European Financial Management, Vol. 9, No. 2, 2003, 251–259

Professional Forum

Simple Construction of the Efficient Frontier

David Feldman

Department of Business Administration, School of Management, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel
email: feldmand@bgumail.bgu.ac.il

Haim Reisman

Davidson Faculty of Management, The Technion – Israel Institute of Technology, Technion City, Haifa 32000, Israel
email: reisman@tx.technion.ac.il

Abstract

We provide simple methods of constructing known results. At the core of our methods is the identification of a simple concise basis that spans the Capital Market Line (CML). We show that a portfolio whose risky assets weights are the product of the inverse variance-covariance matrix of (nonredundant) security rates of return times the vector of the excess expected rates of return over the risk-free rate is a CML portfolio. This portfolio and the risk-free security span the CML. In addition, with this basis, there is immediate construction of the efficient frontier of risky assets (the 'hyperbola'), 'tangency' portfolios, 'reflection' portfolios, and a CAPM relationship. Our method is quick and simple. It is easy to derive, teach, implement, interpret, and remember.

Keywords: *portfolio frontier*; *efficient frontier*; *capital market line*; *asset pricing*.

JEL classification: *G*10, *G*11, *G*12

We thank Shulamith Gross, Shmuel Kandel, and Linda Hutz Pesante for helpful discussions. We also thank the participants of The Financial Education Association Annual Conference, Baltimore, USA, The European Financial Management Association (EFMA) Annual Meeting, London, England, The Annual Conference of the Multinational Finance Society, Paphos, Cyprus, and The Northern Finance Association Annual Conference, Banff, Canada.

1. Introduction

We provide simple methods of constructing known results. At the core of our methods is the identification of a simple concise basis that spans the Capital Market Line (CML). We show that the portfolio whose risky assets weights are the product of the inverse variance-covariance matrix of (nonredundant) security rates of return times the vector of excess expected rates of return over the risk-free rate is a CML portfolio. This portfolio and the risk-free security together span the CML. In addition, with this basis, there is immediate construction of the efficient frontier of risky assets (the 'hyperbola'), 'tangency' portfolios, 'reflection' portfolios, and a CAPM relationship.

In an economy with risky securities and one risk-free security, the CML is the set of mean-variance dominating portfolios, or the Efficient Portfolio Frontier. The CML is one of the most essential tools in the study and practice of finance in general and investments in particular. Investors who hold portfolios that are not on the CML bear risks they are not compensated for. Following Markowitz's (1952) Portfolio Theory, current methods of constructing the CML usually involve several stages. See, for example, Elton and Gruber (1995), Sharpe *et al.* (1999), and Bodie *et al.* (2000), who use a method devised by Elton *et al.* (1976). First, they generate the mean-variance Portfolio Frontier of risky assets by Lagrange constrained optimization methods.[1] Then, they identify the 'tangency portfolio'—the risky frontier portfolio that is also on the CML—or its reflection. Finally, they span the CML using the tangency portfolio and the risk-free security. Ingersoll (1987) and Huang and Litzenberger (1988), for example, following Merton (1972), use Lagrange constrained optimization methods to generate the efficient frontier. The most efficient current procedure that we are aware of is in Benninga (2000).[2] It seems that current procedures of deriving and constructing the CML are not sufficiently simple to be included in most graduate investments books.

Our procedure for constructing the CML, however, does not require the identification or use of the tangency portfolio; it always generates a CML portfolio (unlike the tangency portfolio, which might be on the reflection of the CML); it works where there is no finite tangency portfolio (when the CML is an asymptote to the portfolio frontier of risky assets hyperbola); and it is simpler to derive and implement. To obtain our results we, roughly speaking, convert a three-variable problem to a single variable one. To do that, we first note that there is always a CML portfolio whose 'shadow price' is one. Second, we focus our analysis on the risky securities only, whose weights do not have to add up to one. By focusing on a CML portfolio with desirable properties, we are able to make shortcuts in the derivation and computation. We develop our model in Section 2, with the main result in Proposition 1, and conclude in Section 3.

2. Constructing the CML

We will use the following notational conventions: constants and variables will be typed in italic (slanted) font, operators and functions in regular straight font, and vectors and matrices in boldface (dark) straight font.

