4.1) We begin by looking for the Laurent series for different functions, and use the Laurent series to determine the residue.

a) $\frac{1}{z(z-1)}$ at $z=1$

$$\frac{1}{z(z-1)} = \frac{1}{(z-1)} \cdot \frac{1}{z} = -\frac{1}{1-z} \cdot \frac{1}{z-1+1}$$

for $|1-z|<1$

$$= -\frac{1}{1-z} \cdot \frac{1}{1-(1-z)} = -\frac{1}{1-z}\sum_{n=0}^{\infty}(1-z)^n$$

$$= -\frac{1}{1-z}\left(1+(1-z)+(1-z)^2+\dots\right)$$

$$= -\frac{1}{1-z} + \left(1+(1-z)+(1-z)^2+\dots\right)$$

We see from the Laurent series that the residue is $\underline{-1}$

b) $\dfrac{\sin z}{z^4}$ at $z = 0$

$$\frac{1}{z^4} \cdot \sin(z) = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{z^4} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)$$

$$= \frac{1}{z^3} - \frac{1}{z3!} + \frac{z}{5!} - \cdots$$

We have a pole of order three, with a residue of $-\dfrac{1}{3!} = \underline{-\dfrac{1}{6}}$

c) $\dfrac{1 + \cos(z)}{(z-\pi)^2}$ at $z = \pi$

$$\frac{1+\cos(z)}{(z-\pi)^2} = \frac{1 - \cos(z-\pi)}{(z-\pi)^2} = \frac{1 - \sum_{n=0}^{\infty} (-1)^n (z-\pi)^{2n}/(2n)!}{(z-\pi)^2}$$

$$= \frac{1 - \left(1 - \frac{(z-\pi)^2}{2!} + \frac{(z-\pi)^4}{4!} - \cdots\right)}{(z-\pi)^2}$$

$$= \frac{\frac{(z-\pi)^2}{2!} - \frac{(z-\pi)^4}{4!} + \frac{(z-\pi)^6}{6!} - \cdots}{(z-\pi)^2}$$

$$= \frac{(z-\pi)^0}{2!} - \frac{(z-\pi)^2}{4!} + \frac{(z-\pi)^4}{6!} - \cdots$$

$$= \frac{1}{2} - \frac{(z-\pi)^2}{4!} + \frac{(z-\pi)^4}{6!} - \cdots$$

Which was a removable singularity with no residue

d) $\dfrac{1}{(1-2z)(5z-4)}$ at $z=\frac{1}{2}$ and $z=\frac{4}{5}$

We now want to find the residues at the given points

$\underline{z=\frac{1}{2}:}$

$$\text{Res}\left(\tfrac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left[\frac{(z-\frac{1}{2})}{(1-2z)(5z-4)}\right]$$

$$= \lim_{z\to\frac{1}{2}}\left[\frac{-\frac{1}{2}\cancel{(1-2z)}}{\cancel{(1-2z)}(5z-4)}\right]$$

$$= \lim_{z\to\frac{1}{2}}\left[\frac{-1}{2(5z-4)}\right] = \frac{-1}{2(\frac{5}{2}-4)}$$

$$= \frac{-1}{-2\cdot\frac{3}{2}} = \underline{\underline{\frac{1}{3}}}$$

$\underline{z=\frac{4}{5}:}$

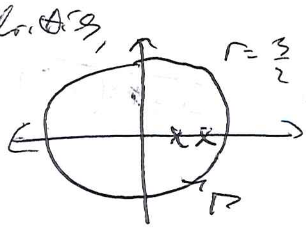
$$\text{Res}\left(\tfrac{4}{5}\right) = \lim_{z\to\frac{4}{5}}\left[\frac{(z-\frac{4}{5})}{(1-2z)(5z-4)}\right]$$

$$= \lim_{z\to\frac{4}{5}}\left[\frac{\cancel{(z-\frac{4}{5})}}{5(1-2z)\cancel{(z-\frac{4}{5})}}\right]$$

$$= \lim_{z\to\frac{4}{5}}\left[\frac{1}{5(1-2z)}\right] = \frac{1}{5(1-\frac{8}{5})}$$

$$= \frac{1}{5-8} = \underline{-\frac{1}{3}}$$

We want to evaluate the integral of a closed curve $|z|=\frac{3}{2}$ centered at the origin. The curve encompasses both singularities, we use the residues to evaluate the integral



$$\oint_{\Gamma}\frac{1}{(1-2z)(5z-4)}\,dz = 2\pi i\left(\text{Res}\left(\tfrac{1}{2}\right)+\text{Res}\left(\tfrac{4}{5}\right)\right)$$

$$= 2\pi i\left(\tfrac{1}{3}-\tfrac{1}{3}\right) = \underline{0}$$

The residues cancel each other!

