Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 8: Detailed Models of Stellar Evolution

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Detailed picture of stellar evolution

- As opposed to the simple treatment we have adopted so far, a general treatment of stellar evolution needs to take into account the details of opacity, nuclear energy production, and equation of state.
- Then the equations of stellar evolution need to be solved *numerically* and due to their non-linearity results that might not be intuitively clear can arise.
- Numerical solutions have been available since the 1950s but we will not go to the details here.
- The goal of the modelling efforts is to explain the observed Herzsprung-Russell diagram, characterised by $(\log T_{\rm eff}, \log L)$ plane, as opposed to $(\log \rho_{\rm c}, \log T_{\rm c})$ on the previous lecture.

Recap: Herzsprung-Russell diagram

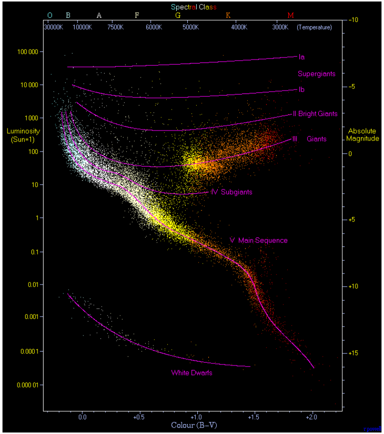
- We saw earlier that there is an observed relation between the luminosity and effective temperature of main sequence stars

$$\log L = \alpha \log T_{\mathrm{eff}} + \mathrm{const.}, \qquad (1)$$

  where the slope α is varies with L.
- Another correlation exists between luminosity and mass:

$$L \propto M^{\nu}, \qquad (2)$$

  with $\nu \approx 3 \ldots 5$.
- The models need to further explain why stars cluster (i.e., spend much of their lifetime) at certain regions in the diagram.



Credits: Richard Powell / Wikipedia

Hayashi zone and pre-main-sequence phase

- Assume a fully convective star of mass M and radius R. Then we can adopt an interior structure corresponding to a polytrope of index $n = 1/(\gamma_{\mathrm{a}} - 1)$,

$$p = K\rho^{1+\frac{1}{n}}. \tag{3}$$

- K is related to M and R via the Lane-Emden equation:

$$K^n = C_n G^n M^{n-1} R^{3-n}, \tag{4}$$

  where C_n depends on the polytropic index n:

$$C_n = \frac{4\pi}{(n+1)^n} \frac{R_n^{n-3}}{M_n^{n-1}}. \tag{5}$$

- R is a free parameter that is fixed by joining the fully convective interior to a radiative photosphere above $r = R$.

Hayashi zone and pre-main-sequence phase

- The photosphere needs to be able to radiate away all of the incoming energy flux. This is determined by the thermodynamic structure, i.e., the drop of p, ρ, and T accross it.
- In Hydrostatic equilibrium

$$\frac{dp}{dr} \approx -\rho \frac{GM}{R^2}, \tag{6}$$

which can be integrated from R to the point where p vanishes

$$p_R = \frac{GM}{R^2} \int_R^\infty \rho dr. \tag{7}$$

- Furthermore, the optical depth of the photosphere, characterised by $T_{\rm eff}$, is of the order on unity and thus $\int_R^\infty \kappa \rho dr = \overline{\kappa} \int_R^\infty \rho dr$, where $\overline{\kappa}$ is the mean opacity in the photosphere.
- Taking $\overline{\kappa} = \kappa(R)$ and assuming it to be a power law in ρ_R and $T_{\rm eff}$ gives:

$$\kappa_0 \rho_R^a T_{\rm eff}^b \int_R^\infty \rho dr = 1. \tag{8}$$

Hayashi zone and pre-main-sequence phase

- Combining Eqs. (7) and (8) gives:

$$p_R = \frac{GM}{R^2 \kappa_0} \rho_R^{-a} T_{\mathrm{eff}}^{-b}. \tag{9}$$

- Yet another relation between the thermodynamic quantities at R is given by the equation of state, here taken to be ideal gas equation:

$$p_R = \frac{\mathcal{R}}{\mu} \rho_R T_{\mathrm{eff}}. \tag{10}$$

- Finally, the temperature at R is related to the luminosity via

$$L = 4\pi R^2 T_{\mathrm{eff}}^4. \tag{11}$$

- Now we have four equations that describe the surface of the star: Eqs.(3) (with Eqs. (4) and (5)), (9), (10), and (11)

