Computing the CMB power spectrum

IV. The CMB and matter power-spectra

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ABSTRACT

Context. The goal of this whole study is to be able to predict the CMB (and matter) fluctuations (described by the so-called power-spectrum) from first principles and learn about all the different physical processes that goes on to be able to explain the results.
Aims. The goal in this work is to make a class/module that computes the CMB power-spectrum.
Methods. A code has been developed, using the template given by Hans A. Winther. In this code, several functions have been implemented to compute the CMB power-spectrum and plot the results.
Results. The matter and CMB power-spectra have been computed, along with a CMB map.
Conclusions. The CMB power-spectrum has been computed with a high accuracy, considering the approximations made except for the low ℓ's, where some mistake in the Doppler component of the source function has given discrepancies with the observational data.

Key words. background cosmology – evolution of the Universe – CMB power-spectrum – CMB map

1. Introduction

Finally, after spending a lot of time on understanding the evolution of the background properties of the universe, its ionization history, and the growth of structure in the universe, the key statistical observable in cosmology has been computed: the CMB and matter power spectrum![1].

The fiducial cosmology used in this work is the best-fit cosmology found from fits to Planck 2018 data [Aghanim et al. (2020)] without neutrinos, reionization, Helium nor photon polarization:

$$h_0 = 0.67,$$
$$T_{\rm CMB,0} = 2.7255\ {\rm K},$$
$$N_{\rm eff} = 3.046,$$
$$\Omega_{\rm B,0} = 0.05,$$
$$\Omega_{\rm CDM,0} = 0.267,$$
$$\Omega_{k,0} = -\frac{kc^2}{H_0^2} = 0,$$
$$\Omega_{\Lambda,0} = 1 - (\Omega_{\rm B,0} + \Omega_{\rm CDM,0} + \Omega_{\rm k,0} + \Omega_{\nu,0} + \Omega_{\gamma,0}),$$
$$Y_p = 0,$$
$$z_{\rm reion} = 0,$$
$$\Delta z_{\rm reion} = 0,$$
$$z_{\rm He,\ reion} = 0,$$
$$\Delta z_{\rm He,\ reion} = 0,$$
$$n_s = 0.965,$$
$$A_s = 2.1 \cdot 10^{-9}.$$

Each quantity will be explained in the following section.

2. CMB power-spectrum

Knowing the ionization history of the universe and knowing how structure have grown from the very earliest epochs, the CMB power-spectrum can be computed now. Pulling it all together is what has to be done in order to derive our preferred observable, namely the CMB power spectrum.

In order to understand how to get to the CMB power spectrum from our computed quantities, let us first recall the definition of the spherical harmonics transform of the CMB temperature field,

$$T(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}), \tag{1}$$

where $\hat{n}$ is the direction on the sky, $a_{\ell m}$ are the spherical harmonics coefficients, and $Y_{\ell m}$ are the spherical harmonics themselves.

⋆ GitHub: https://github.com/ivanvillegas7/CMB_power_spectrum
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[1] Hans A. Winther, Milestone IV: The CMB and matter power-spectra: https://cmb.wintherscoming.no/milestone4.php

The CMB power spectrum, on the other hand, is simply defined as the expectation value of the square of the spherical harmonics coefficients[2],

$$C_\ell \equiv \left\langle |a_{\ell m}|^2 \right\rangle = \left\langle a_{\ell m} a^*_{\ell m} \right\rangle . \tag{2}$$

So, in order to get to the power spectrum, the temperature field we observe around us today $T(\hat{n}, x = 0)$ has to be known. But fortunately, we already have this information is (more or less) available, since the evolution of the temperature field, in the form of $\Theta_\ell(k, x)$, has already been computed. The values of these functions at $x = 0$ have been read off, the resulting coefficients (to get $T(\boldsymbol{n})$ instead of $T(\boldsymbol{k})$) have been Fourier transformed, and the correct values have been read off.

But these functions have only been computed up to $\ell = 6$, and smaller scales than that are surely interesting! Knowing the power spectrum to at least $\ell = 1200$ is what is required in order to compute the CMB power-spectrum. So the code has been rerun, but this time with $\ell_{\max} = 1200$ instead of 6. The only problem with that is that it will take a ***very*** long time to complete.

