TAIPSS, Hands-on exercises 2

$N = \frac{kg\,m}{s^2}$ ①

2. Timescales in the Sun and in a red giant.

Dynamical:

$$\tau_{dyn} = \sqrt{\frac{R^3}{GM}} = \left(\frac{R^3}{R_\odot^3} R_\odot^3 \frac{M_\odot}{M\ M_\odot}\frac{1}{G}\right)^{1/2} = \tau_{dyn}^\odot \left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M_\odot}{M}\right)^{1/2}.$$

$$\tau_{dyn}^\odot = \left(\frac{(7\cdot 10^8)^3 m^{\cancel{3}}}{6.67\cdot 10^{-11}\, N\cancel{m^2}/kg^2 \cdot 2\cdot 10^{30}\cancel{kg}}\right)^{1/2} \approx \left(\frac{10^{24}}{10^{19}}\frac{\cancel{7}^{\cancel{3}\,2}}{2\cdot\cancel{7}}\right)^{1/2}\left(\frac{m\,kg}{N}\right)^{1/2}$$

$$\simeq (2{,}5\cdot 10^6)^{1/2}\left(\frac{m\,kg}{kg\,m}s^2\right)^{1/2} \approx \underline{\underline{1{,}6\cdot 10^3\,s}}$$

$$\tau_{dyn} = \tau_{dyn}^\odot\left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M_\odot}{M}\right)^{1/2}$$ Sun as a red giant?

Thermal (Kelvin-Helmholtz):

$$\tau_{th} = \frac{GM^2}{RL} = G\frac{M^2}{M_\odot^2}M_\odot^2 \frac{R_\odot}{RR_\odot}\frac{L_\odot}{LL_\odot} = \tau_{th}^\odot\left(\frac{M}{M_\odot}\right)^2\frac{R_\odot}{R}\frac{L_\odot}{L}.$$

$$\tau_{th}^\odot \approx \frac{6{,}67\cdot 10^{-11}\frac{Nm^{\cancel{2}}}{\cancel{kg^2}}(2\cdot 10^{30}\cancel{kg})^2}{7\cdot 10^8\cancel{m}\cdot 3{,}83\cdot 10^{26}W} \approx \frac{\cancel{7}\cdot 4}{\cancel{7}\cdot 4}\frac{10^{-11}\,10^{60}}{10^8\,10^{26}}\frac{Nm}{W}$$

$$\approx 10^{15}\frac{kg\,m^2}{s^2}\frac{s^3}{kg\,m^2} = \underline{\underline{10^{15}\,s}}$$ $W = \frac{J}{s} = \frac{kg\,m^2}{s^3}$

One year: $365\cdot 86400\,s \approx 3\cdot 10^2\cdot 10^5 \approx 3\cdot 10^7\,s$

$\Rightarrow \tau_{th}^\odot \approx 3\cdot 10^7\,yr.$

$$\tau_{th} = \tau_{th}^\odot\left(\frac{M}{M_\odot}\right)^2\frac{R_\odot}{R}\frac{L_\odot}{L}$$ Sun as a red giant?

$$L = 4\pi R^2\sigma T_{eff}^4 = L_\odot\left(\frac{R}{R_\odot}\right)^2\left(\frac{T_{eff}}{T_{eff}^\odot}\right)^4.$$

Nuclear timescale:

$$\tau_{nuc} = \frac{\epsilon M c^2}{L} = \tau_{nuc}^{\odot} \frac{M}{M_\odot} \frac{L_\odot}{L}.$$

$\epsilon \simeq 10^{-3}$ (binding energy of a nucleon / rest mass)

$$\tau_{nuc}^{\odot} \approx \frac{10^{-3}\; 2\cdot 10^{30}\,\mathrm{kg}\left(3\cdot 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{3{,}83\cdot 10^{26}\,\mathrm{W}} \approx \frac{\overbrace{2\cdot 9}^{\approx 10}}{4} \frac{10^{-3}\,10^{30}\,10^{16}}{10^{26}} \frac{\mathrm{kg\,m^2}}{\mathrm{W\,s^2}}$$

$$\approx \frac{10^{44}}{10^{26}} \frac{\cancel{\mathrm{kg\,m^2}}}{\cancel{\mathrm{s^2}}} \frac{\mathrm{s}^{\cancel{3}}}{\cancel{\mathrm{kg\,m^2}}} = 10^{18}\,\mathrm{s} \Rightarrow 3\cdot 10^{10}\,\mathrm{yr}$$

