Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 5: Convection in stars versus convection simulations

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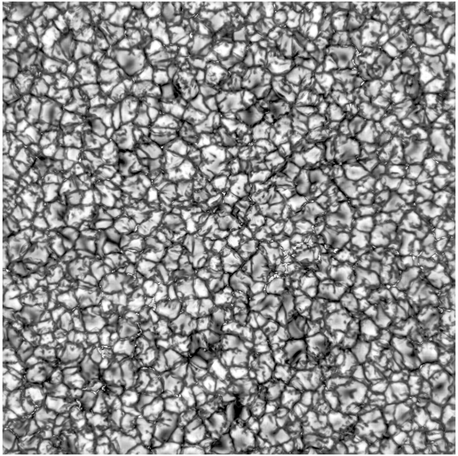
Convection in the Sun

- Several types of convective flows are conjectured to exist in the Sun.
- Granulation is directly observed. Velocity $u \sim 1$ km/s, $\ell \sim 10^3$ km, turnover time $\tau_\mathrm{c} \sim$ a few minutes.



Solar granulation from the Daniel K. Inoue Solar Telescope (DKIST).

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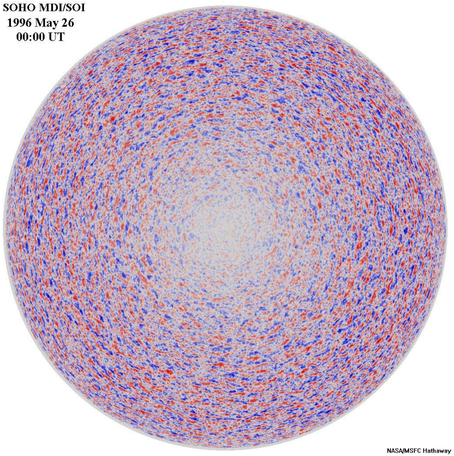
- Supergranulation is observed using the Doppler effect. Velocity $u_h \sim 300$ m/s, $u_v \sim 40$ m/s, $\ell \sim 20 \dots 35$ Mm, turnover time $\tau_c \sim$ about a day.
- Why is the signal so weak near the centre of the disk?



Solar supergranulation from SOHO MDI.

Convection in the Sun

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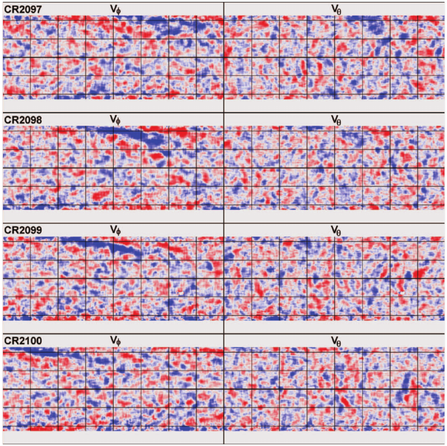
- Giant cells? $u \sim 10$m/s, $\ell \sim 200$ Mm, $\tau_c \sim$ a month.
- Why would we even expect giant cells?



Hathaway et al. (2013), Science, 342, 1217.

Simulations of convection

- Simulations solve the equations of (magneto)hydrodynamics in a coordinate system most suited to the physical problem:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta\mu_0 \boldsymbol{J}),$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}),$$

