2.1) a)
$$f(z) = \frac{-i+2z}{2+iz} = \frac{-i+2(x+iy)}{2+i(x+iy)}$$

$$= \frac{-i+2iy+2x}{2-y+ix}$$

$$= \frac{(2x+2iy-i)(2-y-ix)}{(2-y+ix)(2-y-ix)}$$

$$= \frac{4x-\cancel{2xy}-2ix^2+4iy-2iy^2+\cancel{2xy}-2i+iy-x}{4-2y\cancel{-2ix}-2y+y^2\cancel{+ixy}\cancel{+2ix}\cancel{-ixy}+x^2}$$

$$= \frac{3x+i(-2x^2+4y-2y^2-2+y)}{4-4y+x^2+y^2}$$

$$= \frac{3x}{x^2+(y-2)^2} + \frac{i}{x^2+(y-2)^2}\left(-2x^2-2y^2+5y-2\right)$$

Which means that

$$u(x,y) = \frac{3x}{x^2+(y-2)^2}, \quad \text{and}$$

$$v(x,y) = \frac{-2x^2-2y^2+5y-2}{x^2+(y-2)^2}$$

2.1) b) $f(z) = e^{iz} = e^{i(x+iy)} = e^{ix} \cdot e^{-y}$

$$\underline{= e^{-y} \left(\cos(x) + i \sin(x) \right)}$$

$u(x,y) = \cos(x) \cdot e^{-y}$

$v(x,y) = \sin(x) \cdot e^{-y}$

We know this since sin, cos and e, with real argument, is a real number.

2.2) a) $u(x,y) = \dfrac{y}{(1-x)^2+y^2}$

$$\frac{\partial u}{\partial x} = y \frac{\partial}{\partial x}\left(\left[(1-x)^2+y^2\right]^{-1} \right)$$

$$= -\frac{y}{\left[(1-x)^2+y^2\right]^2} \cdot (2(1-x)) \cdot (-1)$$

$$= \frac{2y(-x+1)}{\left[(1-x)^2+y^2\right]^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2y \frac{\partial}{\partial x}\left((1-x)\left([1-x]^2+y^2 \right)^{-2} \right)$$

$$= \frac{2y \cdot (-1)}{\left([1-x]^2+y^2\right)^2} + \frac{2y(1-x)}{\left([1-x]^2+y^2\right)^3} \cdot (-2) \cdot 2 \cdot (1-x) \cdot (-1)$$

$$= \frac{-2y}{\left([1-x]^2+y^2\right)^2} + \frac{8y(1-x)^2}{\left([1-x]^2+y^2\right)^3}$$

$$= \frac{8y(1-x)^2 - 2y(1-x)^2 - 2y^3}{\left([1-x]^2+y^2\right)^3}$$

$$= \frac{6y(1-x)^2 - 2y^3}{\left([1-x]^2+y^2\right)^3}$$

And we do the same for y

2.2)

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{(1-x)^2+y^2}\right)$$

$$= \frac{1}{(1-x)^2+y^2} + \frac{y}{\left((1-x)^2+y^2\right)^2}\cdot(-1)\cdot 2y$$

$$= \frac{(1-x)^2+y^2}{\left((1-x)^2+y^2\right)^2} \cancel{+} - \frac{2y^2}{\left((1-x)^2+y^2\right)^2}$$

$$= \frac{(1-x)^2-y^2}{\left((1-x)^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2y}{\left((1-x)^2+y^2\right)^2} + \frac{(1-x)^2-y^2}{\left((1-x)^2+y^2\right)^3}(-2)\cdot 2y$$

$$= \frac{(-2y)\left((1-x)^2+y^2\right) - 4y\left((1-x)^2-y^2\right)}{\left((1-x)^2+y^2\right)^3}$$

$$= \frac{-2y\left((1-x)^2+y^2+2(1-x)^2-2y^2\right)}{\left((1-x)^2+y^2\right)^3}$$

$$= \frac{-6y(1-x)^2+2y^3}{\left((1-x)^2+y^2\right)^3}$$

which we see is $-\frac{\partial^2 u}{\partial x^2}$. It will therefore satisfy $\nabla^2 u = 0$.

