Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 11: Solar oscillations and helioseismology

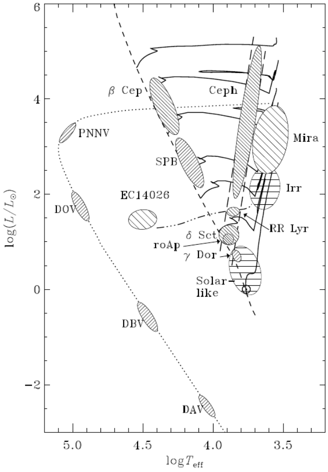
Petri Käpylä Ivan Milić
pkapyla,milic@leibniz-kis.de

Institut für Sonnenphysik - KIS, Freiburg

July 2, 2024

Pulsating stars - Cursory introduction

- A zoo of oscillating stars have been observed.
- Many (but not all) of these occur in stars that have left the main sequence (giant and white dwarf phases).
- Some of these stars (e.g. the Cepheids) are very important in astrophysical distance determinations as *standard candles*. Ivan will talk more about these next week.
- Here we will concentrate only on solar like oscillations.



Credits: Jørgen Christensen-Dalsgaard, lecture notes on stellar oscillations.

Solar oscillations

- The photosphere of the Sun supports so-called 5 minute oscillations. These were discovered in the 1960s.
- In 1974-75 it was discovered that these oscillations have a spectrum of discrete frequencies.
- A wide spectrum of oscillations in the range $2 \ldots 15$ minutes are also observed.
- These are identified as acoustic (sound) waves where the pressure force is the restoring force. Therefore we refer to these as p modes.
- Another type of waves where the gravitational force (gravity waves or g modes) is the restoring force occur in radiative layers.
- There is also a surface gravity wave (f mode) that is observable.
- The discrete spectrum of the p modes is the result of reflecting boundaries and therefore these waves can tell us about the interior structure of the Sun.

Equations of hydrodynamics

- Continuity (mass conservation):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0. \tag{1}$$

- Motion (momentum conservation):

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g}. \tag{2}$$

- Gravity:

$$\boldsymbol{g} = -\boldsymbol{\nabla} \Phi, \quad \nabla^2 \Phi = 4\pi G \rho. \tag{3}$$

- Energy conservation:

$$\rho \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{\gamma_3 - 1} \left(\frac{\mathrm{d}p}{\mathrm{d}t} - \gamma_1 \frac{p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) = \rho \epsilon - \boldsymbol{\nabla} \cdot \boldsymbol{F}. \tag{4}$$

Equations of hydrodynamics

- The different γs are defined via

$$\gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_{\mathrm{ad}}, \quad \frac{\gamma_2 - 1}{\gamma_2} = \left(\frac{\partial \ln T}{\partial \ln p}\right)_{\mathrm{ad}}, \quad \gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{\mathrm{ad}}. \tag{5}$$

- For ideal gas law:

$$\gamma_1 = \gamma_2 = \gamma_3 \equiv \gamma = \tfrac{5}{3}. \tag{6}$$

- Assume small perturbations (primes) around an equilibrium (subscript 0). The latter is:

$$\boldsymbol{u}_0 = 0, \quad \boldsymbol{\nabla} p_0 = \rho_0 \boldsymbol{g}_0, \quad \boldsymbol{g}_0 = -\frac{G m_0}{r^2}\hat{\boldsymbol{e}}_r, \tag{7}$$

$$\rho_0 \epsilon_0 = \boldsymbol{\nabla} \cdot \boldsymbol{F}_0 = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 F_0) = \frac{1}{4\pi r^2}\frac{\mathrm{d}L_0}{\mathrm{d}r}. \tag{8}$$

- Perturbations:

$$p(\boldsymbol{r}, t) = p_0(\boldsymbol{r}, t) + p'(\boldsymbol{r}, t), \quad \text{etc.}, \quad \boldsymbol{u}' = \frac{\partial \delta \boldsymbol{r}}{\partial t}. \tag{9}$$

