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Discrete Calculus - Solution of the Exercises

Springer

Contents

Chapter 1
Problems and Solutions

Problem 1.1. Prove the identities of Proposition 1.7 without using the characteristic functions.

Solution. As an example we verify that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Two sets coincide if and only if they have the same elements: hence, to demonstrate the statement we will verify that if an element x of X belongs to $A \setminus (B \cup C)$, then it belongs to $(A \setminus B) \cap (A \setminus C)$ as well, and, conversely, that if an element x of X belongs to $(A \setminus B) \cap (A \setminus C)$ then it also belongs to $A \setminus (B \cup C)$. Let $x \in A \setminus (B \cup C)$; then x belongs to A, but belongs to neither B, nor C. Thus x belongs both to $A \setminus B$ and to $A \setminus C$, and so belongs to their intersection $(A \setminus B) \cap (A \setminus C)$. Conversely let $x \in (A \setminus B) \cap (A \setminus C)$; then x belongs both to $A \setminus B$ and to $A \setminus C$. Hence x belongs to A, but does not belong to either B or C. In other words x does not belong to $B \cup C$. Therefore x belongs to $A \setminus (B \cup C)$.

Problem 1.2. A store carries 8 different brands of pants. For every brand there are 10 sizes, 6 lengths, and 4 colors. How many different types of pants are there in the store?

Solution. To choose a pair of pants one must specify the brand, size, length, and color: this is a conditional product of multiplicities $(8, 10, 6, 4)$. Hence, by the Multiplication Principle 1.34 there are $8 \times 10 \times 6 \times 4 = 1\,920$ types of of pants.

Problem 1.3. How many four letter words can be formed with an alphabet of 26 letters? How many of these are without a repeated letter?

Solution. The set of 4 letter words is a conditional product with multiplicities $(26, 26, 26, 26)$, or of multiplicities $(26, 25, 24, 23)$ according to whether repetitions are or are not admitted: in the first case there are $26^4 = 456\,976$ such words, while in the second case there are $26 \times 25 \times 24 \times 23 = 358\,800$.

Problem 1.4. Give 8 different books in English, 7 different books in French, and 5 different books in German: in how many ways can one choose three books, one for each language?

Solution. If we denote the sets of books in English, French, and German by E, F and G respectively, then the set of possible choices is given by the cartesian product $E \times F \times G$, that is, by a conditional product with multiplicities $(8,7,5)$. By the Multiplication Principle there are then $8 \times 7 \times 5 = 280$ possible choices.

Problem 1.5. In how many ways can one pick two cards from a deck of 52 playing cards so that:

1. The first card is an ace and the second is not a queen?
2. One is an ace and the other is not a queen?
3. The first card is a spade and the second is not a queen?
4. One is a spade and the other is not a queen?

Solution. 1. We can obtain an ordered pair of cards in which the first is an ace and the second is not a queen by the following procedure in two phases: first we choose an ace, and then among the remaining cards we select one which is not a queen. In the first phase we have 4 choices, and in the second $52 - 4 - 1 = 47$ possibilities. Given an ordered pair of cards obtained with this procedure, it is clear that the first card was selected in the first phase, while the second is chosen in the second phase. Thus we are dealing with a conditional product of multiplicities $(4,47)$ which therefore has cardinality $4 \times 47 = 188$.

2. First we choose an ace, then from the remaining cards we choose one that is not a queen: in this way we certainly obtain two cards that satisfy our requirements. However, we are not always able to recover the outcomes of the two phases starting from knowledge of the two cards obtained. For example, the pair of cards Ace of Hearts and Ace of Spades satisfy the required condition and can be obtained by the procedure described, but we have no way of knowing which ace was chosen in the first phase and which in the second. In this case it is useful to decompose the set X under study into a disjoint union of the set A consisting of the pairs of cards formed by an ace and another card which in neither a queen nor an ace, and the set B of pairs of cards formed by two aces. The set A is a conditional product with multiplicities $(4,44)$. The cardinality of B may be obtained by applying the Division Principle 1.47, explicitly by dividing the number of ordered pairs of aces (which, by the Multiplication Principle, is 4×3) by the number 2 of ordered pairs giving the same hand. Thus $|B| = \dfrac{4 \times 3}{2}$, and therefore $|X| = |A| + |B| = 4 \times 44 + 6 = 182$.

3. The set X whose cardinality we wish to calculate is the disjoint union of the sets A of ordered pairs formed by the queen of spades and a non-queen, and the set B of ordered pairs formed by a spade card different from the queen, and by a non-queen . One has $|A| = 1 \times 48$ and $|B| = 12 \times 47$; so that $|X| = 612$.

4. Here the set X whose cardinality we wish to calculate is the disjoint union of the set A of pairs of cards containing two spades, and the set B of pairs of cards containing a spade card and and a non-spade that is not a queen. One has

$$|X| = |A| + |B| = \frac{13 \times 12}{2} + 13 \times 36 = 78 + 468 = 546.$$

Problem 1.6. In how many ways can one toss two dice, one red and one green, so as to obtain a sum divisible by 3?

Solution. Since the due dice have different colors, we must count ordered pairs of numbers between 1 and 6 with sum equal to 3, 6, 9, or 12. For each number x between 1 and 6 there are exactly two numbers y, w between 1 and 6 such that $x + y$ and $x + w$ are both divisible by 3. Thus we are in the presence of a conditional product of multiplicities $(6, 2)$, which by the Multiplication Principle has cardinality 12.

Problem 1.7. Consider the set X of 5 digit numbers, in other words, the numbers between 10 000 and 99 999.

1. Determine the cardinality of X.
2. How many even numbers are there in X?
3. In how many numbers of X does the digit 3 appear exactly once?
4. How many 5 digit palindromic numbers are there (in other words, how many 5 digit numbers are there which remain unchanged if one inverts the order of its digits, e.g., 15251)?

Solution. 1.: Here we are dealing with a conditional product of multiplicities $(9, 10, 10, 10, 10)$. By the Multiplication Principle 1.34 its cardinality is $90\,000$.

2. In this case the conditional product has multiplicities $(9, 10, 10, 10, 5)$ and the Multiplication Principle 1.34 yields $45\,000$ for its cardinality.

3. The 5-digit numbers which have the 3 only as first digit form a conditional produce of multiplicities $(1, 9, 9, 9, 9)$. The other elements of this set are obtained by selecting the first digit, then the position in which to insert the 3, and then the digits of the other three positions starting from the leftmost "open" position. In all, there are therefore $6\,561 + 23\,328 = 29\,889$ such numbers.

4.: Here the conditional product had multiplicities $(9, 10, 10)$; by the Multiplication Principle 1.34 the cardinality of the set of such palindromic numbers is 900.

Problem 1.8. What is the probability that the two top cards in a deck of 52 cards do not form a pair, that is, are not two cards with the same value (from different suits)?

Solution. The sample space Ω of the ordered pairs of playing cards is formed by equi-probable elements and one has $|\Omega| = 52 \times 51$. The favorable cases constitute a conditional product with multiplicities $(52, 3)$. Therefore, the desired probability is $1 - \frac{52 \times 3}{52 \times 51} = \frac{16}{17}$.

Problem 1.9. A message is spread in a group of 10 people in the following way: the first person telephones a second who in turn telephones a third, and so on in a random way. A person of the group can pass the message to any other member of the group, except the person whose call has just been received.

1. In how many different ways can the message be spread via three phone calls? And via n calls?
2. What is the probability that A receives the third call, if it is known that A made the initial call?
3. What is the probability that A receives the third call, if it is known that A did not make the initial call?

Solution. 1.: With 3 calls one has a conditional product of multiplicities $(10, 9, 8, 8)$; for the case of n calls the multiplicities are $(10, 9, \underbrace{8, 8, ..., 8}_{n-1})$. By the Multiplication Principle 1.34 the message can be spread in $90 \times 8^{n-1}$ ways.

2.: The possible cases are a conditional product with multiplicities $(1, 9, 8, 8)$; the favorable cases form a conditional product with multiplicities $(1, 9, 8, 1)$. Hence the desired probability is $1/8$.

3.: Here the possible cases form a conditional product with multiplicities $(9, 9, 8, 8)$, while the favorable cases are a conditional product with multiplicities $(9, 8, 7, 1)$. Indeed, a favorable 4-tuple is of the type (X, Y, Z, A) with $X \neq A$ since A did not place the first call, $Y \neq X$ because X does not call himself, and $Y \neq A$ since otherwise Z could not call A (who would have just phoned Z), $Z \neq X$ since otherwise Y would call the person who had just called Y, $Z \neq Y$ since Y does not call herself, $Z \neq A$ since Z does not call herself. Thus one has 9 possibilities for X, 8 for Y, 7 for Z, and the probability is indeed $7/72$.

Problem 1.10. How many three letter words without repetition of letters can be made by using the letters a, b, c, d, e, f in such a way that either the letter e or the letter f or both appear?

Solution. If we remove the set of three letter word using the alphabet $\{a, b, c, d\}$ from the set of three letter words without repeated letters using the alphabet $\{a, b, c, d, e, f\}$, the remaining words are those we seek. Hence there are $6!/3! - 4!/1! = 6 \times 5 \times 4 - 4 \times 3 \times 2 = 96$ such words.

Problem 1.11. What is the probability that a natural number between 1 and 10 000 contains both the digits 8 and 9 exactly once?

Solution. We can construct numbers of the type described above via the following procedure: we choose the position for the 9 among the 4 possible places, and then we choose the position for the 8 among the three remaining locations. Finally, we fill the two remaining gaps starting from the leftmost one. Thus we have a conditional product with multiplicities $(4, 3, 8, 8)$. The desired probability is therefore $\dfrac{4 \times 3 \times 8 \times 8}{10\,000} = 0.0768$.

Problem 1.12. An assembly of 20 people must vote by raising hands to choose a president from among 7 candidates A, B, C, D, E, F, G.

1. In how many different ways can the votes of the assembly be cast?

2. How many outcomes of the voting are there in which A and D receive exactly one vote?

Solution. 1.: Here one has a conditional product with multiplicities $(\underbrace{7,7,7,...,7}_{20})$; by the Multiplication Principle 1.34, the assembly can vote in 7^{20} different ways.

2.: An outcome that satisfies the conditions imposed may be obtained by first choosing the person who votes for A and then the person who votes fo D, and then the votes of the other 18 people. Thus one has a conditional product with multiplicities $(20,19,\underbrace{5,5,...,5}_{18})$ and so $20\times 19\times 5^{18}$ possibilities.

Problem 1.13. How many 4 digit numbers divisible by 4 may be formed using the digits 1,2,3,4,5 (with possible repetitions)?

Solution. A number is divisible by 4 if and only if number formed by the last two digits is divisible by 4. Thus the numbers being sought are obtained by choosing the first digit, then the second, and finally either 12 or 24, or 32 or 44 or 52 for the last two positions (since these are the only two possible final two digit numbers among the numbers under consideration which are divisible by 4). Thus we have a conditional product with multiplicities $(5,5,5)$, and hence there are $5^3 = 125$ such numbers.

Problem 1.14. In how many ways can one place two identical rooks in the same row or column of an 8×8 chessboard? What is the result in the case of a chessboard with n rows and m columns?

Solution. Suppose that one of the two rooks is white and one is black. Then there are 64 choices of position for the white rook, and then 14 possible positions for the black one, for a total of 64×14 choices. If we forget the colors of the two rooks, then every positioning of two rooks on the chessboard has been counted twice. Therefore, the two rooks can be positioned in $64\times 7 = 448$ ways.

If the chessboard has n rows and m columns the same reasoning shows that the result is $\dfrac{m\times n\times(n-1+m-1)}{2}$.

Problem 1.15. In how many ways can one place two identical queens on an 8×8 chessboard in such a way that the two queens do not lie in the same row, column, or diagonal?

Solution.

If the two queens were of different colors, one could first place the black queen in any one of the 64 possible positions, and then position the white queen. For the latter, the number of possible choices depends on the square ring in which the first queen was placed. There are 4 square rings: if the black queen was placed in the first square ring (the most external one, consisting of all 28 the squares at the edge of the chessboard) then the white queen may be placed in any one of $64-8-7-7$

positions; if the black queen was placed in one of the 20 squares of the second ring of squares, then there are $64-8-7-9$ possible positions for the white queen; if the black queen is in one of the 12 squares of the third ring of squares, then there are $64-8-7-11$ possible positions for the white queen, and finally, for the black queen placed in one of the innermost four squares there are $64-8-7-13$ possible positions for the second queen. Thus the two queens can be positioned in

$$28\times(64-8-7-7)+20\times(64-8-7-9)+12\times(64-8-7-11)+$$
$$+4\times(64-8-7-13)=2576$$

ways.

If we forget the color of the two queens, then every positioning of them on the chessboard has been counted twice, and so the two queens can be positioned in $2576/2=1288$ ways.

Problem 1.16. In how many ways can one invite friends (at least one!) chosen from 10 people?

Solution. If we line up the 10 friends, for each of them one must decide whether or not to invite his/her. Thus every choice corresponds to a binary 10 sequence, except for the sequence of choices which excludes all 10 friends. Therefore, the number of possibilities is $2^{10}-1$.

Problem 1.17. Following the rules of 'Checkers', in how many ways can one put a white pawn and a black pawn in two black squares of a checkerboard in such a way that the white pawn can jump the black one? Recall that a pawn jumps diagonally, and jumps over the pawn to be taken, and also that pawns can not move backwards.

Solution. Having chosen an orientation for the checkerboard, the white pawn can not occupy any square of the last two rows, since otherwise it could never jump a black pawn. If one places the white pawn in one of the remaining 6 rows, there are two cases in which there are two possible positions for the black pawn, and in the remaining two cases only one position for the black pawn is possible. There are, therefore, $12\times 2+12\times 1=36$ ways in which the two pawns can be positioned.

Chapter 2
Problems and Solutions

Problem 2.1. Prove, directly from the definition of binomial coefficient, the Stifel recursive formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{for } k, n \in \mathbb{N}_{\geq 1}.$$

Solution. We divide the proof into three cases, namely, $n < k$, $n = k$ and $n > k$. If $n < k$ or $n = k$, by the definition of the binomial coefficient the equalities to be proved become respectively the identities $0+0=0$ and $1+0=1$.
Suppose now that $n > k$. If $k = 0$ the identity states that $0+1=1$, and so we suppose that $k \geq 1$. One easily has

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)\cdots(n-k+1)}{(k-1)!} + \frac{(n-1)\cdots(n-k)}{k!} \\
&= \frac{k(n-1)\cdots(n-k+1)}{k!} + \frac{(n-1)\cdots(n-k)}{k!} \\
&= \frac{(k+(n-k))(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{k}.
\end{aligned}$$

Problem 2.2. How many ways are there for ordering the 52 cards of a deck?

Solution. One need only count the 52-sequences of I_{52} without repetitions: there are $S(52,52) = 52!$ of them.

Problem 2.3. How many ways are there to distribute 9 different books among 15 students, if no student receives more than one book?

Solution. The question reduces to counting the 9-sequences in I_{15}: there are $S(15,9) = \dfrac{15!}{6!} = 1816214400$ of them.

Problem 2.4. How many possible anagrams are there for the word INFINITY?

Solution. In the first phase we determine the positions of the three I's; in a second phase we determine the positions of the two N's; then in a third phase we pick the sequence with which we fill the available empty positions with the remaining letters. This corresponds to forming a conditional product with multiplicities $(C(8,3), C(5,2), 3!)$, and so there are $3! \times C(8,3) \times C(5,2) = \dfrac{8!}{2 \times 3!} = 3\,360$ different anagrams.

Problem 2.5. If a coin is tossed 10 times, what is the probability that Heads comes up at least 8 times?

Solution. We take as our sample space the set Ω of the 10 sequences of $\{T, C\}$. The event "Heads comes up at least 8 times" is the disjoint union of the 10 sequences with exactly 8 Heads with those having exactly 9 Heads, and those having exactly 10 Heads. The probability in question is thus

$$\frac{C(10,8)+C(10,9)+C(10,10)}{2^{10}} = \frac{45+10+1}{1\,024} \equiv 0.055 = 5,5\%\,.$$

Problem 2.6. In a weekly lottery, five balls are selected (without replacement) from an urn containing balls numbered from 1 to 90. Calculate the probability that in a given week:

1. The first number chosen is 37;
2. The second number chosen is 37;
3. The first and second numbers chosen are respectively 37 and 51.

Solution.

1. 1/90.
2. 1/90 (sample space: all the 90 possible second numbers).
3. Sample space: 2-sequences without repetition in I_{90}, $P = 1/(90 \times 89)$.

Problem 2.7. How many sequences of 4 numbers are there with only one 8, and without any digit repeated exactly twice? (Sequences starting with 0 are allowed.)

Solution. Sequence of 4 digits with only one 8 may be obtained by first choosing the position for the 8, and then choosing a 3-sequence of numbers different from 8: in all there are $\binom{4}{1} \times 9^3$ possibilities. From these we must remove the sequences in which a digit appears exactly twice. The latter may be obtained by choosing first the position of the 8, then the number to be repeated twice and the two positions it occupies, and finally by choosing another number to appear once: in all there are $\binom{4}{1} \times 9 \times \binom{3}{2} \times 8$. The number of such sequences is $2\,052$.

Problem 2.8. If one writes all the numbers from 1 to 10^5, how many times does one write the digit 5?

Solution. We are dealing with numbers having from one to five digits (100 000 doesn't contain the digit 5); there are the numbers that contain a single digit 5, those containing 5 twice, and so on up to those containing it five times. The first are obtained by deciding where to position the 5, and then choosing the other digits, among which there is also 0. Hence we have a conditional product with multiplicities $(5,9,9,9,9)$. The second type is obtained by deciding first where to put the two 5's, and then choosing the other digits (again including 0), and so we have a conditional product with multiplicities $(\binom{5}{2},9,9,9)$. In the same manner one can settle the other cases, and so the total number of appearances of 5 is

$$1 \times 5 \times 9^4 + 2 \times \binom{5}{2} \times 9^3 + 3 \times \binom{5}{3} \times 9^2 + 4 \times \binom{5}{4} \times 9 + 5;$$

indeed there is one 5 in the numbers of first type, there are two 5's in the numbers of second type, three 5's in the numbers of third type, four 5's in the numbers of fourth type, and finally five 5's in the only number of fifth type.

Problem 2.9. If one throws three distinct dice what is the probability that the highest number is twice the lowest?

Solution. As our sample space Ω we take the cartesian product $I_6 \times I_6 \times I_6$. The ordered triples belonging to the event under consideration are of three types: those composed of a 1, a two and a third number that is either 1 or two; those comprised a two, a four and a number between (or possibly equal to) 2 or 4; and finally, those made up of a 3, a six and a number between 3 and 6 (or equal to one of them). Those of the first type are 6 in number, those of the second type 12, and those of the third type 18. Hence the desired probability is $\frac{6+12+18}{6^3} = \frac{1}{6}$.

Problem 2.10. How many n-sequences of I_3 have exactly 9 digits equal to 1?

Solution. The sequences of the type sought may be obtained using the following procedure in steps: first choose the position of the nine 1's, then fill in the other $n-9$ slots with an $(n-9)$-sequence of $\{2,3\}$. In the first step there are $\binom{n}{9}$ possible outcomes, while in the second there are 2^{n-9}. By the Multiplication Principle 1.34 there are $\binom{n}{9} \times 2^{n-9}$ of the type described.

Problem 2.11. How many possible committees can be formed from a set of 4 men and 6 women if:

1. There are at least two men and twice as many women as men?
2. There are 4 members in all, at least two of whom are women, and Mr. and Mrs. Jones can not simultaneously be members?

Solution. 1.: the committees of the type described can have 2 or 3 men; if they have 2 men, then the can have 4, 5, or 6 women, if they have 3 men then all 6 women must also be members. In all, the number of committees is therefore

$$\binom{4}{2}(\binom{6}{4}+\binom{6}{5}+\binom{6}{6})+\binom{4}{3}\binom{6}{6}=6\times(15+6+1)+4=136.$$

2.: The committees of the type described, except for the condition regarding the Jones family, number $\binom{6}{2}\binom{4}{2}+\binom{6}{3}\binom{4}{1}+\binom{6}{4}=15\times 6+40\times 4+15$. The committees of that type with both Mr. and Mrs. Jones as members number $\binom{5}{2}+\binom{5}{1}\binom{3}{1}=10+15$. Thus, the number of committees satisfying all the conditions imposed is

$$90+80+15-10-15=160.$$

Problem 2.12. There are 6 different books in English, 8 in Russian and 5 in Spanish. In how many ways can one arrange the books in a row on a shelf with all books in the same language grouped together?

Solution. We can place the books following the following two step procedure: first we choose the order in which we will put the languages (3! ways), and then we arrange the books in the first language, then those in the second language, and finally those in the third language. Thus the books can be arranged in $3!\times 6!\times 8!\times 5!$ ways.

Problem 2.13. How many words of 10 different letters can be formed using the 5 vowels and 5 consonants chosen from among the 21 possible consonants of the English alphabet? What is the probability that one of these words does not contain two consecutive consonants?

Solution. Use the following procedure to construct 10-letter words using the 5 vowels and 5 consonants chosen from the 21 possibilities: first choose the positions for the vowels, insert a sequence of vowels, choose the 5 consonants and then place a sequence of these consonants in the remaining slots. In all we create $\binom{10}{5}\times 5!\times\binom{21}{5}\times 5!$ words. Among these words, we count those without consecutive consonants: we set x_1 equal to the number of vowels preceding the first consonant, x_2 equal to the number of vowels between the first and second consonant, ..., and x_6 equal to the number of vowels after the fifth consonant. One has $x_1+x_2+x_3+x_4+x_5+x_6=5$ with $x_1,x_6\geq 0$ and $x_2,x_3,x_4,x_5\geq 1$. This equation has $C(6,1)$ solutions. For each of these solutions, we insert a sequence of vowels and one of consonants into the positions specified. In all, the words without consecutive consonants number $C(6,1)\times 5!\times\binom{21}{5}\times 5!$ The desired probability is

$$\frac{\binom{6}{1} \times 5! \times \binom{21}{5} \times 5!}{\binom{10}{5} \times 5! \times \binom{21}{5} \times 5!} = \frac{1}{42} \approx 0.0238 = 2.38\%\,.$$

Problem 2.14. In how many ways can one distribute 40 identical jelly beans to 4 children in the following cases:

1. Without any restrictions?
2. If each child receives 10 jelly beans?
3. If each child receives at least one jelly bean?

Solution. 1.: Let x_i be the number of jelly beans received by child i. We must count the natural number solutions of $x_1+x_2+x_3+x_4 = 40$: the number is $\binom{43}{3}$.

2.: obviously only 1.

3.: first we give a jelly bean to each child, and the we distribute the others. Thus we must count the solutions of $x_1+x_2+x_3+x_4 = 40-4$, and they number $\binom{39}{3}$.

Problem 2.15. What is the probability that in a sequence without repetitions of $\{a,b,c,d,e,f\}$ one has:

1. a,b in consecutive positions?
2. a appearing before b?

Solution. 1. The requirement that a and b be in consecutive positions corresponds to counting the sequences of $\{(ab),c,d,e,f\}$ and those of $\{(ba),c,d,e,f\}$: in all there are $2\times 5!$. Hence the desired probability is $2\times 5!/6! = 1/3$.

2. If $2 \le i \le 6$, the sequences of the type requested with b in position i number $(i-1)\times 4!$. Therefore the desired probability is $(\sum_{i=2}^{6}(i-1)\times 4!)/6! = 15/30 = 1/2$.

Problem 2.16. A man has n friends and each evening for a year (365 evenings) he invites a different group of 4 of them to his home. How large must n be in order for this to be possible?

Solution. One must have $\binom{n}{4} \ge 365$, and this is only the case for $n \ge 12$.

Problem 2.17. In the first round of a tournament involving $n = 2^m$ players, the n players are divided into $n/2$ pairs each of which then plays a game. The losers are eliminated and the winners participate in the second round, and so on, until there remains a single player, the winner of the tournament.

1. How many outcomes are possible for the first round?
2. How many possible outcomes can the tournament have, if by “outcome of the tournament” we mean complete information on all the rounds?

Solution. 1. In the first round we have $\binom{n}{n/2}$ choices for the winners; we then have $(n/2)!$ ways to couple to each winner with a loser: in all we have

$$\binom{n}{n/2}(n/2)! = \frac{n!}{(n/2)!}$$

possible outcomes for the first round.

2. In the first round we have $\frac{n!}{(n/2)!}$ possible outcomes; in the second round we have $\frac{(n/2)!}{(n/2^2)!}$,..., in the m-th round we have $\frac{(n/2^{m-1})!}{(n/2^m)!} = 2$ possible outcomes. Thus,in all there are

$$\frac{n!}{(n/2)!}\frac{(n/2)!}{(n/2^2)!}\cdots\frac{(n/2^{m-1})!}{(n/2^m)!} = n!$$

possible outcomes for the entire tournament. In the light of the preceding result, it is not difficult to use a direct argument to make the calculations. If we label the participants in the tournament with I_n, we can construct a one to one correspondence between the set of permutations of $(1,...,n)$ and the possible outcomes of the tournament in the following manner: the permutation $(\sigma_1,...,\sigma_n)$ of $(1,...,n)$ represents the outcome in which σ_1 wins the final game playing against σ_2, σ_3 lost in the semifinal against σ_1 and σ_4 lost in the semifinal against σ_2; ...; σ_{2^k+j}, $1 \le j < 2^k$, lost to σ_j in the 2^{m-k}-th round σ_j, and so on.

Problem 2.18. Suppose that a subset of 60 different days of the year is chosen by extraction. What is the probability that there are 5 days for each month in the subset? (For simplicity, assume that there are 12 months of 30 days each.)

Solution. The probability is $\binom{30}{5}^{12} / \binom{360}{60}$.

Problem 2.19. In how many bridge hands do the players North and South have all the spades?

Solution. Choose in order the cards of East, West, North, and South: $\binom{39}{13} \times \binom{26}{13} \times \binom{26}{13}$.

Problem 2.20. What is the probability of choosing at random a 10-sequence of I_{10} without repetitions such that:

1. In the first position there is an odd digit and one of $1,2,3,4,5$ occupies the final position?
2. 5 is not in the first position and 9 is not in the last?

Solution. 1. We make a separate count of the sequences that finish with 1, 3 or 5 and then count the others: in any case, in both situations we first choose the final digit, then the odd digit in the first slot, and then the others:

$$\frac{\binom{3}{1}\times\binom{4}{1}\times 8!+\binom{2}{1}\times\binom{5}{1}\times 8!}{10!}.$$

2. The number of excluded sequences is: $9!+9!-8!$, and so the desired probability is

$$1-8!\frac{17}{10!}=1-\frac{17}{90}=0,81.$$

Problem 2.21. What is the probability that in a hand of 5 cards taken from a deck of 52 there is:

1. At least one of each of the following cards: Ace, King, Queen, Jack?
2. At least one of the following cards: Ace, King, Queen, Jack?
3. The same number of hearts and spades?

Solution. 1. We first count the hands with two Aces, a King, a Queen and a Jack. Such a hand can be built in 4 phases: for the first step one chooses two Aces, in the second phase a King, in the third a Queen and in the fourth a Jack. In all one has $\binom{4}{2}\times 4^3$ hands of this type. There are the same number of hands with one Ace, two Kings, a Queen and a Jack, or of hands with an Ace, a King, two Queens, and a Jack, or hands with an Ace, a King, a Queen, and two Jacks. There are then the hands with exactly one Ace, one King, one Queen, and one Jack, which number $4^4\times 36$. The desired probability is therefore

$$\frac{4\times\binom{4}{2}\times 4^3+4^4\times 36}{\binom{52}{5}}=\frac{10\,752}{2\,598\,960}\approx 0,004=0,4\%\,.$$

2. A 5-card hand without an Ace, King, Queen, or Jack can be chosen in $\binom{36}{5}$ ways. The desired probability is

$$1-\frac{\binom{36}{5}}{\binom{52}{5}}=1-\frac{36!47!1}{31!52!}=1-\frac{36\times 35\times\cdots\times 32}{52\times 51\times\cdots\times 48}=0,855.$$

3. There can be 1 heart and 1 spade, or 2 hearts and 2 spades. The desired probability is

$$\frac{13 \times 13 \times \binom{26}{3} + \binom{13}{2} \times \binom{13}{2} \times 26}{\binom{52}{5}}.$$

Problem 2.22. In how many ways can one form a group (not ordered) of four couples chosen from a set of 30 people?

Solution. There are $\binom{30}{8} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2}$ sequences of 4 couples chosen out of a set of 30. Then by the Division Principle 1.47

$$\frac{1}{4!} \binom{30}{8} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2}$$

is the number sought.

Problem 2.23. Let k be a prescribed natural number satisfying $1 \leq k \leq 17$; then fix 4 numbers chosen between 1 and 20.

1. What is the probability that k appears among the four numbers chosen, and is the smallest of the four?
2. What is the probability that k appears among the four numbers chosen and is the second largest of them?

Solution. 1. The desired probability is $\binom{20-k}{3} / \binom{20}{4}$.

2. The desired probability is $\binom{20-k}{2} \times \binom{k-1}{1} / \binom{20}{4}$.

Problem 2.24. What is the probability that in five tosses of a die only two different numbers come up?

Solution. We choose as our sample space the set of 5-sequences in I_6. The favorable cases may be obtained via the following procedure: in the first phase one chooses two numbers, while in the second phase one constructs a 5-sequence of the set formed by the two numbers chosen, excluding the 5 sequences consisting of a single number. Thus the desired probability is equal to

$$\frac{\binom{6}{2} \times (2^5 - 2)}{6^5} \approx 0.0579 = 5.79\% .$$

Problem 2.25. From a set of $2n$ objects, of which n are identical and the other n all different from each other, how many possible selections of n objects are there?

Solution. There are $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ such selections of n objects: indeed, any given selection is determined by the elements of the set of n different objects which belong to it.

Problem 2.26. In a lake 10 fish are tagged from a population of k. Twenty fish are caught. What is the probability that two of them are tagged?

Solution. The desired probability is $\binom{10}{2} \times \binom{k-10}{18} / \binom{k}{20}$.

Problem 2.27. We wish to organize three dinners on three consecutive evenings to each of which will be invited three friends chosen from among the n schoolmates with whom we are still in contact. In how many ways can the guests for the three evenings be chosen?

Solution. For each evening we must choose 3 friends out of n: therefore there are $\binom{n}{3}^3$ possible choices.

Problem 2.28. We have organized ten dinners on 10 consecutive evenings. To these dinners we wish to invite, among others, the 8 schoolmates with whom we are still in contact, but we are uncertain whether to invite them all for the first evening, or to not invite more than one friend each evening, or to make invitations in some other way (but with no friend invited more than once). How many possible choices are there?

Solution. To start, we choose 8 friends from the group of 15, and for this we have $\binom{15}{8}$ possibilities. We then decide for which dinner they are to be invited, assigning to each of them the index i is they are invited to the i-th dinner: this amounts to fixing an 8-sequence in I_{10}. Hence, in all there are

$$\binom{15}{8} \times 10^8 = 643\,500\,000\,000 \text{ possibilities.}$$

Problem 2.29. What is the probability that in a hand of 5 cards from a deck of 52:

1. There is exactly one pair (not two pairs or three of a kind)?
2. There are at least two cards with the same value?
3. There is at least one spade, one heart, no club or diamond, and the face values of the spade cards are all strictly higher than the face values of the heart cards?

Solution. 1. The probability is

$$13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3 / \binom{52}{5} \approx 0.423 = 42.3\%\,.$$

First choose the face value that appears in the pair, then the suit of the pair, then the face values of the 3 different remaining cards, and finally, starting form the lowest face value, the suits.

2. We count the number of ways one can have a hand without even a pair; to do so, choose 5 distinct numbers between 1 and 13, and assign to each of them a suit: this may be done in $\binom{13}{5} \times 4^5$ ways. The desired probability is then

$$1-\frac{\binom{13}{5}\times 4^5}{\binom{52}{5}}\approx 0.493=49.3\%\,.$$

3. Let $1\leq i\leq 4$ be the number of spade cards. Then a hand of the desired type is found by choosing a 5-collection in I_{13} without repetitions and then taking the i highest values as spades. Thus the desired probability is

$$\frac{\sum_{i=1}^{4}\binom{13}{5}}{\binom{52}{5}}=\frac{4\times\binom{13}{5}}{\binom{52}{5}}\approx 0.002=.2\%\,.$$

Problem 2.30. How many subsets consisting of three distinct natural numbers between 1 and 90 (extremes included) are there if the sum of the three is:

1. An even number?
2. Divisible by 3?
3. Divisible by 4?

Solution. 1. The sum of three integers is even if and only if they are all even or two are odd and one is even. The number of subsets sought is $\binom{45}{3}+\binom{45}{1}\times\binom{45}{2}$.

2. Among the numbers between 1 and 90, there 30 divisible by 3, 30 which leave a remainder of 1 when divided by 3, and another 30 with remainder 2 after division by 3. The sum of three integers is divisible by 3 if and only if (i): all of them are divisible by 3, (ii): all of them have remainder 1 after division by 3, (iii): one is divisible by 3, one has remainder 1 and one has remainder 2, (iv): all of them leave 2 as remainder after by 3. Thus, the number of subsets of the type described is $\binom{30}{3}+\binom{30}{3}+30^3+\binom{30}{3}$.

3. Among the numbers from 1 to 90 there are 22 that are divisible by 4, 23 that give a remainder of 1 after division by 4, 23 that give a remainder of 2, and 22 with remainder 3. The sum of three numbers is divisible by 4 if and only if (i): they are all divisible by 4, (ii): one is divisible by 4 and the other two have remainder 2, (iii): one is divisible by 4, one has remainder 1 and the third has remainder 3, (iv): two have remainder 1 and one has remainder 2, (v): one has remainder 2 and two have remainder 3. Hence the number of subsets to the desired type is

$$\binom{22}{3}+22\times\binom{23}{2}+22\times 23\times 22+\binom{23}{2}\times 23+\binom{23}{1}\times\binom{22}{2}=29\,370.$$

Problem 2.31. In how many ways can one choose 10 coins from a pile of euro-coins consisting of 1 cent, 2 cent, 5 cent, and 10 cent coins?

Solution. Let x_i denote how many of the i cent coins we take; we must then count the natural number solutions to the equation $x_1+x_2+x_5+x_{10}=10$ There are $\binom{13}{3}$ such solutions.

Problem 2.32. We must establish how many places on a commission of 15 congressmen will be awarded to Democrats, Republicans, and Independents. How many possibilities are there if each party must have at least two members of the commission? What if, moreover, no party should by itself comprise the majority of the commission?

Solution. In the first instance we must count the solutions of the $x_D+x_R+x_I=15$ with $x_D, x_R, x_I \geq 2$. These are equal in number to the solutions of $y_1+y_2+y_3=15-6=9$, namely, $\binom{11}{2}$. If no party should have the majority of the committee, one has the further condition $x_D, x_R, x_I < 8$. One must therefore discard the solutions in which one of the x_D, x_R, x_I is ≥ 8. Since it is not possible that two of these be simultaneously ≥ 8 one obtains

$$\binom{11}{2} - 3 \times \binom{5}{2}.$$

Problem 2.33. In how many ways can one distribute 18 chocolate donuts, 12 cinnamon donuts and 14 honey-dip donuts to 4 pupils if each of these requires at least two donuts of each type?

Solution. Each distribution of donuts corresponds to a solution of an equation. The chocolate donuts can be distributed in as many ways as there are solutions of $x_1+x_2+x_3+x_4=18$ con $x_i \geq 2$, namely, $\binom{13}{3}$. Similarly for the other two types of donuts. By the Multiplication Principle 1.34 the donuts can be distributed in $\binom{13}{3} \times \binom{7}{3} \times \binom{9}{3}$ ways.

Problem 2.34. How many integer solutions of $x_1+x_2+x_3=0$ are there with each $x_i \geq -5$?

Solution. Let $y_i = x_i+5$. The integer solutions of the desired type are in one to one correspondence with the natural number solutions of $y_1+y_2+y_3=15$, and so there are $\binom{17}{2}$ such solutions.

Problem 2.35. How many electoral results are possible (number of votes for each candidate) if there are 3 candidates and 30 voters? What if, moreover, some candidate obtains an absolute majority?

Solution. Let x_i denote the number of votes obtained by i. The question reduces to counting the solutions of $x_1+x_2+x_3=30$, and there are $\binom{32}{2}$ of them. If candidate i obtains the absolute majority, then $x_i \geq 16$; in this case the number of solutions is $\binom{16}{2}$. This number must be multiplied by 3 in order to consider three possible winning candidates.

Problem 2.36. How many numbers between 0 and 10000 are such that the sum of their digits is:

1. Equal to 7?
2. Less than or equal to 7?

Solution. Let $x_1x_2x_3x_4$ denote any arbitrary number between 0 and 9 999.

1.: One must have $x_1+x_2+x_3+x_4=7$; and so the number of such numbers is $\binom{10}{3}$.

2.: Here one must have $x_1+x_2+x_3+x_4 \le 7$; the solutions of this inequality are equal in number to the solutions of $x_1+x_2+x_3+x_4+x=7$, namely, $\binom{11}{4}$. To these one must add the number 10000 which also satisfies the required condition, so the total number of such solutions is: $\binom{11}{4}+1$.

Problem 2.37. How many natural number solutions are there for the equation

$$2x_1+2x_2+x_3+x_4=12 \quad ?$$

Solution. The number of such solutions is equal to the number of solutions to the system of equations $\begin{cases} x_1+x_2=i \\ x_3+x_4=12-2i \end{cases}$ with $i \le 6$: that is,

$$\sum_{i=0}^{6}\binom{i+1}{1}\times\binom{12-2i+1}{1}=\sum_{i=0}^{6}(i+1)\times(12-2i+1)=$$

$$=13+2\times 11+3\times 9+\cdots+7\times 1=140.$$

Problem 2.38. How many natural number solutions are there to the system of inequalities

$$\begin{cases} x_1+\cdots+x_6 \le 20 \\ x_1+x_2+x_3 \le 7 \end{cases} \quad ?$$

Solution. There are as many solutions of the system of inequalities as there are solutions of the system of equations $\begin{cases} x_1+x_2+x_3=i \\ x_4+x_5+x_6 \le 20-i \end{cases}$ con $i \le 7$: namely,

$$\sum_{i=0}^{7}\binom{i+2}{2}\times\binom{20-i+3}{3}.$$

Problem 2.39. How many binary sequences are there in which 0 appears n times and 1 appears m times, and having k groups of consecutive 0's?

Solution. Let x_i be the number of 0's that appear in the i-th group of consecutive zeros. We can group the 0's in k groups in as many ways as there are solutions of $x_1+\cdots+x_k=n$, that is, $\binom{n+k-1}{k-1}$. It is now a question of inserting the 1's between one group of zeroes and another: this may be done in as many ways as there are solutions of $y_1+\cdots+y_{k+1}=m$, with $y_2,\ldots,y_k\geq 1$ where y_1 counts how many 1's there are before the first block of 0's, y_2 counts how many 1 there are between the first and the second block of 0's, ..., y_{k+1} counts how many 1's there are after the k-th block of 0's. The 1's can be inserted in $\binom{m-(k-1)+k}{k}$ ways, and so the number of sequences of the desired type is $\binom{n+k-1}{k-1}\times\binom{m-(k-1)+k}{k}$.

Problem 2.40. How many binary sequences of n terms contain the *pattern* 01 exactly m times?

Solution. Between two appearances of the pattern or before the first appearance, or after the last, there can appear only a block of 1's followed by a block of zeroes. Let x_1 and y_1 respectively denote the number of 1's in the block of 1's and the number of 0's in the block of 0's which precedes the first appearance of our pattern; let x_2 and y_2 respectively indicate the number of 1's in the block of 1's and the number of 0's in the block of 0's that lies between the first and second appearances of the pattern; ..., finally, let x_{m+1} and y_{m+1} respectively denote the number of 1's in the block of 1's and the number of 0's in the block of 0's that ends the sequence. One must then have

$$x_1+y_1+\cdots+x_{m+1}+y_{m+1}=n-2m.$$

This equation has $\binom{n-2m+2m+1}{2m+1}=\binom{n+1}{2m+1}$ solutions.

Problem 2.41. Let $m\leq n$ and $s\leq r$ be natural numbers. How many ways are there to distribute r identical balls in n distinct boxes in such a way that the first m boxes contain a total of at least s balls?

Solution. If x_i denotes the number of balls contained in box i, then one must have:

$$x_1+\cdots+x_n=r \text{ and } x_1+\cdots+x_m\geq s.$$

The first equation has $\binom{r+n-1}{r}$ solutions; from these we must eliminate those for which $x_1+\cdots+x_m=i$ con $i<s$ and $x_{m+1}+\cdots+x_n=r-i$; hence the resulting number of solutions is

$$\binom{r+n-1}{r}-\sum_{i=0}^{s-1}\binom{i+m-1}{i}\binom{r-i+n-m-1}{r-i}.$$

Problem 2.42. 1. In how many ways can one seat 8 people in a row of 15 seats of a cinema?

2. In how many of the preceding seating arrangements, do 3 given friends receive adjacent seats?

Solution.

1. $S(15,8) = 15!/7!$.
2. We split the seating procedure into two steps: a) choice of the sequence W of the 3 friends: $3!$ ways; b) placement of W and of the other 5 people in a row with 13 places available (the three count as 1): $S(13,6) = 13!/7!$ ways: thus in all we have $3! \times 13!/7!$ such seating arrangements.

Problem 2.43. If a coin is tossed n times, what is the probability that:

1. The first "Heads" appears after exactly m "Tails";
2. The i-th "Heads" appears after "Tails" has come up m times?

Solution. Consider the sample space Ω of n-sequences in $I_1 = \{H, T\}$; clearly $|\Omega| = 2^n$.

1. The sequences that start with m Tails followed by a Head number 2^{n-m-1}; thus the probability sought is $1/2^{m+1}$.

