Problem set 1 - FYS 3140 - Ivar Haugerud

1.1) a) $\sqrt{2}\, e^{\frac{5\pi}{4} i} = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right)$

$= \sqrt{2}\left(-\sqrt{2} + i(-\sqrt{2})\right)$

$= \underline{-2 - 2i}$

b) $\frac{(1+i)^{48}}{(\sqrt{3}-i)^{25}}$, begin by expressing each term in exponential form

$i+1$: $\quad r = \sqrt{1^2 + i \cdot i^*} = \sqrt{2}$

$\theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\sqrt{3} - i$: $\quad r = \sqrt{\sqrt{3}^2 + (-i)(-i)^*} = \sqrt{3+1} = 2$

$\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

We then have:

$$\frac{(1+i)^{48}}{(\sqrt{3}-i)^{25}} = \frac{\left(\sqrt{2}\, e^{i\pi/4}\right)^{48}}{\left(2 e^{-i\pi/6}\right)^{25}} = \frac{2^{24}}{2^{25}} \frac{e^{i\frac{48\pi}{4}}}{e^{-\frac{i\pi 25}{6}}}$$

$$= \frac{1}{2} \frac{e^{i 12\pi}}{e^{-\frac{i 25\pi}{6}}} = \frac{1}{2} e^{i\left(12\pi - \left(-\frac{25\pi}{6}\right)\right)}$$

$$= \frac{1}{2} e^{\frac{i\pi}{6}(72+25)} = \frac{1}{2} e^{\frac{i\pi 97}{6}}$$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi \cdot 97}{6}\right) + i \sin\left(\frac{\pi \cdot 97}{6}\right)\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{3}{4}} + i\,\frac{1}{2}\right) = \frac{\sqrt{3}}{4} + \frac{i}{4} = \underline{\frac{1}{4}\left(\sqrt{3} + i\right)}$$

1.1)

c) Find all roots of

$(8i\sqrt{3}-8)^{1/4}$. Begin by changing to polar coordinates.

$$r = \sqrt{8^2 + (8\sqrt{3}i)(8\sqrt{3}i)^*} = \underline{16}$$

$$\theta = \arctan\left(\frac{8\sqrt{3}}{-8}\right) = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$$

$$\Rightarrow (-8 + i\sqrt{3}8)^{1/4} = \left(16e^{i2\pi/3 + 2\pi n}\right)^{1/4}, \quad n \in N$$

$$= 2e^{i\left(\frac{\pi}{6} + \frac{\pi n}{2}\right)}$$

$n=0$: $\underline{2e^{i\pi/6}}$

$n=1$: $2e^{i\pi\left(\frac{1}{6}+\frac{1}{2}\right)} = \underline{2e^{i\pi 2/3}}$

$n=2$: $2e^{i\pi\left(\frac{1}{6}+\cancel{1}\right)} = 2e^{i\pi 7/6} = \underline{2e^{-i\pi 5/6}}$

$n=3$: $2e^{i\pi\left(\frac{1}{6}+\frac{3}{2}\right)} = 2e^{i\pi\frac{5}{3}} = \underline{2e^{-i\pi/3}}$

d) First find the cube roots of 8

$$8 = 8e^{i\pi\cdot 0}$$

$$8^{1/3} = \left(8e^{i\pi\cdot 0 + 2\pi n}\right)^{1/3} = 2e^{i2\pi n/3}$$

$n=0$: $2e^0 = \underline{2}$

$n=1$: $\underline{2e^{2\pi i/3}}$

$n=2$: $2e^{i4\pi/3} = \underline{2e^{-i2\pi/3}}$

We calculate the sum

$$2\left(1 + e^{2\pi i/3} + e^{-i2\pi/3}\right) = 2\left(1 + \cos\left(\tfrac{2\pi}{3}\right) + i\sin(2\pi/3) + \cos\left(-\tfrac{2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right)\right)$$

1.1) d) We use that $\sin(-x) = -\sin(x)$ such that the complex numbers cancel. Using that $\cos(x) = \cos(-x)$, and that $\cos(2\pi/3) = -\frac{1}{2}$, we get

$$= 2\left(1 - \frac{1}{2} \cancel{+ i\sin(2\pi/3)} - \frac{1}{2} \cancel{- i\sin(2\pi/3)}\right)$$

$$= 2\left(1 - \frac{1}{2} - \frac{1}{2}\right) = \underline{0}$$

We will now try for a general complex number $z = a + ib = re^{i\theta}$. The roots are given by $\left(re^{i\theta + 2\pi n i}\right)^{1/N} = r^{\frac{1}{N}} e^{i\frac{\theta}{N}} \cdot e^{\frac{2\pi n i}{N}}$

