ON QUOTIENTS OF A MORE GENERAL THEOREM OF WILSON

Ivan V. Morozov
The City College of New York, New York, New York
imorozo000@citymail.cuny.edu

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Abstract

The basis of this work is a simple, extended corollary of Wilson's theorem. This corollary generates many more quotients than those already generated by Wilson's theorem, and it is of interest to derive how they relate to each other and build on the established properties of the original quotients. The main results are expressions for sums of these quotients, modular congruences that extend the results of Lehmer, and generating functions.

1. Introduction

In order to efficiently describe the results presented in this work, we will define $\mathbb{P}_1 := \mathbb{P} \cup \{1\}$, where $\mathbb{P}$ is the set of all primes, as the set of positive non-composite numbers. With this said, Wilson's theorem is a primality test by which

$$W(n) = \frac{1 + (n-1)!}{n}$$

is an integer if and only if $n \geq 1$ is a positive non-composite. Also, $\{W(p) : p \in \mathbb{P}_1\}$ is the set of Wilson quotients. We may, in fact, generalize this notion by observing that $(n-1)! = (n-k-1)! \prod_{m=1}^{k}(n-m)$ implies

$$W(n) = \frac{1 + (n-k-1)! \prod_{m=1}^{k}(n-m)}{n}.$$

The $\prod_{m=1}^{k}(n-m)$ term is a falling factorial, so we can implement an expansion of this product, the coefficients of which are the *Stirling numbers of the first kind* $s(a,b)$, defined by $(x)_n = \prod_{k=0}^{n-1}(x-k) = \sum_{k=0}^{n} s(n,k)x^k$. Thus, $\prod_{m=1}^{k}(n-m) = \frac{1}{n}(n)_{k+1}$ and

$$W(n) = \frac{1 + (n-k-1)! \frac{1}{n}(n)_{k+1}}{n} = \frac{1 + (n-k-1)! \sum_{i=0}^{k+1} s(k+1,i) n^{i-1}}{n},$$

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and since $s(k+1,0)=0$ for $k \geq 0$, we have

$$W(n) = \frac{1+(n-k-1)!\sum_{i=0}^{k+1} s(k+1,i+1)n^i}{n}\,. \tag{1}$$

Observing that $\sum_{i=1}^{k+1} s(k+1,i+1)n^i \equiv 0 \pmod{n}$ and $s(k+1,1) = (-1)^k k!$, Wilson's criterion extends to a corollary with an additional parameter.

Corollary 1. *Given nonnegative integers $n \geq 1$ and $k < n$,*

$$(-1)^k k!(n-k-1)! \equiv -1 \pmod{n}$$

if and only if n is non-composite.

Incidentally, Wilson's theorem is a corollary of its own corollary when $k=0$, and we will thus refer to the more general Wilson-like quotients as

$$M_k(n) = \frac{1+(-1)^k k!(n-k-1)!}{n}\,,$$

where, by Corollary 1, $M_k(n)$ is an integer if and only if $n \in \mathbb{P}_1$. The values of some of these quotients are listed in Table 3 at the end of the paper. Note that $M_0(n) = W(n)$. These quotients yield nonpositive results for odd k, and therefore we can get rid of the signs by defining

$$M_k^+(n) = |M_k(n)| = (-1)^k M_k(n) = \frac{(-1)^k + k!(n-k-1)!}{n}\,. \tag{2}$$

We will broadly refer to the functions $M_k(n)$ and $M_k^+(n)$ as *M-numbers.*

2. Sums of $M_k(n)$ and $M_k^+(n)$

One area of investigation is the study of finite sums of values of the functions M_k and M_k^+, which we will broadly refer to as *Z-numbers.* Considering the sums

$$Z(n) = \sum_{k=0}^{n-1} M_k(n)\,, \quad Z^+(n) = \sum_{k=0}^{n-1} M_k^+(n)\,, \tag{3}$$

we can derive two theorems concerning them.