2. The sequences that belong to the event under consideration are formed by a sequence in which there appear $i-1$ "Heads" and m "Tails", followed by a "Head", and then randomly by "Heads" or "Tails". The number of such sequences is $\binom{m+i-1}{i-1} \times 2^{n-m-i}$, and so the desired probability is $\binom{m+i-1}{i-1} / 2^{m+i}$.

Problem 2.44. In how many ways can one distribute 3 different teddy bears and 9 identical lollipops to four children:

1. Without restrictions?
2. Without having any child receive two or more teddy bears?
3. With each child receiving 3 "gifts"?

Solution. We distribute the gifts to the four children by first distributing the three teddy bears, and then the 9 lollipops.

1.: Here we have a conditional product with multiplicities $(S((4,3)), C((4,9)))$; hence the distribution may be carried out in $4^3 \times \binom{12}{3}$ ways.

2.: In this case we have a conditional product of multiplicities $(S(4,3), C((4,9)))$, and so the distribution may be done in $4! \times \binom{12}{3}$.

3.: Once the teddy bears have been assigned, the distribution of the lollipops is determined. Therefore, the distribution may be carried out in 4^3 ways.

Problem 2.45. Find the number of binary 20-sequences with exactly 15 terms equal to 0 and 5 terms equal to 1. How many sequences with 15 terms of one type and 5 of the other?

Solution. There are $\binom{20}{5} = 15\,504$ sequences of the first type, and twice as many of the second type: 30 108.

Problem 2.46. If n different objects are distributed randomly into n different boxes what is the probability that:

1. No box is empty?
2. Exactly one box is empty?
3. Exactly two boxes are empty?

Solution. By keeping track in order for If we number the objects and boxes, and keep track of the number of the box into which we insert each numbered object, we obtain a sample space Ω consisting of the n-sequences of I_n with possible repetitions, and one has $|\Omega| = n^n$.

1. In this case the sequence of numbers of the boxes has no repetitions, since otherwise there could not be an object in every box. Therefore, the desired probability is $n!/n^n$.

2. First we choose the empty box (n possibilities); then, exactly one box must contain two objects ($n-1$ possibilities for choosing the box and $\binom{n}{2}$ for choosing the objects to put in it); finally we insert an object into each remaining box ($(n-2)!$ possibilities). Hence, the desired probability is

$$\frac{n \times (n-1) \times \binom{n}{2} \times (n-2)!}{n^n} = \frac{n! \binom{n}{2}}{n^n}.$$

3. First we choose the two empty boxes - $\binom{n}{2}$ possibilities; we then count separately the cases in which there is a box with 3 objects, and that in which there are two boxes each receiving two objects:

$$(n-2) \times \binom{n}{3} \times (n-3)! + \binom{n-2}{2} \binom{n}{2} \binom{n-2}{2} \times (n-4)!\,.$$

Therefore the desired probability is

$$\frac{\binom{n}{2} \times \left((n-2) \times \binom{n}{3} \times (n-3)! + \binom{n-2}{2} \binom{n}{2} \binom{n-2}{2} \times (n-4)! \right)}{n^n}.$$

Problem 2.47. In how many ways can one distribute 4 red balls, 5 blue ones, and 7 black balls into:

1. Two boxes?
2. Two boxes neither of which is empty?
3. In how many ways can one place 4 red balls, 6 blue, and 8 black ones into two boxes?

Discuss the case in which the boxes are distinct separately from that in which they are indistinguishable.

Solution. We begin with the case in which the boxes are distinguishable.

1.: The red balls can be distributed between the two boxes in 5 ways: indeed, the first box can receive 0, 2, 3, 4, or 4 balls. Similarly, the one has 6 and 8 ways for distributing the blue and black balls. In all one has $5 \times 6 \times 8 = 240$ ways to place the balls into the two boxes.

2.: From the ways considered in the first part, one must eliminate the two distributions that either assign all the balls to the first box or all to the second box.

Thus altogether we have $240 - 2 = 238$ ways for dividing the balls between the two boxes, without either box being empty.

3.: Reasoning as in (1) one obtains $5 \times 7 \times 9 = 315$ ways to divide the balls between the two boxes.

Let us now analyze the case in which the boxes are indistinguishable.

As far as Points (1) and (2) are concerned, it suffices to divide the previously obtained results by two. Indeed, for each distribution in two indistinguishable boxes there are associated two different distributions is distinct boxes. One notes that the two boxes can never have the same contents. As to Point (3), here we must consider the case in which the two boxes both contain 2 red balls, 3 blue, and 4 black: to such a distribution there is associated only one case even when the two boxes are distinguishable. In this case the balls can be distributed between the two indistinguishable boxes in $(315-1)/2+1 = 158$ ways.

Problem 2.48. In a 4 story house (in addition to the ground floor) an elevator leaves the ground floor with 5 people aboard. No one else gets on, and every person gets off randomly at one of the four (upper) stories. Calculate the probability that the elevator:

1. Arrives empty at the fourth (top) floor;
2. Arrives empty at the third floor;
3. Becomes empty at exactly the third floor;
4. Arrives at the fourth floor carrying 2 people.

Solution. Take as sample space Ω the 5-sequences of I_4, so that $|\Omega| = 4^5$. The various cases of the event A in which we are interested are:

1. $A =$ 5-sequences in I_3, $|A| = 3^5$;
2. $A =$ 5-sequences in I_2, $|A| = 2^5$;
3. $A =$ 5-sequences in I_3 with at least one 3, or equivalently the 5-sequences in I_3 that are not sequences in I_2: $|A| = 3^5 - 2^5$;
4. $A =$ 5-sequences in I_4 with exactly two 4's, $|A| = \binom{5}{2} 3^3$.

In each case $P(A) = |A|/|\Omega|$.

Problem 2.49. We wish to open a locked door. We have a ring of 100 keys, of which exactly 2 open the door in question. The keys are tried successively one by one

1. What is the probability that the 56-th key opens the door?
2. What is the probability that the 56-th key is the second key that opens the door?

Solution. Take as sample space Ω the set of 100-sequences of $\{0,1\}$ with two 1's and ninety-eight 0's. Then $|\Omega| = \binom{100}{2}$. The event A in which we are interested in the various cases are:

1. $A =$ 100-sequences in Ω with 1 at position 56, $|A| = 99$, $P(A) = 1/50$;
2. $A =$ 100-sequences in Ω with 1 in a position between 1 and 55, and a 1 in position 56: $|A| = 55$, $P(A) = 110/(100 \times 99)$.

Problem 2.50. Five marbles are extracted *simultaneously* from an urn containing 10 red marbles and 20 blue ones. Find the probability that only one blue marble is extracted.

Solution. Possible cases: $\binom{30}{5}$; Favorable cases: $\binom{20}{1} \times \binom{10}{4}$. Thus the desired probability is

$$\binom{20}{1} \times \binom{10}{4} \Big/ \binom{30}{5} = \frac{20 \times 10 \times 9 \times 8 \times 7}{30 \times 29 \times 28 \times 27 \times 26} \frac{5!}{4!}.$$

Problem 2.51. How many committees of 5 people with at least two women and at least one man can be formed by choosing from a group of 6 women and 8 men?

Solution. We make separate counts of the committees with 2 women and 3 men, those with 3 women and 2 men, and those of 4 women and 1 man. In all we have: $\binom{6}{2} \times \binom{8}{3} + \binom{6}{3} \times \binom{8}{2} + \binom{6}{4} \times \binom{8}{1}$ committees.

Problem 2.52. Give a combinatorial proof of the equality (see Point 1 of Proposition 2.39)

$$C((n,k)) = C((k+1, n-1)) \quad \forall k \in \mathbb{N},\ n \in \mathbb{N}_{\geq 1}.$$

Solution. Every n-composition $(k_1, \ldots, k_n)$ of k is uniquely determined by the corresponding representation as "roman numerals", that is, by the $(n+k-1)$-sequence of $\{I, +\}$ formed in order by k_1 "I", a "$+$", k_2 "I", a "$+$", ..., a "$+$", k_n "I". This representation contains $n-1$ "$+$"; it is uniquely determined by the $(k+1)$-composition $(n_1, \ldots, n_{k+1})$ of $n-1$ where n_1 is the number of "$+$" before the first "I", n_2 is the number of "$+$" between the first and the second "I's",..., n_k is the number of "$+$" between the $(k-1)$-th and the k-th "I's", and n_{k+1} is the number of "$+$" after the k-th "I. Therefore the number of n-compositions of k equals the number of $(k+1)$-compositions of $n-1$.

Problem 2.53. By using the Binomial Theorem, one sees that $\binom{2n}{n}$ is the coefficient of $x^n y^n$ in the sum giving $(x+y)^{2n}$. Write $(x+y)^{2n}$ in the form $(x+y)^n(x+y)^n$, expand both factors $(x+y)^n$ using the Binomial Theorem, and look for the coefficient of $x^n y^n$ that arises from expanding this product. Show that this procedure leads to an alternative proof for Identity 2 of Proposition 2.56.

Solution. One has

$$(x+y)^n(x+y)^n = (x+y)^n(y+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\sum_{k=0}^{n}\binom{n}{k}y^k x^{n-k}.$$

The coefficient of $x^n y^n$ is $\sum_{k=0}^{n}\binom{n}{k}^2$. Therefore one has

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2.$$

Problem 2.54 (Magnetic colors). Explain the following magic trick, published by N. Gilbreath in [18]:
Take a complete deck of cards out of the case. Ask a spectator to give a few straight cuts, deal off any number of cards into a pile on the table, and then riffle shuffle the pile on the table with the pile still in his hand. You say something like "as you know, red and black cards are magnetically attracted one to each other". Then, ask the spectator to pick the deck up into dealing position, and deal off the top two cards. They will definitely be one red and one black. Deal off the next two cards. Again, one red/one black. Keep going. You'll find that each consecutive pair alternates in color!

Solution. Set up a deck of cards alternating red/black and put them in the card case. Cutting the deck keeps the red/black still alternate. Your instructions are forcing the spectator to perform a Gilbreath shuffle. It follows from Example 2.53 that the resulting sequence of cards is a Gilbreath permutation of the initial one. Therefore the first two cards were initially consecutive, thus one red and one black; the first four cards were initially consecutive so that the third and the fourth are one red and one black, etc.

Problem 2.55. Let k, m, n be natural numbers. Prove the following identity:

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\cdots$$
$$\cdots+\binom{n}{k-1}\binom{m}{1}+\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}.$$

[Hint: one can proceed in a manner similar to the proof of Identity 2 of Proposition 2.56, or use the Binomial Theorem as in as in Problem 2.53.]

Solution. The number $\binom{n+m}{k}$ counts the number of subsets of cardinality k of the set $\{x_1,x_2,\dots,x_n,y_1,\dots,y_m\}$ with $m+n$ elements. Each k-element subset has i

elements of type x and $k-i$ of type y, for some i between 0 and k. Therefore,

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}.$$

Problem 2.56. Prove by induction on k the Point 3 of Proposition 2.56.

Solution. For $k=0$ the identity holds since

$$\binom{n}{0} = 1 = \binom{n+0+1}{0}.$$

We now suppose that the identity holds for a given value of $k \geq 0$, and prove it for $k+1$. In other words we wish to prove that

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+k+2}{k+1}. \tag{2.56.a}$$

The sum of the first k terms of the left hand side coincides with $\binom{n+k+1}{k}$ by the induction hypothesis, and so we may replace the left hand side of the equation with

$$\binom{n+k+1}{k} + \binom{n+k+1}{k+1}.$$

But then, by Proposition 2.22 it coincides with $\binom{n+k+2}{k+1}$.

Solution.

Problem 2.57. Prove Proposition 2.59 by means of Corollary 2.21.
$\left[\text{Hint: for } i<n \text{ multiply } \binom{n}{k}\binom{k}{i} \text{ by } \frac{(n-i)!}{(n-i)!}.\right]$

Solution. If $i=n$ one has immediately

$$\sum_{k=i}^{n} (-1)^k \binom{n}{k} \binom{k}{i} = (-1)^n \binom{n}{n}^2 = (-1)^n.$$

Assume now $i<n$. Multiplying $\binom{n}{k}\binom{k}{i}$ by $\frac{(n-i)!}{(n-i)!} = \frac{((n-k)+(k-i))!}{((n-k)+(k-i))!}$, one obtains

$$\begin{aligned}
\binom{n}{k}\binom{k}{i} &= \frac{n!}{i!} \frac{1}{(n-k)!(k-i)!} \\
&= \frac{n!}{(n-i)!i!} \binom{n-i}{n-k} = \binom{n}{i}\binom{n-i}{n-k} \\
&= \binom{n}{i}\binom{n-i}{k-i}.
\end{aligned}$$

Setting $j = k - i$, since $n - i \geq 1$, by Corollary 2.21 one gets

$$\sum_{k=i}^{n}(-1)^k \binom{n-i}{k-i} = (-1)^i \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} = 0$$

and hence

$$\sum_{k=i}^{n}(-1)^k \binom{n}{k}\binom{k}{i} = \binom{n}{i}\sum_{k=i}^{n}(-1)^k \binom{n-i}{k-i} = 0.$$

Chapter 3
Problems and Solutions

Problem 3.1. A die is tossed six times with the outcome of each toss being recorded. Determine the probability that the six outcomes consist of one 1, three 5's and two 6's.

Solution. The sample space is the set Ω of 6-sequences of I_6 and its cardinality is $S((6,6)) = 6^6$. The favorable cases consist of the 6-sequences of I_6 with occupancy $(1,0,0,0,3,2)$. Therefore, the probability sought is

$$\frac{S(6,6;(1,0,0,0,3,2))}{S((6,6))} = \frac{6!/3!2!}{6^6} \approx 0.001 = 1\% .$$

Problem 3.2.
1. How many 6 digit numbers can be formed with the numbers 3, 5, and 7?
2. How many of the numbers considered above contain two 3's, two 5's, and two 7's?

Solution. 1: $S((3,6)) = 3^6 = 729$.
2: $S(3,6;(2,2,2)) = 6!/(2!)^3 = 90$.

Problem 3.3. Assuming that passwords are generated randomly, what is the probability that a password of 8 digits from 0 to 9 contains two 5's and two 8's, three 2's and a 4?

Solution. The possible passwords constitute the set of all 8-sequences of I_{10} and so there are $S((10,8)) = 10^8$ of them. To calculate the cardinality of the event in which we are interested, one must count the 8-sequences of I_{10} with occupancy $(0,0,3,0,1,2,0,0,2,0)$: there are $S(10,8;(0,0,3,0,1,2,0,0,2,0)) = 1\,680$ such sequences. Therefore, the probability sought is

$$\frac{S(10,8;(0,0,3,0,1,2,0,0,2,0))}{S((10,8))} = \frac{1\,680}{10^8} \approx 0.0000168 = 0.0017\% .$$

Problem 3.4. What is the probability that in randomly distributing 6 indistinguishable objects amongst 9 distinct boxes one obtains a collection with occupancy $[2,2,1,1,0,0,0,0,0]$?

Solution. We number the six indistinguishable object with I_6. Each distribution of the 6 objects then corresponds to a 6-sequence of I_9. The set of 6-sequences of I_9 has cardinality 9^6, and the all distributions are equally probable. The distributions of the type specified are those with occupancy $[2,2,1,1,0,0,0,0,0]$, and there are

$$S(9,6;[2,2,1,1,0,0,0,0,0]) = S(9,6;(2,2,1,1,0,0,0,0,0)) \times P(2,2,1,1,0,0,0,0,0) =$$

$$= \frac{6!}{2\times 2} \times S(3,9;(5,2,2)) = 180 \times \frac{9!}{5!2!2!} = 180 \times 756 = 136\,080$$

of them. The desired probability is therefore

$$\frac{136\,080}{9^6} \approx 0.26 = 26\%\,.$$

Problem 3.5. A waiter has taken orders for four types of drink: Martinis, Manhattans, White Wines, and Ginger Ales. The waiter remembers only that he must serve 3 drinks for two of these types, and 2 drinks for the other two types. What is the probability the he correctly guesses the drinks to be served?

Solution. The possible outcomes consist of the 10-sequences of I_4 with occupancy $[2,2,3,3]$, while the favorable cases are the 10-sequences of I_4 with occupancy $(2,2,3,3)$. The desired probability is therefore

$$S(4,10;(2,2,3,3))/S(4,10;[2,2,3,3]) = 1/P(2,2,3,3) \approx 0.167 = 16.7\%\,.$$

Problem 3.6. In how many ways can one distribute 20 distinct objects amongst 3 distinct boxes with 6 objects in one box and 7 in the two other boxes? What if the objects are indistinguishable?

Solution. For the first question, one must count the 20-sequence of I_3 with occupancy $[6,7,7]$ and these number

$$S(3,20;[6,7,7]) = S(3,20;(6,7,7)) \times P(6,7,7) = 399\,072\,960.$$

If the objects are indistinguishable, one counts the 3-compositions of 20 with occupancy $[6,7,7]$. There are

$$C(3,20;[6,7,7]) = P(6,7,7) = 3$$

of them.

Problem 3.7. A program downloads and randomly distributes 15 different videos in mp4 format in the folders of 5 different users. What is the probability that one folder remains empty, 3 folders have 4 videos, and one folder contains 3?

Solution. The possible cases number 5^{15} while the favorable cases number $S(5,15;[0,4,4,4,3])$. The desired probability is $\approx 0.01 = 1\%$.

Problem 3.8. The components of a group of 30 people travel in a railway car in such a way as to have three of them in compartment 1, six travellers in compartments 2, 4, and 6, five travellers in compartment 3, and four in compartment 5. Represent the outcome of such an arrangement in terms of sequences or collections, and then determine the number of such possible seating arrangements.

Solution. Essentially one is dealing here with the 30-sequences of I_6 with occupancy $(3,6,5,6,4,6)$: there are

$$S(6,30;(3,6,5,6,4,6)) = (30!)/[3!5!4!(6!)^3] = 41\,126\,225\,027\,679\,792\,000$$

such 30-sequences.

Problem 3.9. How many permutations of the digits in 1 224 666 produce numbers less than 3 000 000?

Solution. The number of permutations of the sequence $(1,2,2,4,6,6,6)$ is

$$P(1,2,2,4,6,6,6) = S(4,7;(1,2,1,3)) = \frac{7!}{2!3!}.$$

One obtains a number greater than 3 000 000 only when the sequence begins with 4 or with 6. The number of sequences beginning with 4 is

$$P(1,2,2,6,6,6) = S(3,6;(1,2,3)) = \frac{6!}{2!3!},$$

while the number of those beginning with 6 is

$$P(1,2,2,4,6,6) = S(4,6;(1,2,1,2)) = \frac{6!}{2!2!}.$$

Thus the total number of permutations of the specified type is

$$S(4,7;(1,2,1,3)) - S(3,6;(1,2,3)) - S(4,6;(1,2,1,2)) = \frac{7!}{2!3!} - \frac{6!}{2!3!} - \frac{6!}{2!2!} = 180.$$

Problem 3.10. How many 8 digit numbers are there in which all six of the digits 1, 2, 3, 4, 5, 6 and no others appear? How many 8 digit numbers are there with six different (unspecified) digits?

Solution. The problem is to count the number of 8-sequences of I_6 with occupancy $[1,1,1,1,1,3]$ or $[1,1,1,1,2,2]$:

$$S(6,8;[1,1,1,1,1,3]) + S(6,8;[1,1,1,1,2,2]) =$$

$$= S(6,8;(1,1,1,1,1,3)) \times P(1,1,1,1,1,3) +$$

$$+S(6,8;(1,1,1,1,2,2)) \times P(1,1,1,1,2,2) =$$
$$= \frac{8!}{3!} \times 6 + \frac{8!}{2!2!} \times \binom{6}{2} = 191\,520.$$

As to the second question, divide the problem into steps: first choose the 6 digits to be used from among the 10 available, and then repeat the reasoning used above. One finds then that in all there are

$$\binom{10}{6} \times \left(\frac{8!}{3!} \times 6 + \frac{8!}{2!2!} \times \binom{6}{2} \right) = 210 \times 191\,520 = 40\,219\,200$$

such numbers.

Problem 3.11. Show that $\displaystyle\sum_{\substack{k_1+k_2+k_3=10 \\ k_i \in \mathbb{N}}} S(3,10;(k_1,k_2,k_3)) = 3^{10}$.

Solution. The number $S(3,10;(k_1,k_2,k_3))$ counts the 10-sequences of I_3 with occupancy (k_1,k_2,k_3). As the k_i vary over all possible natural number triples satisfying $k_1+k_2+k_3 = 10$, $k_i \in \mathbb{N}$, one obtains precisely the 10-sequences of I_3, which number 3^{10}.

Problem 3.12. How many words can be formed using seven A's, eight B's, three C's, and six D's if the pairs CC and CA do not appear in immediate succession.

Solution. We are seeking words in which neither the pattern CCCA nor CACC appears. So we must remove the words with the inadmissible patterns (in number $2 \times S(4,21;(6,8,6,1))$) from the set of all words formed by using seven A's, eight B's, three C's and six D's (of which there are $S(4,24;(7,8,3,6))$). Altogether, the words of the type specified number 704 313 890 280.

Problem 3.13. In a bridge hand (a card game with 4 players each receiving 13 cards) what is the probability that:

1. Player West holds 4 spades, 3 hearts, 3 diamonds and 3 clubs?
2. Players North and South each hold 5 spades, West has 2 spades, and East holds 1 spade?
3. A player holds all the Aces?
4. All the players hold 4 cards of one suit and 3 of all the other suits?

Solution. The sample space Ω of the four possible hands has cardinality $S(4,52;(13,13,13,13))$.

1: to choose a hand of the type held by West, one has $\binom{13}{4}\binom{13}{3}\binom{13}{3}\binom{13}{3}$ possibilities; the total possible hands for West number $\binom{52}{13}$ and so the probability sought is $\approx 0.02634 = 2.6\%$.

2: The spade cards may be distributed in the specified manner in $S(4,13;(5,5,2,1))$ ways. The remaining cards may be distributed in $S(4,39;(8,8,11,12))$ ways. The desired probability is therefore $\approx 0.0026 = 0.26\%$.

3: One can choose the player holding the four Aces in 4 ways; the remaining cards can then be distributed in $S(4,48;(9,13,13,13))$ ways. The desired probability is $\approx 0.011 = 1.1\%$.

4: For each player choose the suit of which he holds 4 cards: this may be done in 4! ways. Then deal the spades, followed by the clubs, then the diamonds, and finally the hearts. This may be done in $S(4,13;(4,3,3,3))^4$ ways. Hence the desired probability is $\approx 0.0009 = 0.09\%$.

Problem 3.14. If one tosses a coin 20 times, getting 14 Heads and 6 Tails, what is the probability that two consecutive Tails do not come up?

Solution. The number of possible sequences of this type is $S(2,20;(14,6))$. To count the number of favorable cases we calculate the number of ways it is possible to decide how many Heads come up before the 6 Tails. Let x_i be the number of Heads between the $i-1$-th Tail and the i-th Tail. Then one must have $x_1 + \cdots + x_6 + x_7 = 14$ with $x_2, ..., x_6 \geq 1$: this question has $\binom{15}{6}$ solutions. The desired probability is $\approx 0.129 = 12.9\%$.

Problem 3.15. How many anagrams of MATHEMATICIAN are there in which two *A*'s do not appear consecutively?

Solution. Let x_1 be the number of letters which appear before the first A, x_2 the number of letters between the first and second A's, x_3 the number of letters between the second and third A and x_4 the number of letters after the third A. One must then have $x_1 + \cdots + x_4 = 10$ with $x_2, x_3 \geq 1$, and there are $\binom{11}{3}$ solutions to this equation. Now one takes an arbitrary anagram of MTHEMTICIN (of which there are $S(7,10;(2,2,2,1,1,1,1)) = \frac{10!}{(2!)^3}$) and for each solution (x_1, x_2, x_3, x_4) of the above equation, one inserts the A's into the anagram as specified by the x_i's. Thus in all there are $\binom{11}{3} \times S(7,10;(2,2,2,1,1,1,1))$.

Problem 3.16. How many anagrams are there of ANCESTORS in which each *S* is followed by a vowel?

Solution. Choose 2 vowels ($\binom{3}{2} = 3$ possibilities), for example O and E; then the anagrams of the word (SO)(SE)ANCTR are 7! (where, of course, the blocks (SO) and (SE) are now considered as letters). In all one obtains $3 \times 7!$ anagrams of the desired type.

Problem 3.17. How many anagrams of MISSISSIPPI have no two consecutive S's?

Solution. One may construct an anagram of the desired type in the following way: first, decide how many letters to place before the first *S*, then how many between the first and the second *S*, ..., after the fourth *S*. This may be done in as many ways as there are natural number solutions of the equation $x_1 + \cdots + x_5 = 7$ with $x_2, x_3, x_4 \geq 1$, namely, in $C(8,4)$ ways. One then takes an anagram of MIIIPPI (of which there are $S(3,7;(1,4,2))$). For each solution of the above equation one and each such anagram one inserts the S's into the anagram as prescribed by the solution. In all the number of anagrams of the desired type is 7 350.

Problem 3.18. Among all the anagrams of UNABRIDGED how many have:

1. Four consecutive vowels?
2. At least three consecutive vowels?
3. No two consecutive vowels?

Solution. 1: Begin by deciding the order in which the vowels are to appear: there are 4! choices. From now on treat the block of vowels as if it were a single letter. Then count the anagrams of (for example) NBRGDD(UAIE): there are $S(6,7;(2,1,1,1,1,1)) = \frac{7!}{2!}$ of them, and so $4! \times \frac{7!}{2!}$ anagrams of the type specified.

2: Consider a 3-sequence of the 4 vowels from the 4! possibilities, and, as above, take it as an indivisible block. Count the anagrams of the remaining letters and the block under consideration: there are $S(7,8;(2,1,1,1,1,1,1)) = \frac{8!}{2!}$ of them. As the blocks vary, the anagrams in which the 4 vowels appear consecutively are counted twice under this procedure: for example UAIENBRDGD is an anagram both of (UAI)ENBRDGD and of E(UAI)NBRDGD. Thus it suffices to subtract the number found in poin 1. from $4! \times S(7,8;(2,1,1,1,1,1,1))$. In this way one finds

$$4! \times \big(S(7,8;(2,1,1,1,1,1,1)) - S(6,7;(2,1,1,1,1,1))\big) = 4!\big(\frac{8!}{2} - \frac{7!}{2}\big) = 423\,360$$

anagrams.

3: Take a 4-sequence of the vowels (4! choices). Then let x_1 be the number of consonants before the first vowel, ..., x_4 the number of consonants between the third and fourth vowels and x_5 the number of consonants after the last (fourth) vowel. One must have $x_1 + \cdots + x_5 = 6$ with $x_2, x_3, x_4 \geq 1$. This equation has $\binom{7}{4}$ natural number solutions). Consider now a sequence of consonants (they are $S(5,6;(2,1,1,1,1)) = 6!/2!$) and, for each solution as above, insert the consonant sequence between the vowels according to the rule specified by the solution. Thus, in all there are $4! \times \binom{7}{4} \times \frac{6!}{2!} = 302\,400$ of the type specified.

Problem 3.19. A bartender must serve drinks to 14 people seated at his bar.

1. The bartender does not remember who ordered a given drink: in how many ways could he serve the drinks if 3 people ordered a Martini, 2 a Manhattan, 2 a beer, 3 ordered a glass of Chablis, and 4 did not order anything?
2. The bartender is even more forgetful: he remembers only that Martinis, Manhattans, beers and Chablis have been ordered, and that someone made no order, and also that he must serve 3 drinks of one of these types, 2 of another type, 2

of yet another type, 3 of another type, and 4 of still another type (and here the "types of drink ordered" includes the order for no drink at all). In how many ways can these orders be served?

Solution. 1: The problem amounts to counting the 14-sequences of I_5 with occupancy $(3,2,2,3,4)$. There are $S(5,14;(3,2,2,3,4)) = 25\,225\,200$ such 14-sequences.

2: Here one must count the 14-sequences of I_5 with occupancy $[3,2,2,3,4]$; of these there are

$$S(5,14;[3,2,2,3,4]) = S(5,14;(3,2,2,3,4)) \times P(3,2,2,3,4) = 756\,756\,000.$$

Problem 3.20. Nine people arrive at a restaurant with three empty dining rooms, and each person randomly chooses a dining room. Find the probability that:

1. There are exactly three people in the first dining room;
2. There are three people in each dining room;
3. There is a dining room with 2 people, one with 3 people, and 4 people in the remaining room.

Solution. Label the three dining rooms A,B,C, and consider the sample space Ω of 9-sequences of $\{A,B,C\}$ with $|\Omega| = 3^9$. The events in which we are interested are respectively:

1. $E_a =$ 9-sequences of Ω with exactly 3 A; $|E| = \binom{9}{3} \times 2^6$;
2. $E_b =$ 9-sequences of Ω with occupancy $(3,3,3)$: $S(9;(3,3,3)) = 9!/(3!)^3$;
3. $E_c =$ 9-sequences of Ω with occupancy $[2,3,4]$: $S(9;[2,3,4]) = S(9;(2,3,4)) \times P(2,3,4) = (9!/(2!3!4!)) \times 3!$.

Of course, in each case $P(E) = |E|/|\Omega|$.

Problem 3.21. Determine the probability that in a binary 10-sequence with four 1's and six 0's there are 2 adjacent 1's and the other 1's are not adjacent (for example, $(1,1,0,1,0,1,0,0,0,0)$ fits our requirement).

Solution. The number of binary 10-sequences with four 1's and six 0's is $S(2,10;(4,6))$. The favorable cases may be obtained via the following procedure: Fix a permutation (a,b,c) of $((11),1,1)$. (There are three possibilities.) Then let x_1 be the number of 0's before a, let x_2 be the number of 0's between a and b, x_3 the number of 0's between b and c, and finally let x_4 be the number of 0's after c. One must have $x_1+x_2+x_3+x_4 = 6$ with $x_2, x_3 \geq 1$. There are $\binom{7}{3}$ possibilities. The probability requested is

$$\frac{3 \times \binom{7}{3}}{S(2,10;(4,6))} = 0.5 = 50\%\,.$$

Problem 3.22. A teacher has decided to quiz 8 of his 27 students in English, Latin, and History. In how many ways can he do so if he wishes to quiz 3 students in one subject, 3 in a second subject, and 2 in yet another subject?

Solution. One may choose the 8 students to be tested in $\binom{27}{8}$ ways. To decide the subjects one must choose and 8-sequence of I_3 with occupancy $[3,3,2]$, and for this there are $S(3,8;[3,3,2])$ possibilities. Therefore the testing can take place in 3 729 726 000 different ways.

Problem 3.23. A class of 18 students goes on a trip accompanied by two teachers. The evening accommodation consists of 5 rooms with 4 beds each. The two teachers do not want to sleep in the same room. In how many ways can room assignments satisfying this condition be made?

Solution. First assign the students to the rooms by considering 18-sequences of I_5 with occupancy $[3,3,4,4,4]$. There are $S(5,18;[3,3,4,4,4])$ such 18-sequences. Then assign the remaining two beds to the two teachers. This can be done in two ways. Hence in all, the travelers can be assigned their rooms in 257 297 040 000 ways.

Chapter 4
Problems and Solutions

Problem 4.1. A pastry chef prepares baskets with 6 chocolate eggs; the eggs can be wrapped in tin-foil of any one of 5 different colors: Blue, Green, Red, White, and Yellow. The order in which the eggs are placed in the basket does not matter.

1. What is the maximum number of distinct baskets that can be prepared under these assumptions?
2. The chef has prepared one basket for each of the possible types. A client buys all the baskets in which there is at least one Blue egg or exactly 2 Yellow eggs. How many baskets does the client buy?

Solution. 1. The maximal number of possible baskets is equal to the number of 5-compositions of 6, namely $C(6+4,4) = 210$.

2. Let B denote the set of baskets with at least one blue egg, and Y the set of baskets with exactly two yellow eggs. One has

$$|B \cup G| = |B| + |G| - |B \cap G| = C(5+4,4) + C(4+3,3) - C(3+3,3) = 141.$$

Problem 4.2. In how many ways is it possible to give a child 16 jelly beans when choosing from a large bin containing (at least 13 of each type) lemon, mint, and raspberry jelly beans if the child is to receive exactly 3 of at least one of the flavors?

Solution. Let A_L, A_M, A_R be respectively the distributions of jelly beans in which the child receives exactly three lemon, three mint, or three raspberry beans. The problem is to compute $|A_L \cup A_M \cap A_R|$. One has

$$|A_L \cup A_M \cap A_R| = |A_L| + |A_M| + |A_R| - |A_L \cap A_M| - |A_L \cap A_R| - |A_M \cap A_R| + |A_L \cap A_M \cap A_R|.$$

Now $|A_L| = |A_M| = |A_R|$ is the number of 2-compositions of $16-3 = 13$, namely $C(14,1)$, and similarly $|A_L \cap A_M| = |A_L \cap A_R| = |A_M \cap A_R| = 1$ and $|A_L \cap A_M \cap A_R| = 0$. Therefore,

$$|A_L \cup A_M \cap A_R| = 14 \times 3 - 3 + 0 = 39.$$

Problem 4.3. Determine the number of 13 card hands that one can get from a deck of 52 cards, if the hand has 4 Kings, or 4 Aces, or exactly four spades.

Solution. Let A_R, A_1 and A_P be the hands containing respectively four Kings, four aces, four spades. One has

$$|A_R \cup A_1 \cup A_P| = 2\binom{48}{9} + \binom{13}{4}\binom{39}{9} - \binom{44}{5} - 2\binom{12}{3} \times \binom{36}{6} + \binom{11}{2} \times \binom{33}{3}.$$

Problem 4.4. How many 5 letter words can be formed using an alphabet of 26 letters (with repetitions allowed) if every word must begin or end with a vowel?

Solution. Let B and F denote respectively the set of words that begin with a vowel and those that finish with a vowel. The problem is to compute $|B \cup F|$. One has $|B| = 5 \times 26^4 = |F|$, $|C \cap F| = 5^2 \times 26^3$, and so

$$|C \cup F| = |C| + |F| - |C \cap F| = 5 \times 26^4 + 5 \times 26^4 - 5^2 \times 26^3 = 4\,130\,360.$$

Problem 4.5. In how many ways can one form a sequence of length 5 using an alphabet of 3 letters if the sequence is to have at least two consecutive equal letters?

Solution. For $i = 1, \ldots, 4$ let X_i denote the set of sequences in which the i-th and the $i+1$-th letters coincide. We must compute $|X_1 \cup \cdots \cup X_4|$. One has $|X_i| = 3^{5-1}$; $|X_i \cap X_{i+1}| = 3^{5-2}$; if $i+1 < j$ then $|X_i \cap X_j| = 3^{5-2}$; $|X_i \cap X_{i+1} \cap X_{i+2}| = 3^{5-3}$; $|X_1 \cap X_2 \cap X_4| = |X_1 \cap X_3 \cap X_4| = 3^{5-3}$; $|X_1 \cap X_2 \cap X_3 \cap X_4| = 3$. Thus

$$|X_1 \cup \cdots \cup X_4| = 4 \times 3^{5-1} - \binom{4}{2} \times 3^{5-2} + \binom{4}{3} \times 3^{5-3} - \binom{4}{4} \times 3^{5-4} = 195.$$

Problem 4.6. Suppose that in a bookstore there are 200 books, 70 in French and 100 dealing with a mathematical topic. How many books are there not written in French and not dealing with mathematics if there are 30 books in French dealing with mathematics?

Solution. Let F and M respectively to denote the sets of books in French and those on a mathematical topic. The problem is to calculate $|F^c \cap M^c|$. One has

$$|F^c \cap M^c| = |(F \cup M)^c| = 200 - |F \cup M| = 200 - [|F| + |M| - |F \cap M|] =$$

$$= 200 - 70 - 100 + 30 = 60.$$

Problem 4.7. A group of 200 students are eligible to take three Mathematics courses: Discrete Mathematics, Analysis, and Geometry. Each course has 80 students. Each pair of courses has 30 students in common, and 15 students are taking all three courses.

1. How many students are not taking any of the three mathematics courses?
2. How many students are taking only Discrete Mathematics?

Solution. Let D, A and G respectively denote the sets of students of Discrete Mathematics, Analysis and Geometry.

1. One must compute $|D^c \cap A^c \cap G^c|$, for this one has

$$|D^c \cap A^c \cap G^c| = |(D \cup A \cup G)^c| = 200 - |D \cup A \cup G| =$$

$$= 200 - [|D| + |A| + |G| - |D \cap A| - |D \cap G| - |G \cap A| + |D \cap A \cap G|] =$$

$$= 200 - 3 \times 80 + 3 \times 30 - 15 = 35.$$

2. Here the question is to calculate the cardinality of $S = D \cap A^c \cap G^c$, and in this case

$$80 = |D| = |S| + |D \setminus S| = |S| + [|D \cap A| + |D \cap G| - |D \cap A \cap G| =$$

$$= |S| + [30 + 30 - 15] = |S| + 45.$$

Therefore $|S| = 35$.

Problem 4.8. How many numbers between 1 and 30 are relatively prime to 30?

Solution. Let A_2, A_3 and A_5 be the sets of numbers between 1 and 30 that are divisibile respectively by 2, 3, 5. The problem is to compute $|A_2^c \cap A_3^c \cap A_5^c|$. One has

$$|A_2^c \cap A_3^c \cap A_5^c| = |(A_2 \cup A_3 \cup A_5)^c| =$$

$$= 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1 = 8.$$

Problem 4.9. How many 10-sequences of I_9 are there in which the digits 1, 2, and 3 all appear?

Solution. Let A_i, $i = 1,2,3$, denote the set of 10 sequences in I_9 in which the number i does not appear. The question is then to compute $|A_1^c \cap A_2^c \cap A_3^c|$. Applying the results of this, chapter one has

$$|A_1^c \cap A_2^c \cap A_3^c| = |(A_1 \cup A_2 \cup A_3)^c| = 9^{10} - |(A_1 \cup A_2 \cup A_3)| =$$

$$= 9^{10} - [3 \times 8^{10}] + [\binom{3}{2} \times 7^{10}] - [6^{10}] = 1\,052\,518\,500.$$

Problem 4.10. In how many ways can 20 different people be assigned to 3 rooms if each room must receive at least one person?

Solution. Label the rooms with I_3. Each (complete) room assignment corresponds to a 20-sequence in I_3. Let X_i, $i = 1,2,3$ denote the 20 sequences in I_3 in which the number i does NOT appear. The question amounts to computing $|X_1^c \cap X_2^c \cap X_3^c|$, and for this one has

$$|X_1^c \cap X_2^c \cap X_3^c| = |(X_1 \cup X_2 \cup X_3)^c| = 3^{20} - |X_1 \cup \cdots \cup X_n| =$$

$$= 3^{20} - 3 \times 2^{20} + 3 = 3\,483\,638\,676.$$

Problem 4.11. How many anagrams of SINGER are there in which at least one of the following three conditions holds: (i) S precedes I, (ii) I precedes N, (iii) N precedes G? Here “precedes” means “occurs earlier than”, but not necessarily “immediately before”.

Solution. Let X_{SI}, X_{IN} and X_{NG} denote respectively the sets of words in which S precedes I, I precedes N, N precedes G. The problem is to compute $|X_{SI} \cup X_{IN} \cup X_{NG}|$. A word in X_{SI} is constructed by deciding how many letters to insert before the S, how many between the S and the I and how many after the I, and then inserting a 4-sequence without repetitions of the letters $\{N, G, E, R\}$ into the positions selected Therefore we first calculate the number of solutions of $x_1 + x_2 + x_3 = 4$, namely $\binom{6}{2}$, and then the number of 4-sequences without repetitions in the letters $\{N, G, E, R\}$, namely 4!. Thus one has $|X_{SI}| = |X_{IN}| = |X_{NG}| = \binom{6}{2} \times 4!$ Similarly $|X_{SI} \cap X_{IN}| = \binom{6}{3} \times 3! = |X_{IN} \cap X_{NG}|$ and $|X_{SI} \cap X_{IN} \cap X_{NG}| = \binom{6}{4} \times 2!$. With regard to $|X_{SI} \cap X_{NG}|$ one may proceed as follows: let $\mathscr{P}$ be a word in X_{SI} and $\mathscr{P}'$ the word obtained from $\mathscr{P}$ by interchanging the letters N and G; one and only one of $\mathscr{P}$ and $\mathscr{P}'$ belongs to X_{NG} and so $|X_{SI} \cap X_{NG}| = \frac{1}{2}|X_{SI}|$. Therefore,

$$|X_{SI} \cup X_{IN} \cup X_{NG}| = \binom{6}{2} \times 4! \times 3 - \binom{6}{3} \times 3! \times 2 - \frac{1}{2}\binom{6}{2} \times 4! + \binom{6}{4} \times 2! = 690.$$

Problem 4.12. The Bakers, the Vinsons and the Caseys each have 5 children. If the 15 youngsters camp out in 5 different tents, with three in each tent, and are assigned randomly to the 5 tents, what is the probability that each family has at least two of its children in the same tent?

Solution. Label the tents with I_5. The possible sleeping assignments of the 15 children in the tents number $S(5, 15; (3, 3, 3, 3, 3))$. The excluded assignment are those in which there is a tent to which one child from each family is assigned. Let A_i denote the collection of assignments in which tent i shelters a children from each family. Then one has

$$|A_1 \cup \cdots \cup A_5| = 5 \times [5^3 \times S(4, 12; (3, 3, 3, 3))] - \binom{5}{2} \times [5^3 \times 4^3 \times S(3, 9; (3, 3, 3))] +$$

$$+ \binom{5}{3} \times [5^3 \times 4^3 \times 3^3 \times S(2, 6; (3, 3))] - \binom{5}{4} \times [5^3 \times 4^3 \times 3^3 \times 2^3 \times S(1, 3; (3))] +$$

$$+ \binom{5}{5} \times [5^3 \times 4^3 \times 3^3 \times 2^3 \times 1^3] = 132\,888\,000.$$

In fact, one obtains an assignment to A_i by choosing who to put in tent i (5^3 ways) and then filling the other 4 tents ($S(4, 12; (3, 3, 3, 3))$ ways); an assignment lying in $A_i \cap A_j$, $i < j$, is obtained by choosing who to put in tent i (5^3 ways), who to put in tent j (4^3 ways), and then filling the other 3 tents ($S(3, 9; (3, 3, 3))$ ways); ... and so on.

