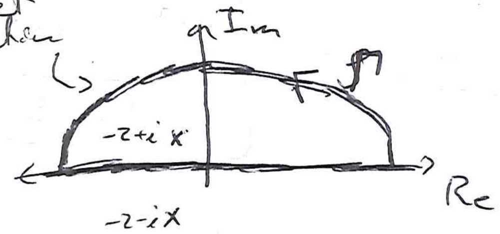
Oblig 5 - Ivar Haugerud - FYS3140

5.1) a) $I = \int_{-\infty}^{\infty} \frac{x \sin(x)}{x^2 + 4x + 5} dx = \mathrm{Im}\left\{ \int_{-\infty}^{\infty} \frac{x e^{ix}}{(x-(-2+i))(x-(-2-i))} dx \right\}$

Since the denominator is of order x^2 and numerator of order $\cancel{x}$ x^1, with an exponential, we know that making the integral into a complex contour integral will not get contribution

$$I = \left\{ \oint_{\Gamma} \frac{z e^{iz}}{(z-(-2+i))(z-(-2-i))} dz \right\}_{imag}$$



We have one singularity inside Γ and can solve the integral by finding the residue

$$\mathrm{Res}(-2+i) = \lim_{z \to -2+i} \left[f(z)\,(z-(-2+i)) \right]$$

$$= \lim_{z \to -2+i} \left[\frac{z e^{iz} \cancel{(z-(-2+i))}}{\cancel{(z-(-2+i))}(z-(-2-i))} \right]$$

$$= \frac{(-2+i)\, e^{i(-2+i)}}{(-2+i+2+i)} = \frac{(-2+i)}{2i} e^{i(-1-2i)}$$

$$= -\frac{i(-2+i)}{2} \cdot \frac{1}{e} \cdot e^{-2i} = \frac{1+2i}{2e}(\cos(2) - i\sin(2))$$

$$I = \mathrm{Im}\{2\pi i \cdot \mathrm{Res}(-2+i)\} = \mathrm{Im}\left\{ \frac{2\pi i(1+2i)}{2e}(\cos(2) - i\sin(2)) \right\}$$

$$= \mathrm{Im}\left\{ \frac{\pi}{e}(i-2)(\cos(2) - i\sin(2)) \right\} = \frac{\pi}{e}(\cos(2) + 2\sin(2))$$

$$\approx \underline{1{,}6208}$$

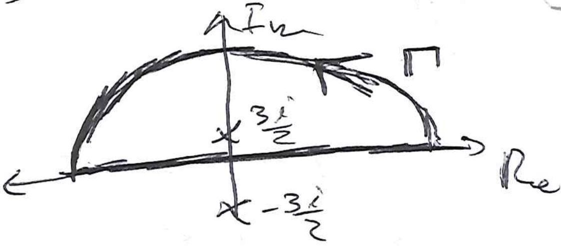
6.1 b) $I = \int_0^\infty \frac{\cos(2x)\,dx}{(4x^2+9)^2}$ Since both cosines and $(4x^2+9)^2$ is symmetric around $x=0$ we can change the integral limits

$$2I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(4x^2+9)^2}\,dx = \mathrm{Re}\left\{\int_{-\infty}^{\infty} \frac{e^{2ix}}{(4x^2+9)^2}\,dx\right\}$$

We find the roots of the polynomial in the denominator

$$2I = \mathrm{Re}\left\{\int_{-\infty}^{\infty} \frac{e^{2ix}}{(x-\frac{3i}{2})^2(x+\frac{3i}{2})^2}\right\}$$

We extend the integral into the complex plane along a contour Γ



Since the denominator is of order 4, and numerator an exponential we see that the complex arc does not contribute to the integral.