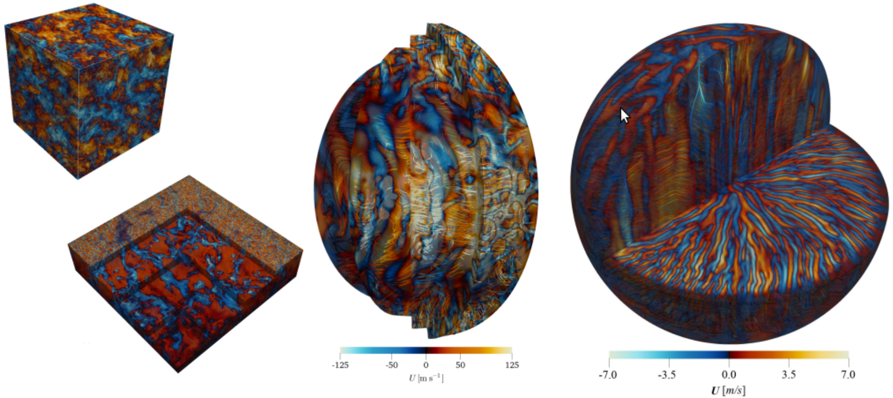
$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) + \rho \boldsymbol{g} - \nabla p - 2\rho \boldsymbol{\Omega}_0 \times \boldsymbol{U} + \boldsymbol{J} \times \boldsymbol{B} + \nabla \cdot \boldsymbol{F}^{\mathrm{visc}},$$

$$\rho T \frac{\partial s}{\partial t} = -\nabla \cdot (\rho s \boldsymbol{u}) + \nabla \cdot \mathcal{F} + \mathcal{H} + 2\nu\rho \mathbf{S}^2 + \eta\mu_0 \boldsymbol{J}^2,$$

- Here, $\boldsymbol{F}_{\mathrm{visc}} = 2\nu\rho\mathbf{S}$, where $\mathsf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla} \cdot \boldsymbol{u}$ is the traceless rate-of-strain tensor, and $\mathcal{F}$ contains the radiative flux $\mathcal{F}_{\mathrm{rad}} = -K\boldsymbol{\nabla} T$.

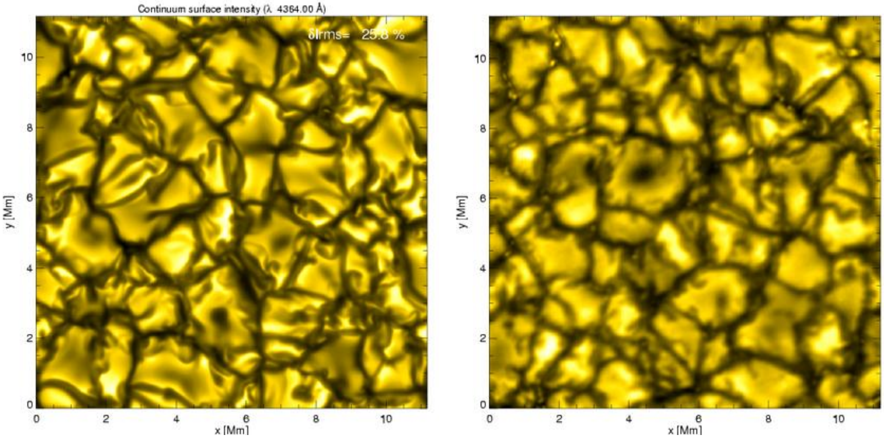
Simulations of convection

- Simulations solve the equations of (magneto)hydrodynamics in a coordinate system most suited to the physical problem:



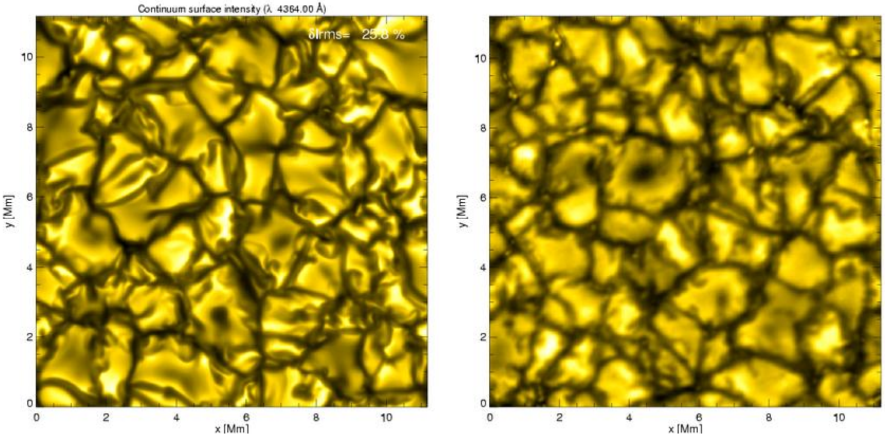
PENCIL CODE, https://pencil-code.org/

Simulations of solar surface convection



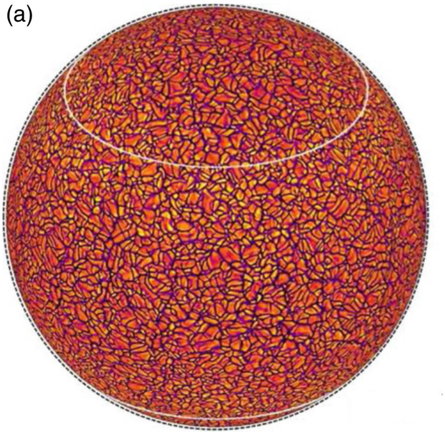
Which one is simulation and which on observation?

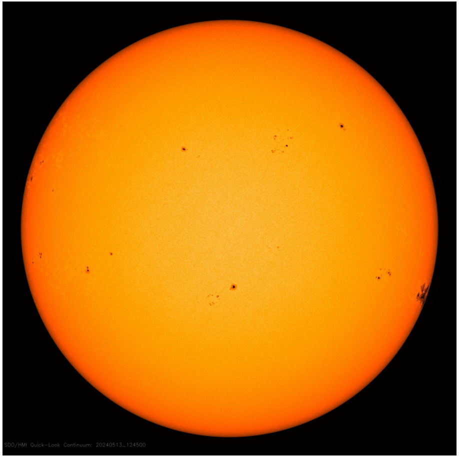
Simulations of solar surface convection



Left: Simulated solar surface convection (credit: Matthias Steffen). Right: Observed granulation from Swedish Solar Telescop (SST) (credit: Mats Carlsson).

Simulations of global solar convection





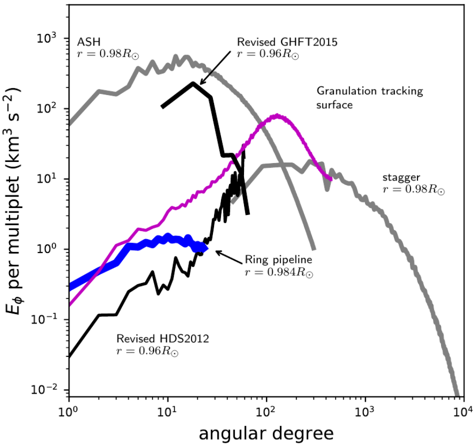
Left: Radial velocity near the surface of a high-resolution global simulation of the Sun (Hotta et al. 2015, Astrophys. J., 798, 51). Right: HMI Intensitygram from Solar Dynamics Observatory (SDO) 13.05.2024.

Convective conundrum

- Convective velocities observed in the Sun can be used to compute the power spectrum of the velocity:

$$E_{\mathrm{K}} = \frac{1}{2}\int E(k)dk, \qquad (1)$$

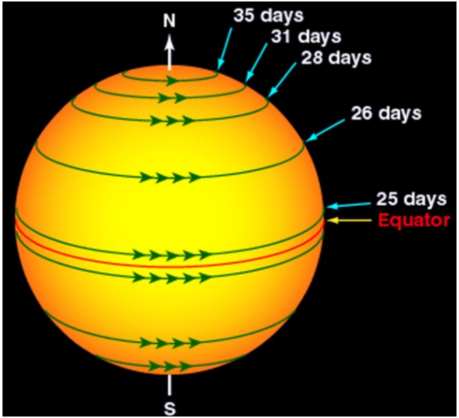
  with $E_{\mathrm{K}} = \frac{1}{2}\boldsymbol{u}^2$ and $E(k) = \sum_k |\hat{\boldsymbol{u}}(k)|^2$.
- While the surface simulations (Stagger) seem to be largely compatible with the granulation tracking observations, global simulations (ASH) suggest velocities that are much larger.
- This is one of the manifestations of the "Convective conundrum," or the discrepancy between models and reality.



