

Theoretical Astrophysics I: Physics of Sun and Stars

Lecture 7: Simple Overview of Stellar Evolution

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The (T, ρ) -plane

- ▶ The evolution of the star is driven by its core. [Why?](#)
- ▶ If we know the chemical composition, density ρ , and the temperature T , all other quantities can be derived.
- ▶ Let us denote the pair of central temperature and density at time t as $[T_c(t), \rho_c(t)]$.
- ▶ Then the evolution of the star is described by a succession of states $[T_c(t_1), \rho_c(t_1)]$, $[T_c(t_2), \rho_c(t_2)]$, $[T_c(t_3), \rho_c(t_3)]$, ..., at times t_1, t_2, t_3, \dots , forming a parametric line $[T_c(t), \rho_c(t)]$.
- ▶ Assuming stars with the same composition, we expect that the $[T_c(t), \rho_c(t)]$ depend on stellar mass M .

Characterization of the $(\log T, \log \rho)$ -plane

- ▶ We anticipate that T and ρ can vary several orders of magnitude depending on the star and/or evolutionary stage, we consider the logarithms $\log T$ and $\log \rho$ in what follows.
- ▶ The $(\log T, \log \rho)$ -plane divides into zones where different equations of state, nuclear reactions, and dynamical instabilities dominate.
- ▶ We begin with the equations of state. The most common state is described by the ideal gas equation:

$$p_{\text{ideal}} = \frac{\mathcal{R}}{\mu} \rho T = K_0 \rho T, \quad (1)$$

where K_0 is a constant. At high density (and not too high temperature), the electrons become degenerate and dominate the pressure according to

$$p_{e,\text{deg}} = K_1 \rho^{5/3}, \quad (2)$$

where K_1 is another constant.

Characterization of the $(\log T, \log \rho)$ -plane

- The border between these lies along a line where $p_{\text{ideal}} = p_{e,\text{deg}}$, or

$$\log \rho = 1.5 \log T + \text{const.} \quad (3)$$

This line separates regions I and II.

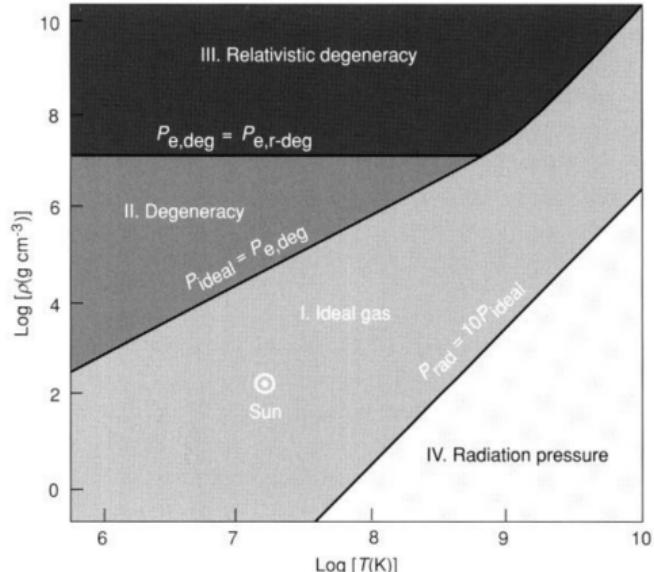
- For relativistic degeneracy

$$p_{e,r-\text{deg}} = K_1 \rho^{4/3}, \quad (4)$$

corresponding to

$$\log \rho = 3 \log T + \text{const.} \quad (5)$$

w.r.t. ideal gas separating regions I and III.



Credits: Prialnik

Characterization of the $(\log T, \log \rho)$ -plane

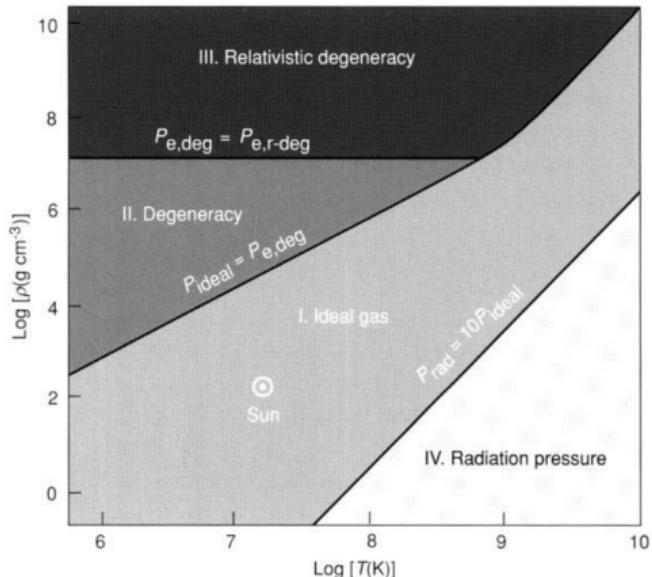
- ▶ Transition from standard to relativistic degeneracy occurs for $\rho \gg (K_2/K_1)^3$ (regions II and III).
- ▶ For sufficiently low pressure and high temperature the radiation pressure with

