Complex analysis

Definitions

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right) \quad \cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

Roots

$$\left(e^{i\theta}\right)^n = \left(\cos\theta + i\sin\theta\right)^n\right) = \cos n\theta + i\sin n\theta$$
$$z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n}e^{i\theta/n}.$$
$$\ln z = \ln re^{i\theta} = \ln r + i\theta.$$

Complex series

Comparison test: If $|Z_n| \leq a_n$ and $\sum a_n$ converges, then $\sum z_n$ converges.
Ratio test: If $\left|\frac{z_{n+1}}{z_n}\right| \leq k$ for all n sufficiently large, and $k < 1$, then $\sum z_n$ converges absolutely.
Divergence check: If z_n does not converge towards zero, then $\sum z_n$ diverges; the complex and imaginary part will diverge seperately. **Complex power series:** The ratio test gives convergence for $|z - z_0| < \left|\frac{a_n}{a_{n+1}}\right| = R$ as $n \to \infty$. We call R radius of convergence, and $|z - z_0| < R$ for the disk of convergence.

Cauchy-Riemann equations

Analytic: $\leftrightarrow$ Has a unique derivative at wanted region. If a function is analytic it has unique derivatives of all orders and is a solution of Laplace's equation.
Regular point: A point where $f(z)$ is analytic.
Singular point or **Singularity:** A point where $f(z)$ is not analytic.
Isolated singularity: If $f(z)$ is analytic in a small circle around, but not at, the given point.
CR-eq. is a tool to check if a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region by requiering existance of a unique derivative

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

must be satisfied in that region.
If a function is analytic within some region it can be expanded as a Taylor series about any point z_0 inside the region. The power series converges inside a circle about z_0 that extends to the nearest singular point.
Harmonic functions: Are solutions to the 2D Laplace equation $\nabla^2\phi = 0$. If a function $f(z) = u(x,y) + iv(x,y)$ is ANALYTIC in a region, then u and v are harmonic functions.
Harmonic conjugate: Given a harmonic function $u(x,y)$, there excists another harmonic functions $v(x,y)$ such that $u + iv$ is an analytic function of z in that region; v is called the HARMONIC CONJUGATE.

- Check that $u(x,y)$ is harmonic
- Find harmonic conjugate through Cauchy-Riemann equations through integration
- Express $u + iv$ in terms of z

Integrals of complex functions

Contours: Finite sequence of directed smooth curves patched together.
Simple closed contours: A contour which does not cross itself.
Positivly oriented contour: A contour with interior to the left and exterior to the right, arrow going counter clockwise.
Loop: A closed contour.
Contour integral:

$$\int_\Gamma f(z)\,\mathrm{d}z = \lim_{n\to\infty}\sum_{k=1}^{n} f(z_k)\Delta z_k.$$

If a function is continous in a domain then

- f has an anti-derivative
- Contour integrals are independent of path
- Any loop integral is zero

If any of these are true, the two others are true.

$$\int_C (z - z_0)^n\,\mathrm{d}z = \delta_{n,-1}.$$

Inside a simply connected region (no singularities) we continously deform contours without changing the integral of analytic functions along these curves.
Cauchy-Integral formula: For an function f, which is analytic inside a simply connected region containg a contour Γ which is simple, closed and positivly oriented, then:

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

$$f(z_0)^{(n)} = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z,\ n \geq 1$$

Liouvilles theorem: A bounded analytic function in the entire complex plane is a constant.

Upper bound estimates

Use the generalized triangle inequality

$$\left|\sum_k z_k\right| \leq \sum_k |z_k| \quad \to \quad |z_1 - z_2| \geq |z_2| - |z_1|.$$

Apply this to Riemann sum

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| = \left|\sum_{k=1}^{n} f(z_k)\Delta z_k\right| \leq \sum_{k=1}^{n} |f(z_k)||\Delta z_k| \leq M\sum_{k=1}^{n}|\Delta z_k|,$$

where M is the maximum value of $|f(z)|$ on CONTOUR. We then use that $\sum_k |\Delta z_k|$ can not be longer than the length of the contour L:

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq ML.$$

Cauchy inequality: A function f which is analytic on and inside a circle with radius R centerd at z_0 satisfy $\left|f^{(n)}\right| \leq \frac{n!M}{R^n}$.

