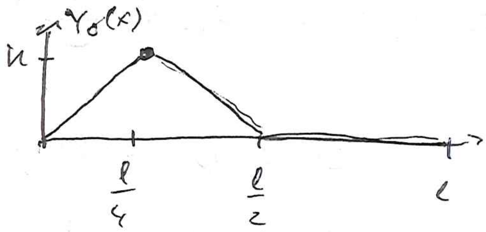
13.4.2)



The string must satisfy the wave-equation $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Assume separation of variables $u(x,t) = X(x)\,T(t)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = T(t)\,X''(x)$ and $\frac{\partial^2 u}{\partial t^2} = X(x)\,\ddot{T}(t)$

Put this into DE: $T(t)\,X''(x) = \frac{1}{c^2} X(x)\,\ddot{T}(t)$

$\frac{1}{X(x)} X''(x) = \frac{1}{c^2}\cdot\frac{1}{T(t)}\ddot{T}(t)$ each side is indep $\Rightarrow$ must be a constant k

$\Rightarrow X''(x) = k\,X(x)$ and $\ddot{T}(t) = c^2 k\,T(t)$ $\quad u(0,t) = u(L,t) = 0$

we have turned the PDE into two ordinary DE.

Our boundary conditions can not be satisfied if k is positive, we therefore write $k = -p^2$, where p is a real number.

We then get $X''(x) = -p^2 X(x) \Rightarrow A\cos(px) + B\sin(px)$

$u(0,t) = T(t)X(0) = 0 = T(t)\cdot(A\cos(0) + B\sin(0)) = A\,T(t) = 0 \Rightarrow \underline{A = 0}$

$u(L,t) = T(t)X(L) = 0 = T(t)\cdot(B\sin(pL)) \Rightarrow pL = n\pi \Rightarrow \underline{p = \frac{n\pi}{L}}$

We can absorb the factor B into $T(t)$, making our solution

$\underline{X(x) = \sin\left(\frac{n\pi x}{L}\right)}$

Do the same for $T(t)$:

$\ddot{T}(t) = -c^2p^2\,T(t) = -c^2\left(\frac{n\pi}{L}\right)^2 T(t) = -\omega_n^2\,T(t)$, $\omega_n = \frac{n\pi c}{L}$

Known solution

$\underline{T(t) = A\cos(\omega_n t) + B\sin(\omega t)}$ Our (infinitely many) solutions is thus

$$\underline{u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A\cos(\omega_n t) + B\sin(\omega_n t)\right)}$$

Now we must find a unique solution from initial conditions

13.4.2) We know that the initial velocity is zero.

$$\dot{u}(x,0) = 0 = \sin\left(\frac{n\pi x}{L}\right)\left(B\cos(\omega_n t) - A\sin(\omega_n t)\right)\Big|_{t=0} \cdot \omega_n$$

$$0 = B\cos(0) - A\sin(0)$$

$$0 = B \Rightarrow \underline{B = 0}$$

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)\cdot A$$

To find the complete solution we must use the initial position of the string. The full solution is written as a super-position of the solutions found with separation of variabels

$$Y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) A_n$$

At $t=0$ the amplitude follows that shown on the previous page:

$$Y(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) A_n = f(x) = \begin{cases} \frac{h4}{\ell}x, & x < \ell/4 \\ \frac{h4}{\ell}\left(\frac{\ell}{2} - x\right), & x \in [\frac{\ell}{4}, \frac{\ell}{2}] \\ 0 & x > \frac{\ell}{2} \end{cases}$$

We can find the coeffs through the fourier series (for sin)

$$A_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\frac{A_n}{2} = \underbrace{\frac{h4}{\ell}\int_0^{\ell/4} x\sin\left(\frac{n\pi x}{\ell}\right)dx}_{} + \frac{h4}{\ell}\int_{\ell/4}^{\ell/2}\left(\frac{\ell}{2} - x\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\int_0^{\ell/4} x\sin\left(\frac{n\pi x}{\ell}\right)dx = -x\cdot\frac{\ell}{n\pi}\cdot\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell/4} + \int_0^{\ell/4} \frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)$$

$$= -\frac{\ell^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{\ell^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{\ell}\right)\right]_0 = \underline{-\frac{\ell^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{\ell^2}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)}$$