$$p = \frac{1}{3}aT^4, \quad (6)$$

becomes dominant. This again leads to:

$$\log \rho = 3 \log T + \text{const.} \quad (7)$$

separating regions I and IV.



Credits: Priplnik

Zones of nuclear burning

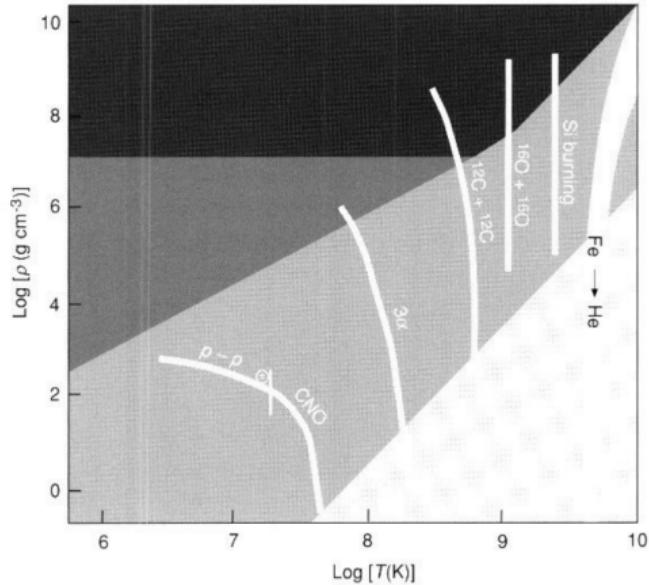
- Nuclear energy generation can be approximated by a power law

$$q = q_0 \rho^m T^n. \quad (8)$$

- A given reaction becomes important when $q > q_{\min}$ (e.g. $0.1 \text{ J km}^{-1} \text{ s}^{-1}$). This leads to

$$\log \rho = -\frac{n}{m} \log T + \frac{1}{m} \log \left(\frac{q_{\min}}{q_0} \right). \quad (9)$$

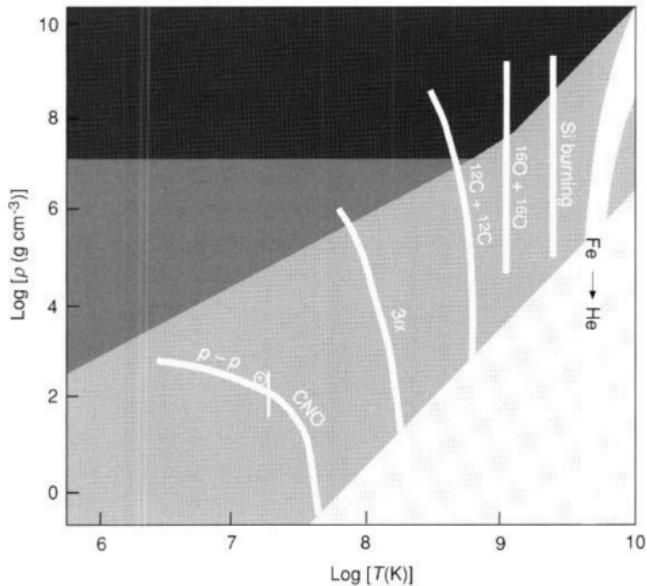
- For most reactions $n \gg m$ so the temperature determines the ignition of nuclear reactions.



Credits: Prialnik

Zones of nuclear burning

- ▶ The Hydrogen burning reactions connect smoothly from the pp chain to the CNO cycle. The latter has much stronger T -dependence and hence the slope in the (T, ρ) plane is steeper.
- ▶ Further reactions include helium, carbon, oxygen, and silicon burning, that are activated at higher T .
- ▶ For T reaching several billion Kelvin, iron cores start to dissociate which plays role at late stages of massive stars.