Laurent series

Let f be analytic in the area between two circles $r < |z - z_0| < R$, then f can be represented uniquely as the sum of two series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z - z_0)^n}$$
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$
$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{-n+1}}\,\mathrm{d}z$$

Derive a_n from using important integral and divide expression for Laurent series with $(z - z_0)^{n+1}$ and integrating. Derive b_n with same method, but multiplying with $(z - z_0)^{n-1}$ instead.
Series of positive powers converges inside some circle $|z - z_0| < R$.
Series of negative powers converges outside some circle $|z - z_0| > R$.
Often usefull to use PARTIAL FRACTION DECOMPOSITION and

$$\frac{1}{1 - w} = \sum_{n=0}^{\infty} w^n, \quad |w| < 1.$$

Residue: Is the value of the coefficient b_1.
A zero of a function: A point z_0 where f is analytic and $f(z_0) = 0$.
A zero of order m: When $f(z_0) = f^{(1)}(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(}m)(z_0) \neq 0$. This can be factorized as $f(z) = (z - z_0)^m\,g(z)$, where $g(z)$ is analytic and non-zero at z_0.
Isolated singularities: Assume $f(z)$ has an isolated singularty at z_0, with a Laurent series, then:

- **Removable singularity:** If all $b_n = 0$
- **Pole of order m:** If $b_m \neq 0$, but $b_k = 0$ for all $k > m$.
- **Essential singularity:** If there are infinitely many b-terms

Residue theory

If Γ is a simplle closed, positivily oriented contour, and f is analytic on and inside Γ, except at the points $z_1, \ldots z_k$ inside Γ, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{N}\mathrm{Res}(f; z_k). \tag{1}$$

True since only $1/(z - z_0)$ term contributes in integrals, we get no other contributions sice we can deform contour around singularities such that the path between each singularity cancels.
Finding residues:

- Find Laurent series around z_0, then $\mathrm{Res}(f; z_0)$ is b_1.
- Evaluate $\mathrm{Res}(f; z_0) = \lim_{b\to z_0}\left[f(z)\,(z - z_0)\right]$, finite answer only if it is a simple pole. Removable singularity gives zero, higher order poles gives infinity.
- For $f(z) = P(z)/Q(z)$ where $P(z_0) \neq 0$ and $Q(z_0)$ is a SIMPLE ZERO, and both analytic at z_0, then $\mathrm{Res}(f; z_0) = P(z_0)/Q'(z_0)$.
- If f has a pole of order m at z_0 then $\mathrm{Res}(f; z_0) = \lim_{z\to z_0}\frac{1}{(M-1)!}\frac{\mathrm{d}^{M-1}}{\mathrm{d}z^{M-1}}\left[(z - z_0)^M f(z)\right]$ where $M \geq m$. If you know the order of the pole us $m = M$, but get correct result by overshooting as well.

