# Theoretical Astrophysics I: Stellar Structure and Evolution
# Lecture 2: Stellar colors and luminosities. Basic equations of stellar structure

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# Recap

- We concluded that stars (and the Sun) emit the light.
- We can measure the recieved flux density at the Earth (i.e. irradiance, $\mathcal{E}$). In observational astrophysics we use the term **apparent magnitude**.
- If we know the distance to the star, and make some reasonable assumptions, we can find the total **luminosity** of the star
$$L = \mathcal{E} 4\pi d^2 [\mathrm{W}]$$
- We know how big the Sun is (cca 700 000 km in radius), so we can estimate its **effective temperature** from Stefan-Boltzmann law:

$$L = F(r = R) 4\pi R^2 = \sigma T^4 4\pi R^2$$

# Stefan-Boltzmann law

- The emissivity (flux density at the surface) is equal to:

$$\frac{dE^{\mathrm{em}}}{dtdS} = F(r = R) = \sigma T^4 \qquad (1)$$

- Then, the luminosity (emitted energy per second, power, flux) of such object is: $L = 4\pi R^2 \sigma T^4$.
- For the Sun If you substitute the numbers, you will get $T = 5777\,\mathrm{K}$.
- This is **a** temperature. The Sun does not have "one" temperature.
- It is basically a measure of solar luminosity!

# The concept of stellar atmosphere

- Energy is generated in the cores of stars and transported outwards (in multiple ways)
- As we move away from the stellar center - the medium becomes less opaque, at some point the photons can escape the star. Then they contribute to the total stellar luminosity that we ultimately detect.
- The layer of the last emission is referred to as $\tau = 1$ layer, where $\tau$ is the so called **optical depth**.
- As the stars are made of plasma - we don't have an obvious surface. $\tau = 1$ layer serves as a loose definition of a solar surface (observationally).
- Today we will describe a star using equations of stellar structure - we will again have to make a definition of solar surface.

# The concept of stellar atmosphere

For some stars, like the Sun, this layer is extremely thin (few 100s to few 1000s km):

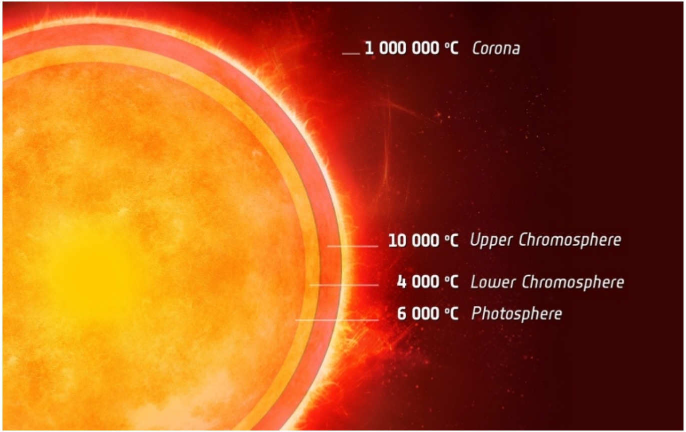



Figure: Scheme of the stellar atmosphere. Credits: ESA

# Interior vs atmosphere relationships

- Stellar structure determines the structure of the atmosphere, and $T_{\rm eff}$.
- Stellar atmosphere oscillates because of the processes in stellar interior.
- At the surface, we measure the magnetic fields generated in the interior.
- Neutrinos produced in the core leave the star directly.
- **The structure of the whole star leaves an observable imprint on its surface.**

# Understanding the stars

- **The structure of the whole star leaves an observable imprint on its surface.**
- We measure these observable imprints.
- We can observe *a lot* of stars - so we can have accurate statistics.
- **We devise theoretical models that try to describe the stellar structure and the evolution and compare the outputs of these models both to the measurable properties of individual stars and to the statistical distributions.**

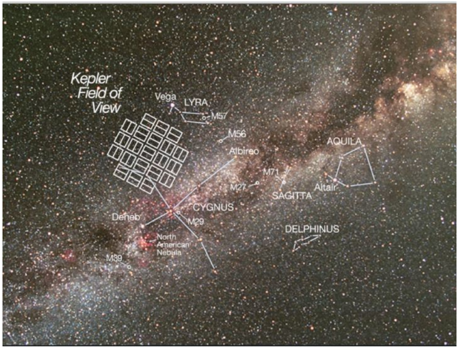



Figure: Kepler mission field of view. Credits: NASA

# Luminosity vs the color

- Effective temperature describes the total luminosity of the star.
- We used a blackbody to approximate solar *surface* - **not the whole star!**.
- Now, different temperatures imply different colors (i.e. different spectral distributions).