Credits: Prialnik

A note on dynamical stability

- ▶ Consider a macroscopic radial perturbation in a gas sphere of mass M in hydrostatic equilibrium. Pressure at any point is the weight per unit area between layers m and M :

$$p = \int_m^M \frac{Gm}{4\pi r^4} dm. \quad (10)$$

The density is given by

$$\rho = \frac{1}{4\pi r^2} \frac{dm}{dr}. \quad (11)$$

- ▶ Consider a small radial compression such that the radii are obtained from

$$r' = r - \epsilon r = r(1 - \epsilon). \quad (12)$$

- ▶ For $\epsilon \ll 1$ the binomial expansion gives:

$$(1 \pm \epsilon)^n \approx 1 \pm n\epsilon. \quad (13)$$

A note on dynamical stability

- ▶ Therefore the new density is:

$$\rho' = \frac{1}{4\pi r^2(1-\epsilon)^2} \frac{dm}{dr} \frac{dr}{dr'} = \frac{\rho}{(1-\epsilon)^3} \approx \rho(1+3\epsilon). \quad (14)$$

- ▶ Assume that the contraction is adiabatic, the perturbed pressure is related to the original one via $(\rho'/\rho)^\gamma$, where γ is the adiabatic exponent. Thus,

$$p'_{\text{gas}} = p(1+3\epsilon)^\gamma \approx p(1+3\gamma\epsilon). \quad (15)$$

- ▶ The new hydrostatic pressure is

$$p'_h = \int_m^M \frac{Gm}{4\pi r^4(1-\epsilon)^4} dm \approx p(1+4\gamma\epsilon). \quad (16)$$

A note on dynamical stability

- ▶ The gas sphere is no longer in hydrostatic equilibrium, i.e., $p'_{\text{gas}} \neq p'_h$. Condition for restoring equilibrium is

$$p'_{\text{gas}} > p'_h, \quad (17)$$

such that the sphere expands to its original state.

- ▶ From Eqs. (15) and (16) we can write this as

$$p(1 + 3\gamma\epsilon) > p(1 + 4\gamma\epsilon). \quad (18)$$

- ▶ From this we can read the condition for *dynamical stability*:

$$\gamma > \frac{4}{3}. \quad (19)$$

- ▶ If $\gamma < \frac{4}{3}$ somewhere (but not throughout) the star, stability is determined by the integral

$$\int \left(\gamma - \frac{4}{3}\right) \frac{p}{\rho} dm. \quad (20)$$

Instability occurs if the integral is negative.

A note on dynamical stability

- ▶ For relativistic-degenerate electron gas $\gamma = \frac{4}{3}$. This is the limiting case corresponding to the Chandrasekhar mass $M_{\text{Ch}} \sim 1.46M_{\odot}$. If the mass is higher a perturbation leads to collapse of the star.
- ▶ If radiation pressure becomes dominant such that $p \approx p_{\text{rad}}$, $\gamma \rightarrow \frac{4}{3}$.
- ▶ For pure photon gas $p/\rho = u_{\text{rad}}/3$. Using the virial theorem, we get

$$-\Omega = 3 \int_0^M \frac{p}{\rho} dM = u_{\text{rad}}. \quad (21)$$

This means that the total energy of the star $E = \Omega + u_{\text{rad}}$ vanishes.

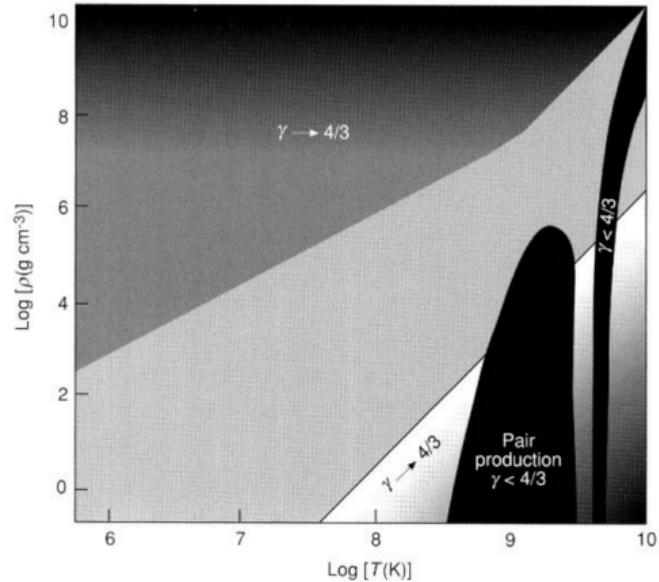
- ▶ This means that the gas sphere is *unbound*.

