Matmet - FYS3140 - Oblig 8 - Ivar Haugerud

8.66) $y'' + 6y' + 9y = 12e^{-x}$

We begin by searching for the particular solution of $y'' + 6y' + 9y = 12e^{-x}$.

Rewrite LHS

$(D+3)(D+3)\,y = 12e^{-x}$

$\alpha_+ = \alpha_- = -3$, while the expont is -1, since α_+ and α_- are different than -1 we guess on solution $Ce^{-x} = y$

$\Rightarrow y'' = Ce^{-x}, \quad y' = -Ce^{-x}, \quad y = Ce^{-x}$

$Ce^{-x} - 6Ce^{-x} + 9Ce^{-x} = 12e^{-x}$ (cancel e^{-x})

$C(1-6+9) = 12$

$4C = 12 \Rightarrow \underline{C = 3} \Rightarrow \underline{y_p = 3e^{-x}}$

We find homogeneus solution for constants coefficients

$(\lambda+3)(\lambda+3) = 0 \Rightarrow \lambda = -3$

Which gives general solution on form

$y_h = Ae^{-3x} + Bxe^{-3x}$

Full solution is then

$$\underline{\underline{y = e^{-3x}(A+Bx) + 3e^{-x}}}$$

8.6.12) $(D^2+4D+12)y = 80 \sin(2x)$

Begin by finding roots $\frac{-4 \pm \sqrt{16-48}}{2} = -2 \pm 2\sqrt{2}\, i = \alpha_\pm$

$(D - \alpha_+)(D - \alpha_-) = \mathrm{Im}\{80\, e^{2ix}\}$

Since α_+ and α_- are different than 2

we guess on particular solution on form of $y_p = C e^{2ix}$

$y_p' = 2i\, C e^{2ix}, \quad y_p'' = -4 C e^{2ix}$

$\left(\frac{d^2}{dx^2} + 4\frac{d}{dx} + 12\right) y = 80\, e^{2ix}$

$(-4 + 8i + 12) C e^{2ix} = 80\, e^{2ix}$

$8(1+i) C = 80$

$C = 10/_{1+i} = \frac{10(1-i)}{(1+i)(1-i)} = \frac{10(1-i)}{2} = 5(1-i)$

$\Rightarrow y_p = 5(1-i)\, e^{2ix} = 5(1-i)(\cos(2x) + i \sin(2x))$

$\underline{\mathrm{Im}\{y_p\} = 5\sin(2x) - 5\cos(2x)}$

We have homogeneous solution for constant coeffs

$y_h = A e^{x(-2+2\sqrt{2}i)} + B e^{x(-2-2\sqrt{2}i)}$

Giving us full solution

$\underline{y = e^{-2x}\left(A e^{2\sqrt{2}ix} + B e^{-2\sqrt{2}ix}\right) + 5(\sin(2x) - \cos(2x))}$

8.6.23) $y'' + y = 2xe^x \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$\Rightarrow \underline{y_h = Ae^{ix} + Be^{-ix}}$

Since $\lambda_\pm \neq 1$ (expont on RHS)

we guess on form $y_p = e^x(Cx + D)$

$y_p' = e^x(Cx + D) + Ce^x = e^x(Cx + C + D)$

$y_p'' = e^x(Cx + C + D) + Ce^x = e^x(Cx + 2C + D)$

Put this in

$\cancel{e^x}((Cx + 2C + D) + \cancel{e^x}(Cx + D) = 2x\cancel{e^x}$

$2Cx + 2C + 2D = 2x$

$Cx + (C + D) = x$

For this to be true for all x

we must have $C = 1$ and $C + D = 0$

$\Rightarrow D = -1$

$y_p = e^x(x - 1)$ giving full solution

$\underline{y = Ae^{ix} + Be^{-ix} + e^x(x - 1)}$

8.7.19 $x^2y'' - 5xy' + 9y = 2x^3$

Use Euler-Cauchy through substitution

$$x = e^z \Rightarrow \frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz} = \frac{1}{\frac{dx}{dz}} \cdot \frac{d}{dz} = \frac{1}{x}\frac{d}{dz}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow \underline{x \cdot \frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x}\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\left(-\frac{dy}{dz}e^{-z} + e^{-z}\frac{d^2y}{dz^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left(-\frac{dy}{dz}\cdot\frac{1}{x} + \frac{1}{x}\frac{d^2y}{dz^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