e) Want to find residue at $3i$ of

$$\frac{z+2}{(z^2+9)(z^2+1)} = \frac{(z+2)/(z^2+1)}{(z^2+9)} = \frac{(z+2)/(z^2+1)}{(z-3i)(z+3i)}$$

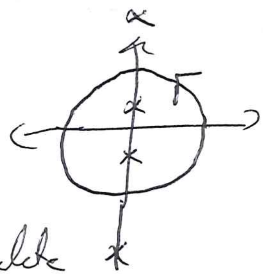
$$\text{Res}(3i) = \lim_{z\to 3i}\left[\frac{\cancel{(z-3i)}(z+2)/(z^2+1)}{\cancel{(z-3i)}(z+3i)}\right]$$

$$= \lim_{z\to 3i}\left[\frac{z+2}{(z^2+1)(z+3i)}\right] = \frac{2+3i}{6i(-9+1)}$$

$$= \frac{2+3i}{6i(-8)} = \frac{i(2+3i)}{48} = \underline{\underline{\frac{-3+2i}{48}}}$$

We want to calculate

$$\oint_\Gamma \frac{z+2}{(z^2+9)(z^2+1)}\,dz \quad \text{around } \Gamma: z = |\tfrac{3}{2}|.$$



We can not use the residue we just found, and need to calculate two more

$$\frac{z+2}{(z^2+9)(z^2+1)} = \frac{z+2}{(z^2+9)(z-i)(z+i)} = \frac{z+2}{(z^2+9)(z\pm i)(z\mp i)}$$

We find the residue at $\pm i$

$$\text{Res}(\pm i) = \frac{\cancel{(z\mp i)}(z+2)}{(z^2+9)(z\pm i)\cancel{(z\mp i)}} = \frac{z+2}{(z^2+9)(z\pm i)}$$

$$\lim_{z\to i}: \frac{2+i}{(-1+9)(2i)} = \frac{2+i}{8(2i)} = -\frac{i(2+i)}{16} = \underline{\frac{1-2i}{16}}$$

$$\lim_{z\to -i}: \frac{2-i}{(-1+9)(-i-i)} = \frac{2-i}{-2i(+8)} = \frac{i(2-i)}{16} = \underline{\frac{1+2i}{16}}$$

Making the integral: $2\pi i\left(\frac{1-2i+1+2i}{16}\right) = 2\pi i \cdot 8 = \underline{\underline{16\pi i}}$

$4.2\ a)\quad I = \int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2}$ we see that this integral is symmetric around $z = \pi$, meaning we can change the limits

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

substitute: $z = e^{i\theta}$

$\frac{dz}{d\theta} = iz \Rightarrow d\theta = \frac{dz}{iz}$

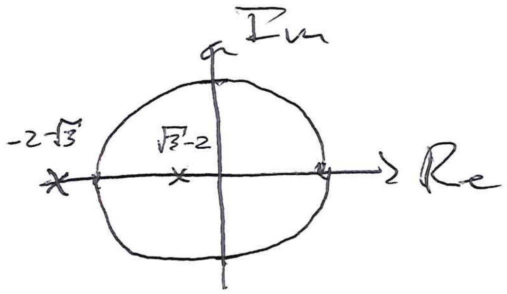
$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\frac{1}{2}(e^{i\theta}+e^{-i\theta}))^2}$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{1}{(2+\frac{1}{2}(z+\frac{1}{z}))^2}\cdot\frac{1}{z}\,dz$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{4z^{\cancel{2}}}{((4+z+\frac{1}{z})z)^2}\,dz = \frac{2}{i}\int_0^{2\pi} \frac{z}{(z^2+4z+1)^2}\,dz$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{z}{[(z-(-2-\sqrt{3}))(z-(\sqrt{3}-2))]^2}\,dz$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{z}{(z+2+\sqrt{3})^2(z+2-\sqrt{3})^2}\,dz$$