M dwarf : $M = 0{,}2\,M_\odot$; $L = 0{,}008\,L_\odot$

Blue supergiant O5V: $M = 60 M_\odot$; $L \approx 8\cdot 10^5 L_\odot$

3. Calculate a lower limit of the gas pressure at the center of the Sun

$$P_c = \int_0^M \frac{G m\, dm}{4\pi r^4} > \int \frac{G m\, dm}{4\pi R^4} = \frac{G M^2}{8\pi R^4}$$

$$P_c^{\odot} \approx \frac{6{,}67\cdot 10^{-11}\,\frac{\mathrm{N}\,\cancel{\mathrm{m}^2}}{\cancel{\mathrm{kg}^2}}\,(2\cdot 10^{30}\,\cancel{\mathrm{kg}})^2}{8\pi\,(7\cdot 10^{8}\,\mathrm{m})^{4}_{\;\mathrm{m^2}}} \approx \frac{\cancel{7}\cdot 4}{\underset{6}{\cancel{24}}\cdot \underset{3}{\cancel{2{,}5}}\cdot 10^{3}} \frac{10^{-11}\cdot 10^{60}}{10^{32}} \frac{\mathrm{N}}{\mathrm{m}^2}$$

$$\approx \frac{10^{49}}{10^{36}} \frac{\mathrm{N}}{\mathrm{m}^2} = 10^{13}\,\frac{\mathrm{N}}{\mathrm{m}^2}$$

Atmospheric pressure $\sim 10^5$ Pa

4. Calculate the mean temperature of the Sun.

Kinetic energy / particle:

$$E_k = \frac{3}{2} kT$$

Ideal gas eos:

$$p = \left(\frac{\rho}{m_g}\right) kT$$

Internal energy per unit mass:

$$u = \frac{3kT}{2m_g} \qquad (1)$$

Total internal energy: $U = \int_0^M u\,dm.$ $\qquad$ (2)

From virial theorem: $U = -\frac{1}{2}\Omega = \int_0^M u\,dm.$

On the other hand: $\Omega = -\alpha \frac{GM^2}{R}$, where α is order unity.

Then we have from virial theorem:

$$U = \alpha \frac{GM^2}{2R}.$$

and from (1) & (2):

Definition of T.

$$U = \int_0^M \frac{3kT}{2m_g}\,dm \equiv \frac{3}{2}\frac{k}{m_g} T M.$$

$Nm = kg\frac{m^2}{s^2} = J$

Therefore:

$$\frac{1}{\cancel{2}}\alpha \frac{GM^{\cancel{2}}}{R} = \frac{3}{\cancel{2}}\frac{k}{m_g} T \cancel{M} \Rightarrow \boxed{T = \frac{\alpha}{3}\frac{G m_g M}{kR}}$$

$$T \approx \frac{1}{\cancel{2}}\frac{1}{3}\frac{6{,}67\cdot 10^{-11}\frac{Nm^{\cancel{2}}}{\cancel{kg^2}}\, 1{,}67\cdot 10^{-27}\cancel{kg}\cdot \cancel{2}\cdot 10^{30}\cancel{kg}}{1{,}38\cdot 10^{-23}J/K\cdot 7\cdot 10^{8}\cancel{m}} \approx \frac{10\cdot 10^{-11}\cdot 10^{-27}\cdot 10^{30}}{3\cdot 10\cdot 10^{-23}\cdot 10^{8}}\frac{\cancel{N}mK}{\cancel{J}}$$

$$\approx 3\cdot\frac{10^{-7}}{10^{-13}} K \approx \underline{\underline{3\cdot 10^{6} K}}$$

$$\underline{\underline{T = T_\odot \frac{R_\odot}{R}\frac{M}{M_\odot}}}$$