2.2) b) The Cauchy – Riemann equations say that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

We know from a) that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2y(1-x)}{((1-x)^2+y^2)^2}$$ and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{(1-x)^2-y^2}{((1-x)^2+y^2)^2}$$

We take the first equation and integrate both sides.

$$\int \frac{\partial v}{\partial y}\,dy = 2(1-x)\int \frac{y}{((1-x)^2+y^2)^2}\,dy$$

$$\boxed{\begin{aligned} u &= (1-x)^2+y^2 \\ \frac{du}{dy} &= 2y \\ dy &= \frac{du}{2y} \end{aligned}}$$

$$v(x,y) = 2(1-x)\int \frac{y}{u^2}\,\frac{du}{2y}$$

$$= \cancel{2}(1-x)\int \frac{1}{u^2}\,du$$

$$= -(1-x)\,\frac{1}{u} + C(x)$$

$$= \frac{(x-1)}{(1-x)^2+y^2} + C(x)$$

where the integration constant can be a function of x.

We then try to solve the second equation.

2.2 b) $-\frac{\partial V}{\partial x} = \frac{(1-x)^2 - y^2}{((1-x)^2+y^2)^2}$

$$-\int \frac{\partial V}{\partial x}\, dx = \int \frac{(1-x)^2 - y^2}{((1-x)^2+y^2)^2}\, dx$$

$u = (1-x)$

$\frac{du}{dx} = -1$

$dx = -du$

$$V(x,y) = \int \frac{u^2 - y^2}{(u^2+y^2)^2}\, du$$

$$= \int \frac{u^2+y^2-y^2-y^2}{(u^2+y^2)^2}\, du$$

$$= \int \frac{u^2+y^2}{(u^2+y^2)^2} - \frac{2y^2}{(u^2+y^2)^2}\, du = \int \frac{1}{u^2+y^2}\, du - 2y^2 \int \frac{1}{(u^2+y^2)^2}\, du$$

$$= \frac{1}{y^2}\int \frac{1}{1+\frac{u^2}{y^2}}\, du - 2y^2 \int \frac{y \cdot \frac{1}{\cos^2(z)}\, dz}{(y^2+y^2\tan^2(z))^2}$$

$\cancel{z = y\tan(u)}$

$\cancel{\frac{dz}{du} = \sec^2(u)}$

$\cancel{du = \frac{dz}{\sec^2(u)}}$

$$= \frac{1}{y^2}\int \frac{1}{1+(\frac{u}{y})^2}\, du - \frac{2y^2}{y^4}\cdot y \int \frac{dz}{\cos^2(z)(1+\tan^2(z))^2}$$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right) + C(x) - \frac{2}{y}\int \frac{dz}{\cos^2(z)\frac{1}{\cos^4(z)}}$$

$u = y\tan(z)$

$\frac{du}{dz} = y\frac{1}{\cos^2(z)}$

$du = \frac{y \cdot dz}{\cos^2(z)}$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right) - \frac{2}{y}\int \cos^2(z)\, dz$$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right) - \frac{1}{y}\left(z + \sin(z)\cdot\cos(z)\right) + C(y)$$

$z = \arctan\left(\frac{u}{y}\right)$

$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$

$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$

$$= \cancel{\frac{1}{y}\arctan\left(\frac{u}{y}\right)} - \cancel{\frac{1}{y}\arctan\left(\frac{u}{y}\right)} - \frac{(\frac{u}{y})}{y(\frac{u^2}{y^2}+1)} + C(y)$$

$$= -\frac{u}{(u^2+y^2)} + C(y) = -\frac{(1-x)}{((1-x)^2+y^2)} + C(y)$$

$$= \frac{x-1}{(1-x)^2+y^2} + C(y).$$

We get the same answer, which we should and find that the integration constant must be zero.

