Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 7: Simple Overview of Stellar Evolution

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The (T, ρ)-plane

- The evolution of the star is driven by its core. Why?

The (T, ρ)-plane

- The evolution of the star is driven by its core. Why?
- If we know the chemical composition, density ρ, and the temperature T, all other quantities can be derived.
- Let us denote the pair of central temperature and density at time t as $[T_\mathrm{c}(t), \rho_\mathrm{c}(t)]$.
- Then the evolution of the star is described by a succession of states $[T_\mathrm{c}(t_1), \rho_\mathrm{c}(t_1)]$, $[T_\mathrm{c}(t_2), \rho_\mathrm{c}(t_2)]$, $[T_\mathrm{c}(t_3), \rho_\mathrm{c}(t_3)]$, ..., at times t_1, t_2, t_3, ..., forming a parametric line $[T_\mathrm{c}(t), \rho_\mathrm{c}(t)]$.
- Assuming stars with the same composition, we expect that the $[T_\mathrm{c}(t), \rho_\mathrm{c}(t)]$ depend on stellar mass M.

Characterization of the $(\log T, \log \rho)$-plane

- We anticipate that T and ρ can vary several orders of magnitude depending on the star and/or evolutionary stage, so we consider the logarithms $\log T$ and $\log \rho$ in what follows.
- The $(\log T, \log \rho)$-plane divides into zones where different equations of state, nuclear reactions, and dynamical instabilities dominate.
- We begin with the equations of state. The most common state is described by the ideal gas equation:

$$p_{\rm ideal} = \frac{\mathcal{R}}{\mu}\rho T = K_0 \rho T, \tag{1}$$

where K_0 is a constant. At high density (and not too high temperature), the electrons become degenerate and dominate the pressure according to (tutorial)

$$p_{\rm e,deg} = K_1 \rho^{5/3}, \tag{2}$$

where K_1 is another constant.

Characterization of the $(\log T, \log \rho)$-plane

- The border between these lies along a line where $p_{\mathrm{ideal}} = p_{\mathrm{e,deg}}$, or

$$\log \rho = 1.5 \log T + \mathrm{const.} \qquad (3)$$

  This line separates regions I and II.

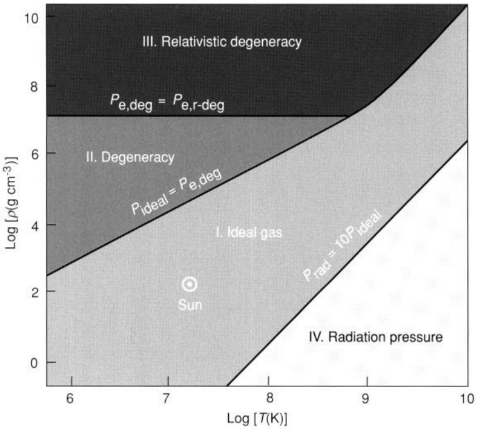
- For relativistic degeneracy

$$p_{\mathrm{e,r-deg}} = K_1 \rho^{4/3}, \qquad (4)$$

  corresponding to

$$\log \rho = 3 \log T + \mathrm{const.} \qquad (5)$$

  w.r.t. ideal gas separating regions I and III.



Credits: Prialnik

Characterization of the $(\log T, \log \rho)$-plane

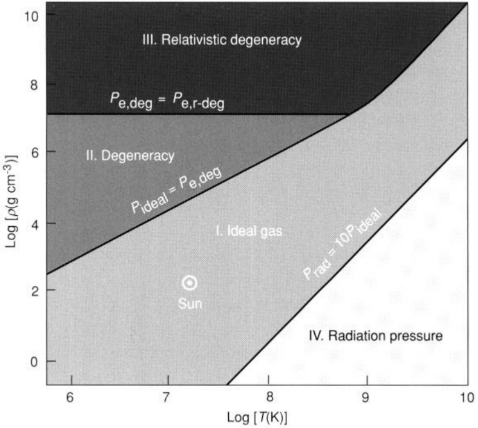
- Transition from standard to relativistic degeneracy occurs for $\rho \gg (K_2/K_1)^3$ (regions II and III).
- For sufficiently low pressure and high temperature the radiation pressure with

$$p = \frac{1}{3} a T^4, \tag{6}$$

becomes dominant. This again leads to:

$$\log \rho = 3 \log T + \text{const.} \tag{7}$$

separating regions I and IV.