Theorem 1. *The function $Z(n)$ is an integer for all $n \in \mathbb{N}$.*

Proof. If $n \in \mathbb{P}_1$, then $M_k(n) \in \mathbb{Z}$ implies $Z(n) \in \mathbb{Z}$ by Corollary 1. Otherwise, consider the term $(-1)^k k!(n-k-1)!$, where n is composite. If it is additionally not equal to 4, we see that

$$k!(n-k-1)! = k!(n-(k+1))(n-(k+2))\dots(n-(n-1))\,,$$

and hence,

$$k!(n-k-1)! \equiv (-1)^{n-k-1}k!(k+1)(k+2)\dots(n-1) \equiv (-1)^{n-k-1}(n-1)! \pmod{n}.$$

Moreover, since $n \neq 4$, it follows that $(n-1)! \equiv 0 \pmod{n}$, implying

$$(-1)^k k!(n-k-1)! \equiv 0 \pmod{n}.$$

It follows that

$$\sum_{k=0}^{n-1} 1 + (-1)^k k!(n-k-1)! \equiv \sum_{k=0}^{n-1} 1 \equiv 0 \pmod{n},$$

which, with the fact that $Z(4) = 1$, shows that $Z(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$. □

Theorem 2. *The function $Z^+(n)$ is a natural number if and only if n is even or non-composite.*

Proof. If $n \in \mathbb{P}_1$, then $M_k^+(n) \in \mathbb{Z}$ implies $Z^+(n) \in \mathbb{Z}$ by Corollary 1. Otherwise, consider the term $(-1)^k k!(n-k-1)!$, where n is composite. If it is additionally not equal to 4, the preceeding proof showed that

$$k!(n-k-1)! \equiv 0 \pmod{n}.$$

This implies that

$$\sum_{k=0}^{n-1} (-1)^k + k!(n-k-1)! \equiv \sum_{k=0}^{n-1} (-1)^k \equiv \frac{1+(-1)^{n-1}}{2} \pmod{n},$$

which, with the fact that $Z^+(4) = 4$, shows that $Z^+(n) \in \mathbb{Z}$ if and only if n is even or non-composite. In this case, we actually have that $Z^+(n) \in \mathbb{N}$ since $Z^+(n) > 0$. □

Some values of $Z(n)$ and $Z^+(n)$ are listed in Tables 1 and 2 in the last section.

2.1. Formula for $Z(n)$

Furthermore, we can derive a closed formula for $Z(n)$. Let $S = \sum_{k=0}^{n-1} (-1)^k k!(n-k-1)!$. Then

$$Z(n) = \left(\sum_{k=0}^{n-1} \frac{1}{n} \right) + \frac{S}{n} = 1 + \frac{S}{n}.$$

Now, dividing S by $(n-1)!$ yields

$$\frac{S}{(n-1)!} = \sum_{k=0}^{n-1} \frac{(-1)^k k!(n-k-1)!}{(n-1)!} = \sum_{k=0}^{n-1} \frac{(-1)^k}{\binom{n-1}{k}}.$$

An identity for the same sum [1] tells us that

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} = \frac{(1+(-1)^n)\,(n+1)}{n+2}\,, \quad n \in \mathbb{N}_0\,,$$

so

$$\frac{S}{(n-1)!} = \frac{\left(1+(-1)^{n-1}\right) n}{n+1}\,,$$

and it follows algebraically that

$$Z(n) = 1 + \frac{\left(1+(-1)^{n-1}\right)(n-1)!}{n+1}\,.$$

We observe that $Z(2n) = 1$ and $Z(2n-1) = 1 + \frac{(2n-2)!}{n}$. The former case is obviously a natural number. In the latter case, $n = 1$ is a trivial case, and since $2n-2 \geq n$ for $n \geq 2$, we have that $n \mid (2n-2)!$ for all $n \in \mathbb{N}$. This is a proof of a stronger version of Theorem 1: $Z(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$.

