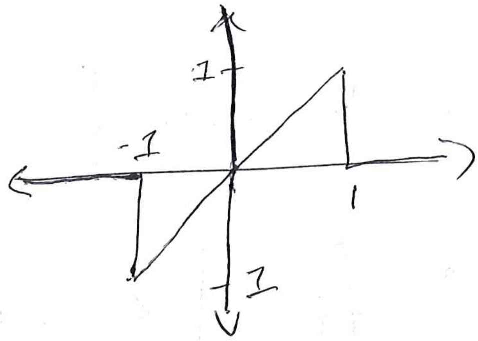
Matmet Oblig 12 - Ivar Haugerud

want to find exponential fourier transform of f(x) and write f(x) as a fourier integral

7.12.6) $f(x) = \begin{cases} x, & |x| < 1 \\ 0 & |x| > 1 \end{cases}$



Begin by finding $g(\alpha)$

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$\int_a^b f' \cdot v = f \cdot v \Big|_a^b - \int f \cdot v'$$

$$f' = e^{-i\alpha x}, \quad f = \frac{e^{-i\alpha x}}{-i\alpha}$$

$$v = x, \quad v' = 1$$

$$= \frac{1}{2\pi}\int_{-1}^{1} x\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{2\pi}\left(x \cdot \frac{e^{-i\alpha x}}{-i\alpha}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-i\alpha x}}{-i\alpha}\, dx \right)$$

$$= \frac{1}{2\pi}\left(-\frac{e^{-i\alpha}}{i\alpha} - \left(-1\,\frac{e^{i\alpha}}{-i\alpha}\right) - \frac{1}{(-i\alpha)^2}\left[e^{-i\alpha x}\right]_{-1}^{1} \right)$$

$$= \frac{1}{2\pi}\left(-\frac{e^{-i\alpha} - e^{i\alpha}}{i\alpha} + \frac{1}{\alpha^2}\left(e^{-i\alpha} - e^{i\alpha}\right)\right)$$

$$= \frac{1}{2\pi}\left(-\frac{2\cos(\alpha)}{i\alpha} - \frac{2i\sin(\alpha)}{\alpha^2} \right)$$

$$= \frac{1}{2\pi}\left(\frac{2\alpha i\cos(\alpha) - 2i\sin(\alpha)}{\alpha^2}\right)$$

$$\boxed{g(\alpha) = \frac{i}{\pi}\left(\frac{\alpha\cos(\alpha) - \sin(\alpha)}{\alpha^2}\right)}$$

from definition

$$\boxed{f(x) = \frac{i}{\pi\alpha^2}\int_{-\infty}^{\infty} (\alpha\cos(\alpha) - \sin(\alpha))\, e^{i\alpha x}\, d\alpha}$$

7.12.18) Now we want to study the same function but using the fourier sine-transform where

$$g_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin(\alpha x)\, dx$$

which should give the same answer since we are representing odd functions.

Put in for $f_s(x) = \begin{cases} x, & |x|<1 \\ 0, & |x|>1 \end{cases}$

$$g_s(x) = \sqrt{\frac{2}{\pi}} \int_0^1 x \sin(\alpha x)\, dx$$

$$\boxed{\begin{aligned} f' &= \sin(\alpha x) \\ f &= -\frac{\cos(\alpha x)}{\alpha} \\ v &= x,\ v' = 1 \end{aligned}}$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{x\cos(\alpha x)}{\alpha} \Big|_0^1 - \int_0^1 -\frac{\cos(\alpha x)}{\alpha} dx \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{\cos(\alpha)}{\alpha} + 0 + \frac{1}{\alpha^2} \Big[\sin(\alpha x) \Big]_0^1 \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\alpha)}{\alpha^2} - \frac{\cos(\alpha)}{\alpha} \right)$$

Insert into def of $f_s(x)$:

$$f_s(x) = \sqrt{\frac{2}{\pi}}^{\,2} \int_0^\infty \left(\frac{\sin(\alpha)}{\alpha^2} - \frac{\cos(\alpha)}{\alpha} \right) \sin(\alpha x)\, d\alpha$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin(\alpha)\sin(\alpha x)}{\alpha^2} - \frac{\sin(\alpha x)\cos(\alpha)}{\alpha}\, d\alpha$$

To check if this is equal to our previous result for $f(x)$ we will try and rewrite it

