UNIVERSITY OF OSLO

FYS3140 - MATHEMATICAL METHODS IN PHYSICS

Midterm exam

Candidate number: —

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Problem 1: Complex analysis

Part A: Cauchy integral formula and harmonic functions

We will begin by studying Cauchy's integral formula for a function $f(z)$ which is analytical inside and on the closed curve C_R defined by

$$C_R : |z - z_0| = R \tag{1}$$

Cauchy's integral formula takes the form

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \tag{2}$$

a)

We let M denote the maximum absolute value of $f(z)$ on C_R such that

$$\left| f(z) \right| \leq M, \tag{3}$$

for all z on the contour. We will use this to find an upper bound on the absolute value of $f(z_0)$

$$\left| f(z_0) \right| = \left| \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| = \frac{1}{2\pi} \left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right|. \tag{4}$$

We begin by writing the absolute value of the integral in terms of Riemann sums

$$\left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| = \lim_{n \to \infty} \left| \sum_{k=1}^{n} \frac{f(z_k)}{z_k - z_0} \right| \Delta z_k. \tag{5}$$

For this sum we will use the generalized triangle inequality, which states that

$$\left| \sum_i = a_i \right| \leq \sum_i |a_i|, \tag{6}$$

which is true for an arbitrary number of terms. Using this we rewrite the Riemann-sum

$$\left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| \leq \lim_{n \to \infty} \sum_{k=1}^{n} \left| \frac{f(z_k)}{z_k - z_0} \right| \Delta z_k = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\left| f(z_k) \right|}{\left| z_k - z_0 \right|} \Delta z_k, \tag{7}$$

where we have used that the absolute value of the fraction is just the absolute value of each factor. We have already defined $\left| f(z) \right|$ in equation (3), and we recognize the denominator as the radius R from the definition of the contour (1). Since both of these are constants we can take them outside of the sum, where we now have a new upper bound estimate

$$\left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| \leq \frac{M}{R} \lim_{n \to \infty} \sum_{k=1}^{n} \Delta z_k. \tag{8}$$

The infinitesimal sums over the changes in z_k will add up to the circomference of the curve, which is just a circle with radius R. Thus the upper bound can be written as

$$\left|\oint_{C_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z\right| \leq \frac{M}{R}\,2\pi R. \tag{9}$$

We cancel the R's leaving us with the simple expression for the upper bound estimate

$$\left|\oint_{C_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z\right| \leq 2\pi M. \tag{10}$$

We insert this expression into our original one (4)

$$\left|f(z_0)\right| = \frac{1}{2\pi}\left|\oint_{C_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z\right| \leq \frac{1}{2\pi}\,2\pi M. \tag{11}$$

Canceling the factors of 2π we find the final expression for the upper bound estimate of the contour integral

$$\left|f(z_0)\right| \leq M. \tag{12}$$

b)

We will try to rewrite Cauchy's integral formula (2) for the special case of a circular contour around a point z_0. We do this by writing the complex number z in terms of the center of the circle plus another terms looping around a cirlce with radius R

$$z = z_0 + Re^{it} \qquad t \in [0, 2\pi]. \tag{13}$$

By taking the derivative of z with respect to time we can solve for the infinitesimal $\mathrm{d}z$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = iRe^{it} \rightarrow \mathrm{d}z = iRe^{it}\,\mathrm{d}t. \tag{14}$$

We use this substitution in Cauchy's integral formula (2)

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_R} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{it})}{z_0+Re^{it}-z_0}\,iRe^{it}\,\mathrm{d}t. \tag{15}$$

We see that the i outside of the integral cancels with the one inside, and by subtracting away the z_0's in the denominator we can cancel the factor Re^{it}, thus

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{it})\,\mathrm{d}t. \tag{16}$$

This expression will only work for circular contours around z_0.

c)

We introduce the function $u(x,y)$, which is harmonic on and inside a circle of radius R centerd at $z_0 = x_0 + iy_0$. We can evaluate the value of u at the center of the circle using Cauchy's integral formula (2)