Thus, the probability sought is

$$1-132\,888\,000/\frac{15!}{3!^5}=1-\frac{3!^5\times 132\,888\,000}{15!}\approx 0.9753=97,53\%.$$

Problem 4.13. What is the probability that a hand of 13 cards taken from a deck of 52 has:

1. At least one missing suit?
2. At least one card of each suit?
3. At least one of each type of face card and at least one Ace (that is, at least one Ace, at least one Jack, at least one Queen, and at least one King)?

Solution. We define A_H, A_D, A_C, A_S to be respectively the collections of all possible hands without hearts, diamonds, clubs, and spades.

1. We must calculate $|A_H\cup A_D\cup A_C\cup A_S|=$

$$=4\times\binom{39}{13}-\binom{4}{2}\times\binom{26}{13}+\binom{4}{3}\binom{13}{13}=32\,427\,298\,180.$$

2. Here we must calculate

$$|(A_C\cup A_Q\cup A_F\cup A_P)^c|=\binom{52}{13}-|A_C\cup A_Q\cup A_F\cup A_P|=602\,586\,261\,420.$$

3. Let X_A, X_K, X_Q, X_J be the 13 card hands without aces, kings, queens, and jacks respectively. One then has

$$|X_A^c\cap X_K^c\cap X_Q^c\cap X_J^c|=|(X_A\cup X_K\cup X_Q\cup X_J)^c|=\binom{52}{13}-|X_A\cup X_K\cup X_Q\cup X_J|=$$

$$=\binom{52}{13}-[4\times\binom{48}{13}-\binom{4}{2}\times\binom{44}{13}+\binom{4}{3}\times\binom{40}{13}-\binom{4}{4}\times\binom{36}{13}]=128\,971\,619\,088.$$

Problem 4.14. How many 9-sequences of I_3 are there in which there appear three 1's, three 2's, and three 3's, but without three consecutive equal numbers?

Solution. Let A_i, $i=1,2,3$, denote the 9 sequences in I_3 for which there are three consecutive i's. The problem is to compute $|A_1^c\cap A_2^c\cap A_3^c|$. Bearing in mind that the sequences in A_1, for example, can ve viewed as 7 sequences of $\{111,2,3\}$ with occupancy $(1,3,3)$, one has

$$|A_1^c\cap A_2^c\cap A_3^c|=|(A_1\cup A_2\cup A_3)^c|=S(3,9;(3,3,3))-|A_1\cup A_2\cup A_3|=$$

$$=\frac{9!}{3!^3}-[3\times S(3,7;(1,3,3))-3\times S(3,5;(1,1,3))+S(3,3;(1,1,1))]=1\,314.$$

Problem 4.15. How many permutations of the 26 letters of the English alphabet are there which do not contain any of the words SOAP, FLY, LENS, GIN?

Solution. Let $X_{LENS}, X_{SOAP}, X_{FLY}, X_{GIN}$ be the set of permutations of $(A,B,C,\ldots,Z)$ containing respectively the sequences LENS, SOAP, FLY, GIN. We must calculate

$$|X^c_{LENS} \cap X^c_{FLY} \cap X^c_{SOAP} \cap X^c_{GIN}| = |(X_{LENS} \cup X_{FLY} \cup X_{SOAP} \cup X_{GIN})^c| =$$
$$= 26! - |X_{LENS} \cup X_{FLY} \cup X_{SOAP} \cup X_{GIN}| =$$
$$= 26! - [|X_{LENS}| + |X_{FLY}| + |X_{SOAP}| + |X_{GIN}| - \cdots].$$

Now one notes that FLY, SOAP, GIN have no letters in common, while LENS has a letter in common with each of the others. A permutation in X_{LENS} is a permutation of the 23 sequence $(LENS, A, B, C, D, F, ...M, O, P, Q, ..., Z)$; therefore X_{LENS} has cardinality 23!. Similarly, $|X_{SOAP}| = 23!$, $|X_{FLY}| = |x_{GIN}| = 24!$ The intersections $X_{LENS} \cap X_{FLY}$, $X_{LENS} \cap X_{GIN}$ are empty; however, the intersection $X_{LENS} \cap X_{SOAP}$ consists of the permutations of the 20-sequence $(LENSOAP, B, C, D, F, ..., R, U, ..., Z)$, which in all number 20!. The intersection $X_{FLY} \cap X_{SOAP}$ consists of the 21! permutations of the 21-sequence $(FLY, SOAP, B, C, D, E, H, ..., Z)$; similarly, one has $|X_{FLY} \cap X_{GIN}| = 22!$ and $|X_{SOAP} \cap X_{GIN}| = 21!$. The threefold intersection of the 4 sets of permutations are empty except for $X_{FLY} \cap X_{SOAP} \cap X_{GIN}$ which consists of the 19! permutations of the 19-sequence $(FLY, SOAP, GIN, B, C, D, E, H, ..., Z)$; finally, the intersection of the 4 sets is empty. Proceeding in this way one finds that

$$|X^c_{LENS} \cap X^c_{FLY} \cap X^c_{SOAP} \cap X^c_{GIN}| = 26! - |X_{SOAP} \cup X_{FLY} \cup X_{LENS} \cup X_{GIN}| =$$
$$= 26! - [(23! + 23! + 24! + 24!) - (20! + 21! + 22! + 21!) + 19!].$$

Problem 4.16. In how many ways can one distribute 25 identical balls in 6 distinct containers so as to have a maximum of 6 balls in any one of the first 3 containers?

Solution. Label the containers with I_6 and let X_i, $i = 1, 2, 3$, be the distributions of the balls with respectively at least 7 balls in container i. The problem amounts to calculating $|X^c_1 \cap X^c_2 \cap X^c_3|$:

$$|X^c_1 \cap X^c_2 \cap X^c_3| = |(X_1 \cup X_2 \cup X_3)^c| = \binom{30}{5} - |X_1 \cup X_2 \cup X_3| =$$
$$= \binom{30}{5} - \left[3 \times \binom{23}{5} - 3 \times \binom{16}{5} + \binom{9}{5}\right] = 54\,537.$$

Problem 4.17. A witch doctor has 5 friends. During a long convention on black magic he went to lunch with each friend 10 times, with every pair of friends 5 times, with every triple of friends 3 times, with every quadruple of friends twice, and only once with all 5 friends. If, moreover, the witch doctor lunched alone 6 times, how many days did the convention last?

Solution. The length of the conference corresponds to the number of Witch Doctor's lunches. Label his friends with I_5, and let X_i, $i \in I_5$, be the subset of his lunches in which friend i was present. The length of the convention is then equal

$$6 + |X_1 \cup \cdots \cup X_5| = 6 + 5 \times 10 - \binom{5}{2} \times 5 + \binom{5}{3} \times 3 - \binom{5}{4} \times 2 + 1 = 27$$

days.

Problem 4.18. Suppose that in a mathematics department there are 10 courses to be assigned to 5 different professors. In how many ways can one assign the 5 professors two courses per year in two successive academic years in such a way that no professor teaches the same 2 courses both years?

Solution. Label the courses with I_{10} and the professors with I_5. The assignment of two courses for the first year amounts to fixing an arbitrary 10-sequence of I_5 with occupancy sequence $(2,2,2,2,2)$. Once the assignments for the first year have been made, one must then make those for the second year. If X_i, $i=1,...,5$ are the assignments of the 10 courses for the second year in which professor i receives the same courses as in the first year, then for the second year we have $|X_1^c \cap \cdots \cap X_5^c|$ assignments of the specified type. One has

$$|X_1^c \cap \cdots \cap X_5^c| = |(X_1 \cup \cdots \cup X_5)^c| = S(5,10;(2,2,2,2,2)) - |X_1 \cup \cdots \cup X_5| =$$
$$= S(5,10;(2,2,2,2,2)) - \left[5 \times S(4,8;(2,2,2,2)) - \binom{5}{2} \times S(3,6;(2,2,2)) + \right.$$
$$\left. + \binom{5}{3} \times S(2,4;(2,2)) - \binom{5}{4} \times S(1,2;(2)) + 1\right] = 101\,644.$$

Therefore, there are $S(5,10;(2,2,2,2,2)) \times 101\,644 = 11\,526\,429\,600$ possible ways of making the assignments for two years.

Problem 4.19. How many permutations of $(1,2,...,n)$ are there in which 1 is not immediately followed by 2, 2 is not immediately followed by 3, ..., n is not immediately followed by 1?

Solution. Let X_i, $1 \le i \le n-1$, be the set of permutations of $(1,2,...,n)$ in which i is followed by $i+1$ and X_n the set of permutations with n followed by 1. We must calculate $|X_1^c \cap \cdots \cap X_n^c|$. Using the results of this chapter we find that

$$|X_1^c \cap \cdots \cap X_n^c| = |(X_1 \cup \cdots \cup X_n)^c| = n! - |X_1 \cup \cdots \cup X_n| =$$
$$= n! - \left[n \times (n-1)! - \binom{n}{2} \times (n-2)! + \cdots + (-1)^{n-1}\binom{n}{n}\right] = D_n$$

where D_n is the number of derangements of $(1,2,...,n)$.

Problem 4.20. In how many ways can one distribute 10 books to 10 boys (one book to each boy), and then collect the books and redistribute them in such a way that every boy gets a new book?

Solution. Label both the books and the boys with I_{10}. Then the distributions of the books are equal in number to the permutations of $(1,2,...,10)$, namely, 10!. The second distribution must be a derangement of the sequence chosen for the first distribution. Hence the two distributions can be made in a total of

$$10! \times D_{10} = 10! \times 10! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\right) = 4\,844\,306\,476\,800$$

ways.

Problem 4.21. In a city 3 newspapers (A, B, and C) are sold. A survey reveals that 47% of the inhabitants read newspaper A, 34% read newspaper B, 12% read newspaper C. Moreover, 8% read both A and B, 5% both A and C, and 4% both B and C. Finally, 4% read all three papers. If one picks an inhabitant of the city at random, find the probability that:

1. She/he does not read any paper;
2. She/he reads only one paper.

Solution. Let X_A, X_B and X_C denote the set of inhabitants of the city who read papers A, B and C respectively, and let X be the set of inhabitants of the city. 1. One has

$$P(X_A^c \cap X_B^c \cap X_C^c) = \frac{|X_A^c \cap X_B^c \cap X_C^c|}{|X|} = 1 - \frac{|X_A \cup X_B \cup X_C|}{|X|} =$$

$$= 1 - [\frac{|X_A|}{|X|} + \frac{|X_B|}{|X|} + \frac{|X_C|}{|X|} - \frac{|X_A \cap X_B|}{|X|} - \frac{|X_A \cap X_C|}{|X|} - \frac{|X_B \cap X_C|}{|X|} + \frac{|X_A \cap X_B \cap X_C|}{|X|} =$$

$$= 1 - 0.47 - 0.34 - 0.12 + 0.08 + 0.05 + 0.04 - 0.04 = 0.2 = 20\%\,.$$

2. From part 1 we know that 80% of the population reads at least one paper. The desired probability is then

$$P(X_A \cup X_B \cup X_C) - P((X_A \cap X_B) \cup (X_A \cap X_C) \cup (X_B \cap X_C)) = 0.8 - [\frac{|(X_A \cap X_B) \cup (X_A \cap X_C) \cup (X_B \cap X_C)|}{|X|} =$$

$$= 0.8 - [\frac{|X_A \cap X_B| + |X_A \cap X_C| + |X_B \cap X_C|}{|X|} - \frac{|X_A \cap X_B \cap X_C|}{|X|} = 0.8 - 0.08 - 0.05 - 0.04 + 0.04 = 0.67 = 67\%\,.$$

Problem 4.22. We must insert 9 distinct numbers between 1 and 90 (including the extreme values) into a table of 4 rows and 6 columns.

1. How many different tables can be created?
2. How many tables have an empty row or 90 in the first row?

Solution. 1. Each table can be constructed in the following two steps: 1) choice of the 9 positions into which numbers are to be inserted; 2) choice of the numbers to be placed in the first, in the second, ..., in the ninth position chosen. Thus in all the number of choices is

$$\binom{24}{9} \times S(90, 9) = 335\,093\,554\,887\,620\,485\,017\,600.$$

2. For $i = 1, 2, 3, 4$, let A_i be the set of tables with empty i-th row and let A_{90} denote the tables con the 90 in the first row.

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_{90}| = \sum_{i=1}^{4} |A_i| + |A_{90}| - (\sum_{1\le i<j\le 4} |A_i \cap A_j| + \sum_{i=1}^{4} |A_i \cap A_{90}) +$$

$$+(\sum_{1\le i<j<k\le 4} |A_i \cap A_j \cap A_k| + \sum_{1\le i<j\le 4} |A_i \cap A_j \cap A_{90}|) - (|A_1 \cap A_2 \cap A_3 \cap A_4| +$$

$$+ \sum_{1\le i<j<k\le 4} |A_i \cap A_j \cap A_k \cap A_{90}|) + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_{90}| =$$

$$= 4 \times \binom{18}{9} \times S(90,9) + 6 \times \binom{23}{8} \times S(89,8) - (\binom{4}{2} \times \binom{12}{9} \times S(90,9) + 3 \times 6 \times \binom{17}{8} \times S(89,8)) +$$

$$+(0 + 3 \times 6 \times \binom{11}{8} \times S(89,8)) - (0+0) + 0 =$$

$$= 56\,643\,733\,686\,692\,382\,531\,840.$$

Problem 4.23. Consider an alphabet composed of 13 symbols.

1. How many 8 letter words containing at least one symbol repeated three times is it possible to write?
2. How many 8 letter words containing at least two distinct symbols repeated exactly 3 times is it possible to write?

Solution. Label the alphabet with I_{13}. Every 8 letter word corresponds to an 8-sequence in I_{13}. Let X_i, $i \in I_{13}$, be the set of words in which the symbol i is repeated exactly three times.

1. Here one must calculate

$$|X_1 \cup \cdots \cup X_{13}| = 13 \times \binom{8}{3} \times 12^5 - \binom{13}{2} \times \binom{8}{3}\binom{5}{3} \times 11^2 = 175\,864\,416.$$

2. Now the calculation is as follows:

$$|\cup_{i<j}(X_i \cup X_j)| = \binom{13}{2} \times \binom{8}{3}\binom{5}{3} \times 11^2 = 5\,285\,280.$$

Problem 4.24. Consider the red cards (13 hearts and 13 diamonds) from a poker deck of 52 cards. The 13 heart cards are distributed to 13 people, and then the diamonds, with one card of each type to each person.

1. How many possible outcomes are there for such a distribution?
2. What is the probability that at least one person receives a pair (two cards with the same face value)?

Solution. Each distribution corresponds to assigning two permutations of the sequence $(1,2,3,...,13)$.

1. There are $13!^2 = 38\,775\,788\,043\,632\,640\,000$ possible distributions.

2. No one receives a pair if and only if the second permutation is a derangement of the first. Thus the desired probability is

$$1-\frac{13!\times D_{13}}{13!^2}=1-\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots-\frac{1}{13!}\right)\approx 0.632=63.2\%\,.$$

Problem 4.25. Prove that the factorial $n!$ and the number of derangements D_n satisfy for $n\geq 3$ the following recurrence formulas:

1. $n!=(n-1)\big((n-1)!+(n-2)!\big)$;
2. $D_n=(n-1)\big(D_{n-1}+D_{n-2}\big)$.

[Hint for Part 2: Carry out every permutation of $(1,\ldots,n)$ in two steps. The first step consists of choosing the position i to which 1 is moved; the second step is to carry out the repositioning of the other digits, distinguishing between the case where i is moved to position 1, and when it is not ...]

Solution. 1.

$$\begin{aligned}(n-1)\,[(n-1)!+(n-2)!]&=n(n-1)!-(n-1)!+(n-1)(n-2)!\\&=n!-(n-1)!+(n-1)!=n!\,.\end{aligned}$$

2. We carry out a derangement of $(1,\ldots,n)$ in two steps:

1. Choose the position $i\neq 1$ to which 1 is moved: there are $n-1$ possible positions;
2. Choose the positions of 2,3,...,n distinguishing two cases:
   a. If we move i to position 1 in the derangement, we much choose the positions of the remaining $n-2$ elements $2,3,\ldots,i-1,i+1,\ldots,n$ in such a way that none of them remain in the same position, and this amounts to carrying out a derangement of the $(n-2)$-sequence $(2,\ldots,i-1,i+1,\ldots,n)$; in all there D_{n-2} such derangements.
   b. If in the derangement one chooses a position different from 1 for i, we must arrange things so that in the derangement of $(1,2,\ldots,n)$:
      - 1 goes to position i;
      - i does not go to the first position.

      This is equivalent to performing a derangement of $(i,2,\ldots,i-1,i+1,\ldots,n)$: this operation can be done in D_{n-1} ways.

For each of the D_{n-1} choices in the first step we have $D_{n-2}+D_{n-1}$ choices for the second step: the Multiplication Principle 1.34 then gives the desired conclusion.

Problem 4.26. Let $n_1,n_2,n_3\in\mathbb{N}_{\geq 1}$ and let $n=n_1+n_2+n_3$. Show, without using Theorem 4.26, that

$$D_n(n_1,n_2,n_3)=\sum_{k=1}^{n_1}\binom{n_2}{k}\binom{n_3}{n_1-k}\binom{n_1}{n_3-n_1+k}\,.$$

[Hint: perform a derangement of the n-sequence

$$(\underbrace{1,\ldots,1}_{n_1},\underbrace{2,\ldots,2}_{n_2},\underbrace{3,\ldots,3}_{n_3}).$$

To obtain a derangement, k elements equal to 1 are placed where the 2's were, the other n_1-k elements equal to 1 are placed where the 3's were. Finally, the remaining free positions where there were 2's are filled by $n_3-(n_1-k)$ elements equal to 3.]

Solution. Here the problem amounts to performing a derangement of

$$(\underbrace{1,...,1}_{n_1},\underbrace{2,...,2}_{n_2},\underbrace{3,...,3}_{n_3})$$

where there are n_1 terms equal to 1, n_2 terms equal to 2, n_3 terms equal to 3. If the derangement has $k \le n_1$ elements equal to 1 that finish in some of the n_2 positions initially occupied by 2's, then the remaining initial positions of the 2's must necessarily be occupied by n_2-k elements equal to 3: the choice of the k positions may be made in $\binom{n_2}{k}$ ways. At this point one must still position exactly n_1-k elements equal to 1 and $n_3-(n_1-k)$ elements equal to 2 in the n_3 positions initially occupied by 3's: this may be done in $\binom{n_3}{n_1-k}$ ways. Finally, there remains the choice of which of the n_1 positions initially occupied by 1's are to be occupied by the $n_3-(n_2-k)$ elements equal to 3 (the other positions are taken by elements equal to 2): this may be done in $\binom{n_1}{n_3-n_1+k}$ ways. The result now follows by the Multiplication Principle 1.34.

Chapter 5
Problems and Solutions

Problem 5.1. In how many ways can one distribute 23 different objects into 8 identical boxes, in such a way as to have no box left empty?

Solution. Use I_{23} to label the different objects. Then every distribution into 8 identical boxes, with no box left empty, corresponds to an 8 partition of I_{23}. Hence, there are

$$\left\{ {23 \atop 8} \right\} = 9\,741\,955\,019\,900\,400$$

different distributions.

Problem 5.2. The members of a group of 30 tourists decide to go for jeep rides; in how many ways can they split up into 6 groups in such a way as to have 3 groups consisting of 6 people, one group of 3 people, one of 4, and one of 5 people?

Solution. The 6-partitions of I_{30} with occupancy collection $[3,4,5,6,6,6]$ are

$$\left\{ {30 \atop 6} \right\} = \frac{1}{6!} \times S(6,30;3,4,5,6,6,6) \times P(3,4,5,6,6,6),$$

that is,

$$\frac{1}{6!} \times \frac{30!}{3!4!5!(6!)^3} \times \frac{6!}{3!}.$$

Problem 5.3. A rescue team of 14 people is searching for a missing person. They decide to split into 5 squads: 3 squads with 2 people and 2 squads with four people. In how many ways is it possible to perform this division into squads?

Solution. The point is to count the 5-partitions of I_{14} with occupancy $[2,2,2,4,4]$:

$$\Pi(5,14;[2,2,2,4,4]) = \frac{1}{5!} S(5,14;[2,2,2,4,4]) = \frac{1}{5!} \frac{14!}{2!^3 \times 4!^2} P(2,2,2,4,4) =$$

$$= \frac{1}{5!} \frac{14!}{2!^3 \times 4!^2} \frac{5!}{3! \times 2!} = 1\,576\,575.$$

Problem 5.4. In how many ways can a deck of 52 cards be divided into:

(a) 4 decks of 13 cards?
(b) 3 decks of 8 cards and 4 decks of 7 cards?

Solution. (a): Here the question is to give the number of 4-partitions of I_{52} with occupancy $[13,13,13,13]$: in all there are $\Pi(4,52;[13,13,13,13]) = \frac{1}{4!}\frac{52!}{(13!)^4}$. (b): Similarly, here the question is to find the number of 7-partitions of I_{52} with occupancy $[7,7,7,8,8,8,8]$: in all there are $\Pi(7,52;[7,7,7,8,8,8,8]) = \frac{1}{7!}\frac{52!}{(7!)^3(8!)^4}\frac{7!}{3!4!}$ such 7-partitions.

Problem 5.5. A class of 18 students goes on a trip accompanied by two teachers. They will spend the night in 5 quadruple rooms; the two teachers do not wish to sleep in the same room. Under this condition, in how many ways can the entire group be split up into 5 groups of exactly 4 people?

Solution. The outcome of the room assignments uniquely determines a 5-partition of the 18 students with occupancy collection $[4,4,4,3,3]$, the first group of 3 students being where the first teacher is assigned, and the second such group being the room assigned to the second teacher. Thus the assignment procedure may be realized in two steps: a) a 5-partition of the students with occupancy collection $[4,4,4,3,3]$ b) assignment of the two teachers to the two groups with 3 people: this may be done in 2 ways. In all, therefore, there are

$$2 \times \Pi(5,18;[4,4,4,3,3]) = 2 \times \frac{1}{5!} \times \frac{18!}{4!^3 \times 3!^2} \times P(4,4,4,3,3) =$$

$$= 2 \times \frac{1}{5!} \times \frac{18!}{4!^3 \times 3!^2} \times \frac{5!}{3! \times 2!}$$

ways of making such room assignments.

Problem 5.6. In how many ways is it possible to split up 20 people into 5 squads, of which two consist of 5 people, two of three people, and one of 4 people?

Solution. The question amounts to counting the 5-partitions of I_{20} with occupancy collection $[5,5,3,3,4]$. There are

$$\Pi(5,20;[5,5,3,3,4]) = \frac{1}{5!}\frac{20!}{5!5!3!3!4!}P(5,5,3,3,4) = \frac{1}{5!}\frac{20!}{5!5!3!3!4!}\frac{5!}{2!2!1!} = \frac{20!}{(5!3!2!)^2 4!}$$

such partitions.

Solution.

Problem 5.7. In how many ways is it possible to fill a Ferris wheel with 30 one-place seats from a school group of 100 people?

Solution. The problem amounts to counting the number of 30-cycles of I_{100}. That number is

$$\frac{1}{30}S(100,30) = \frac{100!}{30 \times 70!} =$$
$$= 259\,703\,237\,901\,926\,829\,152\,907\,749\,975\,910\,714\,678\,610\,923\,356\,160\,000\,000.$$

Problem 5.8. Twenty guests are to be seated around a large circular table. Find the probability that Carlo, Alberto, Elena and Paola are neighbors.

Solution. The number of possible seating arrangements of the twenty guests is given by the number of 20-cycles of I_{20}, namely 19!. Carlo, Alberto, Paola and Elena can be seated (counterclockwise) in 4! ways. Once a seating of these four has been chosen, we consider them to constitute a single block, and count the number of possible seating arrangements around the table of that block and the other 16 guests: that number is 16!. Thus, the desired probability is equal to

$$\frac{4! \times 16!}{19!} = \frac{4}{969} \approx 0.004 = 0.4\%\,.$$

Problem 5.9. In how many ways can one seat 40 people in a banquet hall containing 8 equally sized round tables if no table is to be left empty?

Solution. In the first place we must decide whether or not to consider the tables as distinguishable. If the tables are indistinguishable, then the question amounts to counting the 8-partitions into cycles of I_{40}, and these number $\begin{bmatrix} 40 \\ 8 \end{bmatrix}$. If, on the other hand, the tables are all distinguishable, then one we have obtained an 8-partition into cycles, we must assign each cycle to a table: this can be done in 8! ways. Therefore, if the tables are distinguishable the number of such arrangements turns out to be $8! \times \begin{bmatrix} 40 \\ 8 \end{bmatrix}$.

Problem 5.10. We wish to place 100 students on 4 Ferris wheels with one-place seats, two of the wheels having 25 seats, one with 20 seats and one with 30 seats. Calculate the probability that Nicky, Tommy, and Francies sit in three consecutive places on one of the Ferris wheels.

Solution. Label the students with I_{100}. We carry out a 4-partition into cycles of I_{100} with occupancy $[25,25,20,30]$; each cycle represents the counterclockwise seating arrangements on one of the Ferris wheels. We must then assign to each cycle a suitable wheel: there is only one possibility for both the 30-cycle and the 20-cycle, but there are two possibilities for the 25-cycles. Therefore, there are

$$2 \times \Pi_{cyc}(4,100;[25,25,20,30]) =$$
$$= 2 \times \left[\frac{1}{4!} \times \frac{100!}{25 \times 25 \times 20 \times 30} \times P(25,25,20,30)\right]$$

possible seating arrangements of the students on the Ferris wheels. The number of arrangements with Nicky, Tommy, and Francie seated in that order in consecutive counterclockwise positions are obtained by making separate counts of the cases in which the three youngsters sit in the first 25 place wheel, in the second 25 place wheel, in the 30 place wheel, or in the wheel with 20 places. If the 3 are placed in the first 25 place wheel, we must complete the remaining 22 places on the wheel with a 22-sequence in I_{97} by thinking of the 3 as placed in the first 3 positions on the wheel, and then we must arrange the remaining 75 students in a 3-partition into cycles of I_{75} with occupancy $[25,20,30]$. In all, there are

$$S(97,22) \times \frac{1}{3!} \times \frac{75!}{25 \times 20 \times 30} \times P(25,20,30)$$

different arrangements. Similarly, if the three are placed in the second wheel with 25 places. If, however, the three are placed on the wheel with 20 or that with 30 places one obtains respectively

$$2 \times S(97,17) \times \frac{1}{3!} \times \frac{80!}{25 \times 25 \times 30} \times P(25,25,30) \text{ and}$$

$$2 \times S(97,27) \times \frac{1}{3!} \times \frac{70!}{25 \times 25 \times 20} \times P(25,25,20)$$

different arrangements; the factor 2 in each expression given above accounts for the two choices for placing the two 25-cycles in the two wheels with 25 places. The desired probability is

$$\frac{2 \times \left[\frac{97!}{75!} \times \frac{1}{3!} \times \frac{75!}{25 \times 20 \times 30} \times 3! + \frac{97!}{80!} \times \frac{1}{3!} \times \frac{80!}{25 \times 25 \times 30} \times \frac{3!}{2} + \frac{97!}{70!} \times \frac{1}{3!} \times \frac{70!}{25 \times 25 \times 20} \times \frac{3!}{2}\right]}{2 \times \frac{1}{4!} \times \frac{100!}{25 \times 25 \times 20 \times 30} \times \frac{4!}{2}} =$$

$$\approx 0.0001 = 0.01\% .$$

Problem 5.11. Let $k \in \mathbb{N}_{\geq 1}$ and $b_1,...,b_k$ be natural numbers such that

$$b_1 + 2b_2 + \cdots + kb_k = k.$$

Prove that the number of partitions into cycles of I_k with occupancy collection $[\underbrace{1,...,1}_{b_1},...,\underbrace{k,...,k}_{b_k}]$ is equal to

$$\frac{k!}{(1!)^{b_1} \cdots (k!)^{b_k}} \frac{1}{b_1! \cdots b_k!}.$$

Solution. Indeed, one is necessarily dealing with n-partitions, where we have set $n = b_1 + \cdots + b_k$. By Theorem 5.57 the number of such partitions is

$$\frac{1}{n!} \times \frac{k!}{(1!)^{b_1} \cdots (k!)^{b_k}} \times P(\underbrace{1, \dots, 1}_{b_1}, \dots, \underbrace{k, \dots, k}_{b_k})$$

and moreover

$$P(\underbrace{1, \dots, 1}_{b_1}, \dots, \underbrace{k, \dots, k}_{b_k}) = \frac{(b_1 + \cdots + b_k)!}{b_1! \cdots b_k!} = \frac{n!}{b_1! \cdots b_k!},$$

from which the conclusion follows.

Problem 5.12. Prove the Worpitzky Identity (see Proposition 5.84) by induction on n.

Solution. The statement is true for $n = 1$:

$$\binom{\ell + 0}{1} \left\langle {1 \atop 0} \right\rangle = \ell = \ell^1.$$

Suppose now that it holds for $n \geq 1$. Using Proposition 5.73, 2, one has

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{\ell+i}{n+1} \left\langle {n+1 \atop i} \right\rangle = \\
&= \sum_{i=0}^{n} \binom{\ell+i}{n+1} (i+1) \left\langle {n \atop i} \right\rangle + \sum_{i=0}^{n} \binom{\ell+i}{n+1} (n+1-i) \left\langle {n \atop i-1} \right\rangle \\
&= \sum_{i=0}^{n-1} \binom{\ell+i}{n+1} (i+1) \left\langle {n \atop i} \right\rangle + \sum_{i=0}^{n-1} \binom{\ell+i+1}{n+1} (n-i) \left\langle {n \atop i} \right\rangle \\
&= \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \left[\binom{\ell+i}{n+1} (i+1) + \binom{\ell+i+1}{n+1} (n-i) \right] \\
&= \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{\ell+i}{n} \left[\frac{(i+1)(\ell+i-n) + (n-i)(\ell+i+1)}{n+1} \right] \\
&= \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{\ell+i}{n} \frac{\ell(n+1)}{n+1} \\
&= \ell \sum_{i=0}^{n-1} \binom{\ell+i}{n} \left\langle {n \atop i} \right\rangle = \ell\,\ell^n = \ell^{n+1},
\end{aligned}$$

where the last equality follows from the inductive hypothesis.

Chapter 6
Problems and Solutions

Problem 6.1. Let $a, b \in \mathbb{R}$. Use Gauss method to calculate

$$S = \sum_{k=0}^{n} (a + bk).$$

Solution. We have

$$2S = \sum_{k=0}^{n} (a + bk) + \sum_{k=0}^{n} (a + b(n-k)) =$$

$$= \sum_{k=0}^{n} (a + bk + a + b(n-k)) = \sum_{k=0}^{n} (2a + bn) = (n+1)(2a + bn),$$

from which it follows that $S = (n+1)(a + \frac{1}{2}bn) = \frac{1}{2}bn^2 + (a + \frac{1}{2}b)n + a$.

Problem 6.2. Let x be a given real number. Calculate, using the perturbation method and the result of Example 6.3, the sum

$$T_n = \sum_{k=0}^{n} k^2 x^k.$$

From this deduce, for $|x| < 1$, the sum of the series

$$\sum_{k=0}^{\infty} k^2 x^k.$$

Solution. If $x = 1$ one has

$$T_n = \sum_{k=0}^{n} k^2 = \sum_{k=0}^{n} k \frac{1}{2} \Delta k^{\underline{2}} = \frac{1}{2} \left[k k^{\underline{2}} \right]_{k=0}^{\theta\, n} - \frac{1}{2} \sum_{k=0}^{n} (k+1)^{\underline{2}} =$$

$$=\frac{1}{2}n(n+1)^2-\frac{1}{2}\sum_{k=1}^{n+1}k^{\underline{2}}=\frac{1}{2}n(n+1)^2-\frac{1}{2}\left[\frac{1}{3}k^{\underline{3}}\right]_{k=1}^{n+2}=$$

$$=\frac{1}{2}n(n+1)^2-\frac{1}{6}(n+2)(n+1)n=\frac{1}{2}n(n+1)(n+1-\frac{1}{3}(n+2))=\frac{1}{2}n(n+1)(\frac{2}{3}n+\frac{1}{3}).$$

Suppose now that $x\neq 1$. Then

$$T_n+(n+1)^2x^{n+1}=\sum_{k=0}^{n}(k+1)^2x^{k+1}=x\sum_{k=0}^{n}(k^2+2k+1)x^k=$$

$$=xT_n+2x\sum_{k=0}^{n}kx^k+x\sum_{k=0}^{n}x^k=xT_n+2xs_n+x\frac{1-x^{n+1}}{1-x}$$

with $s_n=\sum_{k=0}^{n}kx^k$. It follows that

$$(1-x)T_n=2xs_n+x\frac{1-x^{n+1}}{1-x}-(n+1)^2x^{n+1};$$

in view of Example 6.3 the latter equals

$$2x\left[x\frac{1-x^{n+1}}{(1-x)^2}-(n+1)\frac{x^{n+1}}{1-x}\right]+x\frac{1-x^{n+1}}{1-x}-(n+1)^2x^{n+1}=$$

$$=\frac{x}{(1-x)^2}\left[-n^2x^{n+2}+(2n^2+2n-1)x^{n+1}-(n+1)^2x^n+x+1\right].$$

Hence, $T_n=\frac{x}{(1-x)^3}\left[-n^2x^{n+2}+(2n^2+2n-1)x^{n+1}-(n+1)^2x^n+x+1\right]$; for $|x|<1$, $\sum_{k=0}^{\infty}k^2x^k=\frac{x(x+1)}{(1-x)^3}$.

Problem 6.3. Let $k\in\mathbb{N}$ and $m,n\in\mathbb{Z}$ with $k\geq m$. Prove that

$$k^{\underline{m+n}}=k^{\underline{m}}(k-m)^{\underline{n}}.$$

Solution. If $k\geq m+n$, then

$$k^{\underline{m+n}}=\frac{k!}{(k-(n+m))!}=\frac{k!}{(k-m)!}\frac{(k-m)!}{((k-m)-n)!}=k^{\underline{m}}(k-m)^{\underline{n}},$$

because $k-m\geq n$; if instead $k<m+n$ then, since $k\geq m$, one has $n>0$ and hence $k^{\underline{m+n}}=0=(k-m)^{\underline{n}}$ from which $k^{\underline{m+n}}=k^{\underline{m}}(k-m)^{\underline{n}}$.

Problem 6.4. We see here how the operators θ and Δ act on a polynomial function. Let $P(X)$ be a polynomial. Prove that:

1. $\theta P(n)=P(n+1)$ and $\deg P(X+1)=\deg P(X)$;

2. $\Delta P(n) = Q(n)$ where $Q(X)$ is a polynomial with $\deg Q(X) = \max\{0, \deg P(X) - 1\}$.

Solution. 1. If $P(X) = a_d X^d + \cdots + a_1 X + a_0$ then

$$\deg P(X+1) = \deg a_d (X+1)^d = d = \deg P(X).$$

2. The claim is obvious if $P(X)$ is a constant. If $\deg P(X) \geq 1$, it is enough to prove the claim for $P(X) = X^d$, $d \in \mathbb{N}_{\geq 1}$. Now

$$\Delta n^d = (n+1)^d - n^d = \sum_{k=0}^{d} \binom{d}{k} n^k - n^d = \sum_{k=0}^{d-1} \binom{d}{k} n^k = Q(n),$$

where

$$Q(X) = \sum_{k=0}^{d-1} \binom{d}{k} X^k$$

is a polynomial of degree $d-1$.

Solution. One has

Problem 6.5. Find a discrete primitive of $k \longmapsto \dfrac{k}{(k+1)(k+2)(k+3)}$.

Solution. It is enough to notice that $k \longmapsto \dfrac{k}{(k+1)(k+2)(k+3)} = k k^{\underline{-3}}$. The summation by parts formula, with $F(k) = k$ and $g(k) = k^{\underline{-3}} = \Delta\left(-\dfrac{1}{2} k^{\underline{-2}}\right)$ shows that a discrete primitive of $k k^{\underline{-3}}$ is

$$-\frac{k}{2} k^{\underline{-2}} + \frac{1}{2} \sum_k k^{\underline{-2}} = -\frac{k}{2} k^{\underline{-2}} - \frac{1}{2} k^{\underline{-1}}.$$

Problem 6.6. Calculate

$$\sum_{0 \leq k < n} k^2 2^k.$$

Solution. Since $\Delta 2^k = 2^k$ one has

$$\sum_{0 \leq k < n} k^2 2^k = \sum_{0 \leq k < n} k^2 \, \Delta 2^k = \left[k^2 2^k\right]_{k=0}^{n} - \sum_{0 \leq k < n} \Delta k^2 \, \theta \, 2^k =$$

$$= n^2 2^n - \sum_{0 \leq k < n} (2k+1) 2^{k+1}.$$

Moreover, $\displaystyle\sum_{0 \leq k < n} (2k+1) 2^{k+1} = \left[(2k+1) 2^{k+1}\right]_{k=0}^{n} - \sum_{0 \leq k < n} \Delta (2k+1) 2^{k+2} =$

$$= (2n+1) 2^{n+1} - 2 - \left[2 \times 2^{k+2}\right]_{k=0}^{n} = (2n-3) 2^{n+1} + 6.$$

Therefore the desired sum is equal to $n^2 2^n - (2n-3)2^{n+1} - 6 = (n^2 - 4n + 6)2^n - 6$.

Problem 6.7. Calculate
$$\sum_{0 \le k < n} k^2 5^k.$$

Solution. Since $\Delta 5^k = 4 \times 5^k$ one has
$$\sum_{0 \le k < n} k^2 5^k = \sum_{0 \le k < n} k^2 \frac{1}{4} \Delta 5^k = \frac{1}{4} \left[k^2 5^k \right]_{k=0}^{n} - \frac{1}{4} \sum_{0 \le k < n} \Delta k^2 \, \theta \, 5^k =$$
$$= \frac{1}{4} \left(n^2 5^n - \sum_{0 \le k < n} (2k+1) 5^{k+1} \right).$$

One has $\sum_{0 \le k < n} (2k+1)5^{k+1} = \frac{1}{4} \left(\left[(2k+1)5^{k+1} \right]_{k=0}^{n} - \sum_{0 \le k < n} \Delta (2k+1) 5^{k+2} \right) =$
$$= \frac{1}{4} \left((2n+1)5^{n+1} - 5 - \frac{1}{4} \times \left[2 \times 5^{k+2} \right]_{k=0}^{n} \right) = \frac{1}{8} \left((4n-3)5^{n+1} + 15 \right).$$

Hence the desired sum is equal to
$$\frac{1}{4} \left(n^2 5^n - \frac{1}{8} \left((4n-3)5^{n+1} + 15 \right) \right) = \frac{1}{32} \left((8n^2 - 20n + 15)5^n - 15 \right).$$

Problem 6.8. Verify that $\Delta \log k! = \log(k+1)$, and then calculate
$$\sum_{0 \le k < n} k \log(k+1).$$

Solution. As to the verification, one has
$$\Delta \log k! = \log(k+1)! - \log k! = \log \frac{(k+1)!}{k!} = \log(k+1).$$

One then has
$$\begin{aligned} \sum_{0 \le k < n} k \log(k+1) &= \sum_{0 \le k < n} k \Delta \log k! \\ &= [k \log k!]_{k=1}^{n} - \sum_{0 \le k < n} (\Delta k) \, \theta (\log k!) \\ &= n \log n! - \log (2!3! \cdots n!). \end{aligned}$$

Solution.

Problem 6.9. Prove the following equality for all $n \ge 1$:
$$\sum_{k=0}^{n} (-1)^k \frac{1}{(k+1)^2} \binom{n}{k} = \frac{1}{n+1} H_{n+1}.$$

Solution. One has

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\frac{1}{(k+1)^2}\binom{n}{k} &= \sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\frac{n!}{(n-k)!(k+1)!}\\
&= \sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\frac{1}{n+1}\frac{(n+1)!}{((n+1)-(k+1))!(k+1)!}\\
&= \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n+1}{k+1}\\
&= \frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k}\binom{n+1}{k} = \frac{1}{n+1}H_{n+1},
\end{aligned}$$

where the last equality follows from Proposition 6.46.

Problem 6.10. We propose an alternative proof for Proposition 5.83. Let $n \geq 1$ and $k \in \mathbb{N}$. Consider the functions

$$m \mapsto m^n, \quad m \mapsto \binom{m+k}{n}.$$

1. For each $i \in \mathbb{N}$ calculate $\Delta^i m^n := (\underbrace{\Delta \circ \cdots \circ \Delta}_{i \text{ times}})m^n$. [Hint: use the representation formula for ordinary powers in terms of descending factorial powers stated in Proposition 6.34.]
2. Calculate $\Delta^i \binom{m+k}{n}$.
3. Use the Worpistzky Identity (Proposition 5.84) to conclude.