Now let's solve the second integral

$$\int_{l/4}^{l/2}\left(\frac{l}{2}-x\right)\sin\left(\frac{n\pi x}{l}\right) = \frac{l}{2}\int_{l/4}^{l/2}\sin\left(\frac{n\pi x}{l}\right)dx - \underbrace{\int_{l/4}^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx}_{\text{same integral as prev. but different limits}}$$

$$= -\frac{l}{2}\left[\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_{l/4}^{l/2} + \frac{xl}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Bigg|_{l/4}^{l/2} - \frac{l^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_{l/4}^{l/2}$$

$$= -\frac{l^2}{2n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right) + \frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{l^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right)$$
$$-\frac{l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)$$

$$= \frac{l^2}{2n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \frac{\cos(n\pi/4)}{2}\right) - \frac{l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)$$

Combing the two expressions our integral becomes

$$\frac{lA_n}{2} = \frac{h4}{l}\left(-\frac{l^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{l^2}{n^2\pi^2}\overset{\sin(n\pi/4)}{\cdot}\right) + \frac{h4}{l}\left(\frac{l^2}{2n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \frac{\cos(n\pi/4)}{2}\right] - \frac{l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)\right]$$

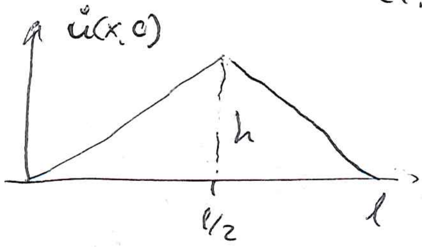
$$A_n = 8h\left[\frac{1}{2n\pi}\left(-\frac{\cos\left(\frac{n\pi}{4}\right)}{2} + \cos\left(\frac{n\pi}{4}\right) - \frac{\cos(n\pi/4)}{2}\right) + \frac{1}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{4}\right)\right)\right]$$

$$A_n = \frac{8h}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$ Making our solution

$$Y(x,t) = \sum \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)A_n$$

$$Y(x,t) = \frac{8h}{\pi^2}\sum \frac{1}{n^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi L t}{c}\right)\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$

13.4.5) We look at same system as previously, but with different initial conditions. Now $u(x,0)=0$ while $\dot{u}(x,0)$ is



Our solution before looking at initial conditions is

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A\cos(\omega_n t) + B\sin(\omega_n t)\right)$$

For $t=0$ we have no displacement $\Rightarrow$

$$u(x,0) = 0 = \sin\left(\frac{n\pi x}{L}\right)(A\cdot 1 + B\cdot 0) \Rightarrow \underline{A=0}$$

$$u(x,t) = B\sin(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)$$ making the velocity

$$\dot{u}(x,t) = \omega_n B\cos(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)$$

These are infinitely many solutions, use the initial velocity to determine exact solution

$$\dot{u}(x,0) = \omega_n B\sin\left(\frac{n\pi x}{L}\right)$$

The exact solution can then be written as

$$Y(x,0) = \sum_{n=1}^{\infty} \omega_n B_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} \frac{2hx}{l} & x<\frac{l}{2} \\ \frac{2h}{l}(l-x) & x>\frac{l}{2} \end{cases}$$

Use a fourier series to determine B_n

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$\frac{B_n\omega_n}{2} = \underbrace{\frac{2h}{l}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx}_{} + \underbrace{\frac{2h}{l}\int_{L/2}^{L}\sin\left(\frac{n\pi x}{L}\right)(L-x)\,dx}_{}$$

same integral, different limits

$$-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} = -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$\to L\int_{L/2}^{L}\sin\left(\frac{n\pi x}{L}\right) - \int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} = \frac{L^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right)-0\right) = \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Continue with the second integral

$$L\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right)dx = L\cdot\frac{L}{n\pi}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} = \frac{L^2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right)-\cos(n\pi)\right)$$

$$\int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L} + \frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L}$$

same integral different limits

$$= -\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left(\sin(n\pi)-\sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{L^2}{n\pi}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{2} - \cos(n\pi)\right) + \frac{L^2}{n^2\pi^2}\left(\sin(n\pi)-\sin\left(\frac{n\pi}{2}\right)\right)$$

