Oblig 2 - Ivar Haugerud - FYS 3140

2.1) a)
$$f(z) = \frac{-i+2z}{2+iz} = \frac{-i+2(x+iy)}{2+i(x+iy)}$$

$$= \frac{-i+2iy+2x}{2-y+ix}$$

$$= \frac{(2x+2iy-i)(2-y-ix)}{(2-y+ix)(2-y-ix)}$$

$$= \frac{4x-\cancel{2xy}-2ix^2+4iy-2iy^2+\cancel{2xy}-2i+iy-x}{4-2y\cancel{-2ix}-2y+y^2\cancel{+ixy}\cancel{+2ix}\cancel{-ixy}+x^2}$$

$$= \frac{3x+i(-2x^2+4y-2y^2-2+y)}{4-4y+x^2+y^2}$$

$$= \frac{3x}{x^2+(y-2)^2} + \frac{i}{x^2+(y-2)^2}\left(-2x^2-2y^2+5y-2\right)$$

Which means that

$$u(x,y) = \frac{3x}{x^2+(y-2)^2}, \quad \text{and}$$

$$v(x,y) = \frac{-2x^2-2y^2+5y-2}{x^2+(y-2)^2}$$

2.1) b) $f(z) = e^{iz} = e^{i(x+iy)} = e^{ix} \cdot e^{-y}$

$$= \underline{e^{-y}(\cos(x) + i\sin(x)}$$

$u(x,y) = \cos(x) \cdot e^{-y}$

$v(x,y) = \sin(x) \cdot e^{-y}$

We know this since sin, cos and e, with real argument, is a real number.

2.2) a) $u(x,y) = \dfrac{y}{(1-x)^2+y^2}$

$$\frac{\partial u}{\partial x} = y \frac{\partial}{\partial x}\left(\left[(1-x)^2+y^2\right]^{-1}\right)$$

$$= -\frac{y}{\left[(1-x)^2+y^2\right]^2}\cdot\left(2(1-x)\right)\cdot(-1)$$

$$= \frac{2y(-x+1)}{\left[(1-x)^2+y^2\right]^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2y\frac{\partial}{\partial x}\left((1-x)\left([1-x]^2+y^2\right)^{-2}\right)$$

$$= \frac{2y\cdot(-1)}{\left([1-x]^2+y^2\right)^2} + \frac{2y(1-x)}{\left([1-x]^2+y^2\right)^3}\cdot(-2)\cdot 2\cdot(1-x)\cdot(-1)$$

$$= \frac{-2y}{\left([1-x]^2+y^2\right)^2} + \frac{8y(1-x)^2}{\left([1-x]^2+y^2\right)^3}$$

$$= \frac{8y(1-x)^2-2y(1-x)^2-2y^3}{\left([1-x]^2+y^2\right)^3}$$

$$= \frac{6y(1-x)^2-2y^3}{\left([1-x]^2+y^2\right)^3}$$

And we do the same for y

2.2) $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{(1-x)^2+y^2}\right)$$

$$= \frac{1}{(1-x)^2+y^2} + \frac{y}{((1-x)^2+y^2)^2}\cdot(-1)\cdot 2y$$

$$= \frac{(1-x)^2+y^2}{((1-x)^2+y^2)^2} - \frac{2y^2}{((1-x)^2+y^2)^2}$$

$$= \frac{(1-x)^2-y^2}{((1-x)^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2y}{((1-x)^2+y^2)^2} + \frac{(1-x)^2-y^2}{((1-x)^2+y^2)^3}(-2)\cdot 2y$$

$$= \frac{(-2y)((1-x)^2+y^2) - 4y((1-x)^2-y^2)}{((1-x)^2+y^2)^3}$$

$$= \frac{-2y((1-x)^2+y^2+2(1-x)^2-2y^2)}{((1-x)^2+y^2)^3}$$

$$= \frac{-6y(1-x)^2+2y^3}{((1-x)^2+y^2)^3}$$

which we see is $-\frac{\partial^2 u}{\partial x^2}$. It will therefore satisfy $\nabla^2 u = 0$.

2.2) b) The Cauchy-Riemann equations say that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

We know from a) that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2y(1-x)}{((1-x)^2+y^2)^2}$$ and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{(1-x)^2 - y^2}{((1-x)^2+y^2)^2}$$

We take the first equation and integrate both sides.

$$\int \frac{\partial v}{\partial y}\,dy = 2(1-x)\int \frac{y}{((1-x)^2+y^2)^2}\,dy$$

$$\boxed{\begin{aligned} u &= (1-x)^2+y^2 \\ \frac{du}{dy} &= 2y \\ dy &= \frac{du}{2y} \end{aligned}}$$

$$v(x,y) = 2(1-x)\int \frac{y}{u^2}\,\frac{du}{2y}$$

$$= \cancel{2}(1-x)\int \frac{1}{u^2}\,du$$

$$= -(1-x)\,\frac{1}{u} + C(x)$$

$$= \frac{(x-1)}{(1-x)^2+y^2} + C(x)$$

where the integration constant can be a function of x.