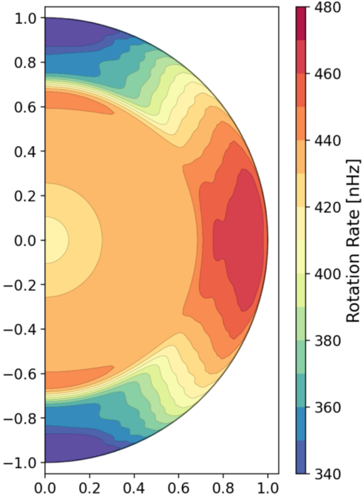
Proxauf (2021), PhD Thesis (Göttingen Univ.)

Differential rotation

The Sun rotates *differentially*, i.e., not like a solid body. Here $\Omega = u_\phi / r \sin\theta$.



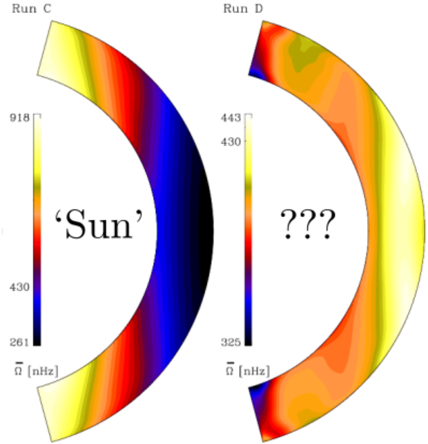
Source: NASA.



Source: Larson and Schou (2018).

Differential rotation in simulations

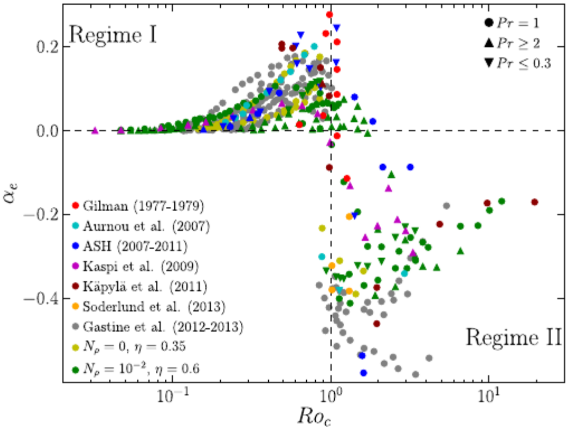
- The interaction of convection and the global solar rotation lead to differential rotation.
- Simulations with solar parameters (only luminosity and rotation rate as we will see below!) very often yield "anti-solar" differential rotation with fast poles and slow equator.
- What simple argument can explain the anti-solar states?



Mean angular velocity $\overline{\Omega} = \overline{u}_\phi / r \sin\theta + \Omega_0$. From Käpylä et al. (2014), Astron. Astrophys., 570, 43

Differential rotation in simulations

- Simulations show that the flip fron anti-solar (Regime II) to solar-like (Regime I) occurs around $\mathrm{Ro}_c \sim \mathrm{Co}^{-1} \sim 1$.
- The Sun appears to be near this transition and simulations tend to underestimate Co.



Source: Gastine et al. (2014), MNRAS, 438, L76.