Making the whole coefficient

$$\frac{2B_n\omega_n}{2} = \frac{2h}{L}\Bigg[\frac{L^2}{n^2\pi^2}\sin(n\pi/2) - \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right)-\cos(n\pi)\right)$$
$$- \frac{L^2}{n\pi}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{2} - \cos(n\pi)\right) - \frac{L^2}{n^2\pi^2}\left(\sin(n\pi)-\sin\left(\frac{n\pi}{2}\right)\right)\Bigg]$$

$$\omega_n B_n = \frac{4h}{L^2}\Bigg[\frac{L^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi) + \sin\left(\frac{n\pi}{2}\right)\right)$$
$$+ \frac{L^2}{n\pi}\left(\cancel{-\frac{1}{2}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\cos\left(\frac{n\pi}{2}\right)} - \cancel{\cos(n\pi)} - \cancel{\frac{\cos\left(\frac{n\pi}{2}\right)}{2}} + \cancel{\cos(n\pi)}\right)$$

$$B_n = \frac{4h}{\omega_n n^2\pi^2}\left(2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)\right)$$

Making the function (put in $\omega_n = \frac{n\pi c}{L}$)

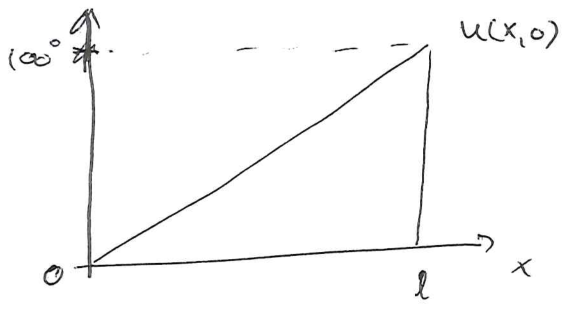
$$Y(x,t) = \frac{4h}{\pi^2}\cdot\frac{L}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)\left(2\sin\left(\frac{n\pi}{2}\right)-\sin(n\pi)\right)$$

$$= \frac{4hL}{\pi^3 c}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\underbrace{\left(2\sin\left(\frac{n\pi}{2}\right)-\sin(n\pi)\right)}_{0 \text{ for all even } n\text{, alternating } \pm \text{ for odd } n}$$

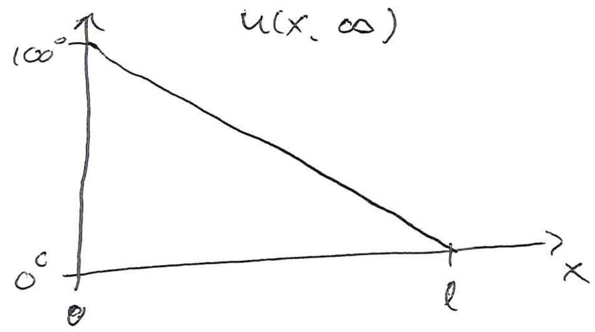
$$= \frac{4hL}{\pi^3 c}\left(2\cos\left(\frac{\pi c t}{L}\right)\sin\left(\frac{\pi x}{L}\right) - \frac{2}{3^3}\cos\left(\frac{3\pi c t}{L}\right)\sin\left(\frac{3\pi x}{L}\right) + \frac{2}{5^2}\cos\left(\frac{5\pi c t}{L}\right)\sin\left(\frac{5\pi x}{L}\right) - \dots\right.$$

$$= \frac{8hL}{\pi^3 c}\left(\cos\left(\frac{\pi c t}{L}\right)\sin\left(\frac{\pi x}{L}\right) - \frac{1}{9}\cos\left(\frac{3\pi c t}{L}\right)\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\cos\left(\frac{5\pi c t}{L}\right)\sin\left(\frac{5\pi x}{L}\right)\dots\right.$$

13.3.3) The initial temperature distribution is



and in infinity



$f(x) = \frac{100x}{l}$ $\qquad$ $g(x) = 100 - \frac{100x}{l}$

We want to find the temperature at some intermediate timestep. To do this we solve the diffusion equation in 1D:

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$, use separation of variables $u(x,t) = X(x)\,\Theta(t)$

$$\Theta(t)\, X''(x) = \frac{1}{\alpha^2} X(x)\, \dot{\Theta}(t)$$

$$\frac{1}{X(x)} X''(x) = \frac{1}{\alpha^2 \Theta(t)} \dot{\Theta}(t)$$

LHS is only x-dependent, while RHS is only t-dependent ⇒ both must equal the same constant k

$$\dot{\Theta}(t) = \alpha^2 k\, \Theta(t)$$

For the temperature to be finite the constant k must be negative: $k = -\beta^2$, which gives us solution

