Computing the CMB power spectrum

III. Evolution of structure in the Universe

Iván Villegas-Pérez[⋆]

High Energy Physics (HEP) Section, Department of Physics, University of Oslo, Sem Sælands vei 24,0371 Oslo, Norway
e-mail: ivanvi@uio.no

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ABSTRACT

Context. The goal of this whole study is to be able to predict the CMB (and matter) fluctuations (described by the so-called power-spectrum) from first principles and learn about all the different physical processes that goes on to be able to explain the results.
Aims. The goal in this work is to make a class/module that evolves the density, velocity, temperature, polarization and neutrinos perturbations from an early time until today.
Methods. A code has been developed, using the template given by Hans A. Winther. In this code, several functions have been implemented to evolve the perturbations and plot the results.
Results. Evolution of the structure of the Universe has been solved. The density and velocity perturbations for CDM, baryonic matter and photon have been computed.
Conclusions. Now that the background cosmology of the Universe, its recombination history and the evolution of it structure have been solved, this study can continue with its next step, studying the evolution of the background properties of the Universe, such as the CMB and its power-spectrum.

Key words. background cosmology – evolution of the Universe – structure evolution – perturbations

1. Introduction

The goal of this study has been to solve the evolution of structure in the Universe: how did small fluctuations in the baryon-photon-dark-matter fluid grow from shortly after inflation until today? The ultimate goal of this part is to construct two-dimensional functions (of time and Fourier scale, x and k) for each of the main physical quantities of interest: $\Phi(x, k)$, $\Psi(x, k)$, $\delta_{\rm CDM}(x, k)$, $\delta_b(x, k)$, $v_{\rm CDM}(x, k)$, $v_B(x, k)$ and $\Theta_\ell(x, k)$[1].

The fiducial cosmology used in this work is the best-fit cosmology found from fits to Planck 2018 data [Aghanim et al. (2020)] without neutrinos, reionization, Helium nor photon polarization:

$$h_0 = 0.67,$$

$$T_{\rm CMB,0} = 2.7255 \text{ K},$$

$$N_{\rm eff} = 0,$$

$$\Omega_{\rm B,0} = 0.05,$$

$$\Omega_{\rm CDM,0} = 0.267,$$

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2} = 0,$$

$$\Omega_{\Lambda,0} = 1 - (\Omega_{\rm B,0} + \Omega_{\rm CDM,0} + \Omega_{\rm k,0} + \Omega_{\nu,0} + \Omega_{\gamma,0}),$$

$$Y_p = 0,$$

$$z_{\rm reion} = 0,$$

$$\Delta z_{\rm reion} = 0,$$

$$z_{\rm He,\ reion} = 0,$$

$$\Delta z_{\rm He,\ reion} = 0.$$

Each quantity will be explained in the following section.

2. Perturbations

During the lectures[2], the linearized Einstein and Boltzmann equations for photons, baryons and dark matter, and their respective inflationary initial conditions have been derived. The goal of the current study is to solve these equations numerically.

But before writing down the equations, there are a few issues that should be pointed out. First, at early times the optical depth, τ, is very large. The ODE we have to solve for the perturbations becomes numerically unstable in this regime and this must be dealed with. Large τ means that electrons at a given place only observe temperature fluctuations that are very nearby. This, in turn, implies that it will only see very smooth fluctuations, since the full system is in thermodynamic equilibrium, and all gradients are efficiently washed out. The only relevant quantities in this regime are therefore

1. The monopole, Θ_0, which measures the mean temperature at the position of the electron.

[⋆] GitHub: https://github.com/ivanvillegas7/CMB_power_spectrum
Orcid: https://orcid.org/0009-0001-2300-603X

[1] Hans A. Winther, Milestone III: Evolution of structure in the Universe: https://cmb.wintherscoming.no/milestone3.php

[2] AST5220 - Cosmology II at the University of Oslo (UiO).

2. The dipole, Θ_1, which is given by the velocity of the fluid due to the Doppler effect.
3. The quadrupole, Θ_2, which is the only relevant source of polarization signals.

The regime where this is the case is called *tight coupling*.

At later times, though, the fluid becomes thinner, and the electrons start seeing further away, and then become sensitive to higher-ordered multipoles, Θ_l. Fortunately, because of a very nice computational trick due to Zaldarriaga and Seljak called *line of sight integration* [Seljak & Zaldarriaga (1996)], only a relatively small number of these (six is enough) needs to be taken into account, and so the system of relevant equations is therefore still tractable[3].