---

[1] This frontier is an hyperbola in the mean-standard deviation space, see Merton (1972).

[2] Jensen (2002) constructs the efficient frontier under short sales and other linear asset allocation constraints using an efficient linear programming procedure.

Assume a market with N risky securities and one risk-free security. Let R_i, $i = 1, \ldots, N$, be the random rates of return of the risky securities $1, \ldots, N$, and let R_F, be the constant rate of return of a risk-free security. We do not specify the probability distributions of the rates of return but assume means and variances that are real finite numbers. We denote the variance-covariance matrix of the risky securities by $\mathbf{V}$, a positive definite matrix. This implies that there are no redundant securities.

Let a portfolio $\mathbf{a}$, of the $N+1$ market securities, be an $N \times 1$ vector of real numbers, a_i, $i = 1, \ldots, N$, where a_i is the 'weight' of (risky) security i in the portfolio. Portfolio $\mathbf{a}$ has some positive level of wealth, and its rate of return is

$$R_{\mathbf{a}} = \left(1 - \sum_1^N a_i\right) R_F + \sum_1^N a_i R_i, \tag{1}$$

or, in matrix notation,

$$R_{\mathbf{a}} = (1 - \mathbf{a}^{\mathbf{T}}\mathbf{1}) R_F + \mathbf{a}^{\mathbf{T}}\mathbf{R}, \tag{2}$$

where $\mathbf{1}$ is an $N \times 1$ vector of ones, $\mathbf{R}$, $\mathbf{R} = (R_1, \ldots, R_N)$, is an $N \times 1$ vector of the risky securities' rates of return, and superscript $\mathbf{T}$ denotes the transpose operator. Thus, $(1 - \Sigma_1^N a_i)$, or $(1 - \mathbf{a}^{\mathbf{T}}\mathbf{1})$, is the weight of the risk-free asset in $\mathbf{a}$; and the weights of all the $N+1$ securities in portfolio $\mathbf{a}$ add up to one.[3] A negative weight indicates a short position, and a weight greater than one indicates an investment that is greater than the net investment in the portfolio.

The expected (mean) rate of return of portfolio $\mathbf{a}$ is

$$\mathrm{E}(R_{\mathbf{a}}) = \left(1 - \sum_{i=1}^N a_i\right) R_F + \sum_{i=1}^N a_i \mathrm{E}(R_i), \tag{3}$$

where $\mathrm{E}(\bullet)$ is the expectations operator; or in matrix notation,

$$\mathrm{E}(R_{\mathbf{a}}) = (1 - \mathbf{a}^{\mathbf{T}}\mathbf{1}) R_F + \mathbf{a}^{\mathbf{T}}\mathbf{E}, \tag{4}$$

where $\mathbf{E}$ is an $N \times 1$ vector of the risky securities expected rates of return, or, simpler

$$\mathrm{E}(R_{\mathbf{a}}) = R_F + \mathbf{a}^{\mathbf{T}}\mathbf{X}, \tag{5}$$

where $\mathbf{X}$ is the $N \times 1$ vector of excess risky security expected rates of return over R_F. That is, $(X_1, X_2, \ldots, X_N) = (\mathrm{E}(R_1) - R_F, \mathrm{E}(R_2) - R_F, \ldots, \mathrm{E}(R_N) - R_F)$, or $\mathbf{X} = \mathbf{E} - \mathbf{1}\boldsymbol{R}_F$.

The variance of the rate of return of portfolio $\mathbf{a}$ is

$$\mathrm{VAR}(R_{\mathbf{a}}) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j \sigma_{ij}, \tag{6}$$

where $\sigma_{ij} \triangleq \mathrm{COV}(R_i, R_j)$, is the covariance of R_i and R_j. As usual, we denote the variance of the random variable R_i by $\sigma_{ii} = \sigma_i^2$, thus $\sigma_{\mathbf{a}}^2 = \mathrm{VAR}(R_{\mathbf{a}})$, and the standard deviation of R_i as σ_i. In matrix notation, the variance of the rate of return of portfolio $\mathbf{a}$ is simply

$$\mathrm{VAR}(R_{\mathbf{a}}) = \mathbf{a}^{\mathbf{T}}\mathbf{V}\mathbf{a}. \tag{7}$$

[3] Our unconventional way of denoting a portfolio that includes $N+1$ securities by only an N vector of risky securities facilitates much simpler presentation and analysis. The *risk-free* security (the $N+1$ security) complements the portfolio weight to 1.

