Computing the CMB power spectrum

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ABSTRACT

Context. The goal of this whole study is to be able to predict the CMB (and matter) fluctuations (described by the so-called power-spectrum) from first principles and learn about all the different physical processes that goes on to be able to explain the results.
Aims. The goals in this work are to make a class/module that takes in the cosmological parameters and has functions for getting the Hubble parameter, conformal time and distance measures as function of scale factor and the variable $x = \ln a$, and to make a class/module that takes in the abundance of Helium in the Universe and has functions for getting the free electron fraction, the optical depth (and its derivatives) and the visibility function (and its derivatives).
Methods. Two codes have been developed, using the template given by Hans A. Winther. In the first code, several functions have been implemented to solve the background cosmology of the Universe and plot the results, identifying each epoch domination (either relativistic particles, non-relativistic matter or dark energy). In the second code, several functions have been implemented to solve the recombination history of the Universe and plot the results.
Results. Background cosmology of the Universe has been solved and each epoch has been determined, identifying in each of them the dominating substance. The age of the Universe has also been computed, being this $t_{\text{Universe}} \approx 13.86$ Gyr. Recombination history of the Universe has been solved. The relevant times of the Universe's recombination history have been computed, as well as the sound horizon at decoupling, being this $r_s \approx 111.677$ Mpc.
Conclusions. Now that the background cosmology of the Universe and its recombination history have been solved, this study can continue with its next step, studying the evolution of structure in the Universe.

Key words. background cosmology – evolution of the Universe – epoch domination – recombination

1. Introduction

The goals of this study has been to solve the evolution of the uniform background cosmology in the Universe and to solve its recombination history. For this first goal, a class/module has been implemented. This class takes the given cosmological parameters sand (with the use of different functions) returns the Hubble parameter (H), conformal time (η) and distance measures as function of the scale factor (a) and the variable x ($x = \ln a$), which will be the main time variable for this work[1].

For the second goal, a different class/module has been implemented. The final goal of this part is to compute the optical depth (τ) as a function of x and its derivatives, and the visibility function, $\tilde{g}$, and its derivatives.[2].

The fiducial cosmology used in this work is the best-fit cosmology found from fits to Planck 2018 data [Aghanim et al. (2020)]:

$$h_0 = 0.67,$$

$$T_{\text{CMB},0} = 2.7255 \text{ K},$$

$$N_{\text{eff}} = 3.046,$$

$$\Omega_{\text{B},0} = 0.05,$$

$$\Omega_{\text{CDM},0} = 0.267,$$

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2} = 0,$$

$$\Omega_{\Lambda,0} = 1 - (\Omega_{\text{B},0} + \Omega_{\text{CDM},0} + \Omega_{\text{k},0} + \Omega_{\nu,0} + \Omega_{\gamma,0}).$$

$$Y_p = 0$$

Each quantity will be explained in the following section.

2. Basic cosmology

2.1. Background Cosmology

Lets review the theory basics. The goal of this work has been computing the expansion history of the universe, and look at the uniform background densities of the various matter and energy components. Lets first define the Friedmann-Lemaître-Robertson-Walker metric (here for a flat space where $k = 0$),

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[1] Hans A. Winther, Milestone I: Background Cosmology: https://cmb.wintherscoming.no/milestone1.php

[2] Hans A. Winther, Milestone II: Recombination History: https://cmb.wintherscoming.no/milestone2.php

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \quad (1)$$

or in Cartesian coordinates

$$ds^2 = a^2(t)\left(d\eta^2 + dx^2 + dy^2 + dz^2\right), \quad (2)$$

where $a(t)$ is the scale factor, which measures the size of the universe relative to today ($a_0 = a_{\text{today}} = 1$), and η is called conformal time. One thing to note: it is called conformal time, but it is usually given in units of length for it to have the same dimension as the spatial coordinates. The conversion factor for this is the speed of light (c). In this work the conformal time is a distance (and the corresponding time is this distance divided by the speed of light). As we will be looking at phenomena that varies strongly over a wide range of time scales, we will mostly be using the logarithm of the scale factor, $x \equiv \ln a$, as the main time variable. A fifth time variable is the redshift, z, which is defined as

$$1 + z = a_0/a(t) \quad (3)$$

Einstein's General Relativity describes how the metric evolves with time, given some matter and density components. The relevant equation for this work purposes is the Friedmann equation, which may be written (when $k = 0$ is not assumed) on the following form

$$H = H_0\sqrt{\Omega_{M,0}a^{-3} + \Omega_{R,0}a^{-4} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}}, \quad (4)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter (dot denotes derivatives with respect to physical time, $\dot{} = d/dt$), and $\Omega_{B,0}$, $\Omega_{CDM,0}$, $\Omega_{\gamma,0}$, $\Omega_{\nu,0}$, and $\Omega_{\Lambda,0}$ are the present day relative densities of baryonic (ordinary) matter, dark matter, radiation, neutrinos and dark energy, respectively. A subscript 0 denotes the value at the present time. The terms $\Omega_{M,0} = \Omega_{B,0} + \Omega_{CDM,0}$ and $\Omega_{R,0} = \Omega_{\nu,0} + \Omega_{\gamma,0}$, stand for cold (non-relativistic) matter and relativistic particles (neutrinos and photons). The term $\Omega_{k,0}$ denotes curvature and acts in the Friedmann equation as if it were a normal matter fluid with equation of state $\omega = -1/3$. This term follows from the other density parameters which can be seen from taking $a = 1$ to get $\Omega_{k,0} = 1 - \Omega_{M,0} - \Omega_{R,0} - \Omega_{\Lambda,0}$. A scaled Hubble parameter, $\mathcal{H} \equiv aH$, is has also been introduced. In this work, curvature has only been implemented when it comes to solving the cosmological background and the fiducial cosmology has $\Omega_k = 0$.

