1. Convince yourself that:

$$\frac{1}{T}\frac{dT}{dr} = -\frac{\nabla}{H_p}, \quad \text{where } \nabla = \frac{p}{T}\frac{dT}{dp} \text{ and } H_p \text{ is the pressure scale height.}$$

Use the chain rule and divide & multiply by p:

$$\underbrace{\frac{p}{p}\frac{1}{T}\frac{dT}{dp}}_{=\nabla}\frac{dp}{dr} = \nabla\frac{1}{p}\frac{dp}{dr}$$

Note that the differential

$$d\ln p = \frac{1}{p}dp, \text{ and } \frac{d\ln p}{dr} = \frac{1}{p}\frac{dp}{dr}$$

Pressure scale height:

$$\frac{d\ln p}{dr} = -\frac{1}{H_p} \Rightarrow \int_{p_0}^{p} d\ln p = -\int_0^r \frac{dr}{H_p}$$

$$\ln\frac{p}{p_0} = -\frac{r}{H_p} \quad \Big\| \exp$$

$$\frac{p}{p_0} = \exp\left\{-\frac{r}{H_p}\right\} \Rightarrow p = p_0\exp\left\{-\frac{r}{H_p}\right\}$$

Therefore from above:

$$\frac{1}{T}\frac{dT}{dr} = \nabla\frac{1}{p}\frac{dp}{dr} = -\frac{\nabla}{H_p}$$

2. Show that:

$$\left.\frac{dT}{dr}\right|_a = -\frac{g}{c_p}.$$

Start again with the chain rule:

$$\left.\frac{dT}{dr}\right|_a = \left.\frac{dT}{dp}\right|_a\frac{dp}{dr} = -\left.\frac{dT}{dp}\right|_a \rho g$$

$$\frac{dp}{dr} = -\rho g$$

On the other hand:

$$\left.\frac{dT}{dp}\right|_a = \underbrace{\left(\frac{p}{T}\frac{dT}{dp}\right)_a}_{=\nabla_a}\frac{T}{p} \quad , \quad \text{and} \quad p = R_{spec}\,\rho T = (c_p - c_v)\rho T \Leftrightarrow \frac{T}{p} = \frac{1}{(c_p - c_v)\rho}$$

$$\Rightarrow \left.\frac{dT}{dp}\right|_a =$$

Substitute above:

$$\left.\frac{dT}{dp}\right|_a \frac{dp}{dr} = -\left.\frac{dT}{dp}\right|_a \rho g = -\frac{\nabla_a}{(c_p - c_v)\cancel{\rho}}\cancel{\rho} g.$$

Recall that $\nabla_a = \frac{\gamma - 1}{\gamma}$ and $\gamma = c_p/c_v$:

$$\left.\frac{dT}{dr}\right|_a = -\frac{\gamma - 1}{\gamma}\frac{1}{(c_p - c_v)} g = -\frac{\cancel{\gamma - 1}}{\gamma}\frac{1}{c_v(\cancel{\gamma - 1})} g = \underline{\underline{-\frac{g}{c_p}}}.$$

3. Use the 2nd law of thermodynamics

$$ds = \frac{dQ}{T}$$

to show that an adiabatic process is isentropic.

Recall that:

$$dQ = du + p\,dV = c_v\,dT - p\frac{d\rho}{\rho^2}.$$

Now we have:

$$ds = c_v\frac{dT}{T} - \frac{p}{T}\frac{d\rho}{\rho^2} \quad \Big\| \cdot \frac{1}{c_p}$$

On the other hand:

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \Rightarrow \frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho}$$

$$\frac{ds}{c_p} = \frac{1}{\gamma}\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - \frac{p}{c_p T}\frac{d\rho}{\rho^2}. \quad ; \quad p = (c_p - c_v)\rho T \Leftrightarrow \frac{p}{T} = (c_p - c_v)\rho$$

$$\frac{ds}{c_p} = \frac{1}{\gamma}\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - \underbrace{\frac{(c_p - c_v)\cancel{\rho}}{c_p}}_{= \gamma - 1/\gamma}\frac{d\rho}{\rho^{\cancel{2}}}$$

$$= \frac{1}{\gamma}\left(\frac{dp}{p} - \cancel{\frac{d\rho}{\rho}}\right) - \left(1 - \cancel{\frac{1}{\gamma}}\right)\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dp}{p} - \frac{d\rho}{\rho}$$