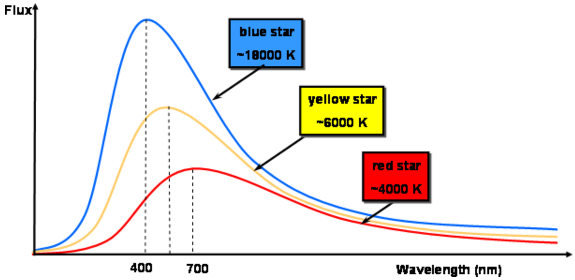
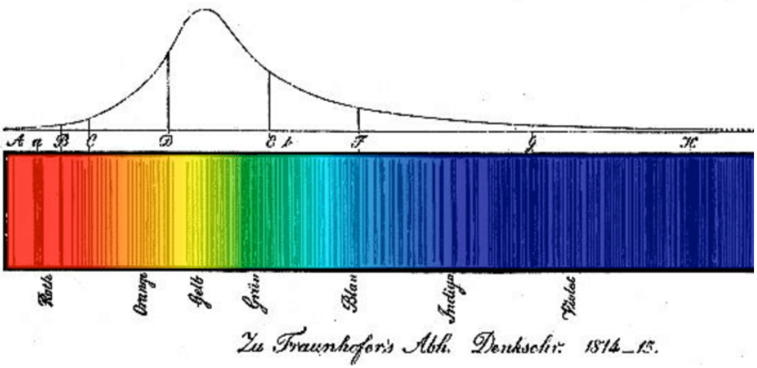



Figure: Planck curves for blackbodies of different temperatures. Credits: University of Swinburne

# Stars are not black bodies!

- Star needs to transport energy outwards. For that a gradient of temperature is needed - not a blackbody.
- Harder question is: can stellar photosphere be a blackbody?
- We leave that for a different discussion, but in short:
- It cannot, although we can roughly approximate it as:
- Blackbody radiation + additional absorption/emission in the stellar *atmosphere*.

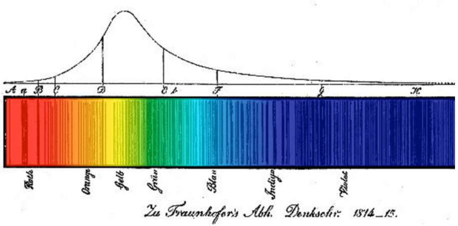
# Solar spectrum is NOT a blackbody spectrum




Credits: Fraunhofer (1814)

- Wollastone (1802) and Fraunhofer (1814) discovered dark lanes in the solar spectrum,
- These lines represent loss of light at specific wavelengths - solar atmosphere is much more opaque at these wavelengths. Why?

# Solar spectrum is NOT a blackbody spectrum




Credits: Fraunhofer (1814)

- These lines represent loss of light at specific wavelengths - solar atmosphere is much more opaque at these wavelengths. Why?
- Kirchoff and Bunsen (1860's) related them to chemical elements.
- Birth of quantum mechanics: Bohr's model can explain hydrogen lines. **Energy jumps between discrete energy levels**.
- Discovery of Helium by Janssen (1868) in the solar spectrum.

# Planck Law

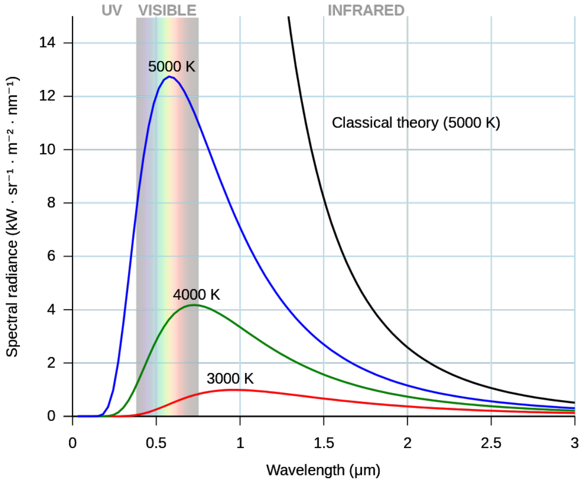
- Kirchoff (again) reckognized the importance of an universal law that connects emission and absorption properties of a medium in an equilibrium state. - *"... It is of utmost importance to derive this law theoretically"*.
- We had some pieces: Stefan-Boltzmann law that we just talked about...
- ... the Wien's law: Emission of bodies of different temperature peaks at different wavelengths (colors).
- People also experimentally captured the dependence $B(T)$, where $B$ is the emitted flux density.
- *Interesting:* Both Boltzmann and Wien (as well as Rayleigh and later Jeans) derived the corresponding laws theoretically. Do you know how?
- Recommendation: "Theoretical Concepts in Physics" (Malcom Longair, 2020, Cambridge)

