Theoretical Astrophysics I: Physics of Sun and Stars
Lecture 6: Radiative Energy Transport

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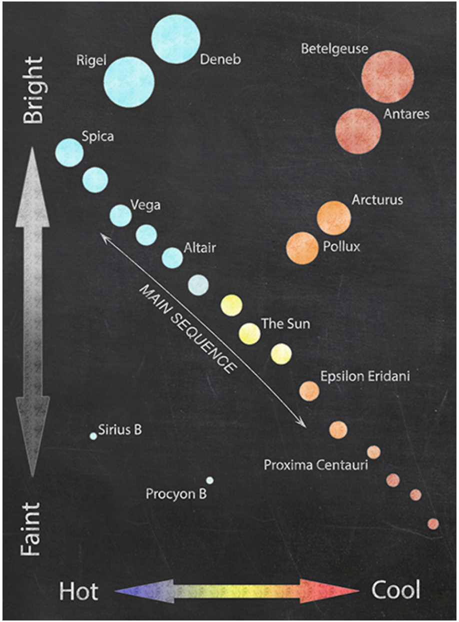
Brief recap

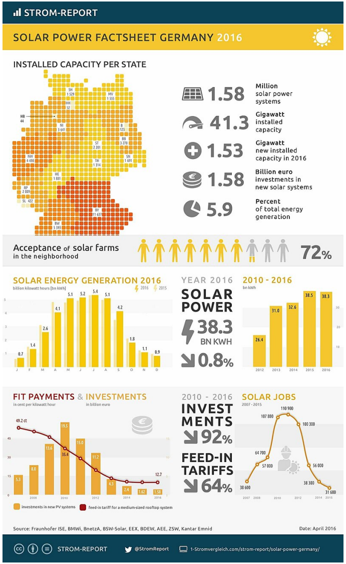
- We started with describing observed properties of the stars. One quantity that we measure (and want to reproduce) is the **luminosity**, and conversely **flux** (or specific flux).
- We wrote down equations that govern stellar structure and evolution. Solving them for proper boundary conditions will yield the structure of a star: $p(r)$, $T(r)$, $\rho(r)$, $F(r)$, etc...
- Analyzing their variation in time allows us to model stellar evolution.
- We spent last two lectures talking about a difficult problem of convection. But there is another way to transport energy: via **radiation**.



HR diagram

Photons vs particles

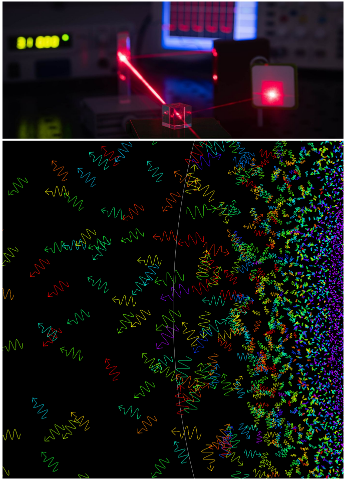
- It is obvious that radiation can carry energy.
- We treat radiation using photons, but they are clearly different from atoms, molecules, ions and electrons.
- Photons do not have mass and move with speed of light.
- The number of photons is not conserved.
- Photons can also be treated like a gas (e.g. see the derivation of Stefan-Boltzmann law by Boltzmann)
- They still observe conservation of energy, momentum, angular momentum, etc.



Credits: Strom Report

Photons vs particles

- It will be essential to understand photon-matter interaction. As Ivan Hubeny said:
- *...In other words radiation in fact determines the structure of the medium yet the medium is probed only by this radiation.*
- Radiation: constituent in energy transport (and equation of state).
- Also: diagnostics that allows us to understand physical properties of the medium.
- Contrary to the lab: we need to treat wavelength and angular dependence of the radiation field.



Credits: LabRoots.com (up), Prof. Rob Rutten (bottom)

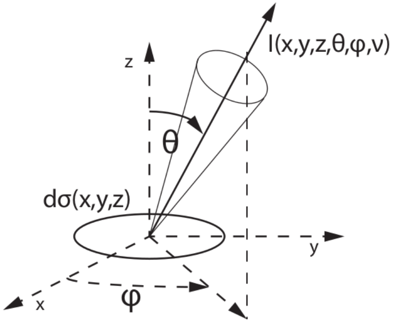
Specific monochromatic intensity

- We need to treat wavelength and angular dependence of the radiation field
- Intensity: energy transported through given area in given time per given solid angle and frequency/wavelength bin (note the deprojection factor $\cos\theta$).

