Computing the CMB power spectrum

II. Recombination history of the Universe

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April 26, 2024

ABSTRACT

Context. The goal of this whole study is to be able to predict the CMB (and matter) fluctuations (described by the so-called power-spectrum) from first principles and learn about all the different physical processes that goes on to be able to explain the results.
Aims. The goal in this work is to make a class/module that takes in the abundance of Helium in the Universe and has functions for getting the free electron fraction, the optical depth (and its derivatives) and the visibility function (and its derivatives).
Methods. A code has been developed, using the template given by Hans A. Winther. In this code, several functions have been implemented to solve the recombination history of the Universe and plot the results.
Results. Recombination history of the Universe has been solved. The relevant times of the Universe's recombination history have been computed, as well as the sound horizon at decoupling, being this $r_s \approx 128.177$ Mpc.
Conclusions. Now that the background cosmology of the Universe and its recombination history have been solved, this study can continue with its next step, studying the evolution of structure in the Universe.

Key words. background cosmology – evolution of the Universe – recombination

1. Introduction

The goal of this study has been to solve the recombination history of the Universe: how did the baryons go from being ionized (consisting only of free electron and protons) to being neutral (consisting of neutral hydrogen atoms)? The final goal of this part is to compute the optical depth as a function of x, the logarithm of the scale factor (a), and its derivatives, and the visibility function, $\tilde{g}$, and its derivatives.[1].

The fiducial cosmology used in this work is the best-fit cosmology found from fits to Planck 2018 data [Aghanim et al. (2020)]:

$$h_0 = 0.67,$$
$$T_{\mathrm{CMB},0} = 2.7255\ \mathrm{K},$$
$$N_{\mathrm{eff}} = 3.046,$$
$$\Omega_{\mathrm{B},0} = 0.05,$$
$$\Omega_{\mathrm{CDM},0} = 0.267,$$
$$\Omega_{k,0} = -\frac{kc^2}{H_0^2} = 0,$$
$$\Omega_{\Lambda,0} = 1 - (\Omega_{\mathrm{B},0} + \Omega_{\mathrm{CDM},0} + \Omega_{\mathrm{k},0} + \Omega_{\nu,0} + \Omega_{\gamma,0}).$$
$$Y_p = 0$$

Each quantity will be explained in the following section.

2. Recombination

The problem of this work is to compute the optical depth, $\tau(x)$, and the so-called visibility function, $g(x)$, which is needed for integrating the Boltzmann-Einstein equations in future works, and computing the CMB power spectrum. Physical descriptions and derivations will not be given here, but can be found in the lecture notes above, in Callin (2006), or Dodelson (2003). Here only the required definitions will be presented.

Light that travels through a medium can be absorbed by the medium. If we have a source emitting an intensity I_0 then, an observer a distance x away from the source will observe an intensity $I(x) = I_0 e^{-\tau(x)}$. The quantity τ is called the optical depth. If $\tau \ll 1$ then the medium does nothing (we say its optically thin) and if $\tau \gg 1$ then we will see nothing (the medium is optically thick). The transition between these two regimes is when $\tau \sim 1$. In cosmology the main 'absorption' is Thompson scattering of photons of free electrons. The optical depth is defined as

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a \mathrm{d}\eta', \tag{1}$$

and quantifies the probability for scattering a photon between some previous time and today (the number of scatterings of photons by electrons per unit time is $cn_e\sigma_T$). The components

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[1] Hans A. Winther, Milestone II: Recombination History: https://cmb.wintherscoming.no/milestone2.php

involved in this expression are: $n_e = n_e(\eta)$; the electron density (the number of free electrons per cubic meter) at time η, $\sigma_T = \frac{8\pi}{3}\frac{\alpha^2\hbar^2}{m_e^2c^2}$; the Thompson cross-section and the scale factor a. The transition between the Universe being optically thick and optically thin happens around recombination when most of the free electrons are captured by free protons to form neutral hydrogen. The expression for τ may also be written on an differential form, such that

$$\tau' = \frac{\mathrm{d}\tau}{\mathrm{d}x} = -\frac{cn_e\sigma_T}{H}. \tag{2}$$