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Zones of nuclear burning

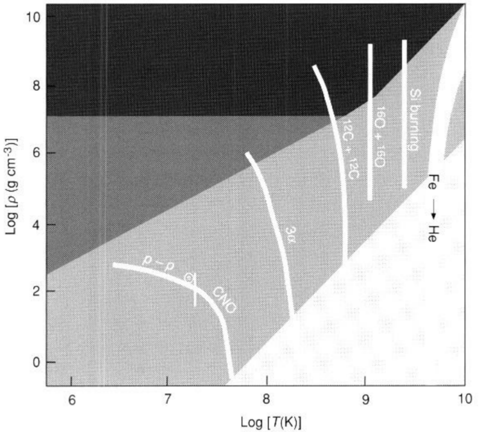
- Nuclear energy generation can be approximated by a power law

$$q = q_0 \rho^m T^n. \tag{8}$$

- A given reaction becomes important when $q > q_{\rm min}$ (e.g. 0.1 J kg^{-1} s^{-1}). This leads to

$$\log \rho = -\frac{n}{m} \log T + \frac{1}{m} \log \left(\frac{q_{\rm min}}{q_0} \right). \tag{9}$$

- For most reactions $n \gg m$ so the temperature determines the ignition of nuclear reactions.



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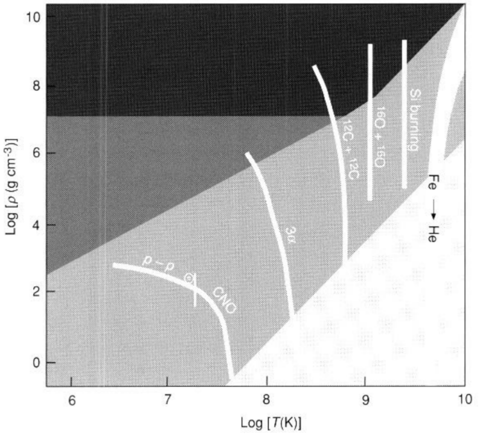
Zones of nuclear burning

- The hydrogen burning reactions connect smoothly from the *pp* chain to the CNO cycle. The latter has much stronger T-dependence and hence the slope in the (T, ρ) plane is steeper.
- Further reactions include helium, carbon, oxygen, and silicon burning, that are activated at higher T.
- When T reaches several billion Kelvin, iron cores start to dissociate which plays a significant role at late stages of massive stars.



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A note on dynamical stability

- Consider a macroscopic radial perturbation in a gas sphere of mass M in hydrostatic equilibrium. Pressure at any point is the weight per unit area between layers m and M:

$$p = \int_m^M \frac{Gm}{4\pi r^4} dm. \tag{10}$$

  The density is given by

$$\rho = \frac{1}{4\pi r^2} \frac{dm}{dr}. \tag{11}$$

- Consider a small radial compression such that the radii are obtained from

$$r' = r - \epsilon r = r(1 - \epsilon). \tag{12}$$

- With $\epsilon \ll 1$ the binomial expansion gives:

$$(1 \pm \epsilon)^n \approx 1 \pm n\epsilon. \tag{13}$$

A note on dynamical stability

- Therefore the new density is:

$$\rho' = \frac{1}{4\pi r^2(1-\epsilon)^2}\frac{dm}{dr}\frac{dr}{dr'} = \frac{\rho}{(1-\epsilon)^3} \approx \rho(1+3\epsilon). \tag{14}$$

- Assume that the contraction is adiabatic, the perturbed pressure is related to the original one via $(\rho'/\rho)^\gamma$, where γ is the adiabatic exponent. Thus,

$$p'_{\mathrm{gas}} = p(1+3\epsilon)^\gamma \approx p(1+3\gamma\epsilon). \tag{15}$$