2.2. Formula for $Z^+(n)$

We can also derive a formula for $Z^+(n)$ by initially setting $S = \sum_{k=0}^{n-1} k!(n-k-1)!$, which produces

$$Z^+(n) = \left(\sum_{k=0}^{n-1} \frac{(-1)^k}{n}\right) + \frac{S}{n} = \frac{1+(-1)^{n-1}}{2n} + \frac{S}{n}\,.$$

Dividing S by $(n-1)!$ produces

$$\frac{S}{(n-1)!} = \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{(n-1)!} = \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}\,.$$

An identity from [1] tells us that

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^n} \sum_{r=0}^{n} \frac{2^r}{r+1}\,, \quad n \in \mathbb{N}_0\,,$$

so

$$\frac{S}{(n-1)!} = \frac{n}{2^{n-1}} \sum_{r=0}^{n-1} \frac{2^r}{r+1}\,,$$

and it follows algebraically that

$$Z^+(n) = \frac{1+(-1)^{n-1}}{2n} + \frac{(n-1)!}{2^n} \sum_{r=1}^{n} \frac{2^r}{r}\,.$$

To deal with the remaining partial sum, we make use of the *Lerch transcendent*, a special function denoted by Φ and defined by $\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}$. It has the property that

$$\Phi(z,s,a)=z^n\Phi(z,s,n+a)+\sum_{r=0}^{n-1}\frac{z^r}{(r+a)^s}$$

for $\Re(a),\Re(s)>0$, $n\in\mathbb{N}$, and $z\in\mathbb{C}$ [2]. We employ this equivalence by setting $a=1$, $s=1$, and $z=2$, which yields

$$\Phi(2,1,1)=2^n\Phi(2,1,n+1)+\sum_{r=0}^{n-1}\frac{2^r}{r+1}\,.$$

We also define the *polylogarithm* to be the function $\mathrm{Li}_s(z)$, commonly given by the Dirichlet series $\mathrm{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}$. Because $\Phi(z,s,1)=\frac{1}{z}\mathrm{Li}_s(z)$ [3], we have

$$\mathrm{Li}_1(2)=-i\pi=2^{n+1}\Phi(2,1,n+1)+\sum_{r=1}^{n}\frac{2^r}{r}\,,$$

and hence,

$$\sum_{r=1}^{n}\frac{2^r}{r}=-i\pi-2^{n+1}\Phi(2,1,n+1)\,.$$

By substitution we derive that

$$Z^+(n)=\frac{1+(-1)^{n-1}}{2n}-(n-1)!\left(2\Phi(2,1,n+1)+2^{-n}i\pi\right)\,. \tag{4}$$

3. Modular Congruences

3.1. Congruences of $M_k(p)$ and $M_k^+(p)$

Bernoulli numbers B_k are signed rational numbers that are ubiquitous in number theory and analysis. We can define them precisely using the exponential generating function $\frac{x}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_kx^k}{k!}$, in which they arise. When p is prime, it is known that

$$M_0(p)\equiv W(p)\equiv B_{2(p-1)}-B_{p-1}\pmod{p}\,,$$

which is obtained from the congruence relation

$$p-1+ptW(p)\equiv pB_{t(p-1)}\pmod{p^2}$$

by subtraction after substituting $t=1$ and $t=2$ [4]. Recalling Equation (1), it is clear that

$$\frac{(p-k-1)!\sum_{i=2}^{k+1}s(k+1,i+1)p^i}{p}\equiv 0\pmod{p}\,,$$

so

$$\begin{aligned} M_0(p) &\equiv \frac{1+(p-k-1)!(s(k+1,1)+ps(k+1,2))}{p} \\ &\equiv M_k(p)+s(k+1,2)(p-k-1)! \pmod{p}. \end{aligned}$$

