# Theoretical Astrophysics I: Physics of Sun and Stars
# Lecture 6: Radiative Energy Transport

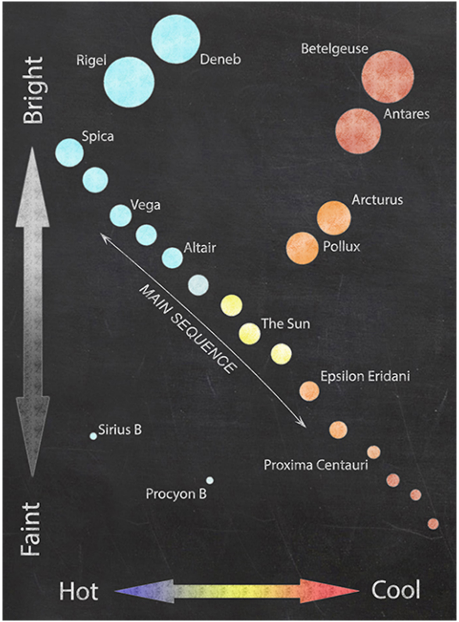
Petri Käpylä Ivan Milić
pkapyla,milic@leibniz-kis.de

Institut für Sonnenphysik - KIS, Freiburg

May 28, 2024

# Brief recap

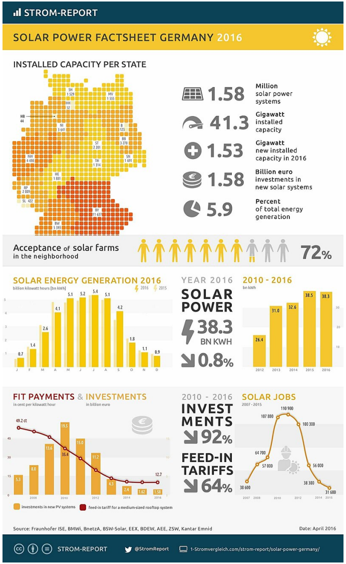
- We started with describing observed properties of the stars. One quantity that we measure (and want to reproduce) is the **luminosity**, and conversely **flux** (or specific flux).
- We wrote down equations that govern stellar structure and evolution. Solving them for proper boundary conditions will yield the structure of a star: $p(r)$, $T(r)$, $\rho(r)$, $F(r)$, etc...
- Analyzing their variation in time allows us to model stellar evolution.
- We spent last two lectures talking about a difficult problem of convection. But there is another way to transport energy: via **radiation**.



HR diagram

# Photons vs particles

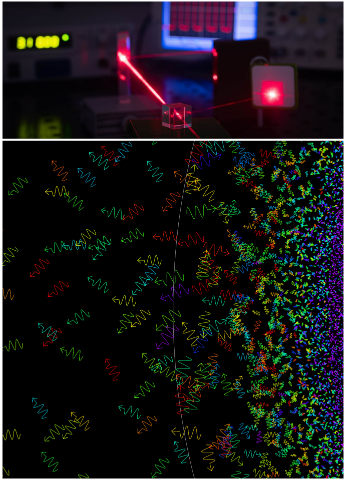
- It is obvious that radiation can carry energy.
- We treat radiation using photons, but they are clearly different from atoms, molecules, ions and electrons.
- Photons do not have mass and move with speed of light.
- The number of photons is not conserved.
- Photons can also be treated like a gas (e.g. see the derivation of Stefan-Boltzmann law by Boltzmann)
- They still observe conservation of energy, momentum, angular momentum, etc.




Credits: Strom Report

# Photons vs particles

- It will be essential to understand photon-matter interaction. As Ivan Hubeny said:
- *...In other words radiation in fact determines the structure of the medium yet the medium is probed only by this radiation.*
- Radiation: constituent in energy transport (and equation of state).
- Also: diagnostics that allows us to understand physical properties of the medium.
- Contrary to the lab: we need to treat wavelength and angular dependence of the radiation field.



