Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 8: Detailed Models of Stellar Evolution

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Detailed picture of stellar evolution

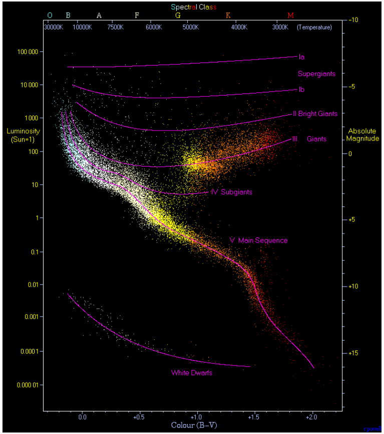
- As opposed to the simple treatment we have adopted so far, a general treatment of stellar evolution needs to take into account the details of opacity, nuclear energy production, and equation of state.
- Then the equations of stellar evolution need to be solved *numerically* and due to their non-linearity results that might not be intuitively clear can arise.
- Numerical solutions have been available since the 1950s but we will not go to the details here.
- The goal of the modelling efforts is to explain the observed Herztsprung-Russell diagram, characterised by $(\log L, \log T_{\rm eff})$ plane, as opposed to $(\log \rho_{\rm c}, \log T_{\rm c})$ in the previous lecture.

Recap: Herzsprung-Russell diagram

- We saw earlier that there is an observed relation between the luminosity and effective temperature of main sequence stars

$$\log L = \alpha \log T_{\text{eff}} + \text{const.}, \tag{1}$$

  where the slope α is varies with L.
- The models need to further explain why stars cluster (i.e., spend much of their lifetime) at certain regions in the diagram.



Credits: Richard Powell / Wikipedia

Hayashi zone and pre-main-sequence phase

- Assume a fully convective star of mass M and radius R. Then we can adopt an interior structure corresponding to a polytrope of index $n = 1/(\gamma_{\rm a} - 1)$,

$$p = K\rho^{1+\frac{1}{n}}. \tag{2}$$

- K is related to M and R via the Lane-Emden equation:

$$K^n = C_n G^n M^{n-1} R^{3-n}, \tag{3}$$

  where C_n depends on the polytropic index n:

$$C_n = \frac{4\pi}{(n+1)^n} \frac{R_n^{n-3}}{M_n^{n-1}}. \tag{4}$$

- R is a free parameter that is fixed by joining the fully convective interior to a radiative photosphere above $r = R$.

Hayashi zone and pre-main-sequence phase

- The photosphere needs to be able to radiate away all of the incoming energy flux. This is determined by the thermodynamic structure, i.e., the drop of p, ρ, and T accross it.
- In Hydrostatic equilibrium
$$\frac{dp}{dr} \approx -\rho \frac{GM}{R^2}, \tag{5}$$
which can be integrated from R to the point where p vanishes
$$p_R = \frac{GM}{R^2} \int_R^\infty \rho dr. \tag{6}$$
- Furthermore, the optical depth of the photosphere, characterised by T_{eff}, is of the order on unity and thus $\int_R^\infty \kappa\rho dr = \overline{\kappa} \int_R^\infty \rho dr$, where $\overline{\kappa}$ is the mean opacity in the photosphere.
- Taking $\overline{\kappa} = \kappa(R)$ and assuming it to be a power law in ρ_R and T_{eff} gives:
$$\kappa_0 \rho_R^a T_{\text{eff}}^b \int_R^\infty \rho dr = 1. \tag{7}$$

Hayashi zone and pre-main-sequence phase

- Combining Eqs. (6) and (7) gives:

$$p_R = \frac{GM}{R^2 \kappa_0} \rho_R^{-a} T_{\rm eff}^{-b}. \tag{8}$$

- Yet another relation between the thermodynamic quantities at R is given by the equation of state, here taken to be ideal gas equation:

$$p_R = \frac{\mathcal{R}}{\mu} \rho_R T_{\rm eff}. \tag{9}$$

- Finally, the temperature at R is related to the luminosity via

$$L = 4\pi R^2 T_{\rm eff}^4. \tag{10}$$

- Now we have four equations that describe the surface of the star: Eqs.(2) (with Eqs. (3) and (4)), (8), (9), and (10)