A note on dynamical stability

- ▶ An ionisation(-type) process can change the adiabatic index and lead to $\gamma < \frac{4}{3}$.
- ▶ The most obvious example is ionisation where a single atom can produce two particles if a suitable amount of energy is absorbed or via a collision with another particle or photon.
- ▶ The reverse process (recombination) occurs at the same time and reduces the number of particles.
- ▶ Because pressure depends on the number of particles irrespective of their nature, ionisation/recombination leads to a dependence of the pressure on volume (density) is different than for ideal gas.
- ▶ As an example, for a pure singly ionized gas $\gamma < \frac{4}{3}$ between 18% and 82% ionization. Therefore only a nearly neutral or nearly completely ionized gas would be dynamically stable.

Zones of instability

- ▶ In regions III and IV $\gamma \rightarrow \frac{4}{3}$ due to relativistic degenerate gas and radiation pressure, respectively. Therefore dynamical instability can also occur.
- ▶ Further regions where dynamical instability occurs include the iron photodisintegration zone and pair production strip which are ionization-like nuclear processes.
- ▶ Therefore the stable part of the (ρ, T) diagram is bounded by high temperatures and high density regions.
- ▶ Furthermore, nuclear burning is *thermally* unstable in degenerate gases.



Credits: Prialnik

Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

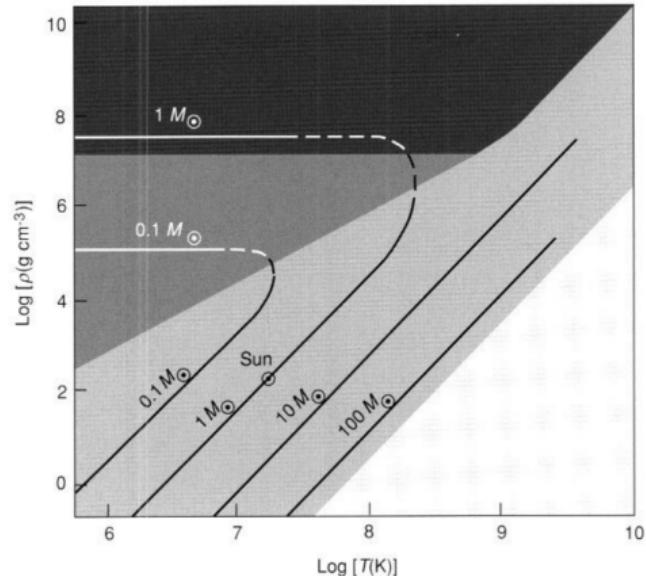
- ▶ Consider a polytropic configuration for a star in hydrostatic equilibrium. Then the central density is

$$p_c = (4\pi)^{1/3} B_n GM^{2/3} \rho_c^{4/3}, \quad (22)$$

where the coefficient B_n is a weak function of the polytropic index in the range $n = 1.5 \dots 3$.

- ▶ Consider a star in zone I where ideal gas equation applies. Then the relation between ρ_c and T_c is

$$\rho_c = \frac{K_0^3}{4\pi B_n^3 G^3} \frac{T_c^3}{M^2}. \quad (23)$$



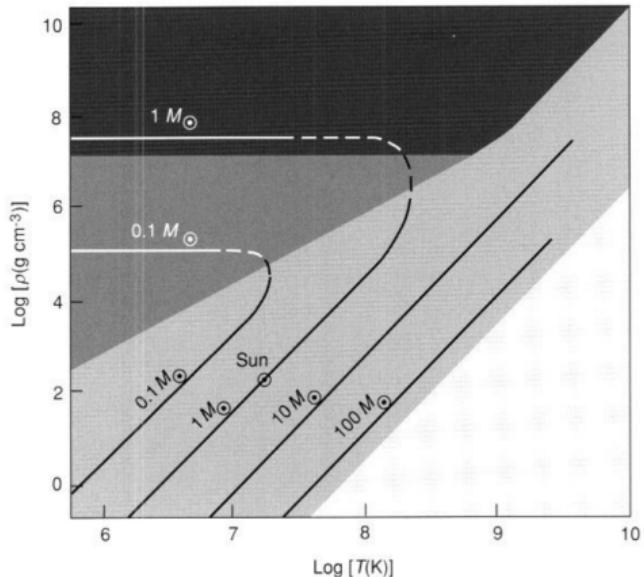
Credits: Prialnik

Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

- ▶ For a given density the temperature increases when the mass of the star increases. This leads to parallel lines whose intersections at a given ρ differ by $\log M$.
- ▶ If the gas at the centre is degenerate, ρ_c is given by