$\underline{\Theta(t) = e^{-\alpha^2\beta^2 t}}$ where $\beta \in \mathbb{R}$. We do the same for the spatial equation

$$X''(x) = -\beta^2 X(x) \implies \underline{X(x) = A\sin(\beta x) + B\cos(\beta x)}$$

At $t=0$ $\Theta(0) = 1$ meaning X will determine the boundary conditions:

$X(0) = A\sin(0) + B\cos(0) = \underline{B = 0}$

$X(x) = A\sin(\beta x)$, and for $t>0$ $X(l) = 0 \implies \sin(\beta l) = 0$

$\implies \beta = \frac{n\pi}{l}$. Making the full solution for X

$X(x) = A_n \sin\left(\frac{n\pi x}{l}\right)$, where A_n is determined from initial conditions.

13.3.3) Thus we find ~~$\beta = ...$~~ $\beta = \frac{\pi n}{\ell}$

Making the spatial solution

$\mathcal{X}(x) = A_n \sin\left(\frac{n\pi x}{\ell}\right)$, the full solution is the

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{\ell^2} t}$$

Evaluated at $t=0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

For these type of problems, with diffent initial and final initial state we have to use $u(x,0) - u(x,\infty)$ to find fourier coeffs

$$u(x,0) - u(x,\infty) = \frac{100x}{\ell} - \left(100 - \frac{100x}{\ell}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{200x}{\ell} - 100 = \sum A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

We determine the fourier coefficients

$$A_n = \frac{2}{\ell}\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\left(\frac{200x}{\ell} - 100\right) dx$$

$$= -\frac{200}{\ell}\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right) dx + \frac{400}{\ell^2}\int_0^\ell x \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= -\frac{200}{\ell}\left[-\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell + \frac{400}{\ell^2}\left(-\frac{x\cdot\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^\ell + \frac{\ell}{n\pi}\int_0^\ell \cos\left(\frac{n\pi x}{\ell}\right) dx\right)$$

$$= -\frac{200}{n\pi}\left(-\cos(n\pi) + 1\right) + \frac{400}{\ell^2}\left(-\frac{\ell^2}{n\pi}\cos(n\pi) + \frac{\ell^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell\right)$$

$$= \frac{200}{n\pi}\left(\cos(n\pi) - 1\right) - \frac{400}{n\pi}\cos(n\pi) + \frac{400}{n^2\pi^2}\left(\sin(n\pi) - 0\right)$$

$$= -\frac{200}{n\pi}\left(1 + \cos(n\pi)\right) + \frac{400}{n^2\pi^2}\underbrace{\sin(n\pi)}_{0} = -\frac{200}{n\pi}\left(1 + \cos(n\pi)\right) = -\frac{400}{n\pi}$$

for even n

Now we have found the fourier coefficients

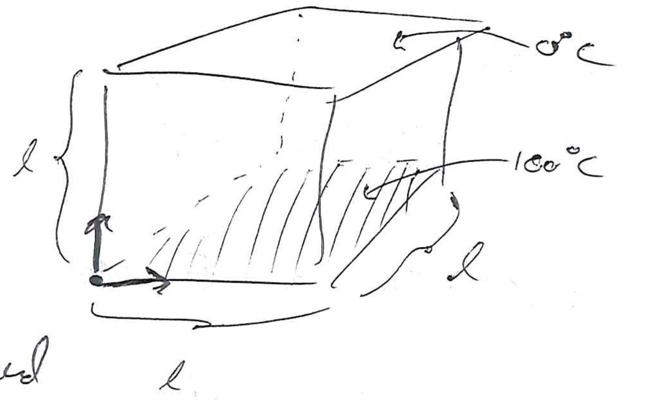
$$A_n = \begin{cases} -\frac{400}{n\pi} & \text{for even } n \\ 0 & \text{for odd } n \end{cases}$$

Thus we find the full solution to be

$$u(x,t) = u_f + \sum_{n=0}^{\infty} A_n e^{-\frac{n^2\pi^2\alpha^2}{l^2}t} \sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,t) = 100 - \frac{100x}{l} - \frac{400}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2\alpha^2}{l^2}t} \sin\left(\frac{n\pi x}{l}\right)$$

13.5.9) We want to find the steady state distribution for the temperature of a cube with fixed $100°C$ at the bottom and $0°C$ on the other sides.