The Friedmann equations also describe how each component evolve with time

$$\dot{\rho} + 3H(\rho + P) = 0 \quad (5)$$

where P is the pressure. It is useful to define the equation of state $\omega \equiv P/\rho$ (which has been considered constant for the fluids considered in this work). In terms of this, the solution reads $\rho \propto a^{-3(1+\omega)}$. For non-relativistic matter we have $\omega = 0$, for relativistic particles we have $\omega = 1/3$ and for a cosmological constant (Λ) we have $\omega = -1$. This gives us:

$$\rho_{CDM} = \rho_{CDM,0}a^{-3} \quad (6)$$

$$\rho_B = \rho_{B,0}a^{-3} \quad (7)$$

$$\rho_\gamma = \rho_{\gamma,0}a^{-4} \quad (8)$$

$$\rho_\nu = \rho_{\nu,0}a^{-4} \quad (9)$$

$$\rho_\Lambda = \rho_{\Lambda,0} \quad (10)$$

Here, quantities with subscripts 0 indicate today's values. The density parameters $\Omega_X(a) = \rho_X/\rho_c$ can be written

$$\Omega_k(a) = \frac{\Omega_{k,0}}{a^2H^2(a)/H_0^2}, \quad (11)$$

$$\Omega_{CDM}(a) = \frac{\Omega_{CDM,0}}{a^3H^2(a)/H_0^2}, \quad (12)$$

$$\Omega_B(a) = \frac{\Omega_{B,0}}{a^3H^2(a)/H_0^2}, \quad (13)$$

$$\Omega_\gamma(a) = \frac{\Omega_{\gamma,0}}{a^4H^2(a)/H_0^2}, \quad (14)$$

$$\Omega_\nu(a) = \frac{\Omega_{\nu,0}}{a^4H^2(a)/H_0^2}, \quad (15)$$

$$\Omega_\Lambda(a) = \frac{\Omega_{\Lambda,0}}{H^2(a)/H_0^2}. \quad (16)$$

Two of the density parameters above follows from the observed temperature of the CMB. we have that $\Omega_{\gamma,0}$ and $\Omega_{\nu,0}$ are given by

$$\Omega_{\gamma,0} = 2 \cdot \frac{\pi^2}{30} \frac{(k_B T_{\mathrm{CMB},0})^4}{\hbar^3 c^5} \cdot \frac{8\pi G}{3H_0}, \tag{17}$$

$$\Omega_{\nu,0} = N_{\mathrm{eff}} \cdot \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \Omega_{\gamma,0}, \tag{18}$$

where $T_{\mathrm{CMB},0} = 2.7255\mathrm{K}$ is the temperature of the CMB today and $N_{\mathrm{eff}} = 3.046$ is the effective number of massless neutrinos (slightly larger than 3 due to the fact that neutrinos had not completely decoupled when electrons and positrons annihilate and the 0.046 accounts for the extra energy pumped into the neutrinos). We define matter-radiation equality as the time when $\Omega_{\mathrm{M}} = \Omega\mathrm{R}$ and the matter-dark energy transition as the time when $\Omega_{\mathrm{M}} = \Omega\Lambda$. The onset of acceleration is the time when $\ddot{a} = 0$.

Another crucial concept for CMB computations is that of the *horizon*. This is simply the distance that light may have travelled since the Big Bang, $t = 0$. If the universe was static, this would simply have been ct, but since the universe also expands, it will be somewhat larger. Note that the horizon is a strictly increasing quantity with time, and we can therefore use it as a time variable. This is often called *conformal time*, and is denoted η.

To find a computable expression for η, note that

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{c}{a}. \tag{19}$$

The left-hand side of this equation may be written into

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}a}\frac{\mathrm{d}\eta}{\mathrm{d}a} = \frac{\mathrm{d}\eta}{\mathrm{d}a} aH, \tag{20}$$

such that

$$\frac{\mathrm{d}\eta}{\mathrm{d}a} = \frac{c}{a^2 H} = \frac{c}{a\mathcal{H}}, \tag{21}$$

and

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\mathrm{d}a}{\mathrm{d}x}\frac{\mathrm{d}\eta}{\mathrm{d}a} = \frac{c}{\mathcal{H}}. \tag{22}$$

This is a differential equation for η, that can either be solved numerically by direct integration, or by plugging the expression into a ordinary differential equation solver. The initial condition is $\eta(-\infty) = 0$. We cannot integrate from $-\infty$ so, in practice, we pick some very early time x_{start} and use the analytical approximation

$$\eta(x_{\mathrm{start}}) = \frac{c}{\mathcal{H}(x_{\mathrm{start}})}. \tag{23}$$

Some distance measures have also been needed. Considering the line-element in spherical coordinates

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right). \tag{24}$$

Photons move on 0-geodesics, $\mathrm{d}s^2 = 0$, so, considering a radially moving photon ($\mathrm{d}\theta = \mathrm{d}\phi = 0$) traveling from (t,r) to us today at $(t = t_{\mathrm{today}}, r = 0)$, we get

$$c\mathrm{d}t = \frac{a\mathrm{d}r}{\sqrt{1-kr}} \implies \int_t^{t_{\mathrm{today}}} \frac{c\mathrm{d}t}{a} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'}}. \tag{25}$$

The left hand side is called the co-moving distance and is closely related to the conformal time

$$\chi = \eta_0 - \eta \tag{26}$$

Evaluating the integral on the right, the full equation above can then be written as

$$r = \begin{cases} \chi \frac{\sin\left(\sqrt{|\Omega_{k,0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k,0}|}H_0\chi/c} & if \quad \Omega_{k,0} < 0 \\ \chi & if \quad \Omega_{k,0} = 0 \\ \chi \frac{\sinh\left(\sqrt{|\Omega_{k,0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k,0}|}H_0\chi/c} & if \quad \Omega_{k,0} > 0 \end{cases} \tag{27}$$

For a flat Universe, we simply have $r = \chi = \eta_0 - \eta$. With this in hand, all the standard distance measures we have in cosmology can be computed. Knowing an object's physical size, D, and its angular size, θ, as viewed from earth then, the angular diameter distance is defined as $d_A = D/\theta$. From the line-element it can be seen that the angular distance when moving in the θ-direction is $\mathrm{d}D = ar\mathrm{d}\theta$ so

$$d_A = \frac{D}{\theta} = ar. \tag{28}$$

Note that for a flat Universe the angular diameter distance reduces to $d_A = a\chi = a \cdot (\eta_0 - \eta)$. The last distance needed is the luminosity distance. Knowing the intrinsic luminosity of a source and measure its flux then we can define the distance to the source via $F = L/4\pi d_L^2$, which is simply

$$d_L = \frac{r}{a} = \frac{d_A}{a^2}. \tag{29}$$

It can be seen that all distance measures are given directly from the conformal time, the scale-factor and the curvature.

