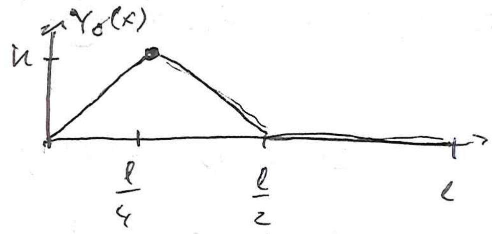
Matmet - oblig 12 - Ivar Haugerud - FYS3140

13.4.2)



The string must satisfy the wave-equation $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Assume separation of variables $u(x,t) = X(x)\,T(t)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = T(t)\,X''(x)$ and $\frac{\partial^2 u}{\partial t^2} = X(x)\,\ddot{T}(t)$

Put this into DE: $T(t)\,X''(x) = \frac{1}{c^2} X(x)\,\ddot{T}(t)$

$\frac{1}{X(x)} X''(x) = \frac{1}{c^2}\cdot\frac{1}{T(t)}\ddot{T}(t)$ each side is indep $\Rightarrow$ must be a constant k

$\Rightarrow X''(x) = k\,X(x)$ and $\ddot{T}(t) = c^2 k\,T(t)$ $u(0,t) = u(L,t) = 0$

we have turned the PDE into two ordinary DE.

Our boundary conditions can not be satisfied if k is positive, we therefore write $k = -p^2$, where p is a real number.

We then get $X''(x) = -p^2 X(x) \Rightarrow A\cos(px) + B\sin(px)$

$u(0,t) = T(t)X(0) = 0 = T(t)\cdot(A\cos(0) + B\sin(0)) = A\,T(t) = 0 \Rightarrow A = 0$

$u(L,t) = T(t)X(L) = 0 = T(t)\cdot(B\sin(pL)) \Rightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L}$

We can absorb the factor B into $T(t)$, making our solution

$X(x) = \sin\left(\frac{n\pi x}{L}\right)$

Do the same for $T(t)$:

$\ddot{T}(t) = -c^2p^2\,T(t) = -c^2\left(\frac{n\pi}{L}\right)^2 T(t) = -\omega_n^2\,T(t)$, $\omega_n = \frac{n\pi c}{L}$

Known solution

$T(t) = A\cos(\omega_n t) + B\sin(\omega t)$

Our (infinitely many) solutions is thus

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A\cos(\omega_n t) + B\sin(\omega_n t)\right)$$

Now we must find a unique solution from initial conditions

13.4.2) We know that the initial velocity is zero.

$$\dot{u}(x,0) = 0 = \sin\left(\frac{n\pi x}{L}\right)\left(B\cos(\omega_n t) - A\sin(\omega_n t)\right)\Big|_{t=0} \cdot \omega_n$$

$$0 = B\cos(0) - A\sin(0)$$

$$0 = B \Rightarrow \underline{B = 0}$$

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)\cdot A$$

To find the complete solution we must use the initial position of the string. The full solution is written as a super-position of the solutions found with seperation of variabels

$$Y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) A_n$$

At $t=0$ the amplitude follows that shown on the previous page:

$$Y(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) A_n = f(x) = \begin{cases} \frac{h4}{\ell}x, & x < \ell/4 \\ \frac{h4}{\ell}\left(\frac{\ell}{2} - x\right), & x \in \left[\frac{\ell}{4}, \frac{\ell}{2}\right] \\ 0 & x > \frac{\ell}{2} \end{cases}$$

We can find the coeffs through the fourier series (for sin)

$$A_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\frac{A_n}{2} = \underbrace{\frac{h4}{\ell}\int_0^{\ell/4} x\sin\left(\frac{n\pi x}{\ell}\right)dx} + \frac{h4}{\ell}\int_{\ell/4}^{\ell/2}\left(\frac{\ell}{2} - x\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\int_0^{\ell/4} x\sin\left(\frac{n\pi x}{\ell}\right)dx = -x\cdot\frac{\ell}{n\pi}\cdot\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell/4} + \int_0^{\ell/4} \frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)$$

$$= -\frac{\ell^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{\ell^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell/4} = \underline{-\frac{\ell^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{\ell^2}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)}$$

Now let's solve the second integral

$$\int_{l/4}^{l/2} \left(\frac{l}{2}-x\right)\sin\left(\frac{n\pi x}{l}\right) = \frac{l}{2}\int_{l/4}^{l/2} \sin\left(\frac{n\pi x}{l}\right)dx - \underbrace{\int_{l/4}^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx}_{\text{same integral as prev. but different limits}}$$

$$= -\frac{l}{2}\left[\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_{l/4}^{l/2} + \frac{x\,l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Bigg|_{l/4}^{l/2} - \frac{l^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_{l/4}^{l/2}$$

$$= -\frac{l^2}{2n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right) + \frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{l^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right)$$
$$-\frac{l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)$$

$$= \frac{l^2}{2n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \frac{\cos(n\pi/4)}{2}\right) - \frac{l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)$$