This is where the method of Zaldarriaga and Seljak comes to our rescue, through what they call the line-of-sight integration approach [Seljak & Zaldarriaga (1996)]. What we really need to know, is $\Theta(k, \mu, x = 0)$. But instead of first expanding the full temperature field in multipoles and then solve the coupled equations, one can start by formally integrating the original equation for $\dot{\Theta}$, and then expand in multipoles at the end. For the details of this process, as in Callin (2006) or Dodelson (2003). The final expression is simply

$$\Theta_\ell(k, x = 0) = \int_{-\infty}^{0} \tilde{S}(k, x) j_\ell[k(\eta_0 - \eta)] dx, \tag{3}$$

where the ***source function*** is defined as

$$\tilde{S}(k, x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Pi\right] + e^{-\tau}\left[\Psi' - \Phi'\right] - \frac{1}{ck}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b)$$
$$+\frac{3}{4c^2k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Pi)\right], \tag{4}$$

where all quantities have been computed in earlier works.

The intuition behind this approach is that the observed CMB radiation in a given direction on the sky, is basically the integral of the local CMB monopole (weighted by the visibility function) along the line of sight from us to infinity. That's the first Θ_0 term in the source function. However, there are a number of corrections to this effect. First, the Ψ term encodes the fact that the photons have to climb out of a gravitational potential, and therefore loose energy on its way to us. Π is a small quadrupolar (plus polarization if it has been included) correction to the original monopole contribution.

The next main term is essentially the so-called Integrated Sachs-Wolfe contribution, which describes the fact that gravitational potentials actually change while the photons are moving. The third term is a Doppler term and the fourth term is a consequence of the angular dependence of Thompson scattering (photons are more likely to be scattered in certain directions than others).

So, this is a much more beautiful approach: Instead of evaluating what every single photon moment is at our position today, the monopole can be computed at all positions and times, and then the integral is done through space. Much faster, and also quite intuitive.

The $j_\ell(x)$'s in the above expression are the so-called spherical Bessel functions, and take into account the projection of the 3D field (characterized by k) onto a 2D sphere (characterized by ℓ).

Now, how to compute $\Theta_\ell(k)$ is known, however going to actual C_ℓ's is still needed. This corresponds to

1. Taking the square of $\Theta_\ell(k)$ (since C_ℓ is the square of the Fourier coefficients).
2. Multiplying with the primordial power spectrum $P_{\text{primordial}}(k)$ coming from inflation (recall that Φ was originally set $\Phi \sim 1$ for all modes; this is now corrected by rescaling everything by $P(k)$ instead, which is perfectly valid, since all the used equations are linear).
3. Integrating over all three spatial directions, instead of just the z direction (but since isotropy is assumed, the same derived functions for all three directions can be used).

The CMB power spectrum then reads

$$C_\ell = \frac{2}{\pi}\int k^2 P_{\text{primordial}}(k)\Theta_\ell^2(k) dk. \tag{5}$$

However, this can be massaged a bit further, by noting that most inflation models predict a so-called Harrison-Zel'dovich spectrum, for which

$$\frac{k^3}{2\pi^2}P_{\text{primordial}}(k) = A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}, \tag{6}$$

where n_s is the spectral index of scalar perturbations ($n_s \sim 0.96$), and expected to be close to unity, k_{pivot} is some scale for which the amplitude is A_s (for $k_{\text{pivot}} = 0.05/\text{Mpc}$ we have $A_s \sim 2 \cdot 10^{-9}$ for our Universe). The final expression for the spectrum is therefore

$$C_\ell = 4\pi\int_0^\infty A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}\Theta_\ell^2(k)\frac{dk}{k}. \tag{7}$$

Two final comments: First, the power spectrum is most often plotted in units of $\ell(\ell+1)/2\pi$ in μK^2, because it's overall trend is to drop as ℓ^{-2}. It is therefore easier to see features when plotted

[2] Note that, in principle, this function should have two subscripts, $C_{\ell m}$, but because we assume that the universe is isotropic, it must have the same power spectrum towards both the x, y and z directions, and this implies full rotational invariance. As a result, there is no m dependence in the power spectrum, and we simply average over m, and only call the spectrum C_ℓ.

in these units (i.e. you multiply C_ℓ by $\frac{\ell(\ell+1)}{2\pi}\left(10^6 T_{\rm CMB0}\right)^2$).