Finally, the relation between cosmic time t and the time-coordinate x. From $H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \rightarrow \mathrm{d}t = \frac{\mathrm{d}a}{aH}$ it can be derived that

$$t(x) = \int_0^a \frac{\mathrm{d}a}{aH} = \int_{-\infty}^x \frac{\mathrm{d}x}{H(x)}. \tag{30}$$

This can be solved as for η, evolving the ODE[3]

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{H}. \tag{31}$$

In the radiation domination era it can be expressed as $t(x) = 1/2H(x)$, so the initial condition is $t(x_{\text{start}}) = 1/2H(x_{\text{start}})$. Evaluating this at $x = 0$ (today) the age of the Universe can be computed.

2.2. Recombination History

The problem of this work is to compute the optical depth, $\tau(x)$, and the so-called visibility function, $g(x)$, which is needed for integrating the Boltzmann-Einstein equations in future works, and computing the CMB power spectrum. Physical descriptions and derivations will not be given here, but can be found in the lecture notes above, in Callin (2006), or Dodelson (2003). Here only the required definitions will be presented.

Light that travels through a medium can be absorbed by the medium. If we have a source emitting an intensity I_0 then, an observer a distance x away from the source will observe an intensity $I(x) = I_0 e^{-\tau(x)}$. The quantity τ is called the optical depth. If $\tau \ll 1$ then the medium does nothing (we say its optically thin) and if $\tau \gg 1$ then we will see nothing (the medium is optically thick). The transition between these two regimes is when $\tau \sim 1$. In cosmology the main 'absorption' is Thompson scattering of photons of free electrons. The optical depth is defined as

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a \mathrm{d}\eta', \tag{32}$$

and quantifies the probability for scattering a photon between some previous time and today (the number of scatterings of photons by electrons per unit time is $cn_e\sigma_T$). The components involved in this expression are: $n_e = n_e(\eta)$; the electron density (the number of free electrons per cubic meter) at time η, $\sigma_T = \frac{8\pi}{3}\frac{\alpha^2\hbar^2}{m_e^2c^2}$; the Thompson cross-section and the scale factor a. The transition between the Universe being optically thick and optically thin happens around recombination when most of the free electrons are captured by free protons to form neutral hydrogen. The expression for τ may also be written on an differential form, such that

$$\tau' = \frac{\mathrm{d}\tau}{\mathrm{d}x} = -\frac{cn_e\sigma_T}{H}. \tag{33}$$

This implies that existing routines can be used for solving differential equations to compute τ, if n_e can only be somehow compute at any time. And that's the difficult part. Instead of actually computing n_e, we rather focus on the fractional electron density, $X_e \equiv n_e/n_H$, where n_H is the proton density. If we will assume that all baryons are protons (no helium or heavier elements), then

$$n_H = n_\mathrm{B} \approx \frac{\rho_\mathrm{B}}{m_H} = \frac{\Omega_{\mathrm{B},0}\rho_{c0}}{m_H a^3}, \tag{34}$$

where $\rho_{c0} \equiv \frac{3H_0^2}{8\pi G}$ is the critical density of the universe today. If we do include Helium then only a (mass) fraction $1 - Y_p$ of the baryons are hydrogen so

$$n_H = (1 - Y_p)n_\mathrm{B} = (1 - Y_p)\frac{\Omega_{\mathrm{B},0}\rho_{c0}}{m_H a^3}, \tag{35}$$

where m_H is the hydrogen mass and Y_p is the Helium fraction and is a new cosmological parameter (BBN[4] gives us $Y_p \approx 0.24$ and observational constraints are consistent with this - the fiducial value in this work is $Y_p = 0$).

Now, there are two different equations available for X_e as a function of temperature and density, namely the so-called Saha and Peebles' equations (Equations 36 and 37 respectively). If we ignore Helium then the first one of these reads

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_\mathrm{B}}\left(\frac{m_e T_\mathrm{B}}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_\mathrm{B}}, \tag{36}$$

where T_B is the baryon temperature of the universe, and $\epsilon = 13.6$ eV is the ionization energy of hydrogen (the energy a photon needs in order to rip an electron away from a proton). In principle, one would have to solve separately for both T_B and the photon temperature, T_γ, but in practice it is an excellent approximation to set these equal. It can therefore be assumed that $T_\mathrm{B} = T_\gamma = T_{\mathrm{CMB},0}/a$. With the above information, Equation 36 (Saha's equation) reduces to a standard second-order equation in X_e, and can be solved directly using the normal formula, $y = (-b \pm \sqrt{b^2 - 4ac})/2a$.

Saha's equation (Equation 36 is an excellent approximation when $X_e \approx 1$. When X_e is noticeably smaller than one, better approximations are required, and one such approximation is the Peebles' equation (Equation 37,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1 - X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right], \tag{37}$$

where H is the Hubble parameter and

$$C_r(T_b) = \frac{\Lambda_{2s\rightarrow1s} + \Lambda_\alpha}{\Lambda_{2s\rightarrow1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \text{ (dimensionless)}, \tag{38}$$

[3] Ordinary Differential Equation.