The diffusion equation becomes a laplace-equation since we are at equilibrium $\frac{\partial u}{\partial t} = 0$.
The equation we need to solve is therefore
$\nabla^2 u(x,y,z) = 0$ the Laplace equation.
We use seperation of variables $u(x,y,z) = X(x)Y(y)Z(z)$ and get

$$\frac{1}{X(x)} X''(x) + \frac{1}{Y(y)} Y''(y) + \frac{1}{Z(z)} Z''(z) = 0$$

These are independent, meaning all must be a constant, we call $\frac{1}{Y(y)} Y''(y) + \frac{1}{Z(z)} Z''(z) = k^2$, giving us

$$X''(x) = -k^2 X(x) \Rightarrow X(x) = A\sin(kx) + B\cos(kx)$$

where we chose $-k^2$ so we get a negative solution.
We define $l^2 = \frac{1}{X(x)} X''(x) + \frac{1}{Z(z)} Z''(z)$, giving us

$$\frac{1}{Y(y)} Y''(y) = -l^2 \Rightarrow Y''(y) = -l^2 Y(y)$$

$$\Rightarrow Y(y) = C\sin(ly) + D\cos(ly)$$

We are now left with $Z(z)$

$$(k^2 + l^2) + \frac{1}{Z(z)} Z''(z) = 0$$

$$\Rightarrow Z''(z) = (k^2+l^2) Z(z)$$, we introduce $\xi = \sqrt{k^2+l^2}$

and get the solution: $Z(z) = Ee^{\xi z} + Fe^{-\xi z}$

(3.5.9) $Z(z) = E \sinh(\xi z) + F \cosh(\xi z)$

where we have BC's $Z(0)=100$, $Z(L)=0$

we make a shift of coordinates

$z' = L - z$ such that $z'=0 \Rightarrow Z(z')=0$

$Z(z') = E\sinh(\xi z') + F\cosh(\xi z')$

$Z(0) = \underline{0 = F}$

$Z(L) = 100 \Rightarrow E \sinh(\xi L) = 100$

$E = \dfrac{100}{\sinh(\xi L)}$, but this does not matter since it can be absorbed in fourier-coeffs.

$\Rightarrow$ $\underline{Z(z) = \cancel{E} \sinh(\xi[L-z])}$

From the BC's in X:

$X(0) = 0 \Rightarrow 0 = B\cos(0) \Rightarrow X(x) = A\sin(kx)$

$X(L) = 0 \Rightarrow \sin(kL) = 0 \Rightarrow k = \frac{2\pi n}{L} \Rightarrow \underline{X(x) = A\sin(\frac{2\pi x n}{L})}$

Same for $Y(y)$ giving us $\underline{Y(y) = C\sin(\frac{2\pi y m}{L})}$

Making $u(x,y,z) = A \cdot L \cdot \sin(\frac{2\pi n x}{L}) \cdot \sin(\frac{2\pi m x}{L}) \cdot \sinh(\xi[L-z])$

where now $\xi = \sqrt{k^2 + l^2} = \frac{2\pi}{L}\sqrt{n^2+m^2}$

At $z=0$ we have a known solution for the temperature gradient $f(x,y,z=0) = 100$, meaning we can use this to find the fourier coeffs for the full solution

$$u(x,y,z) = \sum_{n,m}^{\infty} \underbrace{A_n C_m}_{D_{nm}} \sin\left(\frac{2n\pi x}{L}\right) \sin\left(\frac{2m\pi y}{L}\right) \sinh\left(\frac{2\pi}{L}(L-z)\sqrt{n^2+m^2}\right)$$

giving us

$$u(x,y,0) = 100 = \sum_{n,m}^{\infty} \underbrace{D_{nm} \sinh\left(2\pi\sqrt{n^2+m^2}\right)}_{\text{fourier coefficients} = \eta_{nm}} \sin\left(\frac{2\pi n x}{L}\right) \sin\left(\frac{2\pi m y}{L}\right)$$

$$\Rightarrow \eta_{nm} = \frac{2}{L} \int_0^L 100 \cdot \sin\left(\frac{n\pi x}{L}\right) dx \cdot \frac{2}{L} \int_0^L \sin\left(\frac{m\pi y}{L}\right) dy$$

$$= \frac{400}{L^2} \int_0^L dx \int_0^L dy \; \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$$

$$= \frac{400}{L^2} \left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\right]_0^L \cdot \left[-\frac{L}{m\pi} \cos\left(\frac{m\pi y}{L}\right)\right]_0^L$$

$$= \frac{400}{\pi^2} \left(-\frac{\cos(n\pi)}{n} + \frac{1}{n}\right)\left(-\frac{\cos(m\pi)}{m} + \frac{1}{m}\right)$$