A second issue is the very large value of τ' at early times, which multiplies $(3\Theta_1 + v_B)$ in the Boltzmann equations. The latter factor is very small early on, and the product of the two is therefore numerically extremely unstable. The result is that the standard Boltzmann equation set is completely unstable if one simply implements the full expressions at early times. The solution to this problem is to use a proper approximation for $(3\Theta_1 + v_B)$ at early times. See the appendix for a derivation.

2.1. The full system

Photon temperature multipoles:

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{1}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{2}$$

$$\Theta_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{\Pi\delta_{\ell,2}}{10}\right], \tag{3}$$

where $2 \le \ell < \ell_{\max}$, and

$$\Theta_{\ell_{\max}}' = \frac{ck}{\mathcal{H}}\Theta_{\ell_{\max}-1} - c\frac{\ell_{\max}+1}{\mathcal{H}\eta(x)}\Theta_{\ell_{\max}} + \tau'\Theta_{\ell_{\max}}. \tag{4}$$

Photon polarization multipoles:

$$\Theta_0^{P\prime} = -\frac{ck}{\mathcal{H}}\Theta_{P1} + \tau'\left[\Theta_{P0} - \frac{1}{2}\Pi\right] \tag{5}$$

$$\Theta_\ell^{P\prime} = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1}^P - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1}^P + \tau'\left[\Theta_\ell^P - \frac{\Pi\delta_{\ell,2}}{10}\right], \tag{6}$$

where $1 \le \ell < \ell_{\max}$, and

$$\Theta_{\ell_{\max}}^{P}{}' = \frac{ck}{\mathcal{H}}\Theta_{\ell_{\max}-1}^P - c\frac{\ell_{\max}+1}{\mathcal{H}\eta(x)}\Theta_{\ell_{\max}}^P + \tau'\Theta_{\ell\max}^P. \tag{7}$$

Neutrino multipoles:

$$\mathcal{N}_0' = -\frac{ck}{\mathcal{H}}\mathcal{N}_1 - \Phi', \tag{8}$$

$$\mathcal{N}_1' = \frac{ck}{3\mathcal{H}}\mathcal{N}_0 - \frac{2ck}{3\mathcal{H}}\mathcal{N}_2 + \frac{ck}{3\mathcal{H}}\Psi \tag{9}$$

$$\mathcal{N}_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell+1}, \tag{10}$$

where $2 \le \ell < \ell_{\max,\nu}$, and

$$\mathcal{N}_{\ell_{\max,\nu}}' = \frac{ck}{\mathcal{H}}\mathcal{N}_{\ell_{\max,\nu}-1} - c\frac{\ell_{\max,\nu}+1}{\mathcal{H}\eta(x)}\mathcal{N}_{\ell_{\max,\nu}}. \tag{11}$$

Cold dark matter and baryons:

$$\delta_{\rm CDM}' = \frac{ck}{\mathcal{H}}v_{\rm CDM} - 3\Phi', \tag{12}$$

$$v_{\rm CDM}' = -v_{\rm CDM} - \frac{ck}{\mathcal{H}}\Psi, \tag{13}$$

$$\delta_B' = \frac{ck}{\mathcal{H}}v_B - 3\Phi', \tag{14}$$

$$v_B' = -v_B - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_B). \tag{15}$$

Metric perturbations:

$$\Phi' = \frac{H_0^2}{2\mathcal{H}^2}\left[\frac{\Omega_{\rm CDM0}}{a}\delta_{\rm CDM} + \frac{\Omega_{B0}}{a}\delta_B + 4\frac{\Omega_{\gamma0}}{a^2}\Theta_0 + 4\frac{\Omega_{\nu0}}{a^2}\mathcal{N}_0\right] + \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi, \tag{16}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\left[\Omega_{\gamma0}\Theta_2 + \Omega_{\nu0}\mathcal{N}_2\right]. \tag{17}$$

In the equations above $\Pi = \Theta_2 + \Theta_0^P + \Theta_2^P$ and $R = \frac{4\Omega_{\gamma0}}{3\Omega_{B0}a}$.