Linearized equations

- Continuity:
$$\rho' + \boldsymbol{\nabla} \cdot (\rho_0 \delta \boldsymbol{r}) = 0. \tag{10}$$
- Motion (momentum conservation):
$$\rho_0 \frac{\partial^2 \delta \boldsymbol{r}}{\partial t^2} = \rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla} p' + \rho_0 \boldsymbol{g}' + \rho' \boldsymbol{g}_0. \tag{11}$$
- Poisson equation:
$$\nabla^2 \Phi' = 4\pi G \rho', \quad \boldsymbol{g}' = -\boldsymbol{\nabla} \Phi'. \tag{12}$$
- Adiabaticity:
$$p' = \gamma_{1,0} \frac{p_0}{\rho_0} \rho'. \tag{13}$$

Examples of simple waves: Acoustic waves (p modes)

- *Acoustic waves*: Consider a spatially homogeneous case where the derivatives of the 0-quantities vanish.
- This implies vanishing $\boldsymbol{g}_0$ which does not happen in reality but is a fair approximation if the 0-state varies slowly in comparison to the perturbations.
- For rapid perturbations oppositely signed ρ' nearly cancel and hence Φ' is small.
- Finally, the adiabatic approximation is assumed.
- Equation of motion:

$$\rho_0 \frac{\partial^2 \delta \boldsymbol{r}}{\partial t^2} = -\boldsymbol{\nabla} p', \quad \xrightarrow{\boldsymbol{\nabla}\cdot(\ldots)} \quad \rho_0 \frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla} \cdot \delta \boldsymbol{r}) = -\nabla^2 p'. \tag{14}$$

Examples of simple waves: Acoustic waves (p modes)

- Use Eqs. (10) and (13) to eliminate $\delta\boldsymbol{r}$ and p' gives

$$-\rho_0 \frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla} \cdot \delta\boldsymbol{r}) = \frac{\partial^2 \rho'}{\partial t^2} = \gamma_{1,0} \frac{p_0}{\rho_0} \nabla^2 \rho' \equiv c_0^2 \nabla^2 \rho', \tag{15}$$

- where

$$c_0^2 = \gamma_{1,0} \frac{p_0}{\rho_0}, \tag{16}$$

  has the unit of squared velocity. We identify Eq. (15) as a wave equation and c_0^2 as the *speed of sound*.
- Solutions are in the form of plance waves:

$$\rho' = a \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]. \tag{17}$$

- (The physical solution is the *real part* of the complex solution.)

Examples of simple waves: Acoustic waves (p modes)

- Substituting (17) back to Eq. (15) yields the *dispersion relation* of sound waves:

$$\omega^2 = c_0^2 |\boldsymbol{k}|^2. \tag{18}$$

- What can you conclude about sound waves based on this equation?

- Substituting (17) back to Eq. (15) yields the *dispersion relation* of sound waves:

$$\omega^2 = c_0^2 |\boldsymbol{k}|^2. \tag{19}$$

- What can you conclude about sound waves based on this equation?
- The permissible frequencies depend on the wavenumber, i.e. spatial scale of the cavity where the sound waves propagate. In stars this is bounded by the size of the star. The speed of sound also varies internally such that the combination of c_0 and $|\boldsymbol{k}|$ determines the frequencies.
- For an ideal gas the squared speed of sound is:

$$c_0^2 = \gamma \frac{p_0}{\rho_0} = \gamma \frac{\mathcal{R}}{\mu} T_0 \propto \frac{T_0}{\mu}. \tag{20}$$

- With a suitable choice of phases the (real parts) of solutions read:

$$\rho' = a \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t), \tag{21}$$

$$p' = c_0^2 a \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t), \tag{22}$$

$$\delta \boldsymbol{r} = \frac{c_0^2}{\rho_0 \omega^2} a \cos\left(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \frac{\pi}{2}\right) \boldsymbol{k}. \tag{23}$$

Examples of simple waves: Internal gravity waves (g modes)

- Waves that exist in convectively stable regions instead of convection (radiative interior of the Sun).
- Tutorials. . .

