

Computing the CMB power spectrum

I. Solving the background cosmology

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ABSTRACT

Context. The goal of this whole study is to be able to predict the CMB (and matter) fluctuations (described by the so-called power-spectrum) from first principles and learn about all the different physical processes that goes on to be able to explain the results.
Aims. The goal in this work is to make a class/module that takes in the cosmological parameters and has functions for getting the Hubble parameter, conformal time and distance measures as function of scale factor and the variable $x = \ln a$.
Methods. A code has been developed, using the template given by Hans A. Winther. In this code, several functions have been implemented to solve the background cosmology of the Universe and plot the results, identifying each epoch domination (either relativistic particles, non-relativistic matter or dark energy).
Results. Background cosmology of the Universe has been solved and each epoch has been determined, identifying in each of them the dominating substance. The age of the Universe has also been computed, being this $t_{\text{Universe}} \approx 13.86$ Gyr.
Conclusions. Now that the background cosmology of the Universe has been solved, this study can continue with its next step, studying the recombination history of the Universe.

Key words. background cosmology – evolution of the Universe – epoch domination

1. Introduction

The goal of this study has been to solve the evolution of the uniform background cosmology in the Universe. For this, a class/module has been implemented. This class takes the given cosmological parameters sand (with the use of different functions) returns the Hubble parameter (H), conformal time (η) and distance measures as function of the scale factor (a) and the variable x ($x = \ln a$), which will be the main time variable for this work[1].

The fiducial cosmology used in this work is the best-fit cosmology found from fits to Planck 2018 data [Aghanim et al. (2020)]:

$$h_0 = 0.67,$$
$$T_{\text{CMB},0} = 2.7255 \text{ K},$$
$$N_{\text{eff}} = 3.046,$$
$$\Omega_{\text{B},0} = 0.05,$$
$$\Omega_{\text{CDM},0} = 0.267,$$
$$\Omega_{k,0} = -\frac{kc^2}{H_0^2} = 0,$$
$$\Omega_{\Lambda,0} = 1 - (\Omega_{\text{B},0} + \Omega_{\text{CDM},0} + \Omega_{\text{k},0} + \Omega_{\nu,0} + \Omega_{\gamma,0}).$$

Each quantity will be explained in the following section.

2. Basic cosmology

Lets review the theory basics. The goal of this work has been computing the expansion history of the universe, and look at the uniform background densities of the various matter and energy components. Lets first define the Friedmann-Lemaître-Robertson-Walker metric (here for a flat space where $k = 0$),

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right], \tag{1}$$

or in Cartesian coordinates

$$\mathrm{d}s^2 = a^2(t)\left(\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{2}$$

where $a(t)$ is the scale factor, which measures the size of the universe relative to today ($a_0 = a_{\text{today}} = 1$), and η is called conformal time. One thing to note: it is called conformal time, but it is usually given in units of length for it to have the same dimension as the spatial coordinates. The conversion factor for this is the speed of light (c). In this work the conformal time is a distance (and the corresponding time is this distance divided by the speed of light). As we will be looking at phenomena that varies strongly over a wide range of time scales, we will mostly be using the logarithm of the scale factor, $x \equiv \ln a$, as the main time

⋆ GitHub: https://github.com/ivanvillegas7/CMB_power_spectrum
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[1] Hans A. Winther, Milestone I: Background Cosmology: https://cmb.wintherscoming.no/milestone1.php

variable. A fifth time variable is the redshift, z, which is defined as

$$1 + z = a_0/a(t) \tag{3}$$

Einstein's General Relativity describes how the metric evolves with time, given some matter and density components. The relevant equation for this work purposes is the Friedmann equation, which may be written (when $k = 0$ is not assumed) on the following form

