Oblig 3 - FYS3140 - Ivar Haugerud

Since f is analytic we can taylor-expand the function perfectly, since all the derivatives of f exists, due to it's analyticity.

$$f(z) = \sum_k \frac{f^{(k)}(z-z_0)^k}{k!} \quad \Big| \cdot (z-z_0)^{-n}$$

$$\frac{f(z)}{(z-z_0)^n} = \sum_k \frac{f^{(k)}}{k!}(z-z_0)^{k-n}$$

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^n}\,dz = \sum_k \oint_\Gamma \frac{f^{(k)}}{k!}(z-z_0)^{k-n}\,dz$$

$$\oint \frac{f(z)}{(z-z_0)^n}\,dz = \sum_k \frac{f^{(k)}}{k!} \oint_\Gamma (z-z_0)^{k-n}\,dz$$

$$\oint \frac{f(z)}{(z-z_0)^n}\,dz = \sum_k \frac{f^{(k)}}{k!}\,\delta_{k-n,-1} \cdot 2\pi i$$

$$\boxed{\begin{aligned} &k-n=-1 \\ &\Rightarrow k=-1+n \end{aligned}}$$

$$\oint \frac{f(z)}{(z-z_0)^n}\,dz = \frac{f^{(n-1)}}{(n-1)!}\,2\pi i$$

$$f^{(n-1)} = \frac{(n-1)!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^n}\,dz$$

$$\boxed{f^{(n)} = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

3.1) I will try another derivation using the hint

We use Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0} \, dz$$

and derivate both sides with respect to z_0

$$\frac{\partial f}{\partial z_0} = \frac{1}{2\pi i} \oint_\Gamma dz \, f(z) \frac{\partial}{\partial z_0} \left(\frac{1}{z - z_0} \right)$$

$$= \frac{1}{2\pi i} \oint_\Gamma dz \, f(z) \frac{1}{(z - z_0)^2} \cdot \underbrace{(-1)}_{\text{from expont}} \cdot \underbrace{(-1)}_{\text{from } \frac{\partial(-z_0)}{\partial z_0} \text{ from chain rule}}$$

$$= \frac{1}{2\pi i} \oint_\Gamma dz \, \frac{f(z)}{(z - z_0)^2}$$

$$\frac{\partial^2 f}{\partial z_0^2} = \frac{1}{2\pi i} \oint dz \, f(z) \frac{\partial}{\partial z_0} \left(\frac{1}{(z - z_0)^2} \right)$$

$$= \frac{1}{2\pi i} \oint dz \, f(z) \cdot \frac{1}{(z - z_0)^3} \cdot (-2) \cdot (-1)$$

$$= \frac{2}{2\pi i} \oint dz \, \frac{f(z)}{(z - z_0)^3}$$

We see now that taking the derivative will always result in the sign staying the same, while we for each derivative multiply by the exponent from the chain rule, meaning taking the derivative n-times gives us

$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot n = n!$ Resulting in the generalized cauchy integral formula

$$\frac{n!}{2\pi i} \oint_\Gamma \frac{f(w) \, dw}{(w - z)^{n+1}} = f^{(n)}(z)$$

3.1) We will now use the generalized Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(w)\,dw}{(w-z)^{n+1}}$$

to solve $\oint_{\Gamma} \frac{\sin(2z)}{(6z-\pi)^3}\,dz$, where Γ is

a circle $|z|=2$, positively oriented.

We have $n=2$, $f(w) = \sin(2z)$

$$\oint \frac{\sin(2z)}{(6z-\pi)^3}\,dz = \oint \frac{\sin(2z)}{\left(3\left(2z-\frac{\pi}{3}\right)\right)^3}\,dz = \frac{1}{27} \oint \frac{\sin(2z)}{\left(2z-\frac{\pi}{3}\right)^3}\,dz$$

$$\frac{\sin^{(2)}(2z)\cdot 2\pi i}{2!} = \oint \frac{\sin(2w)\,dw}{\left(2w-\frac{\pi}{3}\right)^3}$$

$$-\frac{4\sin(2z)\cdot 2\pi i}{2} = \oint \frac{\sin(2w)\,dw}{\left(2w-\frac{\pi}{3}\right)^3}$$

$$= -4\pi i \sin\left(\frac{2\pi}{3}\right) = -4\pi i \sqrt{\frac{3}{4}} = \underline{-2\sqrt{3}\pi i}$$

But we have to remember the $\frac{1}{27}$ factor

$$\oint_{\Gamma} \frac{\sin(2z)\,dz}{(6z-\pi)^3} = \underline{-\frac{1}{27}\cdot 2\sqrt{3}\pi i} = \underline{\underline{-\frac{2\pi i}{9\sqrt{3}}}}$$

3.2) a)

$$\oint_\Gamma \frac{\cosh(z)}{2\ln(2) - z}\,dz = -\oint_\Gamma \frac{\cosh(z)\,dz}{z - 2\ln(2)}$$

$$= -2\pi i \cdot \frac{1}{2\pi i}\oint_\Gamma \frac{\cosh(z)\,dz}{z - 2\ln(2)}$$

Since we have a singularity at $z = 2\ln(2) < 3$ the integral is not zero, we use Cauchy's integral formula

$$= -2\pi i \cdot \cosh(2\ln(2))$$

$$= -2\pi i \cdot \frac{1}{2}\left(e^{2\ln(2)} + e^{-2\ln(2)}\right)$$

$$= -\pi i\left(2^2 + \frac{1}{2^2}\right)$$

$$= -i\pi\left(\frac{16+1}{4}\right) = \underline{\underline{\frac{-17 i\pi}{4}}}$$

b) To solve this we will use the generalized cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$$

to solve $\oint_\Gamma \frac{e^{3z}\,dz}{(z-\ln(2))^4}$, where Γ is a ~~sq~~ square with length 1 from the origin in the complex plane and therefore contains the singularity $\ln(2)$.