The integral is determined by the singularity at $\frac{3i}{2}$ which is a pole of second order. To find it's residue we need to calculate

$$\mathrm{Res}\left(\frac{3i}{2}\right) = \lim_{z\to\frac{3i}{2}} \frac{1}{1!}\frac{d}{dz}\left(\frac{(z-\frac{3}{2}i)^2\, e^{2iz}}{(z-\frac{3}{2}i)^2(z+\frac{3}{2}i)^2}\right)$$

$$= \lim_{z\to\frac{3i}{2}} \frac{d}{dz}\left(\frac{e^{2iz}}{(z+\frac{3i}{2})^2}\right)$$

$$= \lim_{z\to\frac{3i}{2}} \left(\frac{2ie^{2iz}}{(z+3i/2)^2} + \frac{e^{2iz}}{(z+\frac{3i}{2})^3}(-2)\cdot 1\right)$$

$$= \lim_{z\to\frac{3i}{2}} \frac{2e^{2iz}\left(i(z+\frac{3i}{2}) - 1\right)}{(z+\frac{3i}{2})^3} = \frac{2e^{-3}(-3-1)}{(6i)^3} = \frac{e^{-3}}{27}$$

Having found the residue it is easy to calculate the integral

$$2I = \mathrm{Re}\left\{ 2\pi i \cdot \mathrm{Res}\left(\frac{3i}{2}\right)\right\}$$

$$I = \frac{1}{2}\mathrm{Re}\left\{ 2\pi i \cdot \frac{e^{-3}}{27 i}\right\}$$

$$\underline{I = \frac{\pi e^{-3}}{27}}$$

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c) $$I = \int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{1-x^2}\, dx$$

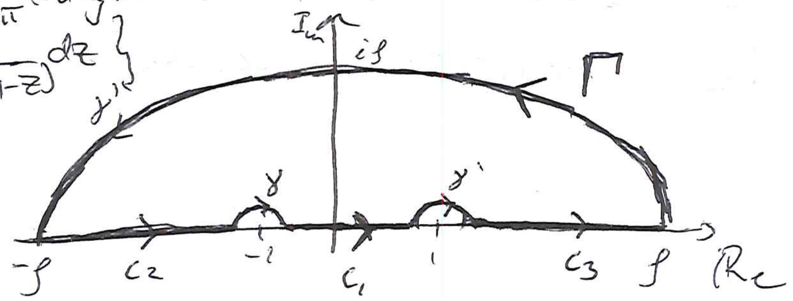
Which has two singularities at ± 1, but we see that $\sin(\pi x)$ also equals zero at $x = \pm 1$

$$= \int_{-\infty}^{\infty} \frac{x \sin(\pi x)\, dx}{(1-x)(1+x)}$$

From jordans Lemma we know that parametrizing a curve in the complex plane would not contribute to the integral

$$I = \oint_{\Gamma} \frac{z \sin(\pi z)\, dz}{(1-z)(1+z)}$$

$$I = \mathrm{Im}\left\{ \oint_{\Gamma} \frac{z e^{iz\pi}}{(1+z)(1-z)}\, dz \right\}$$



We can then use that

$$PV \int_{-\infty}^{\infty} f(x)\, dx = \underbrace{\pi i \sum_n \mathrm{Res}(f, x_n)}_{\text{on real axis}} + \underbrace{2\pi i \sum \mathrm{Res}(f, z_n)}_{\text{in complex plane}}$$

To solve the integral we just need to calculate the residues c)

$$\text{Res}(1) = \lim_{z \to 1} \left[\frac{\cancel{(z-1)} z e^{iz\pi}}{\cancel{(z-1)}(z+1)} \right] = \frac{1 e^{i\pi}}{2} = \frac{e^{i\pi}}{2}$$

$$\text{Res}(-1) = \lim_{z \to -1} \left[\frac{\cancel{(z+1)} z e^{iz\pi}}{(z-1)\cancel{(z+1)}} \right] = \frac{-1 e^{-i\pi}}{-2} = \frac{e^{-i\pi}}{2}$$