Combing the two expressions our integral becomes

$$\frac{l A_n}{2} = \frac{h4}{l}\left(-\frac{l^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{l^2}{n^2\pi^2}\sin(n\pi/4)\right) + \frac{h4}{l}\left(\frac{l^2}{2n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \frac{\cos(n\pi/4)}{2}\right] - \frac{l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)\right]$$

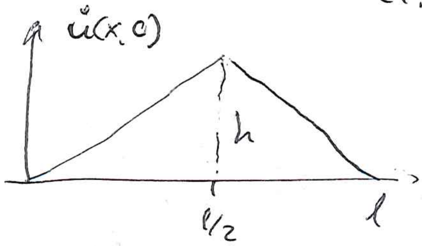
$$A_n = 8h\left[\frac{1}{2n\pi}\left(-\frac{\cos\left(\frac{n\pi}{4}\right)}{2} + \cos\left(\frac{n\pi}{4}\right) - \frac{\cos(n\pi/4)}{2}\right) + \frac{1}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{4}\right)\right)\right]$$

$$A_n = \frac{8h}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$ Making our solution

$$Y(x,t) = \sum \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)\,A_n$$

$$Y(x,t) = \frac{8h}{\pi^2}\sum \frac{1}{n^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi L t}{c}\right)\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$

13.4.5) We look at same system as previously, but with different initial conditions. Now $u(x,0)=0$ while $\dot{u}(x,0)$ is



Our solution before looking at initial conditions is

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A\cos(\omega_n t) + B\sin(\omega_n t)\right)$$

For $t=0$ we have no displacement $\Rightarrow$

$$u(x,0) = 0 = \sin\left(\frac{n\pi x}{L}\right)(A\cdot 1 + B\cdot 0) \Rightarrow \underline{A = 0}$$

$$u(x,t) = B\sin(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)$$ making the velocity

$$\dot{u}(x,t) = \omega_n B\cos(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)$$

Theese are infinitely many solutions, use the initial velocity to determine exact solution

$$\dot{u}(x,0) = \omega_n B\sin\left(\frac{n\pi x}{L}\right)$$

The exact solution can then be written as

$$Y(x,0) = \sum_{n=1}^{\infty} \omega_n B_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} \frac{2hx}{l} & x < \frac{l}{2} \\ \frac{2h}{l}(l - x) & x > \frac{l}{2} \end{cases}$$

Use a fourier series to determine B_n

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$\frac{L B_n\omega_n}{2} = \underbrace{\frac{2h}{l}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx}_{\text{same integral, different limits}} + \frac{2h}{l}\int_{L/2}^{L}\sin\left(\frac{n\pi x}{L}\right)(L-x)\,dx$$

$$-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} = -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$+\frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} = \frac{L^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - 0\right) = \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$L\int_{L/2}^{L}\sin\left(\frac{n\pi x}{L}\right) - \int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)$$

Continue with the second integral

$$L\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = L \cdot \frac{L}{n\pi}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} = \frac{L^2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)$$

$$\int_{L/2}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L} + \frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L}$$

same integral different limits

$$= -\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{L^2}{n\pi}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{2} - \cos(n\pi)\right) + \frac{L^2}{n^2\pi^2}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)$$

Making the whole coefficient

$$\frac{2B_n\omega_n}{2} = \frac{2h}{L}\left[\frac{L^2}{n^2\pi^2}2\sin(n\pi/2) - \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right) - \frac{L^2}{n\pi}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{2} - \cos(n\pi)\right) - \frac{L^2}{n^2\pi^2}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)\right]$$

$$\omega_n B_n = \frac{4h}{L^2}\left[\frac{L^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi) + \sin\left(\frac{n\pi}{2}\right)\right) + \frac{L^2}{n\pi}\left(\cancel{-\frac{1}{2}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\cos\left(\frac{n\pi}{2}\right)} - \cancel{\cos(n\pi)} - \cancel{\frac{\cos\left(\frac{n\pi}{2}\right)}{2}} + \cancel{\cos(n\pi)}\right)\right]$$

$$B_n = \frac{4h}{\omega_n n^2\pi^2}\left(2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)\right)$$

Making the function (put in $\omega_n = \frac{n\pi c}{L}$)