# Planck Law

- Finally, Planck derived this law theoretically.
- After that, multiple other approaces to derivation were made (for example one by Bose and Einstein).

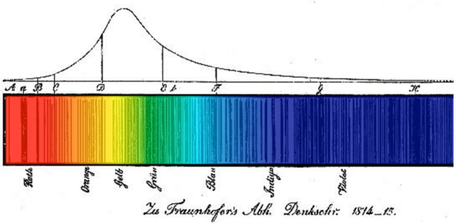
$$B(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \qquad (2)$$

- This equation describes the *intensity* of the light in a state where photons are in equilibrium.




Planck curves for few different temperatures.
Credits: Wikipedia

# Solar spectrum is NOT a blackbody spectrum



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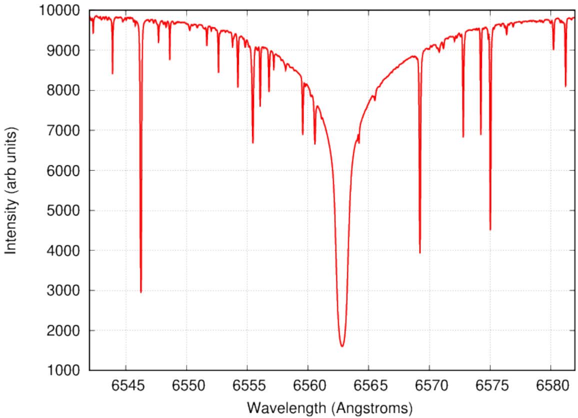
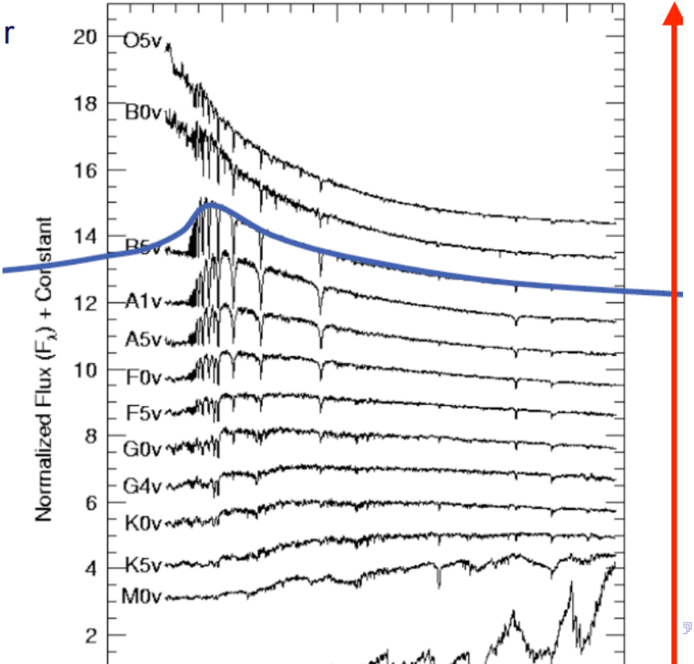
# Today we can do better than 200 years ago ...



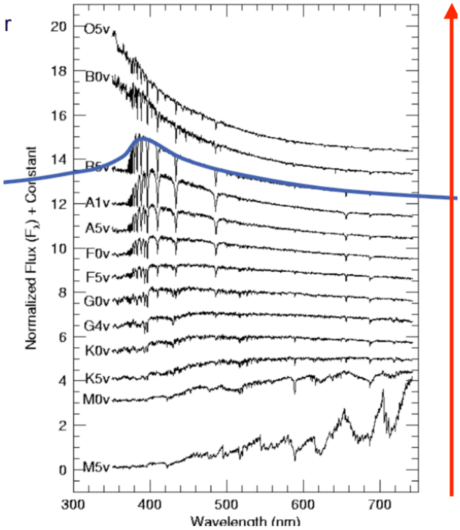

H$\alpha$ spectral line in solar spectrum: notice all the weak spectral lines around!