We then try to solve the second equation.

2.2 b) $-\frac{\partial V}{\partial x} = \frac{(1-x)^2 - y^2}{((1-x)^2+y^2)^2}$

$$-\int \frac{\partial V}{\partial x}\, dx = \int \frac{(1-x)^2 - y^2}{((1-x)^2+y^2)^2}\, dx$$

$u = (1-x)$

$\frac{du}{dx} = -1$

$dx = -du$

$$V(x,y) = \int \frac{u^2-y^2}{(u^2+y^2)^2}\, du$$

$$= \int \frac{u^2+y^2-y^2-y^2}{(u^2+y^2)^2}\, du$$

$$= \int \frac{u^2+y^2}{(u^2+y^2)^2} - \frac{2y^2}{(u^2+y^2)^2}\, du = \int \frac{1}{u^2+y^2}\, du - 2y^2 \int \frac{1}{(u^2+y^2)^2}\, du$$

$$= \frac{1}{y^2}\int \frac{1}{1+\frac{u^2}{y^2}}\, du - 2y^2 \int \frac{y\cdot\frac{1}{\cos^2(z)}\, dz}{(y^2+y^2\tan^2(z))^2}$$

$\cancel{z = y\tan(u)}$

$\cancel{\frac{dz}{du} = \sec^2(u)}$

$\cancel{du = \frac{dz}{\sec^2(u)}}$

$$= \frac{1}{y^2}\int \frac{1}{1+(\frac{u}{y})^2}\, du - \frac{2y^2}{y^4}\cdot y \int \frac{dz}{\cos^2(z)(1+\tan^2(z))^2}$$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right) + C(x) - \frac{2}{y}\int \frac{dz}{\cos^2(z)\frac{1}{\cos^4(z)}}$$

$u = y\tan(z)$

$\frac{du}{dz} = y\frac{1}{\cos^2(z)}$

$du = \frac{y\cdot dz}{\cos^2(z)}$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right) - \frac{2}{y}\int \cos^2(z)\, dz$$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right) - \frac{1}{y}\left(z + \sin(z)\cdot\cos(z)\right) + C(y)$$

$z = \arctan\left(\frac{u}{y}\right)$

$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$

$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$

$$= \cancel{\frac{1}{y}\arctan\left(\frac{u}{y}\right)} - \cancel{\frac{1}{y}\arctan\left(\frac{u}{y}\right)} - \frac{\left(\frac{u}{y}\right)}{y\left(\frac{u^2}{y^2}+1\right)} + C(y)$$

$$= -\frac{u}{(u^2+y^2)} + C(y) = -\frac{(1-x)}{((1-x)^2+y^2)} + C(y)$$

$$= \frac{x-1}{(1-x)^2+y^2} + C(y).$$

We get the same answer, which we should and find that the integration constant must be zero.

2.25)

$$f(x,y) = \frac{y + i(x-1)}{(x-1)^2 + y^2} = \frac{\frac{z-\bar{z}}{2i} + i\left(\frac{z+\bar{z}}{2}\right) - i}{x^2+1-2x + y^2}$$

$$\boxed{\begin{aligned} x &= \frac{z+\bar{z}}{2} \\ y &= \frac{z-\bar{z}}{2i} \end{aligned}}$$

$$= \frac{-\frac{1}{2}i\left(z-\bar{z}\right) + \frac{i}{2}\left(z+\bar{z}\right) - i}{\left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 - 2\left(\frac{z+\bar{z}}{2}\right) + 1}$$

$$= \frac{\frac{i}{2}\left(z+\bar{z} - z + \bar{z}\right) - i}{\frac{1}{4}\left(z^2+\bar{z}^2 + 2z\bar{z}\right) - \left(\frac{z^2+\bar{z}^2 - 2z\bar{z}}{4}\right) - z - \bar{z} + 1}$$

$$= \frac{i\bar{z} - i}{\frac{1}{4}\left(\cancel{z^2} + \cancel{\bar{z}^2} + 2z\bar{z} - \cancel{z^2} - \cancel{\bar{z}^2} + 2z\bar{z}\right) - z - \bar{z} + 1}$$

$$= \frac{i(\bar{z}-1)}{z\bar{z} - z - \bar{z} + 1}$$

$\cancel{(1-z)(1-\bar{z})}$

$$= \frac{i(\bar{z}-1)}{(1-z)(1-\bar{z})} = \frac{-i\cancel{(1-\bar{z})}}{(1-z)\cancel{(1-\bar{z})}} = \frac{-i}{(1-z)}$$

