Oblig 9 - FYS3140 - Ivar Haugerud

9.1) $y'' + P(x)y' + Q(x)y = Dy = R(x)$, where

$$D = \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\frac{d}{dx}$$

Write this in Dirac notation

$$D|y\rangle = |R\rangle$$

identity matrix $= \int dz\, |z\rangle\langle z|$

$$|y\rangle = D^{-1}|R\rangle = D^{-1}\,\mathbb{1}\,|R\rangle + \sum c_i|h_i\rangle$$

$$|y\rangle = \int dz\; D^{-1}|z\rangle\langle z|R\rangle + \sum c_i|h_i\rangle$$

$$\langle x|y\rangle = \int dz\; \langle x|D^{-1}|z\rangle\langle z|R\rangle + \sum c_i\langle x|h_i\rangle$$

where $|h_i\rangle$ is the homogeneous solution with property $D|h_i\rangle = 0$

We define $\langle x|D^{-1}|z\rangle = G(x,z)$ as our Greens function, while $\langle x|y\rangle = y(x)$ and $\langle z|R\rangle = R(z)$, giving us

$$y(x) = \int dz\; G(x,z)\, R(z)\, dz + \sum c_i\, y_h(x)$$

This is the integral we have to solve to find the complete solution, but first we must find the Greens function

From $D|y\rangle = |R\rangle \Rightarrow |y\rangle = D^{-1}\mathbb{1}|R\rangle + \sum c_i|h_i\rangle$

$$D|y\rangle = DD^{-1}\,\mathbb{1}\,|R\rangle + \sum c_i \underbrace{D|h_i\rangle}_{0} = DD^{-1}\,\mathbb{1}\,|R\rangle$$

$$|R\rangle = DD^{-1}\,\mathbb{1}\,|R\rangle$$

$$\langle x|R\rangle = \int dz \langle x|DD^{-1}|z\rangle\langle z|R\rangle = \int dz\,\langle x|D|G\rangle\langle z|R\rangle$$

For this equality to hold $\langle x|D|G\rangle = \langle x|z\rangle = \delta(x-z)$

This means that $D G(x,z) = 0$ for $x < z$ and $D G(x,z) = 0$ for $x > z$. We also need a continuity of the greens-function at $x = z$, and it can be shown that it has to have a discontinuity in it's derivative $\left.\frac{dG}{dx}\right|_{x=z_+} - \left.\frac{dG}{dx}\right|_{x=z_-} = 1$

We will also need the boundary-conditions of the two independent solutions of the homogeneous equation ($Y_1(x)$ & $Y_2(x)$), which satisfy $Y_1(a) = Y_2(b)$.

Using that $DG(x,z) = \delta(x-z)$ we get

$$G = \begin{cases} A(z)\, Y_1(x) + B(z)\, Y_2(x) & \text{for } x < z \\ C(z)\, Y_1(x) + D(z)\, Y_2(x) & \text{for } x > z \end{cases}$$

The greens function must also satisfy the boundary-conditions $G(a,z) = G(b,z) = 0$

$\Rightarrow G(a,z) = A(z) \underbrace{Y_1(a)}_{0} + B(z) Y_2(a) = 0 = B(z) Y_2(a)$

since $a < z$

$\Rightarrow B(z) = 0$

$\Rightarrow G(b,z) = C(z)\, Y_1(b) + D(z) \underbrace{Y_2(b)}_{0} = C(z)\, Y_1(b) = 0$

$\Rightarrow C(z) = 0$

Meaning that the Green's function can be simplified down to

$$G(x,z) = \begin{cases} A(z)\, Y_1(x) & x < z \\ D(z)\, Y_2(x) & x > z \end{cases}$$

We now use the discontinuity in the derivative

$$\left.\frac{dG}{dx}\right|_{x=z_+} - \left.\frac{dG}{dx}\right|_{x=z_-} = D(z)\, Y_2'(z) - A(z)\, Y_1'(z) = 1$$

$$\Rightarrow D(z) = \frac{1 + A(z)\, Y_1'(z)}{Y_2'(z)} \quad (\ast)$$

And then we use the continuity of the Green's function

$$G(x,z)\Big|_{x=z_+} = G(x,z)\Big|_{x=z_-}$$

$$D(z)\, Y_2(z) = A(z)\, Y_1(z) \Rightarrow A(z) = \frac{D(z)\, Y_2(z)}{Y_1(z)} \quad (\ast\ast)$$