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z. \tag{17}$$

In the previous task we showed that such a integral, for a circular contour with radius R around a point z_0 can be rewritten to a integral over a real scalar t from 0 to 2π (16). We use this result, but now for a variable θ over the same interval, to evaluate z_0 at the center of the circel

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) \, \mathrm{d}\theta. \tag{18}$$

This result tells us that to evalue an analytic, and in this case harmonic, function at the center of a circle we can only use values along a circle around the point. Due to the factor $1/2\pi$ outside the integral the functionvalue at the center of the circle is equal to the average value of the function along the contour. It is quite incredible that we can do this for an arbritary radius R (as long as the function is still analytic inside and on the contour), and always be able to evalue the function at a point we never looked at.

d)

For a pair of two dimensional functions $u(x,y)$ and $v(x,y)$ which are each other's harmonic conjugates we have the following relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{19}$$

which is actually the definition of a harmonic conjugate in two dimensions. We can use this to derive the orthogonality of the gradients of the functions, where the two dimensional nabla-operator is defined as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), \tag{20}$$

meaning that the inner product between the two functions can be written as

$$(\boldsymbol{\nabla u}) \cdot (\boldsymbol{\nabla v}) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}. \tag{21}$$

Where we have written the left hand side in bold to emphesise that it is a vector. We then make a substitution from the definition of harmonic conjugates (19) on v so that we only have derivatives acting on u

$$(\boldsymbol{\nabla u}) \cdot (\boldsymbol{\nabla v}) = -\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0. \tag{22}$$

Since both terms are equal, but with oposite sign, they exactly cancel each other. Thus we have showed that the gradient of two functions which are each others harmonic conjugates have to be orthogonal. We did not specify anything about u and v except from them being harmonic conjugates of each other, thus this orthogonality must hold for all pairs of analytical conjugates.

$$(\nabla \boldsymbol{u}) \cdot (\nabla \boldsymbol{v}) = 0. \tag{23}$$

e)

We will now look at concrete example where one of the harmonic functions, $u(x,y)$, is known, and we want to find it's harmonic conjugate $v(x,y)$. The expression for the known harmonic function is

$$u(x,y) = \sin x \cosh y. \tag{24}$$

We begin by finding it's derivatives and second derivatives with respect to both x and y seperately

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos x \cosh y & \qquad \frac{\partial^2 u}{\partial x^2} &= -\sin x \cosh y \\ \frac{\partial u}{\partial y} &= \sin x \sinh y & \qquad \frac{\partial^2 u}{\partial y^2} &= \sin x \cosh y. \end{aligned} \tag{25}$$

We begin by checking that $u(x,y)$ infact is harmonic

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0, \tag{26}$$

here we used the double derivatives calculated in (25), and find that u is harmonic. We now want to find the harmonic conjugate of u, which we can solve for through the definition of harmonic conjugates (19). We begin with the first equality in (19), and put in the derivative of u from (25)

$$\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \sinh y. \tag{27}$$

We multiply each side with $\mathrm{d}y$ and take the integral on both sides

$$\int \mathrm{d}v = \int \cos x \cosh y \, \mathrm{d}y. \tag{28}$$

The left hand side will just be the function $v(x,y)$, while on the right hand side we can take $\cos x$ outside the integral, and the integral of $\cosh y$ is known, leaving us with

$$v(x,y) = \cos x \sinh y + C(x). \tag{29}$$

The term $C(x)$ is the integration constant, which can depend on x since we took the integral over y. We repeat the process for the secound equality in (19)

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{\mathrm{d}u}{\mathrm{d}y} = -\sin x \sinh y. \tag{30}$$