We want to calculate the sum of all N roots for a general N

$$S = \sum_{n=0}^{N-1} r^{\frac{1}{N}} e^{i\frac{\theta}{N}} e^{2\pi n i/N} = r^{\frac{1}{N}} e^{\frac{i\theta}{N}} \sum_{n=0}^{N-1} e^{2\pi i n/N}$$

$$S = r^{\frac{1}{N}} e^{\frac{i\theta}{N}} \sum_{n=0}^{N-1} \left(e^{\frac{i2\pi}{N}}\right)^n,$$ where S is just the sum of the roots.

We also have

$$S \cdot e^{\frac{i2\pi}{N}} = r^{\frac{1}{N}} e^{\frac{i\theta}{N}} \sum_{n=0}^{N-1} \left(e^{\frac{i2\pi}{N}}\right)^{n+1}$$

$$= r^{\frac{1}{N}} e^{\frac{i\theta}{N}} \sum_{n=1}^{N} \left(e^{\frac{i2\pi}{N}}\right)^n$$

We can now calculate the difference

$$S - Se^{i2\pi/N} = r^{\frac{1}{N}} e^{\frac{i\theta}{N}} \underbrace{\left(\sum_{n=0}^{N-1} \left(e^{\frac{i2\pi}{N}}\right)^n - \sum_{n=1}^{N} \left(e^{i2\pi/N}\right)^n\right)}_{\text{every term will cancel except first and last}}$$

1.1) d)

$$S\left(1-e^{\frac{i2\pi}{N}}\right) = r^{\frac{1}{N}} e^{\frac{i\theta}{N}} \left(\left(e^{\frac{i2\pi}{N}}\right)^{n=0} - \left(e^{\frac{i2\pi}{N}}\right)^{n=N} \right)$$

$$S = \frac{r^{\frac{1}{N}} e^{\frac{i\theta}{N}}}{1-e^{i2\pi/N}} \left(e^{0} - e^{i2\pi} \right)$$

$$= \frac{r^{\frac{1}{N}} e^{\frac{i\theta}{N}}}{1-e^{i2\pi/N}} (1-1)$$

$$= \underline{0}$$

So the sum is always zero, when summing over all roots og a general complex number with N roots.

The derivation does not ~~[illegible]~~ hold for N = 1, as we would be dividing by zero, but this is not a problem, as the first root is just the number itself

1.2) a) Sum $= \sum_{n=0}^{\infty} n(n+1)(z-2i)^n$

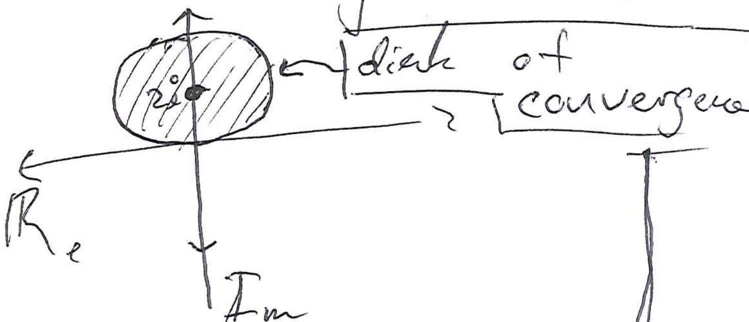
- Take the ratio test

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(n+1)}(n+2)(z-2i)^{n+1}}{n\cancel{(n+1)}(z-2i)^n}\right|$$

$$= \left|\frac{(n+2)}{(n)}(z-2i)\right|$$

$$= \left|\left(1+\frac{2}{n}\right)(z-2i)\right| = |z-2i|$$

- The disk of convergence is a circle with radius 1 around the point $2i$. The radius is 1, as this is the upper limit which guarantees convegence