2.25)

$$f(x,y) = \frac{y + i(x-1)}{(x-1)^2 + y^2} = \frac{\frac{z-\bar z}{2i} + i\left(\frac{z+\bar z}{2}\right) - i}{x^2+1-2x + y^2}$$

$$\boxed{\begin{aligned} x &= \frac{z+\bar z}{2} \\ y &= \frac{z-\bar z}{2i} \end{aligned}}$$

$$= \frac{-\frac{1}{2}i(z-\bar z) + \frac{i}{2}(z+\bar z) - i}{\left(\frac{z+\bar z}{2}\right)^2 + \left(\frac{z-\bar z}{2i}\right)^2 - 2\left(\frac{z+\bar z}{2}\right) + 1}$$

$$= \frac{\frac{i}{2}(z+\bar z - z + \bar z) - i}{\frac{1}{4}(z^2+\bar z^2 + 2z\bar z) - \left(\frac{z^2+\bar z^2 - 2z\bar z}{4}\right) - z - \bar z + 1}$$

$$= \frac{i\bar z - i}{\frac{1}{4}(\cancel{z^2} + \cancel{\bar z^2} + 2z\bar z - \cancel{z^2} - \cancel{\bar z^2} + 2z\bar z) - z - \bar z + 1}$$

$$= \frac{i(\bar z - 1)}{z\bar z - z - \bar z + 1}$$

$$= \frac{i(\bar z - 1)}{(1-z)(1-\bar z)} = \frac{-i\cancel{(1-\bar z)}}{(1-z)\cancel{(1-\bar z)}} = \frac{-i}{(1-z)}$$

$$= \frac{1}{i(1-z)}$$

2.2 c) We want to check that

$$V(x,y) = \frac{(x-1)}{(x-1)^2+y^2}$$ satisfy laplace's

equation, which it should. Let's begin

$$\frac{\partial}{\partial y}\left(\frac{(x-1)}{(x-1)^2+y^2}\right) = a \frac{1}{(a^2+y^2)^2} \cdot (-1) \cdot 2y \quad , \quad a = (x-1)$$

$$= \frac{-2ay}{(a^2+y^2)^2}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{\partial}{\partial y}\left(-\frac{2ay}{(a^2+y^2)^2}\right) = -\frac{2a}{(a^2+y^2)^2}$$

$$-\frac{2ay}{(a^2+y^2)^3} \cdot (-2) \cdot 2y$$

$$= \frac{8ay^2}{(a^2+y^2)^3} - \frac{2a}{(a^2+y^2)^2} = \underline{\frac{6ay^2-2a^3}{(a^2+y^2)^3}}$$

And we do the same for x

$$\boxed{\frac{\partial a}{\partial x} = \frac{\partial}{\partial x}(x-1) = 1}$$

$$\frac{\partial}{\partial x}\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{\partial a}{\partial x}\frac{\partial}{\partial a}\left(\frac{a}{a^2+y^2}\right)$$

$$= \frac{1}{a^2+y^2} + \frac{a}{(a^2+y^2)^2} \cdot (-1) \cdot 2a$$

$$= \frac{1}{a^2+y^2} - \frac{2a^2}{(a^2+y^2)^2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{\partial a}{\partial x}\frac{\partial}{\partial a}\left(\frac{1}{a^2+y^2} - \frac{2a^2}{(a^2+y^2)^2}\right)$$

$$= \frac{1}{(a^2+y^2)^2} \cdot (-1) \cdot 2a - \frac{4a}{(a^2+y^2)^2} - \frac{2a^2}{(a^2+y^2)^3} \cdot (-2) \cdot (2a)$$

$$= \frac{-2a}{(a^2+y^2)^2} - \frac{4a}{(a^2+y^2)^2} + \frac{8a^3}{(a^2+y^2)^3} = \frac{8a^3 - 6a(a^2+y^2)}{(a^2+y^2)^3}$$

$$= \underline{\frac{2a^3-6ay^2}{(a^2+y^2)^3}}$$

As we found the same expression, but with opposite sign, this satisfies laplaces-equation

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2 V}{\partial y^2} = 0$$

2.3) Cauchy's theorem states that the integral over a closed curve for an analytic function on a simply connected domain is allways zero.