Examples of simple waves: Surface gravity waves (f mode)

- Surface gravity waves occur at the interface of between two media where the gravity and buoyancy both try to restore equilibrium (e.g., waves in the sea).
- Tutorials. . .

Oscillations in the Sun

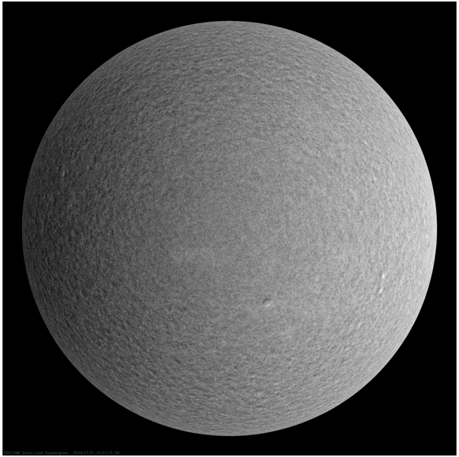
- As already alluded to, waves in the Sun are low amplitude and can therefore be treated as linear perturbations. Their periods are much shorter than the thermal timescale such that the adiabatic approximation is valid.
- The oscillation modes are thought to be excited stochastically by convective motions. The damping of modes is also due to convective flows and turbulent motions. Therefore the oscillations occur on a broad frequency spectrum.
- If the driving is stochastic why doesn't destructive interference wipe out the waves?

Oscillations in the Sun

- As already alluded to, waves in the Sun are low amplitude and can therefore be treated as linear perturbations. Their periods are much shorter than the thermal timescale such that the adiabatic approximation is valid.
- The oscillation modes are thought to be excited stochastically by convective motions. The damping of modes is also due to convective flows and turbulent motions. Therefore the oscillations occur on a broad frequency spectrum.
- If the driving is stochastic why doesn't destructive interference wipe out the waves?
- The observed waves are *standing waves* of the *resonant modes*.
- The amplitudes of the modes are determined by an equilibrium between energy supply and damping.

Oscillations in the Sun: Dopplergram

- Dopplergram of the Sun: radial velocity signal from convection and waves.
- These can be obtained from the full disk or from high-resolution observations of smaller patches of the Sun.
- Here we discuss only global modes and full disk observations.



Credit: Solar Dynamics Observatory

Spherical Harmonics

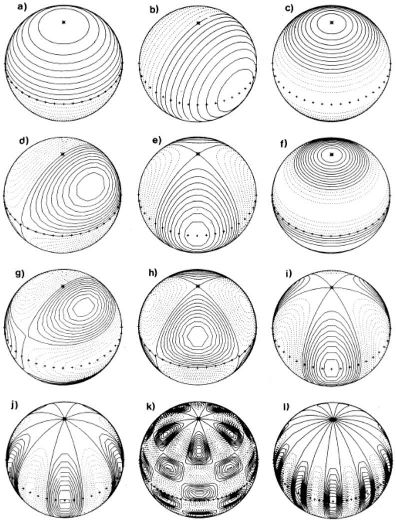
- Global analysis is done by expanding the data in terms of *spherical harmonics* $Y_\ell^m(\theta, \phi)$:

$$f(r, \theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m(t) r^\ell Y_\ell^m(\theta, \phi), \tag{24}$$

where f_ℓ^m is a scalar coefficient and

$$Y_\ell^m(\theta, \phi) = C(m, \ell) P_\ell^m(\cos\theta) e^{im\phi}, \tag{25}$$

where $C(m, \ell)$ is a normalization factor and P_ℓ^m are the associated Legendre polynomials.