Hayashi zone and pre-main-sequence phase

- These read in logarithmic form:

$$n \log p_R = (n-1)\log M + (3-n)\log R + (n+1)\log \rho_R + \text{const.} \tag{11}$$
$$\log p_R = \log M - 2\log R - a\log \rho_R - b\log T_{\rm eff} + \text{const.} \tag{12}$$
$$\log p_R = \log \rho_R + \log T_{\rm eff} + \text{const.} \tag{13}$$
$$\log L = 2\log R + 4\log T_{\rm eff} + \text{const.} \tag{14}$$

- Eliminating $\log R$, $\log \rho_R$, and $\log p_R$ yields:

$$\log L = A\log T_{\rm eff} + B\log M + \text{const.} \tag{15}$$
$$A = \frac{(7-n)(a+1)-4-a+b}{0.5(3-n)(a+1)-1}, \quad B = -\frac{(n-1)(a+1)+1}{0.5(3-n)(a+1)-1}. \tag{16}$$

- This relation traces the *Hayashi track* in the HR diagram. These should not be interpreted as evolutionary tracks but rather as an asymptote.

Hayashi zone and pre-main-sequence phase

- We assume for simplicity that $a = 1$ which is reasonably accurate, such that

$$A = \frac{9-2n+b}{2-n}, \quad B = -\frac{2n-1}{2-n}. \tag{17}$$

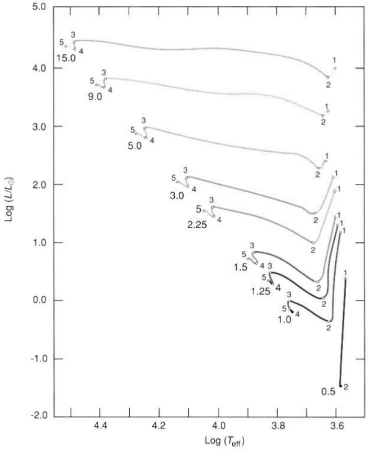
  b varies much more but is usually positive.
- Dynamical stability requires that $n < 3$ and therefore the polytropic index is limited to the range $1.5 \leq n < 3$ because we assumed a fully convective star.
- For $b \approx 4$ and $n = 1.5$ we find that $A = 20$. This means that the Hayashi track is almost vertical in the $(\log L, \log T_{\rm eff})$ plane.
- As a function of mass the tracks are stacked near each other and higher mass leads to a shift toward higher temperatures because A and B have opposite signs.
- The slope changes with composition that can be associated with an effective polytropic index.

Hayashi zone and pre-main-sequence phase

- The signifigance of the Hayashi track can be seen from considering $\overline{\gamma}$ which is an average value of $\gamma = \frac{d \ln p}{d \ln \rho}$ over the whole star. $\overline{\gamma}_{\rm a}$ is the corresponding adiabatic index.
- For a fully convective star $\overline{\gamma} = \overline{\gamma}_{\rm a}$.
- If any part of the star is radiative with $\gamma < \gamma_{\rm a}$, then $\overline{\gamma} < \overline{\gamma}_{\rm a}$. Correspondingly, the average polytropic index $n > n_{\rm a}$ where $n_{\rm a}$ is the adiabatic polytropic index defining the Hayashi track.
- If $\overline{\gamma} > \overline{\gamma}_{\rm a}$, the situation is unstable and therefore such state is "forbidden". In practise in such a situation, convection in the star would very quickly restore near-adiabaticity by transporting any excess heat to the surface because a very small superadiabaticity is enough to transport massive amounts of energy (homework!).