Following Markowitz's (1952), Portfolio Theory, we use the term *Portfolio Frontier* for the set of portfolios that have the lowest standard deviation for any given expected return. Thus, a portfolio **a** is a Portfolio Frontier portfolio if, for some expected return g, **a** is a solution of the following problem:

$$\operatorname*{Min}_{\mathbf{a}} \frac{1}{2}\mathbf{a}^{\mathbf{T}}\mathbf{V}\mathbf{a} \tag{P1}$$

subject to

$$\mathbf{a}^{\mathbf{T}}\mathbf{X} = x,$$

where for simplicity we define x, $x = g - R_F$ to be the excess return of the exogenous desired return, g, over the risk-free rate.

As is customary, we call the set of Portfolio Frontier portfolios with the highest expected return for any given standard deviation, the *Efficient Frontier*. In the presence of a risk-free asset, the Efficient Frontier becomes a straight line—the *Capital Market Line (CML)*. We will call a portfolio on the CML a *CML portfolio*. Thus, the CML portfolios are the efficient frontier portfolios and are called efficient portfolios.

Our main result is the spanning of the CML. The basis for spanning is the risk-free security and a portfolio **a**, whose risky assets weights are the inverse variance-covariance matrix of the risky securities' rates of return times a vector of excess risky securities' expected rates of return over the risk-free rate. That is, $\mathbf{a} = \mathbf{V}^{-1}\mathbf{X}$.

In other words, any CML portfolio with an expected rate of return $\mathrm{E}(R_{\mathbf{p}})$ can be constructed using a combination of the risk-free security and portfolio **a**. That is, there is a weight w, such that

$$R_{\mathbf{p}} = (1 - w)R_F + wR_{\mathbf{a}}. \tag{8}$$

The weight w is

$$w = \frac{\mathrm{E}(R_{\mathbf{p}}) - R_F}{\mathrm{E}(R_{\mathbf{a}}) - R_F}, \tag{9}$$

and we have

$$\mathrm{E}(R_{\mathbf{p}}) = (1 - w)R_F + w\mathrm{E}(R_{\mathbf{a}}), \tag{10}$$

$$\sigma_{\mathbf{p}}^2 = w^2\sigma_{\mathbf{a}}^2, \tag{11}$$

and (note that because **a** is a CML portfolio w is nonnegative)

$$\sigma_{\mathbf{p}} = w\sigma_{\mathbf{a}}. \tag{12}$$

Alternatively, we can write the relation between $\mathrm{E}(R_{\mathbf{p}})$ and $\sigma_{\mathbf{p}}$ using the CML equation:

$$\mathrm{E}(R_{\mathbf{p}}) = R_F + \frac{\mathrm{E}(R_{\mathbf{a}}) - R_F}{\sigma_{\mathbf{a}}}\sigma_{\mathbf{p}}.^4 \tag{13}$$

The following proposition states our main result.

[4] Note that for our solution, portfolio **a**, we can write the CML equation $\mathrm{E}(R_{\mathrm{p}}) = R_F + \sigma_{\mathbf{a}}\sigma_{\mathbf{p}}$.

Proposition 1: *Portfolio* $\mathbf{a}^*$, $\mathbf{a}^* = \mathbf{V}^{-1}\mathbf{X}$, *is a CML portfolio.*[5]

Proof: We need to show that $\mathbf{a}^*$ is a solution to (P1) for some x. We will use a Lagrange constrained optimisation method. The Lagrangean is

$$\mathrm{L}(\mathbf{a}, \lambda) = \frac{1}{2}\mathbf{a}^{\mathbf{T}}\mathbf{V}\mathbf{a} + \lambda(x - \mathbf{a}^{\mathbf{T}}\mathbf{X}). \tag{P2}$$