[4] BigBang Nucleosynthesis

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1},\ \text{(dimension 1/s)}, \tag{39}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}},\ \text{(dimension 1/s)}, \tag{40}$$

$$n_{1s} = (1-X_e)n_H,\ \text{(dimension 1/m}^3\text{)}, \tag{41}$$

$$n_H = (1-Y_p)\frac{3H_0^2\Omega_{b0}}{8\pi G m_H a^3},\ \text{(dimension 1/m}^3\text{)}, \tag{42}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b},\ \text{(dimension 1/s)}, \tag{43}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b},\ \text{(dimension 1/s)}, \tag{44}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b),\ \text{(dimension m}^3\text{/s)}, \tag{45}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b),\ \text{(dimensionless)}, \tag{46}$$

where $\alpha \simeq \frac{1}{137.0359992}$ in the expression for $\alpha^{(2)}$ is the fine-structure constant. This looks a bit scary, but it's not too bad, really. First of all, the various constants are simply physical constants describing simple atomic physics; Peebles' equation takes into account the transition rates between the ground-state ($1s$) and the first excited state ($2s$) of the hydrogen atom. For even higher-accurate work, many more states should be included, and also other atoms, most notably the helium atom. But here the Peebles' equation is enough. Second, the Peebles' equation is simply yet another linear first-order differential equation.

Next, it is necessary to decide when to switch from Saha's equation to Peebles' equation. For simplicity, lets simply say that when $X_e > 0.99$, Saha's has to be used and when $X_e < 0.99$, Peebles' has to be used.

When n_e has been computed, the next step is to spline it, such that it can be evaluated at arbitrary values of x.

Now it is time to compute the optical depth, by solving Equation 33 mentioned above, with $\tau(x_{\text{today}}) = 0$[5] as the initial condition.

[5] Remember that the optical depth at our place in the universe today is precisely zero.

The final item to compute is the so-called visibility function,

$$\tilde{g}(x) = -\tau'(x)e^{-\tau(x)}, \tag{47}$$

which has the property that $\int_{-\infty}^{0}\tilde{g}(x)\mathrm{d}x = 1$. This function is therefore a true probability distribution, and describes the probability density for a given photon to last time having scattered at time x. As it will be seen, this function is sharply peaked around $x = -7$, corresponding to redshifts of $z \sim 1100$. The fact that this is so sharply peaked is the reason why we often refer to the recombination period as the surface of last scattering: This process happened during a very thin shell around us, at a redshift of $z \sim 1100$. The first and second order derivatives of $\tilde{g}$ will be also required, and so these must also be properly splined, just as τ was.

The final thing to compute is the so-called sound-horizon at decoupling (the total distance a sound-wave in the photon-baryon plasma can propagate from the Big Bang and until photons decouple). This quantity is not so relevant at the moment, but it will be important later in this project, since this length scale is imprinted in the CMB and the distribution of dark matter and galaxies in our Universe, so it is important to know how big it is. The sound-speed of the coupled photon-baryon plasma is slightly lower than the sound-speed of photons ($c/\sqrt{3}$) and given by $c_s = c\sqrt{\frac{R}{3(1+R)}}$, where $R = \frac{4\Omega_\gamma}{3\Omega_B a}$, so the sound-horizon is

$$\frac{ds(x)}{dx} = \frac{c_s}{\mathcal{H}} \quad \text{with} \quad s(x_{\text{ini}}) = \frac{c_s(x_{\text{ini}})}{\mathcal{H}(x_{\text{ini}})}. \tag{48}$$

Evaluating it at $x = x_{\text{decoupling}}$ gives us the sound-horizon at decoupling $r_s \equiv s(x_{\text{decoupling}})$.

3. Implementation method

3.1. BackgroundCosmology

A class that takes in all the cosmological parameters (h, $\Omega_{\mathrm{B},0}$, $\Omega_{\mathrm{CDM},0}$,$\Omega_{k,0}$,$T_{\mathrm{CMB},0}$,N_{eff}), computes $\Omega_{\gamma,0}$, $\Omega_{\nu,0}$ and $\Omega_{\Lambda,0}$ from these and stores them has been implemented in a C++ code[6]. Functions that are able to get the cosmological parameters, the Hubble function and $\mathcal{H} = aH$ ("Hp") plus the first two derivatives (these have been computed analytically) together with the different distance measures (co-moving, luminosity and angular diameter distance) have been made. Once this was done, $\eta(x)$ was computed, spline the result was splined and a function that returns this function was made [Callin (2006)].

Then, equality times and acceleration time plus the age of the Universe were computed. This was done in the same way as for the conformal time by solving an ODE and making a spline of $t(x)$ (useful for computing the time at any x later on if needed). Evaluating the spline at $x = 0$ gives the time of the Universe in seconds (if using SI units), but it was converted to a more sensible unit, like gigayears ($109 \cdot 365 \cdot 24 \cdot 60 \cdot 60$ seconds), when presenting the results.

[6] All the used code is fully available in my public GitHub repository.

Once the distance measures were working, the parameter fits to supernova data [Betoule et al. (2014)] was made. First the results of the luminosity distance were plotted, using the fiducial cosmological parameters from Section 1, together with the data points from supernova observations. The fitting routine is a simple Metropolis Monte Carlo Markov Chain (MCMC) sampler. After everything has been implemented, the code was used to get constraints on our cosmological parameters by comparing to data.