Put this into $(\ast)$

$$\Rightarrow D(z) = \frac{1}{Y_2'(x)} + \frac{Y_1'(z)}{Y_2'(z)} \frac{D(z)\, Y_2(z)}{Y_1(z)}$$

$$D(z)\left(1 - \frac{Y_2(z)}{Y_1(z)} \frac{Y_1'(z)}{Y_2'(z)}\right) = \frac{1}{Y_2'(z)}$$

$$D(z) = \frac{1/Y_2'(z)}{1 - \frac{Y_2(z)}{Y_1(z)} \frac{Y_1'(z)}{Y_2'(z)}} = \frac{1}{Y_2'(z) - \frac{Y_2(z)\, Y_1'(z)}{Y_1(z)}}$$

$$= \frac{Y_1(z)}{Y_2'(x)\, Y_1(z) - Y_2(z)\, Y_1'(x)} = \frac{Y_1(z)}{W(z)}$$

Put this into $(\ast\ast)$:

$$A(z) = \frac{Y_1(z)\, Y_2(z)}{W(z)\, Y_1(z)} = \frac{Y_2(z)}{W(z)}$$

We have now found an expression for the Green's function

$$G(x,z) = \begin{cases} \dfrac{Y_2(z)}{W(z)} Y_1(x) & x < z \\ \dfrac{Y_1(z)}{W(z)} Y_2(x) & x > z \end{cases}$$

where $x, z \in [a, b]$

b) We insert our expression for G into $Y(x) = \int_a^b dz\ G(x,z)\ R(z)\ dz$, due to the function G being derivatively discontinuous means that we split the integral at $x = z$

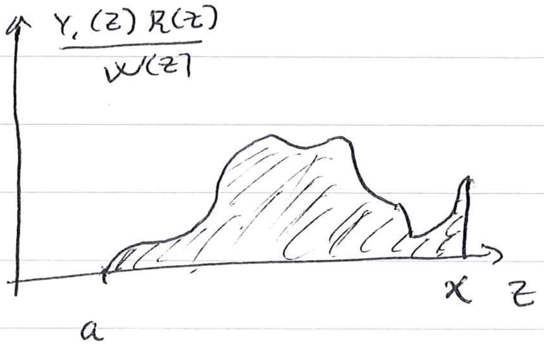
$$Y(x) = \int_a^x \frac{Y_1(z)\, Y_2(x)}{W(z)} R(z)\, dz + \int_b^x \frac{Y_2(z)\, Y_1(x)}{W(z)} R(z)\, dz$$

We moove the x-dependence outside and flip the sign of the latter integral by swaping limits

$$Y(x) = Y_2(x) \int_a^x \frac{Y_1(z) R(z)}{W(z)} dz - Y_1(x) \int_b^x \frac{Y_2(z) R(z)}{W(z)} dz$$

c) We do not know the form of either integrand, but we know it's value at a and b respectively, ~~..~~ from the boundary conditions.

We can illustrate this as



where x can change. We see that taking the integration limits will just give a contribution from the ~~...~~ end point evaluated at ~~z~~ $z=x$, thus

$$\int_a^x \frac{Y_1(z)R(z)}{W(z)}\,dz = \int \frac{Y_1(x)\cdot R(x)}{W(x)}\,dx$$

The exact same argument holds for the integral from b to x, and we can therefore write

$$Y(x) = Y_2(x)\int \frac{Y_1(x)R(x)}{W(x)}\,dx - Y_1(x)\int \frac{Y_2(x)R(x)}{W(x)}\,dx$$

9.2 a) We want to solve

$(x^2+2x)y'' - 2(x+1)y' + 2y = 0$

using a series expansion

$y = \sum a_n x^n, \quad y' = \sum n a_n x^{n-1}, \quad y'' = \sum n(n-1) a_n x^{n-2}$

We insert this into our DE

$$(x^2+2x)\sum n(n-1)a_n x^{n-2} - 2(x+1)\sum n a_n x^{n-1} + 2\sum a_n x^n = 0$$

$$\sum n(n-1)a_n x^n + \sum 2n(n-1)a_n x^{n-1} - \sum 2n a_n x^n - \sum 2n a_n x^{n-1} + \sum 2 a_n x^n = 0$$

This has to be zero for every polynomial, we therefore match coefficients for x^n.

$$n(n-1)a_n + 2n(n+1)a_{n+1} - 2na_n - 2(n+1)a_{n+1} + 2a_n = 0$$
$$a_n\left(n[n-1] - 2n + 2\right) + a_{n+1}\left(2n[n+1] - 2(n+1)\right) = 0$$
$$a_n(n^2 - n - 2n + 2) + a_{n+1}(n+1)(2n-2) = 0$$
$$a_n(n^2 - 3n + 2) = -a_{n+1}\, 2(n+1)(n-1)$$

$$a_{n+1} = \frac{n^2 - 3n + 2}{2(n+1)(n-1)} a_n = \frac{(n-2)(n-1)a_n}{2(n+1)(n-1)}$$

$$\boxed{a_{n+1} = \frac{(n-2)}{2(n+1)} a_n}$$

We see that the sum will terminate for $n = 2$

$\Rightarrow a_0 = a_0$

$a_1 = -a_0$

$a_2 = -\frac{1}{4}a_1 = \frac{a_0}{4}$

$a_3 = 0 \cdot a_2 = 0$

$a_4 = 0$

$\vdots$

$\Rightarrow y = a_0 - a_0 x + \frac{a_0}{4}x^2$

$= a_0(1 - x + \frac{1}{4}x^2)$

Where a_0 is determined from initial conditions!