Because $\mathbf{V}$ is a positive definite matrix, the first-order conditions are necessary and sufficient for a global minimum. The first-order conditions (equating the Lagrangean's first partial derivatives with respect to $\mathbf{a}$ and λ to zero) yield

$$\mathbf{a}^{\mathbf{T}}\mathbf{X} = x, \quad \text{and} \quad \mathbf{V}\mathbf{a} = \lambda\mathbf{X}. \tag{14}$$

For $x = \mathbf{X}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{X}$, the solution[6] to the above equations is

$$\mathbf{a} = \mathbf{V}^{-1}\mathbf{X}, \quad \text{and} \quad \lambda = 1. \tag{15}$$

Thus, $\mathbf{a}^* = \mathbf{V}^{-1}\mathbf{X}$ is a Portfolio Frontier portfolio. In addition, $\mathbf{a}^*$ is a CML portfolio, and not a CML reflection portfolio, because its excess expected return over the risk-free rate $\mathbf{a}^{*\mathbf{T}}\mathbf{X} = \mathbf{X}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{X}$, [see Equation (5)] is positive because $\mathbf{V}$ is positive definite. □

Currently the method of constructing the CML is through the tangency portfolio—the portfolio that is both on the CML, or the CML reflection, and on the portfolio frontier of risky assets hyperbola (see, for example, Bodie *et al.* (2000), Elton and Gruber (1995), and Benninga (2000)). We will now identify the tangency and reflection portfolios within our model. Note that in contrast to these common methods, our method of constructing the CML does not require the use of tangency or reflection portfolios.

Corollary 1. i) [Benninga (2000)[7]] *If* $\mathbf{a}^{\mathbf{T}}\mathbf{1} \neq 0$, *the weights of the tangency portfolio,* $\mathbf{t}_{R_F}$, *are portfolio* $\boldsymbol{a}$*'s weights re-scaled to add up to one. That is,*

$$\mathbf{t}_{R_F} = (\mathbf{a}^{\mathbf{T}}\mathbf{1})^{-1}\mathbf{a}. \tag{16}$$

ii) *If* $\mathbf{a}^{\mathbf{T}}\mathbf{1} > 0$, ($\mathbf{a}^{\mathbf{T}}\mathbf{1} < 0$), *the tangency portfolio is a CML (CML reflection) portfolio.*
iii) *If the sum of weights of the risky securities in* $\mathbf{a}$ *is zero, or equivalently* $\mathbf{a}^{\mathbf{T}}\mathbf{1} = 0$, *there is no finite tangency portfolio (the CML is an asymptote of the efficient frontier hyperbola).*

Proof. The tangency portfolio is the only CML portfolio either with no weight in R_F or, equivalently, with weights in the risky securities that add up to one. This proves (i). To demonstrate (ii), note that because the excess expected return (EER) of portfolio $\mathbf{a}$ over the risk-free rate is positive (see the proof of Proposition 1), the EER of $\mathbf{t}_{R_F}$, which is that of $\mathbf{a}$ divided by $\mathbf{a}^{\mathbf{T}}\mathbf{1}$, is positive (negative) if $\mathbf{a}^{\mathbf{T}}\mathbf{1} > 0$,

---

[5] Compare this result with Huang and Litzenberger (1988), Equation 3.18.1.

[6] We make use of the insights that x could be any constant and that $\mathbf{a}$, and λ are first-degree homogeneous in x. That is, if ($\mathbf{a}$, λ) is a solution for a constraint value x, then the solution for a constraint value cx, where c is some constant, is ($c\mathbf{a}$, $c\lambda$). Thus, there is an x for which $\lambda = 1$.

[7] Benninga (2000), Chapter 9, gets Equation (16) using a different method.

($\mathbf{a}^{\mathbf{T}}\mathbf{1} < 0$), which implies that portfolio $\mathbf{t}_{R_F}$, is on the CML (CML reflection). The proof of (iii) is in Merton (1972). □

In the case where the tangency portfolio is on the CML reflection, one needs to use the following remark in order to identify a CML portfolio.