Hayashi zone and pre-main-sequence phase

- Stars from contrating gas clouds (molecular clouds) through dynamical collapse. These clouds are large (parsecs) and fragment in the process.
- Most of the gas in such clouds is in the form of molecular hydrogen (H_2). The collapse happens in dynamical timescale $\tau_{\rm dyn} \propto \rho^{-0.5}$.
- Gradually the H_2 molecules are dissociated, after which hydrogen and later helium start to be ionised. These processes use up most of the energy from continuing collapse and the temperature stays nearly constant.
- Finally the ionisation is nearly complete and the temperature starts to increase and a hydrostatic equilibrium is restored. The object is now a protostar.



Credits: Iben (1965), Astrophys. J., 141

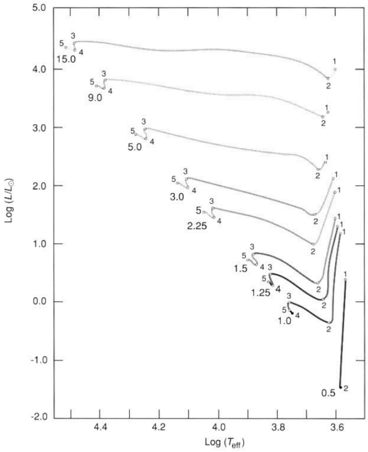
- Estimate of protostellar radius can be obtained by assuming that all of the gravitational energy is spent to dissociate H_2 and ionize H and He. Then,

$$\alpha \frac{GM^2}{R_{\rm ps}} \approx \frac{M}{m_{\rm H}} \left(\frac{X}{2} \chi_{\rm H_2} + X \chi_{\rm H} + \frac{Y}{4} \chi_{\rm He} \right), \quad (18)$$

where $\chi_{\rm H_2} = 4.5$ eV, $\chi_{\rm H} = 13.6$ eV, and $\chi_{\rm He} = 79$ eV.

- Taking $Y \approx 1 - X$ and $\alpha = \frac{1}{2}$ gives

$$\frac{R_{\rm ps}}{R_\odot} \approx \frac{50}{1 - 0.2X} \frac{M}{M_\odot}. \quad (19)$$



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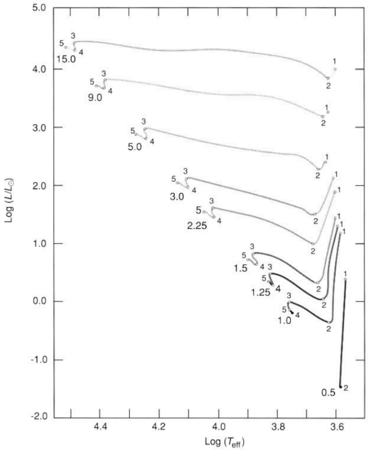
Hayashi zone and pre-main-sequence phase

- Recalling the average temperature from virial theorem and inserting the estimate for $R_{\rm ps}$ with $X = 0.7$ gives:

$$\overline{T} = \frac{\alpha}{3}\frac{\mu}{k}\frac{GMm_{\rm H}}{R_{\rm ps}} \approx 6 \cdot 10^4 \text{ K}. \qquad (20)$$

  Note that the temperature is independent of M.
- At this starting point on the Hayashi track the star is fully convective and the gas is still opaque.
- Contraction continues until all of the gas is ionized. The opacity drops first in the interior and the convection zone recedes. $T_{\rm eff}$ starts to rise slowly.
- Nuclear reactions start gradually when core temperature increases and increase the luminosity. Evolutionary track is complicated by ignition of different branches of hydrogen burning.



Credits: Iben (1965), Astrophys. J., 141

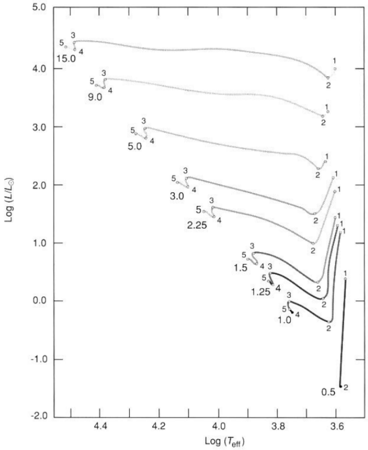
Hayashi zone and pre-main-sequence phase

Table 9.1 Evolutionary lifetimes (years)

