UNIVERSITY OF OSLO

FYS3140 - MATHEMATICAL METHODS IN PHYSICS

# Midterm exam

*Candidate number: —*

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# Problem 1: Complex analysis

## Part A: Cauchy integral formula and harmonic functions

We will begin by studying Cauchy's integral formula for a function $f(z)$ which is analytical inside and on the closed curve $C_R$ defined by

$$C_R : |z - z_0| = R \tag{1}$$

Cauchy's integral formula takes the form

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \tag{2}$$

### a)

We let $M$ denote the maximum absolute value of $f(z)$ on $C_R$ such that

$$\left| f(z) \right| \leq M, \tag{3}$$

for all $z$ on the contour. We will use this to find an upper bound on the absolute value of $f(z_0)$

$$\left| f(z_0) \right| = \left| \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| = \frac{1}{2\pi} \left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right|. \tag{4}$$

We begin by writing the absolute value of the integral in terms of Riemann sums

$$\left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| = \lim_{n \to \infty} \left| \sum_{k=1}^{n} \frac{f(z_k)}{z_k - z_0} \right| \Delta z_k. \tag{5}$$

For this sum we will use the generalized triangle inequality, which states that

$$\left| \sum_i = a_i \right| \leq \sum_i |a_i|, \tag{6}$$

which is true for an arbitrary number of terms. Using this we rewrite the Riemann-sum

$$\left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| \leq \lim_{n \to \infty} \sum_{k=1}^{n} \left| \frac{f(z_k)}{z_k - z_0} \right| \Delta z_k = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\left| f(z_k) \right|}{\left| z_k - z_0 \right|} \Delta z_k, \tag{7}$$

where we have used that the absolute value of the fraction is just the absolute value of each factor. We have already defined $\left| f(z) \right|$ in equation (3), and we recognize the denominator as the radius $R$ from the definition of the contour (1). Since both of these are constants we can take them outside of the sum, where we now have a new upper bound estimate

$$\left| \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| \leq \frac{M}{R} \lim_{n \to \infty} \sum_{k=1}^{n} \Delta z_k. \tag{8}$$

The infinitesimal sums over the changes in $z_k$ will add up to the circomference of the curve, which is just a circle with radius $R$. Thus the upper bound can be written as

$$\left|\oint_{C_R} \frac{f(z)}{z - z_0}\, \mathrm{d}z\right| \leq \frac{M}{R}\, 2\pi R. \tag{9}$$

We cancel the $R$'s leaving us with the simple expression for the upper bound estimate

$$\left|\oint_{C_R} \frac{f(z)}{z - z_0}\, \mathrm{d}z\right| \leq 2\pi M. \tag{10}$$

We insert this expression into our original one (4)

$$\left|f(z_0)\right| = \frac{1}{2\pi}\left|\oint_{C_R} \frac{f(z)}{z - z_0}\, \mathrm{d}z\right| \leq \frac{1}{2\pi}\, 2\pi M. \tag{11}$$

Canceling the factors of $2\pi$ we find the final expression for the upper bound estimate of the contour integral

$$\left|f(z_0)\right| \leq M. \tag{12}$$

**b)**

We will try to rewrite Cauchy's integral formula (2) for the special case of a circular contour around a point $z_0$. We do this by writing the complex number $z$ in terms of the center of the circle plus another terms looping around a cirlce with radius $R$

$$z = z_0 + Re^{it} \qquad t \in [0, 2\pi]. \tag{13}$$

By taking the derivative of $z$ with respect to time we can solve for the infinitesimal $\mathrm{d}z$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = iRe^{it} \rightarrow \mathrm{d}z = iRe^{it}\, \mathrm{d}t. \tag{14}$$

We use this substitution in Cauchy's integral formula (2)

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_R} \frac{f(z)}{z - z_0}\, \mathrm{d}z = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})}{z_0 + Re^{it} - z_0}\, iRe^{it}\, \mathrm{d}t. \tag{15}$$

We see that the $i$ outside of the integral cancels with the one inside, and by subtracting away the $z_0$'s in the denominator we can cancel the factor $Re^{it}$, thus

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\, \mathrm{d}t. \tag{16}$$

This expression will only work for circular contours around $z_0$.