$$H = H_0\sqrt{\Omega_{\mathrm{M},0}a^{-3} + \Omega_{\mathrm{R},0}a^{-4} + \Omega_{\mathrm{k},0}a^{-2} + \Omega_{\Lambda,0}}, \tag{4}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter (dot denotes derivatives with respect to physical time, $\dot{} = \mathrm{d}/\mathrm{d}t$), and $\Omega_{\mathrm{B},0}$, $\Omega_{\mathrm{CDM},0}$, $\Omega_{\gamma,0}$, $\Omega_{\nu,0}$, and $\Omega_{\Lambda,0}$ are the present day relative densities of baryonic (ordinary) matter, dark matter, radiation, neutrinos and dark energy, respectively. A subscript 0 denotes the value at the present time. The terms $\Omega_{\mathrm{M},0} = \Omega_{\mathrm{B},0} + \Omega_{\mathrm{CDM},0}$ and $\Omega_{\mathrm{R},0} = \Omega_{\nu,0} + \Omega_{\gamma,0}$, stand for cold (non-relativistic) matter and relativistic particles (neutrinos and photons). The term $\Omega_{k,0}$ denotes curvature and acts in the Friedmann equation as if it were a normal matter fluid with equation of state $\omega = -1/3$. This term follows from the other density parameters which can be seen from taking $a = 1$ to get $\Omega_{k,0} = 1 - \Omega_{\mathrm{M},0} - \Omega_{\mathrm{R},0} - \Omega_{\Lambda,0}$. A scaled Hubble parameter, $\mathcal{H} \equiv aH$, is has also been introduced. In this work, curvature has only been implemented when it comes to solving the cosmological background and the fiducial cosmology has $\Omega_k = 0$.

The Friedmann equations also describe how each component evolve with time

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{5}$$

where P is the pressure. It is useful to define the equation of state $\omega \equiv P/\rho$ (which has been considered constant for the fluids considered in this work). In terms of this, the solution reads $\rho \propto a^{-3(1+\omega)}$. For non-relativistic matter we have $\omega = 0$, for relativistic particles we have $\omega = 1/3$ and for a cosmological constant (Λ) we have $\omega = -1$. This gives us:

$$\rho_{\mathrm{CDM}} = \rho_{\mathrm{CDM},0}a^{-3} \tag{6}$$

$$\rho_{\mathrm{B}} = \rho_{\mathrm{B},0}a^{-3} \tag{7}$$

$$\rho_{\gamma} = \rho_{\gamma,0}a^{-4} \tag{8}$$

$$\rho_{\nu} = \rho_{\nu,0}a^{-4} \tag{9}$$

$$\rho_{\Lambda} = \rho_{\Lambda,0} \tag{10}$$

Here, quantities with subscripts 0 indicate today's values. The density parameters $\Omega_X(a) = \rho_X/\rho_c$ can be written

$$\Omega_k(a) = \frac{\Omega_{k,0}}{a^2H^2(a)/H_0^2}, \tag{11}$$

$$\Omega_{\mathrm{CDM}}(a) = \frac{\Omega_{\mathrm{CDM},0}}{a^3H^2(a)/H_0^2}, \tag{12}$$

$$\Omega_{\mathrm{B}}(a) = \frac{\Omega_{\mathrm{B},0}}{a^3H^2(a)/H_0^2}, \tag{13}$$

$$\Omega_{\gamma}(a) = \frac{\Omega_{\gamma,0}}{a^4H^2(a)/H_0^2}, \tag{14}$$

$$\Omega_{\nu}(a) = \frac{\Omega_{\nu,0}}{a^4H^2(a)/H_0^2}, \tag{15}$$

$$\Omega_{\Lambda}(a) = \frac{\Omega_{\Lambda,0}}{H^2(a)/H_0^2}. \tag{16}$$

Two of the density parameters above follows from the observed temperature of the CMB. we have that $\Omega_{\gamma,0}$ and $\Omega_{\nu,0}$ are given by