7.12.8) We begin from our result from 7.12.6

$$f(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{(\alpha \cos\alpha - \sin\alpha)}{\alpha^2} e^{i\alpha x} \, d\alpha$$

use $e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$

$$= \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\cos\alpha}{\alpha} - \frac{\sin\alpha}{\alpha^2}\right)\left(\cos(\alpha x) + i\sin(\alpha x)\right) d\alpha$$

$$= \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\alpha)\cos(\alpha x)}{\alpha} + \frac{i\cos(\alpha)\sin(\alpha x)}{\alpha} - \frac{\sin(\alpha)\cos(\alpha x)}{\alpha^2} - \frac{i\sin(\alpha)\sin(\alpha x)}{\alpha^2} \, d\alpha$$

sym · sym · antisym = antisym = 0	sym · antisym · antisym = sym ≠ 0	antisym · sym · sym = antisym = 0	antisym · antisym · sym = sym ≠ 0

$$= \frac{i}{\pi} \int_{-\infty}^{\infty} \cancel{\frac{\cos(\alpha)\cos(\alpha x)}{\alpha}} \; \frac{i\cos(\alpha)\sin(\alpha x)}{\alpha} - \frac{i\sin(\alpha)\sin(\alpha x)}{\alpha^2} \, d\alpha$$

Since the integral symmetric we can change limits

$$= \frac{2i}{\pi} \int_{0}^{\infty} \frac{i\cos(\alpha)\sin(\alpha x)}{\alpha} - \frac{i\sin(\alpha)\sin(\alpha x)}{\alpha^2} \, d\alpha$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin(\alpha)\sin(\alpha x)}{\alpha^2} - \frac{\cos(\alpha)\sin(\alpha x)}{\alpha} \, d\alpha$$

Which ~~that~~ is consistent with our previous result.

7.12.22) $J_1(\alpha) = -\dfrac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}$

From problem 18 we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\right)\sin(\alpha x)\,d\alpha$$

where $f(x) = \begin{cases} x, & |x|<1 \\ 0, & |x|>1 \end{cases}$

We recognize the first factor in the integrand as $J_1(\alpha)$, we rewrite

$$\frac{\pi}{2}f(x) = \int_0^\infty \sin(\alpha x)\,J_1(\alpha)\,d\alpha = \begin{cases} \frac{\pi x}{2}, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

which is what we wanted to show.

8.8.4) We want to derive the Laplace transform of

$f(t) = t \cdot \cos(at)$, we know the Laplace transform of

$\sin(at)$, where $\frac{\partial}{\partial a}(\sin(at)) = t \cdot \cos(at)$

Since the Laplace transform of $\sin(at)$ is known we can

$$\frac{\partial}{\partial a}\int_0^\infty e^{-pt}\sin(at) = \int_0^\infty e^{-pt}\, t\cos(at) = \frac{\partial}{\partial a}\left(\frac{a}{p^2+a^2}\right) = \frac{1}{a^2+p^2} + \frac{a}{(p^2+a^2)^2}\cdot(-1)\cdot 2a$$

$$= \frac{1}{a^2+p^2} - \frac{2a^2}{(a^2+p^2)^2} = \frac{a^2+p^2-2a^2}{(a^2+p^2)^2}$$

$$\int_0^\infty e^{-pt}\, t\cos(at) = \frac{p^2-a^2}{(a^2+p^2)^2}$$

And we have found the ~~[illegible]~~ Laplace-transform of $t\cos(at)$

8.8.13) $$\frac{6-p}{p^2+4p+20} = \cancel{\frac{[illegible]}{p^2+4p+20}}$$

$$= \frac{6}{p^2+4p+20} - \frac{p}{p^2+4p+20}$$

The roots of the denominator are

$p^2+4p+20 = (p-[-2+4i])(p-[-2-4i])$

Where we see that the inverse Laplace transform for

$\frac{1}{(p+a)(p+b)}$ and $\frac{p}{(p+a)(p+b)}$ can be used, with

$a = 2+4i$ and $b = 2-4i$

$$\frac{6-p}{p^2+4p+20} = 6\frac{1}{(p+a)(p+b)} - \frac{p}{(p+a)(p+b)}$$

Since the Laplace transform is a linear operator we find the inverse for each from the table:

8.8.B)

$$6\cdot\left(\frac{e^{-at}-e^{-bt}}{b-a}\right)-\left(\frac{ae^{-at}-be^{-bt}}{a-b}\right)$$