We multilpy each side with dx and take the integral

$$\int \mathrm{d}v = -\int \sin x \sinh y \,\mathrm{d}x. \tag{31}$$

The left hand side will just result in the function $v(x,y)$, while the right hand side is triviall, thus

$$v(x,y) = \cos x \sinh y + C(y). \tag{32}$$

Where $C(y)$ is the integration constant, which can depend on x since the integral was only over y. The two expressions we have found (29, 32) are consistent with one another, as they should be, and we see that the integration constant can not depend on either x or y, and must therefore just be a constant. The final expression for u's harmonic conjugate is therefore

$$v(x,y) = \cos x \sinh y + C. \tag{33}$$

Having found both $u(x,y)$ and $v(x,y)$ we can find the analytic function $f = u + iv$, which has the form

$$f(x,y) = u(x,y) + iv(x,y) = \sin x \cosh y + i \cos x \sinh y. \tag{34}$$

In our calculations we will ignore the integration constant. We begin by writing the trigonometric and hyperbolic functions on exponential form

$$f(x,y) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\frac{1}{2}\left(e^{y} + e^{-y}\right) + i\frac{1}{2}\left(e^{ix} + e^{-ix}\right)\frac{1}{2}\left(e^{y} - e^{-y}\right). \tag{35}$$

We multiply out the parenthesis in each term, and factor out 1/4

$$f(x,y) = \frac{1}{4i}\left(e^{ix-y} + e^{ix+y} - e^{-ix-y} - e^{-ix+y}\right) + i\frac{1}{4}\left(e^{ix+y} - e^{ix-y} + e^{-ix+y} - e^{-ix-y}\right). \tag{36}$$

In the secound term we multiply with i in the denominator and numerator, flipping the sign and making the i appear in the denominator so that we can rewrite the expression to

$$f(x,y) = \frac{1}{4i}\left(e^{ix-y} + e^{ix+y} - e^{-ix-y} - e^{-ix+y} - e^{ix+y} + e^{ix-y} - e^{-ix+y} + e^{-ix-y}\right), \tag{37}$$

where we see that we have four terms canceling

$$f(x,y) = \frac{1}{4i}\left(e^{ix-y} + \cancel{e^{ix+y}} - \cancel{e^{-ix-y}} - e^{-ix+y} - \cancel{e^{ix+y}} + e^{ix-y} - e^{-ix+y} + \cancel{e^{-ix-y}}\right), \tag{38}$$

leaving us with

$$f(x,y) = \frac{1}{4i}\left(2e^{ix-y} - 2e^{-ix+y}\right). \tag{39}$$

We can cancel one factor of 2 and recognize that we can use $z = x + iy$ to rewrite the exponent as $iz = ix - y$

$$f(x,y) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right). \tag{40}$$

We recognize this expression as the exponential form of sinus, thus

$$f(x,y) = \sin z. \tag{41}$$

f)

Now that we have found a pair of two harmonic conjugates we can test the property of the orthogonality of their gradient (23) found earlier. We begin by calculating their derivatives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos x \cosh y & \frac{\partial u}{\partial y} &= \sin x \sinh y \\ \frac{\partial v}{\partial x} &= -\sin x \sinh y & \frac{\partial v}{\partial y} &= \cos x \cosh y. \end{aligned} \tag{42}$$

The inner product between the two gradients is given by

$$(\nabla \boldsymbol{u}) \cdot (\nabla \boldsymbol{v}) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \tag{43}$$

We insert these derivatives into the inner product of the two gradients,

$$(\nabla \boldsymbol{u}) \cdot (\nabla \boldsymbol{v}) = -\cos x \cosh y \sin x \sinh y + \sin x \sinh y \cos x \cosh y = 0, \tag{44}$$

where we see that the two terms are identical, but with opposite sign, and thereby canceling. Our expressions for the harmonic conjugates u and v satisfy the property of orthogonality found previously.

g)

We have studied the orthogonality of the harmonic conjugates, we will now display this graphically for different functions. We begin with the simple case of

$$f(z) = x + iy \rightarrow u = x \qquad v = y. \tag{45}$$

This is displayed in figure 1 on the following page, where the left most figure is the contours of u, the middle is the cotours of v, and the rightmost is both plotted together where u is continous and v is dashed. In the right most figure we can clearly see the orthogonality as all the dashed and continous lines cross orthgonally.
We repeat this for the function

$$f(z) = z^2 = x^2 - y^2 + 2ixy \rightarrow u = x^2 - y^2 \qquad v = 2xy. \tag{46}$$

