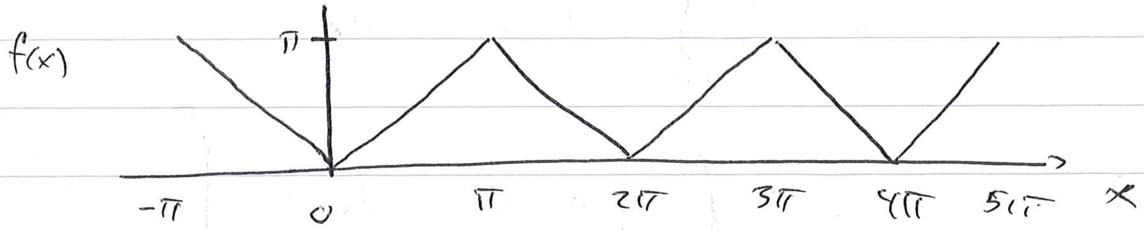
2a) $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$



Find fourier-coeffs

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx,$$ since $f(x)$ is symmetric and $\cos(nx)$ is symmetric

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)$$

$$= \frac{2}{\pi}\left(\left[\frac{x\cdot\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right)$$

$$\boxed{\begin{array}{ll} \int f'\cdot g\,dx = f\cdot g - \int g'\cdot f\,dx & \\ f' = \cos(nx) & g = x \\ f = \frac{1}{n}\sin(nx) & g' = 1 \end{array}}$$

$$= \frac{2}{\pi}\left(\underbrace{\frac{\pi\sin(n\pi)}{n}}_{0} - 0 + \left[\frac{\cos(nx)}{n^2}\right]_0^{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{\cos(n\pi)-1}{n^2}\right) = \begin{cases} -\frac{4}{\pi n^2} & \text{odd } n \\ 0 & \text{even } n \end{cases}$$

We find a_0 explicitly:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(0)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[x^2\right]_0^{\pi}$$

$$= \underline{\pi}$$

Since $f(x)$ is symmetric we wont get sine contribution

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{9} + \frac{\cos(5x)}{25} + \dots\right)$$

2b) Same function $f(x) = \begin{cases} x & x \in [0, \pi] \\ -x & x \in [-\pi, 0] \end{cases}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left(-\int_{-\pi}^{0} x\, e^{-inx}\, dx + \int_{0}^{\pi} x\, e^{inx}\, dx\right)$$

$$= \frac{1}{2\pi}\left(-\left\{\left[\frac{x \cdot e^{-inx}}{-in}\right]_0^{\pi} - \int_0^{\pi} \frac{e^{-inx}}{-in}\, dx\right\} + \left\{\left[\frac{x\, e^{-inx}}{-in}\right]_0^{\pi} - \int_0^{\pi} \frac{e^{-inx}}{-in}\, dx\right\}\right)$$

$$\boxed{\begin{array}{ll} \int f \cdot g' = f \cdot g - \int f' \cdot g & \\ f = x & g' = e^{-inx} \\ f' = 1 & g = \frac{e^{-inx}}{-in} \end{array}}$$

$$= \frac{1}{2\pi}\left(-\cancel{\frac{\pi e^{-in\pi}}{-in}} + \frac{1}{n^2}\left(e^{-in\pi} - e^{in0}\right) + \cancel{\frac{\pi e^{-in\pi}}{-in}} + \frac{1}{n^2}\left(e^{-in\pi} - e^{0}\right)\right)$$

$$= \frac{1}{2\pi n^2}\left(2e^{-in\pi} - 2\right) = \frac{1}{\pi n^2}\left(e^{-in\pi} - 1\right)$$

$$= \begin{cases} -\frac{2}{\pi n^2} & \text{odd } n \\ 0 & \text{even } n \end{cases}$$

differ by a factor of 2, due to negative n. The sine terms will cancel due to n^2.

$$c_n \sin(nx) + c_{-n}\sin(-nx) = \sin(nx)\left(c_n - c_{-n}\right) = \sin(nx)\left(\frac{-2}{\pi n^2} - \frac{-2}{\pi n^2}\right)$$

$= 0. \Rightarrow$ No sine-terms

$$c_n \cos(nx) + c_{-n}\cos(-nx) = \cos(nx)\left(c_n + c_{-n}\right) = \cos(nx)\left(-\frac{2}{\pi n^2} - \frac{2}{\pi n^2}\right)$$

$= -\frac{4}{\pi n^2}\cos(nx)$ ✓ same as previous result.

