Hands-on exercises 7: Degenerate electron gas, properties of white dwarfs, and thermonuclear instability

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Problem 1: Use the Heisenberg uncertainity principle and Pauli exclusion principle to derive the equations of state for non-relativistic and relativistic degenerate electron gas:

$$p_{\mathrm{e,deg}} = \frac{h^2}{20 m_{\mathrm{e}}} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_{\mathrm{H}}^{5/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{5/3}, \tag{1}$$

and

$$p_{\mathrm{e,r-deg}} = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_{\mathrm{H}}^{4/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{4/3}, \tag{2}$$

where $h = 6.626 \cdot 10^{-34}$J Hz^{-1} is the Planck constant, m_{e} is the mass of the electron, m_{H} is the atomic mass unit, and μ_{e}^{-1} is the average number of free electron per nucleon.

Compare the degenerate electron pressure and the the gas pressure from ideal gas equation at the centre of the Sun with solar composition of $\mu = 0.62$ and $\mu_{\mathrm{e}} = 1.17$. Use $\rho_{\mathrm{c}} = 1.6 \cdot 10^5$ kg m^{-3} and $T_{\mathrm{c}} = 1.57 \cdot 10^7$ K for the central density and temperature of the Sun.

Solution: The Heisenberg uncertainity principle states that the position and momentum (p_{m}) of a particle cannot be determined arbitrarily accurately. Quantitatively this is given by:

$$\Delta V \Delta^3 p_{\mathrm{m}} \geq h^3, \tag{3}$$

where h is the Planck constant. On the other hand, according to the Pauli exclusion principle no two particles can occupy the same quantum state (same momentum and spin). Therefore, in a degenerate electron gas an element in phase space (location and momentum) can be occupied by two electrons (spin up and down). In a state of complete degeneration all the quantum states up to a maximum momentum value and Eq. (3) becomes equality. Then the number of electrons with momenta between $(p_{\mathrm{m}}, p_{\mathrm{m}} + dp_{\mathrm{m}})$ per unit volume:

$$n_{\mathrm{e}}(p_{\mathrm{m}}) dp_{\mathrm{m}} = \frac{2}{\Delta V} = \frac{2}{h^3} 4\pi p_{\mathrm{m}}^2 dp_{\mathrm{m}}. \tag{4}$$

Integrating this equation yields

$$\int_0^{p_{\mathrm{m0}}} n_{\mathrm{e}}(p_{\mathrm{m}}) dp_{\mathrm{m}} = \int_0^{p_{\mathrm{m0}}} \frac{2}{h^3} 4\pi p_{\mathrm{m}}^2 dp_{\mathrm{m}}, \tag{5}$$

$$n_{\mathrm{e}} = \frac{8\pi}{3h^3} p_{\mathrm{m0}}^3, \quad \text{or} \quad p_{\mathrm{m0}} = \left(\frac{3 n_{\mathrm{e}} h^3}{8\pi}\right)^{1/3}. \tag{6}$$

Now we use the pressure integral from the lectures:

$$p = \tfrac{1}{3} \int u p_{\mathrm{m}} n(p_{\mathrm{m}}) dp_{\mathrm{m}}, \tag{7}$$

where we now use $u = p_{\mathrm{m}}/m_{\mathrm{e}}$ and $n = n_{\mathrm{e}}$:

$$p_{\mathrm{e,deg}} = \tfrac{1}{3} \int_0^{p_{\mathrm{m0}}} \frac{p_{\mathrm{m}}^2}{m_{\mathrm{e}}} n_{\mathrm{e}}(p_{\mathrm{m}}) dp_{\mathrm{m}}. \tag{8}$$

Substituting n_{e} from Eq.(4) and integrating gives:

$$\begin{aligned} p_{\mathrm{e,deg}} &= \tfrac{1}{3} \int_0^{p_{\mathrm{m0}}} \frac{p_{\mathrm{m}}^2}{m_{\mathrm{e}}} \frac{2}{h^3} 4\pi p_{\mathrm{m}}^2 dp_{\mathrm{m}}. && (9) \\ &= \frac{8\pi}{15 m_{\mathrm{e}} h^3} p_{\mathrm{m0}}^5. && (10) \end{aligned}$$