Solution. 1. We have seen in Proposition 6.34 that

$$m^n = \sum_{j=1}^{n}\left\{ {n \atop j} \right\} m^{\underline{j}} \qquad \forall m \in \mathbb{N}, n \in \mathbb{N}_{\geq 1}.$$

Applying the Δ operator to the function $m \mapsto m^n$ one finds that

$$\Delta m^n = \sum_{j=1}^{n}\left\{ {n \atop j} \right\} j m^{\underline{j-1}};$$

iterating the Δ operator i times yields

$$\Delta^i m^n = \sum_{j=1}^{n}\left\{ {n \atop j} \right\} \frac{j!}{(j-i)!} m^{\underline{j-i}}.$$

2. Applying the operator Δ to the function $m \mapsto \binom{m+k}{n}$ one obtains

$$\Delta\binom{m+k}{n} = \binom{m+1+k}{n} - \binom{m+k}{n} = \binom{m+k}{n-1};$$

iterating the operator Δ i times then yields

$$\Delta^i\binom{m+k}{n} = \binom{m+k}{n-i}.$$

3. The Worpitzky identity gives

$$m^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{m+k}{n};$$

applying the Δ operator to both sides (thought of as functions of the natural number variable m) one obtains

$$\sum_{j=1}^{n} \left\{ {n \atop j} \right\} \frac{j!}{(j-i)!} m^{\underline{j-i}} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{m+k}{n-i}$$

which for $m=0$ becomes

$$\left\{ {n \atop i} \right\} i! = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{k}{n-i} = \sum_{k\in\mathbb{N}} \left\langle {n \atop k} \right\rangle \binom{k}{n-i}.$$

Problem 6.11. For every $n \in \mathbb{N}$ and $s \in \mathbb{R}, s > 1$ let

$$\mathrm{H}_n^{(s)} = 1 + \frac{1}{2^s} + \cdots + \frac{1}{n^s}.$$

Prove that

$$\mathrm{H}_n^{(s)} - 1 \leq \int_1^n \frac{1}{x^s}\,dx \leq \mathrm{H}_n^{(s)} - \frac{1}{n^s};$$

deduce that

$$1 + \frac{1}{2^s} + \cdots + \frac{1}{n^s} \leq \frac{s}{s-1} \qquad \forall s > 1. \tag{6.11.a}$$

Solution. Let $k \in \{1, 2, ..., n-1\}$.
Since $\frac{1}{(k+1)^s} \leq \frac{1}{x^s}$ for each $x \in [k, k+1]$, by integration we get

$$\frac{1}{(k+1)^s} \leq \int_k^{k+1} \frac{1}{x^s}\,dx$$

thus

$$\mathrm{H}_n^{(s)} - 1 = \sum_{k=1}^{n-1} \frac{1}{(}k+1)^s \leq \int_1^n \frac{1}{x^s}\,dx.$$

In particular if $s > 1$ we get

$$\mathrm{H}_n^{(s)} \leq 1 + \int_1^n \frac{1}{x^s}\,dx = 1 + \frac{1}{s-1}(1 - n^{1-s}) \leq 1 + \frac{1}{s-1} = \frac{s}{s-1}.$$

Similarly, since $\frac{1}{x^s} \leq \frac{1}{k^m}$ for all $x \in [k, k+1]$ we get

$$\int_1^n \frac{1}{x^s}\,dx \leq \mathrm{H}_n^{(s)}.$$

Chapter 7
Problems and Solutions

Problem 7.1. Let $A(X)$ be a formal power series and $m \in \mathbb{N}$. Prove that for each $n \geq m \in \mathbb{N}$ one has $[X^n]\left(X^m A(X)\right) = [X^{n-m}]A(X)$.

Solution. Let $A(X) = \sum_{i=0}^{\infty} a_i X^i$; then

$$[X^n](X^m A(X)) = [X^n]\sum_{i=0}^{\infty} a_i X^{i+m} = \begin{cases} a_{n-m} = [X^{n-m}]A(X) & \text{if } n \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 7.2. Let $A(X)$ be a formal power series. Prove that $A'(X) = 0$ if and only if $A(X)$ is a constant.

Solution. Let $A(X) = \sum_{i=0}^{\infty} a_i X^i$; by definition $A'(X) = \sum_{i=1}^{\infty} i a_i X^{i-1}$. Then $A'(X) = 0$ if and only if for each $n \in \mathbb{N}$ one has

$$0 = [X^n]A'(X) = [X^n]\sum_{i=1}^{\infty} i a_i X^{i-1} = (n+1)a_{n+1},$$

that is, if and only if $a_{n+1} = 0$ for each $n \geq 0$. Therefore $A'(X) = 0$ if and only if $A(X) = a_0$.

Problem 7.3. Let $A(X)$ and $B(X)$ be two formal power series and $m, n \in \mathbb{N}$. Prove that

$$\left([X^m]A(X)\right)\left([X^n]B(X)\right) = [X^n]\left(\left([X^m]A(X)\right)B(X)\right).$$

Solution. Put $A(X) = \sum_{k=0}^{\infty} a_k X^k$ and $B(X) = \sum_{k=0}^{\infty} b_k X^k$. One then has

$$\left([X^m]A(X)\right)\left([X^n]B(X)\right) = a_m b_n;$$

and this gives the desired result, since

$$a_m b_n = [X^n] a_m B(X) = [X^n] \left(([X^m]A(X)) B(X) \right) .$$

Problem 7.4. Let $A(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ and $B(X) = \sum_{k=1}^{\infty} (-1)^{k-1} X^k$. Determine a closed form for $A(B(X))$ and calculate the coefficient of X^2.

Solution. One has $A(X) = e^X$ and

$$B(X) = -\left((-X) + (-X)^2 + \cdots\right) = -\left(\frac{1}{1-(-X)} - 1\right) = \frac{X}{1+X}.$$

Then $A(B(X)) = e^{\dfrac{X}{1+X}}$. Let $f(x) = e^{\dfrac{x}{1+x}}$; then

$$[X^2]A(B(X)) = \frac{f''(0)}{2!}.$$

Since $f'(x) = e^{\dfrac{x}{1+x}} \cdot \dfrac{1}{(1+x)^2}$ and

$$f''(x) = e^{\dfrac{x}{1+x}} \cdot \frac{1}{(1+x)^4} + e^{\dfrac{x}{1+x}} \cdot \frac{-2(1+x)}{(1+x)^4} = e^{\dfrac{x}{1+x}} \cdot \left(\frac{-1-2x}{(1+x)^4}\right)$$

one has $[X^2]A(B(X)) = \dfrac{1}{2} e^{\dfrac{0}{1+0}} \cdot \left(\dfrac{-1-2\cdot 0}{(1+0)^4}\right) = -\dfrac{1}{2}$.

Problem 7.5. Calculate a closed form for the formal power series

$$A(X) = \sum_{n=0}^{\infty} nX^n, \quad B(X) = \sum_{n=0}^{\infty} n(n+1)X^n.$$

Solution. Derivation of $\dfrac{1}{1-X} = \sum_{n=0}^{\infty} X^n$ yields

$$\frac{1}{(1-X)^2} = \left(\sum_{n=0}^{\infty} X^n\right)' = \left(\sum_{n=1}^{\infty} X^n\right)' = \sum_{n=1}^{\infty} nX^{n-1}.$$

Therefore one has

$$A(X) = \sum_{n=0}^{\infty} nX^n = X \sum_{n=1}^{\infty} nX^{n-1} = \frac{X}{(1-X)^2}.$$

Then, since $1 + 2 + \cdots + n = (n+1)n/2$, the series $B(X)$ is twice the OGF of the sequence of partial sums of $(n)_n$ and hence

$$B(X) = \sum_{n=0}^{\infty} n(n+1)X^n = \frac{2}{1-X}A(X) = \frac{2X}{(1-X)^3}.$$

Problem 7.6. Let $A(X) = \sum_{n=0}^{\infty} 2^n X^n$; decide whether or not $A(X)$ is invertible in $\mathbb{R}[[X]]$ and, if so, determine its inverse.

Solution. Since $[X^0]A(X) = 1 \neq 0$ the formal power series $A(X)$ is invertible. One then has

$$A(X) = \sum_{n=0}^{\infty} 2^n X^n = 1 + 2X\left(1 + 2X + 2^2X + \cdots\right) = 1 + 2XA(X),$$

and so $A(X)\cdot(1-2X) = 1$, that is, $A^{-1}(X) = 1-2X$.

Problem 7.7. Prove that the derivative of a formal power series has the same properties (as far as sums and products are concerned) as the derivative of a function: if $A(X)$ and $B(X)$ are two formal power series, then

$$(A(X)+B(X))' = A'(X)+B'(X) \text{ and}$$

$$(A(X)B(X))' = A'(X)B(X)+A(X)B'(X).$$

Solution. Let $A(X) = \sum_{i=0}^{\infty} a_i X^i$ and $B(X) = \sum_{i=0}^{\infty} b_i X^i$. Then for each $n \in \mathbb{N}$ one has

$$[X^n](A(X)+B(X))' = [X^n](\sum_{i=0}^{\infty}(a_i+b_i)X^i)' = (n+1)(a_{n+1}+b_{n+1})$$

and

$$[X^n](A'(X)+B'(X)) = [X^n](\sum_{i=1}^{\infty} ia_iX^{i-1} + \sum_{k=1}^{\infty} kb_kX^{k-1}) = (n+1)a_{n+1}+(n+1)b_{n+1}.$$

Similarly, for each $n \in \mathbb{N}$ one has

$$[X^n](A(X)B(X))' = [X^n](\sum_{i=0}^{\infty}(\sum_{j=0}^{i} a_jb_{i-j})X^i)' = (n+1)\sum_{j=0}^{n+1} a_jb_{n+1-j}$$

and

$$[X^n](A'(X)B(X)+A(X)B'(X)) =$$

$$= [X^n](\sum_{i=1}^{\infty} ia_iX^{i-1}\sum_{k=0}^{\infty} b_kX^k) + [X^n](\sum_{i=0}^{\infty} a_iX^i \sum_{k=1}^{\infty} kb_kX^{k-1}) =$$

$$= [X^n]\sum_{\ell=0}^{\infty}(\sum_{j=0}^{\ell}(j+1)a_{j+1}b_{\ell-j})X^\ell + [X^n]\sum_{\ell=0}^{\infty}(\sum_{j=0}^{\ell} a_j(\ell-j+1)b_{\ell-j+1})X^\ell =$$

$$= \sum_{j=0}^{n}(j+1)a_{j+1}b_{n-j} + \sum_{j=0}^{n} a_j(n-j+1)b_{n-j+1} =$$

$$= \sum_{j=0}^{n-1}(j+1)a_{j+1}b_{n-j} + (n+1)a_{n+1}b_0 + (n+1)a_0b_{n+1} + \sum_{j=0}^{n-1} a_{j+1}(n-j)b_{n-j} =$$

$$= (n+1)a_0b_{n+1} + \sum_{j=0}^{n-1} a_{j+1}b_{n-j}(j+1+n-j) + (n+1)a_{n+1}b_0 = (n+1)\sum_{j=0}^{n+1} a_jb_{n+1-j}.$$

Problem 7.8. Let $0 < m \in \mathbb{N}$ and $A(X)$ be a formal power series in $\mathbb{R}[[X]]$. Prove that if $A^m(X) = 1$, then $A(X)$ is constant, equal to 1 if m is odd, and to ± 1 if m is even.

Solution. Since $1 = [X^0]A^m(X) = ([X^0]A(X))^m$, the series $A(X) = \sum_{n=0}^{\infty} a_nX^n$ has a_0 satisfying $a_0^m = 1$. Hence, $a_0 = 1$ if m is odd, while $a_0 = \pm 1$ if m is even. Reasoning by contradiction, suppose that $k > 0$ is the smallest strictly positive natural number such that $a_k \neq 0$. Then $A(X) = a_0 + X^k(a_k + a_{k+1}X + \cdots)$. Hence, on putting $B(X) = a_k + a_{k+1}X + \cdots$ one has

$$0 = [X^k]A^m(X) = [X^k](a_0 + X^kB(X))^m =$$

$$= [X^k]\sum_{j=0}^{m}\binom{m}{j} a_0^j X^{k(m-j)}B^{m-j}(X) = \binom{m}{m-1} a_0^{m-1}a_k = \pm m a_k,$$

contradicting the hypothesis that $a_k \neq 0$.

Problem 7.9. Calculate a closed form for the formal power series

$$A(X) = \sum_{n=2}^{\infty}(n-1)X^n.$$

Solution. One has

$$A(X) = \sum_{n=2}^{\infty}(n-1)X^n = X^2\sum_{n=2}^{\infty}(n-1)X^{n-2} = X^2\sum_{n=1}^{\infty} nX^{n-1} =$$

$$= X^2\left(\sum_{n=0}^{\infty} X^n\right)' = X^2\left(\frac{1}{1-X}\right)' = \frac{X^2}{(1-X)^2}.$$

Solution.
Solution.

Problem 7.10. In view of the upcoming elections, the leader Kingnzi is in the process of choosing candidates. In this first phase it must be decided how many and which positions of the 10 available spots on the ballot should be awarded to various categories of potential candidates; only later will the names of the candidates be selected. The choice will be made respecting the following constraints regarding the four disjoint categories of potential candidates:

1. An even number of party functionaries;
2. An odd number of former council members;
3. At least 5 declaredly traditional conservatives;
4. At most one clearly progressive candidate.

How many possible choices are there in this preliminary phase?

Solution. The problem amounts to counting the number of 10-sequences in I_4 with occupancy (f,v,t,p) where $f \in 2\mathbb{N}$, $v \in 2\mathbb{N}+1$, $t \geq 5$ and $p \leq 1$. These correspond to

$$\left[\frac{X^{10}}{10!}\right](1+\frac{X^2}{2!}+\frac{X^4}{4!}+\cdots)(X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots)(\frac{X^5}{5!}+\frac{X^6}{6!}+\frac{X^7}{7!}+\cdots)(X+\frac{X^2}{2!}+\frac{X^3}{3!}+\cdots)=$$

$$=\left[X^{10}\right]10!(1+\frac{X^2}{2!}+\frac{X^4}{4!}+\cdots+\frac{X^{10}}{10!})(X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots+\frac{X^9}{9!})(\frac{X^5}{5!}+\frac{X^6}{6!}+\frac{X^7}{7!}+\cdots+\frac{X^{10}}{10!})(X+\frac{X^2}{2!}+\frac{X^3}{3!}+\cdots+\frac{X^{10}}{10!})$$

$$=\left[X^{10}\right](30240X^6+35280X^7+25920X^8+24330X^9+7972X^{10}+\cdots)=7972$$

possible choices.

Problem 7.11. How many 6 digit numbers are there having an even number of 7's, an odd number of 9's, and two 5's?

Solution. Here one must count sequences. Let EGF_i denote the characteristic EGF of how many times the digit i may be repeated. We have

$$\mathrm{EGF}_7 = 1+\frac{X^2}{2!}+\frac{X^4}{4!}+\cdots$$

$$\mathrm{EGF}_9 = X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots$$

$$\mathrm{EGF}_5 = \frac{X^2}{2!}$$

$$\mathrm{EGF}_{0,1,2,3,4,6,8} = 1+X+\frac{X^2}{2!}+\cdots,$$

and we must calculate the coefficient of $X^6/6!$ in the product of the EGF's.

$$\left[\frac{X^6}{6!}\right](1+\frac{X^2}{2!}+\frac{X^4}{4!}+\cdots)(X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots)(\frac{X^2}{2!})(1+X+\frac{X^2}{2!}+\cdots)^7=$$

$$=\left[X^6\right]6!(1+\frac{X^2}{2!}+\frac{X^4}{4!}+\cdots)(X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots)(\frac{X^2}{2!})(1+X+\frac{X^2}{2!}+\cdots)^7=22260.$$

If then we do not wish to count the 6-digit numbers that begin with 0, we must remove from these 22260 possibilities the numbers consisting of an initial zero followed by 5 digits with an even number of 7's, an odd number of 9's and two 5's. The number of such sequences is

$$\left[X^5\right] 5!(1+X+X^2/2+X^3/3!+X^4/4!+X^5/5!)^7(1+X^2/2!+X^4/4!)(X+X^3/3!+X^5/5!)(X^2/2!) = 1510,$$

and so in all one has $22260-1510 = 20750$ choices of the desired type.

Problem 7.12. Prove that for each $n \geq 0$ one has

$$\binom{-1/2}{n} = \left(-\frac{1}{4}\right)^n \binom{2n}{n}.$$

Solution.

$$\begin{aligned}
\binom{-1/2}{n} &= \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)}{n!} \\
&= \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!} \\
&= (-1)^n \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2}}{n!} \\
&= (-1)^n \frac{1\times 3\times 5\times\cdots\times(2n-1)}{n!2^n}.
\end{aligned}$$

Multiplying numerator and denominator by

$$2\times 4\times\cdots\times 2n = 2^n n!$$

one obtains

$$\binom{-1/2}{n} = (-1)^n \frac{(2n)!}{n!n!2^n\times 2^n} = \left(-\frac{1}{4}\right)^n \binom{2n}{n}.$$

Problem 7.13. Calculate the number of ways in which it is possible to distribute 25 identical liquorice sticks to Carl, Roberta, Joe, Steve, and Al if Carl wants at least 8 but not more than 12 of them, Roberta no more than 4, and Joe and Al at least 1.

Solution. Consider the sets $I_C = \{8,9,10,11,12\}$, $I_R = \{0,1,2,3,4\}$, $I_J = I_A = \{1,2,3,4,...\}$, $I_S = \mathbb{N}$. The characteristic OGF's of these sets are

$$C(X) = X^8+X^9+X^{10}+X^{11}+X^{12} = X^8\cdot(1+X+X^2+X^3+X^4)$$

$$R(X) = 1+X+X^2+X^3+X^4 \quad J(X) = A(X) = X+X^2+X^3+\cdots$$

$$S(X) = 1+X+X^2+X^3+\cdots.$$

The number of ways of distributing the 25 liquorice sticks is equal to the coefficient of X^{25} in the product of the characteristic OGF's.

$$[X^{25}]C(X)R(X)J(X)A(X)S(X) = 2000.$$

Problem 7.14. A certain pastry shop located in Calle de San Pantalon in Venice displays in its shop windows some "frittelle" with pastry cream (*crema*) filling, some with egg-nog (*zabaione*) filling, and others which are traditional Venetian style "frittelle" with no filling at all. In how many ways can one choose 15 frittelle if one wishes to have an even number ≥ 2 with pastry cream, an odd number ≥ 1 with egg-nog filling, and at least 3 traditional Venetian frittelle? (For the readers' benefit it may be of interest, though of no help in solving the problem, that "frittelle" [singular "frittella"] are a generalization of donuts without holes, but considerably better than their Anglo-Saxon counterpart in the view of many who have tried both.)

Solution. Consider the sets $I_C = \{2,4,6,8,...\}, I_Z = \{1,3,5,7,...\}, I_T = \{5,6,7,8,...\}$. The characteristic OGF's of these sets are

$$C(X) = X^2 + X^4 + X^6 + X^8 + \cdots = X^2 \cdot (1 + X^2 + X^4 + X^6 + \cdots)$$

$$Z(X) = X + X^3 + X^5 + X^7 + \cdots = X \cdot (1 + X^2 + X^4 + X^6 + \cdots)$$

$$T(X) = X^5 + X^6 + X^7 + X^8 + \cdots .$$

The number of ways in which it is possible to choose the 15 frittelle is equal to the coefficient of X^{15} in the product of the characteristic OGF's:

$$[X^{15}]C(X)Z(X)T(X) = 10.$$

Problem 7.15. Calculate the number of ways one can prepare a tray of 12 hors d'oeuvres of five different types with at most four of any given type, and assuming that there must be at least one of each type.

Solution. The problem amounts to calculating

$$[X^{12}](X + X^2 + X^3 + X^4)^5 = 155.$$

Problem 7.16. In how many ways is it possible to assign 20 mint gum drops and 10 licorice ones to 10 children so that each child receives exactly three gum drops.

Solution. Once the distribution of the licorice gum drops has been decided, there is only one way to distribute the mint gum drops in such a way that each child gets exactly 3 gum drops. Thus it is sufficient to calculate the number of ways in which it is possible to give out the licorice gum drops, and this amounts to calculating

$$[X^{10}](1 + X + X^2 + X^3)^{10} = 44\,803.$$

Problem 7.17. In a supermarket a lunatic named Pascal throws items at a man, choosing them at random from the supermarket shelves. In the end, before anyone is able to stop him, Pascal manages to throw 11 items. The witnesses present furnish incompatible descriptions of the items thrown at the man; in the end, however, all agree that the items thrown satisfied the following conditions:

1. An even number (≥ 0) of cans of tomato paste;

2. At least 2 bottles of olive oil;
3. Between 4 and 7 cans of beer;
4. A souvenir of the Cathedral of Milan[1].

Bearing in mind that all possible versions of the story coherent with the conditions given above were actually furnished by witnesses, at least how many people were present in the supermarket besides the man and Pascal?

Solution. We must count the number of collections of Tomato Paste, Olive Oil, Beer, and the Souvenir with occupancy sequence (m_P, m_O, m_B, m_S) where $m_P \geq 0$, $m_O \geq 2$, $4 \leq m_B \leq 7$, $m_S = 1$. The OGF's of the sets in which m_P, m_O, m_B, m_S vary are

$$\begin{aligned} P: &\qquad 1+X^2+X^4+\cdots \\ O: &\qquad X^2+X^3+X^4+\cdots \\ B: &\qquad X^4+X^5+X^6+X^7 \\ S: &\qquad X. \end{aligned}$$

Thus, the question amounts to calculating the coefficient of X^{11} in

$$(1+X^2+X^4+\cdots)(X^2+X^3+X^4+\cdots)(X^4+X^5+X^6+X^7)X.$$

The result is 8.

Problem 7.18. Using Theorem 7.114, compute the coefficients of the formal series

$$\frac{1+X}{1-X-X^2}.$$

Solution. Set

$$P(X) = X+1, \quad Q(X) = 1-X-X^2.$$

The roots of $Q(X)$ are

$$\alpha_1 = \frac{-1-\sqrt{5}}{2}, \quad \alpha_2 = \frac{-1+\sqrt{5}}{2}.$$

Therefore for each n one has

$$s_n : [X^n]\,\frac{1+X}{1-X-X^2} = -\sum_{i=1}^{2} \frac{P(\alpha_i)}{\alpha_i^{n+1} Q'(\alpha_i)}$$

and hence

[1] This exercise was inspired by a true story that took place in Milan on December 13, 2009.

$$s_n = -\frac{P\left(\dfrac{-1-\sqrt{5}}{2}\right)}{\left(\dfrac{-1-\sqrt{5}}{2}\right)^{n+1} Q'\left(\dfrac{-1-\sqrt{5}}{2}\right)} - \frac{P\left(\dfrac{-1+\sqrt{5}}{2}\right)}{\left(\dfrac{-1+\sqrt{5}}{2}\right)^{n+1} Q'\left(\dfrac{-1+\sqrt{5}}{2}\right)}.$$

It is

$$P\left(\frac{-1-\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}}{2}, \quad P\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2}.$$

Since $Q'(X) = -2X-1$, one obtains

$$Q'\left(\frac{-1-\sqrt{5}}{2}\right) = \sqrt{5}, \quad Q'\left(\frac{-1+\sqrt{5}}{2}\right) = -\sqrt{5}$$

and hence

$$s_n = (-1)^n \frac{\dfrac{1-\sqrt{5}}{2}}{\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1}\sqrt{5}} + \frac{\dfrac{1+\sqrt{5}}{2}}{\left(\dfrac{-1+\sqrt{5}}{2}\right)^{n+1}\sqrt{5}}.$$

Multiplying numerator and denominator one has

$$\frac{(1-\sqrt{5})/2}{\left((1+\sqrt{5})/2\right)^{n+1}\sqrt{5}} = \frac{(1-\sqrt{5})/2}{\left((1+\sqrt{5})/2\right)^{n+1}\sqrt{5}} \cdot \frac{\left(\sqrt{5}-1\right)^{n+1}}{\left(\sqrt{5}-1\right)^{n+1}} =$$

$$= -\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^{n+2} \quad \text{and}$$

$$\frac{(1+\sqrt{5})/2}{\left((\sqrt{5}-1)/2\right)^{n+1}\sqrt{5}} = \frac{(1+\sqrt{5})/2}{\left((\sqrt{5}-1)/2\right)^{n+1}\sqrt{5}} \cdot \frac{\left(\sqrt{5}+1\right)^{n+1}}{\left(\sqrt{5}+1\right)^{n+1}} =$$

$$= \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{n+2}.$$

Therefore

$$s_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right).$$

Chapter 8
Problems and Solutions

Problem 8.1. Let $f(x)$ be a closed form for the derivative $A'(X)$ of a formal power series $A(X)$ and let $A(0) = 0$. Then $g(x) = \int_0^x f(t)\,dt$ is a closed form for $A(X)$.

Solution. On putting $A(X) = \sum_{k=0}^{\infty} a_k X^k$, one has $A'(X) = \sum_{k=1}^{\infty} k a_k X^{k-1}$. Since $g(x)$ is a closed form for $A'(X)$, for each $k \geq 1$ one has $k a_k = \frac{g^{(k-1)}(0)}{(k-1)!}$ so that $a_k = \frac{g^{(k-1)}(0)}{k!}$. This gives the desired result because $0 = a_0$ and $f^{(k)}(0) = g^{(k-1)}(0)$ for each $k \geq 1$.

Problem 8.2. Calculate the following sums:

$$\sum_{k=1}^{149} k^5; \quad \sum_{k=1}^{299} k^7; \quad \sum_{k=1}^{9\,999} k^3.$$

Solution. We use the formula

$$\sum_{k=1}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \mathrm{B}_k\, n^{m+1-k},$$

seen in Example 8.52. One finds that

- For $n = 150$ and $m = 5$:

$$\begin{aligned}
\sum_{k=1}^{149} k^5 &= \frac{1}{6} \sum_{k=0}^{5} \binom{6}{k} \mathrm{B}_k\, 150^{6-k} \\
&= \frac{1}{6} \left(150^6 - \frac{1}{2} \times 6 \times 150^5 + \frac{1}{6} \times 15 \times 150^4 - \frac{1}{30} \times 15 \times 150^2 + \frac{1}{42} \right) \\
&= \frac{1}{6} \left(150^6 - 3 \times 150^5 + \frac{15}{6} \times 150^4 - \frac{1}{2} \times 150^2 + \frac{1}{42} \right) = 1\,860\,679\,685\,625.
\end{aligned}$$

- For $n = 300$ and $m = 7$:

$$\sum_{k=1}^{299} k^7 = \frac{1}{8}\sum_{k=0}^{8}\binom{8}{k}\mathrm{B}_k\,300^{8-k}$$
$$= \frac{1}{8}\left(300^8 - 4\times 300^7 + \frac{28}{6}\times 300^6 - \frac{7}{3}\times 300^4 + \frac{2}{3}\times 300^2 - \frac{1}{30}\right)$$
$$= 8092\,325\,247\,637\,507\,500.$$

- For $n = 10000$ and $m = 3$:

$$\sum_{k=1}^{9999} k^3 = \frac{1}{4}\sum_{k=0}^{4}\binom{4}{k}\mathrm{B}_k\,10000^{4-k}$$
$$= \frac{1}{4}\left(10000^4 - 2\times 10000^3 + 10000^2 - \frac{1}{30}\right)$$
$$= 2\,499\,500\,025\,000\,000.$$

Problem 8.3. Use generating formal series to give an alternative proof of the formula (2.57.a):

$$\sum_{k\in\mathbb{Z}}\binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n} \quad \forall m,n,r,s\in\mathbb{N}.$$

Solution. One recognizes that the sum under consideration is the $(m+n)$-th term of the convolution product of the sequence $\binom{r}{k}_k$ with $\binom{s}{k}_k$: by Proposition 7.21 this is the coefficient of X^{m+n} in the product of the corresponding OGF's, which are, respectively, $(1+X)^r$ and $(1+X)^s$. Now

$$(1+X)^r(1+X)^s = (1+X)^{r+s} = \sum_k \binom{r+s}{k}X^k$$

and $[X^{m+n}]\,(1+X)^{r+s} = \binom{r+s}{m+n}$, from which the identity (2.57.a) follows.

Problem 8.4. 1. Prove, from (8.52.a), that

$$\frac{1}{m^m}\sum_{1\le k<m} k^m = \frac{m}{m+1} + \sum_{i=1}^{m}\frac{\mathrm{B}_i}{i!}\left(1+O\left(\frac{1}{m}\right)\right) \quad m\to+\infty.$$

2. Deduce that $\lim_{m\to+\infty}\sum_{0\le k<m}\left(\frac{k}{m}\right)^m = \frac{1}{e-1}$.

Solution. It follows from (8.52.a), in the case $n = m$, that

$$\sum_{0\le k<m} k^m = \frac{1}{m+1}\sum_{i=0}^{m}\binom{m+1}{i}\mathrm{B}_i\,m^{m+1-i}.$$

Dividing both terms of the latter equality by m^m we get

$$\begin{aligned}\frac{1}{m^m}\sum_{0\le k<m} k^m &= \frac{1}{m+1}\sum_{i=0}^{m}\binom{m+1}{i}\mathrm{B}_i\, m^{1-i} \\ &= \frac{m}{m+1}+\sum_{i=1}^{m}\frac{\mathrm{B}_i}{i!}\frac{m(m-1)\cdots(m-i+2)}{m^{i-1}}\end{aligned}.$$

Now, $m(m-1)\cdots(m-i+2)$ is a polynomial function in m of degree $i-1$, and thus

$$m(m-1)\cdots(m-i+2) = m^{i-1}+O\left(m^{i-2}\right),$$

whence

$$\frac{1}{m^m}\sum_{0\le k<m} k^m = \frac{m}{m+1}+\sum_{i=1}^{m}\frac{\mathrm{B}_i}{i!}\left(1+O\left(\frac{1}{m}\right)\right).$$

By passing to the limit for $m\to+\infty$, by means of (8.56.b), we obtain

$$\lim_{m\to+\infty}\frac{1}{m^m}\sum_{0\le k<m} k^m = 1+\sum_{i=1}^{\infty}\frac{\mathrm{B}_i}{i!} = \sum_{i=0}^{\infty}\frac{\mathrm{B}_i}{i!} = \frac{1}{e-1}.$$

Chapter 9
Problems and Solutions

Problem 9.1. Find a recurrence relation for the number of possible distributions of n distinct objects on 5 shelves of a closet. What is the initial condition?

Solution. Let x_n be the number of such distributions. We proceed by first distributing the objects $1, 2, ..., n-1$: this we may do in x_{n-1} ways; we then have 5 possibilities for the last object. By the Multiplication Principle one thus has

$$x_n = 5x_{n-1} \quad n \geq 1.$$

The initial condition is $x_1 = 5$: there are 5 ways to distribute an object on 5 shelves.

Problem 9.2. Find a recurrence relation for the number of sequences of automobiles of 3 different types - Audi, Fiat and Mercedes - in a row of n parking spaces, bearing in mind that an Audi or a Mercedes occupies two parking spaces while a Fiat occupies only one parking space.

Solution. Let x_n be the number of such sequences, with $n \geq 2$. If there is an Audi or a Mercedes in the first position, then there are x_{n-2} ways to fill the other free parking spaces; if instead in the first parking place there is a Fiat then there remain x_{n-1} parking places to be filled. Summing the number of outcomes of the three alternatives thus gives one $x_n = x_{n-1} + 2x_{n-2}$ for $n \geq 2$.

Problem 9.3. Suppose that during every month from the second month on, every pair of rabbits generates a new couple (a male and a female) of rabbits. Find the recurrence relation that describes the number of couples of rabbits month after month (assuming that all the couples survive). If initially there is only a single couple of newly born rabbits, what is the number of pairs of rabbits after 5 months?

Solution. Let x_n be the number of pairs of rabbits after n months. They are in number just as many as the pairs present a month earlier, to which one adds the newly arrived born from the x_{n-2} couples of two months earlier, which are x_{n-2} in number. Therefore, the Fibonacci recurrence holds: $x_n = x_{n-1} + x_{n-2}$ for every $n \geq 2$. If then $x_1 = 1$ one has $x_2 = 1$, from which it follows that $x_3 = 2$, $x_4 = 4 + 1 = 5$, $x_5 = 5 + 2 = 7$.

Problem 9.4. Fix $k \in \mathbb{N}_{\geq 1}$. Find the recursive relation for the number of regions of the plane created by n lines if:

1. Exactly k of these lines are parallel;
2. Each of the $n-k$ non-parallel lines among themselves intersect all the other lines;
3. Three distinct lines never have a point in common.

Find the number of such regions if $n = 9, k = 3$.

Solution. Let x_n be the number of regions created in this fashion. We fix the k parallel lines among themselves and also another $(n-1)-k$ lines among those that are not parallel, leaving out just one of them: in this way we will have singled out x_{n-1} regions of the plane. Adjoining the last line we cut every region through which it passes into two regions, thus creating another n regions (every new intersection being on the boundary of two new regions, and the intersections totaling $n-1$): thus one has $x_n = x_{n-1} + n \quad \forall n > k$. The initial condition is $x_k = k+1$. In particular, if $k = 3$ and $n = 9$:

$$x_3 = 4, x_4 = 4+4 = 8, x_5 = 8+5 = 13, x_6 = 13+6 = 19,$$

$$x_7 = 19+7 = 26, x_8 = 26+8 = 34, x_9 = 34+9 = 43.$$

Problem 9.5. Find a recursive relation for the accumulation of the money deposited in a bank account after n years if the interest is a rate of 6% and in each year 50 euro are added to the bank account.

Solution. Let x_n be the sum of euros accumulated in the bank after n years, $n \geq 10$: one clearly has $x_{n+1} = 1.06x_n + 50$.

Problem 9.6. Find a recurrence relation to count the number of binary n-sequences with at least a pair of consecutive 0 digits.

Solution. Let x_n be the number of such n-sequences, with $n \geq 1$. The binary $(n+1)$-sequences with at least two consecutive 0's are of two types: those that finish with 1 and those that finish with 0. The first type are x_n in number, while the second type are themselves of two types: those whose n-th digit is 0 (and these number 2^{n-1}), and those whose n-th digit is 1 (and these number x_{n-1}): thus one has

$$x_{n+1} = x_n + x_{n-1} + 2^{n-1}, \qquad x_1 = 0, x_2 = 1.$$

Problem 9.7. Find a recurrence relation to count the number of Gilbreath permutations (see Definition 2.48) of a deck of n cards. [Hint: Think at who could be the last card in a Gilbreath permutation.]

Solution. Let x_n, $n \in \mathbb{N}_{\geq 1}$, the number of Gilbreath permutations of a deck of n cards. Clearly $x_1 = 1$. Let $(a_1, \ldots, a_n)$, $n > 1$, be a Gilbreath permutation of $(1, \ldots, n)$. Since both $\{a_1, \ldots, a_{n-1}\}$ and $\{a_1, \ldots, a_{n-1}, a_n\}$ are intervals of $\mathbb{N}$ contained in I_n, necessarily a_n is equal either to n or to 1. The number of Gilbreath permutations with $a_n = 1$ and the number of those with $a_n = n$ are both equal to the number x_{n-1} of Gilbreath permutations of a deck of $n-1$ cards. Therefore we get the recurrence relation

$$x_n = 2x_{n-1}, \quad \forall n \geq 2,$$

with initial datum $x_1 = 1$. The solution is $x_n = 2^{n-1}$ for all $n \in \mathbb{N}_{\geq 1}$.

Problem 9.8. If 500 euros are invested in a fund that gives 8% annual interest, find a formula for calculating the quantity of money accumulated after n years.

Solution. If x_n is the quantity of money (measured in euros) accumulated in n years for $n \geq 0$ one has that

$$x_{n+1} = 1.08x_n, \quad x_0 = 500.$$

Problem 9.9. Prove that the sequence $(a_n)_{n\geq 1}$ defined by setting

$$a_n = \begin{cases} 1 & \text{if } n = 2^m, \\ 2\ell+1 & \text{if } n = 2^m + \ell, 1 \leq \ell < 2^m \end{cases}$$

is a solution of the recurrence relation

$$x_n = \begin{cases} 2x_{n/2} - 1 & \text{if } n \geq 2 \text{ is even,} \\ 2x_{(n-1)/2} + 1 & \text{if } n \geq 3 \text{ is odd} \end{cases}$$

with initial datum $x_1 = 1$.

Solution. Every natural number $n \geq 1$ may be written in the form $n = 2^m + \ell$ with $0 \leq \ell < 2^m$ and $m \geq 0$. Moreover, since $a_1 = 1$, the sequence $(a_n)_n$ satisfies the given initial condition. Now let $n > 1$; one then has $n = 2^m + \ell$ with $0 \leq \ell < 2^m$ and $m \geq 1$. If $\ell = 0$ one has

$$a_{2^m} = 1 = 2 \times 1 - 1 = 2a_{2^{m-1}} - 1$$

and so the recurrence relation is satisfied. Now let $\ell \geq 1$.
If $\ell = 2k$ with $1 \leq \ell < 2^m$, then one has $1 \leq k < 2^{m-1}$ and hence since

$$a_{2^m+\ell} = 2\ell + 1 = 2(2k) + 1 = 2(2k+1) - 1 = 2a_{2^{m-1}+k} - 1,$$

the recurrence relation is indeed verified.
If $\ell = 2k+1$ with $1 \leq \ell < 2^m$, then one has $0 \leq k < 2^{m-1}$ and so since

$$a_{2^m+\ell} = 2\ell + 1 = 2(2k+1) + 1 = 2a_{2^{m-1}+k} + 1$$

the recurrence relation is satisfied.

Problem 9.10. Let $f : [a,b] \to [a,b]$ be continuous. Prove that f has at least a fixed point.

Solution. Set $g(x) = f(x) - x$. Our assumptions imply that

$$g(a) = f(a) - a \geq 0, \qquad g(b) = f(b) - b \leq 0:$$

the Intermediate Zero Theorem yields the existence of $\xi \in [a,b]$ satisfying $g(\xi) = 0$, or equivalently, $f(\xi) = \xi$.

Problem 9.11. Discuss the existence and the value of the limit of the sequence

$$x_{n+1} = \sin x_n + x_n, \quad x_0 = 7\pi/2,$$

whose first iterations are shown in Figure 9.1.

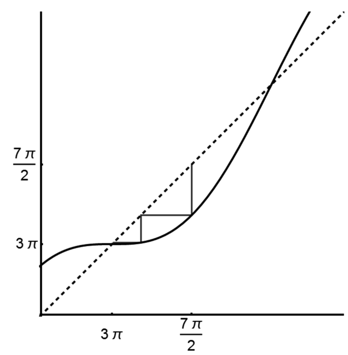

Fig. 9.1: The first terms of the sequence $x_{n+1} = x_n + \sin x_n$, $x_0 = 7\pi/2$.

Solution. The function $f(x) = x + \sin x$ is increasing since $f'(x) = 1 + \cos x \geq 0$ for every x. One then has

$$f(x_0) = f(7\pi/2) = \sin(7\pi/2) + 7\pi/2 = 7\pi/2 - 1 < 7\pi/2 = x_0:$$

and one then applies Point 2 of Proposition 9.24. The fixed points of f are the solutions of $\sin x = 0$, namely the $x = k\pi, k \in \mathbb{Z}$, the closest to $7\pi/2$ are 3π and 4π: the maximal fixed point of f that is strictly less than $7\pi/2$ is 3π, from which it follows that $x_n \downarrow 3\pi$.

Problem 9.12. Let $(a_n)_n$ be the sequence defined by

$$a_0 = \sqrt{2}, \qquad a_{n+1} = \sqrt{2 + a_n}\,.$$

1. Prove that $(a_n)_n$ is bounded above by 2.
2. Study the existence of the limit of the sequence.

Solution. Answer: 2, apply Proposition 9.24.

Problem 9.13. 1. Prove that $f(x) = \dfrac{x+1}{x+2}$ is monotonic on $]-2,+\infty[$ and that $f([0,+\infty[) \subseteq [0,+\infty[$.

2. Study the existence of a limit for the sequence

$$x_0 = 0, \qquad x_{n+1} = \frac{x_n+1}{x_n+2}\,.$$

Solution. f is increasing, since $f(x) = 1 - \dfrac{1}{x+2}$. Of course, if $x \geq 0$ then both $x+1 \geq 0, x+2 \geq 0$ so that $f(x) \geq 0$. The fixed points of f are the solutions to $\dfrac{x+1}{x+2} = x$, i.e., $x^2+x-1 = 0$, which are

$$\alpha := \frac{-1-\sqrt{5}}{2}, \quad \beta := \frac{-1+\sqrt{5}}{2}.$$

Now $f(x_0) = f(0) = \dfrac{1}{2} > 0$: it follows that the sequence $(x_n)_n$ is increasing. Since $\alpha < 0 = x_0 < \beta$ it follows from Proposition 9.24 that the sequence converges to β.

Problem 9.14. Discuss, depending on the value of $\lambda > 0$, the existence of a limit for the sequence defined by

$$x_0 = \lambda, \qquad x_{n+1} = \frac{1}{8}x_n^2 + \frac{1}{8}\,.$$

Solution. $x_n \geq 0$ for all n, and $f(x) = \dfrac{1}{8}x^2 + \dfrac{1}{8}$ is increasing on $[0,+\infty[$; on this interval f has a two fixed points $4 \pm \sqrt{15}$, and $f(x) > x$ on $[0, 4-\sqrt{15}[\cup]4+\sqrt{15},+\infty[$. The application of Proposition 9.24 yields the following cases:

1. $0 < \lambda < 4-\sqrt{15}$: $(x_n)_{n\in\mathbb{N}}$ is increasing since $x_{n+1} = f(x_n) > x_n$ for all n, thus $\lim\limits_{n\to\infty} x_n = 4-\sqrt{15}$;
2. $\lambda = 4-\sqrt{15}$: the sequence is constant, $x_n = 4-\sqrt{15}$ for all n: the sequence converges to $4-\sqrt{15}$;
3. $4-\sqrt{15} < \lambda < 4+\sqrt{15}$: $(x_n)_{n\in\mathbb{N}}$ is decreasing, $4-\sqrt{15} < x_n < 4+\sqrt{15}$ for all n. Thus $\lim\limits_{n\to\infty} x_n = 4-\sqrt{15}$;
4. $\lambda = 4+\sqrt{15}$: the sequence is constant, $x_n = 4+\sqrt{15}$ for all n;
5. $\lambda > 4+\sqrt{15}$: the sequence is increasing, thus $\lim\limits_{n\to\infty} x_n = +\infty$.

Problem 9.15. Let I be an interval of $\mathbb{R}$, and $f : I \to I$ be continuous and decreasing. Let $(x_n)_n$ be the solution of

$$x_0 \in I, \qquad x_{n+1} = f(x_n) \quad \forall n \in \mathbb{N}.$$

Assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $\ell \in \mathbb{R}$. Show that, for any k even, m odd and N larger than both, the point x_N is between x_k and x_m.