$M/M_\odot$	*1–2*	*2–3*	*3–4*	*4–5*
15	6.7(2)	2.6(4)	1.3(4)	6.0(3)
9	1.4(3)	7.8(4)	2.3(4)	1.8(4)
5	2.9(4)	2.8(5)	7.4(4)	6.8(4)
3	2.1(5)	1.0(6)	2.2(5)	2.8(5)
2.25	5.9(5)	2.2(6)	5.0(5)	6.7(5)
1.5	2.4(6)	6.3(6)	1.8(6)	3.0(6)
1.25	4.0(6)	1.0(7)	3.5(6)	1.0(7)
1.0	8.9(6)	1.6(7)	8.9(6)	1.6(7)
0.5	1.6(8)			

Note: powers of 10 are given in parentheses.

Credits: Prialnik.



Credits: Iben (1965), Astrophys. J., 141

- The time that stars spend in the PMS phase depends strongly on the mass.

Main-sequence phase

- The main sequence phase is characterised by hydrogen burning, starting from the epoch when the stellar luminosity L comes solely from nuclear fusion.
- This evolutionary stage is so long that the star "forgets" its earlier structure and evolution. Often considered as the starting point of stellar evolution (Why?).
- The main-sequence lifetime is related to the amount of nuclear fuel (= stellar mass M) and the rate at which energy is radiated away (luminosity):

$$\tau_{\mathrm{MS}} \propto \frac{M}{L}. \tag{21}$$

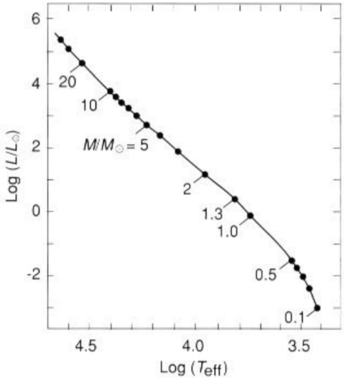
- From the homology relations (previous lecture) we recall that $L \propto M^3$ and thus

$$\tau_{\mathrm{MS}} \propto M^{-2}. \tag{22}$$

- Therefore more massive stars evolve faster and depart the main sequence earlier.

Main-sequence phase

- The figure on the right shows the main sequence for hydrogen-burning stars of solar composition.
- The model results are on a single curve but observed main sequence has a lot of scatter. Why?



Credits: Kippenhahn & Weigert (1990), Stellar Structure and Evolution

Main-sequence phase

- The figure on the right shows the main sequence for hydrogen-burning stars of solar composition.
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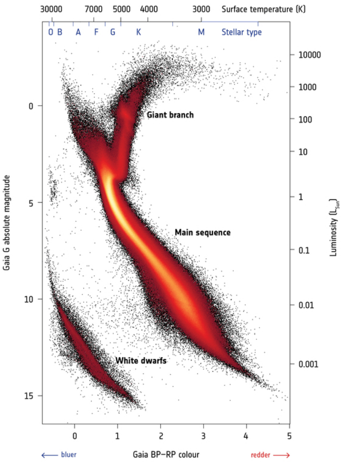
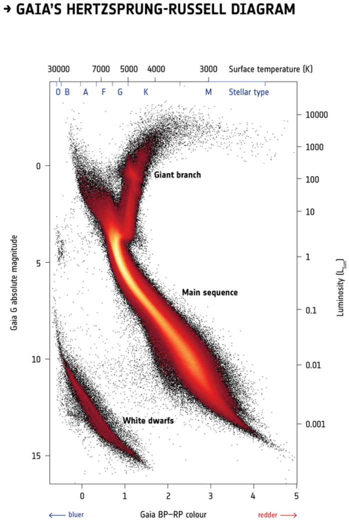
→ GAIA'S HERTZSPRUNG-RUSSELL DIAGRAM