We need to find the residue at $z_0 = \sqrt{3}-2$, this a second order pole, and we will therefore have to calculate the derivative

$$\mathrm{Res}(\sqrt{3}-2) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-z_0)^2\frac{z}{(z-z_0)^2(z+2+\sqrt{3})^2}\right]$$

$$= \frac{d}{dz}\left[\frac{z}{(z+2+\sqrt{3})^2}\right] = \frac{1}{(z+2+\sqrt{3})^2} + \frac{z\cdot(-2)}{(z+2+\sqrt{3})^3}$$

$$= \frac{z+2+\sqrt{3}-2z}{(z+2+\sqrt{3})^3} = \frac{2+\sqrt{3}-z}{(z+2+\sqrt{3})^3}$$

$z = \sqrt{3}-2:$
$$= \frac{2+\sqrt{3}-\sqrt{3}+2}{(\sqrt{3}-2+2+\sqrt{3})^3} = \frac{4}{(2\sqrt{3})^3} = \frac{4}{8\cdot 3\sqrt{3}} = \frac{1}{6\sqrt{3}}$$

Now that we have the residue we can easily find the integral

$$I = \frac{2}{i} \oint \frac{z}{(z+2+\sqrt{3})^2(z+2-\sqrt{3})^2} dz$$

$$= \frac{2}{i} 2\pi i \ \mathrm{Res}(-2+\sqrt{3})$$

$$= 4\pi \cdot \frac{1}{6\sqrt{3}} = \underline{\underline{\frac{2\pi}{3\sqrt{3}}}}$$

which is the integral of the desired function.

it is both real and positive,

4.2 b)

$$I = \int_0^\pi \frac{\sin^3\theta}{13-12\cos\theta}\, d\theta \quad \text{, which is symmetric around } \theta = 0$$

$$2I = \int_{-\pi}^{\pi} \frac{\sin^2\theta}{13-12\cos\theta}\, d\theta, \quad \text{substitute} \quad z = e^{i\theta}$$

$$\frac{dz}{d\theta} = iz \Rightarrow d\theta = \frac{1}{iz}\,dz,$$

$$= \int_{-\pi}^{\pi} \frac{(e^{i\theta}-e^{-i\theta})^2 \cdot \frac{1}{(2i)^2}}{13-12\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})}\, d\theta$$

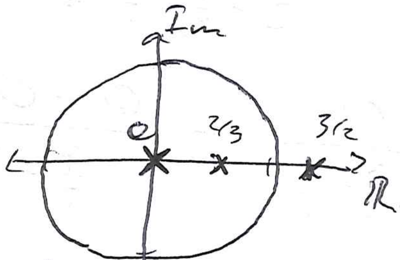
we are now integrating around a complex loop Γ $|z|=1$ around origin

$$= \oint_\Gamma \frac{(z-\frac{1}{z})^2 \frac{1}{(-4)}}{13-6(z+\frac{1}{z})} \cdot \frac{1}{iz}\, dz$$

$$= -\frac{1}{4i} \oint \frac{z^2-2+\frac{1}{z^2}}{z(13-6z-\frac{6}{z})}\, dz = \frac{1}{4i} \oint_\Gamma \frac{z^2-2+\frac{1}{z^2}}{6z^2+6-13z}\, dz$$

$$= \frac{1}{4i} \oint_\Gamma \frac{z^4-2z^2+1}{z^2(6z^2+6-13z)}\, dz = \frac{1}{24i} \oint \frac{z^4-2z^2+1}{z^2(z^2-\frac{13z}{6}+1)}\, dz$$

$$= \frac{1}{24i} \oint_\Gamma \frac{z^4-2z^2+1}{z^2(z-\frac{3}{2})(z-\frac{2}{3})}\, dz$$



The closed curve encompasses two singularities, one of first order, and one of second order. We find the residues of the singularities to determine the ~~residue~~ integral. We start with $z = \frac{2}{3}$

$$\mathrm{Res}\left(\tfrac{2}{3}\right) = \lim_{z\to\frac{2}{3}} \left[\frac{\cancel{(z-\frac{2}{3})}(z^4-2z^2+1)}{z^2(z-\frac{3}{2})\cancel{(z-\frac{2}{3})}} \right] = \lim_{z\to\frac{2}{3}} \left[\frac{z^4-2z^2+1}{z^2(z-\frac{3}{2})} \right]$$

$$= \frac{(\frac{2}{3})^4 - 2(\frac{2}{3})^2+1}{(\frac{2}{3})^2(\frac{2}{3}-\frac{3}{2})} = \frac{\frac{25}{81}}{-\frac{10}{27}} = -\frac{5}{6}$$

For the singularity at $z=0$ we need to use another method, since we have a higher order pole.