The used data was a set of supernova with associated redshift z_i, luminosity distance $d_L^{\text{obs}}(z_i)$ and associated measurement errors σ_i. Under the assumption that the measurements are Gaussian distributed and uncorrelated between different redshifts, the Likelihood function (telling how well the data fit the theory), is given by $\mathcal{L} \propto e^{-\chi^2/2}$ where the chi-squared function is

$$\chi^2(h, \Omega_{\text{M},0}, \Omega_{k,0}) = \sum_{i=1}^{N} \frac{\left[d_L(z_i, h, \Omega_{\text{M},0}, \Omega_{k,0}) - d_L^{\text{obs}}(z_i)\right]^2}{\sigma_i^2}. \tag{49}$$

A low value of χ^2 means a good fit (high likelihood for the choice of parameters). The goal is to basically check all possible values of the parameters to find the best-fitting model and the range of parameters around it which are in agreement with the observed values. One can do this brute-force, but this step is in practice most commonly done by performing a so-called MCMC to randomly sample from the likelihood. The result from performing this step is a chain of values: parameter-points and likelihood values. The set of parameters with the lowest likelihood is the best-fit model. It can be checked if this is really a good fit by comparing the χ^2 to the number of data-points (here $N = 31$). A good fit has $\chi^2/N \sim 1$ (a much higher value denotes a bad fit). When the best-fit has been found, 1σ (68.4%) confidence region can be found by looking at all values that satisfies $\chi^2 - \chi^2_{\text{min}} < 3.53$. It is exactly the same for the 2σ (95.45%) confidence region by looking at all values that satisfies $\chi^2 - \chi^2_{\text{min}} < 8.04$[7].

3.2. RecombinationHistory

A class that takes in recombination parameters and a BackgroundCosmology object, has been implemented in a C++ code[8]. This has been used to solve the recombination history of the Universe.

The code starts by solving for the electron density n_e (X_e) by solving the Saha and Peebles equations (Equations 36 and 37 respectively). The former is easy: it's just a quadratic equation for X_e so it is easy to solve. However, the analytical solution at early times will be on the form "huge" ("huge" which should give approximately 1) so it has been needed to use the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$ when $|x| \ll 1$. The Saha equation is only valid as long as $X_e \approx 1$ so the code switches to the Peebles equation when $X_e < 0.99$. The result has been splined for X_e and n_e as this was needed when computing τ.

[7] Robert Reid, Chi-squared distribution table with sigma values: http://www.reid.ai/2012/09/chi-squared-distribution-table-with.html
[8] All the used code is fully available in my public GitHub repository.

Once n_e has been computed[9], the ODE for τ (Equation 33) has been integrated from $x = 0$ and going back in time. The integration constant has been fixed by ensuring $\tau(x = 0) = 0$.

4. Results

4.1. Solving the background cosmology of the Universe

The evolution of the relative energy densities of the Universe is presented in Figure 1. Here, each component is plotted, together with the combined contribution from all non-relativistic (baryonic/ordinary and cold dark) matter and all the relativistic particles (photons and neutrinos). It can clearly be seen that there are three different regimes, where each component (relativistic particles/radiation, marked with orange, non-relativistic matter, marked with blue, and dark energy/Λ, marked with green). This color coding has been used to better visualize these regimes in all the produced plots.

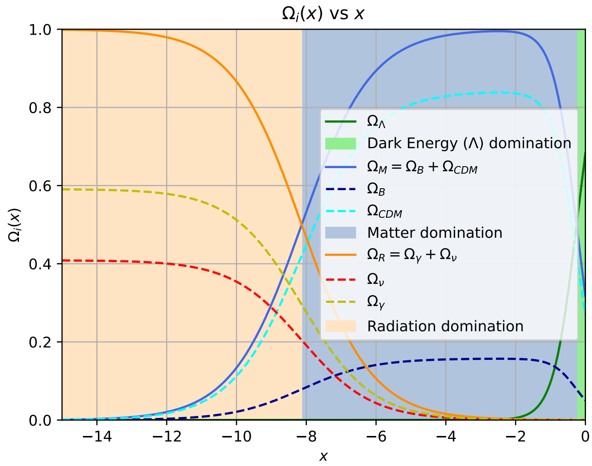


Fig. 1. Plot showing relative density parameters against the natural logarithm of the scale factor ($x = \ln a$). The early Universe was dominated by radiation, marked in orange. Following is the matter dominated era is marked in blue, where the combined density of the dark matter and baryon components is indicated by the blue line. Lastly, the present era is dominated by dark energy, marked in green.

From the splines created to solve the ODEs and compute both the cosmological time (Figure 2) and the conformal time (Figure 3) can be seen how they evolve in the different epoch of domination. From this data, one can compute both the age of the Universe and the conformal time today, being these $t_{\text{Universe}} = t(\text{today}) = t(0) \approx 13.86$ Gyr and $\eta(0)/c \approx 46.32$ Gyr, respectively. It also possible to compute the times when there was a radiation-matter and matter-dark energy equalities, along with the time at which the Universe started to accelerate, being this times summarized in Table 3.

As can be seen in Figure 2, cosmological time grows as the scale factor tends to 1 (if $a = e^x$ and $x \to 0 \Rightarrow a \to 1$). It can be inferred form the plot that cosmological time also grows exponentially, concluding that times is just another dimension (like the three spatial dimensions).

[9] Because n_e varies over many, many orders of magnitude, it is useful to spline $\ln(n_e)$ rather than n_e itself; this function varies much more slowly with x, and is therefore easier to interpolate.

	x	z	t [Gyr]
Radiation-matter	-8.14	3421.55	$5.05 \cdot 10^{-5}$
Matter-dark energy	-0.26	0.29	10.39
Accelerated expansion	-0.44	0.55	8.29

Table 1. Times at which there was radiation-matter equality, being this time expressed as the natural logarithm of the scale factor (x), redshift (z) and the cosmological time (t). The times of at which there was matter-dark energy equality and the time at which the Universe began its accelerated expansion are also shown.

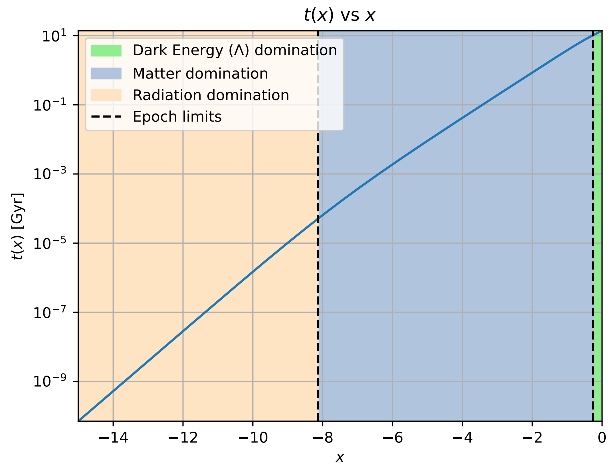


Fig. 2. Cosmological time evolution as a function of the natural logarithm of the scale factor.