Credits: LabRoots.com (up), Prof. Rob Rutten (bottom)

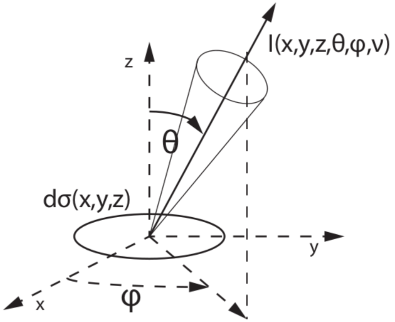
# Specific monochromatic intensity

- We need to treat wavelength and angular dependence of the radiation field
- Intensity: energy transported through given area in given time per given solid angle and frequency/wavelength bin (note the deprojection factor $\cos\theta$).

$$I_\nu = \frac{dE}{dS\, dt\, d\Omega\, d\nu\, \cos\theta} \tag{1}$$

- Going to number of photons:

$$n(\theta, \phi\, \nu) = \frac{I_\nu}{h\nu} \tag{2}$$




Credits: IM thesis (2014, University of Belgrade)

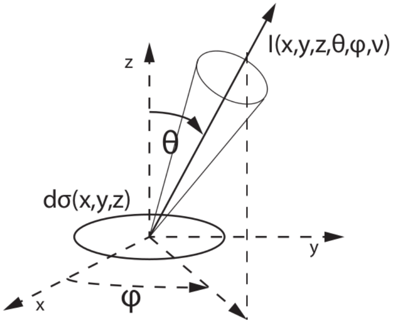
# Other “moments” of the radiation field

- Intensity fully describes the radiation field (without polarization). But often we need some derived quantities:
- Mean intensity:

$$J_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \sin\theta d\theta d\phi \quad (3)$$

- Flux (in $z$ direction):

$$\mathcal{F}_\nu = \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \quad (4)$$




Credits: IM thesis (2014, University of Belgrade)

# Some de-confusion of the term flux

- Typically we say that (spectral, monochromatic) flux is:

$$F_\nu = \frac{dE}{dtdSd\nu} \tag{5}$$

- But in the book typically:

$$F_\nu = \frac{dE}{dtd\nu} \rightarrow F = \frac{dE}{dt} \tag{6}$$

But then, to make situation worse, in stellar atmospheres theory flux is:

$$\mathcal{F}_\nu = \frac{dE}{dtdSd\nu} \tag{7}$$

then astrophysical flux:

$$F_\nu = \frac{1}{\pi}\frac{dE}{dtdSd\nu} \tag{8}$$

and Eddington flux (which the book uses and calls the radiation flux) is:

$$H_\nu = \frac{1}{4\pi}\frac{dE}{dtdSd\nu} = \frac{1}{4\pi}\mathcal{F}_\nu \tag{9}$$

# More moments of the radiation field

- Mean intensity:

$$J_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \sin\theta d\theta d\phi \tag{10}$$

- Radiation flux:

$$\mathcal{H}_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \tag{11}$$

- and the so called K-integral, which is proportional to the pressure of the radiation field:

$$\mathcal{K}_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \cos^2\theta \sin\theta d\theta d\phi = \frac{p_\nu c}{4\pi} \tag{12}$$

- **Note that we can define all these in the frequency/wavelength-integrated form**

# More moments of the radiation field

- Mean intensity:
$$J = \frac{1}{4\pi} \int_0^\infty \oint I_\nu(\theta, \phi) \sin\theta d\theta d\phi d\nu \tag{13}$$
- Radiation flux:
$$\mathcal{H} = \frac{1}{4\pi} \int_0^\infty \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi d\nu \tag{14}$$
- and the so called K-integral, which is proportional to the pressure of the radiation field:
$$\mathcal{K} = \frac{1}{4\pi} \int_0^\infty \oint I_\nu(\theta, \phi) \cos^2\theta \sin\theta d\theta d\phi d\nu = \frac{p\,c}{4\pi} \tag{15}$$
- **Here we integrated over all frequencies**