$$\Omega_{\gamma,0} = 2\cdot\frac{\pi^2}{30}\frac{(k_BT_{\mathrm{CMB},0})^4}{\hbar^3c^5}\cdot\frac{8\pi G}{3H_0}, \tag{17}$$

$$\Omega_{\nu,0} = N_{\mathrm{eff}}\cdot\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\Omega_{\gamma,0}, \tag{18}$$

where $T_{\mathrm{CMB},0} = 2.7255\mathrm{K}$ is the temperature of the CMB today and $N_{\mathrm{eff}} = 3.046$ is the effective number of massless neutrinos (slightly larger than 3 due to the fact that neutrinos had not completely decoupled when electrons and positrons annihilate and the 0.046 accounts for the extra energy pumped into the neutrinos). We define matter-radiation equality as the time when $\Omega_{\mathrm{M}} = \Omega\mathrm{R}$ and the matter-dark energy transition as the time when $\Omega_{\mathrm{M}} = \Omega\Lambda$. The onset of acceleration is the time when $\ddot{a} = 0$.

Another crucial concept for CMB computations is that of the *horizon*. This is simply the distance that light may have travelled since the Big Bang, $t = 0$. If the universe was static, this would simply have been ct, but since the universe also expands, it will be somewhat larger. Note that the horizon is a strictly increasing quantity with time, and we can therefore use it as a time variable. This is often called *conformal time*, and is denoted η.

To find a computable expression for η, note that

$$\frac{d\eta}{dt} = \frac{c}{a}. \tag{19}$$

The left-hand side of this equation may be written into

$$\frac{d\eta}{dt} = \frac{d\eta}{da}\frac{d\eta}{da} = \frac{d\eta}{da}aH, \tag{20}$$

such that

$$\frac{d\eta}{da} = \frac{c}{a^2 H} = \frac{c}{a\mathcal{H}}, \tag{21}$$

and

$$\frac{d\eta}{dx} = \frac{da}{dx}\frac{d\eta}{da} = \frac{c}{\mathcal{H}}. \tag{22}$$

This is a differential equation for η, that can either be solved numerically by direct integration, or by plugging the expression into a ordinary differential equation solver. The initial condition is $\eta(-\infty) = 0$. We cannot integrate from $-\infty$ so, in practice, we pick some very early time x_{start} and use the analytical approximation

$$\eta(x_{\text{start}}) = \frac{c}{\mathcal{H}(x_{\text{start}})}. \tag{23}$$

Some distance measures have also been needed. Considering the line-element in spherical coordinates

$$ds^2 = -c^2dt^2 + a^2\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{24}$$

Photons move on 0-geodesics, $ds^2 = 0$, so, considering a radially moving photon ($d\theta = d\phi = 0$) traveling from (t,r) to us today at $(t = t_{\text{today}}, r = 0)$, we get

$$cdt = \frac{adr}{\sqrt{1-kr}} \Longrightarrow \int_t^{t_{\text{today}}} \frac{cdt}{a} = \int_0^r \frac{dr'}{\sqrt{1-kr'}}. \tag{25}$$

The left hand side is called the co-moving distance and is closely related to the conformal time

$$\chi = \eta_0 - \eta \tag{26}$$

Evaluating the integral on the right, the full equation above can then be written as

$$r = \begin{cases} \chi\dfrac{\sin\left(\sqrt{|\Omega_{k,0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k,0}|}H_0\chi/c} & if \quad \Omega_{k,0} < 0 \\ \chi & if \quad \Omega_{k,0} = 0 \\ \chi\dfrac{\sinh\left(\sqrt{|\Omega_{k,0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k,0}|}H_0\chi/c} & if \quad \Omega_{k,0} > 0 \end{cases} \tag{27}$$