Remark. Let $\boldsymbol{a}$ *be a Portfolio Frontier portfolio. Then* $-\boldsymbol{a}$ *is the 'reflection' portfolio, that is, a Portfolio Frontier portfolio with the same variance but different mean.* Also, $\mathrm{E}(R_{-\mathbf{a}}) = 2R_F - \mathrm{E}(R_{\mathbf{a}})$.

Proof. There is only one Portfolio Frontier portfolio different from $\mathbf{a}$ but with the same variance as $\mathbf{a}$. If $\mathbf{a}$ is on the CML, the reflection portfolio is on the negative slope reflection of the CML and vice versa. Because portfolio $-\mathbf{a}$ has the same variance as portfolio $\mathbf{a}$, $-\mathbf{a}$ must be the reflection portfolio. The mean rate of return of portfolio $-\mathbf{a}$ follows from Equation (5). □

We will demonstrate that a CAPM-type relationship follows immediately from our simple construction. Qualitatively, the above first-order conditions say that the covariances of a CML portfolio with each asset in the market divided by the EER of that asset over the risk-free rate are all equal. This is, in fact, a pricing relation: a measure of risk per unit of EER is the same for all assets. The value of this ratio is also the shadow price of the problem, that is, the marginal change in the variance of the CML portfolio per unit change in its expected return. Quantitatively, the N first-order conditions with respect to $\mathbf{a}$ in Equation (14) (NFOCa) say that for any CML portfolio $\mathbf{a}$ and any risky security i, we have

$$\frac{\mathrm{COV}(R_i, R_{\mathbf{a}})}{\mathrm{E}(R_i) - R_F} = \lambda,\ \forall i,\ i = 1, 2, \ldots, N. \tag{17}$$

In fact, this relation holds for the CML portfolio itself also, that is, for $R_i = R_{\mathbf{a}}$. To obtain this relation, left multiply NFOCa ($\mathbf{Va} = \lambda\mathbf{X}$) by $\mathbf{a}^{\mathbf{T}}$ and get

$$\mathbf{a}^{\mathbf{T}}\mathbf{Va} = \mathbf{a}^{\mathbf{T}}\mathbf{X}\lambda. \tag{18}$$

The left-hand side is the variance of portfolio $\mathbf{a}$, and the right-hand side is the EER of the CML portfolio $\mathbf{a}$ over the risk-free rate (see Equation (5)), times the Lagrange multiplier. After rearranging, Equation (18) becomes

$$\frac{\mathrm{VAR}(R_{\mathbf{a}})}{\mathrm{E}(R_{\mathbf{a}}) - R_F} = \lambda. \tag{19}$$

If we divide equation (19) by equation (17) and rearrange, we get

$$\mathrm{E}(R_i) = R_F + \beta_i[\mathrm{E}(R_{\mathbf{a}}) - R_F],\ \forall i,\ i = 1, 2, \ldots, N, \tag{20}$$

where we define $\beta_i = \dfrac{\mathrm{COV}(R_i, R_{\mathbf{a}})}{\mathrm{VAR}(R_{\mathbf{a}})}, \forall i, i = 1, 2, \ \ldots, N,$

which is a CAPM, or a Security Market Line, type relation, for any market security i with respect to any CML portfolio $\mathbf{a}$. We find it interesting that our very simple structure is sufficient for capturing the pricing essence of the CAPM and constructing a CAPM-type relation. We have thus proved the following Lemma.

Lemma 1. Security expected rates of returns obey a CAPM-type restriction: they are equal to the risk-free rate plus the security beta, with respect to any CML portfolio *a, times portfolio a's excess expected return over the risk-free rate, as specified in Equation (20).* □

Finally, we demonstrate that a simple construction of the efficient frontier of risky assets only (the 'hyperbola'), is a special case of our results. To span the hyperbola we need two distinct hyperbola portfolios. We identify these portfolios by choosing two special cases of our solution (portfolio **a**), where each refers to a particular value of the risk-free rate. One portfolio, $\mathbf{t}_\infty$, is the tangency portfolio (as constructed in Corollary 1) corresponding to our solution under a risk-free rate that is infinitely high (or low). Thus, $\mathbf{t}_\infty$ is the global minimum variance portfolio. The second portfolio, $\mathbf{t}_0$, is the tangency portfolio corresponding to our solution under a risk-free rate of zero. Thus, $\mathbf{t}_0$ is on a tangent line that passes through the origin. If, however, the global minimum variance portfolio $\mathbf{t}_\infty$ has zero expected value (that is, if $\mathbf{1^TV^{-1}E}=0$), $\mathbf{t}_0$ does not exist: there will be no tangent to the hyperbola that goes through the origin.[8] In this case, we can choose a second hyperbola portfolio to be the tangency portfolio corresponding to our solution under any finite, non-zero, risk-free rate value. We state these results in the following corollary.

