LBM of pattern formation in RD system

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(25) $\frac{\partial A}{\partial t} = -B^2 A + 1 - A + \nabla^2 A$ let $\tilde{A} = A$, $\tilde{B} = B$

(26) $\frac{1}{\tau}\frac{\partial B}{\partial t} = \eta B^2 A - B + \frac{1}{\epsilon^2}\nabla^2 B$

① find f.p. $\begin{cases} -B^2A + 1 - A = 0 & (1) \\ \eta B^2 A - B = 0 & (2) \end{cases}$

(2) $\Rightarrow B(\eta BA - 1) = 0$, $B = 0 \Rightarrow A = 1$ is a f.p.

$\eta BA = 1 \Rightarrow B = \frac{1}{\eta A} \Rightarrow$ (1) is $-\left(\frac{1}{\eta A}\right)^2 A + 1 - A = 0$

$\Rightarrow -\frac{1}{\eta^2 A} + 1 - A = 0$

$\Rightarrow -1 + \eta^2 A - \eta^2 A^2 = 0$

$\Rightarrow \eta A^2 - \eta A + \frac{1}{\eta} = 0 \Rightarrow A = \frac{+\eta \pm \sqrt{\eta^2 - 4\eta\frac{1}{\eta}}}{2\eta} = \frac{\eta \pm \sqrt{\eta^2-4}}{2\eta}$ and $B = \frac{1}{\eta A} \Rightarrow \frac{1}{\eta}\frac{2\eta}{\eta \pm \sqrt{\eta^2-4}} = 2\cdot\frac{\eta \mp \sqrt{\eta^2-4}}{\eta^2 - \eta^2 + 4} = \frac{\eta \mp \sqrt{\eta^2-4}}{2}$.

$\Rightarrow$ f.p. $x_1^* = (1, 0)$

$x_{2,3}^* = \left(\frac{\eta \pm \sqrt{\eta^2-4}}{2\eta}, \frac{\eta \mp \sqrt{\eta^2-4}}{2}\right)$

② perturbation + linearization consider non-isotropy?

author assume isotropy (各向同性), system behave the same in each direction) So we can reduce the system to 1D

const

ϕ amplitude

α growth rate

k wave number (单位空间有多少波峰 - 与波长有关)

let $A = \underline{A_e} + \phi_A e^{\alpha t + ikx}$, $B = \underline{B_e} + \phi_B e^{\alpha t + ikx}$

for (25), $\frac{\partial A}{\partial t} = \alpha\phi_A e^{\alpha t+ikx} = -(B_e + \phi_B e^{\alpha t+ikx})^2(A_e + \phi_A e^{\alpha t+ikx}) + 1 - (A_e + \phi_A e^{\alpha t+ikx}) - k^2\phi_A e^{\alpha t+ikx}$

consider term up to O(1), $B^2A = (B_e^2 + 2B_e\phi_B e^{\alpha t+ikx} + o(2))(A_e + \phi_A e^{\alpha t+ikx})$

$= B_e^2 A_e + B_e^2\phi_A e^{\alpha t+ikx} + 2A_eB_e\phi_B e^{\alpha t+ikx}$

collect coefficient of $O(1)\ e^{\alpha t+ikx}$ as constant term gives (A_e, B_e) : (25) $\Rightarrow \alpha\phi_A = -B_e^2\phi_A - 2A_eB_e\phi_B - \phi_A - k^2\phi_A$

$$\boxed{\alpha\phi_A = -(B_e^2 + 1 + k^2)\phi_A - 2A_eB_e\phi_B \quad (3)}$$

for (26), $\frac{1}{\tau}\frac{\partial B}{\partial t} = \frac{1}{\tau}\alpha\phi_B = \eta(B_e^2\phi_A + 2A_eB_e\phi_B) - \phi_B - \frac{k^2}{\epsilon^2}\phi_B$

