Complex analysis

Definitions

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right) \quad \cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

Roots

$$\left(e^{i\theta}\right)^n = \left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta$$
$$z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n}e^{i\theta/n}.$$
$$\ln z = \ln re^{i\theta} = \ln r + i\theta, \quad \theta \in [0, 2\pi]$$

Complex series

Comparison test: If $|Z_n| \leq a_n$ and $\sum a_n$ converges, then $\sum z_n$ converges.
Ratio test: If $\left|\frac{z_{n+1}}{z_n}\right| \leq k$ for all n sufficiently large, and $k < 1$, then $\sum z_n$ converges absolutely.
Divergence check: If z_n does not converge towards zero, then $\sum z_n$ diverges; the complex and imaginary part will diverge seperately. **Complex power series:** The ratio test gives convergence for $|z - z_0| < \left|\frac{a_n}{a_{n+1}}\right| = R$ as $n \to \infty$. We call R radius of convergence, and $|z - z_0| < R$ for the disk of convergence.

Cauchy-Riemann equations

Analytic: $\leftrightarrow$ Has a unique derivative at wanted region. If a function is analytic it has unique derivatives of all orders and is a solution of Laplace's equation.
Regular point: A point where $f(z)$ is analytic.
Singular point or **Singularity:** A point where $f(z)$ is not analytic.
Isolated singularity: If $f(z)$ is analytic in a small circle around, but not at, the given point.
CR-eq. is a tool to check if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a region by requiring existence of a unique derivative

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

must be satisfied in that region.
If a function is analytic within some region it can be expanded as a Taylor series about any point z_0 inside the region. The power series converges inside a circle about z_0 that extends to the nearest singular point.
Harmonic functions: Are solutions to the 2D Laplace equation $\nabla^2\phi = 0$. If a function $f(z) = u(x, y) + iv(x, y)$ is ANALYTIC in a region, then u and v are harmonic functions.
Harmonic conjugate: Given a harmonic function $u(x, y)$, there exists another harmonic functions $v(x, y)$ such that $u + iv$ is an analytic function of z in that region; v is called the HARMONIC CONJUGATE.

- Check that $u(x, y)$ is harmonic
- Find harmonic conjugate through Cauchy-Riemann equations through integration
- Express $u + iv$ in terms of z

Integrals of complex functions

Contours: Finite sequence of directed smooth curves patched together.
Simple closed contours: A contour which does not cross itself.
Positively oriented contour: A contour with interior to the left and exterior to the right, arrow going counter clockwise.
Loop: A closed contour.
Contour integral:

$$\int_\Gamma f(z)\,\mathrm{d}z = \lim_{n\to\infty}\sum_{k=1}^{n} f(z_k)\Delta z_k.$$

If a function is continuous in a domain then

- f has an anti-derivative
- Contour integrals are independent of path
- Any loop integral is zero

If any of these are true, the two others are true.

$$\int_C (z - z_0)^n\,\mathrm{d}z = \delta_{n,-1}.$$

Inside a simply connected region (no singularities) we continuously deform contours without changing the integral of analytic functions along these curves.
Cauchy-Integral formula: For an function f, which is analytic inside a simply connected region containing a contour Γ which is simple, closed and positively oriented, then:

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

$$f(z_0)^{(n)} = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z, n \geq 1$$

Liouville's theorem: A bounded analytic function in the entire complex plane is a constant.

Upper bound estimates

Use the generalized triangle inequality

$$\left|\sum_k z_k\right| \leq \sum_k |z_k| \quad \to \quad |z_1 - z_2| \geq |z_2| - |z_1|.$$

Apply this to Riemann sum

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| = \left|\sum_{k=1}^{n} f(z_k)\Delta z_k\right| \leq \sum_{k=1}^{n}|f(z_k)||\Delta z_k| \leq M\sum_{k=1}^{n}|\Delta z_k|,$$

where M is the maximum value of $|f(z)|$ on CONTOUR. We then use that $\sum_k |\Delta z_k|$ can not be longer than the length of the contour L:

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq ML.$$

Cauchy inequality: A function f which is analytic on and inside a circle with radius R centred at z_0 satisfy $\left|f^{(n)}\right| \leq \frac{n!M}{R^n}$.

Laurent series

Let f be analytic in the area between two circles $r < |z - z_0| < R$, then f can be represented uniquely as the sum of two series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z - z_0)^n}$$
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$
$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{-n+1}}\,\mathrm{d}z$$

Derive a_n from using important integral and divide expression for Laurent series with $(z - z_0)^{n+1}$ and integrating. Derive b_n with same method, but multiplying with $(z - z_0)^{n-1}$ instead.
Series of positive powers converges inside some circle $|z - z_0| < R$.
Series of negative powers converges outside some circle $|z - z_0| > R$.
Often useful to use PARTIAL FRACTION DECOMPOSITION and

$$\frac{1}{1 - w} = \sum_{n=0}^{\infty} w^n, \quad |w| < 1.$$

Residue: Is the value of the coefficient b_1.
A zero of a function: A point z_0 where f is analytic and $f(z_0) = 0$.
A zero of order m: When $f(z_0) = f^{(1)}(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^(m)(z_0) \neq 0$. This can be factorized as $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic and non-zero at z_0.
Isolated singularities: Assume $f(z)$ has an isolated singularity at z_0, with a Laurent series, then:

- **Removable singularity:** If all $b_n = 0$
- **Pole of order m:** If $b_m \neq 0$, but $b_k = 0$ for all $k > m$.
- **Essential singularity:** If there are infinitely many b-terms

Residue theory

If Γ is a simple closed, positively oriented contour, and f is analytic on and inside Γ, except at the points $z_1, \ldots z_k$ inside Γ, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{N}\mathrm{Res}(f; z_k). \qquad (1)$$

True since only $1/(z - z_0)$ term contributes in integrals, we get no other contributions sice we can deform contour around singularities such that the path between each singularity cancels. A function $f(z)$ can have a singularity in infinity if $f(1/z)$ has a pole in $z = 0$, their residues are the same. **Finding residues:**

- Find Laurent series around z_0, then $\mathrm{Res}(f; z_0)$ is b_1.
- Evaluate $\mathrm{Res}(f; z_0) = \lim_{b\to z_0}[f(z)(z - z_0)]$, finite answer only if it is a simple pole. Removable singularity gives zero, higher order poles gives infinity.
- For $f(z) = P(z)/Q(z)$ where $P(z_0) \neq 0$ and $Q(z_0)$ is a SIMPLE ZERO, and both analytic at z_0, then $\mathrm{Res}(f; z_0) = P(z_0)/Q'(z_0)$.
- If f has a pole of order m at z_0 then $\mathrm{Res}(f; z_0) = \lim_{z\to z_0}\frac{1}{(M-1)!}\frac{\mathrm{d}^{M-1}}{\mathrm{d}z^{M-1}}\left[(z - z_0)^M f(z)\right]$ where $M \geq m$. If you know the order of the pole us $m = M$, but get correct result by overshooting as well.

Solving integrals

- **Trigonometric integrals:** Integrals of type $\int_0^{2\pi} u(\cos\theta, \sin\theta)\,\mathrm{d}\theta$, can be made into complex integral by variable substitution $z = e^{i\theta}$ and integrate around $|z| = 1$. Use complex version of $\cos\theta = (z + 1/z)/2$ and $\sin\theta = (z - 1/z)/2i$ and $\mathrm{d}\theta = \mathrm{d}z/(iz)$. Solve the integral with residue theory and take the real value of the final answer. If the integration limits are not 0 to 2π you can use a substitution to change integral limits such that the final limits are 0 to 2π, f.ex $u = 2\pi - \theta$ will change limits from $0 \to \pi$ to $\pi \to 2\pi$.
- **Infinite integrals:** Integrals from $-\infty \to \infty$ can be extended to the complex plane by connecting $\pm\infty$ at the x-axis by a semi-circle with infinite radius giving no contribution. Thus the integral can be solved from residues inside the first and second quadrant, or third and fourth quadrant depending on orientation of semi-circle. Only works if: (1) f is analytic on and above the real axis, except for a finite number of singularities, (2) we can ignore contributions infinitely far away from the origin, we can do this ($f(z) = P(z)/Q(z)$) if $\mathrm{DEG}(Q) \geq \mathrm{DEG}(P) + 2$.
- **Infinite Trigonometric integral:** Two options:
  1. Write the trigonometric function in it's complex form, which will give two integrals, one with e^{imx} which must be closed in the upper half plane, and one with e^{-imx} which must be closed in the lower half plane. Since the last one goes counter clockwise we must flip the sign. We can ignore the semi-circle only if the degree of the polynomial in the denominator is one larger than the degree of the numerator, this is called **Jordan's lemma**. Without the exponential the difference in polynomial degree would be two.
  2. If the integral is real we can write $\cos mx$ as the real part of e^{imx}, or $\sin mx$ as the imaginary part, then solve the integral and take the imaginary or real part of the answer.
- **Singularities on the real axis:** Infinities can *cancel* if approached symmetrically: $\mathrm{PV}\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to 0}\int_a^{c-r} f(x)\,\mathrm{d}x + \int_{c+r}^{b} f(x)\,\mathrm{d}x$, with a singularity at $x = c$. Can also be computed with residue theory:

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i\sum\underbrace{\mathrm{Res}(f; z_k)}_{\text{upper half}} + \pi i\sum\underbrace{\mathrm{Res}(f; z_j)}_{\text{real axis}}$$

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Tensors

Tensors of rank 0 = scalars, rank 1 = vectors, rank 2 = matrix. Tensors represent physical quantities, and physics is independent of choice of coordinate frames, so tensors must transform between coordinate frames. A tensor product is given by

$$\begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix} \otimes \begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix} = \begin{pmatrix}U_1V_1 & U_1V_2 & U_1V_3\\U_2V_1 & U_2V_2 & U_2V_3\\U_3V_1 & U_3V_2 & U_3V_3\end{pmatrix}$$

Cartesian tensors

Transform properly under rotation of a cartesian coordinate $v_i' = A_{ij}v_j$, $v_i = A_{ji}v_j$, where $A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j'$, which implies $A^{-1} = A^T$. We get one A for each index $T_{kl}' = A_{ki}A_{lj}T_{ij}$. Cartesian vectors need only one index.

