Hands-on exercises 7: Degenerate electron gas, properties of white dwarfs, and thermonuclear instability

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Problem 1: Use the Heisenberg uncertainity principle and Pauli exclusion principle to derive the equations of state for non-relativistic and relativistic degenerate electron gas:

$$p_{\mathrm{e,deg}} = \frac{h^2}{20 m_{\mathrm{e}}} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_{\mathrm{H}}^{5/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{5/3}, \tag{1}$$

and

$$p_{\mathrm{e,r-deg}} = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_{\mathrm{H}}^{4/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{4/3}, \tag{2}$$

where $h = 6.626 \cdot 10^{-34}$J Hz^{-1} is the Planck constant, m_{e} is the mass of the electron, m_{H} is the atomic mass unit, and μ_{e}^{-1} is the average number of free electron per nucleon.

Compare the degenerate electron pressure and the "normal" gas pressure at the centre of the Sun with solar composition of $\mu = 0.62$ and $\mu_{\mathrm{e}} = 1.17$. Use ρ_{c} and T_{c} for the central density and temperature of the Sun.

Problem 2: Derive the mass-radius relation of white dwarfs assuming non-relativistic degenerate electron gas using the Lane-Emden equation. Derive the Chandrasekhar mass using the Lane-Emden equation. Hint: recall the for the equation of state of non-relativistic (relativistic) degenerate electron gas the polytropic index is $n = \frac{3}{2}$ ($n = 3$), and assume that hydrogen has been depleted so that $\mu_{\mathrm{e}} = 2$.

Problem 3: Convince yourself that normal stars have a built-in *thermostat* that allows them to maintain thermal stability over very long periods of time. Show also that the opposite is typically true for degenerate electron gas.

Useful physical constants

- $R_\odot = 696 \times 10^6\,\mathrm{m}$
- $M_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$
- $L_\odot = 3.83 \times 10^{26}$ W
- $T_\odot^{\mathrm{eff}} = 5777\,\mathrm{K}$

- $1\,\mathrm{AU} = 1.496 \times 10^8\,\mathrm{km}$
- $c = 2.997 \times 10^8\,\mathrm{m/s}$
- $G = 6.674 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
- $k = 1.38 \cdot 10^{-23}\ \mathrm{J/K}$
- $m_\mathrm{e} = 9.11 \cdot 10^{-31}\ \mathrm{kg}$
- $m_\mathrm{H} = 1.67 \cdot 10^{-27}\ \mathrm{kg}$