Solution. Assume that $x_0 \le x_2 = f^2(x_0)$. It follows from Proposition 9.27 that

$$x_0 \le x_2 \le \cdots \le x_k \le \cdots \le \ell \le \cdots \le x_m \le \cdots \le x_3 \le x_1,$$

so that if N is even we have $x_k \le x_N \le \ell \le x_m$, whereas if N is odd we have $x_k \le \ell \le x_N \le x_m$.

Problem 9.16. Let $[a,b]$ be a closed and bounded interval of $\mathbb{R}$, and $f:[a,b] \to [a,b]$ be continuous and decreasing. Let $(x_n)_n$ be the solution to

$$x_0 \in I, \qquad x_{n+1} = f(x_n) \quad \forall n \in \mathbb{N}.$$

Assume that f has no points of minimum period 2. Show that the sequence $(x_n)_n$ does converge.

Solution. It follows from Proposition 9.27 that the sequences $(x_{2n})_n$ and $(x_{2n+1})_n$ converge, respectively, to some $\alpha \in [a,b]$ and to some $\beta \in [a,b]$. We have

$$\alpha = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} f(x_{2n+1}) = f\left(\lim_{n\to\infty} x_{2n+1}\right) = f(\beta), \quad \text{and}$$

$$\beta = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}) = f\left(\lim_{n\to\infty} x_{2n}\right) = f(\alpha);$$

therefore α and β are periodic of period 2. The assumption implies that $\beta = \alpha$ and the whole sequence converges to α.

Problem 9.17. Let $f: I \to I$ be a function from an interval I into itself. Suppose that p is periodic of period $N \ge 1$. Prove that then the minimum period of p divides N.

Solution. If $N = mq + r$ with $q \in \mathbb{N}$ and $0 < r < m$ one has

$$p = f^N(p) = f^r(f^{mq}(p)) = f^r(p),$$

so that p has period $r < m$, which contradicts the minimality of m. But then $r = 0$, and m divides N.

Problem 9.18. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous, with a periodic point of minimum period 20. Prove that f has a point of minimum period 48.

Solution. In fact $48 = 2^4 \cdot 3$ and $20 = 2^2 \cdot 5$. In the Sarkovskii ordering one has $2^2 \cdot 5 \triangleright 2^4 \cdot 3$: in virtue of Theorem 9.35 f has a point of minimum period 48.

Problem 9.19. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous, with a periodic point of minimum period 40. Does this necessarily imply that f has a point of minimum period 30?

Solution. No. Indeed $30 = 2 \times 15 \triangleright 2^3 \times 5 = 40$: it follows from Remark 9.37 that there may exist a continuous function $f: \mathbb{R} \to \mathbb{R}$ with a point of minimum period 40 but no point of minimum period 30.

Problem 9.20. Let $f : [a,b] \to \mathbb{R}$ be strictly decreasing (as in Figure 9.2). Show that f has at most a single fixed point.

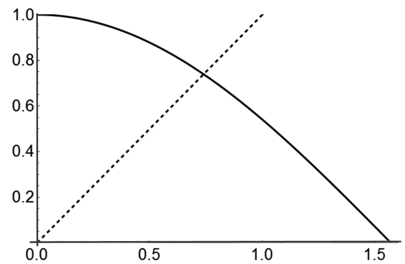

Fig. 9.2: A strictly decreasing function has at most a unique fixed point.

Solution. Indeed, the function $g(x) = f(x) - x$ is strictly decreasing: it assumes the value 0 in at most a single point.

Problem 9.21. In this exercise we give an example of a function that has a point of minimum period 5 but does not have points of minimum period 3. Let $f : [1,5] \to [1,5]$ be such that

$$f(1) = 3,\ f(3) = 4,\ f(4) = 2,\ f(2) = 5,\ f(5) = 1,$$

and be affine on each interval $[1,2], [2,3], [3,4], [4,5]$ (see Figure 9.3).

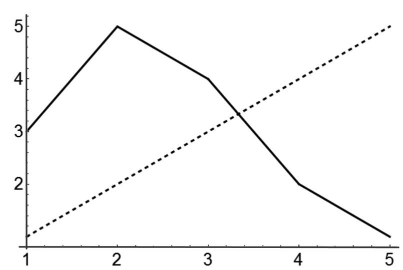

Fig. 9.3: The function f defined in Problem 9.21.

1. Prove that f has a point of minimum period 5.
2. Determine $f^3([1,2]), f^3([2,3]), f^3([4,5])$: deduce that f does not have points of minimum period 3 in the intervals $[1,2], [2,3], [4,5]$.
3. Show that f^3 is strictly decreasing on the interval $[3,4]$, and deduce that f^3 has a unique fixed point in $[3,4]$, and that the uniquely determined point is indeed a fixed point of f.
4. Deduce that f does not have points of minimum period 3.

Solution. Clearly 1 is a point of minimum period 5. One has that

$$f([1,2]) = [3,5],\, f([3,5]) = [1,4],\, f([1,4]) = [2,5].$$

Therefore $f^3([1,2]) = [2,5]$ and $f^3([1,2]) \cap [1,2] = \{2\}$, which has minimum period 5: one deduces that f does not have points of period 3 in the interval $[1,2]$. One concludes in analogous fashion by reasoning with the intervals $[2,3]$ and $[4,5]$. At this point the restrictions of f

$$f : [3,4] \to [2,4],\, f : [2,4] \to [2,5],\, f : [2,5] \to [1,5]$$

are strictly decreasing: from this it follows that the composition f^3 is strictly decreasing on the interval $[3,4]$. Since $f^3([3,4]) = [1,5] \supseteq [3,4]$ one has that f^3 has a fixed point on the interval $[3,4]$ (Lemma 9.39), necessarily unique in view of Exercise 9.20. Now f has a fixed point on the same interval (indeed $f([3,4]) = [3,4]$), and this is the fixed point for f^3; but then the unique fixed point of f^3 does not have minimum period 3 for f.

Chapter 10
Problems and Solutions

Homogeneous recurrences

Problem 10.1. Solve the following recurrence relations:

(a) $x_n = 3x_{n-1} + 4x_{n-2}$, $n \geq 2$, with initial data $x_0 = x_1 = 1$;
(b) $x_n = x_{n-2}$, $n \geq 2$, with initial data $x_0 = x_1 = 1$;
(c) $x_n = 2x_{n-1} - x_{n-2}$, $n \geq 2$, with initial data $x_0 = x_1 = 2$;
(d) $x_n = 3x_{n-1} - 3x_{n-2} + x_{n-3}$, $n \geq 3$, with initial data $x_0 = x_1 = 1$, $x_2 = 2$.

Solution.

a) $x_n = \dfrac{1}{5}\left(3(-1)^n + 2^{1+2n}\right)$, $n \in \mathbb{N}$.
b) $x_n = 1$, $n \in \mathbb{N}$.
c) $x_n = 2$, $n \in \mathbb{N}$.
d) $x_n = \dfrac{1}{2}\left(2 - n + n^2\right)$, $n \in \mathbb{N}$.

Problem 10.2. Find and solve a recurrence relation to compute the number of possible ways of filling a row of n places in a parking using blue cars, red cars and trucks, taking into account that the trucks take up two spaces, whilst the cars will occupy one.

Solution. For $n = 1$ the place can be filled in just two ways: either with a red car or a blue car. For $n = 2$ there are 5 ways: (blue, blue), (red, red), (blue, red), (red, blue) or (truck). For $n \geq 3$ we consider two cases, depending on the item in the last position. If a car, either red or blue, is in the last position, there are x_{n-1} ways to fill the other $n-1$ places; otherwise if the last one is a truck, there are x_{n-2} ways to fill the other free $n-2$ places. We are thus led to the following recurrence relation: $x_n = 2x_{n-1} + x_{n-2}$. With the initial conditions $x_1 = 2$ e $x_2 = 5$ its solution is $x_n = \dfrac{\sqrt{2}\left(1-\sqrt{2}\right)^n + 4\left(1+\sqrt{2}\right)^n + 3\sqrt{2}\left(1+\sqrt{2}\right)^n}{4\left(1+\sqrt{2}\right)}$, $n \in \mathbb{N}$.

Problem 10.3. A multinational pharmaceutical company decides to double the increase in the price of its flagship product every year. Find and solve the recurrence relation for the price p_n of the product in the year n, supposing that $p_0 = 1, p_1 = 4$.

Solution. The assumptions imply $p_{n+1} - p_n = 2(p_n - p_{n-1})$ so that

$$p_{n+1} = 3p_n - 2p_{n-1}, \quad p_0 = 1,\, p_1 = 4,$$

whose solution is $p_n = -2 + 3 \times 2^n$.

Problem 10.4. Assume the recurrence relation $x_n = c_1 x_{n-1} + c_2 x_{n-2}$ $(n \geq 2)$ has general solution $x_n = A\,13^n + B\,26^n$ with $A, B \in \mathbb{R}$: determine the constants c_1, c_2.

Solution. Since $(13^n)_n$ is a solution then

$$13^n = x_n = c_1 x_{n-1} + c_2 x_{n-2} = c_1\,13^{n-1} + c_2\,13^{n-2} \quad \forall n \geq 2,$$

whence

$$13^2 = 13\,c_1 + c_2.$$

Analogously, $(26^n)_n$ is a solution and thus

$$26^2 = 26\,c_1 + c_2.$$

The linear system yields $c_1 = 13 \times 3 = 39$ e $c_2 = 13^2 \times 2$.

Problem 10.5. Find the real and complex solutions to the recurrence

$$x_{n+2} - 6x_{n+1} + 9x_n = 0, \quad n \geq 0.$$

Solution. The characteristic polynomial of the recurrence

$$x_{n+2} - 6x_{n+1} + 9x_n = 0, \quad n \geq 0,$$

is $X^2 - 6X + 9$; it has 3 as root with multiplicity 2. The sequences $(3^n)_n$ and $(n3^n)_n$ are the basis-solutions of the recurrence. The general real (resp. complex) solution is then

$$x_n = A_1 3^n + A_2 n 3^n, \quad n \in \mathbb{N},$$

with the variation of A_1, A_2 among the real (resp. complex) numbers.

Problem 10.6. Compute the general real solution of the recurrence relation

$$x_{n+2} + 4x_{n+1} + 16x_n = 0, \quad n \geq 0.$$

Solution. The characteristic polynomial of the recurrence is

$$X^2 + 4X + 16 = (X+2)^2 + 12 = (X + 2 + 2i\sqrt{3})(X + 2 - 2i\sqrt{3}),$$

whose roots are $-2 \pm 2i\sqrt{3}$. The modulus of such roots is $\sqrt{2^2 + 12} = 4$ and one has

$$-2 \pm 2i\sqrt{3} = 4\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) = 4\left(\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}\right).$$

By Theorem 10.12, the sequences

$$\left(4^n \cos\left(\frac{2\pi n}{3}\right)\right)_n, \left(4^n \sin\left(\frac{2\pi n}{3}\right)\right)_n$$

are the real basis-solutions of the recurrence and thus the general real solution is

$$x_n = A_1 4^n \cos\left(\frac{2\pi n}{3}\right) + A_2 4^n \sin\left(\frac{2\pi n}{3}\right), \quad n \geq 0,$$

with the variation of A_1, A_2 in $\mathbb{R}$.

Problem 10.7. Compute the general real solution of the recurrence relation

$$x_n = x_{n-1} + 8x_{n-2} - 12x_{n-3}, \quad n \geq 3.$$

Solution. The characteristic polynomial of the recurrence is $X^3 - X^2 - 8X + 12 = (X-2)^2(X+3)$, whose roots are $\lambda_1 = 2$ with multiplicity 2 and $\lambda_2 = -3$. The sequences $(2^n)_n$, $(n2^n)_n$ and $((-3)^n)_n$ are the basis-solutions of the recurrence. The general real (resp. complex) solution of the recurrence is then

$$x_n = A_1 2^n + A_2 n 2^n + A_3(-3)^n, \quad n \in \mathbb{N},$$

with the variation of A_1, A_2 and A_3 among the real (resp. complex) numbers.

Problem 10.8. Determine the real solutions of the recurrence

$$x_n - 9x_{n-2} = 0, \quad n \geq 2,$$

with initial data:

1. $x_0 = 6, x_1 = 12$;
2. $x_3 = 324, x_4 = 486$;
3. $x_0 = 6, x_2 = 54$;
4. $x_0 = 6, x_2 = 10$.

Solution. The characteristic polynomial $X^2 - 9$ of the recurrence has roots 3 and -3. The sequences $(3^n)_n$ and $((-3)^n)_n$ are the basis-solutions of the recurrence. The general real solutions of the recurrence is then

$$x_n = A_1 3^n + A_2(-3)^n, \quad n \in \mathbb{N},$$

with the variation of the real constant A_1, A_2.

1. Any linear recurrence relation with assigned sequence of initial data has one and only one solution. In this case, the sequence of initial data is $(x_0 = 6, x_1 = 12)$.

Solving the linear system $\begin{cases} x_0 = A_1 + A_2 \quad\ = 6 \\ x_1 = 3A_1 - 3A_2 = 12 \end{cases}$ in the unknown A_1 and A_2 one obtains $A_1 = 5, A_2 = 1$. The desired solution is thus

$$x_n = 5(3^n) + (-3)^n \quad n \geq 0.$$

2. In this case we have not assigned a sequence of initial data, but anyway two consecutive values. Solving the linear system $\begin{cases} x_3 = 27A_1 - 27A_2 = 324 \\ x_4 = 81A_1 + 81A_2 = 486 \end{cases}$ in the unknown A_1 and A_2 one obtains $A_1 = 9, A_2 = -3$. Thus the required solution is

$$x_n = 9(3^n) - 3(-3)^n \quad n \geq 0.$$

3. Also in this case we have not assigned a sequence of initial data. By $x_0 = 6$ and $x_2 = 54$ one obtains the linear system $\begin{cases} x_0 = A_1 + A_2 \quad\ = 6 \\ x_2 = 9A_1 + 9A_2 = 54 \end{cases}$, which is equivalent to the single equation $A_1 + A_2 = 6$. Thus, in this case the solution is not unique: each sequence of the type $(A3^n + (6-A)(-3)^n)_n$, with the variation of $A \in \mathbb{R}$, satisfies the wished conditions.
4. Again, in this case we have not assigned a sequence of initial data. The two conditions imply $A_1 + A_2 = 6$ and $9A_1 + 9A_2 = 10$; the latter is a linear system with no solutions. Therefore there are no solutions of the recurrence which satisfies the required conditions.

Problem 10.9. Solve the recurrence relation

$$x_n = 3x_{n-1} + 4x_{n-2} - 12x_{n-3}, \quad n \geq 3,$$

with initial data $x_0 = 2, x_1 = 5, x_2 = 13$.

Solution. The characteristic polynomial is $X^3 - 3X^2 - 4X + 12$. By Proposition 10.9, one has to look for its integer roots among the divisors of 12:

$$\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6\}.$$

Computing the value of the polynomial in these points, one gets that the three roots of the characteristic polynomial are $2, -2, 3$: thus the general real solution is

$$x_n = A_1 2^n + A_2(-2)^n + A_3 3^n, \quad n \in \mathbb{N},$$

with the variation of the parameters A_1, A_2 and A_3 in $\mathbb{R}$. The initial data imposes

$$\begin{cases} A_1 + A_2 + A_3 &= 2 \\ 2A_1 - 2A_2 + 3A_3 &= 5 \\ 4A_1 + 4A_2 + 9A_3 &= 13 \end{cases}$$

Solving this linear system one obtains $A_1 = 1, A_2 = 0, A_3 = 1$: thus the wished solution is $x_n = 2^n + 3^n$, $n \geq 0$.

Problem 10.10. Solve the homogeneous linear recurrence of order 3

$$x_{n+3} = 6x_{n+2} - 12x_{n+1} + 8x_n, \quad n \geq 0,$$

with initial data $x_0 = 1$, $x_1 = 0$, $x_2 = 4$.

Solution. The characteristic polynomial is

$$P_{\text{char}}(X) = X^3 - 6X^2 + 12X - 8 = (X-2)^3.$$

The general solution of the recurrence is

$$x_n = 2^n(A_1 + A_2 n + A_3 n^2), \quad n \in \mathbb{N},$$

with the variation of A_1, A_2 and A_3 in $\mathbb{R}$. Imposing the initial data one obtains the linear system:

$$\begin{aligned} A_1 + 0 + 0 &= 1 \\ 2A_1 + 2A_2 + 2A_3 &= 0 \\ 4A_1 + 8A_2 + 16A_3 &= 4. \end{aligned}$$

The first two equalities give $A_3 = -1 - A_2$. Substituting in the third equality one obtains $A_2 = -2$ and hence $A_3 = 1$. Thus, the solution is

$$x_n = (1 - 2n + n^2)2^n = (n-1)^2 2^n, \quad n \geq 0.$$

Particular solutions

Problem 10.11. Determine a particular solution of the recurrence $x_n = cx_{n-1} + h_n$ with $c \in \mathbb{R}$ and:

a) $h_n = 1$;
b) $h_n = n$;
c) $h_n = n^2$;
d) $h_n = q^n$ ($q \in \mathbb{R}$).

Solution. The given sequences $(h_n)_n$ are all of the form $P(n)q^n$ for some polynomial $P(X)$ and $q \in \mathbb{R}$. The form of the particular solutions depends whether the root c of the characteristic polynomial coincides with q.

a) If $c \neq 1$ there is a solution of the form $x_n = A \in \mathbb{R}$, which solves actually the recurrence once $A = cA + 1$, i.e., $A = 1/(1-c)$. If $c = 1$ there is a solution of the form $x_n = An$, and A is given by $An = A(n-1) + 1$, that is $A = 1$.
b) If $c \neq 1$ there is a solution of the form $x_n = An + B$, to find A, B it is enough to solve $An + B = c(A(n-1) + B) + n$, or equivalently $An + B = (cA+1)n + (B - Ac)$ which yields $A(1-c) = 1$, thus $A = 1/(1-c)$ and $(B-A)c = B$; whence $B = c/(1-c)^2$. If $c = 1$ there is a solution of the form $x_n = n(An+B) = An^2 + Bn$, where A, B are given by

$$An^2 + Bn = A(n-1)^2 + B(n-1) + n \quad n \geq 1:$$

for $n = 1$ one finds $A + B = 1$, whereas for $n = 2$ the above equality yields $4A + 2B = A + B + 2$ whence $3A + B = 2$: it follows that $A = B = 1/2$.

c) If $c \neq 1$ there is a solution of the form $x_n = An^2 + Bn + C$; we solve

$$An^2 + Bn + C = c(A(n-1)^2 + B(n-1) + C) + n^2 \quad n \geq 1:$$

for $n = 1$ we get $A + B + C = cC + 1$, for $n = 2$ we have $4A + 2B + C = c(A + B + C) + 4$; the coefficients of n^2 being the same in the above equality we obtain $A = cA + 1$, $A = 1/(1-c)$. At this stage A, B are obtained by solving

$$(2-c)B + (1-c)C = cA - 4A + 4 = -3c/(1-c), \quad B + (1-c)C = 1 - A = -c/(1-c),$$

so that

$$B = -\frac{2c}{(-1+c)^2}, \quad C = \frac{-c-c^2}{(-1+c)^3}.$$

If $c = 1$ there is a solution of the form $x_n = n(An^2 + Bn + C) = An^3 + Bn^2 + Cn$, where A, B, C are given by

$$An^3 + Bn^2 + Cn = A(n-1)^3 + B(n-1)^2 + C(n-1) + n^2 \quad n \geq 1:$$

for $n = 1$ we get $A + B + C = 1$, for $n = 2$ it turns out that $8A + 4B + 2C = A + B + C + 4$; the coincidence of the coefficients of n^2 in the above equality implies that $B = -3A + B + 1$, whence $A = 1/3$. Therefore

$$B + C = 1 - A = 2/3, \quad 3B + C = 4 - 7A = 5/3$$

so that, finally, $B = 1/2, C = 1/6$.

d) If $c \neq q$ there is solution of the form $x_n = \alpha q^n$: one easily finds $\alpha = \dfrac{q}{q-c}$ so that $\left(\dfrac{q^{n+1}}{q-c}\right)_n$ is a solution. If $c = q$ there is a solution of the form $\alpha n q^n$: again one easily finds $\alpha = 1$ so that $(nq^n)_n$ is a particular solution.

Problem 10.12. Assume the characteristic polynomial of a given recurrence relation is $(X-1)^2(X-2)(X-3)^2$; determine the general solution of the induced homogeneous recurrence. Determine the type of a particular solution of the given recurrence if its non homogeneous part h_n is defined by one of the following:

(a) $h_n = 4n^3 + 5n$;
(b) $h_n = 4^n$;
(c) $h_n = 3^n$.

Solution. The characteristic polynomial is $P_{\text{char}}(X) = X^5 - 10X^4 + 38X^3 - 68X^2 + 57X - 18$: the recurrence relation is thus given by

$$x_n - 10x_{n-1} + 38x_{n-3} - 68x_{n-4} + 57x_{n-5} - 18x_{n-6} = 0.$$

a) Since 1 is a root of multiplicity 2 of $P(X)$, the recurrence admits a particular solution of the form $x_n = n^2 Q(n)$ where $Q(X)$ is a polynomial of degree less or equal than 3.
b) Since 4 is not a root of $P(X)$, there is a solution of the form $x_n = A\,4^n$, with $A \in \mathbb{R}$.
c) 3 is a root of multiplicity 2 of $P(X)$, the recurrence is of the form $x_n = An^2 3^n$, with $A \in \mathbb{R}$.

General solutions

Problem 10.13. Let $(h_n)_n$ be a sequence and c a constant. Determine the solution of $x_n = cx_{n-1} + h_n$ $(n \geq 1)$ with initial datum $x_0 = 1$.

Solution. The general solution of the associated homogeneous recurrence is $x_n = Ac^n$, $A \in \mathbb{R}$. The solution $(y_n)_n$ of the given recurrence with initial data equal to 0 is

$$y_1 = h_1,\, y_2 = ch_1 + h_2,\, y_3 = c(ch_1 + h_2) + h_3 = c^2 h_1 + ch_2 + h_3,\, \ldots;$$

it is easy to show, inductively, that

$$y_n = c^{n-1} h_1 + c^{n-2} h_2 + \cdots + ch_{n-1} + h_n \quad n \geq 1.$$

By the Superposition Principle 10.3 the solutions to $x_n = cx_{n-1} + h_n$ $(n \geq 1)$ are given by

$$x_n = Ac^n + y_n :$$

notice that $x_0 = 1$ if and only if $A + y_0 = 1$, i.e., $A = 1$: the solution to our recurrence with the given initial datum is

$$x_n = \begin{cases} 1 & \text{if } n = 0, \\ c^n + c^{n-1} h_1 + c^{n-2} h_2 + \cdots + ch_{n-1} + h_n & \text{if } n \geq 1. \end{cases}$$

Problem 10.14. Solve the following recurrence relations:

(a) $x_n = x_{n-1} + 3(n-1)$, $x_0 = 1$;
(b) $x_n = x_{n-1} + n(n-1)$, $x_0 = 3$;
(c) $x_n = x_{n-1} + 3n^2$, $x_0 = 10$.

Solution.

a) $x_n = \left(2 - 3n + 3n^2\right)/2$, $n \in \mathbb{N}$.
b) $x_n = (9 - n + n^3)/3$, $n \in \mathbb{N}$.
c) $x_n = (20 + n + 3n^2 + 2n^3)/2$, $n \in \mathbb{N}$.

Problem 10.15. Determine $(x_n)_{n\in\mathbb{N}}$ knowing that $x_0 = 3$ and $\dfrac{x_n + x_{n-1}}{2} = 2n + 5$ for each $n \geq 1$.

Solution. The recurrence is $x_n = -x_{n-1} + (4n + 10)$, its general solution is $x_n = 2\,(3 - 3(-1)^n + n) + A\,(-1)^n$ with $A \in \mathbb{R}$; taking into account the initial datum $x_0 = 3$ we get $x_n = 6 - 3(-1)^n + 2n$, $n \in \mathbb{N}$.

Problem 10.16. Find the general solution of the recurrence

$$x_n = 5x_{n-1} - 6x_{n-2} + 6(4^n).$$

Solution. The characteristic polynomial is $X^2 - 5X + 6 = (X-3)(X-2)$. Therefore the general solution of the homogeneous part is the family of sequences $(A2^n + B3^n)_n$ with the variation of the parameters A, B. Since 4 is not a root of the characteristic polynomial, the recurrence has a solution of the form $(\alpha 4^n)_n$; substituting in the recurrence one obtains $\alpha 4^n = 5\alpha 4^{n-1} - 6\alpha 4^{n-2} + 6(4^n)$ and hence $\alpha = 48$. Thus, the general solution is $x_n = A2^n + B3^n + 48(4^n)$, $n \in \mathbb{N}$.

Problem 10.17. Find the general solution of the recurrence

$$x_n = 2x_{n-1} + 2^n + n.$$

Solution. The characteristic polynomial is $X - 2$: therefore the general solution of the homogeneous part is $A2^n$ with the variation of the constant A. We determine separately particular solutions of the recurrences $x_n = 2x_{n-1} + 2^n$ and $x_n = 2x_{n-1} + n$. Since 2 is a root of the characteristic polynomial, the first recurrence has a solution of the form $(\alpha n 2^n)_n$; substituting in the recurrence one obtains $\alpha n 2^n = \alpha(n-1)2^n + 2^n$, or equivalently $\alpha n = \alpha(n-1) + 1$, and hence $\alpha = 1$. Since 1 is not a root of the characteristic polynomial, the second recurrence has a solution of the form $(\alpha_0 + \alpha_1 n)_n$; substituting in the recurrence one obtains $\alpha_0 + \alpha_1 n = 2(\alpha_0 + \alpha_1(n-1)) + n$, or equivalently

$$\alpha_0 + \alpha_1 n = 2\alpha_0 - 2\alpha_1 + (2\alpha_1 + 1)n,$$

and hence $\alpha_1 = -1$, $\alpha_0 = -2$. Thus, the general solution of the recurrence is the family of sequences $(A2^n + n2^n - 2 - n)_n$ with the variation of A among the real numbers.

Problem 10.18. Solve the following recurrence relations:

(a) $x_n = 3x_{n-1} - 2$, $x_0 = 0$;
(b) $x_n = 2x_{n-1} + (-1)^n$, $x_0 = 2$;
(c) $x_n = 2x_{n-1} + n$, $x_0 = 1$;
(d) $x_n = 2x_{n-1} + 2n^2$, $x_0 = 3$.

Solution.

a) $x_n = 1 - 3^n$, $n \in \mathbb{N}$.
b) $x_n = \dfrac{1}{3}\left((-1)^n + 5 \times 2^n\right)$, $n \in \mathbb{N}$.
c) $x_n = -2 + 3 \times 2^n - n$, $n \in \mathbb{N}$.
d) $x_n = -12 + 15 \times 2^n - 8n - 2n^2$, $n \in \mathbb{N}$.

Problem 10.19. Solve the recurrence relation $x_n = 3x_{n-1} - 2x_{n-2} + 3$, with initial data $x_0 = x_1 = 1$.

Solution. $x_n = -2 + 3 \times 2^n - 3n$, $n \in \mathbb{N}$.

Problem 10.20. Find and solve a recurrence relation for the profit of a company if the growth rate of the profit in the n-th year is 10×2^n euros more than the growth rate of the previous year and in the first year the profit is of 20 euros, while in the second year the profit is 1 020 euros.

Solution. If x_n is the profit at year n we have $x_n - x_{n-1} = x_{n-1} - x_{n-2} + 10 \times 2^n$ for $n \geq 2$. We thus obtain

$$x_n = 2x_{n-1} - x_{n-2} + 10 \times 2^n, \quad n \geq 2,$$

with the initial conditions $x_0 = 20$ euros, $x_1 = 1\,020$ euros, whose solution is given by $x_n = 20\left(-1 + 2^{1+n} + 48n\right)$ euros, $n \in \mathbb{N}$.

Problem 10.21. Find the general solution of the recurrence relation

$$x_n - 5x_{n-1} + 6x_{n-2} = 2 + 3n.$$

Solution. $x_n = \dfrac{1}{4}(25 + 6n) + A\,2^n + B\,3^n$, $A, B \in \mathbb{R}$, $n \in \mathbb{N}$.

Problem 10.22. Solve the following recurrence relation with initial datum $y_0 = 1$:

$$y_n^2 = 2y_{n-1}^2 + 1 \quad n \geq 1.$$

(Hint: set $x_n = y_n^2$).

Solution. If we set $x_n = y_n^2$ we get $x_n = 2x_{n-1} + 1$ whence $x_n = -1 + A\,2^n$, $A \in \mathbb{R}$. Now $x_0 = y_0^2 = 1$ if and only if $A = 2$ whence $x_n = -1 + 2^{n+1}$ and $y_n = \pm\sqrt{-1 + 2^{n+1}}$, $n \in \mathbb{N}$. Since $y_0 = 1$, the solution is $y_n = \sqrt{-1 + 2^{n+1}}$, $n \in \mathbb{N}$.

Problem 10.23. Determine the general solution of the recurrence

$$x_n = 4x_{n-1} - 4x_{n-2} + 2^n, \quad n \geq 2.$$

Solution. $x_n = 2^{-1+n}\left(-n + n^2\right) + A\,2^n + B\,n2^n$, $A, B \in \mathbb{R}$, $n \in \mathbb{N}$.

Problem 10.24. Solve the recurrence $x_n = x_{n-1} + 12n^2$, with initial datum $x_0 = 5$.

Solution. $x_n = 5 + 2n + 6n^2 + 4n^3$, $n \in \mathbb{N}$.

Problem 10.25. Determine the general solution of the recurrence

$$x_n = 3x_{n-1} - 4n + 3 \times 2^n \quad n \geq 1.$$

Find the solution with initial datum $x_1 = 8$.

Solution. The general solution is $x_n = -3\left(-1 + 2^{1-n} - 3^n\right) + 2n + A\,3^n$, $A \in \mathbb{R}$, $n \in \mathbb{N}$. The solution with initial datum $x_1 = 8$ is $x_n = 3 - 3 \times 2^{1-n} + 5 \times 3^n + 2n$, $n \in \mathbb{N}$.

Problem 10.26. Find the real solution of the recurrence

$$x_n = -x_{n-2} + 1, \quad x_0 = 0, x_1 = 0.$$

Solution. The roots of the characteristic polynomial X^2+1 are the conjugate complex numbers $i = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ and $-i = 1\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$: the family of sequences

$$\left(A\cos\left(n\frac{\pi}{2}\right) + B\sin\left(n\frac{\pi}{2}\right)\right)_n,$$

with the variation of A and B in $\mathbb{R}$, is the general real solution of the homogeneous part. The non homogeneous term is $1 = 1^n$. Since 1 is not a root of the characteristic polynomial, a constant sequence $(\alpha)_n$ is a particular solution of the recurrence: substituting in the recurrence one obtains $\alpha = -\alpha + 1$ and hence $\alpha = 1/2$. Thus, the general solution of the recurrence is $x_n = A\cos\left(n\frac{\pi}{2}\right) + B\sin\left(n\frac{\pi}{2}\right) + 1/2$, with the variation of A and B in $\mathbb{R}$. Imposing the initial data $x_0 = 0$ and $x_1 = 1$, one has

$$\begin{cases} A + 1/2 = 0, \\ B + 1/2 = 1. \end{cases}$$

Thus, the desired solution is

$$x_n = \frac{1}{2}\left(1 - \cos\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right)\right) = \begin{cases} \dfrac{1-(-1)^k}{2} & \text{if } n = 2k, \\ \dfrac{1+(-1)^k}{2} & \text{if } n = 2k+1. \end{cases}$$

Problem 10.27. Determine the solution of the following recurrences:

1. $x_{n+1} = 5x_n + 2^n\cos\left(n\frac{\pi}{3}\right), \quad x_0 = 0;$
2. $y_{n+1} = 5y_n + 2^n\sin\left(n\frac{\pi}{3}\right), \quad y_0 = 0.$

Solution. We first look for the solution $(z_n)_n$ in $\mathbb{C}$ to

$$z_{n+1} = 5z_n + 2^n\left(\cos\left(n\frac{\pi}{3}\right) + i\sin\left(n\frac{\pi}{3}\right)\right) \qquad z_0 = 0;$$

the sequences $(x_n)_n$ and $(y_n)_n$ that we are looking for are, respectively, the real and imaginary parts of $(z_n)_n$. The solutions of $z_{n+1} = 5z_n$ are $z_n = A\,5^n$, with $A \in \mathbb{C}$. Since

$$q := 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$$

is not a root of the characteristic polynomial $X-5$, the recurrence admits a particular solution of the form $w_n = \alpha\, q^n$, for some $\alpha \in \mathbb{C}$. To find α it is enough to write that

$$\alpha\, q^{n+1} = 5\alpha\, q^n + q^n,$$

from which

$$\alpha = \frac{1}{q-5} = \frac{1}{-4+i\sqrt{3}} = -\frac{4+i\sqrt{3}}{19}.$$

We thus get

$$\begin{aligned}
w_n &= -\frac{4+i\sqrt{3}}{19}\, 2^n \left(\cos\left(n\frac{\pi}{3}\right) + i\sin\left(n\frac{\pi}{3}\right)\right) \\
&= -\frac{2^n}{19}\left(\left(4\cos\left(n\frac{\pi}{3}\right) - \sqrt{3}\sin\left(n\frac{\pi}{3}\right)\right) + i\left(\sqrt{3}\cos\left(n\frac{\pi}{3}\right) + 4\sin\left(n\frac{\pi}{3}\right)\right)\right), n \in \mathbb{N}.
\end{aligned}$$

The general solution of

$$z_{n+1} = 5z_n + q^n$$

is therefore

$$z_n = A\ 5^n + w_n, \quad A \in \mathbb{C}, n \in \mathbb{N}:$$

we have $z_0 = 0$ if and only if $A + w_0 = 0$, thus for

$$A = -w_0 = \frac{1}{19}\left(4 + i\sqrt{3}\right).$$

Therefore

$$\begin{aligned}
z_n &= \frac{5^n}{19}\left(4+i\sqrt{3}\right) + w_n \\
&= \frac{5^n}{19}\left(4+i\sqrt{3}\right) - \frac{2^n}{19}\left(\left(4\cos\left(n\frac{\pi}{3}\right) - \sqrt{3}\sin\left(n\frac{\pi}{3}\right)\right) + i\left(\sqrt{3}\cos\left(n\frac{\pi}{3}\right) + 4\sin\left(n\frac{\pi}{3}\right)\right)\right), n \in \mathbb{N},
\end{aligned}$$

whose real and imaginary parts are, respectively,[1]

$$x_n = 4\ \frac{5^n}{19} - \frac{2^n}{19}\left(4\cos\left(n\frac{\pi}{3}\right) - \sqrt{3}\sin\left(n\frac{\pi}{3}\right)\right), n \in \mathbb{N};$$

$$y_n = \sqrt{3}\ \frac{5^n}{19} - \frac{2^n}{19}\left(\sqrt{3}\cos\left(n\frac{\pi}{3}\right) + 4\sin\left(n\frac{\pi}{3}\right)\right), n \in \mathbb{N}.$$

Problem 10.28. Determine the general solution of the recurrence:

$$x_{n+1} = x_n + (1+n)2^n \sin\left(n\frac{\pi}{3}\right), \quad x_0 = 1.$$

Solution. We first look for a solution in $\mathbb{C}$ of the recurrence

$$z_{n+1} = z_n + (1+n)2^n\left(\cos\left(n\frac{\pi}{3}\right) + i\sin\left(n\frac{\pi}{3}\right)\right) \qquad z_0 = 1;$$

the sequence $(x_n)_n$ that we are looking is just the real part of $(z_n)_n$. The solutions to $z_{n+1} = z_n$ are $z_n = C$, with $C \in \mathbb{C}$, $n \in \mathbb{N}$. Since

[1] It is interesting to note that our excellent CAS yields a very complicated solution for both x_n and y_n; what a satisfaction to be able to solve the recurrence by hands!

$$q := 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$$

is not a root of the characteristic polynomial $X - 1$ of the recurrence, there is a particular solution of the form $w_n = (\alpha + \beta n)\, q^n$, con $\beta \in \mathbb{C}$. In order to find α, β it is enough to impose that

$$(\alpha + \beta(n+1))\, q^{n+1} = (\alpha + \beta n)\, q^n + (1+n)q^n$$

or, equivalently,

$$(\alpha + \beta)q + \beta q n = (\alpha + 1) + (\beta + 1)n \quad \forall n \in \mathbb{N},$$

whence

$$\beta = 1/(q-1), \quad \alpha = -1/(q-1)^2.$$

We thus obtain

$$w_n = \left(-\frac{1}{(q-1)^2} + \frac{1}{q-1}n\right) q^n,\, n \in \mathbb{N}.$$

The general solution of the recurrence

$$z_{n+1} = z_n + (n+1)q^n$$

is

$$z_n = C + w_n,\, n \in \mathbb{N}.$$

We have $z_0 = 1$ if and only if $C + w_0 = 1$, i.e.,

$$C = 1 - w_0 = 1 + \frac{1}{(q-1)^2},$$

whence

$$\begin{aligned} z_n &= 1 + \frac{1}{(q-1)^2} + \left(-\frac{1}{(q-1)^2} + \frac{1}{q-1}n\right) q^n \\ &= \frac{2}{3} + \left(\frac{1}{3} - n\frac{i}{\sqrt{3}}\right) 2^n \left(\cos\left(n\frac{\pi}{3}\right) + i\sin\left(n\frac{\pi}{3}\right)\right),\, n \in \mathbb{N}, \end{aligned}$$

whose real part, our solution, is given by

$$x_n = \frac{2}{3} + 2^n \left(\frac{1}{3}\cos\left(n\frac{\pi}{3}\right) + \frac{n}{\sqrt{3}}\sin\left(n\frac{\pi}{3}\right)\right),\, n \in \mathbb{N}.$$

Problem 10.29. Determine the solution of the recurrence

$$x_{n+2} - 4\sqrt{3}x_{n+1} + 16x_n = 32 \times 4^n \cos\left(n\frac{\pi}{6}\right),$$

with initial data $x_0 = 0, x_1 = 0$.

Solution. Set
$$q = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{3} + 2i,$$
we look for the solutions $(z_n)_n$ in $\mathbb{C}$ of

$$z_{n+2} - 4\sqrt{3}z_{n+1} + 16z_n = 32q^n, \tag{10.29.a}$$

with initial conditions $z_0 = 0, z_1 = 0$: the solution $(x_n)_n$ to the given recurrence is the sequence of the real parts of z_n. The characteristic polynomial of the recurrence is
$$X^2 - 4\sqrt{3}X + 16;$$

since

$$X^2 - 4\sqrt{3}X + 16 = (X - 2\sqrt{3})^2 + 16 - (2\sqrt{3})^2 + 16 = (X - 2\sqrt{3})^2 + 4,$$

then its roots are q and its conjugate $\overline{q} = 2\sqrt{3} \pm 2i$, and

$$2\sqrt{3} \pm 2i = 4\left(\frac{\sqrt{3}}{2} \pm i\frac{1}{2}\right) = 4\left(\cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}\right).$$

The complex solutions to the homogenous recurrence $z_{n+2} - 4\sqrt{3}z_{n+1} + 16z_n = 0$ are thus

$$Aq^n + B\overline{q}^n = A\, 4^n\left(\cos\left(n\frac{\pi}{6}\right) + i\sin\left(n\frac{\pi}{6}\right)\right) + B\, 4^n\left(\cos\left(n\frac{\pi}{6}\right) - i\sin\left(n\frac{\pi}{6}\right)\right),$$

with $A, B \in \mathbb{C}$. Since q is a root of multiplicity 1 of the characteristic polynomial, the non homogeneous recurrence has a solution of the form $w_n = \alpha n q^n$, for some $\alpha \in \mathbb{C}$. To find α we use the fact that $(w_n)_n$ is a solution to the recurrence (10.29.a). Now

$$\begin{aligned}
w_{n+2} - 4\sqrt{3}w_{n+1} + 16w_n &= \alpha\left((n+2)q^{n+2} - 4\sqrt{3}(n+1)q^{n+1} + 16nq^n\right) \\
&= \alpha q^n\left((n+2)q^2 - 4\sqrt{3}(n+1)q + 16n\right) \\
&= \alpha q^n\left(n(q^2 - 4\sqrt{3}q + 16) + (2q^2 - 4\sqrt{3}q)\right) \\
&= \alpha q^n(2q^2 - 4\sqrt{3}q) = -8(1 - i\sqrt{3})\alpha q^n,
\end{aligned}$$

since $q^2 - 4\sqrt{3}q + 16 = 0$ and $2q^2 - 4\sqrt{3}q = -8 + 8i\sqrt{3}$. Therefore

$$w_{n+2} - 4\sqrt{3}w_{n+1} + 16w_n = 32 \times q^n$$

if and only if $(1 - i\sqrt{3})\alpha = -4$, i.e., $\alpha = -(1 + i\sqrt{3})$, so that $w_n = -(1 + i\sqrt{3})nq^n$, $n \in \mathbb{N}$. As a consequence, the general solution of (10.29.a) is

$$z_n = Aq^n + B\overline{q}^n - (1 + i\sqrt{3})nq^n, \quad A, B \in \mathbb{C}, n \in \mathbb{N}.$$

The initial conditions $z_0 = 0$, $z_1 = 0$ yield

$$A + B = 0, \quad Aq + B\overline{q} = q(1 + i\sqrt{3})$$

so $B = -A$ and $A(q - \overline{q}) = q(1 + i\sqrt{3}) = 8i$ whence $A = 2$, $B = -2$. We finally get

$$\begin{aligned}
z_n &= 2(q^n - \overline{q}^n) - (1 + i\sqrt{3})nq^n \\
&= 4i \times 4^n \sin\left(n\frac{\pi}{6}\right) - n(1 + i\sqrt{3})4^n \left(\cos\left(n\frac{\pi}{6}\right) + i\sin\left(n\frac{\pi}{6}\right)\right) \\
&= 4i \times 4^n \sin\left(n\frac{\pi}{6}\right) - n4^n \left(\left(\cos\left(n\frac{\pi}{6}\right) - \sqrt{3}\sin\left(n\frac{\pi}{6}\right)\right) + i\left(\sqrt{3}\cos\left(n\frac{\pi}{6}\right) + \sin\left(n\frac{\pi}{6}\right)\right)\right),
\end{aligned}$$

with $n \in \mathbb{N}$. Its real part, the solution to our problem, is

$$x_n = n4^n \left(\sqrt{3}\sin\left(n\frac{\pi}{6}\right) - \cos\left(n\frac{\pi}{6}\right)\right), \quad n \in \mathbb{N}.$$

Divide and conquer

Problem 10.30. Determine the solution of the following “divide and conquer” recurrences:

(a) $y_n = 2y_{n/2} + 5$, $n = 2^k$, $k \geq 1$ with initial datum $y_1 = 1$;
(b) $y_n = 2y_{n/4} + n$, $n = 4^k$, $k \geq 1$ with initial datum $y_1 = 3$;
(c) $y_n = 2y_{n/2} + 2n$, $n = 2^k$, $k \geq 1$ with initial datum $y_1 = 5$;
(d) $y_n = y_{n/3} + 4$, $n = 3^k$, $k \geq 1$ with initial datum $y_1 = 7$.