# And we can do the same for other stars.



# And we can do the same for other stars.

What is the fundamental parameter that determines the shape of the spectrum and the absence/presence of spectral lines?

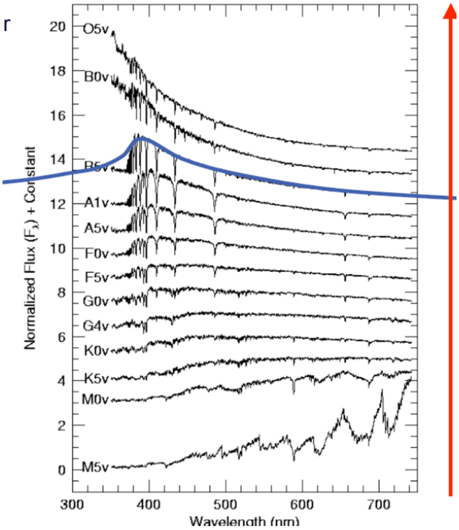



Credits: Adam Burrows, Princeton

# And we can do the same for other stars.

What is the fundamental parameter that determines the shape of the spectrum and the absence/presence of spectral lines?

It is the temperature! It determines the ionization and excitation of the particles and thus their absorption/emission properties.




Credits: Adam Burrows, Princeton

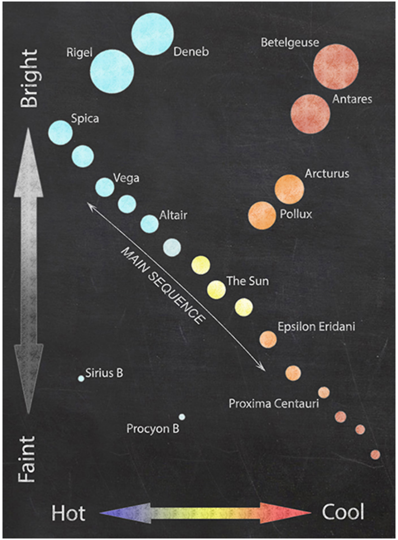
# The Harvard Spectral Classification

| Class | $T_{\mathrm{eff}}$ [K] |
|---|---|
| O | $\geq$ 30000 |
| B | 10000-30000 |
| A | 7500-10000 |
| F | 6000-7500 |
| G | 5000-6000 |
| K | 3000-5000 |
| M | 2000-3000 |

Annie Jump Cannon. Credits: Library of Congress, US

# HR diagram: basics

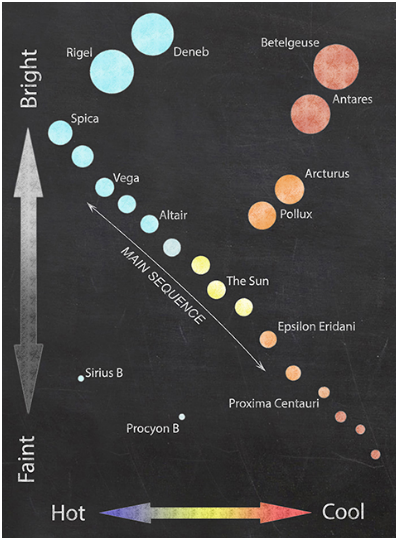
- There are three groups of stars.
- For one (main sequence) the color and brightness are somewhat correlated.
- Other is very hot but faint - white dwarves
- The last is cool but bright - (red) giants (there are also other giants).
- It will turn out these are *phases* in stellar evolution.




Credits: the Open University

# HR diagram: basics

- It will turn out these are *phases* in stellar evolution.
- We **can't** observe stars as they evolve and age.
- But we can observe a lot of stars and try to infer things.
- It is like aliens observed us very briefly: they would need some time that small, weak humans are a first stage in a lifetime of a human being.