This is displayed in figure 2 on the next page, with the same order as the previous figure. We can again see in the right most figure that the dashed and continous lines are orthogonal, as it should be.
Lastly we do this for the function we have just studied

$$f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y \rightarrow u = \sin x \cosh y \qquad v = \cos x \sinh y. \tag{47}$$

This is displayed in figure 3 on page 9, with the same order as the previous figure. We can again see in the right most figure that the dashed and continous lines are orthogonal, as it should be.

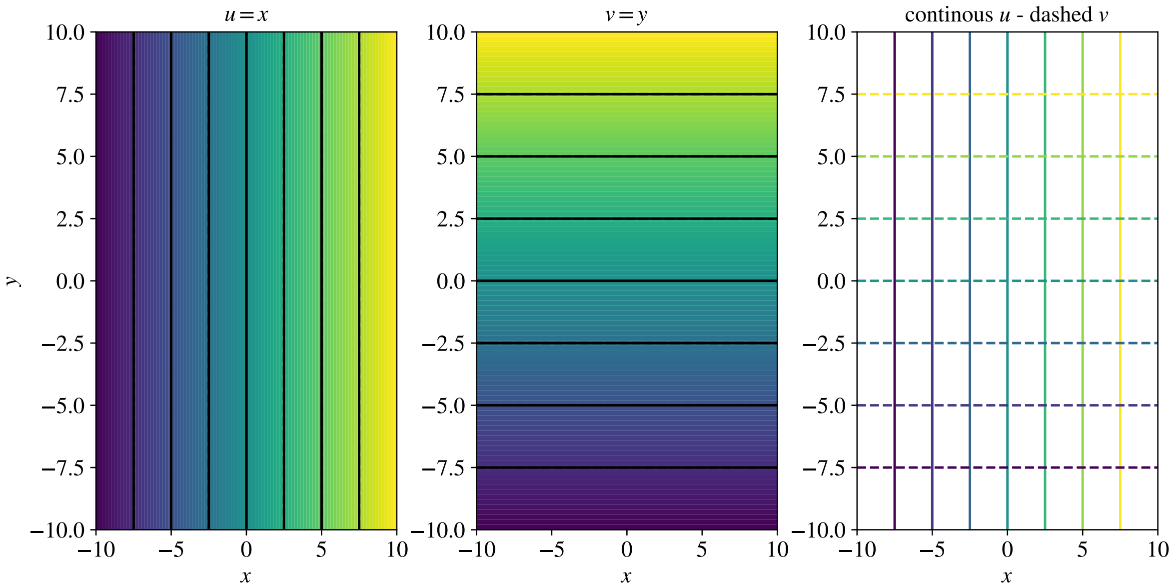

Figure 1: The contour levels of $u = x$ and $v = y$ is displayed in the left and middle figure respectivly. Where the colour indicates the functional value at a given point, and the black lines show the lines with a constant functional value. For the last figure the colour still indicates the function value, and displays the curve of a constant value. The dashed line is for $v = y$, and the continous lines for $u = x$, we see that the dashed and continous lines always cross orthogonally.