$$I_\nu = \frac{dE}{dS\, dt\, d\Omega\, d\nu\, \cos\theta} \tag{1}$$

- Going to number of photons:

$$n(\theta, \phi\, \nu) = \frac{I_\nu}{c\, h\nu} \tag{2}$$



Credits: IM thesis (2014, University of Belgrade)

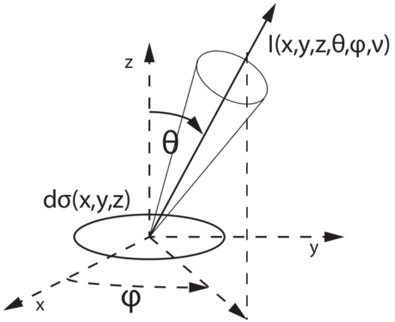
Other "moments" of the radiation field

- Intensity fully describes the radiation field (without polarization). But often we need some derived quantities:
- Mean intensity:

$$J_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \sin\theta d\theta d\phi \quad (3)$$

- Flux (in z direction):

$$\mathcal{F}_\nu = H_\nu = \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \quad (4)$$



Credits: IM thesis (2014, University of Belgrade)

Some de-confusion of the term flux

- Typically we say that (spectral, monochromatic) flux is:

$$F_\nu = \frac{dE}{dtdSd\nu} \tag{5}$$

- But in the book typically:

$$F_\nu = \frac{dE}{dtd\nu} \rightarrow F = \frac{dE}{dt} \tag{6}$$

  But then, to make situation worse, in stellar atmospheres theory flux is (H_nu in the book):

$$\mathcal{F}_\nu = \frac{dE}{dtdSd\nu} = H_\nu \tag{7}$$

  then astrophysical flux:

$$F_\nu = \frac{1}{\pi}\frac{dE}{dtdSd\nu} \tag{8}$$

  and Eddington flux (which the book uses and calls the radiation flux) is:

$$\mathcal{H}_\nu = \frac{1}{4\pi}\frac{dE}{dtdSd\nu} = \frac{1}{4\pi}\mathcal{F}_\nu \tag{9}$$

More moments of the radiation field

- Mean intensity:

$$J_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \sin\theta d\theta d\phi \tag{10}$$

- Radiation flux:

$$H_\nu = \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \tag{11}$$

- and the so called K-integral, which is proportional to the pressure of the radiation field:

$$\mathcal{K}_\nu = \frac{1}{4\pi} \oint I_\nu(\theta, \phi) \cos^2\theta \sin\theta d\theta d\phi = \frac{p_\nu c}{4\pi} \tag{12}$$

- **Note that we can define all these in the frequency/wavelength-integrated form**

More moments of the radiation field

- Mean intensity:

$$J = \frac{1}{4\pi} \int_0^\infty \oint I_\nu(\theta, \phi) \sin\theta d\theta d\phi d\nu \tag{13}$$

- Radiation flux:

$$H = \int_0^\infty \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi d\nu \tag{14}$$

- and the so called K-integral, which is proportional to the pressure of the radiation field:

$$\mathcal{K} = \frac{1}{4\pi} \int_0^\infty \oint I_\nu(\theta, \phi) \cos^2\theta \sin\theta d\theta d\phi d\nu = \frac{p\,c}{4\pi} \tag{15}$$

- **Here we integrated over all frequencies**

A quick question:

- What would be the radiation flux if the intensity was isotropic?

$$H_\nu = \oint I_\nu(\theta, \phi) \cos\theta \sin\theta d\theta d\phi \tag{16}$$

A quick question:

- What would be the radiation flux if the intensity was isotropic?

$$H_\nu = \oint I_\nu(\theta,\phi)\cos\theta\sin\theta d\theta d\phi \tag{17}$$

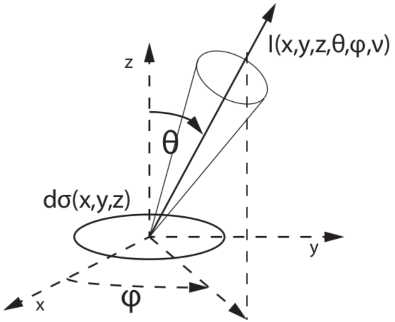
- A common substitute in this case is to integrate ϕ to 2π and then set $\cos\theta = \mu$ (this is again an another μ).

$$H_\nu = 2\pi\int_{-1}^{1} I_\nu(\mu)\mu d\mu = 0 \tag{18}$$

- If the radiation is completely isotropic, there is no energy transport. In order to transport the energy outward toward the surface the radiation has to be slightly anisotropic.