# A quick question:

- What would be the radiation flux if the intensity was isotropic?

$$\mathcal{H}_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \tag{16}$$

# A quick question:

- What would be the radiation flux if the intensity was isotropic?

$$\mathcal{H}_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \tag{17}$$

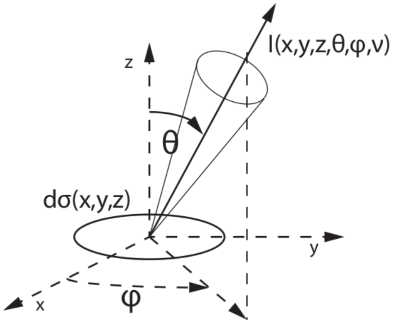
- A common substitute in this case is to integrate $\phi$ to $2\pi$ and then set $\cos\theta = \mu$ (this is again an another $\mu$).

$$\mathcal{H}_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu(\mu) \mu d\mu = 0 \tag{18}$$

- If the radiation is completely isotropic, there is no energy transport. In order to transport the energy outward toward the surface the radiation has to be slightly anisotropic.

# Modeling the radiation field

- Our task is not to model and understand intensity and its relationship with other physical quantities (density, temperature, pressure, chemical composition). For that we need to:
- Understand the interaction between the radiation and matter (absorption, emission, scattering coefficients).
- Mathematically express relationship between these coefficients and the intensity.



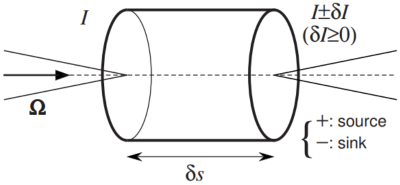

Credits: IM thesis (2014, University of Belgrade)

# Radiative Transfer Equation (RTE)

- This formulation is (more or less) due to Kirchhoff. The change of intensity "along-the-ray" over a distance *ds* is:

$$dI_\nu = \eta_\nu ds - \chi_\nu I_\nu ds \qquad (19)$$

- The terms of the right represent emission and **total** absorption (both true absorption and scattering) per unit volume.




Credit: Anthony B. Davis and Yuri Knyazikhin

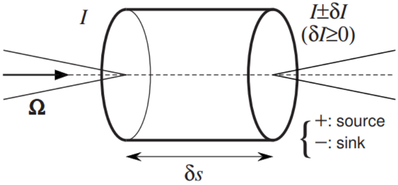
# Radiative Transfer Equation

- Now, it makes sense that the absorption and emission properties of the medium depend on:
- Amount of matter capable of absorbing/emitting
- The inherent properties of the matter at the given temperature ($T$ is very important!)
- So we define:

$$\kappa_\nu = \chi_\nu/\rho \tag{20}$$
$$j_\nu = \eta_\nu/\rho \tag{21}$$

- So our equation becomes:

$$\frac{1}{\rho}\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \tag{22}$$



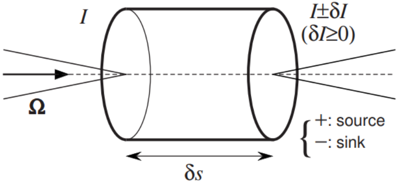

Credit: Anthony B. Davis and Yuri Knyazikhin

# Few remarks

- This equation does not involve any new physical laws, it is a mathematical tool.

$$\frac{1}{\rho}\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \tag{23}$$

- $ds$ is a so-called "ray", its relationship to $dx, dy, dz$ will depend on the geometry we choose and the context.
- We will assume that coefficients of absorption and emission are isotropic, but that they do depend on the frequency / wavelength.
- **The intensity is not isotropic. Can you see why?**.