$$= \frac{1}{i(1-z)}$$

2.2 c) We want to check that

$$V(x,y) = \frac{(x-1)}{(x-1)^2+y^2}$$

satisfy laplace's equation, which it should. Let's begin

$$\frac{\partial}{\partial y}\left(\frac{(x-1)}{(x-1)^2+y^2}\right) = a\,\frac{1}{(a^2+y^2)^2}\cdot(-1)\cdot 2y \quad , \quad a=(x-1)$$

$$= \frac{-2ay}{(a^2+y^2)^2}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{\partial}{\partial y}\left(-\frac{2ay}{(a^2+y^2)^2}\right) = -\frac{2a}{(a^2+y^2)^2} - \frac{2ay}{(a^2+y^2)^3}\cdot(-2)\cdot 2y$$

$$= \frac{8ay^2}{(a^2+y^2)^3} - \frac{2a}{(a^2+y^2)^2} = \underline{\frac{6ay^2-2a^3}{(a^2+y^2)^3}}$$

And we do the same for x

$$\boxed{\frac{\partial a}{\partial x} = \frac{\partial}{\partial x}(x-1) = 1}$$

$$\frac{\partial}{\partial x}\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{\partial a}{\partial x}\frac{\partial}{\partial a}\left(\frac{a}{a^2+y^2}\right)$$

$$= \frac{1}{a^2+y^2} + \frac{a}{(a^2+y^2)^2}\cdot(-1)\cdot 2a$$

$$= \frac{1}{a^2+y^2} - \frac{2a^2}{(a^2+y^2)^2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{\partial a}{\partial x}\frac{\partial}{\partial a}\left(\frac{1}{a^2+y^2} - \frac{2a^2}{(a^2+y^2)^2}\right)$$

$$= \frac{1}{(a^2+y^2)^2}\cdot(-1)\cdot 2a - \frac{4a}{(a^2+y^2)^2} - \frac{2a^2}{(a^2+y^2)^3}\cdot(-2)\cdot(2a)$$

$$= \frac{-2a}{(a^2+y^2)^2} - \frac{4a}{(a^2+y^2)^2} + \frac{8a^3}{(a^2+y^2)^3} = \frac{8a^3-6a(a^2+y^2)}{(a^2+y^2)^3}$$

$$= \underline{\frac{2a^3-6ay^2}{(a^2+y^2)^3}}$$

As we found the same expression, but with opposite sign, this satisfies laplaces-equation

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2 V}{\partial y^2} = 0$$

2.3) In this task we will use

$$f(a) \cdot 2\pi i = \oint \frac{f(z)}{z-a}\, dz$$

to solve the integrals

a) $\oint_{\Gamma} \frac{\sin z \, dz}{2z-\pi}$ we see that $\frac{\sin z}{2z-\pi}$ diverges at $z = \frac{\pi}{2}$, which is inside the circle Γ. We will use Cauchy-integral formula; with $a = \frac{\pi}{2}$, and $f(z) = \sin(z)$

$$\oint_{\Gamma} \frac{\sin z}{2z-\pi}\, dz = \frac{1}{2} \oint \frac{\sin(z)}{z-\frac{\pi}{2}}\, dz = \frac{1}{2} \cdot 2\pi i \sin\left(\frac{\pi}{2}\right)$$

$$= \underline{\pi i}$$

b) Since the singularity is at $z = \frac{\pi}{2} > 1$ our function is analytic inside Γ

$\xRightarrow{}$ the integral is $\underline{0}$

Cauchy's theorem

2.3) c) $\oint_\Gamma \frac{\sin(2z)}{6z-\pi}\,dz = \frac{1}{3}\oint_\Gamma \frac{\sin(2z)}{2z-\frac{\pi}{3}}\,dz$

$u = 2z$
$du = 2\,dz$

$= \frac{1}{3}\oint_\Gamma \frac{\sin(u)}{u-\frac{\pi}{3}}\,\frac{du}{2} = \frac{1}{6}\oint_\Gamma \frac{\sin(u)}{u-\frac{\pi}{3}}$

$= \frac{1}{6}\cdot 2\pi i\,\sin(\pi/3) = \frac{\pi i}{3}\sqrt{\frac{3}{4}} = \underline{\frac{\pi i}{2\sqrt{3}}}$

We use this method since the integrand diverges at $z = \frac{\pi}{6} < 1$

d) The integrand diverges at $z = \ln(2) < 2$, so we use Cauchy integral formula

$\oint_\Gamma \frac{e^{2z}}{z-\ln(2)}\,dz = 2\pi i\, e^{2\cdot\ln(2)} = 2\pi i\,(2)^2 = \underline{8\pi i}$

In these task we do not have to care about the shape of the curves in any of these problems. We only have to care about if the ~~cl~~ closed curve contains singularities or not.