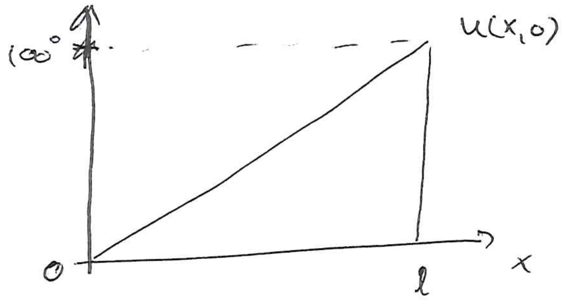
$$Y(x,t) = \frac{4h}{\pi^2}\cdot\frac{L}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)\left(2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)\right)$$

$$= \frac{4hL}{\pi^3 c}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\underbrace{\left(2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)\right)}_{0 \text{ for all even } n\text{, alternating } \pm \text{ for odd } n}$$

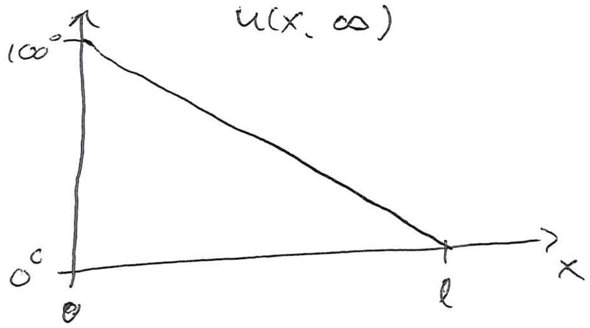
$$= \frac{4hL}{\pi^3 c}\left(2\cos\left(\frac{\pi c t}{L}\right)\sin\left(\frac{\pi x}{L}\right) - \frac{2}{3^3}\cos\left(\frac{3\pi c t}{L}\right)\sin\left(\frac{3\pi x}{L}\right) + \frac{2}{5^2}\cos\left(\frac{5\pi c t}{L}\right)\sin\left(\frac{5\pi x}{L}\right) - \ldots\right)$$

$$= \frac{8hL}{\pi^3 c}\left(\cos\left(\frac{\pi c t}{L}\right)\sin\left(\frac{\pi x}{L}\right) - \frac{1}{9}\cos\left(\frac{3\pi c t}{L}\right)\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\cos\left(\frac{5\pi c t}{L}\right)\sin\left(\frac{5\pi x}{L}\right) \ldots\right.$$

13.3.3) The initial temperature distribution is



and in infinity



$f(x) = \frac{100x}{l}$ $\qquad$ $g(x) = 100 - \frac{100x}{l}$

We want to find the temperature at some intermediate timestep. To do this we solve the diffusion equation in 1D:

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$, use seperation of variables $u(x,t) = X(x)\,\Theta(t)$

$$\Theta(t)\, X''(x) = \frac{1}{\alpha^2} X(x)\, \dot{\Theta}(t)$$

$$\frac{1}{X(x)} X''(x) = \frac{1}{\alpha^2 \Theta(t)} \dot{\Theta}(t)$$

LHS is only x-dependent, while RHS is only t-dependent ⇒ both must equal the same constant k

$$\dot{\Theta}(t) = \alpha^2 k\, \Theta(t)$$

For the temperature to be finite the constant k must be negative: $k = -\beta^2$, which gives us solution $\underline{\Theta(t) = e^{-\alpha^2\beta^2 t}}$ where $\beta \in \mathbb{R}$. We do the same for the spatial equation

$X''(x) = -\beta^2 X(x) \Rightarrow \underline{X(x) = A\sin(\beta x) + B\cos(\beta x)}$

At $t=0$ $\Theta(0) = 1$ meaning X will determine the boundary conditions:

$X(0) = A\sin(0) + B\cos(0) = \underline{B = 0}$

$X(x) = A\sin(\beta x)$, and for $t > 0$ $X(l) = 0 \Rightarrow \sin(\beta l) = 0$

$\Rightarrow \beta = \frac{n\pi}{l}$. Making the full solution for X

$X(x) = A_n \sin\left(\frac{n\pi x}{l}\right)$, where A_n is determined from initial conditions.