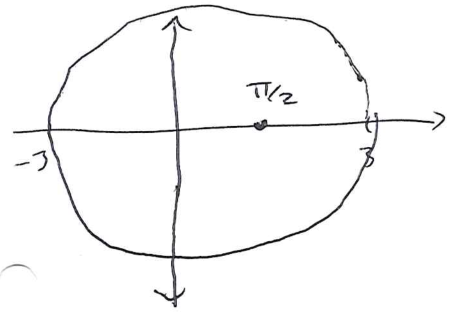
We therefore need to see if the ~~[illegible]~~ functions are analytic inside the closed curves.

a) $\oint_\Gamma \frac{\sin(z)}{2z-\pi}\,dz$, where we diverge at $2z-\pi=0$, $\Rightarrow z=\frac{\pi}{2}$.

Since $\sin(\frac{\pi}{2})=1$, the function is not analytic at that point.

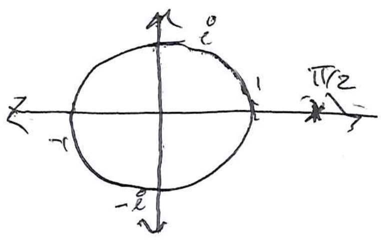
$z=\frac{\pi}{2}$ is within Γ
Γ = circle with radius 3

We see that there are no other singularities within Γ $\Longrightarrow$ $\oint_\Gamma \frac{\sin(z)}{2z-\pi}\,dz = \underline{\underline{2\pi i}}$



b) The same integral $\oint_\Gamma \frac{\sin(z)}{2z-\pi}\,dz$, but now with a curve Γ being a circle with radius 1. Like before, the function is not analytic for $z=\frac{\pi}{2}$ only. ~~[illegible]~~

~~[illegible]~~ Since the radius is 1, and centerd at the origin, and $\frac{\pi}{2}>1$, the poin is outside $\Longrightarrow$ $\int_\Gamma \frac{\sin(z)}{2z-\pi}\,dz = \underline{\underline{0}}$



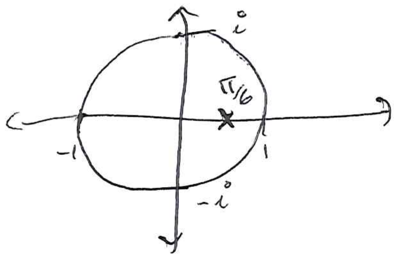
2.3) c) We look at the same circle, but now with another function

$\oint_\Gamma \frac{\sin(2z)}{6z-\pi}$, this has a singularity at $6z-\pi=0 \Rightarrow z=\frac{\pi}{6}$

$\sin\left(\frac{\pi}{3}\right)=\sqrt{\frac{3}{4}}$ (does not cancel the zero)

Since $z=\frac{\pi}{6}\approx 0{,}52$, the singularity is inside the closed curve Γ, we get

$\oint_\Gamma \frac{\sin(2z)}{6z-\pi}\,dz = \underline{\underline{2\pi i}}$, since this is the only singularity:



d) We know look at a Γ defined as a square $\pm 2, \pm 2i$, and an integral:

$\oint_\Gamma \frac{e^{2z}}{z-\ln(2)}\,dz$. This has the singularity at $z-\ln(2)=0 \Rightarrow \underline{z=\ln(2)}$

As the numerator $e^{2\cdot\ln(2)}=4$ is not $\emptyset$, the function has a singularity at $z=\ln(2)\approx 0{,}693$ which inside the square. This is the only non-analytic point in the square, which means

$\oint_\Gamma \frac{e^{2z}}{z-\ln(2)}\,dz = \underline{2\pi i}$