Oblig 6 - Ivar Haugerud - FYS3120

Tensors

10.5.7) a) $\varepsilon_{ijk}\,\varepsilon_{pjq} = \varepsilon_{kij}\,\varepsilon_{qpj} = \varepsilon_{jki}\,\varepsilon_{jqp}$

$$= \delta_{kq}\,\delta_{ip} - \delta_{kp}\,\delta_{iq}$$

b) $\varepsilon_{abc}\,\varepsilon_{pqc} = \varepsilon_{cab}\,\varepsilon_{cpq} = \underline{\delta_{ap}\,\delta_{bq} - \delta_{aq}\,\delta_{bp}}$

10.5.8) $\varepsilon_{ijk}\,\varepsilon_{ijn} = \delta_{jj}\,\delta_{kn} - \delta_{jn}\,\delta_{jk}$

$= \delta_{jj}\,\delta_{kn} - \delta_{nk}$

$= 3\,\delta_{kn} - \delta_{kn}$

$= \underline{2\,\delta_{kn}}$

Since there is an implicit sum (Einstein notation) from $j=1 \to j=3$

* $\varepsilon_{ijk}\,\varepsilon_{ijk}$

$= \delta_{jj}\,\delta_{kk} - \delta_{jk}\,\delta_{jk} = \delta_{jj}\,\delta_{kk} - \delta_{jk}$

* Here we have an implicit sum over all coefficients. The first term will always give a 1, and since our sum has 9 terms this will give us $\underline{9}$ while the second term will kill one of the sums, giving us $\delta_{kk} = 3$

This means that

$\varepsilon_{ijk}\,\varepsilon_{ijk} = 9 - 3 = \underline{\underline{6}}$

since ·-product is sum of product of coeffs

10.5.10 a) $$\underbrace{\vec{A}\cdot(\vec{B}\times\vec{C}) = a_i(\vec{B}\times\vec{C})_i}_{\text{since ·-product is sum of product of coeffs}}$$

$$= a_i(\underbrace{\varepsilon_{ijk}\, b_j c_k}_{\text{identity}}) = a_i b_j c_k\, \varepsilon_{ijk}$$

We compare this to the determinant of a matrix A

$\det(A) = a_{1i}a_{2j}a_{3k}\,\varepsilon_{ijk}$, which is identical ~~...~~ to the expression we found except $a_{1i} = a_i$, $a_{2j} = b_j$, $a_{3k} = c_k$

$$\Rightarrow \vec{A}\cdot(\vec{B}\times\vec{C}) = \det\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

which is equal to what we expected $\boxed{\text{equation 3.2 chapter 6}}$

10.5.11) We will use this result on

$$\vec{A}\cdot(\vec{B}\times\vec{A}) = a_i b_j a_k\, \varepsilon_{ijk}$$

swap k and j, since that is the same

$$= \frac{1}{2}(a_i b_j a_k\, \varepsilon_{ijk} + a_k b_j a_i\, \varepsilon_{kji})$$

$$= \frac{1}{2} a_i b_j a_k(\varepsilon_{ijk} + \varepsilon_{kji})$$

writing it backwards $\Rightarrow -$

$$= \frac{1}{2} a_i b_j a_k(\varepsilon_{ijk} - \varepsilon_{ijk})$$

$$\underline{\underline{= 0}}$$

10.5.13) f) $\nabla \cdot (\varphi \vec{v}) = \varphi (\nabla \cdot \vec{v}) + \vec{\nabla}(\nabla \varphi)$

$$\nabla \cdot (\varphi \vec{v}) = \frac{\partial}{\partial x_i} (\varphi v_i) = \varphi \frac{\partial}{\partial x_i} v_i + v_i \frac{\partial \varphi}{\partial x_i}$$

$$= \underline{\varphi \nabla \cdot \vec{v} + \vec{v} \nabla \varphi}$$

g) $$(\nabla \times (\varphi \vec{v}))_i = \varepsilon_{ijk} \nabla_j (\varphi \vec{v})_k$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varphi v_k) = \varepsilon_{ijk} \left(\varphi \frac{\partial}{\partial x_j} v_k + v_k \frac{\partial \varphi}{\partial x_j} \right)$$

$$= \varepsilon_{ijk} \varphi \frac{\partial v_k}{\partial x_j} + \varepsilon_{ijk} v_k \frac{\partial \varphi}{\partial x_j}$$

$$= \varphi \underbrace{\varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}}_{(\nabla \times \vec{v})_i} + \underbrace{\varepsilon_{ijk} v_k (\nabla \varphi)_j}_{((\nabla \varphi) \times \vec{v})_i}$$

$$= \varphi (\nabla \times \vec{v})_i + ((\nabla \varphi) \times \vec{v})_i$$

$$= \varphi (\nabla \times \vec{v})_i - (\vec{v} \times (\nabla \varphi))_i$$

$$\Rightarrow \underline{\nabla \times (\varphi \vec{v}) = \varphi \nabla \times \vec{v} - \vec{v} \times (\nabla \varphi)}$$