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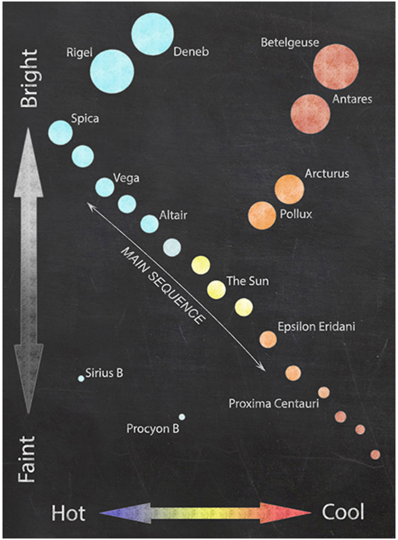
# HR diagram: questions

- Why are the color and the brightness of the stars on the main sequence correlated?
- In which order these evolutionary stages take place?
- What determines when and how the star changes between these stages?
- What are the fundamental differences between these phases?

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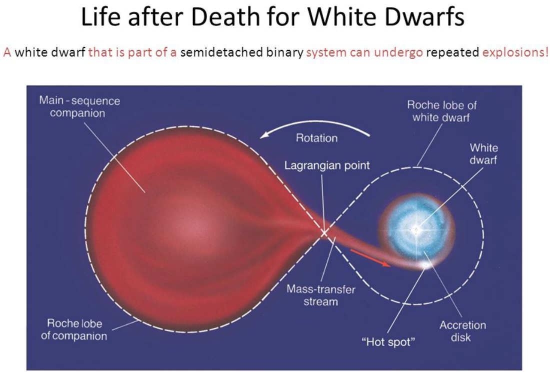
# How can we infer other stellar parameters?

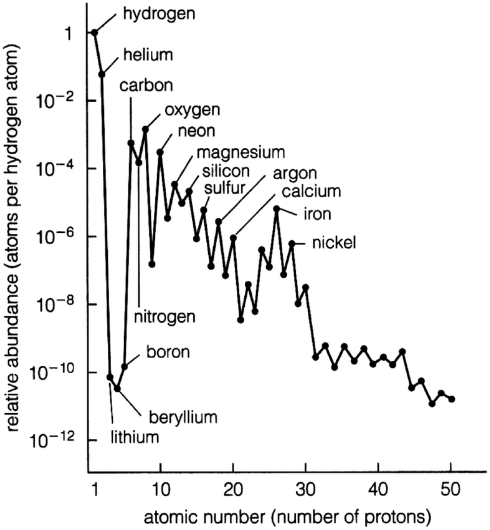
- We saw that we can estimate luminosity from the apparent brightness (irradiance) and distance.
- And that we can infer the effective temperature from the color.
- Combined, these two give us the radius.
- What about the mass? Physical composition? Magnetic field? Rotation rate?

- Are extremely important, yet extremely hard to infer.
- The only accurate opportunity to infer the mass is to observe the star in a dynamical interaction.
- For example in a binary system.
- Very important, a big fraction of the stars are in binary (or multiple systems).
- The stars can also interact with each other.



# Element abundances in the Universe

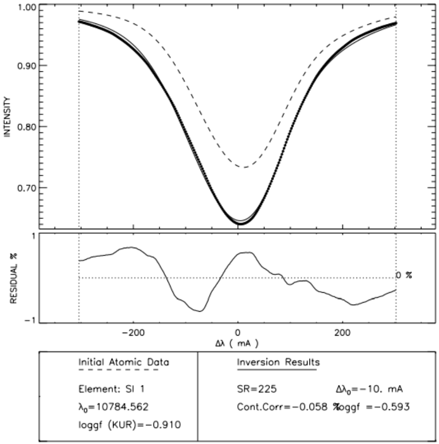
- Most of these elements were generate in stars.
- But they also have implications for generation and evolution of other stars?
- Let's have a relaxed discussion on importance of heavier elements.




Credits: Goldschmidt 1938

# These abundances are inferred from the spectra

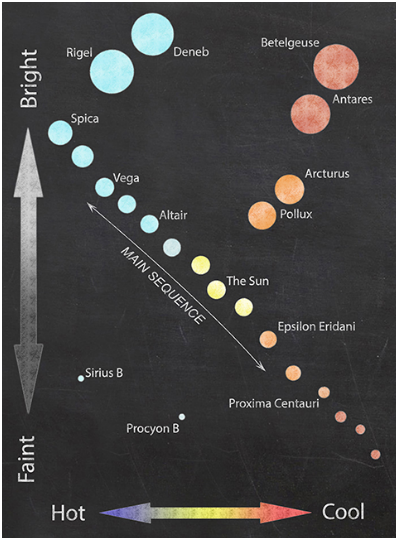
- Determination of the Silicon Abundance by fitting the solar spectra
- Dots: observations; dashed/solid: low/high Silicon abundance.