Solution. (a) $y_n = 6n - 5, n = 2^k$, $k \geq 0$, (b) $y_n = \sqrt{n} + 2n$, $n = 4^k$, $k \geq 0$, (c) $y_n = 5n + 2n\log_2 n$, $n = 2^k$, $k \geq 0$, (d) $y_n = 7 + 4\log_3 n$, $n = 3^k$, $k \geq 0$.

Problem 10.31. Determine the solution of the following “divide and conquer” recurrences:

(a) $y_n = y_{n/3} + 2$, $n = 4 \times 3^k$, $k \geq 1$ with initial datum $y_4 = 5$;
(b) $y_n = 2y_{n/3} + 2$, $n = 3^k$, $k \geq 1$ with initial datum $y_1 = 1$;
(c) $y_n = y_{n/3} + 2n$, $n = 3^k$, $k \geq 1$ with initial datum $y_1 = 5$;
(d) $y_n = 2y_{n/3} + 2n$, $n = 2 \times 3^k$, $k \geq 1$ with initial datum $y_2 = -1$.

Solution. (a) $y_n = 5 + 2\log_3(n/4)$, $n = 4 \times 3^k$, $k \geq 0$, (b) $y_n = 3n^{\log_3 2} - 2$, $n = 3^k$, $k \geq 0$, (c) $y_n = 2 + 3n$, $n = 3^k$, $k \geq 0$, (d) $y_n = -13(n/2)^{\log_3 2} + 6n$, $n = 2 \times 3^k$, $k \geq 0$.

Problem 10.32. Describe an approach of type “divide and conquer” to determine the maximum among the elements of a set of n numbers. Write a recurrence relation for the number of necessary comparisons and solve it.

Solution. If $n = 2^k$ with $k \geq 0$ let y_n be the number of comparison that are needed to find the maximum of a set with n numbers. If $n = 1$ clearly $y_1 = 0$; if $n = 2^k$ with $k \geq 1$, we subdivide the set into two sets of $n/2$ numbers, we take the maximum of each of the sets, and the biggest between the two: this requires $2y_{n/2} + 1$ comparisons. We are thus led to the following divide and conquer recurrence

$$y_n = 2y_{n/2} + 1, \quad n = 2^k, k \geq 1, y_1 = 0.$$

Set $x_k = y_{2^k}$: the induced recurrence is

$$x_k = 2x_{k-1} + 1, k \geq 0, x_0 = 0,$$

which gives $x_k = -1 + 2^k$. Thus,

$$y_n = -1 + 2^{\log_2 n} = -1 + n, \quad n = 2^k, k \geq 0.$$

Problem 10.33. Describe an approach of type "divide and conquer" to determine the first and second largest among the elements of a set of n numbers. Write a recurrence relation for the number of necessary comparisons and solve it.

Solution. If $n = 2^k$ with $k \geq 1$ let y_n be the number of comparisons that are needed in order to find the first and the second largest element on a set of n numbers. If $n = 2$ then $y_2 = 1$: one comparison is enough; if $n = 2^k$ with $k \geq 2$, we split the set into two sets of $n/2$ elements. On each of the two sets we choose the first and second largest ($2y_{n/2}$ comparisons): the greatest element will be the largest between the two biggest elements (1 comparison); the second largest will be the largest among the three remaining numbers (3 comparisons): at the end we are done with $2y_{n/2} + 4$ comparisons. We are thus led to

$$y_n = 2y_{n/2} + 4, \quad n = 2^k, k \geq 2, y_2 = 1.$$

Set $x_k = y_{2^k}$: we obtain the induced recurrence

$$x_k = 2x_{k-1} + 4, k \geq 1, x_1 = 1,$$

whose solution is $x_k = -4 + 5 \times 2^{k-1}$. Thus

$$y_n = -4 + 5 \times 2^{\log_2 n - 1} = -4 + \frac{5n}{2}, \quad n = 2^k, k \geq 1.$$

Problem 10.34. Determine an estimate for the order of magnitude (i.e., $y_n = O(\cdots)$) of the solution $(y_n)_n$ of the "divide and conquer" recurrence

$$y_n = 3y_{n/2} + 4n^2 \quad n = 2^k, \quad k \geq 1,$$

with initial datum $y_1 = 5$ in two different ways: 1) Use Theorem 10.53 2) Solve the recurrence.

Solution. It is a recurrence of the form $y_n = \lambda y_{n/b} + cn^\rho$ with $\lambda = 3, b = 2, \rho = 2$: since $\lambda = 3 < 2^2 = b^\rho$, Theorem 10.53 shows that $y_n = O(n^2)$ for $n = 2^k$, $k \to +\infty$. Alternatively, set $n = 2^k$ and $x_k = y_{2^k}$: the induced recurrence is

$$x_k = 3x_{k-1} + 4 \times 4^k, \quad x_0 = 5,$$

whose solution is $x_k = -11 \times 3^k + 4^{2+k}$. We thus get

$$\begin{aligned} y_n &= -11 \times 3^{\log_2 n} + 4^{2+\log_2 n} \\ &= -11 n^{\frac{\log 3}{\log 2}} + 16n^2 = O(n^2), \quad n = 2^k, k \to +\infty. \end{aligned}$$

Problem 10.35. Determine an estimate for the order of magnitude (i.e., $y_n = O(\cdots)$) of the solution $(y_n)_n$ of the "divide and conquer" recurrence

$$y_n = 2y_{n/2} + 2, \quad n = 2^k, \quad k \geq 1,$$

with initial datum $y_1 = 2$ in two different ways: 1) Use Theorem 10.53 2) Solve the recurrence.

Solution. The recurrence is of the form $y_n = \lambda y_{n/b} + cn^\rho$ with $\lambda = 2, b = 2, \rho = 0$: since $\lambda = 2 > 2^0 = b^\rho$, Theorem 10.53 shows that $y_n = O(n^{\log_2 2}) = O(n)$ for $n = 2^k$, $k \to +\infty$. Alternatively, set $n = 2^k$ and $x_k = y_{2^k}$: the induced recurrence is

$$x_k = 2x_{k-1} + 2, \quad x_0 = 2,$$

whose solution is $x_k = 2\left(-1 + 2^{1+k}\right)$. Thus

$$y_n = 2\left(-1 + 2^{1+\log_2 n}\right) = -2 + 4n = O(n), n = 2^k, k \to +\infty.$$

Problem 10.36. Determine an estimate for the order of magnitude (i.e., $y_n = O(\cdots)$) of the solution $(y_n)_n$ of the "divide and conquer" recurrence

$$y_n = 2y_{n/2} + 2\log_2 n, \quad n = 2^k,\ k \geq 1,$$

with initial datum $y_1 = 1$ in two different ways: 1) Use Theorem 10.53 2) Solve the recurrence.

Solution. The recurrence is of the form $y_n = \lambda y_{n/b} + c\log_b n$ with $\lambda = 2, b = 2$: since $\lambda = 2 > 1$, Theorem 10.53 shows that $y_n = O(n^{\log_2 2}) = O(n)$ for $n = 2^k$, $k \to +\infty$. Alternatively, set $n = 2^k$ and $x_k = y_{2^k}$: the induced recurrence is

$$x_k = 2x_{k-1} + 2k, \quad x_0 = 1,$$

whose solution is $x_k = -4 + 5\ 2^k - 2k$. We thus obtain

$$y_n = -4 + 5n - 2\log_2 n = O(n), \quad n = 2^k, k \to +\infty.$$

Problem 10.37. Determine an estimate for the order of magnitude (i.e., $y_n = O(\cdots)$) of the solution $(y_n)_n$ of the "divide and conquer" recurrence

$$y_n = \frac{1}{2} y_{n/4} + 3\log_4 n, \quad n = 4^k,\ k \geq 1,$$

with initial datum $y_1 = 0$ in two different ways: 1) Use Theorem 10.53 2) Solve the recurrence.

Solution. The recurrence is of the form $y_n = \lambda y_{n/b} + c\log_b n$ with $\lambda = \frac{1}{2}, b = 4$: since $\lambda = \frac{1}{2} < 1$, it follows from Theorem 10.53 that $y_n = O(\log n)$ for $n = 4^k$, $k \to +\infty$. Alternatively set $n = 4^k$ and $x_k = y_{4^k}$: this yields to the induced recurrence given by

$$x_k = \frac{1}{2} x_{k-1} + 3k, \quad x_0 = 0,$$

whose solution is $x_k = 3(2^{1-k} - 2 + 2k)$. We thus obtain

$$y_n = 3(2^{1-\log_4 n} - 2 + 2\log_4 n) = \frac{6}{\sqrt{n}} - 6 + 6\log_4 n = O(\log n),\ n = 4^k,\ k \to +\infty.$$

Problem 10.38. Determine an estimate for the order of magnitude (i.e., $y_n = O(\cdots)$) of the solution $(y_n)_n$ of the "divide and conquer" recurrence

$$y_n = y_{n/2} + 2\log_2 n, \quad n = 2^k,\ k \geq 1,$$

with initial datum $y_1 = 1$ in two different ways: 1) Use Theorem 10.53 2) Solve the recurrence.

Solution. The recurrence is of the form $y_n = \lambda y_{n/b} + c\log_b n$ with $\lambda = 1, b = 2$: since $\lambda = 1$, by Theorem 10.53 we have $y_n = O(\log^2 n)$ for $n = 2^k$, $k \to +\infty$. Alternatively, we set $n = 2^k$ and $x_k = y_{2^k}$: the induced linear recurrence is

$$x_k = x_{k-1} + 2k, \quad x_0 = 1,$$

whose solution is $x_k = 1 + k + k^2$. This leads to

$$y_n = 1 + \log_2 n + \log_2 n^2 = O(\log^2 n), \quad n = 2^k,\ k \to +\infty.$$

Recurrences and generating formal series

Problem 10.39. Solve the following recurrence relation using the generating formal series:

$$x_n = 2x_{n-1} + 1, \quad n \geq 1,$$

with initial datum $x_0 = 1$.

Solution. Denote by $A(X)$ the OGF of the solution we are looking for. By Proposition 10.58 we have

$$A(X) = \frac{S(X) + \sum_{k=1}^{\infty} X^k}{1-2X},$$

with $S(X) = [X^0](1)(a_0) = a_0 = 1$; thus, we get

$$A(X) = \frac{\sum_{k=0}^{\infty} X^k}{1-2X} = \frac{1}{(1-X)(1-2X)}.$$

Decomposing $A(X)$ in simple fractions, we obtain

$$A(X) = \frac{1/2}{(X-1)(X-1/2)} = \frac{1}{X-1} - \frac{1}{X-1/2}.$$

By Theorem 7.114, we get that for each $n \in \mathbb{N}$

$$[X^n]A(X) = -\left(\frac{1}{1^{n+1}} - \frac{1}{(1/2)^{n+1}}\right) = 2^{n+1} - 1$$

and thus the desired solution is the sequence $(2^{n+1}-1)_n$.

Problem 10.40. Solve the following recurrence relation using the generating formal series:

$$x_n = 2x_{n-1} - x_{n-2} + 2x_{n-3}, \quad n \geq 3,$$

with initial data $x_0 = 1, x_1 = 0, x_2 = -1$.

Solution. By Proposition 10.58, the OGF of the solution is

$$A(X) = \frac{S(X)}{1-2X+X^2-2X^3},$$

where

$$S(X) = [X^{\leq 2}](1-2X+X^2)(1-X^2) = 1-2X;$$

hence

$$A(X) = \frac{1-2X}{1-2X+X^2-2X^3}.$$

It is easily seen using Proposition 10.9 that the only real root of $1-2X+X^2-2X^3$ is $1/2$; indeed, dividing by $X - \frac{1}{2}$ one finds

$$1-2X+X^2-2X^3 = (1-2X)(1+X^2).$$

Thus, we have

$$A(X) = \frac{1}{X^2+1}.$$

By Theorem 7.114 with $P(X)=1$ and $Q(X)=1+X^2$, one has that for each $n\in\mathbb{N}$

$$\begin{aligned} A(X) &= -\sum_{\alpha:Q(\alpha)=0}\frac{P(\alpha)}{\alpha^{n+1}Q'(\alpha)} \\ &= -\left(\frac{1}{i^{n+1}(2i)}+\frac{1}{(-i)^{n+1}(-2i)}\right)=\frac{1}{2}i^n+\frac{1}{2}(-i)^n. \end{aligned}$$

Therefore, the solution of the recurrence is the sequence $(a_n)_n$ defined by

$$a_n=\begin{cases}0 & \text{if } n \text{ is odd,}\\ 1 & \text{if } 4|n,\\ -1 & \text{otherwise.}\end{cases}$$

Problem 10.41. Solve the following recurrence relations using the generating formal series:

a) $x_n=-x_{n-1}+6x_{n-2} \quad n>1 \quad x_0=0, x_1=1;$
b) $x_n=3x_{n-1}-4x_{n-2} \quad n>1 \quad x_0=0, x_1=1.$

Solution. Let $A(X)$ be the OGF of the solution to the recurrence.

a) We know, from Proposition 10.58, that

$$A(X)=\frac{S(X)}{1-X-6X^2},$$

with $S(X)=[X^{\leq 1}](X)\left(1+X-6X^2\right)=X$. Now

$$1+X-6X^2=-(-1+2X)(1+3X);$$

the decomposition of $A(X)$ into simple fractions gives

$$A(X)=-\frac{1/10}{X-\frac{1}{2}}-\frac{1/15}{X+\frac{1}{3}}.$$

We deduce from Theorem 7.114 that

$$[X^n]A(X)=\frac{2^n}{5}-\frac{(-1)^n}{5}3^n \quad \forall n\in\mathbb{N}.$$

b) We know, from Proposition 10.58, that

$$A(X)=\frac{S(X)}{1-3X+4X^2},$$

with $S(X)=[X^{\leq 1}](X)\left(1-3X+4X^2\right)=X$. Now

$$1-3X+4X^2 = 4\left(X^2-\frac{3}{4}X+\frac{1}{4}\right) = 4\left(\left(X-\frac{3}{8}\right)^2+\frac{7}{64}\right);$$

the zeroes of $1-3X+4X^2$ are therefore $\dfrac{3}{8}\pm i\dfrac{\sqrt{7}}{8}$. The decomposition of $A(X)$ into simple fractions gives

$$A(X) = \frac{\frac{1}{8}+\frac{3i}{8\sqrt{7}}}{X-\frac{1}{8}\left(3-i\sqrt{7}\right)} + \frac{\frac{1}{8}-\frac{3i}{8\sqrt{7}}}{X-\frac{1}{8}\left(3+i\sqrt{7}\right)}.$$

By Theorem 7.114, for every $n\in\mathbb{N}$ we get

$$\begin{aligned}[X^n]A(X) &= -\frac{\frac{1}{8}+\frac{3i}{8\sqrt{7}}}{\left(\frac{1}{8}\left(3-i\sqrt{7}\right)\right)^{n+1}} - \frac{\frac{1}{8}-\frac{3i}{8\sqrt{7}}}{\left(\frac{1}{8}\left(3+i\sqrt{7}\right)\right)^{n+1}}\\ &= -2\mathrm{Re}\left(\frac{\frac{1}{8}+\frac{3i}{8\sqrt{7}}}{\left(\frac{1}{8}\left(3-i\sqrt{7}\right)\right)^{n+1}}\right).\end{aligned}$$

Problem 10.42. Solve the following recurrence relation using the generating formal series:

$$(n+1)x_{n+1} = (n+100)x_n, \qquad n>0, \quad x_0 = 1.$$

Solution. The recurrence relation is linear with *variable* coefficients, thus we cannot make use of Proposition 10.58. However, following the lines of its proof, we note that if $(x_n)_n$ is a solution to the recurrence, by multiplying formally by X^n we get

$$(n+1)x_{n+1}X^n = (n+100)x_nX^n,$$

and thus

$$\sum_{n=0}^{\infty}(n+1)x_{n+1}X^n = \sum_{n=0}^{\infty}(n+100)x_nX^n.$$

With a change of variable we obtain

$$\sum_{n=1}^{\infty}nx_nX^{n-1} = \sum_{n=1}^{\infty}nx_nX^n + 100\sum_{n=0}^{\infty}x_nX^n.$$

Set $A(X) = \sum_{n=0}^{\infty}x_nX^n$: we thus have

$$(1-X)A'(X) = 100A(X).$$

Now the equation

$$(1-x)y' = 100y, \quad y(0) = 1$$

is solved by

$$f(x) = \frac{1}{(1-x)^{100}} \qquad x \in]-\infty, 1[.$$

It follows from Proposition 7.119 that

$$A(X) = \frac{1}{(1-X)^{100}} = \sum_{n=0}^{\infty} \binom{n+99}{n},$$

whence $x_n = \binom{99}{n}$ for all $n \in \mathbb{N}$.

Problem 10.43. Solve the following recurrence relation using the generating formal series:

$$x_n = 11x_{n-2} - 6x_{n-3}, \quad n > 2, \quad x_0 = 0, x_1 = x_2 = 1.$$

Solution. Let $A(X)$ be the OGF of the solution to the given linear homogeneous recurrence order 3. By Proposition 10.58 we have

$$A(X) = \frac{S(X)}{1 - 11X^2 + 6X^3},$$

where

$$\begin{aligned} S(X) &= [X^{\leq 2}] \left(1 - 11X^2\right) \left(X + X^2\right) \\ &= X + X^2. \end{aligned}$$

Therefore

$$A(X) = \frac{X + X^2}{1 - 11X^2 + 6X^3}.$$

The rational roots of $1 - 11X^2 + 6X^3$ are of the form $1/q$ where q divides 6; it turns out that $1/3$ is a root; dividing by $X - 1/3$ we find

$$6X^3 - 11X + 1 = 6(X - 1/3) \left(X - \frac{1}{4}(3 - \sqrt{17})\right) \left(X - \frac{1}{4}(3 + \sqrt{17})\right).$$

The decomposition into simple fractions yields

$$A(X) = -\frac{1/12}{X - \frac{1}{3}} + \frac{\frac{1}{8} - \frac{5}{8\sqrt{17}}}{X - \frac{1}{4}\left(3 - \sqrt{17}\right)} + \frac{\frac{1}{8} + \frac{5}{8\sqrt{17}}}{X - \frac{1}{4}\left(3 + \sqrt{17}\right)}.$$

From Theorem 7.114, for all $n \in \mathbb{N}$ we have

$$[X^n]A(X) = \frac{3^n}{4} - \frac{\frac{1}{8} - \frac{5}{8\sqrt{17}}}{\left(\frac{1}{4}\left(3 - \sqrt{17}\right)\right)^{n+1}} - \frac{\frac{1}{8} + \frac{5}{8\sqrt{17}}}{\left(\frac{1}{4}\left(3 + \sqrt{17}\right)\right)^{n+1}}$$

Problem 10.44. Solve the following recurrence relation using the generating formal series:

$$x_n = 3x_{n-1} - 3x_{n-2} + x_{n-3}, \quad n > 2, \quad x_0 = x_1 = 0, x_2 = 1.$$

Solution. Let $A(X)$ be the OGF of the solution to the given linear homogeneous recurrence of order 3. By Proposition 10.58 we have

$$A(X)=\frac{S(X)}{1-3X+3X^2-X^3},$$

where

$$S(X)=[X^{\leq 2}]\left(1-3X+3X^2\right)\left(X^2\right)=X^2.$$

Therefore

$$A(X)=\frac{X^2}{1-3X+3X^2-X^3}.$$

Since 1 is a root of $1-3X+3X^2-X^3$, the division with $X-1$ yields

$$1-3X+3X^2-X^3=-(X-1)^3.$$

The decomposition into simple fractions gives

$$\frac{X^2}{1-3X+3X^2-X^3}=-\frac{1}{X-1}-\frac{2}{(X-1)^2}-\frac{1}{(X-1)^3}.$$

From Theorem 7.114 we deduce that for each $n\in\mathbb{N}$

$$\begin{aligned}[X^n]A(X)&=\binom{-1}{n}(-1)^{n+1}(-1)+\binom{-2}{n}(-1)^{n+2}(-2)+\binom{-3}{n}(-1)^{n+3}(-1)\\&=(-1)^n\binom{-1}{n}+(-1)^{n+1}2\binom{-2}{n}+(-1)^n\binom{-3}{n}.\end{aligned}$$

Now, for $n\geq 3$ it turns out that

$$\binom{-1}{n}=\frac{-1(-2)\cdots(-1-n+1)}{n!}=(-1)^n,$$

$$\binom{-2}{n}=\frac{-2(-3)\cdots(-2-n+1)}{n!}=(-1)^n\frac{(n+1)!}{n!}=(-1)^n(n+1),$$

$$\binom{-3}{n}=\frac{-3(-4)\cdots(-3-n+1)}{n!}=(-1)^n\frac{(n+2)!}{2n!}=(-1)^n\frac{(n+1)(n+2)}{2},$$

whence

$$[X^n]A(X)=1-2(n+1)+\frac{(n+1)(n+2)}{2}=\frac{n^2-n}{2}.$$

Problem 10.45. Solve the following recurrence relations using the generating formal series:

a) $x_n=5x_{n-1}-8x_{n-2}+4x_{n-3},\quad n>2,\quad x_0=1,x_1=2,x_2=4;$
b) $x_n=2x_{n-2}-x_{n-4},\quad n>4,\quad x_0=x_1=0,x_2=x_3=1.$

Solution. Let $A(X)$ be the OGF of the solution to the recurrence.

a) From Proposition 10.58 we know that

$$A(X) = \frac{S(X)}{1-5X+8X^2-4X^3},$$

with $S(X) = [X^{\leq 2}]\left(1+2X+4X^2\right)\left(1-5X+8X^2\right) = 1-3X+2X^2$. Since 1 is a root of $1-5X+8X^2-4X^3$, we have

$$1-5X+8X^2-4X^3 = -4\left(X-\frac{1}{2}\right)^2(X-1)$$

and thus $A(X) = -\frac{1/2}{X-\frac{1}{2}}$. From Theorem 7.114 we deduce that

$$[X^n]A(X) = \frac{1/2}{\left(\frac{1}{2}\right)^{n+1}} = 2^n \quad \forall n \in \mathbb{N}.$$

b) From Proposition 10.58 we know that

$$A(X) = \frac{S(X)}{1-2X^2+X^4},$$

with $S(X) = [X^{\leq 3}]\left(X^2+X^3\right)\left(1-2X^2\right) = X^2+X^3$. Since

$$1-2X^2+X^4 = (X^2-1)^2 = (X-1)^2(X+1)^2,$$

we find the following decomposition into simple fractions

$$A(X) = \frac{1}{2(-1+X)^2} + \frac{3}{4(-1+X)} + \frac{1}{4(1+X)}.$$

From Theorem 7.114 we deduce that

$$[X^n]A(X) = \frac{1}{4}\left(-1+(-1)^n+2n\right) \quad \forall n \in \mathbb{N}.$$

Problem 10.46. Let $m \in \mathbb{N}_{\geq 1}$. Solve the following recurrence relation using the generating formal series:

$$\sum_{k=0}^{m}\binom{m}{k}x_{n-k} = 0, \quad n \geq m, \quad x_0 = \cdots = x_{m-2} = 0, x_{m-1} = 1.$$

Solution. Let $A(X)$ be the OGF of the solution to the given recurrence. From Proposition 10.58 we know that

$$A(X) = \frac{S(X)}{\sum_{k=0}^{m} \binom{m}{k} X^k},$$

with

$$S(X) = [X^{\leq m-1}] \left(\sum_{k=0}^{m} \binom{m}{k} X^k \right) X^{m-1} = X^{m-1}.$$

Now $\sum_{k=0}^{m} \binom{m}{k} X^k = (1+X)^m$, and thus

$$A(X) = \frac{X^{m-1}}{(1+X)^m}.$$

For every $n \geq m$ we thus have

$$[X^n]A(X) = [X^{n-(m-1)}] \frac{1}{(1+X)^m} = (-1)^{n-m+1} \binom{n}{n-m+1},$$

thanks to Proposition 7.90.

Problem 10.47. Solve the following recurrence relation using the generating formal series:

$$x_{n+6} - 21x_{n+5} + 175x_{n+4} - 735x_{n+3} + 1624x_{n+2} - 1764x_{n+1} + 720x_n = 0, \quad n \geq 0,$$

with initial data

$$x_0 = 0,\ x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 0,\ x_5 = 1.$$

Solution. The characteristic polynomial is

$$P_{\text{char}}(X) = 720 - 1764X + 1624X^2 - 735X^3 + 175X^4 - 21X^5 + X^6;$$

using suitable CAS to factorise a polynomial, or looking for integer roots following Proposition 10.9, we find

$$P_{\text{char}}(X) = (X-1)(X-2)(X-3)(X-4)(X-5)(X-6).$$

Therefore the family of sequences

$$(A_1 + A_2 2^n + A_3 3^n + A_4 4^n + A_5 5^n + A_6 6^n)_n,$$

with variation of $A_1, A_2, \ldots, A_6$ in $\mathbb{R}$, is the general solution of the recurrence. Imposing the initial data, one finds the linear system

$$\begin{cases} A_1 + A_2 + A_3 + A_4 + A_5 + A_6 & = 0 \\ A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5 + 6A_6 & = 1 \\ A_1 + 2^2A_2 + 3^2A_3 + 4^2A_4 + 5^2A_5 + 6^2A_6 & = 0 \\ A_1 + 2^3A_2 + 3^3A_3 + 4^3A_4 + 5^3A_5 + 6^3A_6 & = 1 \\ A_1 + 2^4A_2 + 3^4A_3 + 4^4A_4 + 5^4A_5 + 6^4A_6 & = 0 \\ A_1 + 2^5A_2 + 3^5A_3 + 4^5A_4 + 5^5A_5 + 6^5A_6 & = 1. \end{cases}$$

With patience - or better, using suitable CAS, one obtains

$$A_1 = -10, A_2 = 35, A_3 = -\frac{105}{2}, A_4 = 42, A_5 = -\frac{35}{2}, A_6 = 3;$$

thus the desired solution is the sequence

$$x_n = -10 + 35 \times 2^n + -\frac{105}{2} \cdot 3^n + 42 \times 4^n - \frac{35}{2} \times 5^n + 3 \times 6^n.$$

One could reach the same solution using generating formal series. By Proposition 10.58, the OGF of the solution is

$$A(X) = \frac{S(X)}{1 - 21X + 175X^2 - 735X^3 + 1624X^4 - 1764X^5 + 720X^6},$$

where $S(X)$ is

$$\begin{aligned} &[X^{\leq 5}](1 - 21X + 175X^2 - 735X^3 + 1624X^4 - 1764X^5)(X + X^3 + X^5) = \\ &= X[X^{\leq 4}](1 - 21X + 175X^2 - 735X^3 + 1624X^4 - 1764X^5)(1 + X^2 + X^4) = \\ &= X\big((1-21X+175X^2-735X^3+1624X^4)+(1-21X+175X^2)X^2+X^4\big) = \\ &= X - 21X^2 + 176X^3 - 756X^4 + 1800X^5; \end{aligned}$$

hence

$$A(X) = \frac{X - 21X^2 + 176X^3 - 756X^4 + 1800X^5}{1 - 21X + 175X^2 - 735X^3 + 1624X^4 - 1764X^5 + 720X^6}.$$

Setting $Q(X) = 1 - 21X + 175X^2 - 735X^3 + 1624X^4 - 1764X^5 + 720X^6$, Theorem 7.114 gives that for each $n \in \mathbb{N}$

$$\begin{aligned} A(X) &= -\sum_{\alpha : Q(\alpha)=0} \frac{P(\alpha)}{\alpha^{n+1} Q'(\alpha)} \\ &= -\sum_{j=1}^{6} j^{n+1} \frac{P(1/j)}{Q'(1/j)}. \end{aligned}$$

At this point you just need a small calculator to find

$$[X^n]A(X) = -10 + \frac{35}{2}\times 2^{n+1} - \frac{35}{2}\times 3^{n+1} + \frac{21}{2}\times 4^{n+1} - \frac{7}{2}5^{n+1} + \frac{1}{2}\times 6^{n+1}.$$

Problem 10.48. Let $t \in \mathbb{R}$, $t \neq 2$. Solve the following recurrence relations using the generating formal series:

$$x_n = n+1+\frac{t}{n}\sum_{k=1}^{n} x_{k-1}\,, \qquad n \geq 1, \quad x_0 = 0,$$

with $t = 2-\varepsilon$ and then with $t = 2+\varepsilon$, for a sufficiently small positive constant ε.

Solution. Let $(x_n)_n$ be the solution to the given recurrence. By multiplying with X^n both members of the equality we obtain

$$x_n X^n = (n+1)X^n + \frac{t}{n}\sum_{k=0}^{n-1} x_k X^n;$$

thus

$$\sum_{n=0}^{\infty} x_n X^n = \sum_{n=0}^{\infty}(n+1)X^n + t\sum_{n=0}^{\infty}\frac{1}{n}\left(\sum_{k=0}^{n-1} x_k\right)X^n.$$

Let $A(X)$ be the OGF of the sequence $(x_n)_n$ and let

$$B(X) = \sum_{n=0}^{\infty}\frac{1}{n}\left(\sum_{k=0}^{n-1} x_k\right)X^n.$$

Recall that OGF $\left(\sum_{k=0}^{n} x_k\right)_n = \frac{1}{1-X}A(X)$; therefore

$$B'(X) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} x_k\right)X^{n-1} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x_k\right)X^n = \frac{1}{1-X}A(X).$$

Moreover

$$\sum_{n=0}^{\infty}(n+1)X^n = \left(\sum_{n=0}^{\infty} X^{n+1}\right)' = \left(\frac{X}{1-X}\right)' = \frac{1}{(X-1)^2}.$$

It follows that

$$A(X) = \frac{1}{(X-1)^2} + tB(X), \quad B'(X) = \frac{1}{1-X}A(X).$$

Deriving again we obtain

$$A'(X) = \left(\frac{1}{(X-1)^2}\right)' + \frac{t}{1-X}A(X) = -\frac{2}{(X-1)^3} + \frac{t}{1-X}A(X).$$

Now the solutions to
$$y' = \frac{t}{1-x}y - \frac{2}{(x-1)^3}$$
are
$$y(x) = -\frac{2}{(t-2)(1-x)^2} + c(1-x)^{-t}, \quad c \in \mathbb{R},$$
and $y(0) = 0$ if and only if $c = \dfrac{2}{t-2}$. By Proposition 7.119 we get
$$A(X) = \frac{2}{2-t}\left(\frac{1}{(1-X)^2} - (1-X)^{-t}\right).$$
Moreover
$$\frac{1}{(1-X)^2} = \sum_{n=0}^{\infty}(n+1)X^n,$$
$$(1-X)^{-t} = \sum_{n=0}^{\infty}(-1)^n\binom{-t}{n}X^n,$$
therefore we obtain
$$A(X) = \frac{2}{2-t}\sum_{n=0}^{\infty}\left(n+1+(-1)^n\binom{-t}{n}\right)X^n,$$
from which we get
$$x_n = \frac{2}{2-t}\left(n+1-(-1)^n\binom{-t}{n}\right) \quad \forall n \in \mathbb{N}.$$

Chapter 11
Problems and Solutions

Problem 11.1. Consider the combinatorial class $\mathrm{SEQ}_{\{\geq 1\}}\{0,1\}$ of the non-*empty* binary sequences with valuation function that associates to each sequence its length. Every such sequence can be either a sequence of length 1 ((0) or (1)), or else a longer sequence beginning with 0 or with 1. Using this fact, prove the following isomorphism of combinatorial classes

$$\mathrm{SEQ}_{\{\geq 1\}}\{0,1\} \cong \{(0)\} \bigoplus \{(1)\} \bigoplus \left(\{0,1\} \times \mathrm{SEQ}_{\{\geq 1\}}\{0,1\}\right).$$

Solution. The bijective mapping

$$\begin{aligned}
\{(0)\} \cup \{(1)\} \cup \left(\{0,1\} \times \mathrm{SEQ}_{\{\geq 1\}}\{0,1\}\right) &\longrightarrow \mathrm{SEQ}_{\{\geq 1\}}\{0,1\} \\
(0) &\longmapsto (0) \\
(1) &\longmapsto (1) \\
((0),(a_1,\ldots,a_k)) &\longmapsto (0,a_1,\ldots,a_k) \\
((1),(a_1,\ldots,a_k)) &\longmapsto (1,a_1,\ldots,a_k)
\end{aligned}$$

conserves the fibres and so is an isomorphism of combinatorial classes.

Problem 11.2. Consider the combinatorial class $\mathrm{SEQ}\{0,1\}$ of the binary sequences with the valuation function that associates to every sequence its length. Prove the following isomorphism of combinatorial classes:

$$\mathrm{SEQ}\{0,1\} \cong \{()\} \bigoplus (\{0,1\} \times \mathrm{SEQ}\{0,1\}).$$

Solution. The bijective map

$$\begin{aligned}
\{()\} \bigoplus (\{0,1\} \times \mathrm{SEQ}\{0,1\}) &\longrightarrow \mathrm{SEQ}\{0,1\} \\
() &\longmapsto () \\
((0),()) &\longmapsto (0) \\
((1),()) &\longmapsto (1) \\
((0),(a_1,...,a_k)) &\longmapsto (0,a_1,...,a_k) \\
((1),(a_1,...,a_k)) &\longmapsto (1,a_1,...,a_k)
\end{aligned}$$

conserves the fibres and thus is an isomorphism of combinatorial classes.

Problem 11.3. Determine the autocorrelation polynomial of the pattern $p = (0,1)$ and then the OGF of the combinatorial class $\mathrm{SEQ}^{\neg p}\{0,1\}$ of the binary sequences which do not contain p. In particular, deduce the number of binary 4-sequences that do not contain p.

Solution. Using the algorithm for the calculation of the autocorrelation polynomial we immediately obtain $C_p(X) = 1$, from which $\mathrm{SEQ}^{\neg p}\{0,1\}(X) = \frac{1}{X^2+(1-2X)} = \frac{1}{(X-1)^2}$. By Proposition 7.90 one has

$$\frac{1}{(X-1)^2} = \sum_{n=0}^{\infty} \binom{1+n}{n} X^n = \sum_{n=0}^{\infty} (n+1)X^n$$

and consequently the number of n-sequences that do not contain the pattern $(0,1)$ is equal to $[X^n]\,\mathrm{SEQ}^{\neg p}\{0,1\}(X) = n+1$. It would have been possible to find this result by hand on observing that a sequence does not contain the given pattern only if one of its 0-terms is never followed by a 1: it then suffices to count in how many ways one can position a strip of 1's at the beginning of the sequence, in all $n+1$ (with no 1, a 1, ..., all 1's).

Problem 11.4. Find the autocorrelation polynomial of the pattern $p = (1,0,1,0)$ and then the OGF of the combinatorial class of the binary sequences that do not contain p. In particular deduce the number of binary 6-sequences which do not contain p.

Solution. Using the algorithm for the calculation of the autocorrelation polynomial we immediately obtain $C_p(X) = 1+X^2$, from which it follows that $\mathrm{SEQ}^{\neg p}\{0,1\}(X) = \frac{X^2+1}{X^4+(1-2X)(X^2+1)} = \frac{X^2+1}{1-2X+X^2-2X^3+X^4}$. Thus one has

$$\mathrm{SEQ}^{\neg p}\{0,1\}(X)(1-2X+X^2-2X^3+X^4) = 1+X^2.$$

We put $\mathrm{SEQ}^{\neg p}\{0,1\}(X) = s_0 + s_1X + s_2X^2 + \cdots$, and we must calculate s_6. By Corollary 11.39 one has,

$$s_0 = 1,\, s_1 = 2^1 = 2,\, s_2 = 2^2 = 4,\, s_3 = 2^3 = 8,\, s_4 = 2^4 - 1 = 15$$

and, for $n > 4$, s_n is a solution of

$$s_n - 2s_{n-1} + s_{n-2} - 2s_{n-3} + s_{n-4} = 0,$$

or equivalently

$$s_n = 2s_{n-1} - s_{n-2} + 2s_{n-3} - s_{n-4},$$

from which it follows that

$$s_5 = 2s_4 - s_3 + 2s_2 - s_1 = 30 - 8 + 8 - 2 = 28,$$

$$s_6 = 2s_5 - s_4 + 2s_3 - s_2 = 56 - 15 + 16 - 4 = 53.$$

Problem 11.5. Find the autocorrelation polynomial of the pattern $p = (0,0,0,0,0)$ and thus the OGF of the combinatorial class of the binary sequences that do not contain p. In particular, deduce the number of binary 7-sequences which do not contain p.

Solution. Using the algorithm for the calculation of the autocorrelation polynomial we immediately obtain $C_p(X) = 1 + X + X^2 + X^3 + X^4$, from which it follows that

$$\begin{aligned}\mathrm{SEQ}^{\neg p}\{0,1\}(X) &= \frac{1 + X + X^2 + X^3 + X^4}{X^5 + (1-2X)(1 + X + X^2 + X^3 + X^4)} \\ &= \frac{1 + X + X^2 + X^3 + X^4}{1 - X - X^2 - X^3 - X^4 - X^5}.\end{aligned}$$

Thus one has

$$\mathrm{SEQ}^{\neg p}\{0,1\}(X)(1 - X - X^2 - X^3 - X^4 - X^5) = 1 + X + X^2 + X^3 + X^4.$$

We set $\mathrm{SEQ}^{\neg p}\{0,1\}(X) = s_0 + s_1X + s_2X^2 + \cdots$, and we must calculate s_7. By Corollary 11.39 one has

$$s_0 = 1,\ s_1 = 2^1 = 2,\ s_2 = 2^2 = 4,\ s_3 = 2^3 = 8,\ s_4 = 2^4 = 16,\ s_5 = 2^5 - 1 = 31$$

and, for $n > 5$, one has that s_n is a solution of

$$s_n - s_{n-1} - s_{n-2} - s_{n-3} - s_{n-4} - s_{n-5} = 0,$$

or equivalently that

$$s_n = s_{n-1} + s_{n-2} + s_{n-3} + s_{n-4} + s_{n-5},$$

from which one has

$$s_6 = s_5 + s_4 + s_3 + s_2 + s_1 = 31 + 16 + 8 + 4 + 2 = 61,$$

$$s_7 = s_6 + s_5 + s_4 + s_3 + s_2 = 61 + 31 + 16 + 8 + 4 = 120.$$

Problem 11.6. What is the number of binary sequences of length 8 which do not contain a strip of 3 consecutive zeroes?

Solution. On setting $p = 000$ we have $C_p(X) = 1+X+X^2$ from which it follows that

$$\begin{aligned}\mathrm{SEQ}^{\neg p}\{0,1\}(X) &= \frac{1+X+X^2}{X^3+(1-2X)(1+X+X^2)} \\ &= \frac{1+X+X^2}{1-X-X^2-X^3}.\end{aligned}$$

Thus, one has

$$\mathrm{SEQ}^{\neg p}\{0,1\}(X)(1-X-X^2-X^3) = 1+X+X^2.$$

We put $\mathrm{SEQ}^{\neg p}\{0,1\}(X) = s_0 + s_1X + s_2X^2 + \cdots$, and we must calculate s_8. By Corollary 11.39 one has,

$$s_0 = 1,\, s_1 = 2^1 = 2,\, s_2 = 2^2 = 4,\, s_3 = 2^3 - 1 = 7$$

and for $n > 3$ s_n is a solution of

$$s_n - s_{n-1} - s_{n-2} - s_{n-3} = 0,$$

that is,

$$s_n = s_{n-1} + s_{n-2} + s_{n-3},$$

from which

$$\begin{aligned} s_4 &= s_3 + s_2 + s_1 = 7+4+2 = 13, \\ s_5 &= s_4 + s_3 + s_2 = 13+7+4 = 24, \\ s_6 &= s_5 + s_4 + s_3 = 24+13+7 = 44, \\ s_7 &= s_6 + s_5 + s_4 = 44+24+13 = 81, \end{aligned}$$

and finally

$$s_8 = s_7 + s_6 + s_5 = 81+44+24 = 149.$$

Thus there are 149 different binary 8-sequences that do not contain the pattern $p = 000$.

Problem 11.7. Determine the number of binary 5-sequences that do not contain the pattern 010.

Solution. The sequence of the coefficients of the autocorrelation polynomial of the pattern of length 3 is

$$(1,0,1)$$

and therefore the autocorrelation polynomial of the pattern is $1+X^2$. The OGF of the number of sequences $\{0,1\}$ that do not contain the pattern is thus

$$\mathrm{SEQ}^{\neg p}\{0,1\}(X) = \frac{1+X^2}{X^3+(1-2X)(1+X^2)} = \frac{1+X^2}{1-2X+X^2-X^3}.$$

Having set $\mathrm{SEQ}^{\neg p}\{0,1\}(X) = \sum_{n=0}^{\infty} s_n X^n$ one then has

$$\left(\sum_{n=0}^{\infty} s_n X^n\right)\left(1-2X+X^2-X^3\right) = 1+X^2,$$

from which it follows that

$$0 = [X^n]\,(1+X^2) = s_n - 2s_{n-1} + s_{n-2} - s_{n-3} \forall n \geq 3,$$

or equivalently that

$$s_n = 2s_{n-1} - s_{n-2} + s_{n-3} \forall n \geq 3.$$

Given that $s_1 = 2^1 = 2$, $s_2 = 2^2 = 4$, $s_3 = 2^3 - 1 = 7$ one immediately obtains

$$s_4 = 2s_3 - s_2 + s_1 = 12, \quad s_5 = 2s_4 - s_3 + s_2 = 21.$$

Problem 11.8. Determine the autocorrelation polynomial of the pattern $p = (1,1)$ of $\Gamma = \{1,2,3\}$ and thus the OGF of the combinatorial class of the sequences that do not contain p. In particular, deduce the number of 4-sequences of Γ which do not contain p.