Convection and rotation

- As we discussed in the last tutorial, the order of magnitude of the convective velocity and length scale are

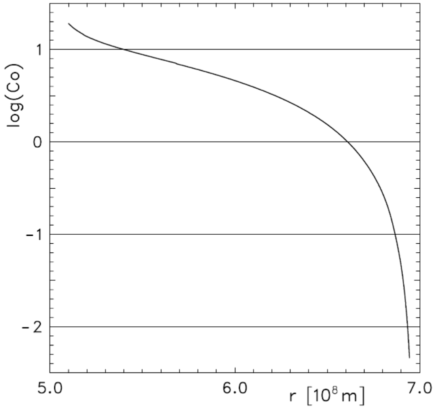
$$u \sim \left(\frac{F_{\rm tot}}{\rho}\right)^{1/3} \equiv u_\star, \text{ and } \ell \sim H_p, \qquad (2)$$

where $F_{\rm tot} = \frac{L}{4\pi r^2}$ and ρ is a reference density.

- We also saw on the previous lecture that according to the mixing length theory the Coriolis number

$$\mathrm{Co} = \frac{2\Omega_\odot \ell}{u}, \qquad (3)$$

varies between 10^{-3} near the surface and around 10 at the base of the convection zone.



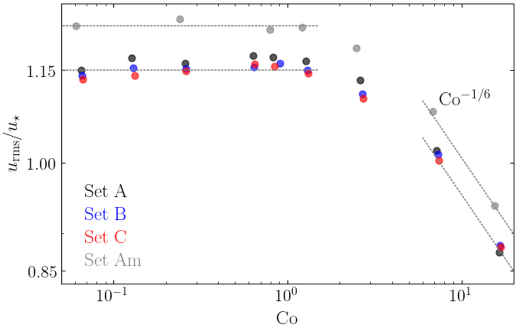
Käpylä et al. (2005), Astron. Astrophys., 438, 403.

Convection and rotation

- When rotation is sufficiently rapid, the rotating mixing length theory (e.g. Stevenson, 1979, Geophys. Astrophys. Fluid Dyn., 12, 139) predicts that the convective velocity changes and is

$$u \sim \left(\frac{F_{\rm tot}}{\rho}\right)^{1/3} \mathrm{Co}^{-1/6} = u_\star \mathrm{Co}^{-1/6}, \quad (4)$$

- Where is the Sun in this diagram?



Convective velocity divided by $u_\star$ from several sets of simulations. From Käpylä (2024), Astron. Astrophys., 683, 221.

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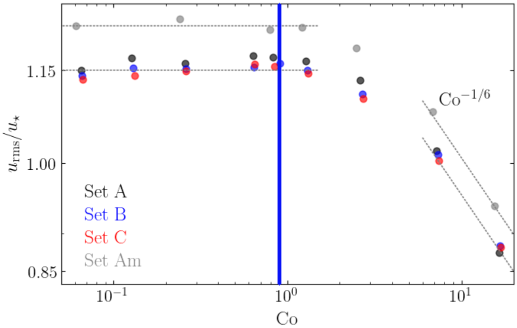
$$u \sim \left(\frac{F_{\rm tot}}{\rho}\right)^{1/3} {\rm Co}^{-1/6} = u_\star {\rm Co}^{-1/6}, \quad (5)$$

- Where is the Sun in this diagram?



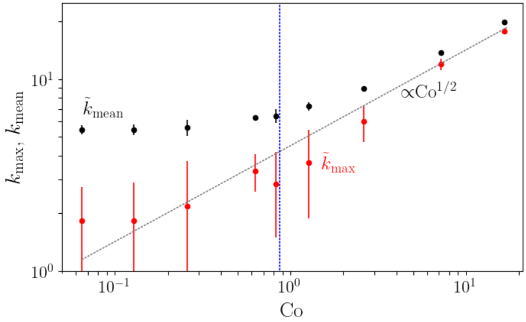
Convective velocity divided by $u_\star$ from several sets of simulations. From Käpylä (2024), Astron. Astrophys., 683, 221.

Convection and rotation

- The convective length scale also changes

$$\ell \sim H_p \mathrm{Co}^{-1/2}. \quad (6)$$

- It appears that convection in the deep parts of solar convection zone are less rotationally constrained than previously thought.