Credits: Borrero et al. 2003

# What is the goal of stellar structure and evolution modeling?

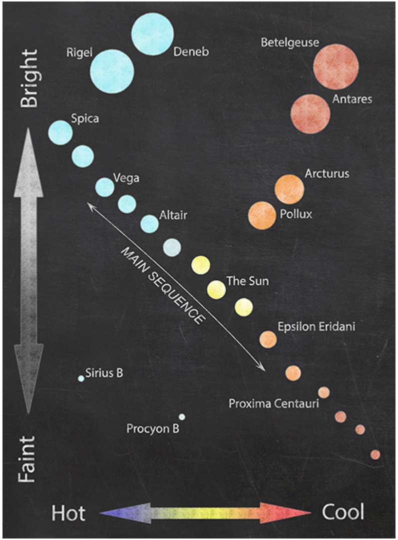
- To explain why are luminosity and temperature correlated.
- To understand what determines these two parameters.
- To predict/reconstruct the evolutionary paths of stars on HR diagram.
- Little bit further down the road: to understand changes in stars on various scales (pulsations, oscillations), stellar magnetic field generation, stellar activity and the interaction between the stars and their environment.




Credits: the Open University

# Basic assumptions - what we are working with

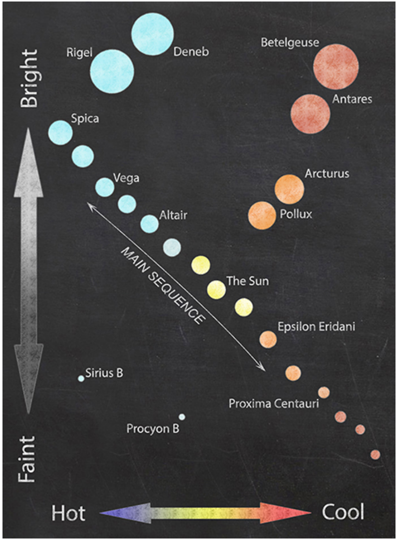
- Star exists and evolves in isolation.
- The star is non-rotating and spherically symmetric.
- The star is uniform in it's initial composition.
- We ignore the effect of magnetic fields.




Credits: the Open University

# How to play the game

- We describe our star as a radial distribution of temperature, pressure and density.
- We also describe it through it's chemical composition.
- These parameters will be connected with some differential equations and equation of state.
- Solving them should give us observable parameters ($T_{\rm eff}$, $L$) for given boundary conditions.
- When we let the system evolve, we should be able to reproduce the evolution of stars on the HR diagram.

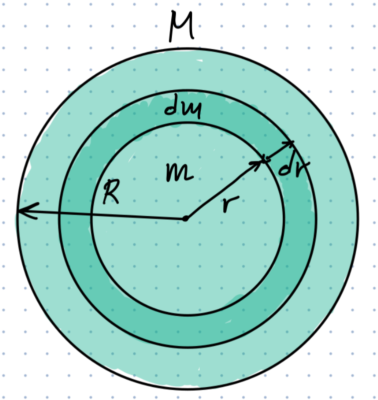



Credits: the Open University

# The coordinate system

- Problem is 1D - it is natural to describe variation of physical quantities with radius: $T(r)$, $\rho(r)$ $p(r)$. I.e. we consider *layers* with thickness $dr$
- Alternatively, we can focus on *elements* that enclose some very small mass $dm$.
- This is Eulerian vs Lagrangian formulation. We will use them both.
- The first one is more intuitive, the second more useful when the star dimensions change.
- They are connected as (conservation of mass):

$$dm = \rho(r)4\pi r^2 dr \tag{3}$$




Coordinate system

# The energy equation

- Energy is flowing through a parcel. It can also be generated in the parcel (if we have a source).