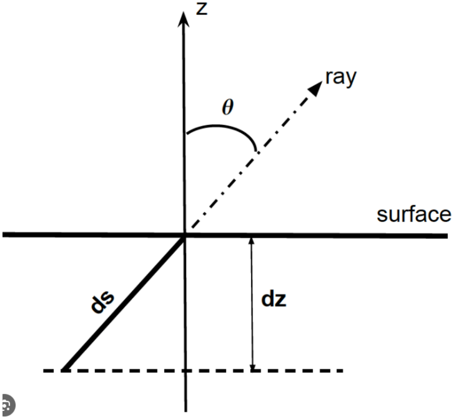

Credit: Anthony B. Davis and Yuri Knyazikhin

# RTE in 1D plane-parallel geometry

- If we assume that we are in 3D Cartesian grid, and nothing depends on $x$ and $y$, we will get:

$$ds = dz/\cos\theta = dr\cos\theta \qquad (24)$$

- If we want to take the sphericity in context, this becomes more complicated, but we won't need it.
- **In 1D, the anisotropy appears because $ds$ depends on the direction ($\theta$).**




Credit: Frederic Paletou

# Optical depth and Source function

- We often do the following:

$$\frac{dI_\nu}{-\rho\kappa_\nu ds} = I_\nu - j_\nu/\kappa_\nu \tag{25}$$

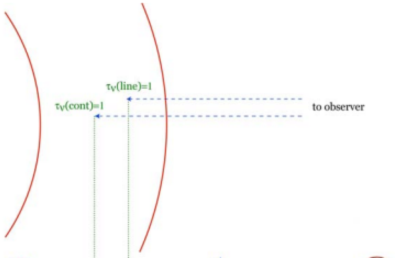
- And we get:

$$\frac{dI_\nu}{-\rho\kappa_\nu ds} = I_\nu - j_\nu/(\rho\kappa_\nu) \tag{26}$$

$$\kappa_\nu = \chi_\nu/\rho;\ j_\nu = \eta_\nu/\rho \tag{27}$$

- So our equation becomes:

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \tag{28}$$




Credit: Rolf Kudritzki

# Optical depth and Source function

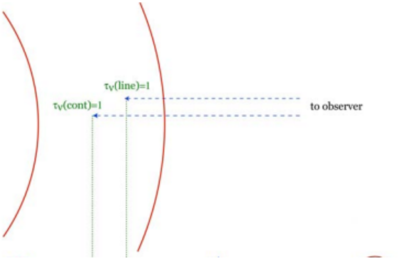
- If I wanted to be pedantic - I would have written:

$$\frac{dI_\nu(\theta)}{d\tau_\nu(\theta)} = I_\nu(\theta) - S_\nu \qquad (29)$$

- A lot of interesting solutions will involve exponents of $\tau_\nu$ - nice that it is dimensionless.
- For example, when there is no emission:

$$I_\nu(\tau_n u) = I_\nu^0 e^{-\tau_\nu} \qquad (30)$$

- Derive this real quick to get used to the orientation of optical depth.




Credit: Rolf Kudritzki

# Kirchhoff Law

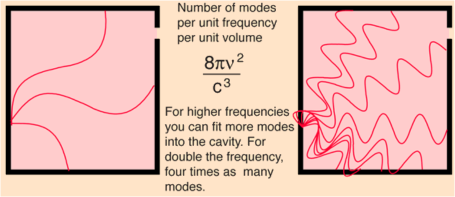
- Kirchhoff first wrote something like this and did a little thought experiment on a blackbody.

$$\frac{dI_\nu(\theta)}{d\tau_\nu(\theta)} = I_\nu(\theta) - S_\nu \qquad (31)$$

- If the body is in a complete equilibrium, $dI_\nu$ is zero on every frequency, and $I_\nu$ is constant in space, and so is $T$. Then:

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = f(T;\nu) = B_\nu(T) \qquad (32)$$

- He argued that it is of utmost importance to find this function.