[3] Note that before 1996 or so, people actually included thousands of variables, to trace multipoles for the full range. Needless to say, this was slow, and other approximations were required.

2.2. The tight coupling regime

Just solving the original ODE system for all times would be ideal, however this does not work as the system is very stiff and numerically unstable when τ is very large. Some slightly different equations in the tight coupling regime therefore have to be solve . In the tight coupling regime, the only differences are:

1. That one should only include $\ell = 0$ and 1 for Θ_ℓ and none for polarization (all higher moments are given by those).
2. That the expressions for Θ_1' and v_B' are quite a bit more involved:

$$\text{Num.} = \left[(1-R)\tau' + (1+R)\tau''\right](3\Theta_1 + v_B) - \frac{ck}{\mathcal{H}}\Psi + \left(1 - \frac{\mathcal{H}'}{\mathcal{H}}\right)\frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Theta_0' \tag{18}$$

$$\text{Den.} = (1+R)\tau' + \frac{\mathcal{H}'}{\mathcal{H}} - 1 \tag{19}$$

$$q = -\frac{\text{Num.}}{\text{Den.}}, \tag{20}$$

$$v_B' = \frac{1}{1+R}\left[-v_B - \frac{ck}{\mathcal{H}}\Psi + R\left(q + \frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Psi\right)\right] \tag{21}$$

$$\Theta_1' = \frac{1}{3}(q - v_B') \tag{22}$$

In the tight coupling regime, we get the same expressions for the higher-ordered photon moments as given by the initial conditions,

$$\Theta_2 = \begin{cases} -\frac{8ck}{15\mathcal{H}\tau'}\Theta_1, & \text{(with polarization)} \\ -\frac{20ck}{45\mathcal{H}\tau'}\Theta_1, & \text{(without polarization)} \end{cases} \tag{23}$$

$$\Theta_\ell = -\frac{\ell}{2\ell+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{\ell-1}, \qquad \ell > 2, \tag{24}$$

$$\Theta_0^P = \frac{5}{4}\Theta_2, \tag{25}$$

$$\Theta_1^P = -\frac{ck}{4\mathcal{H}\tau'}\Theta_2, \tag{26}$$

$$\Theta_2^P = \frac{1}{4}\Theta_2, \tag{27}$$

$$\Theta_\ell^P = -\frac{\ell}{2\ell+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{\ell-1}^P, \qquad \ell > 2. \tag{28}$$

2.3. Initial conditions

The initial conditions are given by:

$$\Psi = -\frac{1}{\frac{3}{2} + \frac{2f_\nu}{5}}, \tag{29}$$

where $f_\nu = \frac{\Omega_{\nu 0}}{\Omega_{\gamma 0} + \Omega_{\nu 0}}$,

$$\Phi = -(1 + \frac{2f_\nu}{5})\Psi, \tag{30}$$

$$\delta_{\text{CDM}} = \delta_B = -\frac{3}{2}\Psi, \tag{31}$$

$$v_{\text{CDM}} = v_B = -\frac{ck}{2\mathcal{H}}\Psi, \tag{32}$$

Photons:

$$\Theta_0 = -\frac{1}{2}\Psi, \tag{33}$$

$$\Theta_1 = \frac{ck}{6\mathcal{H}}\Psi, \tag{34}$$

$$\Theta_2 = \begin{cases} -\frac{8ck}{15\mathcal{H}\tau'}\Theta_1, & \text{(with polarization)} \\ -\frac{20ck}{45\mathcal{H}\tau'}\Theta_1, & \text{(without polarization)} \end{cases} \tag{35}$$

$$\Theta_\ell = -\frac{\ell}{2\ell+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{\ell-1}, \tag{36}$$

Photon polarization:

$$\Theta_0^P = \frac{5}{4}\Theta_2, \tag{37}$$

$$\Theta_1^P = -\frac{ck}{4\mathcal{H}\tau'}\Theta_2, \tag{38}$$

$$\Theta_2^P = \frac{1}{4}\Theta_2, \tag{39}$$

$$\Theta_\ell^P = -\frac{\ell}{2\ell+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{\ell-1}^P, \tag{40}$$