For a flat Universe, we simply have $r = \chi = \eta_0 - \eta$. With this in hand, all the standard distance measures we have in cosmology can be computed. Knowing an object's physical size, D, and its angular size, θ, as viewed from earth then, the angular diameter distance is defined as $d_A = D/\theta$. From the line-element it can be seen that the angular distance when moving in the θ-direction is $dD = ard\theta$ so

$$d_A = \frac{D}{\theta} = ar. \tag{28}$$

Note that for a flat Universe the angular diameter distance reduces to $d_A = a\chi = a \cdot (\eta_0 - \eta)$. The last distance needed is the luminosity distance. Knowing the intrinsic luminosity of a source and measure its flux then we can define the distance to the source via $F = L/4\pi d_L^2$, which is simply

$$d_L = \frac{r}{a} = \frac{d_A}{a^2}. \tag{29}$$

It can be seen that all distance measures are given directly from the conformal time, the scale-factor and the curvature.

Finally, the relation between cosmic time t and the time-coordinate x. From $H = \frac{1}{a}\frac{da}{dt} \rightarrow dt = \frac{da}{aH}$ it can be derived that

$$t(x) = \int_0^a \frac{da}{aH} = \int_{-\infty}^x \frac{dx}{H(x)}. \tag{30}$$

This can be solved as for η, evolving the ODE[2]

$$\frac{dt}{dx} = \frac{1}{H}. \tag{31}$$

In the radiation domination era it can be expressed as $t(x) = 1/2H(x)$, so the initial condition is $t(x_{\text{start}}) = 1/2H(x_{\text{start}})$. Evaluating this at $x = 0$ (today) the age of the Universe can be computed.

[2] Ordinary Differential Equation.

3. Implementation method

A class that takes in all the cosmological parameters (h, $\Omega_{B,0}$, $\Omega_{CDM,0}$,$\Omega_{k,0}$,$T_{CMB,0}$,N_{eff}), computes $\Omega_{\gamma,0}$, $\Omega_{\nu,0}$ and $\Omega_{\Lambda,0}$ from these and stores them has been implemented in a C++ code[3]. Functions that are able to get the cosmological parameters, the Hubble function and $\mathcal{H} = aH$ ("Hp") plus the first two derivatives (these have been computed analytically) together with the different distance measures (co-moving, luminosity and angular diameter distance) have been made. Once this was done, $\eta(x)$ was computed, spline the result was splined and a function that returns this function was made [Callin (2006)].

Then, equality times and acceleration time plus the age of the Universe were computed. This was done in the same way as for the conformal time by solving an ODE and making a spline of $t(x)$ (useful for computing the time at any x later on if needed). Evaluating the spline at $x = 0$ gives the time of the Universe in seconds (if using SI units), but it was converted to a more sensible unit, like gigayears ($109 \cdot 365 \cdot 24 \cdot 60 \cdot 60$ seconds), when presenting the results.

Once the distance measures were working, the parameter fits to supernova data [Betoule et al. (2014)] was made. First the results of the luminosity distance were plotted, using the fiducial cosmological parameters from Section 1, together with the data points from supernova observations. The fitting routine is a simple Metropolis Monte Carlo Markov Chain (MCMC) sampler. After everything has been implemented, the code was used to get constraints on our cosmological parameters by comparing to data.

The used data was a set of supernova with associated redshift z_i, luminosity distance $d_L^{obs}(z_i)$ and associated measurement errors σ_i. Under the assumption that the measurements are Gaussian distributed and uncorrelated between different redshifts, the Likelihood function (telling how well the data fit the theory), is given by $\mathcal{L} \propto e^{-\chi^2/2}$ where the chi-squared function is

$$\chi^2(h, \Omega_{M,0}, \Omega_{k,0}) = \sum_{i=1}^{N} \frac{\left[d_L(z_i, h, \Omega_{M,0}, \Omega_{k,0}) - d_L^{obs}(z_i)\right]^2}{\sigma_i^2}. \quad (32)$$