b) $\sum_{n=1}^{\infty} 2^n (z+i-3)^{2n}$

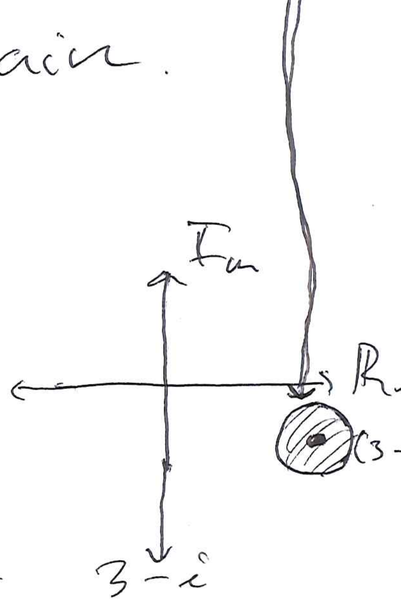
- Use the convergence test again.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(z+i-3)^{2(n+1)}}{2(z+i-3)^{2n}}\right|$$

$$= \lim_{n\to\infty} |2(z+i-3)^2|$$

$$\rho < 1 \implies (z+i-3)^2 < \frac{1}{2}$$

$$(z-(3-i))^2 < \frac{1}{\sqrt{2}}$$



- A circle around the point $3-i$ with a radius of $\frac{1}{\sqrt{2}}$

1.3) a) $\displaystyle\int_0^{2\pi} \sin^2(4x)\,dx = \int_0^{2\pi} \left(\frac{e^{i4x} - e^{-i4x}}{2i}\right)^2 dx$

$$\boxed{\sin(a) = \frac{e^{ia} - e^{-ia}}{2i}} = -\frac{1}{4}\int_0^{2\pi} e^{i8x} + e^{-i8x} - 2 \quad dx$$

$$= -\frac{1}{4}\left[\frac{1}{i8} e^{i8x} - \frac{1}{i8} e^{-i8x} - 2x\right]_0^{2\pi}$$

$$= -\frac{1}{4}\left(\frac{1}{i8}\left(\underbrace{e^{i16\pi}}_{1} - \underbrace{e^{-i16\pi}}_{1} \underbrace{-1+1}_{0}\right) - 4\pi\right)$$

$$= -\frac{1}{32i}\left(\underbrace{1-1}_{0}\right) + \pi$$

$$= \underline{\pi}$$

b) $\sin(2z) = \frac{1}{2i}\left(e^{2iz} - e^{-2iz}\right)$

since the cross-terms cancel

$$= \frac{1}{2i}\left(e^{iz} + e^{-iz}\right)\left(e^{iz} - e^{-iz}\right)$$

$$= 2 \cdot \underbrace{\frac{1}{2}\left(e^{iz} + e^{-iz}\right)}_{\cos(z)} \cdot \underbrace{\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)}_{\sin(z)}$$

$$= \underline{2\cos(z) \cdot \sin(z)}$$

1.3)

c) $\cosh^2 z - \sinh^2 z$

$\cosh(x) = \frac{e^{-x} + e^{x}}{2}$

$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$

$= \left(\frac{e^{-z} + e^{z}}{2}\right)^2 - \left(\frac{e^{z} - e^{-z}}{2}\right)^2$

$= \frac{1}{4}\left(e^{-2z} + e^{2z} + 2e^{0}\right) - \frac{1}{4}\left(e^{2z} + e^{-2z} - 2e^{0}\right)$

$= \frac{1}{4}\left(\cancel{e^{-2z}} - \cancel{e^{-2z}} + \cancel{e^{2z}} - \cancel{e^{2z}} + 2 - (-2)\right)$

$= \underline{\underline{1}}$

d) $\sin\left(i \ln\left(\frac{1-i}{1+i}\right)\right) = \frac{1}{2i}\left(e^{i i \ln\left(\frac{1-i}{1+i}\right)} - e^{-i i \ln\left(\frac{1-i}{1+i}\right)}\right)$

$= \frac{1}{2i}\left(e^{-\ln\left(\frac{1-i}{1+i}\right)} - e^{\ln\left(\frac{1-i}{1+i}\right)}\right)$

$= \frac{1}{2i}\left(\left(\frac{1-i}{1+i}\right)^{-1} - \left(\frac{1-i}{1+i}\right)\right)$

$= \frac{1}{2i}\left(\frac{1+i}{1-i} - \frac{1-i}{1+i}\right)$ Make common factor

$= \frac{1}{2i}\left(\frac{(1+i)(1+i)}{(1-i)(1+i)} - \frac{(1-i)(1-i)}{(1+i)(1+i)}\right)$