The matter power-spectrum has also been computed. This is really easy, only multiplying some numbers together is needed:

$$P(k,x) = |\Delta_M(k,x)|^2 P_{\rm primordial}(k), \tag{8}$$

where $\Delta_M(k,x) \equiv \frac{c^2k^2\Phi(k,x)}{\frac{3}{2}\Omega_{M0}a^{-1}H_0^2}$ and $\Omega_{\rm M,0}$ is the total matter density parameter today. It has only been computed for $x = 0$, i.e. today. This is often plotted as k in units of h/Mpc versus $P(k)$ in units of $({\rm Mpc}/h)^3$. To explain this plot it has been useful to mark in the equality scale $k_{\rm eq} = \frac{a_{\rm eq}H(a_{\rm eq})}{c}$ in the plot where $a_{\rm eq}$ is the scale-factor for matter-radiation equality.

2.1. Generate a CMB map

From the theoretical C_ℓ spectrum a CMB map can be generated. Generating a map is a fairly straight forward thing to do, but the hard part of this is to pixelate the sphere and project that down to a 2D map that we can plot. Luckily there is a great library for this namely the HEALPIX library[3] (for this you also need the CFITSIO library[4]). There is also a Python wrapper for this library called Healpy[5]. This library can be supplied with the C_ℓ's or a realization of the $a_{\ell m}$'s (which can be generated from the C_ℓ's) and have it provide the map.

Healpix has routines for generating maps directly from a power-spectrum or from $a_{\ell m}$'s. If you want to generate your own realization of $T(\hat{n})$ then note that the $a_{\ell m} = x + iy$ with x, y being random numbers drawn from a Gaussian distribution with variance $C_\ell/\sqrt{2}$. A simple way to generate a realization is to draw (for each value of ℓ, m) two random numbers A, θ from a uniform distribution on $[0, 1)$ and set

$$a_{\ell m} = \sqrt{-\log(A)C_\ell}e^{2\pi i\theta}. \tag{9}$$

Note that $T = T^*$ is real, so

$$a_{\ell,-m} = (-1)^m a^*_{\ell m}, \tag{10}$$

and we only need to generate them for $m \geq 0$. The other way around: if you want to estimate the angular power-spectrum from a set of $a_{\ell m}$'s by computing

$$\hat{C}_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}|a_{\ell m}|^2. \tag{11}$$

This should give you back the input power-spectrum for large ℓ (and will suffer from cosmic variance for small ℓ).

[3] https://healpix.sourceforge.io/downloads.php
[4] https://heasarc.gsfc.nasa.gov/fitsio/
[5] https://healpy.readthedocs.io/en/latest/

3. Implementation method

A class that takes in a BackgroundCosmology Recombination-History and a Perturbations object and uses this to compute $\Theta_\ell(k, x = 0)$ for $2 < \ell < 1200$ using line-of-sight integration and integrate this again to obtain the power-spectrum has been implemented in a C++ code[6].

The Perturbations class has been modified, adding a few more lines of code to make sure of the computations and storage of the source-function defined above. This is what needed in order to do the line-of-sight integration. Bessel-functions have also been needed.[7]

There are several ways you can do the line of sight integration:

1. Just evaluate the integral directly as in Callin (2006).
2. Evaluate the integral by treating it as an ODE and using the ODE solver (be careful with the accuracy settings of the ODE solver), i.e.

$$\frac{d\Theta_\ell(k,x)}{dx} = \tilde{S}(k,x)j_\ell[k(\eta_0-\eta)], \quad \Theta_\ell(k,-\infty) = 0 \tag{12}$$

Same goes for the integration of the C_ℓ's.

4. Results

In Figure 1 displays the resulting multipoles obtained from Equation 12 through the line of sight integration method, for four significantly different angular scales in the studied interval. Here it can be seen the oscillatory and dampening effect from the Bessel functions, as the transfer functions look like damped harmonic oscillators.

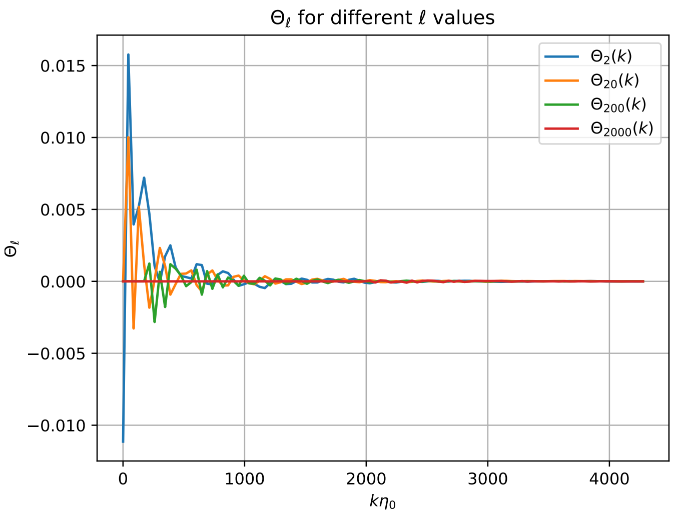

Fig. 1. Transfer function Θ_ℓ obtained from line of sight integration in Equation 12 for four different ℓ's.