Credits: Christensen-Dalsgaard, Lecture notes on stellar oscillations

Fourier transform

- Energy density:

$$E \equiv \int_{-\infty}^{\infty} |x(t)|^2 dt. \tag{26}$$

- Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(\omega)|^2 dt, \tag{27}$$

  where

$$\hat{x}(\omega) \equiv \int_{-\infty}^{\infty} e^{-i\omega t} x(t) dt, \tag{28}$$

  is the *Fourier transform*.

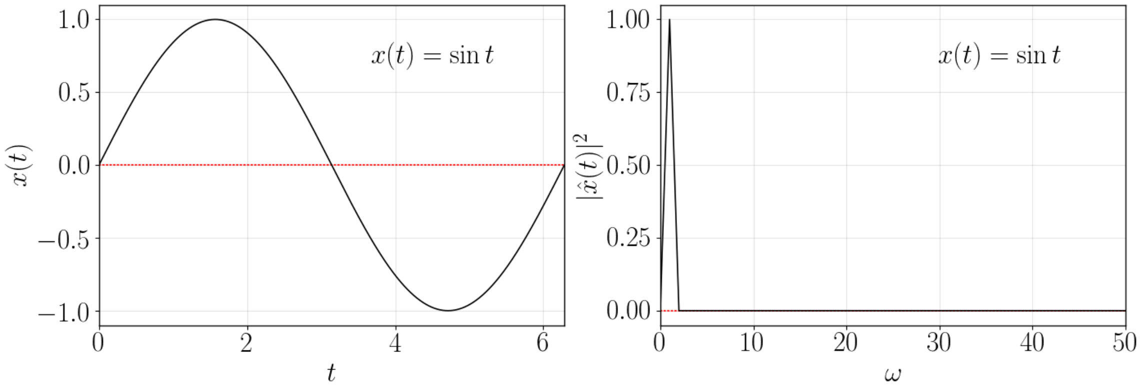
- The *spectral energy density* is given by:

$$E(\omega) = |\hat{x}(\omega)|^2, \tag{29}$$

  which can be used to detect periodic signals.

- Consider a signal $x(t) = \sin(\omega_0 t)$ and its Fourier transform:

$$\hat{x}(\omega) = \frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]. \tag{30}$$

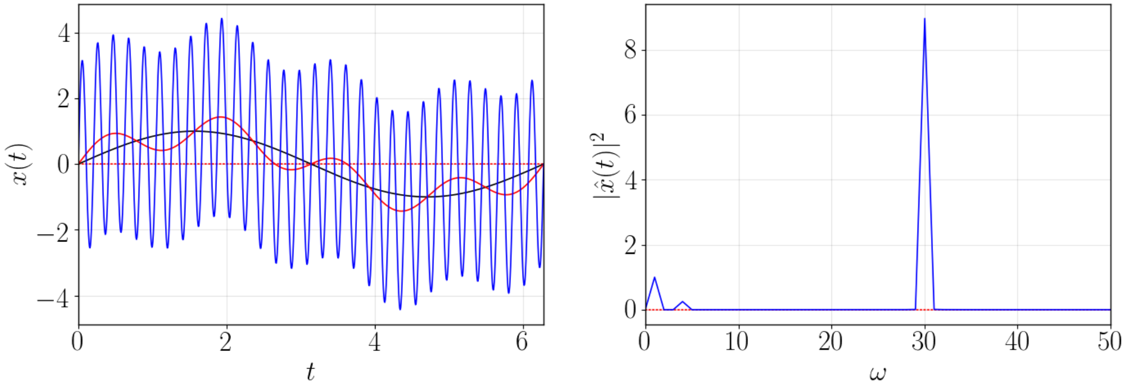


Function x and its *power spectrum*.