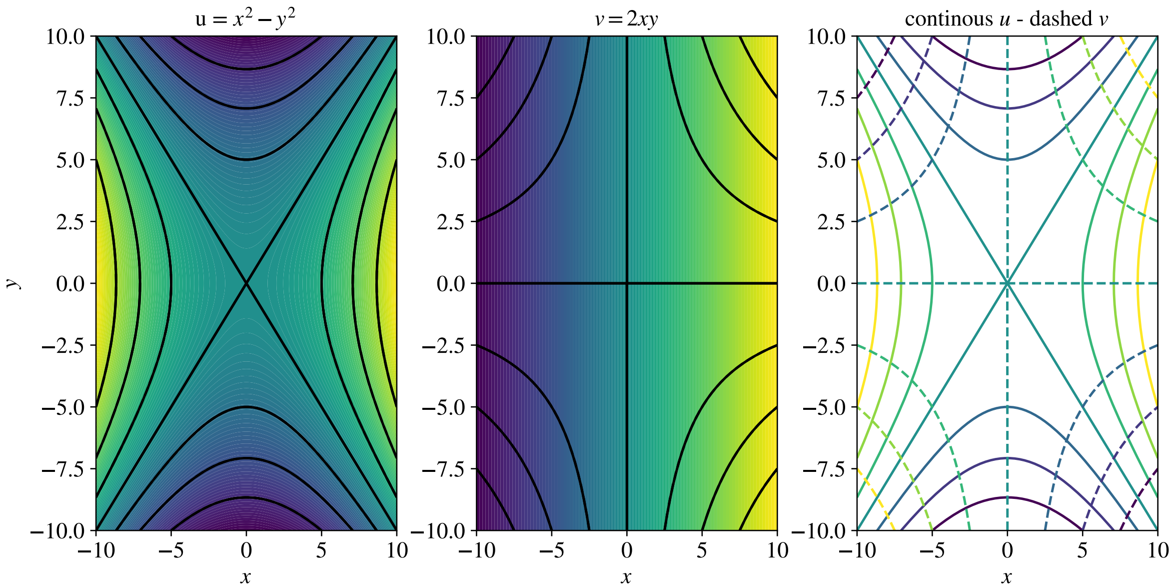

Figure 2: The contour levels of $u = x^2 - y^2$ and $v = 2xy$ is displayed in the left and middle figure respectivly. Where the colour indicates the functional value at a given point, and the black lines show the lines with a constant functional value. For the last figure the colour still indicates the function value, and displays the curve of a constant value. The dashed line is for $v = 2xy$, and the continous lines for $u = x^2 - y^2$, we see that the dashed and continous lines always cross orthogonally.

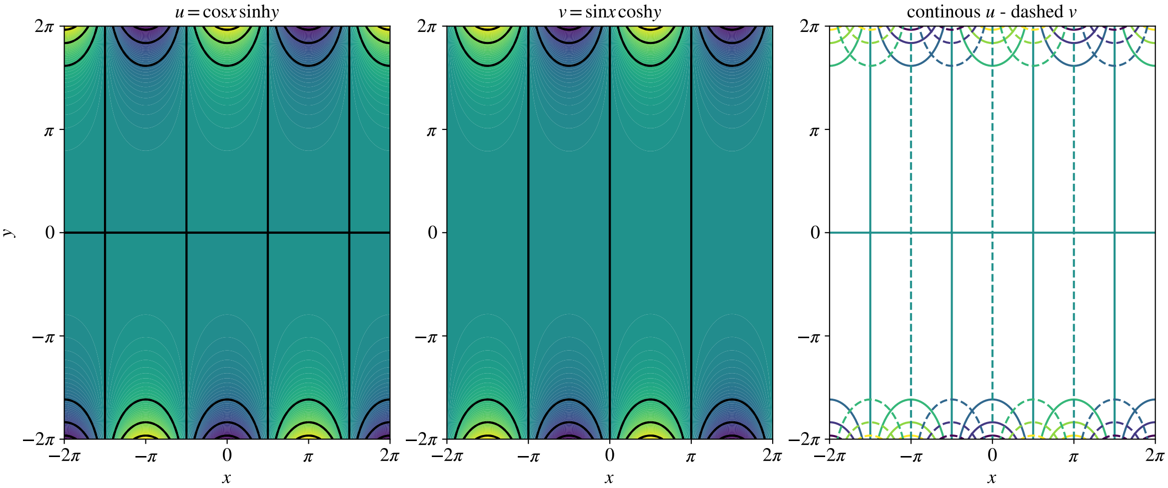

Figure 3: The contour levels of $u = \sin x \cosh y$ and $v = \cos x \sinh y$ is displayed in the left and middle figure respectivly. Where the colour indicates the functional value at a given point, and the black lines show the lines with a constant functional value. For the last figure the colour still indicates the function value, and displays the curve of a constant value. The dashed line is for $v = \cos x \sinh y$, and the continous lines for $u = \sin x \cosh y$, we see that the dashed and continous lines always cross orthogonally.

