Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 3: Equation of state, chemical composition, nuclear reactions

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Equation of state

- Relation between the pressure and the ambient T and ρ of a system of particles with known chemical composition
- Justification of ideal (or perfect) gas equation of state?
- Consider the mean distance between particles:

$$d = \left(\frac{\mathcal{A}m_{\mathrm{H}}}{\overline{\rho}}\right)^{1/3} = \left(\frac{4\pi\mathcal{A}m_{\mathrm{H}}}{3M}\right)^{1/3} R. \tag{1}$$

- Typical Coulomb energy per particle:

$$\epsilon_{\mathrm{C}} = \frac{1}{4\pi\varepsilon_0}\frac{\mathcal{Z}^2 e^2}{d}. \tag{2}$$

Equation of state

- Kinetic energy per particle is of the order of $k\overline{T}$ and therefore

$$\frac{\epsilon_\mathrm{C}}{k\overline{\overline{T}}} \sim \frac{1}{4\pi\varepsilon_0} \frac{\mathcal{Z}^2 e^2}{\mathcal{A}^{4/3} m_\mathrm{H}^{4/3} G M^{2/3}}, \tag{3}$$

  where $\overline{T} = \frac{\alpha}{3}\frac{m_\mathrm{g} G}{k}\frac{M}{R}$ was derived in tutorial last week.
- Assuming pure hydrogen gas ($\mathcal{A} = \mathcal{Z} = 1$) and solar mass ($M = M_\odot$) this ratio is of the order of 10^{-2}. At higher $\mathcal{Z}$, $\mathcal{A} \approx 2\mathcal{Z}$ and $\frac{\epsilon_\mathrm{C}}{k\overline{\overline{T}}} \propto \mathcal{Z}^{2/3}$.
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- We are thus fine in the Sun. When does this approximation break down?
- $\frac{\epsilon_{\mathrm{C}}}{k\overline{T}} \sim 1$ for $M \lesssim 10^{-3} M_\odot$ so planets cannot be considered as a mixture of free gases. For solids $\frac{\epsilon_{\mathrm{C}}}{k\overline{T}} \gg 1$. Therefore the lighter the planet, the more solid it is.

Calculation of the pressure

- In general the pressure is calculated from the pressure integral

$$p = \frac{1}{3} \int_0^\infty v p_\mathrm{m} n(p_\mathrm{m}) dp_\mathrm{m}, \tag{5}$$

where $p_\mathrm{m} = mv$ is the momentum and $n(p_\mathrm{m})dp_\mathrm{m}$ gives the number of particles per unit volume with momenta $p_\mathrm{m}, p_\mathrm{m} + dp_\mathrm{m}$.

- The pressure in stellar interiors has three sources: ions, electrons, and radiation:

$$p = p_\mathrm{I} + p_\mathrm{e} + r_\mathrm{rad} = p_\mathrm{gas} + p_\mathrm{rad}. \tag{6}$$

- Often this is re-written as:

$$p_\mathrm{gas} = \beta p, \quad p_\mathrm{rad} = (1 - \beta) p. \tag{7}$$

Ion pressure

- Ideal ion gas equation of state:

$$p_{\mathrm{I}} = n_{\mathrm{I}} kT, \tag{8}$$

where n_{I} is the number of ions per unit volume. This comes about from Eq. (5) assuming free particle gas in thermodynamic equilibrium with Maxwellian velocity distribution:

$$n(p_{\mathrm{m}})dp_{\mathrm{m}} = \frac{n_{\mathrm{I}} 4\pi p_{\mathrm{m}}^2 dp_{\mathrm{m}}}{(2\pi m_{\mathrm{I}} kT)^{3/2}} e^{-\frac{p_{\mathrm{m}}^2}{2m_{\mathrm{I}} kT}}. \tag{9}$$

- Partial density of the ith nuclear species is $X_i = \rho_i/\rho$. On the other hand, to a good approximation we can write $n_i = \rho_i/(\mathcal{A}_i m_{\mathrm{H}})$, and therefore

$$n_i = \frac{\rho}{m_{\mathrm{H}}} \frac{X_i}{\mathcal{A}_i}, \quad \text{and} \quad X_i = n_i \frac{\mathcal{A}_i}{\rho} m_{\mathrm{H}}. \tag{10}$$