- The new hydrostatic pressure is

$$p'_h = \int_m^M \frac{Gm}{4\pi r^4(1-\epsilon)^4}dm \approx p(1+4\epsilon). \tag{16}$$

A note on dynamical stability

- The gas sphere is no longer in hydrostatic equilibrium, i.e., $p'_{\mathrm{gas}} \neq p'_h$. Condition for restoring equilibrium is

$$p'_{\mathrm{gas}} > p'_h, \tag{17}$$

  such that the sphere expands to it original state.
- From Eqs. (15) and (16) we can write this as

$$p(1 + 3\gamma\epsilon) > p(1 + 4\epsilon). \tag{18}$$

- From this we can read the condition for *dynamical stability*:

$$\gamma > \tfrac{4}{3}. \tag{19}$$

- If $\gamma < \frac{4}{3}$ somewhere (but not throughout) the star, stability is determined by the integral

$$\int \left(\gamma - \tfrac{4}{3}\right) \frac{p}{\rho} dm. \tag{20}$$

  Instability occurs if the integral is negative.

A note on dynamical stability

- For relativistic-degenerate electron gas $\gamma = \frac{4}{3}$ (in reality $\gamma \to \frac{4}{3}$). This corresponds to the Chandrasekhar mass $M_{\rm Ch} \sim 1.46 M_\odot$. If the mass is higher the star collapses.
- If radiation pressure becomes dominant such that $p \approx p_{\rm rad}$, we again have $\gamma \to \frac{4}{3}$.
- For pure photon gas $p/\rho = u_{\rm rad}/3$. Using the virial theorem, we get

$$-\Omega = 3 \int_0^M \frac{p}{\rho} dM = u_{\rm rad}. \tag{21}$$

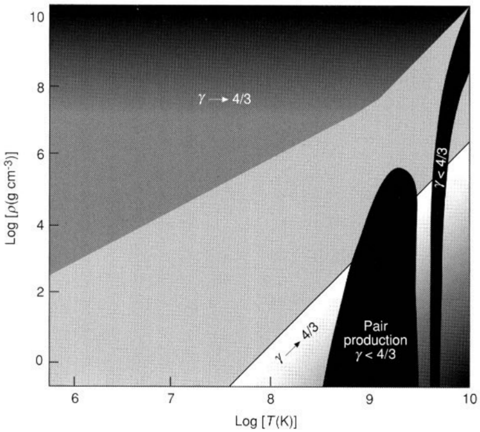
  This means that the total energy of the star $E = \Omega + u_{\rm rad}$ vanishes.
- This means that the gas sphere is *unbound* and the star cannot be held together by gravity.

A note on dynamical stability

- An ionization(-type) process can change the adiabatic index and lead to $\gamma < \frac{4}{3}$.
- The most obvious example is ionization where a single atom can produce two particles if a suitable amount of energy is absorbed or via a collision with another particle or photon.
- The reverse process (recombination) occurs at the same time and reduces the number of particles.
- Because pressure depends on the number of particles irrespective of their nature, ionization/recombination leads to a dependence of the pressure on volume (density) is different than for ideal gas.
- As an example, for a pure singly ionized gas $\gamma < \frac{4}{3}$ between 18% and 82% ionization. Therefore only a nearly neutral or nearly completely ionized gas would be dynamically stable.
- Nuclear processes such as pair production and iron photodissociation are further mechanims that can lead to $\gamma < \frac{4}{3}$.

Zones of instability

- In regions III and IV $\gamma \to \frac{4}{3}$ due to relativistic degeneracy of electrons and radiation presure, respectively. Therefore dynamical instability can also occur.
- Further regions where dynamical instability occurs include the iron photodisintegration zone and pair production strip which are ionization-like nuclear processes.
- Therefore the stable part of the (ρ, T) diagram is bounded by high temperatures and high density regions.
- Furthermore, nuclear burning is *thermally* unstable in degenerate gases (tutorial).