$$\rho_c = 4\pi \left(\frac{B_{1.5} G}{K_1} \right)^3 M^2. \quad (24)$$

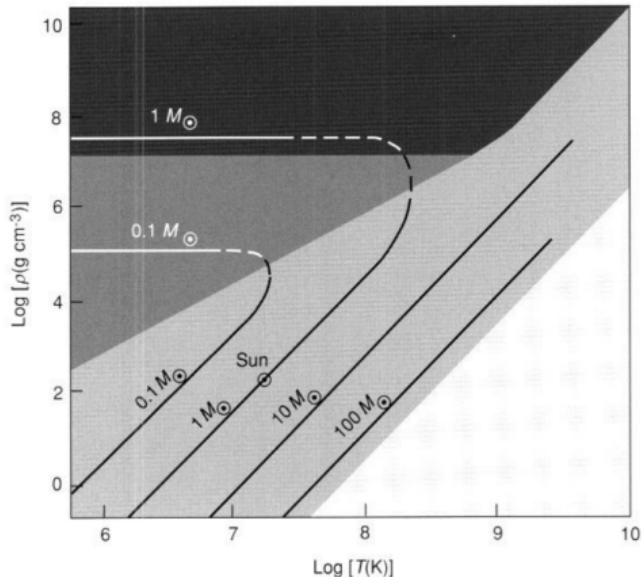
These are horizontal lines in the $(\log T, \log \rho)$ -plane. Note that ρ_c is independent from T_c .



Credits: Prialnik

Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

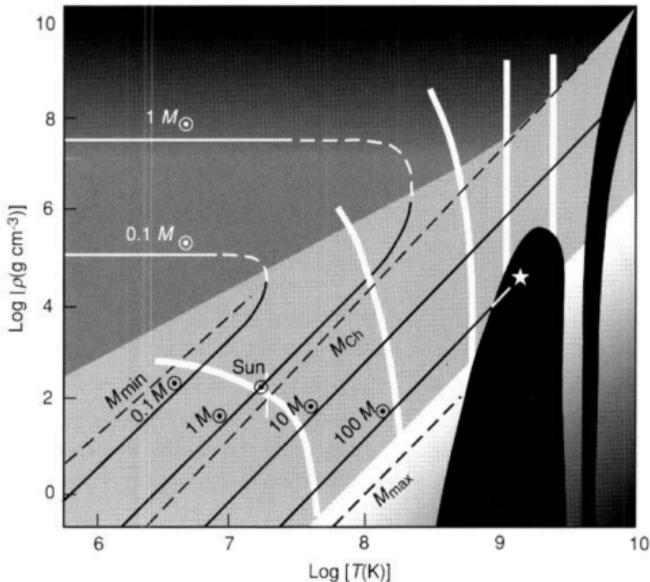
- ▶ The density of degenerate stars tends to infinity as the stellar mass tends to the Chandrasekhar mass M_{Ch} (tutorials).
- ▶ This mass separates the regions I and III and the “knee” curves for $M < M_{\text{Ch}}$ and the straight ones for $M > M_{\text{Ch}}$.



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Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

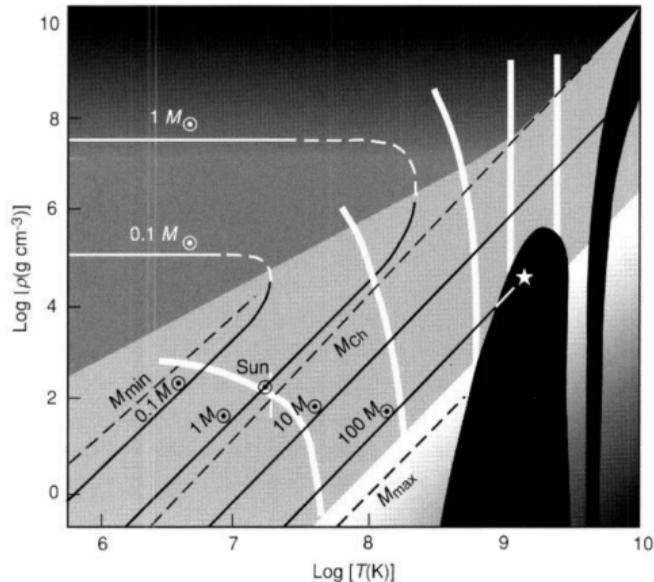
- ▶ Combination of all of the information so far can be used to sketch the evolutionary paths of stars with different masses.
- ▶ This can be used to sketch the general paths of stars of different masses.
- ▶ Stars have a minimum and maximum masses. **Why?**