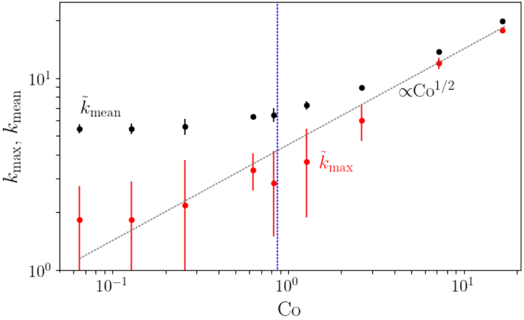
Two measures of the scale ($k = 2\pi/\ell$) of convective structures in simulations as functions of Co. From Käpylä (2023), arXiv:2311.09082.

Convection and rotation

- The convective length scale also changes

$$\ell \sim H_p \mathrm{Co}^{-1/2}. \tag{7}$$

- It appears that convection in the deep parts of solar convection zone are less rotationally constrained than previously thought.
- There is a catch, however, and we will come back to that later.



Two measures of the scale ($k = 2\pi/\ell$) of convective structures in simulations as functions of Co. From Käpylä (2023), arXiv:2311.09082.

Why do simulations struggle to reproduce the Sun?

- Any ideas?

Why do simulations struggle to reproduce the Sun?

- The equations of magnetohydrodynamics can be written in a dimensionless form where a number of dimensionless quantities uniquely defining the system appear (tutorial).

$$
\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta\mu_0 \boldsymbol{J}), \\
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \boldsymbol{u}), \\
\rho \frac{\partial \boldsymbol{u}}{\partial t} &= -\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) + \rho \boldsymbol{g} - \nabla p - 2\rho \boldsymbol{\Omega}_0 \times \boldsymbol{U} + \boldsymbol{J} \times \boldsymbol{B} + \nabla \cdot \boldsymbol{F}^{\mathrm{visc}}, \\
\rho T \frac{\partial s}{\partial t} &= -\nabla \cdot (\rho s \boldsymbol{u}) + \nabla \cdot \mathcal{F} + \mathcal{H} + 2\nu\rho \mathbf{S}^2 + \eta\mu_0 \boldsymbol{J}^2,
\end{aligned}
$$

Why do simulations struggle to reproduce the Sun?

- The equations of magnetohydrodynamics can be written in a dimensionless form where a number of dimensionless quantities uniquely defining the system appear.
- These include the Prandtl numbers, the Rayleigh number, and the Taylor number:

$$\mathrm{Pr} = \frac{\nu}{\chi}, \quad \mathrm{Pr_M} = \frac{\nu}{\eta}, \quad \mathrm{Ra} = \frac{gd^4}{\nu\chi}\left(-\frac{1}{c_p}\frac{ds}{dr}\right), \quad \mathrm{Ta} = \frac{4\Omega^2 d^4}{\nu^2}, \tag{8}$$

  where ν is the kinematic viscosity, η is the magnetic diffusivity, and $\chi = K_{\mathrm{rad}}/c_p\rho$ is the radiative diffusivity.
- We can also estimate the relative magnitudes of different terms in the equations by OOM (order of magnitude) analysis. These give an additional set of (diagnostic) numbers (Reynolds, Péclet, Coriolis):

$$\mathrm{Re} = \frac{ud}{\nu}, \quad \mathrm{Re_M} = \frac{ud}{\eta}, \quad \mathrm{Pe} = \frac{ud}{\chi}, \quad \mathrm{Co} = \frac{2\Omega d}{u}. \tag{9}$$

- We need to compare these quantities in the simulations to those in stars.

Why do simulations struggle to reproduce the Sun?

- Typically these numbers in stars are either $\gg 1$ or $\ll 1$.
- Simulations are limited by available computational resources and can for most parameters do only modest values.