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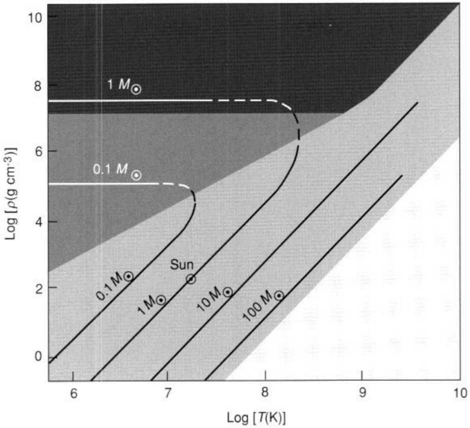
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- Consider a polytropic configuration for a star in hydrostatic equilibrium. Then the central density is

$$p_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}, \qquad (22)$$

  where the coefficient B_n is in the range 0.15 to 0.2 for polytropic indices $n = 1.5 \ldots 3$.
- Consider a star in zone I where ideal gas equation applies. Then the relation between ρ_c and T_c is

$$\rho_c = \frac{K_0^3}{4\pi B_n^3 G^3} \frac{T_c^3}{M^2}. \qquad (23)$$



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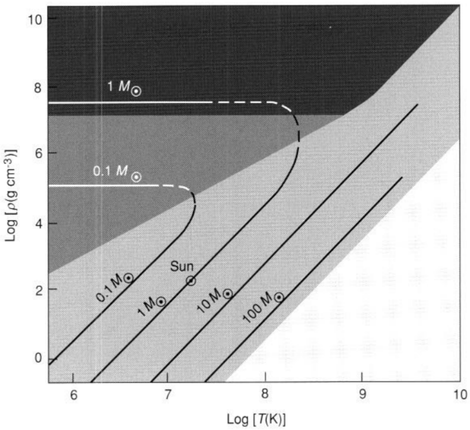
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- For a given density the temperature increasea when the mass of the star increases. This leads to parallel lines whose intersections at a given ρ differ by $\log M$.
- If the gas at the centre is degenerate, ρ_c is given by

$$\rho_c = 4\pi \left(\frac{B_{1.5} G}{K_1} \right)^3 M^2. \qquad (24)$$

These are horizontal lines in the $(\log T, \log \rho)$-plane. Note that ρ_c is independent from T_c.



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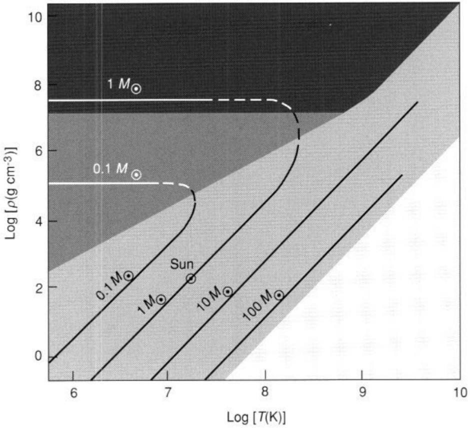
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- The density of degenerate stars tends to infinity as the stellar mass tends to the Chandrasekhar mass $M_{\rm Ch}$ (tutorials).
- This mass separates the regions I and III and the "knee" curves for $M < M_{\rm Ch}$ and the straight ones for $M > M_{\rm Ch}$.



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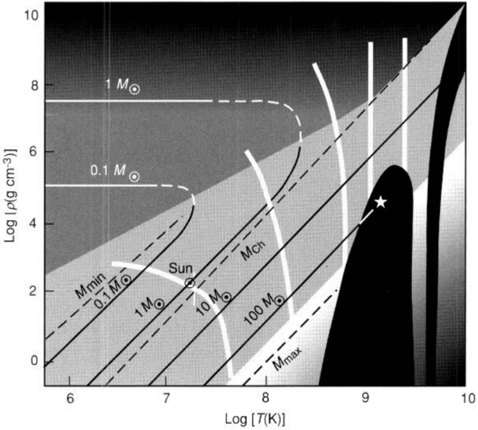
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- Combination of all of the information so far can be used to sketch the positions of stars with different masses.
- This can be used to sketch the general paths of stars of different masses.
- Stars have a minimum and maximum masses. Why?