Ion pressure

- The total number of ions in unit volume is the sum over all ion species

$$n_{\mathrm{I}} = \sum_i n_{\mathrm{i}} = \sum_i \frac{\rho}{m_{\mathrm{H}}} \frac{X_i}{\mathcal{A}_i} \tag{11}$$

The stellar mean atomic mass μ_{I} is defined by

$$\frac{1}{\mu_{\mathrm{I}}} = \sum_i \frac{X_i}{\mathcal{A}_i}, \quad \text{such that} \quad n_{\mathrm{I}} = \frac{\rho}{\mu_{\mathrm{I}} m_{\mathrm{H}}}. \tag{12}$$

This is often approximated by

$$\frac{1}{\mu_{\mathrm{I}}} \approx X + \frac{1}{4} Y + \frac{1 - X - Y}{\langle \mathcal{A} \rangle}, \tag{13}$$

where $\langle \mathcal{A} \rangle$ is the average atomic mass of "heavy" elements that are not hydrogen or helium (a.k.a., "metals").

Ion pressure

- For the Sun $X = 0.707$, $Y = 0.274$, and $\langle \mathcal{A} \rangle \approx 20$, and therefore $\mu_{\rm I} = 1.29$.
- The ideal gas constant is defined as

$$\mathcal{R} \equiv \frac{k}{m_{\rm H}}. \tag{14}$$

- Using Eqs. (12) and (16) gives:

$$p_{\rm I} = \frac{\mathcal{R}}{\mu_{\rm I}} \rho T. \tag{15}$$

- What can you say about other stars in the Universe?

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- What can you say about other stars in the Universe?
- Since the abundances of H and He are still near the ones from Big Bang nucleosynthesis we expect that all stars have nearly the same mean atomic mass (unless iron, uranium, plutonium, etc. stars exist somewhere).

Electron pressure

- Electrons as ideal gas:
$$p_{\mathrm{e}} = n_{\mathrm{e}} kT \tag{18}$$
where n_{e} is the number density of free electrons.
- Let us assume fully ionised gas for now. Then,
$$n_{\mathrm{e}} = \sum_i \mathcal{Z}_i n_{\mathrm{i}} = \frac{\rho}{m_{\mathrm{H}}} \sum_i X_i \frac{\mathcal{Z}_i}{\mathcal{A}_i}. \tag{19}$$
- Number of free electrons per nucleon:
$$\frac{1}{\mu_{\mathrm{e}}} \equiv \sum_i X_i \frac{\mathcal{Z}_i}{\mathcal{A}_i}, \quad \text{and hence} \quad n_{\mathrm{e}} = \frac{\rho}{\mu_{\mathrm{e}} m_{\mathrm{H}}}. \tag{20}$$

Electron pressure

- Using the mass fractions X, and Y:

$$\frac{1}{\mu_{\mathrm{e}}} = X + \tfrac{1}{2}Y + (1 - X - Y)\left\langle \frac{\mathcal{Z}}{\mathcal{A}} \right\rangle . \tag{21}$$

- For the Sun $\langle \frac{\mathcal{Z}}{\mathcal{A}} \rangle \approx \frac{1}{2}$ and thus

$$\frac{1}{\mu_{\mathrm{e}}} = \tfrac{1}{2}(1 + X), \tag{22}$$

  amounting to $\mu_{\mathrm{e}} \approx 1.17$.

- The electron pressure is thus

$$p_{\mathrm{e}} = \frac{\mathcal{R}}{\mu_{\mathrm{e}}}\rho T. \tag{23}$$

Total gas pressure

- The total gas pressure is the sum of (17) and (25):

$$p = p_\mathrm{I} + p_\mathrm{e} = \left(\frac{1}{\mu_\mathrm{I}} + \frac{1}{\mu_\mathrm{e}}\right)\mathcal{R}\rho T = \frac{\mathcal{R}}{\mu}\rho T, \quad \text{with} \quad \frac{1}{\mu} = \frac{1}{\mu_\mathrm{I}} + \frac{1}{\mu_\mathrm{e}}. \tag{24}$$

- For solar abundances we get $\mu \approx 0.61$.
- Sometimes a further simplification (e.g., in 3D simulations of fluid motions inside stars) is made by assuming $Y = 0$ leading to $\mu = 1$ and $p = \mathcal{R}\rho T$.
- We have explicitly assumed lack of interactions between particles and complete ionization. But we made also another assumption implicitly. Which one?