$\Rightarrow \frac{1}{\tau}\alpha\phi_B = \eta B_e^2\phi_A + (2\eta A_eB_e - 1 - \frac{k^2}{\epsilon^2})\phi_B$

$$\Rightarrow \boxed{\alpha\phi_B = \tau\eta B_e^2\phi_A + (2\eta A_eB_e - 1 - \frac{k^2}{\epsilon^2})\tau\phi_B \quad (4)}$$

in matrix vector form:

for some reason author use this order

$$\boxed{\alpha\begin{pmatrix}\phi_B \\ \phi_A\end{pmatrix} = \underbrace{\begin{pmatrix} \tau(\overset{=1}{2\eta A_eB_e} - 1 - \frac{k^2}{\epsilon^2}) & \overset{=\eta B}{\tau\eta B_e^2} \\ -2A_eB_e & -(B_e^2 + 1 + k^2) \end{pmatrix}}_{J}\begin{pmatrix}\phi_B \\ \phi_A\end{pmatrix}}$$

for inhomogeneous

only for $x_{2,3} = \frac{1}{\eta}$

$$\Rightarrow \begin{pmatrix}\phi_B \\ \phi_A\end{pmatrix} = \underbrace{\begin{pmatrix} \tau - \tau\frac{k^2}{\epsilon^2} & \tau\eta B_e^2 \\ -\frac{2}{\eta} & -\eta B - k^2 \end{pmatrix}}_{M}\begin{pmatrix}\phi_B \\ \phi_A\end{pmatrix}$$

③ Stability

at $x_1^* = (1,0) = (A,B)$, $A_1 = J|_{x_1^*} = \begin{pmatrix} \tau(-1-\boxed{\frac{k^2}{\epsilon^2}}) & 0 \\ 0 & -(1+\boxed{k^2}) \end{pmatrix} \begin{pmatrix} \varphi_B \\ \varphi_A \end{pmatrix}$ with respect to t, why? another paper say the same

$\Rightarrow \lambda_1 = -\tau(1+\frac{k^2}{\epsilon^2})$, $\lambda_2 = -(1+k^2)$, $\tau > 0$ (from physics)

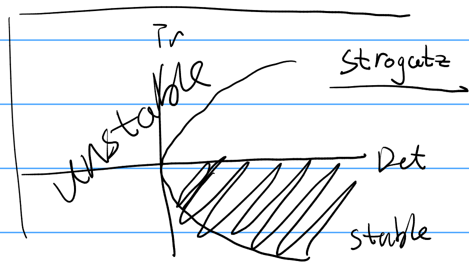
both $\lambda_1, \lambda_2 < 0 \Rightarrow x_1^*$ is stable (only for very small perturbation, $\boxed{\text{unstable for large perturb}}$)

at $x_{2,3}^* = \left(\frac{\eta \pm \sqrt{\eta^2-4}}{2\eta}, \frac{\eta \mp \sqrt{\eta^2-4}}{2}\right)$

$$A_{2,3} = J|_{x_{2,3}^*} = \begin{pmatrix} \tau\left(2\eta \frac{\eta\pm\sqrt{\eta^2-4}}{2\eta} \frac{\eta\mp\sqrt{\eta^2-4}}{2} - 1 - \frac{k^2}{\epsilon^2}\right) & \tau\eta\left(\frac{\eta\mp\sqrt{\eta^2-4}}{2}\right)^2 \\ -2\frac{\eta\pm\sqrt{\eta^2-4}}{2\eta} \frac{\eta\mp\sqrt{\eta^2-4}}{2} & -\left(k^2 + \left(\frac{\eta\mp\sqrt{\eta^2-4}}{2}\right)^2 + 1\right) \end{pmatrix}$$

$$\Rightarrow A_{2,3} = \begin{pmatrix} \tau\left(\frac{1}{2}\left(\eta^2-(\eta^2-4)\right)\right) - 1 - \frac{k^2}{\epsilon^2} & \frac{\tau\eta}{4}\left(\eta^2 \mp 2\eta\sqrt{\eta^2-4} + \eta^2 - 4\right) \\ -\frac{2}{4\eta}\left(\eta^2 - \eta^2 + 4\right) & -k^2 - 1 - \left(\eta^2 \mp 2\eta\sqrt{\eta^2-4} + \eta^2 - 4\right) \end{pmatrix}$$

$$= \begin{pmatrix} \tau\left(1 - \frac{k^2}{\epsilon^2}\right) & \frac{\tau\eta^3}{4}\left(2\eta^2 \mp 2\eta\sqrt{\eta^2-4} - 4\right) \\ -\frac{2}{\eta} & -k^2 - 1 - \left(2\eta^2 \mp 2\eta\sqrt{\eta^2-4} - 4\right) \end{pmatrix}$$