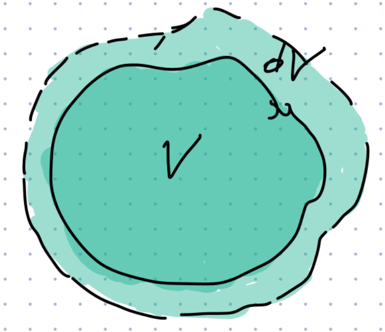
$$\delta(udm) = dm\delta u = \delta Q + \delta W \tag{4}$$

- where $Q$ is amount of heat, and $W$ is work. Then

$$\delta W = -p\delta dV = -p\frac{1}{\rho}\delta dm. \tag{5}$$

$$\delta Q = qdm\delta t + F(m)\delta t - F(m+dm)\delta t \tag{6}$$

$$\delta Q = (q - \frac{\partial F}{\partial m}dm)\delta t \tag{7}$$




Expansion of a small element

# The energy equation

- Now, taking the limit of a small $\delta t$, and putting these two together we have:

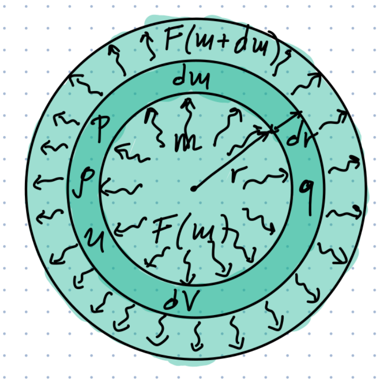
$$\dot{u} + p\left(\frac{1}{\rho}\right)^{\cdot} = q - \frac{dF}{dm} \tag{8}$$

- Which, for the moment, we are going to consider to be static:

$$q = \frac{dF}{dm} \tag{9}$$

- In the absence of sources $F$ is constant. $F$ at surface needs to be $L$. This power is supplied by the nuclear processes (i.e. nuclear luminosity):

$$L_{\mathrm{nuc}} = \int_0^M q dm. \tag{10}$$




Energy flow through a star

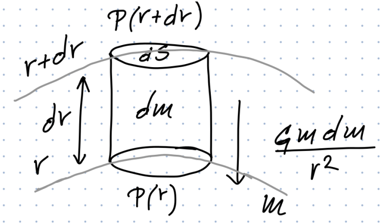
# The equation of motion

- Analyzing the forces acting on the element:

$$\ddot{r} = -G\frac{m}{r^2} - \frac{\partial p}{\partial r}\frac{1}{\rho}. \qquad (11)$$

- Or, if we work in $dm$:

$$\ddot{r} = -G\frac{m}{r^2} - 4\pi r^2 \frac{\partial p}{\partial m}. \qquad (12)$$




Forces acting on a infinitesimal element

# Hydrostatic equilibrium

- If the star is static, there are no accelerations:

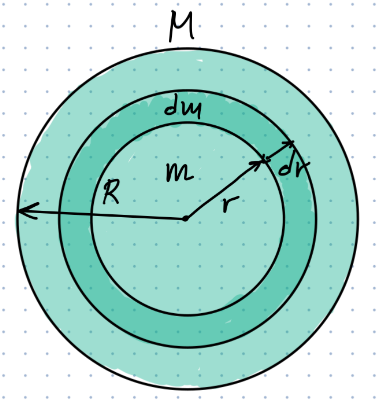
$$\frac{dp}{dr} = -\rho\frac{Gm}{r^2}, \tag{13}$$

- or in $dm$:

$$\frac{dp}{dm} = -G\frac{m}{4\pi r^4} \tag{14}$$

- Estimate the pressure at the center:

$$p_c = \int_0^M \frac{Gmdm}{4\pi r^4} > \int_0^M \frac{Gmdm}{4\pi R^4} \tag{15}$$



# Virial theorem

- In hydrostatic equilibrium, there is a link between graviational potential energy and internal energy (which is, essentially, kinetic energy of particles).
- Multiply HE with $V = \frac{4}{3}\pi r^3$ to get:

$$\int_{p(0)}^{p(R)} VdP = -\frac{1}{3}\int_0^M \frac{Gm\,dm}{r} \tag{16}$$

- Here r.h.s. is a third of the total gravitational potential energy of the star - i.e. energy to assemble the star by bringing matter from infinity. Denoted by $\Omega$.

# Virial theorem

- The l.h.s. is:

$$\int_{p(0)}^{p(R)} Vdp = [PV]|_0^R - \int_0^{V(R)} pdV \qquad (17)$$

- First term is zero (why? think about boundary conditions). So we get:

$$-3\int_0^{V(R)} pdV = -3\int_0^M \frac{p}{\rho} dm = \Omega \qquad (18)$$

- What can we conclude from this?

# Virial theorem

- So:

$$-3\int_0^{V(R)} p dV = -3\int_0^M \frac{p}{\rho} dm = \Omega \qquad (19)$$

- What can we conclude from this:
- The energies have opposite signs, and negative part (gravity) has larger magnitude.
- Hence, the star is **stable** and it won't just fly off in the space.