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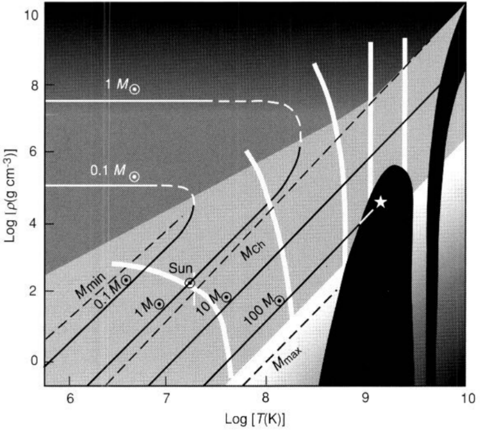
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- Combination of all of the information so far can be used to sketch the evolutionary of stars with different masses.
- This can be used to sketch the general paths of stars of different masses.
- Stars have a minimum and maximum masses. Why?
- The cores of stars with $M \lesssim 0.08 M_\odot$ are not hot enough to ignite nuclear fusion. For $M \gtrsim 100 M_\odot$ the radiation pressure makes the star unbound.



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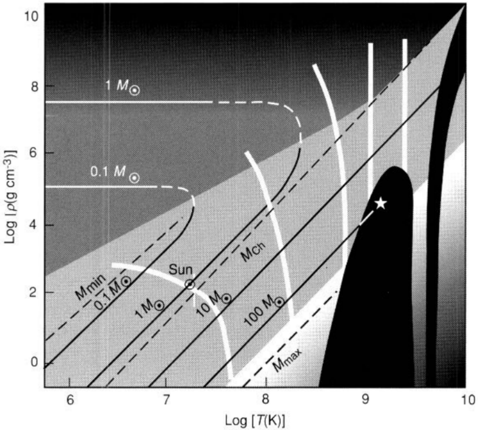
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- Low mass stars contract and heat up until hydrogen burning commences. The stars remain in this evolutionary phase for a very long (nuclear) time.
- When hydrogen is depleted from the core, degeneracy starts to play a role. Therefore stars with $M < M_{\rm Ch}$ end up as white dwarfs and cool over again a long time.
- White dwarfs have a constant radius and density that depend only on the mass M. The larger the mass, the lower the radius (totorial)!
- We would expect two populations of white dwarfs. Why?



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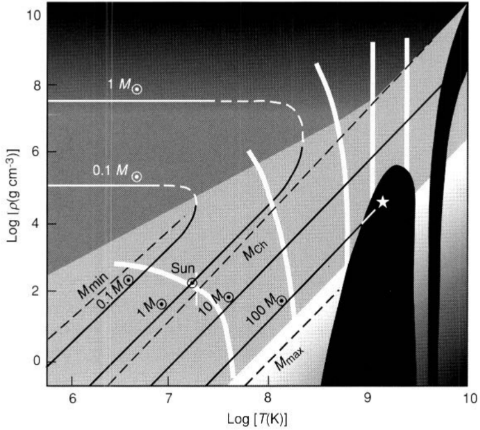
Path of the centre of the star in the $(\log T, \log \rho)$ plane

- Stars near $M = M_{\mathrm{Ch}}$ cross the carbon detonation very near the degeneracy border that can lead to a thermonuclear instability and collase. Why is this quite unlikely for a single star?
- More massive stars burn successively heavier elements as the lighter ones have been consumed. Eventually they hit the iron photodissociation strip where another catastrophic instability happens.
- The most massive stars can hit the pair-production instability strip even earlier than this.