Modeling the radiation field

- Our task is not to model and understand intensity and its relationship with other physical quantities (density, temperature, pressure, chemical composition). For that we need to:
- Understand the interaction between the radiation and matter (absorption, emission, scattering coefficients).
- Mathematically express relationship between these coefficients and the intensity.



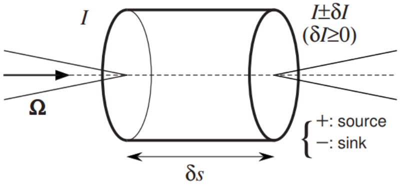
Credits: IM thesis (2014, University of Belgrade)

Radiative Transfer Equation (RTE)

- This formulation is (more or less) due to Kirchhoff. The change of intensity "along-the-ray" over a distance *ds* is:

$$dI_\nu = \eta_\nu ds - \chi_\nu I_\nu ds \tag{19}$$

- The terms of the right represent emission and **total** absorption (both true absorption and scattering) per unit volume.



Credit: Anthony B. Davis and Yuri Knyazikhin

Radiative Transfer Equation

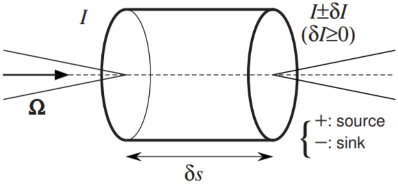
- Now, it makes sense that the absorption and emission properties of the medium depend on:
- Amount of matter capable of absorbing/emitting
- The inherent properties of the matter at the given temperature (T is very important!)
- So we define:

$$\kappa_\nu = \chi_\nu/\rho \tag{20}$$

$$j_\nu = \eta_\nu/\rho \tag{21}$$

- So our equation becomes:

$$\frac{1}{\rho}\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \tag{22}$$



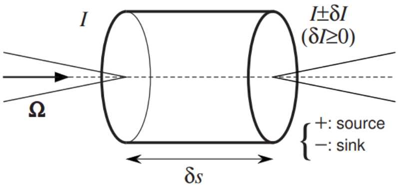
Credit: Anthony B. Davis and Yuri Knyazikhin

Few remarks

- This equation does not involve any new physical laws, it is a mathematical tool.

$$\frac{1}{\rho}\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \qquad (23)$$

- ds is a so-called "ray", its relationship to dx, dy, dz will depend on the geometry we choose and the context.
- We will assume that coefficients of absorption and emission are isotropic, but that they do depend on the frequency / wavelength.
- **The intensity is not isotropic. Can you see why?**.



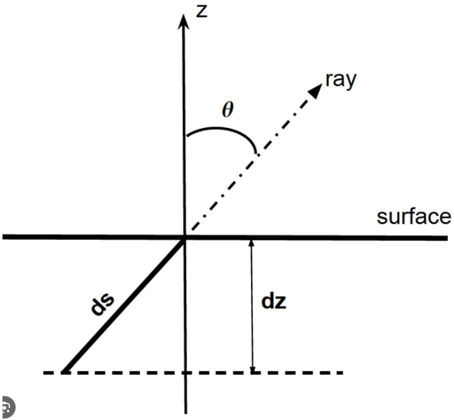
Credit: Anthony B. Davis and Yuri Knyazikhin

RTE in 1D plane-parallel geometry

- If we assume that we are in 3D Cartesian grid, and nothing depends on x and y, we will get:

$$ds = dz/\cos\theta = dr\cos\theta \qquad (24)$$

- If we want to take the sphericity in context, this becomes more complicated, but we won't need it.
- **In 1D, the anisotropy appears because ds depends on the direction (θ).**



Credit: Frederic Paletou

Optical depth and Source function

- We often do the following:

$$\frac{dI_\nu}{-\rho\kappa_\nu ds} = I_\nu - j_\nu/\kappa_\nu \tag{25}$$

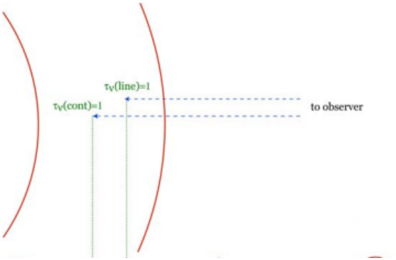
- And we get:

$$\frac{dI_\nu}{-\rho\kappa_\nu ds} = I_\nu - j_\nu/(\rho\kappa_\nu) \tag{26}$$

$$\kappa_\nu = \chi_\nu/\rho;\ j_\nu = \eta_\nu/\rho \tag{27}$$

- So our equation becomes:

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \tag{28}$$



Credit: Rolf Kudritzki

Optical depth and Source function

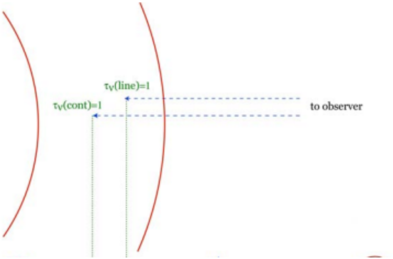
- If I wanted to be pedantic - I would have written:

$$\frac{dI_\nu(\theta)}{d\tau_\nu(\theta)} = I_\nu(\theta) - S_\nu \qquad (29)$$

- A lot of interesting solutions will involve exponents of τ_ν - nice that it is dimensionless.
- For example, when there is no emission:

$$I_\nu(\tau_n u) = I_\nu^0 e^{-\tau_\nu} \qquad (30)$$

- Derive this real quick to get used to the orientation of optical depth.



Credit: Rolf Kudritzki

Kirchhoff Law

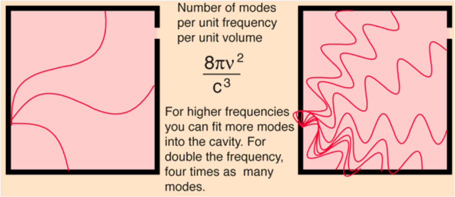
- Kirchhoff first wrote something like this and did a little thought experiment on a blackbody.

$$\frac{dI_\nu(\theta)}{d\tau_\nu(\theta)} = I_\nu(\theta) - S_\nu \qquad (31)$$

- If the body is in a complete equilibrium, dI_ν is zero on every frequency, and I_ν is constant in space, and so is T. Then:

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = f(T;\nu) = B_\nu(T) \qquad (32)$$

- He argued that it is of utmost importance to find this function.



Credit: Hyperphysics

Planck's Law

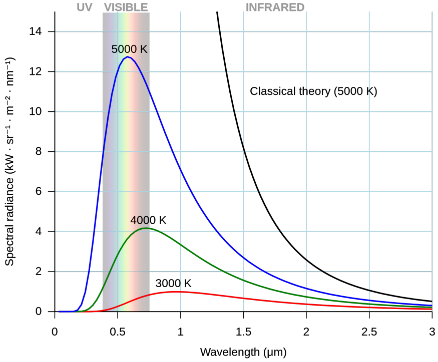
- People measured this function and Planck finally derived it (a nicer derivation is due to Bose and Einstein):

$$B_\nu(T) = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1} \qquad (33)$$

- This is, then, intensity of radiation inside of a blackbody. Integrating in wavelengths yields:

$$B(T) = \mathrm{const}\, T^4 \qquad (34)$$

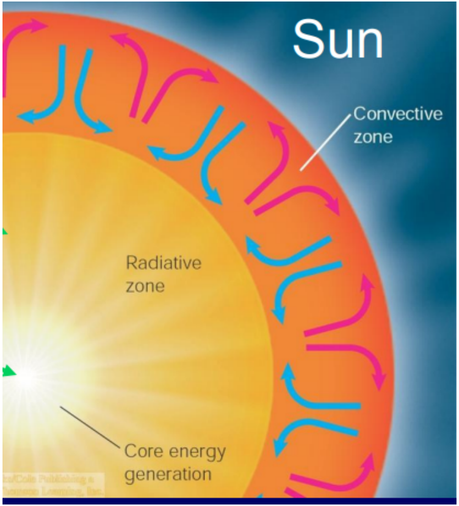
- And finally integrating that in angle yields the good old σT^4.



Credit: Wikipedia

But Ivan, stars are not black bodies

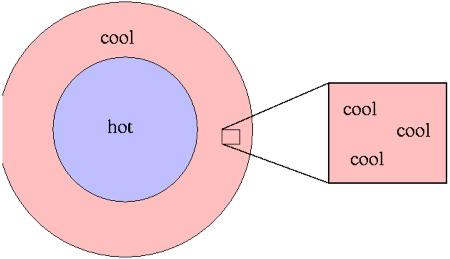
- If stars were blackbodies there would be no transport of energy.
- However, the gradient of temperature in the stars is very small.
- Can you estimate? $(\mathrm{K}/\mathrm{m}?)$



Credits: George Blanford

Local Thermodynamic Equilibrium

- $dT/dr \approx 10^{-2}\mathrm{K}/\mathrm{m}$ - This is incredibly small
- This allows us to presume the so called *local* thermodynamic equilibrium, which states that **matter** obeys:
- Saha distribution over ionization states.
- Boltzmann distribution over excitation states.
- Maxwell distribution over velocities.
- Contrary to what most of the textbooks tell you: *radiation is not in equilibrium with matter* - Intensity has to be out of equilibrium or there is no energy transport.