Solution. Using the algorithm for the calculation of the autocorrelation polynomial we immediately obtain $C_p(X) = 1+X$ from which, since the set Γ is composed of three elements, one has

$$\mathrm{SEQ}^{\neg p}\,\Gamma(X) = \frac{X+1}{X^2+(1-3X)(X+1)} = \frac{X+1}{1-2X-2X^2}.$$

Thus one has

$$\mathrm{SEQ}^{\neg p}\,\Gamma(X)(1-2X-3X^2) = X+1.$$

We put $\mathrm{SEQ}^{\neg p}\,\Gamma(X) = s_0 + s_1 X + s_2 X^2 + \cdots$, and we must calculate s_4. By Corollary 11.39 one has

$$s_0 = 1, s_1 = 3^1 = 3, s_2 = 3^2 - 1 = 8,$$

and, for $n > 2$ s_n is a solution of

$$s_n - 2s_{n-1} - 2s_{n-2} = 0,$$

or equivalently

$$s_n = 2s_{n-1} + 2s_{n-2},$$

from which it follows that

$$s_3 = 2s_2 + 2s_1 = 16 + 6 = 22,\ s_4 = 2s_3 + 2s_2 = 44 + 16 = 60.$$

Problem 11.9. What is the probability that a word of 6 letters written by chance from an alphabet Γ of 26 letters contains the Indian name $p := TATA$? And with the alphabet $\Gamma' = \{A, T\}$?

Solution. The sequence of the coefficients of the autocorrelation polynomial is equal to $(1,0,1,0)$, and so the polynomial $C_p(X) = 1 + X^2$. The OGF $\mathrm{SEQ}^{\neg p}\,\Gamma(X)$ of the succession of the number of sequences which do not contain the name $p := TATA$ formed starting from an alphabet of 26 letters is

$$\mathrm{SEQ}^{\neg p}\,\Gamma(X) = \frac{C_p(X)}{X^4 + (1-26X)C_p(X)} = \frac{1+X^2}{1-26X+X^2-26X^3+X^4}.$$

To find the number of n-sequences of Γ that do not contain the name $TATA$ it now suffices to observe that, having set

$$\mathrm{SEQ}^{\neg p}\,\Gamma(X) = \sum_{n=0}^{\infty} s_n X^n,$$

one has

$$\left(\sum_{n=0}^{\infty} s_n X^n\right)\left(1-26X+X^2-26X^3+X^4\right) = 1 + X^2,$$

from which, on setting equal the terms of degree equaling the terms of degree $n \geq 4$, one obtains

$$s_n - 26s_{n-1} + s_{n-2} - 26s_{n-3} + s_{n-4} = 0,$$

or equivalently,

$$s_n = 26s_{n-1} - s_{n-2} + 26s_{n-3} - s_{n-4}.$$

Given that

$$s_n = 26^n \text{ if } n = 1,2,3, \quad s_4 = 26^4 - 1,$$

one obtains successively

$$s_5 = 11\,881\,324, \quad s_6 = 308\,913\,749.$$

The number of 10-sequences in an alphabet composed by 26 letters is equal to $26^{10} = 141\,167\,095\,653\,376$: the probability that a word of 10 letters written by chance does not contain the name $TATA$ is therefore equal to

$$\frac{308\,913\,749}{141\,167\,095\,653\,376} \approx 2 \times 10^{-6}.$$

Let us now calculate that probability when the alphabet is $\Gamma' = \{T, A\}$. In that case the OGF of the number of sequences which do not contain the name $TATA$ is given by

$$\mathrm{SEQ}^{\neg p}\Gamma'(X) = \frac{C_p(X)}{X^4+(1-4X)C_p(X)} = \frac{1+X^2}{1-4X+X^2-4X^3+X^4}.$$

Having set

$$\mathrm{SEQ}^{\neg p}\Gamma'(X) = \sum_{n=0}^{\infty} s_n X^n,$$

one has

$$\left(\sum_{n=0}^{\infty} s_n X^n\right)\left(1-4X+X^2-4X^3+X^4\right) = 1+X^2,$$

from which, on setting equal the terms of degree $n \geq 4$, one obtains

$$s_n - 4s_{n-1} + s_{n-2} - 4s_{n-3} + s_{n-4} = 0,$$

or, equivalently,

$$s_n = 4s_{n-1} - s_{n-2} + 4s_{n-3} - s_{n-4}.$$

Given that

$$s_n = 2^n \text{ if } n = 1,2,3, \quad s_4 = 2^4 - 1,$$

one then successively recovers that

$$s_5 = 66, \quad s_6 = 277.$$

The number of 10-sequences of an alphabet composed of 2 letters is equal to $2^{10} = 1\,024$: the probability that a word of 10 letters written by chance with the letters A, T does not contain the name $TATA$ is thus equal to

$$\frac{277}{1\,024} \approx 27\%.$$

Problem 11.10. Determine the number of 6-sequences of $\Gamma = \{a,b,c,d,e\}$ which do not contain the pattern $p = abc$.

Solution. The sequence of the coefficients of the autocorrelation polynomial of the pattern of length 3 is

$$(1,0,0)$$

and therefore the autocorrelation polynomial of the pattern is 1. The OGF of the number of sequences of $\Gamma = \{a,b,c,d,e\}$ which do not contain the pattern is thus

$$\mathrm{SEQ}^{\neg p}\Gamma(X) = \frac{1}{X^3+(1-5X)} = \frac{1}{1-5X+X^3}.$$

Having set $\mathrm{SEQ}^{\neg p}\Gamma(X) = \sum_{n=0}^{\infty} s_n X^n$ one then has

$$\left(\sum_{n=0}^{\infty} s_n X^n\right)\left(1-5X+X^3\right) = 1,$$

from which it follows that

$$0 = [X^n]\,1 = s_n - 5s_{n-1} + s_{n-3} \forall n \geq 3,$$

or equivalently that

$$s_n = 5s_{n-1} - s_{n-3} \forall n \geq 3.$$

Given that $s_1 = 5^1 = 5$, $s_2 = 5^2 = 25$, $s_3 = 5^3 - 1 = 124$ one immediately obtains that

$$s_4 = 5s_3 - s_1 = 615, \quad s_5 = 5s_4 - s_2 = 3\,050, \quad s_6 = 5s_5 - s_3 = 15\,126.$$

Problem 11.11. Determine the number of 4-sequences of $\Gamma = \{0,1,2,3\}$ that contain the pattern $p = 11$.

Solution. The sequence of the coefficients of the autocorrelation polynomial of the pattern of length 2 is $(1,1)$, and therefore the autocorrelation polynomial of the pattern is $1+X$. The OGF of the number of sequences of $\{0,1,2,3\}$ that do not contain the pattern is thus

$$\mathrm{SEQ}^{\neg p}\,\Gamma(X) = \frac{1+X}{X^2 + (1-4X)(1+X)} = \frac{1+X}{1-3X-3X^2}.$$

On setting $\mathrm{SEQ}^{\neg p}\,\Gamma(X) = \sum_{n=0}^{\infty} s_n X^n$ one thus has

$$\left(\sum_{n=0}^{\infty} s_n X^n\right)\left(1-3X-3X^2\right) = 1+X,$$

from which it follows that

$$0 = [X^n]\,1 = s_n - 3s_{n-1} - 3s_{n-2} \quad \forall n \geq 2,$$

or equivalently that

$$s_n = 3s_{n-1} + 3s_{n-2} \quad \forall n \geq 2.$$

Given that $s_1 = 4^1 = 4$, $s_2 = 4^2 - 1 = 15$ one immediately obtains

$$s_3 = 3s_2 + 3s_1 = 57, \quad s_4 = 3s_3 + 3s_2 = 216.$$

Thus the sequences that contain the pattern $p = 11$ are $4^4 - 216 = 40$ in number.

Problem 11.12. We toss a coin until when one of the following two patterns appears:

$$p := HTHTTHH, \quad q := THTTHTH.$$

1. Compute the odds in favor of p.
2. Change the pattern p in order to maximize the odds in its favor.

Solution. 1. Let us start computing the correlation polynomials:

$$C_{(p,q)}(X) = 0,\ C_p(X) = 1+X^6,\ C_q(X) = 1+X^5,\ C_{(q,p)}(X) = X^4+X^6.$$

Therefore the odds in favor of p are

$$\frac{C_q(1/2)-C_{(q,p)}(1/2)}{C_p(1/2)-C_{(p,q)}(1/2)} = \frac{1+1/2^5-1/2^4-1/2^6}{1+1/2^6} = \frac{61}{65}.$$

2. We know by Remark 11.54 that the best strategy to maximize the odds in favor of p is choosing $p = p_0THTTHT$ where q_0 minimizes

$$W_q(y) = C_{THTTHTT,yTHTTHT}(1/2)+C_{THTTHTH,yTHTTHT}(1/2).$$

Now we have

$$C_{THTTHTT,TTHTTHT}(X) = X^2+X^5+X^6, \quad C_{THTTHTT,HTHTTHT}(X) = 0,$$

$$C_{THTTHTH,TTHTTHT}(X) = 0, \quad C_{THTTHTH,HTHTTHT}(X) = X^4+X^6.$$

Therefore $W_q(T) = 1/2^2+1/2^5+1/2^6 = 19/64$ and $W_q(H) = 1/2^4+1/2^6 = 5/64$. The best choice is therefore $p = TTHTTHT$. The odds in favor of p are now $11/6$.

Problem 11.13. Prove that a rooted plane tree with n vertices has $n-1$ edges.

Solution. One easily verifies by induction of the number of vertices: if $n=1$ then the tree is trivial and thus there are $1-1=0$ edges. If $n>1$, we use m_1, ..., m_ℓ to denote the number of vertices of the subtrees of the sequence of maximal subtrees. Clearly one has $m_1+\cdots+m_\ell = n-1$. By our inductive hypothesis the number of edges in the i-th maximal subtree is m_i-1; on counting the ℓ edges that join the root of the tree to that of the maximal subtrees, one finds that the edges of the tree are

$$(m_1-1)+\cdots+(m_\ell-1)+\ell = m_1+\cdots+m_\ell = n-1.$$

Problem 11.14. Let $n \geq 1$. Prove that the correspondence between rooted plane trees with $n+1$ vertices and the $2n$-sequences of $\{0,1\}$ established in Remark 11.66 is a bijection between rooted plane trees and the $2n$-sequences of Dyck.

Solution. We observe that the path of the tree described in Observation 11.66 to construct the sequence of $\{0,1\}$ is obtained inductively as follows: if $(A_1,...,A_m)$ is the sequence of maximal subtrees of the tree under consideration, one runs over the edge that joins the root to a root of A_1, if this is not reduced to the unique root one runs over A_1 with the same rule returning to the root of A_1, one returns to the root of the tree; one continues in this way on the branch that leads to A_2, and so on until having run over A_m, to return to its root and finally to return to the root of the departure tree. By Problem 11.13 the tree consists of n edges, each of which is counted twice in the construction of the sequence that has therefore length $2n$: then given that every edge is run over once in one sense, and one in the opposite sense, such a sequence has just as many zeroes as it has ones, and is therefore a sequence of $\{0,1\}$ with occupancy (n,n).

Moreover, given that every edge is first run over by assigning in the sequence of the value 0 and only thereafter is runned over again assigning the value 1, the number of zeroes from the beginning to every point of the sequence is greater than or equal to the number of 1's of the same tract: the sequence obtained is therefore a $2n$-sequence of Dyck. Let us prove by induction on $n \geq 1$ that each $2n$-sequence of Dyck arises in such a mode from a *unique* plain tree with $n+1$ vertices. For $n=1$ the unique 2-sequence of Dyck is $(0,1)$, which necessarily comes in this way from the tree consisting of a root, and a edge that connects it to another vertex. Now let $n \geq 2$ and suppose that every $2(n-1)$-sequence of Dyck arises in this fashion from a root and a edge that joins it to another vertex. Now let $n \geq 2$ and suppose that every $2(n-1)$-sequence of Dyck arises in this way from a *unique* rooted tree having n vertices. Let us consider a $2n$-sequence of Dyck; necessarily the sequence begins with 0 and terminates with 1, Three cases are possible (see Figure 11.1):

1. The sequence σ begins with $(0,1$: in this case the tail of the sequence is a $2(n-1)$-sequence of Dyck, which arises uniquely from a tree A consisting of a root ω conjoined to the maximal subtrees $(A_1,\ldots,A_m)$: necessarily σ arises from the tree that has the same root ω, whose sequence of maximal subtrees is $(\omega',A_1,\ldots,A_m)$, where ω' is a subtree constituted by a single vertex;
2. The sequence σ terminates with $0,1)$: in this case the beginning of the sequence is a $2(n-1)$-sequence of Dyck, which arises uniquely from a tree A consisting of a root ω joined to the maximal subtrees $(A_1,\ldots,A_m)$: necessarily σ arises from the tree that has the same root ω, whose sequence of maximal subtrees is $(A_1,\ldots,A_m,\omega')$, where ω' is a subtree consisting of a solo vertex;
3. The sequence σ begins with $(0,0$ and terminates with $1,1)$: the $2(n-1)$-sequence obtained from σ by removing the first 0 or the last 1 is a sequence of Dyck, which arises from a unique tree A. Necessarily σ arises from the tree consisting of a root ω' and by a branch that joins it to its unique maximal subtree A.

In every case the sequence σ arises in such fashion from a unique rooted tree.

Problem 11.15. Prove that, in a binary tree, the number of external vertices is exactly one plus the number of internal vertices.

Solution. If the binary tree is trivial, the unique vertex is external. Suppose now that the binary tree is not trivial. We use $\mathfrak{l}$, $\mathfrak{e}$ and $\mathfrak{i}$ respectively the number of edges, the number of external vertices, and the number of internal vertices of the tree. By Problem 11.13 one has that

$$\mathfrak{l} = \mathfrak{e} + \mathfrak{i} - 1;$$

since the external vertices have degree 1 and those internal degree 3, except for the root which has degree 2, one has that

$$2\mathfrak{l} = 1\mathfrak{e} + 3(\mathfrak{i} - 1) + 2.$$

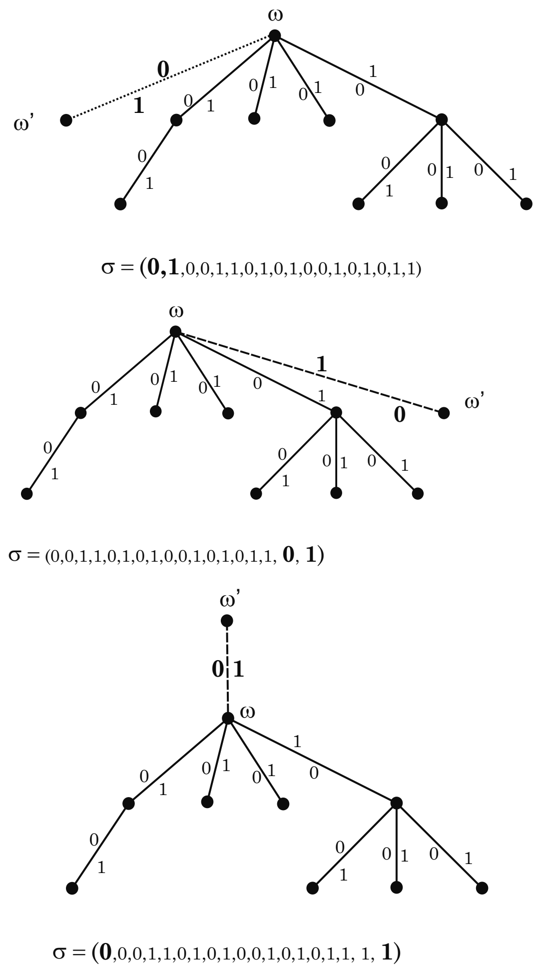

Fig. 11.1: The three types of sequences of Dyck in the solution of Problem 11.14 .

Putting together the two relations found above, one has

$$\mathfrak{e}+3(\mathfrak{i}-1)+2=2\mathfrak{e}+2\mathfrak{i}-2,$$

or equivalently that $\mathfrak{i}+1=\mathfrak{e}$.

Problem 11.16. A ternary tree is a plane tree constituted by a vertex called its root possibly connected by way of three edges to the roots of three ternary trees, as the one depicted in Figure 11.2. We use $\mathscr{T}_{\ni}$ to indicate the combinatorial class formed by the ternary trees, with the valuation equal to the number of vertices.

1. Prove that the class $\mathscr{T}_{\ni}$ is isomorphic to a suitable subclass of $\{\omega\}\times \mathrm{SEQ}\,\mathscr{T}_{\ni}$.

2. Deduce that the OGF of $\mathscr{T}$ satisfies an equation of third degree with coefficients in $\mathbb{R}[[X]]$.
3. Verify the correctness of the result on the number of ternary trees that have 7 vertices.

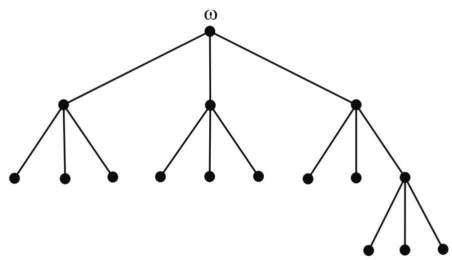

Fig. 11.2: A ternary tree; the root is indicated with ω.

Solution. 1. Given that every ternary tree A is determined by its root and by the sequence (possibly *empty*) (A_1, A_2, A_3) of its 3 maximal subtrees, the correspondence Φ established in the proof of Proposition 11.65 permits one to obtain the following isomorphism of combinatorial classes

$$\mathscr{T}_{\ni} \cong \{\omega\} \times \left(\mathbb{1} \bigoplus \mathscr{T}_{\ni}{}^{3}\right) = \{\omega\} \times \left(\mathbb{1} \bigoplus \mathscr{T}_{\ni}{}^{3}\right).$$

2. Passing to the corresponding OGF, one obtains

$$\mathscr{T}_{\ni}(X) = X\left(1 + \mathscr{T}_{\ni}(X)^{3}\right).$$

¿From this it follows that $\mathscr{T}_{\ni}(X)$ is a solution in $\mathbb{R}[[X]]$ of the equation $X\mathbb{Y}^3 - \mathbb{Y} + X = 0$.
3. In fashion analogous to that used for binary trees, it is easy to see that the number of vertices of a ternary tree is a number that give a remainder of 1 if divided by 3. Having posted $\mathscr{T}_{\ni}(X) = \sum_{n=0}^{\infty} t_n X^n$ one therefore has

$$t_0 = t_2 = t_3 = 0 = t_5 = t_6 = 0;$$

moreover $t_1 = 1$ (the only root) and $t_4 = 1$ (the root connected to 3 vertices by 3 edges). ¿From the relation found above one deduces that

$$\left[X^7\right]\mathscr{T}_{\ni}(X) = \left[X^6\right](1 + \mathscr{T}_{\ni}(X)^3),$$

from which it follows that

$$t_7 = t_1t_1t_4 + t_1t_4t_1 + t_4t_1t_1 = 3t_1^2t_4 = 3.$$

In effect there are only 3 ternary trees with 7 vertices: the root connected to three vertices, from only one of which part 3 edges with 3 other final vertices.

Problem 11.17. A unary–binary tree is a plane tree consisting of a vertex, called its root, possibly connected by 0, 1 or 2 edges to the roots of unary–binary trees (Figure 11.3). Let $\mathscr{U}\lfloor$ indicate the combinatorial class formed by the unary–binary trees, with the valuation equal to the number of vertices.

1. Prove that the class $\mathscr{U}\lfloor$ is isomorphic to a suitable subclass of $\{\omega\} \times \mathrm{SEQ}\,\mathscr{U}\lfloor$.
2. Deduce that the ordinary generating formal series $\mathscr{U}\lfloor(X)$ of $\mathscr{U}\lfloor$ satisfies an equation of second degree with coefficients in $\mathbb{R}[[X]]$.
3. Determine the OGF of $\mathscr{U}\lfloor$.

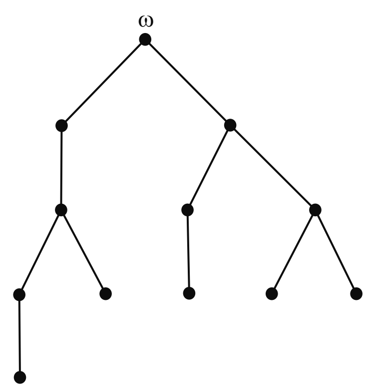

Fig. 11.3: A unary–binary tree; the root is indicated with ω.

Solution. 1. Given that every unary–binary tree A is individuated by its only root, by its maximal subtree, or by the sequence(possibly *empty*) (A_1, A_2) of its 2 maximal subtrees, the correspondence Φ established in the proof of Proposition 11.65 permits one to obtain the following isomorphism of combinatorial classes

$$\mathscr{U}\lfloor \cong \{\omega\} \times \left(\mathbb{1} \bigoplus \mathscr{U}\lfloor \bigoplus \mathscr{U}\lfloor^2\right) = \{\omega\} \times \left(\mathbb{1} \bigoplus \mathscr{U}\lfloor \bigoplus \mathscr{U}\lfloor^2\right).$$

2. Passing to the corresponding OGF, one obtains

$$\mathscr{U}\lfloor(X) = X\left(1 + \mathscr{U}\lfloor(X) + \mathscr{U}\lfloor(X)^2\right).$$

It follows that $\mathscr{U}\lfloor(X)$ is a solution in $\mathbb{R}[[X]]$ of the equation $X\mathbb{Y}^2 + (X-1)\mathbb{Y} + X = 0$.

3. The solutions of $X\mathbb{Y}^2+(X-1)\mathbb{Y}+X=0$ in $\mathbb{R}((X))$ are

$$\mathbb{Y}_1(X)=\frac{1-X+\sqrt{1-2X-3X^2}}{2X},\quad \mathbb{Y}_2(X)=\frac{1-X-\sqrt{1-2X-3X^2}}{2X}.$$

Given that

$$\left[X^0\right]\sqrt{1-2X-3X^2}=\left[X^0\right](1-X(2-3X))^{1/2}=1,$$

one has that X does non divide $1-X+\sqrt{1-2X-3X^2}$, and therefore $\mathbb{Y}_1(X)\notin\mathbb{R}[X]]$. Necessarily one then has

$$\mathscr{U}\lfloor(X)=\mathbb{Y}_2(X)=\frac{1-X-\sqrt{1-2X-3X^2}}{2X}.$$

By developing the first 11 terms of $\mathscr{U}\lfloor(X)$[1], one can see that

$$\left[X^{\leq 10}\right]\mathscr{U}\lfloor(X)=X+X^2+2X^3+4X^4+9X^5+21X^6+51X^7+127X^8+323X^9+835X^{10}.$$

[1] Here we have used a CAS, but a reader armed with good will can verify if by hand.

Chapter 12
Problems and Solutions

Problem 12.1. Let $x_0 \in \mathbb{N}$ and $(x_n)_{n\geq 1}$ be a sequence in $\{0,\ldots,9\}$. Show that the sequence $x_0\,.\,x_1\ldots x_n$ converges.

Solution. Since
$$x_0\,.\,x_1\ldots x_n = x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n},$$
the convergence of the given sequence is equivalent to that of the series
$$\sum_{k=0}^{\infty} \frac{x_k}{10^k}. \tag{12.1.a}$$
Since
$$\left|\frac{x_k}{10^k}\right| \leq 9\frac{1}{10^k} \qquad \forall k \geq 1,$$
it follows from the comparison principle that the series (12.1.a) is absolutely convergent.

Problem 12.2. Let $x_0 \in \mathbb{N}$ and $x_1,\ldots,x_n \in \{0,\ldots,9\}$ and $n \in \mathbb{N}_{\geq 1}$. Prove that
$$x_0\,.\,x_1\ldots x_n + \frac{1}{10^n} = \begin{cases} x_0\,.\,x_1\ldots x_{n-1}(x_n+1) \text{ if } x_n < 9, \\ x_0+1 \text{ if } x_n = x_{n-1} = \cdots = x_1 = 9, \\ x_0\,.\,x_1\ldots x_{i-1}(x_i+1) \text{ if } x_i \neq 9,\ x_{i+1} = \cdots = x_n = 9. \end{cases}$$

Solution. Assume that $x_n < 9$; then
$$\begin{aligned} x_0\,.\,x_1\ldots x_n + \frac{1}{10^n} &= x_0 + \frac{x_1}{10} + \cdots + \frac{x_n+1}{10^n} \\ &= x_0\,.\,x_1\ldots(x_n+1), \end{aligned}$$
since $x_n + 1 \leq 9$. Assume now that $x_n = 9$. Two cases may occur.

- There is $i \geq 1$ with $x_{n-1} = \cdots = x_{i+1} = 9$ and $x_i < 9$. Then

$$
\begin{aligned}
x_0\,.\,x_1 \dots x_n + \frac{1}{10^n} &= x_0 + \frac{x_1}{10} + \cdots + \frac{x_i}{10^i} + \frac{9}{10^{i+1}} + \cdots + \frac{9}{10^n} + \frac{1}{10^n} \\
&= x_0 + \frac{x_1}{10} + \cdots + \frac{x_i}{10^i} + \frac{9}{10^{i+1}}\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-i-1}}\right) + \frac{1}{10^n} \\
&= x_0 + \frac{x_1}{10} + \cdots + \frac{x_i}{10^i} + \frac{9}{10^{i+1}}\frac{1 - \frac{1}{10^{n-i}}}{\frac{9}{10}} + \frac{1}{10^n} \\
&= x_0 + \frac{x_1}{10} + \cdots + \frac{x_i}{10^i} + \frac{1}{10^i}\left(1 - \frac{1}{10^{n-i}}\right) + \frac{1}{10^n} \\
&= x_0 + \frac{x_1}{10} + \cdots + \frac{x_i + 1}{10^i} = x_0\,.\,x_1 \dots (x_i + 1),
\end{aligned}
$$

since $x_i + 1 \leq 9$.

- $x_n = x_{n-1} = \cdots = x_1 = 9$: then

$$
\begin{aligned}
x_0\,.\,x_1 \dots x_n + \frac{1}{10^n} &= x_0 + \frac{9}{10} + \cdots + \frac{9}{10^n} + \frac{1}{10^n} \\
&= x_0 + \frac{9}{10}\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}}\right) + \frac{1}{10^n} \\
&= x_0 + \frac{9}{10}\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} + \frac{1}{10^n} = x_0 + 1.
\end{aligned}
$$

Problem 12.3. Using the Euler-Maclaurin expansion of first-order for a suitable function, estimate the following sums and the error made:

1. $\sum\limits_{1 \leq k < 10} \log k$;
2. $\sum\limits_{1 \leq k < 10} k^{-2}$;
3. $\sum\limits_{0 \leq k < 10} e^{-k}$ (compare this also with the exact sum);
4. $\sum\limits_{1 \leq k < 10} \sqrt{k}$;
5. $\sum\limits_{0 \leq k < 10} k^{3/2}$.

Solution. The sums considered are of the type $\sum\limits_{a \leq k < b} f(k)$ with monotonic f: hence relation (12.27.a) holds. One has:

1.

$$\begin{aligned}\sum_{1\le k<10} \log k &= \int_1^{10} \log x\,dx - \frac{1}{2}\left[\log x\right]_1^{10} + R_1\\ &= \left[x\log x - x\right]_1^{10} - \frac{1}{2}\left[\log x\right]_1^{10} + R_1\\ &= -9 + \frac{19\log 10}{2} + R_1 \approx 12.8746 + R_1,\end{aligned}$$

with $|R_1| \le \left|\frac{1}{2}\left[\log x\right]_1^{10}\right| = \frac{\log 10}{2} \approx 1.15$. Note that the true value of the sum is about 12.8018.

2.

$$\begin{aligned}\sum_{1\le k<10} k^{-2} &= \int_1^{10} x^{-2}\,dx - \frac{1}{2}\left[x^{-2}\right]_1^{10} + R_1\\ &= \left[-x^{-1}\right]_1^{10} - \frac{1}{2}\left[x^{-2}\right]_1^{10} + R_1\\ &= \frac{279}{200} + R_1 \approx 1.395 + R_1,\end{aligned}$$

with $|R_1| \le \left|\frac{1}{2}\left[x^{-2}\right]_1^{10}\right| = \frac{99}{200} \approx 0.495$. Note that the true value of the sum is approximately 1.53977.

3.

$$\begin{aligned}\sum_{0\le k<10} e^{-k} &= \int_0^{10} e^{-x}\,dx - \frac{1}{2}\left[e^{-x}\right]_0^{10} + R_1\\ &= \left[-e^{-x}\right]_0^{10} - \frac{1}{2}\left[e^{-x}\right]_0^{10} + R_1\\ &= \frac{3}{2}\left(1-e^{-10}\right) + R_1 \approx 1.5 + R_1,\end{aligned}$$

with $|R_1| \le \left|\frac{1}{2}\left[e^{-x}\right]_0^{10}\right| = \frac{1}{2}\left(1-e^{-10}\right) \approx 0.5$. The exact value of the sum is

$$1 + e^{-1} + \cdots + (e^{-1})^{10} = \frac{1-e^{-11}}{1-e^{-1}} \approx 1.58.$$

4.

$$\begin{aligned}\sum_{1\le k<10} \sqrt{k} &= \int_1^{10} \sqrt{x}\,dx - \frac{1}{2}\left[\sqrt{x}\right]_1^{10} + R_1\\ &= \left[\frac{2}{3}x^{3/2}\right]_1^{10} - \frac{1}{2}\left[\sqrt{x}\right]_1^{10} + R_1\\ &= \frac{1}{6}\left(-1+37\sqrt{10}\right) + R_1 \approx 19.334 + R_1,\end{aligned}$$

with $|R_1| \le \left|\frac{1}{2}\left[\sqrt{x}\right]_1^{10}\right| = \sqrt{\frac{5}{2}} \approx 1.08$. Note that here the true value of the sum is approximately 19.306.

5.
$$\sum_{0\le k<10} k^{3/2} = \int_0^{10} x^{3/2}\,dx - \frac{1}{2}\left[x^{3/2}\right]_0^{10} + R_1$$
$$= \left[\frac{2}{5}x^{5/2}\right]_0^{10} - \frac{1}{2}\left[x^{3/2}\right]_0^{10} + R_1$$
$$= 35\sqrt{10} + R_1 \approx 110.68 + R_1,$$

with $|R_1| \le \left|\frac{1}{2}\left[x^{3/2}\right]_0^{10}\right| = 5\sqrt{10} \approx 15.81$, so that the true sum yields approximately 111.05.

Problem 12.4. Find the minimum value of n such that the first-order Euler-Maclaurin approximation formula yields an approximation of the sum $\sum_{1\le k<10000} \frac{1}{k^4}$ by means of $\sum_{1\le k<n} \frac{1}{k^4}$ with an error at most equal to 10^{-1}; for such a value compute the approximated value of the given sum.

Solution. Let $f(x) = \frac{1}{x^4}, x \ge 1$. Since $f(\infty) = 0$, it follows from (12.45.a that, for any $n \le 10000$ we can approximate the given sum with

$$\sum_{1\le k<n} \frac{1}{k^4} + \int_n^{10000} \frac{1}{x^4}\,dx - \frac{1}{2}\left(\frac{1}{10000^4} - \frac{1}{n^4}\right)$$

with an error at most equal to $\frac{1}{2n^4}$. We thus have to find n in such a way that

$$\frac{1}{2n^4} \le 10^{-1},$$

and this occurs for $n \ge 2$. With the choice of $n = 2$ we get

$$\sum_{1\le k<10000} \frac{1}{k^4} \approx \sum_{1\le k<2} \frac{1}{k^4} + \int_2^{10000} \frac{1}{x^4}\,dx - \frac{1}{2}\left(\frac{1}{10000^4} - \frac{1}{2^4}\right)$$
$$= \frac{64374999999979997}{60000000000000000} = 1.07291...$$

with an error of at most 10^{-1}. Actually $\sum_{1\le k<10000} \frac{1}{k^4} = 1.08232...$: 1 digit of our approximation is correct.

Problem 12.5. Using the Euler-Maclaurin expansion of second-order estimate the sums and the errors made in the sums of Problem 12.3.

Solution. The sums are all of the type $\sum_{a\le k<b} f(k)$ with monotonic f': thus relation (12.27.b) holds. One then has

1.

$$\sum_{1\le k<10} \log k = \int_1^{10} \log x \, dx - \frac{1}{2}[\log x]_1^{10} + \frac{1}{12}\left[\frac{1}{x}\right]_1^{10} + R_2$$
$$= [x\log x - x]_1^{10} - \frac{1}{2}[\log x]_1^{10} + \frac{1}{12}\left[\frac{1}{x}\right]_1^{10} + R_2$$
$$= -\frac{363}{40} + \frac{19\log 10}{2} + R_2 \approx 12.7996 + R_2,$$

with $|R_2| \le \left|\frac{1}{12}\left[\frac{1}{x}\right]_1^{10}\right| = \frac{3}{40} \approx 0.075$. Note that the true value of the sum is close to 12.8018.

2.

$$\sum_{1\le k<10} k^{-2} = \int_1^{10} x^{-2}\, dx - \frac{1}{2}\left[x^{-2}\right]_1^{10} + \frac{1}{12}\left[-2x^{-3}\right]_1^{10} + R_2$$
$$= \left[-x^{-1}\right]_1^{10} - \frac{1}{2}\left[x^{-2}\right]_1^{10} + \frac{1}{12}\left[-2x^{-3}\right]_1^{10} + R_2$$
$$= \frac{3123}{2000} + R_2 \approx 1.5615 + R_2,$$

with $|R_2| \le \frac{1}{12}\left[-2x^{-3}\right]_1^{10} = \frac{333}{2000} \approx 0.1665$. Observe that the true value of the sum is approximately 1.53977.

3.

$$\sum_{0\le k<10} e^{-k} = \int_0^{10} e^{-x}\, dx - \frac{1}{2}\left[e^{-x}\right]_0^{10} + \frac{1}{12}\left[-e^{-x}\right]_1^{10} + R_2$$
$$= \left[-e^{-x}\right]_0^{10} - \frac{1}{2}\left[e^{-x}\right]_0^{10} + \frac{1}{12}\left[-e^{-x}\right]_0^{10} + R_2$$
$$= \frac{19\left(-1+e^{10}\right)}{12e^{10}} + R_2,$$

with $|R_2| \le \left|\frac{1}{12}[-e^{-x}]_0^{10}\right| = \frac{1}{12}\left(1 - \frac{1}{e^{10}}\right)$.

4.

$$\sum_{1\le k<10} \sqrt{k} = \int_1^{10} \sqrt{x}\, dx - \frac{1}{2}\left[\sqrt{x}\right]_1^{10} + \frac{1}{12}\left[\frac{1}{2}x^{-1/2}\right]_1^{10} + R_2$$
$$= \left[\frac{2}{3}x^{3/2}\right]_1^{10} - \frac{1}{2}\left[\sqrt{x}\right]_1^{10} + \frac{1}{12}\left[\frac{1}{2}x^{-1/2}\right]_1^{10} + R_2$$
$$= \frac{1}{240}(-50 + 1481\sqrt{10}) + R_2 \approx 19.3056 + R_2,$$

with $|R_2| \le \left|\frac{1}{12}\left[\frac{1}{2}x^{-1/2}\right]_0^{10}\right| = \frac{1}{240}\left(-10+\sqrt{10}\right) \approx 0.028$. Note that here the true value of the sum is around 19.306.

5.

$$\begin{aligned}\sum_{0\le k<10} k^{3/2} &= \int_0^{10} x^{3/2}\,dx - \frac{1}{2}\left[x^{3/2}\right]_0^{10} + \frac{1}{12}\left[\frac{3}{2}x^{1/2}\right]_0^{10} + R_2\\ &= \left[\frac{2}{5}x^{5/2}\right]_0^{10} - \frac{1}{2}\left[x^{3/2}\right]_0^{10} + \frac{1}{12}\left[\frac{3}{2}x^{1/2}\right]_0^{10} + R_2\\ &= \frac{281\sqrt{\frac{5}{2}}}{4} + R_2 \approx 111.075 + R_2,\end{aligned}$$

with $|R_2| \le \left|\frac{1}{12}\left[\frac{3}{2}x^{1/2}\right]_0^{10}\right| = \frac{\sqrt{\frac{5}{2}}}{4} \approx 0.395$. Here the true value of the sum is approximately 111.05.

Problem 12.6. Find the minimum value of n such that the second-order Euler-Maclaurin approximation formula yields an approximation of the sum $\sum_{k=1}^{+\infty} \frac{1}{k^4}$ by means of $\sum_{1\le k<n} \frac{1}{k^4}$ with an error at most equal to 10^{-2}; for such a value compute the approximated value of the given sum.

Solution. Let $f(x) = \frac{1}{x^4}$, $x \ge 1$: f is of class $\mathscr{C}^\infty$; moreover f' is monotonic and $f'(\infty) = 0$: (12.45.b) allows to approximate the given sum with an error of at most 10^{-2} if $|\varepsilon_2(n)| \le 10^{-2}$; by Remark 12.47 this occurs if $\frac{1}{12}|f'(n) - f'(\infty)| = \frac{1}{3n^5} \le 10^{-2}$, i.e., for $n \ge \sqrt[5]{\frac{100}{3}} \approx 2.02$. Choosing $n = 3$ we get

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^4} &\approx \sum_{1\le k<3} \frac{1}{k^4} + \int_3^{+\infty} \frac{1}{t^4}\,dt - \frac{1}{2}\left[\frac{1}{t^4}\right]_3^{\infty} + \frac{1}{12}\left[-\frac{4}{t^5}\right]_3^{\infty}\\ &= \frac{12625}{11664} = 1.082390261...\end{aligned}$$

with an error of at most 10^{-2}. Actually $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} = 1.082323234...$: we got 4 digits correct.

Problem 12.7. Prove the existence of the Euler constant for $f(x) := \frac{1}{\sqrt{x}}$, $x \ge 1$; give its first-order approximation with an error less than 10^{-1}.

Solution. The function $x \mapsto f(x) = \frac{1}{\sqrt{x}}$ is decreasing and $f(\infty) = 0$: it follows from Theorem 12.107 that f admits the Euler constant

$$\gamma^f = \lim_{n\to+\infty} \sum_{1\le k<n} \frac{1}{\sqrt{k}} - \int_1^n \frac{1}{\sqrt{x}}\,dx.$$

The first-order approximation of γ^f within10^{-1} is given by

$$\left(\sum_{1\le k<n}\frac{1}{\sqrt{k}}-\int_1^n\frac{1}{\sqrt{x}}\,dx\right)-\frac{1}{2}\left[f\right]_n^\infty$$

if $\dfrac{1}{2\sqrt{n}}\le 10^{-1}$, and this occurs for $n\ge 3$. With $n=3$ we get

$$\begin{aligned}\gamma^f &\approx \left(\sum_{1\le k<3}\frac{1}{\sqrt{k}}-\int_1^3\frac{1}{\sqrt{x}}\,dx\right)+\frac{1}{2\sqrt{3}}\\ &= -\frac{11}{2\sqrt{3}}+3+\frac{1}{\sqrt{2}}=0.53168...\end{aligned}$$

with an error less than 10^{-1}. Actually $\gamma^f=0.53964...$: we got 2 digits correct.

Problem 12.8. Approximate the integral

$$\int_1^8 xe^{-x^{0.8}}\,dx$$

with an error of at most 10^{-2} by means of:

1. The integral approximation of order 2 (12.89.b);
2. The trapezoidal method.

Compare the results with the numerical approximation given by a CAS: which of the two results gives the best approximation?

Solution. Set $g(x)=xe^{-x^{0.8}}, x\in[1,8]$; from (12.89.b), for every $n\in\mathbb{N}_{\ge 1}$ we get

$$\int_1^8 g(x)\,dx=T_n^g-\frac{7^2}{12n^2}\left[g'\right]_1^8+\varepsilon_2(n), |\varepsilon_2(n)|\le\frac{7^3}{12n^2}\|g''\|_\infty.$$

Now

$$g'(x)=e^{-x^{0.8}}-0.8e^{-x^{0.8}}x^{0.8},$$

$$g''(x)=0.64e^{-x^{0.8}}x^{0.6}-\frac{1.44e^{-x^{0.8}}}{x^{0.2}}.$$

A (very) rough estimate of g'' yields

$$\|g''\|_\infty\le e^{-1}\left(1.44+0.64\times 8^{0.6}\right)<1.35$$

(in truth $\|g''\|_\infty=|g''(0)|\approx 0.294$); in order to get $|\varepsilon_2(n)|\le 10^{-2}$ it is thus enough that

$$1.35\frac{7^3}{12n^2}\le 10^{-2},$$

or equivalently,

$$n \geq \sqrt{\frac{100 \times 1.35 \times 7^3}{12}} = \frac{21\sqrt{35}}{2} \approx 62.12;$$

notice that the better estimate $\|g''\| = |g''(0)|$ yields $n > 29$. By taking $n = 63$ we thus get the following approximate value:

$$\int_1^8 xe^{-x^{0.8}}\,dx \approx T_{63}^g - \frac{7^2}{12 \times 63^2}\left[g'\right]_1^8 = 1.30968.$$

The very same estimate in term of the sup-norm of g'' holds for the trapezoidal method; for $n = 63$ the trapezoidal method yields

$$\int_1^8 xe^{-x^{0.8}}\,dx \approx T_{63}^g = 1.30967.$$

By means of a CAS we see that

$$\int_1^8 xe^{-x^{0.8}}\,dx = 1.30976\ldots.$$

Thus (12.89.b) gives a slight better result.