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- We have explicitly assumed lack of interactions between particles and complete ionization. But we made also another assumption implicitly. Which one?
- The gas we have discussed lives in the world of classical physics and is oblivious to quantum effects and relativity. These become important in extreme conditions that are typically found in compact objects but also to some extent in the centres of "normal" stars (we may come back to this later in case there is interest).

Radiation pressure

- Radiation pressure arises from the transfer of momentum when photons interact with gas particles. In thermodynamic equilibrium the photon distribution is isotropic and given by the Planck function

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{d\nu}{e^{\frac{h\nu}{kT}} - 1}. \tag{26}$$

- Inserting this into the pressure integral (5) gives:

$$p_{\rm rad} = \tfrac{1}{3}\int_0^\infty c\frac{h\nu}{c}n(\nu)d\nu = \tfrac{1}{3}aT^4, \quad \text{where} \quad a = \frac{8\pi^5 k^4}{15c^3h^3} = \frac{4\sigma}{c}, \tag{27}$$

  is the radiation constant.
- When does radiation pressure become important? Does it matter for the Sun?

History of stellar energy production research

- Lord Kelvin (William Thomson) originated the idea that the Sun is contracting and it is radiating the gravitational energy away. Thus the lifetime of the Sun would be $\tau_{\mathrm{KH}} \approx 3 \cdot 10^7$ yr.
- However, the theory of evolution by Charles Darwin and geological evidence indicated that the Earth must be much older than τ_{KH}.
- In 1920, Arthur Eddington proposed that stars are powered by nuclear fusion of hydrogen to helium based on measurements of atomic masses.
- In 1928 George Gamow derived the quantum mechanical Gamow factor which gives the probability of two nuclei overcoming the Coulomb barrier via tunneling.
- In 1939 demonstrated (one of) the *pp*-chains that operates in the Sun. Around the same time, him and Friedrich von Weiszäcker discovered CNO cycle.

History of stellar energy production research

If, indeed, the sub-atomic energy in the stars is being freely used to maintain their great furnaces, it seems to bring a little nearer to fulfilment our dream of controlling this latent power for the well-being of the human race – or for its suicide.

Arthur Eddington, 24th August 1920

Evolution of chemical composition

- The chemical composition of a star changes because of nuclear reactions.
- Atomic nuclei are composed of baryons ("heavy ones") such as protons and neutrons. Other particles involved are electrons and neutrinos which are called leptons ("light ones"). Every particle has an associated antiparticle with reversed baryon/lepton number and charge.
- The strong nuclear force keeps the nuclei together and the weak force is relevant for protons and neutrons. Both are short-range forces on a length scale of the order of 10^{-15} m.

Evolution of chemical composition

- Nuclear reactions conserve the baryon and lepton numbers.
- Consider reactants $I(\mathcal{A}_i, \mathcal{Z}_i)$ and $J(\mathcal{A}_k, \mathcal{Z}_j)$ and products $K(\mathcal{A}_k, \mathcal{Z}_k)$ and $L(\mathcal{A}_l, \mathcal{Z}_l)$:

$$I(\mathcal{A}_i, \mathcal{Z}_i) + J(\mathcal{A}_k, \mathcal{Z}_j) \rightleftharpoons K(\mathcal{A}_k, \mathcal{Z}_k) + L(\mathcal{A}_l, \mathcal{Z}_l), \tag{28}$$

with the conservation laws:

$$\mathcal{A}_i + \mathcal{A}_j = \mathcal{A}_k + \mathcal{A}_l, \tag{29}$$

$$\mathcal{Z}_i + \mathcal{Z}_j = \mathcal{Z}_k + \mathcal{Z}_l. \tag{30}$$

- Same considerations apply if electrons/positrons are involved.

- Rate of change of the ith element abundance from all possible nuclear reactions: PJK: the $(1-\delta_{ij})/(1-\delta_{ij})$ below factor makes no sense!

$$\dot{n}_i = -n_i \sum_{j,k} (1+\delta_{ij}) \frac{n_j}{1+\delta_{ij}} R_{ijk} + \sum_{l,k} \frac{n_l n_k}{1+\delta_{lk}} R_{lki}, \tag{31}$$

where δ_{ij} is the Kronecker delta that equals 1 for $i=j$ and 0 for $i \neq j$, and R_{ijk} are the reaction rates (cross-section times velocity).