Neutrinos:

$$\mathcal{N}_0 = -\frac{1}{2}\Psi, \tag{41}$$

$$\mathcal{N}_1 = \frac{ck}{6\mathcal{H}}\Psi, \tag{42}$$

$$\mathcal{N}_2 = -\frac{c^2k^2a^2(\Phi+\Psi)}{12H_0^2\Omega_{\nu 0}}, \tag{43}$$

$$\mathcal{N}_\ell = \frac{ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell-1}, \qquad \ell \geq 3. \tag{44}$$

Note that Ψ is not a dynamical variable in the code and is only used here to set the rest of the variables. Since the equation system is linear, (we are free to choose) the normalization of Ψ can be freely chosen when we solving it (the normalization has be done in the end). The particular normalization used here is such that when computing the power-spectra then $\Psi_{\rm true}^2 = \Psi_{\rm code}^2\mathcal{P}_{\mathcal{R}}$ where $\mathcal{P}_{\mathcal{R}}$ is the usual curvature perturbation power-spectrum, the perturbations set up by inflation, $\mathcal{P}_{\mathcal{R}}(k) = A_s(k/k_{\rm pivot})^{n_s-1}$ with A_s,n_s,$k_{\rm pivot}$ (primordial amplitude, spectral index and the pivot scale) being the same parameters as is standard in the literature (and used in codes like CAMB and CLASS).

3. Implementation method

A class that takes in a BackgroundCosmology and RecombinationHistory object and uses this to evolve the perturbations of the Universe, has been implemented in a C++ code[4].

However one thing that complicates it is that when integrating the perturbations the full system cannot just be integrated directly (it is unstable due to the tight coupling between photons and baryons in the very early Universe). It was therefore needed to start off solving the tight coupling system: here only two photon multipoles Θ_0 and Θ_1 (and no polarization multipoles) had to be included. Once tight coupling ends, it was needed to switch to the full system. When doing this the initial conditions will be settled from the tight coupling solution plus giving a value to the multipoles present in the full system, but the tight-coupling regime (the value of these are given by the same relations as in the initial conditions, e.g. Θ_2 is given by the value of Θ_1) is not included. This means two functions were made to set initial conditions to the ODE vector (one at the start and one after tight coupling end) and two functions that sets the right hand side of the ODE system (one for tight coupling and one for the full system) have been implemented. Once this had been done, a vector of k-values was made, for which the system was solved and the perturbations were integrated from the start until today for all these k-values. The results of the ODE were extracted, stored and splined for all of the quantities (δ_B, v_B, Φ, Θ_0, etc.) that are required in the line-of-sight integrals.

[4] All the used code is fully available in my public GitHub repository.

The vector of the perturbations that we integrate will have between 10 and 30 or so elements depending on how many multipoles are included (and this vector will be different in the two regimes). This means it is very important to keep track over which index in the vector corresponds to which quantity.

4. Results

For all the main physical quantities here, four different scales, or wave numbers, are presented. The values are selected in order to display solutions from each type of scale-regime:

- Large scales, entering early in the radiation dominated regime, with small wave numbers; $k = 0.001$ Mpc^{-1} and $k = 0.0001$ Mpc^{-1}.
- Intermediate scales, entering right before and after radiation-matter-equality, with $k = 0.01$ Mpc^{-1}.
- Small scales, entering early in the radiation dominated regime, with a large wave number, $k = 0.1$ Mpc^{-1}.

In Figure 1 one can see the density perturbations for baryonic (ordinary) matter, cold dark matter and photons. First interpreting the pressure-less CDM which is able to start growing, it can be seen that the perturbations on all scales remain constant until gravity can start contracting them, due to entering the horizon and thus becoming causally connected.

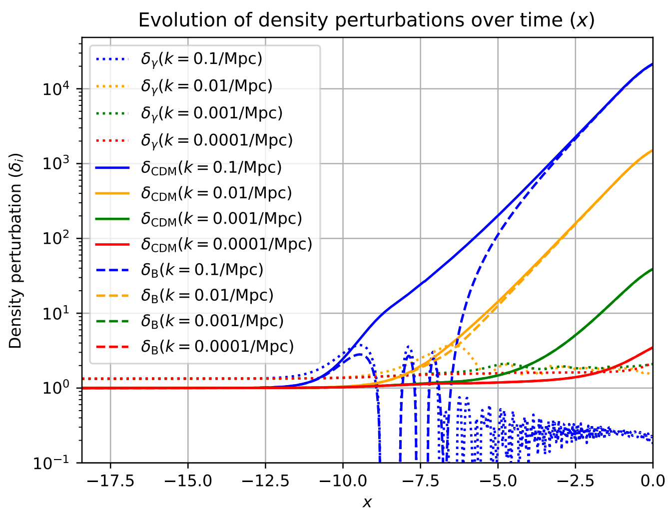

Fig. 1. Density perturbations for baryonic matter ($\delta_{\rm B}$), cold dark matter ($\delta_{\rm CDM}$) and photons (δ_γ) over time (x).