Inertia tensor

A rigid body rotating around a fixed axis: $\boldsymbol{L} = I\boldsymbol{\omega}$, where I is the inertia tensor. Since I is symmetric we can find a coordinate system where I is diagonal. The eigenvectors of I are the principle axis of inertia.
Point masses: $I_{ij} = mr^2\delta_{ij} - mr_ir_j$ where you sum over all masses.
Continuum masses: $I_{ij} = \int mr^2\delta_{ij} - mr_ir_j \,\mathrm{d}m$ where. Examples:
$I_{xx} = \sum_i m_i(r^2 - x^2) = \sum_i m_i(y^2 + z^2) = \int y^2 + z^2 \,\mathrm{d}m$.
$I_{xy} = -\sum_i m_i x_i y_i = -\int xy \,\mathrm{d}m$.

$$[\mathbf{I}] = \begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\I_{yx} & I_{yy} & I_{yz}\\I_{zx} & I_{zy} & I_{zz}\end{bmatrix} = m\begin{bmatrix}y^2+z^2 & -xy & -xz\\-yx & x^2+z^2 & -yz\\-zx & -zy & x^2+y^2\end{bmatrix}$$

Kronecker-delta and Levi-Civita

$$\delta_{ij} = \begin{cases}1 & \text{if } i = j\\0 & \text{if } i \neq j\end{cases}$$

$$\epsilon_{ijk} = \begin{cases}1 & \text{if } ijk = 123, 231, \text{ or } 312\\-1 & \text{if } ijk = 321, 213, \text{ or } 132\\0 & \text{if repeating indices}\end{cases}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$(\mathbf{a} \times \mathbf{b})_i = a_j b_k \epsilon_{ijk}$$

$$\det\begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{1i}a_{2j}a_{3k}\epsilon_{ijk}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{V})_i = \epsilon_{ijk}\frac{\partial V_k}{\partial x_j}$$

$$(\boldsymbol{U} \times \boldsymbol{V})_i = \epsilon_{ijk}U_jV_k$$

Dirac-delta and Heaviside step function

$$\delta(x-a) = \begin{cases}\infty, & x = a\\0, & x \neq a\end{cases}$$

$$\delta(-(x-a)) = \delta(x-a) \qquad \text{symmetric}$$

$$\int_{-\infty}^{\infty} \delta(x-a)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}, \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k$$

$$\mathcal{L}[\delta(t-a)] = e^{-as}$$

$$H(t-a) = \begin{cases}0, & t < a\\1, & t > a\end{cases}$$

$$H'(x-a) = \delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-a)\,\mathrm{d}x = (-1)^{(n)}f(a)$$

Calculus of variations

Given an integral $I = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x$ we want to find the function $x(y)$ or $y(x)$ between (x_1, y_1) and (x_2, y_2) such that I is stationary (locally minimized). The function which satisfies this is

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0.$$

If F has no y dependence we get $\frac{\partial F}{\partial y'}$ = constant, which can be solved for y.
One can change variables such that we have a cyclic coordinate through the substitutions: $y' = 1/x'$ and $\mathrm{d}x = x'\,\mathrm{d}y$.

Tips and tricks

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_n\left(\frac{1}{z}\right)^n = \sum_n\left(\frac{1}{z}\right)^{n+1}, z > 1$$

$$ax^2 + bx + c = 0 \qquad \rightarrow \qquad x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\nabla^2_{pol2D} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\nabla^2_{spher3D} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z-z_0)^k = \frac{k!}{(k-n-1)!}(z-z_0)^{k-n} \quad k \geq n$$

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z-z_0)^{-1} = (-1)^n n!\,(z-z_0)^{-(n+1)}$$

$$\left|e^{x+iy}\right| = e^x$$

DE with exponential solution can be written as $A\sinh x + B\cosh x$.
Rember that $n = 0$ is a possibility when doing boudary conditions for the derivatives of a $A\sin px + B\cos px$ solution.

Steady-state temperatur i en sylinder

Laplace equation in cylindrical coord. with assumption $u = R(r)\Theta(\theta)Z(z)$ gives $\frac{1}{R}\frac{1}{r}r(rRr) + \frac{1}{\Theta}\frac{1}{r^2}{}^2\Theta\theta^2 + \frac{1}{Z}{}^2Zz^2 = 0$

$$\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = K^2; \ \frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = -n^2; \ \frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - n^2 + K^2r^2 = 0$$

where $R(r)$ is solved by the orthogonal bessel functions $J_n(Kr)$.

Legendre

Egenfunksjoner:
Legendrepolynomer $y_n(x) = P_n(x)$ for $x \in [-1, 1]$.
Orthogonalitet: $\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$.
Kompletthet: f(x)$=\sum_{n=0}^{\infty} a_n P_n(x), x \in [-1, 1]$; $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$

Hermite

Egenfunksjoner:
Hermitepolynomer $y_n(x) = H_n(x)$ for $x \in (-\infty, \infty)$.
Orthogonalitet: $\int_{-1}^{1} e^{-x^2}H_n(x)H_m(x)dx = 2^n n!\sqrt{\pi}\delta_{nm}$
Kompletthet: f(x)$=\sum_{n=0}^{\infty} a_n H_n(x), x \in (-\infty, \infty)$; $a_n = \frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{\infty} f(x)e^{-x^2}H_n(x)dx$

Ordinary differential eq.

