Theoretical Astrophysics: Physics of Sun and Stars
Homework 2

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Problem 1: The solar luminosity is $L_\odot = 3.83 \cdot 10^{26}$ W. Assume that all of the energy for this luminosity is provided by the $\mathrm{p-p\,1}$ chain, and that neutrinos carry off 3% of the energy. How many neutrinos are produced per second? What is the neutrino flux (i.e. the number of neutrinos per second per cm^2) at Earth?

The $\mathrm{p-p\,1}$ chain is comprised of the reactions:

$$\begin{aligned} p + p &\rightarrow {}^2\mathrm{D} + e^+ + \nu, \ (1.18 \text{ MeV}) \\ {}^2\mathrm{D} + p &\rightarrow {}^3\mathrm{He} + \gamma, \qquad (5.49 \text{ MeV}) \\ {}^3\mathrm{He} + {}^3\mathrm{He} &\rightarrow {}^4\mathrm{He} + 2p. \quad (12.86 \text{ MeV}) \end{aligned} \tag{1}$$

where the first two reactions need to happen twice for the last reaction to occur, and where ν signifies an emitted neutrino. Energy released (in MeV $= 10^6$ eV) is given in brackets after each reaction.

Solution: Bearing in mind that the first two reactions occur twice each time the chain is completed, $E_{\rm pp} = [2 \times (1.18 + 5.49) + 12.86] = 26.2$ MeV of energy is released. An eV $= 1.602176634 \cdot 10^{-19}$ J, so $E_{\rm pp} \approx 4.198 \cdot 10^{-12}$ J. Because luminosity is energy release per second (unit: W = J/s), the number of reactions per second is $n_{\rm reac} = L_\odot/E_{\rm pp} \approx 9.124 \cdot 10^{37}$. As the first step in the chain happens twice, each time releasing a neutrino, the number of neutrinos produced per second is $n_\nu = 2n_{\rm reac} \approx 1.825 \cdot 10^{38}$.

The radius of the Earth's orbit is $d = 1.5 \cdot 10^{13}$ cm, and the area of a sphere enclosing this is $A_d = 4\pi d^2 \approx 2.827 \cdot 10^{27}$ cm^2. Therefore the number of neutrinos passing per second per cm^2 at the Earth is $n_\nu/A_d \approx 6.45 \cdot 10^{11}$.

Problem 2:

a) Why does convection transport heat radially outward although there is no net mass flux?

Solution: In convecting fluid light warm ($T' > 0$) matter rises ($u_r > 0$) and cool ($T' < 0$) dense matter descends ($u_r < 0$), where $T' = T - \overline{T}$ is the temperature perturbation with respect to the average temperature $\overline{T}$. The energy flux carried by convection is proportional to the product $u_r T'$ which is > 0, i.e., radially outward, for both upflows and downflows. It is also essential the that heat is lost near the surface when radiation becomes efficient.

b) Estimate the superadiabatic temperature gradient (in order of magnitude fashion) in the Sun, by making use of the mixing length expression for $F_{\rm conv}$:

$$F_{\rm conv} = c_p \rho T g^{1/2} \frac{\ell^2}{4\sqrt{2} H_p^{3/2}} (\nabla - \nabla_{\rm ad})^{3/2}. \tag{2}$$

Use the average values of density and temperature of the Sun, and $c_p = 2.07 \cdot 10^4$ J K^{-1} kg^{-1}.

Hint: Recall that $\ell = \alpha_{\rm MLT} H_p$ and bear in mind the definition of the pressure scale height from previous homework. When is this result a good approximation and when not? Why?