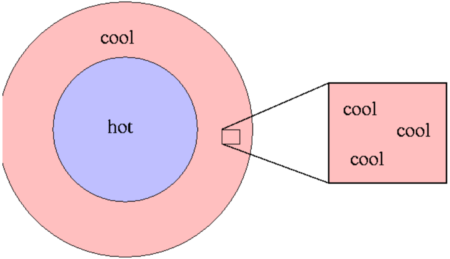
Credits: Michael Richmond

Local Thermodynamic Equilibrium

- *Radiation is not in equilibrium with matter* - Intensity has to be out of equilibrium or there is no energy transport.
- But the source function, that depends only on the matter - is in equilibrium, and is equal to planck function:

$$S_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1} \qquad (35)$$

- So, local emission and absorption properties of the material follow from equilibrium distributions but the radiation departs from it.
- This departure is very slight in the cores of the stars but can be huge in the outer layers.
- Discuss the Sun's radiation that reaches us.



Credits: Michael Richmond

Solving RTE deep in the atmospheres

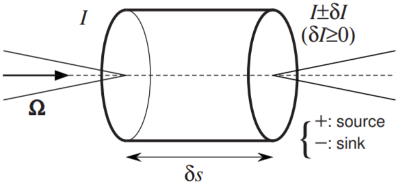
- We will not follow the approach from the book, but rather arguments used in stellar atmosphere modeling. (Still, very similar):

$$\cos\theta \frac{dI_\nu}{-\rho\kappa_\nu dz} = I_\nu - B_\nu \quad (36)$$

- Then we multiply the equation by $\cos\theta$ and integrate over all angles:

$$4\pi \frac{d\mathcal{K}_\nu}{-\rho\kappa_\nu dz} = H_\nu \quad (37)$$

- Here $\mathcal{K}_\nu$ is the so called K-integral, which is proportional to the radiation pressure $p = 4\pi\mathcal{K}/c$.

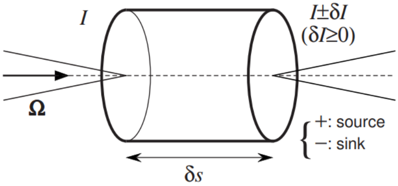


Credit: Anthony B. Davis and Yuri Knyazikhin

$$c\frac{dp_\nu}{-\rho\kappa_\nu dz} = H_\nu \tag{38}$$

- Nice! Flux of the radiation is equal to the gradient in the radiation pressure.
- Now, let's integrate this over the frequencies, and introduce mean opacity $\overline{\kappa}$:

$$H(z) = -\frac{4acT^3}{3\overline{\kappa}\rho}\frac{dT}{dz} \tag{39}$$



Credit: Anthony B. Davis and Yuri Knyazikhin

Radiative flux deep in the atmosphere:

- If we now start from the book definition of F ($F = 4\pi r^2 H$).

$$F = -4\pi r^2 \frac{4acT^3}{3\overline{\kappa}\rho} \tag{40}$$

- Which we can invert to get:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{F}{4\pi r^2} \tag{41}$$

- Or:

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\overline{\kappa}}{T^3}\frac{F}{(4\pi r^2)^2} \tag{42}$$

- Discuss and analyze!
- Obviously $\overline{\kappa}$ plays an important role here. Calculating it accurately as a function of T and chemical compositon, for given ρ is very very hard and important task!

What comes next?

- Now, in principle we could move to upper layers and discuss radiative transfer in stellar atmospheres.
- The situation there is trickier because things are much more anisotropic, and photons start escaping from the star.
- So, **frequency dependence is much more important**.
- To prepare for that, and better understand the $\overline{\kappa}$, we will briefly discuss sources of absorption in the stars.

What comes next?

- In the execises we will see that:

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} dB_\nu/dT d\nu}{\int_0^\infty dB_\nu/dT d\nu} \tag{43}$$

- Therefore, we must understand the wavelength and temperature dependence of κ_ν.
- Let's first talk about what contributes to κ_ν.