A low value of χ^2 means a good fit (high likelihood for the choice of parameters). The goal is to basically check all possible values of the parameters to find the best-fitting model and the range of parameters around it which are in agreement with the observed values. One can do this brute-force, but this step is in practice most commonly done by performing a so-called MCMC to randomly sample from the likelihood. The result from performing this step is a chain of values: parameter-points and likelihood values. The set of parameters with the lowest likelihood is the best-fit model. It can be checked if this is really a good fit by comparing the χ^2 to the number of data-points (here $N = 31$). A good fit has $\chi^2/N \sim 1$ (a much higher value denotes a bad fit). When the best-fit has been found, 1σ (68.4%) confidence region can be found by looking at all values that satisfies $\chi^2 - \chi^2_{min} < 3.53$. It is exactly the same for the 2σ (95.45%) confidence region by looking at all values that satisfies $\chi^2 - \chi^2_{min} < 8.04$[4].

[3] All the used code is fully available in my public GitHub repository.
[4] Robert Reid, Chi-squared distribution table with sigma values: http://www.reid.ai/2012/09/chi-squared-distribution-table-with.html

4. Results

The evolution of the relative energy densities of the Universe is presented in Figure 1. Here, each component is plotted, together with the combined contribution from all non-relativistic (baryonic/ordinary and cold dark) matter and all the relativistic particles (photons and neutrinos). It can clearly be seen that there are three different regimes, where each component (relativistic particles/radiation, marked with orange, non-relativistic matter, marked with blue, and dark energy/Λ, marked with green). This color coding has been used to better visualize these regimes in all the produced plots.

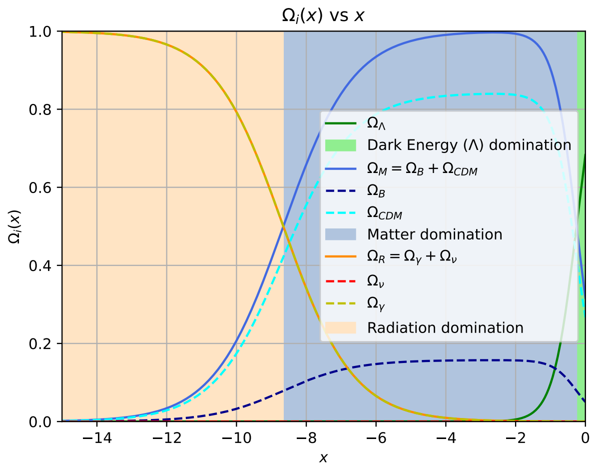


Fig. 1. Plot showing relative density parameters against the natural logarithm of the scale factor ($x = \ln a$). The early Universe was dominated by radiation, marked in orange. Following is the matter dominated era is marked in blue, where the combined density of the dark matter and baryon components is indicated by the blue line. Lastly, the present era is dominated by dark energy, marked in green.

From the splines created to solve the ODEs and compute both the cosmological time (Figure 2) and the conformal time (Figure 3) can be seen how they evolve in the different epoch of domination. From this data, one can compute both the age of the Universe and the conformal time today, being these $t_{\text{Universe}} = t(\text{today}) = t(0) \approx 13.86$ Gyr and $\eta(0)/c \approx 46.52$ Gyr, respectively. It also possible to compute the times when there was a radiation-matter and matter-dark energy equalities, along with the time at which the Universe started to accelerate, being this times summarized in Table 1.

	x	z	t [Gyr]
Radiation-matter	−8.67	5803.51	$2.28 \cdot 10^{-5}$
Matter-dark energy	−0.26	0.29	10.35
Accelerated expansion	−0.42	0.53	8.42

Table 1. Times at which there was radiation-matter equality, being this time expressed as the natural logarithm of the scale factor (x), redshift (z) and the cosmological time (t). The times of at which there was matter-dark energy equality and the time at which the Universe began its accelerated expansion are also shown.

