Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 4: Convective energy transport

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Energy transport mechanisms in stars

- Radiation: photons transport the energy. Mean free path is very short and this can be modeled as a diffusion process down the temperature gradient, i.e.,

$$\mathbf{F}_{\rm rad} = -K_{\rm rad} \boldsymbol{\nabla} T, \tag{1}$$

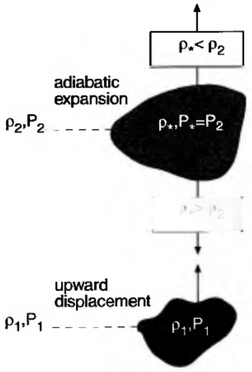
  where $K_{\rm rad}$ is the radiative conductivity.
- Conduction: heat is transported because of collisions of particles. Analogous to 1, this is written as

$$\mathbf{F}_{\rm cd} = -K_{\rm cd} \boldsymbol{\nabla} T. \tag{2}$$

  Typically $K_{\rm rad} \gg K_{\rm cd}$ (Except where?).
- Convection: gas is opaque to radiation, becomes *unstable*, and fluid motions transport the energy. This leads to very complicated dynamics and cannot in general be represented in equally simple terms as radiation or conduction.

Intuitive picture of convective instability

- Consider a fluid element with density ρ_1 and pressure p_1 displaced upward to a level where $\rho = \rho_2$ and $p = p_2$.
- If the density inside the element ($\rho_\star$) is larger (smaller) than the ambient density, it is pulled back (continues to accelerate).
- Assumption: no heat exchange between fluid element and the surroundings.



Source: Prialnik

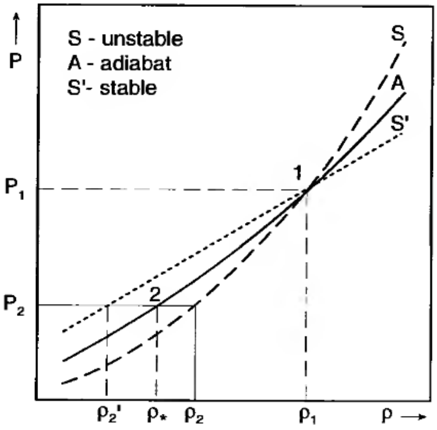
Schwarzschild criterion

- Consider the atmospheres A, S, and S': the stably stratified case corresponds to S', where

$$\frac{\partial p}{\partial \rho} < \left(\frac{\partial p}{\partial \rho}\right)_{\mathrm{a}} \qquad (3)$$

- This form is not particularly useful so we will recast it in terms of a temperature gradient.
- Recall the 1st law of thermodynamics:

$$dQ = du + pdV. \qquad (4)$$



Source: Prialnik

Schwarzschild criterion

- The ideal gas equation can be written as:

$$p = \frac{\mathcal{R}}{\mu}\rho T = (c_p - c_V)\rho T, \tag{5}$$

  where c_p and c_V are specific heat capacities at constant pressure and at constant volume. Their ratio is $\gamma = c_p/c_V = 5/3$.

- Furthermore, with the internal energy $u = c_V T$ this becomes

$$p = \frac{\mathcal{R}}{\mu}\rho T = (\gamma - 1)\rho u, \quad \text{and} \quad u = \frac{1}{\gamma - 1}\frac{p}{\rho}. \tag{6}$$

- For an adiabatic process $dQ = 0$. Furthermore, the specific volume is $V = \rho^{-1}$, and $dV = -d\rho/\rho^2$. Thus,

$$\frac{1}{\gamma - 1}\left(\frac{dp}{\rho} - p\frac{d\rho}{\rho^2}\right) - p\frac{d\rho}{\rho^2} = 0. \tag{7}$$

$$\frac{1}{\gamma - 1}\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - \frac{d\rho}{\rho} = 0. \tag{8}$$

$$\frac{1}{\gamma - 1}\left(\frac{\rho}{p}\frac{dp}{d\rho} - 1\right) - 1 = 0. \tag{9}$$

$$\frac{\rho}{p}\frac{dp}{d\rho} = \left(\frac{\rho}{p}\frac{dp}{d\rho}\right)_{\mathrm{a}} = 1 + (\gamma - 1) = \gamma. \tag{10}$$

Schwarzschild criterion

- Going back to Eq. (3) we can write:

$$\frac{\rho}{p}\frac{dp}{d\rho} < \left(\frac{\rho}{p}\frac{dp}{d\rho}\right)_{\mathrm{a}} = \gamma. \tag{11}$$

- For an ideal gas

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}. \tag{12}$$

- Multiply by p/dp, make use of Eq. (11) and define ∇_{a}:

$$1 = \left(\frac{p}{\rho}\frac{d\rho}{dp} + \frac{p}{T}\frac{dT}{dp}\right)_{\mathrm{a}} \equiv \frac{1}{\gamma} + \nabla_{\mathrm{a}}, \tag{13}$$

or:

$$\nabla_{\mathrm{a}} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma}. \tag{14}$$

Schwarzschild criterion

- Extending the definition of ∇ to the general case

$$\nabla \equiv \frac{p}{T}\frac{dT}{dp}, \tag{15}$$

  we can rewrite the stability condition (11) as

$$\nabla < \nabla_{\mathrm{a}}. \tag{16}$$

- Sometimes the quantity $\Delta\nabla = \nabla - \nabla_{\mathrm{a}}$ (superadiabaticity) is used to denoted whether a layer is convectively stable or not.
- In stellar convection zones (apart from the near-surface layes) $\Delta\nabla$ is very small, e.g., $\mathcal{O}(10^{-3} \ldots 10^{-4})$ in the Sun.

Schwarzschild criterion – reflection

- Violation of Schwarzschild criterion is necessary but not sufficient condition for convection to occur (Why?)

Schwarzschild criterion – reflection

- Violation of Schwarzschild criterion is necessary but not sufficient condition for convection to occur (Why?)
- Internal friction in the gas was been neglected $\rightarrow$ in reality $\Delta\nabla$ has to exceed a finite value $\Delta\nabla_{\rm min}$ (which depends on T, ρ, rotation, magnetic fields, etc.) for convection to ensue.
- Typically this is represented by a Rayleigh number which is a ratio of convective transport to diffusive transport:

$$\mathrm{Ra} = \frac{g d^4}{\nu \chi H_p} \Delta\nabla, \tag{17}$$

  where $g = GM/r^2$, ν is the kinematic viscosity, $\chi = K_{\rm rad}/\rho c_p$ is the radiative diffusivity, and H_p is the pressure scale height.
- Critical value for free convection is $\mathrm{Ra_c} \approx 1500$. In the Sun, $\mathrm{Ra} \approx 10^{20}$.

Mixing length theory

- Assume discrete gas elements that move a vertical distance ℓ before dissolving.
- This distance is called the *mixing length*, and is given by $\ell = \alpha_{\mathrm{MLT}} H_p$, where $\alpha_{\mathrm{MLT}} = \mathcal{O}(1)$.
- Consider the convective energy flux (unit: $\mathrm{W/m^2}$) is:

$$F_{\mathrm{conv}} = c_p \rho u T', \tag{18}$$

where u is the convective velocity, and $T' = T - \overline{T}$.

Mixing length theory

- Average (squared) convective velocity is

$$u^2 = g\delta(\nabla - \nabla_{\mathrm{e}})\frac{\ell^2}{8h_p}. \tag{19}$$

Estimates of convective velocity and temperature fluctuations

- Velocity from convective flux, Eq. (18)

$$u = \left(\frac{F}{\rho}\right)^{1/3} \tag{20}$$