Solution: Solve first for $\nabla - \nabla_{\rm ad}$:

$$\begin{aligned}
(\nabla - \nabla_{\rm ad})^{3/2} &= \frac{4\sqrt{2} H_p^{3/2}}{c_p \rho T g^{1/2} \ell^2} F_{\rm conv}, && (3)\\
(\nabla - \nabla_{\rm ad})^{3/2} &= \frac{4\sqrt{2} H_p^{3/2}}{c_p \rho T g^{1/2} \alpha^2 H_p^2} F_{\rm conv}, && (4)\\
\Delta\nabla \equiv \nabla - \nabla_{\rm ad} &= \frac{(4\sqrt{2})^{2/3}}{(c_p \rho T g^{1/2} \alpha^2 H_p^{1/2})^{2/3}} F_{\rm conv}^{2/3}, && (5)
\end{aligned}$$

where $\ell = \alpha H_p$ was used to eliminate ℓ. Recalling that

$$H_p = \frac{kT}{\mu g} = \mathcal{R} \frac{T}{g}. \tag{6}$$

We further recall that:

$$\mathcal{R} = c_p - c_V = c_p \left(1 - \frac{1}{\gamma}\right) = c_p \nabla_{\rm ad}. \tag{7}$$

Therefore,

$$\Delta\nabla = \frac{(4\sqrt{2})^{2/3}}{c_p \rho^{2/3} T \alpha^{4/3} \nabla_{\rm ad}^{1/3}} F_{\rm conv}^{2/3}, \tag{8}$$

We use the mean solar density $\overline{\rho} \approx 1.4 \cdot 10^3$ kg m^{-3} and the mean temperature $\overline{T} \approx 6 \cdot 10^4$ K from the lectures and recall that $\nabla_{\rm ad} = \frac{2}{5}$. We approximate the mixing length parameter as $\alpha = 1$. Finally, assuming that the solar flux is transported by convection, we obtain at $r = R_\odot$ that $F_{\rm conv} = 6.29 \cdot 10^7$ W m^{-2}. Substituting all the values we obtain

$$\Delta\nabla \approx 4.4 \cdot 10^{-6}. \tag{9}$$

This shows that a very small superadiabaticity is enough to transport all of the solar luminosity.

This value is close to the one we would expect near the base of the solar convection zone. This is because there the density and temperature are large, and the mean density of the Sun is much larger than the density at the base of the convection zone. However, near the surface the temperature and especially the density are much smaller and we would expect $\Delta\nabla$ to be much larger there.

Problem 3: Use the equations of hydrostatic equilibrium and mass conservation:

$$\frac{dp}{dr} = -\rho \frac{Gm}{r^2}, \quad \frac{dm}{dr} = 4\pi r^2 \rho, \tag{10}$$

assuming a polytropic equation of state

$$p = K\rho^{\gamma}, \tag{11}$$

where $\gamma = 1+\frac{1}{n}$ is the polytropic exponent and n is the polytropic index, to derive the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{12}$$

Here θ and ξ are the non-dimensional density and radius defined as

$$\rho = \rho_c\theta^n, \quad \text{and} \quad r = \alpha\xi, \tag{13}$$

and where α^2 equals a constant that arises in the derivation of Eq.(12).

Solution: Start from equations (10) to find that:

$$\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = -Gm, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}m}{\mathrm{d}r} = -4\pi G\rho r^2, \tag{15}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{16}$$

Make use of Eqs.(11) and (13) to write out the pressure and its gradient:

$$p = K\rho^{1+\frac{1}{n}} = K\rho_c^{1+\frac{1}{n}}\theta^{n+1}, \tag{17}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}p}{\alpha\mathrm{d}\xi} = K\rho_c^{1+\frac{1}{n}}(n+1)\theta^n\frac{\mathrm{d}\theta}{\alpha\mathrm{d}\xi}, \tag{18}$$

where we used $\mathrm{d}r = \alpha\mathrm{d}\xi$. Now we can further see that

$$\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\alpha^2\xi^2}{\rho_c\theta^n}K\rho_c^{1+\frac{1}{n}}(n+1)\theta^n\frac{\mathrm{d}\theta}{\alpha\mathrm{d}\xi} = \alpha K(n+1)\rho_c^{\frac{1}{n}}\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}. \tag{19}$$