Hayashi zone and pre-main-sequence phase

- These read in logarithmic form:

$$n \log p_R = (n-1)\log M + (3-n)\log R + (n+1)\log \rho_r + \text{const.} \tag{12}$$
$$\log p_R = \log M - 2\log R - a\log \rho_r - b\log T_{\rm eff} + \text{const.} \tag{13}$$
$$\log p_R = \log \rho_R + \log T_{\rm eff} + \text{const.} \tag{14}$$
$$\log L = 2\log R + 4\log T_{\rm eff} + \text{const.} \tag{15}$$

- Eliminating $\log R$, $\log \rho_R$, and $\log p_R$ yields:

$$\log L = A\log T_{\rm eff} + B\log M + \text{const.} \tag{16}$$
$$A = \frac{(7-n)(a+1)-4-a+b}{0.5(3-n)(a+1)-1}, \quad B = -\frac{(n-1)(a+1)+1}{0.5(3-n)(a+1)-1}. \tag{17}$$

- This relation traces the *Hayashi track* in the HR diagram. These should not be interpreted as evolutionary tracks but rather as an asymptote.

Hayashi zone and pre-main-sequence phase

- We assume for simplicity that $a = 1$ which is reasonably accurate, such that

$$A = \frac{9-2n+b}{2-n}, \quad B = -\frac{2n-1}{2-n}. \tag{18}$$

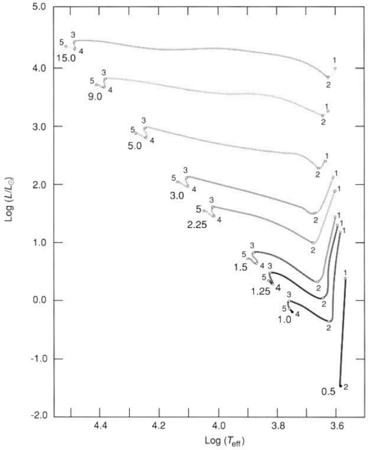
  b varies much more but is usually positive
- Dynamical stability requires that $n < 3$ and therefore the polytropic index is limited to the range $1.5 \leq n < 3$.
- For $b \approx 4$ and $n = 1.5$ we find that $A = 20$. This means that the Hayashi track is almost vertical in the $(\log T_{\rm eff}, \log L)$ plane.
- As a function of mass the tracks are stacked near each other and higher mass leads to a shift toward higher temperatures because A and B have opposite signs.
- The slope changes with composition that can be associated with an effective polytropic index.

Hayashi zone and pre-main-sequence phase

- The signifigance of the Hayashi track can be seen from considering $\overline{\gamma}$ which is an average value of $\gamma = \frac{d \ln p}{d \ln \rho}$ over the whole star. $\overline{\gamma}_{\rm a}$ is the corresponding adiabatic index.
- For a fully convective star $\overline{\gamma} = \overline{\gamma}_{\rm a}$.
- If any part of the star is radiative with $\gamma < \gamma_{\rm a}$, then $\overline{\gamma} < \overline{\gamma}_{\rm a}$. Correspondingly, the average polytropic index $n > n_{\rm a}$ where $n_{\rm a}$ is the adiabatic polytropic index defining the Hayashi track.
- If $\overline{\gamma} > \overline{\gamma}_{\rm a}$, the situation is unstable and therefore such state is "forbidden". In practise in such a situation, convection in the star would very quickly restore near-adiabaticity by transporting any excess heat to the surface because a very small superadiabaticity is enough to transport massive amounts of energy (homework!).

Hayashi zone and pre-main-sequence phase

- Stars from from contrating gas clouds (molecular clouds) through dynamical collapse. These clouds are (parsecs) and fragment in the process.
- Most of the gas in such clouds is in the form of molecular hydrogen (H_2). The collapse happens in dynamical timescale $\tau_{\rm dyn} \propto \rho^{-1.2}$.
- Gradually the H_2 molecules are dissociated, after which hydrogen and later helium start to be ionised. These processes use up most of the energy from continuing collapse and the temperature stays nearly constant.
- Finally the ionisation is nearly complete and the temperature starts to increase and a hydrostatic equilibrium is restored. The object is now a protostar.



