##             [,1]         [,2]        [,3]        [,4]        [,5]
## [1,] -0.02490589  0.000000000 -0.04839790  0.04974944 0.000000000
## [2,]  0.00000000  0.000000000  0.03591079  0.00000000 0.001620448
## [3,]  0.00000000 -0.005418884  0.00000000  0.00000000 0.000000000
## [4,]  0.00000000  0.028130521  0.00000000 -0.06992676 0.000000000
## [5,]  0.00000000  0.000000000  0.00000000  0.01177741 0.000000000
##
## , , 3
##
##            [,1]        [,2] [,3]       [,4]        [,5]
## [1,] -0.0260527  0.00000000    0 0.00000000  0.01571730
## [2,]  0.0000000 -0.02072565    0 0.00000000  0.00501760
## [3,]  0.0000000  0.00000000    0 0.00000000 -0.01671956
## [4,]  0.0000000  0.00000000    0 0.00000000  0.01638764
## [5,]  0.0000000 -0.03799046    0 0.03243367  0.13394702
##
## , , 4
##
##              [,1]        [,2]       [,3]         [,4]       [,5]
## [1,] -0.143487503  0.00000000 0.05787172  0.000000000 0.00000000
## [2,]  0.006972377  0.00000000 0.03893936 -0.002501205 0.00000000
## [3,]  0.039241242  0.01610291 0.00000000 -0.064343499 0.00000000
## [4,] -0.017028797  0.00000000 0.00000000  0.000000000 0.01677271
## [5,]  0.000000000 -0.02087024 0.00000000  0.000000000 0.00000000
```

```
##
## , , 5
##
##              [,1]       [,2]        [,3]        [,4]         [,5]
## [1,] -0.02640785 0.00000000  0.00000000  0.00000000  0.009628652
## [2,]  0.00000000 0.00000000 -0.04089295  0.00000000  0.000000000
## [3,]  0.02145091 0.00000000 -0.16193380 -0.05117872  0.000000000
## [4,]  0.00000000 0.01406469  0.00000000  0.00000000  0.000000000
## [5,] -0.04428628 0.00000000  0.02553156  0.00000000 -0.002725316
##
## , , 6
##
##              [,1]         [,2]        [,3]       [,4]        [,5]
## [1,] -0.00246947 -0.007517499  0.01725877 0.00000000  0.05429237
## [2,]  0.00000000  0.000000000  0.00000000 0.00000000  0.00000000
## [3,]  0.01894134  0.004082285 -0.03897326 0.00000000  0.00000000
## [4,]  0.00000000  0.000000000  0.00000000 0.03937929 -0.00836193
## [5,]  0.00000000  0.000000000  0.00000000 0.00000000  0.00000000
##
## , , 7
##
##              [,1]         [,2]        [,3]      [,4]        [,5]
## [1,] -0.07922060  0.002599051  0.04325946 0.0000000 -0.02342129
## [2,]  0.00000000  0.000000000  0.00000000 0.0000000 -0.02499826
## [3,]  0.08354902 -0.004020094 -0.07812708 0.0000000  0.00000000
## [4,]  0.00000000  0.000000000  0.01848928 0.1236983  0.00000000
## [5,]  0.00000000  0.000000000  0.00000000 0.0000000  0.00000000
##
## , , 8
##
##              [,1]        [,2]         [,3]       [,4]         [,5]
## [1,] -0.02073468 0.000000000  0.000000000 0.00000000 -0.026408926
## [2,] -0.05291022 0.000000000  0.002380055 0.02422195  0.007480266
## [3,]  0.01884365 0.000000000 -0.039703478 0.00000000  0.000000000
## [4,]  0.00000000 0.000000000  0.087955679 0.01507428  0.007124601
## [5,]  0.00000000 0.001324517 -0.002443586 0.00000000  0.000000000
##
## , , 9
##
##              [,1]        [,2]         [,3]       [,4]        [,5]
## [1,]  0.00000000 -0.06535092  0.057110622 0.00000000  0.00000000
## [2,]  0.00000000 -0.03035888  0.000000000 0.00000000  0.00000000
## [3,] -0.01535601 -0.00638247  0.000000000 0.00000000  0.03308409
## [4,]  0.00000000  0.00000000  0.000000000 0.04567596  0.00000000
## [5,]  0.02578865  0.00000000 -0.007673043 0.00000000 -0.03648363
```

```
sparseVAR(scale(stat.Arctic), p = 9, VARpen = "L1", VARlseq = 0.5)$Phihat
```

```
## , , 1
##
##               [,1]        [,2]        [,3]         [,4]          [,5]
## [1,] -0.266959074 -0.08778315 -0.02409943 -0.059183557 -0.0605467231
## [2,]  0.039515845 -0.54336093  0.04429090  0.003924423 -0.0517758150
## [3,]  0.041638716  0.19719540 -0.04670453  0.066064656  0.0000000000
## [4,] -0.047280027  0.02167532  0.02470263  0.728753029 -0.0003887886
```