Credits: ESA

Main-sequence phase

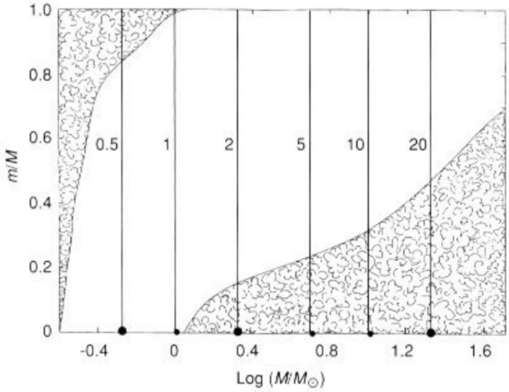
- The figure on the right shows the main sequence for hydrogen-burning stars of solar composition.
- The model results are on a single curve but observed main sequence has a lot of scatter. Why?
- Chemical composition of stars varies leading to different opacity, convection zone depth, radius, luminosity, etc.
- Higher order corrections: Binarity, rotation, magnetic fields?
- Stellar mergers?



Credits: ESA

Main-sequence phase

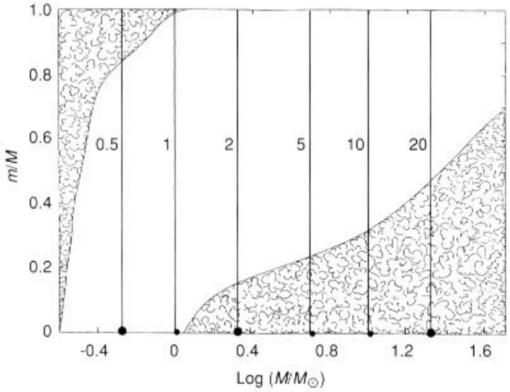
- Low mass stars with $M \lesssim 0.35 M_\odot$ are *fully convective* because of the high opacity of the gas.
- For more massive stars a radiative core develops and the convective envelope shrinks. For the Sun, about 2% of the mass is in the convection zone.
- For $M \gtrsim 1.5 M_\odot$ a convective core develops due to the temperature sensitivity of the CNO-cycle.



Credits: Kippenhahn & Weigert (1990), Stellar Structure and Evolution

Main-sequence phase

- Convection is important for stellar structure because it *mixes* the matter on a very short timescale.
- Matter processed by nuclear reactions in the core can be brought to the surface at a later stage if a deep convective envelope forms at a later evolutionary stage (with observational consequences).
- This is known as the *dredge-up* process.



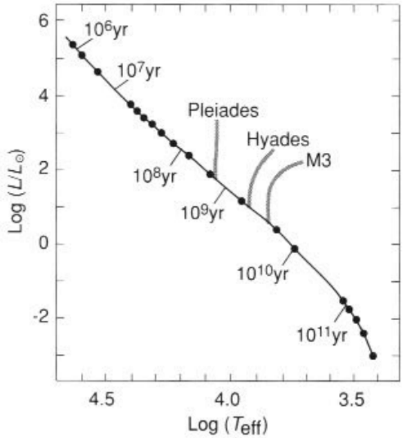
Credits: Kippenhahn & Weigert (1990), Stellar Structure and Evolution

Main-sequence phase

Table 9.2 Main-sequence lifetimes

Mass ($M_\odot$)	*Time (yr)*	α
0.1	6×10^{12}	-2.8
0.5	7×10^{10}	-2.8
1.0	1×10^{10}	
1.25	4×10^{9}	-4.1
1.5	2×10^{9}	-4.0
3.0	2×10^{8}	-3.6
5.0	7×10^{7}	-3.1
9.0	2×10^{7}	-2.8
15	1×10^{7}	-2.6
25	6×10^{6}	-2.3

Credits: Prialnik

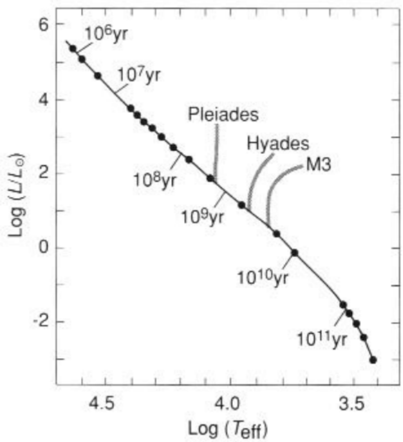


Credits: Prialnik

- Main-sequence lifetimes are reflected by stellar clusters.