- In terms of partial densities this is:

$$\frac{\dot{X}_i}{\mathcal{A}_i} = \frac{\rho}{m_{\mathrm{H}}} \left(-\frac{X_i}{\mathcal{A}_i} \sum_{j,k} (1+\delta_{ij}) \frac{X_j}{\mathcal{A}_j} \frac{R_{ijk}}{1+\delta_{ij}} + \sum_{l,k} \frac{X_l}{\mathcal{A}_l} \frac{X_k}{\mathcal{A}_k} \frac{R_{lki}}{1+\delta_{lk}} \right). \tag{32}$$

This needs to be done for every mass fraction taking part in nuclear reactions.

Evolution of chemical composition

- Consider again the reaction:

$$I(\mathcal{A}_i, \mathcal{Z}_i) + J(\mathcal{A}_k, \mathcal{Z}_j) \rightleftharpoons K(\mathcal{A}_k, \mathcal{Z}_k) + L(\mathcal{A}_l, \mathcal{Z}_l). \tag{33}$$

- Denoting the energy released in this reaction by Q_{ijk} and the mass of the nucleus I by $\mathcal{M}_i$ gives

$$Q_{ijk} = (\mathcal{M}_i + \mathcal{M}_j - \mathcal{M}_k - \mathcal{M}_l)c^2, \tag{34}$$

where the masses of other light particles have been neglected. Using the mass unit m_{H} this becomes

$$Q_{ijk} = [(\mathcal{M}_i - \mathcal{A}_i m_{\mathrm{H}}) + (\mathcal{M}_j - \mathcal{A}_j m_{\mathrm{H}}) - (\mathcal{M}_k - \mathcal{A}_k m_{\mathrm{H}}) - (\mathcal{M}_l - \mathcal{A}_l m_{\mathrm{H}})]c^2, \tag{35}$$

- The difference $\Delta\mathcal{M}(I) = (\mathcal{M}_i - \mathcal{A}_i m_{\mathrm{H}})c^2$ is the mass excess although its unit is that of energy (typically $\mathrm{MeV} = 10^6\ \mathrm{eV} = 1.602 \times 10^{-13}$ J).

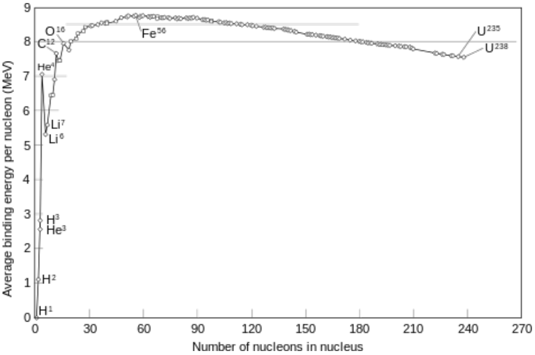
Evolution of chemical composition

- Number of reactions per unit volume and unit time $n_i n_j R_{ijk}$. Then the energy released by such reactions per unit volume and unit time $n_i n_j R_{ijk} Q_{ijk}$.
- Summing over all nuclear reactions and dividing by ρ gives the rate of energy released per unit mass:

$$q = \frac{\rho}{m_{\mathrm{H}}^2} \sum_{ijk} \frac{1}{1+\delta_{ij}} \frac{X_i}{\mathcal{A}_i} \frac{X_j}{\mathcal{A}_j} R_{ijk} Q_{ijk} \tag{36}$$

Nuclear binding energy

- Average binding energy increases for light elements: energy released when fusing nuclei.
- For elements heavier than Fe^{56} the opposite is true: energy is needed to put the nuclei together (and released in fission).