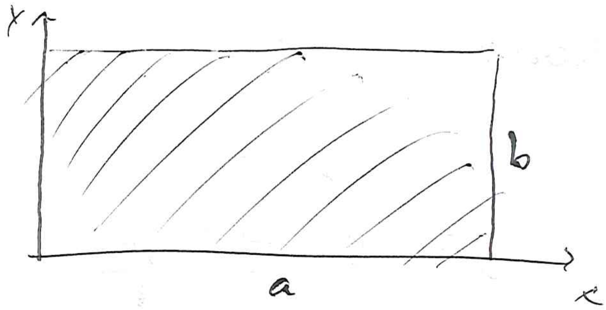
$$= \frac{400}{\pi^2 nm} \left(1 - \cos(n\pi)\right)\left(1 - \cos(m\pi)\right)$$

If either n or m is even we get zero, if both n and m are odd we get $\frac{1600}{\pi^2 nm}$

The full solution is then

$$u(x,y,z) = \sum_{n=\text{odd}}^{\infty} \sum_{m=\text{odd}}^{\infty} \frac{1600}{\pi^2 nm} \cdot \frac{\sinh\left(\frac{2\pi}{L}(L-z)\sqrt{n^2+m^2}\right)}{\sinh\left(2\pi\sqrt{n^2+m^2}\right)} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$$

13.6.3)



$$\nabla^2 z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

Use seperation of variables

$$Z(x,y) = X(x)\,Y(y)\,\Theta(t)$$

This gives us: $\frac{1}{X(x)} X''(x) + \frac{1}{Y(y)} Y''(y) = \frac{1}{v^2\Theta(t)}\ddot{\Theta}(t)$

Each side is independent, and must therefore equal the same constant

$$\frac{1}{X(x)}X''(x) + \frac{1}{Y(y)}Y''(y) = k = -\alpha^2$$ ← must be negative to satisfy that all bounderies are fixed

$$\frac{1}{v^2\Theta(t)}\ddot{\Theta}(t) = k = -\alpha^2$$

$$\ddot{\Theta}(t) = -\alpha^2 v^2 \Theta(t) \Rightarrow \Theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \alpha v$

For the spatial solution we get

$$\frac{1}{X(x)}X''(x) + \frac{1}{Y(y)}Y''(y) = -\alpha^2$$ since they are seperate and indep we get that both must equal a constant

we introduce $\frac{1}{Y(y)}Y''(y) = l^2$

such that $X''(x) = (-\alpha^2 - l^2)X(x) = -(\alpha^2 + k^2)X(x)$

$\Rightarrow X(x) = C\sin(kx) + D\cos(kx)$

Use BC's: $X(0) = 0 = D$

$X(a) = 0 = C\sin(ka) \Rightarrow \boxed{k = \frac{n\pi}{a}}$

Do the same in the y-direction

$\Rightarrow Y(y) = (-\alpha^2 - k^2)Y(y) = -(\alpha^2 + k^2)Y(y)$

$\Rightarrow Y(y) = E\cos(ly) + F\sin(ly)$, and with same BC's

$\Rightarrow Y(y) = F\sin\left(\frac{m\pi}{b}y\right)$ where $l = \frac{m\pi}{b}$

Resulting in . . .

$$-\alpha^2 = -k^2 - l^2$$

$$\alpha^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \Rightarrow \underline{\alpha = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}}$$

Making the time solution:

$$\Theta(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$= A\cos(\alpha v t) + B\sin(\alpha v t)$$