L7 L8

$$\frac{1}{a-b}\left(e^{-at}[-6-a]+e^{-bt}[b+6]\right)$$

$$=\frac{1}{2+4i-(2-4i)}\left(e^{-at}[-6-(2+4i)]+e^{-bt}[2-4i+6]\right)$$

$$=\frac{1}{8i}\left(e^{-at}[-8+4i]+e^{-bt}[8-4i]\right)$$

$$=\frac{1}{8i}\left(4e^{-t(2+4i)}[-2+i]+4e^{-t(2-4i)}[2-i]\right)$$

$$=-\frac{i}{2}\left(-e^{-4it}(2-i)+e^{4it}(2-i)\right)e^{-2t}$$

$$e^{ix}-e^{-ix}=2i\sin(x)$$

$$=\frac{1}{2}(-2i-1)e^{-2t}\left(e^{4it}-e^{-4it}\right)$$

$$=i(-2i-1)e^{-2t}\sin(4t)$$

$$=\underline{\sin(4t)e^{-2t}(2-i)}$$

8.8.21) $f(t)$ $L(f(t))$

$\boxed{L29}$ $e^{-at} g(t)$ $G(p+a) = L(g(t))\Big|_{p+a}$

$\boxed{L11}$ $t \sin(at)$ $\dfrac{2ap}{(p^2+a^2)^2}$

We want to find $L(t e^{-at} \sin(bt))$

~~$L(t \cos(bt))$~~.

We can use $\boxed{L29}$ with $g(t) = t\sin(bt)$

$$L(e^{-at} \sin(bt) \cdot t) = L(\sin(bt)\, t)\Big|_{p+a}$$

$$= \frac{2(p+a)b}{([p+a]^2 + b^2)^2}$$

8.9.10) $y'' - 4y' + 4y = 6e^{2t}$ We take the Laplace-transform on both sides

and use that $L(y'') = p^2 L(y) - p\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} = p^2 L(y)$

and $L(y') = pL(y) - \underbrace{y(0)}_{0} = pL(y)$

from initial conditions

This gives us

$$p^2 L(y) - 4pL(y) + 4L(y) = 6\, L(e^{2t})$$

$$L(y)(p^2 - 4p + 4) = 6\, L(e^{2t}) = 6 \frac{1}{p-2}$$

$$L(y)(p-2)^2 = \frac{6}{(p-2)}$$

$$L(y) = \frac{6}{(p-2)^3}$$ We take the inverse Laplace transform on both sides

$$y = 3 \cdot L^{-1}\left(\frac{2}{(p-2)^3}\right)$$ we see that L6 $\dfrac{k!}{(p+a)^{k+1}}$ works with $a = -2$ and $k = 2$

$$\underline{y(t) = 3 \cdot t^2 e^{2t}}$$

8.9.26) $y'' + 2y' + 10y = -6e^{-t}\sin(3t)$

Laplace transform both sides

$$L(y'') = p^2L(y) - p\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1} = p^2L(y) - 1$$

$$L(y') = pL(y) - \underbrace{y(0)}_{0} = pL(y)$$

$$\Rightarrow p^2L(y) - 1 + 2pL(y) + 10L(y) = L(-6e^{-t}\sin(3t))$$

$$L(y)[p^2 + 2p + 10] - 1 = -6L(e^{-t}\sin(3t))$$

$$L(y)[p^2 + 2p + 10] = 1 - 6\cdot\frac{3}{(p+1)^2+9} = 1 - \frac{18}{(p+1)^2+9}$$

$\boxed{L13 \quad b=3,\ a=1}$

$$L(y) = \frac{1}{p^2+2p+10} - \frac{18}{([p+1]^2+3^2)^2}$$

$$y(t) = \underbrace{\frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{p^2+2p+10}\right]}_{\text{L13 with } b=3 \text{ and } a=1} - \underbrace{\frac{1}{3}\mathcal{L}^{-1}\left[\frac{2\cdot 3^3}{([p+1]^2+3^2)^2}\right]}_{\text{L17 with } a=3 \text{ combined with L29 to shift } p\to p+1}$$

$$= \frac{1}{3}e^{-t}\sin(3t) - \frac{1}{3}e^{-t}\left(\sin(3t) - 3t\cos(3t)\right)$$