Opacity sources

- Reminder, we only talk about opacity (κ_ν) because $j_\nu = B_\nu \kappa_\nu$!
- Some sources are:
- Bound-free transitions (photoionization)
- Free-free processes (inverse bremsstrahlung)
- Bound-bound processes (spectral line - not important in the interior!)
- Thomson and Rayleigh scattering.
- Photodissotiation of $H-$ and molecules.
- Maybe some exotic processes?

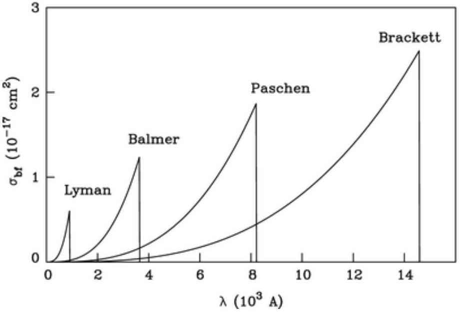
Example: Bound-free processes

- To see the steps needed in the calculation of the opacity, try to solve the following problem
- For the gas of pure hydrogen, with given ρ and T, calculate bound-free opacity at $\lambda = 50\,\mathrm{nm}$ and $500\,\mathrm{nm}$
- First question: can hydrogen absorb these fotons?

Example: Bound-free processes

- For the gas of pure hydrogen, with given ρ and T, calculate bound-free opacity at $\lambda = 50\,\text{nm}$ and $500\,\text{nm}$
- For hydrogen to absorb, energy of the photon must be larger than the binding energy:

$$h\nu \geq E_i \tag{44}$$

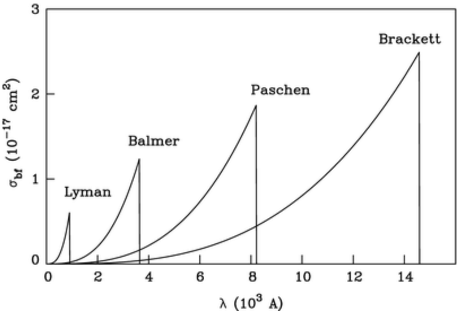


Credit: Walter Maciel, Springer

Example: Bound-free processes

- For the gas of pure hydrogen, with given ρ and T, calculate bound-free opacity at $\lambda = 50\,\mathrm{nm}$ and $500\,\mathrm{nm}$
- For hydrogen to absorb, energy of the photon must be larger than the binding energy.
- But electrons at different bound states have different binding energies (i is the excitation state)!
- Plus, not every wavelength is equally efficient!

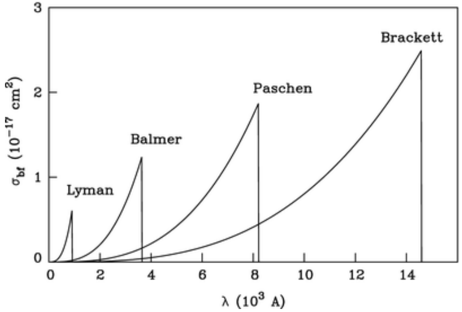
$$h\nu \geq \frac{13.6\mathrm{eV}}{i^2} \tag{45}$$



Credit: Walter Maciel, Springer

Example: Bound-free processes

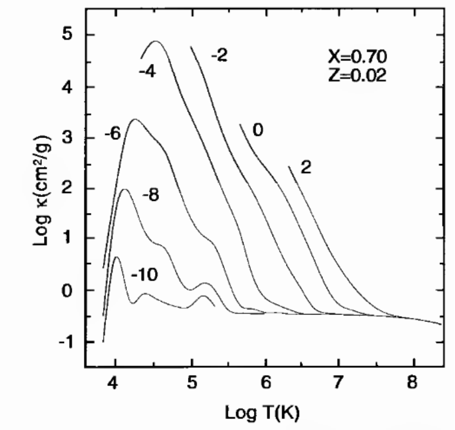
- So, for given ρ and T, we need to find number densities of all hydrogen atoms
- Ionized hydrogen does not contribute.
- Probably better if at this point we rely on the blackboard...
- Plot on the right shows the *cross-section* for various wavelengths.



Credit: Walter Maciel, Springer

Result

- These calculations need to be done for a system of many elements.
- Taking into account electron scattering.
- Taking into account free-free processes (this is not scattering!)
- Eventually, we will get the plot on the right
- What does the plot on the right tell us?



Credit: Iglesias and Rogers (1996)