```
## [5,] -0.002830193 -0.06672277  0.02715458  0.024606540  0.1678797919
##               [,6]        [,7]        [,8]         [,9]       [,10]
## [1,] -0.2303116240 -0.05601861 -0.08432441  0.119310248 -0.06384452
## [2,]  0.0003719175 -0.36894622  0.13259160 -0.063773399  0.02957498
## [3,]  0.0391397809  0.03829012 -0.08469783  0.001341983  0.03780769
## [4,]  0.0085135989  0.06473097  0.07132386 -0.689158793  0.01306756
## [5,] -0.0230955468 -0.09322500 -0.08624377  0.071279414 -0.06995405
##             [,11]       [,12]       [,13]        [,14]       [,15]       [,16]
## [1,] -0.238092828 -0.09758515 -0.04471749 -0.004432824  0.01874220 -0.35718618
## [2,]  0.008044555 -0.37300116  0.05090960  0.069668253  0.04571837  0.09074377
## [3,]  0.069248357  0.05441339 -0.10293150  0.031515561 -0.04503939  0.12539074
## [4,] -0.058337653 -0.06243947  0.00000000  0.514392329  0.05525330 -0.02429832
## [5,]  0.042341081 -0.16096354 -0.03311462  0.042122364  0.22947426 -0.01327016
##            [,17]       [,18]       [,19]        [,20]       [,21]        [,22]
## [1,] -0.12863640  0.01259392  0.00968158 -0.054416226 -0.23976303 -0.139199073
## [2,] -0.30775695  0.15062118 -0.03321722  0.026918676  0.02990234 -0.308460304
## [3,]  0.14792924 -0.03251661 -0.09299303  0.012973040  0.10410436  0.069353096
## [4,]  0.08606659  0.03954597 -0.37466432  0.001311248 -0.03192673  0.001362576
## [5,] -0.11738241 -0.02010167  0.02035260 -0.015288024 -0.05597452  0.007744184
##             [,23]       [,24]       [,25]       [,26]       [,27]        [,28]
## [1,]  0.001217567 -0.01416669  0.04427778 -0.18274698 -0.17641949 -0.009362114
## [2,]  0.010856744 -0.06782975 -0.01691715  0.06847596 -0.26710944  0.076524540
## [3,] -0.229855706 -0.12282959 -0.02284029  0.09859057  0.14319190 -0.056279680
## [4,]  0.023806136  0.23147632 -0.01016076  0.04054952 -0.01181308  0.010233432
## [5,]  0.074649183 -0.04095049 -0.01694816  0.06307101  0.04712380 -0.017418490
##             [,29]       [,30]       [,31]       [,32]       [,33]       [,34]
## [1,] -0.044689205  0.09070253 -0.24791871 -0.08392038  0.01603135  0.07978006
## [2,]  0.124355432  0.01092610  0.08323166 -0.19634594  0.07001997 -0.13263317
## [3,]  0.120634757 -0.03023460  0.18651091  0.04012072 -0.10297696 -0.23644713
## [4,] -0.035347505 -0.04188881 -0.01222620  0.01511969  0.08944528  0.14236104
## [5,]  0.003593723 -0.04388084  0.04923420  0.02664055 -0.04062279 -0.03487150
##            [,35]         [,36]         [,37]       [,38]       [,39]
## [1,] -0.02515585 -0.2239640092 -4.876004e-02  0.03290179 -0.09394686
## [2,] -0.06916262 -0.0314339910 -2.107223e-01  0.09914839  0.15899153
## [3,] -0.03029917  0.1145269190  4.247937e-02 -0.10636333  0.16409873
## [4,]  0.03517802 -0.0007878972  5.545339e-05  0.08447942 -0.05167947
## [5,]  0.05591643  0.0547625587  6.289217e-02 -0.06905248 -0.01620876
##            [,40]        [,41]       [,42]       [,43]        [,44]
## [1,] -0.06144291 -0.135444080 -0.14840959  0.10834297  0.012347842
## [2,]  0.08452974  0.003873051 -0.17585038  0.05674564 -0.020048848
## [3,]  0.02529094  0.042301625  0.03273676 -0.04724625 -0.203575035
## [4,]  0.04062759 -0.017017424  0.01852856  0.01390717  0.128051606
## [5,] -0.04008580  0.050038829  0.01815994 -0.03817078 -0.001503651
##              [,45]
## [1,]  0.0573631650
## [2,]  0.0001016299
## [3,]  0.0336780302
## [4,] -0.0095721825
## [5,] -0.0757582896
```

In the end, we will have 5 rows and 5×VARp columns