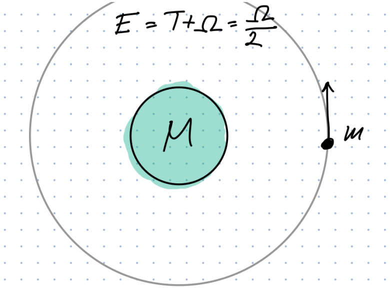
# Virial theorem for ideal gas

- Ideal gas: $p = nkT$, $u = \frac{3}{2}nkT$.
- Or the kinetic energy per unit mass $u = \frac{3}{2}\frac{kT}{m_g}$.

$$-3\int_0^{V(R)} pdV = -3\int_0^M \frac{p}{\rho}dm = \Omega \qquad (20)$$

- **From this we can find that** (also valid for gravitational systems like clusters and galaxies and clusters of galaxies):

$$U = -\frac{1}{2}\Omega \qquad (21)$$




Planets also have roughly $T = -1/2\Omega$

# Total energy of the star

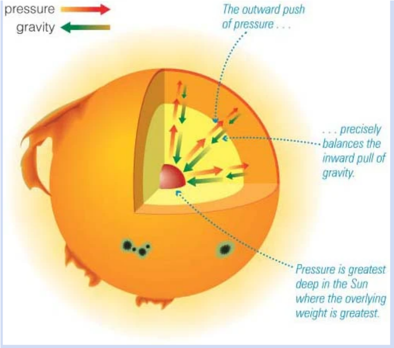
- Here we will go straight to the equation 2.42 from the book:

$$\dot{E} = \dot{U} + \dot{\mathcal{K}} + \dot{\Omega} = L_{nuc} - L \tag{22}$$

- In the static case we will have good old $E = U + \Omega = \frac{1}{2}\Omega$.
- What with $E$ and $U$ when the star contracts? (Remember that $\Omega$ is negative).

# Contracting = losing energy = heating

- What with $E$ and $U$ when the star contracts? (Remember that $\Omega$ is negative).
- Believe it or not: Total energy decreases, but the temperature of the star **increases**.
- This is actually not so unbelievable: when you jump from the hill your total energy decreases but your velocity increases.




Credits: https://scienceatyourdoorstep.com/

# Question to think about

- We said that the effective temperature of the Sun is 5777 K.
- We also said that this temperature is not constant, but must be increasing inward, in order to drive energy flux outward.
- What is the temperature in the core?
- What is the source of energy?

# Answer

- The temperature at the core is $\approx 1.5 \times 10^7$ K. And the source of energy is nuclear fusion of $H$ in $He$.
- How do we know? It took us some time.
- First one to even think about this was Eddington. (Maybe check this paper by H. Kragh).
- The previous idea was graviational contraction.
- The temporal scale for such a process is the **Kelvin-Helmholtz timescale**.

# Kelvin-Helmholtz timescale

- Let's do an order of magnitude estimation:

$$t_{\mathrm{KH}} = \frac{GM^2}{RL} \tag{23}$$

- For the Sun this amounts to 30 million years.
- We knew back then already that Earth is billions of years old.
- It had to be something else!

# Nuclear timescale

- This is a time that it would take the star to spend all the gas in the process of fusion:

$$E = mc^2 \tag{24}$$

- And then:

$$t_{\mathrm{nuc}} = \frac{\epsilon M c^2}{L} \tag{25}$$

- Which for the Sun is a very large number. Turns out that only a fraction of Sun's hydrogen will participate in the fusion, thus bringing this number down to 10 billion years.

# Dynamical timescale

- This is a free-fall time for an object to fall through the star :-)

$$t_{\mathrm{dyn}} \approx \frac{R}{v_{esc}} = \sqrt{\frac{R^3}{GM}} \tag{26}$$

- For the Sun this amounts to roughly 1000 seconds. As $R$ can vary wildly between the stars, so does the dynamical timescale.

# What do these timescales tell us

As a rule of a thumb:

- Dynamical timescale tells us that if we perturb the star in the equilibrium, it will restore (hopefully) in $t_{\rm dyn}$. This would also be a characteristic timescale for star formation ($R$ is much much bigger though).
- Kelvin-Helmholtz timescale tells us how long it takes to reach the thermal equilibrium globally (locally, temperature changes of course).
- The nuclear timescale tells us how much time it would take a star to spend its fuel and enter the next stage of evolution.