Part B: A contour integral

a)

We want to solve the contour integral

$$I = \oint_C \frac{4i\left(z^2+4\right)}{z\left(z^2-16\right)} \sin\left(\frac{5\pi}{z^2+4}\right) \mathrm{d}z, \tag{48}$$

where C is the closed contour integral around $z = 3$ with a radius of 2, $C : |z-3| = 2$. We begin by rewriting the integral by finding the roots of the polynomials to display the singularities more clearly

$$I = \oint_C \frac{4i(z+2i)(z-2i)}{z(z+4)(z-4)} \sin\left(\frac{5\pi}{(z+2i)(z-2i)}\right) \mathrm{d}z. \tag{49}$$

With this rewrite we can see that we have singularities at $z = 0$, $z = 4$ and $z = -4$. We also see that inside the sine we have two points where we divide by zero; $z = 2i$ and $z = -2i$. These singularities are displayed in the complex plane in figure 4 on the next page.
We see from the figure that there is only one singulartity inside the contour, and no singularties on the contour. This singularity is at $z = 4$, and we can see that it is a simple pole since it is of order 1, but we have to be carefull to check that this is not a removable singularity. We must check that for $z = 4$ the sine factor is not equal to zero, which could have removed the singularity. For $z = 4$ the sine-argument is $5\pi/20 = \pi/4$, which will make the sine-factor equal to $\sqrt{2}$. Thus the singularity at $z = 4$ is not removable.

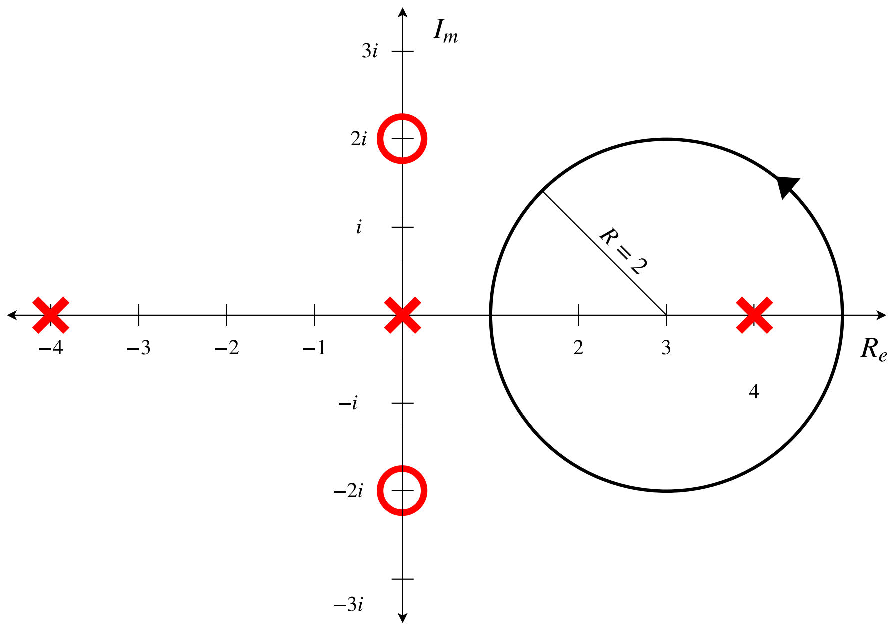

Figure 4: The contour C defined as $C : |z-3| = 2$ is shown in the complex plane together with the singularities of our integral (49). The orientation of the contour is shown by the pointing of the arrow. The singularities are displayed as red crosses, while the singularities inside the sine function are displayed as red circles. We see that inside our contour we only have 1 singularity, which lies at $z = 4$.