The Stirling numbers of the first kind $s(k+1,2)$ can be expressed with harmonic numbers as $s(k+1,2)=(-1)^{k+1}k!H_k$, which yields

$$M_0(p) \equiv M_k(p)+(-1)^{k+1}k!H_k(p-k-1)! \equiv M_k(p)+H_k \pmod{p} \tag{5}$$

by Corollary 1. By subtraction,

$$M_k(p) \equiv B_{2(p-1)}-B_{p-1}-H_k \pmod{p}. \tag{6}$$

From Equation (5), we construct a congruence analogous to Lehmer's:

$$p-1+ptM_k(p) \equiv pB_{t(p-1)}-ptH_k \pmod{p^2}.$$

For the unsigned M-numbers, utilizing Equation (2) produces the congruences

$$M_k^+(p) \equiv (-1)^k M_k(p) \equiv (-1)^k \left(B_{2(p-1)}-B_{p-1}-H_k\right) \pmod{p}, \tag{7}$$

$$(-1)^k(p-1)+ptM_k^+(p) \equiv (-1)^k \left(pB_{t(p-1)}-ptH_k\right) \pmod{p^2}.$$

3.2. Congruences of $Z(n)$ and $Z^+(n)$

Theorem 3. *If n is composite, then $Z(n) \equiv 1 \pmod{n}$. If n is non-composite, then $Z(n) \equiv -1 \pmod{n}$.*

Proof. Recalling the formulae $Z(2n)=1$ and $Z(2n-1)=1+\frac{(2n-2)!}{n}$ for $n \in \mathbb{N}$, observe that if $n=2k-1$ is composite, then we can apply the same argument as in the proof of Theorem 1 to show that $2k-1 \mid (2k-2)!$. Thus, $2k-1 \mid \frac{(2k-2)!}{k}$ since $k \nmid 2k-1$, implying $Z(n) \equiv 1 \pmod{n}$ for $n \notin \mathbb{P}_1$. However, if $n=2k-1$ is non-composite, then $k(Z(2k-1)+1)=2k+(2k-2)! \equiv 1-1 \equiv 0 \pmod{2k-1}$ by Wilson's theorem. Thus, since $k \nmid 2k-1$, it follows that $Z(2k-1)+1 \equiv 0 \pmod{2k-1}$, implying $Z(n) \equiv -1 \pmod{n}$ for $n \in \mathbb{P}_1$, proving our theorem. □

From Equation (6), we have that

$$\begin{aligned} \sum_{k=1}^{p-1} H_k &\equiv \sum_{k=1}^{p-1} (M_0(p)-M_k(p)) \equiv (p-1)M_0(p)-Z(p)+M_0(p) \\ &\equiv pM_0(p)-Z(p) \equiv 2+(p-1)! \pmod{p}. \end{aligned}$$

Since $1 + (p-1)! \equiv 0 \pmod{p}$ by Wilson's theorem,

$$\sum_{k=1}^{p-1} H_k \equiv 1 \pmod{p}$$

for p prime. We can also consider the recursive nature of harmonic numbers given by $H_n = 1 + \frac{1}{n}\sum_{k=1}^{n-1} H_k$, which by recursion yields $H_{n+1} = \frac{1}{n+1} + H_n$. Substituting this into the previous sum implies the following congruence for p prime:

$$H_p \equiv 1 + \frac{1}{p} \pmod{p}.$$

Theorem 4. *We have the congruence $Z(2n-1) \equiv 0 \pmod{n}$ if and only if n is non-composite.*

Proof. Since $Z(2n-1) = 1 + \frac{(2n-2)!}{n} = 1 + (2n-2)(2n-3)\cdots(n+1)(n-1)!$, if $n \in \mathbb{P}_1$, then $Z(2n-1) \equiv 1 - \frac{(2n-2)!}{n!} \pmod{n}$ by Wilson's theorem. However, $\frac{(2n-2)!}{n!} \equiv (n-2)! \pmod{n}$, so $Z(2n-1) \equiv 1 - (n-2)! \pmod{n}$, and thus

$$Z(2n-1) \equiv 0 \pmod{n}$$

by Corollary 1.