$$= \frac{1}{3}e^{-t}\left(\sin(3t) - \sin(3t) - 3t\cos(3t)\right)$$

$$= -te^{-t}\cos(3t)$$

8.10.5) We want to find inverse Laplace transform of $\frac{p}{(p+a)(p+b)^2}$ using the convolution integral

$$\frac{p}{(p+b)^2}\cdot\frac{1}{(p+a)} = L\left(e^{-bt}[1-bt]\right)\cdot L\left(e^{-at}\right)$$

$$= G(p)\,H(p)$$

With ~~G(p)~~ $g(t) = e^{-bt}[1-bt]$ and $h(t) = e^{-at}$

From convolution integral we know

$$G(p)H(p) = L\left[\int_0^t g(\tau)h(t-\tau)\,d\tau\right] = \frac{p}{(p+b)^2(p+a)}$$

$$= L\left(f(t)\right)$$

$$\Rightarrow f(t) = \int_0^t g(\tau)h(t-\tau)\,d\tau$$

$$= \int_0^t e^{-b\tau}[1-b\tau]\, e^{-a(t-\tau)}\,d\tau$$

$$= \int_0^t e^{-at+a\tau-b\tau}(1-b\tau)\,d\tau$$

$$= \int_0^t e^{\tau(a-b)-at}(1-b\tau)\,d\tau = e^{-at}\int_0^t e^{\tau(a-b)}(1-b\tau)\,d\tau$$

$$= e^{-at}\left(\int_0^t e^{\tau(a-b)}\,d\tau - b\int_0^t \tau\, e^{\tau(a-b)}\,d\tau\right)$$

$$= e^{-at}\left(\left.\frac{e^{\tau(a-b)}}{a-b}\right|_0^t - b\left(\left.\frac{\tau e^{\tau(a-b)}}{a-b}\right|_0^t - \int_0^t \frac{e^{\tau(a-b)}}{a-b}\,d\tau\right)\right)$$

$$= e^{-at}\left(\frac{e^{t(a-b)}-1}{a-b} - b\left(\frac{t\,e^{t(a-b)}}{a-b} - \frac{1}{(a-b)^2}\left(e^{t(a-b)}-1\right)\right)\right)$$

$$= \frac{e^{-bt}-e^{-at}}{a-b} - \frac{bte^{-bt}}{a-b} + b\,\frac{e^{-bt}-e^{-at}}{(a-b)^2}$$

$$= \frac{e^{-bt} - e^{-at}}{a-b}\left(1 + \frac{b}{a-b}\right) - \frac{bte^{-bt}}{a-b}$$

Being the inverse Laplace transform for our function

8.10 14) Want to solve

$y'' + 5y' + 6y = e^{-2t}$, Laplace transform both sides

$$p^2 L(y) - \underbrace{py(0)}_{0} - \underbrace{y'(0)}_{0} + 5(pL(y) - \underbrace{y(0)}_{0}) + 6L(y) = L(e^{-2t})$$

$$L(y)(p^2 + 5p + 6) = L(e^{-2t})$$

$$L(y) = \frac{1}{p^2+5p+6} \cdot L(e^{-2t})$$

First try and find inverse of $\frac{1}{p^2+5p+6} = \frac{1}{(p+3)(p+2)}$

this is on the form $\frac{1}{(p+a)(p+b)}$ with $a=3$ and $b=2$

which has an inverse (L7) $\frac{e^{-at} - e^{-bt}}{b-a} = \frac{e^{-3t} - e^{-2t}}{2-3}$

$= e^{-2t} - e^{-3t}$, meaning

$$L(y) = L(e^{-2t} - e^{-3t})L(e^{-2t}) = G(p)H(p)$$

with $g(t) = e^{-2t} - e^{-3t}$ and $h(t) = e^{-2t}$

Can find $y(t)$ from convolution integral

$$L(y(t)) = L\left(\int_0^t g(\tau)h(t-\tau)\,d\tau\right)$$

$$y(t) = \int_0^t g(\tau)h(t-\tau)\,d\tau = \int_0^t (e^{-2\tau} - e^{-3\tau})e^{-2(t-\tau)}\,d\tau$$

$$= e^{-2t}\int_0^t (e^{-2\tau} - e^{-3\tau})e^{2\tau}\,d\tau = e^{-2t}\int_0^t (1 - e^{-\tau})\,d\tau$$