Fourier transform

- Consider now a signal:

$$x(t) = \sin(\omega t) + \frac{1}{2}\sin(3t) + 3\sin(30t). \tag{31}$$

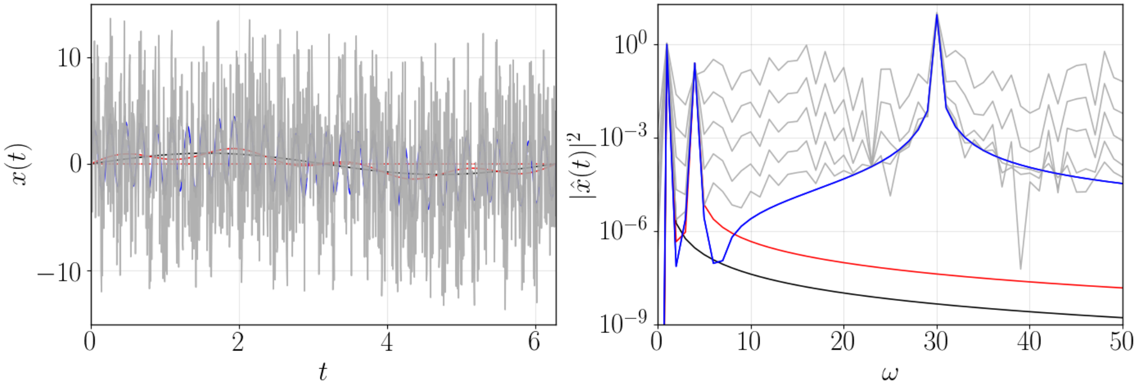


Function x and its *power spectrum*.

Fourier transform

- Consider now a signal:

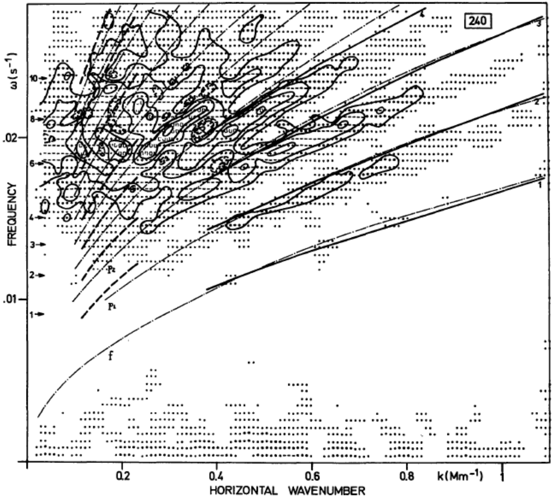
$$x(t) = \sin(\omega t) + \frac{1}{2}\sin(3t) + 3\sin(30t) + \text{noise}. \quad (32)$$



Function x and its *power spectrum*.

Oscillations in the Sun: Experimental verification

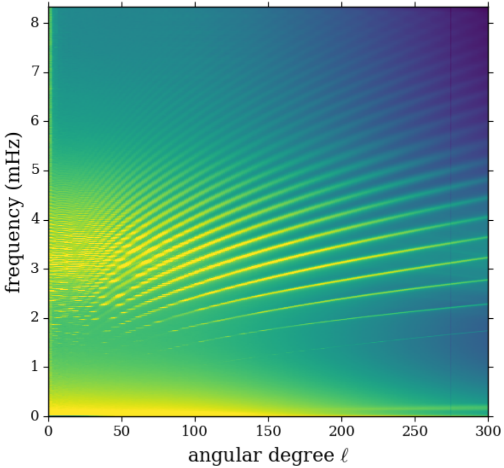
Franz-Ludwig Deubner at Capri, 1974.



Credits: Deubner (1975), A&A, **44**, 371.