This implies that existing routines can be used for solving differential equations to compute τ, if n_e can only be somehow compute at any time. And that's the difficult part. Instead of actually computing n_e, we rather focus on the fractional electron density, $X_e \equiv n_e/n_H$, where n_H is the proton density. If we will assume that all baryons are protons (no helium or heavier elements), then

$$n_H = n_\mathrm{B} \approx \frac{\rho_\mathrm{B}}{m_H} = \frac{\Omega_{\mathrm{B},0}\rho_{c0}}{m_H a^3}, \tag{3}$$

where $\rho_{c0} \equiv \frac{3H_0^2}{8\pi G}$ is the critical density of the universe today. If we do include Helium then only a (mass) fraction $1 - Y_p$ of the baryons are hydrogen so

$$n_H = (1 - Y_p)n_\mathrm{B} = (1 - Y_p)\frac{\Omega_{\mathrm{B},0}\rho_{c0}}{m_H a^3}, \tag{4}$$

where m_H is the hydrogen mass and Y_p is the Helium fraction and is a new cosmological parameter (BBN[2] gives us $Y_p \approx 0.24$ and observational constraints are consistent with this - the fiducial value in this work is $Y_p = 0$).

Now, there are two different equations available for X_e as a function of temperature and density, namely the so-called Saha and Peebles' equations (Equations 5 and 6 respectively). If we ignore Helium then the first one of these reads

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_\mathrm{B}}\left(\frac{m_e T_\mathrm{B}}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_\mathrm{B}}, \tag{5}$$

where T_B is the baryon temperature of the universe, and $\epsilon = 13.6$ eV is the ionization energy of hydrogen (the energy a photon needs in order to rip an electron away from a proton). In principle, one would have to solve separately for both T_B and the photon temperature, T_γ, but in practice it is an excellent approximation to set these equal. It can therefore be assumed that $T_\mathrm{B} = T_\gamma = T_{\mathrm{CMB},0}/a$. With the above information, Equation 5 (Saha's equation) reduces to a standard second-order equation in X_e, and can be solved directly using the normal formula, $y = (-b \pm \sqrt{b^2 - 4ac})/2a$.

Saha's equation (Equation 5 is an excellent approximation when $X_e \approx 1$. When X_e is noticeably smaller than one, better approximations are required, and one such approximation is the Peebles' equation (Equation 6,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1 - X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right], \tag{6}$$

where H is the Hubble parameter and

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \text{ (dimensionless)}, \tag{7}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1}, \text{ (dimension 1/s)}, \tag{8}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}, \text{ (dimension 1/s)}, \tag{9}$$

$$n_{1s} = (1 - X_e)n_H, \text{ (dimension 1/m}^3\text{)}, \tag{10}$$

$$n_H = (1 - Y_p)\frac{3H_0^2\Omega_{b0}}{8\pi G m_H a^3}, \text{ (dimension 1/m}^3\text{)}, \tag{11}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b}, \text{ (dimension 1/s)}, \tag{12}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \text{ (dimension 1/s)}, \tag{13}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b), \text{ (dimension m}^3\text{/s)}, \tag{14}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b), \text{ (dimensionless)}, \tag{15}$$

where $\alpha \simeq \frac{1}{137.0359992}$ in the expression for $\alpha^{(2)}$ is the fine-structure constant. This looks a bit scary, but it's not too bad, really. First of all, the various constants are simply physical constants describing simple atomic physics; Peebles' equation takes into account the transition rates between the ground-state ($1s$) and the first excited state ($2s$) of the hydrogen atom. For even higher-accurate work, many more states should be included, and also other atoms, most notably the helium atom. But here the Peebles' equation is enough. Second, the Peebles' equation is simply yet another linear first-order differential equation.

Next, it is necessary to decide when to switch from Saha's equation to Peebles' equation. For simplicity, lets simply say

[2] BigBang Nucleosynthesis

that when $X_e > 0.99$, Saha's has to be used and when $X_e < 0.99$, Peebles' has to be used.

When n_e has been computed, the next step is to spline it, such that it can be evaluated at arbitrary values of x.

Now it is time to compute the optical depth, by solving Equation 2 mentioned above, with $\tau(x_{\text{today}}) = 0$[3] as the initial condition.

The final item to compute is the so-called visibility function,

$$\tilde{g}(x) = -\tau'(x)e^{-\tau(x)}, \tag{16}$$

which has the property that $\int_{-\infty}^{0} \tilde{g}(x)\mathrm{d}x = 1$. This function is therefore a true probability distribution, and describes the probability density for a given photon to last time having scattered at time x. As it will be seen, this function is sharply peaked around $x = -7$, corresponding to redshifts of $z \sim 1100$. The fact that this is so sharply peaked is the reason why we often refer to the recombination period as the surface of last scattering: This process happened during a very thin shell around us, at a redshift of $z \sim 1100$. The first and second order derivatives of $\tilde{g}$ will be also required, and so these must also be properly splined, just as τ was.