The sum of the residues is then

$$\text{Res}(1) + \text{Res}(-1) = \frac{1}{2}\left(e^{i\pi} + e^{-i\pi}\right)$$

$$= \cos(\pi) = 1$$

The integral is then

$$I = \text{Im}\left\{ 2\pi i \left(\text{Res}(-1) + \text{Res}(1) \right) \cdot \frac{1}{2} \right\}$$ since both singularities at real axis

$$= \text{Im}\left\{ \pi i \cdot 1 \right\} = \underline{\underline{\pi}}$$

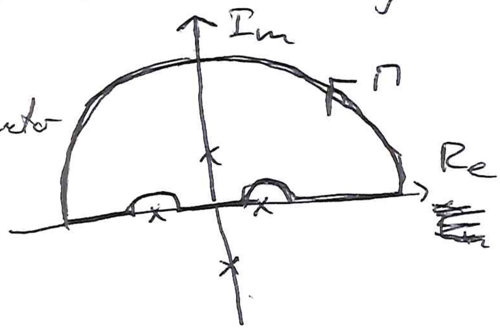
To solve this integral we did not need the principle value.

5.1 d) $I = \int_0^\infty \frac{1}{1-x^4}\,dx$, due to the symmetry of the integrand around $x=0$, we know

$$2I = \int_{-\infty}^{\infty} \frac{1}{1-x^4}\,dx$$ we factorize the denominator

$$= \int_{-\infty}^{\infty} \frac{1}{(x-1)(x+1)(x-i)(x+i)}\,dx$$

We convert this to a closed contour integral in the complex-plane Γ,



as we have $\sim \frac{1}{x^4}$ in the denominator we can ignore the complex arc, and know that the integral can be solved by finding the three residues at $-1, i, +1$, since

$$2I = 2\pi i\left(\tfrac{1}{2}\mathrm{Res}(-1) + \tfrac{1}{2}\mathrm{Res}(1) + \mathrm{Res}(i)\right)$$

Since they are all simple poles we use

$$\mathrm{Res}(-1) = \lim_{z\to -1}\left[\frac{\cancel{(z+1)}}{\cancel{(z+1)}(z-1)(z-i)(z+i)}\right] = \frac{1}{(-2)(-1-i)(-1+i)} = \underline{-\frac{1}{4}}$$

$$\mathrm{Res}(1) = \lim_{z\to 1}\left[\frac{\cancel{(z-1)}}{(z+1)\cancel{(z-1)}(z-i)(z+i)}\right] = \frac{1}{(2)}\cdot\frac{1}{(1-i)(1+i)} = \underline{\frac{1}{4}}$$

$$\mathrm{Res}(i) = \lim_{z\to i}\left[\frac{\cancel{(z-i)}}{(z+1)(z-1)\cancel{(z-i)}(z+i)}\right] = \frac{1}{(i+1)(i-1)2i} = -\frac{1}{4i}$$

$$PV(2I) = 2\pi i\left(\tfrac{1}{2}\left[\tfrac{1}{4} - \tfrac{1}{4}\right] - \frac{1}{4i}\right) = -\frac{\pi}{2}$$

$$\underline{\underline{PV(I) = -\frac{\pi}{4}}}$$ which is the principal value, of the integral

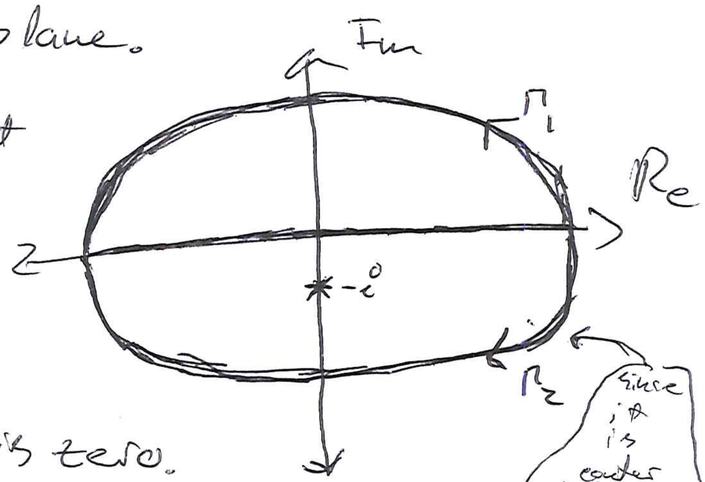
e) $I = \int_{-\infty}^{\infty} \frac{\cos x}{x+i}\, dx$

We can not just take the imaginary part of the integral since we have a complex number in the denominator

$$= \int_{-\infty}^{\infty} \frac{\frac{1}{2}(e^{ix} + e^{-ix})}{x+i}\, dz$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx + \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\, dx$$

Both of these satisfy Jordan's lemma, but have to be folded in different directions on the complex plane.