**c)**

We introduce the function $u(x, y)$, which is harmonic on and inside a circle of radius $R$ centerd at $z_0 = x_0 + iy_0$. We can evaluate the value of $u$ at the center of the circle using Cauchy's integral formula (2)

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} \, \mathrm{d}z. \tag{17}$$

In the previous task we showed that such a integral, for a circular contour with radius $R$ around a point $z_0$ can be rewritten to a integral over a real scalar $t$ from 0 to $2\pi$ (16). We use this result, but now for a variable $\theta$ over the same interval, to evaluate $z_0$ at the center of the circel

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) \, \mathrm{d}\theta. \tag{18}$$

This result tells us that to evalue an analytic, and in this case harmonic, function at the center of a circle we can only use values along a circle around the point. Due to the factor $1/2\pi$ outside the integral the functionvalue at the center of the circle is equal to the average value of the function along the contour. It is quite incredible that we can do this for an arbritary radius $R$ (as long as the function is still analytic inside and on the contour), and always be able to evalue the function at a point we never looked at.

**d)**

For a pair of two dimensional functions $u(x, y)$ and $v(x, y)$ which are each other's harmonic conjugates we have the following relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{19}$$

which is actually the definition of a harmonic conjugate in two dimensions. We can use this to derive the orthogonality of the gradients of the functions, where the two dimensional nabla-operator is defined as

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), \tag{20}$$

meaning that the inner product between the two functions can be written as

$$(\boldsymbol{\nabla u}) \cdot (\boldsymbol{\nabla v}) = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}. \tag{21}$$

Where we have written the left hand side in bold to emphesise that it is a vector. We then make a substitution from the definition of harmonic conjugates (19) on $v$ so that we only have derivatives acting on $u$

$$(\boldsymbol{\nabla u}) \cdot (\boldsymbol{\nabla v}) = -\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0. \tag{22}$$

Since both terms are equal, but with oposite sign, they exactly cancel each other. Thus we have showed that the gradient of two functions which are each others harmonic conjugates have to be orthogonal. We did not specify anything about $u$ and $v$ except from them being harmonic conjugates of each other, thus this orthogonality must hold for all pairs of analytical conjugates.

$$(\nabla \boldsymbol{u}) \cdot (\nabla \boldsymbol{v}) = 0. \tag{23}$$

**e)**

We will now look at concrete example where one of the harmonic functions, $u(x,y)$, is known, and we want to find it's harmonic conjugate $v(x,y)$. The expression for the known harmonic function is

$$u(x,y) = \sin x \cosh y. \tag{24}$$

We begin by finding it's derivatives and second derivatives with respect to both $x$ and $y$ seperately

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos x \cosh y & \frac{\partial^2 u}{\partial x^2} &= -\sin x \cosh y \\ \frac{\partial u}{\partial y} &= \sin x \sinh y & \frac{\partial^2 u}{\partial y^2} &= \sin x \cosh y. \end{aligned} \tag{25}$$

We begin by checking that $u(x,y)$ infact is harmonic

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0, \tag{26}$$

here we used the double derivatives calculated in (25), and find that $u$ is harmonic. We now want to find the harmonic conjugate of $u$, which we can solve for through the definition of harmonic conjugates (19). We begin with the first equality in (19), and put in the derivative of $u$ from (25)

$$\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \sinh y. \tag{27}$$

We multiply each side with $\mathrm{d}y$ and take the integral on both sides

$$\int \mathrm{d}v = \int \cos x \cosh y \,\mathrm{d}y. \tag{28}$$

The left hand side will just be the function $v(x,y)$, while on the right hand side we can take $\cos x$ outside the integral, and the integral of $\cosh y$ is known, leaving us with

$$v(x,y) = \cos x \sinh y + C(x). \tag{29}$$

The term $C(x)$ is the integration constant, which can depend on $x$ since we took the integral over $y$. We repeat the process for the secound equality in (19)

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{\mathrm{d}u}{\mathrm{d}y} = -\sin x \sinh y. \tag{30}$$