Continued...

$$\frac{ds}{c_p} = \frac{1}{\gamma}\frac{dp}{p} - \frac{d\rho}{\rho} = \frac{1}{\gamma}\left(d\ln p - \gamma\, d\ln\rho\right)$$

For an adiabatic process we know that:

$$\frac{\rho}{p}\frac{dp}{d\rho} = \frac{d\ln p}{d\ln\rho} = \gamma \iff d\ln p = \gamma\, d\ln\rho.$$

which leads to:

$$\underline{\underline{\frac{ds}{c_p} = 0}}$$

Show that superadiabatic T-gradient is related to the specific s-grad:

$$-\frac{1}{c_p}\frac{ds}{dr} = \frac{1}{H_p}(\nabla - \nabla_{ad})$$

Start from:

$$ds = c_v\frac{dT}{T} - \frac{p}{T}\frac{d\rho}{\rho^2},$$

and use:

$$\frac{p}{T\rho} = c_p - c_v, \quad \text{and} \quad \frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T},$$

to get

$$ds = \cancel{c_v\frac{dT}{T}} - (c_p - \cancel{c_v})\left(\frac{dp}{p} - \cancel{\frac{dT}{T}}\right) = c_p\frac{dT}{T} - (c_p - c_v)\frac{dp}{p}$$

Divide by c_p and differentiate w.r.t. r:

$$\frac{1}{c_p}\frac{ds}{dr} = \underbrace{\frac{1}{T}\frac{dT}{dr}}_{= -\frac{\nabla}{H_p}} - \underbrace{\frac{c_p - c_v}{c_v}}_{= 1 - \frac{1}{\gamma} = \nabla_a} \underbrace{\frac{1}{p}\frac{dp}{p}}_{= -\frac{1}{H_p}} = -\frac{\nabla}{H_p} + \frac{\nabla_a}{H_p}$$

Therefore:

$$\underline{\underline{-\frac{1}{c_p}\frac{ds}{dr} = \frac{1}{H_p}(\nabla - \nabla_a)}}$$

4. Use the N-S equation and definition of F_{conv} to estim. convective velocity and T-fluctuations in the Sun.

$$\rho \frac{\partial \bar{u}}{\partial t} = -\rho \bar{u}\cdot\nabla\bar{u} - \rho\bar{g} + \nabla p + \bar{F}_{visc} \quad ; \quad F_{conv} = c_p \rho u T'$$

- Assume hydrostatic equilibrium on average:

$$\Rightarrow \frac{d\bar{p}}{dr} = \bar{\rho} g_r .$$

- Then the buoyancy force driving convection is: $\rho' g_r$.
- Assume low internal friction $\Rightarrow \bar{F}_{visc} \approx 0$.
- Balance the remaining terms:

$$\rho\bar{u}\cdot\nabla\bar{u} \sim \rho'\bar{g} \overset{\text{ooM}}{\Longrightarrow} \frac{\bar{\rho} u^2}{H} \sim \rho' g \Leftrightarrow \frac{u^2}{H} \sim g\frac{\rho'}{\rho} \sim g\frac{T'}{T}$$

- Assume convection to transport all of the energy: $F_{conv} = F_{tot} = \frac{L}{4\pi r^2}$.

$$T' = \frac{F_{tot}}{c_p \rho u}$$

- In combination:

$$\frac{u^2}{\cancel{H}} \sim \frac{g F_{tot}}{\underbrace{c_p T}_{=H} \rho u} \Longrightarrow u^3 = \frac{F_{tot}}{\rho} \; ; \text{ or } \; u = \left(\frac{F_{tot}}{\rho}\right)^{1/3} .$$

- Now the T-fluctuation is:

$$T' = \frac{F_{tot}}{c_p \rho u} = \frac{1}{c_p}\left(\frac{F_{tot}}{\rho}\right)^{2/3} .$$

$$F_{tot} \approx \frac{L_\odot}{4\pi r^2}\bigg|_{r=0{,}85R_\odot} \approx 8{,}7\cdot 10^7 \frac{W}{m^2} \; ; \; \rho(r=0{,}85R_\odot) \approx 55\frac{kg}{m^3} \; ; \; c_p \approx 2{,}07\cdot 10^4 \frac{J}{K\cdot kg}$$

$$\Rightarrow u \sim 116\frac{m}{s} \; ; \; T \sim 0{,}66\,K .$$

Think if these make sense!