ORDINARY: Derivatives with respect to only one variable.
LINEAR: Only linear terms y, y' etc. no yy', y^2, $(y'')^3$ etc.
ORDER OF DE: Order of highest derivative.
For second order DE we get exact solution from BC since we can solve DE for y'' and evaluate at BC, and get higher orders from taking the derivative of this equation. From these values we can construct a Taylor series of the function, which uniquely fixes the solution.
A linear combination of two solutions is also a solution.
Criteria for two solutions $y_1(x)$ and $y_2(x)$ being linearly independent:

- Linearly independent if $c_1y_1(x) + c_2y_2(x) = 0$ can only be satisfied when $c_1 = c_2 = 0$ (y_2 is not a multiple of the other).
- Linearly dependent if $y_2(x_0) = Ky_1(x_0)$ AND $y_2'(x_0) = Ky_1(x_0)$ they are linearly dependent.
- Linearly independent if $y_1'(x_0)/y_2'(x_0) = y_1(x_0)/y_2(x_0)$, in other words: if the wronskian is non-zero

$$w(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

Then we can write the full solution as $y(x) = c_1y_1(x) + c_2y_2(x)$.

Separable solutions

A differential equation is separable if it can be written as

$$f(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x$$

then it can be solved by integrating both sides.

Integrating factors

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \to \mathrm{d}y + (Py - Q)\,\mathrm{d}x = 0$$

Multiply with $\mu(x)$ to get an exact solution from.

$$\mu(x) = \exp\left\{\int P\,\mathrm{d}x\right\}$$

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,\mathrm{d}x + C\right)$$

Variation of constants

If $y_1(x)$ is a solution we can find another linearly independent solution $y_2(x)$ by writing $y_2(x) = c(x)y_1(x)$, where we determine $c(x)$ from DE. When finding c we can ignore constants since these constants contribute function equal to $y_1(x)$, and constant factors can be determined outside of solution.

Constant coefficients

$$y'' + ay' + by = 0$$

Always has an exponential solution $y = e^{\lambda x}$, inserting gives $\lambda^2 + a\lambda + b = 0$ which we solve to find λ. This gives three possibilities:

- $\lambda_+ \neq \lambda_-$ and both are real: we get two linearly independent solutions $y(x) = c_1e^{\lambda_+ x} + c_2e^{\lambda_- x}$.
- Double root $a^2 - 4ab = 0$: gives $\lambda_+ = \lambda_- = \lambda = -a/2$. Can be shown from variation of constants that we get two linearly independent solutions $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$.
- Complex roots $a^2 - 4b < 0$: gives $\lambda_\pm = -a/2 \pm i\sqrt{4b - a^2}/2 = -a/2 \pm iw$. Solution is the standard form, but can be written in many different forms:
$y(x) = e^{-ax/2}\left(Ae^{iwx} + Be^{-iwx}\right)$
$y(x) = e^{-ax/2}\,(C\cos wx + D\sin wx)$
$y(x) = e^{-ax/2}\sin(wx + \delta)E$.

Euler-Cauchy equation

$$x^2y'' + a_1xy1 + a_0y = 0 \qquad \text{or} \qquad y'' + \frac{a_1}{x}y' + \frac{a_0}{x^2} = 0$$

Use substitution $x = e^z$ for $x > 0$ and $x = -|x| = -e^z$ for $x < 0$. This gives us $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x\frac{\mathrm{d}y}{\mathrm{d}z}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1/x^2\frac{\mathrm{d}y}{\mathrm{d}z} + 1/x^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}$, inserting this into DE gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (a_1 - 1)\frac{\mathrm{d}y}{\mathrm{d}z} + a_0y = 0$$

This is a differential equation with constant coefficients which we can solve, then substitute back $z = \ln|x|$. Get different coefficients from initial conditions for positive and negative x, no solution for $x = 0$.

Inhomogeneous DEs

$$y'' + P(x)y' + Q(x)y = R(x)$$

The full solution is the sum of the homogeneous and particular equation $y(x) = y_h(x) + y_p(x)$, where y_p has no undetermined coefficients. The homogeneous equation can match boundary conditions if the Wronskian is non-zero.