The integrand from Equation 7 is shown in Figure 2. As it shows essentially the square of the transfer function, one will

[6] All the used code is fully available in my public GitHub repository.
[7] The ComplexBessel library should be installed, to do this check the GitHub page for how to do it). The fallback if you do not install it is to use the GSL, which is quite buggy for large ℓ.

expect to see much of the same trends, with only positive values, with the larger scales even more dominant as it is divided by the wavenumber. However, this is not the case, from where one could infer that there has to be some kind of mistake in the code. Here it could be expected to see how the different scales will affect the power spectrum in the end, and which wavenumbers contributes to the different angular scales. As $k \sim \ell/\eta_0$ this is not so surprising, seeing that the larger scales, low ℓ, have a contribution on small k, and vice versa.

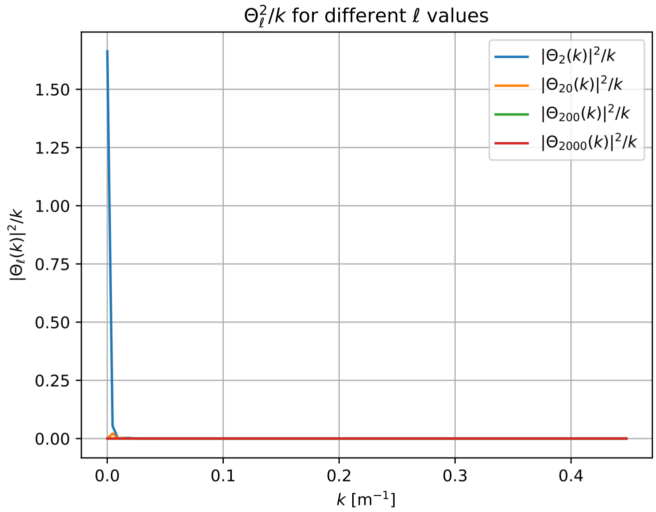

Fig. 2. The integrand Θ_ℓ Equation 7 for four different ℓ's.

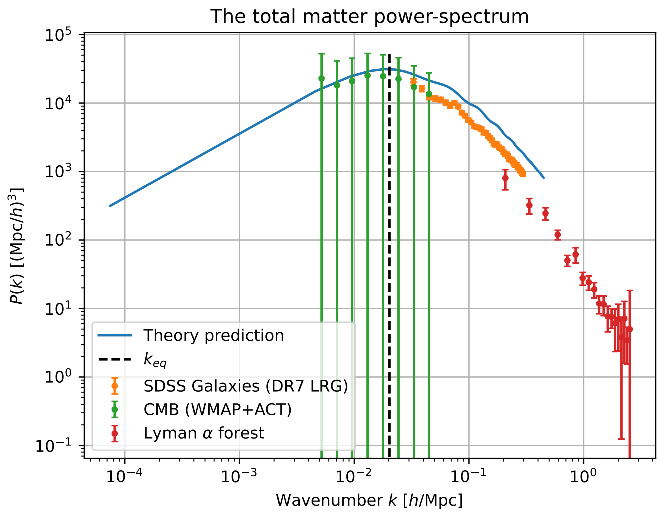

Fig. 3. The matter power spectrum, with the equality scaled (k_{eq}) marked in as the vertical dotted line. For larger scales the spectrum goes as $k^{n_s} \sim k^1$, while small scales being suppressed by the Meszaros effect going as k^{n_s-4}, being suppressed by k^{-4} relative to the larger scales entering after matter-radiation equality.

In Figure 3 the total matter power-spectrum can be seen. For large scales perturbations, it can be seen how the late horizon entry and hence being affected by little causal physics, results in a more or less unprocessed perturbations. Seeing a scale dependence consistent with initial perturbations being set up by a Gaussian random field, where more perturbations of "medium" scale and less of the super-horizon scales are expected, it results in the spectrum proportional to k^{n_s}, where the spectral index n_s is close to unity, being determined by the length of the inflationary period. Going to smaller scales the peak in the spectrum is hit, at the equality scale k_{eq}. This is the scale corresponding to perturbations entering the horizon at the matter-radiation equality. The matter perturbations grows differently with time in the matter and radiation era, and also when being outside and inside the horizon.