Solving integrals

- **Trigonometric integrals:** Integrals of type $\int_0^{2\pi} u(\cos\theta, \sin\theta)\,\mathrm{d}\theta$, can be made into complex integral by variable substitution $z = e^{i\theta}$ and integrate around $|z| = 1$. Use complex version of $\cos\theta = (z + 1/z)/2$ and $\sin\theta = (z - 1/z)/2i$ and $\mathrm{d}\theta = \mathrm{d}z/(iz)$. Solve the integral with residue theory and take the real value of the final answer. If the integration limits are not 0 to 2π you can use a substitution to change integral limits such that the final limits are 0 to 2π, f.ex $u = 2\pi - \theta$ will change limits from $0 \to \pi$ to $\pi \to 2\pi$.
- **Infinte integrals:** Integrals from $-\infty \to \infty$ can be extended to the complex plane by connecting $\pm$ at the x-axis by a semi-circle with infinite radius giving no contribution. Thus the integral can be solved from residues inside the first and secound quadrant, or third and fourth quadrant depending on orientation of semi-sircle. Only works if: (1) f is analytic on and above the real axis, except for a finite number of singularities, (2) we can ignore contributions infinitely far away from the origin, we can do this $(f(z) = P(z)/Q(z))$ if $\mathrm{DEG}(Q) \geq \mathrm{DEG}(P) + 2$.
- **Infinite Trigonometric integral:** Two options:
1. Write the trigonometric function in it's complex form, which will give two integrals, one with e^{imx} which must be closed in the upper half plane, and one with e^{-imx} which must be closed in the lower half plane. Since the last one goes counter clockwise we must flip the sign. We can ignore the semi-circle only if the degree of the polynomial in the denominator is one larger than the degree of the numerator, this is called **Jordan's lemma**. Without the exponential the difference in polynomial degree would be two.
2. If the integral is real we can write $\cos mx$ as the real part of e^{imx}, or $\sin mx$ as the imaginary part, then solve the integral and take the imaginary or real part of the answer.
- **Singularities on the real axis:** Infinities can *cancel* if apprached symmetrically: $\mathrm{PV}\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to 0}\int_a^{c-r} f(x)\,\mathrm{d}x + \int_{c+r}^{b} f(x)\,\mathrm{d}x$, with a singularity at $x = c$. Can also be computed with residue theory:

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i\sum\underbrace{\mathrm{Res}(f; z_k)}_{\text{upper half}} + \pi i\sum\underbrace{\mathrm{Res}(f; z_j)}_{\text{real axis}}$$

.

Tensors

Tensors of rank 0 = scalars, rank 1 = vectors, rank 2 = matrices. Tensors represent physical quantities, should be independent of choice of coordinate frames $v_i' = A_{ij} v_j$, $v_i = A_{ji} v_j$, where $A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j'$, which implies $A^{-1} = A^T$. We get one A for each index $T_{kl}' = A_{ki} A_{lj} T_{ij}$.

Inertia tensor

A rigid body rotating around a fixed axis: $\boldsymbol{L} = I\boldsymbol{\omega}$, where I is the inertia tensor. Since I is symmetric we can find a coordinate system where I is diagonal. The eigenvectors of I are the principle axis of inertia.
Point masses: $I_{ij} = mr^2\delta_{ij} - mr_ir_j$ where you sum over all masses.
Continouum masses: $I_{ij} = \int mr^2\delta_{ij} - mr_ir_j \,\mathrm{d}m$ where.
Examples:
$I_{xx} = \sum_i m_i(r^2 - x^2) = \sum_i m_i(y^2 + z^2) = \int y^2 + z^2 \,\mathrm{d}m$.
$I_{xy} = -\sum_i m_i x_i y_i = -\int xy \,\mathrm{d}m$.

Cronecker-delta and Levi-civita

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, \text{ eller } 312 \\ -1 & \text{if } ijk = 321, 213, \text{ eller } 132 \\ 0 & \text{if repeating indicies} \end{cases}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$(\mathbf{a} \times \mathbf{b})_i = a_j b_k \epsilon_{ijk}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{1i}a_{2j}a_{3k}\epsilon_{ijk}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{V})_i = \epsilon_{ijk}\frac{\partial V_k}{\partial x_j}$$

$$(\boldsymbol{U} \times \boldsymbol{V})_i = \epsilon_{ijk} U_j V_k$$

Calculus of variations

Given an integral $I = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x$ we want to find the function $x(y)$ or $y(x)$ between (x_1, y_1) and (x_2, y_2) such that I is stationary (locally minimized). The function which satisfies this is

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0.$$

If F has no y dependence we get $\frac{\partial F}{\partial y'}$ = constant, which can be solved for y.
One can change variables such that we have a cylic coordinate through the substitutions: $y' = 1/x'$ and $\mathrm{d}x = x'\,\mathrm{d}y$.

