Hands-on exercises 5: Non-dimensionalisation and limits of numerical simulations

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We will take a look at non-dimensionalisation of some of the MHD equations and discuss the limitations of numerical simulations of stellar convection zones.

Problem 1: Take the dimensional Navier-Stokes equation in the Boussinesq approximation, where density differences are ignored everywhere except in the buoyancy term

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} \varpi + \nu \nabla^2 \boldsymbol{u} + \alpha g T \hat{\boldsymbol{z}}, \tag{1}$$

where $\varpi \sim p/\rho$ is a reduced pressure, α is the coefficient of thermal expansion, and $\hat{\boldsymbol{z}}$ is the unit vector in the z direction. Write the equation in non-dimensional form using, e.g., $x = \ell_\mathrm{c}\tilde{x}$, $t = \tau_\mathrm{c}\tilde{t}$, etc., and choose ℓ_c, τ_c, and other units such that the non-dimensional version of Eq. (1) contains system parameters $\mathrm{Ma}^2 = u_c^2/(p_c/\rho_c)$, $\mathrm{Pr} = \nu/\chi$, and $\mathrm{Ra} = g\alpha T_c L^3/(\nu\chi)$.

Problem 2: Estimate the Kolmogorov scale ℓ_ν (scale at which kinetic energy is dissipated to heat) by assuming that the energy transfer rate from large to small scales equals the kinetic energy dissipation rate.

Calculate the ratio ℓ_ν/L where L is the integral (system) scale and relate this to the number of grid points you need in a simulation to fully capture the dynamics (aka a *direct simulation*). Bear in mind the definition of the Reynolds number

$$\mathrm{Re} = \frac{u\ell}{\nu}. \tag{2}$$

For the solar convection zone $\mathrm{Re} \sim 10^{12}$, where $L = 0.3R_\odot$. How many grid points would you need to fully capture this? A typical current simulation has 1000^3 grid points and use around 10^3 CPU cores. How much more would be needed for a direct simulation of the solar convection zone?

The timestep of the simulations is restricted by the Courant-Friedrichs-Levy (CFL) condition that states that

$$\delta t \leq C_\mathrm{CFL} \frac{\Delta x}{u_\mathrm{s}}, \tag{3}$$

where C_CFL is a constant of the order of unity, Δx is the grid spacing, and u_s is the fastest signal propagation speed. Estimate the timestep for a direct simulation of the solar convection zone if the fastest signal is the sound speed at the base ($c_\mathrm{s} \sim 200$ km s^{-1}) or the maximum convective velocity ($u_\mathrm{conv} \sim 2$ km s^{-1}).

Computing a single timestep in the typical simulations mentioned above takes about 0.1 s in wall-clock time. Estimate the time-to-solution for a direct simulation of the solar convection zone over a sunspot cycle (11 years) based on the timestep estimates computed earlier. What can you conclude?

Useful physical constants

- $R_{\odot} = 696 \times 10^{6}\,\mathrm{m}$
- $M_{\odot} = 1.989 \times 10^{30}\,\mathrm{kg}$
- $L_{\odot} = 3.83 \times 10^{26}\ \mathrm{W}$
- $T_{\odot}^{\mathrm{eff}} = 5777\,\mathrm{K}$
- $1\,\mathrm{AU} = 1.496 \times 10^{8}\,\mathrm{km}$
- $c = 2.997 \times 10^{8}\,\mathrm{m/s}$
- $G = 6.674 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
- $k = 1.38 \cdot 10^{-23}\ \mathrm{J/K}$
- $m_{\mathrm{H}} = 1.67 \cdot 10^{-27}\ \mathrm{kg}$