Credits: Prialnik

Theory of the main sequence

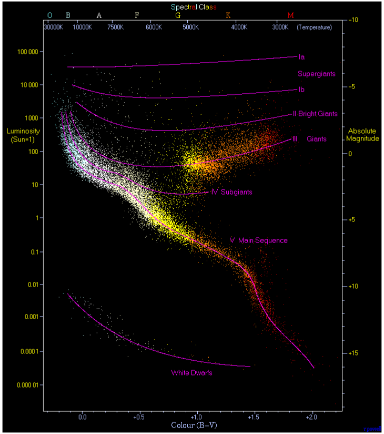
- There is an observed relation between the luminosity and effective temperature of main sequence stars

$$\log L = \alpha \log T_{\rm eff} + \mathrm{const.}, \qquad (25)$$

  where the slope α is different for low and large L.
- Another correlation exists between luminosity and mass:

$$L \propto M^{\nu}, \qquad (26)$$

  with $\nu \approx 3 \ldots 5$.
- The theory of main sequence needs to reproduce the relations (25) and (26).



Credits: Wikipedia

Theory of the main sequence

- Assume stars that have started to burn hydrogen at the centre and are in thermal and hydrostatic equilibrium. We assume uniform chemical composition, radiative equilibrium, neglect radiation pressure and assume analytic (constant) opacity.
- Now the governing equations are:

$$\frac{dp}{dm} = -\frac{Gm}{4\pi r^4}, \quad \frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}, \tag{27}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}, \quad \frac{dF}{dm} = q_o \rho T^n, \quad p = \frac{\mathcal{R}}{\mu}\rho T. \tag{28}$$

- This set of equations needs to be solved for $r(m)$, $p(m)$, $T(m)$, and $F(m)$ in the range $0 \leq m \leq M$ for any valu of M which is a free parameter.
- We would want to get some insight without actually solving the equations. How?

Theory of the main sequence

- The answer is *dimensional analysis* that we already used in some of the tutorials.
- Let us first define a dimensionless variable for the fractional mass:

$$x = \frac{m}{M}. \tag{29}$$

- The functions $r(m)$, $p(m)$, $T(m)$, and $F(m)$ can then be replaced by non-dimensional functions that depend on x: $f_1(x)$, $f_2(x)$, $f_3(x)$, etc.
- We end up with:

$$r = f_1(x) R_\star, \quad p = f_2(x) p_\star, \quad \rho = f_3(x) \rho_\star, \quad T = f_4(x) T_\star, \quad F = f_5(x) F_\star, \tag{30}$$

where the starred quantities have the dimensions of the original functions.

Theory of the main sequence

- As an example, substituting the relations $r = f_1(x)R_\star$, $p = f_2(x)p_\star$, and $\rho = f_3(x)\rho_\star$ into equation of hydrostatic equilibrium yields

$$\frac{p_\star}{M}\frac{df_2}{dx} = -\frac{GMx}{4\pi f_1^4 R_\star^4}. \tag{31}$$

- Since the units of both sides must match and f_i are non-dimensional, $p_\star$ must be proportional to $GM^2/R_\star^4$.
- Setting the coefficient of proportionality to unity leads to:

$$\frac{df_2}{dx} = -\frac{x}{4\pi f_1^4}, \quad p_\star = \frac{GM^2}{R_\star^4}. \tag{32}$$

Theory of the main sequence

- Repeating the exercise for the other four equations:

$$
\begin{aligned}
\frac{df_1}{dx} &= \frac{1}{4\pi f_1^2 f_3}, & \rho_\star &= \frac{M}{R_\star^3}, \\
f_2 &= f_3 f_4, & T_\star &= \frac{\mu p_\star}{\mathcal{R}\rho_\star}, \\
\frac{df_4}{dx} &= -\frac{3f_5}{4f_4^3(4\pi f_1^2)^2}, & F_\star &= \frac{ac}{\kappa}\frac{T_\star^4 R_\star^4}{M}, \\
\frac{df_5}{dx} &= f_3 f_4^n, & F_\star &= q_0 \rho_\star T_\star^n M.
\end{aligned}
\tag{33}
$$

- The equations on the lhs are non-linear differential equations for f_i in the range $0 \leq x \leq 1$. The dimensional quantities on the rhs of Eq. (30) are obtained from the *algebraic* Eqs. (33) (right).
- Combination of these differential and algebraic equations allows for solving the interior structure for any mass M. Importantly the profiles as function of x are the same in all stars, differing only by a constant factor determined by the mass. This property is called *homology*.