Corollary 2. If $\mathbf{1^TV^{-1}E}\neq 0$, *the tangency portfolios* $\mathbf{t}_\infty$, (*which corresponds to* $\mathbf{a}=\mathbf{V^{-1}1}$) *and* $\mathbf{t}_0$, (*which corresponds to* $\mathbf{a}=\mathbf{V^{-1}E}$) *span the efficient frontier of risky assets only—the 'hyperbola'. If* $\mathbf{1^TV^{-1}E}=0$, *then the tangency portfolios* $\mathbf{t}_\infty$ *and* $\mathbf{t_c}$ *(which correspond to* $\mathbf{a}=\mathbf{V^{-1}}(\mathbf{E}-\mathbf{1}c)$, *where c is a finite non-zero constant) span the hyperbola.*[9]

Proof. Merton (1972) showed that any two nonidentical hyperbola portfolios span the hyperbola. By definition, a tangency portfolio is an hyperbola portfolio. We obtain this corollary by substituting the appropriate values for R_F in Corollary 1. In the case of $\mathbf{t}_\infty$, we take the limit of R_F as it goes to infinity (or negative infinity). □

We now offer numerical examples to demonstrate our results.

Example. Let $R_F=0.1$, $\mathrm{E}(R_1)=0.3$, $\mathrm{E}(R_2)=0.2$, $\sigma_1=0.5$, $\sigma_2=0.3$, $\sigma_{12}=0$. Thus, we have,

$$\mathbf{V}=\begin{pmatrix}0.5^2 & 0\\ 0 & 0.3^2\end{pmatrix}, \mathbf{V}^{-1}=\begin{pmatrix}(0.5^2)^{-1} & 0\\ 0 & (0.3^2)^{-1}\end{pmatrix}, \text{ and } \mathbf{X}=\begin{pmatrix}0.2\\ 0.1\end{pmatrix}.$$

- *Determining* **a**: $\mathbf{a}=\mathbf{V}^{-1}\mathbf{X}=\begin{pmatrix}(0.5^2)^{-1} & 0\\ 0 & (0.3^2)^{-1}\end{pmatrix}\begin{pmatrix}0.2\\ 0.1\end{pmatrix}=\begin{pmatrix}0.800\\ 1.111\end{pmatrix}.$
- *The mean and standard deviation of the rate of return of* **a**:
- $\mathrm{E}(R_\mathbf{a})=R_F+\mathbf{a^TX}=0.1+0.8\times 0.2+1.111\times 0.1=0.371$,
  $\sigma_\mathbf{a}=(\mathbf{a^TVa})^{0.5}=(0.8^2\times 0.5^2+1.111^2\times 0.3^2)^{0.5}=0.521.$
- *Spanning the CML*: $\mathrm{E}(R_\mathbf{p})=0.1+\dfrac{0.371-0.1}{0.521}\sigma_\mathbf{p}=0.1+0.521\sigma_\mathbf{p}.$

---

[8] Instead, the hyperbola's two asymptotes pass through the origin, see Merton (1972).

[9] Note that from Corollary 1, $\mathbf{t}_\infty=(\mathbf{1^TV^{-1}1})^{-1}\mathbf{V^{-1}1}$, $\mathbf{t}_0=(\mathbf{1^TV^{-1}E})^{-1}\mathbf{V^{-1}E}$, and $\mathbf{t}_c=[\mathbf{1^TV^{-1}(E-1c)}]^{-1}\mathbf{V^{-1}(E-1c)}$.