As can be seen in Figure 2, cosmological time grows as the scale factor tends to 1 (if $a = e^x$ and $x \to 0 \Rightarrow a \to 1$). It can be inferred form the plot that cosmological time also grows

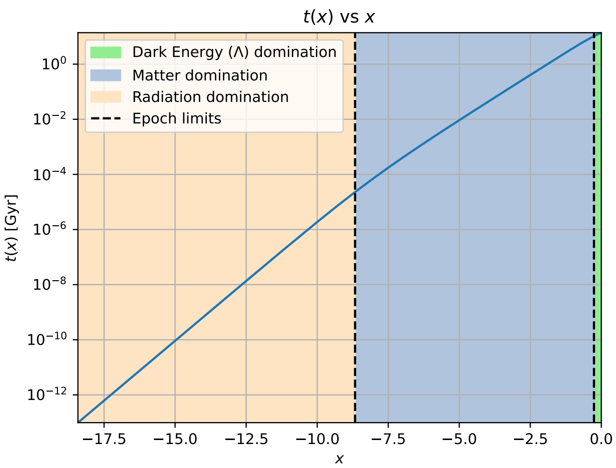


Fig. 2. Cosmological time evolution as a function of the natural logarithm of the scale factor.

exponentially, concluding that times is just another dimension (like the three spatial dimensions).

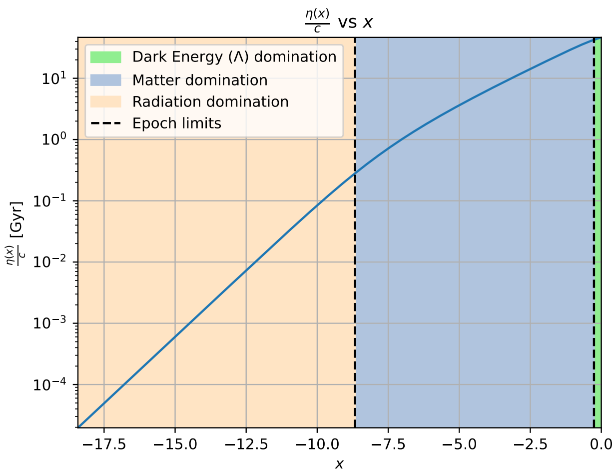
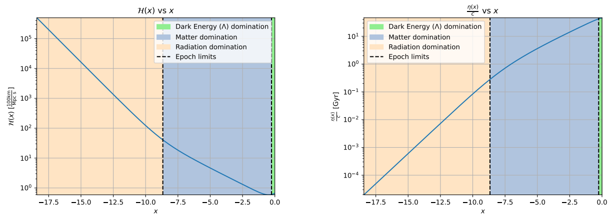


Fig. 3. Cosmological time evolution as a function of the natural logarithm of the scale factor.

The main result from this work is shown in Figure 4. Here are the conformal Hubble parameter (4a) is shown against x. It can be seen how this parameter decreases steeply in the radiation domination epoch, before getting into the matter domination epoch with a slightly gentler slope. The conformal time is, basically, the inverse of the conformal Hubble parameter. While the conformal Hubble parameter has an even steeper decrease in the radiation domination epoch going over to a less steep decrease in the matter domination epoch.

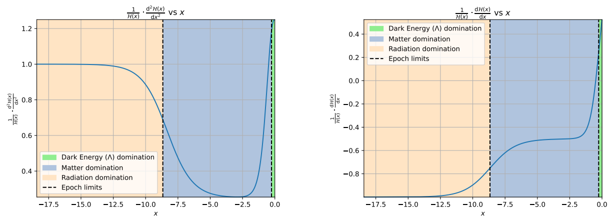
Note that the Hubble prime starts to increase again when getting into the last regime, dominated by dark energy, which corresponds to an accelerated expansion like we observe today.