Looking at the baryonic perturbations much of the same behavior can be seen, with the only exception being scales entering the horizon before the end of recombination not growing beyond a certain value, but instead oscillating. This is due to their tight coupling with the photons and the early baryon-photon fluid. This will be evident in the photon perturbations. The equations for the perturbations can be expressed on the form of a driven harmonic oscillator, with gravity being the attracting force and radiation pressure the repulsive force. This balance will cause the baryonic perturbations, entering before the decoupling of the baryons and photons, to oscillate, first being compressed when the gravitational force is dominating before the radiation pressure starts to dominate. The large scale perturbations, entering after recombination is not affected by this coupled state, and will not oscillate but just follow the dark matter perturbations. This effect is clearly visible on the small scales, oscillating heavily until the end of recombination. At this point the baryons are not coupled to the photons any more, and are free to fall into and follow the large gravitational potential wells set up by the dark matter perturbations. On the intermediate scales, a minor effect of this radiation pressure can be seen, causing a slight delay on the baryons growth relative to the CDM, before catching up again.

For the photon perturbations much of the same behavior is seen, as for the baryon perturbations before recombination, due to their tight coupling. The perturbations are constant before entering the horizon. The small scales that enter early start oscillating heavily together with the baryons, with the oscillations being dampened onward in time until decoupling after recombination. After decoupling, the optical depth has dropped significantly, and the photons are free to move more or less unhindered and no longer collapses together with the baryons. The dampening of the harmonic oscillations cause the perturbations to eventually be more or less constant, until the dark energy dominated regime when expansion causes them to start decay to zero. The intermediate scales are barely affected by the coupled oscillated regime, and undergoes only a single peak before decoupling, while the large scales enter well into the matter dominated regime and is not affected by the coupling.

In Figure 2 one can see the velocity perturbations for baryonic (ordinary) matter, cold dark matter and photons. The velocities of the CDM perturbations increase from close to zero until horizon entry, consistent with the horizon being bigger and bigger and thus the part of the causally connected mass within increasing, pulling more and more, so that the matter falls gradually faster and faster.

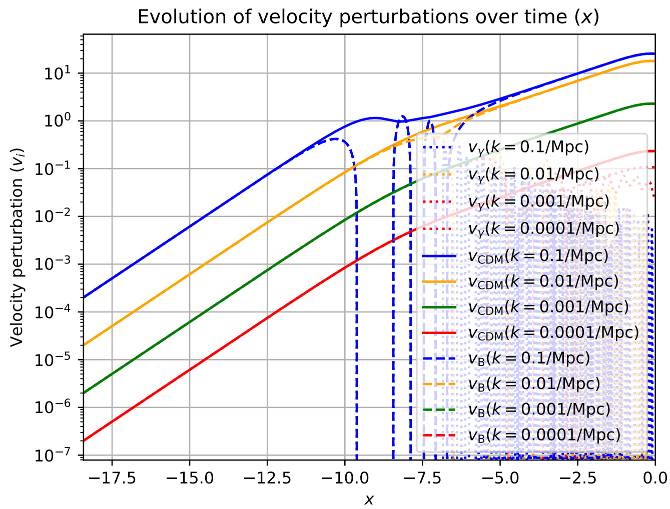

Fig. 2. Velocity perturbations for baryonic matter (v_B), cold dark matter (v_{CDM}) and photons (v_γ) over time (x).

The baryonic matter velocities follow the oscillations, but with a phase shift of $\pi/2$ so that the minimum value of the velocity corresponds to the maximum size of the perturbation, and vise versa, consistent with the harmonic oscillator description. For both the densities and the velocities it can be seen how the early oscillations are dampened over time closer to recombination, due to diffusion dampening caused by the mean free path of the photons increasing with the decreasing optical depth.