Method of undetermined coeffs.

$$y'' + ay' + by = R(x)$$

Method works when $R(x)$ is a function whose derivative resembles the function itself: exponential, trigonometric, polynomial, sine/cosine and combination of these. We then guess on a solution of the same form as $R(x)$, (possibly times x or x^2) with an unknown coefficient which is determined by putting it into the equation:

- $R(x) = Ae^{kx}$. We call the roots of the characteristic equation of the homogeneous equation α and β, then:
  - $k \neq \alpha, \beta : y_p(x) = Be^{kx}$
  - $k = \alpha$ or β, and $\alpha \neq \beta : y_p(x) = Cxe^{kx}$
  - $k = \alpha = \beta : y_p(x) = Dx^2e^{kx}$
- $R(x) = A\sin kx$ or $R(x) = A\cos kx$: $y_p(x) = Ae^{ikx}$ and take real or imaginary part of solution.
- $R(x) = e^{kx}P_nx$ where $P_n(x)$ is a polynomial of degree n
  - $k \neq alpha, beta : y_p = e^{kx}Q_n(x)$
  - $k = \alpha$ or β, and $\alpha \neq \beta : y_p(x) = e^{kx}Q_n(x)x$
  - $k = \alpha = \beta : y_p(x) = e^{kx}Q_n(x)x^2$

If RHS is a sum of two different such functions you take the sum of two guesses (???)

Finding particular solution from factorization

$$y'' + P(x)y' + Q(x)y = R(x)$$

We are given a known solution $u(x)$ of the homogeneous DE. Guess on a particular solution $y_p(x) = u(x)v(x)$, insert into DE and use that u is a solution to the homogeneous equation to reduce equation down to

$$v'' + \left(\frac{2u'}{u} + P\right)v' = \frac{R}{u}$$

Define $w(x) = v'(x)$, getting

$$w + \left(\frac{2u'}{u} + P\right)w = \frac{R}{u}$$

Solve this equation by integrating factors and find v by $v(x) = \int w(x)\,\mathrm{d}x$, no integration constant, get full solution by multiplying $y_p(x) = u(x)v(x)$.

Variation of parameters

Know two linearly independent homogeneous solutions y_1 and y_2, the particular solution is then

$$y_p(x) = -y_1\int\frac{y_2(x)R(x)}{W(x)}\,\mathrm{d}x + y_2\int\frac{y_1(x)R(x)}{W(x)}\,\mathrm{d}x$$

where W is the Wronskian $W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$. Remember to write the DE in standard form before reading of $R(x)$. Derived from $y_p(x) = f_1(x)y_1(x) + f_2(x)y_2(x)$ and impose condition $f_1'y_1 + f_2'y_2 = 0$ to make terms in y_p' vanish. Insert this into DE and use that y_1 and y_2 satisfy homogeneous equation to make two terms disappear. This gives us one equation, which in combination with out imposed condition $(f_1'y_1 + f_2'y_2 = 0)$ gives us the expression for f_1 and f_2 which we use to find y_p from $y_p(x) = f_1(x)y_1(x) + f_2(x)y_2(x)$.

Greens functions

$$\Bigg(\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}}{\mathrm{d}x} + Q(x)}_{D}\Bigg)y = R(x)$$

Assume given DE, homogeneous solution, and BC. LHS and BCs give us the Greens function $G(x, z)$ which gives us $y_h + y_p$ for any $R(x)$ with no arbitrary constants. Begin by writing DE as $Dy(x) = R(x)$, exists D^{-1}, then $y(x) = D^{-1}R(x)$, this is an integral

$$y(x) = \int_a^b G(x, z)R(z)\,\mathrm{d}z, \qquad a \leq x, z \leq b$$

We apply D_x to both sides, giving us

$$D_xy(x) = \int_a^b \underbrace{D_xG(x, z)}_{\delta(x-z)}R(z)\,\mathrm{d}z$$

Have to solve

$$D_xG(x, z) = \delta(x - z)$$

The Greens function satisfy the original DE, but with $R(x) = \delta(x - z)$. We have restrictions on $G(x, z)$

- $G(x, z)$ must satisfy the given BCs given for $y(x)$. Ex: $y(a) = y(b) = 0 \to G(a, z) = G(b, z) = 0$
- $G(x, z)$ is continuous at $x = z$.
- We have a singularity at $x = z$, we integrate the DE for $G(x, z)$ from $z - \epsilon$ to $z + \epsilon$, this gives us that $\frac{\mathrm{d}G(x,z)}{\mathrm{d}x}\Big|_{z-\epsilon}^{z+\epsilon} = 1$

Begin with $DG(x, z) = \delta(x - z)$. For $x < z$ and $x > z$ we get different solutions, which both are equal to the homogeneous solution, but their coefficients are different and dependent on z. Use the boundary conditions, the continuity of G at $x = z$, and the discontinuity of it's derivative $\frac{\mathrm{d}G(x,z)}{\mathrm{d}x}\Big|_{z-\epsilon}^{z+\epsilon} = 1$ to find $G(x, z)$ for $x < z$ and $x > z$ and solve the integral for $y(x)$, which will be two integrals

$$y(x) = \int_a^x G_1(x, z)R(z)\,\mathrm{d}z + \int_x^b G_2(x, z)R(z)\,\mathrm{d}z$$

where G_1 is the greens function for $z < x$ and G_2 is for $z > x$.