13.3.3) Thus we find ~~$\beta = \ldots$~~ $\beta = \frac{\pi n}{l}$

Making the spatial solution

$\mathcal{X}(x) = A_n \sin\left(\frac{n\pi x}{l}\right)$, the full solution is then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$$

Evaluated at $t=0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

For these type of problems, with diffent initial and final initial state we have to use $u(x,0) - u(x,\infty)$ to find fourier coeffs

$$u(x,0) - u(x,\infty) = \frac{100x}{l} - \left(100 - \frac{100x}{l}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{200x}{l} - 100 = \sum A_n \sin\left(\frac{n\pi x}{l}\right)$$

We determine, the fourier coefficients

$$A_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\left(\frac{200x}{l} - 100\right) dx$$

$$= -\frac{200}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right) dx + \frac{400}{l^2}\int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= -\frac{200}{l}\left[-\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{400}{l^2}\left(-\frac{x \cdot l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Big|_0^l + \frac{l}{n\pi}\int_0^l \cos\left(\frac{n\pi x}{l}\right) dx\right)$$

$$= -\frac{200}{n\pi}\left(-\cos(n\pi) + 1\right) + \frac{400}{l^2}\left(-\frac{l^2}{n\pi}\cos(n\pi) + \frac{l^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_0^l\right)$$

$$= \frac{200}{n\pi}\left(\cos(n\pi) - 1\right) - \frac{400}{n\pi}\cos(n\pi) + \frac{400}{n^2\pi^2}\left(\sin(n\pi) - 0\right)$$

$$= -\frac{200}{n\pi}\left(1 + \cos(n\pi)\right) + \frac{400}{n^2\pi^2}\underbrace{\sin(n\pi)}_{0} = -\frac{200}{n\pi}\left(1 + \cos(n\pi)\right) = -\frac{400}{n\pi}$$

for even n

Now we have found the fourier coefficients

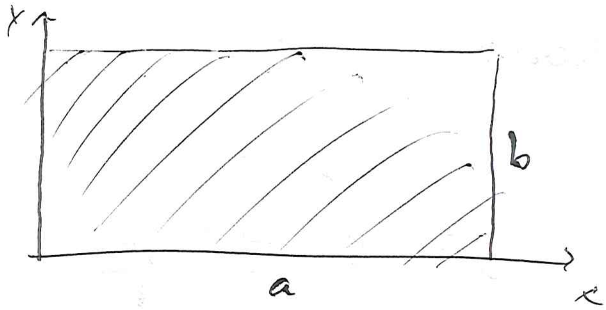
$$A_n = \begin{cases} -\frac{400}{n\pi} & \text{for even } n \\ 0 & \text{for odd } n \end{cases}$$

Thus we find the full solution to be

$$u(x,t) = u_f + \sum_{n=0}^{\infty} A_n e^{-\frac{n^2\pi^2\alpha^2}{\ell^2}t} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$u(x,t) = 100 - \frac{100x}{\ell} - \frac{400}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2\alpha^2}{\ell^2}t} \sin\left(\frac{n\pi x}{\ell}\right)$$

13.6.3)



$$\nabla^2 z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

Use seperation of variables

$$Z(x,y) = X(x)\,Y(y)\,\Theta(t)$$

This gives us: $\frac{1}{X(x)}X''(x) + \frac{1}{Y(y)}Y''(y) = \frac{1}{v^2\Theta(t)}\ddot{\Theta}(t)$

Each side is independent, and must therefore equal the same constant

$$\frac{1}{X(x)}X''(x) + \frac{1}{Y(y)}Y''(y) = k = -\alpha^2$$ ← must be negative to satisfy that all bounderies are fixed

$$\frac{1}{v^2\Theta(t)}\ddot{\Theta}(t) = k = -\alpha^2$$

$$\ddot{\Theta}(t) = -\alpha^2 v^2 \Theta(t) \Rightarrow \Theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\underline{\omega = \alpha v}$

For the spatial solution we get

$$\frac{1}{X(x)}X''(x) + \frac{1}{Y(y)}Y''(y) = -\alpha^2$$ since they are seperate and indep we get that both must equal a constant

we introduce $\frac{1}{Y(y)}Y''(y) = l^2$

such that $X''(x) = (-\alpha^2 - l^2)X(x) = -(\alpha^2 + k^2)X(x)$

$$\Rightarrow X(x) = C\sin(kx) + D\cos(kx)$$

Use BC's: $X(0) = 0 = D$

$X(a) = 0 = C\sin(ka) \Rightarrow \boxed{k = \frac{n\pi}{a}}$

Do the same in the y-direction

$$\Rightarrow Y(y) = (-\alpha^2 - k^2)Y(y) = -(\alpha^2 + k^2)Y(y)$$

$\Rightarrow Y(y) = E\cos(ly) + F\sin(ly)$, and with same BC's

$\Rightarrow Y(y) = F\sin\left(\frac{m\pi}{b}y\right)$ where $l = \frac{m\pi}{b}$