The pole at $z=0$ is a 2nd order pole, we therefore get

$$\mathrm{Res}(0) = \lim_{z\to 0} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}}\left[(z-0)^2 f(z)\right]$$

$$= \frac{d}{dz}\left(\frac{\cancel{z^2}(z^4-2z^2+1)}{\cancel{z^2}(z-\frac{3}{2})(z-\frac{2}{3})}\right)$$

$$= \frac{d}{dz}\left(\frac{z^4-2z^2+1}{(z-\frac{3}{2})(z-\frac{2}{3})}\right)$$

$$= \frac{4z^3-4z}{(z-\frac{3}{2})(z-\frac{2}{3})} + (z^4-2z^2+1)\left(\frac{-1}{(z-\frac{3}{2})^2(z-\frac{2}{3})} - \frac{1}{(z-\frac{3}{2})(z-\frac{2}{3})^2}\right)$$

$$= \frac{4z^3-4z}{(z-\frac{3}{2})(z-\frac{2}{3})} - \frac{z^4-2z^2+1}{(z-\frac{3}{2})(z-\frac{2}{3})}\left(\frac{1}{z-\frac{3}{2}} + \frac{1}{z-\frac{2}{3}}\right)$$

$$= \frac{0}{\frac{3}{2}\cdot\frac{2}{3}} - \frac{0-0+1}{\frac{3}{2}\cdot\frac{2}{3}}\left(\frac{1}{-\frac{3}{2}} + \frac{1}{-\frac{2}{3}}\right)$$

$$= -\frac{1}{1}\left(-\frac{2}{3}-\frac{3}{2}\right) = \underline{\frac{13}{6}}$$

We then find the integral:

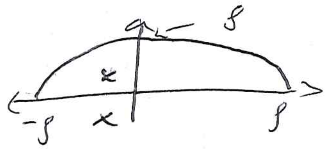
$$I = \frac{1}{48i}\oint_C \frac{z^4-2z^2+1}{z^2(z-\frac{3}{2})(z-\frac{2}{3})}\,dz$$

$$= \frac{2\pi i}{48i}\left(\mathrm{Res}(0) + \mathrm{Res}(\tfrac{2}{3})\right) = \frac{\pi}{24}\left(-\frac{5}{6}+\frac{13}{6}\right)$$

$$= \frac{\pi}{24}\cdot\frac{4}{3} = \underline{\underline{\frac{\pi}{18}}}$$

4.2 c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5}$ ← 0th order ← 2nd order } can neglect ~~[illegible]~~ arc on top of complex plane

$$= \oint_{\Gamma} \frac{dz}{z^2+4z+5}$$ where Γ : ~~[illegible]~~, $\rho \to \infty$



We try and find the roots

$$\frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

We have one divergence inside the integral, we find it's residue:

$$\mathrm{Res}(-2+i) = \lim_{z \to -2+i} \left[\frac{\cancel{(z-(-2+i))}}{\cancel{(z-(-2+i))}(z-(-2-i))} \right]$$

$$= \lim_{z \to -2+i} \left[\frac{1}{z+2+i} \right] = \frac{1}{-2+i+2+i} = \underline{\frac{1}{2i}}$$

Making the integral $2\pi i \cdot \mathrm{Res}(-2+i)$

$$= 2\pi i \cdot \frac{1}{2i} = \underline{\underline{\pi}}$$

$\leftarrow$ symmetric around $x=0$

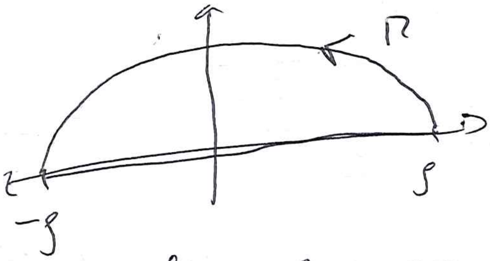
4.2 d) $I = \displaystyle\int_0^\infty \frac{x^2}{x^4+16}\,dx$