$c_0 = a_0$ from previous task due to $c_0 = \int_{-\pi}^{\pi} f(x) = a_0$

$\Rightarrow \frac{a_0}{2} = \frac{\pi}{2} \Rightarrow$ Same series!

2d) $f(x) = x^2$ in $x \in [-\frac{1}{2}, \frac{1}{2}]$

Since $f(x) = f(-x)$ this is an even function Therefore we know that the coefficients are

$$f(x) = \begin{cases} a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx \\ b_n = 0 \end{cases}$$

where $\ell = \frac{1}{2}$, let's calculate a_n

$$a_n = \frac{2}{1/2} \int_0^{1/2} x^2 \cos\left(\frac{n\pi x}{1/2}\right) dx$$

$$= 4 \int_0^{1/2} x^2 \cos(2n\pi x)\, dx$$

Solve this through integration by parts:

$$\boxed{\int f \cdot g' = f \cdot g - \int f' \cdot g}, \quad \boxed{f = x^2, \ f' = 2x, \quad g' = \cos(2n\pi x), \ g = \frac{\sin(2n\pi x)}{2n\pi}}$$

$$\int_0^{1/2} x^2 \cos(2n\pi x)\, dx = x^2 \cdot \frac{\sin(2n\pi x)}{2n\pi}\Bigg|_0^{1/2} - \int_0^{1/2} 2x \cdot \frac{\sin(2n\pi x)}{2n\pi}\, dx$$

$$= \underbrace{\frac{\frac{1}{4}\sin(n\pi)}{2n\pi}}_{0} - 0 - \frac{1}{n\pi} \int_0^{1/2} x \cdot \sin(2n\pi x)\, dx$$

$f = x, \ f' = 1$
$g' = \sin(2n\pi x),$
$g = -\frac{\cos(2n\pi x)}{2n\pi}$

$$= -\frac{1}{n\pi}\left(-x \cdot \frac{\cos(2n\pi x)}{2n\pi}\Bigg|_0^{1/2} - \int_0^{1/2} -\frac{\cos(2n\pi x)}{2n\pi}\, dx \right) = -\frac{1}{n\pi}\left(-\frac{1}{2}\frac{\cos(n\pi)}{2n\pi} + \frac{1}{(2n\pi)^2}\Big[\sin(2n\pi x)\Big]_0^{1/2} \right) = \frac{\cos(n\pi)}{4n^2\pi^2} + 0$$

$$\Rightarrow a_n = \frac{\cos(n\pi)}{n^2\pi^2} \Rightarrow a_n = \begin{cases} \frac{1}{n^2\pi^2} & \text{even } n \\ -\frac{1}{n^2\pi^2} & \text{odd } n \end{cases}$$

We also need to calculate the average value :

$$a_0 = 4 \int_0^{1/2} x^2 \, dx = 4 \cdot \frac{1}{3} \cdot \left[x^3 \right]_0^{1/2}$$

$$= \frac{4}{3} \cdot \frac{1}{8} = \frac{1}{6}$$

Making our series

$$f(x) = \frac{a_0}{2} + \sum_{n=}^{\infty} a_n \cos\left(\frac{n \pi x}{1/2} \right)$$

$$= \frac{1}{12} + \frac{1}{\pi^2} \left(- \cos(2\pi x) + \frac{1}{4} \cos(4\pi x) - \frac{1}{9} \cos(6\pi x) + \ldots \right)$$