Credit: Hyperphysics

# Planck's Law

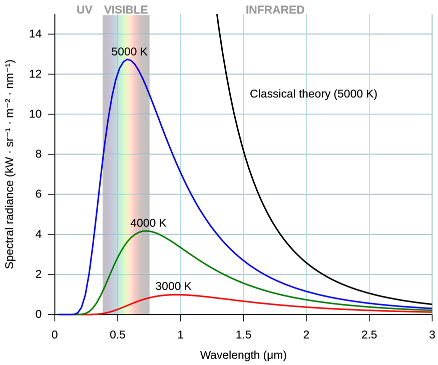
- People measured this function and Planck finally derived it (a nicer derivation is due to Bose and Einstein):

$$B_\nu(T) = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1} \qquad (33)$$

- This is, then, intensity of radiation inside of a blackbody. Integrating in wavelengths yields:

$$B(T) = \mathrm{const}\, T^4 \qquad (34)$$

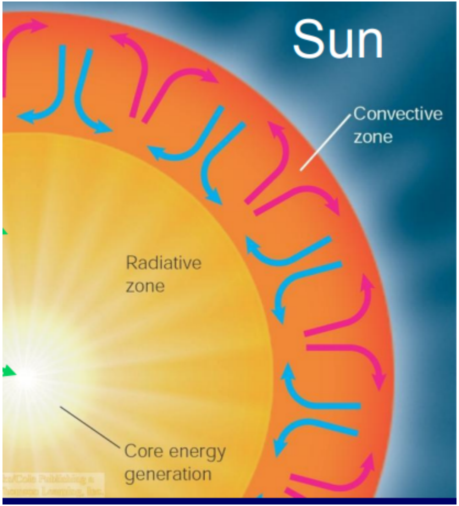
- And finally integrating that in angle yields the good old $\sigma T^4$.




Credit: Wikipedia

# But Ivan, stars are not black bodies

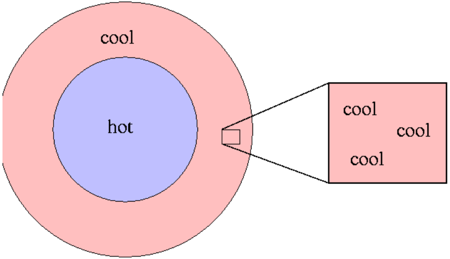
- If stars were blackbodies there would be no transport of energy.
- However, the gradient of temperature in the stars is very small.
- Can you estimate? $(\mathrm{K/m?})$




Credits: George Blanford

# Local Thermodynamic Equilibrium

- $dT/dr \approx 10^{-2}\mathrm{K/m}$ - This is incredibly small
- This allows us to presume the so called *local* thermodynamic equilibrium, which states that **matter** obeys:
- Saha distribution over ionization states.
- Boltzmann distribution over excitation states.
- Maxwell distribution over velocities.
- Contrary to what most of the textbooks tell you: *radiation is not in equilibrium with matter* - Intensity has to be out of equilibrium or there is no energy transport.



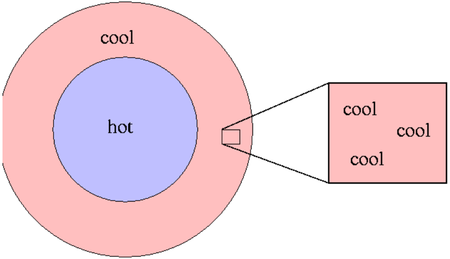

Credits: Michael Richmond

# Local Thermodynamic Equilibrium

- *Radiation is not in equilibrium with matter* - Intensity has to be out of equilibrium or there is no energy transport.
- But the source function, that depends only on the matter - is in equilibrium, and is equal to planck function:

$$S_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1} \qquad (35)$$

- So, local emission and absorption properties of the material follow from equilibrium distributions but the radiation departs from it.
- This departure is very slight in the cores of the stars but can be huge in the outer layers.
- Discuss the Sun's radiation that reaches us.