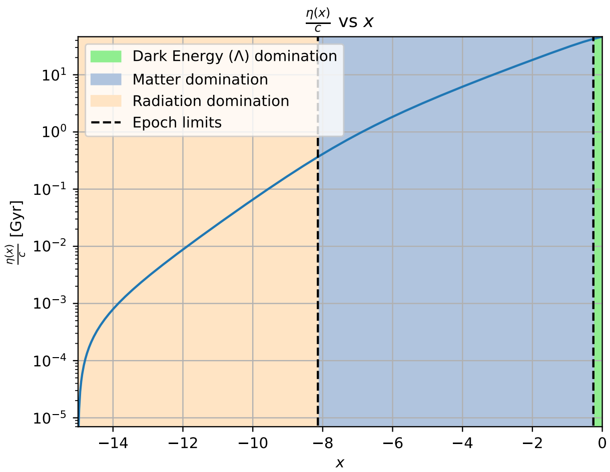
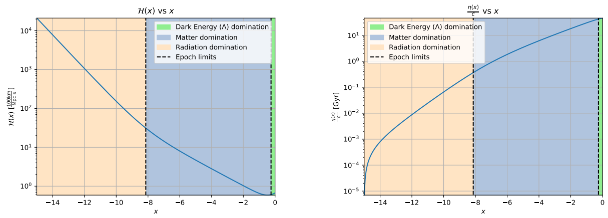


Fig. 3. Cosmological time evolution as a function of the natural logarithm of the scale factor.

The main result from this work is shown in Figure 4. Here are the conformal Hubble parameter (4a) is shown against x. It can be seen how this parameter decreases steeply in the radiation domination epoch, before getting into the matter domination epoch with a slightly gentler slope. The conformal time is, basically, the inverse of the conformal Hubble parameter. While the conformal Hubble parameter has an even steeper decrease in the radiation domination epoch going over to a less steep decrease in the matter domination epoch.

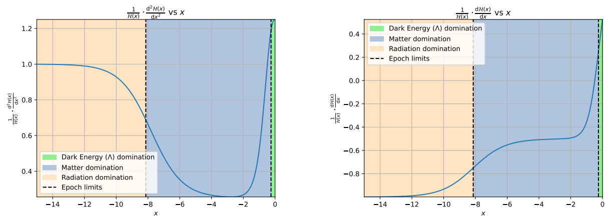
Note that the Hubble prime starts to increase again when getting into the last regime, dominated by dark energy, which corresponds to an accelerated expansion like we observe today.



(a) Conformal Hubble parameter. (b) Conformal time.

Fig. 4. Plots showing the evolution of the conformal Hubble parameter (4a) and the conformal time (4b).

Some ways of testing the code is studying the evolution of the (first, 5b, and second, 5a) derivatives of the conformal Hubble parameter with respect to x and the conformal Hubble parameter itself, as can be seen in Figure 5.



(a) $\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2}/\mathcal{H}$ ratio. (b) $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x}/\mathcal{H}$ ratio.

Fig. 5. 5a: Ratio between the first derivative of $\mathcal{H}$ with respect to x and $\mathcal{H}$. 5b: Ratio between the first derivative of $\mathcal{H}$ with respect to x and $\mathcal{H}$.

Other check one can do is plotting the conformal time times the conformal Hubble parameter, which in the radiation domination epoch tends to 1. As can be seen from Figure 6, the further to the left, the closest to 1, so it can be assumed that when the curve is prolonged far enough it will asymptotically tend to 1.

Once it is clear that the code works (except for that mistake when computing the time at which the accelerated expansion of the Universe starts), the next step to check how good is the code is to try to get the parameters that fit better to some observations. For this, the data from Betoule et al. (2014) has been studied and fitted with a MCMC. The results of the fit are shown in Figure 7.

In Figure 8, the 1σ and 2σ deviation from the best fit parameters can be seen, showing in a dashed black line the set of parameters corresponding to a flat Universe. This deviations have been computed by choosing the parameters with a χ^2 greater by 3.53 (for the 1σ deviation) and by 8.04 (for the 2σ deviation) than the minimum value of χ^2 (the one of the best fitting parameters). A summary of values of the best fitting parameters can be found in Table 2.

In Figure 9, a collection of histograms, showing the Gaussian distribution, of the generated parameters for the fitting

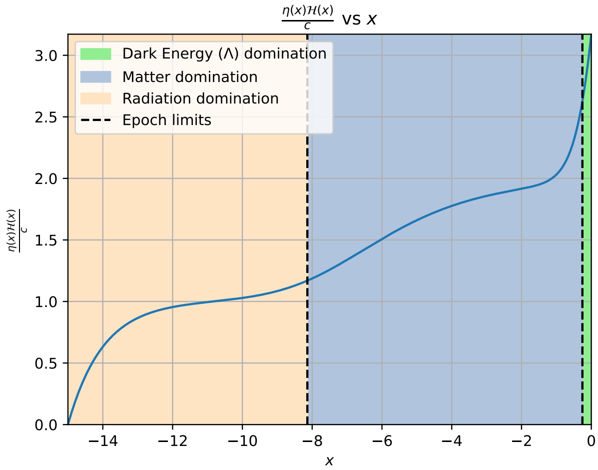


Fig. 6. Plot showing the relation between the conformal time and the conformal Hubble factor, showing that their product tends to 1 during the radiation domination epoch. It can also be seen that it is always of the order of 10^0.