We put this in

$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 2x^3$$

$$\ddot{y} - \dot{y} - 5\dot{y} + 9y = 2e^{3z}, \quad \dot{y} = \frac{dy}{dz}$$

$$\underline{\ddot{y} - 6\dot{y} + 9y = 2e^{3z}}$$

$$\underline{(D-3)(D-3)y = 2e^{3z}}$$

$y_h = Ae^{3z} + Be^{3z}\cdot z$ Since $3=3=3$ we guess on $y_p = z^2Ce^{3z}$

$$y_p = z^2\cdot C\cdot e^{3z}, \quad \dot{y}_p = (2z + 3z^2)Ce^{3z}$$

$$\ddot{y}_p = (2 + 12z + 9z^2)Ce^{3z}$$

$$\Rightarrow (2 + 12z + 9z^2)\cancel{Ce^{3z}} - 6(2z + 3z^2)\cancel{Ce^{3z}} + 9z^2\cancel{Ce^{3z}} = 2\cancel{e^{3z}}$$

$$C\left[z^2\underbrace{(9 - 18 + 9)}_{0} + z\underbrace{(12 - 12)}_{0} + 2\right] = 2$$

$2C = 2 \Rightarrow C = 1$ giving us

$$y(z) = Ae^{3z} + Bze^{3z} + z^2e^{3z}$$

$$\underline{y(x) = Ax^3 + Bx^3\ln(x) + \ln(x)^2x^3}$$

where A and B are different for $x<0$ and $x>0$

8.2a) $x^2 y'' - 2xy' + 2y = x \ln x$

with solutions $y_1 = x$, $y_2 = x^2$, we rewrite

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = \frac{1}{x}\ln(x)$$

Can find solution by solving

$$y_p = -y_1 \int \frac{y_2 \cdot g(x)}{W} dx + y_2 \int \frac{y_1 \cdot g(x)}{W} dx,$$

where $g(x) = \frac{1}{x} \ln(x)$, and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = \underline{x^2}$$

$$\Rightarrow y_p = -x \int \frac{x^2 \cdot \frac{1}{x} \ln(x)}{x^2} dx + x^2 \int \frac{x \cdot \frac{1}{x} \ln(x)}{x^2} dx$$

$$= -x \int \frac{1}{x} \ln(x)\, dx + x^2 \int \frac{\ln(x)}{x^2} dx$$

$$= -x \int \frac{d}{dx}\left(\frac{1}{2}\ln(x)^2\right) dx + x^2 \int \frac{d}{dx}\left(-\frac{\ln x + 1}{x}\right) dx$$

$$= -\frac{x}{2} \ln^2(x) - x(\ln x + 1)$$

$$= \underline{-x\left(\frac{\ln^2(x)}{2} + \ln(x) + 1\right)}$$

6.2b) $(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2$

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = (x^2+1)$$

with known solution $y_1 = x$, $y_2 = 1-x^2$

$g(x) = x^2+1$, particular solution on form

$$y_p = -y_1 \int \frac{y_2 g}{w}\,dx + y_2 \int \frac{y_1 g}{w}\,dx$$

$$w = \begin{vmatrix} x & 1-x^2 \\ 1 & -2x \end{vmatrix} = -2x^2 - 1 + x^2 = -(1+x^2)$$

$$y_p = -x \int \frac{(1-x^2)\cancel{(x^2+1)}}{-\cancel{(1+x^2)}}dx + (1-x^2)\int \frac{x\cancel{(x^2+1)}}{-\cancel{(1+x^2)}}dx$$

$$= x\int 1-x^2\,dx - (1-x^2)\int x\,dx$$

$$= x\left(x - \frac{1}{3}x^3\right) - (1-x^2)x^2$$

$$= x^2\left(1 - \frac{1}{3}x - 1 + x^2\right)$$

$$= x^2\left(x^2 - \frac{1}{3}x\right) = \underline{x^3\left(x - \frac{1}{3}\right)}$$

8.3) a) $\int_0^{\pi} \sin(x)\, \delta(x-\frac{\pi}{2})\,dx = \sin(\frac{\pi}{2}) = 1$

b) $\int_0^{\pi} \sin(x)\, \delta(x+\frac{\pi}{2})\,dx = \sin(-\frac{\pi}{2}) = -1$

c) $\int_{-1}^{1} e^{3x}\, \delta'(x)\,dx = -\left(e^{3x}\right)'\Big|_{x=0} = -3e^{3x}\Big|_{x=0} = \underline{\underline{-3}}$

d) $\int_0^{\pi} \cosh(x)\, \delta''(x-1)\,dx$

$= (-1)^2 \left(\cosh(x)\right)''\Big|_{x=1} = 1 \cdot \cosh(x)\Big|_{x=1}$

$= \underline{\cosh(1) = \frac{1}{2}\left(e + \frac{1}{e}\right)}$