For the other direction, observe that $\frac{(2n-2)!}{n} \equiv (n-1)!(n-2)! \equiv -1 \pmod{n}$ implies that n is non-composite by Corollary 1. This is because if n were otherwise composite, then $(n-1)!(n-2)! \equiv 0 \pmod{n}$. □

Theorem 5. *For even $n \neq 2$, the function $Z^+(n)$ obeys two congruences:*

$$Z^+(2n) \equiv 0 \pmod{2n}, \quad n \notin \mathbb{P}_1 \setminus \{2\}, \tag{8}$$

$$Z^+(2n) \equiv n+1 \pmod{2n}, \quad n \in \mathbb{P} \setminus \{2\}. \tag{9}$$

Proof. Recalling Equation (3),

$$Z^+(2n) = \sum_{k=0}^{2n-1} \frac{(-1)^k + k!(2n-k-1)!}{2n} = \frac{1}{n}\sum_{k=0}^{n-1} k!(2n-k-1)!$$

due to the even number of terms and consequential symmetry of the summands. Evidently, since $n \notin \mathbb{P}_1 \setminus \{2\}$ and if $n \neq 2$ and $n \neq 4$, we have $n^2 \mid (2n-k-1)!$ for all $0 \leq k \leq n-1$. Additionally, $2 \mid k!$ for $2 \leq k \leq n-1$ and

$$\frac{(2n-1)! + (2n-2)!}{2n^2} = \frac{(2n-2)!}{n} \in \mathbb{N}$$

for $n \in \mathbb{N}$. With the cases $Z^+(4) = 4$ and $Z^+(8) = 1536$, this proves Equation (8).

For the second congruence, again consider $Z^+(2n) = \frac{1}{n}\sum_{k=0}^{n-1} k!(2n-k-1)!$. Let us select distinct $0 \le k_1 < k_2 \le n-1$ and take the sum of two terms:

$$k_1!(2n-k_1-1)! + k_2!(2n-k_2-1)! = k_1!(2n-k_2-1)!\left(\frac{(2n-k_1-1)!}{(2n-k_2-1)!} + \frac{k_2!}{k_1!}\right).$$

Setting $k_2 = k_1 + 1$, this becomes $k_1!(2n-k_1-2)!(2n)$. Clearly, $2n^2 \mid k_1!(2n-k_1-2)!(2n)$. Because n is an odd prime, the sum for $Z^+(2n)$ will have an odd number of terms, allowing us to pair every other term with its succeeding term as described, leaving only the last one, $(n-1)!n!$, unpaired. Therefore, $nZ^+(2n) \equiv -n! \pmod{2n^2}$ by Wilson's theorem, and since Wilson quotients for odd primes are always odd, $2n^2 \mid n(1+(n-1)!+n)$. A simple algebraic manipulation allows us to see that $-n! \equiv n^2+n \pmod{2n^2}$, which proves Equation (9). □

Theorem 6. *We have the congruence* $Z^+(p) \equiv B_{2(p-1)} - B_{p-1} - \frac{1}{2}H_{\frac{p-1}{2}} \pmod{p}$, *where* p *is an odd prime.*

Proof. If p is prime, we invoke Equation (7) to observe that

$$\begin{aligned} Z^+(p) &\equiv \sum_{k=0}^{p-1}(-1)^k\left(B_{2(p-1)} - B_{p-1} - H_k\right) \\ &\equiv B_{2(p-1)} - B_{p-1} - \sum_{k=1}^{p-1}(-1)^k H_k \pmod{p}. \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{k=1}^{p-1}(-1)^k H_k &= -1 + \left(1+\frac{1}{2}\right) - \left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots + \left(1+\ldots+\frac{1}{p-1}\right) \\ &= \frac{1}{2}\sum_{n=1}^{\frac{p-1}{2}}\frac{1}{n} = \frac{1}{2}H_{\frac{p-1}{2}} \end{aligned}$$

when p is an odd prime, so we get our desired congruence. □

If $p = 2$, then $Z^+(2) = 1 \equiv B_2 - B_1 + 1 \pmod{2}$. It is worth noting that this congruence bears a close resemblance to the one of $M_{\frac{p-1}{2}}(p)$. Thus, by combining Equation (6) with the above congruence, we obtain the interesting relation

$$Z^+(p) \equiv \frac{1}{2}\left(M_0(p) + M_{\frac{p-1}{2}}(p)\right) \pmod{p}.$$

4. Generating Functions

In this section we derive the exponential generating functions of our M-numbers and Z-numbers.