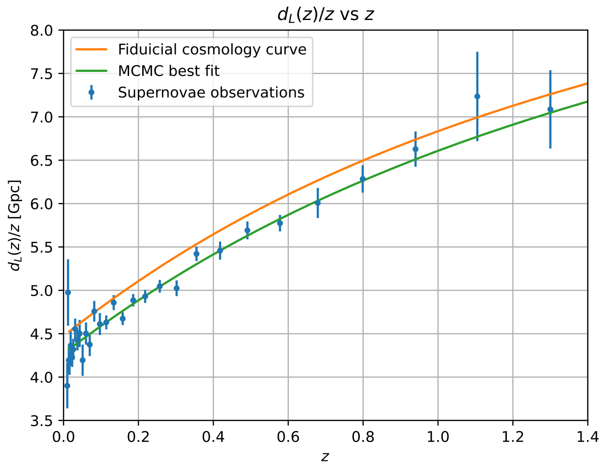


Fig. 7. Plot showing the observational data from Betoule et al. (2014) and the fit computed with the fiducial cosmology and with the MCMC fit, selecting the best parameters by choosing those corresponding to the lowest χ^2.

parameters, along with a dashed black line showing the best fitting parameters.

χ^2	Ω_M	Ω_k	Ω_Λ	H_0
29.33	0.26	0.08	0.66	70.14

Table 2. Best fitting parameters from the supernova fitting via MCMC.

One could be surprised to see that some parameters are negative or greater than one, but if the sum of all the Ωs were to be computed, it will give a result of exactly 1. All Ωs should be greater or equal than 0 and smaller or equal to 1, except for $\Omega_k < 0$, which is perfectly a physical result since $\Omega_k \propto k$, and k can be negative is the Universe is openly curved (hyperbolic). From the histograms it can be expected to live in a Universe with around 26% matter, around 66% dark energy (Λ) and a flat (or slightly closely curved/spherical) Universe ($\Omega_k = 0.08$).

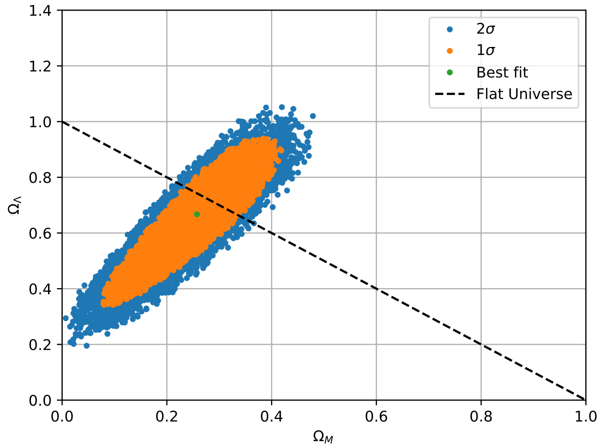
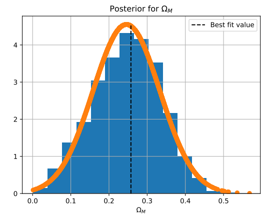
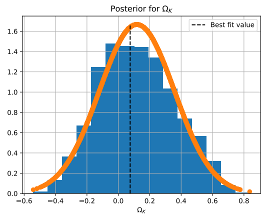


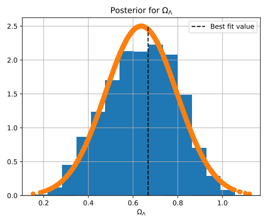
Fig. 8. Plot showing 1σ and 2σ deviation from the best fit parameters, showing in a dashed black line the set of parameters corresponding to a flat Universe.



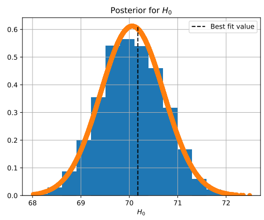
(a) Ω_M histogram.



(b) Ω_k histogram.



(c) Ω_Λ histogram.



(d) H_0 histogram.

Fig. 9. Histograms of the generated parameters: Ω_M (9a), Ω_k (9b), Ω_Λ (9c) and H_0 (9d) The best fitting parameters are shown in Table 2.

As can bee seen in Figure 9c, the fit of the data do not allow a negative cosmological constant, which implies that most of our Universe is unknown to us and further research is needed. Focusing now in Figure 9d, one can see that the best fitting parameter is slightly bigger than the fiducial H_0 (H_0 = 67 km/s/Mpc). This discrepancy is what is called the Hubble tension, whose cause is not known. This discrepancy arises when comparing the computed H_0 with "early" and "late" Universe observations.

4.2. Solving the recombination history of the Universe

First of all, the different times for decoupling, last scattering, half-way recombination (using only Saha's equation and using both Saha's and Peebles') and the recombination time have been computed, being these times summarized in Table 3.

	x	z	t [Myr]
Decoupling	-7.370	1586.94	0.196
Last scattering	-6.988	1082.30	0.376
Half-way rec.	-7.162	1288.29	0.280
Half-way rec (Saha)	-7.230	1379.48	0.250
Recombination	-6.986	1079.91	0.378

Table 3. Summary of all the computed times in both redshift (z) and cosmological time (t).

The evolution of the free electron fraction is shown in Figure 10. Here it can be seen how the free electron fraction stay constant at 1 in the early Universe, before recombination happens. Then, at $x = x_{\text{recombination}} \approx -6.986$ we enter the Peebles' regime, and shortly after it can be seen that the free electron fraction start to evolve quite rapidly. At this point, the temperature of the Universe has dropped significantly, so free electrons and protons can start to form atoms of neutral hydrogen. As can be inferred from Table 3, at $x = -7.163$ we are half way through recombination (using the complete solution)[10]. Equation 36 has a exponential drop, which can be recognized in Figure 10, the $X_{e,\text{Saha}}$ solution drops off to zero with decreasing temperature. The full solution does not fall off exponentially, and instead flattens out to a stable value. This flattening can be described by two phases. First the free electrons decouple from the rest of the Universe as their interaction rate with the free protons drops below the expansion rate of the Universe. Second the free electrons freeze out, where the free electron fraction stops evolving and becomes, approximately, constant.