We multilpy each side with d$x$ and take the integral

$$\int \mathrm{d}v = -\int \sin x \sinh y \, \mathrm{d}x. \tag{31}$$

The left hand side will just result in the function $v(x,y)$, while the right hand side is triviall, thus

$$v(x,y) = \cos x \sinh y + C(y). \tag{32}$$

Where $C(y)$ is the integration constant, which can depend on $x$ since the integral was only over $y$. The two expressions we have found (29, 32) are consistent with one another, as they should be, and we see that the integration constant can not depend on either $x$ or $y$, and must therefore just be a constant. The final expression for $u$'s harmonic conjugate is therefore

$$v(x,y) = \cos x \sinh y + C. \tag{33}$$

Having found both $u(x,y)$ and $v(x,y)$ we can find the analytic function $f = u + iv$, which has the form

$$f(x,y) = u(x,y) + iv(x,y) = \sin x \cosh y + i \cos x \sinh y. \tag{34}$$

In our calculations we will ignore the integration constant. We begin by writing the trigonometric and hyperbolic functions on exponential form

$$f(x,y) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\frac{1}{2}\left(e^{y} + e^{-y}\right) + i\frac{1}{2}\left(e^{ix} + e^{-ix}\right)\frac{1}{2}\left(e^{y} - e^{-y}\right). \tag{35}$$

We multiply out the parenthesis in each term, and factor out 1/4

$$f(x,y) = \frac{1}{4i}\left(e^{ix-y} + e^{ix+y} - e^{-ix-y} - e^{-ix+y}\right) + i\frac{1}{4}\left(e^{ix+y} - e^{ix-y} + e^{-ix+y} - e^{-ix-y}\right). \tag{36}$$

In the secound term we multiply with $i$ in the denominator and numerator, flipping the sign and making the $i$ appear in the denominator so that we can rewrite the expression to

$$f(x,y) = \frac{1}{4i}\left(e^{ix-y} + e^{ix+y} - e^{-ix-y} - e^{-ix+y} - e^{ix+y} + e^{ix-y} - e^{-ix+y} + e^{-ix-y}\right), \tag{37}$$

where we see that we have four terms canceling

$$f(x,y) = \frac{1}{4i}\left(e^{ix-y} + \cancel{e^{ix+y}} - \cancel{e^{-ix-y}} - e^{-ix+y} - \cancel{e^{ix+y}} + e^{ix-y} - e^{-ix+y} + \cancel{e^{-ix-y}}\right), \tag{38}$$

leaving us with

$$f(x,y) = \frac{1}{4i}\left(2e^{ix-y} - 2e^{-ix+y}\right). \tag{39}$$

We can cancel one factor of 2 and recognize that we can use $z = x + iy$ to rewrite the exponent as $iz = ix - y$

$$f(x,y) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right). \tag{40}$$

We recognize this expression as the exponential form of sinus, thus

$$f(x,y) = \sin z. \tag{41}$$

**f)**

Now that we have found a pair of two harmonic conjugates we can test the property of the orthogonality of their gradient (23) found earlier. We begin by calculating their derivatives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos x \cosh y & \frac{\partial u}{\partial y} &= \sin x \sinh y \\ \frac{\partial v}{\partial x} &= -\sin x \sinh y & \frac{\partial v}{\partial y} &= \cos x \cosh y. \end{aligned} \tag{42}$$

The inner product between the two gradients is given by

$$(\nabla \boldsymbol{u}) \cdot (\nabla \boldsymbol{v}) = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0 \tag{43}$$

We insert these derivatives into the inner product of the two gradients,

$$(\nabla \boldsymbol{u}) \cdot (\nabla \boldsymbol{v}) = -\cos x \cosh y \sin x \sinh y + \sin x \sinh y \cos x \cosh y = 0, \tag{44}$$

where we see that the two terms are identical, but with opposite sign, and thereby canceling. Our expressions for the harmonic conjugates $u$ and $v$ satisfy the property of orthogonality found previously.