This function has a period T of

$$\alpha v T = 2\pi \Rightarrow T = \frac{2\pi}{\alpha v}$$

and a frequency of

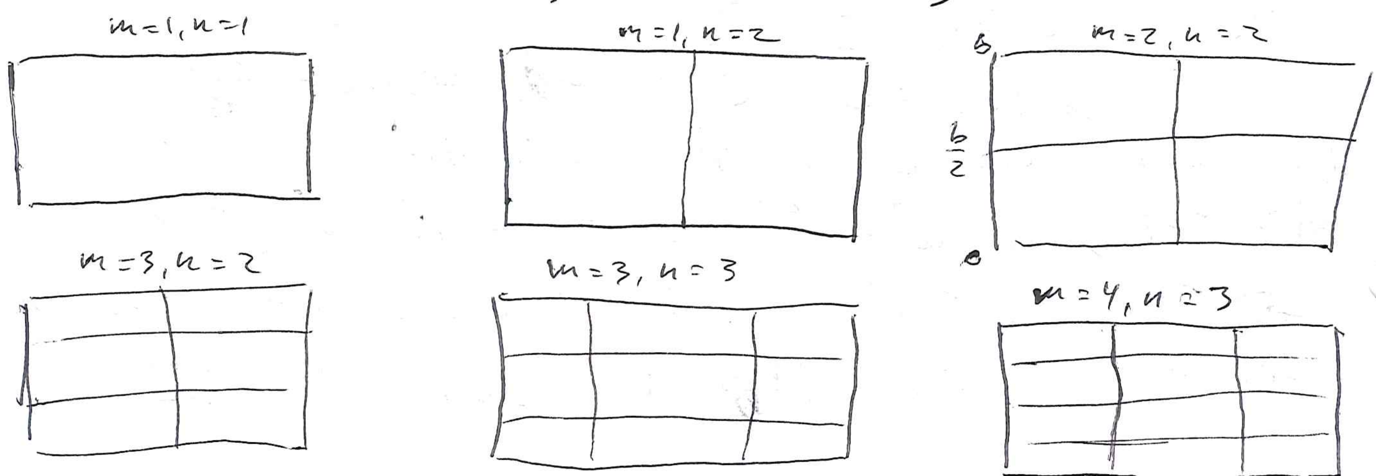
$$\nu = \frac{1}{T} = \frac{\alpha v}{2\pi} = \frac{v}{2\pi}\,\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

$$\underline{\nu = \frac{v}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}}$$

The full solution is

$$u = \sum_{m,n}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) F_m \sin\left(\frac{m\pi y}{b}\right)\left(A\cos\left(v\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\,t\right) + B\sin\left(v\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\,t\right)\right)$$

Let's draw the ~~zero~~ modal lines



If the rectangle is a square: $a = b$

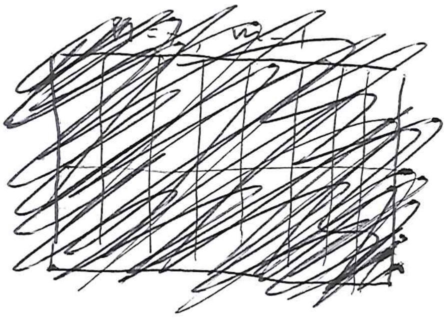
the frequencies are

$$\nu = \frac{v}{2a}\sqrt{n^2 + m^2}$$ where n and m are positive integers

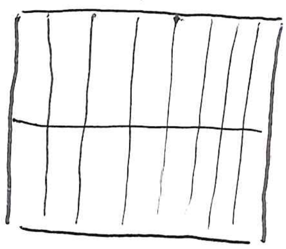
We notice that we get a degeneracy of frequencies. For example will

$n = 7$ & $m = 1$ and $n = 5$ & $m = 5$ give the same frequency. ~~The~~

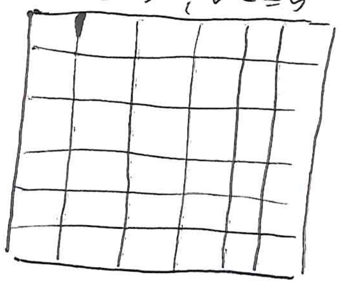
We have different eigenfunctions with the same frequency, resulting in the same eigenvalue. This is what we call degeneracy.



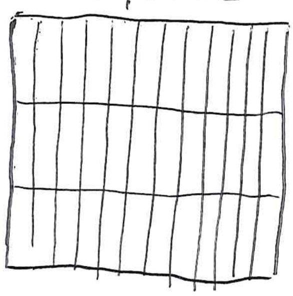
$n = 7,\ m = 1$



$n = 5,\ m = 5$



$n = 11,\ m = 2$



$n = 10,\ m = 5$