$$= e^{-2t}\left[\tau + e^{-\tau}\right]_0^t = e^{-2t}(t + e^{-t} - 0 - 1) = te^{-2t} + e^{-3t} - e^{-2t}$$

$$= \underline{e^{-2t}(t - 1 + e^{-t})}$$ solution to DE

8.11.8) $y'' + 4y' + 5y = \delta(t-t_0)$ with $y(0) = y'(0) = 0$

Laplace transform both sides

$$\mathcal{L}[y''] = p^2\mathcal{L}(y) - p\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} = p^2\mathcal{L}(y)$$

$$\mathcal{L}[y'] = p\mathcal{L}(y) - \underbrace{y(0)}_{0} = p\mathcal{L}(y)$$

assume $t_0 > 0$

$$\text{DE} \Rightarrow p^2\mathcal{L}(y) + 4p\mathcal{L}(y) + 5\mathcal{L}(y) = \mathcal{L}(\delta(t-t_0)) = e^{-t_0 p}$$

$$\mathcal{L}(y)(p^2 + 4p + 5) = e^{-t_0 p}$$

$$\mathcal{L}(y) = \frac{e^{-t_0 p}}{p^2+4p+5}$$ Try partial fraction decomp:

$$\frac{1}{p^2+4p+5} = \frac{1}{(p+a)(p+b)}$$ where $a = 2+i$ & $b = 2-i$ we then get

$$\frac{1}{(p+a)(p+b)} = \frac{A}{(p+a)} + \frac{B}{(p+b)} \Rightarrow 1 = A(p+b) + B(p+a)$$

$p=-a$) $1 = A(-a+b) = A(-2-i+2-i) = -2iA$

$$\Rightarrow A = \frac{-1}{2i} = \frac{i}{2}$$

$p=-b$) $1 = B(-b+a) = B(-2+i+2+i) = 2iB$

$$\Rightarrow B = \frac{1}{2i} = -\frac{1}{2}i$$

$$\frac{1}{(p+a)(p+b)} = \frac{i}{2(p+2+i)} - \frac{i}{2(p+2-i)}$$ we then get

$$\mathcal{L}(y) = \frac{i}{2}e^{-t_0 p}\left(\frac{1}{p+2+i} - \frac{1}{p+2-i}\right)$$

Here we can use $\mathcal{L}^{-1}\left[\frac{1}{p+a}\right] = e^{-at}$ with the exponential factor introducing a heavyside step function factor of $H(t - t_0)$ making the whole expression and shifts $t \to t - t_0$ will write t for short

$$Y(t) = H(t - t_0)\,\frac{i}{2}\left(e^{-t(2+i)} - e^{-t(2-i)}\right)$$

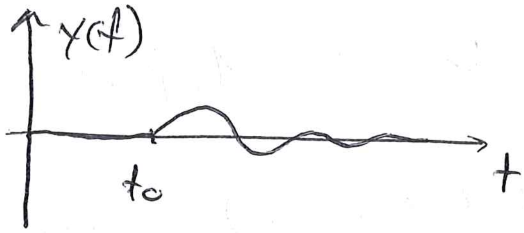
$$= H(t-t_0)\,\frac{i}{2}\left(e^{-it} - e^{it}\right)e^{-2t}$$

$$= H(t-t_0)\,e^{-2t}\,\frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

put in $t = t - t_0$

$$= H(t-t_0)\,e^{-2t}\sin(t) = H(t-t_0)\,e^{-2(t-t_0)}\sin(t-t_0)$$

$$Y(t) = \begin{cases} 0 & 0 < t < t_0 \\ e^{-2(t-t_0)}\sin(t-t_0) & t > t_0 \end{cases}$$



This is a damped harmonic oscillator with push at $t = t_0$ which stays at rest before, makes sense.