Credits: Prialnik

Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

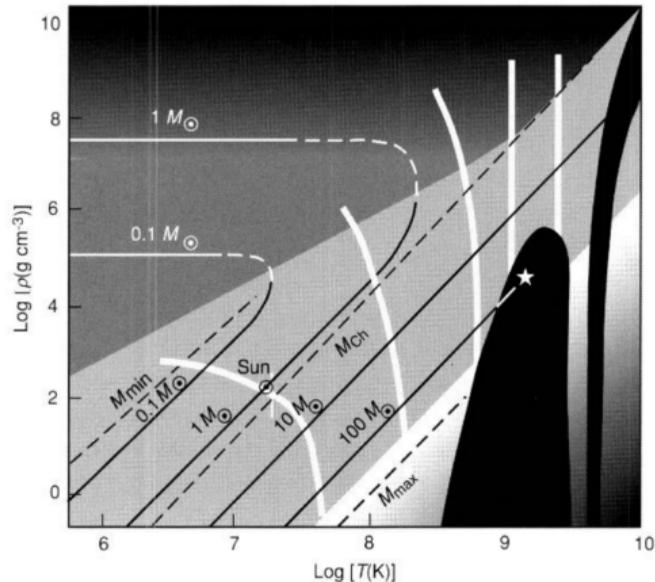
- ▶ Combination of all of the information so far can be used to sketch the evolutionary paths of stars with different masses.
- ▶ This can be used to sketch the general paths of stars of different masses.
- ▶ Stars have a minimum and maximum masses. **Why?**
- ▶ The cores of stars with $M \lesssim 0.08M_\odot$ are not hot enough to ignite nuclear fusion. For $M \gtrsim 100M_\odot$ the radiation pressure makes the star unbound.



Credits: Prialnik

Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

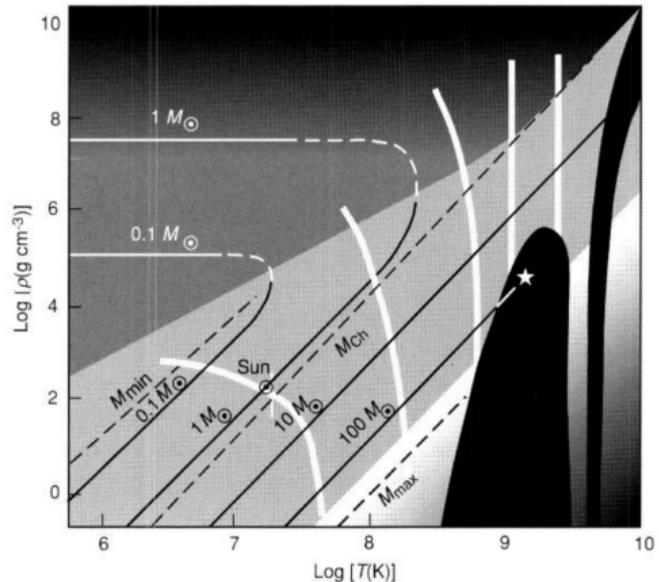
- ▶ Low mass stars contract and heat up until hydrogen burning commences. The stars remain in this evolutionary phase for a very long (nuclear) time.
- ▶ When hydrogen is depleted from the core, degeneracy starts to play a role. Therefore stars with $M < M_{\text{Ch}}$ end up as white dwarfs and cool over again a long time.
- ▶ White dwarfs have a constant radius and density that depend only on the mass M . The larger the mass, the lower the radius.
- ▶ We would expect two populations of white dwarfs. [Why?](#)



Credits: Prialnik

Evolutionary path of the centre of the star in the $(\log T, \log \rho)$ plane

- ▶ Stars near $M = M_{\text{Ch}}$ cross the carbon detonation very near the degeneracy border that can lead to a thermonuclear instability and collapse. **Why is this very unlikely for a single star?**
- ▶ More massive stars burn successively heavier elements as the lighter ones have been consumed. Eventually they hit the iron photodissociation strip where another catastrophic instability happens.
- ▶ The most massive stars can hit the pair-production instability strip even earlier than this.