The main result, the CMB power spectrum, is shown in Figure 4. Reading the plot from large scales to small, we see the plateau on the largest scales, with the late ISW effect contributing to the increase on the very largest scales in $\ell \lesssim 10$. This is due to the dark energy dominated regime, where the accelerated expansion halts the growing modes and thus the derivatives of the gravitational potentials are large. Increasing the dark energy density parameter would cause the dark energy to dominate earlier, and we would have a larger late ISW effect. Moving on, the first peak, called the acoustic peak, can be found. This is a remnant from the tight coupled regime between baryons and photons in the early perturbations. The baryon-photon fluid traveling as one in the tight coupled regime, before the photons decouple leaving the baryons behind. The baryons are then affected by the gravity of dark matter perturbations, and the baryons affecting the dark matter. The result is a high peak at the $\ell \sim 200$ scale. On the smaller scales it can be seen how the power spectrum starts to oscillate, called acoustic oscillations, and eventually being exponentially suppressed. Looking at the monopole oscillations from the previous work[8] the oscillations can be understood. The equation for the monopole can be characterized as a driven harmonic oscillator. The peaks in the C_ℓ corresponds to scales for which the effective monopole oscillations have a maximum or minimum at the time of recombination, or at the last scattering surface (LSS). The troughs in the power spectrum corresponds to scales where the effective monopole is zero at LSS. The photon perturbations characterized by the driven harmonic oscillator equation can be thought of as a simple mass less ball on a spring in a gravitational field, oscillating up and down in a driven motion. If photon perturbations were considered, these oscillations would have been even around the zero point. But as the baryon and photons are heavily coupled, and the baryons having mass in the gravitational field, we can instead think of the oscillator as a ball on a spring, but with mass. The oscillations being due to the balance between gravity and radiation pressure, adding the baryons with mass enhances the compression modes, but not so much the rarification. Thus we get an uneven oscillation, every compression being enhanced, which corresponds to the odd numbered peaks in the power spectrum. This we can clearly see in Figure 4, where the first and third peak being larger than the second and fourth and so on. For higher number peaks this is harder to make out, as the exponential suppression of the power spectrum kicks in. This gives us another insight into determining the cosmological parameters. Measuring the relative height of the even and odd numbered peaks we can infer restrictions on the baryon density parameter. Lastly, the exponential suppression on the smallest scales can be seen. This effect is due to photon diffusion, caused by the photons moving in a so called random walk motion through the photon-baryon fluid due to the photons scattering on the free electrons. The

[8] III: Evolution of structure in the Universe

average distance the photons travel between each scattering is called the mean free path, $\lambda_{\rm mfp}$, being dependent on the free electron density. Following the central limit theorem, after n scatterings the photons would have moved on average a distance $\lambda_D \sim \sqrt{n}\lambda_{\rm mfp} \sim 1/\sqrt{n_e \sigma_T H}$. or perturbations on scales smaller than this distance, the over- and underdensities are able to mix dampening the oscillations. This effect is especially prominent for scales for which the finite thickness of the LSS is not negligible relative perturbation size. This is called diffusion dampening, or silk dampening, exponentially suppressing the anisotropies on scales $\ell \lesssim 500$. This effect is also dependent on the baryon density parameter because of the scattering, as well as the helium fraction and the number of relativistic particles in our Universe. The latter being determined by the number of massless neutrinos.

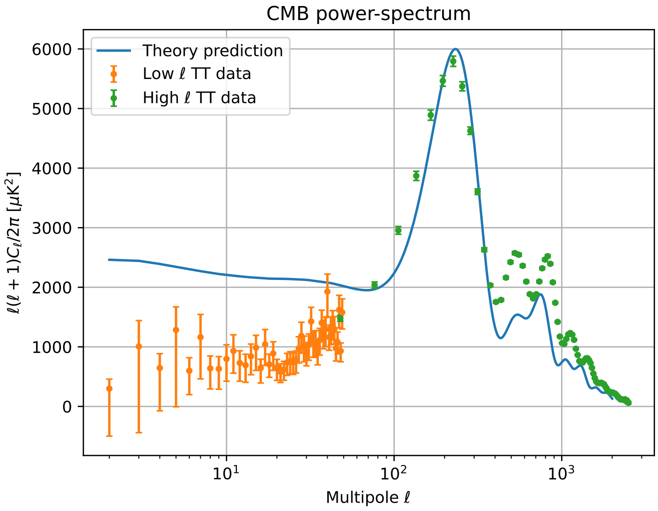

Fig. 4. CMB power-spectrum with the observational data.