**g)**

We have studied the orthogonality of the harmonic conjugates, we will now display this graphically for different functions. We begin with the simple case of

$$f(z) = x + iy \rightarrow u = x \qquad v = y. \tag{45}$$

This is displayed in figure 1 on the following page, where the left most figure is the contours of $u$, the middle is the cotours of $v$, and the rightmost is both plotted together where $u$ is continous and $v$ is dashed. In the right most figure we can clearly see the orthogonality as all the dashed and continous lines cross orthgonally.
We repeat this for the function

$$f(z) = z^2 = x^2 - y^2 + 2ixy \rightarrow u = x^2 - y^2 \qquad v = 2xy. \tag{46}$$

This is displayed in figure 2 on the next page, with the same order as the previous figure. We can again see in the right most figure that the dashed and continous lines are orthogonal, as it should be.
Lastly we do this for the function we have just studied

$$f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y \rightarrow u = \sin x \cosh y \qquad v = \cos x \sinh y. \tag{47}$$

This is displayed in figure 3 on page 9, with the same order as the previous figure. We can again see in the right most figure that the dashed and continous lines are orthogonal, as it should be.

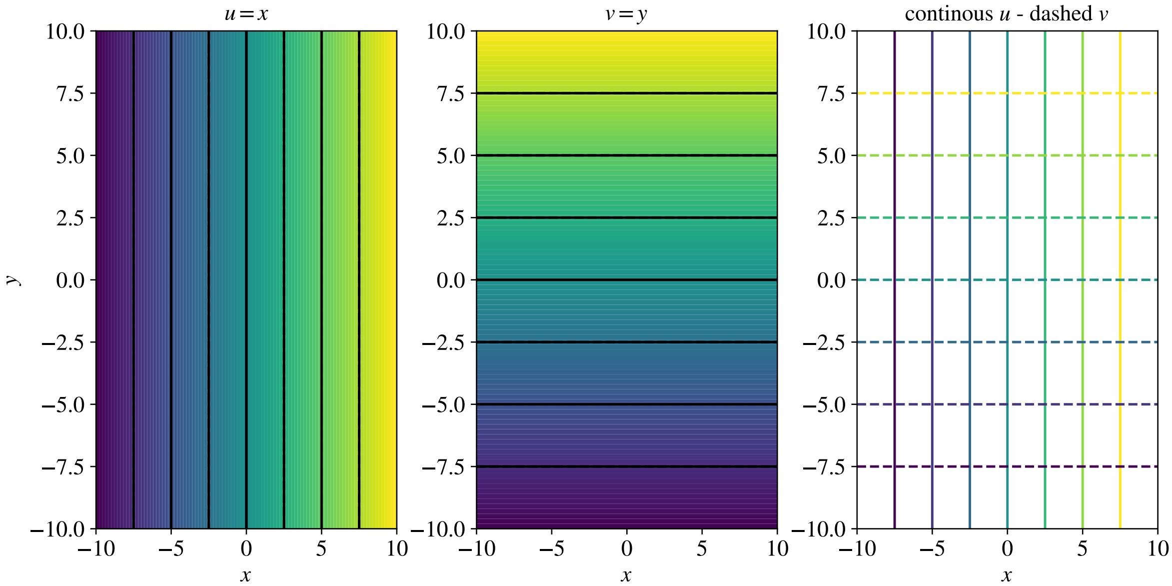


**Figure 1:** The contour levels of $u = x$ and $v = y$ is displayed in the left and middle figure respectivly. Where the colour indicates the functional value at a given point, and the black lines show the lines with a constant functional value. For the last figure the colour still indicates the function value, and displays the curve of a constant value. The dashed line is for $v = y$, and the continous lines for $u = x$, we see that the dashed and continous lines always cross orthogonally.