$Tr(A_{2,3}) = \tau - \frac{k^2}{\epsilon}\tau - k^2 - 1 - \left(2\eta^2 \mp 2\eta\sqrt{\eta^2-4} - 4\right)$

$Det(A_{2,3}) = -\tau\left(1 - \frac{k^2}{\epsilon^2}\right)\left(k^2 + 1 + \left(2\eta^2 \mp 2\eta\sqrt{\eta^2-4} - 4\right)\right)$
$+ \frac{2}{\eta} \frac{\tau\eta^3}{4}\left(2\eta^2 \mp 2\eta\sqrt{\eta^2-4} - 4\right)$

Anther consider going from stable to saddle (maybe possible from stable to unstable?)

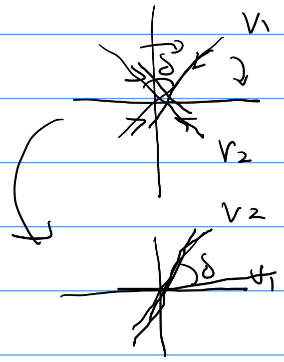
Author find the formula for k s.t. $Det(A_{2,3}) = 0$, k is wave number,

Unstable solution of nonautonomous Linear DE

we want to know why stable f.p. can be unstable

Real Case: $\lambda_1 < \lambda_2 < 0$ with all real entry B

set $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $v_2 = \begin{pmatrix}\cos\delta\\ \sin\delta\end{pmatrix}$



$0<\delta<\frac{\pi}{2}$

Since $PBP^{-1} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ (eigen decomposition)

then $P = (v_1\ v_2) = \begin{pmatrix}1 & \cos\delta\\ 0 & \sin\delta\end{pmatrix}$

to get $B = P\ \mathrm{diag}(\lambda_1,\lambda_2)\ P^{-1}$.

then $P^{-1} = \frac{1}{\sin\delta}\begin{pmatrix}\sin\delta & -\cos\delta\\ 0 & 1\end{pmatrix}$

$\Rightarrow B = \begin{pmatrix}\lambda_1 & \lambda_2-\lambda_1\cos\delta\\ 0 & \lambda_2\end{pmatrix}$

we demand $\frac{d\|x\|}{dt} < 0$ (by def of stable)

then $= \frac{d}{dt}(x^Tx) = 2x^Tx'$, and $\boxed{x' = A(t)x}$ this is the system we are studying

$= 2x^TAx$

So, we want to know when $x^TAx > 0$ for a stable f.p. (so distance is increasing)

then $x^TBx = P(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + (\lambda_2-\lambda_1)x_1x_2\cot(\delta)$

author switch to unit circle so $x_1^2+x_2^2=1$, (rule out influence of factor

let $x = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ so by rotate θ, we have access of all possible combination of $\bar{x}$.

then $\frac{dr}{d\theta} = (\lambda_2-\lambda_1)\cos(\delta-2\theta)\csc(\delta)$

$\frac{dr}{d\theta} = 0 \Leftrightarrow \cos(\delta-2\theta)=0 \Rightarrow \theta = \frac{\delta}{2}-\frac{\pi}{4}$, to check it is max, consider $\theta<\theta_{max}<\theta_2$,

compare derivative

let $\lambda_2/\lambda_1 = p$, $0\le p\le 1$, then $x^TBx > 0$ for some $x \Leftrightarrow \sin(\delta) < \frac{1-p}{1+p}$

conclusion if $\sin(\delta) \ll 1$, then sign of eigenvalue doesn't work any more

Complex

skip

for non autonomous system, eigenvalue negative for all t is not enough, it must contract fast enough for system to be stable