Main-sequence phase

Credits: NASA, ESA, AURA/Caltech, Palomar Observatory, Public Domain, https://commons.wikimedia.org/w/index.php?curid=7805481

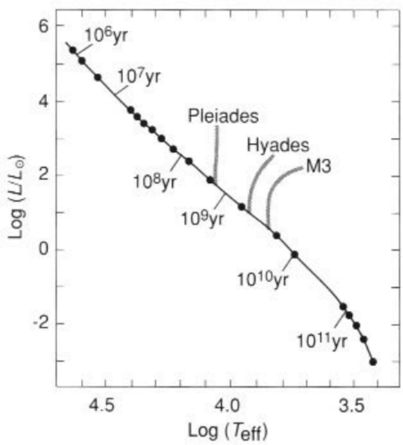


Credits: Prialnik

- Main-sequence lifetimes are reflected by stellar clusters.

Main-sequence phase

- Age of the cluster can be estimated from the *turn-off point* of the main-sequence.
- The distance to the cluster can be obtained from *main-sequence fitting*.
- All stars in the universe with mass $M \lesssim 0.7 M_\odot$ are still in the main sequence.



Credits: Prialnik

Red Giant phase

- Let us consider a very simplified argument for envelope expansion when the core collapses.
- Consider two mass elements Δm_1 and Δm_2 at some radius r_0 from the centre of the star. Treat $m(r_0)$ as a point mass.
- Assume that the mass elements move to new positions $r_1 < r_0$ and $r_2 > r_0$ and that gravitational energy is conserved.
- The new distances are related via

$$\tilde{r}_2 = \frac{1}{(2 - \tilde{r}_1^{-1})}, \tag{23}$$

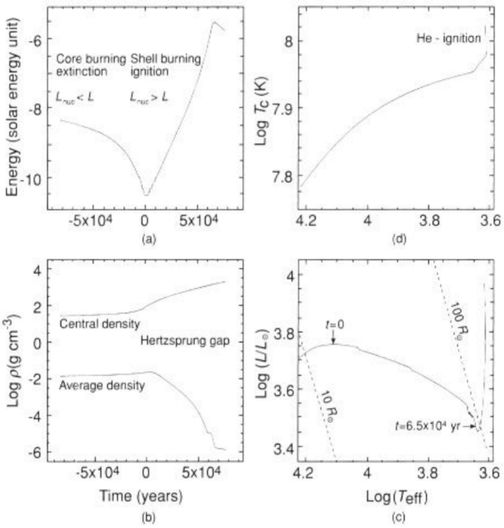
  where the tildes refer to normalization by r_0.
- Now a change of r_1 by 20% leads to an increase of r_2 by 33% and a change of r_1 by 50% leads to $r_2 \rightarrow \infty$ (reality more complex, why?).

Red Giant phase

- Ultimately hydrogen is depleted in the core of the star and burning proceeds in the surrounding shell.
- Due to the lack of energy sources the flux through the inert core approaches zero and the temperature gradient decreases.
- As the hydrogen burning shell moves outwards, the growing core becomes isothermal.
- The isothermal core has a maximum stable mass ($M_\mathrm{c}/M \approx 0.13$) above which an instability (Schönberg-Chandrasekhar instability) sets in leading to collapse.