- *Identifying the tangency portfolio*: Let $\mathbf{t}_{R_F}$ be the tangency portfolio. Then,
- $\mathbf{t}_{R_F} = (\mathbf{a}^{\mathbf{T}}\mathbf{1})^{-1}\mathbf{a} = \dfrac{1}{0.8+1.111}\begin{pmatrix}0.800\\1.111\end{pmatrix} = \begin{pmatrix}0.419\\0.581\end{pmatrix}$.
- *Spanning the efficient frontier of risky assets only, the 'hyperbola'*:

$$\mathbf{V^{-1}1} = \begin{pmatrix}(0.5^2)^{-1} & 0\\ 0 & (0.3^2)^{-1}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4.000\\11.111\end{pmatrix},$$

$$\mathbf{V^{-1}E} = \begin{pmatrix}(0.5^2)^{-1} & 0\\ 0 & (0.3^2)^{-1}\end{pmatrix}\begin{pmatrix}0.3\\0.2\end{pmatrix} = \begin{pmatrix}1.200\\2.222\end{pmatrix}, \text{ thus (see footnote 9),}$$

$$\mathbf{t}_\infty = \frac{1}{4+11.111}\begin{pmatrix}4.000\\11.111\end{pmatrix} = \begin{pmatrix}0.265\\0.735\end{pmatrix}, \text{ and } \mathbf{t}_0 = \frac{1}{1.2+2.222}\begin{pmatrix}1.200\\2.222\end{pmatrix} = \begin{pmatrix}0.351\\0.649\end{pmatrix}.$$

3. Conclusion

Our approach always identifies a CML portfolio while the 'tangency portfolio approach' (TPA) might identify a CML reflection portfolio, making the procedure a bit more cumbersome. In addition, the TPA does not handle well the case where the sum of weights of the portfolio risky securities is zero. Thus, though the popular TPA is an *ex ante* natural approach, it appears to have no advantage over our approach.

The difference between our approach and the one that Huang and Litzenberger (HL) choose (following Merton (1972)) is as follows. HL obtain a closed form formula for a CML portfolio with any given expected return, whereas we, first, identify a simple basis for spanning any CML portfolio, and then use a two fund basis to obtain a particular CML portfolio. Our basis is easy to derive and remember, especially in comparison to the formulas in HL. In identifying a particular CML portfolio, our method uses two very easy steps while HL's requires a single more difficult step.

In a market with N risky securities and one risk-free security, we demonstrated a simple and concise way to construct the Capital Market Line, the 'tangency' and 'reflection' portfolios, and a CAPM-type relation in terms of any CML portfolio. We also demonstrated an easy method for constructing the efficient frontier of risky assets only, the 'hyperbola', as a special case of our results. The simplicity of the derivations and the methods provides economic insight and, hopefully, makes these procedures more accessible to students and practitioners alike.

References

Benninga, S., *Financial Modeling,* (Cambridge MA: MIT Press, 2nd ed., 2000).

Bodie, Z., Kane, A. and Marcus, A., *Investments* (New York: McGraw-Hill, 4th ed., 2000).

Elton, E. J. and Gruber, M. J., *Modern Portfolio Theory And Investment Analysis* (New York: Wiley, 5th ed., 1995).

Elton, E. J., Gruber, M. J. and Padberg, M. D., 'Simple criteria for optimal portfolio selection', *Journal of Finance*, Vol. 31, 1976, pp. 1341–1357.

Huang, C-F. and Litzenberger, R., *Foundations for Financial Economics* (New York: North-Holland, 1988).

Ingersoll, J. E., *Theory of Financial Decision Making* (Lanham MD: Rowman & Littlefield, 1987).

Jensen, B. A., 'Mean variance efficient portfolios by linear programming', *Working Paper,* (Copenhagen Business School, 2002).

Markowitz, H., 'Portfolio selection', *Journal of Finance*, Vol. 7, 1952, pp. 77–91.

Merton, R., 'An analytical derivation of the efficient portfolio frontier', *Journal of Financial and Quantitative Analysis*, Vol. 7, 1972, pp. 1851–1872.

Sharpe, W. F., Alexander, G. J. and Bailey, J. V., *Investments* (Englewood Cliffs NJ: Prentice-Hall, 6th ed., 1999).