Now we make use of Eq.(6) and recast the electron number density in terms of the gas density (lectures)

$$n_{\mathrm{e}} = \frac{\rho}{\mu_{\mathrm{e}} m_{\mathrm{H}}}. \tag{11}$$

Substituting these in Eq. (10) gives

$$\begin{aligned} p_{\mathrm{e,deg}} &= \frac{8\pi}{15 m_{\mathrm{e}} h^3} \left(\frac{3 n_{\mathrm{e}} h^3}{8\pi} \right)^{5/3}, && (12) \\ &= \frac{8\pi}{15 m_{\mathrm{e}} h^3} \left(\frac{3 h^3}{8\pi} \frac{\rho}{\mu_{\mathrm{e}} m_{\mathrm{H}}} \right)^{5/3}. && (13) \end{aligned}$$

Rearranging this yields the desired formula:

$$p_{\mathrm{e,deg}} = \frac{h^2}{20 m_{\mathrm{e}}} \left(\frac{3}{\pi} \right)^{2/3} \frac{1}{m_{\mathrm{H}}^{5/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}} \right)^{5/3} \equiv K_1 \rho^{5/3}. \tag{14}$$

In the (ultra-)relativistic case the the velocity of the electrons approaches the speed of light. Then we replace u in the pressure integral (7) by c

$$p_{\mathrm{e,r-deg}} = \tfrac{1}{3} \int_0^{p_{\mathrm{m0}}} c p_{\mathrm{m}} n_{\mathrm{e}}(p_{\mathrm{m}}) dp_{\mathrm{m}}. \tag{15}$$

Substituting (4) and integrating yields:

$$\begin{aligned} p_{\mathrm{e,r-deg}} &= \tfrac{1}{3} \int_0^{p_{\mathrm{m0}}} c p_{\mathrm{m}} \frac{2}{h^3} 4\pi p_{\mathrm{m}}^2 dp_{\mathrm{m}}. && (16) \\ &= \frac{2\pi c}{3 h^3} p_{\mathrm{m0}}^4. && (17) \end{aligned}$$

We again make use of Eqs.(6) and (11):

$$\begin{aligned} p_{\mathrm{e,r-deg}} &= \frac{2\pi c}{3 h^3} \left(\frac{3 n_{\mathrm{e}} h^3}{8\pi} \right)^{4/3}, && (18) \\ &= \frac{2\pi c}{3 h^3} \left(\frac{3 h^3}{8\pi} \frac{\rho}{\mu_{\mathrm{e}} m_{\mathrm{H}}} \right)^{4/3}. && (19) \end{aligned}$$

Rearranging this yields the desired formula:

$$p_{\mathrm{e,r-deg}} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{1}{m_{\mathrm{H}}^{4/3}}\left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{4/3} \equiv K_2\rho^{4/3}. \tag{20}$$

To compare the gas pressure and degenerate electron pressure in the Sun, we compute the ratio $p_{\mathrm{e,deg}}/p_{\mathrm{gas}}$, where

$$p_{\mathrm{gas}} = \frac{\mathcal{R}}{\mu}\rho T, \tag{21}$$

and where the specific gas constant is given by $\mathcal{R} = k/m_{\mathrm{H}}$,

$$\frac{p_{\mathrm{e,deg}}}{p_{\mathrm{gas}}} = \frac{h^2}{20m_{\mathrm{e}}}\left(\frac{3}{\pi}\right)^{2/3}\frac{1}{m_{\mathrm{H}}^{5/3}}\frac{\mu}{\mu_{\mathrm{e}}^{5/3}}\frac{\rho^{2/3}}{\mathcal{R}T}. \tag{22}$$

Inserting the numerical values yields

$$\frac{p_{\mathrm{e,deg}}}{p_{\mathrm{gas}}} \approx 0.11, \tag{23}$$

which means that electron degeneracy is beginning to have an effect in the solar centre, although gas pressure still dominates.