To calculate the integral we only need to find the integrand's residue at $z = 4$. We do this by using the following procedure

$$\mathrm{Res}(z_0) = \lim_{z \to z_0} \Big[(z - z_0) f(z) \Big], \tag{50}$$

where $f(z)$ is the integrand. We calculate this for $z_0 = 4$ and $f(z)$ being the integrand in our integral (49)

$$\mathrm{Res}(4) = \lim_{z \to 4} \left[\cancel{(z-4)} \frac{4i(z+2i)(z-2i)}{z(z+4)\cancel{(z-4)}} \sin\left(\frac{5\pi}{(z+2i)(z-2i)} \right) \right]. \tag{51}$$

We cancel the $z - 4$ terms, and the value now becomes well defined. We calculate the products and find

$$\mathrm{Res}(4) = \lim_{z \to 4} \left[\frac{4i\left(z^2+4\right)}{z(z+4)} \sin\left(\frac{5\pi}{z^2+4} \right) \right]. \tag{52}$$

By evaluating the limit of z we get

$$\mathrm{Res}(4) = \frac{4i(16+4)}{4(4+4)} \sin\left(\frac{5\pi}{16+4} \right) = \frac{80i}{32} \sin\left(\frac{5\pi}{20} \right) = \frac{5i}{2} \sin\left(\frac{\pi}{4} \right) = \frac{5i}{\sqrt{2}}. \tag{53}$$

Where we have used that $\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$. Since this is the value of the only residue inside our contour we can use it to calculate the integral

$$I = 2\pi i \operatorname{Res}(4) = 2\pi i \frac{5i}{\sqrt{2}} = -5\pi\sqrt{2} \tag{54}$$

Problem 2: Variational calculus

Fermat's principle states that light travels through media with an index of refraction n such that

$$P = \int n \, \mathrm{d}s \tag{55}$$

is stationary, where stationary means that the value of the integral is minimized when the integrationlimits are fixed. In this expression $\mathrm{d}s$ is an infinitesimal line element, and as we are studying the path of light in a two dimensional plane it is defined as

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + \left(\mathrm{d}y\right)^2}. \tag{56}$$

We will in this problem assume that the index of refraction is only a function of y-coordinate, $n = n(y)$.

a)

To find the integrand K which makes the integral stationary we will insert the integrand into the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}K}{\mathrm{d}\dot{q}}\right) - \frac{\mathrm{d}K}{\mathrm{d}q} = 0, \tag{57}$$

and solve for one of the coordinates. In this expression t is the integration-variable, and q is the function variable of n, and $\dot{q}$ is the derivative of the q with respect to the integration-variable t. With the definition of $\mathrm{d}s$ (56) we can either choose to integrate over x or y, depending on which one we factorize out of the square root. One option is favorable over the other. We want to choose the integration variable such that one term in the Euler-Lagrange equation (59) dissapears. Since the index of refraction is only a function of y we see that the right most term in the Euler-Lagrange equation (59) will dissapear if $q = x$. We will therefore factorize out the $\mathrm{d}y^2$ in $\mathrm{d}s$ such that the integrand is only a function of y and $\dot{x}$, where $\dot{x} = \mathrm{d}x/\mathrm{d}y$. Thus we rewrite our integral to the form

$$P = \int n(y)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y = \int \underbrace{n(y)\sqrt{1 + \dot{x}^2}}_{F}\,\mathrm{d}y. \tag{58}$$

We put this expression for F into the Euler-Lagrange equation (59), now with y instead of t and x instead of q and see that the secound term dissapears

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}}{\mathrm{d}\dot{x}}\left(n(y)\sqrt{1+\dot{x}^2}\right)\right)-\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(n(y)\sqrt{1+\dot{x}^2}\right)}_{0}=0. \tag{59}$$