Power series

$$y'' + P(x)y' + Q(x)y = 0$$

Useful for DEs whose solution can not be written in terms of elementary functions.
Represent $P(x)$ and $Q(x)$ as power series (if they are not already polynomials), and assume solution on the form

$$y(x) = \sum_{n=1}^{\infty} a_nx^n, \qquad y'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$$

$$y''(x) = \sum_{n=1}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^{n-1}$$

Insert into DE and equate power by power in x to determine coefficients a_n.
EXISTENCE OF SOLUTION: If $P(x)$ and $Q(x)$ are analytic at $x = x_0$, then every solution of the differential equation is analytic at $x = x_0$ and thus can be represented by a power series in $(x - x_0)$ with some radius of convergence $R > 0$.

$$\text{Legendre functions: } (1 - x^2)y'' - 2xy' + l(l+1)y = 0$$

Fröbenious method

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

where $b(x)$ and $c(x)$ are analytic at $x = 0$, has at least one solution that can be represented as

$$y(x) = x^s\sum_{n=1}^{\infty} a_nx^n$$

By construction $a_0 \neq 0$, and s may be non-integer, real or complex. Determine s from matching terms for lowest possible power, this gives **indicial equation**, which is a 2nd order equation for s with at least one solution. Write DE as

$$x^2y'' + xb(x)y' + c(x)y = 0$$

with b and c written as polynomials or infinite series, then equate terms with x^s which gives INDICIAL EQUATION, this gives three possible scenarios:

- Two different roots, and their difference is not an integer. Then we have two linearly independent solutions.
- Two different roots, and their difference is an integer. Find coefficients for the smallest root first, since it might give the full solution. Get full solution if a_1 is undetermined, as well as a_0 which is always undtermined. If it does not give the full solution then finding the coefficients with the other root will. Sometimes the the smallest root does not give a solution, then find solution of largest root and use variation of constants $y_2(x) = f(x)y_1(x)$.
- Double root. Find coefficient with this root, and find second solution by variation of constants $y_2(x) = f(x)y_1(x)$.

Fuchs' theorem: A differential equation $y'' + f(x)y' + g(x)y = 0$ has a non-essential singularity at the origin if $xf(x)$ and $x^2g(x)$ is expandable in convergent power series. The solution will then either be (1) two Frobenius series, or (2) one solution $S_1(x)$ and another $S_1(x)\ln x + S_2(x)$ where S_1 and S_2 are the Frobenius solutions.

Fourier series

Series representation of periodic functions expanded in sin , cos or e rather than in power series. Can represent functions with cusps and discontinuities.
Orthogonality relations:

$$\int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = 0 \qquad n \text{ is integer}$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, \mathrm{d}x = 0 \qquad n, \, m \text{ is integer}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \, \mathrm{d}x = \begin{cases} 0; & m \neq n \\ 1/2; & m = n \neq 0 \\ 0; & \text{m,m} = 0 \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \, \mathrm{d}x = \begin{cases} 0; & m \neq n \\ 1/2; & m = n \neq 0 \\ 1; & m, m = 0 \end{cases}$$

Period of 2π:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1} a_n \cos nx + \sum_{n=1} b_n \sin nx$$

Find coefficients by using orthogonality of sin and cos

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x$$

Can also write as exponentials:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x$$

Dirichlets:
If $f(x)$ is periodic with period of 2π between $-\pi$ and π, it is single valued, has a finite number of maximum and minimum values, finite number of discontinuities, $\int_{-\pi}^{\pi} |f(x)| \, \mathrm{d}x$ is finite, then the Fourier series converges. For jumps the Fourier series converges to the midpoint.
Other basic intervals:
For basic interval of length $2l$, $[-l, l]$, we can do the transform $x \to n\pi/l$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} \to \frac{1}{l}\int_{-l}^{l}$.

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) \mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) \exp\left\{\frac{-in\pi x}{l}\right\} \mathrm{d}x$$

For basic interval of length $2l$ on $[0, 2l]$ or $[3l, 5l]$, we only change the integration limits to that same interval, while the form of the coefficients is the same.
For basic interval $[0, l]$ we can find:

- a series with periodicity of l instead of $2l$, meaning $l \to l/2$: $a_n = \frac{1}{l/2}\int_0^l f(x) \cos\left(\frac{n\pi x}{l/2}\right)$.
- Define an even extension to a periodicity of $2l$, this gives a cos-series.
- Define an odd extension of the period to a length of $2l$, this gives a sine-series.

Even functions: $f(-x) = f(x)$
Odd functions: $f(-x) = -f(x)$
odd · odd = even · even = even
even · odd = odd · even = odd

$$\int_{-L}^{L} f(x) \, \mathrm{d}x = \begin{cases} 0; & f(x) \text{ is odd} \\ 2\int_0^L f(x) \, \mathrm{d}x; & f(x) \text{ is even} \end{cases}$$

The Fourier series of an even function of period $2l$ is a cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \quad a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) \mathrm{d}x$$

The Fourier series of an odd function of period $2l$ is a sine series ($f(0) = 0$ when f is odd)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x$$

Perseval's theorem: For a complex Fourier series $f(x)$ we can evaluate infinite series. Can use complex or real Fourier series

$$\frac{1}{2l}\int_{-l}^{l} |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$\frac{1}{2l}\int_{-l}^{l} |f(x)|^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

Fourier integrals

$F(k)$ is the Fourier transform of $f(x)$, and $f(x)$ is the inverse Fourier transform of $F(k)$.

