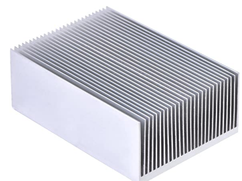
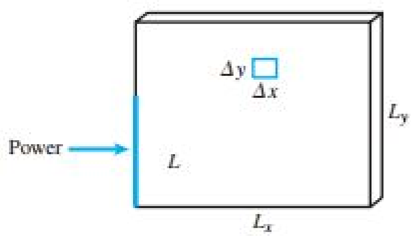
Project 3 in numerical analysis

Heatsinks are an ubiquitous design used to dissipate heat and cool certain objects which do not handle high temperatures. Examples are a car's radiator and the cooling system of a microprocessor. A typical design for a heatsink is as shown here:



which is composed of a number of thin plates. In this project we focus on one such plate and study how it reacts to being subjected to heat on one side. The plate is rectangular with dimensions L_x and L_y, and its small thickness is δ. Heat enters the plate on its left-hand side as shown here:



Our goal is to find the long-term equilibrium of the system that is compute the temperature distribution after an infinite amount of time. One can show that at an interior point the temperature $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2H}{K\delta} u$$

The constant K is the thermal conductivity of the material while H is the convective heat transfer coefficient, they relate to both ways that heat can dissipate. Again δ is the thickness of the plate. On top of the equation we have boundayr conditions which encode convection into the ambient air. More precisely:

$$\frac{\partial u}{\partial n} = \frac{H}{K} u$$

which is the outward normal derivative on the sides of the plate. Since the plate is rectangular the normal derivative is simply parallel to the x and y axes with opposite directions on opposite sides:

$$\begin{aligned} \frac{\partial u}{\partial n} &= -\frac{\partial u}{\partial y} && \text{bottom} \\ \frac{\partial u}{\partial n} &= \frac{\partial u}{\partial y} && \text{top} \\ \frac{\partial u}{\partial n} &= -\frac{\partial u}{\partial x} && \text{left} \\ \frac{\partial u}{\partial n} &= \frac{\partial u}{\partial x} && \text{right} \end{aligned}$$

Finally we have a separate boundary condition where heat enters the plate:

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} = \frac{P}{L\delta K}$$

where P is the power og L the length along which power is applied.

Discretization and first solution

On the x-axis we divide the interval $[0, L_x]$ in $m-1$ subintervals of length h with

$$0 = x_1 < x_2 < \cdots < x_{m-1} < x_m = L_x$$

and likewise $[0, L_y]$ is divided in $n-1$ subintervals of length k:

$$0 = y_1 < y_2 < \cdots < y_{n-1} < y_n = L_y$$

and we let $u_{i,j}$ be an approximation for the temperature at the point (x_i, y_j). Altogether we need to compute mn values.

1. Discretize the partial differential equation together with the boundary conditions. Show your calculations and explain which formulas are used. The boundary conditions are of Robin type.

Two points might be troublesome:

- Power is entering the plate on the left side along some portion of the side. There are therefore two possible boundary conditions on that side.
- A smaller issue is which condition to use on the four corners since we could both consider vertical and horizontal diffusion. We will consider that the normal derivative is along the x-axis in the corners to get the full effect of the power intake. This however doesn't change have much of an effect.

2. Rewrite the discretized equations from question 1 to form a linear system for mn variables $v_1, \cdots, v_{mn}$ defined such that

$$v_{i+(j-1)m} = u_{i,j}$$

Write it in the form $A\mathbf{v} = \mathbf{b}$ where A is a $mn \times mn$ matrix.

3. We use the following parameters:

- Size of the plate: $L_x = L_y = 2\,\mathrm{cm}$, $\delta = 0.1\,\mathrm{cm}$
- Power: $P = 5\,\mathrm{W}$ along the whole left side (including the corners) þ.e. $L = 2\,\mathrm{cm}$.
- Material: $K = 1.68\,\mathrm{W/cm}°C$ (for aluminum), $H = 0.005\,\mathrm{W/cm^2}°C$ (for air)
- External temperature $20°C$. Note that the equations assume that it is rather $0°$ which warrants a small correction afterwards.

Solve the system for question 2 for these parameters with $m = n = 10$ steps in each direction. Draw a 3D picture showing the temperature distribution on the plate, for instance using mesh in Matlab.

Check. Highest temperature is $164.9626°C$, both in $(0, 0)$ and $(0, L_y)$. Small differences can be explained by different interpretations of the boundary conditions in the corners.

Error Analysis

We aim to find suitables values for m and n to minimize errors while keeping the running time down. We shall then use these values for the remainder of the project.

4. Solve the system for $m = n = 100$ and keep the answer - it will be used as a reference. Running time should to be too high for these values to be practical.

Solve then the system for $m, n \in \{10, 20, \cdots, 90\}$ a total of 9^2 times. Each time compute the deviation from the reference $m = n = 100$ and save the answers in a matrix. It might be technically difficult to compare vectors and different lengths, for instance you might want to look at the value at $(0, 0)$ for instance.

Which matters more, an increase in m or an increase in n? Why? Choose then the best values for m and n which satisfy:

- Running time less than $0.5\,\mathrm{s}$ (rather much lower)
- Deviation from the reference less than $0.01°\mathrm{C}$

Explain your choice.

Varying power intake

The plate has now dimensions $L_x = L_y = 4\,\mathrm{cm}$. The power intake is now located in the lower half of the left-hand side, like in the picture in the introduction. In other words we keep $L = 2$. This could be a model for a $2\,\mathrm{cm}$ processor alongside a $4 \times 4\,\mathrm{cm}^2$ heatsink.

5. Solve the system under these new conditions and plot the temperature distribution. Use the values of m and n computed earlier.

6. The processor doesn't need to be placed at $y \in [0, 2]$. Find temperature distributions for a range of different positions. Which one is the best, that is minimizes the highest temperature?

7. We use the position found in question 6. How large may the power inntake be if the plate temperature may never exceed $100°$ C? Use for instance the bisection method to find P with 2 significant digits.

Varying material

8. We assume the thermal conductivity K is variable from 1 to $5\,\mathrm{W/cm}°C$. For each value of K we may compute the highest power intake which keeps the plate under 100° C as in question 7. Plot it as a function of K on the interval $[1, 5]$ and interpret the results. You may choose the position of the plate as discussed in question 6.

9. We use again $K = 1.68\,\mathrm{W/cm}°C$. Cut a small rectangular notch off the plate in the upper right corner. Change the boundary conditions so that they reflect the situation and repeat questions 6 and 7. Interpret your results and compare both shapes.

Independent work

Perform one or more experiments of your own conception. Here are a few ideas:

- Effect of the constant H which encodes how fast heat dissipates in the environing material, for instance for water we have $H = 0.1\,\mathrm{W/cm^2\,°C}$.
- Changes in shape and sizes of the plate
- Non-constant power for instance most in the middle.
- Power intake on two sides of the plate.