For the e^{ix} we use Γ_1, but for the e^{-ix} we use Γ_2.

We see imidiatly that Γ_1 contains no singularities, and thus the $\oint$ integral is zero.

The curve Γ_2 contains a simple pole, and we need to find it's residue.

since it is counter clockwise we get a minus

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\, dx = 2\pi i \cdot \lim_{z \to -i} \left[- \frac{(x+i)\, e^{-ix}}{(x+i)} \right]$$

$$= -2\pi i \cdot e^{-i i \cdot (-1)} = \underline{-2\pi e^{-1} i}$$

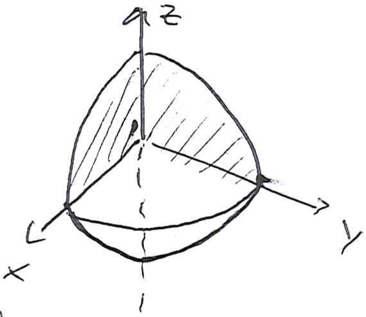
Making the integral

$$I = \underbrace{\frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx}_{0} + \underbrace{\frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\, dx}_{-2\pi i e^{-1}}$$

$\underline{I = -\pi i e^{-1}} \Rightarrow \underline{\underline{I = -\frac{\pi i}{e}}}$

5.2 a) Want to find inertia tensor for sphere with $x>0$, $y>0$, with a radius of 1 and a density of 1. This makes our integration limits in spherical coordinates

$$r \in [0,1], \quad \varphi \in [0, \tfrac{\pi}{2}], \quad \theta \in [0,\pi]$$



We will use

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r\cos\theta,$$

and to calculate the inertia we need to solve an integral on the form

$$I_{xx} = \int_V dv\,(y^2+z^2) = \int_V dv\,(r^2 - x^2) \quad \text{for the diagonals}$$

$$I_{xy} = I_{yx} = \int_V -xy\,dv \quad \text{for the non-diagonals}$$

Let's begin

$$I_{xy} = \int_0^1 r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{\pi/2} d\varphi \cdot (-r^2 \sin^2\theta \cos\varphi \sin\varphi)$$

$$= -\underbrace{\int_0^1 r^4\, dr}_{1/5} \underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{4/3} \underbrace{\int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi}_{1/2}$$

$$= -\frac{1}{5}\cdot\frac{4}{3}\cdot\frac{1}{2} = -\frac{2}{15}$$

$$I_{xz} = \int_0^1 r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{\pi/2} d\varphi\, (-r^2 \sin\theta\cos\theta \cos\varphi)$$

$$= -\int_0^1 r^4\, dr \underbrace{\int_0^\pi \sin^2\theta \cos\theta\, d\theta}_{0} \int_0^{\pi/2} \cos\varphi = 0$$

We see that I_{yz} will also get the factor $\int_0^\pi \sin^2\theta\cos\theta\, d\theta$, and thus will become zero $I_{yz} = 0$

2a) We will now try and calculate the diagonal elements of the matrix

$$I_{xx} = \int_0^1 r^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{\pi/2} d\varphi \; (r^2 - r^2 \sin^2\theta \cos^2\varphi)$$

$$= \underbrace{\int_0^1 r^4 dr}_{1/5} \left(\underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{\pi/2} d\varphi}_{\pi/2} - \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} \underbrace{\int_0^{\pi/2} \cos^2\varphi \, d\varphi}_{\pi/4} \right)$$

$$= \frac{1}{5}\left(\pi - \frac{\cancel{2}\pi}{3} \right) = \underline{\underline{\frac{2\pi}{15}}}$$

$$I_{yy} = \int_0^1 r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{\pi/2} d\varphi \; (r^2 - r^2 \sin^2\theta \sin^2\varphi)$$

$$= \underbrace{\int_0^1 r^4 dr}_{1/5} \left(\underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{\pi/2} d\varphi}_{\pi/2} - \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} \underbrace{\int_0^{\pi/2} \sin^2\varphi \, d\varphi}_{\pi/4} \right)$$