Some ways of testing the code is studying the evolution of the (first, 5b, and second, 5a) derivatives of the conformal Hubble parameter with respect to x and the conformal Hubble parameter itself, as can be seen in Figure 5.



(a) Conformal Hubble parameter. (b) Conformal time.

Fig. 4. Plots showing the evolution of the conformal Hubble parameter (4a) and the conformal time (4b).



(a) $\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2}/\mathcal{H}$ ratio. (b) $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x}/\mathcal{H}$ ratio.

Fig. 5. 5a: Ratio between the first derivative of $\mathcal{H}$ with respect to x and $\mathcal{H}$. 5b: Ratio between the first derivative of $\mathcal{H}$ with respect to x and $\mathcal{H}$.

Other check one can do is plotting the conformal time times the conformal Hubble parameter, which in the radiation domination epoch tends to 1. As can be seen from Figure 6, the further to the left, the closest to 1, so it can be assumed that when the curve is prolonged far enough it will asymptotically tend to 1.

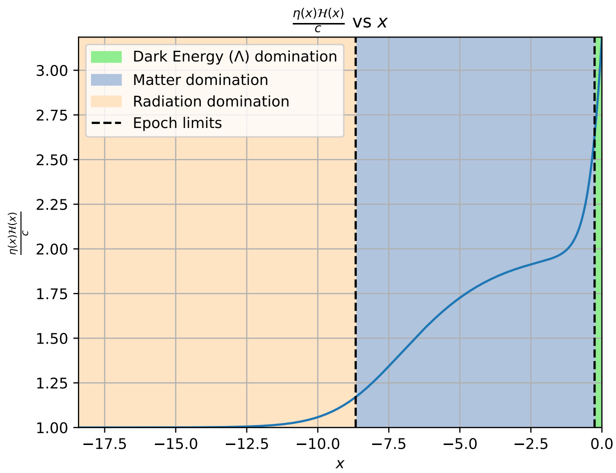


Fig. 6. Plot showing the relation between the conformal time and the conformal Hubble factor, showing that their product tends to 1 during the radiation domination epoch. It can also be seen that it is always of the order of 10^0.

Once it is clear that the code works (except for that mistake when computing the time at which the accelerated expansion of the Universe starts), the next step to check how good is the code is to try to get the parameters that fit better to some observations. For this, the data from Betoule et al. (2014) has been studied and fitted with a MCMC. The results of the fit are shown in Figure 7.

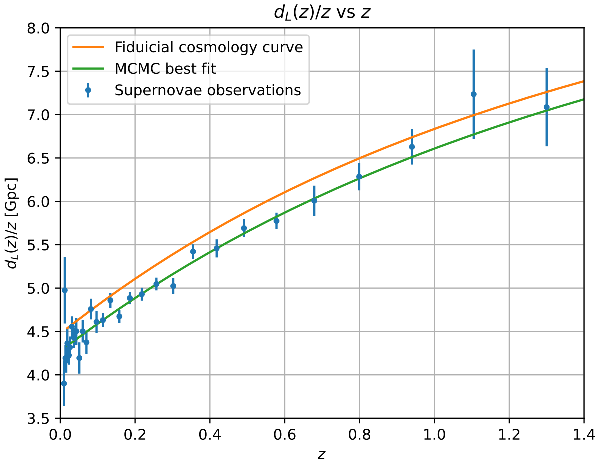


Fig. 7. Plot showing the observational data from Betoule et al. (2014) and the fit computed with the fiducial cosmology and with the MCMC fit, selecting the best parameters by choosing those corresponding to the lowest χ^2.

In Figure 8, the 1σ and 2σ deviation from the best fit parameters can be seen, showing in a dashed black line the set of parameters corresponding to a flat Universe. This deviations have been computed by choosing the parameters with a χ^2 greater by 3.53 (for the 1σ deviation) and by 8.04 (for the 2σ deviation) than the minimum value of χ^2 (the one of the best fitting parameters). A summary of values of the best fitting parameters can be found in Table 2.