Resulting in . . .

$$-\alpha^2 = -h^2 - l^2$$

$$\alpha^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \Rightarrow \underline{\alpha = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}}$$

Making the time solution:

$$G(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$= A\cos(\alpha v t) + B\sin(\alpha v t)$$

This function has a period T of

$$\alpha v T = 2\pi \Rightarrow T = \frac{2\pi}{\alpha v}$$

and a frequency of

$$\nu = \frac{1}{T} = \frac{\alpha v}{2\pi} = \frac{v}{2\pi}\,\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

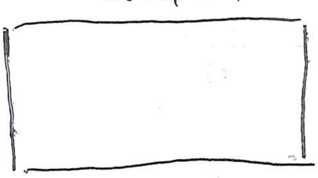
$$\underline{\nu = \frac{v}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}}$$

The full solution is

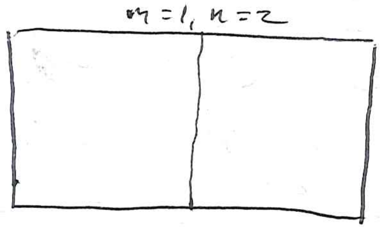
$$u = \sum_{m,n}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) F_m \sin\left(\frac{m\pi y}{b}\right)\left(A\cos\left(v\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\,t\right) + B\sin\left(v\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\,t\right)\right)$$

Let's draw the modal lines

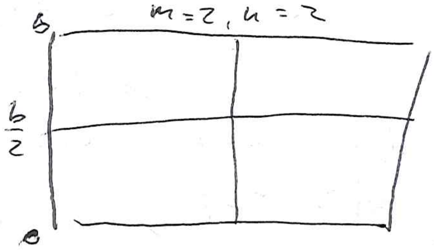
$m=1, n=1$



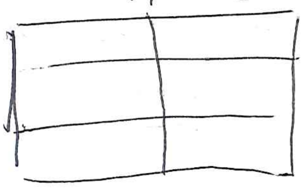
$m=1, n=2$



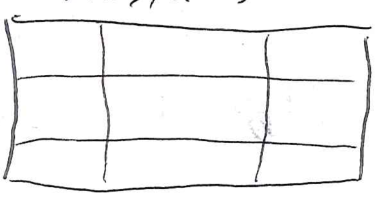
$m=2, n=2$



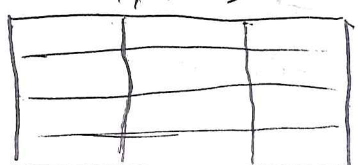
$m=3, n=2$



$m=3, n=3$



$m=4, n=3$

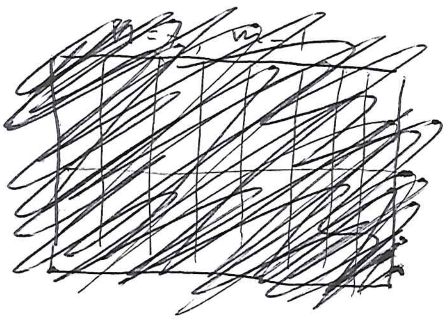


If the rectangle is a square: $a = b$
the frequencies are

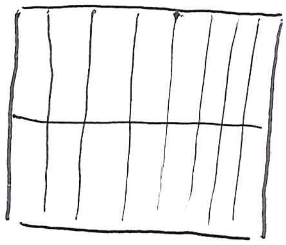
$$\nu = \frac{v}{2a}\sqrt{n^2 + m^2}$$ where n and m are positive integers

We notice that we get a degeneracy of frequencies. For example will
$n = 7$ & $m = 1$ and $n = 5$ & $m = 5$ give the same frequency. ~~The~~

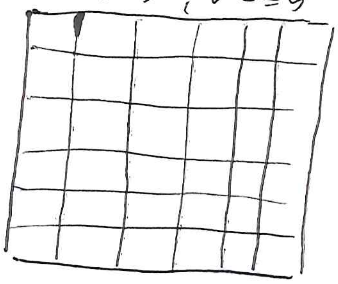
We have different eigenfunctions with the same frequency, resulting in the same eigenvalue. This is what we call degeneracy.



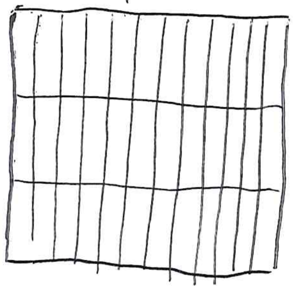
$n=7,\ m=1$



$n=5,\ m=5$



$n=11,\ m=2$



$n=10,\ m=5$