Oscillations in the Sun

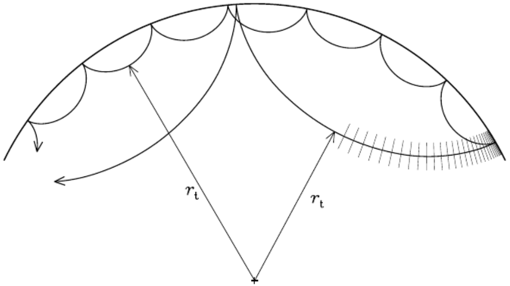
- Modern k-ω diagram shows many ridges that can be identified as p modes.
- Figure on the right shows the power spectrum of medium angular degree ℓ solar oscillations, computed for 144 days of data from the MDI instrument aboard SOHO.
- Long time series is needed to resolve the "ridges" (the longer the better).
- Preferably the time series should not have gaps $\Rightarrow$ continuous observations from space (SOHO, SDO) or from networks of telescopes around the world (e.g. BISON, GONG).



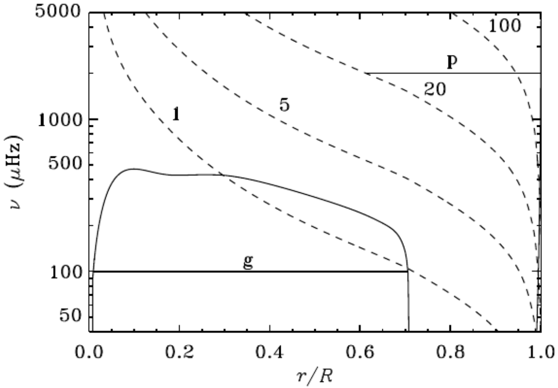
Credits: SOHO/MDI.

Oscillations in the Sun: Mode trapping

- Modes with different spherical harmonic degree ℓ and frequency ν are *trapped* at different depths.



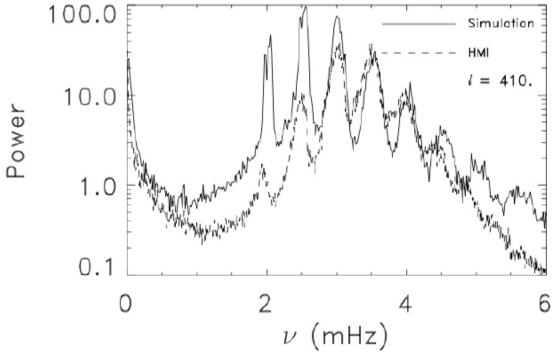
Credits: Christensen-Dalsgaard, Lecture notes on stellar oscillations



Credits: Christensen-Dalsgaard, Lecture notes on stellar oscillations. Numbers refer to ℓ.

Oscillations in the Sun vs. in simulations

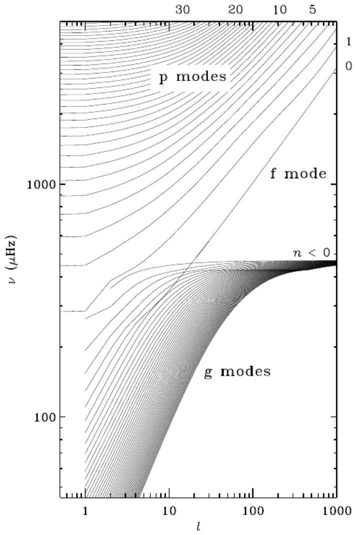
- Modes in solar surface convection simulations agree well with solar observations.
- This supports the idea that the oscillations are excited by convection.
- Simulations can be used to probe effects of magnetic fields, convection zone structure, etc. on oscillation modes.



Credits: Stein et al. (2024), arXiv:2405.02483

How good are solar and stellar structure models?

- Oscillation frequencies from a *model Sun*, i.e., from a model that is a solution of the stellar structure equations for the solar age.