$$f(x) = \mathcal{F}^{-1}[F(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k) e^{ikx} \, \mathrm{d}k$$

$$F(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x$$

The sign in the exponent and the $1/\sqrt{2\pi}$ is a convention.
Ok to use Fourier integrals if: $f(x)$ satisfy Dirichlet conditions on any finite interval, $\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x$ is finite, at discontinuities in $f(x)$ this formula gives the midpoint of the jump.
The Fourier transform of a Gaussian is a Gaussian.
Fourier integrals be used to go from a DE to an algebraic equation which you can solve for $\mathcal{F}[f(x)]$ and take the inverse Fourier transform

$$\mathcal{F}\left[f^{(n)}(x)\right] = (ik)^n \mathcal{F}[f(x)]$$

Perseval's theorem (v2): For the Fourier transform $F(k)$ of $f(x)$ satisfying Dirichlet conditions we have

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} |F(k)|^2 \, \mathrm{d}k$$

Even and odds: Even functions are represented by cos-transform
Odd functions are represented by sin-transform
If $f(x)$ is even/odd then $F(k)$ is even/odd. Making the Fourier transform for even and odd functions respectively:

$$F(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_e(x) \cos x \, \mathrm{d}x, \, f_e(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(k) \cos kx \, \mathrm{d}k$$

$$F(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_o(x) \sin x \, \mathrm{d}x, \, f_o(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(k) \sin kx \, \mathrm{d}k$$

Laplace transforms

An integral transform which reduces DE to algebraic equations where initial value problems are solved directly. The Fourier transform of $f(t)$ is written as $F(s)$ and is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) \, \mathrm{d}t \qquad f(t) = \mathcal{L}^{-1}[F(s)]$$

Inverse Laplace transforms are found from tables

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n \mathcal{L}[f(t)] - s^{n-1}\mathcal{L}[f(0)] - s^{n-2}\mathcal{L}\left[f'(0)\right] - \dots$$

$$\text{s-shifting: } \mathcal{L}^{-1}[F(s-a)] = e^{at}\mathcal{L}^{-1}[F(s)]$$

$$\text{t-shifting: } \mathcal{L}^{-1}\left[e^{-as}F(s)\right] = f(t-a)\underbrace{H(t-a)}_{\text{Heaviside}}$$

$$\mathcal{L}^{-1}[H(t-a)] = \frac{1}{s}e^{-as}$$

$$\mathcal{L}[f(t) \cdot t] = -F'(s) \text{ or } \mathcal{L}^{-1}\left[F'(s)\right] = -tf(t)$$

Convolution: We are given $H(s) = F(s)G(s)$ and know $f(t) = \mathcal{L}^{-1}[F(s)]$ and $g(t) = \mathcal{L}^{-1}[G(s)]$, and want to use tihs to find $h(t) = \mathcal{L}^{-1}[H(s)]$ in terms of $f(t)$ and $g(t)$, we call this the CONVOLUTION of f and g, and is written as, and can be found by

$$h(t) = \mathcal{L}^{-1}[H(s)] = (f * g)(t) = \int_0^t f(\tau)g(t-\tau) \, \mathrm{d}\tau$$

it does not matter which function is f and which is g, choose g as the simplest function of the two $f * g = g * f$ and $\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g]$.
Bromwich integral: Laplacetransform can be related to fourier transform by allowing s to be complex, $s \to x + iy$, and can then find an explicit formula for the inverse laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z) e^{zt} \, \mathrm{d}z, \quad t > 0$$

Can solve integral by finding residues of all singularities of the integrand. Close contour with a semi-circle with no contribution as $R \to \infty$ and get $f(t) = \sum \mathrm{res}(F(z)e^{zt})$.

Partial DE

Separation of variables

Transform a PDE with n variables to n ordinary DE. Find a solution of those DEs satisfying boundary conditions, this gives infinitely many solutions, this is not the most general solution but forms a basis of solutions which we can expand the full solution in. Superposition of these such that initial conditions are satisfied (a Fourier series).
Write $u(x, t)$ as $F(x)$ or $G(t)$. Use separation of variable and solve their ordinary DE separately. Quantize constant such that they can satisfy BCs. This gives

$$u(x, t) = \sum_{n=1}^{\infty} A_n f_n(x) g_n(t)$$

and impose initial conditions, which should give a Fourier series f.ex

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \to A_n = \frac{2}{L}\int_0^L u(x, 0) \sin\left(\frac{n\pi x}{L}\right)$$