Parameter	Sun ($M_\odot$)	O9 (20 $M_\odot$)	Simulations
Ra	10^{20}	10^{24}	10^{9}
Pr	10^{-6}	10^{-5}	$10^{-1}\dots 10$
$\mathrm{Pr_M}$	10^{-3}	10^{3}	$10^{-1}\dots 10$
Re	10^{13}	10^{11}	10^{4}
Pe	10^{7}	10^{6}	10^{4}
$\mathrm{Re_M}$	10^{10}	10^{14}	10^{4}
$\Delta\rho$	10^{6}	3	10^{2}
Ro	$0.1\dots 1$	1[1]	$10^{-2}\dots 10^{3}$

Some dimensionless parameters in the Sun and in a $20M_\odot$ O9 star in comparison to typical simulations. From Käpylä et al. (2023), Space Science Rev., 219, 58.

Only the Rossby number (Co^{-1}) can be reproduced!

Why do simulations struggle to capture the Sun?

- More precisely, another Coriolis number can be defined as

$$\mathrm{Co}_\star = \frac{2\Omega H_p}{u_\star}, \tag{10}$$

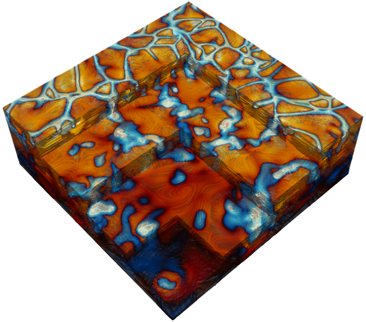
that does not depend on any actual velocity or length scale. It turns out that

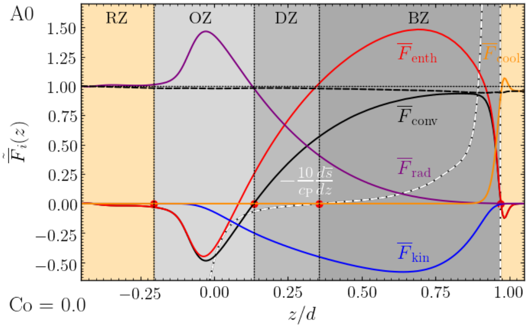
$$\mathrm{Co}_\star = (\mathrm{Ra}_\mathrm{F}^\star)^{-1/3}, \quad \text{where} \quad \mathrm{Ra}_\mathrm{F}^\star = \frac{\mathrm{Ra}_\mathrm{F}}{\mathrm{Pr}^2\mathrm{Ta}^{3/2}}. \tag{11}$$

- The catch mentioned earlier is that it is not enough to reproduce *one* of the system parameters if all the other ones are (too much) off.
- We will come back to this in the tutorial...

Non-canonical models of convection

- Heating from the bottom or cooling at the top?
- Instead of $\nabla - \nabla_{\rm a} > 0$ everywhere (as in mixing length theory), perhaps the lower part of the convection zone is *stably stratified*? This happens in the case of surface cooling driven convection.





Left: snapshot of the velocity. Right: Energy fluxes and the negative of the entropy gradient. Grey areas mixed by convection!

Why should we care?

- The evolution of the magnetic field is governed by

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B} - \eta\mu_0 \boldsymbol{J}), \tag{12}$$

  where $\boldsymbol{J} = \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{B}$ is the current density.
- If $\boldsymbol{u} = 0$, the induction equation reduces to a *diffusion equation*:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}. \tag{13}$$

- Why is this relevant?

Why should we care?

- The evolution of the magnetic field is governed by

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  where $\boldsymbol{J} = \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{B}$ is the current density.
- If $\boldsymbol{u} = 0$, the induction equation reduces to a *diffusion equation*:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}. \tag{15}$$

- Why is this relevant?
- The field cannot be generated without flows. To understand the generation of, e.g., the solar magnetic field, we must now the velocity field. *Study of magnetic phenomena is by far the most interesting branch of solar physics!*