Theory of the main sequence

- Now we can skip solving the differential equations if we are just interested in the dependence of physical properties on M.
- Solving for $T_\star$ gives:
$$T_\star = \frac{\mu G}{\mathcal{R}} \frac{M}{R_\star}. \tag{34}$$
Using this in the first equation for $F_\star$ gives:
$$F_\star = \frac{ac}{\kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^3. \tag{35}$$
- Since this relation is valid for any value of x, the surface value scales as
$$L \propto M^3, \tag{36}$$
which compares rather well with observations.

Theory of the main sequence

- Furthermore, we can now solve for (exercises) $R_\star$ and $\rho_\star$:

$$R_\star \propto M^{\frac{n-1}{n+3}}, \quad \rho_\star \propto M^{2\left(\frac{3-n}{3+n}\right)}. \tag{37}$$

What can be concluded based on these relations?

Theory of the main sequence

- Furthermore, we can now solve for (exercises) $R_\star$ and $\rho_\star$:

$$R_\star \propto M^{\frac{n-1}{n+3}}, \quad \rho_\star \propto M^{2\frac{3-n}{3+n}}. \tag{38}$$

What can be concluded based on these relations?

- $n \approx 16$ for the CNO cycle so stellar radius is roughly proportional to mass

$$R_\star \propto M. \tag{39}$$

- For pp chain $n \approx 4$ and

$$R_\star \propto M^{3/7}. \tag{40}$$

Therefore radius always increases with mass unlike for white dwarfs!

- Furthermore, since $n > 3$ density decreases with increasing M again opposite to white dwarfs.

Theory of the main sequence

- We are now in a position to compare with the observed HR diagram. The relation between luminosity and effective temperature is given by:

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4. \tag{41}$$

- Using the mass-luminosity and mass-radius relations to eliminate $R_\star$ and M, we arrive at

$$L^{1-\frac{2(n-1)}{3(n+3)}} \propto T_{\mathrm{eff}}^4. \tag{42}$$

- For $n = 4$ this yields

$$\log L = 5.6 \log T_{\mathrm{eff}} + \mathrm{const.}, \tag{43}$$

  and for $n = 16$ we get

$$\log L = 8.4 \log T_{\mathrm{eff}} + \mathrm{const.} \tag{44}$$

- These again match the observations fairly well.

Theory of the main sequence

- The main sequence lifetime is epected to be proportional to the nuclear fuel (stellar mass) and inversely proportional to the energy release rate (luminosity):

$$\tau_{\mathrm{MS}} \propto \frac{M}{L} \propto M^{-2}. \tag{45}$$

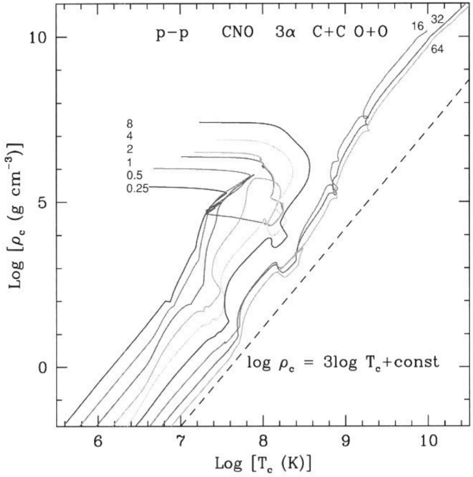
- This confirms that more massive stars evolve faster and the main sequence shortens gradually from the more massive end.

Limitations of the simple model

- Any ideas?

Limitations of the simple model

- Applies to uniform chemical composition: strictly valid for *zero-age main sequence*.
- Convection is ignored.
- Homogeneity of the star is not a good approximation in advanced stages of stellar evolution.
- More realistic models indicate that mass loss is important, e.g., stars with masses much higher than $M_{\rm Ch}$ end up as white dwarfs.



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