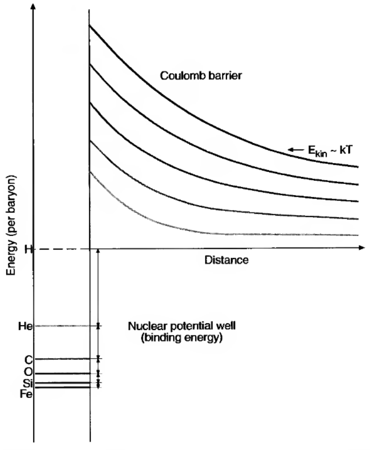


Credit: Wikipedia

Coulomb barrier

- Nuclei have to be brought in range for the strong nuclear force ($10^{-15}m$!).
- This is opposed by the long-distance electrostatic force between positively charged ions.
- To overcome this barrier the kinetic energy of the particles has to exceed the electrostatic potential. This happens at:

$$d = \frac{1}{4\pi\varepsilon_0} \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{\frac{1}{2} m_{\mathrm{H}} v^2}. \qquad (37)$$



Credit: Prialnik

For typical mean stellar temperatures d is about 10^3 larger than the range of the strong nuclear force!

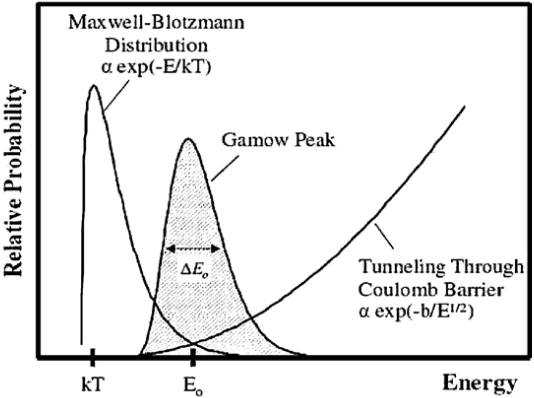
Gamov peak

- Quantum mechanical tunneling through the Coulomb barrier was calculated by George Gamov in 1928.
- The penetration probability is proportional to

$$p_q \propto \exp\left\{\frac{-\pi \mathcal{Z}_i \mathcal{Z}_j e^2}{\varepsilon_0 h v}\right\}. \quad (38)$$

- The total probability is

$$p_t \propto \exp\left\{\frac{-\pi \mathcal{Z}_i \mathcal{Z}_j e^2}{\varepsilon_0 h v}\right\} \exp\left\{\frac{-m_g v^2}{2kT}\right\} \quad (39)$$



Credit: Trache (2010), Rom. J. Phys., 52, 823

This function has a maximum known as the Gamow peak which enables fusion in stars.

pp chain

- Making stable helium out of hydrogen requires three or four protons to be brought to within 10^{-15} m. This is theoretically possible but incredibly unlikely.
- Thus the process H $\rightarrow$ He has to go through several steps in a chain.
- One such chain (*pp*1, four in total are known to exist) is

$$\begin{aligned} p + p &\rightarrow {}^{2}\mathrm{D} + e^{+} + \nu, \\ {}^{2}\mathrm{D} + p &\rightarrow {}^{3}\mathrm{He} + \gamma, \\ {}^{3}\mathrm{He} + {}^{3}\mathrm{He} &\rightarrow {}^{4}\mathrm{He} + 2p. \end{aligned} \tag{40}$$

- The *pp*1 branch works at the lowest temperatures and is dominant, e.g., in the Sun.
- The energy released in the *pp* chain is $Q_{pp} = 4\Delta\mathcal{M}({}^{1}\mathrm{H}) - \Delta\mathcal{M}({}^{4}\mathrm{He}) = 26.73$ MeV.
- Rate of energy release is roughly proportional to

$$q_{pp} \propto \rho T^{4}. \tag{41}$$

CNO cycle

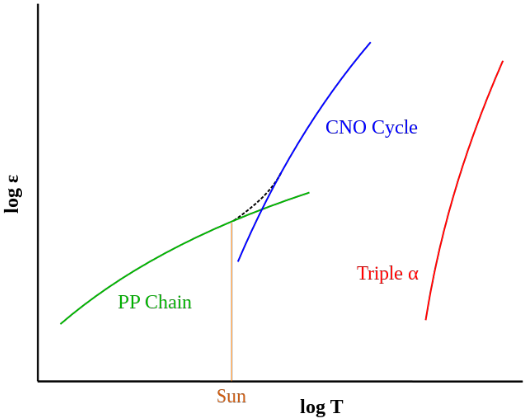
- Mediocre shit shit

Triple α reaction

- Awesome shit

Summary of the most common fusion processes

- *pp* chain dominates in low mass stars such as the Sun.
- CNO cycle is operating in more massive main-sequence stars.
- The first stars got their energy from triple α reactions.



Fusion processes as a function of T Credit: PLACEHOLDER