Credits: Michael Richmond

# Solving RTE deep in the atmospheres

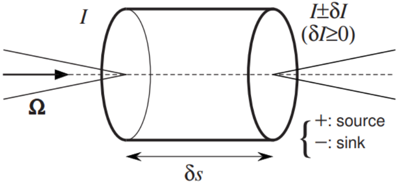
- We will not follow the approach from the book, but rather arguments used in stellar atmosphere modeling. (Still, very similar):

$$\cos\theta \frac{dI_\nu}{-\rho\kappa_\nu dz} = I_\nu - B_\nu \tag{36}$$

- Then we multiply the equation by $\cos\theta$ and integrate over all angles:

$$\frac{dK_\nu}{-\rho\kappa_\nu dz} = \mathcal{F}_\nu \tag{37}$$

- Here $K_\nu$ is the so called K-integral, which is proportional to the radiation pressure.



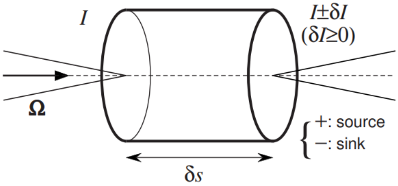

Credit: Anthony B. Davis and Yuri Knyazikhin

# Solving RTE deep in the atmospheres

$$\frac{dp_\nu}{-\rho\kappa_\nu dz} = \mathcal{F}_\nu \tag{38}$$

- Nice! Flux of the radiation is equal to the gradient in the radiation pressure.
- Now, let's integrate this over the frequencies, and introduce mean opacity $\overline{\kappa}$:

$$H(z) = \frac{1}{4\pi}\mathcal{F}(z) = -\frac{4acT^3}{3\overline{\kappa}\rho}\frac{dT}{dz} \tag{39}$$




Credit: Anthony B. Davis and Yuri Knyazikhin

# Radiative flux deep in the atmosphere:

- If we now start from the book definition of $F$ ($F = 4\pi r^2 H$).

$$F = -4\pi r^2 \frac{4acT^3}{3\overline{\kappa}\rho} \tag{40}$$

- Which we can invert to get:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{F}{4\pi r^2} \tag{41}$$

- Or:

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\overline{\kappa}}{T^3}\frac{F}{(4\pi r^2)^2} \tag{42}$$

- Discuss and analyze!
- Obviously $\overline{\kappa}$ plays an important role here. Calculating it accurately as a function of $T$ and chemical compositon, for given $\rho$ is very very hard and important task!

# What comes next?

- Now, in principle we could move to upper layers and discuss radiative transfer in stellar atmospheres.
- The situation there is trickier because things are much more anisotropic, and photons start escaping from the star.
- So, **frequency dependence is much more important**.
- To prepare for that, and better understand the $\overline{\kappa}$, we will briefly discuss sources of absorption in the stars.

# What comes next?

- In the execises we will see that:

$$\frac{1}{\overline{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} dB_\nu / dT d\nu}{\int_0^\infty dB_\nu / dT d\nu} \tag{43}$$

- Therefore, we must understand the wavelength and temperature dependence of $\kappa_\nu$.
- Let's first talk about what contributes to $\kappa_\nu$.

# Opacity sources

- Reminder, we only talk about opacity ($\kappa_\nu$) because $j_\nu = B_\nu \kappa_\nu$!
- Some sources are:
- Bound-free transitions (photoionization)
- Free-free processes (inverse bremsstrahlung)
- Bound-bound processes (spectral line - not important in the interior!)
- Thomson and Rayleigh scattering.
- Photodissotiation of $H-$ and molecules.
- Maybe some exotic processes?