Credits: Christensen-Dalsgaard, Lecture notes on stellar oscillations

Duvall's law and sound speed inversion

- Duvall's law (Duvall 1982, Nature, 300, 242):

$$F\left(\frac{\omega}{L}\right) = \int_{r_t}^{R} \left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{1/2} \frac{\mathrm{d}r}{c} = \frac{[n+\alpha(\omega)]\pi}{\omega}, \quad L = \ell + \tfrac{1}{2}. \tag{33}$$

- This equation is an implicit equation for c which is the sound speed and allows to solve for it *without resorting to any model of the solar interior*.

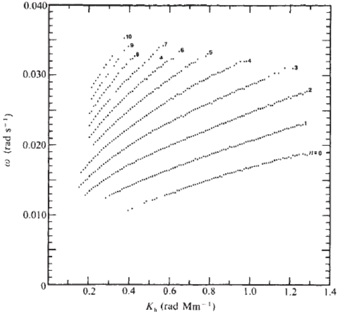


Fig. 1 Mode frequencies derived by a centroid analysis from a two-dimensional (ω, k_h) power spectrum of velocity observations made at Kitt Peak in May 1980 (ref. 1). The curves are labelled with the radial harmonic number n. Solar rotation has been cancelled by averaging the frequencies ω for eastward and westward propagating waves.

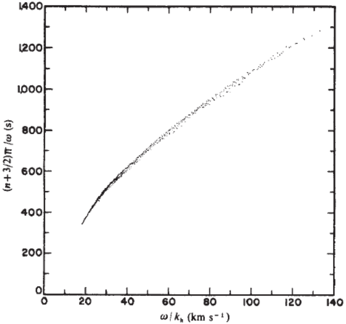


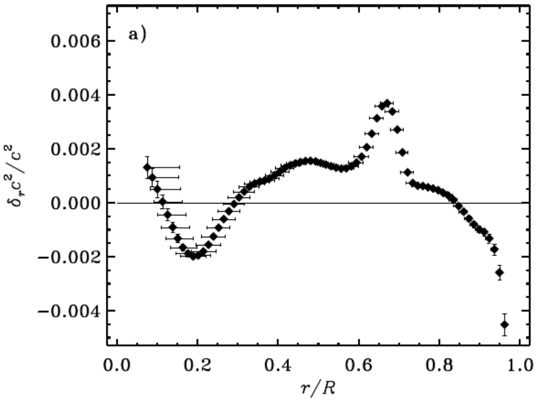
Fig. 2 Plot of $(n+3/2)\pi/\omega$ against ω/k_h for the mode positions of Fig. 1. The $n=0$ modes are excluded.

How good are solar and stellar structure models?

- Compare sound speed profile from helioseismology to that from a standard solar model.
- How well do you think the model fares?

How good are solar and stellar structure models?

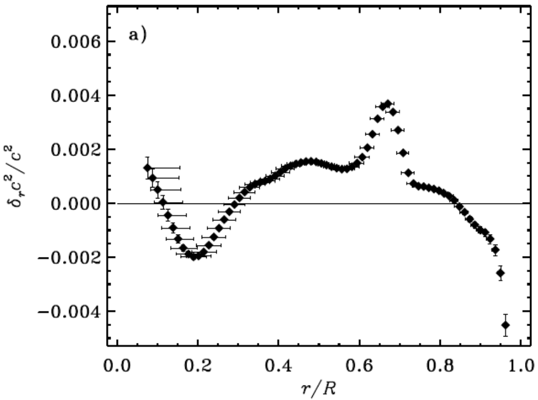
- Compare sound speed profile from helioseismology to that from a standard solar model.
- How well do you think the model fares?
- Current solar models (solving the full set of stellar structure equations) reproduces the squared sound speed profile within 0.4 per cent.
- What are the most significant sources of the discrepancy?



Credits: Basu et al. (1996)

How good are solar and stellar structure models?

- Compare sound speed profile from helioseismology to that from a standard solar model.
- How well do you think the model fares?
- Current solar models (solving the full set of stellar structure equations) reproduces the squared sound speed profile within 0.4 per cent.
- What are the most significant sources of the discrepancy?
- Lack of data to probe the center, problems with parameterization of convection at the base and near the surface of the convection zone.