Credits: Prialnik

Theory of the main sequence

- ▶ There is an observed relation between the luminosity and effective temperature of main sequence stars

$$\log L = \alpha \log T_{\text{eff}} + \text{const.}, \quad (25)$$

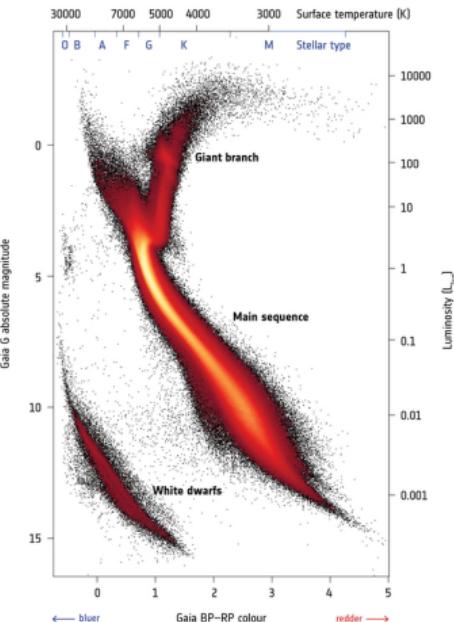
where the slope α is different for low and large L .

- ▶ Another apparent correlation exists between luminosity and mass:

$$L \propto M^\nu. \quad (26)$$

- ▶ Based on our theory, we make a hypothesis that main sequence stars burn hydrogen in their cores and are correspondingly placed on the $(\log T, \log \rho)$ plane. This theory needs to reproduce the relations (25) and (26).

→ GAIA'S HERTZSPRUNG-RUSSELL DIAGRAM



Credits: ESA

Theory of the main sequence

- ▶ Assume stars that have started to burn hydrogen at the centre and are in thermal and hydrostatic equilibrium. We assume uniform chemical composition, radiative equilibrium, neglect radiation pressure and assume analytic (constant) opacity.
- ▶ Now the governing equations are:

$$\frac{dp}{dm} = -\frac{Gm}{4\pi r^4}, \quad \frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}, \quad (27)$$

$$\frac{dT}{dm} = -\frac{3}{4ac} \frac{\kappa}{T^3} \frac{F}{(4\pi r^2)^2}, \quad \frac{dF}{dm} = q_o \rho T^n, \quad p = \frac{\mathcal{R}}{\mu} \rho T. \quad (28)$$

- ▶ This set of equations needs to be solved for $r(m)$, $p(m)$, $T(m)$, and $F(m)$ in the range $0 \leq m \leq M$ for any value of M which is a free parameter.
- ▶ We would want to get some insight without actually solving the equations. [How?](#)

Theory of the main sequence

- ▶ The answer is *dimensional analysis* that we already used on some of the tutorials.
- ▶ Let us first define a dimensionless variable for the fractional mass:

$$x = \frac{m}{M}. \quad (29)$$

- ▶ The functions $r(m)$, $p(m)$, $T(m)$, and $F(m)$ can then be replaced by non-dimensional functions that depend on x : $f_1(x)$, $f_2(x)$, $f_3(x)$, etc.
- ▶ We end up with:

$$r = f_1(x)R_\star, \quad p = f_2(x)p_\star, \quad \rho = f_3(x)\rho_\star, \quad T = f_4(x)T_\star, \quad F = f_5(x)F_\star, \quad (30)$$

where the starred quantities have the dimensions of the original functions.

Theory of the main sequence

- ▶ As an example, substituting the relations $r = f_1(x)R_\star$, $p = f_2(x)p_\star$, and $\rho = f_3(x)\rho_\star$ into equation of hydrostatic equilibrium yields

$$\frac{p_\star}{M} \frac{df_2}{dx} = -\frac{GMx}{4\pi f_1^4 R_\star^4}. \quad (31)$$

- ▶ Since the units of both sides must match and f_i are non-dimensional, p_\star must be proportional to GM^2/R_\star^4 .
- ▶ Setting the coefficient of proportionality to unity leads to:

$$\frac{df_2}{dx} = -\frac{x}{4\pi f_1^4}, \quad p_\star = \frac{GM^2}{R_\star^4}. \quad (32)$$

Theory of the main sequence

- ▶ Repeating the exercise for the other four equations:

$$\begin{aligned}\frac{df_1}{dx} &= \frac{1}{4\pi f_1^2 f_3}, & \rho_\star &= \frac{M}{R_\star^3}, \\ f_2 &= f_3 f_4, & T_\star &= \frac{\mu p_\star}{\mathcal{R} \rho_\star}, \\ \frac{df_4}{dx} &= -\frac{3f_5}{4f_4^3 (4\pi f_1^2)^2}, & F_\star &= \frac{ac}{\kappa} \frac{T_\star^4 R_\star^4}{M}, \\ \frac{df_5}{dx} &= f_3 f_4^n, & F_\star &= q_0 \rho_\star T_\star^n M.\end{aligned}\tag{33}$$

- ▶ The equations on the lhs are non-linear differential equations for f_i in the range $0 \leq x \leq 1$. The dimensional quantities on the rhs of Eq. (30) are obtained from the *algebraic* Eqs. (33) (right).
- ▶ Combination of these differential and algebraic equations allows for solving the interior structure for any mass M . Importantly the profiles as function of x are the same in all stars, differing only by a constant factor determined by the mass. This property is called *homology*.