Credits: Iben (1965), Astrophys. J., 141

Hayashi zone and pre-main-sequence phase

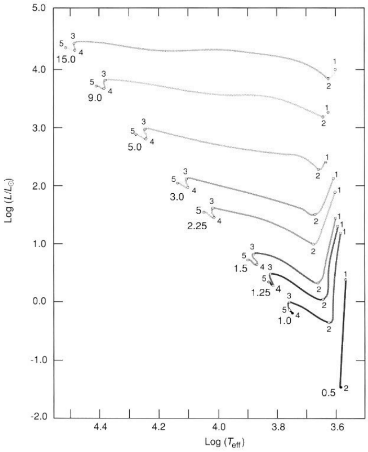
- Estimate of protostellar radius can be obtained by assuming that all of the gravitational energy is spent to dissociate H_2 and ionize H and He. Then,

$$\alpha \frac{GM^2}{R_{\rm ps}} \approx \frac{M}{m_{\rm H}} \left(\frac{X}{2}\chi_{\rm H_2} + X\chi_{\rm H} + \frac{Y}{4}\chi_{\rm He} \right), \quad (19)$$

where $\chi_{\rm H_2} = 4.5$ eV, $\chi_{\rm H} = 13.6$ eV, and $\chi_{\rm He} = 79$ eV.

- Taking $Y \approx 1 - X$ and $\alpha = \frac{1}{2}$ gives

$$\frac{R_{\rm ps}}{R_\odot} \approx \frac{50}{1 - 0.2X} \frac{M}{M_\odot}. \quad (20)$$



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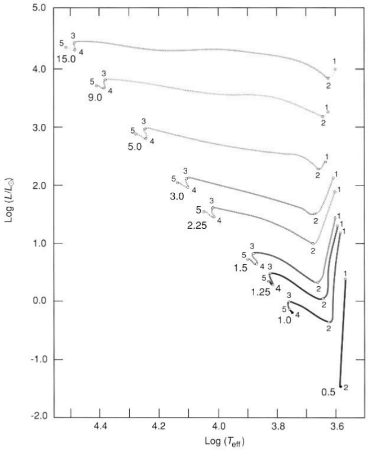
Hayashi zone and pre-main-sequence phase

- Recalling the average temperature from virial theorem and inserting the estimate for $R_{\rm ps}$ with $X = 0.7$ gives:

$$\overline{T} = \frac{\alpha}{3}\frac{\mu}{k}\frac{GMm_{\rm H}}{R_{\rm ps}} \approx 6\cdot 10^4\ \mathrm{K}. \qquad (21)$$

  Note that the temperature is independent of M.

- At this starting point on the Hayashi track the star is fully convective and the gas is still opaque.
- Contraction continues until all of the gas is ionized. The opacity drops first in the interior and the convection zone recedes. $T_{\rm eff}$ starts to rise slowly.
- Nuclear reactions start gradually when core temperature increases and increase the luminosity. Evolutionary track is complicated by ignition of different branches of hydrogen burning.



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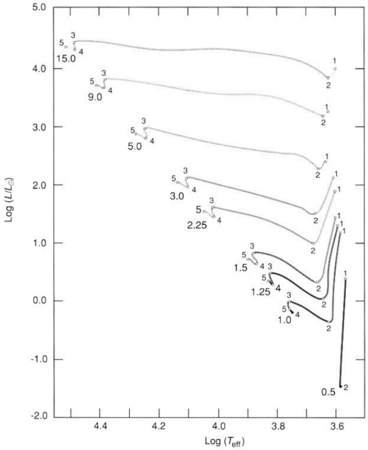
Hayashi zone and pre-main-sequence phase

Table 9.1 Evolutionary lifetimes (years)

$M/M_\odot$	*1–2*	*2–3*	*3–4*	*4–5*
15	6.7(2)	2.6(4)	1.3(4)	6.0(3)
9	1.4(3)	7.8(4)	2.3(4)	1.8(4)
5	2.9(4)	2.8(5)	7.4(4)	6.8(4)
3	2.1(5)	1.0(6)	2.2(5)	2.8(5)
2.25	5.9(5)	2.2(6)	5.0(5)	6.7(5)
1.5	2.4(6)	6.3(6)	1.8(6)	3.0(6)
1.25	4.0(6)	1.0(7)	3.5(6)	1.0(7)
1.0	8.9(6)	1.6(7)	8.9(6)	1.6(7)
0.5	1.6(8)			

Note: powers of 10 are given in parentheses.

Credits: Prialnik.



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- The time that stars spend in the PMS phase depends strongly on the mass.

Main-sequence phase

- TBA

Main-sequence phase

- TBA

Main-sequence phase

- TBA

Red Giant phase

- TBA

Red Giant phase

- TBA

Red Giant phase

- TBA