$$2I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+16}\,dx$$

$\leftarrow$ order x^2
$\leftarrow$ order x^4

} can ignore integral contribution from complex arc

$$= \oint_\Gamma \frac{z^2}{z^4+16}\,dz$$



$\lim \rho \to \infty$

We find the roots of z^4+16

$z^4+16=0$, $\quad z^4 = -16 = 16\, e^{i\pi} e^{2\pi i n}$

$$z = 2 e^{i\pi/4} e^{\frac{\pi i n}{2}} = 2\, e^{\frac{i\pi}{4}(1+2n)}$$

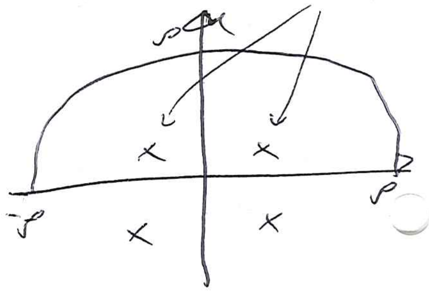
$z_0 = 2 e^{\frac{i\pi}{4}} = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \underline{\sqrt{2}(1+i)}$

$z_1 = e^{\frac{3\pi i}{4}} = 2\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \underline{\sqrt{2}(-1+i)}$

$z_2 = e^{\frac{5\pi i}{2}} = 2\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \underline{-\sqrt{2}(1+i)}$

$z_3 = e^{\frac{7i\pi}{4}} = 2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \underline{\sqrt{2}(1-i)}$

inside Γ



We have two singularities inside Γ, we need to find their residues.

4.2 d) $\oint_{\Gamma} \underbrace{\frac{z^2}{(z-\sqrt{2}(1+i))(z-\sqrt{2}(1-i))(z-\sqrt{2}(-1+i))(z-\sqrt{2}(-1-i))}}_{f(z)} dz$

And to solve this integral we need to find it's residues at $\sqrt{2}(1+i)$ and $\sqrt{2}(1-i)$

$$\Rightarrow \text{Res}(\sqrt{2}(1+i)) = \lim_{z\to\sqrt{2}(1+i)} \left[(z-\sqrt{2}(1+i)) f(z) \right]$$

$$= \left. \frac{z^2}{(z-\sqrt{2}(1-i))(z-\sqrt{2}(-1-i))(z-\sqrt{2}(-1+i))} \right|_{z=\sqrt{2}(1+i)}$$

$$= \frac{2(1+i)^2}{\sqrt{2}(1+i-1+i)\sqrt{2}(1+i+1+i)\sqrt{2}(1+i+1-i)}$$

$$= \frac{2(1+2i-1)}{2\sqrt{2}(2i)(2+2i)(2)} = \frac{4i}{16\sqrt{2}i(1+i)} = \underline{\frac{1}{4\sqrt{2}(1+i)}}$$

Do the same for $\sqrt{2}(1-i)$:

$$= \frac{(\sqrt{2}(1-i))^2}{\sqrt{2}(1-i-1-i)\sqrt{2}(1-i+1+i)\sqrt{2}(1-i+1-i)}$$

$$= \frac{2(1-2i+1)}{2\sqrt{2}(-2i)(2)(2-2i)} = \frac{-4i}{16\sqrt{2}(-i)(1-i)} = \frac{1}{4\sqrt{2}(1-i)}$$

The integral is then $2\pi i \left(\frac{1}{4\sqrt{2}(1+i)} + \frac{1}{4\sqrt{2}(1-i)} \right)$

$$= \frac{2\pi i}{4\sqrt{2}} \left(\frac{1-i}{(1+i)(1-i)} + \frac{1+i}{(1-i)(1+i)} \right) = \frac{\pi i}{2\sqrt{2}} \left(\frac{1-i+1+i}{1+1} \right)$$

$$= \frac{\pi i}{2\sqrt{2}}$$

this answer is wrong. I want a real number for a real integral.
By looking for the integral online i find that the only thing different than the real answer is the factor of "i" should not be there. I can not find my error in the calculation, but I expect the error might result from the wrong sign in the second residue, since this would remove the i factor, and I went a bit too fast in the calculations. I hope the method is correct at least