Now we are in a position to write Eq. (16) with the new variables:

$$\frac{1}{\alpha^2\xi^2}\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\alpha K(n+1)\rho_c^{\frac{1}{n}}\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right] = -4\pi G\rho_c\theta^n, \tag{20}$$

$$\frac{K(n+1)}{4\pi G\alpha^2}\rho_c^{\frac{1}{n}-1}\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{21}$$

With the choice:

$$\alpha^2 = \frac{K(n+1)}{4\pi G}\rho_c^{\frac{1}{n}-1} \tag{22}$$

we recover the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{23}$$

The boundary conditions at $\xi = 0$ for the Lane-Emden equation are:

$$\theta = 1, \text{ and } \frac{d\theta}{d\xi} = 0. \tag{24}$$

What do these boundary conditions correspond to?

Solution: The first condition simply states that the density is non-dimensionalized by the central density, i.e.,

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}. \tag{25}$$

The latter condition indicates that there is no density or pressure gradients at $r = \xi = 0$. This means that the gravity must also vanish at $r = \xi = 0$.

Solve the Lane-Emden equation for $n = 0, 1,$ and 5. For these values the equation can be solved *analytically.*

Solution: Case $n = 0$: The Lane-Emden equation reduces to:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -1. \tag{26}$$

Integrating once gives:

$$\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\tfrac{1}{3}\xi^3 - C. \tag{27}$$

Second integration gives:

$$\theta = D - \frac{C}{\xi} - \tfrac{1}{6}\xi^2. \tag{28}$$

We want a finite solution at $\xi = 0$ and therefore we need to have $C = 0$. Furthermore, the condition $\theta = 1$ at ξ requires that $D = 1$. Therefore the solution for $n = 0$ reads

$$\theta_0 = 1 - \tfrac{1}{6}\xi^2. \tag{29}$$

Case $n = 1$: Make first a transformation:

$$\theta = \frac{\chi}{\xi}, \tag{30}$$

so that the Lane-Emden equation reduces to:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\xi^2} = -\frac{\chi^n}{\xi^{n-1}}. \tag{31}$$

For $n = 1$ this reduces further to:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\xi^2} = -\chi. \tag{32}$$

General solution of this equation is

$$\chi = C\sin(\xi - \delta), \tag{33}$$

where C and δ are integration constants. From Eq. (30) we get

$$\theta = \frac{C\sin(\xi - \delta)}{\xi}. \tag{34}$$

If $\delta \neq 0$ we again have a singularity at $\xi = 0$ and therefore we put $\delta = 0$. Furthermore, the boundary condition $\theta = 1$ at $\xi = 0$ stipulates that $C = 1$ because at the limit of small ξ, $\sin\xi/\xi \longrightarrow 1$. Therefore,

$$\theta_1 = \frac{\sin\xi}{\xi}. \tag{35}$$

Case $n = 5$: We simply state the final result here:

$$\theta_5 = \frac{1}{(1 + \frac{1}{3}\xi^2)^{1/2}}. \tag{36}$$

Detailed derivation can be found, e.g., in Chandrasekhar, S. (1939): "An Introduction to the Study of Stellar Structure" (Dover Publications), p. 93.

Solve the Lane-Emden equation for $n = 1.5$ and $n = 3$, and compare the resuls with the ones found in the textbook by D. Prialnik. For this you will have to solve the equation *numerically*.

Solution: See script in the `homework/python` folder.

Useful physical constants

- $R_\odot = 696 \times 10^6\,\mathrm{m}$
- $M_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$
- $L_\odot = 3.83 \times 10^{26}$ W
- $T_\odot^{\mathrm{eff}} = 5777\,\mathrm{K}$
- $1\,\mathrm{AU} = 1.496 \times 10^8\,\mathrm{km}$
- $c = 2.997 \times 10^8\,\mathrm{m/s}$
- $G = 6.674 \times 10^{-11}$ $\mathrm{Nm^2/kg^2}$
- $k = 1.38 \cdot 10^{-23}$ J/K
- $m_\mathrm{H} = 1.67 \cdot 10^{-27}$ kg
- $h = 6.626 \times 10^{-34}$ J s.
- $k = 1.38 \times 10^{-23}$ J/K.