The final thing to compute is the so-called sound-horizon at decoupling (the total distance a sound-wave in the photon-baryon plasma can propagate from the Big Bang and until photons decouple). This quantity is not so relevant at the moment, but it will be important later in this project, since this length scale is imprinted in the CMB and the distribution of dark matter and galaxies in our Universe, so it is important to know how big it is. The sound-speed of the coupled photon-baryon plasma is slightly lower than the sound-speed of photons ($c/\sqrt{3}$) and given by $c_s = c\sqrt{\frac{R}{3(1+R)}}$, where $R = \frac{4\Omega_{\gamma,0}}{3\Omega_{B,0}a}$, so the sound-horizon is

$$\frac{ds(x)}{dx} = \frac{c_s}{\mathcal{H}} \quad \text{with} \quad s(x_{\text{ini}}) = \frac{c_s(x_{\text{ini}})}{\mathcal{H}(x_{\text{ini}})}. \tag{17}$$

Evaluating it at $x = x_{\text{decoupling}}$ gives us the sound-horizon at decoupling $r_s \equiv s(x_{\text{decoupling}})$.

3. Implementation method

A class that takes in recombination parameters and a BackgroundCosmology object, has been implemented in a C++ code[4]. This has been used to solve the recombination history of the Universe.

The code starts by solving for the electron density n_e (X_e) by solving the Saha and Peebles equations (Equations 5 and 6 respectively). The former is easy: it's just a quadratic equation for X_e so it is easy to solve. However, the analytical solution at early times will be on the form "huge" ("huge" which should give approximately 1) so it has been needed to use the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$ when $|x| \ll 1$. The Saha equation is only valid as long as $X_e \approx 1$ so the code switches to the Peebles equation when $X_e < 0.99$. The result has been splined for X_e and n_e as this was needed when computing τ.

Once n_e has been computed[5], the ODE for τ (Equation 2) has been integrated from $x = 0$ and going back in time. The integration constant has been fixed by ensuring $\tau(x = 0) = 0$.

4. Results

First of all, the different times for decoupling, last scattering, half-way recombination (using only Saha's equation and using both Saha's and Peebles') and the recombination time have been computed, being these times summarized in Table 1.

	x	z	t [Myr]
Decoupling	-7.370	1586.94	0.0.218
Last scattering	-6.987	1081.31	0.409
Half-way rec.	-7.164	1290.67	0.306
Half-way rec (Saha)	-7.230	1379.48	0.275
Recombination	-6.989	1083.10	0.408

Table 1. Summary of all the computed times in both redshift (z) and cosmological time (t).

The evolution of the free electron fraction is shown in Figure 1. Here it can be seen how the free electron fraction stay constant at 1 in the early Universe, before recombination happens. Then, at $x = x_{\text{recombination}} \approx -6.987$ we enter the Peebles' regime, and shortly after it can be seen that the free electron fraction start to evolve quite rapidly. At this point, the temperature of the Universe has dropped significantly, so free electrons and protons can start to form atoms of neutral hydrogen. As can be inferred from Table 1, at $x = -7.164$ we are half way through recombination (using the complete solution)[6]. Equation 5 has a exponential drop, which can be recognized in Figure 1, the $X_{e,\text{Saha}}$ solution drops off to zero with decreasing temperature. The full solution does not fall off exponentially, and instead flattens out to a stable value. This flattening can be described by two phases. First the free electrons decouple from the rest of the Universe as their interaction rate with the free protons drops below the expansion rate of the Universe. Second the free electrons freeze out, where the free electron fraction stops evolving and becomes, approximately, constant.

In Figure 2, the optical depth and its derivatives have been plotted. In the dense primordial Universe, the mean free path of photons was really low due to Thompson's scattering on the abundant free electrons. This can be seen in the high optical depth, and high free electron fraction. When $\tau \gg 1$, the Universe was opaque. During radiation domination era, Universe expands proportionally to $\sqrt{t}$, so the optical "thicknes" of the Universe is only decreasing along with the expansion. When recombination starts and X_e drops, it follows with a drop in the optical depth and its derivatives. This is consistent with less free electrons,

[3] Remember that the optical depth at our place in the universe today is precisely zero.

[4] All the used code is fully available in my public GitHub repository.

[5] Because n_e varies over many, many orders of magnitude, it is useful to spline $\ln(n_e)$ rather than n_e itself; this function varies much more slowly with x, and is therefore easier to interpolate.

[6] Note that if only the Saha approximation is used, we will be half way through recombination at $x = -7.230$.

