Hands-on exercises 10 solution: Gravity waves

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Problem 1: Add gravity to the system that was used to demonstrate sound waves on the lectures and derive the equations governing internal gravity waves or g modes.

Solution: We first recall that we still have the pressure gradient which led to the sound waves. We will also assume that the gradients of the 0-quantities are negligible compared to the gradients of the fluctuations. The fluctuating contribution to the gravitational potential is also omitted.

Then the linearized equations read:

$$\rho' + \boldsymbol{\nabla} \cdot (\rho_0 \delta \boldsymbol{r}') = 0, \tag{1}$$

$$\rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} = -\boldsymbol{\nabla} p' + \rho_0' \boldsymbol{g}. \tag{2}$$

We assume from the start that the solution is a plane wave:

$$\exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]. \tag{3}$$

Let the r coordinate increase outwards so that $\boldsymbol{g}_0 = -g_0 \hat{\boldsymbol{e}}_r$, and

$$\boldsymbol{\nabla} p_0 = \frac{\mathrm{d}p_0}{\mathrm{d}r} \hat{\boldsymbol{e}}_r, \quad \text{and} \quad \boldsymbol{\nabla} \rho_0 = \frac{\mathrm{d}\rho_0}{\mathrm{d}r} \hat{\boldsymbol{e}}_r. \tag{4}$$

Let us further separate the directions along (radial) and perpendicular (horizontal) to the gravity:

$$\delta \boldsymbol{r} = \xi_r \hat{\boldsymbol{e}}_r + \boldsymbol{\xi}_\mathrm{h}, \tag{5}$$

$$\boldsymbol{k} = k_r \hat{\boldsymbol{e}}_r + \boldsymbol{k}_\mathrm{h}. \tag{6}$$

Inserting these into Eq. (2) give:

$$-\rho_0 \omega^2 \xi_r = -i k_r p' - \rho' g_0, \tag{7}$$

$$-\rho_0 \omega^2 \boldsymbol{\xi}_\mathrm{h} = -i \boldsymbol{k}_\mathrm{h} p'. \tag{8}$$

The continuity equation (1) gives:

$$\rho' + \rho_0 i k_r \xi_r + \rho_0 i \boldsymbol{k}_\mathrm{h} \cdot \boldsymbol{\xi}_\mathrm{h} = 0. \tag{9}$$

The two previous equations combine to:

$$p' = \frac{\omega^2}{k_\mathrm{h}^2} (\rho' + \rho_0 i k_r \xi_r). \tag{10}$$

Using this in Eq. (7) yields:

$$-\rho_0\omega^2\xi_r = -i\frac{k_r}{k_\mathrm{h}^2}\omega^2\rho' + \omega^2\rho_0\frac{k_r^2}{k_\mathrm{h}^2}\xi_r - \rho' g_0. \tag{11}$$

For very small frequencies ω the first term proportional to ρ' is small compared to the buoyancy term and we therefore omit it. Then the equation reduces to:

$$\rho_0\omega^2\left(1-\frac{k_r^2}{k_\mathrm{h}^2}\right)\xi_r = \rho' g_0. \tag{12}$$

The term on the right hand side is the driving of density perturbations by the buoyancy force, whereas the left hand side gives the vertical acceleration. Note the term with the wavenumbers: this arises from the pressure force because a radial displacement means that matter has to be pushed aside horizontally which adds up to the total inertia. This effect is particularly large for large horizontal length scales with $k_\mathrm{h} \ll k_r$.

To arrive at the dispersion relation we again make use of the adiabatic relation:

$$\delta p = \gamma_{1,0}\frac{p_0}{\rho_0}\delta\rho. \tag{13}$$

Note that the fluctuations are Lagrangian, i.e., they follow the fluid elements and in Eulerian form where we sit in a fixed position they read $\delta p = p' + \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}p_0$ and $\delta\rho = \rho' + \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}\rho_0$. Therefore,

$$\rho' = c_0^{-2}p' + \rho_0\delta\boldsymbol{r}\cdot\left(\frac{1}{\gamma_{1,0}p_0}\boldsymbol{\nabla}p_0 - \frac{1}{\rho_0}\boldsymbol{\nabla}\rho_0\right). \tag{14}$$

Using Eq. (10) we find that

$$\frac{c_0^{-2}p'}{\rho'} \approx \frac{\omega^2}{c_0^2 k_\mathrm{h}}, \tag{15}$$

where $c_0^2 k_\mathrm{h}$ is the frequency corresponding to sound waves. The frequencies ω that we are interested in are much lower so we can therefore omit the term proportional to p'. Substitution of Eq. (14) without this term to Eq. (12) yields:

$$\omega^2\left(1+\frac{k_r^2}{k_\mathrm{h}^2}\right)\xi_r = N^2\xi_r, \tag{16}$$

where

$$N^2 = g_0\left(\frac{1}{\gamma_{1,0}}\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}r}\right), \tag{17}$$

is the squared *Brunt-Väisälä* frequency. The Brunt-Väisälä frequency related to the convective stability of the fluid: for $N^2 > 0$ the stratification is stable and for $N^2 < 0$ it is unstable. This is more clearly seen from the dispersion relation of gravity waves:

$$\omega^2 = \frac{N^2}{1+k_r^2/k_\mathrm{h}^2}. \tag{18}$$

The frequency of the gravity waves coincides with N^2 (> 0) when $k_\mathrm{h} \to \infty$. For larger k_h the frequency is decreased. If $N^2 < 0$, the solution grows exponentially in time corresponding to the convective instability.

Useful physical constants

- $R_\odot = 696 \times 10^6\,\mathrm{m}$
- $M_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$
- $L_\odot = 3.83 \times 10^{26}\ \mathrm{W}$
- $T_\odot^{\mathrm{eff}} = 5777\,\mathrm{K}$
- $1\,\mathrm{AU} = 1.496 \times 10^8\,\mathrm{km}$
- $c = 2.997 \times 10^8\,\mathrm{m/s}$
- $G = 6.674 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
- $k = 1.38 \cdot 10^{-23}\ \mathrm{J/K}$
- $m_\mathrm{e} = 9.11 \cdot 10^{-31}\ \mathrm{kg}$
- $m_\mathrm{H} = 1.67 \cdot 10^{-27}\ \mathrm{kg}$