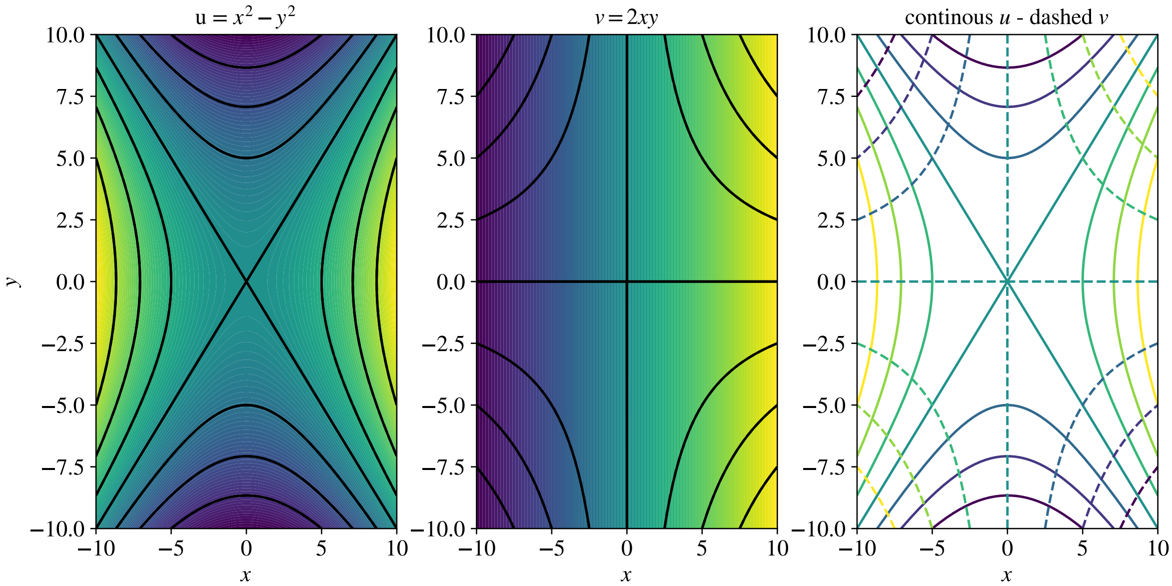


**Figure 2:** The contour levels of $u = x^2 - y^2$ and $v = 2xy$ is displayed in the left and middle figure respectivly. Where the colour indicates the functional value at a given point, and the black lines show the lines with a constant functional value. For the last figure the colour still indicates the function value, and displays the curve of a constant value. The dashed line is for $v = 2xy$, and the continous lines for $u = x^2 - y^2$, we see that the dashed and continous lines always cross orthogonally.

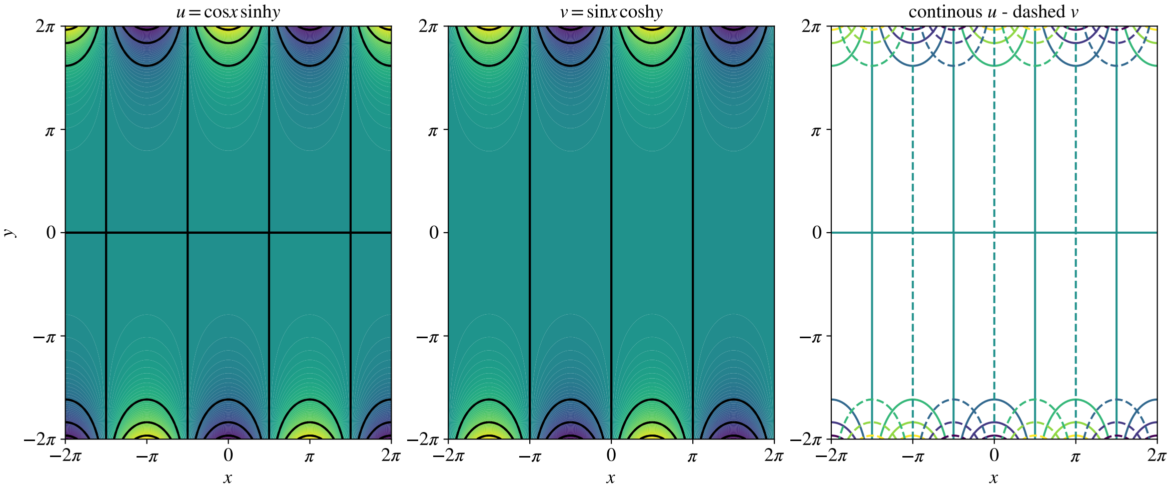


**Figure 3:** The contour levels of $u = \sin x \cosh y$ and $v = \cos x \sinh y$ is displayed in the left and middle figure respectivly. Where the colour indicates the functional value at a given point, and the black lines show the lines with a constant functional value. For the last figure the colour still indicates the function value, and displays the curve of a constant value. The dashed line is for $v = \cos x \sinh y$, and the continous lines for $u = \sin x \cosh y$, we see that the dashed and continous lines always cross orthogonally.