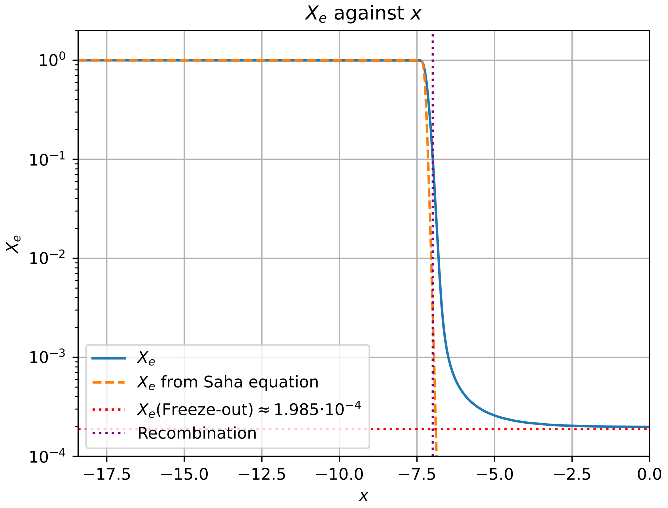

Fig. 1. Plot showing the free electron fraction X_e. The solution obtained using only Saha's equation, which drops off exponentially to zero once the solution deviates from equilibrium, has been included.

so less Thompson scattering and longer mean free path. Now the optical "thickness" of the Universe has changed due to the change in the constituents of the Universe, not only the expansion.

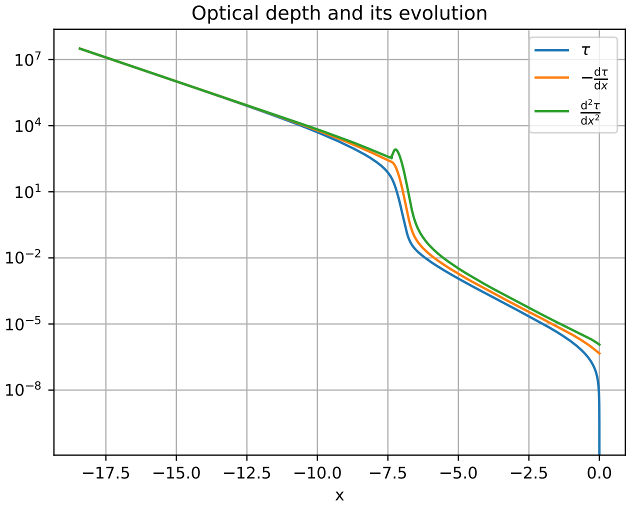

Fig. 2. Plot showing the optical depth (τ) and its two first derivatives.

Figure 3 shows the visibility function and its derivatives. The visibility function and its derivatives have been scaled to fit into the same plot, with the first and second derivative having much larger values than the visibility function it self. The visibility function describes the probability that a observed photon was last scattered at time x, going backwards in time from today. In Figure 3 it can be seen that this probability is more or less zero both before and after recombination. This can be understood as a dense and opaque Universe before recombination, where photons scattered constantly, and thus the probability of a photon observed today scattered last time before recombination is almost zero. After recombination the Universe is transparent, huge and "empty". Here Thompson's scattering rate is lower than the expansion's, as $\tau' < 1$, and so the chance for scattering is really low.

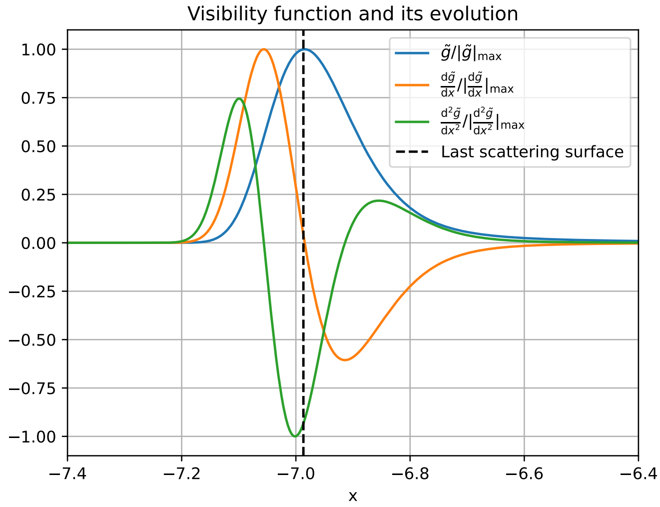

Fig. 3. Plot showing the visibility function ($\tilde{g}$) and its two first derivatives.

The sound horizon at decoupling has also been computed, being this $r_s \approx 128.177$ Mpc.

5. Conclusions

Recombination history of the Universe has been solved, computing different important times during recombination, summarized in Table 1. The sound horizon at decoupling has also been computed, being this $r_s \approx 128.177$ Mpc.

References

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