Extra problems FYS3140
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8.8.5) Want to verify ~~L12~~ L19

~~$L12:\quad \mathcal{L}[t\cos(at)] = \frac{p^2-a^2}{(p^2+a^2)^2}$~~

$$\underline{L19:}\quad \mathcal{L}\left[\frac{\sin(at)}{t}\right] = \arctan\left(\frac{a}{p}\right)$$

We do this by integrating $\cos(at)$ from L4 wrt a on both sides

$$\int \mathcal{L}[\cos(at)]\,da = \int \frac{p}{p^2+a^2}\,da$$

$$\mathcal{L}\left[\int \cos(at)\,da\right] = \frac{p}{p^2}\int \frac{1}{1+\frac{a^2}{p^2}}\,da$$

$$u = \frac{a}{p},\quad \frac{du}{da} = \frac{1}{p} \Rightarrow da = p\,du$$

$$\mathcal{L}\left[\frac{\sin(at)}{t} + \underbrace{C}_{\text{set to zero...}}\right] = \frac{1}{p}\int \frac{1}{1+u^2}\,p\,du$$

$$\mathcal{L}\left[\frac{\sin(at)}{t}\right] = \int \frac{1}{1+u^2}\,du = \arctan(u) =$$

$$\underline{\mathcal{L}\left[\frac{\sin(at)}{t}\right] = \arctan\left(\frac{a}{p}\right)}$$

Which is what we wanted to show

8.9.31) $y'' + z'' - z' = 0$ Laplacetransform both sides

$$\mathcal{L}(y'') + \mathcal{L}(z'') - \mathcal{L}(z') = 0$$

$$p^2\mathcal{L}(y) - p\underbrace{y_0}_{0} - \underbrace{y_0'}_{1} + p^2\mathcal{L}(z) - p\underbrace{z_0}_{1} - \underbrace{z_0}_{1} - (p\mathcal{L}(z) - \underbrace{z_0}_{1}) = 0$$

use initial conditions $y_0 = 0,\ y_0' = 1,\ z_0 = 1,\ z_0' = 1$

$$p^2\mathcal{L}(y) - 1 + p^2\mathcal{L}(z) - p - \cancel{1} - p\mathcal{L}(z) + \cancel{1} = 0$$

$$p^2\mathcal{L}(y) = 1 - p^2\mathcal{L}(z) + p + p\mathcal{L}(z) \qquad (1)$$

Do the same for our second DE

$y' + z' - 2z = 1 - e^{-t}$ Laplace transform both sides

$$\mathcal{L}(y') + \mathcal{L}(z') - 2\mathcal{L}(z) = \mathcal{L}(1) - \mathcal{L}(e^{-t})$$

$$p\mathcal{L}(y) - \underbrace{y_0}_{0} + p\mathcal{L}(z) - \underbrace{z_0}_{1} - 2\mathcal{L}(z) = \frac{1}{p} - \frac{1}{p+1}$$

L1 L2

$$p\mathcal{L}(y) + p\mathcal{L}(z) - 2\mathcal{L}(z) = \frac{1}{p} - \frac{1}{p+1} + 1$$

$$\mathcal{L}(y) = \frac{2\mathcal{L}(z)}{p} - \mathcal{L}(z) + \frac{1}{p^2} - \frac{1}{p^2+p} + \frac{1}{p}$$ insert into (1)

$$2p\mathcal{L}(z) - \cancel{p^2\mathcal{L}(z)} + \cancel{1} - \frac{p^2}{p^2+p} = \cancel{1} - \cancel{p^2\mathcal{L}(z)} + \cancel{p} + p\mathcal{L}(z) - \cancel{p}$$

$$2p\mathcal{L}(z) - \frac{p}{p+1} = p\mathcal{L}(z)$$

$$p\mathcal{L}(z) = \frac{p}{p+1} \Rightarrow \mathcal{L}(z) = \frac{1}{p+1} \Rightarrow \mathcal{L}^{-1}(\mathcal{L}(z)) = e^{-t}$$

Use this result in (1)

$$p^2\mathcal{L}(y) = 1 - p^2\frac{1}{p+1} + p + \frac{p}{p+1}$$

$$\Rightarrow \quad \mathcal{L}(Y) = \frac{1}{p^2} - \frac{1}{p+1} + \frac{1}{p} + \frac{1}{p(p+1)}$$

$$Y(t) = \underbrace{\mathcal{L}^{-1}\left(\frac{1}{p^2}\right)}_{L5 \to t} - \underbrace{\mathcal{L}^{-1}\left(\frac{1}{p+1}\right)}_{L2 \to e^{-t}} + \underbrace{\mathcal{L}^{-1}\left(\frac{1}{p}\right)}_{L1} + \underbrace{\mathcal{L}^{-1}\left(\frac{1}{p(p+1)}\right)}_{\substack{L7 \text{ with } a=0 \\ \text{and } b=1 \\ \to 1-e^{-t}}}$$