4.1. M-numbers

Considering the sequences of quotients $M_k(n)$ for $n \in \mathbb{N}$, its exponential generating function[1] for $|x| \leq R$, where $R > 0$ is the radius of convergence, is derived as follows:

$$\mathrm{EG}(M_k(n); x) = \sum_{n=0}^{\infty} M_k(n+k+1)\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left[\frac{1+(-1)^k k!n!}{n+k+1}\right]\frac{x^n}{n!}\,.$$

Note that the first term appearing in the summand is $M_k(k+1)$, as $M_k(n)$ is undefined for $n \leq k$. Also observe that

$$\sum_{n=0}^{\infty}\left[\frac{1+(-1)^k k!n!}{n+k+1}\right]\frac{x^n}{n!} < \sum_{n=0}^{\infty}\left(1+(-1)^k k!\right)x^n\,,$$

which has a radius of convergence $R = 1$. Therefore, $\mathrm{EG}(M_k(n); x)$ has a positive radius of convergence of at least 1. This allows the summation to be split as

$$\mathrm{EG}(M_k(n); x) = \sum_{n=0}^{\infty}\frac{x^n}{n!(n+k+1)} + (-1)^k k! \sum_{n=0}^{\infty}\frac{x^n}{n+k+1}\,,$$

where, from the second summation term above, the radius of convergence, R, equals 1. As dictated by the definitions of the lower incomplete gamma function $\gamma(a,x) = x^a \sum_{n=0}^{\infty}\frac{(-x)^n}{n!(n+a)}$ and the Lerch transcendent $\Phi(x,s,a) = \sum_{n=0}^{\infty}\frac{x^n}{(n+a)^s}$ [2], we derive the exponential generating function by setting $a = k+1$ and $s = 1$,

$$\mathrm{EG}(M_k(n); x) = (-1)^k\left(k!\Phi(x,1,k+1) - \frac{\gamma(k+1,-x)}{x^{k+1}}\right)\,.$$

For the exponential generating function of the signless $M_k^+(p)$, it is not difficult to see from Equation (2) that

$$\mathrm{EG}(M_k^+(n); x) = (-1)^k\,\mathrm{EG}(M_k(n); x) = k!\Phi(x,1,k+1) - \frac{\gamma(k+1,-x)}{x^{k+1}}\,,$$

since the summations are independent of k.

4.2. Z-numbers

We can also derive the exponential generating function of $Z(n)$ for $n \in \mathbb{N}$ and $|x| \leq R$, where $R > 0$ is the radius of convergence,

$$\mathrm{EG}(Z(n); x) = \sum_{n=0}^{\infty} Z(n+1)\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left[1 + \frac{(1+(-1)^n)\,n!}{n+2}\right]\frac{x^n}{n!}\,.$$

[1]Note that ordinary generating functions are of little utility in this instance, as their radius of convergence is zero.