(0.5.13) h)

$$\vec{\nabla} \cdot (\vec{u} \times \vec{v}) = \nabla_i \; (\vec{u} \times \vec{v})_i$$

$$= \nabla_i \; \varepsilon_{ijk} u_j v_k = \varepsilon_{ijk} \; \nabla_i (u_j v_k)$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_i} (u_j v_k) = \varepsilon_{ijk} \left(\frac{\partial u_j}{\partial x_i} v_k + u_j \frac{\partial v_k}{\partial x_i} \right)$$

$$= \varepsilon_{ijk} \; v_k \frac{\partial u_j}{\partial x_i} + \varepsilon_{ijk} \; u_j \frac{\partial v_k}{\partial x_i}$$

$$= \left(\varepsilon_{ijk} \frac{\partial u_j}{\partial x_i} \right) v_k + \left(\varepsilon_{ijk} \frac{\partial v_k}{\partial x_i} \right) u_j$$

$$= \left(\varepsilon_{kij} \frac{\partial u_j}{\partial x_i} \right) v_k + \left(- \varepsilon_{jik} \frac{\partial v_k}{\partial x_i} \right) u_j$$

$$= \underbrace{\left(\varepsilon_{kij} \; \nabla_i u_j \right)}_{(\nabla \times \vec{u})_k} v_k - \underbrace{\left(\varepsilon_{jik} \; \nabla_i v_k \right)}_{(\nabla \times \vec{v})_j} u_j$$

$$= (\nabla \times \vec{u})_k v_k - (\nabla \times \vec{v})_j u_j$$

$$= (\nabla \times \vec{u}) \cdot \vec{v} - (\nabla \times \vec{v}) \cdot \vec{u}$$

$$= \underline{\vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})}$$

9.2.3) $\displaystyle\int_{x_1}^{x_2} x\sqrt{1-y'^2}\,dx$

$$L(x,y') = x\sqrt{1-y'^2}$$

$$\frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial y'} = x\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1-y'^2}}\cdot 2y' = \frac{xy'}{\sqrt{1-y'^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \frac{\partial L}{\partial y'} = c \leftarrow \text{some constant}$$

$$\frac{xy'}{\sqrt{1-y'^2}} = c$$

$$x^2y'^2 = c^2(1-y'^2)$$

$$x^2y'^2 + c^2y'^2 = c^2$$

$$y'^2(c^2+x^2) = c^2$$

$$y'^2 = \frac{c^2}{x^2+c^2}$$

$$y' = \frac{1}{\sqrt{1+\left(\frac{x}{c}\right)^2}} = \frac{dy}{dx}$$

$$dy = \frac{1}{\sqrt{1+\left(\frac{x}{c}\right)^2}}\,dx$$

This is the form of $y(x)$ which makes the integral stationary

$$\int dy = \int \frac{1}{\sqrt{1+\left(\frac{x}{c}\right)^2}}\,dx = c\cdot \operatorname{arcsinh}\left(\frac{x}{c}\right)$$

9.2.5) $\int_{x_1}^{x_2} (y'^2 + y^2)\, dx$ Since the integrand is independant of x we will make a substitution

$$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'} \Rightarrow y' = \frac{1}{x'}$$

$dx = \frac{dx}{dy} dy = x' \, dy$, we put this in

$$\int_{y(x_1)}^{y(x_2)} \left(\frac{1}{x'^2} + y^2\right) x' \, dy = \int_{y(x_1)}^{y(x_2)} \left(\frac{1}{x'} + x' y^2\right) dy$$

$$\Rightarrow L = \frac{1}{x'} + x' y^2$$

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial x'} = -\frac{1}{x'^2} + y^2$$

some constant

$$\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial x'} = c$$

$$\Rightarrow -\frac{1}{x'^2} + y^2 = c$$

$$\frac{1}{x'^2} = y^2 - c \Rightarrow x' = \frac{1}{\sqrt{y^2 - c}}$$

$$dx = \frac{1}{\sqrt{y^2 - c}} dy$$

$$\int dx = \int \frac{1}{\sqrt{y^2 - c}} dy = \ln\left(\sqrt{y^2 - c} + y\right) + k$$

$$\boxed{x(y) = \ln\left(\sqrt{y^2 - c} + y\right) + k}$$

9.3. f) Since I did a substitution in the previous task I will try and solve it by not using a substitution

$$\int_{x_1}^{x_2} (y'^2 + y^2)\,dx \quad , \quad L = y'^2 + y^2$$

$$\frac{\partial L}{\partial y} = 2y \, , \quad \frac{\partial L}{\partial y'} = 2y' \, , \quad \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y'}\right) = 2y''$$

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$$

$$2y'' - 2y = 0 \implies y'' = y$$

A solution to this differential eq.

$$\underline{y(x) = A\sinh(x) + B\cosh(x)}$$

Which will make the integral stationary. If we were to solve

$x(y) = \ln\left(\sqrt{y^2 - c^2} + y\right) + K$ for y

I assume we would find the same solution