Our equation is thereby simplified down to

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}}{\mathrm{d}\dot{x}}\left(n(y)\sqrt{1+\dot{x}^2}\right)\right)=0. \tag{60}$$

When the derivative of a function is zero we know that the function must be equal to a constant, which we will call K, we can therefore rewrite the equation to

$$\frac{\mathrm{d}}{\mathrm{d}\dot{x}}\left(n(y)\sqrt{1+\dot{x}^2}\right)=K, \tag{61}$$

which we want to solve for $\dot{x}$. To do this we begin by calculating the derivative using the chain rule

$$\frac{n(y)\dot{x}}{\sqrt{1+\dot{x}^2}}=K. \tag{62}$$

We devide with the index of refraction before squaring on both sides

$$\frac{\dot{x}^2}{1+\dot{x}^2}=\frac{K^2}{n^2}. \tag{63}$$

We divide the the numerator and denominator on the left hand side with $\dot{x}^2$ to only get one term containing $\dot{x}$

$$\frac{1}{1+\dot{x}^{-2}}=\frac{K^2}{n^2}. \tag{64}$$

We flip each fraction and subtract one from each side singeling out $\dot{x}^2$

$$\frac{1}{\dot{x}^2}=\frac{n^2}{K^2}-1=\frac{n^2-K^2}{K^2}. \tag{65}$$

We flip the fractions back and the square root on both sides leaving us with an expression for $\dot{x}$

$$\dot{x}=\pm\frac{K}{\sqrt{n^2-K^2}}. \tag{66}$$

By knowing the derivative of the x-coordinate of the light ray as a function of the index of refraction we can determine the path of the light given initial conditions.

b)

We will try and solve equation (66) for $y(x)$ to explain the path of light close to the hot asphalt on a warm day. The closer the air is to the hot asphalt, the hotter the air is. When the air is hotter it becomes less dense, and thereby effecting the index of refraction. We will model this by linearizing the index of refraction

$$n\left(y\right) = n_0\left(1 + \gamma y\right), \tag{67}$$

where γ is a positive with units of one over length. Our coordinate system is set such that $y = 0$ corresponds to the ground. From this we see that the index of refraction is equal to n_0 on the ground, and increases the further away from the ground you are. We will use this linearization for the index of refraction and solve for the path of the light, starting from (66)

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{K}{\sqrt{n_0^2\left(1 + \gamma y\right)^2 - K^2}}. \tag{68}$$

We multiply both sides with $\mathrm{d}y$ and divide with K upstairs and downstairs on the right hand side

$$\mathrm{d}x = \frac{\mathrm{d}y}{\sqrt{\left(n_0\left(1 + \gamma y\right)/K\right)^2 - 1}}. \tag{69}$$

Now we only have y dependence on the right hand side, and x dependence on the left hand side. We can therefore take the integral on each side

$$\int \mathrm{d}x = \int \frac{1}{\sqrt{\left(n_0\left(1 + \gamma y\right)/K\right)^2 - 1}}\,\mathrm{d}y. \tag{70}$$

The left most integral is simply x, while to the left integral we will begin by substituting the ugly expression in the denominator

$$\frac{n_0\left(1 + \gamma y\right)}{K} = u \to \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{n_0\gamma}{K} \to \mathrm{d}y = \frac{K}{n_0\gamma}\,\mathrm{d}u. \tag{71}$$

We put in for u and $\mathrm{d}u$ in the integral, thus becoming

$$x = \int \frac{1}{\sqrt{u^2 - 1}}\frac{K}{n_0\gamma}\,\mathrm{d}u, \tag{72}$$

Where we can move $K/n_0\gamma$ outside the integral as they are constants, and move them other to the other side of the equals sign. The integral we are left with is therefore simplified down to integral

$$\frac{n_0\gamma x}{K} = \int \frac{1}{\sqrt{u^2 - 1}}\,\mathrm{d}u. \tag{73}$$

which we can solve. To simplify we multiply the constants over to the other side