$= \frac{1}{2i}\left(\frac{1 + 2i - 1 - (1 - 2i - 1)}{1 + i - i + 1}\right)$

$= \frac{1}{4i}(4i) = \underline{1}$

e) $(-e)^{i\pi} = (-1)^{i\pi} \cdot e^{i\pi}$

$$= \left(e^{i\pi}\right)^{i\pi} e^{i\pi}$$

$$= e^{i^2\pi^2} e^{i\pi}$$

$$= e^{-\pi^2} e^{i\pi}$$

$$= e^{-\pi^2} \cdot (-1)$$

$$= -e^{-\pi^2} \approx -5{,}17 \cdot 10^{-5}$$

f) Want to find arctanh(x)

$y = \operatorname{arctanh}(x)$

$\tanh(y) = x$ want to solve for y

$$\frac{e^y - e^{-y}}{e^y + e^y} = x$$

$$= \frac{e^y}{e^y}\left(\frac{e^y - e^{-y}}{e^y + e^{-y}}\right) = \frac{e^{2y} - 1}{e^{2y} + 1} = x$$

$$e^{2y} - 1 = xe^{2y} + x$$

$$e^{2y} - xe^{2y} = x + 1$$

$$e^{2y}(1 - x) = x + 1$$

$$e^{2y} = \frac{x+1}{1-x}$$

$$2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

So this is the expression for arctanh(x)

Extra problem

2.17.30)

$$e^{x(1+i)} = \sum_{n=0}^{\infty} \frac{(x(1+i))^n}{n!} = \sum_{n=0} \frac{x^n}{n!}(1+i)^n$$

$$\boxed{1+i = \sqrt{2}\, e^{i\pi/4}} \longrightarrow = \sum_{n=0} \frac{x^n}{n!}(\sqrt{2}\, e^{i\pi/4})^n$$

$$= \sum_{n=0} \frac{x^n \sqrt{2}^n}{n!} e^{\frac{i\pi n}{4}}$$

$$e^x \cdot \cos(x) = \frac{e^x}{2}\left(e^{ix} + e^{-ix}\right) = \frac{1}{2}\left(e^{x(1+i)} + e^{x(1-i)}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n\sqrt{2}^n}{n!} e^{i\pi n/4} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n \sqrt{2}^n}{n!} e^{-\frac{i\pi n}{4}} \qquad \boxed{1-i = \sqrt{2}e^{-i\pi n/4}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n\sqrt{2}^n}{n!}\left(e^{\frac{i\pi n}{4}} + e^{-\frac{i\pi n}{4}}\right) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n\sqrt{2}^n}{n!}\left(\cos\left(\frac{\pi n}{4}\right) + i\sin\left(\frac{\pi n}{4}\right) + \cos\left(-\frac{\pi n}{4}\right) + i\sin\left(-\frac{\pi n}{4}\right)\right)$$

Use that cos is symmetric and sin antisymmetric

$$= \sum_{n=0}^{\infty} \frac{x^n\sqrt{2}^n}{n!}\cos\left(\frac{\pi n}{4}\right),$$ we see that if the argument of cos is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, the term vanishes in the series.

The values of n which vanishes the terms are therefore: $n=2$, $n=6$, $n=10$, $n=14$...

$n = 2+4k$, k is a whole number > 0.

Extra Problem

Do the same for

$$e^x \cdot \sin(x) = \frac{e^x}{2i}\left(e^{ix} - e^{-ix}\right) = \frac{1}{2i}\left(e^{x(1+i)} - e^{x(1-i)}\right)$$

This is exactly the same as we had for $e^x \cdot \cos(x)$, but now the cosines canal, leaving us with:

$$e^x \sin(x) = \frac{1}{2i}\sum \frac{x^n \sqrt{2}^n}{n!}\left(i\sin\left(\frac{\pi n}{4}\right) - \left(-i \sin\left(\frac{\pi n}{4}\right)\right)\right)$$

$$= \sum \frac{x^n \sqrt{2}^n}{n!} \sin\left(\frac{\pi n}{4}\right)$$

When sin is zero the terms vanishes from the series, this happens when

$\frac{\pi n}{4} = \pi k \Rightarrow n = 4k, \quad k \in \mathbb{N}^+$

So the terms are zero for $n=0$, $n=4$, $n=8$, $n=12 \dots$