$$= \frac{1}{5}\left(\pi - \frac{\pi}{3} \right) = \underline{\underline{\frac{2\pi}{15}}}$$

$$I_{zz} = \int_0^1 r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{\pi/2} d\varphi \; (r^2 - r^2 \cos^2\theta)$$

$$= \underbrace{\int_0^1 r^4 dr}_{1/5} \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} \underbrace{\int_0^{\pi/2} d\varphi}_{\pi/2}$$

$$= \frac{1}{5} \cdot \frac{4}{3} \cdot \frac{\pi}{2} = \underline{\underline{\frac{2\pi}{15}}}$$

2a) Thus we find that the inertia tensor is

$$\mathbb{I} = \begin{pmatrix} \frac{2\pi}{15} & -\frac{2}{15} & 0 \\ -\frac{2}{15} & \frac{2\pi}{15} & 0 \\ 0 & 0 & \frac{2\pi}{15} \end{pmatrix} = \frac{2}{15}\begin{pmatrix} \pi & -1 & 0 \\ -1 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$$

To find the principle moments of inertia, and the principle axis, we need to calculate the eigenvalues and their corresponding eigenvector

$$\det(\mathbb{I} - \lambda \mathbb{1}) = 0$$

$$\det\begin{pmatrix} \frac{2\pi}{15}-\lambda & -\frac{2}{15} & 0 \\ -\frac{2}{15} & \frac{2\pi}{15}-\lambda & 0 \\ 0 & 0 & \frac{2\pi}{15}-\lambda \end{pmatrix} = \left(\frac{2\pi}{15}-\lambda\right)\cdot\left(\frac{2\pi}{15}-\lambda\right)\left(\frac{2\pi}{15}-\lambda\right) + \frac{2}{15}\left(-\frac{2}{15}\left(\frac{2\pi}{15}-\lambda\right)\right) = 0$$

$$\left(\frac{2\pi}{15}-\lambda\right)^3 - \frac{4}{225}\left(\frac{2\pi}{15}-\lambda\right) = 0$$

$$\left(\frac{2\pi}{15}-\lambda\right)\left[\left(\frac{2\pi}{15}-\lambda\right)^2 - \frac{4}{225}\right] = 0$$

$$\left(\frac{2\pi}{15}-\lambda\right)\left(\left[\frac{2\pi}{15}-\lambda-\frac{\cancel{4}2}{\sqrt{225}}\right]\left[\frac{2\pi}{15}-\lambda+\frac{2}{\sqrt{225}}\right]\right) = 0$$

$$\left(\frac{2\pi}{15}-\lambda\right)\left(\frac{2}{15}(\pi-1)-\lambda\right)\left(\frac{2}{15}(\pi+1)-\lambda\right) = 0$$

And we see that the three eigenvalues are distinct and equal to

$$\lambda_1 = \frac{2\pi}{15}, \quad \lambda_2 = \frac{2}{15}(\pi-1), \quad \lambda_3 = \frac{2}{15}(\pi+1)$$

2a) We try and find their corresponding eigenvectors by solving

$$\underline{I}\,\vec{x_i} = \lambda_i \vec{x_i}$$

We start with $\underline{\lambda_1:}$

$$\frac{2}{15}\begin{pmatrix} \pi & -1 & 0 \\ -1 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2\pi}{15}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left.\begin{aligned} \pi x - y &= \pi x \\ -x + \pi y &= \pi y \end{aligned}\right\} \text{These imply } x = y = 0 \Rightarrow \underline{\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\pi z = \pi z$$

$\underline{\lambda_2:}$

$$\frac{2}{15}\begin{pmatrix} \pi & -1 & 0 \\ -1 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2}{15}(\pi - 1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\pi x - y = (\pi - 1)x \Rightarrow \boxed{-y = -x}$$

$$-x + \pi y = (\pi - 1)y$$

$$\pi z = (\pi - 1)z \Rightarrow \boxed{z = 0}$$

$$\Rightarrow \underline{\underline{\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}}$$

$\underline{\lambda_3:}$

$$\frac{2}{15}\begin{pmatrix} \pi & -1 & 0 \\ -1 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2}{15}(\pi + 1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\pi x - y = (\pi + 1)x \Rightarrow \boxed{-y = x}$$

$$-x + \pi y = (\pi + 1)y \Rightarrow \boxed{-x = y}$$

$$\pi z = (\pi + 1)z \Rightarrow \boxed{z = 0}$$

$$\Rightarrow \underline{\underline{\vec{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}}$$

And we have now found the principle axis of rotation, and their corresponding principle moments of inertia.