Theory of the main sequence

- ▶ Now we can skip solving the differential equations if we are just interested in the dependence of physical properties on M .
- ▶ Solving for T_* gives:

$$T_* = \frac{\mu G}{\mathcal{R}} \frac{M}{R_*}. \quad (34)$$

Using this in the first equation for F_* gives:

$$F_* = \frac{ac}{\kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^3. \quad (35)$$

- ▶ Since this relation is valid for any value of x , the surface value scales as

$$L \propto M^3, \quad (36)$$

which can be compared with observations.

Theory of the main sequence

- ▶ Furthermore, we can now solve for (exercises) R_* and ρ_* :

$$R_* \propto M^{\frac{n-1}{n+3}}, \quad \rho_* \propto M^{2\frac{3-n}{3+n}}. \quad (37)$$

What can be concluded based on these relations?

Theory of the main sequence

- ▶ Furthermore, we can now solve for (exercises) R_\star and ρ_\star :

$$R_\star \propto M^{\frac{n-1}{n+3}}, \quad \rho_\star \propto M^{2\frac{3-n}{3+n}}. \quad (38)$$

What can be concluded based on these relations?

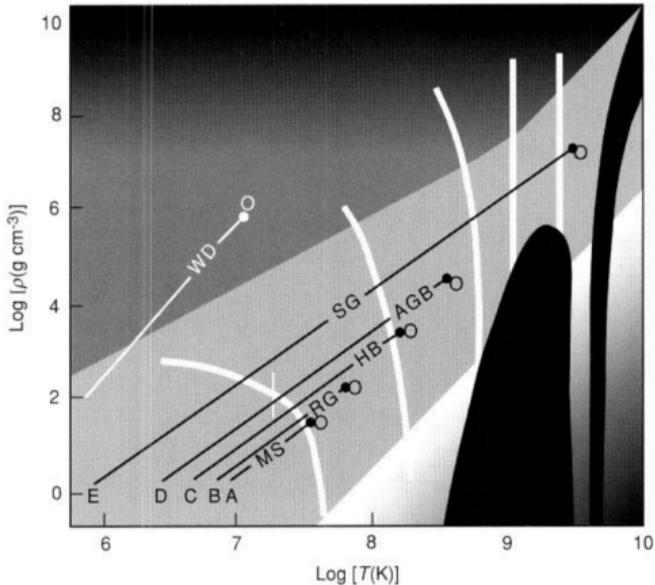
- ▶ $n \approx 16$ for the CNO cycle so stellar radius is almost independent of mass. For *pp* chain $n \approx 4$ and $R_\star \propto M^{3/7}$. Radius always increases with mass unlike for white dwarfs!
- ▶ Since $n > 3$, density decreases with increasing M again opposite to white dwarfs.

Theory of the main sequence

- ▶ TBA

Evolutionary phases of a $10M_{\odot}$ star

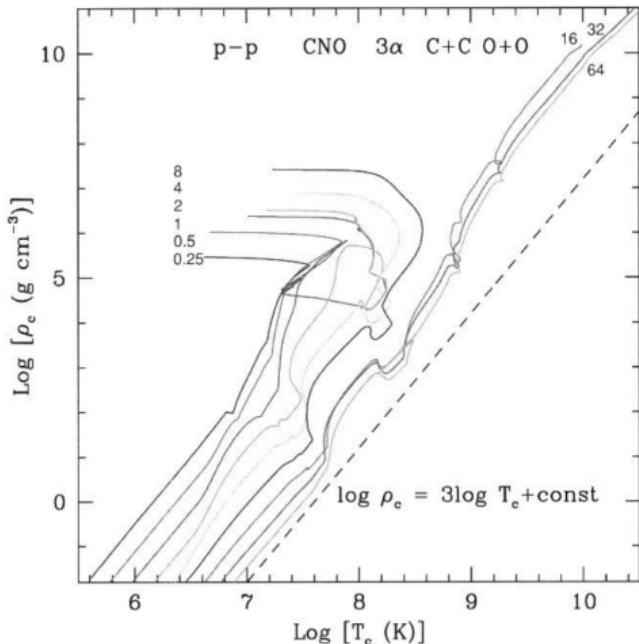
► TBA



Credits: Prialnik

Limitations of the simple model

► TBA



Credits: Prialnik