Similarly, the sequence begins at $Z(1)$, as $Z(0)$ is undefined by Equation (3). Since

$$\sum_{n=0}^{\infty}\left[1+\frac{(1+(-1)^n)\,n!}{n+2}\right]\frac{x^n}{n!}<\sum_{n=0}^{\infty}3x^n\,,$$

which has a radius of convergence $R=1$, we have that $\mathrm{EG}(Z(n);x)$ has a positive radius of convergence of at least 1. Additionally

$$\sum_{n=0}^{\infty}\frac{(1+(-1)^n)\,x^n}{n+2}=2\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n+1}\,.$$

This allows the summation to be split as

$$\mathrm{EG}(Z(n);x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^{2n}}{n+1}\,,$$

where the radius of convergence is clearly $R=1$. Of course, $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$, and because the natural logarithm has the Taylor expansion $\ln(z)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{n}$ for $z\in(0,2]$, we see that

$$\ln(1-x^2)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-x^2)^n}{n}=-\sum_{n=1}^{\infty}\frac{x^{2n}}{n}=-x^2\sum_{n=1}^{\infty}\frac{x^{2n-2}}{n}=-x^2\sum_{n=0}^{\infty}\frac{x^{2n}}{n+1}\,.$$

Therefore, we obtain the exponential generating function,

$$\mathrm{EG}(Z(n);x)=e^x-\frac{1}{x^2}\ln\left(1-x^2\right)\,.$$

The exponential generating function for $Z^+(n)$ is a pinch more involved. However, by invoking Equation (4) for $n\in\mathbb{N}$ and $|x|\le R$, we see that

$$\begin{aligned}\mathrm{EG}(Z^+(n);x)&=\sum_{n=0}^{\infty}Z^+(n+1)\frac{x^n}{n!}\\&=\sum_{n=0}^{\infty}\left[\frac{1+(-1)^n}{2n+2}-n!\left(2\Phi(2,1,n+2)+2^{-n-1}i\pi\right)\right]\frac{x^n}{n!}\,.\end{aligned}$$

Since $|\Phi(2,1,n+2)|<1$ for $n\ge 1$, we can say $\mathrm{EG}(Z^+(n);x)<\sum_{n=0}^{\infty}7x^n$, which has a radius of convergence of 1, and thus our generating function converges for $R\ge 1$. Thus, after splitting, simplification, and removal of null terms, we see that

$$\mathrm{EG}(Z^+(n);x)=2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!(2n+2)}-2\sum_{n=0}^{\infty}\Phi(2,1,n+2)x^n-\frac{i\pi}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n\,.$$

For the first summation, let us take the series expansions of hyperbolic trigonometric functions $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ and $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ to derive that

$$\cosh x - 1 = -1 + \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{(2n+2)!}\,,$$

and thus

$$x \sinh x - \cosh x + 1 = \sum_{n=0}^{\infty} \left[\frac{x^{2n+2}}{(2n+1)!} - \frac{x^{2n+2}}{(2n+2)!} \right] = x^2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!(2n+2)}\,,$$

which is precisely our series. Additionally, it is useful to recall that $\sum_{n=0}^{\infty} x^n$ generates $\frac{1}{1-x}$. Putting these together, we obtain the exponential generating function

$$\mathrm{EG}(Z^+(n); x) = \frac{2}{x^2}(x \sinh x - \cosh x + 1) + \frac{i\pi}{x-2} - 2\,\mathrm{G}(\Phi(2,1,n+2); x)\,,$$

where $\mathrm{G}(\Phi(2,1,n+2); x)$ is the generating function of $\Phi(2,1,n+2)$. For the sake of notation, let us define $\tau(z,s,\alpha) := \mathrm{G}(\Phi(z,s,\alpha); x)$, so

$$\mathrm{EG}(Z^+(n); x) = \frac{2}{x^2}(x \sinh x - \cosh x + 1) + \frac{i\pi}{x-2} - 2\tau(2,1,n+2)\,.$$

5. Tables

Below are tables of values for the Z-numbers and unsigned M-numbers.

$n:$	1	2	3	4	5	6	7	8	9	10	11	12	13
$*$	2	1	2	1	9	1	181	1	8065	1	604801	1	68428801

$n:$	14	15	16	17	18	19	20
$*$	1	10897286401	1	2324754432001	1	640237370572801	1

Table 1: Values of $Z(n)$ with n varying from 1 to 20.