2 b) To find the moments of inertia we use $\vec{x}_1 = (1, 1, -2)$ and $\vec{x}_2 = (1, 1, 1)$ with corresponding mass $m_1 = 1$, and $m_2 = 2$

Since we have discrete values we calculate the inertia with a sum

$$I_{xx} = \sum_{i=1}^{2} m_i (y_i^2 + z_i^2) = 1(1^2 + 2^2) + 2(1^2 + 1^2)$$
$$= 5 + 4 = \underline{9}$$

$$I_{yy} = \sum_{i=1}^{2} m_i (x_i^2 + z_i^2) = 1(1^2 + (-2)^2) + 2(1^2 + 1^2)$$
$$= 5 + 4 = \underline{9}$$

$$I_{zz} = \sum_{i=1}^{2} m_i (x_i^2 + \cancel{z}_i^2) = 1(1^2 + 1^2) + 2(1^2 + 1^2)$$
$$= 2 + 4 = \underline{6}$$

$$I_{xy} = -\sum_i m_i x_i y_i = -1(1 \cdot 1) - 2(1 \cdot 1)$$
$$= -1 - 2 = \underline{-3}$$

$$I_{xz} = -\sum m_i x_i z_i = -1(1 \cdot (-2)) - 2(1 \cdot 1)$$
$$= 2 - 2 = \underline{0}$$

$$I_{yz} = -\sum_i m_i y_i z_i = -1(1 \cdot (-2)) - 2 \cdot (1 \cdot 1)$$
$$= 2 - 2 = \underline{0}$$

Making the tensor

$$I = \begin{pmatrix} 9 & -3 & 0 \\ -3 & 9 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 3 \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We try and find it's eigenvalues

$$\det(I - \lambda \mathbb{1}) = \det\left(3\begin{pmatrix} 3-\frac{\lambda}{3} & -1 & 0 \\ -1 & 3-\frac{\lambda}{3} & 0 \\ 0 & 0 & 2-\frac{\lambda}{3} \end{pmatrix}\right) \qquad u = \frac{\lambda}{3}$$

$$= 3\det\begin{pmatrix} 3-u & -1 & 0 \\ -1 & 3-u & 0 \\ 0 & 0 & 2-u \end{pmatrix} = 0$$

$$= (3-u)(3-u)(2-u) + 1\left(-1(2-u)\right) = 0$$

$$= (3-u)^2(2-u) - (2-u) = 0$$

$$= (2-u)\left([3-u]^2 - 1\right) = 0$$

$$= (2-u)\left([u-2][u-4]\right) = 0$$

$$= (u-2)^2(u-4) = 0 \qquad u_2 = u_1 = 2,\ u_2 = 4$$

$$\Rightarrow \boxed{\lambda_1 = 6 = \lambda_2,\ \lambda_3 = 12}$$

We calculate the corresponding eigenvectors

$\lambda_1 = 6$: $3\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\left.\begin{aligned} 3x - y &= 2x \quad \Rightarrow x = y \\ -x + 3y &= 2y \quad \Rightarrow x = y \\ 2z &= 2z \quad \Rightarrow z = \text{free} \end{aligned}\right\} \rightarrow \vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = 12$: $3\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 12\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\left.\begin{aligned} 3x - y &= 4x \quad \Rightarrow x = -y \\ -x + 3y &= 4y \quad \Rightarrow x = -y \\ 2z &= 4z \quad \Rightarrow z = 0 \end{aligned}\right\} \rightarrow \vec{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

And we have found the principle values of inertia and axes of rotation.