1D wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $u(x, t) = F(x)G(t)$, get $F'' = kF$ and $\ddot{G} = kc^2 G$. To satisfy BC ($u(0, t) = u(L, t) = 0$): $k = -p^2$, gives $F_n = \sin(p_n x)$ and $G_n = A_n \cos p_n ct + B_n \sin p_n ct$ with $p_n = n\pi/L$. Combine to get $u_n(x, t) = (A_n \cos p_n ct + B_n \sin p_n ct) \sin p_n x$, where A_n and B_n are fixed by initial conditions: $u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin p_n t \to A_n = 2/L \int_0^L f(x) \sin p_n x \, \mathrm{d}x$. And $\dot{u}(x, 0) = g(x) = \sum_{n=1}^{\infty} \lambda_n B_n \sin p_n x \to B_n \lambda_n = 2/L \int_0^L g(x) \sin p_n x \, \mathrm{d}x$
1D heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $u(x, t) = F(x)G(t)$, get $F'' = kF$ and $\dot{G} = kc^2 G$, to satisfy BC $k = -p^2$, gives $F = A_n \cos p_n x + B_n \sin p_n x$ and $G = e^{-c^2 p_n^2 t}$. Impose BC $u_n(x, t) = B_n \sin p_n x e^{-c^2 p_n^2 t}$. Use initial conditions $B_n = \frac{2}{L}\int_0^L u(x, 0) \sin n\pi x/L$.
2D wave equation: First separate time and space $u = F(x, y)G(t)$, then separate $F(x, y) = H(x)Q(y)$. Get $\ddot{G} = -c^2\nu^2 G$, $H'' = -k^2 H$, $Q'' = -p^2 Q$ with $p^2 + k^2 = \nu^2$. Get $F_n(x, y) = \sin n\pi x/a \sin m\pi x/b$. We get a double Fourier series $u(x, y, t) = \sum_{n,m=1}^{\infty} [\alpha_{mn} \cos cp_{nm} t + \beta_{mn} \sin p_{nm} ct] \sin n\pi x/a \sin m\pi x/b$. If $u(x, y, 0) = f(x, y)$ we get $\alpha_{mn} = \frac{4}{\alpha\beta}\int_0^a \int_0^b f(x, y) \sin n\pi x/a \sin m\pi x/b \, \mathrm{d}x \, \mathrm{d}y$

Integral transforms

Laplace transform: Laplace transform wrt. one of the variables, and we get an ordinary DE for the other variable which we solve to find the Laplace transform of the solution, which we can take the inverse transform of to find the actual solution.
If we have a function $u(x, t)$ and take the Laplace transform wrt. t we use $\mathcal{L}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial U}{\partial x}$ and $\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = sU - u(x, 0)$. Solve ordinary DE, and impose initial conditions $\mathcal{L}[u(x, 0)] = U(0, s)$. When $U(x, s)$ is found the inverse Laplace transform is done.
Fourier transform: Fourier transform with respect to one of the variables (in this case x), and use $\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -k^2 U(k, t)$. Impose initial conditions $U(k, 0) = \mathcal{F}[u(x, 0)]$, solve the ordinary DE for $U(k, t)$ and take the inverse transform.

Orthogonal functions

Differential equation with certain BCs have set of solutions that are orthogonal and can expand any function, this goes for DE on the form

$$P(x)y'' + P'(x)y' + [q(x) + \lambda r(x)]\, y = 0$$

with some BC at $x = a$ and $x = b$ and $r(x) > 0$ on $[a, b]$. If there exists a non-zero solution for a given λ the $y(x)$ is called an eigenfunction, and λ is called the eigenvalue of the problem. Typically BCs lead to a series of discrete eigenvalues λ_m and eigenfunctions $y_m(x)$. Examples: (1) Fourier $y'' + (n\pi/L)^2 y = 0$, (2) Legendre $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. The eigenfunctions being orthogonal means

$$\int_a^b r(x) y_n(x) y_m^*(x) \, \mathrm{d}x$$

and we use this orthogonality relation to expand any function through the eigenfunctions by $f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$ by multiplying each side with $r(x)y_m^*(x)$ and integrating from a to b, giving us

$$a_m = \int_a^b f(x) r(x) y_m^*(x) \, \mathrm{d}x$$

D is hermitian if it satisfies

$$\int_a^b y_n(x)^* D y_m(x) dx = \int_a^b y_m(x) D y_n(x)^* dx$$

for boundary conditions, then: real eigenvalues, eigenfunctions are orthogonal over a, b, and the eigenfunctions forms a complete sett for a, b. Where $D = p(x)\frac{\partial^2}{\partial x^2} + p'(x)\frac{\partial}{\partial x} + q(x)$.
Examples:
Legendre: $x \in [-1, 1]$, $(1-x^2)y'' - 2xy' + n(n+1)y = 0$
where $\lambda = n(n+1)$, $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$.
Fourier: $x \in [-L, L]$, $y'' + (n\pi/L)^2 y = 0$
where $\lambda = (n\pi/L)^2$, $p(x) = 1$, $q(x) = 0$, $r(x) = 1$.
Hermite: $x \in (-\infty, \infty)$ $y'' - 2xy' + 2ny = 0$
multiply with e^{-x^2} $e^{-x^2}y'' - 2xe^{-x^2}y' + 2ne^{-x^2}y = 0$
where $\lambda = 2n$, $p(x) = e^{-x^2}$, $q(x) = 0$, $r(x) = e^{-x^2}$.
Laguerre: $x \in [0, \infty)$.
$xy'' + (1-x)e^{-x}y' + \lambda e^{-x}y = 0$ multiply with e^{-x}
$xe^{-x}y'' + (1-x)e^{-x}y' + \lambda e^{-x}y = 0$
where $p(x) = xe^{-x}$, $q(x) = 0$, $r(x) = e^{-x}$.