We see that $z_0 = \ln(2)$, $n = 3$, $f(z) = e^{3z}$, first we find it's derivative

$f'(z) = 3e^{3z}$, $f''(z) = 9e^{3z}$, $f'''(z) = 27e^{3z}$, meaning that

$$\frac{27 \cdot e^{3\cdot\ln(2)} \cdot 2\pi i}{4!} = \oint_\Gamma \frac{e^{3z}\,dz}{(z-\ln(z))^4}$$

$$= \frac{27 \cdot 2^3 \cdot 2\pi i}{24} = \boxed{18\pi i = \oint_\Gamma \frac{e^{3z}\,dz}{(z-\ln(2))^4}}$$

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3.3) We begin by rewriting $f(z)$

$$f(z) = \frac{z-1}{z^2(z-2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{(z-2)}$$

$$\Rightarrow \quad z-1 = Az(z-2) + B(z-2) + Cz^2$$

$z=2$: $\quad 1 = 0 + 0 + 4C \Rightarrow \underline{C = \frac{1}{4}}$

$z=0$: $\quad -1 = 0 - 2B + 0 \Rightarrow \underline{B = \frac{1}{2}}$

$z=1$: $\quad 0 = -A - B + C$

$$A = C - B = \frac{1}{4} - \frac{1}{2} \Rightarrow \underline{A = -\frac{1}{4}}$$

$$f(z) = -\frac{1}{4z} + \frac{1}{2z^2} + \frac{1}{4(z-2)}$$

We will use $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ for $|w| < 1$

$$\frac{1}{4} \cdot \frac{1}{z-2} = -\frac{1}{4} \cdot 2^{-1} \cdot \frac{1}{1-\frac{z}{2}} = \underline{-\frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n}$$

Giving us the series

$$f(z) = \frac{1}{2z^2} - \frac{1}{4z} - \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \frac{1}{2z^2} - \frac{1}{4z} - \frac{1}{8} - \frac{z}{16} - \frac{z^2}{32} - \frac{z^3}{64} - \dots$$

Which is valid for $|z| < 2$, for $z \neq 0$

b) The terms $-\frac{1}{4z}$ and $\frac{1}{2z^2}$ converge for $|z| > 2$, so we need to change the term $\frac{1}{4(z-2)}$ so that it becomes valid for $|z| > 2$

$$\frac{1}{4} \cdot \frac{1}{z-2} = \frac{1}{4z} \cdot \frac{1}{1-\frac{2}{z}} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

which converges for $\frac{2}{|z|} < 1 \Rightarrow \underline{|z| > 2}$

Meaning that our series is

$$f(z) = \frac{1}{2z^2} - \frac{1}{4z} + \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

c) The residue is the coefficient b_1 of the $\frac{1}{z-z_0}$ term in the Laurent series.

Our Laurent series for $|z| < 2$ was

$$f(z) = \frac{1}{2z^2} - \frac{1}{4z} - \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

We therefore have $\underline{b_1 = -\frac{1}{4}}$ and $z_0 = 0$

The residue at the origin is $\underline{-\frac{1}{4}}$

3.4 a) The Taylor expansion of $\sin(z)$ around $z=0$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

With this we get

$$\frac{\sin z}{3z} = \frac{1}{3z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$$= \frac{1}{3}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)$$

We see that $z=0$ is a regular point, the apparent singularity is removable.

b) $$\frac{\cos(z)}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$$= \frac{1}{z^4} - \frac{1}{z3!} + \frac{z}{5!} - \dots$$

As the first (lowest) exponent is 4, we have a pole of order 4

c) $\dfrac{z^3-1}{(z-1)^3}$ we try to use partial fraction decomposition to rewrite the expression

$$\frac{z^3-1}{(z-1)^3} = A + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

$$z^3-1 = A(z-1)^3 + B(z-1)^2 + C(z-1)$$

$$
\begin{array}{llr}
z=0: & -1 = -A + B - C & \text{I} \\
z=-1: & -2 = -8A + 4B - 2C & \text{II} \\
z=2: & 7 = A + B + C & \text{III}
\end{array}
$$

$\text{III} - \text{I}: \quad 8 = 2A + 2C$

$\text{III} + \text{I}: \quad 6 = 2B \Rightarrow \underline{B = 3}$

$\text{III}: \quad 4 = A + C \Rightarrow A = 4 - C$

$\text{II}: \quad -2 = -8(4-C) + 4\cdot 3 - 2C$

$$-2 = -32 + 8C + 12 - 2C$$

$$18 = 6C \Rightarrow \underline{C = 3}$$

$$A = 4 - C \Rightarrow \underline{A = 1}$$

$$\frac{z^3-1}{(z-1)^3} = 1 + \frac{3}{z-1} + \frac{3}{(z-1)^2}$$

Here we see that we have a pole of order 2, since this is the largest exponent.

d) $\frac{e^2}{(z-1)}$ Since the numerat is not zero for $z=1$ we see that we have a pole of order 1, a simple pole.

We can see this from looking at the taylor expansion around $z=1$,

$$e^z = e + e(z-1) + \frac{e(z-1)^2}{2} + \frac{e(z-1)^3}{6}$$

Leaving us with

$$\frac{e^z}{(z-1)} = \frac{e + e(z-1) + \frac{e(z-1)^2}{2} + \frac{e(z-1)^3}{6}}{(z-1)}$$

$$= \frac{e}{(z-1)} + e + \frac{e(z-1)}{2} + \frac{e(z-1)^2}{6}$$

And we again see that this is a pole of order 1, with a residue of $\underline{e}$