$n:$	1	2	3	4	5	6	7	8	9	10	11	12
$*$	2	1	2	4	13	52	259	1536	$\frac{95617}{9}$	84096	750371	7453440

$n:$	13	14	15	16	17
$*$	81566917	974972160	$\frac{189550368001}{15}$	176504832000	2642791002353

Table 2: Values of $Z^+(n)$ with n varying from 1 to 17.

k	$n:1$	2	3	4	5	6	7	8	9	10
0	2	1	1	$\frac{7}{4}$	5	$\frac{121}{6}$	103	$\frac{5041}{8}$	$\frac{40321}{9}$	$\frac{362881}{10}$
1	$*$	0	0	$-\frac{1}{4}$	-1	$-\frac{23}{6}$	-17	$-\frac{719}{8}$	$-\frac{5039}{9}$	$-\frac{40319}{10}$
2	$*$	$*$	1	$\frac{3}{4}$	1	$\frac{13}{6}$	7	$\frac{241}{8}$	$\frac{1441}{9}$	$\frac{10081}{10}$
3	$*$	$*$	$*$	$-\frac{5}{4}$	-1	$-\frac{11}{6}$	-5	$-\frac{143}{8}$	$-\frac{719}{9}$	$-\frac{4319}{10}$
4	$*$	$*$	$*$	$*$	5	$\frac{25}{6}$	7	$\frac{145}{8}$	$\frac{577}{9}$	$\frac{2881}{10}$
5	$*$	$*$	$*$	$*$	$*$	$-\frac{119}{6}$	-17	$-\frac{239}{8}$	$-\frac{719}{9}$	$-\frac{2879}{10}$
6	$*$	$*$	$*$	$*$	$*$	$*$	103	$\frac{721}{8}$	$\frac{1441}{9}$	$\frac{4321}{10}$
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$-\frac{5039}{8}$	$-\frac{5039}{9}$	$-\frac{10079}{10}$
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\frac{40321}{9}$	$\frac{40321}{10}$
9	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$-\frac{362879}{10}$

k	$n:11$	12	13	14	15
0	329891	$\frac{39916801}{12}$	36846277	$\frac{6227020801}{14}$	$\frac{87178291201}{15}$
1	-32989	$-\frac{3628799}{12}$	-3070523	$-\frac{479001599}{14}$	$-\frac{6227020799}{15}$
2	7331	$\frac{725761}{12}$	558277	$\frac{79833601}{14}$	$\frac{958003201}{15}$
3	-2749	$-\frac{241919}{12}$	-167483	$-\frac{21772799}{14}$	$-\frac{239500799}{15}$
4	1571	$\frac{120961}{12}$	74437	$\frac{8709121}{14}$	$\frac{87091201}{15}$
5	-1309	$-\frac{86399}{12}$	-46523	$-\frac{4838399}{14}$	$-\frac{43545599}{15}$
6	1571	$\frac{86401}{12}$	39877	$\frac{3628801}{14}$	$\frac{29030401}{15}$
7	-2749	$-\frac{120959}{12}$	-46523	$-\frac{3628799}{14}$	$-\frac{25401599}{15}$
8	7331	$\frac{241921}{12}$	74437	$\frac{4838401}{14}$	$\frac{29030401}{15}$
9	-32989	$-\frac{725759}{12}$	-167483	$-\frac{8709119}{14}$	$-\frac{43545599}{15}$
10	329891	$\frac{3628801}{12}$	558277	$\frac{21772801}{14}$	$\frac{87091201}{15}$
11	$*$	$-\frac{39916799}{12}$	-3070523	$-\frac{79833599}{14}$	$-\frac{239500799}{15}$
12	$*$	$*$	36846277	$\frac{479001601}{14}$	$\frac{958003201}{15}$
13	$*$	$*$	$*$	$-\frac{6227020799}{14}$	$-\frac{6227020799}{15}$
14	$*$	$*$	$*$	$*$	$\frac{87178291201}{15}$

Table 3: Values of $M_k(n)$ with n varying from 1 to 15 and k varying from 0 to 14.

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