Again, it can be seen that the photon velocities are following the density in a similar fashion as for the baryons, being phase shifted by $\pi/2$.

In Figure 3 the photon quadrupoles can be seen, related with the polarization source, which will be relevant in a future work.

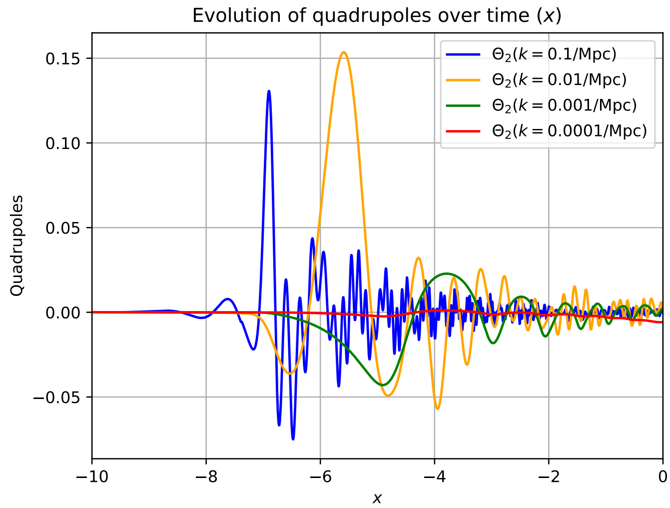

Fig. 3. Photon quadrupole perturbations over time (x).

Lastly, the gravitational potentials are presented in Figure 4. Here it is evident that $\Psi \approx -\Phi$, and thus we will keep to discussing Φ alone where it is inferred that Ψ behaves similarly. Again are all scales constant before entering the horizon, and behaves quite differently depending on the time of entry.

In our understanding of these gravitational potentials we think of them as a type of Newtonian potential. On small scales, entering the horizon and being causally connected in the radiation dominated regime, the potentials are dominated by radiation energy, which does not cluster efficiently. Thus the potentials decay rapidly when the monopole increases, but start to oscillate due to the baryon-photon-fluid oscillations until after decoupling.

When entering into the matter dominated regime, the dominating energy contributor to the potentials are baryons and the CDM, which do indeed cluster and causes the potentials to be constant with growing matter perturbations. The larger scales have less time to collapse, as can be seen in the difference between the small scales, and thus holds a larger constant value through the matter dominated regime.

For the intermediate scales we see the slight effect of the one oscillation of the monopole before the matter perturbations

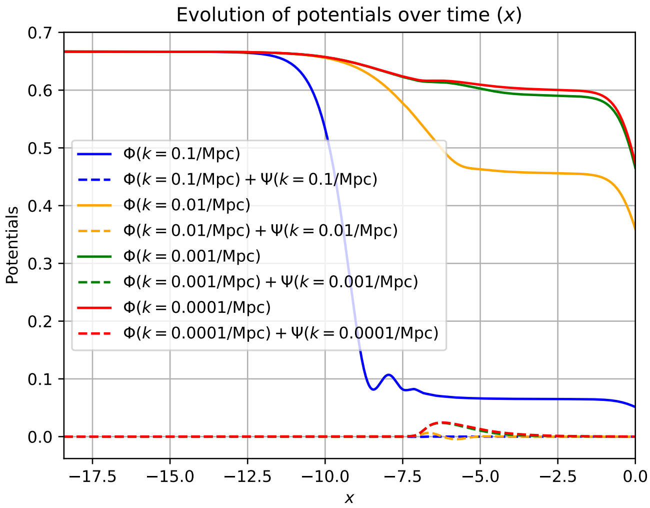

Fig. 4. Evolution of potentials over time (x).

are too dominating, resulting in a minor drop off from the initial value, flattening out after recombination.

The largest scales entering well inside the matter dominated regime are nearly unaffected from their initial conditions, and it can be shown to drop of by a factor of $9/10$ into the matter dominated regime, which is consistent with the way our solutions for how the large scales behave. Note that the potentials all starts to decay into the dark energy regime, where expansion causes even the largest potentials to be smoothed out.

5. Conclusions

The evolution of the structure in the Universe has been solved and the different kind of perturbations have been determined, showing that the metric in our Universe deviates from the Friedmann-Lemaître-Robertson-Walker (FLWR), implying that our Universe is not homogeneous nor isotropic.

References

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