Oblig 7 - FYS3140 - Ivar Haugerud

7.1) a) $dy + (\overbrace{2x}^{P(x)}y - \overbrace{xe^{-x^2}}^{Q(x)})dx = 0$

We begin by finding the integrating factor $\mu(x)$

$\mu(x) = e^{\int P dx}$, where $P = 2x \Rightarrow \int P(x)\,dx = x^2$

$\Rightarrow \underline{\mu(x) = e^{x^2}}$

We find y by solving

$$\int \mu(x)\, Q(x)\, dx = \int x e^{-x^2} e^{x^2} dx = \int x\, dx$$

$$= \frac{1}{2}x^2$$

This makes

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)\, Q(x)\, dx + C\right)$$

$$= \underline{e^{-x^2}(x + C)}$$

b) $\frac{dy}{dx} + y\cos(x) = \sin(2x)$

$$\frac{dy}{dx} + y\cos(x) - \sin(2x) = 0$$

$$dy + (y \underbrace{\cos(x)}_{P(x)} - \underbrace{\sin(2x)}_{Q(x)})dx = 0$$

We begin by finding the integrating factor

$$\mu(x) = e^{\int P\,dx} = e^{\int \cos(x)\,dx} = e^{\sin(x)}$$

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)\,Q(x)\,dx + C\right)$$

$$= e^{-\sin(x)}\left(\int e^{\sin(x)} \sin(2x)\,dx + C\right)$$

So let's do the integral

$$\int e^{\sin x} \sin(2x)\,dx = 2\int e^{\sin x} \sin x \cdot \cos x\,dx$$

$$u = \sin x, \quad \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$$

$$\Rightarrow 2\int e^u u\,du = 2(e^u \cdot u - \int e^u\,du)$$

$$= 2e^u(u-1)$$

$$= 2e^{\sin x}(\sin x - 1)$$

integration by parts
$\int f' \cdot g\,dx = f \cdot g - \int f \cdot g'$
with $f = e^u$, $g = u$

We put this in for y

$$y = e^{-\sin(x)}\left(2e^{\sin x}(\sin x - 1) + C\right) = \underline{2(\sin x - 1) + Ce^{-\sin}}$$

c) $y' \cdot \cos x + y = \cos^2 x$

$$y' + \frac{y}{\cos x} - \cos x = 0$$

$$\frac{dy}{dx} + y \cdot \frac{1}{\cos(x)} - \cos(x) = 0$$

$$dy + \Big(\mu \cdot \overbrace{\frac{1}{\cos(x)}}^{P(x)} - \overbrace{\cos(x)}^{Q(x)}\Big) = 0$$

We begin py finding the integegrating constant $\mu(x)$ by solving $\int p(x)\,dx = \int \frac{1}{\cos(x)}\,dx$

$$= \int \frac{\tan(x) + \frac{1}{\cos(x)}}{\cos(x)\left(\tan(x) + \frac{1}{\cos(x)}\right)}\,dx = \int \frac{\tan(x) + \frac{1}{\cos(x)}}{\sin(x) + 1}\,dx$$

$$u = \tan(x) + \frac{1}{\cos x}, \quad \frac{du}{dx} = \frac{1}{\cos^2 x}\left(1 + \sin(x)\right)$$

$$\Rightarrow dx = \frac{\cos^2 x}{1 + \sin(x)}\,du$$

$$\int \frac{u}{\sin(x)+1} \cdot \frac{\cos^2 x}{1+\sin x}\,du = \int \left(\frac{\cos}{1+\sin x}\right)^2 u\,du$$

$$= \int \frac{1}{\left(\frac{1}{\cos x} + \tan x\right)^2}\,u\,du = \int \frac{u}{u^2}\,du$$

$$= \int \frac{1}{u}\,du = \ln(u) = \ln\left(\tan(x) + \frac{1}{\cos(x)}\right)$$