Main-sequence phase

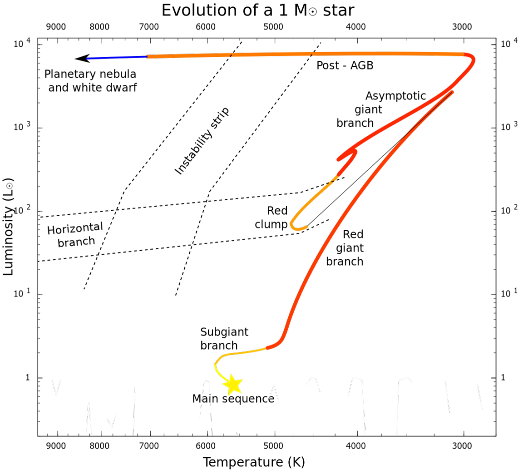
- An isothermal core develops for stars with $M \gtrsim 2M_\odot$.
- The core collapses due to the Schönberg-Chandrasekhar instability but ultimately the temperature increses and hydrostatic equilibrium is restored. Core contraction continues but now in Kelvin-Helmholtz timescale.
- Hydrogen burning commences in a shell surrouding the core. This is due to the CNO-cycle which is highly T-dependent.
- At sufficiently high T helium burning commences.
- Gas cools and opacity increases in envelope: convection ensues (dredge-up).



Evolution of a $7M_\odot$ star crossing the Hertzsprung gap Credits: Prialnik.

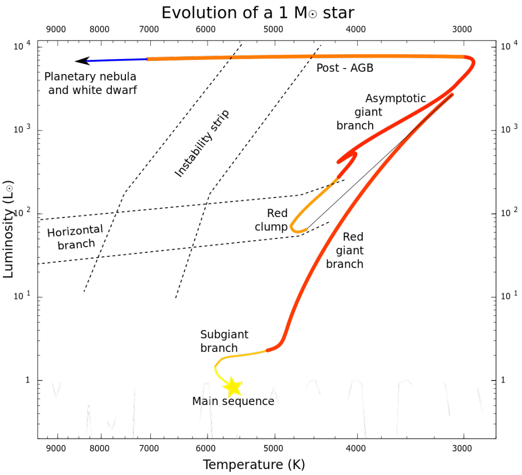
Red Giant phase and beyond

- In stars with $M \lesssim 2M_{\odot}$ the electron degeneracy pressure becomes high enough that the core can grow beyond the Schönberg-Chandrasekhar limit.
- Nuclear reactions and degenerate electron gas are an explosive combination because of the thermal instability discussed on the previous lecture.
- This leads to a *helium flash* where large amounts of helium are fused into carbon in a very short period of time. Enormous energy release ($10^{11} L_{\odot}$!) goes to bring gas out of degeneration and absorbed by the envelope.



Red Giant phase and beyond

- This is followed by less volatile core helium burning in the horizontal branch.
- In the early asymptotic giant branch (AGB) phase the core of the star consists of mostly carbon and oxygen, while helium and hydrogen continue to be burned in separated shells.
- At later AGB stage the star undergoes thermal pulsing that is related to unstable helium and hydrogen shell burning. This is associated with strong stellar winds and sometimes expulsion of some of the envelope material.



Post-AGB phase: superwind and planetary nebula

- At the end of the AGB the outer envelope of the star cools down and recombination, molecule, and dust formation starts to happen.
- The outward radiation pressure starts to become important especially for the dust grains (Why?). The wind picks up and drives strong mass loss ($\dot{M} \sim 10^{-4} M_\odot$, *superwind*).
- Due to the superwind, stars from the mass range $1M_\odot \leq M \lesssim 9M_\odot$ end up with cores (that develop to white dwarfs) in the range $0.6M_\odot$ and $1.1M_\odot$. Most end up around $0.6M_\odot$ (Why?)
- When the mass loss ends the core continues to contract and heat up. At sufficiently high temperature ($\sim 3 \cdot 10^4$ K) the radiation is strong enough to ionize matter in the nearly spherically symmetric ejecta envelope.
- This is then observed as a *planetary nebula*.

Post-AGB phase: superwind and planetary nebula

- Planetary nebula is a misnomer because originally these objects were thought to be forming (instead of dying) solar systems.
- The planetary nebula phase is relatively short-lived and because the ejected shell expands and cools and because the nuclear reactions in the core ultimately cease.
- The leftover is just the core of the star which is now a white dwarf.

Helix Nebula NGC7293 (Credit: Public Domain, https://commons.wikimedia.org/w/index.php?curid=1