Problem 2: Derive the mass-radius relation of white dwarfs assuming non-relativistic degenerate electron gas using the Lane-Emden equation. Derive the Chandrasekhar mass using the Lane-Emden equation. Hint: recall the for the equation of state of non-relativistic (relativistic) degenerate electron gas the polytropic index is $n = \frac{3}{2}$ ($n = 3$), and assume that hydrogen has been depleted so that $\mu_{\mathrm{e}} = 2$.

Solution: Let us first recall the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{24}$$

where $\xi = \alpha r$ and $\rho = \rho_{\mathrm{c}}\theta^n$ where n is the polytropic index related to the adiabatic index via

$$\gamma = 1 + \frac{1}{n}. \tag{25}$$

The mass of the star is given by:

$$M = \int_0^R 4\pi r^2 \rho dr. \tag{26}$$

We now make a substitution of variables $r = \xi/\alpha$, $dr = d\xi/\alpha$, and $\rho = \rho_{\mathrm{c}}\theta^n$, and set the integration limits to $(0, \xi_{\mathrm{s}})$ where ξ_{s} is the surface corresponding to the first zero-crossing of the solution of Eq. (24). We obtain:

$$M = \frac{4\pi\rho_{\mathrm{c}}}{\alpha^3}\int_0^{\xi_{\mathrm{s}}} \xi^2\theta^n d\xi. \tag{27}$$

Now we substitute the Lane-Emden equation, Eq. (24), to find:

$$M = -\frac{4\pi\rho_{\mathrm{c}}}{\alpha^3}\int_0^{\xi_{\mathrm{s}}} \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) d\xi. \tag{28}$$

We may also write $\xi_\mathrm{s} = \alpha R$, where R is the radius of the star to eliminate α. Furthermore, it is easy to integrate Eq. (28) bearing in mind that $d\theta/d\xi = 0$ at $\xi = 0$:

$$M = -4\pi\rho_\mathrm{c}\left(\frac{R}{\xi_\mathrm{s}}\right)^3 \int_0^{\xi_\mathrm{s}} \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi = -4\pi\rho_\mathrm{c}\left(\frac{R}{\xi_\mathrm{s}}\right)^3\left(\xi_\mathrm{s}^2\left.\frac{d\theta}{d\xi}\right|_{\xi_\mathrm{s}}\right) = 4\pi\rho_\mathrm{c}R^3\left(-\frac{1}{\xi_\mathrm{s}}\left.\frac{d\theta}{d\xi}\right|_{\xi_\mathrm{s}}\right). \quad (29)$$

Now we recall from the homework that

$$\alpha^2 = \frac{4\pi G}{K(n+1)}\rho_\mathrm{c}^{1-\frac{1}{n}}, \quad \text{and} \quad p = K\rho^{1-\frac{1}{n}}. \quad (30)$$

We eliminate α with the definition of ξ_s and ρ_c using Eq. (29):

$$\frac{\xi_\mathrm{s}^2}{R^2} = \frac{4\pi G}{K(n+1)}\left[\frac{M}{4\pi R^3\left(-\frac{1}{\xi_\mathrm{s}}\left.\frac{d\theta}{d\xi}\right|_{\xi_\mathrm{s}}\right)}\right]^{1-\frac{1}{n}}. \quad (31)$$

Assuming that we have solved the Lane-Emden equation we know ξ_s and $-\frac{1}{\xi_\mathrm{s}}\left.\frac{d\theta}{d\xi}\right|_{\xi_\mathrm{s}}$ and identifying that everything except M and R are constants, we find:

$$\begin{aligned} R^{-2} &\propto \frac{M^{1-\frac{1}{n}}}{R^{3-\frac{3}{n}}}, \\ R &\propto M^{\frac{n-1}{n-3}}. \end{aligned} \quad (32)$$

Inserting $n = \frac{3}{2}$ for non-relativistic electron gas gives

$$R \propto M^{-1/3}, \quad (33)$$

which shows that the radius of the white dwarf decreases with mass. If we use $n = 3$ corresponding to relativistic electron gas, we see that the radius approaches zero suggesting that something interesting happens at this limit.