1c) $2xy'' - y' + 2y = 0$

$y = \sum_n a_n x^{n+s}$, $y' = \sum_n (n+s) a_n x^{n+s-1}$

$y'' = \sum_n (n+s)(n+s-1) a_n x^{n+s-2}$

insert

$$\sum_n 2(n+s)(n+s-1) a_n x^{n+s-1} - \sum_n (n+s) a_n x^{n+s-1} + \sum_n 2 a_n x^{n+s} = 0$$

Multiply each side with x

$$\sum_n 2(n+s)(n+s-1) a_n x^{n+s} - \sum_n (n+s) a_n x^{n+s} + \sum_n 2 a_n x^{n+s+1} = 0$$

Look at x^s term:

$2s(s-1)a_0 - s a_0 + 0 = 0$ $\quad (a_0 \neq 0)$

$2s^2 - 2s - s = 0$

$s(2s-3) = 0 \Rightarrow s_+ = \frac{3}{2}, s_- = 0$

They do not differ by an integer

$\Rightarrow$ two linearly indep solution

For $s = 0$ we get

$$\sum_n 2n(n-1)a_n x^{n-1} - \sum_n n a_n x^{n-1} + \sum_n 2 a_n x^n = 0$$

We match coefficients

$2(n+1)n a_{n+1} - (n+1) a_{n+1} + 2a_n = 0$

$-2a_n = a_{n+1}(2n^2 + 2n - n - 1)$

$a_n = -\frac{1}{2} a_{n+1}(2n^2 + n - 1)$

$a_n = -a_{n+1}(n - \frac{1}{2})(n+1)$

$$\boxed{a_{n+1} = -\frac{a_n}{(n+1)(n-\frac{1}{2})}}$$

$a_1 = -\frac{a_0}{-\frac{1}{2}} = 2a_0$

$a_2 = -\frac{a_1}{2 \cdot \frac{1}{2}} = -a_1 = -2a_0$

$a_3 = -\frac{a_2}{3 \cdot \frac{3}{2}} = \frac{4a_0}{9}$, $a_4 = -\frac{a_3}{4 \cdot \frac{5}{2}} = -\frac{2a_0}{45}$

2e) We want to solve $\sum_{n=1} \frac{1}{n^4}$ using

Parseval's theorem, which states that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2L}\int_{-L}^{L} |f(x)|^2\,dx$$

In problem 9.9 we found the coefficients for $f(x) = x^2$ in $x\,[-\frac{1}{2}, \frac{1}{2}]$, where we found $a_n = \frac{\cos(n\pi)}{n^2\pi^2}$ and $b_n = 0$ and $a_0 = \frac{1}{12}$, since $\cos(n\pi) = \pm 1$ we get

$$|a_n|^2 = \left(\frac{1}{n^2\pi^2}\right)^2 = \frac{1}{\pi^4 n^4}$$

$$\sum_{n=0}^{\infty} |a_n|^2 = |a_0|^2 + \sum_{n=1}^{\infty} \frac{1}{\pi^4 n^4} = \left|\frac{1}{12}\right|^2 + \frac{1}{\pi^4}\sum_{n=1} \frac{1}{n^2}$$

which has to be equal to

$$\frac{1}{2L}\int_{-L}^{L} x^4\,dx = \frac{1}{2\cdot\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^4\,dx = \frac{1}{5}\left(\left(\frac{1}{2}\right)^5 - \left(-\frac{1}{2}\right)^5\right)$$

$$= \frac{2}{5}\left(\frac{1}{2}\right)^5 = \frac{1}{5}\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{16\cdot 5} = \frac{1}{80}$$

$$\frac{1}{80} = \frac{1}{144} + \frac{1}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi^4\left(\frac{1}{80} - \frac{1}{144}\right) = \frac{\pi^4}{180}$$

This is wrong by a factor of 2, I am not sure why... I tried to solve this ~~~~ using a_n not c_n, and therefore have a sum from ~~~~ to ∞ instead of $-\infty$ to ∞, maybe thats the reason.