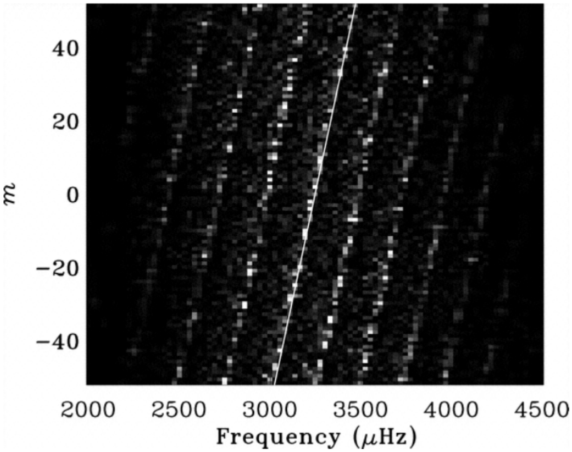
Credits: Basu et al. (1996)

Frequency splitting

- Standing wave results from two waves travelling in opposite directions between two nodes.
- In a rotating sytem waves travelling to opposite directions are advected by the flow.
- Hence there will be a Doppler shift of frequency by the amount

$$\frac{\Delta\omega_\pm}{\omega} = \frac{u_\pm}{u_\mathrm{p}}, \tag{34}$$

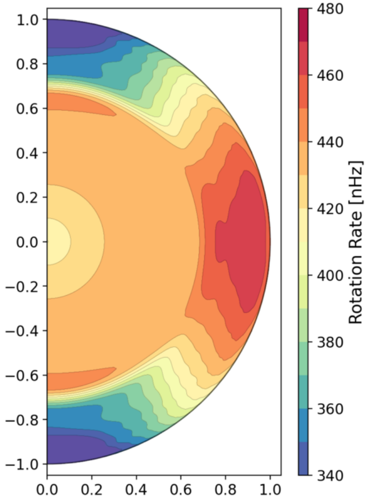
  where $u_\pm$ are the velocities of the waves and u_p is the phase velocity of the wave.
- The total shift is $2\Delta\omega$ which leads the standing wave pattern to drift in the opposite direction of the rotation.



Rotation induces a dependence on azimuthal order m.

Internal rotation of the Sun

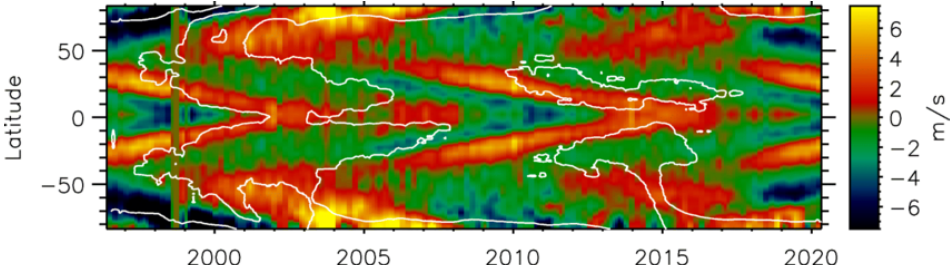
- The frequency splitting can be used to deduce the internal rotation profile of the Sun.
- The surface differential rotation extends essentially to the entire convection zone.
- Theoretical models prior to helioseismic era predicted that angular velocity Ω would decrease with radius. In reality the radial gradient is small in the bulk and large only near the boundaries of the convection zone.
- This is a very important constraint for theories regarding stellar differential rotation and the global solar dynamo.



Solar differential rotation from helioseismology.
Credits: Larson & Schou (2018).

Internal rotation of the Sun: torsional oscillations

- *Torsional oscillations* (faster and slower bands of rotation) trace the magnetic activity (white contours).



Credits: Harra et al. (2021)