## Part B: A contour integral

### a)

We want to solve the contour integral

$$I = \oint_C \frac{4i\left(z^2+4\right)}{z\left(z^2-16\right)} \sin\left(\frac{5\pi}{z^2+4}\right) \mathrm{d}z, \tag{48}$$

where $C$ is the closed contour integral around $z = 3$ with a radius of 2, $C : |z-3| = 2$. We begin by rewriting the integral by finding the roots of the polynomials to display the singularities more clearly

$$I = \oint_C \frac{4i\,(z+2i)\,(z-2i)}{z\,(z+4)\,(z-4)} \sin\left(\frac{5\pi}{(z+2i)\,(z-2i)}\right) \mathrm{d}z. \tag{49}$$

With this rewrite we can see that we have singularities at $z = 0$, $z = 4$ and $z = -4$. We also see that inside the sine we have two points where we divide by zero; $z = 2i$ and $z = -2i$. These singularities are displayed in the complex plane in figure 4 on the next page.

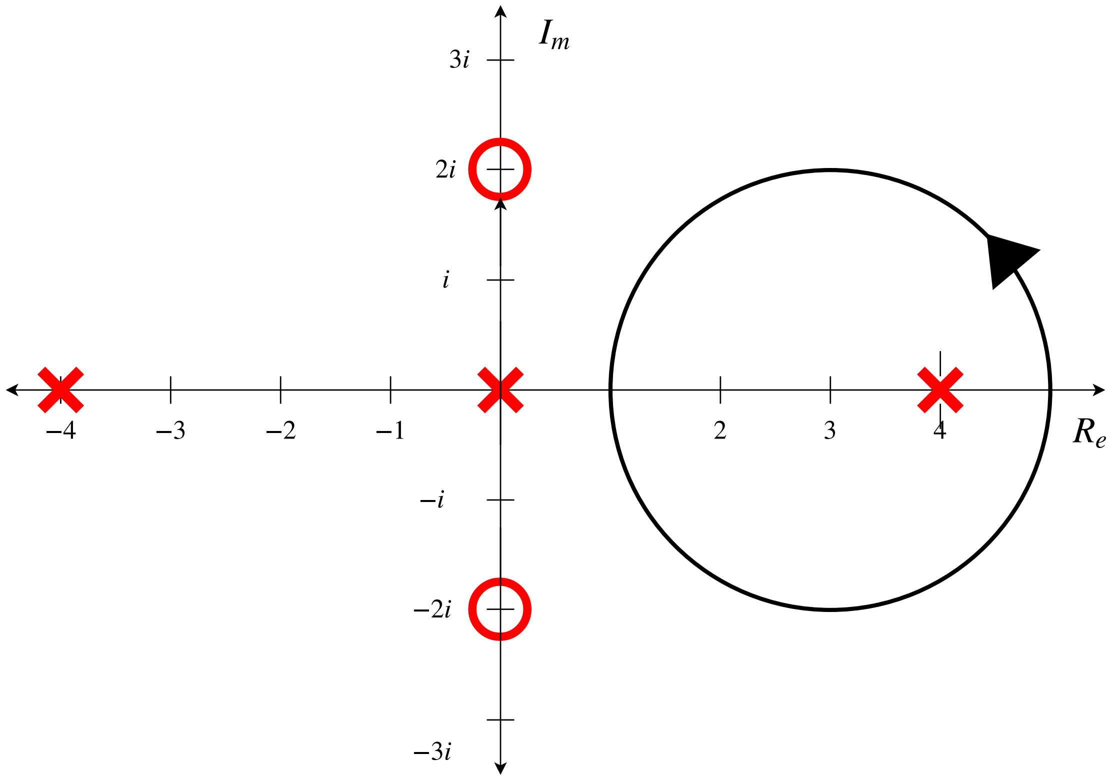


**Figure 4:** The contour $C$ defined as $C: |z-3| = 2$ is shown in the complex plane together with the singularities of our integral (**??**). The true singularities are displayed as red crosses, while the singularities inside the sine function is displayed as red circles. We see that inside our contour we only have 1 singulart, which lies at $z = 4$.