$$= t - e^{-t} + 1 + 1 - e^{-t}$$

$$\underline{Y(t) = 2 + t - 2e^{-t}}$$

and $Z(t) = e^{-t}$

8.9.38) $$\int_0^\infty e^{-t}(1-\cos(2t))\,dt = \mathcal{L}\left[1-\cos(2t)\right]$$ $p=1$

$$= \underset{\substack{L15 \text{ with} \\ a=2}}{=} \frac{a^2}{p(p^2+a^2)} = \frac{4}{p(p^2+4)} \underset{p=1}{=} \frac{4}{1(1+4)} = \underline{\underline{\frac{4}{5}}}$$

8.11.11) $\dfrac{d^4Y}{dt^4} - Y = \delta(t-t_0)$ Laplace transform both sides

$$\mathcal{L}(Y''''(t)) - \mathcal{L}(Y) = \mathcal{L}(\delta(t-t_0)) = e^{-pt_0}$$

Since $Y(0) = Y'(0) = Y''(0) = Y'''(0) = 0 \Rightarrow \mathcal{L}(Y^{(4)}(t)) = p^4\mathcal{L}(Y)$

this gives us

$$p^4\mathcal{L}(Y) - \mathcal{L}(Y) = e^{-pt_0}$$

$$\mathcal{L}(Y) = \frac{e^{-pt_0}}{p^4-1} =$$

The exponential factor will give us a heavyside factor $H(t-t_0)$ and shift arguments from $t \to t-t_0$

~~To find the inverse we will use L6~~

~~L6 is $\mathcal{L}^{-1}\left[\frac{n!}{(p+a)^{n+1}}\right]$~~

Will try to write the fraction using partial fraction decomposition

$$\frac{1}{(p+1)(p-1)(p+i)(p-i)} = \frac{A}{(p+1)} + \frac{B}{(p-1)} + \frac{C}{(p+i)} + \frac{D}{(p-i)}$$

$$1 = A(p-1)(p+i)(p-i) + B(p+1)(p+i)(p-i) + C(p+1)(p-1)(p-i)$$
$$+ D(p+1)(p-1)(p+i)$$

$p=-1)$ $\quad 1 = A(-2)(-1+i)(-1-i) = A(-2)(1+i-i+1) \Rightarrow A = -\frac{1}{4}$

$p=1)$ $\quad 1 = B(2)(1+i)(1-i) = 4B \Rightarrow B = \frac{1}{4}$

$p=-i)$ $\quad 1 = C(1-i)(-1-i)(-2i) = -2iC(-1-i+i-1) = 4iC \Rightarrow C = \frac{1}{4i}$

$p=i)$ $\quad 1 = D(1+i)(i-1)2i = 2iD(i-1-1-i) = -4iD \Rightarrow D = -\frac{1}{4i}$

$$\Rightarrow \frac{1}{p^4-1} = \frac{1}{4}\left(\frac{1}{p-1} - \frac{1}{p+1} - \frac{i}{p+i} + \frac{i}{p-i}\right)$$

$$\Rightarrow Y(t) = \frac{1}{4}\mathcal{L}^{-1}\left[e^{-pt_0}\left[\frac{1}{p-1} - \frac{1}{p+1} - \frac{i}{p+i} + \frac{i}{p-i}\right]\right], \quad \xi = t - t_0$$

$$= \frac{1}{4}H(\xi)\Big(\underbrace{e^{\xi} - e^{-\xi}}_{2\cosh(\xi)} - ie^{-i\xi} + ie^{i\xi}\Big)$$

$$= H\frac{1}{4}\Big(2\cosh(\xi) + i\big(\underbrace{e^{i\xi} - e^{-i\xi}}_{2i\sin(\xi)}\big)\Big) = \frac{1}{2}(\cosh\xi - \sin\xi)\,H(\xi)$$

~~$= H(t-t_0)\frac{1}{2}(\cosh\xi - \sin\xi)$~~

$$= H(t-t_0)\frac{1}{2}\big(\cosh(t-t_0) - \sin(t-t_0)\big) \quad // \quad \begin{cases} 0 & 0 < t < t_0 \\ \frac{1}{2}[\cosh(t-t_0) - \sin(t-t_0)] & t > t_0 \end{cases}$$

discont at t_0...