In Figure 5, the contribution of each term to the final CMB power-spectrum can bee seen. Comparing with Figure 6 in Baumann (2018) (Figure 6 in this work) one can see that there is a discrepancy in the Doppler term, possibly being this the reason why there is a discrepancy between the computed CMB power-spectrum and the observational data plotted in Figure 4.

Additionally, a map of the CMB power-spectrum as been generated, being shown in Figure 7.

5. Conclusions

The matter and CMB power-spectra have been computed with high accuracy, except for the low ℓ's in the CMB power-spectrum due to some mistake in the Doppler term of the source function.

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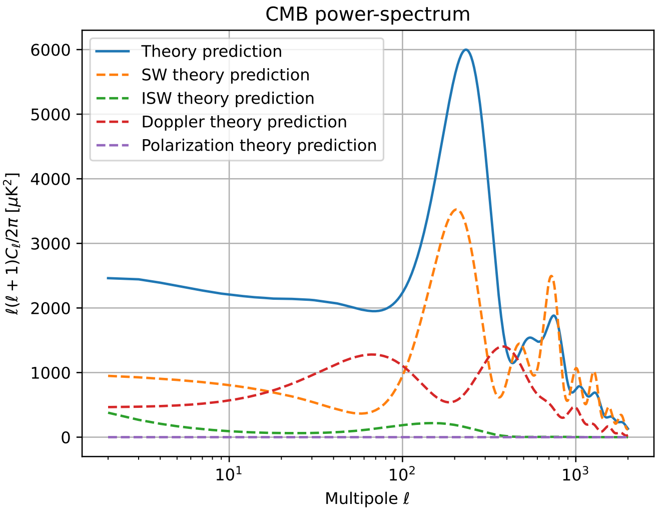

Fig. 5. Contribution of each term to the final CMB power-spectrum

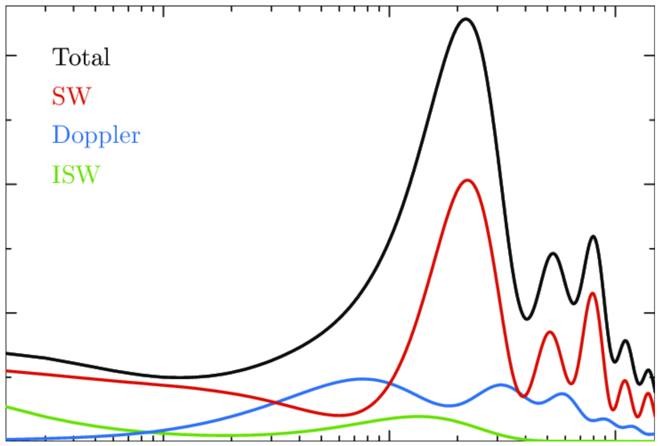

Fig. 6. Contribution of each term to the final CMB power-spectrum from Baumann (2018).

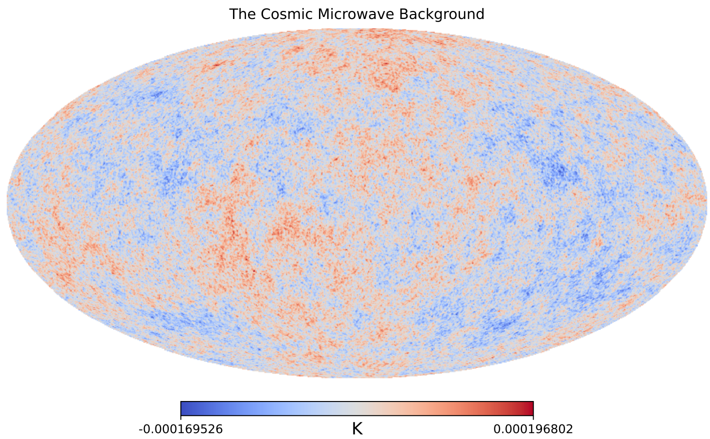

Fig. 7. CMB map generated using Healpy.