$$\Rightarrow \mu = e^{\int p\,dx} = \tan(x) + \frac{1}{\cos(x)}$$

~~De~~ Put this in for ~~...~~ y

$$y = \frac{1}{\mu(x)}\left(\int \mu(x) \cdot Q(x) + C \right)$$

$$= \frac{1}{\tan(x) + \frac{1}{\cos(x)}} \left(\int \cos(x) \cdot \left(\frac{\sin(x)}{\cos(x)} + \frac{1}{\cos(x)} \right. + C \right)$$

$$= \frac{\cos(x)}{\sin(x)+1} \left(\int \sin x + 1 \, dx + C \right)$$

$$= \frac{\cos(x)}{\sin(x)+1} \left(-\cos(x) + x + C \right)$$

7.2) a) $4y'' + 12y' + 9y = 0$

Need to solve characteristic equation

$$4\lambda^2 + 12\lambda + 9 = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-12 \pm \sqrt{144 - 144}}{2 \cdot 4} = \frac{-12 \pm 0}{8}$$

$$= -\frac{3}{2}$$

We then have one solution $y_1(x) = e^{-\frac{3}{2}x}$

We will assume $y_2(x) = c(x)\, y_1(x) = c(x)\, e^{\lambda x}$
And put it into our equation

$$\frac{dy_2}{dx} = c' e^{\lambda x} + c\lambda e^{\lambda x} = e^{\lambda x}(c' + c\lambda)$$

$$\frac{d^2y_2}{dx^2} = \lambda e^{\lambda x}(c' + c\lambda) + e^{\lambda x}(c'' + c'\lambda)$$

$$= e^{\lambda x}(c'\lambda + c\lambda^2 + c'' + c'\lambda)$$

$$= e^{\lambda x}(c\lambda^2 + 2c'\lambda + c'')$$

$$4e^{\lambda x}(c\lambda^2 + 2c'\lambda + c'') + 12e^{\lambda x}(c' + c\lambda) + 9e^{\lambda x}c$$

$$= e^{\lambda x}\left(4c\lambda^2 + 8c'\lambda + 4c'' + 12c' + 12c\lambda + 9c\right), \quad \lambda = -\frac{3}{2}$$

$$= e^{\lambda x}\left(9c + -12c' + 4c'' + 12c' + -18c + 9c\right)$$

$$= e^{\lambda x}(4c'') = 0 \implies c'' = 0 \implies c = Ax + B$$

$$\implies y_2(x) = e^{-\frac{3}{2}x}(Ax + B)$$

$$\implies y(x) = e^{-\frac{3x}{2}}(Ax + B)$$

b) ~~...~~ $y'' - 4y' + 13y = 0$

$\lambda^2 - 4\lambda + 13 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$

$$= \underline{2 \pm 3i}$$

Our solution is then on the form

$$y(x) = A e^{x(2+3i)} + B e^{x(2-3i)}$$

$$= e^{2x}\left(A e^{3xi} + B e^{-3xi}\right)$$

$$= e^{2x}\left(A[\cos(3x) + i\sin(3x)] + B[\cos(3x) - i\sin(3x)]\right)$$

$$= e^{2x}\left(\overbrace{[A+B]}^{\tilde{A}}\cos(3x) + \overbrace{i[A-B]}^{\tilde{B}}\sin(3x)\right)$$

$$= \underline{e^{2x}\left(\tilde{A}\cos(3x) + \tilde{B}\sin(3x)\right)}$$

7.3) a) $xy'' - 2(x+1)y' + (x+2)y = 0$

We guess on a solution on the form $y = u(x)\,v(x)$, with $v(x) = e^x$

$$y' = u'e^x + ue^x = e^x(u'+u)$$
$$y'' = e^x(u'+u) + e^x(u''+u')$$
$$= e^x(u'' + 2u' + u)$$