Problem 12.9. Prove that

$$\left(n + \frac{1}{2n}\right)(n^2+1) = n^3 + \frac{3n}{2} + o(1) \quad n \to +\infty.$$

Solution. Developing this relation, one finds that

$$\begin{aligned}\left(n + \frac{1}{2n}\right)(n^2+1) &= n^3 + \frac{n}{2} + n + \frac{1}{2n} = n^3 + \frac{3n}{2} + \frac{1}{2n} \\ &= n^3 + \frac{3n}{2} + o(1) \quad n \to +\infty,\end{aligned}$$

given that $1/n \to 0$ for $n \to +\infty$.

Problem 12.10. Show that

$$1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right) = \left(1 + \frac{2}{n}\right)\left(1 + O\left(\frac{1}{n^2}\right)\right) \quad n \to +\infty.$$

Solution. One has

$$\begin{aligned}\left(1 + \frac{2}{n}\right)\left(1 + O\left(\frac{1}{n^2}\right)\right) &= 1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^3}\right) \\ &= 1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad n \to +\infty,\end{aligned}$$

given that $\frac{1}{n^3} = O\left(\frac{1}{n^2}\right)$.

Problem 12.11. Let a_n and b_n be two sequences, and suppose furthermore that $b_n \neq 0$ for every n. Prove that $a_n = O(b_n)$ if and only if there exists a $C > 0$ such that $|a_n| \leq C|b_n|$ for every n.

Solution. If $|a_n| \leq C|b_n|$ for every n then it is clear that $a_n = O(b_n)$. Vice versa, we suppose that $a_n = O(b_n)$: there exist $M > 0$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leq M|b_n|$ for every $n \geq n_0$: now setting $C = \max\{M, |a_1|/|b_1|, \ldots, |a_{n_0-1}|/|b_{n_0-1}|\}$ one obtains that $|a_n| \leq C|b_n|$ for every n.

Problem 12.12. In each row, which of the two statements implies the other, for $n \to +\infty$?

$$1.a)\ a_n = n^2 + O(n \log n) \quad 1.b)\ a_n = n^2 + 2n \log n + O(n).$$

$$2.a)\ a_n = n^2 + o(n^2) \quad 2.b)\ a_n = n^2 + 2n \log n - 6n + O(n).$$

Solution. One has $1.b) \Rightarrow 1.a)$, given that $n = O(n \log n)$ and $n \log n = O(n \log n)$ for $n \to +\infty$; thus, one has $2.b) \Rightarrow 2.a)$, given that $n \log n = o(n^2)$ and $n = o(n^2)$ for $n \to +\infty$.

Problem 12.13. Let $\alpha, \beta \in \mathbb{R}$, $a > 0$ and $a_n \sim \alpha n^a$ for $n \to +\infty$ and $b_n \sim \beta n^a$ for $n \to +\infty$, with $\alpha + \beta \neq 0$. Prove that $a_n + b_n \sim (\alpha + \beta) n^a$ for $n \to +\infty$.

Solution. One has $a_n = \alpha n^a + o(n^a), b_n = \beta n^a + o(n^a)$ for $n \to +\infty$ from which it follows that $a_n + b_n = (\alpha + \beta) n^a + o(n^a)$; since $\alpha + \beta \neq 0$ one immediately obtains the conclusion sought.

Problem 12.14. Let a_n be a sequence such that $\lim_{n \to +\infty} a_n = \ell \in \mathbb{R}$. Then $a_n = \ell + O(1)$ for $n \to +\infty$.

Solution. Indeed, $a_n - \ell = o(1) \subseteq O(1)$ for $n \to +\infty$.

Problem 12.15. Let $f : [a, +\infty[\to \mathbb{R}$ be locally integrable, and assume that the limit $\ell := \lim_{x \to +\infty} f(x)$ exists in $\mathbb{R} \cup \{\pm\infty\}$. Prove that the limit $\lim_{\substack{n \to +\infty \\ n \in \mathbb{N}}} \int_a^n f(t)\, dt$ and the generalized integral $\int_a^{+\infty} f(t)\, dt$ have the same behavior: both exist and are equal or both do not exist.

Solution. If $\ell \neq 0$ the limit and the integral do both diverge; assume that $\ell = 0$ and fix $\varepsilon > 0$. There is $b \in \mathbb{N}$ such that $|f(x)| \leq \varepsilon$ for all $x \geq b$. As a consequence, for all $x \geq b$ we have

$$\left| \int_a^x f(t)\, dt - \int_a^{[x]} f(t)\, dt \right| \leq \int_{[x]}^{[x]+1} |f(t)|\, dt \leq \varepsilon,$$

whence the conclusion.

Problem 12.16. The function $\dfrac{\sin x}{x^2}$ is summable on $[1,+\infty[$, due to the fact that $\left|\dfrac{\sin x}{x^2}\right| \le 1/x^2 \in L^1([1,+\infty[)$. Set $a := \displaystyle\int_1^{+\infty} \frac{\sin t}{t^2}\,dt$; it can be shown that $a \approx 0.504067$. Let $f(x) := \displaystyle\int_1^x \frac{\sin t - a}{t^2}\,dt$, for $x > 1$. Study the convergence of the series $\displaystyle\sum_{k=1}^{\infty} f(k)$:

1. Prove that f is not monotonic, $f(\infty) = 0$, and $f' \in L^1([1,+\infty[)$;
2. Deduce that $\displaystyle\sum_{k=1}^{\infty} f(k)$ and $\displaystyle\lim_{n\to+\infty}\int_1^n f(x)\,dx$ have the same behavior;
3. Prove that $\displaystyle\lim_{n\to+\infty}\int_1^n f(x)\,dx = +\infty$.

Solution. It is clear that $\displaystyle\lim_{x\to+\infty} f(x) = 0$ and that f is not monotonic, since the sign of $f'(x)$ is that of $\sin x - a$. Since $|f'(x)| \le 2/x^2$ then $f' \in L^1([1,+\infty[)$. It follows from Theorem 12.78 that the series $\displaystyle\sum_{k=1}^{\infty} f(k)$ and the limit $\displaystyle\lim_{b\to+\infty}\int_1^b f(x)\,dx$ have the same behavior, in this case they do both diverge to $+\infty$. Indeed, for $b > 1$ we have

$$\begin{aligned}
F(b) := \int_1^b f(x)\,dx &= \int_1^b \int_1^x \frac{\sin t - a}{t^2}\,dt\,dx \\
&= \int_{1\le t\le x\le b} \frac{\sin t - a}{t^2}\,dx\,dt = \int_1^b \int_t^b \frac{\sin t - a}{t^2}\,dx\,dt \\
&= \int_1^b (b-t)\frac{\sin t - a}{t^2}\,dt = b\int_1^b \frac{\sin t - a}{t^2}\,dt - \int_1^b \frac{\sin t}{t}\,dt + a\log t.
\end{aligned}$$

Now, the integral $\displaystyle\int_1^{+\infty} \frac{\sin t}{t}$ is finite, and since $\displaystyle\int_1^{+\infty} \frac{\sin t - a}{t^2}\,dt = 0$ we get (the idea is due to G. De Marco)

$$\int_1^b \frac{\sin t - a}{t^2}\,dt = -\int_b^{+\infty} \frac{\sin t - a}{t^2}\,dt$$

whence

$$\left| b\int_1^b \frac{\sin t - a}{t^2}\,dt \right| \le |b| \int_b^{+\infty} \frac{|\sin t| + |a|}{t^2}\,dt \le |b| \int_b^{+\infty} \frac{|a| + 1}{t^2}\,dt = |a| + 1,$$

so that $b\displaystyle\int_1^b \frac{\sin t - a}{t^2}\,dt$ is bounded. It follows that $\displaystyle\lim_{b\to+\infty} F(b) = +\infty$.

Problem 12.17. Prove relation (12.65.c) in the case $f'(\infty) \in \{0, \pm\infty\}$.

Solution. From the Euler-Maclaurin formula (12.27.b) one gets

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx - \frac{f(n)}{2} + \frac{f(a)}{2} + \frac{1}{12}\left[f'\right]_a^n + R_2(n) =$$

$$= \int_a^n f(x)\,dx - \frac{f(n)}{2} + \frac{1}{12}f'(n) + \frac{f(a)}{2} - \frac{1}{12}f'(a) + R_2(n) \qquad (12.17.\text{a})$$

where $R_2(n) = -\dfrac{1}{2}\displaystyle\int_a^n f''(x)\,\mathrm{B}_2(x-[x])\,dx$. Since $|\,\mathrm{B}_2(x)| \le \dfrac{1}{6}$ for $x \in [0,1]$, one has that $|\,\mathrm{B}_2(x-[x])| \le \dfrac{1}{6}$ for every $x \ge a$ and thus

$$|R_2(n)| \le \frac{1}{12}\int_a^n |f''(x)|\,dx. \qquad (12.17.\text{b})$$

We consider the two following alternatives:

i) $f'(\infty) = \pm\infty$ (in this case there exists $M > 0$ such that $|f'| \ge M$ definitively);
ii) $f'(\infty) = 0$.

Suppose that i) holds. Since f' is definitively monotonic, for some $b \ge a$ one has that f'' has a constant sign on the interval $[b,+\infty[$; therefore for $n \ge b$ one has

$$\int_b^n \left|f''(x)\right|\,dt = \left|\int_b^n f''(x)\,dt\right| = \left|f'(n) - f'(b)\right| \le |f'(n)| + |f'(b)|. \qquad (12.17.\text{c})$$

From relation (12.17.b) one then deduces that for $n \ge b$ the following estimate holds:

$$\begin{aligned} |R_2(n)| &\le \frac{1}{12}\int_a^n |f''(x)|\,dx \le \frac{1}{12}\int_a^b |f''(x)|\,dx + \frac{1}{12}\int_b^n |f''(x)|\,dx \\ &\le c + \frac{1}{12}|f'(n)|, \end{aligned}$$

where $c = \dfrac{1}{12}\left(\displaystyle\int_a^b |f''(x)|\,dx + |f'(b)|\right)$ is a constant that depends only on f. Obviously $|f'(n)|$ and every constant is $O(f'(n))$ for $n \to +\infty$: from relation (12.17.a) one thus obtains that

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx - \frac{f(n)}{2} + O(f'(n)) \qquad n \to +\infty,$$

which is relation (12.65.c) with $C^2 = 0$.

Now let us suppose instead that condition ii) holds. Then the integral $\displaystyle\int_a^{+\infty} |f''(x)|\,dt$ exists and is finite: indeed, since f' is definitively monotonic, for some $b \ge a$ one finds that f'' has a constant sign on the interval $[b,+\infty[$; therefore, since $\lim\limits_{x\to+\infty} f'(x) = 0$ one has that

$$\int_a^{+\infty} \left|f''(x)\right|\,dt = \int_a^b \left|f''(x)\right|\,dt + \int_b^{+\infty} \left|f''(x)\right|\,dt =$$

$$= \int_a^b \left|f''(x)\right| dt + \left|\int_b^{+\infty} f''(x)\, dt\right| = \int_a^b \left|f''(x)\right| dt + \left|f'(b)\right|. \tag{12.17.d}$$

Then, from the boundedness of $\mathrm{B}_2(x)$ on $[0,1]$ it follows that $f''(x)\,\mathrm{B}_2(x-[x])$ is absolutely integrable on $[a,+\infty[$ and hence that the integral $\int_a^{+\infty} f''(x)\,\mathrm{B}_2(x-[x])\,dx$ exists and is finite. Hence we can then write

$$\begin{aligned} R_2(n) &= -\frac{1}{2}\int_a^n f''(x)\,\mathrm{B}_2(x-[x])\,dx \\ &= -\frac{1}{2}\int_a^{+\infty} f''(x)\,\mathrm{B}_2(x-[x])\,dx + \frac{1}{2}\int_n^{+\infty} f''(x)\,\mathrm{B}_2(x-[x])\,dx; \end{aligned}$$

relation (12.17.a) then yields

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx + C^2 - \frac{f(n)}{2} + \frac{f'(n)}{12} + \widetilde{R}(n), \tag{12.17.e}$$

where we have set

$$C^2 = -\frac{1}{2}\int_a^{+\infty} f''(x)\,\mathrm{B}_2(x-[x])\,dx + \frac{1}{2}f(a) - \frac{f'(a)}{12},$$

$$\widetilde{R}(n) = \frac{1}{2}\int_n^{+\infty} f''(x)\,\mathrm{B}_2(x-[x])\,dx.$$

Finally, for (12.17.d) with n used in the place of b, since $|\,\mathrm{B}_2(x)| \le \dfrac{1}{6}$ for $x \in [0,1]$, it follows that

$$|\widetilde{R}(n)| \le \frac{1}{12}\left|\int_n^{+\infty} f''(x)\,dx\right| = \frac{1}{12}|f'(n)| = O(f'(n)) \qquad n \to +\infty.$$

From (12.17.e) one then obtains

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx + C^2 - \frac{f(n)}{2} + O(f'(n)).$$

Problem 12.18. Furnish a first-order asymptotic estimate for:

1. $\sum_{1\le k<n} \log k$;
2. $\sum_{1\le k<n} k^{-2}$;
3. $\sum_{0\le k<n} e^{-k}$;
4. $\sum_{0\le k<n} \sqrt{k}$;
5. $\sum_{0\le k<n} k^{3/2}$.

Solution. It is a question of sums of the type $\sum_{a \le k < n} f(k)$ with monotonic f: we can apply formula (12.61.c).

1.
$$\sum_{1 \le k < n} \log k = \int_1^n \log x \, dx + C + O(\log n) \quad n \to +\infty$$
$$= n \log n - n + C' + O(\log n) \quad n \to +\infty,$$
where C and C' are constants that do not depend on n.

2.
$$\sum_{1 \le k < n} k^{-2} = \int_1^n x^{-2} \, dx + C + O(n^{-2}) \quad n \to +\infty$$
$$= -\frac{1}{n} + C' + O(n^{-2}) \quad n \to +\infty,$$
where C and C' are constants that do not depend on n.

3.
$$\sum_{0 \le k < n} e^{-k} = \int_0^n e^{-x} \, dx + C + O(e^{-n}) \quad n \to +\infty$$
$$= -e^{-n} + C' + O(e^{-n}) = C'' + O(e^{-n}) \quad n \to +\infty,$$
where C, C', C'' are constants that do not depend on n.

4.
$$\sum_{0 \le k < n} \sqrt{k} = \int_0^n \sqrt{x} \, dx + C + O(\sqrt{n}) \quad n \to +\infty$$
$$= \frac{2}{3} n^{3/2} + C + O(\sqrt{n}) \quad n \to +\infty,$$
where C is a constant that does not depend on n.

5.
$$\sum_{0 \le k < n} k^{3/2} = \int_0^n x^{3/2} \, dx + C + O(n^{3/2}) \quad n \to +\infty$$
$$= \frac{2}{5} n^{5/2} + C + O(n^{3/2}) \quad n \to +\infty,$$
where C is a constant that does not depend on n.

Problem 12.19. Using the asymptotic versions of the Euler-Maclaurin formula (Theorem 12.61), provide approximations for $\sum_{1 \le k < n} k \log k$.

Solution. The first-order asymptotic formula (Theorem 12.61) with $f(x) = x \log x$ yields

$$\begin{aligned}
\sum_{1\le k<n} k\log k &= \int_1^n x\log x\,dx + C - \frac{n\log n}{2} + O(\log n)\\
&= \left[-\frac{x^2}{4}+\frac{1}{2}x^2\log x\right]_1^n - \frac{n\log n}{2} + O(\log n)\\
&= -\frac{n^2}{4}+\frac{1}{2}n^2\log n - \frac{n\log n}{2} + O(\log n), \quad n\to+\infty.
\end{aligned}$$

Solution.

Problem 12.20. Let $p>0$. Show that

$$\sum_{1\le k<n} \frac{\log^p(k)}{k} = \frac{1}{p+1}\log^{p+1}(n) + O\left(\frac{\log^p(n)}{n}\right) \quad n\to+\infty.$$

Solution. Let $f(x)=\dfrac{\log^p(x)}{x}$, $x\ge 1$. The function f tends to 0 at infinity. Since

$$f'(x) = \frac{p\log^{p-1}(x)}{x^2} - \frac{\log^p(x)}{x^2} = \frac{\log^p(x)}{x^2}\left(\frac{p}{\log x}-1\right) > 0 \quad \forall x > e^p$$

then f is definitely monotonic. It follows from Theorem 12.61 that

$$\sum_{1\le k<n} \frac{\log^p(k)}{k} = \int_1^n \frac{\log^p(x)}{x}\,dx + O\left(\frac{\log^p(n)}{n}\right), \quad n\to+\infty.$$

Now, the change of variables $t=\log x$ yields

$$\int_1^n \frac{\log^p(x)}{x}\,dx = \int_0^{\log n} t^p\,dt = \frac{1}{p+1}\log^{p+1}(n).$$

Problem 12.21. Assume that $f:[a,+\infty[\to\mathbb{R}$ is of class $\mathscr{C}^1$, definitively monotonic and tends to 0 at infinity, and moreover that $f'(n)=O(f(n))$ as $n\to+\infty$. If (12.65.c) holds true, deduce the validity of (12.61.c).

Solution. Since $f(\infty)=0$ and $f'=O(f(n))$ as $n\to+\infty$ then $f'(\infty)=0$: (12.65.c) implies that

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx + C_2^f - \frac{f(n)}{2} + O\big(f'(n)\big) \quad n\to+\infty.$$

The fact that $f'=O(f(n))$ as $n\to+\infty$ yields

$$\begin{aligned}
\sum_{a\le k<n} f(k) &= \int_a^n f(x)\,dx + C_2^f - \frac{f(n)}{2} + O\left(f(n)\right)\\
&= \int_a^n f(x)\,dx + C_2^f + O\left(f(n)\right) \quad n\to+\infty.
\end{aligned}$$

Since $f(\infty) = 0$ we deduce that

$$C_2^f = \lim_{n \to +\infty} \left(\sum_{a \le k < n} f(k) - \int_a^n f(x)\,dx \right) = \gamma^f,$$

thus proving the validity of (12.61.c).

Chapter 13
Problems and Solutions

Problem 13.1. Prove that the Bernoulli polynomial functions $\mathrm{B}_m(x)$ with $m \geq 2$ have the following properties:

1. If m is divisible by 4 then there exists a real number a such that $a < 1/2$ and

$$\begin{cases} \mathrm{B}_m(x) < 0 \text{ on } [0, a[; \\ \mathrm{B}_m(x) > 0 \text{ on }]a, 1-a[; \\ \mathrm{B}_m(x) < 0 \text{ on }]1-a, 1]. \end{cases}$$

2. If $m-1$ is divisible by 4 then

$$\begin{cases} \mathrm{B}_m(x) < 0 \text{ on }]0, 1/2[; \\ \mathrm{B}_m(x) > 0 \text{ on }]1/2, 1[. \end{cases}$$

3. If $m-2$ is divisible by 4 then there exists $a < 1/2$ such that

$$\begin{cases} \mathrm{B}_m(x) > 0 \text{ on } [0, a[; \\ \mathrm{B}_m(x) < 0 \text{ on }]a, 1-a[; \\ \mathrm{B}_m(x) > 0 \text{ on }]1-a, 1]. \end{cases}$$

4. If $m-3$ is divisible by 4 then

$$\begin{cases} \mathrm{B}_m(x) > 0 \text{ on }]0, 1/2[; \\ \mathrm{B}_m(x < 0 \text{ on }]1/2, 1[. \end{cases}$$

Solution. Here we make use of the properties established in Proposition 13.4. We begin by treating the case $m = 2$. The function $\mathrm{B}_2(x) = x^2 - x + 1/6$ has graph that is symmetric with respect to the axis $x = 1/2$, and with derivative $\mathrm{B}_2'(x) = 2x - 1$ vanishing at $1/2$, negative on $[0, 1/2[$ and positive on $]1/2, 1[$. One then has $\mathrm{B}_2(1/2) < 0$, $\mathrm{B}_2(0) = \mathrm{B}_2(1) = 1/6$: B_2 vanishes in a point $a \in]0, 1/2[$ and also at $1-a$, so that $\mathrm{B}_2 > 0$ on $[0, a[$, while $\mathrm{B}_2 < 0$ on $]a, 1-a[$, and $\mathrm{B}_2 > 0$ on $]1-a, 1]$.

We proceed by induction on m, and suppose that $m \geq 2$ and that the assertion is valid up to a given m. Given that $\mathrm{B}'_{m+1} = (m+1)\,\mathrm{B}_m$ the properties of B_{m+1} may be deduced from those of B_m:

1. Suppose that $(m+1)-3$ is divisible by 4, so that $m = 2$ or $m-2 \geq 2$ is divisible by 4. From what has been checked in the case $m = 2$ or by the inductive hypothesis there is an $a \in]0,1/2[$ such that
$$\begin{cases} \mathrm{B}_m(x) > 0 \text{ on } [0,a[, \\ \mathrm{B}_m(x) < 0 \text{ on }]a,1-a[, \\ \mathrm{B}_m(x) > 0 \text{ on }]1-a,1]. \end{cases}$$
Given that $\mathrm{B}_{m+1}(0) = 0$ one finds that $\mathrm{B}_{m+1} > 0$ on $]0,1/2[$ and $\mathrm{B}_{m+1} < 0$ on $]1/2,1[$.
2. Suppose that $m+1$ is divisible by 4, so that $m = 3$ (treated in the preceding case) or $m-3 \geq 2$ is divisible by 4. In view of what we have seen just above if $m = 3$, or by the inductive hypothesis if $m \geq 5$ one finds that
$$\begin{cases} \mathrm{B}_m(x) > 0 \text{ on }]0,1/2[, \\ \mathrm{B}_m(x) < 0 \text{ on }]1/2,1[. \end{cases}$$
Therefore B_{m+1} is increasing on $[0,1/2]$ and symmetrically decreasing on $[1/2,1]$. By Proposition 13.4 we know that B_{m+1} vanishes in two points a and $1-a$ that are symmetrical with respect to $x = 1/2$ and hence $\mathrm{B}_{m+1} < 0$ on $[0,a[$, $\mathrm{B}_{m+1} > 0$ on $]a,1-a[$, $\mathrm{B}_{m+1} < 0$ on $]1-a,1]$.
3. Suppose now that $(m+1)-1 = m$ is divisible by 4. By the inductive hypothesis there is an $a \in]0,1/2[$ such that
$$\begin{cases} \mathrm{B}_m(x) < 0 \text{ on } [0,a[, \\ \mathrm{B}_m(x) > 0 \text{ on }]a,1-a[, \\ \mathrm{B}_m(x) < 0 \text{ on }]1-a,1]. \end{cases}$$
Given that $\mathrm{B}_{m+1}(0) = 0$ one then has $\mathrm{B}_{m+1} < 0$ on $]0,1/2[$ and $\mathrm{B}_{m+1} > 0$ on $]1/2,1[$.
4. Suppose in this case that $(m+1)-2 = m-1$ is divisible by 4. In view of the inductive hypothesis one now has that
$$\begin{cases} \mathrm{B}_m(x) < 0 \text{ on }]0,1/2[, \\ \mathrm{B}_m(x) > 0 \text{ on }]1/2,1[. \end{cases}$$
Therefore, B_{m+1} decreases on $[0,1/2]$ and increases symmetrically on $[1/2,1]$. By Proposition 13.4 we know that B_{m+1} vanishes in two points a and $1-a$ which are symmetric with respect to the point $x = 1/2$ and so $\mathrm{B}_{m+1} > 0$ on $[0,a[$, $\mathrm{B}_{m+1} < 0$ on $]a,1-a[$, $\mathrm{B}_{m+1} > 0$ on $]1-a,1]$.

Thus our assertions are completely proved.

Problem 13.2. Find the Euler-Maclaurin expansions of order 2 and 4 of $f(x) = 1/x^3$ on $[1,100]$, and give in both cases an estimate of their difference with $S := \sum_{1\le k<100} \frac{1}{k^3}$.

Solution. The Euler-Maclaurin expansion of f of order 2 equals

$$\left[-1/(2x^2) - 1/(2x^3) - (1/(4x^4))\right]_1^{100} \approx 1.250.$$

Taking into account Remark 13.18, the remainder R_2 in (13.18.a) is such that

$$|R_2| \le \frac{|\mathrm{B}_2|}{2!}(f'(100) - f'(1)) = \frac{1}{12}(1 - 1/100^4) \approx 0.083.$$

The Euler-Maclaurin of order 4 is

$$\left[-1/(2x^2) - 1/(2x^3) - (1/(4x^4)) + (1/(12x^6))\right]_1^{100} \approx 1.167.$$

Taking into account Remark 13.18, the remainder R_4 in (13.18.a) is such that

$$|R_4| \le \frac{|\mathrm{B}_4|}{4!}(f^{(3)}(100) - f^{(3)}(1)) = \frac{1}{12}(1 - 1/100^6) \approx 0.083$$

It is interesting to remark that a CAS gives $\sum_{1\le k<100} \frac{1}{k^3} \approx 1.202$.

Problem 13.3. Let $f : [a,+\infty[\to \mathbb{R}$ a function of class $\mathscr{C}^m$ for some $m \ge 1$. Suppose moreover that the integral $\int_a^{+\infty} |f^{(m)}(x)|\,dx$ exists and is finite and that the series $\sum_{k=a}^{\infty} f(k)$ converges. Prove that there exists a constant c_m^f, depending only on m and on f, such that

$$\int_a^n f(x)\,dx - \left(c_m^f - \sum_{i=1}^{m} \frac{\mathrm{B}_i}{i!} f^{(i-1)}(n)\right) \to 0 \qquad \text{for } n \to +\infty.$$

Solution. In view of (13.45.a), for some C_m^f in $\mathbb{R}$ one has

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx + C_m^f + \sum_{i=1}^{m} \frac{\mathrm{B}_i}{i!} f^{(i-1)}(n) + O\left(\int_n^{+\infty} |f^{(m)}(x)|\,dx\right) \qquad n \to +\infty.$$

Furthermore, from the convergence of the given series one has

$$\sum_{a\le k<n} f(k) = \sum_{k=a}^{\infty} f(k) + o(1) \qquad n \to +\infty.$$

On setting $c_m^f := \sum_{k=a}^{\infty} f(k) - C_m^f$, and bearing in mind Remark 13.46, one immediately obtains the desired conclusion.

Chapter 14
Problems and Solutions

Problem 14.1. Provide asymptotic estimates for the following Cauchy sums:

1. $\sum_{1\le k<n} e^{-k/n^2}$;
2. $\sum_{1\le k<n} e^{-k/\sqrt{n}}$;
3. $\sum_{1\le k<n} \dfrac{n}{(k+n^{1/2})(k+2n^{1/2})}$;
4. $\sum_{1\le k<n} \dfrac{n^4}{(k+n^2)(k+2n^2)}$.

Solution.

1. The sum coincides with $\sum_{1\le k<n} g\left(k/n^2\right)$ where $g(x) = e^{-x}$; applying relation (14.18.a) one obtains

$$\sum_{1\le k<n} e^{-k/n^2} = ng(0) + O(n) = n + O(n) \quad n \to +\infty.$$

2. This sum coincides with $\sum_{1\le k<n} g\left(k/n^{1/2}\right)$ where once again $g(x) = e^{-x}$: note that g and g' are integrable in the generalized sense on the interval $[0,+\infty[$: applying relation (14.13.a) one obtains

$$\sum_{1\le k<n} e^{-k/\sqrt{n}} = n^{1/2}\int_0^{+\infty} e^{-t}\,dt + o(n^{1/2}) = \sqrt{n} + o(\sqrt{n}) \quad n \to +\infty.$$

3. On dividing the numerator and the denominator by n one has

$$\sum_{1\le k<n} \frac{n}{(k+n^{1/2})(k+2n^{1/2})} = \sum_{1\le k<n} \frac{1}{(k/n^{1/2}+1)(k/n^{1/2}+2)} :$$

it is a question involving $\sum_{1\le k<n} g\left(k/n^{1/2}\right)$ with $g(x)=\dfrac{1}{(x+1)(x+2)}$: note that both g and $|g'|$ are integrable in the generalized sense on the interval $[0,+\infty[$: applying relation (14.13.a) one obtains

$$\begin{aligned}\sum_{1\le k<n}\frac{1}{(k/n^{1/2}+1)(k/n^{1/2}+2)} &= n^{1/2}\int_0^{+\infty}\frac{1}{(t+1)(t+2)}\,dt+o(n^{1/2})\\ &= n^{1/2}\left[\log\frac{1+t}{2+t}\right]_0^{+\infty}+o(n^{1/2})\\ &= n^{1/2}\log 2+o(n^{1/2})\quad n\to+\infty.\end{aligned}$$

4. On dividing both the numerator and the denominator by n^4 one finds that

$$\sum_{1\le k<n}\frac{n^4}{(k+n^2)(k+2n^2)}=\sum_{1\le k<n}\frac{1}{(k/n^2+1)(k/n^2+2)}:$$

it is a question of the sum $\sum_{1\le k<n} g\left(k/n^2\right)$ with $g(x)=\dfrac{1}{(x+1)(x+2)}$: applying relation (14.18.a) one gets

$$\sum_{1\le k<n}\frac{1}{(k/n^2+1)(k/n^2+2)}=ng(0)+O(1)=\frac{n}{2}+O(1),\,n\to+\infty.$$

Problem 14.2. Let $(a_{k,n})_n$ and $(b_{k,n})_n$ be two families of sequences, with $b_{k,n}\neq 0$ for all k,n, and such that $\lim_{n\to+\infty}\dfrac{a_{k,n}}{b_{k,n}}=0$ uniformly for $k\le k_n$, where k_n is a sequence of positive terms. Prove that $O(a_{k,n})=O(b_{k,n})$ for $n\to+\infty$, uniformly for $k\le k_n$, or, in other words, that if $c_{k,n}=O(a_{k,n})$ for $n\to+\infty$ uniformly for $k\le k_n$ then $c_{k,n}=O(b_{k,n})$ for $n\to+\infty$ uniformly for $k\le k_n$.

Solution. Given that in the case at hand one has $a_{k,n}=O(b_{k,n})$ uniformly for $k\le k_n$ the statement follows immediately from the transitivity of the relation of type "big O" seen in Proposition 14.39.

Problem 14.3 (Ramanujan R-distribution). Ramanujan R-distribution is

$$\forall n,k\in\mathbb{N},\quad k\le n,\qquad R(n,k)=\frac{n!n^k}{(n+k)!}.$$

Show that
$$\frac{n!n^k}{(n+k)!}=e^{-k^2/(2n)}\left(1+O\left(\frac{1}{n}\right)\right)\quad n\to+\infty.$$

Solution. One has

$$R(n,k)=\frac{n^k}{(n+k)(n+k-1)\cdots(n+1)}=\frac{1}{\left(1+\frac{k}{n}\right)\cdots\left(1+\frac{1}{n}\right)},$$

from which it follows that

$$\log R(n,k) = -\left[\log\left(1+\frac{k}{n}\right)+\cdots+\log\left(1+\frac{1}{n}\right)\right].$$

The proof then proceeds in a fashion completely analogous to that used in Theorem 14.53.

Problem 14.4. Prove Proposition 14.15 using the Euler-Maclaurin expansion of first rank (that is, of order 1) to replace the role played by the trapezoidal method.

Solution.

1. We start by proving relation (14.15.a). The Euler-Maclaurin formula 12.27.a of first rank (that is, of order 1 from Theorem 12.27) yields

$$\sum_{a\le k<n} g\left(\frac{k}{n}\right) = \int_a^n g\left(\frac{x}{n}\right)dx - \frac{1}{2}\left(g(1)-g\left(\frac{a}{n}\right)\right)+R_1(n), \qquad (14.4.a)$$

with

$$|R_1(n)| \le \frac{1}{2n}\int_a^n \left|g'\left(\frac{x}{n}\right)\right| dx = \frac{1}{2}\int_{a/n}^1 |g'(t)|\,dt \le \frac{1}{2}\int_0^1 |g'(t)|\,dt.$$

Now

$$\int_a^n g\left(\frac{x}{n}\right)dx = n\int_{a/n}^1 g(t)\,dt = n\int_0^1 g(t)\,dt - n\int_0^{a/n} g(t)\,dt.$$

Given that both g and g' are bounded on $[0,1]$, inasmuch as they are continuous on a closed and bounded interval, all the terms that follow the integral in (14.4.a) are bounded. Then, since

$$\left|n\int_0^{a/n} g(t)\,dt\right| \le n\int_0^{a/n} \|g\|_{L^\infty(0,1)}\,dt \le \|g\|_{L^\infty(0,1)},$$

one has

$$\sum_{a\le k<n} g\left(\frac{k}{n}\right) = n\int_0^1 g(x)\,dx + O(1) \qquad n\to+\infty.$$

2. We prove relation (14.15.b). We put $f(x)=g(x/n)$. The Euler-Maclaurin formula (12.27.b) yields

$$\sum_{a\le k<n} f(k) = \int_a^n f(x)\,dx - \frac{f(n)-f(a)}{2} + \frac{1}{12}\left[f'\right]_a^n + R_2(n),$$

with $|R_2(n)| \le \dfrac{1}{12}\displaystyle\int_a^n \left|f''(x)\right| dx$. Now one has

$$f'(x)=\frac{1}{n}g'\left(\frac{x}{n}\right), \quad f''(x)=\frac{1}{n^2}g''\left(\frac{x}{n}\right);$$

and hence

$$\begin{aligned}\frac{f(n)-f(a)}{2}&=\frac{g(1)-g(a/n)}{2}=\frac{g(1)-g(0)}{2}+\frac{g(0)-g(a/n)}{2}\\&=\frac{g(1)-g(0)}{2}+O\left(\frac{1}{n}\right) \qquad n\to+\infty\end{aligned}$$

given that, on using the Taylor series expansion with the Peano remainder, one has

$$g(a/n)=g(0)+g'(0)\frac{a}{n}+O\left(\frac{1}{n}\right) \quad n\to+\infty.$$

Moreover, one has

$$\begin{aligned}\left[f'\right]_a^n&=\frac{g'(1)-g'(a/n)}{n}=\frac{g'(1)-g'(0)}{n}+\frac{g'(0)-g'(a/n)}{n}\\&=\frac{g'(1)-g'(0)}{n}+O\left(\frac{1}{n}\right) \qquad n\to+\infty,\end{aligned}$$

given that use of the Taylor series expansion with Peano remainder for the function g' yields

$$g'(a/n)=g'(0)+g''(0)\frac{a}{n}+O\left(\frac{1}{n}\right)=g'(0)+O\left(\frac{1}{n}\right) \quad n\to+\infty.$$

By way of the usual change of variable $t=x/n$ in the integrals one then has

$$\int_a^n f(x)\,dx=n\int_{a/n}^1 g(t)\,dt=n\int_0^1 g(t)\,dt-n\int_0^{a/n} g(t)\,dt. \tag{14.4.b}$$

Applying the Taylor series expansion to the function $F(x)=\int_0^x g(t)\,dt$ in the neighborhood of $x=0$ one finds that

$$\int_0^x g(t)\,dt=g(0)x+O(x^2) \quad x\to 0.$$

Indeed, one has $F'(x)=g(x)$. Hence, one has

$$\int_0^{a/n} g(t)\,dt=g(0)\frac{a}{n}+O\left(\frac{1}{n^2}\right) \quad n\to+\infty$$

henceforth

$$n\int_0^{a/n} g(t)\,dt=ag(0)+O\left(\frac{1}{n}\right) \quad n\to+\infty,$$

so that relation (14.4.b) yields

$$\int_a^n f(x)\,dx = n\int_0^1 g(t)\,dt - ag(0) + O\left(\frac{1}{n}\right) \quad n\to+\infty.$$

The conclusion is now immediate.

Problem 14.5. Give asymptotic estimates for both the first- and second- order for the following Riemann sums:

1. $\sum_{1\le k<n} e^{-k/n}$;
2. $\sum_{1\le k<n} \frac{n^2}{(k+n)(k+2n)}$.

Solution.

1. The first sum coincides with $\sum_{1\le k<n} g(k/n)$ where $g(x)=e^{-x}$; on applying relation (14.18.a) one gets

$$\sum_{1\le k<n} e^{-k/n} = n\int_0^1 g(x)\,dx + O(1) = n(1-e^{-1}) + O(1) \quad n\to+\infty.$$

Applying relation (14.15.b) then yields

$$\begin{aligned}\sum_{1\le k<n} e^{-k/n} &= n\int_0^1 g(x)\,dx - \frac{1}{2}\,[g]_0^1 - g(0) + O(1/n)\\ &= n(1-e^{-1}) - \frac{1}{2}(e^{-1}-1) - 1 + O(1/n)\\ &= n(1-e^{-1}) - \frac{1}{2e} - \frac{1}{2} + O(1/n) \quad n\to+\infty.\end{aligned}$$

Note that the given sum has exact value $\frac{(-1+e)e^{-1+\frac{1}{n}}}{-1+e^{\frac{1}{n}}}$.

2. This sum coincides with $\sum_{1\le k<n} g(k/n)$ where $g(x)=\frac{1}{(1+x)(2+x)}$; applying relation (14.18.a) one gets

$$\begin{aligned}\sum_{1\le k<n} \frac{n^2}{(k+n)(k+2n)} &= n\int_0^1 g(x)\,dx + O(1)\\ &= n\left[\log\frac{1+x}{2+x}\right]_0^1 + O(1) = n\log\frac{4}{3} + O(1) \quad n\to+\infty.\end{aligned}$$

On applying relation (14.15.b) one obtains

$$\begin{aligned}\sum_{1\le k<n}\frac{n^2}{(k+n)(k+2n)} &= n\int_0^1 g(x)\,dx-\frac{1}{2}\,[g]_0^1-g(0)+O(1/n)\\ &= n\log\frac{4}{3}-\frac{1}{2}\left(\frac{1}{6}-\frac{1}{2}\right)-\frac{1}{2}+O(1/n)\\ &= n\log\frac{4}{3}-\frac{1}{3}+O(1/n) \quad n\to+\infty.\end{aligned}$$

Problem 14.6. Furnish an estimate of $\sum_{0\le k<n}\frac{1}{n^2+k^2}$ by means of (14.15.a) and (14.15.b). Compare, for $n=10$, the approximations with the actual value of the sum.

Solution. We observe that on setting $g(x)=\frac{1}{1+x^2}$, one has

$$\frac{1}{n^2+k^2}=\frac{1}{n^2}g\left(\frac{k}{n}\right).$$

(14.15.a) yields

$$\sum_{0\le k<n}g\left(\frac{k}{n}\right)\sim n\int_0^1 g(x)\,dx=n\int_0^1\frac{1}{1+x^2}\,dx=n\arctan 1=n\frac{\pi}{4},$$

from which it follows that

$$\sum_{0\le k<n}\frac{1}{n^2+k^2}=\frac{1}{n^2}\sum_{0\le k<n}g\left(\frac{k}{n}\right)\sim\frac{\pi}{4n} \qquad n\to+\infty. \tag{14.6.a}$$

Since g is of class $\mathscr{C}^2$ one can use relation (14.15.b) from which it follows that

$$\begin{aligned}\sum_{0\le k<n}g\left(\frac{k}{n}\right) &= n\int_0^1 g(x)\,dx-\frac{g(1)-g(0)}{2}+O\left(\frac{1}{n}\right)\\ &= n\frac{\pi}{4}+\frac{1}{4}+O\left(\frac{1}{n}\right) \qquad n\to+\infty\end{aligned}$$

and so

$$\sum_{0\le k<n}\frac{1}{n^2+k^2}=\frac{1}{n^2}\sum_{k=1}^{n}g\left(\frac{k}{n}\right)=\frac{\pi}{4n}+\frac{1}{4n^2}+O\left(\frac{1}{n^3}\right) \quad n\to+\infty. \tag{14.6.b}$$

Consider now the case $n=10$. The formula (14.6.a) then yields

$$\sum_{0\le k<10}\frac{1}{10^2+k^2}\approx\frac{\pi}{4\times 10}\approx 0.0785398,$$

while relation (14.6.b) yields

$$\frac{\pi}{4 \times 10} + \frac{1}{4 \times 10^2} \approx 0.0810398.$$

Actually, carrying out the calculations, one finds that

$$\sum_{0 \le k < 10} \frac{1}{10^2 + k^2} \approx 0.0809981.$$

One sees how relation (14.6.b) turns out to be decidedly more precise than relation (14.6.a).

Problem 14.7. Estimate the following sums to at worst $O(1/n)$:

1. $\displaystyle\sum_{0 \le k < n} \frac{1}{1 + k^2/n^2}$;
2. $\displaystyle\sum_{0 \le k < n} \frac{1}{1 + k^3/n^3}$.

Solution.

1. The first sum coincides with $\displaystyle\sum_{1 \le k < n} g(k/n)$ where $g(x) = \dfrac{1}{1+x^2}$; on applying relation (14.15.b) one obtains

$$\begin{aligned}
\frac{1}{1+\frac{k^2}{n^2}} &= n \int_0^1 g(x)\, dx - \frac{1}{2}\, [g]_0^1 + O(1/n) \\
&= n \arctan 1 - \frac{1}{2}\left(\frac{1}{2} - 1\right) + O(1/n) \\
&= n\frac{\pi}{4} + \frac{1}{4} + O(1/n) \quad n \to +\infty.
\end{aligned}$$

2. Here the sum coincides with $\displaystyle\sum_{1 \le k < n} g(k/n)$ where $g(x) = \dfrac{1}{1+x^3}$; on applying relation (14.15.b) one obtains

$$\begin{aligned}
\frac{1}{1+\frac{k^3}{n^3}} &= n \int_0^1 g(x)\, dx - \frac{1}{2}\, [g]_0^1 + O(1/n) \\
&= n \left[\frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}} + \frac{1}{3}\log(1+x) - \frac{1}{6}\log\left[1 - x + x^2\right] \right]_0^1 - \\
&\quad - \frac{1}{2}\left(\frac{1}{2} - 1\right) + O(1/n) \\
&= \frac{n}{18}\left(2\sqrt{3}\pi + \log 64\right) + \frac{1}{4} + O(1/n) \quad n \to +\infty.
\end{aligned}$$

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