Tips and tricks

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_n \left(\frac{1}{z}\right)^n = \sum_n \left(\frac{1}{z}\right)^{n+1}, z > 1$$

$$ax^2 + bx + c = 0 \qquad \rightarrow \qquad x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ordinary differential eq.

Ordinary: Derivatives with respect to only one variable.
Linear: Only linear terms y, y' etc. no yy', y^2, $(y'')^3$ etc.
Order of DE: Order of highest derivative.
For secound order DE we get exact solution from BC since we can solve DE for y'' and evaluate at BC, and get higher orders from taking the derivative of this equation. From these values we can construct a taylor series of the function, which uniquely fixes the solution.
A linear combination of two solutions is also a solution.
Criteria for two solutions $y_1(x)$ and $y_2(x)$ being linearly independent:

- Linearly independent if $c_1 y_1(x) + c_2 y_2(x) = 0$ can only be satisfied when $c_1 = c_2 = 0$ (y_2 is not a multiple of the other).
- Linearly dependent if $y_2(x_0) = K y_1(x_0)$ AND $y_2'(x_0) = K y_1(x_0)$ they are linearly dependent.
- Linearly independent if $y_1'(x_0)/y_2'(x_0) = y_1(x_0)/y_2(x_0)$, in other words: if the wronskian is non-zero

$$w(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

Then we can write the full solution as $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Integrating factors

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \to \mathrm{d}y + (Py - Q)\,\mathrm{d}x = 0$$

Multiply with $\mu(x)$ to get an exact solution from.

$$\mu(x) = \exp\left\{\int P\,\mathrm{d}x\right\}$$

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,\mathrm{d}x + C\right)$$

Variation of constants

If $y_1(x)$ is a solution we can find another linearly independent solution $y_2(x)$ by writing $y_2(x) = c(x)y_1(x)$, where we determine $c(x)$ from DE. When finding c we can ignore constants since these constants contribute function equal to $y_1(x)$, and constant factors can be determined outside of solution.

Constant coefficients

$$y'' + ay' + by = 0$$

Always has an exponential solution $y = e^{\lambda x}$, inserting gives $\lambda^2 + a\lambda + b = 0$ which we solve to find λ. This gives three possibilities:

- $\lambda_+ \neq \lambda_-$ and both are real: we get two linearly independent solutions $y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$.
- Double root $a^2 - 4ab = 0$: gives $\lambda_+ = \lambda_- = \lambda = -a/2$. Can be shown from variation of constants that we get two linearly independent solutions $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$.
- Complex roots $a^2 - 4b < 0$: gives $\lambda_\pm = -a/2 \pm i\sqrt{4b - a^2}/2 = -a/2 \pm iw$. Solution is the standard form, but can be written in many different forms:
  $y(x) = e^{-ax/2}\left(Ae^{iwx} + Be^{-iwx}\right)$
  $y(x) = e^{-ax/2}\left(C\cos wx + D\sin wx\right)$
  $y(x) = e^{-ax/2}\sin(wx + \delta)E$.

Euler-Cauchy equation

$$x^2 y'' + a_1 xy1 + a_0 y = 0 \qquad \text{or} \qquad y'' + \frac{a_1}{x}y' + \frac{a_0}{x^2} = 0$$

Use substitution $x = e^z$ for $x > 0$ and $x = -|x| = -e^z$ for $x < 0$. This gives us $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x\frac{\mathrm{d}y}{\mathrm{d}z}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1/x^2\frac{\mathrm{d}y}{\mathrm{d}z} + 1/x^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}$, inserting this into DE gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (a_1 - 1)\frac{\mathrm{d}y}{\mathrm{d}z} + a_0 y = 0$$

This is a differential equation with constant coefficients which we can solve, then substitute back $z = \ln|x|$. Get different coefficients from initial conditions for positive and negative x, no solution for $x = 0$.