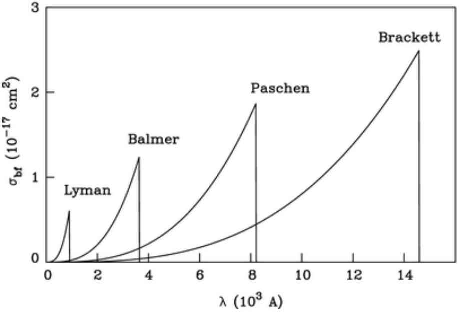
# Example: Bound-free processes

- To see the steps needed in the calculation of the opacity, try to solve the following problem
- For the gas of pure hydrogen, with given $\rho$ and $T$, calculate bound-free opacity at $\lambda = 50\,\mathrm{nm}$ and $500\,\mathrm{nm}$
- First question: can hydrogen absorb these fotons?

# Example: Bound-free processes

- For the gas of pure hydrogen, with given $\rho$ and $T$, calculate bound-free opacity at $\lambda = 50\,\text{nm}$ and $500\,\text{nm}$
- For hydrogen to absorb, energy of the photon must be larger than the binding energy:

$$h\nu \geq E_i \tag{44}$$

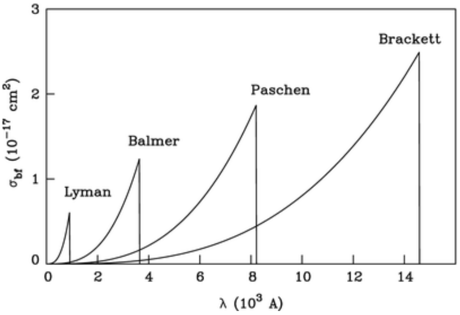



Credit: Walter Maciel, Springer

# Example: Bound-free processes

- For the gas of pure hydrogen, with given $\rho$ and $T$, calculate bound-free opacity at $\lambda = 50\,\mathrm{nm}$ and $500\,\mathrm{nm}$
- For hydrogen to absorb, energy of the photon must be larger than the binding energy.
- But electrons at different bound states have different binding energies ($i$ is the excitation state)!
- Plus, not every wavelength is equally efficient!

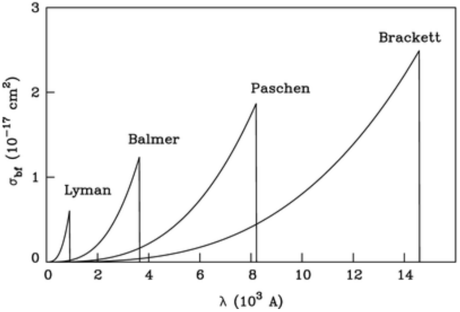
$$h\nu \geq \frac{13.6\mathrm{eV}}{i^2} \tag{45}$$




Credit: Walter Maciel, Springer

# Example: Bound-free processes

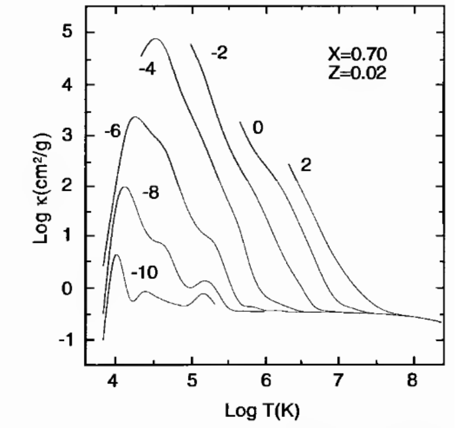
- So, for given $\rho$ and $T$, we need to find number densities of all hydrogen atoms
- Ionized hydrogen does not contribute.
- Probably better if at this point we rely on the blackboard...
- Plot on the right shows the *cross-section* for various wavelengths.




Credit: Walter Maciel, Springer

# Result

- These calculations need to be done for a system of many elements.
- Taking into account electron scattering.
- Taking into account free-free processes (this is not scattering!)
- Eventually, we will get the plot on the right
- What does the plot on the right tell us?




Credit: Iglesias and Rogers (1996)