Going back to Eq. (31) and substituting $n = 3$, we find that R drops out from the equation. Furthermore, solving for M yields

$$M = 4\pi\left(\frac{K}{\pi G}\right)^{3/2}\left(-\xi_\mathrm{s}^2\left.\frac{d\theta}{d\xi}\right|_{\xi_\mathrm{s}}\right). \quad (34)$$

Identifying that $K = K_2$ from Eq. (20) and that for a polytrope of $n = 3$, $-\xi_\mathrm{s}^2\left.\frac{d\theta}{d\xi}\right|_{\xi_\mathrm{s}} = 2.02$, we get

$$M \approx 2.89 \cdot 10^{30}\ \mathrm{kg} = 1.45 M_\odot \equiv M_\mathrm{Ch}. \quad (35)$$

Problem 3: Convince yourself that normal stars have a built-in *thermostat* that allows them to maintain thermal stability over very long periods of time. Show also that the opposite is typically true for degenerate electron gas.

Solution: Assume a star in hydrostatic equilibrium. Then the central pressure from a polytropic model is given by

$$p_\mathrm{c} = (4\pi)^{1/3}B_n G M^{2/3}\rho_\mathrm{c}^{4/3}, \quad (36)$$

where B_n is a constant depending on the polytropic index n. The relation between p_c and ρ_c can be recast as

$$\frac{dp_\mathrm{c}}{p_\mathrm{c}} = \frac{4}{3}\frac{d\rho_\mathrm{c}}{\rho_\mathrm{c}}. \tag{37}$$

The pressure, density, and temperature are linked through the equation of state

$$\frac{dp_\mathrm{c}}{p_\mathrm{c}} = a\frac{d\rho_\mathrm{c}}{\rho_\mathrm{c}} + b\frac{dT_\mathrm{c}}{T_\mathrm{c}}. \tag{38}$$

Combining these equations gives:

$$\left(\frac{4}{3} - a\right)\frac{d\rho_\mathrm{c}}{\rho_\mathrm{c}} = b\frac{dT_\mathrm{c}}{T_\mathrm{c}}. \tag{39}$$

From here we see that for $a < \frac{4}{3}$, the signs of density and temperature changes are the same. That it, a compression (expansion) leads to an increase (decrease) of temperature. This is the case for ideal gas where $a = b = 1$.

In this case a compression would lead to an increase in ρ_c and T_c and hence increased nuclear energy production and further increase in T_c. However, then the star expands lowering ρ_c and T_c such that a new equilibrium is reached. This is the thermostat mechanims working in non-degenerate stars.

In degenerate stars $a \gtrsim \frac{4}{3}$ and $0 < b \ll 1$. If the star expands because its internal energy increases, e.g., by nuclear reactions the density decreases but the temperature *increases*. This leads to stronger nuclear energy release, even higher temperature, and a runaway thermonuclear instability.

This is observed in some close binary systems where a main-sequence or giant star orbits a white dwarf. The former fills its Roche lobe and mass (mostly hydrogen) accretes on the white dwarf and forms an atmosphere. The atmosphere heats up and at some point a critical point is exceeded and runaway fusion happens. This can either expel the atmosphere or destroy the white dwarf. Repeated outbursts are sometimes observed from the former type with periods of decades. These stars are referred to as recurrent novae. An example is the star T Coronae Borealis that should have an outburst this year! The latter case where the white dwarf is destroyed are classified as type Ia supernovae.

Useful physical constants

- $R_\odot = 696 \times 10^6\,\mathrm{m}$
- $M_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$
- $L_\odot = 3.83 \times 10^{26}\ \mathrm{W}$
- $T_\odot^\mathrm{eff} = 5777\,\mathrm{K}$
- $1\,\mathrm{AU} = 1.496 \times 10^8\,\mathrm{km}$
- $c = 2.997 \times 10^8\,\mathrm{m/s}$
- $G = 6.674 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
- $k = 1.38 \cdot 10^{-23}\ \mathrm{J/K}$

- $m_{\mathrm{e}} = 9.11 \cdot 10^{-31}$ kg
- $m_{\mathrm{H}} = 1.67 \cdot 10^{-27}$ kg