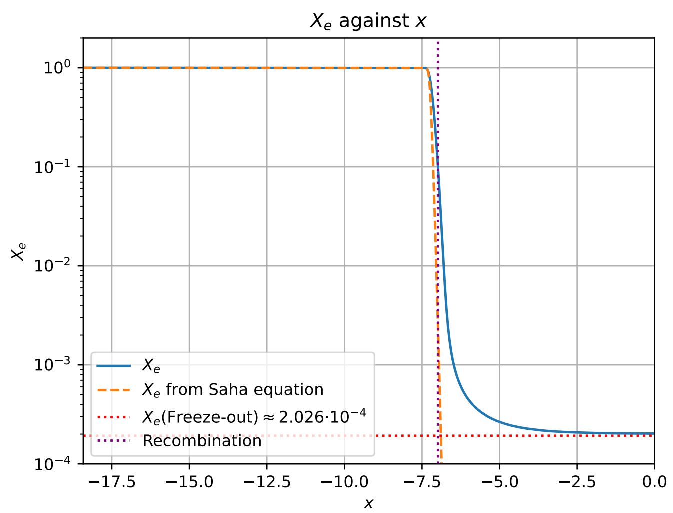


Fig. 10. Plot showing the free electron fraction X_e. The solution obtained using only Saha's equation, which drops off exponentially to zero once the solution deviates from equilibrium, has been included.

In Figure 11, the optical depth and its derivatives have been plotted. In the dense primordial Universe, the mean free path of photons was really low due to Thompson's scattering on the abundant free electrons. This can be seen in the high optical depth, and high free electron fraction. Where $\tau \gg 1$, the Universe was opaque. During radiation domination era, Universe expands proportionally to $\sqrt{t}$, so the optical "thickness" of the Universe is only decreasing along with the expansion. When recombination starts and X_e drops, it follows with a drop in the optical depth and its derivatives. This is consistent with less free electrons, so less Thompson scattering and longer mean free path. Now the optical "thickness" of the Universe has changed due to the change in the constituents of the Universe, not only the expansion.

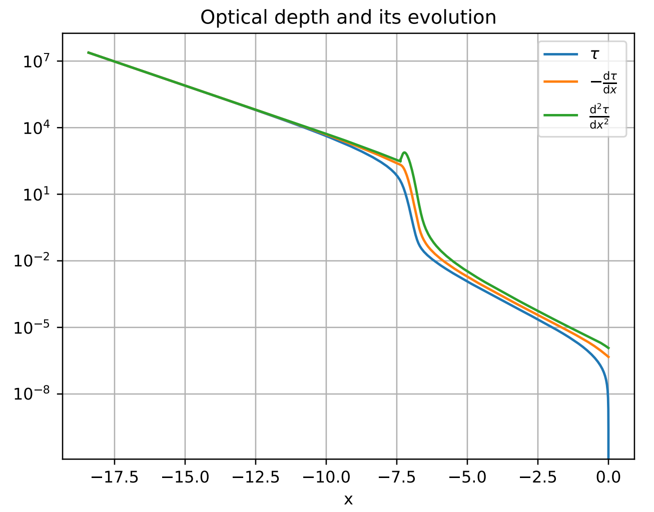


Fig. 11. Plot showing the optical depth (τ) and its two first derivatives.

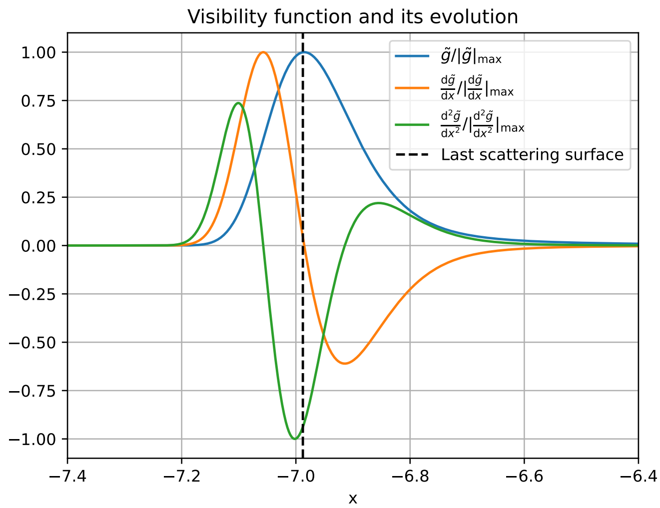


Fig. 12. Plot showing the visibility function ($\tilde{g}$) and its two first derivatives.

Figure 12 shows the visibility function and its derivatives. The visibility function and its derivatives have been scaled to fit into the same plot, with the first and second derivative having much larger values than the visibility function it self. The visibility function describes the probability that a observed photon was last scattered at time x, going backwards in time from today. In Figure 12 it can be seen that this probability is

[10] Note that if only the Saha approximation is used, we will be half way through recombination at $x = -7.230$.

more or less zero both before and after recombination. This can be understood as a dense and opaque Universe before recombination, where photons scattered constantly, and thus the probability of a photon observed today scattered last time before recombination is almost zero. After recombination the Universe is transparent, huge and "empty". Here Thompson's scattering rate is lower than the expansion's, as $\tau' < 1$, and so the chance for scattering is really low.

The sound horizon at decoupling has also been computed, being this $r_s \approx 111.677$ Mpc.

5. Conclusions

Background cosmology of the Universe has been solved and each epoch has been determined, identifying in each of them the dominating substance. The age of the Universe has also been computed, being this $t_{\,\mathrm{Universe}} \approx 13.86$ Gyr.

Recombination history of the Universe has been solved, computing different important times during recombination, summarized in Table 3. The sound horizon at decoupling has also been computed, being this $r_s \approx 111.677$ Mpc.

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