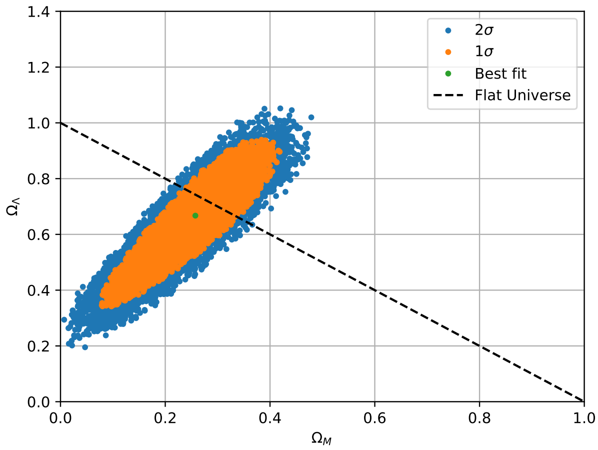
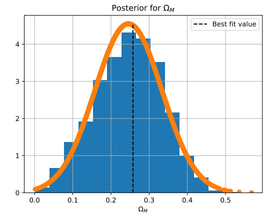
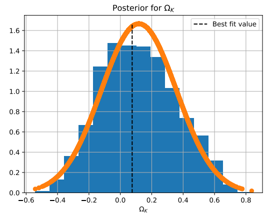


Fig. 8. Plot showing 1σ and 2σ deviation from the best fit parameters, showing in a dashed black line the set of parameters corresponding to a flat Universe.

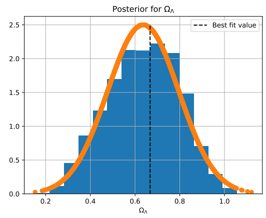
In Figure 9, a collection of histograms, showing the Gaussian distribution, of the generated parameters for the fitting parameters, along with a dashed black line showing the best fitting parameters.



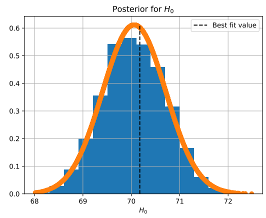
(a) Ω_M histogram.



(b) Ω_k histogram.



(c) Ω_Λ histogram.



(d) H_0 histogram.

Fig. 9. Histograms of the generated parameters: Ω_M (9a), Ω_k (9b), Ω_Λ (9c) and H_0 (9d) The best fitting parameters are shown in Table 2.

χ^2	Ω_M	Ω_k	Ω_Λ	H_0
29.30	0.26	0.08	0.67	70.18

Table 2. Best fitting parameters from the supernova fitting via MCMC.

One could be surprised to see that some parameters are negative or greater than one, but if the sum of all the Ωs were to be computed, it will give a result of exactly 1. All Ωs should be greater or equal than 0 and smaller or equal to 1, except for $\Omega_k < 0$, which is perfectly a physical result since $\Omega_k \propto k$, and k can be negative is the Universe is openly curved (hyperbolic). From the histograms it can be expected to live in a Universe with around 26% matter, around 67% dark energy (Λ) and a flat (or slightly closely curved/spherical) Universe ($\Omega_k = 0.08$).

As can bee seen in Figure 9c, the fit of the data do not allow a negative cosmological constant, which implies that most of our Universe is unknown to us and further research is needed. Focusing now in Figure 9d, one can see that the best fitting parameter is slightly bigger than the fiducial H_0 ($H_0 = 67$ km/s/Mpc). This discrepancy is what is called the Hubble tension, whose cause is not known. This discrepancy arises when comparing the computed H_0 with "early" and "late" Universe observations.

5. Conclusions

Background cosmology of the Universe has been solved and each epoch has been determined, identifying in each of them the dominating substance. The age of the Universe has also been computed, being this $t_\mathrm{Universe} \approx 13.86$ Gyr.

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