Hands-on exercises 7: Degenerate electron gas, properties of white dwarfs, and thermonuclear instability

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Problem 1: Use the Heisenberg uncertainity principle and Pauli exclusion principle to derive the equations of state for non-relativistic and relativistic degenerate electron gas:

$$p_{\rm e,deg} = \frac{h^2}{20 m_{\rm e}} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_{\rm H}^{5/3}} \left(\frac{\rho}{\mu_{\rm e}}\right)^{5/3}, \tag{1}$$

and

$$p_{\rm e,r-deg} = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_{\rm H}^{4/3}} \left(\frac{\rho}{\mu_{\rm e}}\right)^{4/3}, \tag{2}$$

where $h = 6.626 \cdot 10^{-34}$J Hz^{-1} is the Planck constant, $m_{\rm e}$ is the mass of the electron, $m_{\rm H}$ is the atomic mass unit, and $\mu_{\rm e}^{-1}$ is the average number of free electron per nucleon.

Compare the degenerate electron pressure and the the gas pressure from ideal gas equation at the centre of the Sun with solar composition of $\mu = 0.62$ and $\mu_{\rm e} = 1.17$. Use $\rho_{\rm c} = 1.6 \cdot 10^5$ kg m^{-3} and $T_{\rm c} = 1.57 \cdot 10^7$ K for the central density and temperature of the Sun.

Problem 2: Derive the mass-radius relation of white dwarfs assuming non-relativistic degenerate electron gas using the Lane-Emden equation. Derive the Chandrasekhar mass using the Lane-Emden equation. Hint: recall the for the equation of state of non-relativistic (relativistic) degenerate electron gas the polytropic index is $n = \frac{3}{2}$ ($n = 3$), and assume that hydrogen has been depleted so that $\mu_{\rm e} = 2$.

Problem 3: Convince yourself that normal stars have a built-in *thermostat* that allows them to maintain thermal stability over very long periods of time. Show also that the opposite is typically true for degenerate electron gas.

Useful physical constants

- $R_\odot = 696 \times 10^6\,$m
- $M_\odot = 1.989 \times 10^{30}\,$kg
- $L_\odot = 3.83 \times 10^{26}$ W
- $T_\odot^{\rm eff} = 5777\,$K

- $1\,\mathrm{AU} = 1.496 \times 10^{8}\,\mathrm{km}$
- $c = 2.997 \times 10^{8}\,\mathrm{m/s}$
- $G = 6.674 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
- $k = 1.38 \cdot 10^{-23}\ \mathrm{J/K}$
- $m_{\mathrm{e}} = 9.11 \cdot 10^{-31}\ \mathrm{kg}$
- $m_{\mathrm{H}} = 1.67 \cdot 10^{-27}\ \mathrm{kg}$