Put this in

$$xe^x(u''+2u'+u) - 2(x+1)(u'+u)e^x + (x+2)ue^x = 0$$

$$e^x\left(xu'' + 2u'x + xu - 2xu' - 2u' - 2xu - 2u + ux + 2u\right) = 0$$

$$e^x\left(xu'' + u'[2x - 2x - 2] + u(x - 2x - 2 + x + 2)\right) = 0$$

$$e^x(xu'' - 2u') = 0$$

$$\Rightarrow \quad xu'' - 2u' = 0 \quad \Rightarrow \quad u'' = \frac{2u'}{x}$$

By inspection we see

$$u(x) = Ax^3 \Rightarrow u' = 3Ax^2 \Rightarrow u'' = 6Ax = \frac{2}{x}u'$$

A solution to the differential equation is therefore

$$y(x) = 3Ax^2 e^x + Bx^2 e^x$$

$$xe^x - 2(x+1)e^x + (x+2)e^x = 0$$
$$e^x(x - 2x - 2 + x + 2) = 0$$
$$e^x(2x - 2x + 2 - 2) = 0 \quad \Rightarrow \quad e^x \text{ is a solution}$$

7.3b) $x^2 y'' + (x+1)y' - y = 0$, $\quad y = x+1$

$y' = 1$

$0 + (x+1)\cdot 1 - (x+1) = 0 \qquad y'' = 0$

$y_1 = x+1$ is a solution,

assume secondary solution $y_2(x) = u(x)\, y_1(x)$

$y_2(x) = u(x+1)$

$y_2'(x) = u'(x+1) + u$

$y_2''(x) = u''(x+1) + u' + u' = u''\cdot(x+1) + 2u'$

Put this in

$x^2(u''[x+1] + 2u') + (x+1)(u'(x+1) + u) - u(x+1) = 0$

$= u''(x^3+x^2) + 2x^2u' + u'(x+1)^2$

$= u''(x^3+x^2) + u'(2x^2 + (x+1)^2)$

$= u''(x^3+x^2) + u'(2x^2 + x^2 + 2x + 1)$

$= u''(x^3+x^2) + u'(3x^2+2x+1) = 0$

$f = u'$, $\quad f' = u''$

$f' + f\cdot\dfrac{(3x^2+2x+1)}{(x^3+x^2)} = 0$, $\qquad P(x) = \dfrac{3x^2+2x+1}{x^3+x^2}$

$Q(x) = 0$

We need to solve

$\int \dfrac{3x^2+2x+1}{x^2(x+1)}\, dx$

7.3b) $\int \frac{3x^2+2x+1}{x^2(x+1)}\,dx = \int \frac{3x^2+2x+1-2x^2+2x^2}{x^2(x+1)}\,dx$

$= \int \frac{x^2+2x+1}{x^2(x+1)}\,dx + \int \frac{2x^2}{x^2(x+1)}\,dx$

$= \int \frac{(x+1)^2}{x^2(x+1)}\,dx + 2\int \frac{1}{x+1}\,dx$

$= \int \frac{x+1}{x^2}\,dx + 2\ln(x+1) + C$

$= \int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx + 2\ln(x+1) + C$

$= \ln(x) - \frac{1}{x} + 2\ln(x+1) + C$

We have found $f = u' = \ln(x) - \frac{1}{x} + 2\ln(x+1) + C$

$\Rightarrow u = \int \ln(x) - \frac{1}{x} + 2\ln(x+1) + C\,dx$

$= x(\ln x - 1) - \ln(x) + 2(x+1)(\ln(x+1) - 1) + Cx + D$

$= x\ln(x) - x - \ln(x) + (2x+2)\ln(x+1) - 2x - 2 + Cx + D$

$= x\ln(x) + (2x+2)\ln(x+1) - \ln(x) - 3x - 2 + Cx + D$

$= \ln(x)(x-1) + \ln(x+1)(2x+2) + x(C-3) + (D-2)$

The solution is therefore equal to

$Y(x) = (x+1)\left[\ln(x)(x-1) + \ln(x+1)(2x+2) + x(C-3) + (D-2)\right]$